Geometry of spaces with a synthetic lower curvature bound

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Abstract

In recent years, several major breakthroughs in the field of geometry have been closely linked to the investigation of two pivotal topics: curvature and singularities. The objective of this thesis is to explore the geometric and structural properties of singular spaces with a synthetic lower curvature bound. The crucial tool employed in this research field is the theory of *optimal transport*, which allows to develop an intrinsic approach to curvature bounds and to study many singular spaces from an abstract viewpoint.

In the seminal papers [Stu06a, Stu06b, LV09], Sturm and Lott–Villani introduced a synthetic notion of curvature-dimension bounds in the non-smooth setting of *metric measure spaces*. This condition, called $\mathsf{CD}(K, N)$, is formulated in terms of the optimal transport interpolation of measures and consists in a convexity property of the Rényi entropy functionals along *Wasserstein geodesics*. The $\mathsf{CD}(K, N)$ condition represents a lower (Ricci) curvature bound by K and an upper bound on the dimension by N, and it is coherent with the Riemannian setting.

The $\mathsf{CD}(K, N)$ spaces enjoy different metric and geometric features, nevertheless, their study by means of classical analytic tools presents non-trivial difficulties, mainly due to their complex geometric structure. In Paper 1 we present several examples of singular $\mathsf{CD}(K, N)$ spaces, having different dimensions in different regions. As a consequence, we show how basic rigidity properties, such as weak *non-branching* conditions, may fail in this setting, despite the curvature-dimension bound.

One of the main merits of the $\mathsf{CD}(K, N)$ condition is that it is sufficient to deduce geometric inequalities that hold in the smooth setting. A notable example is the generalized *Brunn–Minkowski inequality*, called $\mathsf{BM}(K, N)$. In Papers 2 and 3, we obtain two partial results in the direction of proving the equivalence between $\mathsf{BM}(K, N)$ and $\mathsf{CD}(K, N)$. Firstly, we prove it in the setting of *weighted Riemannian manifolds*. Secondly, we show that, in the general framework of *essentially non-branching* metric measure spaces, the $\mathsf{CD}(K, N)$ condition is equivalent to a more stringent version of the $\mathsf{BM}(K, N)$ inequality, that we call *strong Brunn-Minkowski* $\mathsf{SBM}(K, N)$.

While the $\mathsf{CD}(K, N)$ condition is equivalent to a lower curvature bound in the Riemannian and Finsler settings, a similar result does not hold for *sub-Riemannian* and *sub-Finsler* manifolds. In Papers 4 and 5, we show how the $\mathsf{CD}(K, N)$ condition is not well-suited to characterize curvature in these frameworks. On the one hand, we show that every *almost-Riemannian* manifold, with dimension 2 or *strongly regular*, equipped with a *smooth measure*, does not satisfy the $\mathsf{CD}(K, N)$ condition for every K and N. On the other hand, we prove the failure of the $\mathsf{CD}(K, N)$ condition for *smooth* sub-Finsler manifolds and in the specific case of the *Heisenberg group*, under weaker regularity assumptions.

In Papers 6 and 7, we study the extension of the $\mathsf{CD}(K, N)$ condition where the dimensional bound N is negative (introduced by Ohta in [Oht16]), considering metric measure spaces with *quasi-Radon* reference measures, as a natural framework for its analysis. In particular, we prove the stability of the $\mathsf{CD}(K, N)$ condition with respect to a suitably refined measured Gromov-Hausdorff convergence, that controls the behavior of singular points of the reference measures. Moreover, we prove two remarkable features for the $\mathsf{CD}(K, N)$ condition (with negative N), namely the existence of *optimal transport maps* and the *local-to-global* property.

Contents

Summary

In Riemannnian manifolds, curvature represents a fundamental tool for many results in geometric analysis. In particular, a lower bound on the Ricci curvature tensor allows to deduce heat kernel estimates, isoperimetric and Sobolev inequalities and many other geometric properties. However, the family of Riemannian manifolds, satisfying a given Ricci lower bound, is not closed under Gromov-Hausdorff convergence, nor with respect to any other reasonable notion of convergence. This phenomenon is caused by the possible arising of singularities at the limit structure, which would prevent to evaluate the Ricci curvature. This observation, led to the challenge to establish a generalized notion of lower Ricci curvature bounds for singular spaces.

In their seminal papers [Stu06a, Stu06b, LV09], Sturm and Lott–Villani introduced a synthetic notion of *curvature-dimension bounds*, denoted by $\mathsf{CD}(K,N)$, with $K \in \mathbb{R}$, $N \in (1,\infty)$, in the non-smooth setting of metric measure spaces, i.e. complete and separable metric spaces endowed with a locally finite Borel measure. This property is formulated using the theory of *optimal transport* and, in the setting of Riemannian manifolds, is equivalent to having Ricci curvature bounded below by K and dimension bounded above by N. More precisely, the $\mathsf{CD}(K,N)$ condition consists in a convexity property of the Rényi entropy functionals along Wasserstein geodesics.

In the setting of metric measure spaces, the $\mathsf{CD}(K,N)$ condition is stable with respect to the (pointed) measured Gromov-Hausdorff convergence, cf [Stu06b, LV09, GMS15]. Moreover, $\mathsf{CD}(K,N)$ spaces (i.e. spaces satisfying the $\mathsf{CD}(K,N)$ condition) enjoy different geometric features which hold in the smooth setting, such as the scaling [Stu06b], tensorzation [BS10] and local-to-global properties [BS10, CM21]. Nevertheless, the study of $\mathsf{CD}(K,N)$ spaces by means of classical analytic tools presents non-trivial difficulties, mainly due to their complex geometric structure.

In Paper 1, we present different examples of singular $\mathsf{CD}(K,N)$ spaces, highlighting how basic rigidity properties could fail in this setting. In particular, we obtain examples of $\mathsf{CD}(0,N)$ spaces having different (topological and Hausdorff) dimensions in different regions, observing in particular that the topological splitting may fail for $\mathsf{CD}(0,N)$ spaces. As other remarkable consequences, we deduce that any reasonable *non-branching* condition may fail in $\mathsf{CD}(0,N)$ spaces and that the existence of an *optimal transport map*, between two absolutely continuous marginals, is not guaranteed by the $\mathsf{CD}(0,N)$ condition, without requiring a non-branching assumption. Finally, we are able to draw several notable conclusions regarding the so-called *strict* CD condition.

One of the most important merits of the $\mathsf{CD}(K,N)$ condition is that it is sufficient to deduce geometric and functional inequalities that hold in the smooth setting. A remarkable example is the so-called *Brunn–Minkowski inequality*, whose classical version in $\mathbb{R}^n$ states that

$$\mathscr{L}^n\left((1-t)A+tB\right)^{\frac{1}{n}} \geq (1-t)\mathscr{L}^n(A)^{\frac{1}{n}} + t\mathscr{L}^n(B)^{\frac{1}{n}}, \quad \forall t \in [0,1],$$

for every two nonempty compact sets $A,B \subset \mathbb{R}^n$. It was already observed by McCann [McC94, McC97] that, in the Euclidean setting, this inequality can be proved using the convexity property of the Rényi entropy in the Wasserstein space. Following the same argument, Sturm [Stu06b] proved that a $\mathsf{CD}(K,N)$ space supports a generalized version of the Brunn–Minkowski inequality,

denoted $\mathsf{BM}(K,N)$, where the Minkowski sum of A and B is replaced with the set of t-midpoints $M_t(A,B)$ and the convex interpolation is done accordingly to the distortion coefficients $\tau_{K,N}^{(t)}$.

While the curvature-dimension bound $\mathsf{CD}(K,N)$ allows to deduce the Brunn-Minkowski inequality $\mathsf{BM}(K,N)$, it is less clear whether assuming the validity of $\mathsf{BM}(K,N)$ is sufficient to deduce the $\mathsf{CD}(K,N)$ condition. The interest in proving this equivalence stems from the fact that it would provide a characterization of the curvature-dimension condition *without* the need of optimal transport techniques. In fact, a remarkable feature of the Brunn–Minkowski inequality is that its formulation is very simple and does not refer to the Wasserstein interpolation of measures. The equivalence between $\mathsf{CD}(K,N)$ and $\mathsf{BM}(K,N)$ would also provide an alternative proof of the *globalization theorem* by Cavalletti and Milman [CM21]. Indeed, according to [CM17c, Theorem 1.2], the local validity of the $\mathsf{CD}(K,N)$ condition is enough to the deduce the (global) Brunn–Minkowski inequality $\mathsf{BM}(K,N)$ with sharp coefficients. In the general setting of metric measure spaces, the equivalence between the Brunn-Minkowski inequality $\mathsf{BM}(K,N)$ and the $\mathsf{CD}(K,N)$ condition is still an open question. However, in this thesis, we present two notable partial results in this direction.

In Paper 2 we show the equivalence between $\mathsf{CD}(K,N)$ and $\mathsf{BM}(K,N)$ in the setting of *weighted Riemannian manifolds*, i.e. metric measure spaces $(M, \mathsf{d}_{\mathrm{g}}, e^{-V} \mathrm{vol}_{\mathrm{g}})$ where (M,g) is an n-dimensional Riemannian manifold and V is a C^2 potential. In particular, this result shows that, in this setting, the Brunn-Minkowski inequality $\mathsf{BM}(K,N)$ is equivalent to the generalized Ricci lower bound

$$\mathrm{Ric}^{N,V} := \mathrm{Ric} + \nabla^2 V - \frac{\nabla V \otimes \nabla V}{N-n} \geq K \cdot g. \tag{0.0.1}$$

As a fundamental tool for the proof, which also has independent interest, we present a refined study of the set of t-midpoints $M_t(A,B)$, for suitable sets A and B, in terms of the Riemann curvature tensor.

In Paper 3, we introduce a natural strengthened version of the $\mathsf{BM}(K,N)$ inequality, called strong Brunn-Minkowski inequality $\mathsf{SBM}(K,N)$, which is itself a consequence of the $\mathsf{CD}(K,N)$ condition. Then, we show that, in the general framework of essentially non-branching metric measure spaces, the $\mathsf{CD}(K,N)$ condition is equivalent to the $\mathsf{SBM}(K,N)$ inequality. This result provides a first significant step in the direction of proving the equivalence between $\mathsf{BM}(K,N)$ and $\mathsf{CD}(K,N)$ in a general setting.

In Riemannian manifolds, the $\mathsf{CD}(K,N)$ condition is well-suited to characterize curvature, however similar results cannot be obtained in *sub-Riemannian* setting. Sub-Riemannian geometry is a broad generalization of Riemannian geometry where, given a smooth manifold M, a (smoothly varying) scalar product is only defined on a subset of *horizontal* directions $\mathcal{D}_p \subset T_pM$ (called distribution) at each point $p \in M$. Under the so-called *Hörmander condition*, M is horizontally-path connected, and the usual length-minimization procedure yields a well-defined distance d_{SR}. In particular, differently from what happens in Riemannian geometry, the rank of the distribution $r(p) := \dim \mathcal{D}_p$ may be strictly less than the dimension of the manifold and may vary with the point.

It was proved by Juillet in [Jui21] that a complete sub-Riemannian manifold M, equipped with a positive smooth measure $\mathfrak{m}$ and such that the rank of the distribution $r(p)$ is everywhere smaller than $\dim M - 1$, cannot satisfy the $\mathsf{CD}(K,N)$ condition for any choice of the parameters $K \in \mathbb{R}$ and $N \in (1,\infty)$. Despite being quite general, this statement does not include *almost-Riemannian manifolds*, i.e. sub-Riemannian manifolds where the rank of the distribution coincides with the dimension, at almost every point. In Paper 4 we extend Juillet's result, proving that a complete almost-Riemannian manifold, with dimension 2 or strongly regular, equipped with a positive smooth measure, cannot satisfy the $\mathsf{CD}(K,N)$ condition for any $K \in \mathbb{R}$ and $N \in (1,\infty)$. The main idea behind the proof is to exploit the one-dimensional characterization of the CD condition:

$$\mathsf{CD}(K,N) \quad \Rightarrow \quad \mathsf{CD}^1(K,N),$$

proved by Cavalletti and Mondino in [CM17b], and disprove the $\mathsf{CD}^1(K,N)$ condition. In order to do so, we take advantage of a truly sub-Riemannian phenomenon, namely the existence of *characteristic points*. Most recently, Rizzi and Stefani [RS23] were able to demonstrate the failure of the $\mathsf{CD}(K,N)$ condition in the sub-Riemannian setting, in full generality.

In Paper 5, we aim at extending this results to the setting of *sub-Finsler* manifolds. On a smooth manifold M, a sub-Finsler structure induces a (smoothly varying) *norm*, which needs not be induced by a scalar product, on the distribution $\mathcal{D}_p \subset T_pM$, at each point $p \in M$. As in the sub-Riemannian setting, the Hörmander condition ensures that the length-minimization procedure among admissible curves gives a well-defined distance d_{SF}. Replacing the scalar product with a (possibly singular) norm is not merely a technical choice, as the metric structure of a sub-Finsler manifold reflects the singularities of the reference norm. A particularly relevant class of sub-Finsler manifolds is the one of *sub-Finsler Carnot groups*, which, loosely speaking, are sub-Finsler manifolds possessing a Lie group structure. In fact, they naturally arise as metric tangents of metric measure spaces.

In Paper 5, we conjecture that a sub-Finsler Carnot group, endowed with a positive smooth measure $\mathfrak{m}$ does not satisfy the $\mathsf{CD}(K,N)$ condition for any $K \in \mathbb{R}$ and $N \in (1,\infty)$. Subsequently, we establish three partial results that support and validate this conjecture. The first is the *smooth* sub-Finsler analogous of Juillet's statement and, in particular, it proves the conjecture for every *smooth* sub-Finsler Carnot group. Then, we show how, in the specific case of the *Heisenberg group*, it is possible to weaken the regularity assumption on the generating norm. On the one hand, we prove that the sub-Finsler Heisenberg group, equipped with a $C^{1,1}$ (and strictly convex) norm and with a positive smooth measure, cannot satisfy the $\mathsf{CD}(K,N)$ condition. On the other hand, we show the failure of the weaker *measure contraction property* $\mathsf{MCP}(K,N)$ in the sub-Finsler Heisenberg group, equipped with a non-C^1 (and strictly convex) norm and with a positive smooth measure. This result has independent interest because it stands in contrast to what typically happens in the sub-Riemannian setting, in fact, the sub-Riemannian Heisenberg group satisfies the $\mathsf{MCP}(0,5)$ condition.

Traditionally, the parameter N, which represents an upper bound on the dimension, has usually been taken positive. In Papers 6 and 7 we discuss the extension of the $\mathsf{CD}(K,N)$ condition where the dimensional bound N is a negative number. Admitting N to be negative may sound strange and artificial (if one thinks to N as an upper bound on the dimension), however, in the smooth setting the Ricci lower bound (0.0.1) makes sense also when N is negative and it actually provides a weaker inequality. In this setting, the Ricci bound with negative parameter N naturally arises in information geometry and statistical mechanics. Motivated by this, Ohta [Oht16] introduced the $\mathsf{CD}(K,N)$ condition for negative values of the dimensional parameter N, properly adapting Sturm's approach (of [Stu06b]).

The main difference with the classical $\mathsf{CD}(K,N)$ definition (with positive N) is that, when the dimensional parameter is negative, the $\mathsf{CD}(K,N)$ condition does not guarantee any control on the reference measure. This potential singular behavior is highlighted in Paper 6, where we consider metric measure spaces with *quasi-Radon* reference measures, as a natural framework for the study of $\mathsf{CD}(K,N)$ spaces, with negative N. In the same work we prove the stability of the $\mathsf{CD}(K,N)$ condition with respect to a suitably refined measured Gromov-Hausdorff convergence, that controls the singularity of the reference measures.

Finally, in Paper 7, we prove two important properties for the $\mathsf{CD}(K,N)$ condition with negative dimensional bound. The first is that this condition is sufficient to deduce the existence of an *optimal transport map*, whenever the first marginal is absolutely continuous with respect to the reference measure. The second is the so-called local-to-global property, which states that a local version of the $\mathsf{CD}(K, N)$ condition is sufficient to deduce the global one. The proof of both this results is a highly non-trivial adaptation of the classical arguments (cf. [Bac10, Gig12]), as we need to deal with the singularities of the reference measure.

Overview of articles

Paper 1: Magnabosco, Mattia. *Examples of* $\mathsf{CD}(0, N)$ *spaces with non-constant dimension.* 2023. preprint: arXiv:2310.05738 (`https://arxiv.org/abs/2310.05738`).

Paper 2: Magnabosco, Mattia; Portinale, Lorenzo; Rossi, Tommaso. *The Brunn–Minkowski inequality implies the CD condition in weighted Riemannian manifolds.* 2022. preprint: arXiv:2209.13424 (`https://arxiv.org/abs/2209.13424`)

Paper 3: Magnabosco, Mattia; Portinale, Lorenzo; Rossi, Tommaso. *The strong Brunn–Minkowski inequality and its equivalence with the CD condition.* 2022. preprint: arXiv:2210.01494 (`https://arxiv.org/abs/2210.01494`)

Paper 4: Magnabosco, Mattia; Rossi, Tommaso. *Almost-Riemannian manifolds do not satisfy the curvature-dimension condition.* Calc. Var. 62, 123 (2023). DOI: `https://doi.org/10.1007/s00526-023-02466-x`.

Paper 5: Magnabosco, Mattia; Rossi, Tommaso. *Failure of the curvature-dimension condition in sub-Finsler manifolds.* 2023. preprint: arXiv:2307.01820 (`https://arxiv.org/abs/2307.01820`)

Paper 6: Magnabosco, Mattia; Rigoni, Chiara; Sosa, Gerardo. *Convergence of metric measure spaces satisfying the CD condition for negative values of the dimension parameter.* Nonlinear Analysis, Volume 237, 2023. DOI: `https://doi.org/10.1016/j.na.2023.113366`.

Paper 7: Magnabosco, Mattia; Rigoni, Chiara. *Optimal maps and local-to-global property in negative dimensional spaces with Ricci curvature bounded from below.* 2021. preprint: arXiv:2105.12017 (`https://arxiv.org/abs/2105.12017`)

Introduction

In recent years, several major breakthroughs in the field of geometry have been related to the study of two fundamental topics: curvature and singularities. This thesis is aimed at studying geometric and structural properties of singular spaces satisfying a synthetic lower curvature bound. A crucial tool in this research field is the *optimal transport* theory, which allows an intrinsic approach to curvature. In this way, many singular spaces can be studied in a general framework, from an abstract viewpoint.

The theory of curvature-dimension bounds for nonsmooth spaces arises from the observation that, in the Riemannian setting, having a uniform lower bound on the Ricci curvature tensor can be equivalent characterized using the theory of optimal transport. In particular, consider a Riemannian manifold (M, g) of dimension n and call d_g and vol_g the distance and the volume measure, respectively. In this case, the growth of Riemannian volumes along the geodesic flow is ruled by an ODE involving the Ricci tensor Ric. Instead, if we consider the weighted measure $e^{-V}\,\mathrm{vol}_g$ as reference measure, where the potential V is in $C^2(M)$, volume growth is controlled by the generalized Ricci tensor

$$\mathrm{Ric}^{N,V} := \mathrm{Ric} + \nabla^2 V - \frac{\nabla V \otimes \nabla V}{N-n}. \tag{I.0.2}$$

On the other hand, solving the optimal transport problem, we can define the Wasserstein distance on the set $\mathscr{P}_2(M)$ of Borel probability measures with finite second order moment, as

$$W_2^2(\mu_0, \mu_1) := \inf_{\pi} \int \mathsf{d}^2(x, y)\, \mathrm{d}\pi(x, y),$$

where the infimum is taken over all admissible plans π (i.e. probability measures on $M \times M$ having μ_0 and μ_1 as marginals).

The fundamental observation, from which the Lott–Sturm–Villani theory arose, is that the Ricci lower bound

$$\mathrm{Ric}^{N,V}(x) \geq K \cdot g(x) \qquad \text{for every } x \in M, \tag{I.0.3}$$

for some $K \in \mathbb{R}$, is equivalent to a convexity property of suitable entropy functionals in the Wasserstein space $(\mathscr{P}_2(M), W_2)$. While the first condition involves a differential object, the Ricci tensor, the second can be formulated without explicitly using the underlying smooth structure of the manifold. In particular, it only requires a distance, generating a Wasserstein distance, and a reference measure, with respect to which the entropy functionals are defined. This observation led Sturm [Stu06a, Stu06b] and Lott–Villani [LV09] to introduce the curvature-dimension condition $\mathsf{CD}(K, N)$, with $K \in \mathbb{R}$ and $N \in (1, \infty]$, in the non-smooth setting of metric measure spaces, i.e. complete metric spaces endowed with a locally finite Borel measure. They considered the entropy convexity property, which is equivalent to the Ricci bound (I.0.3) in the Riemannian setting, they formulated it for metric measure spaces and took it as definition of curvature-dimension bound $\mathsf{CD}(K, N)$. In particular, the $\mathsf{CD}(K, N)$ condition is a synthetic notion representing a lower (Ricci) curvature bound by K and an upper bound on the dimension by N. For example, when the curvature parameter K is equal to 0 and the dimensional one N is equal to ∞, the

CD condition requires the geodesic convexity in $(\mathscr{P}_2(\mathsf{X}), W_2)$ of the Boltzmann-Shannon entropy functional $\mathrm{Ent} : \mathscr{P}_2(\mathsf{X}) \to \mathbb{R} \cup \{+\infty\}$, defined as

$$\mathrm{Ent}(\mu) := \begin{cases} \int \rho \log \rho \, \mathrm{d}\mathfrak{m} & \text{if } \mu \ll \mathfrak{m} \text{ and } \mu = \rho\mathfrak{m} \\ +\infty & \text{otherwise} \end{cases}.$$

This meaning that, for every pair of measures $\mu_0, \mu_1 \in \mathscr{P}_{ac}(\mathsf{X}, \mathfrak{m})$, there exists a constant speed W_2-geodesic (i.e. a length minimizing curve) $(\mu_t)_{t\in[0,1]}$ such that

$$\mathrm{Ent}(\mu_t) \leq (1-t)\,\mathrm{Ent}(\mu_0) + t\,\mathrm{Ent}(\mu_1), \quad \forall t \in [0,1].$$

The definition for general values of the parameters K and N is given using the Rényi entropy functionals and the distortion coefficients $\tau_{K,N'}^{(t)}$ (see [Stu06b])

The CD condition is therefore a synthetic notion which consistently generalizes a Riemannian curvature-dimension bound to the non-smooth setting of metric measure spaces. Moreover, it was proved by Ohta [Oht09] that the relation between curvature and CD condition holds also in the context of Finsler manifolds. Remarkably, $\mathsf{CD}(K, N)$ spaces (i.e. spaces satisfying the $\mathsf{CD}(K, N)$ condition) enjoy different geometric properties which hold in the smooth setting. Some of them are expected and in a way necessary for a reasonable curvature-dimension bound, like the scaling [Stu06b] and tensorzation [BS10, DS11] properties or the monotonicity with respect to the parameters [Stu06b], i.e.

$$\mathsf{CD}(K', N') \implies \mathsf{CD}(K'', N'') \quad \text{if } K' \geq K'' \text{ and } N' \leq N'',$$

for every $K', K'' \in \mathbb{R}$ and $N', N'' \in (1, \infty]$. Others are completely non-trivial and highlight some notable geometric features. Between those, we cite the Bonnet–Myers diameter bound, the Bishop–Gromov inequality that provides an estimate on the volume growth of concentric balls and the Brunn–Minkowski inequality, which will be studied in Papers 2 and 3. The $\mathsf{CD}(K, N)$ condition also enjoys the so-called local-to-global property [BS10, DS11, CM21], meaning that, if the $\mathsf{CD}(K, N)$ condition holds locally around every point, then it also holds for the whole space. Another fundamental property of the $\mathsf{CD}(K, N)$ condition is its the stability with respect to the (pointed) measured Gromov-Hausdorff convergence [Stu06b, LV09, GMS15]. This is a notion of convergence for metric measure spaces that basically combines the Hausdorff convergence for the metric side and the weak convergence for the reference measures. The stability of the $\mathsf{CD}(K, N)$ condition is especially relevant in view of the fact that the measured Gromov-Hasdorff limit of Riemannian manifolds with a uniform curvature-dimension bound (i.e. a Ricci limit space), is not necessarily a Riemannian manifold. In particular, $\mathsf{CD}(K, N)$ spaces constitutes a class containing all Riemannian and Finsler manifolds with a lower curvature bound, which is also closed with respect to the measured Gromov-Hausdorff convergence.

The $\mathsf{CD}(K, N)$ condition has several consequences that have allowed the development of a solid theory. Nevertheless, the study of $\mathsf{CD}(K, N)$ spaces by means of classical analytic tools presents non-trivial difficulties, mainly due to their complex geometric structure. To overcome these issues, Ambrosio, Gigli and Savaré [AGS14b] (in the case $N = \infty$) and subsequently Gigli [Gig15] (in the case $N < \infty$) considered $\mathsf{CD}(K, N)$ metric measure spaces having linear heat flow, the so called $\mathsf{RCD}(K, N)$ spaces (see also [AGS14a, AGS15]). The extra assumption, named *infinitesimal Hilbertianity* in [Gig15], asks the Sobolev space $W^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ (defined according to Cheeger [Che99]) to be Hilbert and forces the space to be Riemannian-like.

I.1 Examples of singular CD spaces

In recent years, different refined calculus tools has been developed in $\mathsf{RCD}(K, N)$ spaces (see [Gig18]), and they allowed in particular to equivalently characterize the curvature-dimension bound in terms of a suitable version of the Bochner inequality

$$\frac{1}{2}\Delta|\nabla u|^2 - \langle \nabla u, \nabla \Delta u \rangle \geq K \cdot |\nabla u|^2 + \frac{1}{N} \cdot |\Delta u|^2.$$

It was then possible to study $\mathsf{RCD}(K, N)$ spaces with different analytic techniques, often resembling the classical (smooth) ones, that allowed to prove many geometric measure theory results. A remarkable example of this, is the work by Brué and Semola [BS20], where the authors proved the constancy of dimension for $\mathsf{RCD}(K, N)$ spaces, generalizing the result by Colding and Naber [CN12] for Ricci-limit spaces. In particular, calling $\mathcal{R}_k$ the set of k-regular points (i.e. points in which the tangent space is the k-dimensional Euclidean space), they proved that there exist only one $n \leq N$ such that $\mathcal{R}_n$ has positive measure.

Similar rigidity results are not known for $\mathsf{CD}(K, N)$ spaces, moreover the same advanced calculus tools are not available without the infinitesimal Hilbertianity assumption. In the first paper, we present different examples of singular $\mathsf{CD}(K, N)$ spaces, showing in particular that some rigidity properties are not guaranteed by the curvature-dimension condition.

I.1.1 Paper 1: Examples of $\mathsf{CD}(K, N)$ spaces with non-constant dimension [Mag]

In this work, we present a strategy that allows to demonstrate the validity of the $\mathsf{CD}(0, N)$ condition in a class of singular metric measure spaces. In particular, we obtain examples of $\mathsf{CD}(0, N)$ spaces having different (topological and Hausdorff) dimensions in different regions. This work is a twofold generalization of [Mag22a], where an example of a highly branching $\mathsf{CD}(0, \infty)$ space with non-constant dimension is constructed. On the one hand, we extend the result of [Mag22a] to $\mathsf{CD}(0, N)$ spaces, having a finite dimensional bound N. This generalization is somewhat expected but far from trivial, in fact the $\mathsf{CD}(0, N)$ condition implies some properties which are not guaranteed in $\mathsf{CD}(0, \infty)$ spaces (for example the Bishop-Gromov inequality). On the other hand, we prove the $\mathsf{CD}(0, N)$ condition for a class of metric measure spaces which is considerably larger than the one considered in [Mag22a]. This allows to highlight other types of singular behaviour that are proved to be possible in CD spaces.

From the considered examples, we are then able to draw some conclusion, regarding the failure of rigidity properties in $\mathsf{CD}(K, N)$ spaces.

Conclusion I.1.1. *The constancy of dimension may fail in* $\mathsf{CD}(0, N)$ *spaces.*

In particular, this paper presents an example of a $\mathsf{CD}(0, N)$ space (contained in $\mathbb{R}^2$) which is the union of a line segment L and of a two-dimensional set C, as represented in Figure I.1. This set is metrized with the distance d_∞ induced by the l^∞ norm on $\mathbb{R}^2$. Moreover, L is equipped with the Hausdorff measure $\mathcal{H}^1$, while C is equipped with a measure, absolutely continuous with respect to the Lebesgue measure $\mathscr{L}^2$, having density that degenerates when approaching the line segment.

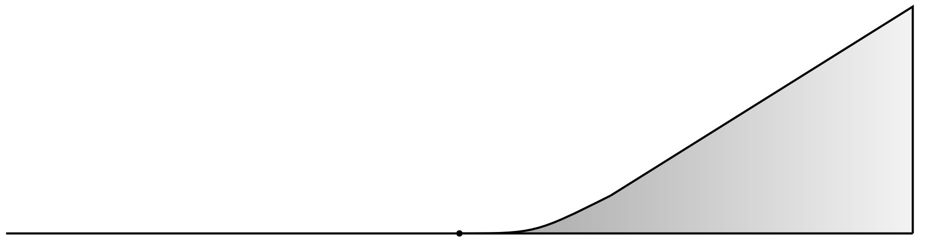

Figure I.1: The metric measure space $(X_f, \mathsf{d}_\infty, \mathfrak{m}_{f,K})$.

This example shows that the infinitesimal Hilbertianity assumption is fundamental in the work of Brué and Semola [BS20]. Moreover, having chosen the distance d_∞, it can be observed that geodesics from L to C are branching, therefore:

Conclusion I.1.2. *Any reasonably meaningful non-branching condition may fail in* $\mathsf{CD}(0, N)$ *spaces.*

There are different ways to require convexity of the entropy along Wasserstein geodesics and they originate different curvature-dimension conditions. In particular, the (classical) $\mathsf{CD}(K, N)$ condition is the weakest version, because it just requires the existence of a geodesic along which the entropy is convex. Remarkably, stronger CD condition are enough to deduce some non-branching conditions, as shown in the works by Rajala-Sturm [RS14] and Schultz [Sch18]. In this sense, the following statement is not surprising, in view of Conclusion I.1.2.

Conclusion I.1.3. *The strict* $\mathsf{CD}(0, N)$ *condition (see Paper 1 for the definition) is strictly stronger than the classical* $\mathsf{CD}(0, N)$ *one, i.e. there exists a* $\mathsf{CD}(0, N)$ *space which does not satisfy the strict* $\mathsf{CD}(0, N)$ *condition. Moreover, the strict* $\mathsf{CD}(0, N)$ *condition is not stable with respect to the measured Gromov-Hausdorff convergence.*

Indeed, the metric measure space represented in Figure I.1 is the measured Gromov-Hausdorff limit of a sequence of strict $\mathsf{CD}(0, N)$ spaces.

Non-branching of geodesics is also often closely related to the existence of optimal transport maps between absolutely continuous marginals. In fact, many results in this direction assume non-branching, in addition to a curvature-dimension bound, see for example [Gig12], [CM17a] and [MR21]. From the observations leading to Conclusion I.1.2, it is possible to deduce that every optimal transport plan between a probability measure concentrated on L and one concentrated in C cannot be induced by a map.

Conclusion I.1.4. *The existence of an optimal transport map between two absolutely continuous marginals is not guaranteed in* $\mathsf{CD}(0, N)$ *spaces, without assuming a non-branching condition.*

It is also possible to find a non-compact version of the example represented in Figure I.1, where L is replaced by a half-line and C by the set $\{0 \leq y \leq kx\}$, for a sufficiently small $k > 0$. This metric measure space provides an example of a $\mathsf{CD}(0, N)$ spaces having a metric measure tangent with a singular structure. Moreover, it allows to draw this last conclusion:

Conclusion I.1.5. *The topological splitting may fail in* $\mathsf{CD}(0, N)$ *spaces, i.e. there exists a* $\mathsf{CD}(0, N)$ *space containing a subset isometric to* $\mathbb{R}$*, which does not topologically split as the product of* $\mathbb{R}$ *with another space.*

I.2 The Brunn–Minkowski inequality and its relation with the CD condition

As previously mentioned, one of the most important merits of the CD condition is that it is sufficient to deduce geometric and functional inequalities that hold in the smooth setting. The example, which is the main focus of this section, is the Brunn–Minkowski inequality, whose classical version in $\mathbb{R}^n$ (see e.g. [Gar02]) states that

$$\mathscr{L}^n\left((1-t)A + tB\right)^{\frac{1}{n}} \geq (1-t)\mathscr{L}^n(A)^{\frac{1}{n}} + t\mathscr{L}^n(B)^{\frac{1}{n}}, \quad \forall t \in [0,1],$$

for every two nonempty compact sets $A, B \subset \mathbb{R}^n$. It was already observed by McCann in [McC94] and [McC97] that, in the Euclidean setting, this inequality can be proved using the convexity property of the Rényi entropy in the Wasserstein space. In [Stu06b], Sturm improved this result, proving that a $\mathsf{CD}(K,N)$ space supports a generalized version of the Brunn–Minkowski inequality, denoted $\mathsf{BM}(K,N)$. In the setting of metric measure spaces the Minkowski sum of A and B is replaced with the set of t-midpoints, i.e.

$$M_t(A,B) := \{\gamma(t) \,:\, \gamma \text{ constant-speed geodesic of } (\mathsf{X},\mathsf{d}),\, \gamma(0)\in A,\, \gamma(1)\in B\}.$$

Moreover, in order to take into account curvature and dimension, the convex interpolation is done according to the distortion coefficients $\tau_{K,N}^{(t)}$. In particular, a metric measure space $(\mathsf{X},\mathsf{d},\mathfrak{m})$ is said to satisfy the Brunn–Minkowski inequality $\mathsf{BM}(K,N)$ if for every pair of nonempty Borel sets $A,B\subset\mathsf{X}$ it holds that

$$\mathfrak{m}(M_t(A,B))^{\frac{1}{N'}} \geq \tau_{K,N'}^{(1-t)}(\Theta(A,B))\cdot\mathfrak{m}(A)^{\frac{1}{N'}} + \tau_{K,N'}^{(t)}(\Theta(A,B))\cdot\mathfrak{m}(B)^{\frac{1}{N'}},$$

for every $N'\geq N$ and every $t\in[0,1]$, where

$$\Theta(A,B) := \begin{cases} \inf\limits_{x\in A,\, y\in B} \mathsf{d}(x,y) & \text{if } K\geq 0\,, \\ \sup\limits_{x\in A,\, y\in B} \mathsf{d}(x,y) & \text{if } K<0\,. \end{cases}$$

While the curvature-dimension bound $\mathsf{CD}(K,N)$ is sufficient to deduce the Brunn–Minkowski inequality $\mathsf{BM}(K,N)$, it is less clear whether assuming the validity of the $\mathsf{BM}(K,N)$ inequality is sufficient to deduce the $\mathsf{CD}(K,N)$ condition. The interest in proving the equivalence is twofold. On the one hand, it would provide an alternative proof of the *globalization theorem*, cf. [CM21]. Indeed, according to [CM17c, Theorem 1.2], the local curvature dimension condition, denoted by $\mathsf{CD}_{\mathrm{loc}}(K,N)$, is enough to the deduce the (global) Brunn–Minkowski inequality with sharp coefficients. On the other hand, it would provide a characterization of the curvature-dimension condition without the need of optimal transport techniques. In fact, a remarkable feature of the Brunn–Minkowski inequality is that its formulation does not refer to the Wasserstein interpolation of measures.

In the general setting of metric measure spaces, the equivalence between the Brunn–Minkowski inequality $\mathsf{BM}(K,N)$ and the $\mathsf{CD}(K,N)$ condition is still an open question. However, in two joint works with Lorenzo Portinale and Tommaso Rossi, we have obtained two partial results in this direction, which are described in the following subsections.

I.2.1 Paper 2: The Brunn–Minkowski inequality is equivalent to the CD condition in Riemannian setting [MPR22a]

In this paper, we prove that the $\mathsf{CD}(K,N)$ condition and the Brunn–Minkowski inequality $\mathsf{BM}(K,N)$ are equivalent in the setting of (weighted) Riemannian manifolds.

Theorem I.2.1. *Let (M,g) be a complete Riemannian manifold of dimension n, endowed with the reference measure $\mathfrak{m}=e^{-V}\,\mathrm{vol}_g$, where $V\in C^2(M)$. Suppose that the metric measure space $(M,\mathsf{d}_{\mathrm{g}},\mathfrak{m})$ satisfies $\mathsf{BM}(K,N)$ for some $K\in\mathbb{R}$ and $N>1$. Then, it is a $\mathsf{CD}(K,N)$ space and in particular the two conditions are equivalent.*

The main idea behind the proof is to demonstrate, arguing by contradiction, the validity of the Ricci lower bound (I.0.3), which is equivalent to the $\mathsf{CD}(K,N)$ condition. Thus, we assume the existence of $v_0\in T_{x_0}M$, with $x_0\in M$, such that

$$\mathrm{Ric}^{N,V}(v_0,v_0) < K\|v_0\|^2,$$

and consider the geodesic γ starting at x_0 with velocity v_0. The infinitesimal volume distortion around the geodesic can be estimated through the quantity $\mathrm{Ric}^{N,V}(v_0, v_0)$. Therefore, we are able to find two (small) sets $A, B \subset M$ around γ such that

$$\mathfrak{m}\left(T_{\frac{1}{2}}(A)\right)^{\frac{1}{N}} < \tau_{K,N}^{(\frac{1}{2})}(\Theta(A,B))\left(\mathfrak{m}(A)^{\frac{1}{N}} + \mathfrak{m}(B)^{\frac{1}{N}}\right), \tag{I.2.1}$$

where $T_{\frac{1}{2}}$ is the interpolating optimal transport map at time $\frac{1}{2}$, between the marginals $\frac{\mathfrak{m}|_A}{\mathfrak{m}(A)}$ and $\frac{\mathfrak{m}|_B}{\mathfrak{m}(B)}$. The final, and most challenging, step is to compare the measure of $T_{\frac{1}{2}}(A)$ with the measure of $M_{\frac{1}{2}}(A, B)$, the set of midpoints between A and B. This is done by choosing as A a specific cube oriented according to the Riemann curvature tensor at x_0. We are then able to obtain

$$\mathfrak{m}\left(M_{\frac{1}{2}}(A,B)\right) \approx \mathfrak{m}(T_{\frac{1}{2}}(A)), \tag{I.2.2}$$

which, together with (I.2.1), gives a contradiction to the Brunn–Minkowski inequality $\mathsf{BM}(K, N)$. Remarkably, the proof of (I.2.2) requires a second-order expansion capturing the local geometry of the manifold which, in particular, involves the Riemann curvature tensor at x_0.

Theorem I.2.1 allows to equivalently characterize both the $\mathsf{CD}(K, N)$ and the Ricci bound (I.0.3), without using neither the optimal transport nor the differential structure of the manifold. Moreover, it shows that the Brunn–Minkowski inequality $\mathsf{BM}(K, N)$ is itself a reasonable synthetic curvature-dimension bound, as it agrees with (I.0.3) in the Riemannian setting. This observation is particularly relevant when compared to what happens for the measure contraction property $\mathsf{MCP}(K, N)$, introduced by Ohta in [Oht07]. In fact, this condition may fail to be equivalent to the Ricci bound (I.0.3), for a weighted Riemannian manifold $(M, \mathsf{d}_\mathrm{g}, e^{-V}\mathrm{vol}_\mathrm{g})$ (with non-trivial potential V). Finally, as a corollary of Theorem I.2.1 and in light of the work by Bacher [Bac10], the Brunn–Minkowski inequality $\mathsf{BM}(K, N)$ is equivalent to a modified Borell–Brascamp–Lieb inequality, see [Bac10, Definition 1.1] for the precise definition.

I.2.2 Paper 3: The strong Brunn–Minkowski inequality and its equivalence with the CD condition [MPR22b]

In this paper, we introduce a more stringent version of the $\mathsf{BM}(K, N)$ inequality, that we called strong Brunn–Minkowski $\mathsf{SBM}(K, N)$, and we prove its equivalence with the $\mathsf{CD}(K, N)$ condition, in the general framework of essentially non-branching metric measure spaces.

Definition I.2.2. A metric measure space $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ is said to satisfy the *strong Brunn–Minkowski inequality* $\mathsf{SBM}(K, N)$ if, for every pair of Borel sets $A, B \subset \mathrm{spt}(\mathfrak{m})$ such that $0 < \mathfrak{m}(A), \mathfrak{m}(B) < \infty$, there exists a Wasserstein geodesic $(\mu_t)_{t\in[0,1]}$, connecting $\mathfrak{m}_A := \frac{\mathfrak{m}|_A}{\mathfrak{m}(A)}$ and $\mathfrak{m}_B := \frac{\mathfrak{m}|_B}{\mathfrak{m}(B)}$, such that the following inequality holds

$$\mathfrak{m}(\mathrm{spt}(\mu_t))^{\frac{1}{N'}} \geq \tau_{K,N'}^{(1-t)}(\Theta(A,B)) \cdot \mathfrak{m}(A)^{\frac{1}{N'}} + \tau_{K,N'}^{(t)}(\Theta(A,B)) \cdot \mathfrak{m}(B)^{\frac{1}{N'}},$$

for every $N' \geq N$ and every $t \in [0, 1]$.

The strong Brunn–Minkowski inequality $\mathsf{SBM}(K, N)$ is a very natural strengthening of the Brunn–Minkowski inequality $\mathsf{BM}(K, N)$. In fact, the proof of the latter (starting from the $\mathsf{CD}(K, N)$ condition) relies on the well-known inclusion of the support of any Wasserstein t-midpoint, between $\mathfrak{m}_A$ and $\mathfrak{m}_B$, in the set of t-midpoints $M_t(A, B)$. In particular, the strong Brunn–Minkowski inequality $\mathsf{SBM}(K, N)$ is itself implied by the $\mathsf{CD}(K, N)$ condition. If we add the essentially non-branching assumption (see [RS14] for the definition) on the underline space, we are actually able to deduce the equivalence between $\mathsf{SBM}(K, N)$ and $\mathsf{CD}(K, N)$.

Theorem I.2.3. *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *be an essentially non-branching metric measure space, then* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *satisfies the strong Brunn–Minkowski inequality* $\mathsf{SBM}(K, N)$ *if and only if it satisfies the* $\mathsf{CD}(K, N)$ *condition.*

Theorem I.2.3 represents a first remarkable step in the direction of proving the equivalence between the Brunn–Minkowski inequality $\mathsf{BM}(K, N)$ and the $\mathsf{CD}(K, N)$ condition in a general setting. The missing piece to show it is a good understanding of the relation between the set $M_t(A, B)$ of t midpoints and the support of a Wasserstein t-midpoint of $\mathfrak{m}_A$ and $\mathfrak{m}_B$ (cf. (I.2.2)). In the general setting of metric measure spaces, the problem of finding A, B and a Wassertein geodesic connecting $\mathfrak{m}_A$ and $\mathfrak{m}_B$, such that the sets $M_t(A, B)$ and $\operatorname{spt}(\mu_t)$ are comparable, is very challenging. In Paper 5 we will discuss a strategy developed by Juillet [Jui21], upon the introduction of the *inverse geodesic map*, which has proved to be effective in the sub-Riemannian and sub-Finsler settings.

I.3 The CD condition in sub-Riemannian and sub-Finsler geometry

While the $\mathsf{CD}(K, N)$ condition is well-suited to characterize curvature in the Riemannian setting, a similar result does not hold in the sub-Riemannian setting. Sub-Riemannian geometry is a far-reaching generalization of Riemannian geometry where, given a smooth manifold M, we consider a smoothly varying scalar product only on a subset of *horizontal* directions $\mathcal{D}_p \subset T_pM$ (called distribution) at each point $p \in M$. Under the so-called Hörmander condition, M is horizontally-path connected, and the usual length-minimization procedure yields a well-defined distance d_{SR}. In particular, differently from what happens in Riemannian geometry, the rank of the distribution $r(p) := \dim \mathcal{D}_p$ may be strictly less than the dimension of the manifold and may vary with the point. This may influence the behavior of geodesics, emphasizing singularities of the distance d_{SR}.

I.3.1 Paper 4: Almost-Riemannian manifolds do not satisfy the curvature-dimension condition [MR23a]

While the measure contraction property $\mathsf{MCP}(K, N)$ is well-suited to study curvature in sub-Riemannian manifolds (see [Jui09, Rif13, Riz16, BR18, BR20]), the $\mathsf{CD}(K, N)$ condition is too strong and it fails to hold in this setting. The first result in this direction has been proved by Juillet in [Jui21].

Theorem I.3.1 ([Jui21])**.** *Let M be a complete sub-Riemannian manifold with* $\dim M \geq 3$, *equipped with a smooth positive (i.e. with strictly positive density) measure* $\mathfrak{m}$. *Assume that the possibly varying rank of the distribution is everywhere smaller than* $\dim M - 1$. *Then,* $(M, \mathsf{d}_{SR}, \mathfrak{m})$ *does not satisfy the* $\mathsf{CD}(K, N)$ *condition for any* $K \in \mathbb{R}$ *and* $N \in (1, \infty)$.

While being quite general, this result does not include many cases of interest, such as *almost-Riemannian geometry.* Roughly speaking, an almost-Riemannian manifold is a sub-Riemannian manifold where the rank of the distribution coincides with the dimension of M, outside a null set called *singular region* $\mathcal{Z} = \{r(p) < \dim(M)\}$. In this paper, we prove the following theorem, which extends Juillet's result to the setting of almost-Riemannian manifolds.

Theorem I.3.2. *Let M be a complete almost-Riemannian manifold and let $\mathfrak{m}$ be any smooth positive measure on M. Assume M is either of dimension 2 or strongly regular (see [PRS18, CPR19]). Then, the metric measure space* $(M, \mathsf{d}_{SR}, \mathfrak{m})$ *does not satisfy the* $\mathsf{CD}(K, N)$ *condition for any* $K \in \mathbb{R}$ *and* $N \in (1, +\infty)$.

Loosely speaking, the technical assumption of being strongly regular, forces the singular region $\mathcal{Z}$ to be an hypersurface and controls the behavior of the distribution, making the computations feasible. Remarkably, for 2-dimensional almost-Riemannian manifolds, we do not need require any additional assumption on the structure of the singular region $\mathcal{Z}$.

The main idea behind the proof of Theorem I.3.2 is to exploit the one-dimensional characterization of the $\mathsf{CD}(K,N)$ condition:

$$\mathsf{CD}(K,N) \quad \Rightarrow \quad \mathsf{CD}^1(K,N),$$

proved by Cavalletti and Mondino in [CM17b] (see also [Cav14, CM20, CM21]), and to contradict the $\mathsf{CD}^1(K,N)$ condition. On a metric measure space $(\mathsf{X},\mathsf{d},\mathfrak{m})$, given a 1-Lipschitz function $u \in \mathsf{Lip}(\mathsf{X})$, it is possible to partition X in one-dimensional *transport rays*, associated with u, and disintegrate the measure $\mathfrak{m}$ accordingly. Then, the $\mathsf{CD}^1(K,N)$ condition asks for the validity of the $\mathsf{CD}(K,N)$ condition along the transport rays of the disintegration associated with u, for every choice of $u \in \mathsf{Lip}(\mathsf{X})$. The main advantage in dealing with one-dimensional $\mathsf{CD}(K,N)$ spaces is related to a differential characterization of $\mathsf{CD}(K,N)$ densities, which is easier to disprove compared with the convexity of the Rényi entropy. Indeed, in an almost-Riemannian manifold, equipped with a smooth positive measure $\mathfrak{m}$, we are able to explicitly compute the disintegration of $\mathfrak{m}$, induced by a suitable $u \in \mathsf{Lip}(\mathsf{X})$, and verify that the *one-dimensional* $\mathsf{CD}(K,N)$ condition along the rays does not hold, for any $K \in \mathbb{R}$ and $N \in (1,\infty)$.

A crucial tool for proving Theorem I.3.2 is a truly sub-Riemannian phenomenon, namely the existence of *characteristic points*. Given an embedded hypersurface $\Sigma \subset M$, a point $p \in M$ is called characteristic if the distribution is tangent to Σ in p, i.e. $\mathcal{D}_p \subset T_p\Sigma$. Of course, such points do not exist in Riemannian geometry, but they can appear as soon as the rank of the distribution $r(p)$ is strictly less than $\dim M$, for some $p \in M$. Usually, characteristic points are source of subtle technical problems, mostly related to the low regularity of the (signed) distance function δ_Σ from Σ. Indeed, although being 1-Lipschitz with respect to d_{SR}, δ_Σ is not smooth around characteristic points (and not even Lipschitz in coordinates). In the proof of Theorem I.3.2, we choose a suitable hypersurface Σ, we (locally) build the disintegration of $\mathfrak{m}$ associated with δ_Σ and we exploit its singular behavior to contradict the differential characterization of the one-dimensional $\mathsf{CD}(K,N)$ condition, along the associated transport rays. In particular, Σ is chosen to be *transverse* to the singular region of M in such a way that $\Sigma \cap \mathcal{Z}$ exhibits characteristic points. We can then exploit the Riemannian structure at the points of $\Sigma \setminus \mathcal{Z}$, to describe the degeneration of δ_Σ in the disintegration of $\mathfrak{m}$.

Most recently, Rizzi and Stefani [RS23] were able to demonstrate the failure of the $\mathsf{CD}(K,N)$ condition in the sub-Riemannian setting, in full generality.

Theorem I.3.3. *Let M be a complete sub-Riemannian manifold, equipped with a positive smooth measure $\mathfrak{m}$. Then, the metric measure space $(M,\mathsf{d}_{SR},\mathfrak{m})$ does not satisfy the $\mathsf{CD}(K,N)$ condition, for any $K \in \mathbb{R}$ and $N \in (1,\infty]$.*

Differently from the strategies proposed in [Jui21] and in Paper 5, Rizzi and Stefani pursue the "Eulerian" approach to curvature-dimension bounds. In particular, they observe that, in a sub-Riemannian manifold, the validity of the Bochner inequality

$$\frac{1}{2}\Delta\left(\|\nabla f\|^2\right) \geq g(\nabla f, \nabla\Delta f) + K\|\nabla f\|^2, \qquad \forall f \in C_c^\infty(M), \tag{I.3.1}$$

implies the existence of enough isometries on the metric tangent to force it to be Euclidean at *each point*. However, since sub-Riemannian manifolds are infintesimally Hilbertian, the Bochner inequality (I.3.1) is equivalent to the $\mathsf{CD}(K,\infty)$ condition.

I.3.2 Paper 5: Failure of the curvature-dimension condition in sub-Finsler manifolds [MR23b]

The aim of this paper, is to extend Theorem I.3.3 to the setting of sub-Finsler manifolds, which widely generalize both sub-Riemannian and Finsler geometry. Given a smooth manifold M, a sub-Finsler structure induces a smoothly varying *norm* (which needs not be induced by a scalar product) on the distribution $\mathcal{D}_p \subset T_pM$, at each point $p \in M$. As in the sub-Riemannian setting, $\mathcal{D}$ must satisfy the Hörmander condition, and consequently the length-minimization procedure among admissible curves gives a well-defined distance d_{SF}. Replacing the scalar product with a (possibly singular) norm is not merely a technical choice, as the metric structure of a sub-Finsler manifold reflects the singularities of the reference norm. In this regard, sub-Finsler manifolds provide an interesting example of smooth structures which present both the typical sub-Riemannian and Finsler singular behavior.

A particularly relevant class of sub-Finsler manifolds is the one of *sub-Finsler Carnot groups*, which basically are sub-Finsler manifolds possessing a Lie group structure. Motivated from the results presented in the previous section, and especially from the ones obtained in the present work, we formulate the following conjecture.

Conjecture I.3.4. *Let G be a sub-Finsler Carnot group, endowed with a positive smooth measure $\mathfrak{m}$. Then, the metric measure space $(G, \mathsf{d}_{SF}, \mathfrak{m})$ does not satisfy the $\mathsf{CD}(K, N)$ condition for any $K \in \mathbb{R}$ and $N \in (1, \infty)$.*

Our interest in Carnot groups stems from the fact that they are the only metric spaces that are locally compact, geodesic, isometrically homogeneous and self-similar (i.e. admitting a dilation) [LD15]. According to this property, sub-Finsler Carnot groups naturally arise as metric tangents of metric measure spaces.

Theorem I.3.5 (Le Donne [LD11])**.** *Let $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ be a geodesic metric measure space, equipped with a doubling measure $\mathfrak{m}$. Assume that, for $\mathfrak{m}$-almost every $x \in \mathsf{X}$, the set $\mathrm{Tan}(\mathsf{X}, x)$ of all metric tangent spaces at x contains only one element. Then, for $\mathfrak{m}$-almost every $x \in \mathsf{X}$, the element in $\mathrm{Tan}(\mathsf{X}, x)$ is a sub-Finsler Carnot group G.*

In particular, this result applies to $\mathsf{CD}(K, N)$ spaces, where the validity of the doubling property is guaranteed by the Bishop–Gromov inequality. Moreover, the measured Gromov–Hausdorff stability of the CD condition ensures that the metric measure tangents of a $\mathsf{CD}(K, N)$ space are $\mathsf{CD}(0, N)$. Therefore, the study of the $\mathsf{CD}(K, N)$ condition in sub-Finsler Carnot groups, and especially the validity of Conjecture I.3.4, has the potential to provide deep insights on the structure of tangents of $\mathsf{CD}(K, N)$ spaces. This could be of significant interest, particularly in connection with Bate's recent work [Bat22], which establishes a criterion for rectifiability of metric measure spaces, based on the structure of metric tangents.

Our first result is the sub-Finsler analogue of I.3.1.

Theorem I.3.6. *Let M be a complete sub-Finsler manifold with $r(p) < n := \dim M$ for every $p \in M$, equipped with a smooth, strictly convex norm and with a positive smooth measure $\mathfrak{m}$. Then, the metric measure space $(M, \mathsf{d}_{SF}, \mathfrak{m})$ does not satisfy the $\mathsf{CD}(K, N)$ condition, for any $K \in \mathbb{R}$ and $N \in (1, \infty)$.*

The strategy for proving this theorem follows the blueprint of [Jui21], however the adaptation to our setting is non-trivial and requires the development of many intermediate results of independent interest. In particular, we establish the existence of geodesics *without abnormal sub-segments* and we study the volume contraction rate along these geodesics.

Observe that, since sub-Finsler Carnot groups are equiregular (and thus $r(p) < n, \forall p \in G$) and complete, we immediately obtain the following consequence of Theorem I.3.6, which constitutes a significant step forward towards the proof of Conjecture I.3.4.

Theorem I.3.7. *Let G be a sub-Finsler Carnot group, equipped with a smooth, strictly convex norm and with a positive smooth measure $\mathfrak{m}$. Then, the metric measure space $(G, \mathsf{d}_{SF}, \mathfrak{m})$ does not satisfy the $\mathsf{CD}(K, N)$ condition, for any $K \in \mathbb{R}$ and $N \in (1, \infty)$.*

In the proof of Theorem I.3.6, the smoothness of the norm plays a pivotal role in establishing the correct volume contraction rate along geodesics. When the norm is less regular, it is not clear how to achieve an analogue behavior in full generality. Nonetheless, we are able to recover such a result in the context of the *sub-Finsler Heisenberg group* $\mathbb{H}$, equipped with a possibly singular norm. Working in this setting is advantageous since, assuming strict convexity of the norm, the geodesics and the cut locus are completely described [Ber94] and there exists an explicit expression for them in terms of convex trigonometric functions [Lok21].

For the sub-Finsler Heisenberg group, we prove two different results, with the first addressing the case of $C^{1,1}$ reference norms and thus substantially relaxing the smoothness assumption of Theorem I.3.6.

Theorem I.3.8. *Let $\mathbb{H}$ be the sub-Finsler Heisenberg group, equipped with a strictly convex and $C^{1,1}$ norm and with a positive smooth measure $\mathfrak{m}$. Then, the metric measure space $(\mathbb{H}, \mathsf{d}_{SF}, \mathfrak{m})$ does not satisfy the $\mathsf{CD}(K, N)$, for any $K \in \mathbb{R}$ and $N \in (1, \infty)$.*

The proof of this statement follows the same lines of Juillet's strategy. However, the low regularity of the norm, and thus of geodesics, prevent us to exploit the same differential tools developed for Theorem I.3.6. Nonetheless, using the explicit expression of geodesics and of the exponential map, we can still recover an analogue result. Remarkably, for every $C^{1,1}$ strictly convex reference norm, we obtain the exact same contraction rate, equal to the geodesic dimension $\mathcal{N} = 5$, that characterizes the sub-Riemannian Heisenberg group.

Our second result in the sub-Finsler Heisenberg group deals with the case of singular (i.e. non-C^1) reference norms.

Theorem I.3.9. *Let $\mathbb{H}$ be the sub-Finsler Heisenberg group, equipped with a strictly convex norm which is not C^1, and let $\mathfrak{m}$ be a positive smooth measure on $\mathbb{H}$. Then, the metric measure space $(\mathbb{H}, \mathsf{d}_{SF}, \mathfrak{m})$ does not satisfy the measure contraction property $\mathsf{MCP}(K, N)$ for any $K \in \mathbb{R}$ and $N \in (1, \infty)$.*

Observe that this theorem also shows the failure of the $\mathsf{CD}(K, N)$ condition, which is stronger than the measure contraction property $\mathsf{MCP}(K, N)$. However, Theorem I.3.9 has an interest that goes beyond this consequence, as it reveals a phenomenon that stands in contrast to what typically happens in the sub-Riemannian setting. In fact, the sub-Riemannian Heisenberg group satisfies the $\mathsf{MCP}(0, 5)$ condition. Therefore, Theorem I.3.9 shows that a singularity of the reference norm can cause the failure of the measure contraction property $\mathsf{MCP}(K, N)$. The proof of Theorem I.3.9 also highlights a remarkable geometric property of the space $(\mathbb{H}, \mathsf{d}_{SF}, \mathfrak{m})$, where geodesics can branch, even though they are unique. This has independent interest, as examples of branching spaces usually occur when geodesics are not unique.

We conclude by highlighting that the combination of Theorem I.3.8 and Theorem I.3.9 proves Conjecture I.3.4 for a large class of sub-Finsler Heisenberg groups. This is particularly interesting as the sub-Finsler Heisenberg groups are the unique sub-Finsler Carnot groups with Hausdorff dimension less than 5 (or with topological dimension less than or equal to 3), up to isometries.

I.4 CD spaces with negative dimension parameter

Up to now the constant N, which represents an upper bound on the dimension, has always been taken positive. In this section we discuss the extension of the $\mathsf{CD}(K, N)$ condition where the dimensional bound N is a negative number. Admitting N to be negative may sound strange and artificial (if one thinks to N as an upper bound on the dimension), however, in the smooth case, the Ricci lower bound (I.0.3) makes sense also when N is negative (see (I.0.2)) and it is actually a weaker requirement. In this setting, the Ricci bound with negative parameter N naturally arises when considering suitable generalizations of the entropy functional, stemming from the Bregman divergence in information geometry, which is closely related to the Rényi entropies in statistical mechanics (cf. [OT11] and [OT13]).

Motivated by this, Ohta [Oht16] introduced the $\mathsf{CD}(K, N)$ condition for negative values of the dimensional parameter N, properly adapting Sturm's approach (of [Stu06b]). He proved that his definition is coherent with the differential approach in the smooth setting and, in particular, that any $\mathsf{CD}(K, N)$ space, for some $K \in \mathbb{R}$ and $N > 1$, is a $\mathsf{CD}(K, N')$ space for every $N' < 0$. The main difference with the classical definition (with positive N) is that, when the dimensional parameter N is negative, the $\mathsf{CD}(K, N)$ condition does not guarantee any nice behavior of the reference measure. Indeed, already in some simple examples on the real line (see [Oht16, Example 2.4]), the reference measure fails to be locally finite.

I.4.1 Paper 6: Convergence of metric measure spaces satisfying the CD condition for negative values of the dimension parameter [MRS23]

The objective of this paper is to address whether the curvature-dimension condition, with negative value of generalized dimension, is stable under convergence in a suitable topology. Special attention has to be payed to establishing an appropriate setting. In fact, the classical one of Polish spaces equipped with Radon measures is not fit for the purpose, as it does not allow to study singularties of the reference measure. For this reason we introduce the notion of *pointed generalized metric measure space*, as a structure $(\mathsf{X}, \mathsf{d}, \mathfrak{m}, \mathcal{C}, p)$ where:

- (X, d) is a Polish length metric space and $p \in \mathsf{X}$ is a distinguished point,
- $\mathfrak{m}$ is a non-null *quasi-Radon* measure on X.
- $\mathcal{C} \subset \mathsf{X}$ is a closed set with empty interior and $\mathfrak{m}(\mathcal{C}) = 0$ (which will represent the set of singular points for the measure).

A quasi-Radon measure $\mathfrak{m}$ can be seen as a measure which is Radon outside the (closed) set of singular points:

$$\mathcal{S}_{\mathfrak{m}} := \{x \in \mathsf{X} \,:\, \mathfrak{m}(U) = \infty \text{ for every open neighborhood } U \text{ of } x\}.$$

We define the distance $\mathsf{d}_{\mathsf{iKRW}}$ between pointed generalized metric measure spaces and we prove that the $\mathsf{CD}(K, N)$ condition (with negative N) is stable with respect to the induced topology. The distance $\mathsf{d}_{\mathsf{iKRW}}$ is defined starting from the case of spaces with finite mass. In particular, given $\mathbb{X}_1 := (\mathsf{X}_1, \mathsf{d}_1, \mathfrak{m}_1, \mathcal{C}_1, p_1)$, $\mathbb{X}_2 := (\mathsf{X}_2, \mathsf{d}_2, \mathfrak{m}_2, \mathcal{C}_2, p_2)$ pointed generalized metric measure spaces with finite mass, we set

$$\begin{aligned} \mathsf{d}_{\mathsf{iKRW}}(\mathbb{X}_1, \mathbb{X}_2) := & \left| \log \left(\frac{\mathfrak{m}_1(\mathsf{X}_1)}{\mathfrak{m}_2(\mathsf{X}_2)} \right) \right| \\ & + \inf \Big\{ \mathsf{d}\big(i_1(p_1), i_2(p_2)\big) + \mathsf{d}_H\big(i_1(\mathcal{C}_1), i_2(\mathcal{C}_2)\big) + W_{\mathsf{c}}\big((i_1)_\sharp \bar{\mathfrak{m}}_1, (i_2)_\sharp \bar{\mathfrak{m}}_2\big) \Big\}, \end{aligned} \tag{I.4.1}$$

where the infimum is taken over all isometric embeddings $i_j \colon (\mathsf{X}_j, \mathsf{d}_j) \to (\mathsf{X}, \mathsf{d})$ onto a Polish metric space. In (I.4.1), d_H is the Hausdorff distance, $\bar{\mathfrak{m}}_j := \frac{\mathfrak{m}_j}{\mathfrak{m}_j(\mathsf{X}_j)}$ is the normalization of the measure $\mathfrak{m}_j$, for $j = 1, 2$, and W_c denotes the Wasserstein distance with respect to a concave cost c. Given instead any pair $\mathbb{X}_1 := (\mathsf{X}_1, \mathsf{d}_1, \mathfrak{m}_1, \mathcal{C}_1, p_1)$, $\mathbb{X}_2 := (\mathsf{X}_2, \mathsf{d}_2, \mathfrak{m}_2, \mathcal{C}_2, p_2)$ of pointed generalized metric measure spaces, we define

$$\mathsf{d}_{\mathsf{iKRW}}(\mathbb{X}_1, \mathbb{X}_2) := \sum_{k\in\mathbb{N}} \frac{1}{2^k} \min\Big\{1, \mathsf{d}_{\mathsf{iKRW}}(\mathbb{X}_1^k, \mathbb{X}_2^k)\Big\},$$

where $\mathbb{X}_i^k = (\mathsf{X}_i, \mathsf{d}_i, f_i^k \mathfrak{m}_i, \mathcal{C}_i, p_i)$ is the k-th cut of $\mathbb{X}_i$, for $i \in \{1, 2\}$. In particular, $(f_i^k)_{k\in\mathbb{N}}$ is a sequence of cut-off functions, vanishing inside consecutively smaller neighborhoods of $\mathcal{S}_{\mathfrak{m}_i}$ and outside consecutively bigger balls centred in p_i.

The distance $\mathsf{d}_{\mathsf{iKRW}}$ is a natural generalization of the distance $\mathrm{p}\mathbb{G}_{\mathrm{W}}$ on pointed metric measure spaces (equipped with a Radon measure), introduced by Gigli, Mondino and Savaré in [GMS15]. As a matter of fact, in their setting the singular set is always the empty set, and the two distances coincide. Moreover, in [GMS15] the authors prove that the distance $\mathrm{p}\mathbb{G}_{\mathrm{W}}$ metrizes the (pointed) measured Gromov-Hausdorff topology. Therefore, the convergence induced by the distance $\mathsf{d}_{\mathsf{iKRW}}$ is a consistent adaptation of the (pointed) measured Gromov-Hausdorff convergence to pointed generalized metric measure spaces.

The main theorem of the paper shows that the $\mathsf{CD}(K, N)$ condition (with negative N) is stable with respect to the topology induced by $\mathsf{d}_{\mathsf{iKRW}}$:

Theorem I.4.1. *Given $K \in \mathbb{R}$ and $N \in (-\infty, 0)$, let $\big\{(\mathsf{X}_n, \mathsf{d}_n, \mathfrak{m}_n, \mathcal{S}_{\mathfrak{m}_n}, p_n)\big\}_{n\in\mathbb{N}}$ be a sequence of pointed generalized metric measure spaces converging to $(\mathsf{X}_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, \mathcal{S}_{\mathfrak{m}_\infty}, p_\infty)$ with respect to the distance $\mathsf{d}_{\mathsf{iKRW}}$. Assume that:*

- *(i)* $(\mathsf{X}_n, \mathsf{d}_n, \mathfrak{m}_n)$ *is a* $\mathsf{CD}(K, N)$ *space for every* $n \in \mathbb{N}$*;*
- *(ii) there exists* $\omega : \mathbb{N} \times \mathbb{N} \times \mathbb{R}^+ \to [0, 1]$*, for which* $(\mathsf{X}_n, \mathsf{d}_n, \mathfrak{m}_n)$ *is* ω*-uniformly convex, for every* $n \in \mathbb{N}$*;*
- *(iii)* $\sup_{n\in\mathbb{N}\cup\{\infty\}} \mathrm{diam}(\mathsf{X}_n, \mathsf{d}_n) < \pi\sqrt{\frac{1}{|K|}}$*, if* $K < 0$*.*

Then $(\mathsf{X}_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty)$ *is a* $\mathsf{CD}(K, N)$ *space.*

Assumption *(ii)* (see Paper 6 for a precise definition) is a technical requirement that controls the behaviour of the singular sets, uniformly through the sequence, in order to avoid a degenerate structure at the limit.

I.4.2 Paper 7: Optimal maps and local-to-global property in negative dimensional spaces with Ricci curvature bounded from below [MR21]

In [Oht16], Ohta introduced also the *reduced curvature-dimension condition* $\mathsf{CD}^*(K, N)$ for negative values of the dimensional parameter N, following the approach developed by Bacher and Sturm [BS10], for the positive dimensional case. This condition is a minor modification of the classical $\mathsf{CD}(K, N)$ one, which uses the distortion coefficients $\sigma_{K,N}^{(t)}$ in place of $\tau_{K,N}^{(t)}$, we refer to [Oht16, Definition 4.5] for the precise definition.

In this work, we prove two important properties for metric measure spaces satisfying the $\mathsf{CD}^*(K, N)$ condition for negative values of the parameter N: the existence of a transport map between two absolutely continuous marginals and the so-called local-to-global property. The proof of these features for $\mathsf{CD}(K, N)$ spaces, with positive dimensional bound N, heavily rely on the lower semicontinuity of the characterizing entropy functional. Unfortunately, with a quasi-Radon reference measure, the lower semicontinuity of the entropy functional does not hold on the whole $\mathscr{P}_2(\mathsf{X})$. The main proofs are based on suitable adaptations of these classical arguments, in which we have to pay particular attention to the set of singular points of the reference measure.

The problem of addressing existence and/or uniqueness of optimal transport maps between two given marginals has a long history, since it represents the original formulation of the optimal transport problem by Monge. The first positive answers were given in classical settings by Brenier [Bre87], McCann [McC01], Ambrosio-Rigot [AR04], Figalli-Rifford [FR10] and Bertrand [Ber08]. In the context of metric measure spaces, most of the results are proven under a non-branching assumption and a metric curvature bound. In particular, we recall the works by Gigli [Gig12], Rajala-Sturm [RS14], Gigli-Rajala-Sturm [GRS16] and Cavalletti-Mondino [CM17a]. Moreover, it has been shown by Rajala in [Raj16] that the non-branching assumption is necessary to prove the uniqueness of the transport map.

In this paper, we solve Monge's problem in essentially non-branching spaces satisfying the $\mathsf{CD}^*(K,N)$-condition for negative N:

Theorem I.4.2. *Let $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be an essentially non-branching $\mathsf{CD}^*(K,N)$ space, for some $K\in\mathbb{R}$ and $N<0$. Then, for every $\mu_0,\mu_1\in\mathscr{P}_2(\mathsf{X})$ which are absolutely continuous with respect to $\mathfrak{m}$ there exists a unique optimal transport plan $\pi\in\mathsf{Adm}(\mu_0,\mu_1)$ and it is induced by a map.*

The local-to-global property of the reduced curvature-dimension condition is an important and remarkable feature. In fact, it shows that the synthetic curvature-dimension bound $\mathsf{CD}(K,N)$ is a local requirement, as it happens in the case of Ricci curvature bounds in Riemannian setting. In the positive dimensional case, the local-to-global property was proved by Sturm [Stu06b] for the $\mathsf{CD}(K,\infty)$-condition, by Villani [Vil09] for the $\mathsf{CD}(0,N)$-condition and then by Bacher-Sturm [BS10] for the general $\mathsf{CD}^*(K,N)$ one. Thereafter, the globalization of the $\mathsf{CD}(K,N)$ condition was proved by Cavalletti and Milman [CM21] with a much more sophisticated argument, which allowed them to demonstrate other remarkable properties of the $\mathsf{CD}(K,N)$ condition.

In this paper we prove the analogous of the result by Bacher and Sturm [BS10], for $\mathsf{CD}^*(K,N)$ spaces with negative dimensional bound N.

Theorem I.4.3. *Let $K,N\in\mathbb{R}$ with $N<0$ and let $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be a locally compact, essentially non-branching metric measure space such that $\mathscr{P}^*_N(\mathsf{X},\mathfrak{m})$ is a geodesic space. If $(\mathsf{X},\mathsf{d},\mathfrak{m})$ satisfies the condition $\mathsf{CD}^*(K-,N)$ locally, then it satisfies the condition $\mathsf{CD}^*(K-,N)$ globally.*

In the statement, $\mathsf{CD}^*(K-,N)$ denotes that the $\mathsf{CD}^*(K',N)$ condition holds for every $K'<K$, while $\mathscr{P}^*_N(\mathsf{X},\mathfrak{m})$ is the set of probabilities having finite N-Rényi entropy.

Paper 1

Examples of $\mathsf{CD}(0, N)$ *spaces with non-constant dimension*

In this work, we generalize the results obtained in [Mag22a], presenting some examples of $\mathsf{CD}(0, N)$ *spaces having different dimensions in different regions, deducing in particular that the topological splitting may fail in* $\mathsf{CD}(0, N)$ *spaces. We also observe that any reasonable non-branching condition may fail in* $\mathsf{CD}(0, N)$ *spaces and that the existence of an optimal transport map, between two absolutely continuous marginals, is not guaranteed by the* $\mathsf{CD}(0, N)$ *condition, without requiring a non-branching assumption. Moreover, we show that the strict* $\mathsf{CD}(0, N)$ *condition is strictly stronger than the classical* $\mathsf{CD}(0, N)$ *one and it is not stable with respect to the measured Gromov-Hausdorff convergence.*

1.1 Introduction

In their seminal papers [Stu06a, Stu06b, LV09], Sturm and Lott–Villani introduced the so-called $\mathsf{CD}(K, N)$ condition, a synthetic notion representing a lower curvature bound by K and an upper bound on the dimension by N, formulated in the non-smooth setting of metric measure spaces. Their works are based on the observation that, for a (weighted) Riemannian manifold, having Ricci curvature bounded below and dimension bounded above, can be equivalently characterized in terms of a convexity property of the Rényi entropy functional, along Wasserstein geodesics. In particular, this property relies on the theory of optimal transport and does not require the smooth underlying structure, therefore it can be taken as definition of curvature dimension bound for a metric measure space.

In this paper we show different examples of $\mathsf{CD}(0, N)$ spaces (i.e. spaces satisfying the $\mathsf{CD}(0, N)$ condition), having different singularities in their metric measure structure. This work is a twofold generalization of [Mag22a], where an example of a highly branching $\mathsf{CD}(0, \infty)$ space with non-constant (topological) dimension is constructed. On the one hand, we extend the result of [Mag22a] to $\mathsf{CD}(0, N)$ spaces, having a finite dimensional bound N. This generalization is somewhat expected but far from trivial, in fact the finite dimensional CD condition implies some properties which are not guaranteed in $\mathsf{CD}(0, \infty)$ spaces (for example the Bishop-Gromov inequality). On the other hand, we prove the $\mathsf{CD}(0, N)$ condition for a class of metric measure spaces which is considerably larger than the one considered in [Mag22a]. This allows to highlight other types of singular behaviour that are proved to be possible in CD spaces.

The main conclusions, which can be drawn from the examples of CD spaces considered in this paper, are the following:

- The constancy of dimension may fail in $\mathsf{CD}(0,N)$ spaces, i.e. there exists a $\mathsf{CD}(0,N)$ space having different topological and Hausdorff dimensions in different regions (see Section 1.4.1). This is particularly interesting if seen in relation with the work of Brué and Semola [BS20], that proves constancy of dimension for $\mathsf{RCD}(K,N)$ spaces.
- The strict $\mathsf{CD}(0,N)$ condition (see Definition 1.4.4) is not stable with respect to the measured Gromov-Hausdorff convergence. Moreover, the strict $\mathsf{CD}(0,N)$ condition is strictly stronger than the classical $\mathsf{CD}(0,N)$ one, i.e. there exists a $\mathsf{CD}(0,N)$ space which does not satisfy the strict $\mathsf{CD}(0,N)$ condition (see Section 1.4.2).
- Any reasonably meaningful non-branching condition may fail in $\mathsf{CD}(0,N)$ spaces (see Section 1.4.2).
- The existence of an optimal transport map between two absolutely continuous marginals is not guaranteed in $\mathsf{CD}(0,N)$ spaces, without assuming a non-branching condition (see Section 1.4.2).
- The topological splitting may fail in $\mathsf{CD}(0,N)$ spaces, i.e. there exists a $\mathsf{CD}(0,N)$ space containing a subset isometric to $\mathbb{R}$, which does not topologically split as the product of $\mathbb{R}$ with another space (see Section 1.4.3).

This work, as it is for [Mag22a], is inspired by the work of Ketterer and Rajala [KR15], where similar conclusions were drawn for spaces satisfying the measure contraction property $\mathsf{MCP}(0,N)$. We consider metric measure measure spaces having metric structure and singularities similar to the ones consider in [KR15], but equipped with a more complicated measure that allows to achieve the CD condition (cfr. [KR15, Remark 1]). This idea was already developed in [Mag22a], were the $\mathsf{CD}(0,\infty)$ condition was proved in a space with non-constant topological dimension. In this paper, we improve the strategy of [Mag22a], extending it to a larger class of spaces and refining different estimates and computations to prove the $\mathsf{CD}(0,N)$ condition. This generalization relies on the relation between the $(0,N)$-convexity of interpolating densities along geodesics and the $\mathsf{CD}(0,N)$ condition (see [Stu06b, Proposition 4.2]). The combination of this observation with a suitable version of the Jacobi equation proved by Rajala in [Raj16], that allows to compute interpolating densities, results in Proposition 1.2.11, which will be fundamental to prove the $\mathsf{CD}(0,N)$ condition.

1.2 Preliminaries

1.2.1 CD spaces

A triple $(X,\mathsf{d},\mathfrak{m})$ is called metric measure space if (X,d) is a complete and separable metric space and $\mathfrak{m}$ is a locally finite Borel measure on it. Denote by $\mathscr{P}(X)$ the set of Borel probability measures on X and by $\mathscr{P}_2(X)\subset\mathscr{P}(X)$ the set of those having finite second moment. We endow the space $\mathscr{P}_2(X)$ with the Wasserstein distance W_2, defined by

$$W_2^2(\mu_0,\mu_1):=\inf_{\pi\in\mathsf{Adm}(\mu_0,\mu_1)}\int \mathsf{d}^2(x,y)\,\mathrm{d}\pi(x,y), \tag{1.2.1}$$

where $\mathsf{Adm}(\mu_0,\mu_1)$ is the set of all the admissible transport plans between μ_0 and μ_1, namely all the measures in $\mathscr{P}(X^2)$ such that $(\mathsf{p}_1)_\sharp\pi=\mu_0$ and $(\mathsf{p}_2)_\sharp\pi=\mu_1$. We say that $\pi\in\mathsf{Adm}(\mu_0,\mu_1)$ is an optimal transport plan between μ_0 and μ_1 if it realizes the infimum in (1.2.1). Moreover, we say that an optimal transport plan π is induced by a map if there exists T measurable such that $\pi = (\mathrm{id}, T)_\# \mu_0$, in this case T is said to be an optimal transport map. For every $N > 1$, define the Rényi entropy functional

$$S_N(\mu) = -\int \rho^{1-\frac{1}{N}} \, \mathrm{d}\mathfrak{m},$$

where ρ denotes the density of the absolutely continuous part of μ with respect to the reference measure $\mathfrak{m}$, i.e. $\mu = \rho\mathfrak{m} + \mu^s$ with $\mu^s \perp \mathfrak{m}$. Call $\mathscr{P}_{ac}(X, \mathfrak{m})$ the set of all probability measures in $\mathscr{P}_2(X)$ which are absolutely continuous with respect to the reference measure $\mathfrak{m}$.

Definition 1.2.1. Given $N > 1$, the metric measure space $(X, \mathsf{d}, \mathfrak{m})$ is said to be a $\mathsf{CD}(0, N)$ space (or to satisfy the $\mathsf{CD}(0, N)$ condition) if for every pair of measures $\mu_0, \mu_1 \in \mathscr{P}_{ac}(X, \mathfrak{m})$ there exists a constant speed W_2-geodesic $(\mu_t)_{t\in[0,1]} \subset \mathscr{P}_{ac}(X, \mathfrak{m})$ connecting them, along which the entropy functional $S_{N'}$ is convex for every $N' \geq N$, i.e.

$$S_{N'}(\mu_t) \leq (1-t) S_{N'}(\mu_0) + t S_{N'}(\mu_1), \quad \forall t \in [0, 1].$$

Remark 1.2.2. Every constant speed W_2-geodesic $(\mu_t)_{t\in[0,1]}$ can be represented by a probability measure η on the space $\mathrm{Geo}(X)$ of constant speed geodesics of X (parameterized on $[0, 1]$), meaning that $(e_t)_\# \eta = \mu_t$ for every $t \in [0, 1]$, where e_t is the evaluation map at time t.

As every convexity property, the $\mathsf{CD}(0, N)$ condition can be equivalently characterized just looking at midpoints instead of whole geodesics (cfr. [Mag22a, Proposition 1.8]):

Proposition 1.2.3. *The metric measure space* $(X, \mathsf{d}, \mathfrak{m})$ *is a* $\mathsf{CD}(0, N)$ *space if for every pair* $\mu_0, \mu_1 \in \mathscr{P}_{ac}(X, \mathfrak{m})$ *there exists a midpoint* $\nu \in \mathscr{P}_{ac}(X, \mathfrak{m})$ *of* μ_0 *and* μ_1*, such that for every* $N' \geq N$

$$S_{N'}(\nu) \leq \frac{1}{2} S_{N'}(\mu_0) + \frac{1}{2} S_{N'}(\mu_1).$$

The CD condition can be defined also when the dimensional parameter takes the value ∞, requiring convexity of the entropy functional $\mathrm{Ent} : \mathscr{P}_2(X) \to \mathbb{R} \cup \{+\infty\}$, defined as

$$\mathrm{Ent}(\mu) := \begin{cases} \int \rho \log \rho \, \mathrm{d}\mathfrak{m} & \text{if } \mu \ll \mathfrak{m} \text{ and } \mu = \rho\mathfrak{m} \\ +\infty & \text{otherwise} \end{cases}, \tag{1.2.2}$$

which is the limit of $N + N S_N$ as $N \to \infty$.

Definition 1.2.4. A metric measure space $(X, \mathsf{d}, \mathfrak{m})$ is said to be a $\mathsf{CD}(0, \infty)$ space (or to satisfy the $\mathsf{CD}(0, \infty)$ condition) if for every pair of measures $\mu_0, \mu_1 \in \mathscr{P}_{ac}(X, \mathfrak{m})$ there exists a constant speed W_2-geodesic $(\mu_t)_{t\in[0,1]}$ connecting them, along which the entropy functional Ent is convex, i.e.

$$\mathrm{Ent}(\mu_t) \leq (1-t) \mathrm{Ent}(\mu_0) + t \, \mathrm{Ent}(\mu_1), \quad \forall t \in [0, 1].$$

Remark 1.2.5. Observe that the entropy functionals Ent an S_N (for some $N > 1$) have different behaviour on singular measures, in fact $\mathrm{Ent}(\mu) = +\infty$ whenever $\mu \not\ll \mathfrak{m}$, while the singular part of μ does not contribute to the value of $S_N(\mu)$. This substantial difference is the reason why, in the definition of the $\mathsf{CD}(0, \infty)$ condition there is no need to require the Wasserstein geodesic to be contained in $\mathscr{P}_{ac}(X, \mathfrak{m})$.

The $\mathsf{CD}(K, N)$ condition can be defined for every real value of the dimensional parameter K, but its formulation becomes more complicated as it involves the distortion coefficients $\tau_{K,N}^{(t)}$. We refer to [Stu06b] for the precise definition of the $\mathsf{CD}(K, N)$ condition (with $K \neq 0$), this general notion will not be used in this work. Among other nice properties, the $\mathsf{CD}(K, N)$ condition has the monotonicity in the parameters that we expect from a requirement that represents a lower curvature bound and an upper dimensional bound, i.e.

$$\mathsf{CD}(K',N') \implies \mathsf{CD}(K'',N'') \quad \text{if } K' \geq K'' \text{ and } N' \leq N'',$$

for every $K', K'' \in \mathbb{R}$ and $N', N'' \in (1,\infty]$.

1.2.2 (K,N)-convexity of functions

Given $K \in \mathbb{R}$, $N > 0$ and an interval $I \subseteq \mathbb{R}$, a function $g \in C^2(I,\mathbb{R})$ is said to be (K,N)-convex if

$$g''(x) \geq K + \frac{1}{N}(g'(x))^2 \quad \text{for every } x \in I.$$

In working with (K,N)-convex is often convenient to use the following equivalent characterization: $g \in C^2(I,\mathbb{R})$ is (K,N)-convex if and only if the function $g_N := e^{-g/N}$ satisfies

$$g_N''(x) \leq -\frac{K}{N} g_N.$$

As a consequence, we also deduce that:

$$g \in C^2(I,\mathbb{R}) \text{ is } (0,N)\text{-convex if and only if } g_N \text{ is concave.} \tag{1.2.3}$$

Example 1.2.6. The function $-\log : (0,\infty) \to \mathbb{R}$ is $(0,1)$-convex, while for every $K > 0$ the function

$$[0,1] \ni x \mapsto Kx^2$$

is $(0,2K)$-convex.

Remark 1.2.7. The $(0,N)$-convexity is invariant by linear reparametrization, meaning that, if $g \in C^2(I,\mathbb{R})$ is $(0,N)$-convex, then for every $\alpha \in \mathbb{R}$ and $\beta \neq 0$ the function

$$t \mapsto g(\alpha + \beta t)$$

is $(0,N)$-convex (where defined). More in general, if $g \in C^2(I,\mathbb{R})$ is (K,N)-convex, then for every $\alpha \in \mathbb{R}$ and $\beta \neq 0$ the function $t \mapsto g(\alpha + \beta t)$ is $(\beta^2 K, N)$-convex.

The (K,N)-convexity enjoys different nice properties, among these the following additivity for the constants will be particularly important in section 1.3.

Lemma 1.2.8. *If $g \in C^2(I,\mathbb{R})$ is (K_1,N_1)-convex and $h \in C^2(I,\mathbb{R})$ is (K_2,N_2)-convex, then $g+h$ is (K_1+K_2, N_1+N_2)-convex,*

For a proof of this lemma we refer to [EKS15, Lemma 2.10].

We conclude the subsection with a technical lemma that will be useful to prove the main result (Theorem 1.3.1).

Lemma 1.2.9. *Given $A \in [0,\infty)$ and $\delta \in \left(-\frac{1}{2^{11}}, \frac{1}{2^{11}}\right)$, there exists a function $h : [0,1] \to \mathbb{R}$ strictly positive on $[0,1)$, with $h(0) = 1$, $h(1) = A$ and $h(\frac{1}{2}) = 1 + (\frac{1}{2} + \delta)(A-1)$, such that $t \mapsto -\log(h(t))$ is $(-2^{21}\,\delta^2, 2)$-convex*

Proof. Let $\phi : [0,1] \to [0,1]$ be a C^2 function such that $\phi(\frac{1}{2}) = 1$, $\phi(t) = 0$ if $t \in [0,\frac{1}{4}] \cup [\frac{3}{4},1]$ and $|\phi'| \leq 2^4$, $|\phi''| \leq 2^7$ on $[0,1]$. Define h as

$$h(t) = 1 + t(A-1) + \delta\phi(t)(A-1),$$

observe that
$$h'(t) = (A-1)[1+\delta\phi'(t)] \qquad \text{and} \qquad h''(t) = \delta\phi''(t)(A-1).$$
Now, we want to prove that
$$\inf_{t\in[0,1]} (-\log(h(t)))'' - \frac{1}{2}[(-\log(h(t)))']^2 = \inf_{t\in[0,1]} \frac{1}{2}\frac{h'(t)^2}{h(t)^2} - \frac{h''(t)}{h(t)} \geq -2^{21}\,\delta^2, \tag{1.2.4}$$
to this aim we divide the problem in two cases. First of all, we prove (1.2.4) when $|A-1| \geq 2^{11}\delta$. In this case we have that
$$\begin{aligned}|h(t)h''(t)| &= |\phi''(t)\big[\delta(A-1)+\delta(A-1)^2(1+\delta\phi(t))\big]| \\ &\leq 2^7\left[\frac{1}{2^{11}}|A-1|^2 + \frac{1}{2^{10}}|A-1|^2\right] \leq \frac{1}{4}|A-1|^2\end{aligned}$$
while, on the other hand,
$$h'(t)^2 \geq \frac{1}{2}|A-1|^2.$$
Putting together these two inequalities, we conclude that
$$\inf_{t\in[0,1]} (-\log(h(t)))'' - \frac{1}{2}[(-\log(h(t)))']^2 = \inf_{t\in[0,1]} \frac{h'(t)^2 - 2h(t)h''(t)}{2h(t)^2} \geq 0,$$
which in particular implies (1.2.4). Assume now that $|A-1| < 2^{11}\delta$. Notice that
$$\inf_{t\in[0,1]} (-\log(h(t)))'' - \frac{1}{2}[(-\log(h(t)))']^2 \geq -\sup_{t\in[0,1]} \frac{h''(t)}{h(t)} = -\sup_{t\in[\frac14,\frac34]} \frac{h''(t)}{h(t)}$$
where the last equality is true because ϕ is constant on $\left[0,\frac14\right] \cup \left[\frac34,1\right]$. Moreover, observe that, since $|\delta| < \frac{1}{2^{11}}$, we have $h(t) \geq \frac18$ on $\left[\frac14,\frac34\right]$, therefore we deduce
$$\inf_{t\in[0,1]} (-\log(h(t)))'' - \frac{1}{2}[(-\log(h(t)))']^2 \geq -8\sup_{t\in[\frac14,\frac34]} |h''(t)| \geq -8\sup_{t\in[0,1]} \delta|\phi''(t)||A-1| \geq -2^{21}\,\delta^2,$$
concluding the proof. □

1.2.3 Definition of the Metric Measure Spaces

In this section we define the metric measure spaces that will be studied in the following. For every $0<k<\frac14$, introduce the class
$$\mathscr{F}_k := \{f \in C^2([-1,1]) \,:\, 0<f<3k,\ |f'| \leq k,\ |f''| \leq 1\}. \tag{1.2.5}$$
Then, for every $f \in \mathscr{F}_k$ define the set
$$X_f = \{(x,y) \in \mathbb{R}^2 \,:\, x \in [-1,1] \text{ and } 0 \leq y \leq f(x)\}.$$
This space will be equipped by the distance d_∞ induced by the l_∞ norm on $\mathbb{R}^2$, that is
$$\mathsf{d}_\infty((x_1,y_1),(x_2,y_2)) = \max\{|x_2-x_1|,|y_2-y_1|\}.$$
Observe that, since we imposed k to be less than $\frac14$, d_∞ is a geodesic distance on X_f. Finally, for every $K \geq 1$, define the measure $\mathfrak{m}_{f,K}$ on X_f as
$$\mathfrak{m}_{f,K} = m_{f,K}(x,y)\cdot\mathscr{L}^2|_{X_f} := \frac{1}{f(x)}\exp\left(-K\left(\frac{y}{f(x)}\right)^2\right)\cdot\mathscr{L}^2|_{X_f}.$$

A simple computation shows that for every $f \in \mathscr{F}_k$ and $K \geq 1$ it holds that

$$(\mathtt{p}_x)_\# \mathfrak{m}_{f,K} = C_K \cdot \chi_{\{-1 \leq x \leq 1\}} \cdot \mathcal{H}^1,$$

where $C_K = \int_0^1 e^{-Ky^2}\, \mathrm{d}y$ and $\mathtt{p}_x$ denotes the projection on the x-axis.

In section 1.3 we will prove that it is possible to find constants $k \in (0, \frac{1}{4})$, $K \geq 1$ and $N > 1$ such that, for every $f \in \mathscr{F}_k$, the metric measure space $(X_f, \mathsf{d}_\infty, \mathfrak{m}_{f,K})$ satisfies the $\mathsf{CD}(0,N)$ condition. In the following, we will assume to have fixed an $f \in \mathscr{F}_k$ and we will develop an argument that only uses the properties of f, see (1.2.5), proving in particular the result for the whole class of functions. Moreover, in order to ease the notation, we will usually denote the space $(X_f, \mathsf{d}_\infty, \mathfrak{m}_{f,K})$ simply by $(X, \mathsf{d}, \mathfrak{m})$.

1.2.4 How to Prove Convexity of the Entropy

In this section we prove an important result (Proposition 1.2.11) that will be a fundamental ingredient in proving the CD condition. The proof of Proposition 1.2.11 relies on the possibility to compute the density of a pushforward measure, through Jacobi equation. For example, take two measures $\mu_0, \mu_1 \in \mathscr{P}_2(\mathbb{R}^2)$ absolutely continuous with respect to the Lebesgue measure $\mathscr{L}^2$, with densities ρ_0 and ρ_1. If there exists a smooth one-to-one map $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T_\# \mu_0 = \mu_1$, then Jacobi equation ensures that

$$\rho_1(T(x,y)) J_T(x,y) = \rho_0(x,y), \tag{1.2.6}$$

for μ_0-almost every (x,y). The assumptions on the map T can be relaxed in different ways, in this work we are particularly interested in the following version, which is a straightforward consequence of [Raj16, Proposition 2.1].

Proposition 1.2.10. *Let $\mu_0, \mu_1 \in \mathscr{P}_2(\mathbb{R}^2)$ be absolutely continuous with respect to the Lebesgue measure $\mathscr{L}^2$ and assume that there exists a map $T = (T_1, T_2)$ which is injective outside a μ_0-null set, such that $T_\# \mu_0 = \mu_1$. Suppose also that T_1 locally does not depend on the y coordinate and it is increasing in x, while T_2 is increasing in y for every fixed x. Then the Jacobi equation* (1.2.6) *is satisfied with $J_T = \frac{\partial T_1}{\partial x} \frac{\partial T_2}{\partial y}$.*

This proposition shows in particular that the Jacobi equation can be properly adapted to our setting.

Using Jacobi equation we can deduce the following criterion to prove convexity of the entropy, by looking at local quantities instead of global ones.

Proposition 1.2.11. *Let $\mu_0, \mu_1 \in \mathscr{P}_{ac}(X, \mathfrak{m})$ and $T : X \to X$ be an optimal transport map between μ_0 and μ_1, in particular $T_\# \mu_0 = \mu_1$. Consider a midpoint $\mu_{1/2} \in \mathscr{P}_{ac}(X, \mathfrak{m})$ of μ_0 and μ_1, assume that $\mu_{1/2} = [M \circ (\mathrm{id}, T)]_\# \mu_0$ where the map $M : X \times X \to X$ is a (measurable) midpoint selection. Suppose also that the maps T and $M \circ (\mathrm{id}, T) : X \to X$ are injective outside a μ_0-null set and they satisfy the Jacobi equation* (1.2.6), *with suitable Jacobian functions J_T and $J_{M\circ(\mathrm{id},T)}$. If*

$$\Big(m(M((x,y),T(x,y)))J_{M\circ(\mathrm{id},T)}(x,y)\Big)^{\frac{1}{N}} \geq \frac{1}{2}\left(m(T(x,y))J_T(x,y)\right)^{\frac{1}{N}} + \frac{1}{2}(m(x,y))^{\frac{1}{N}}$$

for μ_0-almost every (x,y), then

$$S_N(\mu_{1/2}) \leq \frac{1}{2} S_N(\mu_0) + \frac{1}{2} S_N(\mu_1). \tag{1.2.7}$$

Proof. Set $\mu_0 = \rho_0\mathfrak{m} = \tilde{\rho}_0\mathscr{L}^2$, $\mu_1 = \rho_1\mathfrak{m} = \tilde{\rho}_1\mathscr{L}^2$ and $\mu_{1/2} = \rho_{1/2}\mathfrak{m} = \tilde{\rho}_{1/2}\mathscr{L}^2$, observe that, in order to prove (1.2.7), it is sufficient to prove that

$$\rho_{1/2}(M((x,y),T(x,y)))^{-\frac{1}{N}} \geq \frac{1}{2}\rho_1(T(x,y))^{-\frac{1}{N}} + \frac{1}{2}\rho_0(x,y)^{-\frac{1}{N}}, \tag{1.2.8}$$

for μ_0-almost every (x,y). On the other hand, our assumption on the validity of Jacobi equation for T ensures that

$$\tilde{\rho}_1(T(x,y))J_T(x,y) = \tilde{\rho}_0(x,y),$$

for μ_0-almost every (x,y), and thus that

$$\rho_1(T(x,y))m(T(x,y))J_T(x,y) = \rho_0(x,y)m(x,y).$$

for μ_0-almost every (x,y). Analogously, since the Jacobi equation holds for $M \circ (\mathrm{id}, T)$, we can deduce that

$$\rho_{1/2}(M((x,y),T(x,y)))m(M((x,y),T(x,y)))J_{M\circ(\mathrm{id},T)}(x,y) = \rho_0(x,y)m(x,y),$$

for μ_0-almost every (x,y). Therefore, (1.2.8) is equivalent to

$$\left(\frac{\rho_0(x,y)m(x,y)}{m(M((x,y),T(x,y)))J_{M\circ(\mathrm{id},T)}(x,y)}\right)^{-\frac{1}{N}} \geq \frac{1}{2}\left(\frac{\rho_0(x,y)m(x,y)}{m(T(x,y))J_T(x,y)}\right)^{-\frac{1}{N}} + \frac{1}{2}(\rho_0(x,y))^{-\frac{1}{N}}.$$

Some easy rearrangements show that this last equation is equivalent to

$$\Big(m(M((x,y),T(x,y)))J_{M\circ(\mathrm{id},T)}(x,y)\Big)^{\frac{1}{N}} \geq \frac{1}{2}\left(m(T(x,y))J_T(x,y)\right)^{\frac{1}{N}} + \frac{1}{2}(m(x,y))^{\frac{1}{N}},$$

concluding the proof. □

In the proof of the main result (Theorem 1.3.1) we use Proposition 1.2.11, together with Proposition 1.2.3, to prove the $\mathsf{CD}(0,N)$ condition. This will is possible because the maps T and $M \circ (\mathrm{id}, T)$, that we are going to consider, satisfy the assumptions of Proposition 1.2.10.

1.2.5 Definition of the Midpoint

According to Proposition 1.2.3, in order to prove $\mathsf{CD}(0,N)$ condition, it is sufficient to show entropy convexity in a suitable midpoint of any pair of absolutely continuous measures. Observe that in highly branching metric measure spaces, like $(X,\mathsf{d},\mathfrak{m}) = (X_f,\mathsf{d}_\infty,\mathfrak{m}_{f,K})$, the choice of a midpoint can be done with great freedom. This is because, in general, both the optimal transport map and the geodesic interpolation are not unique. In this section, we present the midpoint selection used in [Mag22a]. In particular, for any pair $\mu_0,\mu_1 \in \mathscr{P}_{ac}(X,\mathfrak{m})$, we select a suitable optimal transport map T between them and then we identify a Wasserstein midpoint with a suitable midpoint interpolation map M. In order to define both the optimal transport map T and the midpoint interpolation map M, we introduce the sets $V, D, H, H_0, H_{\frac{1}{2}}, H_1 \subset \mathbb{R}^2 \times \mathbb{R}^2$ as:

$$V := \Big\{((x_0,y_0),(x_1,y_1)) \in \mathbb{R}^2 \times \mathbb{R}^2 : |x_0 - x_1| < |y_0 - y_1|\Big\},$$

$$D := \Big\{((x_0,y_0),(x_1,y_1)) \in \mathbb{R}^2 \times \mathbb{R}^2 : |x_0 - x_1| = |y_0 - y_1|\Big\},$$

$$H := \Big\{((x_0,y_0),(x_1,y_1)) \in \mathbb{R}^2 \times \mathbb{R}^2 : |x_0 - x_1| > |y_0 - y_1|\Big\} = H_0 \cup H_1,$$

where

$$H_0 := \left\{((x_0,y_0),(x_1,y_1)) \in \mathbb{R}^2 \times \mathbb{R}^2 : \frac{1}{2}|x_0 - x_1| \geq |y_0 - y_1|\right\},$$

$$H_1 := \left\{ \left((x_0, y_0) , (x_1, y_1) \right) \in \mathbb{R}^2 \times \mathbb{R}^2 : |x_0 - x_1| > |y_0 - y_1| > \frac{1}{2} |x_0 - x_1| \right\}.$$

First of all we present our optimal transport map selection, which follows the work of Rajala [Raj16]. In particular, given two absolutely continuous measures $\mu_0, \mu_1 \in \mathscr{P}(\mathbb{R}^2)$, he was able to select, via consecutive minimizations, an optimal transport map with different nice properties, which are summarized in the following statement.

Proposition 1.2.12. *Given two measures $\mu_0, \mu_1 \in \mathscr{P}(\mathbb{R}^2)$ which are absolutely continuous with respect to the Lebesgue measure $\mathscr{L}^2$, there exists a measurable optimal transport map $T = (T_1, T_2)$ between μ_0 and μ_1, injective outside a μ_0-null set, with the following properties. For μ_0-almost every (x, y), we have that*

$$\begin{array}{l} T_1 \text{ is locally constant in } y, \text{ if } ((x,y), T(x,y)) \in H \text{ and} \\ T_2 \text{ is locally constant in } x, \text{ if } ((x,y), T(x,y)) \in V \end{array} .$$

Moreover, the function $T_1(x, y)$ is increasing in x for every fixed y and the function $T_2(x, y)$ is increasing in y for every fixed x, therefore for μ_0-almost every (x, y) it holds

$$\frac{\partial T_1}{\partial x} \geq 0 \text{ and } \frac{\partial T_2}{\partial y} \geq 0, \text{ if } ((x,y), T(x,y)) \in H \cup V.$$

Now fix two measures $\mu_0, \mu_1 \in \mathscr{P}_{ac}(X, \mathfrak{m})$, observe that, since they are absolutely continuous with respect to the reference measure $\mathfrak{m}$, they are absolutely continuous also with respect to the Lebesgue measure $\mathscr{L}^2$. Call T the optimal transport map between μ_0 and μ_1, identified by Proposition 1.2.12. In order to identify a midpoint of μ_0 and μ_1, we need to select a proper midpoint interpolation map, i.e. a measurable map $M : X \times X \to X$ such that

$$\mathsf{d}_\infty(M(z,w), z) = \mathsf{d}_\infty(M(z,w), w) = \frac{1}{2} \mathsf{d}_\infty(z, w) \quad \text{for every } (z, w) \in X \times X,$$

the desired midpoint will be $M_\#((\mathrm{id}, T)_\# \mu_0) = [M \circ (\mathrm{id}, T)]_\# \mu_0$.

The midpoint interpolation map M that we are going to use in the following is defined in different ways on the sets V, D, H_0 and H_1. In particular, the precise definition is the following:

- If $\big((x_0, y_0), (x_1, y_1)\big) \in V \cup D$

$$M\big((x_0, y_0), (x_1, y_1)\big) := \left(\frac{x_0 + x_1}{2}, \frac{y_0 + y_1}{2} \right)$$

- If $\big((x_0, y_0), (x_1, y_1)\big) \in H_0$,

$$M\big((x_0, y_0), (x_1, y_1)\big) = \left(\frac{x_0 + x_1}{2}, \frac{1}{2} \left(\frac{y_0}{f(x_0)} + \frac{y_1}{f(x_1)} \right) f\left(\frac{x_0 + x_1}{2} \right) \right).$$

- If $\big((x_0, y_0), (x_1, y_1)\big) \in H_1$, with $x_0 < x_1$ and $y_0 < y_1$, introduce the quantity

$$\tilde{y}(x_0, x_1, y_0) = \frac{1}{2} \left(\frac{y_0}{f(x)} + \frac{y_0 + \frac{x_1 - x_0}{2}}{f(x_1)} \right) f\left(\frac{x_1 + x_0}{2} \right) - y_0,$$

  and consequently define

$$\begin{aligned} & M\big((x_0, y_0), (x_1, y_1)\big) \\ & \qquad = \left(\frac{x_0 + x_1}{2}, y_0 + \tilde{y}(x_0, x_1, y_0) + \left(\frac{x_1 - x_0}{2} - \tilde{y}(x_0, x_1, y_0) \right) \left(2 \frac{y_1 - y_0}{x_1 - x_0} - 1 \right) \right). \end{aligned}$$

  In the other cases where $\big((x_0, y_0), (x_1, y_1)\big) \in H_1$, M can be defined analogously, every proof from now on will be done only taking care of this case, implying it can be easily adapted to the other cases.

The next statement, which combines Proposition 5.3 and Proposition 5.4 in [Mag22a], ensures that M actually provides a midpoint selection and interacts well with the selected optimal transport map T.

Proposition 1.2.13. *For k sufficiently small, the map M is a midpoint interpolation map and the map $M \circ (\mathrm{id}, T)$ is injective outside a μ_0-null set.*

Remark 1.2.14. The proof Proposition 5.3 and Proposition 5.4 in [Mag22a] is done for a quite specific choice of the function f which is defining the profile of $X = X_f$. However the proof only requires f to satisfy the properties which define the set $\mathscr{F}_k$ (see (1.2.5)), therefore the statement is true for every $f \in \mathscr{F}_k$.

1.3 Proof of CD Condition

In this section we prove the main result of this work, showing the validity of the $\mathsf{CD}(0, N)$ condition for metric measure spaces of the type $(X_f, \mathsf{d}_\infty, \mathfrak{m}_{f,K})$, with $f \in \mathscr{F}_k$. The proof follows the strategy developed in [Mag22a], refining it in order to obtain the CD condition with finite dimensional parameter.

Theorem 1.3.1. *For k sufficiently small and K sufficiently large, there exists $N > 1$ such that for every $f \in \mathscr{F}_k$ the metric measure space $(X_f, \mathsf{d}_\infty, \mathfrak{m}_{f,K})$ is a $\mathsf{CD}(0, N)$ space.*

Before going into the proof of Theorem 1.3.1, we need the following preliminary lemma, which is an improvement of [Mag22a, Lemma 3.3].

Lemma 1.3.2. *Having fixed the constant $K \geq 1$ and given another constant $H > 0$, it is possible to find k sufficiently small such that the following holds. Given any C^2 function $y : I = [x_0, x_1] \to \mathbb{R}^+$ such that $y'(x) \geq \frac{1}{4}$ and $y''(x) \leq H \frac{k}{f(x)}$ for every $x \in I$, and calling f_I the maximum of f on the interval $I = [x_0, x_1]$ (i.e. $f_I = \max_{x_0 \leq x \leq x_1} f(x)$), the function $-\log(m(x, y(x)))$ is $(\frac{K}{32 f_I^2}, 32K)$-convex.*

Proof. From the proof of [Mag22a, Lemma 3.3], we know that

$$\frac{\partial}{\partial x}\big(- \log(m(x, y(x)))\big) = \frac{f'(x)}{f(x)} + 2K \frac{y(x)}{f(x)} \left(\frac{y'(x)}{f(x)} - \frac{y(x) f'(x)}{f(x)^2} \right),$$

in particular, keeping in mind that $\left|\frac{y(x)}{f(x)}\right| \leq 1$ and $|f'(x)| \leq k$, we deduce that

$$\begin{aligned}\left|\frac{\partial}{\partial x}\big(- \log(m(x, y(x)))\big)\right| &\leq \left|\frac{f'(x)}{f(x)}\right| + 2K \left|\frac{y(x) y'(x)}{f(x)^2}\right| + 2K \left|\frac{y(x) f'(x)}{f(x)^3}\right| \\ &\leq \frac{k}{f(x)} + 2K \frac{y'(x)}{f(x)} + 2K \frac{k}{f(x)} \leq 4K \frac{y'(x)}{f(x)},\end{aligned}$$

where the last inequality holds for k sufficiently small. Moreover, from the computations done in the proof of [Mag22a, Lemma 3.3], we deduce that for k sufficiently small

$$\frac{\partial^2}{\partial x^2}\big(- \log(m(x, y(x)))\big) \geq \frac{K}{f(x)^2}\, y'(x)^2.$$

We can then conclude that

$$\begin{aligned}\frac{\partial^2}{\partial x^2}\big(- \log(m(x, y(x)))\big) \geq \frac{K}{f(x)^2}\, y'(x)^2 &\geq \frac{K}{32 f_I^2} + \frac{1}{32K} \left[4K \frac{y'(x)}{f(x)} \right]^2 \\ &\geq \frac{K}{32 f_I^2} + \frac{1}{32K} \left[\frac{\partial}{\partial x}\big(- \log(m(x, y(x)))\big) \right]^2,\end{aligned}$$

which proves the $(\frac{K}{32f_I^2}, 32K)$-convexity. □

Proof of Theorem 1.3.1. Fix any $f \in \mathscr{F}_k$ and consider the metric measure space $(X,\mathsf{d},\mathfrak{m}) = (X_f, \mathsf{d}_\infty, \mathfrak{m}_{f,K})$. We are going to prove that, for k sufficiently small and K sufficiently large, $(X,\mathsf{d},\mathfrak{m})$ satisfies the $\mathsf{CD}(0,N)$ condition, for a suitable $N > 1$. Moreover, the whole argument will not depend on the specific choice of f, but only on the properties defining the set $\mathscr{F}_k$ (which f satisfies). In particular, the choice of the parameters k, K and N will be independent on $f \in \mathscr{F}_k$ and this will be sufficient to prove the statement.

Let $\mu_0, \mu_1 \in \mathscr{P}_{ac}(X,\mathfrak{m})$, then, according to Proposition 1.2.3, it is sufficient to prove that, for every $N' \geq N$, we have

$$S_{N'}(\mu_{1/2}) \leq (1-t)S_{N'}(\mu_0) + tS_{N'}(\mu_1),$$

where $\mu_{1/2} = [M \circ (\mathrm{id}, T)]_\# \mu_0$ is the midpoint selected in Section 1.2.5. Given Proposition 1.2.11, it is enough check that for every $N' \geq N$

$$\Big(m(M((x,y),T(x,y)))J_{M\circ(\mathrm{id},T)}(x,y)\Big)^{\frac{1}{N'}} \geq \frac{1}{2}\left(m(T(x,y))J_T(x,y)\right)^{\frac{1}{N'}} + \frac{1}{2}(m(x,y))^{\frac{1}{N'}}, \quad (1.3.1)$$

for μ_0-almost every (x,y). We are going to prove (1.3.1), in different cases depending on the pair $((x,y),T(x,y))$. Observe that, since we have to prove (1.3.1) for μ_0-almost every (x,y), we can assume that the conclusions of Proposition 1.2.12 hold for every (x,y) we are considering.

Let us start by proving (1.3.1) for the points (x,y) such that $((x,y),T(x,y)) \in H_0$, recall that in this case we have

$$M \circ (\mathrm{id},T)(x,y) = \left(\frac{x+T_1}{2}, \frac{1}{2}\Big(\frac{y}{f(x)} + \frac{T_2}{f(T_1)}\Big) f\left(\frac{x+T_1}{2}\right)\right).$$

In particular, keeping in mind Proposition 1.2.13 we can easily see that $M \circ (\mathrm{id},T)$ satisfies the assumption of Proposition 1.2.10, therefore it holds that

$$J_{M\circ(\mathrm{id},T)}(x,y) = \frac{1}{2}\left(1 + \frac{\partial T_1}{\partial x}\right)\frac{1}{2}\Big(\frac{1}{f(x)} + \frac{\frac{\partial T_2}{\partial y}}{f(T_1)}\Big) f\left(\frac{x+T_1}{2}\right).$$

Moreover, we have that

$$m(M((x,y),T(x,y))) = f\left(\frac{x+T_1}{2}\right)^{-1} \exp\left(\frac{-K}{4}\Big(\frac{y}{f(x)} + \frac{T_2}{f(T_1)}\Big)^2\right),$$

thus, putting together these last two equation, we deduce that

$$\begin{aligned}
&-\log\Big(m(M((x,y),T(x,y)))J_{M\circ(\mathrm{id},T)}(x,y)\Big) \\
&\quad = -\log\left(\frac{1}{2}\left(1+\frac{\partial T_1}{\partial x}\right)\frac{1}{2}\Big(\frac{1}{f(x)} + \frac{\frac{\partial T_2}{\partial y}}{f(T_1)}\Big)\exp\left(\frac{-K}{4}\Big(\frac{y}{f(x)} + \frac{T_2}{f(T_1)}\Big)^2\right)\right) \\
&\quad = -\log\left(\frac{1}{2}\left(1+\frac{\partial T_1}{\partial x}\right)\right) - \log\left(\frac{1}{2}\Big(\frac{1}{f(x)} + \frac{\frac{\partial T_2}{\partial y}}{f(T_1)}\Big)\right) + K\Big(\frac{1}{2}\Big(\frac{y}{f(x)} + \frac{T_2}{f(T_1)}\Big)\Big)^2.
\end{aligned}$$

On the other hand, we have that

$$-\log(m(x,y)) = -\log\left(\frac{1}{f(x)}\exp\left(-K\Big(\frac{y}{f(x)}\Big)^2\right)\right) = -\log(1) - \log\left(\frac{1}{f(x)}\right) + K\Big(\frac{y}{f(x)}\Big)^2$$

and, applying once again Proposition 1.2.10, this time to the map T,

$$-\log\left(m(T(x,y))J_T(x,y)\right) = -\log\left(\frac{\partial T_1}{\partial x}\frac{\partial T_2}{\partial y}\frac{1}{f(T_1)}\exp\left(-K\Big(\frac{T_2}{f(T_1)}\Big)^2\right)\right)$$
$$= -\log\left(\frac{\partial T_1}{\partial x}\right) - \log\left(\frac{\frac{\partial T_2}{\partial y}}{f(T_1)}\right) + K\Big(\frac{T_2}{f(T_1)}\Big)^2.$$

Observe now that, combining the statements of Example 1.2.6, Remark 1.2.7 and Lemma 1.2.8, we deduce that the function

$$[0,1]\ni t\mapsto -\log\left((1-t)+t\frac{\partial T_1}{\partial x}\right) - \log\left((1-t)\frac{1}{f(x)}+t\frac{\frac{\partial T_2}{\partial y}}{f(T_1)}\right) + K\Big((1-t)\frac{y}{f(x)}+t\frac{T_2}{f(T_1)}\Big)^2.$$

is $(0, 2K+2)$-convex and therefore $(0,N')$-convex for every $N'\geq 2K+2$. Then, according to (1.2.3), we deduce that (1.3.1) holds for every $N'\geq 2K+2$.

We now proceed to prove (1.3.1) for (μ_0-almost) every (x,y) such that $((x,y),T(x,y))\in V$, the strategy is similar but requires Lemma 1.3.2. In this case the midpoint interpolation map is trivial, therefore it holds that

$$J_{M\circ(\mathrm{id},T)}(x,y) = \frac{1}{2}\left(1+\frac{\partial T_1}{\partial x}\right)\cdot\frac{1}{2}\left(1+\frac{\partial T_2}{\partial y}\right).$$

In particular, we have that

$$-\log\Big(m\big(M((x,y),T(x,y))\big)J_{M\circ(\mathrm{id},T)}(x,y)\Big)$$
$$= -\log\left(\frac{1}{2}\left(1+\frac{\partial T_1}{\partial x}\right)\right) - \log\left(\frac{1}{2}\left(1+\frac{\partial T_2}{\partial y}\right)\right) - \log\Big(m\big(M((x,y),T(x,y))\big)\Big)$$

and, on the other hand,

$$-\log\left(m(T(x,y))J_T(x,y)\right) = -\log\left(\frac{\partial T_1}{\partial x}\right) - \log\left(\frac{\partial T_2}{\partial y}\right) - \log\left(m(T(x,y))\right). \tag{1.3.2}$$

Now, we can assume without loss of generality that $x<T_1$ and define the function $z:[x,T_1]\to\mathbb{R}$ parameterizing the segment connecting (x,y) and $T(x,y)$, i.e. $z(t) = y + (T_2-y)\frac{t-x}{T_1-x}$ for every $t\in[x,T_1]$. Then, applying Lemma 1.3.2, we deduce that the function $-\log(m(t,z(t)))$ is $(\frac{K}{32f^2_{[x,T_1]}}, 32K)$-convex (on $[x,T_1]$), thus also $(0,32K)$-convex. (Observe that when $x>T_1$ we can use an analogous argument, while, when $x=T_1$ the direct computation yields the same convexity.) Then, we can proceed as in the previous case and conclude that (1.3.1) holds for every $N'\geq 32K+2$. The case of (x,y) such that $((x,y),T(x,y))\in D$ can be solved following the same strategy, after a change of variable, as done in [Raj16].

We are left with the last case, which consists in proving (1.3.1) for (μ_0-almost) every (x,y) such that $((x,y),T(x,y))\in H_1$ (with $x<T_1(x,y)$ and $y<T_2(x,y)$). Before developing the argument, we notice that

$$f(T_1)\geq T_2\geq y+\frac{T_1-x}{2}\geq\frac{T_1-x}{2},$$

therefore, since $|f'|\leq k$, we deduce that

$$f_I := f_{[x,T_1]} = \max_{x\leq r\leq T_1} f(r) \leq f(T_1)+k(T_1-x)\leq (1+2k)f(T_1)\leq 2f(T_1), \tag{1.3.3}$$

for k sufficiently small. Back to the argument, consider the map

$$(S_1, S_2)(x, y) := M \circ (\mathrm{id}, T)(x, y) \\ = \left(\frac{x + T_1}{2}, y + \tilde{y}(x, T_1, y) + \left(\frac{T_1 - x}{2} - \tilde{y}(x, T_1, y)\right)\left(2\frac{T_2 - y}{T_1 - x} - 1\right)\right).$$

Proceeding as before, Proposition 1.2.10 and Proposition 1.2.13 ensure that

$$J_{M\circ(\mathrm{id},T)}(x, y) = \frac{\partial S_1}{\partial x}\frac{\partial S_2}{\partial y},$$

and therefore we have

$$-\log\Big(m(M((x, y), T(x, y)))\,J_{M\circ(\mathrm{id},T)}(x, y)\Big) \\ = -\log\left(\frac{\partial S_1}{\partial x}\right) - \log\left(\frac{\partial S_2}{\partial y}\right) - \log\Big(m(M((x, y), T(x, y)))\Big),$$

while, as before, (1.3.2) holds. On the one hand, we easily have that

$$\frac{\partial S_1}{\partial x} = \frac{1}{2}\left(1 + \frac{\partial T_1}{\partial x}\right),$$

and, on the other hand, from the proof of Theorem 6.1 in [Mag22a] we know that

$$\frac{\partial S_2}{\partial y} = 1 + \frac{\partial}{\partial y}\tilde{y}(x, T_1, y)\left(2 - 2\frac{T_2 - y}{T_1 - x}\right) + \left(\frac{\partial T_2}{\partial y} - 1\right)\left(\frac{1}{2} - \frac{\tilde{y}(x, T_1, y) - \frac{T_1 - x}{4}}{\frac{T_1 - x}{2}}\right).$$

Moreover, the computations in the proof of Theorem 6.1 in [Mag22a] show that

$$\left|\frac{\tilde{y}(x, T_1, y) - \frac{T_1 - x}{4}}{\frac{T_1 - x}{2}}\right| \leq \frac{[2k^2 + 4k]\frac{T_1 - x}{2}}{f(T_1)} < \frac{1}{2^{12}}\frac{\frac{T_1 - x}{2}}{f(T_1)} \leq \frac{1}{2^{11}}\frac{\frac{T_1 - x}{2}}{f_I},$$

where the second inequality holds for a sufficiently small k and the third follows from (1.3.3). Now, suppose that

$$\frac{\partial}{\partial y}\tilde{y}(x, T_1, y) = \frac{1}{2}\frac{f\left(\frac{x + T_1}{2}\right)}{f(x)} + \frac{1}{2}\frac{f\left(\frac{x + T_1}{2}\right)}{f(T_1)} - 1 \geq 0.$$

Then, after noticing that $\frac{T_1 - x}{2} \leq f(T_1) \leq f_I$, we can apply Lemma 1.2.9 and find a function $h : [0, 1] \to \mathbb{R}$, with $h(0) = 1$, $h(1) = \frac{\partial T_2}{\partial y}$ and

$$h(1/2) = 1 + \left(\frac{\partial T_2}{\partial y} - 1\right)\left(\frac{1}{2} - \frac{\tilde{y}(x, T_1, y) - \frac{T_1 - x}{4}}{\frac{T_1 - x}{2}}\right) \leq \frac{\partial S_2}{\partial y} \tag{1.3.4}$$

such that

$$-\log(h(t)) \text{ is } \left(-2^{21}\left[\frac{T_1 - x}{2^{11} f_I}\right]^2, 2\right) = \left(-\frac{1}{2}\left[\frac{T_1 - x}{f_I}\right]^2, 2\right)\text{-convex.} \tag{1.3.5}$$

On the other hand, following the argument in the proof of Theorem 6.1 in [Mag22a], we can find a function $z : [x, T_1] \to R$ satisfying the assumption of Lemma 1.3.2 such that

$$\left(\frac{x + T_1}{2}, z\Big(\frac{x + T_1}{2}\Big)\right) = M((x, y), T(x, y))$$

In particular, according to Lemma 1.3.2 and Remark 1.2.7 the function

$$t \mapsto -\log\Big(m\Big((1 - t)x + tT_1, z((1 - t)x + tT_1)\Big)\Big)$$

is $(\frac{K}{32f_I^2}(T_1-x)^2, 32K)$-convex. Then, according to the Lemma 1.2.8 and keeping in mind (1.3.5), whenever K is sufficiently large (i.e. $K \geq 16$) we have that the function

$$t \mapsto -\log\left(1+t\left(\frac{\partial T_1}{\partial x}-1\right)\right) - \log(h(t)) - \log\Big(m\Big((1-t)x+tT_1, z\big((1-t)x+tT_1\big)\Big)\Big)$$

is $(0, 32K+3)$-convex. Proceeding as in the first case and keeping in mind the inequality in (1.3.4), we prove that (1.3.1) holds for every $N' \geq 32K+3$.

If instead

$$\frac{\partial}{\partial y}\tilde{y}(x,T_1,y) = \frac{1}{2}\frac{f(\frac{x+T_1}{2})}{f(x)} + \frac{1}{2}\frac{f(\frac{x+T_1}{2})}{f(T_1)} - 1 < 0,$$

the argument can be adapted following the same strategy developed in the proof of Theorem 6.1 in [Mag22a]. We are then able to prove (1.3.1) for μ_0-almost every (x,y) in every case, concluding the proof. □

Now, we want to combine Theorem 1.3.1 with the stability of the $\mathsf{CD}(0,N)$ condition with respect to the measured Gromov-Hausdorff convergence, in order to prove the $\mathsf{CD}(0,N)$ condition for singular spaces. To this aim, we introduce the set

$$\overline{\mathscr{F}_k} := \{f \in C^2([-1,1]) \,:\, 0 \leq f < 3k,\, |f'| \leq k,\, |f''| \leq 1\} \supset \mathscr{F}_k,$$

and we extend the definition of the metric measure space $(X_f, \mathsf{d}_\infty, \mathfrak{m}_{f,K})$ to all functions $f \in \overline{\mathscr{F}_k}$, in analogy to what did in Section 1.2.3. In particular, given any $\overline{\mathscr{F}_k}$, the definition of the space X_f is the same as before, i.e.

$$X_f = \{(x,y) \in \mathbb{R}^2 \,:\, x \in [-1,1] \text{ and } 0 \leq y \leq f(x)\}.$$

while the measure $\mathfrak{m}_{f,K}$ becomes singular:

$$\mathfrak{m}_{f,K} := \mathbb{1}_{\{f(x)=0\}} \cdot C_K \cdot \mathcal{H}^1|_{y=0} + \mathbb{1}_{\{f(x)>0\}} \cdot \frac{1}{f(x)} \exp\left(-K\left(\frac{y}{f(x)}\right)^2\right) \cdot \mathscr{L}^2|_{X_f},$$

where, as before, $C_K = \int_0^1 e^{-Ky^2}\,\mathrm{d}y$.

Remark 1.3.3. The measured Gromov-Hausdorff convergence is a notion of convergence for metric measure spaces that basically combines the Hausdorff convergence for the metric side and the weak convergence for the reference measures. It has different equivalent definitions (see [GMS15]), but for the purpose of this paper it is sufficient to consider the definition given in Villani's book [Vil09, Definition 27.30]. The CD condition is stable with respect to the measure Gromov-Hausdorff convergence, see Theorem 29.24 and Theorem 29.25 in [Vil09].

Corollary 1.3.4. *For k sufficiently small and K sufficiently large, there exists $N > 1$ such that for every $f \in \overline{\mathscr{F}_k}$ the metric measure space $(X_f, \mathsf{d}_\infty, \mathfrak{m}_{f,K})$ is a $\mathsf{CD}(0,N)$ space.*

Proof. For suitable k and K, Theorem 1.3.1 guarantees the existence of $N > 1$ such that $(X_g, \mathsf{d}_\infty, \mathfrak{m}_{g,K})$ is a $\mathsf{CD}(0,N)$ space for every $g \in \mathscr{F}_k$. Now, given $f \in \overline{\mathscr{F}_k}$, take a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive numbers converging to 0, notice that definitely $f + \varepsilon_n \in \mathscr{F}_k$ and thus $(X_{f+\varepsilon_n}, \mathsf{d}_\infty, \mathfrak{m}_{f+\varepsilon_n,K})$ is a $\mathsf{CD}(0,N)$ space. Moreover, it is easy to realize that

$$(X_{f+\varepsilon_n}, \mathsf{d}_\infty, \mathfrak{m}_{f+\varepsilon_n,K}) \longrightarrow (X_f, \mathsf{d}_\infty, \mathfrak{m}_{f,K}) \quad \text{as } n \to \infty, \tag{1.3.6}$$

with respect to the the measured Gromov-Hausdorff convergence. From the stability of the $\mathsf{CD}(0,N)$ condition (see Remark 1.3.3), we conclude that $(X_f, \mathsf{d}_\infty, \mathfrak{m}_{f,K})$ is a $\mathsf{CD}(0,N)$ space. □

Remark 1.3.5. For a formal proof of the measured Gromov-Hausdorff convergence (1.3.6) we refer to the proof of Theorem 7.1 in [Mag22a], the setting of [Mag22a] is less general but the strategy developed in that case works also in the context of this work.

Remark 1.3.6. Observe that, the fact that the constant $N > 1$ in Theorem 1.3.1 does not depend on the specific choice of the function $f \in \mathscr{F}_k$, is crucial in the proof of Corollary 1.3.4.

1.4 Conclusions

1.4.1 Examples of singular $\mathsf{CD}(0, N)$ spaces

Using Corollary 1.3.4 we can construct interesting examples of singular $\mathsf{CD}(0, N)$ spaces, in fact, given a function $f \in \overline{\mathscr{F}_k} \setminus \mathscr{F}_k$, the measured Gromov-Hausdorff limit procedure in the proof of Corollary 1.3.4 makes the y-dimension collapse in $\{f(x) = 0\}$. For example, taking $f \in \overline{\mathscr{F}_k}$ increasing and such that $\{f = 0\} = [-1, 0]$, the metric measure space $(X_f, \mathsf{d}_\infty, \mathfrak{m}_{f,K})$, which is $\mathsf{CD}(0, N)$, is similar to the representation in Figure 1.1.

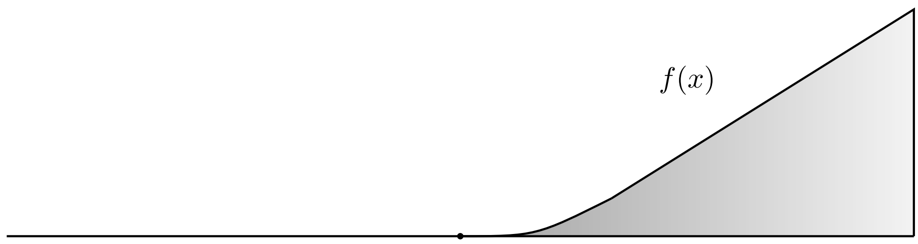

Figure 1.1: The metric measure space $(X_f, \mathsf{d}_\infty, \mathfrak{m}_{f,K})$.

Observe that this space has different topological and Hausdorff dimensions in different regions, in fact $X_f \cap ([-1, 0] \times \mathbb{R})$ has (topological and Hausdorff) dimension 1, while $X_f \cap ([0, 1] \times \mathbb{R})$ has (topological and Hausdorff) dimension 2. This observation generalizes a result by Ketterer and Rajala [KR15], who constructed a space with non-constant dimension, satisfying the so-called measure contraction property MCP (see [Oht07] for the definition). The extension of this result to $\mathsf{CD}(0, N)$ spaces is not obvious, since the CD condition is strictly stronger than the measure contraction property and forces the space to be a little bit more rigid.

Moreover, as highlighted in the introduction, the proved possible non-constancy of the dimension for $\mathsf{CD}(0, N)$ spaces is especially interesting in relation to what happens in the context of RCD spaces, that are CD spaces which are also infinitesimally Hilbertian (cfr. [Gig15, AGS14b]). In fact, it was proved by Brué and Semola in [BS20] that every $\mathsf{RCD}(K, N)$ space has constant dimension. The example presented in this section proves that the same is not true for $\mathsf{CD}(0, N)$ spaces, showing in particular that, as expected, the infinitesimal Hilbertianity assumption is necessary in [BS20].

The function $f \in \overline{\mathscr{F}_k} \setminus \mathscr{F}_k$ we have considered up to now in this section identifies a set X_f, which is basically the same as the one considered in [Mag22a]. However, the more general approach adopted in this work allows to identify other examples of CD spaces with different shapes. In particular, Corollary 1.3.4 proves that the $\mathsf{CD}(0, N)$ condition can be verified also by spaces having the shapes represented in Figure 1.2. This shows that one-dimensional and two-dimensional parts can be alternated along $\{y = 0\}$ and the $\mathsf{CD}(0, N)$ can still be true.

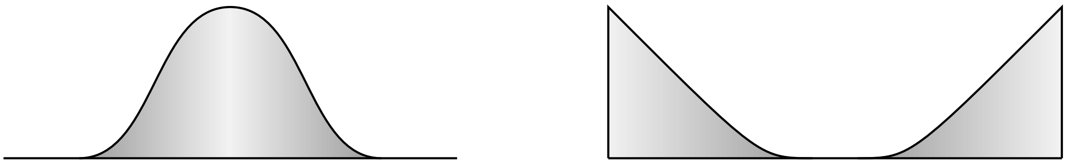

Figure 1.2: Possible shapes for a metric measure space of the type $(X_f, \mathsf{d}_\infty, \mathfrak{m}_{f,K})$ with $f \in \overline{\mathscr{F}_k}$.

1.4.2 Stronger CD conditions

There are different ways to require convexity of the entropy along Wasserstein geodesics and they originate different CD conditions. In particular, the CD condition defined in Section 1.2.1 is the weakest version, because it just requires the existence of a geodesic along which the entropy is convex. The most natural strengthening brings to the definition of the strong CD condition.

Definition 1.4.1. A metric measure space $(X, \mathsf{d}, \mathfrak{m})$ is said to be a strong $\mathsf{CD}(0, \infty)$ space (or to satisfy the strong $\mathsf{CD}(0, \infty)$ condition) if the entropy functional Ent (see (1.2.2)) is convex along every constant speed W_2-geodesic connecting two measures $\mu_0, \mu_1 \in \mathscr{P}_{ac}(X, \mathfrak{m})$.

Remark 1.4.2. The strong CD condition is usually defined with the dimensional parameter equal to ∞ for a technical reason related to Remark 1.2.5.

This strengthening of the CD condition is sufficient to show the following result on the geodesic structure of the space, which was proved by Rajala and Sturm in [RS14].

Theorem 1.4.3. *Every strong $\mathsf{CD}(0, \infty)$ metric measure space $(X, \mathsf{d}, \mathfrak{m})$ is essentially non-branching, i.e. for every pair $\mu_0, \mu_1 \in \mathscr{P}_{ac}(X, \mathfrak{m})$, every $\eta \in \mathscr{P}(\mathrm{Geo}(X))$ representing a geodesic (see Remark 1.2.2) connecting them, is concentrated on a non-branching set of geodesics.*

It is easy to realize that the same result cannot be true for CD spaces, an example for this is the metric measure space $(\mathbb{R}^2, \mathsf{d}_\infty, \mathscr{L}^2)$, which is a $\mathsf{CD}(0, 2)$ space but it is not an essentially non-branching space. The same example shows that the strong CD condition is not stable with respect to the measured Gromov-Hausdorff convergence, and this constitutes a major flaw. It is interesting to wonder whether there exists a stronger version of the CD condition, which is stable with respect to the measured Gromov-Hausdorff convergence and still guarantees some additional properties on the space. To this purpose we consider the definition of the strict CD condition.

Definition 1.4.4. Given $N > 1$, the metric measure space $(X, \mathsf{d}, \mathfrak{m})$ is said to be a strict $\mathsf{CD}(0, N)$ space (or to satisfy the strict $\mathsf{CD}(0, N)$ condition) if for every pair of measures $\mu_0, \mu_1 \in \mathscr{P}_{ac}(X, \mathfrak{m})$ there exists $\eta \in \mathscr{P}(\mathrm{Geo}(X))$ representing a constant speed W_2-geodesic $(\mu_t)_{t\in[0,1]} \subset \mathscr{P}_{ac}(X, \mathfrak{m})$ connecting them, such that, for every bounded measurable function $f : \mathrm{Geo}(X) \to \mathbb{R}^+$ with $\int f \, \mathrm{d}\eta = 1$ and every $N' \geq N$, the entropy functional $S_{N'}$ is convex along the geodesic $t \mapsto (e_t)_\#(f\eta)$.

Analogously, it is possible to define the strict $\mathsf{CD}(0, \infty)$ condition, which was proved not to be stable with respect to the measured Gromov-Hausdorff convergence in [Mag22a]. It is then not surprising that the same holds for the strict $\mathsf{CD}(0, N)$, in fact proceeding as in [Mag22a] (see in particular Section 7) we can prove the following result.

Proposition 1.4.5. *Fix constant k and K such that the conclusions of Theorem 1.3.1 and Corollary 1.3.4 hold with $N > 1$.*

1. *For every $f \in \mathscr{F}_k$ the metric measure space $(X_f, \mathsf{d}_\infty, \mathfrak{m}_{f,K})$ satisfy the strict $\mathsf{CD}(0,N)$ condition.*
2. *Given $f \in \overline{\mathscr{F}_k}$ increasing and such that $\{f = 0\} = [-1, 0]$, the metric measure space $(X_f, \mathsf{d}_\infty, \mathfrak{m}_{f,K})$ does not satisfy the strict $\mathsf{CD}(0,N)$ condition.*

In particular, according to what we did in the proof of Corollary 1.3.4, we conclude that the strict $\mathsf{CD}(0,N)$ condition is not stable with respect to the measured Gromov-Hausdorff convergence.

Observe also that point 2 in this last proposition, combined with Corollary 1.3.4, shows that the strict $\mathsf{CD}(0,N)$ condition is actually strictly stronger than the (classical) $\mathsf{CD}(0,N)$ condition.

It is also possible to find a requirement which is intermediate between the strong CD condition and the strict CD one. This is called very strict CD condition and was introduced and studied by Schultz in [Sch18]. The original definition [Sch18, Definition 1] is given with the dimensional parameter equal to ∞, but it can be easily adapted to the finite dimensional case following Definition 1.4.4. Generalizing the work of Rajala and Sturm, Schultz proved that the very strict CD condition is sufficient to deduce a non-branching property on the space.

Theorem 1.4.6. *Every very strict $\mathsf{CD}(0,\infty)$ space is weakly essentially non-branching, i.e. for every pair $\mu_0, \mu_1 \in \mathscr{P}_{ac}(X, \mathfrak{m})$ there exists $\eta \in \mathscr{P}(\mathrm{Geo}(X))$ representing a geodesic connecting them, that is concentrated on a non-branching set of geodesics.*

Being weakly essentially non-branching, is possibly the weakest meaningful non-branching condition that can be defined in metric measure spaces. However, as a consequence of Corollary 1.3.4, we can observe that is not satisfied in every $\mathsf{CD}(0,N)$ space. In fact, taking $f \in \overline{\mathscr{F}_k}$ increasing and such that $\{f = 0\} = [-1, 0]$, it is not difficult to see that every $\eta \in \mathscr{P}(\mathrm{Geo}(X))$, representing a geodesic connecting a marginal $\mu_0 \in \mathscr{P}_{ac}(X, \mathfrak{m})$, concentrated in the one dimensional part $X_f \cap ([-1, 0] \times \mathbb{R})$, and a marginal $\mu_1 \in \mathscr{P}_{ac}(X, \mathfrak{m})$, concentrated in the two dimensional part $X_f \cap ([0, 1] \times \mathbb{R})$, cannot be concentrated on a non-branching set of geodesics. Moreover, it is possible to observe that it cannot exist an optimal transport plan between μ_0 and μ_1, which is induced by a map. This proves that the essentially non-branching assumption is necessary to guarantee the existence of an optimal transport map, between absolutely continuous marginals in $\mathsf{CD}(0,N)$ spaces (cfr. [Gig12, RS14, CM17a, MR21]).

Finally, we point out that Proposition 1.4.5 suggest that also the the very strict CD condition should not be stable with respect to the measured Gromov-Hausdorff convergence. However, in [Mag22b] a stability result for the very strict CD condition is proved, under some additional metric assumptions on the converging sequence and on the limit space.

1.4.3 Non-compact version

In this last section we present a non-compact example of a singular $\mathsf{CD}(0,N)$, which structure is substantially analogous to the one of the space considered in Section 1.4.1. In particular, given k sufficiently small, introduce the metric measure space $(\mathsf{X}_k, \mathsf{d}_\infty, \mathfrak{m}_{k,K})$ defined as

$$\mathsf{X}_k = \{0\} \times (-\infty, 0] \ \cup \ \{(x,y) \in \mathbb{R}^2 \,:\, x \in [0, +\infty) \text{ and } 0 \leq y \leq kx\} =: L \cup C^k,$$

$$\mathfrak{m}_{k,K} := \mathbb{1}_{\{x \leq 0\}} \cdot C_K \cdot \mathcal{H}^1|_{\{y=0\}} + \mathbb{1}_{\{x>0\}} \cdot \frac{1}{kx} \exp\left(-K \left(\frac{y}{kx}\right)^2\right) \cdot \mathscr{L}^2|_{X_f},$$

where $C_K = \int_0^1 e^{-Ky^2}\,\mathrm{d}y$.

Proposition 1.4.7. *For suitable constants k, K and $N > 1$, the metric measure space $(\mathsf{X}_k, \mathsf{d}_\infty, \mathfrak{m}_{k,K})$ satisfies the $\mathsf{CD}(0,N)$ condition.*

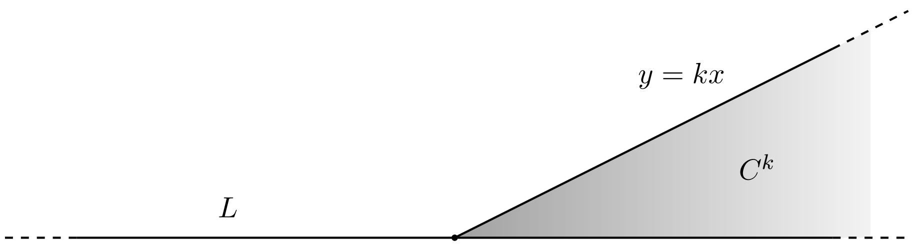

Figure 1.3: The metric measure space $(\mathsf{X}_k, \mathsf{d}_\infty, \mathfrak{m}_{k,K})$.

Sketch of the proof. With the exact same strategy developed in Section 1.3, it is possible to prove that, for suitable constants k, K and $N > 1$, $(C^k, \mathsf{d}_\infty, \mathfrak{m}_{k,K}|_{C^k})$ is a $\mathsf{CD}(0, N)$ space. In fact, all the steps can be repeated with minor changes and they give the same results. In particular, every computation in Section 1.3 that needed the assumption $f < 3k$ in 1.2.5, can be done also in this specific case, taking advantage of the fact that $(kx)'' = 0$. On the other hand, the space $(L, \mathsf{d}_\infty, \mathfrak{m}_{k,K}|_L)$ satisfies the $\mathsf{CD}(0, N)$ as well.

Now consider the full space $\mathsf{X}_k = L \cup C^k$, take a pair $\mu_0, \mu_1 \in \mathscr{P}_{ac}(\mathsf{X}_k, \mathfrak{m}_{k,K})$ and an optimal transport plan π between them. The plan π will send part of the mass of μ_0 in L in part of the mass of μ_1 in L, part of the mass of μ_0 in C^k in part of the mass of μ_1 in C^k and part of the mass of μ_0 in L in part of the mass of μ_1 in C^k (or vice versa). It is possible to show that it is sufficient to prove entropy convexity on each of this "sub-transport". In particular, for the first two, this follows from the first part of the proof, therefore it is enough to prove entropy convexity for transports from L to C^k. This is not trivial, but can be done finding a clever geodesic selection in accordance to the transport plan, following some ideas developed in [KR15] and [Mag22a, Section 7] (see also Remark 1 in [KR15]). $\square$

Proposition 1.4.7 allows to extend Theorem 3 of [KR15] to the setting of $\mathsf{CD}(0, N)$ spaces, proving the failure of topological splitting. Moreover, the metric measure space $(\mathsf{X}_k, \mathsf{d}_\infty, \mathfrak{m}_{k,K})$ is invariant with respect to rescalings centred in the origin, thus it is the (unique) metric measure tangent of itself in the origin. Therefore $(\mathsf{X}_k, \mathsf{d}_\infty, \mathfrak{m}_{k,K})$ provides an example of a $\mathsf{CD}(0, N)$ spaces having a metric measure tangent with a singular structure.

Paper 2

The Brunn–Minkowski inequality implies the CD condition in weighted Riemannian manifolds

with Lorenzo Portinale and Tommaso Rossi

The curvature dimension condition $\mathsf{CD}(K,N)$*, pioneered by Sturm and Lott–Villani in [Stu06a, Stu06b, LV09], is a synthetic notion of having curvature bounded below and dimension bounded above, in the non-smooth setting. This condition implies a suitable generalization of the Brunn–Minkowski inequality, denoted* $\mathsf{BM}(K,N)$*. In this paper, we address the converse implication in the setting of weighted Riemannian manifolds, proving that* $\mathsf{BM}(K,N)$ *is in fact equivalent to* $\mathsf{CD}(K,N)$*. Our result allows to characterize the curvature dimension condition without using neither the optimal transport nor the differential structure of the manifold.*

All authors of this paper contributed equally to all results.

2.1 Introduction

In their seminal papers [Stu06a, Stu06b, LV09], Sturm and Lott–Villani introduced a synthetic notion of curvature dimension bounds, in the non-smooth setting of metric measure spaces, usually denoted by $\mathsf{CD}(K,N)$, with $K \in \mathbb{R}$, $N \in (1,\infty]$. They observed that, in a (weighted) Riemannian manifold, the *differential* notion of having Ricci curvature bounded below, and dimension bounded above, can be equivalently characterized in terms of a convexity property of the Rényi entropy functional, along Wasserstein geodesics. In particular, the latter property relies on the theory of optimal transport and does not require the smooth underlying structure. Therefore, for a metric measure space, it can be taken as a *synthetic* definition of having a curvature dimension bound.

Among its many merits, the $\mathsf{CD}(K,N)$ condition is sufficient to deduce geometric and functional inequalities that hold in the smooth setting. An example is the so-called Brunn–Minkowski inequality, whose classical version in $\mathbb{R}^n$ (see e.g. [Gar02]) states that

$$\mathscr{L}^n\left((1-t)A+tB\right)^{\frac{1}{n}} \geq (1-t)\mathscr{L}^n(A)^{\frac{1}{n}} + t\mathscr{L}^n(B)^{\frac{1}{n}}, \quad \forall t\in[0,1],$$

for every two nonempty compact sets $A, B \subset \mathbb{R}^n$. In [Stu06b], Sturm proved that a $\mathsf{CD}(K,N)$ space supports a generalized version of the Brunn–Minkowski inequality, denoted $\mathsf{BM}(K,N)$, replacing the Minkowski sum of A and B with the set of midpoints and employing the so-called *distortion coefficients*, see Definition 2.2.7 for further details.

In this paper, we address the converse implication: indeed there is a general belief in the optimal transport community that the Brunn–Minkowski inequality $\mathsf{BM}(K,N)$ is sufficient to deduce the $\mathsf{CD}(K,N)$ condition. This work provides a first positive partial answer to this problem in the setting of weighted Riemannian manifolds.

Theorem 2.1.1 ($\mathsf{BM}(K,N) \Rightarrow \mathsf{CD}(K,N)$)**.** *Let (M,g) be a complete Riemannian manifold of dimension n, endowed with the reference measure $\mathfrak{m} = e^{-V}\,\mathrm{vol}$, where $V \in C^2(M)$. Suppose that the metric measure space $(M,\mathsf{d}_\mathrm{g},\mathfrak{m})$ satisfies $\mathsf{BM}(K,N)$ for some $K \in \mathbb{R}$ and $N > 1$. Then, it is a $\mathsf{CD}(K,N)$ space (and in particular the two conditions are equivalent).*

As mentioned before, in a weighted Riemannian manifolds $(M,\mathsf{d}_\mathrm{g},e^{-V}\,\mathrm{vol})$, the $\mathsf{CD}(K,N)$ condition is equivalent to the lower Ricci bound $\mathrm{Ric}^{N,\mathfrak{m}} \geq K\mathrm{g}$, see Definition 2.2.1. Our result allows us to characterize both conditions, without using neither the optimal transport nor the differential structure of the manifold. Moreover, as a corollary of Theorem 2.1.1 and in light of [Bac10], $\mathsf{BM}(K,N)$ is equivalent to a modified Borell–Brascamp–Lieb inequality. see [Bac10, Definition 1.1] for the precise definition.

Relations with the MCP condition

In [Oht07], the author introduced the so-called *measure contraction property*, $\mathsf{MCP}(K,N)$ for short, for a general metric measure space. This condition for a *non-weighted* Riemannian manifold is equivalent to having the (standard) Ricci tensor bounded below, see [Oht07, Theorem 3.2]. However, in general, the $\mathsf{MCP}(K,N)$ condition is strictly weaker than the $\mathsf{CD}(K,N)$ condition, and this is also the case for weighted Riemannian manifolds. Theorem 2.1.1 confirms that $\mathsf{BM}(K,N)$ is much closer to the $\mathsf{CD}(K,N)$ condition than the $\mathsf{MCP}(K,N)$ condition is.

We mention that in [BR19], in ideal sub-Riemannian manifolds, a different version of the Brunn–Minkowski inequality has been studied. When $K = 0$, this turns out to be equivalent to the $\mathsf{MCP}(0,N)$ condition, thus it is strictly weaker than $\mathsf{BM}(0,N)$.

Strategy of the proof of Theorem 2.1.1

The idea of the proof is to deduce the differential characterization of the $\mathsf{CD}(K,N)$ condition, arguing by contradiction. Thus, we assume there exists $v_0 \in T_{x_0}M$, with $x_0 \in M$ such that

$$\mathrm{Ric}^{N,\mathfrak{m}}_{x_0}(v_0,v_0) < K\|v_0\|^2, \tag{2.1.1}$$

and then find two subsets $A, B \subset M$ contradicting $\mathsf{BM}(K,N)$. The first step is to build a suitable optimal transport map T moving the mass in a neighborhood of x_0 in the direction v_0, see Section 2.3.2. The second step is to estimate the infinitesimal volume distortion around the geodesic $\gamma(t) = \exp_{x_0}(tv_0)$, joining x_0 and $T(x_0)$, cf. Proposition 2.3.4. By means of a comparison principle for ordinary differential equations (cf. Lemma 2.3.3), the condition (2.1.1) implies that

$$\mathfrak{m}\left(T_{\frac{1}{2}}(A)\right)^{\frac{1}{N}} < \tau^{(\frac{1}{2})}_{K,N}(\Theta(A,B))\left(\mathfrak{m}(A)^{\frac{1}{N}} + \mathfrak{m}(B)^{\frac{1}{N}}\right), \tag{2.1.2}$$

where $T_{\frac{1}{2}}$ is the interpolating optimal transport map, $A \subset M$ is any sufficiently small neighborhood of x_0 and $B := T(A)$. For the technical definitions of the distortion coefficients $\tau^{(t)}_{K,N}(\cdot)$ and $\Theta(A,B)$, see Definitions 2.2.3 and 2.2.7. The final, and most challenging, step is to compare the measure of $T_{\frac{1}{2}}(A)$ with the measure of $M_{\frac{1}{2}}(A,B)$, the set of midpoints between A and B, cf. Definition 2.2.6. This is done through a careful analysis of the behavior of the map T and choosing as A a specific cube oriented according to the Riemann curvature tensor at x_0, cf. Section 2.3.4. We then obtain

$$\mathfrak{m}\left(M_{\frac{1}{2}}(A,B)\right) \approx \mathfrak{m}(T_{\frac{1}{2}}(A)), \tag{2.1.3}$$

which, together with (2.1.2), gives a contradiction with $\mathsf{BM}(K,N)$. The relation (2.1.3) is made rigorous in Proposition 2.3.6, whose proof is based on the linearization of the map $T_{\frac{1}{2}}$. Remarkably, a second-order expansion capturing the local geometry of the manifold (involving in particular the Riemann curvature tensor at x_0) is needed.

Open problems

It would be relevant to extend Theorem 2.1.1 to general *essentially non-branching* metric measure spaces. Indeed, on the one hand this would produce an equivalent characterization of the CD condition without the need of optimal transport. On the other hand, it would provide an alternative proof of the *globalization theorem*, cf. [CM21], using [CM17c, Theorem 1.2]. In [MPR22b], we prove that in essentially non-branching metric measure spaces, the $\mathsf{CD}(K,N)$ condition is in fact equivalent to a stronger version of $\mathsf{BM}(K,N)$, denominated *strong Brunn–Minkowski condition*. However, the equivalence between the two, at this level of generality, seems to be out of reach with the techniques developed up to now. Nonetheless, one may hope to adapt our strategy either to the setting of Finsler manifolds or to the one of RCD spaces, where a second-order calculus is available.

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2.2 Preliminaries

In this section, we introduce the general framework of interest, recalling basic facts about optimal transport and Riemannian manifolds. Moreover, we present the Brunn–Minkowski inequality and state our main result.

2.2.1 Riemannian manifolds

Let (M, g) be a Riemannian manifold of dimension $n \in \mathbb{N}$. Let TM denote the tangent bundle of M, and for $x \in M$, T_xM the tangent space at x. For simplicity of notation, whenever it does not create ambiguity, we write for $x \in M$ and $v, w \in T_xM$

$$\langle v, w\rangle := \mathrm{g}_x(v,w) \quad \text{ and } \quad \|v\|^2 := \mathrm{g}_x(v,v)\,.$$

Let d_g the Riemannian distance associated with g, defined by length–minization procedure, and we say that (M, g) is complete if $(M, \mathsf{d}_\mathrm{g})$ is a complete metric space. Furthermore, for $x \in M$, we denote by $\exp_x : T_xM \to M$ the exponential map, i.e. $\exp_x(v) = \gamma_{x,v}(1)$, where $\gamma_{x,v}$ denotes the geodesic on M such that $\gamma_{x,v}(0) = 0$ and $\dot\gamma_{x,v}(0) = v \in T_xM$ (whenever it is well-defined for $t = 1$). Denote by ∇ the associated Levi-Civita connection on (M, g) and, for $X \in TM$, by ∇_X the covariant derivative along the vector field X. Then the Riemann curvature tensor is defined as

$$\mathrm{Riem}(X,Y) := \nabla_X\nabla_Y - \nabla_Y\nabla_X + \nabla_{[X,Y]}\,, \quad X, Y \in TM\,. \tag{2.2.1}$$

The Ricci tensor is obtained by taking suitable traces of the Riemann tensor (more precisely it is the trace of the sectional curvature tensor). In details, one has that

$$\mathrm{Ric} : TM \times TM \to \mathbb{R}\,, \quad \mathrm{Ric}(Y,Z) := \mathrm{tr}\big(X \mapsto \mathrm{Riem}(X,Y)Z\big)\,.$$

Both the Riemann and the Ricci tensor naturally appears in the study of the volume deformation along geodesics, see Section 2.3.1.

In particular, the Ricci tensor is closely related to convexity properties of entropy functionals along the geodesics of optimal transport. In the framework of weighted Riemannian manifolds a similar role is played by a modified version of the Ricci tensor, which depends on a dimensional parameter $N \in \mathbb{N}$ and a reference measure $\mathfrak{m}$.

Definition 2.2.1 (Modified Ricci tensor)**.** Let (M, g) be a Riemannian manifold, let $V \in C^2(M)$ and consider the measure $\mathfrak{m} = e^{-V}\,\mathrm{vol}$ on M, where vol is the Riemannian measure. Fix $N \geq n$. Then the $(N, \mathfrak{m})$-modified Ricci tensor is given by

$$\mathrm{Ric}^{N,\mathfrak{m}} := \mathrm{Ric} + \nabla^2 V - \frac{\nabla V \otimes \nabla V}{N-n}\,.$$

Here $\nabla^2 V$ denotes the Hessian of V, suitably identified to a bilinear form. With the convention that $0 \cdot \infty = 0$, if $N = n$ then necessarily $\nabla V = 0$, and thus $\mathrm{Ric}^{N,\mathfrak{m}} = \mathrm{Ric}$.

2.2.2 Optimal transport and curvature

Let $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ be a metric measure space, i.e. (X, d) is a complete and separable metric space and $\mathfrak{m}$ is a non-negative Borel measure on X, finite on bounded sets. Denote by $C([0,1], \mathsf{X})$ the space of continuous curves from $[0,1]$ to X, and define the *t-evaluation map* as $e_t : C([0,1],\mathsf{X}) \to \mathsf{X}$; $e_t(\gamma) := \gamma(t)$, for $\gamma \in C([0,1],\mathsf{X})$. A curve $\gamma \in C([0,1],\mathsf{X})$ is called *geodesic* if

$$\mathsf{d}(\gamma(s),\gamma(t)) = |t-s| \cdot \mathsf{d}(\gamma(0),\gamma(1)) \quad \text{for every } s,t \in [0,1]\,.$$

We denote by $\mathrm{Geo}(\mathsf{X}) \subset C([0,1],\mathsf{X})$ the space of constant speed geodesics in (X, d). The metric space (X, d) is said to be *geodesic* if every two points are connected by a curve in $\mathrm{Geo}(\mathsf{X})$. Note that any complete Riemannian manifold is a geodesic metric space.

Denote by $\mathscr{P}(\mathsf{X})$ the set of all Borel probability measures on X and by $\mathscr{P}_2(\mathsf{X}) \subset \mathscr{P}(\mathsf{X})$ the set of all probability measures with finite second moment. The 2-Wasserstein distance $\mathbb{W}_2$ is a distance on the space $\mathscr{P}_2(\mathsf{X})$ defined by

$$\mathbb{W}_2^2(\mu,\nu) := \inf_{\pi \in \mathrm{Adm}(\mu,\nu)} \int_{\mathsf{X}\times\mathsf{X}} \mathsf{d}(x,y)^2 \,\mathrm{d}\pi(x,y)\,, \tag{2.2.2}$$

for μ, $\nu \in \mathscr{P}_2(M)$, where $\mathrm{Adm}(\mu,\nu) := \{\pi \in \mathscr{P}(\mathsf{X}\times\mathsf{X}) : (\mathsf{p}_1)_\#\pi = \mu,\ (\mathsf{p}_2)_\#\pi = \nu\}$ is the set of admissible plans. Here $\mathsf{p}_i : \mathsf{X}\times\mathsf{X} \to \mathsf{X}$ denotes the projection on the i-th factor. The infimum in (2.2.2) is always attained, the admissible plans realizing it are called *optimal transport plans* and are denoted by $\mathrm{Opt}(\mu,\nu)$. Whenever a plan $\pi \in \mathrm{Opt}(\mu,\nu)$ is induced by a map $T : \mathsf{X} \to \mathsf{X}$ if $\pi = (\mathrm{id}, T)_\#\mu$, we say that T is an *optimal transport map*. It turns out that $\mathbb{W}_2$ defines a complete and separable distance on $\mathscr{P}_2(\mathsf{X})$ and, moreover $(\mathscr{P}_2(\mathsf{X}), \mathbb{W}_2)$ is geodesic if and only if (X, d) is.

In this paper, we work in the setting of weighted Riemannian manifolds, namely considering the metric measure space $(M, \mathsf{d}_\mathrm{g}, \mathfrak{m})$, where $\mathfrak{m} = e^{-V}\,\mathrm{vol}$. In this framework, whenever μ is absolutely continuous with respect to vol, then the optimal plan is unique and induced by an optimal transport map $T : M \to M$, see [BB00], [McC01]. Moreover, T is driven by the gradient of the so-called *Kantorovich potential* $-\psi : M \to \mathbb{R}$ via the exponential map as

$$T(x) = \exp_x(\nabla\psi(x))\,, \quad \forall x \in M\,,$$

where ψ is a semiconvex function, cf. [Vil09, Definition 10.10].

Since the seminal works of Sturm [Stu06a, Stu06b] and Lott–Villani [LV09], it is known that lower bounds on the (modified) Ricci curvature tensor can be recast in a synthetic way in terms of a suitable entropy convexity property. The latter is formulated in terms of the Rényi entropy functional and the distortion coefficients.

Definition 2.2.2 (Rényi entropy functional)**.** Let $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ be a metric measure space and fix $N > 1$. The *N-Rényi entropy functional* on $\mathscr{P}_2(\mathsf{X})$ is defined as

$$\mathcal{E}_N(\mu) = -\int_{\mathsf{X}} \rho(x)^{1-\frac{1}{N}} \, \mathrm{d}\mathfrak{m}(x) \qquad \forall \mu \in \mathscr{P}_2(\mathsf{X}),$$

where ρ is the density of the absolutely continuous part of μ, with respect to $\mathfrak{m}$.

Definition 2.2.3 (Distortion coefficients)**.** For every $K \in \mathbb{R}$, $N > 0$, we define for $\theta \geq 0$

$$\sigma_{K,N}^{(t)}(\theta) := \begin{cases} +\infty & N\pi^2 \leq K\theta^2 \\ \dfrac{\sin(t\alpha)}{\sin(\alpha)} & 0 < K\theta^2 < N\pi^2 \\ t & K = 0 \\ \dfrac{\sinh(t\alpha)}{\sinh(\alpha)} & K < 0 \end{cases}, \quad \alpha := \theta\sqrt{\frac{|K|}{N}},$$

while for $K \in \mathbb{R}$ and $N > 1$ we introduce the *distortion coefficients* for $t \in [0, 1]$ as

$$\tau_{K,N}^{(t)}(\theta) := t^{\frac{1}{N}} \sigma_{K,N-1}^{(t)}(\theta)^{1-\frac{1}{N}}.$$

Definition 2.2.4 ($\mathsf{CD}(K, N)$ space)**.** Given $K \in \mathbb{R}$ and $N > 1$, a metric measure space $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ is said to be a $\mathsf{CD}(K, N)$ *space* (or to satisfy the $\mathsf{CD}(K, N)$ condition) if for every pair of measures $\mu_0 = \rho_0\mathfrak{m}$, $\mu_1 = \rho_1\mathfrak{m} \in \mathscr{P}_2(\mathsf{X})$, there exists $\eta \in \mathscr{P}(\mathrm{Geo}(\mathsf{X}))$ such that $\mu_t := (e_t)_\#\eta =: \mu_t \ll \mathfrak{m}$ is $\mathbb{W}_2$-geodesic from μ_0 and μ_1 which satisfies the following inequality, for every $N' \geq N$ and every $t \in [0, 1]$:

$$\mathcal{E}_{N'}(\mu_t) \leq -\int_{\mathsf{X}\times\mathsf{X}} \left[\tau_{K,N'}^{(1-t)}(\mathsf{d}(x, y))\rho_0(x)^{-\frac{1}{N'}} + \tau_{K,N'}^{(t)}(\mathsf{d}(x, y))\rho_1(y)^{-\frac{1}{N'}}\right] \mathrm{d}\pi(x, y),$$

where $\pi = (e_0, e_1)_\#\eta$ is the optimal plan between μ_0 and μ_1 induced by η.

Note that in the case $K = 0$, the distortion coefficients are linear in t, and the CD condition simply becomes convexity of the Rényi entropy functional along $\mathbb{W}_2$-geodesics. We end this section stating an equivalence result between the CD condition and a Ricci bound for weighted Riemannian manifolds, which plays a crucial role in the sequel.

Theorem 2.2.5 (Equivalence theorem [Stu06b], [LV09])**.** *Let $K \in \mathbb{R}$ and $N > 1$ and let (M, g) be a complete Riemannian manifold of dimension n. For $N \geq n$, the metric measure space $(M, \mathsf{d}_g, e^{-V}\mathrm{vol}_{\mathrm{g}})$ is a $\mathsf{CD}(K, N)$ space if and only if*

$$\mathrm{Ric}^{N,\mathfrak{m}} \geq K \qquad (\textit{thus: } \mathrm{Ric}_x^{N,\mathfrak{m}}(v, v) \geq K\|v\|^2, \quad \forall x \in M,\, v \in T_xM). \tag{2.2.3}$$

2.2.3 Brunn–Minkowski inequality

We now introduce a generalized version of the Brunn–Minkowski inequality, tailored to a curvature parameter and a dimensional one. As proven by Sturm in [Stu06b], this is a consequence of the CD condition.

Definition 2.2.6 (t-midpoints)**.** Let (X,d) be a metric space. Let $A, B \subset \mathsf{X}$ be two Borel subsets. Then for $t \in (0,1)$, we defined the set of t-midpoints between A and B as

$$M_t(A,B) := \{x \in \mathsf{X} \,:\, x = \gamma(t)\,,\, \gamma \in \mathrm{Geo}(\mathsf{X})\,,\, \gamma(0) \in A\,,\ \text{and}\ \gamma(1) \in B\}\,.$$

Definition 2.2.7 (Brunn–Minkowski inequality)**.** Let $K \in \mathbb{R}$, $N > 1$. Then we say that a metric measure space $(\mathsf{X},\mathsf{d},\mathfrak{m})$ satisfies the Brunn–Minkowski inequality $\mathsf{BM}(K,N)$ if for every nonempty $A, B \subset \mathrm{spt}(\mathfrak{m})$ Borel subsets, $t \in (0,1)$, we have

$$\mathfrak{m}(M_t(A,B)))^{\frac{1}{N}} \geq \tau_{K,N}^{(1-t)}(\Theta(A,B)) \cdot \mathfrak{m}(A)^{\frac{1}{N}} + \tau_{K,N}^{(t)}(\Theta(A,B)) \cdot \mathfrak{m}(B)^{\frac{1}{N}}\,, \tag{2.2.4}$$

where

$$\Theta(A,B) := \begin{cases} \inf\limits_{x\in A,\, y\in B} \mathsf{d}(x,y) & \text{if } K \geq 0\,, \\ \sup\limits_{x\in A,\, y\in B} \mathsf{d}(x,y) & \text{if } K < 0\,. \end{cases} \tag{2.2.5}$$

Remark 2.2.8. In general the set $M_t(A,B)$ is not Borel measurable, even if the sets A and B are Borel. Therefore, when $M_t(A,B)$ is not measurable, the left-hand side of (2.2.4) has to be intended with the outer measure $\bar{\mathfrak{m}}$ associated with $\mathfrak{m}$, in place of $\mathfrak{m}$ itself.

Proposition 2.2.9 ([Stu06b, Proposition 2.1])**.** *Let $K \in \mathbb{R}$, $N > 1$ and let $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be a metric measure space satisfying the $\mathsf{CD}(K,N)$ condition. Then, $(\mathsf{X},\mathsf{d},\mathfrak{m})$ satisfies $\mathsf{BM}(K,N)$.*

We point out that in Euclidean spaces, the relation between the convexity of entropies and the Brunn–Minkowski inequality was already observed in the works of McCann [McC94], [McC97]. A remarkable feature of Proposition 2.2.9 lies in the sharp dependence of the Brunn–Minkowski inequality on the curvature exponent $K \in \mathbb{R}$ and the dimensional parameter $N > 1$. In a weighted Riemannian manifold, using the equivalence result of Theorem 2.2.5, we prove that the *sharp* Brunn–Minkowski inequality is enough to deduce the CD condition, with the *same constants*. We are in position to state our main result, which is a rephrasing of Theorem 2.1.1, in view of Theorem 2.2.5.

Theorem 2.2.10 ($\mathsf{BM}(K,N) \Rightarrow \mathrm{Ric}^{N,\mathfrak{m}} \geq K$)**.** *Let (M,g) be a complete Riemannian manifold of dimension n, endowed with the reference measure $\mathfrak{m} = e^{-V}\,\mathrm{vol}$, where $V \in C^2(M)$. Suppose that the metric measure space $(M,\mathsf{d}_{\mathrm{g}},\mathfrak{m})$ satisfies $\mathsf{BM}(K,N)$ for some $K \in \mathbb{R}$ and $N > 1$. Then $\mathrm{Ric}^{N,\mathfrak{m}} \geq K$, in the sense of* (2.2.3).

Remark 2.2.11. In [Stu06b, Theorem 1.7] and [Oht09, Theorem 1.2], the authors prove the implication $\mathsf{CD}(K,N) \Rightarrow \mathrm{Ric}^{N,\mathfrak{m}} \geq K$. Their argument is based on an inequality involving the t-midpoints, which would imply Theorem 2.2.10. However, they do not address the problem of comparing the latter set with the support of the interpolating optimal transport, cf. Proposition 2.3.6, which is the crucial and most challenging step. We stress that our choice of sets is more involved, precisely with the aim of obtaining a better control on the t-midpoints, which was lacking in the previous constructions. Finally, we point out that their argument works verbatim replacing the t-midpoints with the support of the interpolating optimal transport, nonetheless it does not provide an effective strategy for Theorem 2.2.10.

2.3 Brunn–Minkowski implies CD

Our strategy to prove Theorem 2.2.10 is to proceed by contradiction: let $(M,\mathsf{d}_{\mathrm{g}},\mathfrak{m})$ support $\mathsf{BM}(K,N)$, and assume there exist $\delta > 0$, $x_0 \in M$, and $v_0 \in T_{x_0}M$, with $\|v_0\| = 1$, such that

$$\mathrm{Ric}_{x_0}^{N,\mathfrak{m}}(v_0,v_0) < (K - 3\delta)\,. \tag{2.3.1}$$

More precisely, taking λ small enough, we can assume

$$\gamma^\lambda(t) := \exp_{x_0}(t\lambda v_0) \text{ is a geodesic on } (M,g) \text{ from } x_0 \text{ to } \gamma^\lambda(1)\,, \tag{2.3.2}$$

$$\mathrm{Ric}^{N,\mathfrak{m}}_{\gamma^\lambda(t)}(\dot\gamma^\lambda(t),\dot\gamma^\lambda(t)) < (K-2\delta)\lambda^2\,, \quad \text{for every } t\in[0,1]\,. \tag{2.3.3}$$

The idea is to exploit the fact that the generalized Ricci tensor controls the infinitesimal distortion of volumes around γ^λ, to build explicitly two sets A, a neighborhood of x_0, and B, a neighborhood of $\gamma^\lambda(1)$, contradicting the Brunn–Minkowski inequality $\mathsf{BM}(K,N)$.

2.3.1 Infinitesimal volume distortion

In this section, we recall general facts regarding infinitesimal volume distortion around a geodesic starting at any point $x \in M$, see [Vil09, Chapter 14]. In the next section, we specialize the results to the geodesic γ^λ.

To capture the infinitesimal volume distortion given by the (generalized) Ricci tensor, consider a transport map

$$T\colon M\to M, \qquad T(z) = \exp_x(\nabla\psi(z))\,,$$

where $\psi \in C^3_c(M)$. In analogous way, the transport map interpolating the identity and T is given by $T_t(z) = \exp_x(t\nabla\psi(z))$. Fix an orthonormal basis $\{e_1,\dots,e_n\}$ of T_xM and let $Q_\varepsilon(x)\subset M$ be the image, via the exponential map $\exp_x$, of the cube of size $\varepsilon>0$ centered at the point x with sides given by the e_i. Define the (weighted) *Jacobian determinant* by

$$\mathcal{J}_x(t) := \lim_{\varepsilon\to 0}\frac{\mathfrak{m}(T_t(Q_\varepsilon(x)))}{\mathfrak{m}(Q_\varepsilon(x))} = \frac{e^{-V(T_t(x))}}{e^{-V(x)}}\lim_{\varepsilon\to0}\frac{\mathrm{vol}(T_t(Q_\varepsilon(x)))}{\mathrm{vol}(Q_\varepsilon(x))} = \frac{e^{-V(T_t(x))}}{e^{-V(x)}}\tilde{\mathcal{J}}_x(t)\,.$$

It turns out that the function $\tilde{\mathcal{J}}_x$ has a useful geometric interpretation. Let $\gamma\colon[0,1]\to M$ be a geodesic connecting x with $T(x)$ (thus with velocity $\nabla\psi(x)$ in $t=0$) and let $\{e_1(t),\dots,e_n(t)\}$ be a parallel orthonormal frame along γ with $e_i(0)=e_i$. For any $i=1,\dots,n$, consider the Jacobi vector field $J_i(t) := J_i(t,x)$ solving

$$\ddot J(t) + \mathrm{Riem}(\dot\gamma_x(t),J(t))\dot\gamma_x(t) = 0\,, \qquad J(0)=e_i\,, \quad \dot J(0) = \nabla^2\psi(x)e_i\,, \tag{2.3.4}$$

where Riem is defined in (2.2.1). For $x\in M$ and $t\in[0,1]$, we define the $n\times n$ matrix $\mathsf{J}_x(t) := (J_1(t,x)|\dots|J_n(t,x))$. The relation with $\tilde{\mathcal{J}}_x(t)$ is then given by the identity

$$\tilde{\mathcal{J}}_x(t) = \det \mathsf{J}_x(t)\,.$$

Due to our interest in the Brunn–Minkowski inequality, the natural quantity to look at is given by $\mathcal{D}_x(t) = \mathcal{J}_x(t)^{1/N}$. Differentiating the determinant and using (2.3.4), one can prove that $\mathcal{D}_x(t)$ satisfies a Riccati–type equation involving the generalized Ricci tensor, given by

$$-N\frac{\mathcal{D}_x''(t)}{\mathcal{D}_x(t)} = \mathrm{Ric}^{N,\mathfrak{m}}_{\gamma(t)}(\dot\gamma(t),\dot\gamma(t)) + \mathcal{E}_x(t)\,, \qquad \forall\, t\in[0,1]\,,\; x\in M\,.$$

The remainder term $\mathcal{E}_x(t)$ depends on the transport map and is defined as follows: set $U_x(t) = \partial_t\mathsf{J}(t,x)\mathsf{J}^{-1}(t,x)$, then we have an explicit expression given by

$$\mathcal{E}_x(t) = \left\|U_x(t) - \frac{\mathrm{tr}\,U_x(t)}{n}\,\mathrm{Id}\right\|^2_{\mathrm{HS}} + \frac{n}{N(N-n)}\left|\frac{N-n}{n}\,\mathrm{tr}\,U_x(t) + \mathrm{g}_{\gamma(t)}(\dot\gamma(t),\nabla V(\gamma(t)))\right|^2,$$

where $\|\cdot\|_{\mathrm{HS}}$ denotes the Hilbert–Schmidt norm of a matrix.

Remark 2.3.1. The term $\mathcal{E}_x(t)$ enjoys good behavior under reparametrization of the curve γ. Indeed, if we see the map $\mathcal{E}_x$ as a function of t and $v := \dot\gamma(0)$, then we have that

$$\mathcal{E}_x(t) = \mathcal{E}_x(t,v) = \|v\|^2 \mathcal{E}_x\left(\|v\|t, \frac{v}{\|v\|}\right), \qquad \forall t \in [0,1].$$

In particular, for any given $\lambda \in [0,1]$, if we consider the curve $\gamma^\lambda(s) := \gamma(\lambda s)$ for $s \in [0,1]$, then the corresponding functional $\mathcal{E}_{x,\lambda}$, obtained via the Jacobi vector fields along the reparametrized curve γ^λ, satisfies

$$\mathcal{E}_{x,\lambda}(s) = \mathcal{E}_x(\lambda s, \lambda v) = \lambda^2\|v\|^2 \mathcal{E}_x\left(\lambda^2\|v\|s, \frac{v}{\|v\|}\right), \qquad \forall\, s \in [0,1]. \tag{2.3.5}$$

Notation 2.3.2. From now on, once a curve γ is fixed, whenever we consider a reparametrization $\gamma^\lambda(s) := \gamma(\lambda s)$, all the quantities defined in this section, and associated with γ^λ, are denoted with a subscript λ.

2.3.2 Choice of the Kantorovich potential

In order to exploit the upper bound in (2.3.3), we would like to control the volume distortion along the direction of v_0, thus we choose a Kantorovich potential to suitably *drive* the transport along that direction. In particular, fix $\psi \in C^3_c(M)$ such that

$$\begin{cases} \nabla\psi(x_0) = v_0\,, \\ \nabla^2\psi(x_0) = \alpha_0 \,\mathrm{Id}\,, \\ \Delta\psi(x_0) = n\,\alpha_0 = -\frac{n}{N-n}\mathrm{g}(\nabla V(x_0), v_0)\,. \end{cases} \tag{2.3.6}$$

In addition, define $\psi_\lambda = \lambda\psi$, for any $\lambda \in [0,1]$. Note that $\|\psi_\lambda\|_{C^2} \sim \lambda$, hence, for λ sufficiently small, we can apply [Vil09, Theorem 13.5] and deduce that ψ_λ is $\frac{d^2}{2}$-convex. As a consequence, the following map

$$T^\lambda(x) := \exp_x(\nabla\psi_\lambda(x)) = \exp_x(\lambda\nabla\psi(x)) \tag{2.3.7}$$

is optimal, by [McC01, Theorem 8]. We also denote by $T^\lambda_t(x) := \exp_x(t\nabla\psi_\lambda(x))$, for any $x \in M$ and $t \in [0,1]$, the interpolating optimal map between the identity and $T^\lambda_1 = T^\lambda$.

The unique geodesic joining x_0 and $T^1(x_0)$ is exactly given by $\gamma(t) := \exp_{x_0}(tv_0)$ and, by definition, $U_{x_0}(0) = \nabla^2\psi(x_0)$. Thus, we have that $\mathcal{E}_{x_0}(0) = 0$. This choice of ψ is closely related to the Bochner inequality and its equivalence to lower bounds on the modified Ricci tensor, see e.g. [Vil09, Theorem 14.8]. In particular, since $\mathcal{E}_{x_0}(\cdot)$ is a continuous function on $[0,1]$, there exists $\bar\lambda \in (0,1]$ such that

$$\mathcal{E}_{x_0}(t) \le \delta, \qquad \forall t \in [0,\bar\lambda]. \tag{2.3.8}$$

Now, for any $\lambda \le \bar\lambda$ we reparametrize γ on the interval $[0,\lambda]$ obtaining γ^λ as in (2.3.2). Then, according to the notation introduced in the previous section, denoting $\mathcal{D}_{x_0,\lambda}$ the associated (power of the) Jacobian determinant and with $\mathcal{E}_{x_0,\lambda}$ the remainder as in Remark 2.3.1, from (2.3.3) and (2.3.5), with $v = \lambda v_0$ we obtain that

$$-N\frac{\mathcal{D}''_{x_0,\lambda}(s)}{\mathcal{D}_{x_0,\lambda}(s)} = \mathrm{Ric}^{N,\mathfrak{m}}_{\gamma^\lambda(s)}(\dot\gamma^\lambda(s), \dot\gamma^\lambda(s)) + \mathcal{E}_{x_0,\lambda}(s) < (K-\delta)\lambda^2\,, \qquad \forall s \in (0,1). \tag{2.3.9}$$

The next step is to provide a suitable one-dimensional comparison result for $\mathcal{D}_{x_0,\lambda}$, as a solution of the ordinary differential inequality (2.3.9).

2.3.3 One-dimensional comparison

The following lemma is in the same spirit of [Vil09, Theorem 14.28], although concerning the reverse inequality.

Lemma 2.3.3. *Let $\Lambda < \pi^2$ and $f \in C([0,1]) \cap C^2(0,1)$, with $f \geq 0$. Then the following are equivalent:*

1. *$\ddot{f} + \Lambda f \geq 0$ in $(0,1)$;*
2. *For all $t, s_0, s_1 \in [0,1]$,*

$$f((1-t)s_0 + ts_1) \leq \sigma^{(1-t)}(|s_0 - s_1|)f(s_0) + \sigma^{(t)}(|s_0 - s_1|)f(s_1)\,, \tag{2.3.10}$$

where $\sigma^{(t)}(\cdot) = \sigma^{(t)}_{\Lambda,1}(\cdot)$, according to the notation in Definition 2.2.3.

Proof. (1) $\Rightarrow$ (2). Let $\Lambda \neq 0$, the case $\Lambda = 0$ being trivial. Set $f(t) := f((1-t)s_0 + ts_1)$, then by assumption it satisfies

$$f''(t) + \bar{\Lambda} f(t) \geq 0, \qquad \forall\, t \in [0,1]\,, \quad \bar{\Lambda} := \Lambda |s_1 - s_0|^2\,.$$

Let g be the right-hand side of (2.3.10), as a function of t. In particular, g solves the same equation of f with an equal sign, i.e. $g'' + \bar{\Lambda} g = 0$, and $g(0) = f(0)$, $g(1) = f(1)$. Let $w : [0,1] \to \mathbb{R}_+$ be any solution to the problem

$$w''(t) + \bar{\Lambda} w(t) > 0 \quad \text{ and } \quad w > 0 \quad \text{on} \quad (0,1)\,.$$

If $\Lambda > 0$, then we can choose $w \equiv 1$; if $\Lambda < 0$, we can choose e.g. $w(t) = \exp(t\sqrt{-\bar{\Lambda} + 1})$. Then for every $a \in \mathbb{R}_+$, we can define $f_a := f + aw$ and let g_a be the solution to $g_a'' + \bar{\Lambda} g_a = 0$ with $g_a(0) = f_a(0)$, $g_a(1) = f_a(1)$. We note that $f_a > 0$ on $(0,1)$ by construction and $f_a \to f$, $g_a \to g$ uniformly, as $a \to 0$. Moreover, we also have that

$$f_a''(t) + \bar{\Lambda} f_a(t) > 0, \qquad \forall\, t \in [0,1]\,. \tag{2.3.11}$$

Therefore, without loss of generality, we can assume $f, g > 0$ and $f/g \in C^2$ in $(0,1)$ and (2.3.11) is satisfied with f instead of f_a. In order to prove (2.3.10), we shall prove that f/g attains its maximum in $\{0,1\}$. By contradiction, let $t_0 \in (0,1)$ be a maximum for f/g, hence $(f/g)'(t_0) = 0$ and $(f/g)''(t_0) \leq 0$. An elementary computation shows that

$$\left(\frac{f}{g}\right)'' = \frac{f'' + \bar{\Lambda} f}{g} - \frac{f}{g^2}(g'' + \bar{\Lambda} g) - 2\frac{g'}{g}\left(\frac{f}{g}\right)'.$$

Evaluating at $t = t_0$, since $g > 0$, and using the equation solved by g, we find that $f''(t_0) + \bar{\Lambda} f(t_0) \leq 0$, which is a contradiction.
(2) $\Rightarrow$ (1). Consider the Taylor expansion of $\sigma^{(\frac{1}{2})}(\cdot)$ at $\theta = 0$, namely

$$\sigma^{(\frac{1}{2})}(\theta) = \frac{1}{2}\left(1 + \frac{\Lambda}{8}\theta^2\right) + o(\theta^3)\,. \tag{2.3.12}$$

Analogously, we have the following Taylor expansion for f,

$$\frac{f(s_0) + f(s_1)}{2} = f\left(\frac{s_0 + s_1}{2}\right) + \frac{1}{2}\ddot{f}\left(\frac{s_0 + s_1}{2}\right)\left|\frac{s_1 - s_0}{2}\right|^2 + o(|s_0 - s_1|^2)\,. \tag{2.3.13}$$

Now, fix $t \in (0,1)$ and let $s_0, s_1 \to t$ in such a way $t = \frac{s_0 + s_1}{2}$. From (2.3.13), we obtain

$$\frac{f(s_0) + f(s_1)}{2} = f(t) + \frac{1}{8}\ddot{f}(t)\,|s_1 - s_0|^2 + o(|s_0 - s_1|^2)\,.$$

Moreover, evaluating (2.3.12) at $\theta = |s_0 - s_1|$, we rewrite

$$\sigma^{\left(\frac{1}{2}\right)}(|s_0 - s_1|) = \frac{1}{2}\left(1 + \frac{\Lambda}{8}|s_0 - s_1|^2\right) + o(|s_0 - s_1|^3)\,.$$

Finally, putting all together we obtain that

$$\sigma^{\left(\frac{1}{2}\right)}(|s_0 - s_1|)\,(f(s_0) + f(s_1)) - f(t) = \frac{1}{8}|s_0 - s_1|^2\left(\Lambda f(t) + \ddot{f}(t) + o(1)\right) \geq 0\,.$$

Taking the limit as $s_0, s_1 \to t$ leads to the conclusion. □

As a consequence of Lemma 2.3.3, we prove the following upper bound. Recall the definition of $\bar{\lambda} \in (0,1]$ in (2.3.8).

Proposition 2.3.4. *There exist $\lambda_1 \in (0, \bar{\lambda})$ and $c > 0$ not depending on $\bar{\lambda}$, such that, whenever $\lambda \leq \lambda_1$, we have that*

$$\mathcal{D}_{x_0,\lambda}\left(\frac{1}{2}\right) \leq \tau_{K,N}^{\left(\frac{1}{2}\right)}(\lambda)\,(\mathcal{D}_{x_0,\lambda}(0) + \mathcal{D}_{x_0,\lambda}(1)) - c\lambda^2\,. \tag{2.3.14}$$

Proof. Setting $f(t) := \mathcal{D}_{x_0,\lambda}(t)$ and $\Lambda := N^{-1}(K-\delta)\lambda^2$, we can choose $\rho \in \mathbb{R}_+$ small enough such that, for any $\lambda \leq \rho$, conditions (2.3.2) and (2.3.3) are satisfied and $\Lambda < \pi^2$. The estimate obtained in (2.3.9) shows that the hypothesis of Lemma 2.3.3 is satisfied. Therefore, using that $\sigma^{(s)}_{K-\delta,N} \leq \tau^{(s)}_{K-\delta,N}$, for any $s \in [0,1]$, we have

$$\mathcal{D}_{x_0,\lambda}(t) \leq \tau^{(1-t)}_{K-\delta,N}(\lambda)\mathcal{D}_{x_0,\lambda}(0) + \tau^{(t)}_{K-\delta,N}(\lambda)\mathcal{D}_{x_0,\lambda}(1)\,, \qquad \forall t \in [0,1]\,. \tag{2.3.15}$$

Recalling Definition 2.2.3, we consider the Taylor expansion of $\tau^{(t)}_{K,N}(\lambda)$, as $\lambda \to 0$, obtaining

$$\tau^{(t)}_{K,N}(\lambda) = t\left(1 + (1-t^2)\frac{K}{6N}\lambda^2\right) + o(\lambda^2)\,, \qquad \forall t \in [0,1]\,.$$

Then, using this expansion, we can deduce that, as $\lambda \to 0$,

$$\tau^{(t)}_{K-\delta,N}(\lambda) = \tau^{(t)}_{K,N}(\lambda) - t(1-t^2)\frac{\delta}{6N}\lambda^2 + o(\lambda^2)\,, \qquad \forall t \in [0,1]. \tag{2.3.16}$$

Combining (2.3.15) and (2.3.16) for $t = 1/2$, and noting that $\mathcal{D}_{x_0,\lambda}(0) + \mathcal{D}_{x_0,\lambda}(1) \geq \mathcal{D}_{x_0,\lambda}(1) = 1$, we obtain

$$\begin{aligned}\mathcal{D}_{x_0,\lambda}\left(\frac{1}{2}\right) &\leq \tau^{\left(\frac{1}{2}\right)}_{K,N}(\lambda)\,(\mathcal{D}_{x_0,\lambda}(0) + \mathcal{D}_{x_0,\lambda}(1)) - \frac{\delta}{16N}\lambda^2 + o(\lambda^2) \\ &\leq \tau^{\left(\frac{1}{2}\right)}_{K,N}(\lambda)\,(\mathcal{D}_{x_0,\lambda}(0) + \mathcal{D}_{x_0,\lambda}(1)) - c\lambda^2,\end{aligned}$$

where the last inequality holds definitely as $\lambda \to 0$, for a suitable positive constant $c > 0$, independent on λ. This concludes the proof. □

Proposition 2.3.4 is a step forward towards the contradiction of the Brunn–Minkowski inequality. Indeed, on one side, $\mathcal{D}_{x_0,\lambda}$ measures the infinitesimal volume distortion given by the transport map T^λ around the geodesic γ^λ. On the other side, the inequality (2.3.14) goes in the opposite direction with respect to the Brunn–Minkowski inequality. The next step is to find an initial set A, as a suitable infinitesimal cube generated by an orthonormal basis $\{e_1, \ldots, e_n\}$, such that the distortion steered by $\mathcal{D}_{x_0,\lambda}$ allows to estimate the mass of the midpoints between A and $T^\lambda(A)$.

2.3.4 The choice of the basis

For a fixed point $y \in M$, consider the squared distance function $\mathsf{d}_y^2(x) := \mathsf{d}_{\mathrm{g}}^2(x,y)$, for $x \in M$. If $\mathrm{Cut}(y) \subset M$ denotes the cut-locus set of y (i.e. the set of points $x \in M$ where $t \mapsto \exp_y(tx)$ loses minimality), then for $x \in M \setminus \mathrm{Cut}(y)$, the gradient (or Levi-Civita covariant derivative) of d_y^2 is given by

$$\nabla \mathsf{d}_y^2(x) = -2\log_x(y)\,, \tag{2.3.17}$$

where $\log_x = \exp_x^{-1}$. The Hessian of the squared distance is defined as

$$\nabla^2 \mathsf{d}_y^2(x_0)(v,w) = V(W\mathsf{d}_y^2) - (\nabla_V W)\mathsf{d}_y^2\,, \qquad v,w \in T_{x_0}M\,,$$

where V and W are any extension to a vector field of v and w, respectively. In particular, as a quadratic form, we have

$$\nabla^2 \mathsf{d}_y^2(x_0)(v,v) = \frac{d^2}{dt^2}\bigg|_{t=0} \mathsf{d}_y^2(\exp_{x_0}(tv)), \qquad v \in T_{x_0}M\,.$$

Thus, using [Loe09, Theorem 3.8], we deduce (see also e.g. [Pen17]),

$$\frac{1}{2}\nabla^2 \mathsf{d}_y^2(x_0) = \mathrm{id} - \frac{1}{3}R_y(x_0) + O\big(\mathsf{d}_{\mathrm{g}}^4(x_0,y)\big)\,, \tag{2.3.18}$$

where $R_y(x_0)$ denotes the symmetric $(0,2)$-tensor given by

$$(v,w) \in T_{x_0}M \times T_{x_0}M \mapsto \mathrm{g}_{x_0}\big(\mathrm{Riem}\,(\log_{x_0}(y),v)\log_{x_0}(y),w\big)\,.$$

Notice that (2.3.18) is a statement between bilinear forms on the finite-dimensional vector space $T_{x_0}M$, thus the norm of the tensor

$$\frac{1}{2}\nabla^2 \mathsf{d}_y^2(x_0) - \mathrm{id} + \frac{1}{3}R_y(x_0)\,,$$

goes to 0 as $y \to x_0$ (with order 4), for any choice of operator norm on $T_{x_0}M \times T_{x_0}M$. From the symmetry of the Riemann tensor, we know that the tensor $R_y(x_0)$ is symmetric. Therefore, for every reference frame of $T_{x_0}M$, the matrix representation of $R_y(x_0)$ is self-adjoint, hence it is diagonalizable with orthogonal eigenspaces. We choose $\{e_1,\dots,e_n\} \subset T_{x_0}M$ to be an orthonormal basis of eigenvectors of $R_y(x_0)$.

Remark 2.3.5. Observe that for $v \in T_{x_0}M$, such that $\|v\| = 1$ and $\langle v, \log_{x_0}(y)\rangle = 0$, $R_y(x_0)(v,v)$ is the sectional curvature of the plane generated by v and $\log_{x_0}(y)$. Thus, the choice of $Q_\varepsilon(x_0)$ encodes information of the sectional curvature of M at x_0. This could present a possible issue for extensions of this technique to non-smooth settings.

We claim that, with this particular choice, the set of $\frac{1}{2}$-midpoints between $Q_\varepsilon(x_0)$ and $T^\lambda(Q_\varepsilon(x_0))$, where T^λ is defined in (2.3.7), is quantitatively close (in measure) to $T^\lambda_{\frac{1}{2}}(Q_\varepsilon(x_0))$, for λ sufficiently small. Recall that the t-midpoints between two sets A, B are given by

$$M_t(A,B) := \{y = \exp_x(tv)\,:\, x \in A\,,\ \exp_x v \in B\}\,.$$

Proposition 2.3.6 (Control for the measure of the midpoints)**.** *Let $x_0 \in M$, $v_0 \in T_{x_0}M$ be a point satisfying* (2.3.1)*. Recall the definition of T^λ in* (2.3.7) *and set $y_0 := T^\lambda(x_0)$. Let $\{e_1,\dots,e_n\}$ to be an orthonormal basis of eigenvectors of $R_{y_0}(x_0)$ and for $\varepsilon \in (0,1)$, let $Q_\varepsilon(x_0)$*

be the corresponding cube, as described in Section 2.3.1. Then there exists $\lambda_2 \in (0, \bar\lambda)$, where $\bar\lambda$ is defined in (2.3.8), and $\bar\varepsilon \in (0,1)$, such that, whenever $\lambda \le \lambda_2$ and $\varepsilon \le \bar\varepsilon$, we have that

$$\mathfrak{m}\left(M_{\frac{1}{2}}(Q_\varepsilon(x_0), T^\lambda(Q_\varepsilon(x_0)))\right)^{\frac{1}{N}} \le (1 + C(\lambda^4 + \varepsilon))\mathfrak{m}\left(T^\lambda_{\frac{1}{2}}(Q_\varepsilon(x_0))\right)^{\frac{1}{N}} ,$$

where $C \in \mathbb{R}_+$ is a constant that does not depend on λ and ε.

Remark 2.3.7. Recalling Remark 2.2.8, since $\exp_x$ is a local diffeomorphism around x, the set $M_t(Q_\varepsilon(x_0), T^\lambda(Q_\varepsilon(x_0)))$ is measurable for any $t \in [0,1]$.

In order to prove this proposition, we need the following preliminary result.

Lemma 2.3.8. *Let $\psi \in C^3(M)$ such that $T(x) := \exp_x(\nabla\psi(x))$ is optimal and define $T_t(x) := \exp_x(t\nabla\psi(x))$. Then,*

1. *We have that $\nabla\psi(x) = -\nabla \mathsf{d}^2_y(x)/2$, where $y = T(x)$.*
2. *For every $x \in M$, and for every $t \in [0,1]$, in normal coordinates centered at x, we have that*

$$\mathrm{d}_x T_t = \nabla^2\left(\frac{1}{2}\mathsf{d}^2_{z_t} + t\psi\right)(x) , \quad z_t := T_t(x) .$$

The proof of this lemma closely follows the one of [CEMS01, Proposition 4.1], for completeness we report here a concise proof of this fact.

Proof. The proof of (1) follows from [CEMS01, Lemma 3.3(b)]. We now prove (2): without loss of generality, we can prove the claimed equality for $t = 1$, the general case follows by suitable rescaling. We fix $x \in M$ and set $y := T(x) = \exp_x(\nabla\psi(x))$. Define

$$h(z) := \frac{1}{2}\mathsf{d}^2_y(z) + \psi(z) , \quad \forall z \in M .$$

Let $u \in T_xM$ and set, for any $s \in [0,1]$, $x_s := \exp_x(su)$, $y_s := T(x_s) = \exp_{x_s}(\nabla\psi(x_s))$, where in particular $x_0 = x$, $y_0 = y$ (see Figure 2.1). We also define

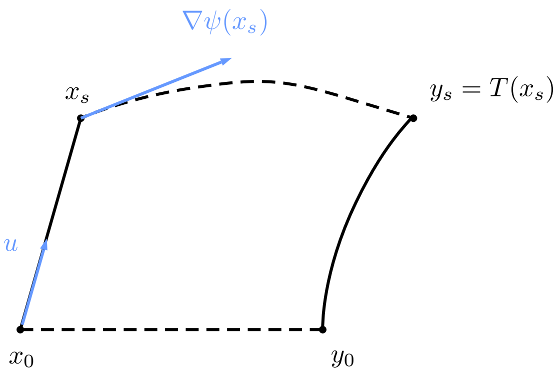

Figure 2.1: Picture of $s \mapsto x_s, y_s$ in normal coordinates centered at x.

$$F(z,v) := \exp_z(v) , \quad G(z) := -\frac{1}{2}\nabla\mathsf{d}^2_y(z) , \qquad \forall\, v \in T_zM,\ z \in M$$

and note that by construction

$$F(z, G(z)) = y , \quad \forall\, z \in M . \tag{2.3.19}$$

We also define $w_s := \nabla h(x_s)$ and observe that $w_0 = 0$, using point (1) of the statement, and by construction $y_s = F(x_s, \nabla\psi(x_s)) = F(x_s, G(x_s) + w_s)$. Therefore, by computing the time derivative in s, at $s = 0$, we find that

$$\begin{aligned}\dot{y}_s|_{s=0} &= \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} F(x_s, G(x_s) + w_s) = \mathrm{d}_x F(\cdot, G(\cdot))\, \dot{x}_s|_{s=0} + \mathrm{d}_{G(x)} F(x,\cdot)\, \dot{w}_s|_{s=0} \\ &= \mathrm{d}_{G(x)} F(x,\cdot)\, \dot{w}_s|_{s=0} = \left(\mathrm{d}_{G(x)} \exp_x \right) \dot{w}_s|_{s=0}\,,\end{aligned}$$

where we used that $w_0 = 0$ and $\mathrm{d}_x F(\cdot, G(\cdot)) = 0$, which follows from (2.3.19). Working in normal coordinates centered at x, the exponential map (with base point x) is the identity, thus $\mathrm{d}_{G(x)} \exp_x = \mathrm{id}$. Consequently, we conclude that

$$\mathrm{d}T_x(u) = \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} T(x_s) = \dot{y}_s|_{s=0} = \dot{w}_s|_{s=0} = \nabla^2 h(x)(u)\,,$$

for every $u \in T_x M$, which concludes the proof. □

We are ready to prove Proposition 2.3.6.

Proof of Proposition 2.3.6. We set $A_\varepsilon := Q_\varepsilon(x_0)$ and $B_\varepsilon := T^\lambda(Q_\varepsilon(x_0))$. The idea is that, for if T^λ were linear and (2.3.18) were without the fourth–order error in the distance, we would be able to prove an exact set equality between $M_{\frac{1}{2}}(A_\varepsilon, B_\varepsilon)$ and $T^\lambda_{\frac{1}{2}}(A_\varepsilon)$. Therefore, we linearize around x_0 and quantitatively study the error of this procedure.

From now on, we assume to work in normal coordinates centered at x_0 , in a neighborhood $U \subset M$. With slight abuse of notation, we do not change names of quantities when written in coordinates.

Recall that $\log_x = \exp_x^{-1}$. We define the map $F(\cdot,\cdot)$ as

$$F(x,y) := \exp_x\left(\frac{1}{2}\log_x y\right) = \exp_x\left(-\frac{1}{2}\nabla\left(\frac{1}{2}\mathsf{d}_y^2(\cdot)\right)(x)\right)\,, \quad \text{for } x, y \in M\,,$$

where the second equality follows from (2.3.17), and observe that, by definition of midpoints,

$$M_{\frac{1}{2}}(A_\varepsilon, B_\varepsilon) = F(A_\varepsilon \times B_\varepsilon)\,.$$

Step 1: expansions around x_0. A simple computation shows that, in normal coordinates centered at x_0, the differential of F with respect to the second variable reads

$$\mathrm{d}_y F(x_0,\cdot) = \frac{1}{2}\,\mathrm{id}\,, \quad \forall y \in U\,. \tag{2.3.20}$$

To compute the differential with respect to the first variable, we set $\tilde{\psi} := -\frac{1}{2}\mathsf{d}^2_{y_0}$ and note that $F(x, y_0) = \tilde{T}_{\frac{1}{2}}(x) := \exp_x\left(\nabla\tilde{\psi}(x)/2\right)$. Therefore, thanks to Lemma 2.3.8, we obtain that

$$\mathrm{d}_{x_0} F(\cdot, y_0) = \nabla^2\left(\frac{1}{2}\mathsf{d}^2_{\tilde{T}_{\frac{1}{2}}(x_0)} + \frac{1}{2}\tilde{\psi}\right)(x_0) = \nabla^2\left(\frac{1}{2}\mathsf{d}^2_{z_0} - \frac{1}{4}\mathsf{d}^2_{y_0}\right)(x_0)\,, \tag{2.3.21}$$

where $\tilde{T}_{\frac{1}{2}}(x_0) = T^\lambda_{\frac{1}{2}}(x_0) =: z_0$. Note that, by $\log_{x_0}(z_0) = \frac{1}{2}\log_{x_0}(y_0)$ and and the homogeneity of the Riemann tensor, we get that $R_{z_0}(x_0) = \frac{1}{4}R_{y_0}(x_0)$. From the expansion (2.3.18), we then find that

$$\begin{aligned}\mathrm{d}_{x_0} F(\cdot, y_0) &= \mathrm{id} - \frac{1}{3}R_{z_0}(x_0) - \frac{1}{2}\left(\mathrm{id} - \frac{1}{3}R_{y_0}(x_0)\right) + O(\lambda^4) \\ &= \frac{1}{2}\left(\mathrm{id} + \frac{1}{6}R_{y_0}(x_0)\right) + O(\lambda^4)\,.\end{aligned}$$

Similarly, by $R_{T_t^\lambda(x_0)}(x_0) = t^2 R_{y_0}(x_0)$, and applying Lemma 2.3.8, we find that

$$\mathrm{d}_{x_0} T_t^\lambda = \mathrm{id} - \frac{t^2}{3} R_{y_0}(x_0) + t\nabla^2\psi_\lambda(x_0) + O(\lambda^4)\,, \quad \forall t \in [0,1]\,, \tag{2.3.22}$$

as $\lambda \to 0$. We compute the Taylor expansion of the map

$$A_\varepsilon \times A_\varepsilon \ni (x, x') \mapsto F(x, T^\lambda(x'))$$

at the point (x_0, x_0). Using the differentiability of F in $(x, y) = (x_0, y_0)$, from (2.3.20), (2.3.21), and (2.3.22) with $t = 1$, for $x, x' \in A_\varepsilon$ – recall that $\operatorname{diam}(A_\varepsilon) = O(\varepsilon)$ – in coordinates, as $\varepsilon \to 0$, we obtain

$$\begin{aligned} F(x, T^\lambda(x')) &= F(x_0, y_0) + \mathrm{d}_{x_0}F(\cdot, y_0)(x - x_0) + \mathrm{d}_{y_0}F(x_0, \cdot)(T^\lambda(x') - y_0) + O(\varepsilon^2) \\ &= z_0 + \frac{1}{2}(T^\lambda(x') - y_0) + \frac{1}{2}\left(\mathrm{id} + \frac{1}{6}R_{y_0}(x_0)\right)(x - x_0) + O(\varepsilon^2 + \varepsilon\lambda^4) \\ &= z_0 + \frac{1}{2}\left(\mathrm{id} - \frac{1}{3}R_{y_0}(x_0) + \nabla^2\psi_\lambda(0)\right)(x' - x_0) \\ &\qquad\qquad + \frac{1}{2}\left(\mathrm{id} + \frac{1}{6}R_{y_0}(x_0)\right)(x - x_0) + O(\varepsilon^2 + \varepsilon\lambda^4). \end{aligned}$$

We remark that, to deduce the error terms in the first and third equality, we have performed a Taylor expansion of F and exploited the fact that

$$\|T^\lambda\|_{C^2(U)} \le L\left(1 + \|T^1\|_{C^2(U)}\right), \tag{2.3.23}$$

with $L > 0$ independent on λ. To prove (2.3.23), it is enough to use Lemma 2.3.8, together with the expansion (2.3.18). Similarly, using (2.3.22) with $t = \frac{1}{2}$, we have that, for any $x'' \in A_\varepsilon$,

$$\begin{aligned} T^\lambda_{\frac{1}{2}}(x'') &= z_0 + \mathrm{d}_{x_0}T^\lambda_{\frac{1}{2}}(x'' - x_0) + O(\varepsilon^2) \\ &= z_0 + \left(\mathrm{id} - \frac{1}{12}R_{y_0}(x_0) + \frac{1}{2}\nabla^2\psi_\lambda(x_0)\right)(x'' - x_0) + O(\varepsilon^2 + \varepsilon\lambda^4)\,. \end{aligned}$$

Taking into account condition (2.3.6), we can then write

$$\begin{aligned} F(x, T^\lambda(x')) &= z_0 + \frac{1}{2}\Big(M_1(x - x_0) + M_2(x' - x_0)\Big) + O(\varepsilon^2 + \varepsilon\lambda^4)\,, \\ T^\lambda_{\frac{1}{2}}(x'') &= z_0 + M_3(x'' - x_0) + O(\varepsilon^2 + \varepsilon\lambda^4)\,, \end{aligned} \tag{2.3.24}$$

for x, x', $x'' \in A_\varepsilon$, where the linear operators M_i are given by

$$\begin{aligned} M_1 &:= \mathrm{id} + \frac{1}{6}R_{y_0}(x_0)\,, \quad M_2 := (1 + \lambda\alpha_0)\,\mathrm{id} - \frac{1}{3}R_{y_0}(x_0)\,, \\ M_3 &:= \left(1 + \lambda\frac{\alpha_0}{2}\right)\mathrm{id} - \frac{1}{12}R_{y_0}(x_0) = \frac{1}{2}(M_1 + M_2)\,. \end{aligned} \tag{2.3.25}$$

Step 2: solution to the linear problem. We set $T^\lambda_{\frac{1}{2},\mathrm{lin}}(x) := z_0 + M_3(x - x_0)$. We claim

$$M_{\frac{1}{2}}(A_\varepsilon, B_\varepsilon) \subset \mathrm{B}_{\rho_{\varepsilon,\lambda}}\Big(T^\lambda_{\frac{1}{2},\mathrm{lin}}(A_\varepsilon)\Big)\,, \quad \rho_{\varepsilon,\lambda} := C(\varepsilon^2 + \varepsilon\lambda^4)\,, \quad C \in \mathbb{R}_+\,, \tag{2.3.26}$$

where $\mathrm{B}_\rho(D)$ denotes the Euclidean ρ-enlargement of a set D, in coordinates. It suffices to prove that, for every x, $x' \in A_\varepsilon$, we can solve the problem

$$\text{find } x'' \in A_\varepsilon \text{ such that } \quad M_3 x'' = \frac{1}{2}\left(M_1 x + M_2 x'\right). \tag{2.3.27}$$

Indeed, fix $z \in M_{\frac{1}{2}}(A_\varepsilon, B_\varepsilon)$, then by definition $z = F(x, T^\lambda(x'))$, for some $x, x' \in A_\varepsilon$. Let $x'' \in A_\varepsilon$ solving (2.3.27), then, thanks to (2.3.24),

$$z = F(x, T^\lambda(x')) = T^\lambda_{\frac{1}{2},\mathrm{lin}}(x'') + O(\varepsilon^2 + \varepsilon\lambda^4)\,,$$

thus proving (2.3.26). In order to prove claim (2.3.27), note there exists λ_2 sufficiently small such that, for $\lambda \le \lambda_2$, the matrices M_i are positive definite, since $\|R_{y_0}(x_0)\|_{\mathrm{HS}} \le C\lambda^2$. In particular, the matrix M_3 is invertible. It follows that the problem (2.3.27) is solved as soon as we can ensure that

$$\forall x,\, x' \in A_\varepsilon\,, \quad M_3^{-1}\left(\frac{1}{2}\left(M_1 x + M_2 x'\right)\right) \in A_\varepsilon\,. \tag{2.3.28}$$

This is a consequence of the fact that A_ε is a cube of eigenvectors of $R_{y_0}(x_0)$. Indeed, let $\{\mu_i^1\}_i$, $\{\mu_i^2\}_i$, $\{\mu_i^3\}_i$ be the corresponding (positive) eigenvalues associated with $\{e_i\}_i$ of the matrices M_1, M_2, M_3, respectively (note that they share the same eigenspaces). By definition of normal coordinates and A_ε, every x, $x' \in A_\varepsilon$ can be written as

$$x = x_0 + \sum_{i=1}^n x_i e_i\,, \quad x' = x_0 + \sum_{i=1}^n x'_i e_i\,, \quad x_j\,,\, x'_j \in [0,\varepsilon]\,, \quad \forall j = 1,\dots,n\,.$$

Therefore, we have that

$$M_3^{-1}\left(\frac{1}{2}\left(M_1 x + M_2 x'\right)\right) - x_0 = M_3^{-1}\left(\frac{1}{2}\sum_{i=1}^n (\mu_i^1 x_i + \mu_i^2 x'_i) e_i\right) = \sum_{i=1}^n \left(\frac{\mu_i^1 x_i + \mu_i^2 x'_i}{2\mu_i^3}\right) e_i\,.$$

Recall that $M_3 = (M_1 + M_2)/2$, which in particular implies that, for every $i = 1,\dots,n$,

$$\frac{\mu_i^1 + \mu_i^2}{2\mu_i^3} = 1 \quad \Longrightarrow \quad \frac{\mu_i^1 x_i + \mu_i^2 x'_i}{2\mu_i^3} \in [0,\varepsilon]\,.$$

This concludes the proof of (2.3.28), and hence of (2.3.27).

Step 3: measure comparison. In the remaining part of the proof, the constant $C \in \mathbb{R}_+$ does not depend on λ and ε, and might change line by line. Taking advantage of (2.3.22) once again, recalling the definition of M_3 in (2.3.25), we infer

$$\mathrm{d}_x T^\lambda_{\frac{1}{2}} = M_3 + O(\lambda^4 + \varepsilon)\,, \quad \forall x \in A_\varepsilon\,.$$

By the Jacobi's formula for the derivative of the determinant, we deduce that

$$\det\left(\mathrm{d}_x T^\lambda_{\frac{1}{2}}\right) = \det M_3 \left(1 + O(\lambda^4 + \varepsilon)\right)\,, \quad \forall x \in A_\varepsilon\,. \tag{2.3.29}$$

Let $f \in C^2$ be the density of $\mathfrak{m}$ in coordinates and set $f^\lambda := f \circ T^\lambda_{\frac{1}{2}}$, $f^\lambda_{\mathrm{lin}} := f \circ T^\lambda_{\frac{1}{2},\mathrm{lin}}$. We expand f at $T^\lambda_{\frac{1}{2},\mathrm{lin}}(x)$ for $x \in A_\varepsilon$, and evaluate the expansion at $T^\lambda_{\frac{1}{2}}(x)$, obtaining

$$f^\lambda(x) = f^\lambda_{\mathrm{lin}}(x)\Big(1 + O\Big(\|T^\lambda_{\frac{1}{2}} - T^\lambda_{\frac{1}{2},\mathrm{lin}}\|_{L^\infty(A_\varepsilon)}\Big)\Big) = f^\lambda_{\mathrm{lin}}(x)\Big(1 + O(\varepsilon^2 + \varepsilon\lambda^4)\Big)\,, \tag{2.3.30}$$

where we used (2.3.24) and the fact that f is of the form $e^{-\tilde{V}}$, for some $\tilde{V} \in C^2$ depending on V and the metric g. Thus, by (2.3.29) and (2.3.30), we have that

$$\begin{aligned}
\mathfrak{m}\left(T^\lambda_{\frac{1}{2}}(A_\varepsilon)\right) &= \int_{A_\varepsilon} \left|\det\left(\mathrm{d}T^\lambda_{\frac{1}{2}}\right)\right| f^\lambda(x)\,\mathrm{d}x \\
&= \int_{A_\varepsilon} |\det M_3| \left(1 + O(\lambda^4 + \varepsilon)\right) f^\lambda_{\mathrm{lin}}(x)\Big(1 + O(\varepsilon^2 + \varepsilon\lambda^4)\Big)\,\mathrm{d}x \qquad (2.3.31) \\
&= \left(1 + O(\lambda^4 + \varepsilon)\right) \mathfrak{m}\left(T^\lambda_{\frac{1}{2},\mathrm{lin}}(A_\varepsilon)\right)\,.
\end{aligned}$$

Moreover, by means of basic properties of the linear maps, it is easy to see that

$$\mathrm{B}_{\rho_{\varepsilon,\lambda}}\Big(T^\lambda_{\frac12,\mathrm{lin}}(A_\varepsilon)\Big) \subset T^\lambda_{\frac12,\mathrm{lin}}\Big(\mathrm{B}_{c\rho_{\varepsilon,\lambda}}(A_\varepsilon)\Big)\,, \quad c\in\mathbb{R}_+\,, \tag{2.3.32}$$

where $\rho_{\varepsilon,\lambda}$ is defined in (2.3.26) and c is an upper bound for $\|M_3\|_{\mathrm{op}}$ uniform in λ. The next step consists of studying the ratio between the measure of the image, via the affine map $T^\lambda_{\frac12,\mathrm{lin}}$, of the set A_ε and its enlargement. Denote by $\mathfrak{m}_{\mathrm{lin}}\in\mathcal{M}_+(U)$ the measure with density f^λ_{lin} in coordinates. Arguing as in (2.3.31), we find that

$$\frac{\mathfrak{m}\Big(T^\lambda_{\frac12,\mathrm{lin}}\Big(\mathrm{B}_{c\rho_{\varepsilon,\lambda}}(A_\varepsilon)\Big)\Big)}{\mathfrak{m}\Big(T^\lambda_{\frac12,\mathrm{lin}}(A_\varepsilon)\Big)} = \frac{\mathfrak{m}_{\mathrm{lin}}\Big(\mathrm{B}_{c\rho_{\varepsilon,\lambda}}(A_\varepsilon)\Big)}{\mathfrak{m}_{\mathrm{lin}}(A_\varepsilon)} = \frac{\mathscr{L}^n\Big(\mathrm{B}_{c\rho_{\varepsilon,\lambda}}(A_\varepsilon)\Big)}{\mathscr{L}^n(A_\varepsilon)}\frac{⨍_{\mathrm{B}_{c\rho_{\varepsilon,\lambda}}(A_\varepsilon)} f^\lambda_{\mathrm{lin}}(x)\,\mathrm{d}x}{⨍_{A_\varepsilon} f^\lambda_{\mathrm{lin}}(x)\,\mathrm{d}x}\,.$$

On the one hand, an application of the Minkowski–Steiner formula for convex bodies, together with $\mathscr{H}^{n-1}(A_\varepsilon)=C\varepsilon^{-1}\mathscr{L}^n(A_\varepsilon)$, yields

$$\mathscr{L}^n\Big(\mathrm{B}_{c\rho_{\varepsilon,\lambda}}(A_\varepsilon)\Big) \le \Big(1+C\varepsilon^{-1}\rho_{\varepsilon,\lambda}\Big)\mathscr{L}^n(A_\varepsilon) = \Big(1+C(\lambda^4+\varepsilon)\Big)\mathscr{L}^n(A_\varepsilon)\,.$$

On the other hand, reasoning as in (2.3.30), we perform a Taylor expansion of f^λ_{lin} at x_0, obtaining the following two-sided bound:

$$\Big(1-C(\varepsilon+\rho_{\varepsilon,\lambda})\Big) \le \frac{f^\lambda_{\mathrm{lin}}(x)}{f^\lambda_{\mathrm{lin}}(x_0)} \le \Big(1+C(\varepsilon+\rho_{\varepsilon,\lambda})\Big)\,, \quad \forall x\in\mathrm{B}_{c\rho_{\varepsilon,\lambda}}(A_\varepsilon)\,.$$

Therefore, using that $\varepsilon+\rho_{\varepsilon,\lambda}\le 2\varepsilon+\lambda^4$, we obtain the upper bound

$$\mathfrak{m}\Big(T^\lambda_{\frac12,\mathrm{lin}}\Big(\mathrm{B}_{c\rho_{\varepsilon,\lambda}}(A_\varepsilon)\Big)\Big) \le \Big(1+C(\lambda^4+\varepsilon)\Big)\mathfrak{m}\Big(T^\lambda_{\frac12,\mathrm{lin}}(A_\varepsilon)\Big)\,. \tag{2.3.33}$$

In conclusion, putting together (2.3.26), (2.3.31), (2.3.32), and (2.3.33), we get that

$$\begin{aligned}\mathfrak{m}\Big(M_{\frac12}(A_\varepsilon,B_\varepsilon)\Big) &\overset{(2.3.26)}{\le} \mathfrak{m}\Big(\mathrm{B}_{\rho_{\varepsilon,\lambda}}\Big(T^\lambda_{\frac12,\mathrm{lin}}(A_\varepsilon)\Big)\Big) \overset{(2.3.32)}{\le} \mathfrak{m}\Big(T^\lambda_{\frac12,\mathrm{lin}}\Big(\mathrm{B}_{c\rho_{\varepsilon,\lambda}}(A_\varepsilon)\Big)\Big)\\ &\overset{(2.3.33)}{\le} (1+C(\lambda^4+\varepsilon))\mathfrak{m}\Big(T^\lambda_{\frac12,\mathrm{lin}}(A_\varepsilon)\Big)\\ &\overset{(2.3.31)}{\le} (1+C(\lambda^4+\varepsilon))\mathfrak{m}\Big(T^\lambda_{\frac12}(A_\varepsilon)\Big)\,.\end{aligned}$$

We take the power $\frac{1}{N}$ on both side: up to changing the constant C once again and considering λ and ε sufficiently small, we conclude the proof. □

2.3.5 Proof of the main result

We are ready to prove our main result, Theorem 2.2.10.

Proof of Theorem 2.2.10. By contradiction, we assume there exists $x_0\in M$, $v_0\in T_{x_0}M$ such that (2.3.1) is satisfied, for some $\delta>0$. Consider λ sufficiently small, and let T^λ be as in (2.3.7) and $y_0:=T^\lambda(x_0)$. In addition, let $\{e_1,\dots,e_n\}$ be an orthonormal basis of eigenvectors of $R_{y_0}(x_0)$ and, for $\varepsilon\in(0,\bar\varepsilon)$, let $Q_\varepsilon(x_0)$ be the corresponding cube, as described in Section 2.3.1. Thanks to Proposition 2.3.4 and Proposition 2.3.6, choosing $\lambda\le\min\{\lambda_1,\lambda_2\}$, we know

$$\mathcal{D}_{x_0,\lambda}\Big(\frac12\Big) \le \tau^{(\frac12)}_{K,N}(\lambda)\left(\mathcal{D}_{x_0,\lambda}(0)+\mathcal{D}_{x_0,\lambda}(1)\right) - c\lambda^2\,, \tag{2.3.34}$$

$$\mathfrak{m}\Big(M_{\frac12}(Q_\varepsilon(x_0),T^\lambda(Q_\varepsilon(x_0)))\Big)^{\frac1N} \le (1+C(\lambda^4+\varepsilon))\mathfrak{m}\Big(T^\lambda_{\frac12}(Q_\varepsilon(x_0))\Big)^{\frac1N}\,, \tag{2.3.35}$$

where $\mathcal{D}_{x_0,\lambda}$ is the volume distortion function associated with γ^λ, i.e.

$$\mathcal{D}_{x_0,\lambda}(t) := \lim_{\varepsilon\to 0} \frac{\mathfrak{m}(T_t^\lambda(Q_\varepsilon(x_0)))^{\frac{1}{N}}}{\mathfrak{m}(Q_\varepsilon(x_0))^{\frac{1}{N}}}\,, \quad t\in[0,1]\,.$$

We select the marginal sets $A_\varepsilon := Q_\varepsilon(x_0)$, $B_\varepsilon := T^\lambda(Q_\varepsilon(x_0))$. Recalling the definition of $\Theta(A,B)$ in (2.2.5), we have that $\Theta(A_\varepsilon,B_\varepsilon) = \lambda + O(\varepsilon)$, as $\varepsilon\to 0$, hence we deduce that

$$\tau_{K,N}^{(\frac{1}{2})}(\lambda) = \tau_{K,N}^{(\frac{1}{2})}(\Theta(A_\varepsilon,B_\varepsilon)) + O(\varepsilon)\,, \qquad \text{as } \varepsilon\to 0\,. \tag{2.3.36}$$

Combining (2.3.36) with (2.3.34), we infer that there exists $\varepsilon_0 = \varepsilon_0(\lambda)$ and $\bar{c} > 0$, such that, for every $\varepsilon \le \varepsilon_0$, we have the inequality

$$\mathfrak{m}(T^\lambda_{\frac{1}{2}}(A_\varepsilon))^{\frac{1}{N}} \le \tau_{K,N}^{(\frac{1}{2})}(\Theta(A_\varepsilon,B_\varepsilon))\left(\mathfrak{m}(A_\varepsilon)^{\frac{1}{N}} + \mathfrak{m}(B_\varepsilon)^{\frac{1}{N}}\right) + \left(-\frac{c}{2}\lambda^2 + \bar{c}\varepsilon\right)\mathfrak{m}(A_\varepsilon)^{\frac{1}{N}}\,, \tag{2.3.37}$$

where we have also used (2.3.23) to estimate $\mathfrak{m}(B_\varepsilon)$ in terms of $\mathfrak{m}(A_\varepsilon)$. As a consequence, if $\varepsilon \le \min\{\varepsilon_0,\lambda^4\}$, from (2.3.35) and (2.3.37), we obtain that

$$\begin{aligned}\mathfrak{m}\left(M_{\frac{1}{2}}(A_\varepsilon,B_\varepsilon)\right)^{\frac{1}{N}} &\le \tau_{K,N}^{(\frac{1}{2})}(\Theta(A_\varepsilon,B_\varepsilon))\left(\mathfrak{m}(A_\varepsilon)^{\frac{1}{N}} + \mathfrak{m}(B_\varepsilon)^{\frac{1}{N}}\right)\\ &\qquad + \left(-\frac{c}{2}\lambda^2 + \bar{c}\lambda^4\right)\mathfrak{m}(A_\varepsilon)^{\frac{1}{N}} + 2C\lambda^4\mathfrak{m}(T^\lambda_{\frac{1}{2}}(A_\varepsilon))^{\frac{1}{N}}\\ &\le \tau_{K,N}^{(\frac{1}{2})}(\Theta(A_\varepsilon,B_\varepsilon))\left(\mathfrak{m}(A_\varepsilon)^{\frac{1}{N}} + \mathfrak{m}(B_\varepsilon)^{\frac{1}{N}}\right)\\ &\qquad + \left(-\frac{c}{2}\lambda^2 + 2RC\lambda^4 + \bar{c}\lambda^4\right)\mathfrak{m}(A_\varepsilon)^{\frac{1}{N}}\,,\end{aligned}$$

where we have used that $\mathfrak{m}(T^\lambda_{\frac{1}{2}}(A_\varepsilon)) \le R\mathfrak{m}(A_\varepsilon)$, for some $R>0$, which is a consequence once again of (2.3.23). Therefore, if λ was chosen in such a way that $\lambda^2 < c(4RC+2\bar{c})^{-1}$, then for any $\varepsilon\le\min\{\varepsilon_0,\lambda^4\}$, we finally conclude that

$$\mathfrak{m}\left(M_{\frac{1}{2}}(A_\varepsilon,B_\varepsilon)\right)^{\frac{1}{N}} < \tau_{K,N}^{(\frac{1}{2})}(\Theta(A_\varepsilon,B_\varepsilon))\left(\mathfrak{m}(A_\varepsilon)^{\frac{1}{N}} + \mathfrak{m}(B_\varepsilon)^{\frac{1}{N}}\right)\,, \tag{2.3.38}$$

which is a contradiction to the space satisfying $\mathsf{BM}(K,N)$. This concludes the proof. ☐

Remark 2.3.9. The estimate of Proposition 2.3.4 gives *at best* a negative error in λ of order 2. Therefore, in Proposition 2.3.6 an estimate with a second–order precision would not be enough as the two errors would compete without having a definitive sign in the limit. Thus, we need to push the estimate of Proposition 2.3.6 to a fourth–order precision, in order to conclude that the error term is *definitively negative.* Remarkably the estimate of Proposition 2.3.6 does not involve a term of order 2 in λ and we were finally able to prove (2.3.38) carefully choosing λ.

Paper 3

The strong Brunn–Minkowski inequality and its equivalence with the CD *condition*

with Lorenzo Portinale and Tommaso Rossi

In the setting of essentially non-branching metric measure spaces, we prove the equivalence between the curvature dimension condition $\mathsf{CD}(K,N)$*, in the sense of Lott–Sturm–Villani [Stu06a, Stu06b, LV09], and a newly introduced notion that we call strong Brunn–Minkowski inequality* $\mathsf{SBM}(K,N)$*. This condition is a reinforcement of the generalized Brunn–Minkowski inequality* $\mathsf{BM}(K,N)$*, which is known to hold in* $\mathsf{CD}(K,N)$ *spaces. Our result is a first step towards providing a full equivalence between the* $\mathsf{CD}(K,N)$ *condition and the validity of* $\mathsf{BM}(K,N)$*, which has been recently proved in [MPR22a] in the framework of weighted Riemannian manifolds.*

All authors of this paper contributed equally to all results.

3.1 Introduction

In their seminal papers [Stu06a, Stu06b, LV09], Lott–Sturm–Villani introduced a synthetic notion of curvature dimension bounds, in the non-smooth setting of metric measure spaces, usually denoted by $\mathsf{CD}(K,N)$, with $K \in \mathbb{R}$, $N \in (1,\infty]$. This property is formulated using the theory of *optimal transport* and, in the setting of Riemannian manifolds, is equivalent to having Ricci curvature bounded below and dimension bounded above. More precisely, the $\mathsf{CD}(K,N)$ condition consists in a convexity property of the so-called Rényi entropy functional along Wasserstein geodesics.

In the Riemannian setting, curvature bounds are sufficient to deduce many geometric and functional inequalities. Similar results can be obtained in the framework of a metric measure space $(\mathsf{X},\mathsf{d},\mathfrak{m})$ as a consequence of the curvature-dimension condition. A celebrated geometric inequality that can be deduced from the $\mathsf{CD}(K,N)$ condition, in a generalized form, is the *Brunn–Minkowski inequality*: the convexity of the Rényi entropy translates to a concavity property of the mass of the t-midpoints, namely of the function

$$[0,1] \ni t \mapsto \mathfrak{m}(M_t(A,B))^{\frac{1}{N}}, \qquad A,B \subset \mathsf{X} \text{ Borel sets},$$

see Definition 3.3.6. This was already observed in the first papers on the $\mathsf{CD}(K,N)$ condition, see in particular [Stu06b, Proposition 2.1]. On the one hand, a remarkable feature of the Brunn–Minkowski inequality is that its formulation does not invoke optimal transport. On the other hand, its proof relies on the well-known inclusion of the support $D_t(A,B)$ of the (unique) Wasserstein t-midpoint, between the normalized uniform distributions on A and B, in the set of t-midpoints $M_t(A,B)$. Therefore, a natural strengthening of the Brunn–Minkowski inequality is to require that the aforementioned concavity property holds, not for the whole set of t-midpoints, but only for the support of the Wasserstein t-midpoint. This leads to main novelty of this paper: the introduction of the *strong Brunn–Minkowski inequality*, which we denote by $\mathsf{SBM}(K,N)$, cf. Definition 3.3.9. Despite being still dependent on the optimal transport, this inequality is reminiscent of the Brunn–Minkowski one. Our main result is that $\mathsf{SBM}(K,N)$ is equivalent to $\mathsf{CD}(K,N)$, in the setting of essentially non-branching metric measure spaces. We refer to Sections 3.2 and 3.3 for the precise definitions.

Theorem 3.1.1. *Let $K \in \mathbb{R}$ and $N > 1$ and let $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be an essentially non-branching metric measure space supporting $\mathsf{SBM}(K,N)$. Then, $(\mathsf{X},\mathsf{d},\mathfrak{m})$ is a $\mathsf{CD}(K,N)$ space. In particular,*

$$(\mathsf{X},\mathsf{d},\mathfrak{m}) \textit{ supports } \mathsf{SBM}(K,N) \textit{ if and only if it satisfies } \mathsf{CD}(K,N).$$

This theorem is a first step towards the complete equivalence between the Brunn–Minkowski inequality and the curvature dimension condition, in the setting of essentially non-branching spaces. The interest in the aforementioned equivalence is twofold: on one side, it would provide a characterization of the curvature dimension condition *without* the need of optimal transport. On the other side, it would provide an alternative proof of the *globalization theorem*, cf. [CM21]. Indeed, according to [CM17c, Theorem 1.2], the local curvature dimension condition, denoted by $\mathsf{CD}_{\mathrm{loc}}(K,N)$, is enough to the deduce the (global) Brunn–Minkowski inequality with sharp coefficients. Note that, should $\mathsf{CD}_{\mathrm{loc}}(K,N)$ imply $\mathsf{SBM}(K,N)$, Theorem 3.1.1 would already be enough to deduce the globalization theorem. However, this implication does not easily follow from the arguments in [CM17c, Theorem 1.2]. In fact, their technique is based on a suitable localization argument, built upon the L^1-optimal transport and thus it does not immediately imply the validity of the strong Brunn–Minkowski inequality, which on the contrary is based on the L^2-optimal transport.

In [MPR22a], we prove the equivalence between the Brunn–Minkowski inequality and the curvature dimension condition, in the setting of weighted Riemannian manifolds. In this framework, using the *full* Riemann curvature tensor, we are able to identify pairs of sets A and B, for which the sets $M_t(A,B)$ and $D_t(A,B)$ are comparable in measure. In the setting of essentially non-branching metric measure spaces, the relation between $D_t(A,B)$ and $M_t(A,B)$ is in general less clear. A better understanding of this problem is most likely required to close the gap between the Brunn–Minkowski inequality and the $\mathsf{CD}(K,N)$ condition, cf. Section 3.5.2 for a more detailed discussion.

Organization of the paper

In Section 3.2, we introduce the basic tools of metric measure spaces and optimal transport. In Section 3.3, we recall the definition of a $\mathsf{CD}(K,N)$ space and we prove an equivalence criterion, which we believe is of independent interest, cf. Proposition 3.3.5. Finally, we introduce the strong Brunn–Minkowski inequality $\mathsf{SBM}(K,N)$, cf. Definition 3.3.9, and we show its interplay with the Brunn–Minkowski inequality. In Section 3.4, we prove Theorem 3.1.1: the most challenging step is to find a large class of measures for which $\mathsf{SBM}(K,N)$ implies $\mathsf{CD}(K,N)$, cf. Theorem 3.4.3. At last, in Section 3.5, we show that the Brunn–Minkowski inequality implies the so-called measure contraction property, cf. Proposition 3.5.3, and we add some final remarks.

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3.2 Preliminaries

A metric measure space is a triple $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$, where (X, d) is a Polish metric space (i.e. complete and separable) and $\mathfrak{m}$ is a positive Borel measure on X, finite on bounded sets. We call $\mathscr{M}_+(\mathsf{X})$ the set of all positive and finite Borel measures on a metric space (X, d). We denote by $\mathscr{P}(\mathsf{X}) \subset \mathscr{M}_+(\mathsf{X})$ the set of all Borel probability measures on (X, d), and by $\mathscr{P}_2(\mathsf{X}) \subset \mathscr{P}(\mathsf{X})$ the set of all probability measures with finite second moment. Moreover, the set of probability measures in $\mathscr{P}_2(\mathsf{X})$, which are absolutely continuous with respect to the reference measure $\mathfrak{m}$, will be denoted by $\mathscr{P}^{ac}(\mathsf{X}, \mathfrak{m})$. On the space $\mathscr{P}_2(\mathsf{X})$, we introduce the 2-Wasserstein distance

$$W_2^2(\mu, \nu) := \inf_\pi \int_{\mathsf{X}\times\mathsf{X}} \mathsf{d}(x, y)^2 \, \mathrm{d}\pi(x, y), \tag{3.2.1}$$

where the infimum is taken over all *admissible plans*, that is over all $\pi \in \mathscr{P}(\mathsf{X} \times \mathsf{X})$ such that $(\mathsf{p}_1)_\# \pi = \mu$ and $(\mathsf{p}_2)_\# \pi = \nu$. Here $\mathsf{p}_i : \mathsf{X} \times \mathsf{X} \to \mathsf{X}$ denotes the projection on the i-th factor. The infimum in (3.2.1) is always attained, the admissible plans realizing it are called *optimal transport plans* and the set that contains all of them is denoted by $\mathrm{Opt}(\mu, \nu)$. The optimality of a transport plan can be equivalently characterized with the notion of d^2-cyclical monotonicity (see [ABS21, Definition 3.10] for the definition). In particular, an admissible plan $\pi \in \mathscr{P}(\mathsf{X} \times \mathsf{X})$ is optimal between its marginals if and only if it is concentrated on a d^2-cyclically monotone (σ-compact) set $\Gamma \in \mathsf{X} \times \mathsf{X}$, cf. [ABS21, Theorem 4.2].

It is well known that W_2 is a complete and separable distance on $\mathscr{P}_2(\mathsf{X})$. Moreover, the convergence with respect to the Wasserstein distance is characterized in the following way, cf. [ABS21, Theorem 8.8]:

$$\mu_n \xrightarrow{W_2} \mu \quad \iff \quad \mu_n \rightharpoonup \mu \text{ and } \int \mathsf{d}(x_0, x)^2 \, \mathrm{d}\mu_n \to \int \mathsf{d}(x_0, x)^2 \, \mathrm{d}\mu \quad \forall x_0 \in \mathsf{X}.$$

In the last formula the symbol $\rightharpoonup$ denotes the weak convergence of measures, i.e. the one in duality with the space of continuous and bounded functions $C_b(\mathsf{X})$. This description shows in particular that, for a sequence of probability measures with uniformly bounded support, the W_2-convergence is equivalent to the weak one. Moreover, as a consequence of Riesz and Banach–Alaoglu theorems, if (X, d) is compact then $\mathscr{P}(\mathsf{X}) = \mathscr{P}_2(\mathsf{X})$ is compact as well, with respect to the Wasserstein distance W_2. For the same reason, every family of measures in $\mathscr{P}_2(\mathsf{X})$, having the same compact support, will be W_2-precompact (even if X is not compact).

Let $C([0, 1], \mathsf{X})$ be the set of continuous functions from $[0, 1]$ to X and define the *t-evaluation map* as $e_t : C([0, 1], \mathsf{X}) \to \mathsf{X}$; $e_t(\gamma) := \gamma(t)$, for $\gamma \in C([0, 1], \mathsf{X})$, for any $t \in [0, 1]$. A curve $\gamma \in C([0, 1], \mathsf{X})$ is called *geodesic* if

$$\mathsf{d}(\gamma(s), \gamma(t)) = |t - s| \cdot \mathsf{d}(\gamma(0), \gamma(1)) \quad \text{for every } s, t \in [0, 1].$$

We will denote by $\mathrm{Geo}(\mathsf{X})$ the space of constant speed geodesics in (X, d) parametrized on $[0, 1]$. The metric space (X, d) is said to be *geodesic* if every pair of points is connected by a geodesic, Notice that every measure $\eta \in \mathscr{P}(C([0, 1], \mathsf{X}))$ induces the curve $[0, 1] \ni t \mapsto \mu_t = (e_t)_\# \eta$ in the space of probability measures $\mathscr{P}(\mathsf{X})$. If (X,d) is a geodesic metric space then $(\mathscr{P}_2(\mathsf{X}),W_2)$ is a geodesic metric space as well. More precisely, given two measures $\mu,\nu\in\mathscr{P}_2(\mathsf{X})$, the curve $\{\mu_t\}_{t\in[0,1]}\subset\mathscr{P}_2(\mathsf{X})$ connecting μ and ν is a Wasserstein geodesic if and only if there exists $\eta\in\mathscr{P}(C([0,1],\mathsf{X}))$ inducing $\{\mu_t\}_{t\in[0,1]}$ (that is $\mu_t=(e_t)_\#\eta$ for every $t\in[0,1]$), which is concentrated on $\mathrm{Geo}(\mathsf{X})$ and satisfies $(e_0,e_1)_\#\eta\in\mathrm{Opt}(\mu,\nu)$. In this case it is said that η is an *optimal geodesic plan* between μ and ν and this will be denoted as $\eta\in\mathrm{OptGeo}(\mu,\nu)$.

Definition 3.2.1 (Essentially non-branching metric space)**.** In a metric space (X,d), a subset $G\subset\mathrm{Geo}(\mathsf{X})$ is called *non-branching* if for any pair of geodesics $\gamma_1,\gamma_2\in G$ such that $\gamma_1\neq\gamma_2$, it holds that

$$\mathrm{restr}_0^t\gamma_1\neq\mathrm{restr}_0^t\gamma_1\qquad\text{for every } t\in(0,1).$$

A metric measure space $(\mathsf{X},\mathsf{d},\mathfrak{m})$ is said to be *essentially non-branching* if for every absolutely continuous measures $\mu_0,\mu_1\in\mathscr{P}^{ac}(\mathsf{X},\mathfrak{m})$, every optimal geodesic plan η connecting them is concentrated on a non-branching set of geodesics.

Definition 3.2.2 (Midpoint set)**.** Let (X,d) be a metric space. For every $t\in[0,1]$ and any pair of sets $A,B\subset\mathsf{X}$ we define the set of t-midpoints between A and B as

$$M_t(A,B):=e_t(\{\gamma\in\mathrm{Geo}(\mathsf{X})\,:\,\gamma(0)\in A,\,\gamma(1)\in B\})$$

We will adopt the notation $M_t(x,A):=M_t(\{x\},A)$ and $M_t(A,x):=M_t(A,\{x\})$ for every $x\in\mathsf{X}$ and $A\subset\mathsf{X}$.

Observe that in general the set $M_t(A,B)$ is not Borel measurable, even if the sets A and B are Borel. In the following we will need to evaluate the measure of $M_t(A,B)$, for this reason we introduce the measure $\bar{\mathfrak{m}}$ as the outer measure associated to $\mathfrak{m}$. This measure will not play a significant role and will only be used when dealing with sets of the form $M_t(A,B)$. In particular, having a control on the measure of some suitable sets of t-midpoints, is sufficient to deduce some nice properties regarding the structure of W_2-geodesics, as shown in Proposition 3.2.5.

Definition 3.2.3 ([Kel17, Definition 5.1])**.** A measure $\mathfrak{m}$ on a metric space (X,d) is said to be *qualitatively non-degenerate* if for every $R>0$ and $\bar{x}\in\mathsf{X}$ there exists a function $f_{R,\bar{x}}:(0,1)\to(0,\infty)$ with

$$\limsup_{t\to 0} f_{R,\bar{x}}(t)>\frac{1}{2},$$

such that for every $x\in\mathsf{B}_R(\bar{x})$ and every Borel subset $A\subset\mathsf{B}_R(\bar{x})$

$$\bar{\mathfrak{m}}(M_t(A,x))\geq f_{R,\bar{x}}(t)\cdot\mathfrak{m}(A).$$

Definition 3.2.4 ([Kel17, Definition 3.1])**.** We say that a metric measure space $(\mathsf{X},\mathsf{d},\mathfrak{m})$ has the *good transport behaviour* if, for every pair $\mu_0,\mu_1\in\mathscr{P}_2(\mathsf{X})$ with $\mu_0\ll\mathfrak{m}$, any optimal transport plan between μ_0 and μ_1 is induced by a map. We say that $(\mathsf{X},\mathsf{d},\mathfrak{m})$ has the *strong interpolation property* if for every pair $\mu_0,\mu_1\in\mathscr{P}_2(\mathsf{X})$ with $\mu_0\ll\mathfrak{m}$, there exists a unique optimal geodesic plan $\eta\in\mathrm{OptGeo}(\mu_0,\mu_1)$ and is induced by a map and such that $(e_t)_\#\eta\ll\mathfrak{m}$ for every $t\in[0,1)$.

Recall that a metric space (X,d) is said to be *proper* if every closed and bounded set is compact.

Proposition 3.2.5 ([Kel17, Theorem 5.8, Corollary 5.9])**.** *Assume* $(\mathsf{X},\mathsf{d},\mathfrak{m})$ *is a proper, geodesic, essentially non-branching metric measure space and* $\mathfrak{m}$ *is qualitatively non-degenerate. Then,* $(\mathsf{X},\mathsf{d},\mathfrak{m})$ *has both the good transport behaviour and the strong interpolation property.*

3.3 The CD condition and the Brunn-Minkowski inequality

Before going through the definition of the CD condition, we introduce the two object which are necessary to do it: the distortion coefficient and the Rényi entropy functional. For every $K \in \mathbb{R}$ and $N > 0$ we define

$$\sigma_{K,N}^{(t)}(\theta) := \begin{cases} +\infty & N\pi^2 \leq K\theta^2 \\ \frac{\sin\left(t\theta\sqrt{\frac{K}{N}}\right)}{\sin\left(\theta\sqrt{\frac{K}{N}}\right)} & 0 < K\theta^2 < N\pi^2 \\ t & K = 0 \text{ or } N = \infty \\ \frac{\sinh\left(t\theta\sqrt{\frac{-K}{N}}\right)}{\sinh\left(\theta\sqrt{\frac{-K}{N}}\right)} & K < 0 \end{cases},$$

while for $K \in \mathbb{R}$ and $N > 1$ we introduce

$$\tau_{K,N}^{(t)}(\theta) := t^{\frac{1}{N}} \sigma_{K,N-1}^{(t)}(\theta)^{1-\frac{1}{N}}.$$

These coefficients have nice monotonicity properties, in particular for every fixed $t \in (0,1)$, $K \in \mathbb{R}$ and $N > 1$

$$\text{the function } \ \theta \mapsto \tau_{K,N}^{(t)}(\theta) \ \text{ is } \begin{array}{ll} \text{nondecreasing} & \text{if } K \geq 0 \\ \text{nonincreasing} & \text{if } K < 0 \end{array}. \tag{3.3.1}$$

The N-Rényi entropy functional on $\mathscr{P}_2(\mathsf{X})$ is defined as

$$\mathcal{E}_N(\mu) = -\int_{\mathsf{X}} \rho(x)^{1-\frac{1}{N}} \, \mathrm{d}\mathfrak{m}(x) \qquad \forall \mu \in \mathscr{P}_2(\mathsf{X}),$$

where ρ is the density of the absolutely continuous part of μ, with respect to $\mathfrak{m}$. It is well known (see for instance [Stu06a, Lemma 4.1]) that the N-Rényi entropy is lower semicontinuous in $(\mathscr{P}_2(\mathsf{X}), W_2)$, if the reference measure $\mathfrak{m}$ has finite total mass. In general, for metric measure spaces with possibly infinite mass, the lower semicontinuity holds for W_2-converging sequences of measures concentrated on the same bounded set.

Definition 3.3.1 ($\mathsf{CD}(K,N)$ condition)**.** Given $K \in \mathbb{R}$ and $N > 1$, a metric measure space $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ is said to satisfy the *curvature dimension* condition $\mathsf{CD}(K,N)$ (or simply to be a $\mathsf{CD}(K,N)$ space) if for every pair of measures $\mu_0 = \rho_0\mathfrak{m}, \mu_1 = \rho_1\mathfrak{m} \in \mathscr{P}^{ac}(\mathsf{X},\mathfrak{m})$, there exists a W_2-geodesic $\eta \in \mathscr{P}(\mathrm{Geo}(\mathsf{X}))$ connecting them, such that $(e_t)_\# \eta =: \mu_t = \rho_t \mathfrak{m} \ll \mathfrak{m}$, for every $t \in [0,1]$, and the following inequality holds for every $N' \geq N$ and every $t \in [0,1]$:

$$\mathcal{E}_{N'}(\mu_t) \leq -\int_{\mathsf{X}\times\mathsf{X}} \left[\tau_{K,N'}^{(1-t)}(\mathsf{d}(x,y))\rho_0(x)^{-\frac{1}{N'}} + \tau_{K,N'}^{(t)}(\mathsf{d}(x,y))\rho_1(y)^{-\frac{1}{N'}}\right] \mathrm{d}\pi(x,y), \tag{3.3.2}$$

where $\pi = (e_0, e_1)_\# \eta \in \mathrm{Opt}(\mu_0, \mu_1)$.

Notation 3.3.2. In the following, in order to ease the notation we will sometimes denote by $T_{K,N'}^{(t)}(\pi|\mathfrak{m})$ the right hand side of (3.3.2), that is

$$T_{K,N'}^{(t)}(\pi|\mathfrak{m}) = -\int_{\mathsf{X}\times\mathsf{X}} \left[\tau_{K,N'}^{(1-t)}(\mathsf{d}(x,y))\rho_0(x)^{-\frac{1}{N'}} + \tau_{K,N'}^{(t)}(\mathsf{d}(x,y))\rho_1(y)^{-\frac{1}{N'}}\right] \mathrm{d}\pi(x,y).$$

Notice that it is not necessary to explicit the dependence of the integral on the densities ρ_0 and ρ_1, because this information is already encoded in π and $\mathfrak{m}$, in fact $(\mathsf{p}_1)_\#\pi = \mu_0 = \rho_0\mathfrak{m}$ and $(\mathsf{p}_2)_\#\pi = \mu_1 = \rho_1\mathfrak{m}$.

We now want to state a sufficient criterion to verify the $\mathsf{CD}(K,N)$ condition, allowing to test the definition only on suitable pairs of marginals. To this aim, we introduce the notion of bounded probability measure.

Definition 3.3.3 (Bounded probability measure)**.** A probability measure $\mu \in \mathscr{P}^{ac}(\mathsf{X},\mathfrak{m})$ is said to be *bounded* if it has bounded support and density bounded from above and below away from zero. A subset $A \subset \mathscr{P}^{ac}(\mathsf{X},\mathfrak{m})$ is said to be *uniformly bounded* if there exist a bounded set K and two constants $C > c > 0$ such that for every $\mu = \rho\mathfrak{m} \in A$, $\mathrm{spt}(\mu) = K$ and $c \le \rho \le C$ $\mathfrak{m}$-almost everywhere on K.

Notation 3.3.4. Given a Borel set $A \subset \mathsf{X}$ such that $0 < \mathfrak{m}(A) < \infty$, we will denote by $\mathfrak{m}_A$ the normalized restriction of the reference measure to the set A, that is

$$\mathfrak{m}_A = \frac{\mathfrak{m}|_A}{\mathfrak{m}(A)}. \tag{3.3.3}$$

Proposition 3.3.5. *Let $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be a proper, geodesic and essentially non-branching metric measure space and assume $\mathfrak{m}$ is qualitatively non-degenerate. Then, $(\mathsf{X},\mathsf{d},\mathfrak{m})$ satisfies the $\mathsf{CD}(K,N)$ condition if and only if the requirements of Definition 3.3.1 hold for any pair of bounded probability measures.*

Proof. Suppose that the $\mathsf{CD}(K,N)$ condition holds for every pair of bounded marginals and fix $\mu_0 = \rho_0\mathfrak{m}$, $\mu_1 = \rho_1\mathfrak{m} \in \mathscr{P}^{ac}(\mathsf{X},\mathfrak{m})$. According to Proposition 3.2.5, both the good transport behaviour and the strong interpolation property hold for $(\mathsf{X},\mathsf{d},\mathfrak{m})$. Then, let η be the unique optimal geodesic plan connecting μ_0 and μ_1 and $\pi = (e_0,e_1)_\#\eta$ the unique optimal transport plan between them. Moreover, $(e_t)_\#\eta \ll \mathfrak{m}$ for any $t \in [0,1]$ and we denote by ρ_t its density. Fix $x_0 \in \mathsf{X}$ and define (up to null sets) the following sets

$$A_n := \{x \in \mathsf{B}_n(x_0) \,:\, 1/n \le \rho_0(x) \le n\} \quad \text{and} \quad B_n := \{x \in \mathsf{B}_n(x_0) \,:\, 1/n \le \rho_1(x) \le n\}.$$

Then, we introduce the set

$$G_n := \{\gamma \in \mathrm{Geo}(\mathsf{X}) \,:\, \gamma(0) \in A_n,\, \gamma(1) \in B_n\}$$

and the measures

$$\eta_n := \eta_{G_n} \in \mathscr{P}(\mathrm{Geo}(\mathsf{X})) \quad \text{and} \quad \pi_n := (e_0,e_1)_\#\eta_n \in \mathscr{P}(\mathsf{X}\times\mathsf{X}).$$

Note that, since $\eta \in \mathrm{OptGeo}(\mu_0,\mu_1)$, its restriction η_n is still optimal between $\mu_0^n := (e_0)_\#\eta_n$ and $\mu_1^n := (e_1)_\#\eta_n$ and, as a consequence, $\pi_n \in \mathrm{Opt}(\mu_0^n,\mu_1^n)$. In addition, according to Proposition 3.2.5, η_n is the unique optimal geodesic plan between μ_0^n and μ_1^n, and π_n is the unique optimal plan. Observe that, thanks to the good transport behaviour, both π and $\pi^{-1} := (e_1,e_0)_\#\eta$ are induced by a map, thus, defining $\tilde{A}_n := e_0(G_n \cap \mathrm{spt}(\eta)) \subset A_n$ and $\tilde{B}_n := e_1(G_n \cap \mathrm{spt}(\eta)) \subset B_n$, it holds that

$$\mu_0^n = \frac{\mu_0|_{\tilde{A}_n}}{\eta(G_n)} \quad \text{and} \quad \mu_1^n = \frac{\mu_1|_{\tilde{B}_n}}{\eta(G_n)}. \tag{3.3.4}$$

This shows in particular that μ_0^n and μ_1^n are bounded for every n. Moreover, by definition $\eta(G_n)\cdot\eta_n \le \eta$, thus denoting by ρ_t^n the density of $(e_t)_\#\eta_n$ (with respect to $\mathfrak{m}$) and setting $\tilde{\rho}_t^n := \eta(G_n)\cdot\rho_t^n$ for every $t \in [0,1]$, we have

$$\tilde{\rho}_t^n \le \rho_t \qquad \mathfrak{m}\text{-a.e., } \forall\, t \in [0,1]. \tag{3.3.5}$$

On the other hand, the families $\{A_n\}_{n\in\mathbb{N}}$ and $\{B_n\}_{n\in\mathbb{N}}$ exhaust the supports of μ_0 and μ_1 respectively, hence $\eta(G_n) \to 1$ and $\pi(\mathsf{X}\times\mathsf{X}\setminus(\tilde{A}_n\times\tilde{B}_n)) \to 0$ as $n\to\infty$. Applying the $\mathsf{CD}(K,N)$ condition for the bounded marginals μ_0^n and μ_1^n, we have, for every $N'\geq N$ and for every $t\in[0,1]$,

$$\begin{aligned}(\eta(G_n))^{\frac{1}{N'}-1}\int(\tilde{\rho}_t^n)^{1-\frac{1}{N'}}\,\mathrm{d}\mathfrak{m} &= \int(\rho_t^n)^{1-\frac{1}{N'}}\,\mathrm{d}\mathfrak{m}\\ &\geq \int_{\mathsf{X}\times\mathsf{X}}\Big[\tau_{K,N'}^{(1-t)}(\mathsf{d}(x,y))\rho_0^n(x)^{-\frac{1}{N'}}+\tau_{K,N'}^{(t)}(\mathsf{d}(x,y))\rho_1^n(y)^{-\frac{1}{N'}}\Big]\,\mathrm{d}\pi_n(x,y)\\ &= (\eta(G_n))^{\frac{1}{N'}-1}\int_{\tilde{A}_n\times\tilde{B}_n}\Big[\tau_{K,N'}^{(1-t)}(\mathsf{d}(x,y))\rho_0(x)^{-\frac{1}{N'}}+\tau_{K,N'}^{(t)}(\mathsf{d}(x,y))\rho_1(y)^{-\frac{1}{N'}}\Big]\,\mathrm{d}\pi(x,y),\end{aligned}$$

where the last equality follows from (3.3.4) and from the fact that $\pi_{\tilde{A}_n\times\tilde{B}_n}$ coincides with π_n. Simplifying the term $\eta(G_n)$ (which is definitely strictly greater than 0) and using (3.3.5), we obtain for every $N'\geq N$ and for every $t\in[0,1]$

$$\begin{aligned}\int\rho_t^{1-\frac{1}{N'}}\,\mathrm{d}\mathfrak{m} &\geq \int(\tilde{\rho}_t^n)^{1-\frac{1}{N'}}\,\mathrm{d}\mathfrak{m}\\ &\geq \int_{\tilde{A}_n\times\tilde{B}_n}\Big[\tau_{K,N'}^{(1-t)}(\mathsf{d}(x,y))\rho_0(x)^{-\frac{1}{N'}}+\tau_{K,N'}^{(t)}(\mathsf{d}(x,y))\rho_1(y)^{-\frac{1}{N'}}\Big]\,\mathrm{d}\pi(x,y),\end{aligned}$$

and taking the limit as $n\to\infty$, we conclude that

$$\int\rho_t^{1-\frac{1}{N'}}\,\mathrm{d}\mathfrak{m} \geq \int_{\mathsf{X}\times\mathsf{X}}\Big[\tau_{K,N'}^{(1-t)}(\mathsf{d}(x,y))\rho_0(x)^{-\frac{1}{N'}}+\tau_{K,N'}^{(t)}(\mathsf{d}(x,y))\rho_1(y)^{-\frac{1}{N'}}\Big]\,\mathrm{d}\pi(x,y),$$

which is exactly (3.3.2) for μ_0 and μ_1. This concludes the proof. □

We introduce now a generalized version of the classical Brunn–Minkowski inequality to the non-smooth setting. Similarly to CD condition, this inequality takes into account dimensional and curvature parameters.

Definition 3.3.6 (Brunn–Minkowski inequality)**.** Let $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be a metric measure space and let $K\in\mathbb{R}$ and $N>1$. We say that $(\mathsf{X},\mathsf{d},\mathfrak{m})$ supports the *Brunn–Minkowski inequality* $\mathsf{BM}(K,N)$ if, for every pair of nonempty Borel sets $A,B\subset\mathrm{spt}(\mathfrak{m})$, the following inequality holds for every $N'\geq N$ and every $t\in[0,1]$:

$$\bar{\mathfrak{m}}(M_t(A,B)))^{\frac{1}{N'}} \geq \tau_{K,N'}^{(1-t)}(\Theta(A,B))\cdot\mathfrak{m}(A)^{\frac{1}{N'}}+\tau_{K,N'}^{(t)}(\Theta(A,B))\cdot\mathfrak{m}(B)^{\frac{1}{N'}}, \tag{3.3.6}$$

where

$$\Theta(A,B) := \begin{cases}\inf_{x\in A,\,y\in B}\mathsf{d}(x,y) & \text{if } K\geq 0,\\ \sup_{x\in A,\,y\in B}\mathsf{d}(x,y) & \text{if } K<0.\end{cases} \tag{3.3.7}$$

Similarly as for the t-midpoints, we adopt the notation $\Theta(A,x) := \Theta(A,\{x\})$ and $\Theta(x,A) := \Theta(\{x\},A)$, for every $x\in\mathsf{X}$ and $A\subset\mathsf{X}$.

Lemma 3.3.7. *Let $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be a metric measure space supporting $\mathsf{BM}(K,N)$. Then, $(\mathrm{spt}(\mathfrak{m}),\mathsf{d})$ is a Polish, geodesic and proper metric space. Moreover, $\mathfrak{m}$ is a Radon measure.*

Proof. Since (X,d) is Polish and $\mathrm{spt}(\mathfrak{m})$ is closed, the metric space $(\mathrm{spt}(\mathfrak{m}),\mathsf{d})$ is Polish as well. Moreover, from the proof of [Stu06b, Theorem 2.3], $\mathsf{BM}(K,N)$ implies that $(\mathrm{spt}(\mathfrak{m}),\mathsf{d},\mathfrak{m})$ satisfies a Bishop–Gromov inequality, and thus $\mathfrak{m}$ is a doubling measure. By a standard argument, this means that $(\mathrm{spt}(\mathfrak{m}),\mathsf{d})$ is a doubling metric space, i.e. any bounded set is totally bounded, therefore it is proper and also σ-compact. As a consequence, $\mathfrak{m}$ is Radon, being a locally finite measure on a locally compact, second countable space (see for example [Fol99, Thoerem 7.8]). We prove now that $(\mathrm{spt}(\mathfrak{m}), \mathsf{d})$ is length: let $x, y \in \mathrm{spt}(\mathfrak{m})$, $\varepsilon > 0$ and fix $A_\varepsilon := \mathsf{B}_\varepsilon(x) \cap \mathrm{spt}(\mathfrak{m})$ and $B_\varepsilon := \mathsf{B}_\varepsilon(y) \cap \mathrm{spt}(\mathfrak{m})$. Applying $\mathsf{BM}(K,N)$ we deduce that $\bar{\mathfrak{m}}(M_{1/2}(A_\varepsilon, B_\varepsilon)) > 0$, therefore there exists $z \in M_{1/2}(A_\varepsilon, B_\varepsilon) \cap \mathrm{spt}(\mathfrak{m})$. In particular, by construction, this implies that:

$$\mathsf{d}(x,z), \mathsf{d}(z,y) \leq \frac{1}{2}\mathsf{d}(x,y) + \varepsilon.$$

Since x, y and ε are arbitrary, we can conclude that $(\mathrm{spt}(\mathfrak{m}), \mathsf{d})$ is a length space (see [Bal95, Proposition 1.4]). Finally, a complete, proper and length space is geodesic. □

Applying the previous lemma, we can deduce that a metric measure space supporting the Brunn–Minkowski inequality has the properties of Definition 3.2.4.

Corollary 3.3.8. *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *be an essentially non-branching metric measure space supporting the Brunn–Minkowski inequality* $\mathsf{BM}(K,N)$*. Then,* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *has the good transport behavior and the strong interpolation property.*

Proof. First of all, we restrict ourselves to the support of $\mathfrak{m}$ and consider the metric measure space $(\mathrm{spt}(\mathfrak{m}), \mathsf{d}, \mathfrak{m})$. From Lemma 3.3.7, this is a proper and geodesic metric space. Second of all, letting $R > 0$, $\bar{x} \in \mathrm{spt}(\mathfrak{m})$ and $A \subset \mathsf{B}_R(\bar{x}) \cap \mathrm{spt}(\mathfrak{m})$ Borel, we may apply $\mathsf{BM}(K,N)$ to A and $\bar{x}$, obtaining

$$\bar{\mathfrak{m}}(M_t(A, \bar{x})) \geq \tau_{K,N}^{(1-t)}(\Theta(A, \bar{x}))^N \mathfrak{m}(A),$$

for any $t \in [0,1]$. This shows that $\mathfrak{m}$ is qualitatively non-degenerate on its support. Finally, applying Proposition 3.2.5 to the metric measure space $(\mathrm{spt}(\mathfrak{m}), \mathsf{d}, \mathfrak{m})$ we conclude the proof. Note that the good transport behavior and the strong interpolation property are only related to optimal transport, which in turn depends on the metric measure structure of $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ only on the support of $\mathfrak{m}$. □

In this paper, we study a stronger version of the Brunn–Minkowski inequality, which is more sensitive to the optimal transport interpolation.

Definition 3.3.9 (Strong Brunn–Minkowski inequality)**.** Let $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ be a metric measure space and let $K \in \mathbb{R}$ and $N > 1$. We say that $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ supports the *strong Brunn–Minkowski inequality* $\mathsf{SBM}(K,N)$ if, for every pair of Borel sets $A, B \subset \mathrm{spt}(\mathfrak{m})$ such that $0 < \mathfrak{m}(A), \mathfrak{m}(B) < \infty$, there exists $\eta \in \mathrm{OptGeo}(\mathfrak{m}_A, \mathfrak{m}_B)$, where $\mathfrak{m}_A, \mathfrak{m}_B$ are as in (3.3.3), such that the following inequality holds for every $N' \geq N$ and every $t \in [0,1]$

$$\mathfrak{m}(\mathrm{spt}((e_t)_\# \eta))^{\frac{1}{N'}} \geq \tau_{K,N'}^{(1-t)}(\Theta(A,B)) \cdot \mathfrak{m}(A)^{\frac{1}{N'}} + \tau_{K,N'}^{(t)}(\Theta(A,B)) \cdot \mathfrak{m}(B)^{\frac{1}{N'}}, \tag{3.3.8}$$

where $\Theta(A,B)$ is defined in (3.3.7).

Proposition 3.3.10. *Given* $K \in \mathbb{R}$ *and* $N > 1$*, if the metric measure space* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *supports the strong Brunn–Minkowski inequality* $\mathsf{SBM}(K,N)$*, it also supports the Brunn–Minkowski inequality* $\mathsf{BM}(K,N)$*.*

Proof. Fix $N' \geq N$ and $t \in [0,1]$. Let $A, B \subset \mathrm{spt}(\mathfrak{m})$ be two Borel sets with $0 < \mathfrak{m}(A), \mathfrak{m}(B) < \infty$. Then, for every $\eta \in \mathrm{OptGeo}(\mathfrak{m}_A, \mathfrak{m}_B)$ it holds that $\mathrm{spt}((e_t)_\#\eta) \subset M_t(A,B)$ up to a $\mathfrak{m}$-null set. Therefore, $\mathsf{SBM}(K,N)$ implies (3.3.6) for sets with finite and positive measure. Moreover, since the proof of Lemma 3.3.7 only relies on $\mathsf{BM}(K,N)$ for sets of finite and positive measure, we can deduce that $(\mathrm{spt}(\mathfrak{m}), \mathsf{d})$ is proper, geodesic and $\mathfrak{m}$ is Radon. Let us prove now $\mathsf{BM}(K,N)$ for any compact sets $A, B \subset \mathrm{spt}(\mathfrak{m})$, with possibly zero measure. If $\mathfrak{m}(A) = \mathfrak{m}(B) = 0$, there's nothing to prove, hence we may assume $\mathfrak{m}(A) > 0$ and $\mathfrak{m}(B) = 0$. Moreover, for the time being, assume also that $B = \{x\}$, where $x \in \text{spt}(\mathfrak{m})$. Applying the Brunn–Minkowski inequality with the sets A and $\mathsf{B}_r(x)$ (for $r > 0$) we obtain

$$\mathfrak{m}\big(M_t(A, \mathsf{B}_r(x))\big)^{\frac{1}{N'}} \geq \tau_{K,N'}^{(1-t)}(\Theta_r) \cdot \mathfrak{m}(A)^{\frac{1}{N'}} + \tau_{K,N'}^{(t)}(\Theta_r) \cdot \mathfrak{m}(\mathsf{B}_r(x))^{\frac{1}{N'}}, \tag{3.3.9}$$

where $\Theta_r := \Theta(A, \mathsf{B}_r(x))$. On the other hand, it can be proven that

$$\bigcap_{r>0} M_t(A, \mathsf{B}_r(x)) = M_t(A, x).$$

The $\supset$ inclusion is obvious, while to prove $\subset$ we take $w \in \bigcap_{r>0} M_t(A, \mathsf{B}_r(x))$ and we observe that, given a sequence $\{r_n\}_{n\in\mathbb{N}}$ converging to 0, there exist $a_n \in A$ and $x_n \in \mathsf{B}_{r_n}(x)$ such that w is a t-midpoint of a_n and x_n. Since A is compact, up to subsequences $a_n \to a \in A$ and then, by Ascoli–Arzelà theorem, w is a t-midpoint of a and x, thus $w \in M_t(A, x)$. At this point, noting that the sets $M_t(A, \mathsf{B}_r(x))$ are decreasing as $r \to 0$, we can pass to the limit (3.3.9) and obtain

$$\mathfrak{m}\big(M_t(A, x)\big)^{\frac{1}{N'}} \geq \tau_{K,N'}^{(1-t)}(\Theta(A, x)) \cdot \mathfrak{m}(A)^{\frac{1}{N'}}. \tag{3.3.10}$$

Now, let $B \subset \text{spt}(\mathfrak{m})$ any compact set with $\mathfrak{m}(B) = 0$. Then for any $x \in B$, we have

$$M_t(A, x) \subset M_t(A, B) \qquad \text{and} \qquad \tau_{K,N'}^{(1-t)}(\Theta(A, x)) \geq \tau_{K,N'}^{(1-t)}(\Theta(A, B)), \tag{3.3.11}$$

where the inequality follows from (3.3.1). Thus, we may apply (3.3.10) and (3.3.11), obtaining

$$\begin{aligned} \mathfrak{m}\big(M_t(A, B)\big)^{\frac{1}{N'}} &\geq \mathfrak{m}\big(M_t(A, x)\big)^{\frac{1}{N'}} \\ &\geq \tau_{K,N'}^{(1-t)}(\Theta(A, x)) \cdot \mathfrak{m}(A)^{\frac{1}{N'}} \geq \tau_{K,N'}^{(1-t)}(\Theta(A, B)) \cdot \mathfrak{m}(A)^{\frac{1}{N'}}. \end{aligned} \tag{3.3.12}$$

In order to prove (3.3.6) for any Borel sets $A, B \subset \text{spt}(\mathfrak{m})$, with possibly zero or infinite measure, we use the inner regularity of $\mathfrak{m}$. In particular, there exist two sequences of compact sets $\{A_n\}_{n\in\mathbb{N}}$ and $\{B_n\}_{n\in\mathbb{N}}$ such that

$$A_n \subset A,\ \mathfrak{m}(A_n) \to \mathfrak{m}(A) \qquad \text{and} \qquad B_n \subset B,\ \mathfrak{m}(B_n) \to \mathfrak{m}(B).$$

For the sets A_n and B_n, inequality (3.3.12) holds, therefore, using the monotonicity of the t-midpoints set and of the distortion coefficients as in (3.3.11), we obtain that

$$\bar{\mathfrak{m}}\big(M_t(A, B))\big)^{\frac{1}{N'}} \geq \tau_{K,N'}^{(1-t)}(\Theta(A, B)) \cdot \mathfrak{m}(A_n)^{\frac{1}{N'}} + \tau_{K,N'}^{(t)}(\Theta(A, B)) \cdot \mathfrak{m}(B_n)^{\frac{1}{N'}}. \tag{3.3.13}$$

Passing to the limit the right-hand side of (3.3.13), we finally conclude that $\mathsf{BM}(K, N)$ holds also for A and B. □

Remark 3.3.11. From Corollary 3.3.8 and the previous proposition, an essentially non-branching metric measure space $(X, \mathsf{d}, \mathfrak{m})$ supporting $\mathsf{SBM}(K, N)$ has the good transport behavior and the strong interpolation property, cf. Definition 3.2.4. Thus, given $A, B \subset \text{spt}(\mathfrak{m})$ Borel sets with finite and positive measure, there exists a unique $\eta \in \text{OptGeo}(\mathfrak{m}_A, \mathfrak{m}_B)$, depending only on the sets A and B. Hence, we can introduce the following notation without ambiguity:

$$D_t(A, B) := \text{spt}\big((e_t)_\#\eta\big), \qquad \forall\, t \in [0, 1].$$

In particular, the inequality (3.3.8) now reads as follows:

$$\mathfrak{m}\big(D_t(A, B)\big)^{\frac{1}{N'}} \geq \tau_{K,N'}^{(1-t)}(\Theta(A, B)) \cdot \mathfrak{m}(A)^{\frac{1}{N'}} + \tau_{K,N'}^{(t)}(\Theta(A, B)) \cdot \mathfrak{m}(B)^{\frac{1}{N'}}.$$

It was already noticed in the first works about CD spaces (see in particular [Stu06b]) that the $\mathsf{CD}(K,N)$ condition implies the Brunn–Minkowski inequality $\mathsf{BM}(K,N)$. Following the exact same proof is actually possible to deduce that the $\mathsf{CD}(K,N)$ condition implies the strong Brunn–Minkowski inequality $\mathsf{SBM}(K,N)$. We point out that in Euclidean spaces, the relation between the convexity of entropies and the Brunn–Minkowski inequality was already investigated in the works of McCann [McC94],[McC97]. In the following we provide a quick proof of this fact, in order to be self-contained and avoid confusion.

Proposition 3.3.12. *Let* $(\mathsf{X},\mathsf{d},\mathfrak{m})$ *be a* $\mathsf{CD}(K,N)$ *space, for some* $K\in\mathbb{R}$ *and* $N>1$*. Then, it supports the strong Brunn–Minkowski inequality* $\mathsf{SBM}(K,N)$*.*

Proof. Given any pair of Borel sets $A,B\subset \operatorname{spt}(\mathfrak{m})$ such that $0<\mathfrak{m}(A),\mathfrak{m}(B)<\infty$, take the optimal geodesic plan $\eta\in\operatorname{OptGeo}(\mathfrak{m}_A,\mathfrak{m}_B)$ satisfying (3.3.2). In particular, letting $\pi=(e_0,e_1)_\#\eta$, for every $N'\geq N$ it holds that

$$\begin{aligned}\mathcal{E}_{N'}((e_t)_\#\eta) &\leq -\int_{\mathsf{X}\times\mathsf{X}}\Big[\tau_{K,N'}^{(1-t)}(\mathsf{d}(x,y))\mathfrak{m}(A)^{\frac{1}{N'}}+\tau_{K,N'}^{(t)}(\mathsf{d}(x,y))\mathfrak{m}(B)^{\frac{1}{N'}}\Big]\,\mathrm{d}\pi(x,y)\\ &\leq -\Big[\tau_{K,N'}^{(1-t)}(\Theta(A,B))\cdot\mathfrak{m}(A)^{\frac{1}{N'}}+\tau_{K,N'}^{(t)}(\Theta(A,B))\cdot\mathfrak{m}(B)^{\frac{1}{N'}}\Big],\end{aligned}\tag{3.3.14}$$

using (3.3.1). On the other hand, Jensen's inequality ensures that

$$\mathcal{E}_{N'}((e_t)_\#\eta)=-\int_{\operatorname{spt}((e_t)_\#\eta)}\rho_t(x)^{1-\frac{1}{N'}}\,\mathrm{d}\mathfrak{m}(x)\geq -\mathfrak{m}(\operatorname{spt}((e_t)_\#\eta))^{\frac{1}{N'}},\tag{3.3.15}$$

where ρ_t denotes the density of $(e_t)_\#\eta$ with respect to $\mathfrak{m}$, for every $t\in[0,1]$. Putting together (3.3.14) and (3.3.15), we obtain (3.3.8), concluding the proof. ☐

3.4 Proof of the main theorem

In this section, we prove our main result, Theorem 3.1.1. For the convenience of the reader, we recall its statement.

Theorem 3.4.1. *Let* $(\mathsf{X},\mathsf{d},\mathfrak{m})$ *be an essentially non-branching metric measure space supporting* $\mathsf{SBM}(K,N)$ *for some* $K\in\mathbb{R}$ *and* $N>1$*. Then,* $(\mathsf{X},\mathsf{d},\mathfrak{m})$ *is a* $\mathsf{CD}(K,N)$ *space. In particular,* $(\mathsf{X},\mathsf{d},\mathfrak{m})$ *supports* $\mathsf{SBM}(K,N)$ *if and only if it satisfies* $\mathsf{CD}(K,N)$*.*

A key idea in our argument is to prove the $\mathsf{CD}(K,N)$ condition for a suitable subclass of bounded probability measures, called step measures. By an approximation strategy, we then extend the result to *all* bounded measures and finally apply Proposition 3.3.5, to conclude.

Definition 3.4.2. We say that a measure $\mu\in\mathscr{P}_2(\mathsf{X})$ is a *step measure* if it can be written as finite sum of measures with constant density with respect to $\mathfrak{m}$, that is

$$\mu=\sum_{i=1}^{N}\lambda_i\mathfrak{m}_{A_i},$$

where, for every $i=1,\dots,N$, $\lambda_i\in\mathbb{R}$ and A_i is a Borel set with $0<\mathfrak{m}(A_i)<\infty$. Moreover, we assume the sets $\{A_i\}_{i=1,\dots,N}$ to be mutually disjoint.

Note that the entropy $\mathcal{E}_N$ of a measure $\nu\in\mathscr{P}^{ac}(\mathsf{X},\mathfrak{m})$ equals to $\mathfrak{m}(\operatorname{spt}(\nu))^{1/N}$ if and only if ν has constant density. Thus, letting $A,B\subset\operatorname{spt}(\mathfrak{m})$ with finite and positive measure, the $\mathsf{SBM}(K,N)$ inequality would translate directly to an information on the entropy of the t-midpoint

μ_t between $\mathfrak{m}_A$ and $\mathfrak{m}_B$, only if μ_t had constant density. However, we can not expect this to be true in general. The previous discussion suggests that, in order to promote $\mathsf{SBM}(K,N)$ to an inequality on the entropy, an argument based on a subsequently refined partition of the support of the marginals is needed, built in accordance with the optimal transport coupling. Indeed, using the partition argument, the t-midpoint between $\mathfrak{m}_A$ and $\mathfrak{m}_B$ can be approximated in entropy with a step measure, which by definition has locally constant density. In addition, the $\mathsf{SBM}(K,N)$ inequality, applied to each element of the partition, controls the entropy of the step measure approximant. The partition argument works also when replacing the measures $\mathfrak{m}_A$ and $\mathfrak{m}_B$ with general step measures as marginals. The advantage of proving the $\mathsf{CD}(K,N)$ inequality for the class of step measures is that the latter is sufficiently large to deduce $\mathsf{CD}(K,N)$ for all bounded measures, by approximation.

Theorem 3.4.3. *Let $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be an essentially non-branching supporting $\mathsf{SBM}(K,N)$ for some $K\in\mathbb{R}$ and $N>1$ and let $\mu_0,\mu_1\in\mathscr{P}^{ac}(\mathsf{X},\mathfrak{m})$ be two step measures with bounded support. Then, there exists $\eta\in OptGeo(\mu_0,\mu_1)$ such that* (3.3.2) *holds.*

Proof. Combining Corollary 3.3.8 and Proposition 3.3.10, we deduce that $(\mathsf{X},\mathsf{d},\mathfrak{m})$ has the good transport behavior and the strong interpolation property. Therefore, letting μ_0 and μ_1 be step measures, i.e.

$$\mu_0=\sum_{i=1}^{N_0}\lambda_i^0\mathfrak{m}_{A_i}\quad\text{and}\quad\mu_1=\sum_{i=1}^{N_1}\lambda_i^1\mathfrak{m}_{B_i},$$

there exists $\eta\in\mathrm{OptGeo}(\mu_0,\mu_1)$, the unique optimal geodesic plan connecting μ_0 and μ_1. Then, $\pi:=(e_0,e_1)_\#\eta\in\mathrm{Opt}(\mu_0,\mu_1)$ is the unique optimal transport plan between μ_0 and μ_1 and it is induced by a map T, that is $\pi=(\mathrm{id},T)_\#\mu_0$. Suppose for now that T is continuous, we will get rid of this assumption later in the proof. In this case, since $\mathrm{spt}(\mu_0)$ is compact (see Lemma 3.3.7), for every $\varepsilon>0$, it is possible to find a finite partition in Borel sets $\{P_j^\varepsilon\}_{j=1,\dots,L_\varepsilon}$ of it, that is $\cup_{j=1}^{L_\varepsilon}P_j^\varepsilon=\mathrm{spt}(\mu_0)$ up to an $\mathfrak{m}$-null set and $P_i^\varepsilon\cap P_j^\varepsilon=\emptyset$ if $i\neq j$, with the following properties:

(i) $\mathfrak{m}(P_j^\varepsilon)>0$, for every $j=1,\dots,L_\varepsilon$,

(ii) $\mathrm{diam}\left(P_j^\varepsilon\right)<\varepsilon$ and $\mathrm{diam}\left(T(P_j^\varepsilon)\right)<\varepsilon$, for every $j=1,\dots,L_\varepsilon$,

(iii) for every $j=1,\dots,L_\varepsilon$, there exists $i(j)$ such that $P_j^\varepsilon\subset A_{i(j)}$,

(iv) for every $j=1,\dots,L_\varepsilon$, there exists $\iota(j)$ such that $T(P_j^\varepsilon)\subset B_{\iota(j)}$.

For example, consider the sets $\{P_{i,j}\}_{i,j}$ defined as $P_{i,j}:=A_i\cap T^{-1}(B_j)$, which already satisfy the properties (iii) and (iv). The sought partition can then be found as a suitable refinement of the partition $\{P_{i,j}\}_{i,j}$, ensuring the property (ii) using the equicontinuity of the map T on the compact set $\mathrm{spt}(\mu_0)$, and condition (i) by neglecting the sets with zero $\mathfrak{m}$-measure.

We observe that the good transport behavior implies that the unique optimal map T^{-1} from μ_1 to μ_0 is such that $T^{-1}\circ T=\mathrm{id}$ μ_0-almost everywhere. In particular,

$$\mu_0(P_j^\varepsilon)=\mu_0(T^{-1}\circ T(P_j^\varepsilon))=T_\#\mu_0(T(P_j^\varepsilon))=\mu_1(T(P_j^\varepsilon)).\tag{3.4.1}$$

Now we define the measures $\mu_0^{\varepsilon,j}$, $\mu_1^{\varepsilon,j}\in\mathscr{M}_+(\mathsf{X})$ as

$$\mu_0^{\varepsilon,j}:=\mu_0(P_j^\varepsilon)\cdot\mathfrak{m}_{P_j^\varepsilon}\quad\text{and}\quad\mu_1^{\varepsilon,j}:=\mu_1(T(P_j^\varepsilon))\cdot\mathfrak{m}_{T(P_j^\varepsilon)}=\mu_0(P_j^\varepsilon)\cdot\mathfrak{m}_{T(P_j^\varepsilon)}\,,$$

where the last equality follows from (3.4.1). Property (iii) of the partition ensures that $\mu_0^{\varepsilon,j}$ and $\mu_0|_{P_j^\varepsilon}$ are both measures of constant density with respect to $\mathfrak{m}$ and with equal mass. Therefore

$\mu_0^{\varepsilon,j} = \mu_0|_{P_j^\varepsilon}$, and, as a consequence

$$\mu_0 = \sum_{j=1}^{L_\varepsilon} \mu_0|_{P_j^\varepsilon} = \sum_{j=1}^{L_\varepsilon} \mu_0^{\varepsilon,j}.$$

Similarly, by (3.4.1) and property (iv) we conclude that $T_\# \mu_0^{\varepsilon,j} = \mu_1|_{T(P_j^\varepsilon)} = \mu_1^{\varepsilon,j}$, hence

$$\mu_1 = \sum_{j=1}^{L_\varepsilon} \mu_1^{\varepsilon,j}.$$

Defining $\eta_j^\varepsilon := \eta|_{e_0^{-1}(P_j^\varepsilon)} \in \mathscr{M}_+(\mathrm{Geo}(\mathsf{X}))$, it holds that $\eta = \sum_{j=1}^{L_\varepsilon} \eta_j^\varepsilon$. Note that $\bar{\eta}_j^\varepsilon := \frac{\eta_j^\varepsilon}{\mu_0(P_j^\varepsilon)} \in \mathscr{P}(\mathrm{Geo}(\mathsf{X}))$. Moreover, since it holds that $(e_0, e_1)_\# \eta = \pi = (\mathrm{id}, T)_\# \mu_0$, by (3.4.1) we deduce that, for every $j = 1, \dots, L_\varepsilon$,

$$\{\bar{\eta}_j^\varepsilon\} = \mathrm{OptGeo}\left(\frac{(e_0)_\# \eta_j^\varepsilon}{\mu_0(P_j^\varepsilon)}, \frac{(e_1)_\# \eta_j^\varepsilon}{\mu_0(P_j^\varepsilon)} \right) = \mathrm{OptGeo}(\mathfrak{m}_{P_j^\varepsilon}, \mathfrak{m}_{T(P_j^\varepsilon)}). \tag{3.4.2}$$

Thus, for every j, the curve $t \mapsto \bar{\mu}_t^{\varepsilon,j} := (e_t)_\# \bar{\eta}_j^\varepsilon$ is the unique Wasserstein geodesic connecting $\mathfrak{m}_{P_j^\varepsilon}$ and $\mathfrak{m}_{T(P_j^\varepsilon)}$, hence

$$D_t(P_j^\varepsilon, T(P_j^\varepsilon)) = \mathrm{spt}(\bar{\mu}_t^{\varepsilon,j}), \quad \forall t \in [0,1], \tag{3.4.3}$$

where the set $D_t(\cdot,\cdot)$ is defined in Remark 3.3.11. As a consequence of the strong interpolation property, for every j and t the measure $\bar{\mu}_t^{\varepsilon,j}$ is absolutely continuous with respect to $\mathfrak{m}$, with density $\bar{\rho}_t^{\varepsilon,j}$. Moreover, by definition

$$\bar{\rho}_t^{\varepsilon,j} > 0 \quad \bar{\mu}_t^{\varepsilon,j}\text{-almost everywhere on } D_t(P_j^\varepsilon, T(P_j^\varepsilon)). \tag{3.4.4}$$

In addition, we can apply the strong Brunn–Minkowski inequality $\mathsf{SBM}(K,N)$ and deduce that for every $j = 1, \dots, L_\varepsilon$, $N' \geq N$, and $t \in [0,1]$ it holds that

$$\mathfrak{m}(D_t(P_j^\varepsilon, T(P_j^\varepsilon)))^{\frac{1}{N'}} \geq \tau_{K,N'}^{(1-t)}(\Theta_j) \cdot \mathfrak{m}(P_j^\varepsilon)^{\frac{1}{N'}} + \tau_{K,N'}^{(t)}(\Theta_j) \cdot \mathfrak{m}(T(P_j^\varepsilon))^{\frac{1}{N'}}, \tag{3.4.5}$$

where we use the shorthand notation $\Theta_j := \Theta(P_j^\varepsilon, T(P_j^\varepsilon))$.

The next goal is to find a suitable approximately t-intermediate point $\tilde{\mu}_t^\varepsilon$ between μ_0 and μ_1. We claim that this can be achieved by considering a family of measures $\tilde{\mu}_t^{\varepsilon,j} \in \mathscr{M}_+(\mathsf{X})$ supported on the sets $D_t(P_j^\varepsilon, T(P_j^\varepsilon))$ and having constant density, then gluing them together. This would allow us to use (3.4.5) on each set P_j^ε and provide a lower bound for the entropy of $\tilde{\mu}_t^\varepsilon$. Precisely, we define, for every $t \in [0,1]$,

$$\tilde{\mu}_t^{\varepsilon,j} := \mu_0(P_j^\varepsilon) \cdot \mathfrak{m}_{D_t\left(P_j^\varepsilon, T(P_j^\varepsilon)\right)} \text{ for every } j = 1, \dots, L_\varepsilon \quad \text{and} \quad \tilde{\mu}_t^\varepsilon := \sum_{j=1}^{L_\varepsilon} \tilde{\mu}_t^{\varepsilon,j}.$$

Note that, for every $t \in [0,1]$, $D_t(P_j^\varepsilon, T(P_j^\varepsilon))$ has positive measure by (3.4.5). Moreover, since $D_t(P_j^\varepsilon, T(P_j^\varepsilon))$ is bounded (being contained in the t-midpoints of two bounded sets), it also has finite measure, therefore $\tilde{\mu}_t^{\varepsilon,j}$ is well defined.

Since $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ is essentially non-branching, one can prove that $D_t(P_j^\varepsilon, T(P_j^\varepsilon)) \cap D_t(P_i^\varepsilon, T(P_i^\varepsilon))$ is a $\mathfrak{m}$-null measure set, whenever $i \neq j$, see for example [MR21, Proposition 2.7]. Let $\tilde{\rho}_t^\varepsilon$ be the density of $\tilde{\mu}_t^\varepsilon$ with respect to $\mathfrak{m}$, i.e. $\tilde{\mu}_t^\varepsilon = \tilde{\rho}_t^\varepsilon \mathfrak{m} \in \mathscr{P}(\mathsf{X})$. Then, for any $t \in [0,1]$ and $N' \geq N$, the entropy $\mathcal{E}_{N'}$ of $\tilde{\mu}_t^\varepsilon$ is given by

$$\begin{aligned}\mathcal{E}_{N'}(\tilde{\mu}_t^\varepsilon) = -\int_{\mathsf{X}} (\tilde{\rho}_t^\varepsilon)^{1-\frac{1}{N'}} \,\mathrm{d}\mathfrak{m} &= -\sum_{j=1}^{L_\varepsilon} \int_{D_t\left(P_j^\varepsilon, T(P_j^\varepsilon)\right)} (\tilde{\rho}_t^\varepsilon)^{1-\frac{1}{N'}} \,\mathrm{d}\mathfrak{m} \\ &= -\sum_{j=1}^{L_\varepsilon} \mu_0(P_j^\varepsilon)^{1-\frac{1}{N'}} \mathfrak{m}\big(D_t(P_j^\varepsilon, T(P_j^\varepsilon))\big)^{\frac{1}{N'}}\end{aligned} \tag{3.4.6}$$

The combination of (3.4.5) and (3.4.6) gives the following estimate, where $\pi_j := \pi|_{P_j^\varepsilon \times T(P_j^\varepsilon)}$:

$$\begin{aligned}\mathcal{E}_{N'}(\tilde{\mu}_t^\varepsilon) &\leq -\sum_{j=1}^{L_\varepsilon} \mu_0(P_j^\varepsilon)^{1-\frac{1}{N'}} \Big[\tau_{K,N'}^{(1-t)}(\Theta_j) \cdot \mathfrak{m}(P_j^\varepsilon)^{\frac{1}{N'}} + \tau_{K,N'}^{(t)}(\Theta_j) \cdot \mathfrak{m}(T(P_j^\varepsilon))^{\frac{1}{N'}}\Big] \\ &= -\sum_{j=1}^{L_\varepsilon} \int \Big[\tau_{K,N'}^{(1-t)}(\Theta_j)\rho_0^{-\frac{1}{N'}}(x) + \tau_{K,N'}^{(t)}(\Theta_j)\rho_1^{-\frac{1}{N'}}(y)\Big] \,\mathrm{d}\pi_j(x,y) \\ &\leq -\sum_{j=1}^{L_\varepsilon} \int \Big[\tau_{K,N'}^{(1-t)}(\mathsf{d}(x,y) \mp \varepsilon)\rho_0^{-\frac{1}{N'}}(x) + \tau_{K,N'}^{(t)}(\mathsf{d}(x,y) \mp \varepsilon)\rho_1^{-\frac{1}{N'}}(y)\Big] \,\mathrm{d}\pi_j(x,y) \\ &= -\int \Big[\tau_{K,N'}^{(1-t)}(\mathsf{d}(x,y) \mp \varepsilon)\rho_0^{-\frac{1}{N'}}(x) + \tau_{K,N'}^{(t)}(\mathsf{d}(x,y) \mp \varepsilon)\rho_1^{-\frac{1}{N'}}(y)\Big] \,\mathrm{d}\pi(x,y).\end{aligned} \tag{3.4.7}$$

Here the symbol $\mp$ means that the estimate holds with the minus if $K \geq 0$ and with the plus if $K < 0$. Note that the first equality follows by properties (iii) and (iv) of the partition, since π_j is concentrated on $P_j^\varepsilon \times T(P_j^\varepsilon)$, whereas the second inequality follows from the monotonicity of the distortion coefficients (3.3.1) and the diameter bounds (ii).

Recall the definition of the measure $\bar{\mu}_t^{\varepsilon,j}$, its density $\bar{\rho}_t^{\varepsilon,j}$, and their properties, in particular (3.4.3) and (3.4.4). For every fixed $s \in [0,1]$, we can then define the measure

$$\tilde{\eta}_j^\varepsilon := \frac{1}{\mathfrak{m}(D_s(P_j^\varepsilon, T(P_j^\varepsilon)))} \cdot \frac{\bar{\eta}_j^\varepsilon(\,\mathrm{d}\gamma)}{\bar{\rho}_s^{\varepsilon,j}(e_s(\gamma))} \in \mathscr{M}_+(\mathrm{Geo}(\mathsf{X})).$$

By construction $(e_s)_\# \tilde{\eta}_j^\varepsilon = \mathfrak{m}_{D_s\left(P_j^\varepsilon, T(P_j^\varepsilon)\right)}$, which in particular shows that $\tilde{\eta}_j^\varepsilon$ is a probability measure. Moreover, $\tilde{\eta}_j^\varepsilon$ is concentrated on $\mathrm{Geo}(\mathsf{X})$ and $(e_0,e_1)_\#\tilde{\eta}_j^\varepsilon$ is concentrated on the same d^2-cyclically monotone set as $(e_0,e_1)_\#\bar{\eta}_j^\varepsilon$. For these reasons, we have that

$$\tilde{\eta}_j^\varepsilon \in \mathrm{OptGeo}\big((e_0)_\#\tilde{\eta}_j, (e_1)_\#\tilde{\eta}_j\big)\,, \quad \forall j = 1, \dots, L_\varepsilon.$$

On the other hand, the measures defined by $\nu_0^{\varepsilon,j} := (e_0)_\#\tilde{\eta}_j^\varepsilon$ and $\nu_1^{\varepsilon,j} := (e_1)_\#\tilde{\eta}_j^\varepsilon$ are concentrated on P_j^ε and $T(P_j^\varepsilon)$, respectively. Hence, recalling that,

$$W_2(\mu,\nu) \leq \mathrm{diam}\big(\mathrm{spt}(\mu) \cup \mathrm{spt}(\nu)\big), \qquad \forall\, \mu,\nu \in \mathscr{P}_2(\mathsf{X}), \tag{3.4.8}$$

property (ii) of the partition and the triangle inequality allow us to conclude that

$$\begin{aligned}W_2\big(\mathfrak{m}_{P_j^\varepsilon}, \mathfrak{m}_{D_s(P_j^\varepsilon, T(P_j^\varepsilon))}\big) &\leq W_2\big(\mathfrak{m}_{P_j^\varepsilon}, \nu_0^{\varepsilon,j}\big) + W_2\big(\nu_0^{\varepsilon,j}, \mathfrak{m}_{D_s(P_j^\varepsilon, T(P_j^\varepsilon))}\big) \\ &\leq \varepsilon + s \cdot W_2\big(\nu_0^{\varepsilon,j}, \nu_1^{\varepsilon,j}\big) \\ &\leq 3\varepsilon + s \cdot W_2\big(\mathfrak{m}_{P_j^\varepsilon}, \mathfrak{m}_{T(P_j^\varepsilon)}\big),\end{aligned} \tag{3.4.9}$$

where we repeatedly used (3.4.8), and analogously

$$W_2\big(\mathfrak{m}_{D_s(P_j^\varepsilon, T(P_j^\varepsilon))}, \mathfrak{m}_{T(P_j^\varepsilon)}\big) \leq 3\varepsilon + (1-s) \cdot W_2\big(\mathfrak{m}_{P_j^\varepsilon}, \mathfrak{m}_{T(P_j^\varepsilon)}\big),$$

for every $j = 1, \dots, L_\varepsilon$. On the other hand, recalling that $\mu_0 = \sum_{j=1}^{L_\varepsilon} \mu_0(P_j^\varepsilon)\mathfrak{m}_{P_j^\varepsilon}$ and $\tilde{\mu}_s^\varepsilon = \sum_{j=1}^{L_\varepsilon} \mu_0(P_j^\varepsilon)\mathfrak{m}_{D_s(P_j^\varepsilon, T(P_j^\varepsilon))}$, the convexity of W_2^2 gives us

$$W_2^2(\mu_0, \tilde{\mu}_s^\varepsilon) \leq \sum_{j=1}^{L_\varepsilon} \mu_0(P_j^\varepsilon) \cdot W_2^2\big(\mathfrak{m}_{P_j^\varepsilon}, \mathfrak{m}_{D_s(P_j^\varepsilon, T(P_j^\varepsilon))}\big).$$

In addition, thanks to the optimality of the map T and the properties of the partition, cf. (3.4.2), we have that

$$\sum_{j=1}^{L_\varepsilon} \mu_0(P_j^\varepsilon) W_2^2(\mathfrak{m}_{P_j^\varepsilon}, \mathfrak{m}_{T(P_j^\varepsilon)}) = W_2^2(\mu_0, \mu_1).$$

Hence, by summing (3.4.9) on j, we obtain the estimate

$$\begin{aligned} W_2^2(\mu_0, \tilde{\mu}_s^\varepsilon) &\leq \sum_{j=1}^{L_\varepsilon} \mu_0(P_j^\varepsilon)\Big(3\varepsilon + s \cdot W_2(\mathfrak{m}_{P_j^\varepsilon}, \mathfrak{m}_{T(P_j^\varepsilon)})\Big)^2 \\ &= 9\varepsilon^2 + 6\varepsilon s \sum_{j=1}^{L_\varepsilon} \mu_0(P_j^\varepsilon) W_2(\mathfrak{m}_{P_j^\varepsilon}, \mathfrak{m}_{T(P_j^\varepsilon)}) + s^2 \sum_{j=1}^{L_\varepsilon} \mu_0(P_j^\varepsilon) W_2^2(\mathfrak{m}_{P_j^\varepsilon}, \mathfrak{m}_{T(P_j^\varepsilon)}) \\ &\leq 9\varepsilon^2 + 6\varepsilon s D + s^2 \cdot W_2^2(\mu_0, \mu_1), \end{aligned} \tag{3.4.10}$$

where,in the third line, we introduced the quantity $D := \operatorname{diam}(\operatorname{spt}(\mu_0) \cup \operatorname{spt}(\mu_1))$ and applied (3.4.8). Analogously, we have

$$W_2(\tilde{\mu}_s^\varepsilon, \mu_1) \leq 9\varepsilon^2 + 6\varepsilon(1-s)D + (1-s)^2 \cdot W_2^2(\mu_0, \mu_1). \tag{3.4.11}$$

Observe that we have proven (3.4.10) and (3.4.11) for every $s \in [0,1]$.

We now pass to the limit as $\varepsilon \to 0$. Notice that, since μ_0 and μ_1 have bounded support and the space $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ is proper by Lemma 3.3.7, for every $t \in [0,1]$, all the measures in the family $\{\tilde{\mu}_t^\varepsilon\}_{\varepsilon>0}$ are concentrated on a common compact set. In particular, for every fixed $t \in [0,1]$, the family $\{\tilde{\mu}_t^\varepsilon\}_{\varepsilon>0}$ is W_2-precompact (see Section 3.2). Thus we can find a sequence $\{\varepsilon_m\}_{m\in\mathbb{N}}$ converging to 0 such that

$$\tilde{\mu}_t^{\varepsilon_m} \xrightarrow{W_2} \mu_t \in \mathscr{P}_2(\mathsf{X}) \quad \text{as } m \to \infty.$$

Now we can pass (3.4.10) and (3.4.11) to the limit as $m \to \infty$ and obtain that

$$W_2(\mu_0, \mu_t) \leq t \cdot W_2(\mu_0, \mu_1) \quad \text{and} \quad W_2(\mu_t, \mu_1) \leq (1-t) \cdot W_2(\mu_0, \mu_1).$$

As a consequence, we deduce that μ_t is the unique t-midpoint between μ_0 and μ_1, and

$$\tilde{\mu}_t^\varepsilon \xrightarrow{W_2} \mu_t \in \mathscr{P}_2(\mathsf{X}) \quad \text{as } \varepsilon \to 0,$$

without extracting a subsequence. Repeating the argument for every $t \in [0,1]$, we deduce that the curve $t \mapsto \mu_t$ is the unique Wasserstein geodesic connecting μ_0 and μ_1. Then, we can pass to the limit (3.4.7), using the monotone convergence theorem for the right-hand side and the lower semicontinuity of $\mathcal{E}_{N'}$ for the left-hand side and obtain, for every $t \in [0,1]$ and $N' \geq N$,

$$\mathcal{E}_{N'}(\mu_t) \leq -\int_{\mathsf{X}\times\mathsf{X}} \Big[\tau_{K,N'}^{(1-t)}(\mathsf{d}(x,y))\rho_0(x)^{-\frac{1}{N'}} + \tau_{K,N'}^{(t)}(\mathsf{d}(x,y))\rho_1(y)^{-\frac{1}{N'}}\Big] \,\mathrm{d}\pi(x,y),$$

which is the desired inequality. Note that we were able to exploit the lower semicontinuity of $\mathcal{E}_{N'}$, because for every $t \in [0,1]$ the measures of the family $\{\tilde{\mu}_t^\varepsilon\}_{\varepsilon>0} \cup \{\mu_t\}$ are concentrated on a common bounded set.

So far we have proven that, if the optimal map between two step measures with bounded support is continuous, the unique geodesic connecting them satisfies the entropy convexity inequality (3.3.2). We now have to get rid of the assumption on the continuity of the optimal transport map T. If T is not continuous, recalling that $\mathfrak{m}$ is Radon thanks to Lemma 3.3.7, we may apply Lusin theorem: for every $\epsilon > 0$, there exists a compact set $A_\epsilon \subset \operatorname{spt}(\mu_0)$, such that T is continuous on A_ϵ and $\mu_0(\operatorname{spt}(\mu_0) \setminus A_\epsilon) < \epsilon$. Then, define the measures

$$\eta^\epsilon := \frac{\eta|_{e_0^{-1}(A_\epsilon)}}{\mu_0(A_\epsilon)} \in \mathscr{P}(\mathrm{Geo}(\mathsf{X})) \quad \text{and} \quad \mu_0^\epsilon := (e_0)_\# \eta,\ \mu_1^\epsilon := (e_1)_\# \eta \in \mathscr{P}^{ac}(\mathsf{X}, \mathfrak{m}).$$

Note also that η^ϵ is the unique optimal geodesic plan in $\mathrm{OptGeo}(\mu_0^\epsilon, \mu_1^\epsilon)$. Moreover, exploiting the good transport behaviour as done in the first part of the proof, we deduce that the set $B_\epsilon := T(A_\epsilon) \subset \operatorname{spt}(\mu_1)$ is such that $\mu_1^\epsilon = \mu_1|_{B_\epsilon}/\mu_1(B_\epsilon)$, while by definition is clear that $\mu_0 = \mu_0|_{A_\epsilon}/\mu_0(A_\epsilon)$. In particular, μ_0^ϵ and μ_1^ϵ are step measures with bounded support, $T|_{A_\epsilon}$ is the optimal map between them and it is continuous. Therefore (3.3.2) holds for μ_0^ϵ, μ_1^ϵ and η^ϵ. Repeating the same argument used in the proof of Proposition 3.3.5 replacing the sets A_n and B_n with A_ϵ and B_ϵ, respectively, we can pass to the limit as $\epsilon \to 0$ and deduce that (3.3.2) holds for μ_0, μ_1 and η. This concludes the proof. □

At this point, we proceed by approximation and use Theorem 3.4.3 to prove $\mathsf{CD}(K, N)$ for every pair of bounded marginals. The next two lemmas serve for this purpose, providing a suitable sequence of step measures converging to a given bounded measure and an upper semicontinuity result for the functional $T_{K,N}^{(t)}(\cdot|\mathfrak{m})$.

Lemma 3.4.4. *Let $\mu = \rho\mathfrak{m} \in \mathscr{P}^{ac}(\mathsf{X}, \mathfrak{m})$ be bounded, then there exists a sequence of step measures $\{\mu_n = \rho_n \mathfrak{m}\}_{n\in\mathbb{N}}$ W_2-convergent to μ, such that $\{\mu_n\}_{n\in\mathbb{N}} \cup \{\mu\}$ is uniformly bounded and $\rho_n^{-1/N'} \to \rho^{-1/N'}$ in $L^1(\mathfrak{m})$ for every $N' > 1$.*

Proof. Let $K \subset \mathsf{X}$ be the (compact) support of μ and $c > 0$ as in Definition 3.3.3. Let $\{\tilde\rho_n\}_{n\in\mathbb{N}}$ be a sequence of step functions such that $0 \le \tilde\rho_n \le \tilde\rho := \rho - c\chi_K$ and

$$\lim_{n\to\infty} \int \tilde\rho_n \,\mathrm{d}\mathfrak{m} = \int \tilde\rho \,\mathrm{d}\mathfrak{m}.$$

We then define for every $n \in \mathbb{N}$, the measure $\mu_n := \rho_n \mathfrak{m} \in \mathscr{P}^{ac}(\mathsf{X}, \mathfrak{m})$, where ρ_n is the step function defined as follows:

$$\rho_n := \frac{\tilde\rho_n + c\chi_K}{\|\tilde\rho_n + c\chi_K\|_{L^1(\mathfrak{m})}}.$$

Note that $\|\tilde\rho_n + c\chi_K\|_{L^1(\mathfrak{m})} \to \|\rho\|_{L^1(\mathfrak{m})} = 1$ and, in particular, we observe that

$$\int |\rho_n - \rho| \,\mathrm{d}\mathfrak{m} \le \|\tilde\rho_n - \tilde\rho\|_{L^1(\mathfrak{m})} + \left(\frac{1}{\|\tilde\rho_n + c\chi_K\|_{L^1(\mathfrak{m})}} - 1\right)\|\tilde\rho_n + c\chi_K\|_{L^1(\mathfrak{m})} \to 0$$

as $n \to \infty$, thus $\rho_n \to \rho$ in $L^1(\mathfrak{m})$. Moreover, the sequence $\{\mu_n\}_{n\in\mathbb{N}}$ is by construction uniformly bounded, and in particular for every $N' \in \mathbb{N}$, the sequence $\{\rho_n^{-1/N'}\}_{n\in\mathbb{N}}$ is uniformly bounded from below and above. Furthermore, since $\rho_n \to \rho$ in $L^1(\mathfrak{m})$, up to a (non-relabeled) subsequence, $\rho_n \to \rho$ pointwise $\mathfrak{m}$-almost everywhere (see e.g. [Rud87, Theorem 3.12]). Trivially, this also shows that $\rho_n^{-1/N'} \to \rho^{-1/N'}$ pointwise $\mathfrak{m}$-almost everywhere, hence an application of the dominated convergence theorem implies that $\rho_n^{-1/N'} \to \rho^{-1/N'}$ in $L^1(\mathfrak{m})$ and conclude the proof. □

Lemma 3.4.5. *Let $\mu_0 = \rho_0\mathfrak{m}, \mu_1 = \rho_1\mathfrak{m} \in \mathscr{P}^{ac}(\mathsf{X},\mathfrak{m})$ be bounded and π be the unique optimal transport plan between them. Let $\{\mu_0^n = \rho_0^n\mathfrak{m}\}_{n\in\mathbb{N}}$, $\{\mu_1^n = \rho_1^n\mathfrak{m}\}_{n\in\mathbb{N}} \subset \mathscr{P}^{ac}(\mathsf{X},\mathfrak{m})$ be the approximating sequences provided by Lemma 3.4.4. Then, letting π_n be the unique optimal transport plan between μ_0^n and μ_1^n, it holds that*

$$\limsup_{n\to\infty} T^{(t)}_{K,N'}(\pi_n|\mathfrak{m}) \le T^{(t)}_{K,N'}(\pi|\mathfrak{m}),$$

for every $K \in \mathbb{R}$, $N' > 1$ and $t \in [0,1]$.

Proof. We follow a strategy similar to the one developed in [MRS23, Proposition 4.10]. Recall that the sequence $\{\pi_n\}_{n\in\mathbb{N}}$ weakly converges to $\pi \in \mathrm{Opt}(\mu_0,\mu_1)$ (see for example [ABS21, Theorem 6.8]). Moreover, we observe that it is sufficient to prove

$$\liminf_{n\to\infty} \int \tau^{(1-t)}_{K,N'}(\mathsf{d}(x,y))\rho_0^n(x)^{-\frac{1}{N'}}\,\mathrm{d}\pi_n(x,y) \ge \int \tau^{(1-t)}_{K,N'}(\mathsf{d}(x,y))\rho_0(x)^{-\frac{1}{N'}}\,\mathrm{d}\pi(x,y), \tag{3.4.12}$$

since the other term can be treated analogously. The space of continuous and bounded functions $C_b(\mathsf{X})$ is dense in $L^1(\mathfrak{m})$ (see for example [Rud87, Theorem 3.14]), thus for every $\varepsilon > 0$ we can find $g^\varepsilon \in C_b(\mathsf{X})$ such that $||\rho_0^{-1/N'} - g^\varepsilon||_{L^1(\mathfrak{m})} < \varepsilon$. Furthermore, for every $n \in \mathbb{N}$ big enough we have that $||(\rho_0^n)^{-1/N'} - g^\varepsilon||_{L^1(\mathfrak{m})} < 2\varepsilon$. In general, the function $(x,y) \mapsto \tau^{(1-t)}_{K,N'}(\mathsf{d}(x,y))$ is continuous but not bounded, for this reason, for every $M > 0$ we introduce the function

$$f_M(x,y) = \tau^{(1-t)}_{K,N'}(\mathsf{d}(x,y)) \wedge M.$$

The function f_M is continuous and bounded above by M and therefore

$$\begin{aligned}
&\liminf_{n\to\infty} \int f_M(x,y)\rho_0^n(x)^{-\frac{1}{N'}}\,\mathrm{d}\pi_n(x,y)\\
&\qquad\ge \liminf_{n\to\infty} \int f_M(x,y)g^\varepsilon(x)\,\mathrm{d}\pi_n(x,y) - M\int |g^\varepsilon - (\rho_0^n)^{-1/N'}|\,\mathrm{d}\mu_0^n\\
&\qquad\ge \liminf_{n\to\infty} \int f_M(x,y)g^\varepsilon(x)\,\mathrm{d}\pi_n(x,y) - CM\int |g^\varepsilon - (\rho_0^n)^{-1/N'}|\,\mathrm{d}\mathfrak{m}\\
&\qquad\ge \liminf_{n\to\infty} \int f_M(x,y)g^\varepsilon(x)\,\mathrm{d}\pi_n(x,y) - 2\varepsilon CM\\
&\qquad= \int f_M(x,y)g^\varepsilon(x)\,\mathrm{d}\pi(x,y) - 2\varepsilon CM\\
&\qquad\ge \int f_M(x,y)\rho_0(x)^{-\frac{1}{N'}}\,\mathrm{d}\pi(x,y) - 3\varepsilon CM,
\end{aligned}$$

where the equality holds because $(x,y) \mapsto f_M(x,y)g^\varepsilon(x)$ is continuous and bounded and $\pi_n \rightharpoonup \pi$, while the constant $C > 0$ represents the uniform upper bound on $\{\rho_0^n\}_{n\in\mathbb{N}} \cup \{\rho_0\}$. Since this last inequality holds for every $\varepsilon > 0$, we can conclude that

$$\liminf_{n\to\infty} \int f_M(x,y)\rho_0^n(x)^{-\frac{1}{N'}}\,\mathrm{d}\pi_n(x,y) \ge \int f_M(x,y)\rho_0(x)^{-\frac{1}{N'}}\,\mathrm{d}\pi(x,y).$$

Taking into account this semicontinuity property we deduce that for every $M > 0$

$$\begin{aligned}
\liminf_{n\to\infty} \int \tau^{(1-t)}_{K,N'}(\mathsf{d}(x,y))\rho_0^n(x)^{-\frac{1}{N'}}\,\mathrm{d}\pi_n(x,y) &\ge \liminf_{n\to\infty} \int f_M(x,y)\rho_0^n(x)^{-\frac{1}{N'}}\,\mathrm{d}\pi_n(x,y)\\
&\ge \int f_M(x,y)\rho_0(x)^{-\frac{1}{N'}}\,\mathrm{d}\pi(x,y),
\end{aligned}$$

taking now the limit as $M \to \infty$, the monotone convergence theorem allows to prove (3.4.12), concluding the proof. □

Proof of Theorem 3.4.1. According to Proposition 3.3.5, we can limit ourselves to prove the $\mathsf{CD}(K,N)$ condition for bounded marginals. On the other hand, for every pair of bounded marginals $\mu_0, \mu_1 \in \mathscr{P}^{ac}(\mathsf{X},\mathfrak{m})$, there exist two approximating sequences $\{\mu_0^n\}_{n\in\mathbb{N}}$ and $\{\mu_1^n\}_{n\in\mathbb{N}}$ satisfying the requirements of Lemma 3.4.4. For every $n \in \mathbb{N}$ call η_n the unique optimal geodesic plan in $\mathrm{OptGeo}(\mu_0^n, \mu_1^n)$, let $\pi_n := (e_0, e_1)_\# \eta_n$ and $\mu_t^n := (e_t)_\# \eta_n$. For every $n \in \mathbb{N}$, μ_0^n and μ_1^n are step measures with bounded support, thus Theorem 3.4.3 ensures that

$$\mathcal{E}_{N'}(\mu_t^n) \leq T^{(t)}_{K,N'}(\pi_n|\mathfrak{m}) \qquad \text{for every } N' \geq N \text{ and } t \in [0,1]. \tag{3.4.13}$$

Now we want to pass to the limit as $n \to \infty$. Notice that, since the families $\{\mu_0^n\}_{n\in\mathbb{N}} \cup \{\mu_0\}$ and $\{\mu_1^n\}_{n\in\mathbb{N}} \cup \{\mu_1\}$ are uniformly bounded and the space $(\mathsf{X},\mathsf{d},\mathfrak{m})$ is proper, for every fixed $t \in [0,1]$, all the measures in the family $\{\mu_t^n\}_{n\in\mathbb{N}}$ are concentrated on the same compact set. In particular, for every fixed $t \in [0,1]$, the family $\{\mu_t^n\}_{n\in\mathbb{N}}$ is W_2-precompact. We can then extract a (non-relabeled) subsequence such that

$$\mu_t^n \xrightarrow{W_2} \mu_t \in \mathscr{P}_2(\mathsf{X}) \quad \text{as } n \to \infty.$$

For every $n \in \mathbb{N}$, the measure μ_t^n is a t-midpoint between μ_0^n and μ_1^n, moreover $\mu_0^n \to \mu_0$ and $\mu_1^n \to \mu_1$ with respect to W_2, thus μ_t is the unique t-midpoint between μ_0 and μ_1. We can then pass (3.4.13) to the limit as $n \to \infty$, thanks to the lower semicontinuity of the entropy functional $\mathcal{E}_{N'}$ and to Lemma 3.4.5, and obtain

$$\mathcal{E}_{N'}(\mu_t) \leq T^{(t)}_{K,N'}(\pi|\mathfrak{m}), \qquad \text{for every } N' \geq N \text{ and } t \in [0,1],$$

which is (3.3.2). This concludes the proof. ☐

3.5 Final comments

3.5.1 The $\mathsf{BM}(K,N)$ inequality and the $\mathsf{MCP}(K,N)$ condition

In this section, we explore the relation between the (strong) Brunn–Minkowski inequality and the so-called measure contraction property. We recall here its definition, firstly introduced in [Oht07], and an equivalent characterization proved therein.

Definition 3.5.1 ($\mathsf{MCP}(K,N)$ condition)**.** Given $K \in \mathbb{R}$ and $N > 1$, a metric measure space $(\mathsf{X},\mathsf{d},\mathfrak{m})$ is said to satisfy the *measure contraction property* $\mathsf{MCP}(K,N)$ if for every $x \in \mathrm{spt}(\mathfrak{m})$ and a Borel set $A \subset \mathsf{X}$ with $0 < \mathfrak{m}(A) < \infty$, there exists $\eta \in \mathrm{OptGeo}(\delta_x, \mathfrak{m}_A)$ such that, for every $t \in [0,1]$,

$$\frac{1}{\mathfrak{m}(A)}\mathfrak{m} \geq (e_t)_\# \Big(\tau^{(t)}_{K,N}\big(\mathsf{d}(\gamma_0,\gamma_1)\big)^N \eta(\,\mathrm{d}\gamma) \Big).$$

Lemma 3.5.2 ([Oht07, Lemma 2.3])**.** *Let $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be a metric measure space. Assume that, for every $x \in \mathrm{spt}(\mathfrak{m})$ and $A \subset \mathsf{X}$ a Borel set with $0 < \mathfrak{m}(A) < \infty$, there exists a measurable selection $\Phi\colon A \to \mathrm{Geo}(\mathsf{X})$ satisfying $e_0 \circ \Phi \equiv x$ and $e_1 \circ \Phi = \mathrm{id}_A$ such that,*

$$\mathfrak{m}\big(e_t(\Phi(A'))\big) \geq \int_{A'} \tau^{(t)}_{K,N}\big(\mathsf{d}(x,y)\big)^N \,\mathrm{d}\mathfrak{m}(y), \qquad \textit{for any Borel } A' \subset A. \tag{3.5.1}$$

Then, $(\mathsf{X},\mathsf{d},\mathfrak{m})$ satisfies the $\mathsf{MCP}(K,N)$ condition.

Proposition 3.5.3. *Let $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be an essentially non-branching metric measure space supporting the Brunn–Minkowski inequality $\mathsf{BM}(K,N)$. Then, $(\mathsf{X},\mathsf{d},\mathfrak{m})$ has the measure contraction property $\mathsf{MCP}(K,N)$.*

Proof. Let $x \in \operatorname{spt}(\mathfrak{m})$ and $A \subset \mathsf{X}$ a Borel set with $0 < \mathfrak{m}(A) < \infty$; we can assume, without loss of generality, that $A \subset \operatorname{spt}(\mathfrak{m})$. Then, applying Corollary 3.3.8, and in particular the strong interpolation property to the marginals $\mathfrak{m}_A$ and δ_x, there exists a unique geodesic connecting a and x, for $\mathfrak{m}$-a.e. $a \in A$. Thus, we have a well-defined measurable selection Φ satisfying the requirements of Lemma 3.5.2 and

$$\mathfrak{m}(M_t(x, A')) = \mathfrak{m}(e_t(\Phi(A'))), \qquad \text{for every Borel } A' \subset A.$$

Let $\varepsilon > 0$ and $A' \subset A$ be a Borel set. Define a partition $\{A_n^\varepsilon\}_{n\in\mathbb{N}}$ of A' as follows:

$$A_n^\varepsilon := A' \cap C_n^\varepsilon \qquad \text{where} \qquad C_n^\varepsilon := \{y \in \mathsf{X} \,:\, n\varepsilon < \mathsf{d}(y,x) \le (n+1)\varepsilon\}.$$

Applying the $\mathsf{BM}(K,N)$ inequality for the sets $\{x\}$ and A_n^ε we obtain that

$$\begin{aligned}\mathfrak{m}(M_t(x, A_n^\varepsilon)) &\ge \tau_{K,N}^{(t)}\big(\Theta(x, A_n^\varepsilon)\big)^N \mathfrak{m}(A_n^\varepsilon)\\ &= \int_{A_n^\varepsilon} \tau_{K,N}^{(t)}\big(\Theta(x, A_n^\varepsilon)\big)^N \,\mathrm{d}\mathfrak{m}(z) \ge \int_{A_n^\varepsilon} \tau_{K,N}^{(t)}(\mathsf{d}(x,z) \mp \varepsilon)^N \,\mathrm{d}\mathfrak{m}(z).\end{aligned} \tag{3.5.2}$$

Note that $M_t(x, A_n^\varepsilon) \cap M_t(x, A_m^\varepsilon) \neq \emptyset$ whenever $n \neq m$, since by construction every $z \in M_t(x, A_n^\varepsilon)$ is such that $\mathsf{d}(x,z) \in (tn\varepsilon, t(n+1)\varepsilon]$. Therefore, we can sum the inequalities (3.5.2) over all $n \in \mathbb{N}$, obtaining

$$\mathfrak{m}(M_t(x, A')) \ge \int_{A'} \tau_{K,N}^{(t)}(\mathsf{d}(x,z) \mp \varepsilon)^N \,\mathrm{d}\mathfrak{m}(z).$$

Passing to the limit as $\varepsilon \to 0$ and using Fatou lemma, we deduce inequality (3.5.1) for A'. By the arbitrariness of $t \in (0,1]$ and $A' \subset A$, we conclude the proof. □

Proposition 3.5.3 shows that the $\mathsf{BM}(K,N)$ inequality, intended as a curvature dimension bound, is stronger than the $\mathsf{MCP}(K,N)$ condition. The heuristic reason behind this difference is that, while the $\mathsf{BM}(K,N)$ inequality controls the behavior of the set geodesics joining *any* two sets, $\mathsf{MCP}(K,N)$ controls only the interpolation between a constant density measure and a Dirac delta (which, in the case of essentially non-branching spaces, corresponds to a control on geodesics spreading out from a point). This is also confirmed by the existence of weighted Riemannian manifolds where $\mathsf{MCP}(K,N)$ does not imply $\mathsf{BM}(K,N)$, with the same (sharp) constants. In conclusion, the Brunn–Minkowski inequality $\mathsf{BM}(K,N)$ is closer to the $\mathsf{CD}(K,N)$ condition than the measure contraction property $\mathsf{MCP}(K,N)$ is.

3.5.2 Relation between $M_t(A,B)$ and $D_t(A,B)$

In Theorem 3.4.1, we proved that $\mathsf{SBM}(K,N)$ is equivalent to $\mathsf{CD}(K,N)$, for essentially non-branching metric measure spaces. In principle, we would like to improve the equivalence, including $\mathsf{BM}(K,N)$, as shown for weighted Riemannian manifolds in [MPR22a]. The strategy proposed in the proof of Theorem 3.4.3 could be adapted to deduce stronger the implication $\mathsf{BM}(K,N) \Rightarrow \mathsf{CD}(K,N)$, if we were able to control the difference between (the measure of) the sets $M_t(A,B)$ and $D_t(A,B)$. In particular, we can not expect them to be equal for any couple of sets, also in elementary examples. Consider for instance the metric measure space $(\mathbb{R}^2, |\cdot|, \mathscr{L}^2)$ and the sets $A, B \subset \mathbb{R}^2$, as in Figure 3.1. In this case, one has

$$\mathscr{L}^2(M_{1/2}(A,B)) > \mathscr{L}^2(D_{1/2}(A,B)).$$

However, partitioning A and B as in the right-hand side of Figure 3.1, the problem is remedied, indeed

$$M_t(A_1,B_1) = D_t(A_1,B_1) \quad \text{and} \quad M_t(A_2,B_2) = D_t(A_2,B_2), \tag{3.5.3}$$

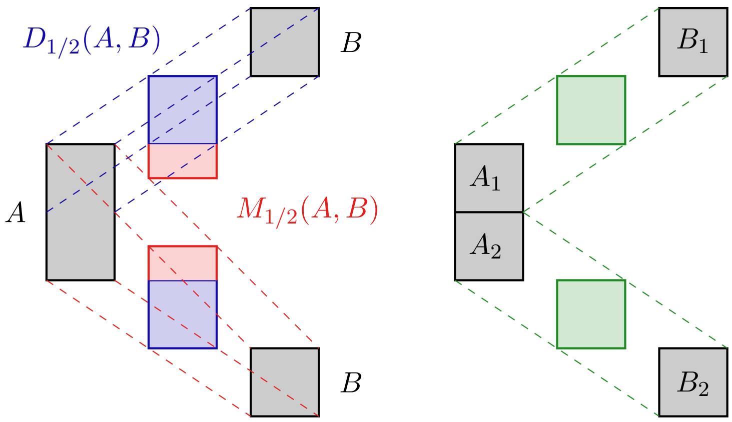

Figure 3.1: On the left-hand side, the sets $D_{1/2}(A,B)$ and $M_{1/2}(A,B)$ differ. On the right-hand side, after partitioning A and B, the two coincide.

and now the strategy of the proof of Theorem 3.4.3 can be employed without changes. The delicate issue is that, in general metric measure spaces, we are not able to find a suitable partition for arbitrary Borel sets A and B, in such a way (3.5.3) is verified (up to $\mathfrak{m}$-null sets) for *each element* of the partition.

Paper 4

Almost-Riemannian manifolds do not satisfy the curvature-dimension condition

with Tommaso Rossi

The Lott-Sturm-Villani curvature-dimension condition $\mathsf{CD}(K,N)$ *provides a synthetic notion for a metric measure space to have curvature bounded from below by* K *and dimension bounded from above by* N*. It was proved by Juillet in [Jui21] that a large class of sub-Riemannian manifolds do not satisfy the* $\mathsf{CD}(K,N)$ *condition, for any* $K \in \mathbb{R}$ *and* $N \in (1,\infty)$*. However, his result does not cover the case of almost-Riemannian manifolds. In this paper, we address the problem of disproving the* CD *condition in this setting, providing a new strategy which allows us to contradict the one-dimensional version of the* CD *condition. In particular, we prove that* 2*-dimensional almost-Riemannian manifolds and strongly regular almost-Riemannian manifolds do not satisfy the* $\mathsf{CD}(K,N)$ *condition for any* $K \in \mathbb{R}$ *and* $N \in (1,\infty)$*.*

All authors of this paper contributed equally to all results.

4.1 Introduction

In their seminal works Lott-Villani [LV09] and Sturm [Stu06a, Stu06b] introduced a synthetic notion of curvature-dimension bounds, which is heavily based on the theory of Optimal Transport. They noticed that, in a Riemannian manifold, a uniform lower bound on the Ricci curvature, together with an upper bound on the dimension, is equivalent to a convexity property of the Rényi entropy functionals in the Wasserstein space. This allowed them to define a consistent notion of curvature-dimension bounds for metric measure spaces, known as CD condition. While in the Riemannian setting, the CD condition is equivalent to having bounded geometry, an analogue result does not hold in the sub-Riemannian setting. Sub-Riemannian geometry is a far-reaching generalization of Riemannian geometry: given a smooth manifold M, we define a smoothly varying scalar product only on a subset of *horizontal* directions $\mathcal{D}_p \subset T_pM$ (called distribution) at each point $p \in M$. Under the so-called Hörmander condition, M is horizontally-path connected, and the usual length-minimization procedure yields a well-defined distance d. In particular, differently from what happens in Riemannian geometry, the rank of the distribution $r(p) = \dim \mathcal{D}_p$ may be strictly less than the dimension of the manifold and may vary with the point. In general, we can not expect the CD condition to hold for *truly* sub-Riemannian manifolds. This statement is confirmed by the following result by Juillet.

Theorem 4.1.1 ([Jui21, Cor. 1.2])**.** *Let M be a complete sub-Riemannian manifold with* $\dim M \geq 3$*, equipped with a smooth positive (i.e. with strictly positive density) measure* $\mathfrak{m}$*. Assume that the possibly varying rank of the distribution is smaller than* $\dim M - 1$*. Then,* $(M, \mathsf{d}, \mathfrak{m})$ *does not satisfy the* $\mathsf{CD}(K, N)$ *condition for any* $K \in \mathbb{R}$ *and* $N \in (1, \infty)$*.*

While this result is quite general, it does not include many cases of interest, such as *almost-Riemannian geometry*. Roughly speaking, an almost-Riemannian manifold is a sub-Riemannian manifold where the rank of the distribution coincides with the dimension of M, at almost every point[1]. For this reason, the technique used to prove Theorem 4.1.1 can not be adapted to this setting. Indeed, it relies on the construction of two Borel subsets for which the Brunn-Minkowski inequality does not hold, namely, for all $R, \varepsilon > 0$, one can find $A, B \subset M$ such that $\mathrm{diam}(A \cup B) < R$, $\mathfrak{m}(A) \approx \mathfrak{m}(B)$, and such that there exists $t \in (0, 1)$ for which

$$\mathfrak{m}(Z_t(A, B)) \leq \frac{1}{2^{\mathcal{N} - \dim M}} \mathfrak{m}(B)(1 + \varepsilon), \tag{4.1.1}$$

where $Z_t(A, B)$ denotes the t-intermediate set between A and B and $\mathcal{N}$ is the so-called *geodesic dimension* of M, see [Riz16] or [ABR18, Def. 5.47] for a precise definition. The inequality (4.1.1) allows to contradict the Brunn-Minkowski inequality if and only if the geodesic dimension $\mathcal{N}$ is strictly greater than $\dim M$. However, in the almost-Riemannian setting, $\mathcal{N} = \dim M$ almost everywhere, making this construction inconclusive. We mention that Juillet in [Jui10] disproved the CD condition in the simple example of the standard Grushin plane (cf. Example 4.2.6) equipped with the Lebesgue measure, by direct computations. Heuristically, disproving the CD condition in almost-Riemannian manifolds is a more challenging task, since they behave in some sense like non-complete Riemannian manifolds. Thus, a new strategy is needed.

Our idea is to exploit the one-dimensional characterization of the CD condition:

$$\mathsf{CD}(K, N) \quad \Rightarrow \quad \mathsf{CD}^1(K, N), \tag{4.1.2}$$

proven by Cavalletti and Mondino in [CM17b], and contradict the $\mathsf{CD}^1(K, N)$ condition. For any 1-Lipschitz function u, the latter relies on a disintegration of the reference measure, associated with u, in one-dimensional transport rays and requires the $\mathsf{CD}(K, N)$ condition to hold along them. The main advantage in dealing with one-dimensional CD spaces is related to a differential characterization of the CD densities, (cf. Lemma 4.3.2), which is easier to disprove compared with the convexity of the Rényi entropy. In Section 4.3.2, we present a local version of the one-dimensional characterization (4.1.2) (cf. Proposition 4.3.7), which permits to exploit the local structure of sub-Riemannian manifolds. Then, in the case of an almost-Riemannian manifold, equipped with a smooth positive measure $\mathfrak{m}$, we are able to explicitly compute the disintegration and verify that the *one-dimensional* $\mathsf{CD}(K, N)$ condition along the rays does not hold for any $K \in \mathbb{R}$ and $N \in (1, \infty)$. Our main result is the following, cf. Theorems 4.5.4 and 4.6.3. We refer to Sections 4.2 and 4.6 for precise definitions.

Theorem 4.1.2. *Let M be a complete almost-Riemannian manifold and let $\mathfrak{m}$ be any smooth positive (i.e. with strictly positive density) measure on M. Assume M is either of dimension* 2 *or strongly regular. Then, the metric measure space* $(M, \mathsf{d}, \mathfrak{m})$ *does not satisfy the* $\mathsf{CD}(K, N)$ *condition for any* $K \in \mathbb{R}$ *and* $N \in (1, +\infty)$*.*

Remarkably, for 2-dimensional almost-Riemannian manifolds, we do not require any additional assumption on the structure of the singular region $\mathcal{Z}$, see (4.2.2) for the precise definition. However, as soon as the dimension of the manifold increases, the complexity of the computations prevents us to treat the general case and we need an auxiliary control on the behavior of the distribution.

[1]But not at every point, otherwise the structure would be Riemannian.

Nonetheless, we stress that our procedure is algorithmic and can be applied to any explicit example of almost-Riemannian manifold. This algorithmic procedure has been implemented in the software *Mathematica*, see [MR22].

A crucial tool for proving Theorem 4.1.2 will be a truly sub-Riemannian phenomenon, namely the existence of *characteristic points*. For an embedded hypersurface $\Sigma \subset M$, a characteristic point is a point where the distribution is tangent to Σ. Of course, such points do not exist in Riemannian geometry, but as soon as the rank of the distribution $r(p) < \dim M$ for some $p \in M$, they can appear. Usually, characteristic points are source of subtle technical problems, mostly related to the low regularity of the (signed) distance δ_Σ from Σ. Indeed, although being 1-Lipschitz with respect to d, δ_Σ is not smooth around characteristic points (and not even Lipschitz in coordinates). In the proof of Theorem 4.1.2, we choose a suitable hypersurface Σ, we build the disintegration of $\mathfrak{m}$ associated with a localized version of δ_Σ and we exploit its singular behavior to contradict the differential characterization of the one-dimensional $\mathsf{CD}(K,N)$ condition. In particular, Σ is chosen to be *transverse* to the singular region of M in such a way $\Sigma \cap \mathcal{Z}$ exhibits characteristic points; we can then exploit the Riemannian structure at points of $\Sigma \setminus \mathcal{Z}$ to describe the degeneration of δ_Σ in the disintegration of $\mathfrak{m}$. For example, in the standard Grushin plane, where the singular region is $\mathcal{Z} = \{x = 0\}$, a suitable transverse hypersurface is $\Sigma = \{y = 0\}$.

It is worth mentioning that there exists a weaker synthetic notion of curvature bounds, introduced by Ohta in [Oht07], called *measure contraction property* or MCP condition. This property seems to be more suited to sub-Riemannian geometry, see for example [BR18, BR19, BKS19, BR20]. Finally, we refer to [Mil21] for a relaxation of the CD condition, called *quasi-curvature-dimension* condition, which holds for a certain class of sub-Riemannian manifolds. However, it is not known whether these weaker conditions hold for a general almost-Riemannian manifold.

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After submitting this work, Rizzi and Stefani proved in [RS23] that every sub-Riemannian manifold does not satisfy the $\mathsf{CD}(K,\infty)$ condition, using different techniques.

4.2 Preliminaries

4.2.1 Almost-Riemannian geometry

We recall some basic facts about almost-Riemannian geometry, following [ABB20].

Definition 4.2.1. Let M be a smooth, connected manifold. A *sub-Riemannian structure* on M is a triple $(\mathbb{U}, \xi, (\cdot|\cdot))$ satisfying the following conditions:

i) $\pi_\mathbb{U} \colon \mathbb{U} \to M$ is a Euclidean bundle of rank k with base M, namely for all $p \in M$, the fiber $\mathbb{U}_p$ is a vector space equipped with a scalar product $(\cdot|\cdot)_p$, which depends smoothly on p;

ii) The map $\xi \colon \mathbb{U} \to TM$ is a morphism of vector bundles, i.e. ξ is smooth and such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{U} & \xrightarrow{\ \xi\ } & TM \\ & \searrow^{\pi_\mathbb{U}} & \downarrow^{\pi_M} \\ & & M \end{array}$$

where $\pi_M : TM \to M$ denotes the canonical projection of the tangent bundle.

iii) The distribution $\mathcal{D} = \{\xi(\sigma) \mid \sigma : M \to \mathbb{U} \text{ smooth section}\} \subset TM$ satisfies the *Hörmander condition* (also known as bracket-generating condition), namely

$$\mathrm{Lie}_p(\mathcal{D}) = T_pM, \qquad \forall\, p \in M.$$

With a slight abuse of notation, we say that M is a sub-Riemannian manifold.

Let $(\mathbb{U}, \xi, (\cdot|\cdot))$ be a sub-Riemannian structure on M. We can define the sub-Riemannian norm on $\mathcal{D}$ as

$$\|v\|_p^2 = \inf\{(u|u)_p \mid u \in \mathbb{U}_p,\ \xi(u) = v\}, \qquad \forall\, v \in \mathcal{D}_p,\ p \in M. \tag{4.2.1}$$

The norm (4.2.1) is well-defined since the infimum is actually a minimum and it induces a scalar product g_p on $\mathcal{D}_p$ by polarization. Notice that different sub-Riemannian structures on M may define the same distributions and induced norms. This is the case for equivalent sub-Riemannian structures.

Definition 4.2.2. Let $(\mathbb{U}_1, \xi_1, (\cdot|\cdot)_1)$, $(\mathbb{U}_2, \xi_2, (\cdot|\cdot)_2)$ be two sub-Riemannian structures on M. These are said to be *equivalent* if the following conditions hold:

i) There exists a Euclidean bundle $(\mathbb{V}, (\cdot|\cdot)_{\mathbb{V}})$ and two surjective bundle morphisms $p_i : \mathbb{V} \to \mathbb{U}_i$ such that the following diagram is commutative

$$\begin{array}{ccc} \mathbb{V} & \xrightarrow{p_1} & \mathbb{U}_1 \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle \xi_1} \\ \mathbb{U}_2 & \xrightarrow[\xi_2]{} & TM \end{array}$$

ii) The projections p_i's are compatible with the scalar products defined on $\mathbb{U}_i$, namely

$$(u|u)_i = \min\{(v|v)_{\mathbb{V}} \mid p_i(v) = u\}, \qquad \forall\, u \in \mathbb{U}_i, \quad i = 1, 2.$$

Definition 4.2.3. Let M be a sub-Riemannian manifold. The *minimal bundle rank* is the infimum of the rank of Euclidean bundles inducing equivalent structures on M. For $p \in M$, the *local minimal bundle rank* of M at p is the minimal bundle rank of the structure when restricted to a sufficiently small neighborhood $\mathcal{U}_p$.

Definition 4.2.4 (Almost-Riemannian structure)**.** Let M be a connected, smooth manifold of dimension $n+1$ and let $(\mathbb{U}, \xi, (\cdot|\cdot))$ be a sub-Riemannian structure on M. We say that M is an *almost-Riemannian manifold* if the local minimal bundle rank of the structure is $n+1$.

We denote by $\mathcal{Z}$ the set of *singular points*, namely those points where the distribution has not full rank:

$$\mathcal{Z} = \{p \in M \mid \dim(\mathcal{D}_p) < n+1\}. \tag{4.2.2}$$

Notice that $\mathcal{Z}$ is closed, since the rank of the distribution is lower semi-continuous. We say that a point is *Riemannian* if it belongs to $M \setminus \mathcal{Z}$.

Remark 4.2.5. If the singular set is empty, then the structure on M is Riemannian. Therefore, we will always tacitly assume that $\mathcal{Z} \neq \emptyset$.

A local orthonormal frame for the distribution is the image through ξ of a local orthonormal frame for $\mathbb{U}$. Consequently, by definition of almost-Riemannian manifold of dimension $n+1$, it consists of exactly $n+1$ vector fields which are linearly independent only at Riemannian points. In particular, local orthonormal frames are standard Riemannian orthonormal frames around Riemannian points.

Example 4.2.6 (Grushin plane)**.** Let $M = \mathbb{R}^2$ and consider the sub-Riemannian structure given by $\mathbb{U} = \mathbb{R}^2 \times \mathbb{R}^2$ with the standard Euclidean scalar product on fibers and

$$\xi \colon \mathbb{U} \to T\mathbb{R}^2; \qquad \xi(x, z, u_1, u_2) = (x, z, u_1, xu_2).$$

As one can check, the resulting distribution is generated by the orthonormal vector fields $X = \partial_x$, $Y = x\partial_z$. The local minimal bundle rank is equal to 2, thus the structure is almost-Riemannian. In this case the singular region is $\mathcal{Z} = \{x = 0\}$ and $\{X, Y\}$ is a (global) orthonormal frame.

Remark 4.2.7. Any truly sub-Riemannian structure (meaning that is not Riemannian) of rank 2 on a 2-dimensional manifold is always almost-Riemannian, in the sense of Definition 4.2.4.

4.2.2 Almost-Riemannian distance

Let $(\mathbb{U}, \xi, (\cdot|\cdot))$ be an almost-Riemannian structure on M. We say that $\gamma : [0,T] \to M$ is a *horizontal curve*, if it is absolutely continuous and

$$\dot\gamma(t) \in \mathcal{D}_{\gamma(t)}, \qquad \text{for a.e.}\, t \in [0,T].$$

This implies that there exists a measurable function $u : [0,T] \to \mathbb{U}$, such that

$$\pi_{\mathbb{U}}(u(t)) = \gamma(t), \qquad \dot\gamma(t) = \xi(u(t)), \qquad \text{for a.e.}\, t \in [0,T].$$

Moreover, we have that $u \in L^\infty([0,T], \mathbb{U})$, see [ABB20, Lemma 3.12], therefore the map $t \mapsto \|\dot\gamma(t)\|$ is integrable on $[0,T]$. We define the *length* of a horizontal curve as follows:

$$\ell(\gamma) = \int_0^T \|\dot\gamma(t)\|\, \mathrm{d}t.$$

The *almost-Riemannian distance* on M is defined, for any $p, q \in M$, by

$$\mathsf{d}(p,q) = \inf\{\ell(\gamma) \mid \gamma \text{ horizontal curve between } p \text{ and } x\}.$$

By Chow-Rashevskii theorem (see for example [AS04, Thm. 5.9]), the bracket-generating assumption ensures that the distance $\mathsf{d} \colon M \times M \to \mathbb{R}$ is finite and continuous. Furthermore it induces the same topology as the manifold one. We say that M is *complete*, if the metric space (M, d) is.

4.2.3 Geodesics and Hamiltonian flow

A *geodesic* is a horizontal curve $\gamma\colon [0,T] \to M$, parameterized with constant speed, such that any sufficiently short segment is length-minimizing. The *almost-Riemannian Hamiltonian* is the function on the cotagent space $H \in C^\infty(T^*M)$ defined by

$$H(\lambda) = \frac{1}{2}\sum_{i=0}^{n} \langle \lambda, X_i\rangle^2, \qquad \lambda \in T^*M, \tag{4.2.3}$$

where $\{X_0, \dots, X_n\}$ is a local orthonormal frame for the almost-Riemannian structure, and $\langle \lambda, \cdot\rangle$ denotes the action of covectors on vectors. The *Hamiltonian vector field* $\vec{H}$ on T^*M is defined by

$\varsigma(\cdot,\vec{H}) = dH$, where $\varsigma \in \Lambda^2(T^*M)$ is the canonical symplectic form. Solutions $\lambda\colon [0,T] \to T^*M$ to the *Hamilton equations*

$$\dot{\lambda}(t) = \vec{H}(\lambda(t)), \tag{4.2.4}$$

are called *normal extremals*. Their projections $\gamma(t) = \pi(\lambda(t))$ on M, where $\pi\colon T^*M \to M$ is the bundle projection, are locally length-minimizing horizontal curves parameterized with constant speed, and are called *normal geodesics*. If γ is a normal geodesic with normal extremal λ, then its speed is given by $\|\dot\gamma\|_g = \sqrt{2H(\lambda)}$. In particular

$$\ell(\gamma|_{[0,t]}) = t\sqrt{2H(\lambda(0))}, \qquad \forall\, t \in [0,T].$$

There is another class of length-minimizing curves in sub-Riemannian geometry, called *abnormal* or *singular*. As for the normal case, to these curves it corresponds an extremal lift $\lambda(t)$ on T^*M, which however may not follow the Hamiltonian dynamics (4.2.4). Here we only observe that an abnormal extremal lift $\lambda(t) \in T^*M$ satisfies

$$\langle \lambda(t), \mathcal{D}_{\pi(\lambda(t))}\rangle = 0 \quad \text{and} \quad \lambda(t) \neq 0, \qquad \forall\, t \in [0,T], \tag{4.2.5}$$

that is $H(\lambda(t)) \equiv 0$, therefore abnormal geodesics are always contained in the singular region $\mathcal{Z}$. A geodesic may be abnormal and normal at the same time.

Definition 4.2.8. Let M be an almost-Riemannian manifold and let $p \in M$. Then, the *almost-Riemannian exponential map* is

$$\exp_p(\lambda) = \pi \circ e^{\vec{H}}(\lambda), \qquad \forall \lambda \in T_p^*M, \tag{4.2.6}$$

where H denotes the almost-Riemannian Hamiltonian (4.2.3) and $e^{\vec{H}}(\lambda)$ is the solution to (4.2.4) at time $t = 1$, with initial datum $\lambda \in T_p^*M$.

Note that, in general, $\exp_p$ may not be defined on the whole cotangent space, but if M is complete, then $\vec{H}$ is a complete vector field and (4.2.6) is well-posed.

4.2.4 Length-minimizers to a hypersurface

Let $\Sigma \subset M$ be a smooth hypersurface and fix $q_0 \in \Sigma$. Moreover, let $v \in C^\infty(M)$ be a local defining function for Σ around q_0, namely there exists an open neighborhood $\Sigma_{q_0} \subset \Sigma$ of q_0 such that

$$\Sigma_{q_0} \subset \{v = 0\} \qquad \text{and} \qquad dv_{|\Sigma_{q_0}} \neq 0. \tag{4.2.7}$$

We define the *local signed distance function* from Σ around q_0 as follows:

$$\delta_v := \operatorname{sgn}(v(p)) \cdot \mathsf{d}(p, \{v = 0\}), \qquad \forall\, p \in M. \tag{4.2.8}$$

Let $\gamma\colon [0,T] \to M$ be a horizontal curve, parameterized with constant speed, such that $\gamma(0) \in \Sigma$, $\gamma(T) = p \in M \setminus \Sigma$ and assume γ is a minimizer for $\mathsf{d}(\cdot,\Sigma)$, that is $\ell(\gamma) = \mathsf{d}(p,\Sigma)$. In particular, γ is a geodesic and any corresponding normal or abnormal lift, say $\lambda : [0,T] \to T^*M$, must satisfy the transversality conditions, cf. [AS04, Thm 12.13],

$$\langle \lambda(0), w\rangle = 0, \qquad \forall\, w \in T_{\gamma(0)}\Sigma. \tag{4.2.9}$$

Equivalently, the initial covector $\lambda(0)$ must belong to the *annihilator bundle* $\mathcal{A}\Sigma$ of Σ with fiber $\mathcal{A}_q\Sigma = \{\lambda \in T_q^*M \mid \langle \lambda, T_q\Sigma\rangle = 0\}$, for any $q \in \Sigma$. The restriction of $\exp_q$ to the annihilator bundle of Σ allows to build (locally) a smooth tubular neighborhood around non-characteristic points. Recall that $q \in \Sigma$ is a *characteristic point*, and we write $q \in C(\Sigma)$, if $\mathcal{D}_q \subset T_q\Sigma$.

Lemma 4.2.9. *Let $\Sigma \subset M$ be a smooth hypersurface, let $q_0 \in \Sigma \setminus C(\Sigma)$ be a non-characteristic point and $v \in C^\infty(M)$ as in (4.2.7). Then, there exist $\varepsilon_{q_0} > 0$ and a neighborhood $\mathcal{O}_{q_0} \subset \Sigma_{q_0}$ of q_0 such that the map*

$$G\colon (-\varepsilon_{q_0}, \varepsilon_{q_0}) \times \mathcal{O}_{q_0} \to M, \qquad G(s,q) = \exp_q(s\lambda(q)), \tag{4.2.10}$$

is a diffeomorphism on its image, where $\lambda(q)$ is the unique element (up to a sign) of $\mathcal{A}_q\Sigma$ such that $2H(\lambda(q)) = 1$. Moreover, δ_v is smooth in $G((-\varepsilon_{q_0}, \varepsilon_{q_0}) \times \mathcal{O}_{q_0})$ and[2]

$$G_*\partial_{s|(s,q)} = \nabla\delta_v(G(s,q)), \qquad \forall\, (s,q) \in (-\varepsilon_{q_0}, \varepsilon_{q_0}) \times \mathcal{O}_{q_0}. \tag{4.2.11}$$

Remark 4.2.10. It is known that if Σ has no characteristic points, the signed distance is smooth in a tubular neighborhood of Σ, cf. [FPR20, Prop. 3.1]. This lemma can be regarded as its local version and its proof is a straightforward adaptation of the aforementioned result. Moreover, note that $C(\Sigma) \subset \mathcal{Z}$ and so the Riemannian points of Σ are non-characteristic. Finally, if Σ contains characteristic points, the parameter ε_{q_0}, as well as $\mathcal{O}_{q_0}$, can not be chosen uniformly.

Remark 4.2.11. By condition (4.2.11), for any $q \in \mathcal{O}_{q_0}$, we have

$$(-\varepsilon_{q_0}, \varepsilon_{q_0}) \in s \mapsto G(s,q) \in M$$

is the unique minimizing geodesic (parameterized by unit-speed) from Σ passing through q. Moreover, notice that the initial covector $\lambda(q)$ in (4.2.10) is unique up to a sign: the only requirement is to choose this covector in such a way it defines a continuous section of the annihilator bundle.

4.3 The curvature-dimension condition

A triple $(X, \mathsf{d}, \mathfrak{m})$ is called metric measure space if (X, d) is a complete and separable metric space and $\mathfrak{m}$ is a locally finite Borel measure on it. In the following $C([0,1], X)$ will stand for the space of continuous curves from $[0,1]$ to X. A curve $\gamma \in C([0,1], X)$ is called *minimizing geodesic* if

$$\mathsf{d}(\gamma_s, \gamma_t) = |t - s| \cdot \mathsf{d}(\gamma_0, \gamma_1) \quad \text{for every } s, t \in [0,1],$$

we denote by $\mathrm{Geo}(X)$ the space of minimizing geodesics on X. The metric space (X, d) is said to be geodesic if every pair of points $x, y \in X$ can be connected with a curve $\gamma \in \mathrm{Geo}(X)$. For any $t \in [0,1]$ we define the evaluation map $e_t\colon C([0,1], X) \to X$ by setting $e_t(\gamma) := \gamma_t$ and the stretching/restriction operator restr_r^s in $C([0,1], X)$, defined, for all $0 \le r < s \le 1$, by

$$[\mathrm{restr}_r^s(\gamma)]_t := \gamma_{r+t(s-r)}, \qquad t \in [0,1].$$

We denote by $\mathscr{P}(X)$ the set of Borel probability measures on X and by $\mathscr{P}_2(X) \subset \mathscr{P}(X)$ the set of those having finite second moment. We endow the space $\mathscr{P}_2(X)$ with the Wasserstein distance W_2, defined by

$$W_2^2(\mu_0, \mu_1) := \inf_{\pi \in \mathsf{Adm}(\mu_0, \mu_1)} \int \mathsf{d}^2(x,y)\, \mathrm{d}\pi(x,y),$$

where $\mathsf{Adm}(\mu_0, \mu_1)$ is the set of all the admissible transport plans between μ_0 and μ_1, namely all the measures in $\mathscr{P}(X^2)$ such that $(\mathsf{p}_1)_\sharp\pi = \mu_0$ and $(\mathsf{p}_2)_\sharp\pi = \mu_1$. The metric space $(\mathscr{P}_2(X), W_2)$ is itself complete and separable, moreover, if (X, d) is geodesic, then $(\mathscr{P}_2(X), W_2)$ is geodesic as well. In particular, every geodesic $(\mu_t)_{t\in[0,1]}$ in $(\mathscr{P}_2(X), W_2)$ can be represented with a measure $\eta \in \mathscr{P}(\mathrm{Geo}(X))$, meaning that $\mu_t = (e_t)_\# \eta$. A subset $G \subset \mathrm{Geo}(X)$ is called non-branching if for any pair $\gamma_1, \gamma_2 \in G$ such that $\gamma_1 \neq \gamma_2$, it holds that

$$\mathrm{restr}_0^t(\gamma_1) \neq \mathrm{restr}_0^t(\gamma_2) \qquad \text{for every } t \in (0,1).$$

[2]The *horizontal gradient* of $f \in C^\infty(M)$ is defined by $g_p(\nabla f, v) = d_p f(v), \forall\, v \in \mathcal{D}_p$ and $p \in M$.

A metric measure space $(X, \mathsf{d}, \mathfrak{m})$ is said to be essentially non-branching if for every two measures $\mu_0, \mu_1 \in \mathscr{P}_2(X)$ which are absolutely continuous with respect to the reference measure $\mathfrak{m}$ ($\mu_0, \mu_1 \ll \mathfrak{m}$), every W_2-geodesic connecting them is concentrated on a non-branching set of geodesics.

4.3.1 CD spaces

In this subsection we introduce the CD condition, pioneered by Sturm and Lott-Villani [Stu06a, Stu06b, LV09]. This condition aims to generalize, to the context metric measure spaces, the notion of having Ricci curvature bounded from below by K and dimension less than or equal to N. In particular, in the Riemannian setting it is possible to characterize this two bounds in terms of a property whose definition involves only the distance and the (volume) measure. This property, which is stated in Definition 4.3.1, is given in terms of the following distortion coefficients: for every $K \in \mathbb{R}$ and $N \in (1, +\infty)$

$$\tau_{K,N}^{(t)}(\theta) := t^{\frac{1}{N}} \left[\sigma_{K,N-1}^{(t)}(\theta)\right]^{1-\frac{1}{N}},$$

where

$$\sigma_{K,N}^{(t)}(\theta) := \begin{cases} \dfrac{\sin(t\theta\sqrt{K/N})}{\sin(\theta\sqrt{K/N})} & \text{if } N\pi^2 > K\theta^2 > 0, \\ t & \text{if } K = 0, \\ \dfrac{\sinh(t\theta\sqrt{-K/N})}{\sinh(\theta\sqrt{-K/N})} & \text{if } K < 0. \end{cases}$$

Definition 4.3.1. A metric measure space $(X, \mathsf{d}, \mathfrak{m})$ is said to be a $\mathsf{CD}(K, N)$ space (or to satisfy the $\mathsf{CD}(K, N)$ condition) if for every pair of measures $\mu_0 = \rho_0\mathfrak{m}, \mu_1 = \rho_1\mathfrak{m} \in \mathscr{P}_2(X)$, absolutely continuous with respect to $\mathfrak{m}$, there exists a W_2-geodesic $(\mu_t)_{t\in[0,1]}$ connecting them and induced by $\eta \in \mathscr{P}(\mathrm{Geo}(X))$, such that for every $t \in [0, 1]$ $\mu_t = \rho_t\mathfrak{m} \ll \mathfrak{m}$ and the following inequality holds for every $N' \geq N$ and every $t \in [0, 1]$

$$\int_X \rho_t^{1-\frac{1}{N'}} \,\mathrm{d}\mathfrak{m} \geq \int_{X\times X} \left[\tau_{K,N'}^{(1-t)}(\mathsf{d}(x, y))\rho_0(x)^{-\frac{1}{N'}} + \tau_{K,N'}^{(t)}(\mathsf{d}(x, y))\rho_1(y)^{-\frac{1}{N'}}\right] \mathrm{d}\pi(x, y), \tag{4.3.1}$$

where $\pi = (e_0, e_1)_\#\eta$.

In general, the CD condition is not very easy to disprove, however when the reference space is an interval $I \subseteq \mathbb{R}$ the following lemma, whose proof can be find in [CM21, Lemma A.5], provides a nice strategy.

Lemma 4.3.2. *Let $I \subset \mathbb{R}$ be an interval and let $h : I \to \mathbb{R}$ be a measurable function such that $(I, |\cdot|, h\mathscr{L}^1)$ is a $\mathsf{CD}(K, N)$ space. Then at any point x in the interior of I where h is twice differentiable it holds that*

$$(\log h)''(x) + \frac{1}{N-1}\left((\log h)'(x)\right)^2 \leq -K. \tag{4.3.2}$$

Remark 4.3.3. This lemma holds also for $N = +\infty$, where now the left-hand side of (4.3.2) has to be intended as $(\log h)''(x)$ (for the definition of $\mathsf{CD}(K, \infty)$ space, see [Stu06a]).

In fact, in order to disprove that the space $(I, |\cdot|, h\mathscr{L}^1)$ satisfies $\mathsf{CD}(K, N)$ is sufficient to find a point x in the interior of I such that h is twice differentiable in x and

$$(\log h)''(x) + \frac{1}{N-1}\left((\log h)'(x)\right)^2 > -K.$$

Notice also that, if we manage to prove that

$$(\log h)''(x) > -K, \tag{4.3.3}$$

we automatically show that $(I, |\cdot|, h\mathscr{L}^1)$ does not satisfy $\mathsf{CD}(K,N)$ for every $N \in (1,+\infty]$. This observation will be fundamental in the following, especially in combination with the one-dimensional localization results we are now going to present.

4.3.2 One-dimensional localization

In this subsection we present a suitable adaptation of the one-dimensional characterization of the CD condition. This property, called $\mathsf{CD}^1(K,N)$ condition, has been studied in the general framework of essentially non-branching metric measure spaces with a curvature-dimension bound in [Cav14, CM17b, CM20, CM21]. We provide a local version of such characterization, that allows us to take advantage of the local structure of almost-Riemannian manifolds.

We recall a general result regarding disintegration of measures. Given a measurable space $(R, \mathscr{R})$, and a function $\mathfrak{Q} : R \to Q$ to a general set Q, we endow Q with the push forward σ-algebra $\mathscr{Q}$ of $\mathscr{R}$, i.e. the biggest σ-algebra on Q such that $\mathfrak{Q}$ is measurable. Moreover, given a finite (non-null) measure ρ on $(R, \mathscr{R})$, consider the measure $\mathfrak{q} := \mathbb{Q}_{\#}\rho$ on $(Q, \mathscr{Q})$.

Definition 4.3.4. A disintegration of ρ consistent with $\mathfrak{Q}$ is a map $Q \ni q \mapsto \rho_q \in \mathscr{P}(R)$ such that the following hold:

1. for all $B \in \mathscr{R}$, $\rho_\cdot(B)$ is $\mathfrak{q}$-measurable,
2. for all $B \in \mathscr{R}$, $C \in \mathscr{Q}$, we have

$$\rho\left(B \cap \mathfrak{Q}^{-1}(C)\right) = \int_C \rho_q(B)\, \mathrm{d}\mathfrak{q}(q).$$

A disintegration is called *strongly consistent* with respect to $\mathfrak{Q}$ if, in addition, for all $q \in Q$ it holds that $\rho_q(\mathfrak{Q}^{-1}(q)) = 1$.

Theorem 4.3.5 ([CM17b, Thm. 2.8])**.** *Let $(R, \mathscr{R})$ be a countably generated measurable space and ρ be a finite measure on it. Assume there exists a partition of R as*

$$R = \bigcup_{q \in Q} R_q,$$

denote by $\mathfrak{Q} : R \to Q$ the quotient map and by $(Q, \mathscr{Q}, \mathfrak{q})$ the quotient measure space. If $(Q, \mathscr{Q}) = (X, \mathscr{B}(X))$ where X is a Polish space and $\mathscr{B}(X)$ denotes its Borel σ-algebra, then there exists a unique strongly consistent disintegration $q \mapsto \rho_q$ with respect to $\mathfrak{Q}$.

Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space and fix an open subset $\Omega \subset X$ with $0 < \mathfrak{m}(\Omega) = \mathfrak{m}(\bar\Omega) < \infty$. Let $u : \bar\Omega \to \mathbb{R}$ be a 1-Lipschitz function, define

$$\Gamma_u := \{(x,y) \in \bar\Omega \times \bar\Omega \mid u(x) - u(y) = \mathsf{d}(x,y)\}$$

and its transpose $\Gamma_u^{-1} := \{(x,y) \in \bar\Omega \times \bar\Omega \mid (y,x) \in \Gamma_u\}$. Consequently, we introduce the transport relation R_u and the transport set T_u as

$$R_u := \Gamma_u \cup \Gamma_u^{-1} \quad \text{and} \quad T_u := \mathtt{p}_1(R_u \setminus \{(x,y) \in \bar\Omega \times \bar\Omega \mid x = y\}),$$

where p_1 denotes the projection on the first factor. Although this is not always the case, if we assume that R_u is an equivalence relation, we may partition the set $\bar{\Omega}$. Letting Q be the set of equivalence classes and $\mathfrak{Q} : \bar{\Omega} \to Q$ the quotient map, we can write

$$\bar{\Omega} = \bigcup_{q \in Q} \gamma_q,$$

where $\gamma_q := \{x \in \bar{\Omega} \mid \mathfrak{Q}(x) = q\}$ for every $q \in Q$. With the quotient map we can endow Q with the quotient σ-algebra $\mathscr{Q}$, that is the finest σ-algebra on Q for which $\mathfrak{Q}$ is measurable. We introduce the following definition to obtain a local version of the one-dimensional localization of [CM17b], which better fits the setting of almost-Riemannian geometry, where we have a good local description of geodesics. In [CM17b], the authors define a global partition starting from a globally defined 1-Lipschitz function, see Remark 4.3.8.

Definition 4.3.6. We say that a 1-Lipschitz function $u : \bar{\Omega} \to \mathbb{R}$ induces a *one-dimensional partition* of $\bar{\Omega}$ if

1. R_u is an equivalence relation and $\mathfrak{m}(\bar{\Omega} \setminus T_u) = 0$,
2. for every $q \in Q$, the set $\gamma_q \subset \bar{\Omega}$ is the image of a geodesic of (X, d),
3. for every $q \in Q$ there exists $x \in \bar{\Omega}$ such that $\mathfrak{Q}(x) = q$ and $u(x) = 0$.

If u induces a one-dimensional partition, then, in particular, we can choose $Q = \{u = 0\}$. Indeed, the point $x \in \bar{\Omega}$ satisfying (3) of Definition 4.3.6 is unique. Define the ray map

$$g : \mathrm{Dom}(g) \subset Q \times \mathbb{R} \to \bar{\Omega}$$

by imposing that

$$\mathrm{graph}(g) := \{(q, t, x) \in Q \times \mathbb{R} \times \bar{\Omega} \mid \mathfrak{Q}(x) = q, u(x) = t\}.$$

The ray map g is Borel and bijective, its inverse is

$$\bar{\Omega} \ni x \mapsto g^{-1}(x) := (\mathfrak{Q}(x), u(x)).$$

Moreover, for every $q \in Q$ the map $t \mapsto g(q, \cdot)$ is an isometry, and consequently $\mathcal{H}^1 \llcorner \gamma_q = \mathcal{H}^1 \llcorner \{g(q,t) \mid t \in I_q\} = g(q, \cdot)_\# \mathscr{L}^1$, where $I_q := \mathrm{Dom}(g(q, \cdot))$. Theorem 4.3.5 ensures that there exists a unique strongly consistent disintegration of $\mathfrak{m} \llcorner \Omega$:

$$\mathfrak{m} \llcorner \Omega = \int_Q \mathfrak{m}_q \, \mathrm{d}\mathfrak{q}(q),$$

where $\mathfrak{m}_q$ is a measure concentrated on γ_q and we recall that $\mathfrak{q} := \mathfrak{Q}_\#(\mathfrak{m} \llcorner \Omega)$.

Proposition 4.3.7. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a essentially non-branching metric measure space satisfying the $\mathsf{CD}(K, N)$ condition, for some $K \in \mathbb{R}$ and $N \in (1, \infty)$. Let $\Omega \subset X$ be open and such that $0 < \mathfrak{m}(\Omega) = \mathfrak{m}(\bar{\Omega}) < \infty$ and let $u : \bar{\Omega} \to \mathbb{R}$ be a 1-Lipschitz function providing a one-dimensional partition. Then:*

1. *for $\mathfrak{q}$-a.e. $q \in Q$, the measure $\mathfrak{m}_q$ is absolutely continuous with respect to $\mathcal{H}^1 \llcorner \gamma_q$, namely there exists $h_q : I_q \to [0, \infty]$ such that $\mathfrak{m}_q = g(q, \cdot)_\# \, (h_q \cdot \mathscr{L}^1)$,*
2. *for $\mathfrak{q}$-a.e. $q \in Q$, $(I_q, |\cdot|, h_q \mathscr{L}^1)$ is a $\mathsf{CD}(K, N)$ space.*

Proof. The proof of this proposition can be done by adapting the classical global approach to the space $(\bar\Omega, \mathsf{d}, \mathfrak{m}\llcorner\Omega)$, see in particular [Cav14, Sec. 6] for point (1) and [CM17b, Thm. 4.2] for point (2). We point out that this space is not necessarily geodesic, hence we can not conclude that it satisfies the $\mathsf{CD}(K,N)$ condition and simply apply the known results. However, observe that, in order to deduce properties of the disintegration induced by u, it is enough to study Wasserstein geodesics that follow its transport rays. Since u induces a one-dimensional partition in $\bar\Omega$ in the sense of Definition 4.3.6, all the transport rays are contained in $\bar\Omega$ and the CD condition (4.3.1) holds along such Wasserstein geodesics. For this reason, we can repeat the standard arguments verbatim, obtaining the result. □

Remark 4.3.8. In the classical theory of [CM17b], starting from a *globally defined* 1-Lipschitz function on a $\mathsf{CD}(K,N)$ essentially non-branching metric measure space, the authors build a one-dimensional partition of the whole space, up to a negligible set, and disintegrate the measure accordingly. Then, the densities in the disintegration satisfy the $\mathsf{CD}(K,N)$ condition, providing the one-dimensional characterization $\mathsf{CD}^1(K,N)$. In this setting, there is no need for the additional properties of Definition 4.3.6 on the 1-Lipschitz function u.

In particular, given a 1-Lipschitz function $u : X \to \mathbb{R}$, we can introduce the transport relation R_u and the transport set T_u as before (with X in place of $\bar\Omega$) and denote by $\Gamma_u(x)$ the section of Γ_u through x in the first coordinate. Then, we define the set of forward and backward branching points as

$$A^+ := \{x \in T_u \mid \exists y, z \in \Gamma_u(x), (y,z) \notin R_u\}$$
$$A^- := \{x \in T_u \mid \exists y, z \in \Gamma_u^{-1}(x), (y,z) \notin R_u\}.$$

Finally, we define the non-branched transport set and the non-branched transport relation as

$$T_u^{nb} := T_u \setminus (A^+ \cup A^-), \qquad \text{and} \qquad R_u^{nb} := R_u \cap (T_u^{nb} \times T_u^{nb}).$$

On the one hand, as shown in [Cav14], the essentially non-branching assumption ensures that R_u^{nb} is an equivalence relation on T_u^{nb} and for $\mathfrak{q}$-a.e. $q \in Q$, γ_q is isometric to a closed interval of $\mathbb{R}$. On the other hand, if $(X, \mathsf{d}, \mathfrak{m})$ also satisfies the $\mathsf{CD}(K,N)$ condition, the set $T_u \setminus T_u^{nb}$ is $\mathfrak{m}$-negligible, cf. [CM17b, Thm. 3.4]. It is then possible to obtain a global result, analogous to Proposition 4.3.7.

Remark 4.3.9. Note that showing (4.3.3) for $K \in \mathbb{R}$ actually implies that h_q can not be a $\mathsf{CD}(K,\infty)$ density. However, for a metric measure space $(X, \mathsf{d}, \mathfrak{m})$, it is not known whether the $\mathsf{CD}(K,\infty)$ condition can be characterized with one-dimensional disintegrations.

4.4 A general strategy for disproving the CD condition

The 1-Lipschitz function whose disintegration allows us to disprove the CD condition will be a localized version of the (signed) distance function from a hypersurface Σ. Indeed with this choice we are able to compute explicitly the one-dimensional marginals and to exploit the existence of characteristic points.

4.4.1 Existence of normal coordinates

Let M be an almost-Riemannian manifold of dimension $n+1$. We build a convenient set of coordinates around a point in the singular region. This result can be regarded as a generalization of [ABB20, Prop. 9.8].

Lemma 4.4.1 (Normal coordinates)**.** *Let M be an almost-Riemannian manifold of dimension $n+1$, let $\mathcal{Z} \subset M$ the set of singular points and let $p_0 \in \mathcal{Z}$. Then, there exists a set of coordinates $\varphi = (x, z) \colon \mathcal{U} \to M$ centered at p_0 and a local orthonormal frame $X_0, \dots, X_n$ for the almost-Riemannian structure on $\mathcal{U}$ such that:*

$$X_0 = \partial_x, \qquad X_i = \sum_{j=1}^{n} a_{ij}(x,z)\partial_{z_j}, \quad \forall\, i = 1, \dots, n, \tag{4.4.1}$$

where a_{ij} are smooth functions in $\mathcal{U}$. Moreover, denoting by $A(x,z) = (a_{ij}(x,z))_{i.j}$, we have $\det A(0, \mathbf{0}) = 0$.

The proof of Lemma 4.4.1 follows from the existence of a tubular neighborhood around non-characteristic points. We require a preliminary result.

Lemma 4.4.2. *Let M be an almost-Riemannian manifold and let $p_0 \in M$. Then, there exists a hypersurface $W \subset M$ such that $p_0 \in W \setminus C(W)$.*

Proof. We assume by contradiction that $p_0 \in M$ is a characteristic point for any hypersurface $W \subset M$ passing through p_0. By definition of characteristic point, this means that $\mathcal{D}_{p_0} \subset T_{p_0}W$ for any such W. In turn, this implies that $\mathcal{D}_{p_0} = \{0\}$, contradicting the bracket-generating assumption. □

Proof of Lemma 4.4.1. Using Lemma 4.4.2, we find an embedded hypersurface $W \subset M$ such that $p_0 \in W \setminus C(W)$. We use the almost-Riemannian normal exponential map to define the desired coordinates. Indeed, let $v \in C^\infty(M)$ be a local defining function as in (4.2.7). Then, by Lemma 4.2.9, there exist $\varepsilon_{p_0} > 0$ and a neighborhood $\mathcal{O}_{p_0} \subset W$ of p_0 such that

$$G \colon (-\varepsilon_{p_0}, \varepsilon_{p_0}) \times \mathcal{O}_{p_0} \to M, \qquad G(s,p) = \exp_p(s\lambda(p)),$$

is a diffeomorphism on its image, where $\lambda(p)$ satisfies (4.2.9) with $2H(\lambda(p)) = 1$. Moreover, the local signed distance function δ_v is smooth in $G((-\varepsilon_{p_0}, \varepsilon_{p_0}) \times \mathcal{O}_{p_0})$ and

$$G_* \partial_{s|(s,p)} = \nabla \delta_v(G(s,p)), \qquad \forall\, (s,p) \in (-\varepsilon_{p_0}, \varepsilon_{p_0}) \times \mathcal{O}_{p_0}.$$

Thus, fixing any set of coordinates $(z_1, \dots, z_n)$ for $\mathcal{O}_{p_0}$, and relabelling $s = x$, the coordinates (x, z) satisifies (4.4.1). Finally, since $p_0 \in \mathcal{Z}$, the vector fields $\{X_0, \dots, X_n\}$ are linearly dependent at p_0, meaning that the matrix $A(x,z)$ has zero determinant at $(0, \mathbf{0})$. □

Remark 4.4.3. From now on, without loss of generality, whenever we fix a set of coordinates, we will assume that the domain of the chart is the whole $\mathbb{R} \times \mathbb{R}^n$.

4.4.2 Assumptions on the almost-Riemannian structure

Let M be an almost-Riemannian manifold of dimension $n+1$, and let $\mathcal{Z} \subset M$ be the set of singular points. Let $p_0 \in \mathcal{Z}$ and let $\Sigma \subset M$ be a hypersurface. To proceed with our general construction, we need two assumptions on the almost-Riemannian structure: in coordinates $(x,z) \in \mathbb{R} \times \mathbb{R}^n$ centered at p_0 given by Lemma 4.4.1, we require that

i) the hypersurface Σ consists of Riemannian points except when $x = 0$ and has a characteristic point at the origin, i.e.

$$\Sigma \cap \mathcal{Z} \subset \{x = 0\} \quad \text{and} \quad (0, \mathbf{0}) \in C(\Sigma); \tag{\textbf{H1}}$$

ii) let $\mathfrak{m}$ be any smooth positive measure on M, then

$$\mathfrak{m}(\mathcal{Z}) = 0. \tag{\textbf{H2}}$$

Here by smooth positive measure, we mean a measure with strictly positive and smooth density with respect to the Lebesgue measure in coordinates.

Remark 4.4.4. Let us comment on why we need these assumptions. The first one is necessary to have a good local description of the marginals *outside* the set $\{x = 0\}$, in order to exploit the presence of a characteristic point *only* at the origin, cf. Lemma 4.2.9. The second one is necessary in order to ensure the essentially non-branching property, cf. Lemma 4.4.7, and to characterize the marginals in the disintegration, cf. (4.4.5).

Remark 4.4.5. We remark that assumption (**H2**) is not always guaranteed as the next example shows. Consider $C \subset [0, 1]$ a closed subset with positive Lebesgue measure and empty interior and let $f \in C^\infty(\mathbb{R})$ such that $f_{|C} \equiv 0$. Then, define the structure on $\mathbb{R}^4$ with global orthonormal frame:

$$X_0 = \partial_x, \qquad X_1 = \partial_{z_1} - \frac{z_2}{2}\partial_{z_3}, \qquad X_2 = \partial_{z_2} + \frac{z_1}{2}\partial_{z_3}, \qquad X_3 = f(x)\partial_{z_3}.$$

As one can check, the Hörmander condition is verified and the local minimal bundle rank is always 4, since C has empty interior. Thus, the structure is almost-Riemannian. Now fix $\mathfrak{m} = \mathscr{L}^4$, then the singular set has infinite measure indeed, by (4.4.2)

$$\mathcal{Z} = \{(x, z) \in \mathbb{R}^4 \mid \det A(x, z) = 0\} = \{(x, z) \in \mathbb{R}^4 \mid f(x) = 0\} = C \times \mathbb{R}^3.$$

Here, the matrix A is given by

$$A(x, z) = \begin{pmatrix} 1 & & \\ & 1 & \\ -\frac{z_2}{2} & \frac{z_1}{2} & f \end{pmatrix}.$$

In an analogue way, one can build an example where (**H1**) is not verified. Indeed, in the construction above, it is enough to consider a closed set C with empty interior and with an accumulation point at the origin. Then, the hypersurface $\Sigma = \{z_3 = 0\}$ has a characteristic point at the origin, however it intersects the singular region in $C \times \mathbb{R}^2$.

Notice that, in coordinates (4.4.1), the singular set can be described by the matrix $A = (a_{ij})_{i,j}$, indeed

$$(x, z) \in \mathcal{Z} \quad \text{if and only if} \quad \det A(x, z) = 0. \tag{4.4.2}$$

In particular, along the hypersurface Σ, by (**H1**), we have

$$x \neq 0 \quad \Rightarrow \quad \det A(x, z_1, \dots, z_{n-1}, 0) \neq 0, \tag{4.4.3}$$

since the set $\Sigma \cap \{x \neq 0\}$ consists of Riemannian points. As a consequence, since a Riemannian point is never a characteristic one, $C(\Sigma) \subset \{x = 0\}$. Actually, it is always possible to ensure that $\Sigma = \{z_n = 0\}$ satisfies $(0, \mathbf{0}) \in C(\Sigma)$, so the only condition one should check is (4.4.3).

Lemma 4.4.6. *Let M be an almost-Riemannian manifold and let $p_0 \in \mathcal{Z}$. Then, there exists an hypersurface $\Sigma \subset M$ such that $p_0 \in C(\Sigma)$. Moreover, in coordinates (x, z) as in (4.4.1), up to a rotation, we can choose $\Sigma = \{z_n = 0\}$.*

Proof. Assume by contradiction that $p_0 \in M$ is not a characteristic point for every hypersurface $W \subset M$ passing through p_0. Then, by definition of characteristic point, we deduce that $\mathcal{D}_{p_0}$ must be transversal to $T_{p_0}W$, for every such W, or equivalently

$$\mathcal{D}_{p_0} + T_{p_0}W = T_{p_0}M,$$

for every $W \subset M$ passing trough p_0. As a consequence, $\mathcal{D}_{p_0} = T_{p_0}M$ and thus $r(p_0) = n+1$. This gives a contradiction, since $r(p_0) < n+1$, being $p_0 \in \mathcal{Z}$. Let us show that in coordinates Σ can be chosen as $\{z_n = 0\}$: since $\det A(p_0) = 0$, there exists an invertible matrix $M \in \mathrm{GL}(n+1, \mathbb{R})$ such that the last column of the matrix $A(p_0)M$ consists of zeroes. Then, we introduce the following change of coordinates

$$\psi\colon (x, z) \mapsto (x, \tilde{z}) = (x, M^{\mathsf{T}} z).$$

In the new coordinates, the generating family for the distribution has the following expression:

$$X_0 = \partial_x, \qquad X_i = \sum_{j=1}^{n} a_{ij}(x, z)\partial_{z_j} = \sum_{k,j=1}^{n} a_{ij}(x, \psi^{-1}(z)) m_{jk} \partial_{\tilde{z}_k},$$

having denoted by $M = (m_{ij})_{i,j}$. Thus, (4.4.1) is still valid and, when evaluated at p_0, the matrix describing the generating family has the last column consisting of zeroes. Finally, this implies that the hypersurface $\Sigma = \{\tilde{z}_n = 0\}$ has a characteristic point at p_0. Indeed,

$$\nabla \tilde{z}_n(p_0) = \sum_{i=1}^{n} X_i(\tilde{z}_n) X_i(p_0) = \sum_{i,j=1}^{n} a_{ij}(0, \mathbf{0}) m_{jn} X_i(p_0) = \sum_{i=1}^{n} (A(p_0)M)_{in} X_i(p_0) = 0,$$

since the last column of the matrix $A(p_0)M$ is zero, implying that $\mathcal{D}_{p_0} \subset T_{p_0}\Sigma$. □

Lemma 4.4.7. *Let M be an almost-Riemannian manifold, equipped with a smooth positive measure $\mathfrak{m}$ and satisfying assumptions* (**H2**)*. Then $(M, \mathsf{d}, \mathfrak{m})$ is essentially non-branching.*

Proof. Let $\gamma\colon [0, 1] \to M$ be a minimizing geodesic. Then γ is abnormal if there exists an abnormal extremal lift $\lambda(t) \neq 0$, satisfying (4.2.5). But this implies that

$$\gamma(t) \subset \mathcal{Z}, \qquad \forall\, t \in [0, 1].$$

Hence, if γ is a minimizing geodesic with endpoints in the Riemannian region, i.e.

$$\gamma(0), \gamma(1) \in M \setminus \mathcal{Z}, \tag{4.4.4}$$

then γ must be strictly normal. As showed in [MR20, Cor. 6], a strictly normal geodesic $\gamma\colon [0, 1] \to M$ is branching for some positive time $t \in (0, 1)$ if and only if it contains a non-trivial abnormal subsegment that starts at time 0. Thus, a minimizing geodesic satisfying (4.4.4) can not branch for positive times since $\mathcal{Z}$ is closed. Now, let $\eta \in \mathscr{P}(\mathrm{Geo}(X))$ be a W_2-geodesic joining the measures $\mu_0, \mu_1 \in \mathscr{P}_2(X)$, which are absolutely continuous with respect to the reference measure $\mathfrak{m}$ ($\mu_0, \mu_1 \ll \mathfrak{m}$). In particular, notice that $(e_0)_\# \eta = \mu_0$ and $(e_1)_\# \eta = \mu_1$ and therefore, by (**H2**),

$$\eta(e_0^{-1}(\mathcal{Z})) = \mu_0(\mathcal{Z}) = 0 \quad \text{ and } \quad \eta(e_1^{-1}(\mathcal{Z})) = \mu_1(\mathcal{Z}) = 0.$$

Consequently, the measure η is concentrated on $\mathrm{Geo}(X) \setminus (e_0^{-1}(\mathcal{Z}) \cup e_1^{-1}(\mathcal{Z}))$, which is a non-branching set of geodesics, according to the first part of the proof. □

Remark 4.4.8. Notice that it is possible to build examples of almost-Riemannian manifolds where (**H2**) is verified but there exist branching geodesics. Indeed, consider $\mathbb{R}^4$, with the global orthonormal frame given by

$$X_0 = \partial_x, \qquad X_1 = \partial_{z_1}, \qquad X_2 = \partial_{z_2} + B(z_1, z_2)\partial_{z_3}, \qquad X_3 = x\partial_{z_3},$$

where B is a smooth magnetic potential, defined as in [MR20], namely

$$B(z_1, z_2) = z_1\theta(z_2) + z_2^2\theta(1 - z_2)$$

and $\theta \in C^\infty(\mathbb{R})$ such that $0 \leq \theta \leq 1$, $\theta(r) = 0$ for $r \leq 0$ and $\theta(r) = 1$ for $r \geq 1$. In this situation, we have strictly normal branching geodesic in the singular region. Nevertheless, thanks to Lemma 4.4.7, $(M, \mathsf{d}, \mathfrak{m})$ is essentially non-branching. On the other hand, if the measure of the singular set is positive, it is unclear whether an almost-Riemannian manifold is essentially non-branching.

4.4.3 Choice of the local disintegration

Let us fix coordinates $(x, z) \in \mathbb{R} \times \mathbb{R}^n$ as in (4.4.1). Let $\mathfrak{m}$ be a smooth positive measure on M and let $\Sigma = \{z_n = 0\}$ be defined as Section 4.4.2. Under the assumptions (**H1**) and (**H2**), we consider the signed distance function δ_v from Σ, as defined in (4.2.8), with $v := z_n$. For ease of notation, we denote δ_v simply by δ. By triangle inequality, δ is always 1-Lipschitz on M with respect to d. However, δ develops singularities at a characteristic point, indeed it is only Hölder (and not Lipschitz) with respect to the Euclidean distance of the chart, see [ACS18, Thm. 4.2]. Roughly speaking, such a singularity is related to the fact that the horizontal gradient of δ, which exists almost everywhere (see, [FHK99, Thm. 8]), becomes tangent to Σ as the base point approaches a characteristic point. Our idea is to exploit this behavior to prove that the disintegration associated with δ does not produce $\mathsf{CD}(K, N)$ densities along the transport rays, for any $K \in \mathbb{R}$, $N \geq 1$.

Starting from δ, we build a suitable open and bounded set Ω and we consider the local disintegration of $\mathfrak{m} \llcorner \Omega$ induced by δ, cf. Section 4.3.2. Set $B := B_r((0, \mathbf{0}))$ for some $r > 0$ and define the open and bounded set

$$\Omega := \bigcup_{q_0 \in (\Sigma \setminus C(\Sigma)) \cap B} \gamma_{q_0}, \qquad \text{where} \quad \gamma_{q_0} = G((-f(q_0), f(q_0)) \times \{q_0\}),$$

where G is the map defined in Lemma 4.2.9 and $f \in C^\infty(\Sigma)$ such that $0 < f(q_0) < \varepsilon_{q_0}$, for every $q_0 \in \Sigma \setminus C(\Sigma)$. Note that G is a local diffeomorphism on Ω and, with this choice of f, $\mathfrak{m}(\Omega) = \mathfrak{m}(\bar{\Omega})$.

Then, δ is a 1-Lipschitz function on $\bar{\Omega}$ inducing a one-dimensional partition in the sense of Definition 4.3.6. Indeed $Q = \{\delta = 0\} \cap \bar{\Omega} = \Sigma \cap \bar{B}$ and, for every $q \in Q$, the transport ray γ_q of the disintegration coincides with the minimizing geodesics for δ, which exist by completeness. Moreover, $T_\delta = \bar{\Omega} \setminus C(\Sigma)$ and therefore, $\mathfrak{m}(\bar{\Omega} \setminus T_\delta) = 0$. The quotient map $\mathfrak{Q} \colon \bar{\Omega} \to Q$ can be regarded as a projection on the foot of a geodesic[3], thus $\mathfrak{Q}$ is the inverse of the exponential map, namely

$$G\left(\delta(p), \mathfrak{Q}(p)\right) = p, \qquad \forall\, p \in \bar{\Omega},$$

and G is indeed the ray map associated to the partition. Finally, since Ω is defined by the smooth function f, the measure $\mathfrak{q} = \mathfrak{Q}_\#(\mathfrak{m} \llcorner \Omega)$ is smooth on Σ.

4.4.4 Coordinate expression for the marginals in the disintegration

Using Lemma 4.2.9 and, in particular, the diffeomorphism (4.2.10), we can conveniently represent the one-dimensional densities in the disintegration. Indeed, consider a Riemannian point $q_0 = (\bar{x}, \bar{z}) \in \Sigma$, with $\bar{z}_n = 0$. In particular, thanks to (4.4.3), it is enough to assume $\bar{x} \neq 0$. Then, for every Borel set $C \subset \Omega$, on the one hand we have that

$$\int_{\Omega \cap C} \mathrm{d}\mathfrak{m} = \int_\Sigma \int_{-f(q)}^{f(q)} \chi_{G^{-1}(C)} \, \mathrm{d}(G^*(\mathfrak{m} \llcorner \Omega)),$$

while, on the other hand, making the disintegration explicit and recalling that G is the ray map, we conclude that

$$\begin{aligned} \int_{\Omega \cap C} \mathrm{d}\mathfrak{m} &= \int_\Sigma \mathfrak{m}_q(\Omega \cap C) \, \mathrm{d}\mathfrak{q} \\ &= \int_\Sigma \int_{-f(q)}^{f(q)} \chi_C(G(s, q)) \, \mathrm{d}\mathfrak{m}_q \, \mathrm{d}\mathfrak{q} = \int_\Sigma \int_{-f(q)}^{f(q)} \chi_{G^{-1}(C)}(s, q) h_q(s) \, \mathrm{d}s \, \mathrm{d}\mathfrak{q}. \end{aligned}$$

[3]For any $p \in \bar{\Omega}$, there exists a unique point $\mathsf{f}(p) \in \Sigma \cap \bar{\Omega}$ for which $|\delta(p)| = \mathsf{d}(p, \mathsf{f}(p))$

Thus, having fixed a frame for $T\Sigma$, say $\{v_1,\dots,v_n\}$, the density $h_q(s)$ is given by:

$$\begin{aligned} h_q(s) = h_q(s)ds(\partial_s)\frac{\mathfrak{q}(v_1,\dots,v_n)}{\mathfrak{q}(v_1,\dots,v_n)} &= \frac{G^*\mathfrak{m}(\partial_s,v_1,\dots,v_n)}{\mathfrak{q}(v_1,\dots,v_n)} \\ = \frac{\mathfrak{m}(G_*\partial_s,G_*v_1,\dots,G_*v_n)}{\mathfrak{q}(v_1,\dots,v_n)} &= \frac{\mathfrak{m}(\nabla\delta,G_*v_1,\dots,G_*v_n)}{\mathfrak{q}(v_1,\dots,v_n)}, \end{aligned} \tag{4.4.5}$$

for any $(s,q)\in G^{-1}(\Omega)$, and having used (4.2.11) in the last equality.

Remark 4.4.9. Notice that, from (4.4.5), the one-dimensional densities $h_q(s)$ are smooth functions of $(s,q)\in G^{-1}(\Omega)$. Moreover, they do not depend on the choice of coordinates.

We are going to study the second logarithmic derivative of $h_q(s)$, at $s=0$ and as $q\to 0$, in order to obtain a contradiction with the differential characterization of Lemma 4.3.2. Firstly, notice that since we are performing derivatives in s, we can disregard constant functions in s. Secondly, by definition $\mathfrak{m}$ is a smooth positive measure, i.e.

$$\mathfrak{m} = m(x,z)dxdz, \qquad \text{with} \quad m\in C^\infty(\mathbb{R}\times\mathbb{R}^n), \quad c\le m\le C, \tag{4.4.6}$$

for some $C,c>0$. Moreover, in (4.4.5), as a frame for $T\Sigma$, we can choose the vector fields $\{\partial_x,\partial_{z_1},\dots,\partial_{z_{n-1}}\}$. In conclusion, we obtain the following expression for the one-dimensional density associated with the disintegration:

$$h_q(s) \propto m(G(s,q))dxdz(\nabla\delta, G_*\partial_x,\dots,G_*\partial_{z_{n-1}})_{|(s,q)}. \tag{4.4.7}$$

Then, defining the matrix

$$B_q(s) := (\nabla\delta \mid G_*\partial_x \mid \dots \mid G_*\partial_{z_{n-1}}),$$

where the columns are expressed in coordinates $\{\partial_x,\dots,\partial_{z_n}\}$, the second logarithm derivative at $s=0$ is given by:

$$\begin{aligned} (\log(h_q(s)))''_{|s=0} &= (\log(m(G(s,q)))''_{|s=0} + (\log\det(B_q(s))))''_{|s=0} \\ &= (\log(m(G(s,q)))''_{|s=0} + \operatorname{tr}\left(-(B_q^{-1}(0)B_q'(0))^2 + B_q^{-1}(0)B_q''(0)\right), \end{aligned} \tag{4.4.8}$$

having used Jacobi formula for the determinant of a smooth curve of invertible matrices:

$$\det(B(s))' = \det(B(s))\operatorname{tr}\left(B^{-1}(s)B'(s)\right).$$

Remark 4.4.10. We stress that, for any $q\in Q\setminus C(\Sigma)$, $h_q(\cdot)$ is defined on an open interval I_q containing 0. Thus, the derivative in (4.4.8) makes sense.

4.4.5 Computations for the matrix $B_q(s)$

Proceeding with hindsight, we analyze the term of (4.4.8) involving $B_q(s)$, as in general it will be more singular than the other one. We expand in s its columns and we deduce an expression for the coefficients of the expansion, using the almost-Riemannian Hamiltonian system.

An expression for the trace term in (4.4.8)

We look for an explicit expression for the matrix $B_q(s)$. We may regard $\nabla\delta\in\mathbb{R}^{n+1}$ and $G\colon\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^{n+1}$, therefore expanding in s, there exists smooth functions $\mathfrak{f},\mathfrak{h}\in C^\infty(\Sigma\cap\Omega)$ such that,

$$\begin{aligned} G(s,q) &= q + \nabla\delta(q)s + \frac{1}{2}\mathfrak{f}(q)s^2 + o(s^2), \\ \nabla\delta(G(s,q)) &= \nabla\delta(q) + \mathfrak{f}(q)s + \frac{1}{2}\mathfrak{h}(q)s^2 + o(s^2), \end{aligned} \tag{4.4.9}$$

as $s \to 0$. The relation between the two expansions comes from (4.2.11). Therefore, we obtain the following formulas for $B_q(s)$ and its derivatives at $s = 0$: for the zero order term, we have

$$B_q(0) = (\nabla\delta \mid \partial_x \mid \partial_{z_1} \mid \ldots \mid \partial_{z_{n-1}})_{|q} = \left(\begin{array}{c|c} \nabla\delta(q) & \begin{array}{c} \mathrm{Id}_{n\times n} \\ \hline 0 \ \cdots \ 0 \end{array} \end{array} \right).$$

For the first derivative of $B_q(s)$, we differentiate component by component. Notice that we have to take into account the quantities $\partial_{z_i} G(0, q)$, with $i = 0, \ldots, n-1$[4], therefore we have to differentiate the expansion (4.4.9), namely:

$$\partial_{z_i} G(s, q) = \partial_{z_i} + \partial_{z_i}\nabla\delta(q)s + \frac{1}{2}\partial_{z_i}\mathfrak{f}(q)s^2 + o(s^2),$$

as $s \to 0$, for any $i = 0, \ldots, n-1$, where the derivatives have to be interpreted component by component. Therefore, we obtain:

$$B_q'(0) = (\mathfrak{f} \mid \partial_x \mid \partial_{z_1} \mid \ldots \mid \partial_{z_{n-1}})_{|q} + \sum_{i=0}^{n-1} (\nabla\delta \mid \partial_x \mid \ldots \mid \partial_{z_i}\nabla\delta \mid \ldots \mid \partial_{z_{n-1}})_{|q}.$$

Analogously, we can deduce the expression for the second-order derivative of $B_q(s)$ at $s = 0$:

$$\begin{aligned} B_q''(0) = {} & (\mathfrak{h} \mid \partial_x \mid \partial_{z_1} \mid \ldots \mid \partial_{z_{n-1}})_{|q} + \sum_{i=0}^{n-1} (\nabla\delta \mid \partial_x \mid \ldots \mid \partial_{z_i}\mathfrak{f} \mid \ldots \mid \partial_{z_{n-1}})_{|q} \\ & + 2\sum_{i=0}^{n-2}\sum_{j=i+1}^{n-1} \Big(\nabla\delta \mid \partial_x \mid \ldots \mid \partial_{z_i}\nabla\delta \mid \ldots \mid \partial_{z_j}\nabla\delta \mid \ldots \mid \partial_{z_{n-1}}\Big)_{|q} \\ & + 2\sum_{i=0}^{n-1} (\mathfrak{f} \mid \partial_x \mid \ldots \mid \partial_{z_i}\nabla\delta \mid \ldots \mid \partial_{z_{n-1}})_{|q}. \end{aligned}$$

Inserting the above formulas in the trace term in (4.4.8), we obtain the desired expression, in terms of the quantities $\nabla\delta$, $\mathfrak{f}$ and $\mathfrak{h}$.

Explicit expression for $\nabla\delta$, $\mathfrak{f}$ and $\mathfrak{h}$

In order to obtain an explicit expression for $\nabla\delta$, $\mathfrak{f}$ and $\mathfrak{h}$, we study the Hamiltonian system associated with the almost-Riemannian Hamiltonian

$$H(\lambda) = \frac{1}{2}\sum_{i=0}^{n} \langle \lambda, X_i \rangle^2, \qquad \forall\, \lambda \in T^*M,$$

where $\{X_0, \ldots, X_n\}$ is the local orthonormal frame for the distribution defined in (4.4.1). In coordinates (x, z), the almost-Riemannian metric g on the Riemannian region is represented by the matrix

$$\left(\begin{array}{c|c} 1 & \\ \hline & (A^\intercal A)^{-1} \end{array} \right).$$

Therefore the Hamiltonian in canonical coordinates induced by (x, z) is

$$H(p_x, p_z; x, z) = \frac{1}{2}p_x^2 + \frac{1}{2}p_z^\intercal A^\intercal A(x, z) p_z,$$

[4]Here and below, with a slight abuse of notation, for $i = 0$, we set $z_0 = x$ and $\partial_{z_0} = \partial_x$.

where p_z is a shorthand for $(p_{z_1},\dots,p_{z_n})$. The Hamiltonian system then becomes

$$\begin{cases} \dot{x} = \dfrac{\partial H}{\partial p_x} = p_x & \dot{p}_x = -\dfrac{\partial H}{\partial x} = -\dfrac{1}{2} p_z^{\intercal} \partial_x \left(A^{\intercal} A\right) p_z \\ \dot{z} = \dfrac{\partial H}{\partial p_z} = A^{\intercal} A p_z & \dot{p}_z = -\dfrac{\partial H}{\partial z} = -\dfrac{1}{2} p_z^{\intercal} \partial_z \left(A^{\intercal} A\right) p_z \end{cases} \tag{4.4.10}$$

From Lemma 4.2.9 (cf. also Remark 4.2.11) we know that the unique minimizing geodesic for δ with initial point $q \in \Sigma \cap \Omega$ has unique (up to a sign) initial covector such that:

$$\langle \lambda(q), T_q\Sigma \rangle = 0 \qquad \text{and} \qquad 2H(\lambda(q)) = 1. \tag{4.4.11}$$

Since $T_q\Sigma = \operatorname{span}\{\partial_x, \partial_{z_1}, \dots, \partial_{z_{n-1}}\}$, the first condition in (4.4.11) implies that $\lambda(q) = p_{z_n} dz_n$. In addition, the second condition in (4.4.11) forces $\lambda(q)$ to be of the form:

$$\lambda(q) = \frac{1}{\beta(q)} dz_n, \qquad \text{with} \quad \beta(q)^2 = \sum_{k=1}^{n} a_{kn}(q)^2, \tag{4.4.12}$$

where we choose β to be positive. Thus, denoting by $(x^q(s), z^q(s); p_x^q(s), p_z^q(s))$ the solution to (4.4.10) with initial datum $(\lambda(q); q)$, the minimizing geodesic for δ starting at q is given by:

$$I_q \ni s \mapsto G(s,q) = (x^q(s), z^q(s)), \tag{4.4.13}$$

where I_q is an open interval containing the origin. By (4.2.11) and (4.4.13), we deduce that

$$\nabla\delta(q) = \partial_s G_{|(0,q)} = \left(\dot{x}^q(0), \dot{z}^q(0)\right).$$

Computing derivatives along s of the equality (4.2.11) and recalling the definition of $\mathfrak{f}, \mathfrak{h}$ in (4.4.9), we analogously obtain higher-order expression in terms of the solution to (4.4.10), precisely:

$$\mathfrak{f}(q) = \left(\ddot{x}^q(0), \ddot{z}^q(0)\right), \qquad \mathfrak{h}(q) = \left(\dddot{x}^q(0), \dddot{z}^q(0)\right). \tag{4.4.14}$$

We refer to Appendix 4.7 for the explicit expression of $\mathfrak{f}$ and $\mathfrak{h}$.

4.4.6 Contradicting the CD condition

The idea is to exploit the presence of a characteristic point for Σ at the origin to conclude that

$$\left(\log(h_q(s))\right)''_{|s=0} \xrightarrow{(x,\mathbf{0})\to(0,\mathbf{0})} +\infty, \tag{4.4.15}$$

proving (4.3.3) for every $K \in \mathbb{R}$, up to taking x sufficiently small. Keeping in mind (4.4.8), we anticipate that the term providing the desired pathology will be

$$\left(\log\det(B_q(s))\right)''_{|s=0}.$$

Observe that, according to Section 4.3, in order to disprove the $\mathsf{CD}(K,N)$ condition for every $K \in \mathbb{R}$ and $N \in (1,+\infty)$ we need to show that, given any $K \in \mathbb{R}$, it holds

$$\left(\log(h_q(s))\right)''_{|s=0} > -K,$$

for every q in a $\mathfrak{q}$-positive set. However, since the function $(s,q) \to h_q(s)$ is smooth (see Remark 4.4.9) it is sufficient to prove (4.4.15).

Notice that the initial covector for the minimizing geodesic from Σ in (4.4.12) is singular at $q=(0,\mathbf{0})$. More precisely, since $(0,\mathbf{0})\in C(\Sigma)$ and Σ is the level set of $v(x,z)=z_n$, we have that

$$0=\nabla z_{n|(0,\mathbf{0})}=\sum_{i=1}^{n}X_i(z_n)X_{i|(0,\mathbf{0})}=\sum_{i,j=1}^{n}a_{ij}(0,\mathbf{0})a_{in}(0,\mathbf{0})\partial_{z_j},$$

meaning that the function $\beta(q)$ defined in (4.4.12) vanishes at q *if and only if* $q\in C(\Sigma)$. In particular, it vanishes at the origin, making the initial covector singular at $q=(0,\mathbf{0})$. Moreover, solving the Hamiltonian system, we deduce that

$$\nabla\delta(q)=(\dot{x}^q(0),\dot{z}^q(0))=(0,A^\intercal A\beta^{-1}(q)\partial_{z_n})=\left(0,\beta^{-1}(q)\sum_{k=1}^{n}a_{ki}(q)a_{kn}(q)\right). \tag{4.4.16}$$

Remark 4.4.11. On the one hand, all the components of $\nabla\delta(q)$, but the first and last, are singular at $q=(0,\mathbf{0})$, as fast as the initial covector (4.4.12). On the other hand, since the last component of $\nabla\delta(q)$ is exactly $\beta(q)$ which tends to 0 as $q\to(0,\mathbf{0})$, formally $\nabla\delta(q)$ becomes tangent to Σ at the characteristic point.

In particular, we see that $\nabla\delta(q)$ is singular at the origin and the same goes for the functions $\mathfrak{f}(q)$, $\mathfrak{h}(q)$. Replacing their explicit expressions in (4.4.8), we will be able to prove (4.4.15).

Remark 4.4.12. The procedure described in this section for disproving the CD condition is constructive and the algorithm has been implemented in the software *Mathematica*. The code is available online, see [MR22].

4.5 2-dimensional almost-Riemannian manifolds do not satisfy CD

In this section, we apply our general strategy to show that 2-dimensional almost-Riemannian manifolds do not satisfy any curvature-dimension condition. The reason why we are able to perform explicit computations is related to the better regularity properties of δ, when $\dim M=2$, cf. Remark 4.4.11.

Let M be an almost-Riemannian manifold of dimension 2, with non-empty singular region $\mathcal{Z}\subset M$. We recall the following local description of a general 2-dimensional almost-Riemannian manifold which holds *without* any assumption on the structure of the singular set, see [ABS08, Lem. 17].

Lemma 4.5.1. *Let M be an almost-Riemannian manifold. Then, for every point $q_0\in M$, there exists a set of coordinates $\varphi=(x,z)\colon\mathcal{U}\to M$, centered at q_0, such that a local orthonormal frame for the distribution is given by*

$$X=\partial_x,\qquad Y=f(x,z)\partial_z.$$

where $f\colon\mathcal{U}\to\mathbb{R}$ is a smooth function. Moreover,

i) the integral curves of X are normal extremals, as in (4.2.4)*;*

ii) let s be the step of the structure at q_0. If $s=1$ then $f(0,0)\neq0$. If $s\geq2$, we have

$$f(0,0)=0,\ \dots,\ \frac{\partial^{s-2}f}{\partial x^{s-2}}(0,0)=0,\ \frac{\partial^{s-1}f}{\partial x^{s-1}}(0,0)\neq0. \tag{4.5.1}$$

Remark 4.5.2. This Lemma improves Lemma 4.4.1 since we can give additional condition on the function $f(x,z)=\det A(x,z)$, using the Hörmander condition.

In the 2-dimensional case, assumption (**H2**) is always verified, see [ABB20, Thm. 9.14]. For what concerns assumption (**H1**), we have the following lemma.

Lemma 4.5.3. *Let M be a 2-dimensional almost-Riemannian manifold and let $q_0 \in \mathcal{Z}$. Consider the curve in normal coordinates $\Sigma = \{z = 0\}$. Then, up to restricting the chart, $\Sigma \cap \mathcal{Z} = C(\Sigma) = \{(0,0)\}$.*

Proof. Recall that if $\Sigma = \{v = 0\}$, for $v \in C^\infty$ with never-vanishing differential, then

$$p \in C(\Sigma) \quad \Leftrightarrow \quad \nabla v(p) = 0,$$

where ∇u denotes the horizontal gradient of v. In particular, in the normal coordinates given by Lemma 4.5.1, the singular region is $\mathcal{Z} \cap \mathcal{U} = \{(x,z) \mid f(x,z) = 0\}$, thus setting $v(x,z) = z$,

$$p = (x,0) \in C(\Sigma) \quad \Leftrightarrow \quad f(x,0) = 0.$$

Since $q_0 \in \mathcal{Z}$, then $\dim(\mathcal{D}_{q_0}) < 2$ and the almost-Riemannian structure has step $s \geq 2$ at q_0. Thus $f(0,0) = 0$ and consequently $p = q_0 \in C(\Sigma)$. On the other hand, if $0 < |x| < \varepsilon$, $f(x,0) \neq 0$. Indeed, by the vanishing condition (4.5.1) on f, we can expand $f(x,0)$ as a Taylor series at $x = 0$, obtaining

$$f(x,0) = \frac{\partial^{s-1} f}{\partial x^{s-1}}(0,0)x^{s-1} + o(x^{s-1}), \qquad \text{as } x \to 0.$$

where the leading term is not zero. This implies that there exists a smooth function $r \in C^\infty(-\varepsilon,\varepsilon)$, such that $r(x) \neq 0$, for every $x \in (-\varepsilon,\varepsilon)$ and $f(x,0) = r(x)x^{s-1}$, which never vanishes on $\Sigma \cap \mathcal{U} \setminus \{q_0\}$, up to restricting the domain of the chart $\mathcal{U}$. □

Now, thanks to Lemmas 4.5.1 and 4.5.3, we can follow the general strategy (cf. Section 4.4) to disprove the CD condition. First of all, notice that the matrix $A = (f(x,z))$ has only one entry, so the Hamiltonian system is greatly simplified. More precisely, the initial covector (4.4.12) becomes:

$$\lambda(x) = \frac{1}{f(x,0)} dz, \qquad \forall\, x \neq 0.$$

Thus, as one can check using (4.4.10), (4.4.14) and (4.4.16) we have

$$\begin{aligned} \nabla\delta(x) = (0, f(x,0)), \qquad \mathfrak{f}(x) &= \left(-\frac{\partial_x f}{f}, f\partial_z f \right)_{|(x,0)}, \\ \mathfrak{h}(x) &= \left(\star, -\frac{2(\partial_x f)^2}{f} + f(\partial_z f)^2 + f^2 \partial_z^2 f \right)_{|(x,0)}. \end{aligned} \tag{4.5.2}$$

Here we have omitted the first component of $\mathfrak{h}(x)$, since we will not need it.

Second of all, we can replace the quantities (4.5.2) in the matrix $B_q(s)$, defined in (4.4.7). After a long but routine computation, we obtain the following expression for the logarithmic second derivative of $\det B_q(s)$ at $s = 0$, namely

$$(\log\det B_x(s))''(0) = \left(f\partial_z^2 f + \frac{(\partial_x f)^2 - f\partial_x^2 f}{f^2} \right)_{|(x,0)}.$$

We are in position to prove the main result of this section.

Theorem 4.5.4. *Let M be a complete 2-dimensional almost-Riemannian manifold and let $\mathfrak{m}$ be any smooth positive measure on M. Then, the metric measure space $(M, \mathsf{d}, \mathfrak{m})$ does not satisfy the $\mathsf{CD}(K,N)$ condition for any $K \in \mathbb{R}$ and $N \in (1,+\infty)$.*

Proof. As explained in Section 4.4, we have to show that the quantity $(\log h_x(s))''(0)$ diverges at $+\infty$ as $x \to 0$. Recall that, by the proof of Lemma 4.5.3, there exists a never-vanishing function $r \in C^\infty(-\varepsilon, \varepsilon)$ such that

$$f(x,0) = r(x)x^{s-1}, \qquad \text{with} \quad f, r \in C^\infty(\mathcal{U}).$$

Therefore, using the smoothness of both f and r, we deduce that

$$\begin{aligned}(\log \det B_x(s))''(0) &= \frac{(\partial_x f(x,0))^2 - f(x,0)\partial_x^2 f(x,0)}{f(x,0)^2} + O(1) \\ &= \left(\frac{s-1}{x^2} + \frac{\partial_x r(x)^2 - r(x)\partial_x^2 r(x)}{r(x)^2}\right) + O(1) \\ &= \frac{s-1}{x^2} + O(1),\end{aligned} \tag{4.5.3}$$

which diverges to $+\infty$ as $x \to 0$, since $(0,0) \in \mathcal{Z}$ and therefore $s > 1$. Moreover, let us remark that the singularity is polynomial of order -2. We are left to take care of the first term in (4.4.8): by a direct computation and using (4.4.6), one can check that

$$\begin{aligned}\left|(\log m(G(s,x)))''(0)\right| \leq C_0\left(|\partial_s G(0,x)|_e^2 + |\partial_s^2 G(0,x)|_e\right) \leq C_1 + C_2\left|\frac{\partial_x f(x,0)}{f(x,0)}\right| \\ = C_1 + C_2\left|\frac{s-1}{x} + \frac{\partial_x r(x)}{r(x)}\right|,\end{aligned} \tag{4.5.4}$$

where $|\cdot|_e$ denotes the Euclidean norm of $\mathbb{R}^2$. Since the singularity in (4.5.4) is polynomial of order 1, it is negligible compared to the one in (4.5.3), and we conclude that:

$$(\log \det h_x(s))''(0) \xrightarrow{x \to 0} +\infty,$$

disproving the $\mathsf{CD}(K,N)$ condition for any $K \in \mathbb{R}$ and $N \in (0,+\infty)$, as desired. □

Remark 4.5.5. A similar argument can be carried out for *generic* 3-dimensional almost-Riemannian manifolds, in the sense of [ABS08, Def. 2]. Indeed, in this situation we have a convenient description of a local orthonormal frame, and of the matrix A, cf. [BCGM15, Thm. 2].

4.6 Strongly regular almost-Riemannian manifolds do not satisfy CD

In this section, we prove that strongly regular almost-Riemannian manifolds do not satisfy any curvature-dimension condition. Strongly regular almost-Riemannian manifolds have been studied in [PRS18, CPR19]. In this setting, we can deal with the complexity of the computations thanks to a nice local description of the singularities of the structure. We recall the following definition.

Definition 4.6.1. Let M be a n-dimensional almost-Riemannian manifold. Assume that the singular set $\mathcal{Z} \subset M$ is an embedded hypersurface without characteristic points. Then, for any $q_0 \in \mathcal{Z}$, there exist local coordinates (x,z) centered at q_0 such that $\mathcal{Z} = \{x = 0\}$ in coordinates, and condition (4.4.1) is verified, namely a local orthonormal frame for the distribution is given by

$$X_0 = \partial_x, \qquad X_i = \sum_{j=1}^n a_{ij}(x,z)\partial_{z_j}, \quad \forall\, i = 1,\dots,n,$$

for some smooth functions a_{ij}, so that, denoting by $A = (a_{ij})_{i,j}$,

$$\det A(x,z) = 0 \qquad \text{if and only if} \qquad x = 0.$$

We say that M is a *strongly regular* almost-Riemannian manifold, if there exists $l \in \mathbb{N}$ such that

$$a_{ij}(x,z) = x^l \hat{a}_{ij}(x,z) \qquad \text{with} \quad \det(\hat{a}_{ij})(0,z) \neq 0, \tag{4.6.1}$$

for all $(0,z)$ in the domain of the chart.

Remark 4.6.2. Although being formulated in coordinates, the notion of a strongly regular almost-Riemannian structure on M is intrinsic. In particular, condition (4.6.1), as well as the order l, do not depend neither on the choice of $q_0 \in \mathcal{Z}$ nor on the coordinates (x,z), see [PRS18] for further details.

In order to apply our general strategy, we have to ensure that conditions (**H1**) and (**H2**) are verified. The former is a consequence of the very definition of strongly regular almost-Riemannian structure and we pick Σ as in Lemma 4.4.6 so that also the latter condition is satisfied. We proceed by computing the second logarithmic derivative of the one-dimensional densities,

$$\begin{aligned}(\log\det B_q(s))''_{|s=0} = \frac{1}{\beta}\Bigg[&\mathfrak{h}_n + \beta\partial_x\mathfrak{f}_0 + \sum_{i=1}^{n-1}(\beta\partial_{z_i}\mathfrak{f}_i - \beta_i\partial_{z_i}\mathfrak{f}_n) - 2\mathfrak{f}_0\partial_x\beta \\ &+ 2\sum_{i=1}^{n-1}(\mathfrak{f}_n\partial_{z_i}\beta_i - \mathfrak{f}_i\partial_{z_i}\beta) + 2\sum_{0<i<j<n}\det\begin{pmatrix}\partial_{z_i}\beta_i & \partial_{z_j}\beta_i & \beta_i \\ \partial_{z_i}\beta_j & \partial_{z_j}\beta_j & \beta_j \\ \partial_{z_i}\beta & \partial_{z_j}\beta & \beta\end{pmatrix}\Bigg] \\ &- \frac{1}{\beta^2}\left(\mathfrak{f}_n + \sum_{i=1}^{n-1}(\partial_{z_i}\beta_i\beta - \beta_i\partial_{z_i}\beta)\right)^2,\end{aligned} \tag{4.6.2}$$

where β_i, $\mathfrak{f}_i$, $\mathfrak{h}_i$ denote the components of $\nabla\delta$, $\mathfrak{f}$ and $\mathfrak{h}$ respectively. This computation follows from the trace term in (4.4.8), using the property that the first component of $\nabla\delta$ is identically zero, cf. (4.4.16).

Theorem 4.6.3. *Let M be a complete strongly regular almost-Riemannian manifold and let $\mathfrak{m}$ be any smooth positive measure on M. Then, the metric measure space $(M, \mathsf{d}, \mathfrak{m})$ does not satisfy the $\mathsf{CD}(K,N)$ condition for any $K \in \mathbb{R}$ and $N \in (1,+\infty)$.*

Proof. As in the proof of Theorem 4.5.4, we have to show that the quantity $(\log h_q(s))''(0)$ diverges at $+\infty$ as $q \to 0$ along Σ. To do that, the idea is to highlight the most singular terms in x, namely those where a derivative in x appears. Let us discuss the order in x of the quantities in (4.6.2), using as well formulas from Appendix 4.7. Firstly, since M is strongly regular, (4.6.1) holds and we have

$$\beta(x,z)^2 = \sum_{i=1}^n a_{kn}^2(x,z) = x^{2l}\sum_{i=1}^n \hat{a}_{kn}^2(x,z) = x^{2l}\hat{\beta}(x,z)^2, \tag{4.6.3}$$

with $\hat{\beta}(0,z) \neq 0$. Thus, β has order l in x. Similarly, the components β_i of $\nabla\delta$ are given by (4.7.1),

$$\beta_i(x,z) = \frac{\alpha_i(x,z)}{\beta(x,z)} = \frac{1}{\beta(x,z)}\sum_{k=1}^n a_{ki}(x,z)a_{kn}(x,z) = \frac{x^l}{\hat{\beta}(x,z)}\sum_{k=1}^n \hat{a}_{ki}(x,z)\hat{a}_{kn}(x,z).$$

Therefore, also β_i's have order l in x. A crucial remark before moving forward is that, thanks to the strongly regular assumption on M, computing derivatives along z-directions does not change the order in x of the quantities. Thus, for example,

$$\mathrm{ord}_x \partial_{z_j} \beta_i(x,z) = l, \qquad \forall\, i,j = 1,\dots,n.$$

Reasoning in this way, for the functions $\mathfrak{f}_i$ defined in (4.7.2), we have:

$$\mathrm{ord}_x \mathfrak{f}_i(x,z) = 2l, \qquad \forall\, i = 1,\dots,n,$$

and the same is true for any derivative in z-directions. For what concerns $\mathfrak{f}_0$, recall that

$$f_0(x,z) = -\frac{\partial_x \beta(x,z)}{\beta(x,z)} \qquad \Rightarrow \qquad \mathrm{ord}_x \mathfrak{f}_0(x,z) = -1. \tag{4.6.4}$$

From (4.6.4), it is clear that derivatives in the x-direction encode all the possible singularities of second logarithmic derivatives of $h_q(s)$. Finally, using (4.7.3), we see that

$$\mathrm{ord}_x \mathfrak{h}_n(x,z) = l-2, \qquad \text{and} \qquad \mathfrak{h}_n(x,z) = -\frac{2(\partial_x\beta(x,z))^2}{\beta(x,z)} + O(x^{3l}).$$

Finally, we can evaluate the order in x of the functions in (4.6.2): the lowest order is -2 coming from the terms $\mathfrak{h}_n\beta^{-1}$, $\partial_x\mathfrak{f}_0$ and $\beta^{-1}\mathfrak{f}_0\partial_x\beta$. Thus, denoting by $z' = (z_1,\dots,z_{n-1},0)$, we obtain:

$$\begin{aligned}(\log\det B_q(s))''_{|s=0} &= \left(\frac{\mathfrak{h}_n(x,z') - 2\mathfrak{f}_0\partial_x\beta(x,z')}{\beta(x,z')} + \partial_x\mathfrak{f}_0(x,z')\right) + O(1)\\ &= \left(\frac{-\partial_x^2\beta(x,z')\beta(x,z') + (\partial_x\beta(x,z'))^2}{\beta^2(x,z')}\right) + O(1).\end{aligned}$$

Now using (4.6.3), we can reason as in the 2-dimensional case, cf. (4.5.3), to conclude that

$$(\log\det B_q(s))''(0) \xrightarrow{q\to(0,\mathbf{0})} +\infty.$$

Once again, also in this situation, the singularity in x of the quantity $(\log\det B_q(s))''(0)$ is polynomial of order -2. Finally, using the same argument used in (4.5.4) for the 2-dimensional case, we can show that the density of the measure $\mathfrak{m}$ produces a polynomial singularity of order -1, which is negligible as $q\to(0,\mathbf{0})$. Finally, we obtain

$$(\log\det h_q(s))''(0) \xrightarrow{q\to(0,\mathbf{0})} +\infty,$$

disproving the $\mathsf{CD}(K,N)$ condition for any $K\in\mathbb{R}$ and $N\in(1,+\infty)$, as desired. □

Remark 4.6.4. We stress once again that, thanks to the strongly regular assumption on M, the order of the structure (and thus the order of β) controls the orders in x, not only of the functions β_i, $\mathfrak{f}_i$ and $\mathfrak{h}_n$, but also of their derivatives in the z-directions. Below, we provide an example of regular (but not strongly regular) structure where the orders of the derivatives are not controlled by the order of β. Nevertheless our strategy to disprove the CD condition works.

In full generality, it is possible to prove that $(\log\det h_x(s))''(0)$ actually diverges, however there is no criterion of determining the *sign* of the leading order, without requiring some additional regularity on the structure. On the other hand, as characteristic points encode the truly sub-Riemannian behavior of almost-Riemannian manifolds, we believe that our strategy should always be effective.

Example 4.6.5. Let $M = \mathbb{R}^4$ and in coordinates (x, z_1, z_2, z_3) consider the almost-Riemannian structure defined by the global vector fields

$$X_0 = \partial_x, \quad X_1 = \partial_{z_1} - \frac{z_2}{2}\partial_{z_3}, \quad X_2 = \partial_{z_2} + \frac{z_1}{2}\partial_{z_3}, \quad X_3 = x\partial_{z_3}.$$

The singular region is given by $\mathcal{Z} = \{x = 0\}$ and is an embedded hypersurface without characteristic points. Notice that M is regular, see [PRS18, Def. 7.10] for the precise definition, but not strongly regular, thus we can not apply Theorem 4.6.3. Nevertheless, if we consider $\Sigma = \{z_3 = 0\}$, assumptions (**H1**) and (**H2**) are verified, therefore, we can apply our general strategy. Setting $\mathfrak{m} = \mathscr{L}^4$, an explicit computation leads to

$$(\log\det h_q(s))''(0) = \frac{8x^2 - 4(z_1^2 + z_2^2)}{(4x^2 + z_1^2 + z_2^2)^2}, \tag{4.6.5}$$

which diverges at $+\infty$ along the curve $(x, 0, 0, 0)$ as $x \to 0$, disproving the $\mathsf{CD}(K, N)$ condition for any $K \in \mathbb{R}$ and $N \geq 1$. A few remarks are in order: first of all, the function $\beta(x, z_1, z_2) = 4x^2 + z_1^2 + z_2^2$ has order 2 in x but this is not true for its derivatives in the z-directions. Second of all, the numerator of (4.6.5) does not have a sign, highlighting the difficulties of the general case of determining the behavior of the leading term.

4.7 Appendix: Explicit expression for $\nabla\delta$, $\mathfrak{f}$ and $\mathfrak{h}$

In order to obtain an explicit expression for $\nabla\delta$, $\mathfrak{f}$ and $\mathfrak{h}$, we study the Hamiltonian system associated with the sub-Riemannian Hamiltonian. Recall that in canonical coordinates induced by (x, z), given by (4.4.1), the Hamiltonian is

$$H(p_x, p_z; x, z) = \frac{1}{2}p_x^2 + \frac{1}{2}p_z^{\mathsf{T}}A^{\mathsf{T}}A(x, z)p_z,$$

where p_z is a shorthand for $(p_{z_1}, \dots, p_{z_n})$. For the hypersurface $\Sigma \subset M$, given by (**H1**), from Lemma 4.2.9, we know that the unique minimizing geodesic for δ with initial point $q \in \Sigma \setminus C(\Sigma)$ has unique (up to a sign) initial covector:

$$\lambda(q) = \frac{1}{\beta(q)}dz_n, \qquad \text{with} \quad \beta(q)^2 = \sum_{k=1}^{n} a_{kn}(q)^2$$

Thus, if $(x(s), z(s), p_x(s), p_z(s))$[5] is the solution to (4.4.10) with initial datum $(\lambda(q); q)$, we deduce that

$$\beta_0(q) = \dot{x}(0) = 0, \qquad \beta_i(q) = \dot{z}_i(0) = \frac{1}{\beta(q)}\sum_{k=1}^{n} a_{ki}(q)a_{kn}(q) = \frac{\alpha_i(q)}{\beta(q)}, \quad \forall\, i = 1, \dots, n, \tag{4.7.1}$$

having denoted $\nabla\delta(q) = (\beta_0(q), \dots, \beta_n(q))$. Moreover, notice that by definition $\beta_n(q) = \beta(q)$. In an analogous way, we can compute $\mathfrak{f} = (\mathfrak{f}_0, \dots, \mathfrak{f}_n)$:

$$\begin{aligned} \mathfrak{f}_0(q) &= \ddot{x}(0) = -\frac{\partial_x\beta(q)}{\beta(q)} \\ \mathfrak{f}_i(q) &= \ddot{z}_i(0) = \frac{1}{\beta^2(q)}\left[\sum_{l=1}^{n}\partial_{z_l}\alpha_i(q)\alpha_l(q) - \frac{1}{2}\sum_{j,k=1}^{n} a_{ki}(q)a_{kj}(q)\partial_{z_j}\beta^2(q)\right]. \end{aligned} \tag{4.7.2}$$

[5]We drop the superscript q only in this section to ease the notation.

Finally, taking the third-order derivatives in s of the solution to (4.4.10), we obtain $\mathfrak{h}$. Notice, however, that we only need the n-th component of $\mathfrak{h}$ in (4.6.2), thus:

$$\mathfrak{h}_n = \dddot{z}_n(0) = \frac{1}{\beta^3}\Bigg[-\frac{(\partial_x \beta^2)^2}{2} + \sum_{j,r,l=1}^{n} \alpha_l \alpha_r \partial^2_{z_l z_r}(\beta^2) + \sum_{j,l=1}^{n} \beta^2 \partial_{z_l}(\beta^2)\mathfrak{f}_l \\ - \sum_{j,l=1}^{n} \alpha_l \partial_{z_l} \alpha_j \partial_{z_j}(\beta^2) - \frac{1}{2}\sum_{j,l=1}^{n} \alpha_j \left(\alpha_l \partial_{z_j z_l}(\beta^2) - \partial_{z_j} \alpha_l \partial_{z_l}(\beta^2) \right) \Bigg]. \quad (4.7.3)$$

Paper 5

Failure of the curvature-dimension condition in sub-Finsler manifolds

with Tommaso Rossi

The Lott–Sturm–Villani curvature-dimension condition $\mathsf{CD}(K,N)$ *provides a synthetic notion for a metric measure space to have curvature bounded from below by* K *and dimension bounded from above by* N*. It has been recently proved that this condition does not hold in sub-Riemannian geometry for every choice of the parameters* K *and* N*. In this paper, we extend this result to the context sub-Finsler geometry, showing that the* $\mathsf{CD}(K,N)$ *condition is not well-suited to characterize curvature in this setting. Firstly, we show that this condition fails in (strict) sub-Finsler manifolds equipped with a smooth strictly convex norm and with a positive smooth measure. Secondly, we focus on the sub-Finsler Heisenberg group, proving that curvature-dimension bounds can not hold also when the reference norm is less regular, in particular when it is of class* $C^{1,1}$*. The strategy for proving these results is a non-trivial adaptation of the work of Juillet [Jui21], and it requires the introduction of new tools and ideas of independent interest. Finally, we demonstrate the failure of the (weaker) measure contraction property* $\mathsf{MCP}(K,N)$ *in the sub-Finsler Heisenberg group, equipped with a singular strictly convex norm and with a positive smooth measure. This result contrasts with what happens in the sub-Riemannian Heisenberg group, which instead satisfies* $\mathsf{MCP}(0,5)$*.*

All authors of this paper contributed equally to all results.

5.1 Introduction

In the present paper, we address the validity of the Lott–Sturm–Villani curvature-dimension (in short $\mathsf{CD}(K,N)$) condition in the setting of sub-Finsler geometry. In particular, we prove that this condition can not hold in a large class of sub-Finsler manifolds. Thus, on the one hand, this work shows that the $\mathsf{CD}(K,N)$ condition is not well-suited to characterize curvature in sub-Finsler geometry. On the other hand, we discuss how our results could provide remarkable insights about the geometry of $\mathsf{CD}(K,N)$ spaces.

5.1.1 Curvature-dimension conditions

In their groundbreaking works, Sturm [Stu06a, Stu06b] and Lott–Villani [LV09] introduced independently a synthetic notion of curvature-dimension bounds for non-smooth spaces, using Optimal Transport. Their theory stems from the crucial observation that, in the Riemannian setting, having a uniform lower bound on the Ricci curvature and an upper bound on the dimension, can be equivalently characterized in terms of a convexity property of suitable entropy functionals in the Wasserstein space. In particular, it was already observed in [vRS05] that the Ricci bound $\mathrm{Ric} \geq K \cdot g$ holds if and only if the Boltzmann–Shannon entropy functional is K-convex in the Wasserstein space. More generally, let (M, g) be a complete Riemannian manifold, equipped with a measure of the form $\mathfrak{m} = e^{-V} \mathrm{vol}_g$, where vol_g is the Riemannian volume and $V \in C^2(M)$. Given $K \in \mathbb{R}$ and $N \in (n, +\infty]$, Sturm [Stu06b] proved that the (generalized) Ricci lower bound

$$\mathrm{Ric}_{N,V} := \mathrm{Ric} + \nabla^2 V - \frac{\nabla V \otimes \nabla V}{N - n} \geq K \cdot g, \tag{5.1.1}$$

holds if and only if a (K, N)-convexity inequality holds for Rényi entropy functionals, defined with respect to the reference measure $\mathfrak{m}$. While (5.1.1) involves a differential object, the Ricci tensor, entropy convexity can be formulated relying solely upon a reference distance and a reference measure, without the need of the underlying smooth structure of the Riemannian manifold. Therefore, it can be introduced in the non-smooth setting of metric measure spaces and taken as definition of curvature-dimension bound. This condition is called $\mathsf{CD}(K, N)$ and represents a synthetic lower bound on the (Ricci) curvature by $K \in \mathbb{R}$ and a synthetic upper bound on the dimension by $N \in (1, \infty]$, see Definition 5.2.2. In this sense, according to the discussion above, the $\mathsf{CD}(K, N)$ condition is coherent with the Riemannian setting. Moreover, it was proved by Ohta [Oht09] that the relation between curvature and $\mathsf{CD}(K, N)$ condition holds also in the context of Finsler manifolds.

Remarkably, $\mathsf{CD}(K, N)$ spaces (i.e. spaces satisfying the $\mathsf{CD}(K, N)$ condition) enjoy several geometric properties which hold in the smooth setting. Some of them are expected (and in a way necessary) for a reasonable curvature-dimension bound, such as the scaling [Stu06b], tensorization [BS10] and globalization [CM21] properties or the monotonicity with respect to the parameters [Stu06b], i.e.

$$\mathsf{CD}(K', N') \implies \mathsf{CD}(K, N) \qquad \text{if } K' \geq K \text{ and } N' \leq N.$$

Others are completely non-trivial and highlight some notable geometric features. Among them, we mention the Bonnet–Myers diameter bound and the Bishop–Gromov inequality, that provides an estimate on the volume growth of concentric balls. Particularly interesting in the context of this work is the Brunn–Minkowski inequality $\mathsf{BM}(K, N)$, which, given two sets A and B in the reference metric measure space $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$, provides a lower estimate on the measure of the set of t-midpoints

$$M_t(A, B) = \{x \in \mathsf{X} \,:\, \mathsf{d}(a, x) = t\mathsf{d}(a, b),\, \mathsf{d}(x, b) = (1 - t)\mathsf{d}(a, b) \text{ for some } a \in A, b \in B\},$$

in terms of $\mathfrak{m}(A)$ and $\mathfrak{m}(B)$, for every $t \in [0, 1]$, cf. (5.2.2). The notable feature of the $\mathsf{BM}(K, N)$ inequality is that its formulation does not invoke optimal transport, or Wasserstein interpolation, and because of that, it is easier to handle than the $\mathsf{CD}(K, N)$ condition. Nonetheless, it contains a strong information about the curvature of the underlying space, to the extent that it is equivalent to the $\mathsf{CD}(K, N)$ condition in the Riemannian setting, cf. [MPR22a]. In particular, in the proof of Theorem 5.1.5 and Theorem 5.1.7, we show the failure of the $\mathsf{CD}(K, N)$ condition by contradicting the Brunn–Minkowski inequality $\mathsf{BM}(K, N)$.

Finally, another fundamental property of the $\mathsf{CD}(K, N)$ condition is its stability with respect to the (pointed) measured Gromov–Hausdorff convergence [Stu06b, LV09, GMS15]. This notion of convergence for metric measure spaces essentially combines the Hausdorff convergence for the metric side and the weak convergence for the reference measures. As in a metric measure space, the tangent spaces at a point are identified with a measured Gromov–Hausdorff limit procedure of suitably rescalings of the original space, the stability of the curvature-dimension condition implies that the metric measure tangents of a $\mathsf{CD}(K,N)$ space is a $\mathsf{CD}(0,N)$ space.

In the setting of metric measure spaces, it is possible to define other curvature-dimension bounds, such as the so-called measure contraction property (in short $\mathsf{MCP}(K,N)$), introduced by Ohta in [Oht07]. In broad terms, the $\mathsf{MCP}(K,N)$ condition can be interpreted as the Brunn–Minkowski inequality where one of the two sets degenerates to a point. In particular, it is implied by (and strictly weaker than) the $\mathsf{BM}(K,N)$ inequality, and therefore it is also a consequence of the $\mathsf{CD}(K,N)$ condition.

5.1.2 The curvature-dimension condition in sub-Riemannian geometry

While in the Riemannian setting the $\mathsf{CD}(K,N)$ condition is equivalent to having bounded geometry, a similar result does not hold in the sub-Riemannian setting. Sub-Riemannian geometry is a generalization of Riemannian geometry where, given a smooth manifold M, we define a smoothly varying scalar product only on a subset of *horizontal* directions $\mathcal{D}_p \subset T_pM$ (called distribution) at each point $p \in M$. Under the so-called Hörmander condition, M is horizontally-path connected, and the usual length-minimization procedure yields a well-defined distance d_{SR}. In particular, differently from what happens in Riemannian geometry, the rank of the distribution $r(p) := \dim \mathcal{D}_p$ may be strictly less than the dimension of the manifold and may vary with the point. This may influence the behavior of geodesics, emphasizing singularities of the distance d_{SR}. For this reason, we can not expect the $\mathsf{CD}(K,N)$ condition to hold for *truly* sub-Riemannian manifolds. This statement is confirmed by a series of papers, most notably [Jui21, MR23a, RS23], that contributed to the proof of the following result.

Theorem 5.1.1. *Let M be a complete sub-Riemannian manifold, equipped with a positive smooth measure $\mathfrak{m}$. Then, the metric measure space $(M,\mathsf{d}_{SR},\mathfrak{m})$ does not satisfy the $\mathsf{CD}(K,N)$ condition, for any $K \in \mathbb{R}$ and $N \in (1,\infty)$.*

In [Jui21], Juillet proved Theorem 5.1.1 for sub-Riemannian manifolds where the rank of the distribution $r(p)$ is strictly smaller than the topological dimension $n := \dim M$, for every $p \in M$. His strategy relies on the construction of two Borel subsets for which the Brunn–Minkowski inequality $\mathsf{BM}(K,N)$ does not hold. Namely, for all $R,\varepsilon > 0$, one can find $A,B \subset M$ such that $\operatorname{diam}(A \cup B) < R$, $\mathfrak{m}(A) \approx \mathfrak{m}(B)$, and such that there exists $t \in (0,1)$ for which

$$\mathfrak{m}(M_t(A,B)) \leq \frac{1}{2^{\mathcal{N}-n}}\,\mathfrak{m}(B)(1+\varepsilon), \tag{5.1.2}$$

where $\mathcal{N}$ is the so-called *geodesic dimension* of M, see [ABR18, Def. 5.47] for a precise definition. The sets A and B are metric balls of small radius, centered at the endpoints of a short segment of an *ample geodesic*, see [ABR18] for details. The inequality (5.1.2) allows to contradict the Brunn–Minkowski inequality $\mathsf{BM}(K,N)$ if and only if the geodesic dimension $\mathcal{N}$ is strictly greater than n, which is the case if $r(p) < n$, for every $p \in M$.

While Julliet's result is quite general, it does not include almost-Riemannian geometry. Roughly speaking, an almost-Riemannian manifold is a sub-Riemannian manifold where the rank of the distribution coincides with the dimension of M, at almost every point. In [MR23a], we addressed this issue, proposing a new strategy for proving Theorem 5.1.1 in this setting. Our idea is to exploit the following one-dimensional characterization of the $\mathsf{CD}(K,N)$ condition:

$$\mathsf{CD}(K,N) \quad \Rightarrow \quad \mathsf{CD}^1(K,N), \tag{5.1.3}$$

proved by Cavalletti and Mondino in [CM17b], and contradict the $\mathsf{CD}^1(K,N)$ condition. On a metric measure space $(\mathsf{X},\mathsf{d},\mathfrak{m})$, given a 1-Lipschitz function $u \in \operatorname{Lip}(\mathsf{X})$, it is possible to partition X

in one-dimensional transport rays, associated with u, and disintegrate the measure $\mathfrak{m}$ accordingly. Then, the $\mathsf{CD}^1(K,N)$ condition asks for the validity of the $\mathsf{CD}(K,N)$ condition along the transport rays of the disintegration associated with u, for any choice of $u \in \mathrm{Lip}(\mathsf{X})$. In [MR23a, Thm. 1.2], when M is either strongly regular or $\dim M = 2$, we are able to explicitly build a 1-Lipschitz function, and compute the associated disintegration, showing that the $\mathsf{CD}(K,N)$ condition along the rays does not hold for any $K \in \mathbb{R}$ and $N \in (1,\infty)$.

Most recently, Rizzi and Stefani [RS23] proposed yet another strategy to prove Theorem 5.1.1. Differently from the strategies presented above, they pursue the "Eulerian" approach to curvature-dimension bounds, based on a suitable Gamma calculus, see [BGL14] for details. This approach can be adopted for metric measure spaces that satisfy the *infinitesimal Hibertian* condition (cf. [AGS14b, Gig15]) which forces the space to be *Riemannian-like* and ensures the linearity of the heat flow. According to [AGS15], an infinitesimally Hilbertian $\mathsf{CD}(K,N)$ space supports the so-called Bakry–Émery inequality $\mathsf{BE}(K,\infty)$, which, in the sub-Riemannian setting reads as

$$\frac{1}{2}\Delta\left(\|\nabla f\|^2\right) \geq g(\nabla f, \nabla\Delta f) + K\|\nabla f\|^2, \qquad \forall\, f \in C_c^\infty(M), \tag{5.1.4}$$

where ∇ is the horizontal gradient and Δ is the sub-Laplacian. In [RS23], the authors show that (5.1.4) implies the existence of enough isometries on the metric tangent to force it to be Euclidean at *each point*, proving Theorem 5.1.1 (including also the case $N = \infty$).

5.1.3 Other curvature-dimension bounds in sub-Riemannian geometry

Given that the $\mathsf{CD}(K,N)$ condition does not hold in sub-Riemannian geometry, considerable efforts have been undertaken to explore potential curvature-dimension bounds that may hold in this class. A first observation in this direction is that the weaker $\mathsf{MCP}(K,N)$ condition does hold in many examples of sub-Riemannian manifolds. In particular, it was proved by Juillet [Jui09] that the sub-Riemannian Heisenberg group satisfies the $\mathsf{MCP}(0,5)$ condition, where the curvature-dimension parameters can not be improved. Moreover, in [ABR18] it was observed that the optimal dimensional parameter for the measure contraction property coincides with the geodesic dimension of the sub-Riemannian Heisenberg group (i.e. $\mathcal{N} = 5$). This result has been subsequently extended to a large class of sub-Riemannian manifolds, including ideal Carnot groups [Rif13], corank-1 Carnot groups [Riz16], generalised H-type Carnot groups [BR18] and two-step analytic sub-Riemannian structures [BR20]. In all these cases, the $\mathsf{MCP}(0,N)$ condition holds with the dimensional parameter N greater than or equal to the geodesic dimension $\mathcal{N}$.

Another attempt is due to Milman [Mil21], who introduced the quasi curvature-dimension condition, inspired by the interpolation inequalities along Wasserstein geodesics in ideal sub-Riemannian manifolds, proved by Barilari and Rizzi [BR19]. Finally, these efforts culminated in the recent work by Barilari, Mondino and Rizzi [BMR22], where the authors propose a unification of Riemannian and sub-Riemannian geometries in a comprehensive theory of synthetic Ricci curvature lower bounds. In the setting of gauge metric measure spaces, they introduce the $\mathsf{CD}(\beta,n)$ condition, encoding in the distortion coefficient β finer geometrical information of the underlying structure. Moreover they prove that the $\mathsf{CD}(\beta,n)$ condition holds for compact fat sub-Riemannian manifolds, thus substantiating the definition.

5.1.4 Sub-Finsler manifolds and Carnot groups

In the present paper, we focus on sub-Finsler manifolds, which widely generalize both sub-Riemannian and Finsler geometry. Indeed, in this setting, given a smooth manifold M, we prescribe a smoothly varying *norm* (which needs not be induced by a scalar product) on the distribution $\mathcal{D}_p \subset T_pM$, at each point $p \in M$. As in the sub-Riemannian setting, $\mathcal{D}$ must satisfy the Hörmander condition, and consequently the length-minimization procedure among admissible curves gives a well-defined distance d_{SF}. Note that, on the one hand, if the $\mathcal{D}_p = T_pM$ for every $p \in M$, we recover the classical Finsler geometry. On the other hand, if the norm on $\mathcal{D}_p$ is induced by a scalar product for every $p \in M$, we fall back into sub-Riemannian geometry.

Replacing the scalar product with a (possibly singular) norm is not merely a technical choice, as the metric structure of a sub-Finsler manifold reflects the singularities of the reference norm. Indeed, even though sub-Finsler manifolds can still be investigated by means of classical control theory [AS04], deducing finer geometrical properties is more delicate compared to what happens in the sub-Riemannian setting, as the Hamiltonian function has a low regularity, cf. Section 5.3. In this regard, sub-Finsler manifolds provide an interesting example of smooth structures which present both the typical sub-Riemannian and Finsler singular behavior. A particularly relevant class of sub-Finsler manifolds is the one of *sub-Finsler Carnot groups*.

Definition 5.1.2 (Carnot group)**.** A Carnot group is a connected, simply connected Lie group G with nilpotent Lie algebra $\mathfrak{g}$, admitting a stratification

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k,$$

where $\mathfrak{g}_{i+1} = [\mathfrak{g}_1, \mathfrak{g}_i]$, for every $i = 1, \ldots, k-1$, and $[\mathfrak{g}_1, \mathfrak{g}_k] = \{0\}$.

Given a Carnot group G, if we equip the first layer $\mathfrak{g}_1$ of its Lie algebra with a norm, we naturally obtain a left-invariant sub-Finsler structure on G. We refer to the resulting manifold as a sub-Finsler Carnot group.

Motivated from the results presented in the previous section, cf. Theorem 5.1.1, and especially from the ones obtained in the present work (see Section 5.1.5), we formulate the following conjecture.

Conjecture 5.1.3. *Let G be a sub-Finsler Carnot group, endowed with a positive smooth measure $\mathfrak{m}$. Then, the metric measure space $(G, \mathsf{d}_{SF}, \mathfrak{m})$ does not satisfy the $\mathsf{CD}(K,N)$ condition for any $K \in \mathbb{R}$ and $N \in (1,\infty)$.*

Our interest in Carnot groups stems from the fact that they are the only metric spaces that are locally compact, geodesic, isometrically homogeneous and self-similar (i.e. admitting a dilation) [LD15]. According to this property, sub-Finsler Carnot groups naturally arise as metric tangents of metric measure spaces.

Theorem 5.1.4 (Le Donne [LD11])**.** *Let $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ be a geodesic metric measure space, equipped with a doubling measure $\mathfrak{m}$. Assume that, for $\mathfrak{m}$-almost every $x \in \mathsf{X}$, the set $\mathrm{Tan}(\mathsf{X}, x)$ of all metric tangent spaces at x contains only one element. Then, for $\mathfrak{m}$-almost every $x \in \mathsf{X}$, the element in $\mathrm{Tan}(\mathsf{X}, x)$ is a sub-Finsler Carnot group G.*

In particular, this result applies to $\mathsf{CD}(K,N)$ spaces, where the validity of the doubling property is guaranteed by the Bishop–Gromov inequality. Moreover, as already mentioned, the metric measure tangents of a $\mathsf{CD}(K,N)$ space are $\mathsf{CD}(0,N)$. Therefore, the study of the $\mathsf{CD}(K,N)$ condition in sub-Finsler Carnot groups, and especially the validity of Conjecture 5.1.3, has the potential to provide deep insights on the structure of tangents of $\mathsf{CD}(K,N)$ spaces. This could be of significant interest, particularly in connection with Bate's recent work [Bat22], which establishes a criterion for rectifiability in metric measure spaces, based on the structure of metric tangents.

5.1.5 Main results

The aim of this paper is to show the failure of the $\mathsf{CD}(K,N)$ condition in the sub-Finsler setting, with a particular attention to Conjecture 5.1.3. Our results offer an advance into two different directions: on the one hand we deal with general sub-Finsler structures, where the norm is smooth, cf. Theorem 5.1.5 and Theorem 5.1.6, and, on the other hand, we deal with the sub-Finsler Heisenberg group, equipped with more general norms, cf. Theorem 5.1.8 and Theorem 5.1.7.

In order to extend the sub-Riemannian result of Theorem 5.1.1 to the sub-Finsler setting, one can attempt to adapt the strategies discussed in Section 5.1.2, however this can present major difficulties. Specifically, the argument developed in [RS23] has little hope to be generalized, because the infinitesimal Hilbertianity assumption does not hold in Finsler-like spaces, see [OS12]. It is important to note that this is not solely a “regularity" issue, in the sense that it also occurs when the norm generating the sub-Finsler structure is smooth, but not induced by a scalar product. Instead, the approach proposed in [MR23a] could potentially be applied to sub-Finsler manifolds as it relies on tools developed in the non-smooth setting, see (5.1.3). However, adapting the sub-Riemannian computations that led to a contradiction of the $\mathsf{CD}^1(K,N)$ condition seems non-trivial already when the reference norm is smooth. Finally, the strategy illustrated in [Jui21] hinges upon geometrical constructions and seems to be well-suited to generalizations to the sub-Finsler setting. In this paper, we build upon this observation and adapt the latter strategy to prove our main theorems.

Our first result is about the failure of the $\mathsf{CD}(K,N)$ condition in *smooth* sub-Finsler manifolds, cf. Theorem 5.4.26.

Theorem 5.1.5. *Let M be a complete sub-Finsler manifold with $r(p) < n := \dim M$ for every $p \in M$, equipped with a smooth, strictly convex norm $\|\cdot\|$ and with a positive smooth measure $\mathfrak{m}$. Then, the metric measure space $(M, \mathsf{d}_{SF}, \mathfrak{m})$ does not satisfy the $\mathsf{CD}(K,N)$ condition, for any $K \in \mathbb{R}$ and $N \in (1,\infty)$.*

This result is the sub-Finsler analogue of [Jui21, Cor. 1.2]. Although the strategy of its proof follows the blueprint of [Jui21], the adaptation to our setting is non-trivial and requires many intermediate results of independent interest. First of all, we establish the existence of geodesics *without abnormal sub-segments*, cf. Theorem 5.4.11, proposing a construction that is new even in the sub-Riemannian framework and relies on the regularity properties of the distance function from the boundary of an open set. Note that, while these properties are well-known in the sub-Riemannian context (cf. [FPR20, Prop. 3.1]), inferring them in the sub-Finsler setting becomes more challenging due to the low regularity of the Hamiltonian, which affects the regularity of the normal exponential map. Nonetheless, we settle a weaker regularity result that is enough for our purposes, cf. Theorem 5.4.8. Second of all, we prove an analogue of the sub-Riemannian theorem, indicating that the volume contraction along ample geodesics is governed by the geodesic dimension, see [ABR18, Thm. D]. Indeed, in a smooth sub-Finsler manifold, we establish that the volume contraction rate along geodesics without abnormal sub-segments is bigger than $\dim M + 1$, cf. Theorem 5.4.22. Finally, we mention that these technical challenges lead us to a simplification of Juillet's argument (cf. Theorem 5.4.26), which revealed itself to be useful also in the proof of Theorem 5.1.7.

Observe that, since sub-Finsler Carnot groups are equiregular (and thus $r(p) < n$, for every $p \in G$) and complete, we immediately obtain the following consequence of Theorem 5.1.5, which constitutes a significant step forward towards the proof of Conjecture 5.1.3.

Theorem 5.1.6. *Let G be a sub-Finsler Carnot group, equipped with a smooth, strictly convex norm $\|\cdot\|$ and with a positive smooth measure $\mathfrak{m}$. Then, the metric measure space $(G, \mathsf{d}_{SF}, \mathfrak{m})$ does not satisfy the $\mathsf{CD}(K,N)$ condition, for any $K \in \mathbb{R}$ and $N \in (1,\infty)$.*

In the proof of Theorem 5.1.5, the smoothness of the norm plays a pivotal role in establishing the correct volume contraction rate along geodesics. When the norm is less regular, it is not clear how to achieve an analogue behavior in full generality. Nonetheless, we are able to recover such a result in the context of the *sub-Finsler Heisenberg group* $\mathbb{H}$, equipped with a possibly singular norm (see Section 5.5). Working in this setting is advantageous since, assuming strict convexity of the norm, the geodesics and the cut locus are completely described [Ber94] and there exists an explicit expression for them in terms of convex trigonometric functions [Lok21] (see also [BBLDS17] for an example of the non-strictly convex case).

For the sub-Finsler Heisenberg group, we prove two different results, with the first addressing the case of $C^{1,1}$ reference norms and thus substantially relaxing the smoothness assumption of Theorem 5.1.5, cf. Theorem 5.5.24.

Theorem 5.1.7. *Let $\mathbb{H}$ be the sub-Finsler Heisenberg group, equipped with a strictly convex and $C^{1,1}$ norm and with a positive smooth measure $\mathfrak{m}$. Then, the metric measure space $(\mathbb{H}, \mathsf{d}_{SF}, \mathfrak{m})$ does not satisfy the $\mathsf{CD}(K,N)$, for any $K \in \mathbb{R}$ and $N \in (1,\infty)$.*

The proof of this statement follows the same lines of [Jui21, Cor. 1.2]. However, the low regularity of the norm, and thus of geodesics, prevent us to exploit the same differential tools developed for Theorem 5.1.5. Nonetheless, using the explicit expression of geodesics and of the exponential map, we can still recover an analogue result. In particular, guided by the intuition that a contraction rate along geodesics, similar to the one appearing in the smooth case, should still hold, we thoroughly study the Jacobian determinant of the exponential map. Building upon a fine analysis of convex trigonometric functions, cf. Section 5.5.1 and Proposition 5.5.13, we obtain an estimate on the contraction rate of the Jacobian determinant of the exponential map, but only for a large (in a measure-theoretic sense) set of covectors in the cotangent space. This poses additional challenges that we are able to overcome with a delicate density-type argument, together with an extensive use of the left-translations of the group, cf. Theorem 5.5.24 and also Remark 5.5.25. Remarkably, for every $C^{1,1}$ reference norm, we obtain the exact same contraction rate, equal to the geodesic dimension $\mathcal{N} = 5$, that characterizes the sub-Riemannian Heisenberg group.

Our second result in the sub-Finsler Heisenberg group deals with the case of singular (i.e. non-C^1) reference norms, cf. Theorem 5.5.26.

Theorem 5.1.8. *Let $\mathbb{H}$ be the sub-Finsler Heisenberg group, equipped with a strictly convex norm $\|\cdot\|$ which is not C^1, and let $\mathfrak{m}$ be a positive smooth measure on $\mathbb{H}$. Then, the metric measure space $(\mathbb{H}, \mathsf{d}_{SF}, \mathfrak{m})$ does not satisfy the measure contraction property $\mathsf{MCP}(K,N)$ for any $K \in \mathbb{R}$ and $N \in (1,\infty)$.*

Observe that this theorem also shows the failure of the $\mathsf{CD}(K,N)$ condition, which is stronger than the measure contraction property $\mathsf{MCP}(K,N)$. However, Theorem 5.1.8 has an interest that goes beyond this consequence, as it reveals a phenomenon that stands in contrast to what typically happens in the sub-Riemannian setting. In fact, as already mentioned in section 5.1.3, the $\mathsf{MCP}(K,N)$ condition holds in many sub-Riemannian manifolds, and, in the particular case of the sub-Riemannian Heisenberg group, holds with parameters $K = 0$ and $N = 5$. Therefore, Theorem 5.1.8 shows that a singularity of the reference norm can cause the failure of the measure contraction property $\mathsf{MCP}(K,N)$. A similar phenomenon is highlighted in the recent paper by Borza and Tashiro [BT23], where the authors prove that the Heisenberg group equipped with the l^p-norm cannot satisfy the $\mathsf{MCP}(K,N)$ condition if $p > 2$.

Our strategy to show Theorem 5.1.8 consists in finding a set $A \subset \mathbb{H}$, having positive $\mathfrak{m}$-measure, such that the set of t-midpoints $M_t(\{\mathrm{e}\}, A)$ (where e denotes the identity in $\mathbb{H}$) is $\mathfrak{m}$-null for every t sufficiently small. This construction is based on a remarkable geometric property of the space $(\mathbb{H}, \mathsf{d}_{SF}, \mathfrak{m})$, where geodesics can branch, even though they are unique. This has independent interest, as examples of branching spaces usually occur when geodesics are not unique.

We conclude this section highlighting that the combination of Theorem 5.1.7 and Theorem 5.1.8 proves Conjecture 5.1.3 for a large class of sub-Finsler Heisenberg groups. This is particularly interesting as the sub-Finsler Heisenberg groups are the unique sub-Finsler Carnot groups with Hausdorff dimension less than 5 (or with topological dimension less than or equal to 3), up to isometries.

Structure of the paper

In Section 5.2 we introduce all the necessary preliminaries. In particular, we present the precise definition of the $\mathsf{CD}(K,N)$ condition with some of its consequences, and we introduce the notion of sub-Finsler structure on a manifold. Section 5.3 is devoted to the study of the geometry of sub-Finsler manifolds. For the sake of completeness, we include generalizations of various sub-Riemannian results, especially regarding the characterizations of normal and abnormal extremals and the exponential map. In Section 5.4, we present the proof of Theorem 5.1.5. We start by developing the building blocks for it, namely the existence of a geodesic without abnormal sub-segments and the regularity of the distance function. Then, we estimate the volume contraction rate, along the previously selected geodesic. Finally, in Section 5.4.4, we adapt Juillet's strategy to obtain our first main theorem. Section 5.5 collects our results about the failure of the $\mathsf{CD}(K,N)$ condition in the sub-Finsler Heisenberg group. After having introduced the convex trigonometric functions in Section 5.5.1, we use them to provide the explicit explicit expression of geodesics, cf. Section 5.5.2. We conclude by proving Theorem 5.1.7 in Section 5.5.3 and Theorem 5.1.8 in Section 5.5.4.

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5.2 Preliminaries

5.2.1 The $\mathsf{CD}(K,N)$ condition

A metric measure space is a triple $(\mathsf{X},\mathsf{d},\mathfrak{m})$ where (X,d) is a complete and separable metric space and $\mathfrak{m}$ is a locally finite Borel measure on it. In the following $C([0,1],\mathsf{X})$ will stand for the space of continuous curves from $[0,1]$ to X. A curve $\gamma\in C([0,1],\mathsf{X})$ is called *geodesic* if

$$\mathsf{d}(\gamma(s),\gamma(t)) = |t-s|\cdot \mathsf{d}(\gamma(0),\gamma(1)) \quad \text{for every } s,t\in[0,1],$$

and we denote by $\mathrm{Geo}(\mathsf{X})$ the space of geodesics on X. The metric space (X,d) is said to be geodesic if every pair of points $x,y\in\mathsf{X}$ can be connected with a curve $\gamma\in\mathrm{Geo}(\mathsf{X})$. For any $t\in[0,1]$ we define the evaluation map $e_t\colon C([0,1],\mathsf{X})\to\mathsf{X}$ by setting $e_t(\gamma):=\gamma(t)$. We denote by $\mathscr{P}(\mathsf{X})$ the set of Borel probability measures on X and by $\mathscr{P}_2(\mathsf{X})\subset\mathscr{P}(\mathsf{X})$ the set of those having finite second moment. We endow the space $\mathscr{P}_2(\mathsf{X})$ with the Wasserstein distance W_2, defined by

$$W_2^2(\mu_0,\mu_1) := \inf_{\pi\in\mathsf{Adm}(\mu_0,\mu_1)} \int \mathsf{d}^2(x,y)\,\mathrm{d}\pi(x,y),$$

where $\mathsf{Adm}(\mu_0,\mu_1)$ is the set of all the admissible transport plans between μ_0 and μ_1, namely all the measures in $\mathscr{P}(\mathsf{X}\times\mathsf{X})$ such that $(\mathtt{p}_1)_\sharp\pi=\mu_0$ and $(\mathtt{p}_2)_\sharp\pi=\mu_1$. The metric space $(\mathscr{P}_2(\mathsf{X}),W_2)$

is itself complete and separable, moreover, if (X,d) is geodesic, then $(\mathscr{P}_2(\mathsf{X}),W_2)$ is geodesic as well. In particular, every geodesic $(\mu_t)_{t\in[0,1]}$ in $(\mathscr{P}_2(\mathsf{X}),W_2)$ can be represented with a measure $\eta\in\mathscr{P}(\mathrm{Geo}(\mathsf{X}))$, meaning that $\mu_t=(e_t)_\#\eta$.

We are now ready to introduce the $\mathsf{CD}(K,N)$ condition, pioneered by Sturm and Lott–Villani [Stu06a, Stu06b, LV09]. As already mentioned, this condition aims to generalize, to the context metric measure spaces, the notion of having Ricci curvature bounded from below by $K\in\mathbb{R}$ and dimension bounded above by $N>1$. In order to define the $\mathsf{CD}(K,N)$ condition, let us introduce the following distortion coefficients: for every $K\in\mathbb{R}$ and $N\in(1,\infty)$,

$$\tau_{K,N}^{(t)}(\theta):=t^{\frac{1}{N}}\left[\sigma_{K,N-1}^{(t)}(\theta)\right]^{1-\frac{1}{N}}, \tag{5.2.1}$$

where

$$\sigma_{K,N}^{(t)}(\theta):=\begin{cases}\dfrac{\sin(t\theta\sqrt{K/N})}{\sin(\theta\sqrt{K/N})} & \text{if } N\pi^2>K\theta^2>0,\\ t & \text{if } K=0,\\ \dfrac{\sinh(t\theta\sqrt{-K/N})}{\sinh(\theta\sqrt{-K/N})} & \text{if } K<0.\end{cases}$$

Remark 5.2.1. Observe that for every $K\in\mathbb{R}$, $N\in(1,\infty)$ and $t\in[0,1]$ we have

$$\lim_{\theta\to0}\sigma_{K,N}^{(t)}(\theta)=t \qquad \text{and} \qquad \lim_{\theta\to0}\tau_{K,N}^{(t)}(\theta)=t.$$

Definition 5.2.2. A metric measure space $(\mathsf{X},\mathsf{d},\mathfrak{m})$ is said to be a $\mathsf{CD}(K,N)$ space (or to satisfy the $\mathsf{CD}(K,N)$ condition) if for every pair of measures $\mu_0=\rho_0\mathfrak{m},\mu_1=\rho_1\mathfrak{m}\in\mathscr{P}_2(\mathsf{X})$, absolutely continuous with respect to $\mathfrak{m}$, there exists a W_2-geodesic $(\mu_t)_{t\in[0,1]}$ connecting them and induced by $\eta\in\mathscr{P}(\mathrm{Geo}(\mathsf{X}))$, such that for every $t\in[0,1]$, $\mu_t=\rho_t\mathfrak{m}\ll\mathfrak{m}$ and the following inequality holds for every $N'\geq N$ and every $t\in[0,1]$

$$\int_{\mathsf{X}}\rho_t^{1-\frac{1}{N'}}\,\mathrm{d}\mathfrak{m}\geq\int_{\mathsf{X}\times\mathsf{X}}\left[\tau_{K,N'}^{(1-t)}(\mathsf{d}(x,y))\rho_0(x)^{-\frac{1}{N'}}+\tau_{K,N'}^{(t)}(\mathsf{d}(x,y))\rho_1(y)^{-\frac{1}{N'}}\right]\mathrm{d}\pi(x,y),$$

where $\pi=(e_0,e_1)_\#\eta$.

One of the most important merits of the $\mathsf{CD}(K,N)$ condition is that it is sufficient to deduce geometric and functional inequalities that hold in the smooth setting. An example which is particularly relevant for this work is the so-called Brunn–Minkowski inequality, whose definition in the metric measure setting requires the following notion.

Definition 5.2.3. Let (X,d) be a metric space and let $A,B\subset\mathsf{X}$ be two Borel subsets. Then for $t\in(0,1)$, we defined the set of *t-midpoints* between A and B as

$$M_t(A,B):=\{x\in\mathsf{X}\,:\,x=\gamma(t)\,,\,\gamma\in\mathrm{Geo}(\mathsf{X})\,,\,\gamma(0)\in A\,,\text{ and }\gamma(1)\in B\}\,.$$

We can now introduce the metric measure version of the Brunn–Minkowski inequality, whose formulation is stated in terms of the distortion coefficients (5.2.1).

Definition 5.2.4. Given $K\in\mathbb{R}$ and $N\in(1,\infty)$, we say that a metric measure space $(\mathsf{X},\mathsf{d},\mathfrak{m})$ satisfies the *Brunn–Minkowski inequality* $\mathsf{BM}(K,N)$ if, for every nonempty $A,B\subset\mathrm{spt}(\mathfrak{m})$ Borel subsets, $t\in(0,1)$, we have

$$\mathfrak{m}(M_t(A,B)))^{\frac{1}{N}}\geq\tau_{K,N}^{(1-t)}(\Theta(A,B))\cdot\mathfrak{m}(A)^{\frac{1}{N}}+\tau_{K,N}^{(t)}(\Theta(A,B))\cdot\mathfrak{m}(B)^{\frac{1}{N}}\,, \tag{5.2.2}$$

where

$$\Theta(A,B) := \begin{cases} \inf_{x\in A,\, y\in B} \mathsf{d}(x,y) & \text{if } K \geq 0\,, \\ \sup_{x\in A,\, y\in B} \mathsf{d}(x,y) & \text{if } K < 0\,. \end{cases}$$

As already mentioned, the Brunn–Minkowski inequality is a consequence of the $\mathsf{CD}(K,N)$ condition, in particular we have that

$$\mathsf{CD}(K,N) \implies \mathsf{BM}(K,N),$$

for every $K \in \mathbb{R}$ and every $N \in (1,\infty)$. In Sections 5.4 and 5.5, we are going to disprove the $\mathsf{CD}(K,N)$ condition for every choice of the parameters $K \in \mathbb{R}$ and $N \in (1,\infty)$, by contradicting the Brunn–Minkowski inequality $\mathsf{BM}(K,N)$. A priori, this is a stronger result than the ones stated in Theorem 5.1.5 and Theorem 5.1.7, since the Brunn–Minkowski inequality is (in principle) weaker than the $\mathsf{CD}(K,N)$ condition. However, recent developments (cf. [MPR22a, MPR22b]) suggest that the Brunn–Minkowski $\mathsf{BM}(K,N)$ could be equivalent to the $\mathsf{CD}(K,N)$ condition in a wide class of metric measure spaces.

Another curvature-dimension bound, which can be defined for metric measure spaces, is the so-called measure contraction property (in short $\mathsf{MCP}(K,N)$), that was introduced by Ohta in [Oht07]. The idea behind it is basically to require the $\mathsf{CD}(K,N)$ condition to hold when the first marginal degenerates to δ_x, a delta-measure at $x \in \operatorname{spt}(\mathfrak{m})$, and the second marginal is $\frac{\mathfrak{m}|_A}{\mathfrak{m}(A)}$, for some Borel set $A \subset \mathsf{X}$ with $0 < \mathfrak{m}(A) < \infty$.

Definition 5.2.5 ($\mathsf{MCP}(K,N)$ condition)**.** Given $K \in \mathbb{R}$ and $N \in (1,\infty)$, a metric measure space $(\mathsf{X},\mathsf{d},\mathfrak{m})$ is said to satisfy the *measure contraction property* $\mathsf{MCP}(K,N)$ if for every $x \in \operatorname{spt}(\mathfrak{m})$ and a Borel set $A \subset \mathsf{X}$ with $0 < \mathfrak{m}(A) < \infty$, there exists a Wasserstein geodesic induced by $\eta \in \mathscr{P}(\mathrm{Geo}(\mathsf{X}))$ connecting δ_x and $\frac{\mathfrak{m}|_A}{\mathfrak{m}(A)}$ such that, for every $t \in [0,1]$,

$$\frac{1}{\mathfrak{m}(A)}\mathfrak{m} \geq (e_t)_\# \Big(\tau_{K,N}^{(t)} \big(\mathsf{d}(\gamma(0),\gamma(1))\big)^N \eta(\mathrm{d}\gamma) \Big). \tag{5.2.3}$$

Remark 5.2.6. For our purposes, we will use an equivalent formulation of the inequality (5.2.3), which holds whenever geodesics are unique, cf. [Oht07, Lemma 2.3] for further details. More precisely, let $x \in \operatorname{spt}(\mathfrak{m})$ and a Borel set $A \subset \mathsf{X}$ with $0 < \mathfrak{m}(A) < \infty$. Assume that for every $y \in A$, there exists a unique geodesic $\gamma_{x,y} : [0,1] \to \mathsf{X}$ joining x and y. Then, (5.2.3) is verified for the measures δ_x and $\frac{\mathfrak{m}|_A}{\mathfrak{m}(A)}$ if and only if

$$\mathfrak{m}(M_t(\{x\},A'))) \geq \int_{A'} \tau_{K,N}^{(t)}(\mathsf{d}(x,y))^N \,\mathrm{d}\mathfrak{m}(y), \qquad \text{for any Borel } A' \subset A. \tag{5.2.4}$$

The $\mathsf{MCP}(K,N)$ condition is weaker than the $\mathsf{CD}(K,N)$ one, i.e.

$$\mathsf{CD}(K,N) \implies \mathsf{MCP}(K,N),$$

for every $K \in \mathbb{R}$ and every $N \in (1,\infty)$. In Theorem 5.4.26 (for the case of non-ample geodesics) and Theorem 5.5.26 (cf. Theorem 5.1.8) in the Heisenberg group, equipped with singular norms, we contradict the $\mathsf{MCP}(K,N)$ condition. More precisely, we find a counterexample to (5.2.4).

5.2.2 Sub-Finsler structures

Let M be a smooth manifold of dimension n and let $k \in \mathbb{N}$. A *sub-Finsler structure* on M is a couple $(\xi, \|\cdot\|)$ where $\|\cdot\| : \mathbb{R}^k \to \mathbb{R}_+$ is a strictly convex norm on $\mathbb{R}^k$ and $\xi : M \times \mathbb{R}^k \to TM$ is a morphism of vector bundles such that:

(i) each fiber of the (trivial) bundle $M \times \mathbb{R}^k$ is equipped with the norm $\|\cdot\|$;

(ii) The set of horizontal vector fields, defined as

$$\mathcal{D} := \{\xi \circ \sigma \,:\, \sigma \in \Gamma(M \times \mathbb{R}^k)\} \subset \Gamma(TM),$$

is a *bracket-generating* family of vector fields (or it satisfies the Hörmander condtion), namely setting

$$\mathrm{Lie}_q(\mathcal{D}) := \{X(q) \,:\, X \in \mathrm{span}\{[X_1, \dots, [X_{j-1}, X_j]] \,:\, X_i \in \mathcal{D}, j \in \mathbb{N}\}\}, \qquad \forall\, q \in M,$$

we assume that $\mathrm{Lie}_q(\mathcal{D}) = T_qM$, for every $q \in M$.

We say that M is a *smooth sub-Finsler manifold*, if the norm of the sub-Finsler structure $(\xi, \|\cdot\|)$ is smooth, namely $\|\cdot\| \in C^\infty(\mathbb{R}^k \setminus \{0\})$.

Remark 5.2.7. Although this definition is not completely general in sub-Finsler context, since it does not allow the norm to vary on the fiber of $M \times \mathbb{R}^k$, it includes sub-Riemannian geometry (where $\|\cdot\|$ is induced by a scalar product), as every sub-Riemannian structure is equivalent to a free one, cf. [ABB20, Sec. 3.1.4].

At every point $q \in M$ we define the *distribution* at q as

$$\mathcal{D}_q := \{\xi(q, w) \,:\, w \in \mathbb{R}^k\} = \{X(q) \,:\, X \in \mathcal{D}\} \subset T_qM. \tag{5.2.5}$$

This is a vector subspace of T_qM whose dimension is called *rank* (of the distribution) and denoted by $r(q) := \dim \mathcal{D}_q \le n$. Moreover, the distribution is described by a family of horizontal vector fields. Indeed, letting $\{e_i\}_{i=1,\dots,k}$ be the standard basis of $\mathbb{R}^k$, the *generating frame* is the family $\{X_i\}_{i=1,\dots,k}$, where

$$X_i(q) := \xi(q, e_i) \qquad \forall\, q \in M, \quad \text{for } i = 1, \dots, k.$$

Then, according to (5.2.5), $\mathcal{D}_q = \mathrm{span}\{X_1(q), \dots, X_k(q)\}$. On the distribution we define the *induced norm* as

$$\|v\|_q := \inf\{\, \|w\| \,:\, v = \xi(q, w)\} \qquad \text{for every } v \in \mathcal{D}_q.$$

Since the infimum is actually a minimum, the function $\|\cdot\|_q$ is a norm on $\mathcal{D}_q$, so that $(\mathcal{D}_q, \|\cdot\|_q)$ is a normed space. Moreover, the norm depends smoothly on the base point $q \in M$. A curve $\gamma : [0, 1] \to M$ is *admissible* if its velocity $\dot\gamma(t)$ exists almost everywhere and there exists a function $u = (u_1, \dots, u_k) \in L^2([0, 1]; (\mathbb{R}^k, \|\cdot\|))$ such that

$$\dot\gamma(t) = \sum_{i=1}^k u_i(t) X_i(\gamma(t)), \qquad \text{for a.e. } t \in [0, 1]. \tag{5.2.6}$$

The function u is called *control*. Furthermore, given an admissible curve γ, there exists $\bar u = (\bar u_1, \dots, \bar u_k) : [0, 1] \to \mathbb{R}^k$ such that

$$\dot\gamma(t) = \sum_{i=1}^k \bar u_i(t) X_i(\gamma(t)), \qquad \text{and} \qquad \|\dot\gamma(t)\|_{\gamma(t)} = \|\bar u(t)\|, \qquad \text{for a.e. } t \in [0, 1]. \tag{5.2.7}$$

The function $\bar u$ is called *minimal control*, and it belongs to $L^2([0, 1]; (\mathbb{R}^k, \|\cdot\|))$, cf. [ABB20, Lem. 3.12]. We define the *length* of an admissible curve:

$$\ell(\gamma) := \int_0^1 \|\dot\gamma(t)\|_{\gamma(t)}\, \mathrm{d}t \in [0, \infty).$$

We can rewrite the length of a curve as the L^1-norm of the associated minimal control, indeed by (5.2.7),

$$\ell(\gamma) = \int_0^1 \|\bar{u}(t)\| \, \mathrm{d}t = \|\bar{u}\|_{L^1([0,1];(\mathbb{R}^k,\|\cdot\|))} \,. \tag{5.2.8}$$

For every couple of points $q_0, q_1 \in M$, define the *sub-Finsler distance* between them as

$$\mathsf{d}_{SF}(q_0, q_1) = \inf \left\{ \ell(\gamma) \,:\, \gamma \text{ admissible, } \gamma(0) = q_0 \text{ and } \gamma(1) = q_1 \right\}.$$

Since every norm on $\mathbb{R}^k$ is equivalent to the standard scalar product on $\mathbb{R}^k$, it follows that the sub-Riemannian structure on M given by $(\xi, \langle\cdot,\cdot\rangle)$ induces an equivalent distance. Namely, denoting by d_{SR} the induced sub-Riemannian distance, there exist constants $C > c > 0$ such that

$$c\,\mathsf{d}_{SR} \leq \mathsf{d}_{SF} \leq C\mathsf{d}_{SR}, \qquad \text{on } M \times M. \tag{5.2.9}$$

Thus, as a consequence of the classical Chow–Rashevskii Theorem in sub-Riemannian geometry, we obtain the following.

Proposition 5.2.8 (Chow–Rashevskii)**.** *Let M be a sub-Finsler manifold. The sub-Finsler distance is finite, continuous on $M \times M$ and the induced topology is the manifold one.*

From this proposition, we get that (M, d_{SF}) is a locally compact metric space. The local existence of minimizers of the length functional can be obtained as in the sub-Riemannian setting, in particular, one can repeat the proof of [ABB20, Thm. 3.43]. Finally, if (M, d_{SF}) is complete, then it is also a geodesic metric space.

5.3 The geometry of smooth sub-Finsler manifolds

5.3.1 The energy functional and the optimal control problem

Let $\gamma : [0,1] \to M$ be an admissible curve. Then, we define the *energy* of γ as

$$J(\gamma) = \frac{1}{2} \int_0^1 \|\dot{\gamma}(t)\|^2_{\gamma(t)} \, \mathrm{d}t.$$

By definition of admissible curve, $J(\gamma) < +\infty$. In addition, a standard argument shows that $\gamma : [0,1] \to M$ is a minimum for the energy functional if and only if it is a minimum of the length functional with constant speed.

Remark 5.3.1. The minimum of J is not invariant under reparametrization of γ, so one needs to fix the interval where the curve is defined. Here and below, we choose $[0,1]$.

The problem of finding geodesics between two points $q_0, q_1 \in M$ can be formulated using the energy functional as the following costrained minimization problem:

$$\begin{cases} \gamma : [0,1] \to M, \quad \text{admissible}, \\ \gamma(0) = q_0 \text{ and } \gamma(1) = q_1, \\ J(\gamma) = \dfrac{1}{2} \displaystyle\int_0^1 \|\dot{\gamma}(t)\|^2_{\gamma(t)} \, \mathrm{d}t \to \min. \end{cases} \tag{P}$$

The problem (P) can be recasted as an optimal control problem. First of all, a curve is admissible if and only if there exists a control in $L^2([0,1]; (\mathbb{R}^k, \|\cdot\|))$ satisfying (5.2.6). Second of all, we can consider the energy as a functional on the space of controls. Indeed, as in (5.2.8), given an admissible curve $\gamma : [0,1] \to M$, we let $\bar{u} \in L^2([0,1]; (\mathbb{R}^k, \|\cdot\|))$ be its minimal control, as in (5.2.7). Then, we have

$$J(\gamma) = \frac{1}{2}\int_0^1 \|\bar{u}(t)\|^2 \, \mathrm{d}t = \frac{1}{2}\|\bar{u}\|^2_{L^2([0,1];(\mathbb{R}^k,\|\cdot\|))} . \tag{5.3.1}$$

Hence, we regard the energy as a functional on $L^2([0,1]; (\mathbb{R}^k, \|\cdot\|))$, namely

$$J : L^2([0,1]; (\mathbb{R}^k, \|\cdot\|)) \to \mathbb{R}_+; \qquad J(u) := \frac{1}{2}\|u\|^2_{L^2([0,1];(\mathbb{R}^k,\|\cdot\|))} ,$$

and we look for a constrained minimum of it. Thus, the problem (P) becomes:

$$\begin{cases} \dot\gamma(t) = \displaystyle\sum_{i=1}^k u_i(t) X_i(\gamma(t)), \\ \gamma(0) = q_0 \text{ and } \gamma(1) = q_1, \\ J(u) = \displaystyle\frac{1}{2}\int_0^1 \|u(t)\|^2 \, \mathrm{d}t \to \min . \end{cases} \tag{P$'$}$$

Note that, by (5.3.1), the solutions of (P) and (P′) coincide. An application of Pontryagin Maximum Principle (see [AS04, Thm. 12.10]) yields necessary conditions for optimality. For every $u \in \mathbb{R}^k$ and $\nu \in \mathbb{R}$, introduce the following Hamiltonian:

$$h_u^\nu(\lambda) := \langle \lambda, \xi(\pi(\lambda), u)\rangle + \frac{\nu}{2}\|u\|^2 , \qquad \forall \lambda \in T^*M. \tag{5.3.2}$$

Recall that for $h \in C^1(T^*M)$, its Hamiltonian vector field $\vec{h} \in \mathrm{Vec}(T^*M)$ is defined as the unique vector field in T^*M satisfying

$$d_\lambda h = \sigma(\cdot, \vec{h}(\lambda)), \qquad \forall\, \lambda \in T^*M,$$

where σ is the canonical symplectic form on T^*M.

Theorem 5.3.2 (Pontryagin Maximum Principle)**.** *Let M be a sub-Finsler manifold and let $(\gamma, \bar{u})$ be a solution of* (P′)*. Then, there exists $(\nu, \lambda_t) \neq 0$, where $\nu \in \mathbb{R}$ and $\lambda_t \in T^*_{\gamma(t)}M$ for every $t \in [0,1]$, such that*

$$\begin{cases} \dot\lambda_t = \vec{h}^\nu_{\bar{u}(t)}(\lambda_t) & \text{for a.e. } t \in [0,1], \\ h^\nu_{\bar{u}(t)}(\lambda_t) = \displaystyle\max_{v\in\mathbb{R}^k} h^\nu_v(\lambda_t) & \text{for a.e. } t \in [0,1], \\ \nu \leq 0. \end{cases} \tag{H}$$

Definition 5.3.3. If $\nu < 0$ in (H), $(\lambda_t)_{t\in[0,1]}$ is called *normal extremal.* If $\nu = 0$ in (H), $(\lambda_t)_{t\in[0,1]}$ is called *abnormal extremal.*

Remark 5.3.4. By homogeneity of the Hamiltonian system, if $\nu \neq 0$ in (H) we can fix $\nu = -1$.

5.3.2 Characterization of extremals and the exponential map

In this section, we recall some characterizations of normal and abnormal extremal, which are well-known in sub-Riemannian geometry. We include the proofs in our case, for the sake of completeness.

Recall that the annihilator $\mathrm{Ann}(\mathcal{D}) \subset T^*M$ is defined by

$$\mathrm{Ann}(\mathcal{D})_q := \{\lambda \in T^*_qM \,:\, \langle \lambda, w\rangle = 0, \forall\, w \in \mathcal{D}_q\}, \qquad \forall\, q \in M.$$

Lemma 5.3.5. *Let M be a sub-Finsler manifold. Let $(\gamma, \bar{u})$ be a non-trivial solution to* (P′) *and let $(\lambda_t)_{t\in[0,1]}$ be its lift. Then, $(\lambda_t)_{t\in[0,1]}$ is an abnormal extremal if and only if $\lambda_t \neq 0$ and $\lambda_t \in \mathrm{Ann}(\mathcal{D})_{\gamma(t)}$ for every $t \in [0,1]$, where $\gamma(t) := \pi(\lambda_t)$.*

Proof. The claim is an easy consequence of the maximization property of the Hamiltonian along the dynamic. More precisely, by (H), we have

$$h^{\nu}_{\bar{u}(t)}(\lambda_t) = \max_{v\in\mathbb{R}^k} h^{\nu}_v(\lambda_t), \qquad \text{for a.e. } t \in [0,1], \tag{5.3.3}$$

where the function h^{ν}_u is defined in (5.3.2). Assume that $\nu = 0$, then (5.3.3) reads as

$$\langle \lambda_t, \xi(\gamma(t), \bar{u}(t))\rangle = \max_{v\in\mathbb{R}^k} \langle \lambda_t, \xi(\gamma(t), v)\rangle, \qquad \text{for a.e. } t \in [0,1],$$

with $\lambda_t \neq 0$ for every $t \in [0,1]$. Now, since ξ is linear in the controls, the right-hand side is $+\infty$ unless $\lambda_t \in \mathrm{Ann}(\mathcal{D})_{\gamma(t)}$ for a.e. $t \in [0,1]$. By continuity of $t \mapsto \lambda_t$, this is true for every $t \in [0,1]$. Conversely, assume that $\lambda_t \in \mathrm{Ann}(\mathcal{D})_{\gamma(t)}$, then the maximization condition (5.3.3) becomes

$$\frac{\nu}{2}\|\bar{u}(t)\|^2 = \max_{v\in\mathbb{R}^k} \frac{\nu}{2}\|v\|^2.$$

Since $\nu \leq 0$, we may distinguish two cases, either $\nu = 0$ and the extremal is abnormal, or $\nu = -1$ and the extremal is normal. In the second case, the optimal control must be 0, so that $\dot{\lambda}_t = 0$ and the extremal is constant and constantly equal to λ_0. Since we are assuming $(\lambda_t)_{t\in[0,1]}$ to be non-constant the latter can not happen. ☐

The *sub-Finsler (or maximized) Hamiltonian* is defined as

$$H(\lambda) := \max_{u\in\mathbb{R}^k} h^{-1}_u(\lambda_t) = \max_{u\in\mathbb{R}^k} \left(\sum_{i=1}^k \langle \lambda, u_i X_i(\pi(\lambda))\rangle - \frac{\|u\|^2}{2} \right) \tag{5.3.4}$$

The sub-Finsler Hamiltonian can be explicitly characterized in terms of the norm $\|\cdot\|_*$, which denotes the dual norm of $\|\cdot\|$. To this aim, we prove the following lemma describing the dual element of $v \in (\mathbb{R}^k, \|\cdot\|)$, when $\|\cdot\|$ is a C^1 norm. Recall that its dual vector $v^* \in (\mathbb{R}^k, \|\cdot\|)^*$ is uniquely characterized by

$$\|v^*\|_* = \|v\| \qquad \text{and} \qquad \langle v^*, v\rangle = \|v\|^2, \tag{5.3.5}$$

where $\langle \cdot, \cdot \rangle$ is the dual coupling.

Lemma 5.3.6. *Let $(\mathbb{R}^k, \|\cdot\|)$ be a normed space and assume $\|\cdot\| : \mathbb{R}^k \to \mathbb{R}_+$ is a strictly convex C^1 norm, i.e. $\|\cdot\| \in C^1(\mathbb{R}^k \setminus \{0\})$. Then, for every non-zero vector $v \in \mathbb{R}^k$, it holds that*

$$v^* = \|v\| \cdot d_v\|\cdot\|.$$

Proof. Set $\lambda := d_v\|\cdot\| \in (\mathbb{R}^k, \|\cdot\|)^*$, where we recall that

$$d_v\|\cdot\|(u) := \lim_{t\to 0} \frac{\|v + tu\| - \|u\|}{t}, \qquad \forall\, u, v \in \mathbb{R}^k.$$

Then, on the one hand, it holds that

$$\langle \lambda, v\rangle = d_v\|\cdot\|(v) = \lim_{t\to 0} \frac{\|v + tv\| - \|v\|}{t} = \|v\|\,.$$

On the other hand, we have

$$\|\lambda\|_* = \sup_{u\in B_1}\langle\lambda,u\rangle = \sup_{u\in B_1} d_v\|\cdot\|(u) = \sup_{u\in B_1}\lim_{t\to 0}\frac{1}{t}\big(\,\|v+tu\|-\|v\|\,\big) \le \sup_{u\in B_1}\|u\| = 1,$$

where $B_1 := B_1^{\|\cdot\|}(0)\subset\mathbb{R}^k$ is the ball of radius 1 and centered at 0, with respect to the norm $\|\cdot\|$. The converse inequality can be obtained by taking $u=\frac{v}{\|v\|}$. Finally, the conclusion follows by homogeneity of the dual norm. □

Lemma 5.3.7. *Let M be a smooth sub-Finsler manifold. Given $\lambda\in T^*M$, define $\hat\lambda=(\hat\lambda_i)_{i=1,\dots,k}$ where $\hat\lambda_i := \langle\lambda, X_i(\pi(\lambda))\rangle$ for every $i=1,\dots,k$. Then, $H^{-1}(0)=\mathrm{Ann}(\mathcal{D})$ and*

$$H(\lambda)=\frac{1}{2}\|\hat\lambda\|_*^2,\qquad \forall\lambda\in T^*M\setminus\mathrm{Ann}(\mathcal{D}).$$

where $\|\cdot\|_*$ is the dual norm to $\|\cdot\|$ in $\mathbb{R}^k$. Moreover, $H\in C^\infty(T^*M\setminus\mathrm{Ann}(\mathcal{D}))\cap C^1(T^*M)$.

Proof. Let $q\in M$. Assume that $\lambda\in T_q^*M\setminus\mathrm{Ann}(\mathcal{D})_q$ and set

$$F(u) := \langle\hat\lambda,u\rangle - \frac{\|u\|^2}{2},\qquad \forall u\in\mathbb{R}^k.$$

Since $\|\cdot\|$ is smooth, its square is a C^1-function, thus $F\in C^1(\mathbb{R}^k)$. Moreover, by homogeneity of the norm, $F(u)\to-\infty$ as $\|u\|\to\infty$, hence F admits a maximum. We compute its differential:

$$d_uF = \hat\lambda - \|u\|\cdot d_u\|\cdot\| = \hat\lambda - u^*, \tag{5.3.6}$$

according to Lemma 5.3.6. Therefore, F has a unique critical point (which is also the unique point of maximum) given by $u=u^{**}=\hat\lambda^*$. Finally, using also (5.3.5), this implies that

$$H(\lambda)=\max_{u\in\mathbb{R}^k}F(u) = F(\hat\lambda^*) = \langle\hat\lambda,\hat\lambda^*\rangle - \frac{1}{2}\|\hat\lambda\|_*^2 = \frac{1}{2}\|\hat\lambda\|_*^2.$$

To conclude, observe that if $\lambda\in\mathrm{Ann}(\mathcal{D})_q$, then $\hat\lambda=0$ and

$$H(\lambda)=\max_{u\in\mathbb{R}^k}\left(-\frac{\|u\|^2}{2}\right)=0.$$

Conversely, if $H(\lambda)=0$ we must have $\lambda\in\mathrm{Ann}(\mathcal{D})$. Indeed, if this is not the case, $\hat\lambda\neq 0$ and hence $\|\hat\lambda\|_*\neq 0$, giving a contradiction. This proves that $H^{-1}(0)=\mathrm{Ann}(\mathcal{D})$.

Finally, we prove the regularity of H. Note that $\|\cdot\|_*$ is a smooth norm itself. Indeed, as $\|\cdot\|$ is smooth and strictly convex, the dual map of Lemma 5.3.6, which is

$$v^* = \|v\|\,d_v\|\cdot\| = \frac{1}{2}d_v(\|\cdot\|^2) =: N(v),$$

is smooth on $v\neq 0$, invertible and with invertible differential on $v\neq 0$. Thus, by the inverse function theorem, $N^{-1}\in C^\infty(\mathbb{R}^k\setminus\{0\})$. But now the dual norm satisfies (5.3.5), hence

$$\|\hat\lambda\|_* = \|N^{-1}(\hat\lambda)\|,$$

and the claim follows. Therefore, we deduce that $H\in C^\infty(T^*M\setminus H^{-1}(0))\cap C^1(T^*M) = C^\infty(T^*M\setminus\mathrm{Ann}(\mathcal{D}))\cap C^1(T^*M)$. □

Corollary 5.3.8. *Let $(\lambda_t)_{t\in[0,1]}$ be a normal extremal for the problem* (H)*, then the associated control is given by*

$$\bar u(t) = \hat\lambda_t^*,\qquad \textit{for a.e. } t\in[0,1]. \tag{5.3.7}$$

Proof. This is again a consequence of the maximality condition in (H), together with the characterization of the sub-Finsler Hamiltonian. In particular, we must have

$$h^{-1}_{\bar{u}(t)}(\lambda_t) = \max_{u \in \mathbb{R}^k} h^{-1}_u(\lambda_t) = H(\lambda_t), \qquad \forall\, t \in [0,1],$$

From this and (5.3.6), we deduce that the control associated with $(\lambda_t)_{t\in[0,1]}$ satisfies the identity (5.3.7). □

The next result relates the system (H) with the Hamiltonian system associated with the sub-Finsler Hamiltonian (5.3.4). A similar statement can be found in [AS04, Prop. 12.3]. The main difference is the regularity of the Hamiltonian function, which in the classical statement is assumed to be smooth outside the zero section.

Proposition 5.3.9. *Let M be a smooth sub-Finsler manifold. Let $H \in C^\infty(T^*M \setminus \mathrm{Ann}(\mathcal{D})) \cap C^1(T^*M)$ be the sub-Finsler Hamiltonian defined in* (5.3.4)*. If $(\lambda_t)_{t\in[0,1]}$ is a normal extremal, and $\lambda_0 \in \mathrm{Ann}(\mathcal{D})$, then $\lambda_t \equiv \lambda_0$. If $\lambda_0 \in T^*M \setminus \mathrm{Ann}(\mathcal{D})$, then*

$$\dot{\lambda}_t = \vec{H}(\lambda_t). \tag{5.3.8}$$

Conversely, if $(\lambda_t)_{t\in[0,1]}$ is a solution of (5.3.8) *with initial condition $\lambda_0 \in T^*M \setminus \mathrm{Ann}(\mathcal{D})$, then there exists $\bar{u} \in L^2([0,1]; (\mathbb{R}^k, \|\cdot\|))$ such that $(\lambda_t)_{t\in[0,1]}$ is a normal extremal with control $\bar{u}$ (i.e. the pair $(-1, (\lambda_t))$ is a solution of* (H)*).*

Proof. If $(\lambda_t)_{t\in[0,1]}$ is a normal extremal, there exists an optimal control $\bar{u}$ such that the pair $(-1, (\lambda_t))$ is a solution to (H). Now, if the initial covector $\lambda_0 \in \mathrm{Ann}(\mathcal{D})$, then $\lambda_t \in \mathrm{Ann}(\mathcal{D})$ for all $t \in [0,1]$, as the sub-Finsler Hamiltonian is constant along the motion and $H(\lambda_0) = 0$. Using Corollary 5.3.8, this implies that the control $\bar{u} \equiv 0$ and that $\lambda_t \equiv \lambda_0$ as claimed. If $\lambda_0 \in T^*M \setminus \mathrm{Ann}(\mathcal{D})$, we follow the blueprint of [AS04, Prop. 12.3]. By the definition of sub-Finsler Hamiltonian (5.3.4), we have

$$H(\lambda) \geq h_{\bar{u}(t)}(\lambda), \qquad \forall\, \lambda \in T^*M,\ t \in [0,1],$$

with equality along the dynamic $t \mapsto \lambda_t$. This means that the function $T^*M \ni \lambda \mapsto H(\lambda) - h_{\bar{u}(t)}(\lambda)$ has a maximum at λ_t. Therefore, using that $H \in C^1(T^*M)$, we deduce that

$$d_{\lambda_t} H = d_{\lambda_t} h_{\bar{u}(t)}, \qquad \forall\, t \in [0,1].$$

Such an equality immediately implies that the Hamiltonian vector fields are equal along the dynamic, namely

$$\vec{H}(\lambda_t) = \vec{h}_{\bar{u}(t)}(\lambda_t), \qquad \forall\, t \in [0,1].$$

For the converse implication, recall that the Hamiltonian is constant along the motion. So, if $(\lambda_t)_{t\in[0,1]}$ is a solution to (5.3.8) with initial condition $\lambda_0 \in T^*M \setminus \mathrm{Ann}(\mathcal{D})$, then $H(\lambda_0) = H(\lambda_t)$ and $\lambda_t \in T^*M \setminus \mathrm{Ann}(\mathcal{D})$, for every $t \in [0,1]$. Since H is smooth outside the annihilator bundle of $\mathcal{D}$, we deduce that $(\lambda_t)_{t\in[0,1]}$ is uniquely determined by λ_0 and, repeating verbatim the argument of [AS04, Prop. 12.3], we conclude the proof. □

Fix $t \in \mathbb{R}$ and consider the *(reduced) flow of* $\vec{H}$ on $T^*M \setminus \mathrm{Ann}(\mathcal{D})$:

$$e^{t\vec{H}}_{\mathrm{r}} : \mathscr{A}_t \to T^*M \setminus \mathrm{Ann}(\mathcal{D}),$$

where $\mathscr{A}_t \subset T^*M \setminus \mathrm{Ann}(\mathcal{D})$ is the set of covectors such that the associated maximal solution $(\lambda_s)_{s\in I}$, with $I \subset \mathbb{R}$ such that $0 \in I$, is defined up to time t. Under the assumption of completeness of (M, d_{SF}), $\vec{H}$ is complete as a vector field on $T^*M \setminus \mathrm{Ann}(\mathcal{D})$ (and thus $\mathscr{A}_t = T^*M \setminus \mathrm{Ann}(\mathcal{D})$). We state below this result without proof as the latter is analogous to the classical sub-Riemannian proof, cf. [ABB20, Prop. 8.38], in view of Lemma 5.3.6 and Lemma 5.3.7.

Proposition 5.3.10. *Let M be a smooth complete sub-Finsler manifold. Then any normal extremal $t \mapsto \lambda_t = e_{\mathrm{r}}^{t\vec{H}}(\lambda_0)$, with $\lambda_0 \in T^*M \setminus \mathrm{Ann}(\mathcal{D})$, is extendable to $\mathbb{R}$.*

Definition 5.3.11 (Sub-Finsler exponential map)**.** Let (M, d_{SF}) be a complete smooth sub-Finsler manifold and let $q \in M$. Then, the *sub-Finsler exponential map* at q is defined as

$$\exp_q(\lambda) := \begin{cases} \pi \circ e_{\mathrm{r}}^{\vec{H}}(\lambda) & \text{if } \lambda \in T_q^*M \setminus \mathrm{Ann}(\mathcal{D})_q, \\ q & \text{if } \lambda \in \mathrm{Ann}(\mathcal{D})_q. \end{cases}$$

Remark 5.3.12. The exponential map is smooth in $T_q^*M \setminus \mathrm{Ann}(\mathcal{D})_q$ and, by homogeneity, we also have $\exp_q \in C^1(T_q^*M)$. However, note that the we do not have spatial regularity. More precisely, setting $\mathcal{E} : T^*M \to M$ to be $\mathcal{E}(q,\lambda) = \exp_q(\lambda)$, then

$$\mathcal{E} \in C(T^*M) \cap C^\infty(T^*M \setminus \mathrm{Ann}(\mathcal{D})),$$

and we can not expect a better spatial regularity as the vector field $\vec{H}$ is only continuous on the annihilator bundle.

5.3.3 The end-point map

Let M be a sub-Finsler manifold, consider $u \in L^2([0,1]; (\mathbb{R}^k, \|\cdot\|))$ and fix $t_0 \in [0,1]$. Define the non-autonomous vector field

$$\xi_{u(t)}(q) := \xi(q, u(t)) = \sum_{i=1}^{k} u_i(t) X_i(q), \qquad \forall\, q \in M, t \in [0,1],$$

and denote by $P^u_{t_0,t} : M \to M$ its flow. This means that, for $q_0 \in M$, the curve $t \mapsto P^u_{t_0,t}(q_0)$ is the unique maximal solution to the Cauchy problem

$$\begin{cases} \dot\gamma(t) = \displaystyle\sum_{i=1}^{m} u_i(t) X_i(\gamma(t)), \\ \gamma(t_0) = q_0. \end{cases}$$

Moreover, we denote by $\gamma_u : I \to M$, the trajectory starting at q_0 and corresponding to u, namely $\gamma_u(t) := P^u_{0,t}(q_0)$, for every $t \in I$, which is the maximal interval of definition of γ_u.

Definition 5.3.13 (End-point map)**.** Let M be a sub-Finsler manifold and let $q_0 \in M$. We call $\mathcal{U}_{q_0} \subset L^2([0,1]; (\mathbb{R}^k, \|\cdot\|))$ the open set of controls for which the corresponding trajectory γ_u is defined on the interval $[0,1]$. We define the *end-point map* based at q_0 as

$$E_{q_0} : \mathcal{U}_{q_0} \to M, \qquad E_{q_0}(u) = \gamma_u(1).$$

Lemma 5.3.14 (Differential of the end-point map, [ABB20, Prop. 8.5])**.** *Let M be a sub-Finsler manifold. The end-point map is smooth on $\mathcal{U}_{q_0}$. Moreover, for every $u \in \mathcal{U}_{q_0}$ the differential $d_u E_{q_0} : L^2([0,1]; (\mathbb{R}^k, \|\cdot\|)) \to T_{E_{q_0}(u)}M$ has the expression*

$$d_u E_{q_0}(v) = \int_0^1 \left(P^u_{t,1}\right)_* \xi_{v(t)}(E_{q_0}(u))\, \mathrm{d}t, \qquad \text{for every } v \in L^2([0,1]; (\mathbb{R}^k, \|\cdot\|)).$$

Using the explicit expression for the differential of the end-point map we deduce the following characterization for normal and abnormal extremals in sub-Finsler geometry, see [ABB20, Prop. 8.9] for the analogous sub-Riemannian result.

Proposition 5.3.15. *Let M be a smooth sub-Finsler manifold and let (γ, u) be a non-trivial solution to* (P′)*. Then, there exists $\lambda_1 \in T^*_{q_1}M$, where $q_1 = E_{q_0}(u)$, such that the curve $(\lambda_t)_{t\in[0,1]}$, with*

$$\lambda_t := (P^u_{t,1})^* \lambda_1 \in T^*_{\gamma(t)}M, \qquad \forall\, t \in [0,1], \tag{5.3.9}$$

is a solution to (H)*. Moreover, one of the following conditions is satisfied:*

(i) $(\lambda_t)_{t\in[0,1]}$ *is a normal extremal if and only if u satisfies*

$$\langle \lambda_1, d_u E_{q_0}(v)\rangle = \langle u^*, v\rangle \qquad \text{for every } v \in L^2([0,1]; (\mathbb{R}^k, \|\cdot\|)); \tag{5.3.10}$$

(ii) $(\lambda_t)_{t\in[0,1]}$ *is an abnormal extremal if and only if u satisfies*

$$\langle \lambda_1, d_u E_{q_0}(v)\rangle = 0 \qquad \text{for every } v \in L^2([0,1]; (\mathbb{R}^k, \|\cdot\|)).$$

Proof. Firstly, (5.3.9) is a well-known consequence of the Pontryagin maximum principle. We prove *(i)*, the proof of *(ii)* is analogous. For every $v \in L^2([0,1]; (\mathbb{R}^k, \|\cdot\|))$, using Lemma 5.3.14 we deduce that

$$\begin{aligned}\langle \lambda_1, d_u E_{q_0}(v)\rangle &= \int_0^1 \langle \lambda_1, (P^u_{t,1})_* \xi_{v(t)}(E_{q_0}(u))\rangle \,\mathrm{d}t = \int_0^1 \langle (P^u_{t,1})^*\lambda_1, \xi_{v(t)}(\gamma(t))\rangle \,\mathrm{d}t \\ &= \int_0^1 \langle \lambda_t, \xi_{v(t)}(\gamma(t))\rangle \,\mathrm{d}t = \int_0^1 \sum_{i=1}^k \langle \lambda_t, \xi_i(\gamma(t))\rangle v_i(t) \,\mathrm{d}t.\end{aligned} \tag{5.3.11}$$

Assume $(\lambda_t)_{t\in[0,1]}$ is a normal extremal, then, by Corollary 5.3.8, the associated optimal control u satisfies $\langle \lambda_t, \xi_i(\gamma(t))\rangle = u_i^*(t)$ for $i = 1, \dots, k$. Therefore, using (5.3.11) we deduce that for every $v \in L^2([0,1]; (\mathbb{R}^k, \|\cdot\|))$ it holds

$$\langle \lambda_1, d_u E_{q_0}(v)\rangle = \int_0^1 \sum_{i=1}^k \langle \lambda_t, \xi_i(\gamma(t))\rangle v_i(t) \,\mathrm{d}t = \int_0^1 \langle u^*(t), v(t)\rangle \,\mathrm{d}t = \langle u^*, v\rangle.$$

This proves (5.3.10).

Conversely, assume that the control u satisfies (5.3.10). We are going to prove that $(\lambda_t)_{t\in[0,1]}$ is a normal extremal. Using (5.3.10) and (5.3.11) we deduce that for every $v \in L^2([0,1]; (\mathbb{R}^k, \|\cdot\|))$

$$\langle u^*, v\rangle_{L^2} = \langle \lambda_1, d_u E_{q_0}(v)\rangle = \int_0^1 \sum_{i=1}^k \langle \lambda_t, \xi_i(\gamma(t))\rangle v_i(t) \,\mathrm{d}t,$$

As a consequence, since v is arbitrary, we conclude that $\langle \lambda_t, \xi_i(\gamma(t))\rangle = u_i^*(t)$ for $i = 1, \dots, k$. □

5.4 Failure of the $\mathsf{CD}(K,N)$ condition in smooth sub-Finsler manifolds

In this section, we prove our main result regarding smooth sub-Finsler manifolds, cf. Theorem 5.1.5. One of the crucial ingredients for the proof is the construction of a geodesic, enjoying good regularity properties, cf. Theorem 5.4.11.

5.4.1 Construction of a geodesic without abnormal sub-segments

This section is devoted to the construction of a geodesic without abnormal sub-segments, in smooth sub-Finsler manifolds. The main idea is to choose a short segment of a normal geodesic that minimizes the distance from a hypersurface without characteristic points. We recall the definition of strongly normal geodesic and of geodesic without abnormal sub-segments.

Definition 5.4.1. Let M be a sub-Finsler manifold and let $\gamma : [0,1] \to M$ be a normal geodesic. Then, we say that γ is

(i) *left strongly normal*, if for all $s \in [0,1]$, the restriction $\gamma|_{[0,s]}$ is not abnormal;

(ii) *right strongly normal*, if for all $s \in [0,1]$, the restriction $\gamma|_{[s,1]}$ is not abnormal;

(iii) *strongly normal*, if γ is left and right strongly normal.

Finally, we say that γ does not admit abnormal sub-segments if any restriction of γ is strongly normal.

Let $\Sigma \subset M$ be a hypersurface and let $\gamma \colon [0,T] \to M$ be a horizontal curve, parameterized with constant speed, such that $\gamma(0) \in \Sigma$, $\gamma(T) = p \in M \setminus \Sigma$. Assume γ is a minimizer for $\mathsf{d}_{SF}(\cdot, \Sigma)$, that is $\ell(\gamma) = \mathsf{d}_{SF}(p, \Sigma)$. Then, γ is a geodesic and any corresponding normal or abnormal lift, say $\lambda : [0,T] \to T^*M$, must satisfy the transversality conditions, cf. [AS04, Thm 12.13],

$$\langle \lambda_0, w \rangle = 0, \qquad \forall\, w \in T_{\gamma(0)}\Sigma. \tag{5.4.1}$$

Equivalently, the initial covector λ_0 must belong to the annihilator bundle $\mathrm{Ann}(\Sigma)$ of Σ with fiber $\mathrm{Ann}(\Sigma)_q = \{\lambda \in T_q^*M \mid \langle \lambda, T_q\Sigma \rangle = 0\}$, for any $q \in \Sigma$.

Remark 5.4.2. In the sub-Riemannian setting, the normal exponential map E, defined as the restriction of the exponential map to the annihilator bundle of Σ, allows to build (locally) a smooth tubular neighborhood around non-characteristic points, cf. [FPR20, Prop. 3.1]. This may fail in the sub-Finsler setting as E is not regular at Σ, cf. Remark 5.3.12. Nonetheless, we are able to deduce a weaker result that is enough for our construction, see Theorem 5.4.8.

Recall that $q \in \Sigma$ is a *characteristic point*, and we write $q \in C(\Sigma)$, if $\mathcal{D}_q \subset T_q\Sigma$. As it happens in the sub-Riemannian case, also in the sub-Finsler setting, minimizers of $\mathsf{d}_{SF}(\cdot, \Sigma)$ whose initial point is a non-characteristic point, can not be abnormal geodesics.

Lemma 5.4.3. *Let M be a smooth sub-Finsler manifold. Let $p \in M \setminus \Sigma$ and let $\gamma : [0,1] \to M$ be a horizontal curve such that*

$$\gamma(0) \in \Sigma, \quad \gamma(1) = p \quad \textit{and} \quad \ell(\gamma) = \mathsf{d}_{SF}(p, \Sigma).$$

Then, $\gamma(0) \in C(\Sigma)$ if and only if γ is an abnormal geodesic.

Proof. The proof is a straightforward adaptation of the analogous result in the sub-Riemannian setting. We sketch here the argument for completeness. By the Pontryagin maximum principle, cf. Theorem 5.3.2, there exists a lift $\lambda : [0,1] \to T^*M$ verifying the system (H) with the additional condition (5.4.1). Using the characterization of Lemma 5.3.5, λ is an abnormal lift if and only if $\lambda_0 \in \mathrm{Ann}(\mathcal{D})_{\gamma(0)}$. The latter, combined with the transversality condition concludes the proof. □

From now on, we assume that Σ is the boundary of an open set $\Omega \subset M$. As our results are local in nature, this assumption is not necessary, however it makes the presentation easier. Let $\Omega \subset M$ be a non-characteristic domain in M, so that $\partial\Omega$ is compact and without characteristic points. Then, there exists a never-vanishing smooth section of $\mathrm{Ann}(\partial\Omega)$, i.e. a smooth map $\lambda^+ : \partial\Omega \to \mathrm{Ann}(\partial\Omega)$ such that

$$\lambda^+(q) \in \mathrm{Ann}_q(\partial\Omega) \qquad \text{and} \qquad 2H(\lambda^+) = 1, \tag{5.4.2}$$

which is uniquely determined, up to a sign. Define the *normal exponential map* as the restriction of the sub-Finsler exponential map to the annihilator bundle, namely

$$E : D \to M, \qquad E(q,\lambda) = \exp_q(\lambda),$$

where $D \subset \mathrm{Ann}(\partial\Omega)$ is the largest open sub-bundle where E is defined. Furthermore, we define the *distance function from* $\partial\Omega$ as

$$\delta : M \to [0,\infty), \qquad \delta(p) := \mathsf{d}_{SF}(p, \partial\Omega).$$

Lemma 5.4.4. *Let M be a smooth sub-Finsler manifold. There exists $\epsilon > 0$ such that on the sub-bundle*

$$D_\epsilon := \{(q,\lambda) \in \mathrm{Ann}(\partial\Omega) : E(q,\lambda) \in \Omega \ \text{ and } \ 0 < \sqrt{2H(\lambda)} < \epsilon\} \subset D$$

the map $E|_{D_\epsilon}$ is injective and $E(D_\epsilon) = \{0 < \delta < \epsilon\} \cap \Omega$.

Proof. Without loss of generality, we assume that M is complete, so that $D = \mathrm{Ann}(\partial\Omega)$. We may proceed by contradiction and assume that there does not exist a choice of $\epsilon > 0$ so that $E|_{D_\epsilon}$ is injective. Hence, we can find sequences $\{(q_n,\lambda_n)\}, \{(q_n',\lambda_n')\} \subset \mathrm{Ann}(\partial\Omega)$ such that

$$(q_n,\lambda_n) \neq (q_n',\lambda_n') \qquad E(q_n,\lambda_n) = E(q_n',\lambda_n'), \qquad \text{and} \qquad H(\lambda_n),\, H(\lambda_n') \to 0. \tag{5.4.3}$$

Note that, as $\partial\Omega$ has no characteristic points, the sub-Finsler Hamiltonian is a norm on the fibers of $\mathrm{Ann}(\partial\Omega)$. Therefore, by compactness, $(q_n,\lambda_n) \to (q,0)$ and $(q_n',\lambda_n') \to (q',0)$, up to subsequences. Thus, recalling that E is continuous on D, passing to the limit in (5.4.3), we get that $E(q,0) = E(q',0)$, meaning that $q = q'$. As a consequence $\lambda_n, \lambda_n' \in \mathrm{Ann}(\partial\Omega)_q$ so they are multiple of the section defined in (5.4.2), namely

$$\lambda_n = t_n \lambda^+(q), \qquad \lambda_n' = t_n' \lambda^+(q),$$

where $t_n, t_n' \to 0$ and their signs agree. Finally, recall that the length of the normal curve $[0,1] \ni t \mapsto E(q,t\lambda)$ is exactly $\sqrt{2H(\lambda)}$. This forces $t_n = t_n'$ which is a contradiction with (5.4.3). We are left to prove the last part of the statement. Fix $\epsilon > 0$ so that $E|_{D_\epsilon}$ is injective, then $E(D_\epsilon) \subset \{0 < \delta < \epsilon\} \cap \Omega$. For the converse inclusion, pick $p \in \{0 < \delta < \epsilon\} \cap \Omega$ and let $\gamma : [0,1] \to M$ be a geodesic joining $\gamma(0) \in \partial\Omega$ and $p = \gamma(1)$ such that $\ell(\gamma) = \delta(p)$. Then, $\gamma(0)$ is a non-characteristic point, therefore γ is a normal geodesic, whose lift satisfies (5.4.1), according to Lemma 5.4.3. Hence, there exists $0 \neq \lambda \in \mathrm{Ann}(\partial\Omega)_{\gamma(0)}$ such that $\gamma(t) = E(\gamma(0), t\lambda)$ and $\ell(\gamma) = \sqrt{2H(\lambda)} < \epsilon$. Thus, $(q,\lambda) \in D_\epsilon$ concluding the proof. □

We state here a useful lemma regarding the regularity of the distance function from a boundary. Recall that a function $f : M \to \mathbb{R}$ is said to be *locally semiconcave* if, for every $p \in M$, there exist a coordinate chart $\varphi : U \subset M \to \mathbb{R}^n$, with $p \in U$, and a constant $C \in \mathbb{R}$ such that

$$F : \mathbb{R}^n \to \mathbb{R}; \qquad F(x) := f \circ \varphi^{-1}(x) - C\frac{|x|^2}{2},$$

is concave, where $|\cdot|$ denotes the Euclidean norm.

Lemma 5.4.5. *Let M be a smooth sub-Finsler manifold. Let $\Omega \subset M$ be an open and bounded subset. Assume that $\partial\Omega$ is smooth and without characteristic points. Then, the distance function from $\partial\Omega$, δ is locally semiconcave in Ω.*

Proof. We do not report here a complete proof, since it follows the same arguments of [ACS18], with the obvious modification for the sub-Finsler case. In particular, applying Lemma 5.4.3, we deduce there are no abnormal geodesic joining points of Ω to its boundary and realizing δ. Thus, the proof of [ACS18, Thm. 3.2] shows that δ is locally Lipschitz in coordinates, meaning that the function δ written in coordinates is Lipschitz with respect to the Euclidean distance. Then, using [ACS18, Thm. 4.1] implication (3) $\Rightarrow$ (2), we conclude. ☐

Since δ is locally semiconcave, Alexandrov's theorem ensures that δ is differentiable two times $\mathscr{L}^n$-a.e. (in coordinates) and, letting $\mathcal{U} \subset \Omega$ be the set where δ is differentiable, the function $d\delta : \mathcal{U} \to T^*M$ is differentiable $\mathscr{L}^n$-a.e., cf. [ABS21, Thm. 6.4] for the precise statement of Alexandrov's theorem. This observation, combined with Lemma 5.4.6 below, gives us an alternative description of geodesics joining $\partial\Omega$ and differentiablity points of δ in $\{0 < \delta < \epsilon\} \cap \Omega$.

Lemma 5.4.6. *Let M be a smooth sub-Finsler manifold. Let $p, q \in M$ be distinct points and assume there is a function $\phi : M \to \mathbb{R}$ differentiable at p and such that*

$$\phi(p) = \frac{1}{2}\mathsf{d}^2_{SF}(p, q) \qquad \text{and} \qquad \frac{1}{2}\mathsf{d}^2_{SF}(z, q) \geq \phi(z), \quad \forall\, z \in M. \tag{5.4.4}$$

Then, the geodesic joining p and q is unique, has a normal lift and is given by $\gamma : [0, 1] \to M$; $\gamma(t) = \exp_p(-t d_p\phi)$.

Proof. This is a well-known result in sub-Riemannian geometry, cf. [Rif14, Lem. 2.15]. The same proof can be carried out without substantial modifications in the setting of sub-Finsler manifolds, in light of Proposition 5.3.15. ☐

Corollary 5.4.7. *Let M be a smooth sub-Finsler manifold and let $\Omega \subset M$ be an open and bounded subset. Assume that $\partial\Omega$ is smooth and without characteristic points. Let $p \in \{0 < \delta < \epsilon\} \cap \Omega$ be a differentiability point of δ. Then, the unique geodesic $\gamma : [0, 1] \to M$ joining p and $\partial\Omega$ and such that $\delta(p) = \ell(\gamma)$ is defined by $\gamma(t) = \exp_p\big(-\frac{t}{2} d_p\delta^2\big)$.*

Proof. Since $p \in \{0 < \delta < \epsilon\} \cap \Omega$, from Lemma 5.4.4 we know that the geodesic joining p and $\partial\Omega$, and realizing δ is normal and unique. Let $q \in \partial\Omega$ be its endpoint and define

$$\phi : M \to \mathbb{R}; \qquad \phi(z) := \frac{1}{2}\delta^2(z).$$

Note that ϕ is differentiable at the point p, $\phi(p) = \frac{1}{2}\ell(\gamma)^2 = \frac{1}{2}\mathsf{d}^2_{SF}(p, q)$ and, since $q \in \partial\Omega$, it also satisfies the inequality in (5.4.4). Thus, we may apply Lemma 5.4.6 and conclude the proof. ☐

Collecting all the previous results, we are in position to prove the following theorem concerning the regularity of the normal exponential map.

Theorem 5.4.8. *Let M be a smooth sub-Finsler manifold. The restriction of the sub-Finsler normal exponential map to D_ϵ, namely $E|_{D_\epsilon} : D_\epsilon \to \{0 < \delta < \epsilon\} \cap \Omega$, defines a diffeomorphism on an open and dense subset $\mathcal{O} \subset D_\epsilon$. Moreover, δ is smooth on $E(\mathcal{O}) \subset \{0 < \delta < \epsilon\} \cap \Omega$, which is open and with full-measure.*

Proof. We are going to show that $d_{(q,\lambda)}E$ is invertible for every (q, λ) in a suitable subset of D_ϵ. By Corollary 5.4.7, letting $U \subset \{0 < \delta < \epsilon\} \cap \Omega$ be the set where δ is twice-differentiable, the map

$$\Phi : U \to \mathrm{Ann}(\partial\Omega); \qquad \Phi(p) = e^{-\vec{H}}(d_p\delta)$$

is a right-inverse for the normal exponential map, namely $E\circ\Phi = \mathrm{id}_U$. Note that $\Phi(U) \subset \mathrm{Ann}(\partial\Omega)$ by Corollary 5.4.7, in combination with the transversality condition (5.4.1). Moreover, recalling that the Hamiltonian is constant along the motion, we also have:

$$\sqrt{2H(\Phi(p))} = \ell(\gamma) = \delta(p) \in (0, \epsilon),$$

so that $\Phi(U) \subset D_\epsilon$. But now by the choice of the set U, δ is twice-differentiable on this set and it has a Taylor expansion up to order 2. Thus, expanding the identity $E \circ \Phi = \mathrm{id}_U$ at a point $p = E(q, \lambda)$, we deduce that $d_{(q,\lambda)}E$ must be invertible for every $(q, \lambda) \in \Phi(U) \subset D_\epsilon$, and thus E is a local diffeomorphism around every point in $\Phi(U)$. Furthermore, observing that $\Phi(U)$ is dense in D_ϵ, we see that E is a local diffeomorphism everywhere on a open and dense subset $\mathcal{O} \subset D_\epsilon$, containing $\Phi(U)$. Hence, we conclude that $E|_{\mathcal{O}}$ is a diffeormorphism onto its image, being a local diffeomorphism that is also invertible, thanks to Lemma 5.4.4. Finally, in order to prove that δ is smooth on $E(\mathcal{O}) \subset \{0 < \delta < \epsilon\} \cap \Omega$, it is enough to observe that, by construction,

$$\delta(E(q, \lambda)) = \sqrt{2H(\lambda)}, \qquad \forall\, (q, \lambda) \in D_\epsilon.$$

On D_ϵ, H is smooth, hence we conclude that δ is smooth on $E(\mathcal{O})$. Now $U \subset E(\mathcal{O})$ so that $E(\mathcal{O})$ is open, dense and has full-measure in $\{0 < \delta < \epsilon\} \cap \Omega$. □

An immediate consequence of the previous theorem is the existence of many geodesics that are strongly normal in the sense of Definition 5.4.1.

Corollary 5.4.9. *Let M be a smooth sub-Finsler manifold and let $\Omega \subset M$ be an open and bounded subset. Assume that $\partial\Omega$ is smooth and without characteristic points. Let $p \in E(\mathcal{O}) \subset \{0 < \delta < \epsilon\} \cap \Omega$ and let $\gamma : [0, 1] \to M$ be the unique geodesic joining p and $\partial\Omega$ and realizing δ. Then, γ is strongly normal.*

Proof. Let $q := \gamma(0) \in \partial\Omega$ the endpoint of γ on the boundary of Ω. Then, since q is non-characteristic point, Lemma 5.4.3 ensures that $\gamma|_{[0,s]}$ can not have an abnormal lift. Hence, γ is left strongly normal. In order to prove that γ is also right strongly normal, we reason in a similar way but with $\{\delta = \delta(p)\}$ in place of $\partial\Omega$. Indeed, since δ is smooth on the open set $E(\mathcal{O})$ by Theorem 5.4.8 and $d_{\bar{p}}\delta$ is not vanishing for every $\bar{p} \in E(\mathcal{O})$ as a consequence of Corollary 5.4.7, the set $\Sigma := \{\delta = \delta(p)\}$ defines a smooth hypersurface in a neighborhood of the point p. In addition, $\delta_\Sigma(q) := \mathsf{d}_{SF}(q, \Sigma) = \delta(p)$ and γ is the unique geodesic realizing δ_Σ. Finally, applying once again Lemma 5.4.3, we also deduce that $p \notin C(\Sigma)$, so repeating the argument we did before, we conclude that γ must be also right strongly normal. This concludes the proof. □

Remark 5.4.10. Since $E(\mathcal{O})$ has full measure in $\{0 < \delta < \epsilon\} \cap \Omega$, we can find $(q, \lambda) \in \mathcal{O}$ such that, denoting by $\gamma : [0, 1] \to M$ the corresponding geodesic minimizing δ, we have that $\gamma(t) \in E(\mathcal{O})$ for $\mathscr{L}^1$-a.e. $t \in [0, 1]$. This means that $\mathscr{L}^1$-almost every level set defines locally a hypersurface and, recalling that restrictions of abnormal geodesics are still abnormal, the proof of Corollary 5.4.9 can be repeated to show that the curve γ does not contain abnormal sub-segments.

Theorem 5.4.11 (Existence of strongly normal geodesics without abnormal sub-segments)**.** *Let M be a smooth sub-Finsler manifold. Then, there exists a strongly normal geodesic $\gamma : [0, 1] \to M$, which does not contain abnormal sub-segments.*

Proof. Note that Theorem 5.4.8 was stated for a hypersurface that is the boundary of non-characteristic domain Ω. However, without substantial modifications, one can prove that an analogous result holds locally around a non-characteristic point of a given smooth hypersurface $\Sigma \subset M$. In particular, letting $q \in \Sigma \setminus C(\Sigma)$, there exists $V_q \subset \Sigma$ open neighborhood of q, and

$\epsilon > 0$ such that, denoting by

$$\tilde{D}_\epsilon := \{(\bar{q}, \lambda) : \bar{q} \in V_q,\, 0 < \sqrt{2H(\lambda)} < \epsilon\},$$

the map $E|_{\tilde{D}_\epsilon} : \tilde{D}_\epsilon \to E(\tilde{D}_\epsilon) \subset \{0 < \delta_\Sigma < \epsilon\}$ is a diffeomorphism on an open and dense subset $\mathcal{O} \subset \tilde{D}_\epsilon$ and δ_Σ is smooth on $E(\mathcal{O})$. Now, Corollary 5.4.9 shows that there exists a point $p \in E(\mathcal{O})$ such that the unique geodesic $\gamma : [0, 1] \to M$ minimizing δ is strongly normal and, also according to Remark 5.4.10, it does not contain abnormal sub-segments. In order to conclude, we need to show that there exists a hypersurface Σ with $\Sigma \setminus C(\Sigma) \neq \emptyset$. But this is a consequence of the Hörmander condition, indeed if $\mathcal{D}_q \subset \Sigma_q$ for every $q \in \Sigma$, then Frobenius' theorem would ensure $\mathcal{D}$ be involutive and thus it would not be bracket-generating. □

5.4.2 Regularity of the distance function

We state below the definition of conjugate and cut loci in a sub-Finsler manifold, following the blueprint of the sub-Riemannian setting, cf. [ABB20, Chap. 11] or [BR19].

Definition 5.4.12 (Conjugate point)**.** Let M be a smooth sub-Finsler manifold and let $\gamma : [0, 1] \to M$ be a normal geodesic with initial covector $\lambda \in T_p^*M$, that is $\gamma(t) = \exp_p(t\lambda)$. We say that $q = \exp_p(\bar{t}\lambda)$ is a *conjugate point* to p along γ if $\bar{t}\lambda$ is a critical point for $\exp_p$.

Definition 5.4.13 (Cut locus)**.** Let M be a smooth sub-Finsler manifold and let $p \in M$. We say that $q \in M$ is a *smooth point* with respect to p, and write $q \in \Sigma_p$, if there exists a unique geodesic $\gamma : [0, 1] \to M$ joining p and q, which is not abnormal and such that q is not conjugate along p. Define the *cut locus* of $p \in M$ as $\mathrm{Cut}(p) := M \setminus \Sigma_p$. Finally, the cut locus of M is the set

$$\mathrm{Cut}(M) := \{(p, q) \in M \times M : q \in \mathrm{Cut}(p)\} \subset M \times M.$$

Remark 5.4.14. In the sub-Riemannian setting, according to [Agr09], the set of smooth points with respect to p is open and dense. However, it is an open question to understand whether its complement, that is the cut locus, is negligible.

Outside the cut locus of a sub-Finsler manifold, we can define the *t-midpoint map*, for $t \in [0, 1]$, as the map $\phi_t : M \times M \setminus \mathrm{Cut}(M) \to M$ assigning to (p, q) the t-midpoint of the (unique) geodesic $\gamma_{p,q}$ joining p and q. More precisely, for every $(p, q) \in M \times M \setminus \mathrm{Cut}(M)$,

$$\phi_t(p, q) := e_t(\gamma_{p,q}) = \exp_q((t-1)\lambda_p), \qquad \text{where } \lambda_p \in T_q^*M \text{ such that } p = \exp_q(-\lambda_p). \tag{5.4.5}$$

Note that, by definition of cut locus, the t-midpoint map is well-defined since the geodesic joining p and q for $q \notin \mathrm{Cut}(p)$ is unique and strictly normal, i.e. without abnormal lifts.

We report a useful result relating the regularity of the squared distance function on a sub-Finsler manifold M with the cut locus. Such result can be proved repeating verbatim the proof of [ABB20, Prop. 11.4], in light of Proposition 5.3.15 and Lemma 5.4.6. For every $p \in M$, let $\mathfrak{f}_p := \frac{1}{2}\mathsf{d}_{SF}^2(\cdot, p)$.

Proposition 5.4.15. *Let M be a smooth sub-Finsler manifold and let $p, q \in M$. Assume there exists an open neighborhood $\mathcal{O}_q \subset M$ of q such that $\mathfrak{f}_p$ is smooth. Then, $\mathcal{O}_q \subset \Sigma_p$ and*

$$\phi_t(p, z) = \exp_z((t-1)d_z\mathfrak{f}_p), \qquad \forall\, z \in \mathcal{O}_q.$$

Thanks to Proposition 5.4.15, the regularity of the squared distance ensures uniqueness of geodesics and smoothness of the t-midpoint map. Thus, it is desirable to understand where the squared distance is smooth. In this regard, [ABR18, Thm. 2.19] proves the regularity of the squared distance function along left strongly normal geodesics. We refer to [ABR18, App. A] for further details.

Theorem 5.4.16 ([ABR18, Thm. 2.19])**.** *Let M be a smooth sub-Finsler manifold and let $\gamma : [0,1] \to M$ be a left strongly normal geodesic. Then there exists $\epsilon > 0$ and an open neighborhood $U \subset M \times M$ such that:*

(i) $(\gamma(0), \gamma(t)) \in U$ *for all* $t \in (0, \epsilon)$*;*

(ii) For any $(p,q) \in U$ *there exists a unique (normal) geodesic joining* p *and* q*, shorter than* ϵ*;*

(iii) The squared distance function $(p,q) \mapsto \mathsf{d}_{SF}^2(p,q)$ *is smooth on* U*.*

The regularity of the squared distance function can be "propagated" along geodesics that do not admit abnormal sub-segments, applying the previous theorem for every sub-segment.

Corollary 5.4.17. *Let M be a smooth sub-Finsler manifold and let $\gamma : [0,1] \to M$ be a geodesic that does not admit abnormal sub-segments. Then, for every $s \in [0,1]$, there exists $\epsilon > 0$ and an open neighborhood $U \subset M \times M$ such that:*

(i) $(\gamma(s), \gamma(t)) \in U$ *for all* $t \in [0,1]$ *such that* $0 < |t - s| < \epsilon$*;*

(ii) For any $(p,q) \in U$ *there exists a unique (normal) geodesic joining* p *and* q*, shorter than* ϵ*;*

(iii) The squared distance function $(p,q) \mapsto \mathsf{d}_{SF}^2(p,q)$ *is smooth on* U*.*

5.4.3 Volume contraction rate along geodesics

Our goal is to quantify the contraction rate of small volumes along geodesics. To do this, we combine the smoothness of the t-midpoint map, with a lower bound on the so-called geodesic dimension. The latter has been introduced in [ABR18] for sub-Riemannian manifolds and in [Riz16, Def. 5.47] for general metric measure spaces. We recall below the definition.

Let M be a smooth sub-Finsler manifold. Given a point $p \in M$ and a Borel set $\Omega \subset M \setminus \mathrm{Cut}(p)$, we define the *geodesic homothety* of Ω with center p and ratio $t \in [0,1]$ as

$$\Omega_t^p := \{\phi_t(p,q) : q \in \Omega\}.$$

In the sequel, we say that $\mathfrak{m}$ is a *smooth measure* if, in coordinates, is absolutely continuous with respect the Lebesgue measure of the chart with a smooth and positive density. We will consider the metric measure space $(M, \mathsf{d}_{SF}, \mathfrak{m})$.

Definition 5.4.18. Let M be a smooth sub-Finsler manifold, equipped with a smooth measure $\mathfrak{m}$. For any $p \in M$ and $s > 0$, define

$$C_s(p) := \sup \left\{ \limsup_{t \to 0} \frac{1}{t^s} \frac{\mathfrak{m}(\Omega_t^p)}{\mathfrak{m}(\Omega)} : \Omega \subset M \setminus \mathrm{Cut}(p) \text{ Borel, bounded and } \mathfrak{m}(\Omega) \in (0, +\infty) \right\}, \tag{5.4.6}$$

We define the *geodesic dimension* of $(M, \mathsf{d}_{SF}, \mathfrak{m})$ at $p \in M$ as the non-negative real number

$$\mathcal{N}(p) := \inf\{s > 0 : C_s(p) = +\infty\} = \sup\{s > 0 : C_s(p) = 0\},$$

with the conventions $\inf \emptyset = +\infty$ and $\sup \emptyset = 0$.

Remark 5.4.19. In [Riz16], the definition of geodesic dimension is given for metric measure spaces with *negligible cut loci*. In this work, we adapted the definition by taking the supremum (5.4.6) over sets Ω which are outside the cut locus $\mathrm{Cut}(p)$.

We now prove a fundamental theorem which relates the geodesic and topological dimensions of a sub-Finsler manifold M. This result is a suitable adaptation of [Riz16, Thm. 4] to our setting.

Proposition 5.4.20. *Let M be a smooth sub-Finsler manifold, equipped with a smooth measure $\mathfrak{m}$. Assume that $r(p) < n := \dim M$ for every $p \in M$. Then,*

$$\mathcal{N}(p) \geq n+1, \qquad \forall\, p \in M.$$

Proof. Let d_{SR} be a sub-Riemannian distance on the manifold M, equivalent to d_{SF} (see (5.2.9)). The Ball-Box theorem, cf. [Jea14, Cor. 2.1], ensures that for every $p \in M$ there exist $n_p \geq n+1$ and a positive constant C_p such that

$$\mathfrak{m}(B_r^{SR}(p)) \leq C_p \cdot r^{n_p} \qquad \text{for } r \text{ sufficiently small.} \tag{5.4.7}$$

Since d_{SF} and d_{SR} are equivalent, up to changing the constant, the same estimate holds for sub-Finsler balls, in particular

$$\limsup_{r\to 0} \frac{\mathfrak{m}(B_r^{SF}(p))}{r^k} = 0 \tag{5.4.8}$$

for every $k < n+1$. Take any $\Omega \subset M \setminus \mathrm{Cut}(p)$ Borel, bounded and with $\mathfrak{m}(\Omega) \in (0,+\infty)$ and consider $R > 0$ such that $\Omega \subset B_R^{SF}(p)$. Note that $\Omega_t^p \subset B_{tR}^{SF}(p)$ and thus for every $k < n+1$ we have that

$$\limsup_{t\to 0} \frac{\mathfrak{m}(\Omega_t^p)}{t^k \mathfrak{m}(\Omega)} \leq \limsup_{t\to 0} \frac{\mathfrak{m}(B_{tR}^{SF}(p))}{t^k \mathfrak{m}(\Omega)} = \limsup_{t\to 0} \frac{\mathfrak{m}(B_{tR}^{SF}(p))}{(tR)^k} \cdot \frac{R^k}{\mathfrak{m}(\Omega)} = 0,$$

where we used (5.4.8) for the last equality. Since Ω was arbitrary, we deduce that $C_k(p) = 0$ for every $k < n+1$ and then $\mathcal{N}(p) \geq n+1$. □

Remark 5.4.21. For an equiregular sub-Finsler manifold, with the same proof, it is possible to improve the estimate of Proposition 5.4.20. In fact, in this case the Ball-Box theorem provides the estimate (5.4.7) with n_p equal to the Hausdorff dimension $\dim_H(M)$, for every p, and consequently $\mathcal{N}(p) \geq \dim_H(M)$, cf. [ABR18, Prop. 5.49].

By construction, the geodesic dimension controls the contraction rate of volumes along geodesics. This information can be transferred to the t-midpoint map, provided that is smooth. By invoking Theorem 5.4.16, we can always guarantee the smoothness of the t-midpoint map for a sufficiently short segment of a geodesic without abnormal sub-segments.

Theorem 5.4.22. *Let M be a smooth sub-Finsler manifold equipped with a smooth measure $\mathfrak{m}$ and such that $r(p) < n := \dim M$ for every $p \in M$. Let $\gamma : [0,1] \to M$ be a geodesic that does not admit abnormal sub-segments, with endpoints p and q. Assume that (p,q) belongs to the open set U, found in Theorem 5.4.16. Then, either $|\det(d_q\phi_t(p,\cdot))|$ has infinite order at $t=0$ or*

$$|\det(d_q\phi_t(p,\cdot))| \sim t^{m_p}, \qquad \text{as } t \to 0 \tag{5.4.9}$$

for some integer $m_p \geq \mathcal{N}(p) \geq n+1$.

Proof. Since, by assumption $(p,q) \in U$, we can apply item *(iii)* of Theorem 5.4.16, deducing the regularity of the distance function. Combining this with Proposition 5.4.15 and the homogeneity of the Hamiltonian flow, there exists an open neighborhood $V \subset M$ of q, such that the function

$$[0,1) \times V \ni (t,z) \mapsto d_z\phi_t(p,\cdot) = d_z\left(\exp_z((t-1)d_z\mathfrak{f}_p)\right)$$

is smooth. Thus, we can compute the Taylor expansion of its determinant in the t-variable at order $N := \lceil \mathcal{N}(p) \rceil - 1 < \mathcal{N}(p)$, obtaining:

$$\det(d_z\phi_t(p,\cdot)) = \sum_{i=0}^{N} a_i(z)t^i + t^{N+1}R_N(t,z), \qquad \forall\, z \in V,$$

where the functions a_i and R_N are smooth. Arguing by contradiction, we assume that there exists $j \leq N$ such that $a_j(q) \neq 0$ and define

$$m := \min\{i \leq N \,:\, \exists\, z \in V \text{ such that } a_i(z) \neq 0\}.$$

Note that $m \leq j$ since $a_j(q) \neq 0$ and thus $m \leq N$. Without loss of generality, we can assume that V and p are contained in the same coordinate chart and that $a_m > 0$ on an open subset $\tilde{V} \subset V$ with positive measure. Then, in charts, it holds that

$$\mathscr{L}^n(\tilde{V}_t^p) = \int_{\tilde{V}} \left|\det\left(d_z\phi_t(p,\cdot)\right)\right| \mathrm{d}z = \int_{\tilde{V}} a_m(z)\,\mathrm{d}z \cdot t^m + o(t^m) \qquad \text{as } t \to 0.$$

Therefore, recalling that $\mathfrak{m}$ is a smooth measure, there exists a constant $a > 0$ such that

$$\mathfrak{m}(\tilde{V}_t^p) \geq a \cdot t^m,$$

for every t sufficiently small. As a consequence, taking any $s \in (N, \mathcal{N}(p))$ we have that

$$\limsup_{t\to 0} \frac{1}{t^s}\frac{\mathfrak{m}(\tilde{V}_t^p)}{\mathfrak{m}(\tilde{V})} \geq \limsup_{t\to 0} \frac{1}{\mathfrak{m}(\tilde{V})}\frac{a\cdot t^m}{t^s} = +\infty,$$

and therefore we deduce $C_s(p) = +\infty$, which in turn implies $\mathcal{N}(p) \leq s$, giving a contradiction. ☐

Theorem 5.4.22 motivates the following definition.

Definition 5.4.23 (Ample geodesic)**.** Let M be a smooth sub-Finsler manifold and let $\gamma : [0,1] \to M$ be a strictly normal geodesic not admitting abnormal sub-segments. We say that γ is *ample* if, for every couple of distinct points $p, q \in \gamma([0,1])$, $\left|\det\left(d_q\phi_t(p,\cdot)\right)\right|$ exists and has finite order in $t = 0$.

Remark 5.4.24. The concept of ample geodesic in the sub-Riemannian setting has been introduced in [ABR18] and it differs from Definition 5.4.23. However, we remark that, in sub-Riemannian manifolds, for ample geodesics in the sense of [ABR18], $\left|\det\left(d_q\phi_t(p,\cdot)\right)\right|$ has finite order equal to the geodesic dimension at p, cf. [ABR18, Lem. 6.27]. Thus, our definition is weaker, but enough for our purposes.

5.4.4 Proof of Theorem 5.1.5

Let M be a smooth sub-Finsler manifold and let ϕ_t the t-midpoint map, defined as in (5.4.5). For ease of notation, set

$$\mathcal{M}(p,q) := \phi_{1/2}(p,q), \qquad \forall\,(p,q) \in M \times M \setminus \mathrm{Cut}(M), \tag{5.4.10}$$

be the 1/2-midpoint map or simply midpoint map. Reasoning as in [Jui21, Prop. 3.1], we obtain the following result as a consequence of Corollary 5.4.17 and Theorem 5.4.22. This argument hinges upon Theorem 5.4.11, which establishes the existence of a geodesic without abnormal sub-segments in a sub-Finsler manifold.

Proposition 5.4.25. *Let M be a smooth sub-Finsler manifold equipped with a smooth measure $\mathfrak{m}$ and such that $r(p) < n := \dim M$ for every $p \in M$. Let $\gamma : [0,1] \to M$ be the geodesic identified in Theorem 5.4.11 and let $\varepsilon > 0$. Then, there exist $0 \leq a < b \leq 1$ such that, letting $\bar{p} := \gamma(a)$, $\bar{q} := \gamma(b)$, the following statements hold:*

(i) $\bar{p} \notin \mathrm{Cut}(\bar{q})$, $\bar{q} \notin \mathrm{Cut}(\bar{p})$ and, for every $t \in (a,b)$, we have $\bar{p}, \bar{q} \notin \mathrm{Cut}(\gamma(t))$. Moreover, for every $t \in (a,b)$, $\mathfrak{f}_{\gamma(t)}$ is smooth in a neighborhood of $\bar{p}$ and in a neighborhood $\bar{q}$.

(ii) If, in addition, γ is ample, the midpoint map satisfies

$$|\det d_{\bar q}\mathcal{M}(\bar p,\cdot)| \le (1+\varepsilon)2^{-m_{\bar p}}, \qquad |\det d_{\bar p}\mathcal{M}(\cdot,\bar q)| \le (1+\varepsilon)2^{-m_{\bar q}} \tag{5.4.11}$$

where $m_{\bar p}$ and $m_{\bar q}$ are defined by (5.4.9) and $m_{\bar p}, m_{\bar q} \ge n+1$.

Given $z \in M$, define the *inverse geodesic map* $\mathcal{I}_z : M \setminus \mathrm{Cut}(z) \to M$ as

$$\mathcal{I}_z(p) = \exp_z(-\lambda) \qquad \text{where } \lambda \in T_z^*M \text{ such that } p = \exp_z(\lambda). \tag{5.4.12}$$

We may interpret this map as the one associating to p the point $\mathcal{I}_z(p)$ such that z is the midpoint of x and $\mathcal{I}_z(p)$.

We prove now the main theorem of this section, which also implies Theorem 5.1.5. Our strategy is an adaptation to the sub-Finsler setting of the one proposed in [Jui21].

Theorem 5.4.26. *Let M be a complete smooth sub-Finsler manifold equipped with a smooth measure $\mathfrak{m}$ and such that $r(p) < n := \dim M$ for every $p \in M$. Then, the metric measure space $(M, \mathsf{d}_{SF}, \mathfrak{m})$ does not satisfy the Brunn–Minkowski inequality $\mathsf{BM}(K,N)$, for every $K \in \mathbb{R}$ and $N \in (1,\infty)$.*

Proof. Fix $\varepsilon > 0$, $K \in \mathbb{R}$ and $N \in (1,\infty)$. Let $\gamma : [0,1] \to M$ be the geodesic identified by Theorem 5.4.11 and assume it is contained in a coordinate chart with (sub-Finsler) diameter $D > 0$. Up to restricting the domain of the chart and the geodesic, we can also assume that

$$(1-\varepsilon)\mathscr{L}^n \le \mathfrak{m} \le (1+\varepsilon)\mathscr{L}^n \qquad \text{and} \qquad \tau_{K,N}^{(1/2)}(\theta) \ge \frac{1}{2} - \varepsilon, \quad \forall \theta \le D, \tag{5.4.13}$$

where the second inequality can be fulfilled, according to Remark 5.2.1. Moreover, let $0 \le a < b \le 1$ be as in Proposition 5.4.25. We proceed by contradiction and assume that $(M, \mathsf{d}_{SF}, \mathfrak{m})$ satisfies the $\mathsf{BM}(K,N)$.

First of all, suppose that γ is not ample. According to [MPR22b, Prop. 5.3], the Brunn–Minkowski inequality $\mathsf{BM}(K,N)$ implies the $\mathsf{MCP}(K,N)$ condition[1]. Therefore, $(M, \mathsf{d}_{SF}, \mathfrak{m})$ satisfies the $\mathsf{MCP}(K,N)$ condition and, for the moment, assume $K = 0$. Set $\bar p := \gamma(a)$ and $\bar q := \gamma(b)$ and let $\Omega_\varrho := B_\varrho(\bar q)$ for $\varrho > 0$. From the $\mathsf{MCP}(0,N)$ condition we get

$$\mathfrak{m}(\Omega_{\varrho,t}^{\bar p}) \ge t^N \mathfrak{m}(\Omega_\varrho), \qquad \forall\, t \in [0,1],\ \varrho > 0. \tag{5.4.14}$$

If ϱ is sufficiently small, then $\Omega_{\varrho,t}^{\bar p} = \phi_t(\bar p, \Omega_\varrho)$ for $t \in [0,1)$, therefore, employing the first estimate in (5.4.13), the inequality (5.4.14) can be reformulated as follows:

$$\frac{1+\varepsilon}{1-\varepsilon} \fint_{\Omega_\varrho} \left|\det\left(d_z\phi_t(\bar p,\cdot)\right)\right| \mathrm{d}z \ge \frac{\mathfrak{m}(\Omega_{\varrho,t}^{\bar p})}{\mathfrak{m}(\Omega_\varrho)} \ge t^N, \qquad \forall\, t \in [0,1),\ \varrho > 0.$$

Taking the limit as $\varrho \to 0$, and then the limit as $t \to 0$, we find that the order of $\left|\det\left(d_{\bar q}\phi_t(\bar p,\cdot)\right)\right|$ should be smaller than or equal to N, giving a contradiction. Finally, if $K \ne 0$, observe that the behavior of the distortion coefficients, as $t \to 0$, is comparable with t, namely there exists a constant $C = C(K, N, \mathsf{d}(\bar p, \bar q)) > 0$ such that

$$\tau_{K,N}^{(t)}(\theta) \ge Ct, \qquad \text{as } t \to 0, \qquad \forall\, \theta \in (\mathsf{d}(\bar p,\bar q) - \varrho, \mathsf{d}(\bar p,\bar q) + \varrho).$$

Therefore, repeating the same argument that we did for the case $K = 0$, we obtain the sought contradiction.

[1] In that paper, the proposition is proved for essentially non-branching metric measure spaces. However, what is needed is a measurable selection of geodesics which we have locally around the curve γ.

Suppose instead that the geodesic γ is ample and let m be the unique midpoint between $\bar{p} = \gamma(a)$ and $\bar{q} = \gamma(b)$. According to item *(i)* of Proposition 5.4.25, the map $\mathcal{I}_m$ is well-defined and smooth in a neighborhood of $\bar{p}$ and $\bar{q}$, moreover by definition $\mathcal{I}_m(\bar{q}) = \bar{p}$ and $\mathcal{I}_m(\bar{p}) = \bar{q}$. Note that $\mathcal{I}_m \circ \mathcal{I}_m = \mathrm{id}$ (where defined), thus

$$|\det(d_{\bar{p}}\mathcal{I}_m)| \cdot |\det(d_{\bar{q}}\mathcal{I}_m)| = \big|\det\big(d_{\bar{q}}(\mathcal{I}_m \circ \mathcal{I}_m)\big)\big| = 1.$$

Therefore, at least one between $|\det(d_{\bar{q}}\mathcal{I}_m)|$ and $|\det(d_{\bar{p}}\mathcal{I}_m)|$ is greater than or equal to 1, without loss of generality we assume

$$|\det(d_{\bar{q}}\mathcal{I}_m)| \geq 1.$$

Let $B_\varrho := B^{eu}_\varrho(\bar{q})$ the (Euclidean) ball of radius $\varrho > 0$ centered in $\bar{q}$. Introduce the function $F : B_\varrho \times B_\varrho \to M$, defined as

$$B_\varrho \times B_\varrho \ni (x, y) \mapsto F(x, y) := \mathcal{M}(\mathcal{I}_m(x), y).$$

Observe that, for ϱ small enough, F is well-defined and by construction $F(x, x) = m$ for every $x \in B_\varrho$. Therefore, we deduce that for every vector $v \in T_{\bar{q}}M \cong \mathbb{R}^n$, the following holds:

$$0 = d_{(\bar{q},\bar{q})}F(v, v) = \big(d_{\bar{p}}\mathcal{M}(\cdot, \bar{q}) \circ d_{\bar{q}}\mathcal{I}_m\big)\, v + d_{\bar{q}}\mathcal{M}(\bar{p}, \cdot)\, v.$$

Since the former identity is true for every vector $v \in \mathbb{R}^n$, we can conclude that

$$d_{\bar{p}}\mathcal{M}(\cdot, \bar{q}) \circ d_{\bar{q}}\mathcal{I}_m + d_{\bar{q}}\mathcal{M}(\bar{p}, \cdot) = 0,$$

and consequently, for every $v, w \in \mathbb{R}^n$, we have

$$d_{(\bar{q},\bar{q})}F(v, w) = \big(d_{\bar{p}}\mathcal{M}(\cdot, \bar{q}) \circ d_{\bar{q}}\mathcal{I}_m\big)\, v + d_{\bar{q}}\mathcal{M}(\bar{p}, \cdot)\, w = d_{\bar{q}}\mathcal{M}(\bar{p}, \cdot)\,(w - v).$$

In particular, we obtain a Taylor expansion of the function F at the point $(\bar{q}, \bar{q})$ that in coordinates takes the form:

$$\|F(\bar{q} + v, \bar{q} + w) - m - d_{\bar{q}}\mathcal{M}(\bar{p}, \cdot)\,(w - v)\|_{eu} = o(\|v\|_{eu} + \|w\|_{eu}), \qquad \text{as } v, w \to 0.$$

Then, as v and w vary in $B^{eu}_\varrho(0)$, $v - w$ varies in $B^{eu}_{2\varrho}(0)$, and we obtain that

$$F(B_\varrho, B_\varrho) \subseteq m + d_{\bar{q}}\mathcal{M}(\bar{p}, \cdot)\,\big(B^{eu}_{2\varrho}(0)\big) + B^{eu}_{\omega(\varrho)}(0), \tag{5.4.15}$$

where $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is such that $\omega(r) = o(r)$ when $r \to 0^+$. Now, consider $A_\varrho := \mathcal{I}_m(B_\varrho)$ and note that by definition $M_{1/2}(A_\varrho, B_\varrho) = F(B_\varrho, B_\varrho)$, then using (5.4.15) we conclude that, as $\varrho \to 0$,

$$\begin{aligned}\mathscr{L}^n(M_{1/2}(A_\varrho, B_\varrho)) &= \mathscr{L}^n(F(B_\varrho, B_\varrho)) \leq \mathscr{L}^n\Big(d_{\bar{q}}\mathcal{M}(p, \cdot)\,\big(B^{eu}_{2\varrho}(0)\big)\Big) + o(\varrho^n)\\ &= \big|\det(d_{\bar{q}}\mathcal{M}(p, \cdot))\big| \cdot \omega_n 2^n \varrho^n + o(\varrho^n) \leq (1 + \varepsilon)2^{n - m_q}\omega_n\varrho^n + o(\varrho^n) \leq \frac{1}{2}(1 + \varepsilon)\omega_n\varrho^n + o(\varrho^n)\end{aligned}$$

where $\omega_n = \mathscr{L}^n(B^{eu}_1(0))$ and the two last inequalities follow from (5.4.11) and $m_{\bar{q}} \geq n + 1$. On the other hand, it holds that $\mathscr{L}^n(B_\varrho) = \omega_n\varrho^n$ and, as $\varrho \to 0$,

$$\mathscr{L}^n(A_\varrho) = \mathscr{L}^n(\mathcal{I}_m(B_\varrho)) = \big(|\det(d_{\bar{q}}\mathcal{I}_m)| + O(\varrho)\big)\, \mathscr{L}^n(B_\varrho) \geq \omega_n\varrho^n + o(\varrho^n).$$

Taking into account the first estimate of (5.4.13), we deduce the following inequalities for the measure $\mathfrak{m}$, as $\varrho \to 0$,

$$\mathfrak{m}(M_{1/2}(A_\varrho, B_\varrho)) \leq \frac{1}{2}(1 + \varepsilon)^2\omega_n\varrho^n + o(\varrho^n),$$
$$\mathfrak{m}(A_\varrho) \geq (1 - \varepsilon)\omega_n\varrho^n + o(\varrho^n) \qquad \text{and} \qquad \mathfrak{m}(B_\varrho) \geq (1 - \varepsilon)\omega_n\varrho^n.$$

Finally, if ε is small enough we can find ϱ sufficiently small such that

$$\begin{aligned}\mathfrak{m}(M_{1/2}(A_\varrho, B_\varrho))^{\frac{1}{N}} &< \left(\frac{1}{2} - \varepsilon\right) \mathfrak{m}(A_\varrho)^{\frac{1}{N}} + \left(\frac{1}{2} - \varepsilon\right) \mathfrak{m}(B_\varrho)^{\frac{1}{N}} \\ &\leq \tau_{K,N}^{(1/2)}(\Theta(A_\varrho, B_\varrho))\, \mathfrak{m}(A_\varrho)^{\frac{1}{N}} + \tau_{K,N}^{(1/2)}(\Theta(A_\varrho, B_\varrho))\, \mathfrak{m}(B_\varrho)^{\frac{1}{N}},\end{aligned}$$

which contradicts the Brunn–Minkowski inequality $\mathsf{BM}(K, N)$. □

Remark 5.4.27. Observe that the argument presented in this section is local, around the geodesic without abnormal sub-segments. Thus, repeating the same proof, we can extend Theorem 5.4.26 if the assumption on the rank holds on an open set $V \subset M$, namely $r(p) < n$ for every $p \in V$.

5.5 Failure of the $\mathsf{CD}(K, N)$ condition in the sub-Finsler Heisenberg group

In this section, we disprove the curvature-dimension condition in the sub-Finsler Heisenberg group, cf. Theorem 5.1.8. Our strategy relies on the explicit expression of geodesics in terms of convex trigonometric functions, found in [Lok21].

5.5.1 Convex trigonometry

In this section, we recall the definition and main properties of the convex trigonometric functions, firstly introduced in [Lok19]. Let $\Omega \subset \mathbb{R}^2$ be a convex, compact set, such that $O := (0, 0) \in \mathrm{Int}(\Omega)$ and denote by $\mathbb{S}$ its surface area.

Definition 5.5.1. Let $\theta \in \mathbb{R}$ denote a generalized angle. If $0 \leq \theta < 2\mathbb{S}$ define P_θ as the point on the boundary of Ω, such that the area of the sector of Ω between the rays Ox and OP_θ is $\frac{1}{2}\theta$ (see Figure 5.1). Moreover, define $\sin_\Omega(\theta)$ and $\cos_\Omega(\theta)$ as the coordinates of the point P_θ, i.e.

$$P_\theta = \big(\sin_\Omega(\theta), \cos_\Omega(\theta)\big).$$

Finally, extend these trigonometric functions outside the interval $[0, 2\mathbb{S})$ by periodicity (of period $2\mathbb{S}$), so that for every $k \in \mathbb{Z}$.

$$\cos_\Omega(\theta) = \cos_\Omega(\theta + 2k\mathbb{S}), \quad \sin_\Omega(\theta) = \sin_\Omega(\theta + 2k\mathbb{S}) \quad \text{and} \quad P_\theta = P_{\theta + 2k\mathbb{S}}.$$

Observe that by definition $\sin_\Omega(0) = 0$ and that when Ω is the Euclidean unit ball we recover the classical trigonometric functions.

Consider now the polar set:

$$\Omega^\circ := \{(p, q) \in \mathbb{R}^2 \, : \, px + qy \leq 1 \text{ for every } (x, y) \in \Omega\},$$

which is itself a convex, compact set such that $O \in \mathrm{Int}(\Omega^\circ)$. Therefore, we can consider the trigonometric functions $\sin_{\Omega^\circ}$ and $\cos_{\Omega^\circ}$. Observe that, by definition of polar set, it holds that

$$\cos_\Omega(\theta)\cos_{\Omega^\circ}(\psi) + \sin_\Omega(\theta)\sin_{\Omega^\circ}(\psi) \leq 1, \qquad \text{for every } \theta, \psi \in \mathbb{R}. \tag{5.5.1}$$

Definition 5.5.2. We say that two angles $\theta, \psi \in \mathbb{R}$ *correspond* to each other and write $\theta \overset{\Omega}{\longleftrightarrow} \psi$ if the vector $Q_\psi := (\cos_{\Omega^\circ}(\psi), \sin_{\Omega^\circ}(\psi))$ determines a half-plane containing Ω (see Figure 5.2).

By the bipolar theorem [Roc70, Thm. 14.5], it holds that $\Omega^{\circ\circ} = \Omega$ and this allow to prove the following symmetry property for the correspondence just defined.

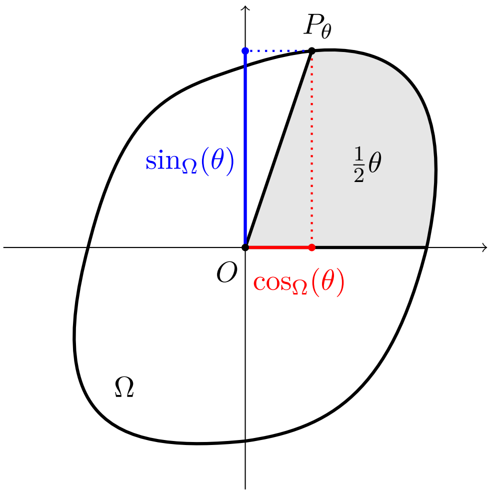

Figure 5.1: Values of the generalized trigonometric functions $\cos_\Omega$ and $\sin_\Omega$.

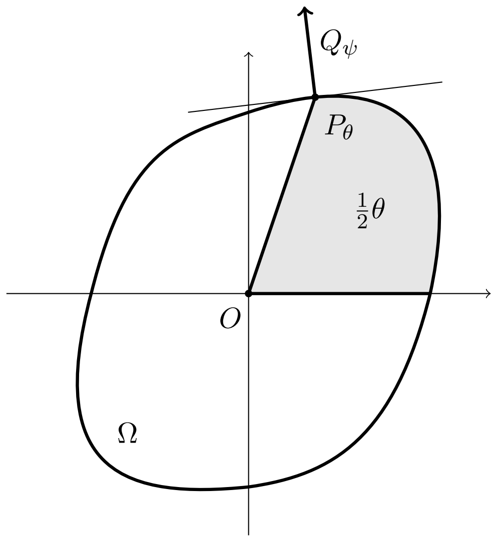

Figure 5.2: Representation of the correspondence $\theta \overset{\Omega}{\longleftrightarrow} \psi$.

Proposition 5.5.3. *Let $\Omega \subset \mathbb{R}^2$ be a convex and compact set, with $O \in Int(\Omega)$. Given two angles $\theta, \psi \in \mathbb{R}$, $\theta \overset{\Omega}{\longleftrightarrow} \psi$ if and only if $\psi \overset{\Omega^\circ}{\longleftrightarrow} \theta$. Moreover, the following analogous of the Pythagorean equality holds:*

$$\theta \overset{\Omega}{\longleftrightarrow} \psi \qquad \text{if and only if} \qquad \cos_\Omega(\theta)\cos_{\Omega^\circ}(\psi) + \sin_\Omega(\theta)\sin_{\Omega^\circ}(\psi) = 1. \tag{5.5.2}$$

The correspondence $\theta \overset{\Omega}{\longleftrightarrow} \psi$ is not one-to-one in general, in fact if the boundary of Ω has a corner at the point P_θ, the angle θ corresponds to an interval of angles (in every period). Nonetheless, we can define a monotone multi-valued map C° that maps an angle θ to the maximal closed interval containing angles corresponding to θ. This function has the following periodicity property:

$$C^\circ(\theta + 2\mathbb{S}k) = C^\circ(\theta) + 2\mathbb{S}^\circ k \qquad \text{for every } k \in \mathbb{Z},$$

where $\mathbb{S}^\circ$ denotes the surface area of Ω°. If Ω is strictly convex, then the map C° is strictly monotone, while if the boundary of Ω is C^1, then C° is a (single-valued) map from $\mathbb{R}$ to $\mathbb{R}$ and it is continuous. Analogously, we can define the map $C_\circ$ associated to the correspondence $\psi \overset{\Omega^\circ}{\longleftrightarrow} \theta$. Proposition 5.5.3 guarantees that $C_\circ \circ C^\circ = C^\circ \circ C_\circ = \mathrm{id}$.

Proposition 5.5.4. *Let $\Omega \subset \mathbb{R}^2$ as above. The trigonometric functions $\sin_\Omega$ and $\cos_\Omega$ are Lipschitz and therefore differentiable almost everywhere. At every differentiability point θ of both functions, there exists a unique angle ψ corresponding to θ and it holds that*

$$\sin_\Omega'(\theta) = \cos_{\Omega^\circ}(\psi) \qquad \text{and} \qquad \cos_\Omega'(\theta) = -\sin_{\Omega^\circ}(\psi).$$

Naturally, the analogous result holds for the trigonometric functions $\sin_{\Omega^\circ}$ and $\cos_{\Omega^\circ}$.

As a corollary of the previous proposition, we obtain the following convexity properties for the trigonometric functions.

Corollary 5.5.5. *The functions $\sin_\Omega$ and $\cos_\Omega$ are concave in every interval in which they are non-negative and convex in every interval in which they are non-positive.*

This convexity properties of the trigonometric functions will play a small but fundamental role in Section 5.5.3 in the form of the following corollaries.

Corollary 5.5.6. *Given a non-null constant $k \in \mathbb{R}$ and every angle θ, consider the function*

$$g : \mathbb{R} \to \mathbb{R}; \qquad g(t) := \sin_\Omega(\theta)\cos_\Omega(\theta + kt) - \cos_\Omega(\theta)\sin_\Omega(\theta + kt). \tag{5.5.3}$$

If $k > 0$ this function is convex for positive values of t and concave for negative values of t, locally around 0. Vice versa, If $k < 0$ it is concave for positive values of t and convex for negative values of t, locally around 0.

Proof. The function $g(t)$ can be seen as a scalar product of two vectors in $\mathbb{R}^2$, therefore it is invariant by rotations. In particular, we consider the rotation that sends θ to 0: this maps P_θ to the the positive x-axis and the set Ω to a convex, compact set $\tilde{\Omega} \subset \mathbb{R}^2$. Then, $g(t)$ in (5.5.3) is equal to the function

$$t \mapsto -\cos_{\tilde{\Omega}}(0)\sin_{\tilde{\Omega}}(kt).$$

The conclusion immediately follows from Corollary 5.5.5. □

Corollary 5.5.7. *Given a non-null constant $k \in \mathbb{R}$ and every angle ψ, the function*

$$h : \mathbb{R} \to \mathbb{R}; \qquad h(t) := 1 - \sin_{\Omega^\circ}(\psi)\sin_\Omega\big((\psi + kt)_\circ\big) - \cos_{\Omega^\circ}(\psi)\cos_\Omega\big((\psi + kt)_\circ\big).$$

is non-decreasing for positive values of t and non-increasing for negative values of t, locally around 0.

Proof. Note that h is the derivative of the function

$$\mathbb{R} \ni t \mapsto kt + \sin_{\Omega^\circ}(\psi)\cos_{\Omega^\circ}(\psi + kt) - \cos_{\Omega^\circ}(\psi)\sin_{\Omega^\circ}(\psi + kt), \tag{5.5.4}$$

divided by k. The thesis follows from Corollary 5.5.6, since the function (5.5.4) is the sum of a linear function and of a function of the type (5.5.3). □

In the following we are going to consider the trigonometric functions associated to the unit ball of a strictly convex norm $\|\cdot\|$ on $\mathbb{R}^2$, i.e. $\Omega := B_1^{\|\cdot\|}(0)$. In this case, the polar set Ω° is the unit ball $B_1^{\|\cdot\|_*}(0)$ of the dual norm $\|\cdot\|_*$. Moreover, according to the Pythagorean identity (5.5.2), if $\theta \overset{\Omega}{\leftrightarrow} \psi$ $\|\cdot\|$ then Q_ψ is a dual vector of P_θ. In particular, if $\|\cdot\|$ is a C^1 norm, Lemma 5.3.6 ensures that

$$(\cos_{\Omega^\circ}(\psi), \sin_{\Omega^\circ}(\psi)) = Q_\psi = d_{P_\theta}\|\cdot\| = d_{(\cos_\Omega(\theta),\sin_\Omega(\theta))}\|\cdot\|.$$

We conclude this section by recalling a well-known result on the relation between a norm $\|\cdot\|$ and its dual $\|\cdot\|_*$. This will be employed in the subsequent sections, as the geodesics of the sub-Finsler Heisenberg group, equipped with the norm $\|\cdot\|$, follow the shape of the boundary of $B_1^{\|\cdot\|_*}(0)$, cf. Theorem 5.5.9.

Proposition 5.5.8. *Let $\|\cdot\|$ be a norm on $\mathbb{R}^2$, and let $\|\cdot\|_*$ be its dual norm, then:*

(i) $\|\cdot\|_*$ *is a strictly convex norm if and only if* $\|\cdot\|$ *is a* C^1 *norm;*

(ii) $\|\cdot\|_*$ *is a strongly convex norm if and only if* $\|\cdot\|$ *is a* $C^{1,1}$ *norm.*

5.5.2 Geodesics in the Heisenberg group

We present here the sub-Finsler Heisenberg group and study its geodesics. Let us consider the Lie group $M = \mathbb{R}^3$, equipped with the non-commutative group law, defined by

$$(x,y,z) \star (x',y',z') = \Big(x + x', y + y', z + z' + \frac{1}{2}(xy' - x'y)\Big), \qquad \forall\, (x,y,z),(x',y',z') \in \mathbb{R}^3,$$

with identity element e $=(0,0,0)$. In the notation of Section 5.2.2, we define the following morphism of bundles

$$\xi : M\times\mathbb{R}^2\to TM, \qquad \xi(x,y,z;u_1,u_2)=\Big(x,y,z;u_1,u_2,\frac{1}{2}(u_2x-u_1y)\Big).$$

The associated distribution of rank 2 is spanned by the following left-invariant vector fields:

$$X_1=\partial_x-\frac{y}{2}\partial_z, \qquad X_2=\partial_y+\frac{x}{2}\partial_z,$$

namely $\mathcal{D}=\operatorname{span}\{X_1,X_2\}$. It can be easily seen that $\mathcal{D}$ is bracket-generating. Then, letting $\|\cdot\|:\mathbb{R}^2\to\mathbb{R}_+$ be a norm, the *sub-Finsler Heisenberg group* $\mathbb{H}$ is the Lie group M equipped with the sub-Finsler structure $(\xi,\|\cdot\|)$. By construction, also the resulting norm on the distribution is left-invariant, so that the left-translations defined by

$$L_p:\mathbb{H}\to\mathbb{H}; \qquad L_p(q):=p\star q, \tag{5.5.5}$$

are isometries for every $p\in\mathbb{H}$.

In this setting, the geodesics were originally studied in [Bus47] and [Ber94] for the three-dimensional case and in [Lok21] for general left-invariant structures on higher-dimensional Heisenberg groups. We recall below the main results of [Ber94] for strictly convex norms.

Theorem 5.5.9 ([Ber94, Thm. 1, Thm. 2])**.** *Let $\mathbb{H}$ be the sub-Finsler Heisenberg group, equipped with a strictly convex norm. Then, the following statements hold:*

(i) for any $q\in\mathbb{H}\setminus\{x=y=0\}$, there exists a unique geodesic $\gamma:[0,1]\to\mathbb{H}$ joining the origin and q.

(ii) $\gamma:[0,T]\to\mathbb{H}$ is a geodesic starting at the origin if and only if it satisfies the Pontryagin's maximum principle for the time-optimal control problem:

$$\begin{cases}\dot\gamma(t)=u_1(t)X_1(\gamma(t))+u_2(t)X_2(\gamma(t)),\\ u(t)\in B_1^{\|\cdot\|}(0), \quad \gamma(0)=q_0, \quad \textit{and} \quad \gamma(T)=q_1,\\ T\to\min.\end{cases}$$

Remark 5.5.10. Note that the geodesics in [Ber94] are found solving the Pontryagin maximum principle of Theorem 5.3.2 for the time-optimal problem. The latter is an equivalent formulation of (P), however it produces arc-length parameterized geodesics.

The next step is to compute explicitly the exponential map. In [Lok21], the author provides an explicit expression for geodesics starting at the origin, using the convex trigonometric functions functions presented in Section 5.5.1. Since therein the author solves the time-optimal problem, we prefer to solve explicitly the Hamiltonian system (H), in the case of the sub-Finsler Heisenberg group.

Proposition 5.5.11 ([Lok21, Thm. 4])**.** *Let $\mathbb{H}$ be the sub-Finsler Heisenberg group, equipped with a strictly convex norm. Let $\gamma:[0,1]\to\mathbb{H}$ be the projection of a (non-trivial) normal extremal $(\lambda_t)_{t\in[0,1]}$ starting at the origin, then $\gamma(t)=(x(t),y(t),z(t))$, with*

$$\begin{cases}x(t)=\dfrac{r}{w}\left(\sin_{\Omega^\circ}(\phi+\omega t)-\sin_{\Omega^\circ}(\phi)\right),\\ y(t)=-\dfrac{r}{w}\left(\cos_{\Omega^\circ}(\phi+\omega t)-\cos_{\Omega^\circ}(\phi)\right),\\ z(t)=\dfrac{r^2}{2\omega^2}\left(\omega t+\cos_{\Omega^\circ}(\phi+\omega t)\sin_{\Omega^\circ}(\phi)-\sin_{\Omega^\circ}(\phi+\omega t)\cos_{\Omega^\circ}(\phi)\right),\end{cases}$$

for some $\phi \in [0, 2\mathbb{S}^\circ)$, $\omega \in \mathbb{R}\setminus\{0\}$ *and* $r > 0$. *If* $\omega = 0$, *then*

$$\begin{cases} x(t) = (r\cos_\Omega(\phi^\circ))\, t, \\ y(t) = (r\sin_\Omega(\phi^\circ))\, t, \\ z(t) = 0. \end{cases} \tag{5.5.6}$$

Proof. Firstly, we characterize the sub-Finsler Hamiltonian in the sub-Finsler Heisenberg group. Note that, without assuming additional regularity on $\|\cdot\|$, we can not apply directly Lemma 5.3.7. Nevertheless, we can still obtain an analogous result by means of convex trigonometry. Indeed, let $h_i(\lambda) := \langle \lambda, X_i(\pi(\lambda))\rangle$ for $i = 1, 2$, then

$$H(\lambda) := \max_{u\in\mathbb{R}^2}\left(\sum_{i=1}^{2} u_i h_i(\lambda) - \frac{\|u\|}{2}\right), \qquad \forall\,\lambda \in\ T^*\mathbb{H}.$$

We introduce polar coordinates on $\mathbb{R}^2$ associated with $\|\cdot\|$ and its dual norm $\|\cdot\|_*$, namely $(u_1, u_2) \mapsto (\rho, \theta)$ and $(h_1, h_2) \mapsto (\zeta, \psi)$ where

$$\begin{cases} u_1 = \rho\cos_{\Omega^\circ}(\theta), \\ u_2 = \rho\sin_{\Omega^\circ}(\theta), \end{cases} \qquad \text{and} \qquad \begin{cases} h_1 = \zeta\cos_{\Omega^\circ}(\psi), \\ h_2 = \zeta\sin_{\Omega^\circ}(\psi). \end{cases} \tag{5.5.7}$$

Hence, the sub-Finsler Hamiltonian becomes

$$\begin{aligned} H(\lambda) &= \max_{u\in\mathbb{R}^2}\left(\sum_{i=1}^{2} u_i h_i(\lambda) - \frac{\|u\|}{2}\right) \\ &= \max_{\substack{\theta\in[0,2\mathbb{S})\\ \rho>0}}\left(\rho\zeta\left(\cos_\Omega(\theta)\cos_{\Omega^\circ}(\psi) + \sin_\Omega(\theta)\sin_{\Omega^\circ}(\psi)\right) - \frac{\rho^2}{2}\right) \le \max_{\rho>0}\left(\rho\zeta - \frac{\rho^2}{2}\right) = \frac{\zeta^2}{2}, \end{aligned} \tag{5.5.8}$$

where the last inequality is a consequence of (5.5.1). Moreover, we attain the equality in (5.5.8) if and only if $\rho = \zeta$ and $\psi = C^\circ(\theta)$. Therefore, since $\zeta = \|\hat\lambda\|_*$ with $\hat\lambda = (h_1(\lambda), h_2(\lambda))$, we conclude that

$$H(\lambda) = \frac{1}{2}\|\hat\lambda\|_*^2, \qquad \forall\,\lambda \in T^*\mathbb{H}\setminus \mathrm{Ann}(\mathcal{D}),$$

and the maximum is attained at the control $u = \hat\lambda^*$. Furthermore, $H \in C^1(T^*M)$ by strict convexity of $\|\cdot\|$, cf. Proposition 5.5.8. We write the system (5.3.8) in coordinates $(x, y, z; h_1, h_2, h_3)$ for the cotangent bundle, where $h_3(\lambda) := \langle\lambda, \partial_z\rangle$. The vertical part of (5.3.8) becomes

$$\begin{cases} \dot h_1(t) = \|\hat\lambda_t\|_* d_{\hat\lambda_t}\|\cdot\|_* \cdot (0, -h_3(t))\,, \\ \dot h_2(t) = \|\hat\lambda_t\|_* d_{\hat\lambda_t}\|\cdot\|_* \cdot (h_3(t), 0)\,, \\ \dot h_3(t) = 0. \end{cases} \tag{5.5.9}$$

Let $(\lambda_t)_{t\in[0,1]}$ be a normal extremal with associated maximal control given by $t \mapsto u(t)$, then we use Lemma 5.3.6 to deduce that $\|\hat\lambda_t\|_* d_{\hat\lambda_t}\|\cdot\|_* = \hat\lambda_t^* = u(t)$. Therefore, letting $h_3(t) \equiv \omega \in \mathbb{R}$, we may rewrite (5.5.9) as

$$\begin{cases} \dot h_1(t) = -\omega\, u_2(t), \\ \dot h_2(t) = \omega\, u_1(t). \end{cases} \tag{5.5.10}$$

To solve this system, we use the polar coordinates (5.5.7): letting $t \mapsto (\rho(t), \psi(t))$ be the curve representing $\hat\lambda_t = (h_1(t), h_2(t))$, we deduce that $\rho(t)$ and $\psi(t)$ are absolutely continuous and satisfy

$$\rho(t) = \|\hat\lambda_t\|_*, \qquad \dot\psi(t) = \frac{h_1(t)\dot h_2(t) - \dot h_1(t) h_2(t)}{\rho^2(t)}.$$

We may compute explicitly $\dot\rho(t)$ and $\dot\psi(t)$, using once again Lemma 5.3.6, the system (5.5.10) and identity (5.5.2):

$$\dot\rho(t) = d_{\hat\lambda_t}\|\cdot\|_* \cdot (\dot h_1(t), \dot h_2(t)) = \frac{\omega}{\|\hat\lambda_t\|_*} u(t) \cdot (-u_2(t), u_1(t)) = 0, \qquad \dot\psi(t) = \omega.$$

Thus, integrating the above identities, we obtain $\rho(t) \equiv r$ and $\psi(t) = \omega t + \phi$ for some $r > 0$ and $\phi \in [0, 2\mathbb{S}^\circ)$. Finally, we find an explicit expression for the maximal control:

$$u(t) = \left(r \cos_\Omega(C_\circ(\phi + \omega t)), r \sin_\Omega(C_\circ(\phi + \omega t))\right).$$

From this, we may explicitly integrate the horizontal part of the Hamiltonian system, obtaining the desired expression. In particular, if $\omega = 0$ we immediately obtain (5.5.6). If $\omega \neq 0$, we may employ Proposition 5.5.4 to conclude. □

As $(\mathbb{H}, \mathsf{d}_{SF})$ is complete, normal extremals can be extended to $\mathbb{R}$, according to Proposition 5.3.10. Thus, we may define the (extended) exponential map at the origin on the whole $T_0^*\mathbb{H} \times \mathbb{R}$:

$$\begin{aligned} G : ([0, 2\mathbb{S}^\circ) \times \mathbb{R} \times [0, \infty)) \times \mathbb{R} &\longrightarrow \mathbb{H}, \\ (\phi, \omega, r; t) &\longmapsto (x(\phi, \omega, r; t), y(\phi, \omega, r; t), z(\phi, \omega, r; t)), \end{aligned} \tag{5.5.11}$$

where $(x(\phi, \omega, r; t), y(\phi, \omega, r; t), z(\phi, \omega, r; t))$ correspond to the curve $(x(t), y(t), z(t))$ defined by Proposition 5.5.11 with initial datum (ϕ, ω, r) and with the understanding that $G(\phi, \omega, 0; t) \equiv 0$. By the properties of the convex trigonometric functions, G is a C^1 map for $\omega \neq 0$. Moreover, thanks to Theorem 5.5.9, for every initial datum (ϕ, ω, r), the curve $t \mapsto G(\phi, \omega, r; t)$ is a geodesic between its endpoints for sufficiently small times. More precisely, it is minimal for $|t| < t^* = t^*(\phi, \omega, r)$, where $t^* > 0$ is the first positive time such that $G(\phi, \omega, r; t^*)$ lies on the z-axis. In particular, a direct computation shows that

$$t^* = \begin{cases} \dfrac{2\mathbb{S}^\circ}{|\omega|}, & \text{if } \omega \neq 0, \\ \infty, & \text{if } \omega = 0. \end{cases}$$

We conclude this section by highlighting a property of geodesics in the Heisenberg group that will be relevant in our analysis. For the sake of notation, denote by $\Omega^\circ_{(\phi,\omega,r)}$ the following transformation of $\Omega^\circ = B_1^{\|\cdot\|_*}(0)$:

$$\Omega^\circ_{(\phi,\omega,r)} := R_{-\pi/2}\left[\frac{r}{\omega}(\Omega^\circ - (\cos_{\Omega^\circ}(\phi), \sin_{\Omega^\circ}(\phi)))\right],$$

where $R_{-\pi/2}$ is counter-clockwise rotation in the plane of angle $-\pi/2$.

Proposition 5.5.12 ([Ber94, Thm. 1])**.** *Let $\mathbb{H}$ be the sub-Finsler Heisenberg group, equipped with a strictly convex norm and let $\gamma : [0, 1] \to \mathbb{H}$ be a geodesic starting at the origin, with $\gamma(t) = (x(t), y(t), z(t))$. Then, the curve $t \mapsto (x(t), y(t))$ is either a straight line or belongs to the boundary of $\Omega^\circ_{(\phi,\omega,r)}$. Moreover, for every $t \in [0, 1]$, $z(t)$ equals the oriented area that is swept by the vector joining $(0, 0)$ with $(x(s), y(s))$, for $s \in [0, t]$.*

5.5.3 Failure of the $\mathsf{CD}(K, N)$ condition for $C^{1,1}$-norms

In this section we contradict the validity of the $\mathsf{CD}(K, N)$ condition in the sub-Finsler Heisenberg group, equipped with a strictly convex and $C^{1,1}$ norm and with a smooth measure. The strategy follows the blueprint of the one presented in Section 5.4.4. The main issue we have to address here is the low regularity (cf. Remark 5.5.25) of the midpoint and inverse geodesic maps of (5.4.5) and

(5.4.12). Nevertheless, using the explicit expression of geodesics presented in Proposition 5.5.11, we successfully overcome these challenges through a series of technical lemmas, culminating in Corollary 5.5.20, Proposition 5.5.23 and Theorem 5.5.24.

Let $\mathbb{H}$ be the sub-Finsler Heisenberg group, equipped with a $C^{1,1}$ and strictly convex norm $\|\cdot\|$. According to Proposition 5.5.8, the dual norm $\|\cdot\|_*$ is C^1 and strongly convex. Thus, in the notations of Section 5.5.1, the correspondences C° and $C_\circ$ are continuous functions. In order to ease the notation, in this section we sometimes use the shorthands:

$$\theta^\circ = C^\circ(\theta) \quad \text{and} \quad \psi_\circ = C_\circ(\psi), \qquad \forall\, \theta, \psi \in \mathbb{R}.$$

Alexandrov's theorem ensures that the dual norm $\|\cdot\|_*$ has a second derivative and a second-order Taylor expansion almost everywhere, we call $D_* \subset \mathbb{R}^2$ the set of twice differentiability of it.

Proposition 5.5.13. *Let $\psi \in [0, 2\mathbb{S}^\circ)$ be an angle such that $Q_\psi \in D_*$, then the function $C_\circ$ is differentiable at ψ with positive derivative.*

Proof. Consider a vector $v \in \mathbb{R}^2$ orthogonal to $d_{Q_\psi}\|\cdot\|_*$ such that $\|v\|_{eu} = 1$. Then, since $Q_\psi \in D_*$, there exists a constant $C \in \mathbb{R}$ such that

$$\|Q_\psi + sv\|_* = 1 + Cs^2 + o(s^2), \qquad \text{as } s \to 0. \tag{5.5.12}$$

Observe that, since the norm $\|\cdot\|_*$ is strongly convex, the constant C is strictly positive. Consider the curve

$$s \mapsto x(s) := \frac{Q_\psi + sv}{\|Q_\psi + sv\|_*},$$

which by definition is a parametrization of an arc of the unit sphere $S_1^{\|\cdot\|_*}(0) = \partial\Omega^\circ$. Call $A(s)$ the signed area of the sector of Ω° between the rays OQ_ψ and $Ox(s)$ (see Figure 5.3). As a consequence of (5.5.12), we deduce that

$$A(s) = \frac{1}{2} ks + o(s^2), \qquad \text{as } s \to 0,$$

where k is the scalar product between Q_ψ and $v^\perp$, that is the vector obtained by rotating v with an angle of $-\frac{\pi}{2}$. In fact, the first-order term $\frac{1}{2}ks$ is the area of the triangle of vertices O, Q_ψ and $Q_\psi + sv$, while the error term is controlled by the area of the triangle of vertices $x(s)$, Q_ψ and $Q_\psi + sv$. The latter is an $o(s^2)$ as $s \to 0$, thanks to (5.5.12). In particular, letting $\psi(s)$ be the angle such that $x(s) = Q_{\psi(s)}$, by definition of generalized angles, it holds that

$$\psi(s) - \psi = 2A(s) = ks + o(s^2), \qquad \text{as } s \to 0.$$

Up to substituting the vector v with $-v$, we can assume $k > 0$. Then, in order to conclude, it is enough to prove that the function $s \mapsto C_\circ(\psi(s))$ is differentiable in $s = 0$ with positive derivative.

First of all, by our choice of $k > 0$, $s \mapsto C_\circ(\psi(s))$ is monotone non-decreasing close to $s = 0$, being a composition of monotone non-decreasing functions. Second of all, we can show that it has a first-order expansion. To this aim, note that the curve

$$s \mapsto y(s) := d_{x(s)}\|\cdot\|_*$$

is a parametrization of an arc of the sphere $S_1^{\|\cdot\|}(0) = \partial\Omega$ (cf. Lemma 5.3.6). Moreover, recalling that $Q_\psi \in D_*$ and using the homogeneity of the norm, we have that

$$y(s) = d_{Q_\psi + sv}\|\cdot\|_* = d_{Q_\psi}\|\cdot\|_* + a\, s + o(s), \qquad \text{as } s \to 0, \tag{5.5.13}$$

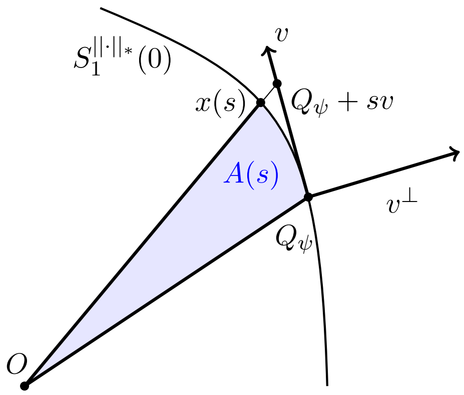

Figure 5.3: Definition of $A(s)$.

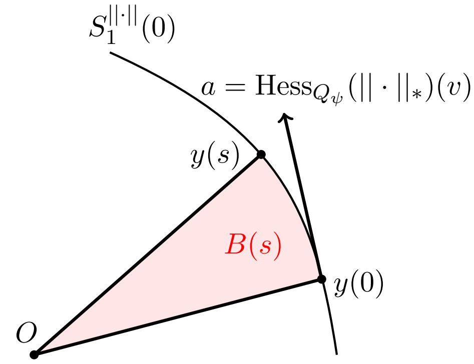

Figure 5.4: Definition of $B(s)$.

where $a := \mathrm{Hess}_{Q_\psi}(\|\cdot\|_*)(v)$. Observe that $a \neq 0$ because $\|\cdot\|_*$ is strongly convex and $\|Q_\psi\|_* = 1$. Then, call $B(s)$ the (signed) area of the sector of Ω between the rays $Oy(0)$ and $Oy(s)$ (see Figure 5.4). Reasoning as we did for $A(s)$, from (5.5.13) we deduce that

$$B(s) = \frac{1}{2}\langle y(0), a^\perp\rangle s + o(s^2), \qquad \text{as } s \to 0.$$

On the other hand, by definition

$$C_\circ(\psi(s)) - C_\circ(\psi(0)) = 2B(s) = \langle y(0), a^\perp\rangle s + o(s^2), \qquad \text{as } s \to 0. \tag{5.5.14}$$

This shows that the function $s \mapsto C_\circ(\psi(s))$ is differentiable in $s = 0$ with derivative $\langle y(0), a^\perp\rangle$. In addition, since $C_\circ \circ \psi$ is non-decreasing close to $s = 0$, (5.5.14) also implies that $\langle y(0), a^\perp\rangle \geq 0$. We are left to show that $\langle y(0), a^\perp\rangle$ is strictly positive. If $\langle y(0), a^\perp\rangle = 0$ then a is parallel to $y(0)$, however, according to (5.5.13), the vector a is tangent to the sphere $S_1^{\|\cdot\|}(0)$ at $y(0)$ and therefore we obtain a contradiction. □

Remark 5.5.14. Since the norm is invariant by homotheties, then also D_* is so, thus the set of angles ψ such that $Q_\psi \notin D_*$ has null $\mathscr{L}^1$-measure. In particular, the function $C_\circ$ is differentiable with positive derivative $\mathscr{L}^1$-almost everywhere, as a consequence of Proposition 5.5.13.

As already mentioned, the strategy to prove the main theorem of this section is the same of Section 5.4.4. In particular, it is fundamental to prove estimates on the volume contraction along geodesic homotheties. To this aim, we consider the Jacobian determinant of the exponential map (5.5.11):

$$J(\phi,\omega,r;t) := \left|\det\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \omega} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \omega} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \omega} \end{pmatrix}(\phi,\omega,r;t)\right|$$

where we recall $x(\phi,\omega,r;t)$, $y(\phi,\omega,r;t)$, $z(\phi,\omega,r;t)$ are defined in Proposition 5.5.11. In order to study this, we will use the following formulation:

$$J(\phi,\omega,r;t) = \left|\frac{\partial z}{\partial \omega}(\phi,\omega,r;t)\det(M_1) - \frac{\partial z}{\partial \phi}(\phi,\omega,r;t)\det(M_2) + \frac{\partial z}{\partial r}(\phi,\omega,r;t)\det(M_3)\right| \tag{5.5.15}$$

where

$$M_1 := \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} \end{pmatrix}(\phi,\omega,r;t), \quad M_2 := \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \omega} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \omega} \end{pmatrix}(\phi,\omega,r;t), \quad M_3 := \begin{pmatrix} \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \omega} \\ \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \omega} \end{pmatrix}(\phi,\omega,r;t).$$

We are particularly interested in studying the behaviour of $J(\phi,\omega,r;t)$ as $t\to 0$. In the following lemmas we estimate the behaviour of every term in (5.5.15) as $t\to 0$.

Notation 5.5.15. Let $I\subset\mathbb{R}$ be an interval containing 0. Given a function $f:I\to\mathbb{R}$ and $n\in\mathbb{N}$, we write

$$f(t)\sim t^n,\qquad \text{as } t\to 0,$$

if there exists a constant $C\neq 0$ such that $f(t)=Ct^n+o(t^n)$, as $t\to 0$.

Lemma 5.5.16. *Let $\phi\in[0,2\mathbb{S}^\circ)$ be a differentiability point for the map $C_\circ$, $r>0$ and $\omega\neq 0$, then*

$$\det\big(M_1(\phi,\omega,r;t)\big)\sim t^2,\qquad \text{as } t\to 0, \tag{5.5.16}$$

while

$$\det\big(M_2(\phi,\omega,r;t)\big),\,\det\big(M_3(\phi,\omega,r;t)\big)=O(t^3),\qquad \text{as } t\to 0. \tag{5.5.17}$$

Proof. Let us begin by proving (5.5.16). Firstly, since the function $C_\circ$ is differentiable at ϕ, we can compute the following Taylor expansions as $t\to 0$, using Proposition 5.5.4:

$$\begin{aligned}\cos_\Omega\big((\phi+\omega t)_\circ\big)&=\cos_\Omega\big(\phi_\circ\big)-t\omega C_\circ'(\phi)\sin_{\Omega^\circ}(\phi)+o(t),\\ \sin_\Omega\big((\phi+\omega t)_\circ\big)&=\sin_\Omega\big(\phi_\circ\big)+t\omega C_\circ'(\phi)\cos_{\Omega^\circ}(\phi)+o(t).\end{aligned}$$

Therefore, we may expand the entries of M_1 as $t\to 0$:

$$\begin{aligned}\frac{\partial x}{\partial r}(\phi,\omega,r;t)&=\frac{1}{\omega}\left(\sin_{\Omega^\circ}(\phi+\omega t)-\sin_{\Omega^\circ}(\phi)\right)=\cos_\Omega(\phi_\circ)t+o(t),\\ \frac{\partial y}{\partial r}(\phi,\omega,r;t)&=-\frac{1}{\omega}\left(\cos_{\Omega^\circ}(\phi+\omega t)-\cos_{\Omega^\circ}(\phi)\right)=\sin_\Omega(\phi_\circ)t+o(t),\\ \frac{\partial x}{\partial \phi}(\phi,\omega,r;t)&=\frac{r}{\omega}\left(\cos_\Omega\big((\phi+\omega t)_\circ\big)-\cos_\Omega\big(\phi_\circ\big)\right)=-trC_\circ'(\phi)\sin_{\Omega^\circ}(\phi)+o(t),\\ \frac{\partial y}{\partial \phi}(\phi,\omega,r;t)&=\frac{r}{\omega}\left(\sin_\Omega\big((\phi+\omega t)_\circ\big)-\sin_\Omega\big(\phi_\circ\big)\right)=trC_\circ'(\phi)\cos_{\Omega^\circ}(\phi)+o(t),\end{aligned} \tag{5.5.18}$$

where we used once again Proposition 5.5.4. Finally, the determinant has the following Taylor expansion as $t\to 0$:

$$\begin{aligned}\det(M_1)=\frac{\partial x}{\partial r}\frac{\partial y}{\partial \phi}-\frac{\partial y}{\partial r}\frac{\partial x}{\partial \phi}&=t^2rC_\circ'(\phi)\big(\sin_{\Omega^\circ}(\phi)\sin_\Omega\big(\phi_\circ\big)+\cos_{\Omega^\circ}(\phi)\cos_\Omega\big(\phi_\circ\big)\big)+o(t^2)\\ &=t^2rC_\circ'(\phi)+o(t^2),\end{aligned}$$

where, in the last equality, we used Proposition 5.5.3. This proves (5.5.16), keeping in mind Proposition 5.5.13, which guarantees that $C_\circ'(\phi)>0$.

Now we prove (5.5.17) for $\det(M_2)$, the proof for $\det(M_3)$ is analogous. As a first step, reasoning as before, we can Taylor expand at second-order the following quantities, as $t\to 0$:

$$\begin{aligned}\cos_{\Omega^\circ}(\phi+\omega t)&=\cos_{\Omega^\circ}(\phi)-\omega t\sin_\Omega(\phi_\circ)-\frac{1}{2}(\omega t)^2C_\circ'(\phi)\cos_{\Omega^\circ}(\phi)+o(t^2),\\ \sin_{\Omega^\circ}(\phi+\omega t)&=\sin_{\Omega^\circ}(\phi)+\omega t\cos_\Omega(\phi_\circ)-\frac{1}{2}(\omega t)^2C_\circ'(\phi)\sin_{\Omega^\circ}(\phi)+o(t^2).\end{aligned}$$

Hence, we deduce the expansion for the derivative of x in the ω direction, as $t\to 0$:

$$\begin{aligned}\frac{\partial x}{\partial \omega}(\phi,\omega,r;t)&=-\frac{r}{\omega^2}\big(\sin_{\Omega^\circ}(\phi+\omega t)-\sin_{\Omega^\circ}(\phi)\big)+\frac{rt}{\omega}\cos_\Omega\big((\phi+\omega t)_\circ\big)\\ &=-\frac{r}{\omega}\big(t\cos_\Omega(\phi_\circ)-\frac{1}{2}\omega t^2C_\circ'(\phi)\sin_{\Omega^\circ}(\phi)\big)+\frac{rt}{\omega}\big(\cos_\Omega\big(\phi_\circ\big)-t\omega C_\circ'(\phi)\sin_{\Omega^\circ}(\phi)\big)+o(t^2)\\ &=-\frac{1}{2}rt^2C_\circ'(\phi)\sin_{\Omega^\circ}(\phi)+o(t^2).\end{aligned} \tag{5.5.19}$$

An analogous computation shows that the derivative of y in ω has the ensuing expansion as $t \to 0$:

$$\frac{\partial y}{\partial \omega}(\phi, \omega, r; t) = \frac{1}{2} r t^2 C_\circ'(\phi) \cos_{\Omega^\circ}(\phi) + o(t^2). \tag{5.5.20}$$

Note that, on the one hand, (5.5.19) and (5.5.20) imply that

$$\frac{\partial x}{\partial \omega} = O(t^2) \qquad \text{and} \qquad \frac{\partial y}{\partial \omega} = O(t^2), \tag{5.5.21}$$

as $t \to 0$. On the other hand, by (5.5.18), we can deduce the following behavior, as $t \to 0$:

$$\frac{\partial x}{\partial r} = O(t) \qquad \text{and} \qquad \frac{\partial y}{\partial r} = O(t). \tag{5.5.22}$$

Thus, (5.5.21) and (5.5.22) prove the claimed behavior of $\det(M_2)$ as $t \to 0$, since

$$\det(M_2) = \frac{\partial x}{\partial r}\frac{\partial y}{\partial \omega} - \frac{\partial y}{\partial r}\frac{\partial x}{\partial \omega}.$$

□

In the next lemmas, we study the derivatives of z. These are the most delicate to estimate, since the second-order Taylor polynomial of z is zero and higher-order derivatives may not exist.

Notation 5.5.17. Let $g : \mathbb{R} \to \mathbb{R}$ be a function. We write

$$g(t) = C(1 + O(\varepsilon)) f(t), \qquad \forall\, t \in [-\rho, \rho].$$

if there exists a constant $K > 0$ and a function $f : \mathbb{R} \to \mathbb{R}$ such that, for every $\varepsilon > 0$, there exist positive constants $C = C(\varepsilon), \rho = \rho(\varepsilon) > 0$ for which the following holds

$$C(1 - K\varepsilon) f(t) < g(t) < C(1 + K\varepsilon) f(t), \qquad \forall\, t \in [-\rho, \rho].$$

Lemma 5.5.18. *Given $\varepsilon > 0$ sufficiently small, for $\mathscr{L}^1$-almost every $\phi \in [0, 2\mathbb{S}^\circ)$, every $r > 0$ and $\omega \neq 0$, there exist two positive constants $k = k(r)$ and $\rho = \rho(\phi, \omega, r)$ such that*

$$\frac{\partial z}{\partial \omega}(\phi, \omega, r; t) = (1 + O(\varepsilon)) k t^3, \qquad \forall\, t \in [-\rho, \rho]. \tag{5.5.23}$$

Proof. First of all, we compute that

$$\begin{aligned} \frac{\partial z}{\partial \omega}(\phi, \omega, r; t) = {} & \frac{r^2 t}{2\omega^2}\big(1 - \sin_{\Omega^\circ}(\phi) \sin_\Omega\left((\phi + \omega t)_\circ\right) - \cos_{\Omega^\circ}(\phi) \cos_\Omega\left((\phi + \omega t)_\circ\right)\big) \\ & - \frac{r^2}{\omega^3}\big(\omega t + \sin_{\Omega^\circ}(\phi) \cos_{\Omega^\circ}(\phi + \omega t) - \cos_{\Omega^\circ}(\phi) \sin_{\Omega^\circ}(\phi + \omega t)\big). \end{aligned} \tag{5.5.24}$$

In order to evaluate this quantity, fix an angle $\psi \in [0, 2\mathbb{S}^\circ)$, for which Proposition 5.5.13 holds, and consider the function f_ψ, defined as

$$s \mapsto f_\psi(s) := 1 - \sin_{\Omega^\circ}(\psi + s) \sin_\Omega(\psi_\circ) - \cos_{\Omega^\circ}(\psi + s) \cos_\Omega(\psi_\circ).$$

Notice that (5.5.2) ensures that $f_\psi(0) = 0$, moreover direct computations show that

$$f_\psi'(0) = 0 \qquad \text{and} \qquad f_\psi''(0) = C_\circ'(\psi) > 0.$$

Consequently, it holds that

$$f_\psi(s) = C_\circ'(\psi) \cdot s^2 + o(s^2), \qquad \text{as } s \to 0. \tag{5.5.25}$$

For every $n\in\mathbb{Z}$ and $m\in\mathbb{N}$ define the set of angles

$$E_{n,m}:=\Big\{\psi\in[0,2\mathbb{S}^\circ)\,:\,(1+\varepsilon)^{n-1}s^2<f_\psi(s)<(1+\varepsilon)^{n+1}s^2,\ \text{for every } s\in\Big[-\frac{1}{m},\frac{1}{m}\Big]\Big\}.$$

Observe that, by (5.5.25), we have that $E:=\bigcup_{n\in\mathbb{Z},m\in\mathbb{N}}E_{n,m}$ covers all differentiability points of $C_\circ$, in particular E has full $\mathscr{L}^1$-measure, cf. Remark 5.5.14.

Now, fix $\omega>0$, $r>0$ and take $\phi\in[0,2\mathbb{S}^\circ)$ to be a density point[2] for the set $E_{n,m}$, for some $n\in\mathbb{Z}$, $m\in\mathbb{N}$. We are going to prove the statement (5.5.23) for our choice of parameters and for positive times. The cases with $\omega<0$ and negative times are completely analogous. Let $0<\rho(\phi,\omega,r)<\frac{1}{2\omega m}$ be sufficiently small such that for every $t\in(0,\rho]$

$$\mathscr{L}^1(E_{n,m}\cap[\phi-2\omega t,\phi+2\omega t])>4\omega t(1-\varepsilon/4). \tag{5.5.26}$$

Introduce the set

$$F_{n,m}:=\{s\in\mathbb{R}\,:\,\phi+\omega s\in E_{n,m}\}.$$

Observe that, from (5.5.26), we can deduce that for every $t\in(0,\rho]$,

$$\mathscr{L}^1(F_{n,m}\cap[-2t,2t])>4t(1-\varepsilon/4). \tag{5.5.27}$$

Now, given every $t\in(0,\rho]$, (5.5.27) ensures that there exists $\bar{s}\in[t(1-\varepsilon),t]$ such that $\bar{s}\in F_{n,m}$. Then, thanks to Corollary 5.5.7, we obtain that

$$\begin{aligned}1-\sin_{\Omega^\circ}(\phi)\sin_\Omega\big((\phi+\omega t)_\circ\big)&-\cos_{\Omega^\circ}(\phi)\cos_\Omega\big((\phi+\omega t)_\circ\big)\\&\geq 1-\sin_{\Omega^\circ}(\phi)\sin_\Omega\big((\phi+\omega\bar{s})_\circ\big)-\cos_{\Omega^\circ}(\phi)\cos_\Omega\big((\phi+\omega\bar{s})_\circ\big)\\&=f_{\phi+\omega\bar{s}}(-\omega\bar{s})\geq(1+\varepsilon)^{n-1}(\omega\bar{s})^2\geq(1-\varepsilon)^2(1+\varepsilon)^{n-1}(\omega t)^2,\end{aligned} \tag{5.5.28}$$

where the second to last inequality holds by our choice of the parameter ρ and because $\phi+\omega\bar{s}\in E_{n,m}$. With an analogous argument, we can find an element in $[t,t(1+\varepsilon)]\cap F_{n,m}$ and deduce the estimate:

$$1-\sin_{\Omega^\circ}(\phi)\sin_\Omega\big((\phi+\omega t)_\circ\big)-\cos_{\Omega^\circ}(\phi)\cos_\Omega\big((\phi+\omega t)_\circ\big)\leq(1+\varepsilon)^2(1+\varepsilon)^{n+1}(\omega t)^2. \tag{5.5.29}$$

Combining (5.5.28) and (5.5.29), we conclude that, on $(0,\rho]$, the following holds

$$1-\sin_{\Omega^\circ}(\phi)\sin_\Omega\big((\phi+\omega t)_\circ\big)-\cos_{\Omega^\circ}(\phi)\cos_\Omega\big((\phi+\omega t)_\circ\big)=(1+O(\varepsilon))(1+\varepsilon)^n(\omega t)^2, \tag{5.5.30}$$

in the Notation 5.5.17. Consequently, we deduce:

$$\frac{r^2t}{2\omega^2}\big(1-\sin_{\Omega^\circ}(\phi)\sin_\Omega\big((\phi+\omega t)_\circ\big)-\cos_{\Omega^\circ}(\phi)\cos_\Omega\big((\phi+\omega t)_\circ\big)\big)=(1+O(\varepsilon))(1+\varepsilon)^n\frac{r^2t^3}{2}, \tag{5.5.31}$$

for $t\in(0,\rho]$. To estimate the second term in (5.5.24), observe that

$$\begin{aligned}\frac{\partial}{\partial s}\big(\omega s&+\sin_{\Omega^\circ}(\phi)\cos_{\Omega^\circ}(\phi+\omega s)-\cos_{\Omega^\circ}(\phi)\sin_{\Omega^\circ}(\phi+\omega s)\big)\\&=\omega\big(1-\sin_{\Omega^\circ}(\phi)\sin_\Omega\big((\phi+\omega s)_\circ\big)-\cos_{\Omega^\circ}(\phi)\cos_\Omega\big((\phi+\omega s)_\circ\big)\big).\end{aligned}$$

[2] We say that $r\in\mathbb{R}$ is a *density point* for a measurable set $J\subset\mathbb{R}$ if

$$\lim_{s\to0^+}\frac{\mathscr{L}^1(J\cap[r-s,r+s])}{2s}=1.$$

In particular, since in $s=0$ this quantity is equal to 0, we have that for $t \in (0,\rho]$

$$\begin{aligned}\omega t &+ \sin_{\Omega^\circ}(\phi)\cos_{\Omega^\circ}(\phi+\omega t) - \cos_{\Omega^\circ}(\phi)\sin_{\Omega^\circ}(\phi+\omega t)\\ &= \omega\int_0^t \left(1-\sin_{\Omega^\circ}(\phi)\sin_\Omega\left((\phi+\omega s)_\circ\right) - \cos_{\Omega^\circ}(\phi)\cos_\Omega\left((\phi+\omega s)_\circ\right)\right) \mathrm{d}s\\ &= \omega\int_0^t (1+O(\varepsilon))(1+\varepsilon)^n(\omega t)^2\,\mathrm{d}s = (1+O(\varepsilon))(1+\varepsilon)^n\frac{(\omega t)^3}{3},\end{aligned}$$

where the second equality follows from (5.5.30). Then, we obtain that:

$$\frac{r^2}{\omega^3}\left(\omega t + \sin_{\Omega^\circ}(\phi)\cos_{\Omega^\circ}(\phi+\omega t) - \cos_{\Omega^\circ}(\phi)\sin_{\Omega^\circ}(\phi+\omega t)\right) = (1+O(\varepsilon))(1+\varepsilon)^n\frac{r^2t^3}{3}. \tag{5.5.32}$$

Finally, putting together (5.5.31) and (5.5.32), we conclude that

$$\frac{\partial z}{\partial\omega}(\phi,\omega,r;t) = (1+O(\varepsilon))(1+\varepsilon)^n\frac{r^2t^3}{6}, \qquad \forall\, t\in(0,\rho],$$

that is (5.5.23) with $k=k(r):=(1+\varepsilon)^n\frac{r^2}{6}$. To conclude, observe that we proved the statement for (every $r>0$, $\omega\neq 0$ and) every $\phi\in[0,2\mathbb{S}^\circ)$ which is a density point of some $E_{n,m}$ and the set of such angles has full $\mathscr{L}^1$-measure in $[0,2\mathbb{S}^\circ)$. Indeed, $E=\bigcup_{n\in\mathbb{Z},m\in\mathbb{N}}E_{n,m}$ has full $\mathscr{L}^1$-measure in $[0,2\mathbb{S}^\circ)$ and almost every point of a measurable set is a density point. □

Lemma 5.5.19. *Let $\phi\in[0,2\mathbb{S}^\circ)$ be a differentiability point for the map $C_\circ$, $r>0$ and $\omega\neq0$, then*

$$\frac{\partial z}{\partial r}(\phi,\omega,r;t),\ \frac{\partial z}{\partial\phi}(\phi,\omega,r;t) = o(t^2), \qquad \text{as } t\to 0.$$

Proof. We start by proving the statement for $\frac{\partial z}{\partial r}$. We have that

$$\frac{\partial z}{\partial r}(\phi,\omega,r;t) = \frac{r}{\omega^2}\left(\omega t + \cos_{\Omega^\circ}(\phi+\omega t)\sin_{\Omega^\circ}(\phi) - \sin_{\Omega^\circ}(\phi+\omega t)\cos_{\Omega^\circ}(\phi)\right).$$

Direct computations show that, on the one hand,

$$\begin{aligned}\frac{\partial}{\partial t}\Big|_{t=0}&\left(\omega t + \cos_{\Omega^\circ}(\phi+\omega t)\sin_{\Omega^\circ}(\phi) - \sin_{\Omega^\circ}(\phi+\omega t)\cos_{\Omega^\circ}(\phi)\right)\\ &= \omega - \omega\sin_\Omega(\phi_\circ)\sin_{\Omega^\circ}(\phi) - \omega\cos_\Omega(\phi_\circ)\cos_{\Omega^\circ}(\phi) = 0,\end{aligned}$$

where we applied Proposition 5.5.3, and, on the other hand,

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\Big|_{t=0}&\left(\omega t + \cos_{\Omega^\circ}(\phi+\omega t)\sin_{\Omega^\circ}(\phi) - \sin_{\Omega^\circ}(\phi+\omega t)\cos_{\Omega^\circ}(\phi)\right)\\ &= -\omega^2\cos_{\Omega^\circ}(\phi)\sin_{\Omega^\circ}(\phi)C'_\circ(\phi) + \omega^2\sin_{\Omega^\circ}(\phi)\cos_{\Omega^\circ}(\phi)C'_\circ(\phi) = 0.\end{aligned}$$

Consequently, we conclude the proof of the first part of the statement:

$$\frac{\partial z}{\partial r}(\phi,\omega,r;t) = \frac{r}{\omega^2}\cdot o(t^2) = o(t^2), \qquad \text{as } t\to 0.$$

In order to prove the statement for $\frac{\partial z}{\partial\omega}$, we use a geometric argument based on Proposition 5.5.12. First of all, recall that $d_{Q_\phi}\|\cdot\|_*$ identifies a half-plane tangent at Q_ϕ and containing Ω°. Thus, we can find a rigid transformation $R:\mathbb{R}^2\to\mathbb{R}^2$, such that $R(Q_\phi)=(0,0)$ and $R(\Omega^\circ)$ is contained in $\{y\geq 0\}\subset\mathbb{R}^2$, see Figure 5.5. Then, as $\|\cdot\|_*$ is $C^{1,1}$, the image of the unit sphere

$R(\partial\Omega^\circ)$, can be described (locally around O) as the graph of a non-negative function $f \in C^{1,1}(\mathbb{R})$ with $f(0) = 0$. In addition, by our choice of $\phi \in [0, 2\mathbb{S}^\circ)$, f is twice differentiable in 0 with strictly positive second derivative $f''(0) := c > 0$. Now consider the function p defined in a neighborhood of ϕ as

$$p(\psi) := \mathtt{p}_x(R(Q_\psi)),$$

where $\mathtt{p}_x : \mathbb{R}^2 \to \mathbb{R}$ denotes the projection on the x-axis, i.e. $\mathtt{p}_x(a, b) = a$.

Second of all, for $s_1, s_2 \in \mathbb{R}$, call $F(s_1, s_2)$ the signed area between the segment connecting $(s_1, f(s_1))$ and $(s_2, f(s_2))$ and the graph of f (intended positive if $s_1 < s_2$ and negative if $s_1 > s_2$), see Figure 5.6. Proposition 5.5.12 ensures that for ψ in a neighborhood of ϕ it holds that

$$z(\psi, \omega, r; t) = \frac{r^2}{\omega^2} F\big(p(\psi), p(\psi + \omega t)\big).$$

In particular, we obtain that

$$\frac{\omega^2}{r^2} \frac{\partial z}{\partial \phi}(\phi, \omega, r; t) = \frac{\partial}{\partial s_1} F(0, p(\phi + \omega t)) \cdot p'(\phi) + \frac{\partial}{\partial s_2} F(0, p(\phi + \omega t)) \cdot p'(\phi + \omega t). \tag{5.5.33}$$

We now proceed to compute the terms in the last formula, starting from the ones involving p'. To this aim, consider the point $(x_0, y_0) := R(O)$ and, for every q in a neighborhood of 0, call $A(q)$ the signed area inside $R(\partial\Omega^\circ)$ between the segments $(x_0, y_0)O$ and $(x_0, y_0)(q, f(q))$. Observe that

$$A'(q) = \frac{1}{2}\langle (1, f'(q)), (y_0 - f(q), q - x_0)\rangle = \frac{1}{2} y_0 + O(q), \qquad \text{as } q \to 0.$$

Note that, in the last equality, we have used that $f(0) = f'(0) = 0$ and $f \in C^{1,1}(\mathbb{R})$. Consequently, since $A(0) = 0$, we have that

$$A(q) = \frac{1}{2} y_0 q + O(q^2), \qquad \text{as } q \to 0. \tag{5.5.34}$$

On the other hand, by the definition of angle it holds that $2A(p(\phi+\vartheta)) = \vartheta$ for every ϑ sufficiently small and therefore, invoking (5.5.34) and observing that $p \in C^1$, we obtain that

$$p(\phi + \vartheta) = \frac{1}{y_0}\vartheta + o(\vartheta) \quad \text{and} \quad p'(\phi + \vartheta) = \frac{1}{y_0} + o(1), \qquad \text{as } \vartheta \to 0. \tag{5.5.35}$$

Now we compute the partial derivatives of the function F. Observe that F can be calculated in the following way

$$F(s_1, s_2) = \frac{1}{2}(f(s_1) + f(s_2))(s_2 - s_1) - \int_{s_1}^{s_2} f(x)\,\mathrm{d}x.$$

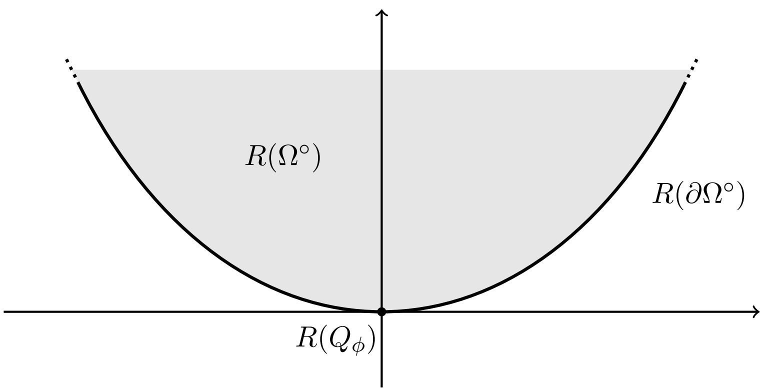

Figure 5.5: Image of Q_ϕ and Ω° through the rigid transformation R.

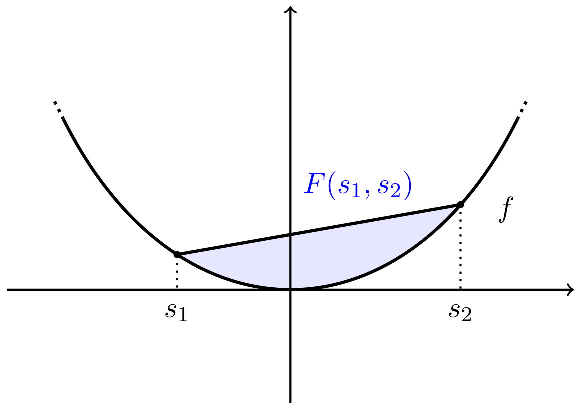

Figure 5.6: Definition of the function $F(s_1, s_2)$.

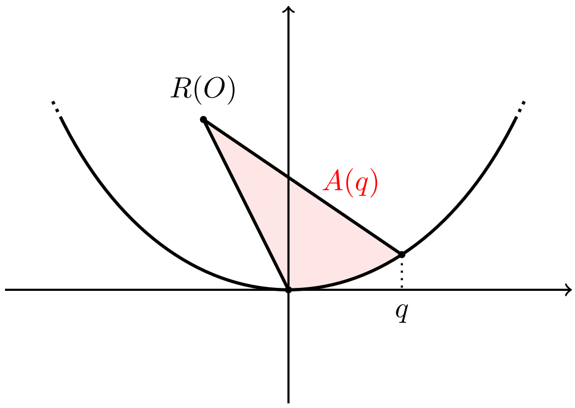

Figure 5.7: Definition of the function $A(q)$.

As a consequence, we compute that

$$\frac{\partial}{\partial s_1} F(s_1, s_2) = \frac{1}{2} f'(s_1)(s_2 - s_1) + \frac{1}{2}(f(s_1) - f(s_2))$$
$$\frac{\partial}{\partial s_2} F(s_1, s_2) = \frac{1}{2} f'(s_2)(s_2 - s_1) + \frac{1}{2}(f(s_1) - f(s_2)).$$

Combining these two relations with (5.5.33), we conclude that

$$\frac{\omega^2}{r^2}\frac{\partial z}{\partial \phi}(\phi, \omega, r; t) = -\frac{1}{2} f(p(\phi + \omega t)) \cdot p'(\phi),$$
$$+ \frac{1}{2}[f'(p(\phi + \omega t))p(\phi + \omega t) - f(p(\phi + \omega t))] \cdot p'(\phi + \omega t).$$

Now, recall that f is twice differentiable in 0 with positive second derivative c, therefore we have that

$$f(x) = \frac{1}{2}cx^2 + o(x^2) \qquad \text{and} \qquad f'(x) = cx + o(x).$$

Using these relations, together with (5.5.35), we can conclude that

$$\begin{aligned}
\frac{\omega^2}{r^2}\frac{\partial z}{\partial \phi}(\phi, \omega, r; t) &= \frac{1}{2} f'(p(\phi + \omega t))p(\phi + \omega t)p'(\phi + \omega t) - \frac{1}{2} f(p(\phi + \omega t))[p'(\phi) + p'(\phi + \omega t)] \\
&= \frac{1}{2}[cp(\phi + \omega t) + o(p(\phi + \omega t))]\, p(\phi + \omega t)p'(\phi + \omega t) \\
&\quad - \frac{1}{4}[cp(\phi + \omega t)^2 + o(p(\phi + \omega t)^2)][p'(\phi) + p'(\phi + \omega t)] \\
&= \frac{c}{2y_0^3}(\omega t)^2 - \frac{c}{2y_0^3}(\omega t)^2 + o(t^2) = o(t^2).
\end{aligned}$$

This concludes the proof. □

As a consequence of the these lemmas, we obtain the following estimate of the quantity $J(\phi, \omega, r; t)$, as $t \to 0$.

Corollary 5.5.20. *Given $\varepsilon > 0$ sufficiently small, for $\mathscr{L}^1$-almost every $\phi \in [0, 2\mathbb{S}^\circ)$, every $r > 0$ and $\omega \neq 0$, there exist two positive constants $C = C(\phi, \omega, r)$ and $\rho = \rho(\phi, \omega, r)$ such that*

$$J(\phi, \omega, r; t) = C(1 + O(\varepsilon))|t|^5, \qquad \forall\, t \in [-\rho, \rho],$$

in the Notation 5.5.17.

Proof. Let $\phi \in [0, 2\mathbb{S}^\circ)$ be a differentiability point for the map $C_\circ$ and such that the conclusion of Lemma 5.5.18 holds, and fix $r > 0$ and $\omega \neq 0$. Observe that, on the one hand, as a consequence of Lemma 5.5.16 and Lemma 5.5.19, we have that

$$\left|\frac{\partial z}{\partial \phi}\det(M_2)\right|(\phi,\omega,r;t),\ \left|\frac{\partial z}{\partial r}\det(M_3)\right|(\phi,\omega,r;t) = o(t^5), \qquad \text{as } t \to 0.$$

On the other hand, Lemma 5.5.16 and Lemma 5.5.18 ensure that there exist positive constants $C = C(\phi,\omega,r), \rho = \rho(\phi,\omega,r) > 0$ such that

$$\left|\frac{\partial z}{\partial \omega}\det(M_1)\right|(\phi,\omega,r;t) = C(1+O(\varepsilon))|t|^5, \qquad \forall\, t \in [-\rho,\rho]$$

where, in particular, ρ has to be smaller than the constant identified by Lemma 5.5.18. Up to taking a smaller ρ and keeping in mind (5.5.15), we may conclude that

$$J(\phi,\omega,r;t) = C(1+O(\varepsilon))|t|^5, \qquad \forall\, t \in [-\rho,\rho].$$

□

Remark 5.5.21. Note that, in the sub-Riemannian Heisenberg group, the contraction rate of volumes along geodesic is exactly t^5, cf. [ABR18]. In our setting, we are able to highlight the same behavior for the Jacobian determinant of the exponential map $J(\phi,\omega,r;t)$, as $t \to 0$.

Now that we know the behaviour of $J(\phi,\omega,r;t)$ as $t \to 0$, in the next proposition, we obtain a statement similar to Proposition 5.4.25, which will allow us to disprove the CD(K,N) condition in the Heisenberg group. In particular, the proof of the following proposition uses Corollary 5.5.20 and some ideas developed in [Jui21, Prop. 3.1].

In our setting, we define the *midpoint map* as:

$$\mathcal{M}(p,q) := e_{\frac{1}{2}}(\gamma_{pq}), \qquad \text{if } p \star q^{-1} \notin \{x = y = 0\}, \tag{5.5.36}$$

where $\gamma_{pq} : [0,1] \to \mathbb{H}$ is the unique geodesic joining p and q, given by Theorem 5.5.9. Similarly, we define the *inverse geodesic map* I_m (with respect to $m \in \mathbb{H}$) as:

$$I_m(q) = p, \qquad \text{if there exists } x \in \mathbb{H} \text{ such that } \mathcal{M}(p,q) = m. \tag{5.5.37}$$

Remark 5.5.22. Recall the definition of midpoint map in (5.4.10) and inverse geodesic map in (5.4.12). Both maps were defined using the differential structure of a smooth sub-Finsler manifold, however they are characterized by the metric structure of the space. In particular, if the norm is sufficently regular, they coincide with (5.5.36) and (5.5.37).

Proposition 5.5.23. *Let $\mathbb{H}$ be the sub-Finsler Heisenberg group, equipped with a strictly convex and $C^{1,1}$ norm. For $\mathscr{L}^1$-almost every $\phi \in [0, 2\mathbb{S}^\circ)$, every $r > 0$ and $\omega \neq 0$, there exists a positive constant $\rho = \rho(\phi,\omega,r)$ such that for every $t \in [-\rho,\rho]$:*

(i) the inverse geodesic map I_e is well-defined and C^1 in a neighborhood of $G(\phi,\omega,r;t)$;

(ii) the midpoint map $\mathcal{M}$ is well-defined and C^1 in a neighborhood of $(\mathrm{e}, G(\phi,\omega,r;t))$, moreover

$$|\det d_{G(\phi,\omega,r;t)}\mathcal{M}(\mathrm{e},\cdot)| \leq \frac{1}{2^4}. \tag{5.5.38}$$

Proof. Take ε sufficiently small, let ϕ be an angle for which the conclusion of Corollary 5.5.20 holds. Fix $r > 0$ and $\omega \neq 0$, and let $\rho = \rho(\phi,\omega,r)$ be the (positive) constant identified by Corollary 5.5.20. Let $t \in [-\rho,\rho]$ and consider the map $E_t : T^*_{\mathrm{e}}\mathbb{H} \to \mathbb{H}$ defined as

$$E_t(\phi,\omega,r) := G(\phi,\omega,r;t) = \big(x(\phi,\omega,r;t), y(\phi,\omega,r;t), z(\phi,\omega,r;t)\big), \tag{5.5.39}$$

where G is defined in (5.5.11). Note that $J(\phi,\omega,r;t)$ is the Jacobian of $E_t(\phi,\omega,r)$ and in particular, since $t\in[-\rho,\rho]$, Corollary 5.5.20 ensures that $J(\phi,\omega,r;t)>0$. Then, from the inverse function theorem, we deduce that E_t is locally invertible in a neighborhood $B_t\subset\mathbb{H}$ of $E_t(\phi,\omega,r)$ with C^1 inverse $E_t^{-1}: B_t\to T^*_{\mathrm{e}}\mathbb{H}$. Then, according to Theorem 5.5.9 and Proposition 5.5.11, the curve $[-t,t]\ni s\mapsto G(\phi,\omega,r;s)$ is the unique geodesic connecting $G(\phi,\omega,r;-t)$ and $G(\phi,\omega,r;-t)$, and such that $G(\phi,\omega,r;0)=\mathrm{e}$, provided that ρ is sufficiently small. Hence, we can write the map $I_{\mathrm{e}}: B_t\to\mathbb{R}^3$ as

$$I_{\mathrm{e}}(q)=E_{-t}(E_t^{-1}(q)),\qquad \forall\, q\in B_t.$$

Therefore, the map I_{e} is C^1 on B_t, being a composition of C^1 functions, proving item *(i)*.

With an analogous argument, the midpoint map (with first entry e), $\mathcal{M}_{\mathrm{e}}(\cdot):=\mathcal{M}(\mathrm{e},\cdot): B_t\to\mathbb{R}^3$, can be written as

$$\mathcal{M}_{\mathrm{e}}(q)=E_{t/2}(E_t^{-1}(q)),\qquad \forall\, q\in B_t. \tag{5.5.40}$$

As before, we deduce this map is well-defined and C^1. To infer regularity of the midpoint map in a neighborhood of $(\mathrm{e},G(\phi,\omega,r;t))$, we take advantage of the underline group structure, in particular of the left-translations (5.5.5), which are isometries. Indeed, note that

$$\mathcal{M}(p,q)=L_p\left(\mathcal{M}_{\mathrm{e}}(L_{p^{-1}}(q))\right),\qquad \forall\, p,q\in\mathbb{H},$$

and, for every (p,q) in a suitable neighborhood of $(\mathrm{e},G(\phi,\omega,r;t))$, we have $L_{p^{-1}}(q)\in B_t$, therefore $\mathcal{M}$ is well-defined and C^1. Finally, keeping in mind (5.5.40) and applying Corollary 5.5.20, we deduce that

$$\begin{aligned}\left|\det d_{G(\phi,\omega,r;t)}\mathcal{M}_{\mathrm{e}}(\cdot)\right| &= \left|\det d_{(\phi,\omega,r)}E_{t/2}\right|\cdot\left|\det d_{(\phi,\omega,r)}E_t\right|^{-1}\\ &= J(\phi,\omega,r;t/2)\cdot J(\phi,\omega,r;t)^{-1}\\ &= \frac{C(1+O(\varepsilon))|t/2|^5}{C(1+O(\varepsilon))|t|^5}=\frac{1}{2^5}\left(1+O(\varepsilon)\right)\le\frac{1}{2^4},\end{aligned}$$

where the last inequality is true for ε sufficiently small. This concludes the proof of item *(ii)*. □

Theorem 5.5.24. *Let $\mathbb{H}$ be the sub-Finsler Heisenberg group, equipped with a strictly convex and $C^{1,1}$ norm and with a smooth measure $\mathfrak{m}$. Then, the metric measure space $(\mathbb{H},\mathsf{d}_{SF},\mathfrak{m})$ does not satisfy the Brunn–Minkowski inequality $\mathsf{BM}(K,N)$, for every $K\in\mathbb{R}$ and $N\in(1,\infty)$.*

Proof. Take an angle ϕ for which the conclusion of Proposition 5.5.23 holds, fix $r>0$, $\omega\neq0$ and call γ the curve

$$\mathbb{R}\ni s\mapsto\gamma(s):=G(\phi,\omega,r;s).$$

Fix $t\in(0,\rho]$, where $\rho=\rho(\phi,\omega,r)$ is the positive constant identified by Proposition 5.5.23. Recall the map E_t (see (5.5.39)) from the proof of Proposition 5.5.23. E_t is invertible, with C^1 inverse, in a neighborhood $B_t\subset\mathbb{H}$ of $E_t(\phi,\omega,r)=\gamma(t)$. Consider the function

$$s\mapsto\Phi(s):=\mathtt{p}_1\big[E_t^{-1}\big(L_{\gamma(s)^{-1}}(\gamma(t+s))\big)\big],$$

where $\mathtt{p}_1$ denotes the projection onto the first coordinate. Observe that, for s sufficiently small, $L_{\gamma(s)^{-1}}(\gamma(t+s))\in B_t$, thus, Φ is well-defined and C^1 (being composition of C^1 functions) in an open interval $I\subset\mathbb{R}$ containing 0. Moreover, note that $\Phi(s)$ is the initial angle for the geodesic joining e and $\gamma(s)^{-1}\star\gamma(t+s)$. Now, we want to prove that there exists an interval $\tilde{I}\subset I$ such that, for $\mathscr{L}^1$-almost every $s\in\tilde{I}$, $\Phi(s)$ is an angle for which the conclusion of Proposition 5.5.23 holds. We have two cases, either $\Phi'\equiv0$ in I or there is $\bar{s}\in I$ such that $\Phi'(\bar{s})\neq0$. In the first case, since by definition $\Phi(0)=\phi$, we deduce that $\Phi(s)\equiv\phi$, thus the claim is true. In the second case, since Φ is C^1, we can find an interval $\tilde I \subset I$ such that $\Phi'(s) \neq 0$ for every $s \in \tilde I$. Then, consider

$$J := \{\psi \in \Phi(\tilde I) : \psi \text{ is an angle for which Proposition 5.5.23 holds}\} \subset \Phi(\tilde I)$$

and observe that J has full $\mathscr{L}^1$-measure in $\Phi(\tilde I)$. Therefore, the set $\tilde J := \Phi^{-1}(J) \subset \tilde I$ has full $\mathscr{L}^1$-measure in $\tilde I$, it being the image of J through a C^1 function with non-null derivative. Thus the claim is true also in this second case.

At this point, let $\bar s \in \tilde I$ such that $\Phi(\bar s)$ is an angle for which the conclusion of Proposition 5.5.23 holds and consider

$$\bar\rho := \rho\big(E_t^{-1}(L_{\gamma(\bar s)^{-1}}(\gamma(t+\bar s)))\big) > 0.$$

For every $s \in [-\bar\rho,\bar\rho]\backslash\{0\}$, from Proposition 5.5.23, we deduce that the inverse geodesic map I_{e} and the midpoint map $\mathcal{M}$ are well-defined and C^1 in a neighborhood of $G\big(E_t^{-1}(L_{\gamma(\bar s)^{-1}}(\gamma(t+\bar s))); s\big)$ and $\big(\mathrm{e}, G(E_t^{-1}(L_{\gamma(\bar s)^{-1}}(\gamma(t+\bar s))); s)\big)$, respectively. Moreover, we have that

$$|\det d_{G(E_t^{-1}(L_{\gamma(\bar s)^{-1}}(\gamma(t+\bar s)));s)}\mathcal{M}(\mathrm{e},\cdot)| \leq \frac{1}{2^4}.$$

Observe that, since the left-translations are smooth isometries, the inverse geodesic map $I_{\gamma(\bar s)}$ is well-defined and C^1 in a neighborhood of $\gamma(\bar s + s)$, in fact it can be written as

$$I_{\gamma(\bar s)}(p) = L_{\gamma(\bar s)}\big[I_{\mathrm{e}}(L_{\gamma(\bar s)^{-1}}(p))\big],$$

and $L_{\gamma(\bar s)^{-1}}(\gamma(\bar s+s)) = G\big(E_t^{-1}(L_{\gamma(\bar s)^{-1}}(\gamma(t+\bar s))); s\big)$. Similarly, we can prove that the midpoint map is well-defined and C^1 in a neighborhood of $(\gamma(\bar s), \gamma(\bar s + s))$, with

$$|\det d_{\gamma(\bar s+s)}\mathcal{M}(\gamma(\bar s),\cdot)| \leq \frac{1}{2^4}.$$

In conclusion, up to restriction and reparametrization, we can find a geodesic $\eta : [0,1] \to \mathbb{H}$ with the property that, for $\mathscr{L}^1$-almost every $\bar s \in [0,1]$, there exists $\lambda(\bar s) > 0$ such that, for every $s \in [\bar s - \lambda(\bar s), \bar s + \lambda(\bar s)] \cap [0,1] \setminus \{\bar s\}$, the inverse geodesic map $I_{\eta(\bar s)}$ and the midpoint map $\mathcal{M}$ are well-defined and C^1 in a neighborhood of $\eta(s)$ and $(\eta(\bar s), \eta(s))$ respectively, and in addition

$$|\det d_{\eta(s)}\mathcal{M}(\eta(\bar s),\cdot)| \leq \frac{1}{2^4}.$$

Set $\lambda(s) = 0$ on the (null) set where this property is not satisfied and consider the set

$$T := \Big\{(s,t) \in [0,1]^2 \,:\, t \in [s-\lambda(s), s+\lambda(s)]\Big\}.$$

Observe that, introducing for every $\epsilon > 0$ the set

$$D_\epsilon := \{(s,t) \in [0,1]^2 \,:\, |t-s| < \epsilon\},$$

we have that

$$\frac{\mathscr{L}^2(T\cap D_\epsilon)}{\mathscr{L}^2(D_\epsilon)} = \frac{\mathscr{L}^2(T\cap D_\epsilon)}{2\epsilon-\epsilon^2} \to 1, \qquad \text{as } \epsilon \to 0. \tag{5.5.41}$$

On the other hand, we can find $\delta > 0$ such that the set $\Lambda_\delta := \{s \in [0,1] \,:\, \lambda(s) > \delta\}$ satisfies $\mathscr{L}^1(\Lambda_\delta) > \frac{3}{4}$. In particular, for every $\epsilon < \delta$ sufficiently small we have that

$$\mathscr{L}^2\left(\left\{(s,t) \in [0,1]^2 \,:\, \frac{s+t}{2} \notin \Lambda_\delta\right\} \cap D_\epsilon\right) < \frac{1}{2}\epsilon. \tag{5.5.42}$$

Therefore, putting together (5.5.41) and (5.5.42), we can find $\epsilon < \delta$ sufficiently small such that

$$\mathscr{L}^2\left(T \cap D_\epsilon \cap \left\{(s,t) \in [0,1]^2 \,:\, \frac{s+t}{2} \in \Lambda_\delta\right\}\right) > \frac{1}{2}\mathscr{L}^2(D_\epsilon).$$

Then, since the set D_ϵ is symmetric with respect to the diagonal $\{s = t\}$, we can find $\bar{s} \neq \bar{t}$ such that

$$(\bar{s},\bar{t}), (\bar{t},\bar{s}) \in T \cap D_\epsilon \cap \left\{(s,t) \in [0,1]^2 \,:\, \frac{s+t}{2} \in \Lambda_\delta\right\}.$$

In particular, this tells us that:

(i) $\bar{t} \in [\bar{s} - \lambda(\bar{s}), \bar{s} + \lambda(\bar{s})]$ and $\bar{s} \in [\bar{t} - \lambda(\bar{t}), \bar{t} + \lambda(\bar{t})]$;

(ii) $|\bar{t} - \bar{s}| < \epsilon < \delta$;

(iii) $\frac{\bar{s}+\bar{t}}{2} \in \Lambda_\delta$.

Now, on the one hand, (i) ensures that the midpoint map $\mathcal{M}$ is well-defined and C^1 in a neighborhood of $(\eta(\bar{s}), \eta(\bar{t}))$ with

$$\left|\det d_{\eta(\bar{t})}\mathcal{M}(\eta(\bar{s}), \cdot)\right| \leq \frac{1}{2^4} \qquad \text{and} \qquad \left|\det d_{\eta(\bar{s})}\mathcal{M}(\cdot, \eta(\bar{t}))\right| \leq \frac{1}{2^4}.$$

While, on the other hand, the combination of (ii) and (iii) guarantees that the inverse geodesic map $I_{\eta(\frac{\bar{s}+\bar{t}}{2})}$ is well-defined and C^1 in a neighborhood of $\eta(\bar{s})$ and in a neighborhood of $\eta(\bar{t})$ respectively. Indeed, we have:

$$\bar{s},\bar{t} \in \left[\frac{\bar{s}+\bar{t}}{2} - \delta, \frac{\bar{s}+\bar{t}}{2} + \delta\right] \subset \left[\frac{\bar{s}+\bar{t}}{2} - \lambda\left(\frac{\bar{s}+\bar{t}}{2}\right), \frac{\bar{s}+\bar{t}}{2} + \lambda\left(\frac{\bar{s}+\bar{t}}{2}\right)\right],$$

and, by the very definition of $\lambda(\cdot)$, we obtain the claimed regularity of the inverse geodesic map.

Once we have these properties, we can repeat the same strategy used in the second part of the proof of Theorem 5.4.26 and contradict the Brunn–Minkowski inequality $\mathsf{BM}(K,N)$ for every $K \in \mathbb{R}$ and every $N \in (1,\infty)$. □

Remark 5.5.25. If we want to replicate the strategy of Theorem 5.4.26, we ought to find a short geodesic $\gamma : [0,1] \to \mathbb{H}$ such that

(i) the midpoint map $\mathcal{M}$ is C^1 around $(\gamma(0), \gamma(1))$ and satisfies a Jacobian estimates at $\gamma(1)$ of the type (5.5.38);

(ii) the midpoint map $\mathcal{M}$ satisfies a Jacobian estimates at $\gamma(0)$ of the type (5.5.38);

(iii) the inverse geodesic map $\mathcal{I}_{\gamma(1/2)}$, with respect to $\gamma(1/2)$, is C^1 around $\gamma(0)$ and $\gamma(1)$.

Proposition 5.5.23 guarantees the existence of a large set $\mathscr{A} \subset T^*_{\gamma(0)}\mathbb{H}$ of initial covectors for which the corresponding geodesic γ satisfies (i). The problem arises as the set $\mathscr{A}$ of "good" covectors depends on the base point and is large only in a measure-theoretic sense. A simple "shortening" argument, mimicking the strategy of the smooth case, is sufficient to address (ii) . However, once the geodesic is fixed, we have no way of ensuring that (iii) is satisfied. In particular, it may happen that the map $\mathcal{I}_{\gamma(1/2)}$ does not fit within the framework of Proposition 5.5.23 item *(i)*, as the corresponding initial covector may fall outside the hypothesis. To overcome such a difficulty, we use a density-type argument to choose *simultaneously* an initial point and an initial covector in such a way that (i)–(iii) are satisfied.

5.5.4 Failure of the $\mathsf{MCP}(K, N)$ condition for singular norms

In this section we prove Theorem 5.1.7, showing that the measure contraction property (see Definition 5.2.5) can not hold in a sub-Finsler Heisenberg group, equipped with a strictly convex, singular norm. Our strategy is based on the observation that, in this setting, geodesics exhibit a branching behavior, despite being unique (at least for small times).

Theorem 5.5.26. *Let $\mathbb{H}$ be the sub-Finsler Heisenberg group, equipped with a strictly convex norm $\|\cdot\|$ which is not C^1, and let $\mathfrak{m}$ be a smooth measure on $\mathbb{H}$. Then, the metric measure space $(\mathbb{H}, \mathsf{d}_{SF}, \mathfrak{m})$ does not satisfy the measure contraction property $\mathsf{MCP}(K, N)$ for every $K \in \mathbb{R}$ and $N \in (1, \infty)$.*

Proof. For simplicity, we assume $\mathfrak{m} = \mathscr{L}^3$. As it is apparent from the proof, the same argument can be carried out in the general case.

According to Proposition 5.5.8, since $\|\cdot\|$ is not C^1, its dual norm $\|\cdot\|_*$ is not strictly convex. In particular, there exists a straight segment contained in the sphere $S_1^{\|\cdot\|_*}(0) = \partial\Omega^\circ$. Since the differential structure of the Heisenberg group is invariant under rotations around the z-axis, we can assume without losing generality that this segment is vertical in $\mathbb{R}^2 \cap \{x > 0\}$, i.e. there exists $\bar{x} \in \mathbb{R}$ and an interval $I := [y_0, y_1] \subset \mathbb{R}$ such that

$$\{\bar{x}\} \times I \subset \partial\Omega^\circ.$$

Moreover, we can take the interval I to be maximal, namely for every $y \notin I$ we have $(\bar{x}, y) \notin \Omega$ (see Figure 5.8). Let $\psi_0 \in [0, 2\mathbb{S}^\circ)$ be such that $Q_{\psi_0} = (\bar{x}, y_0)$, then it holds that

$$(\bar{x}, y) = Q_{\psi_0 + (y - y_0)\bar{x}}, \qquad \text{for every } y \in I. \tag{5.5.43}$$

As a consequence, we have that

$$\cos_{\Omega^\circ}(\psi_0 + (y - y_0)\bar{x}) = \bar{x} \quad \text{and} \quad \sin_{\Omega^\circ}(\psi_0 + (y - y_0)\bar{x}) = y, \qquad \text{for } y \in I. \tag{5.5.44}$$

Let $y_2 = \frac{1}{2}(y_0 + y_1)$ and $\phi_0 = \psi_0 + \frac{1}{2}(y_1 - y_0)\bar{x}$, so that $(\bar{x}, y_2) = Q_{\phi_0}$ by (5.5.43). Moreover, take $\phi_1 > \psi_1 := \psi_0 + (y_1 - y_0)\bar{x}$ sufficiently close to ψ_1 (so that Q_{ϕ_1} is not in the flat part of $\partial\Omega^\circ$) and call $\bar{r} = \phi_1 - \phi_0 > 0$. We are now going to prove that there exists a suitably small neighborhood $\mathscr{A} \subset T_0^*\mathbb{H} \cong [0, 2\mathbb{S}^\circ) \times \mathbb{R} \times [0, \infty)$ of the point $(\phi_0, \bar{r}, \bar{r})$[3] such that

$$\mathscr{L}^3(G(\mathscr{A}; 1)) > 0. \tag{5.5.45}$$

For proving this claim, one could argue directly by computing the Jacobian of the map $G(\cdot, 1)$ at the point $(\phi_0, \bar{r}, \bar{r})$, however the computations are rather involved and do not display the geometrical features of the space. Thus, we instead prefer to present a different strategy, which highlights the interesting behaviour of geodesics.

Consider the map

$$F(\phi, \omega, r) := \big(x(\phi, \omega, r; 1), y(\phi, \omega, r; 1)\big),$$

where $x(\phi, \omega, r; t), y(\phi, \omega, r; t)$ are defined as in (5.5.11), and observe that

$$F(\phi_0, \bar{r}, \bar{r}) = (\sin_{\Omega^\circ}(\phi_1) - \sin_{\Omega^\circ}(\phi_0), \cos_{\Omega^\circ}(\phi_0) - \cos_{\Omega^\circ}(\phi_1)) = (\sin_{\Omega^\circ}(\phi_1) - y_2, \bar{x} - \cos_{\Omega^\circ}(\phi_1)).$$

Proceeding with hindsight, let $\varepsilon > 0$ such that $\varepsilon < \min\{\frac{1}{2}(\phi_1 - \psi_1), \frac{1}{4}(\psi_1 - \phi_0)\}$ and consider the intervals $I_\phi = [\phi_0 - \varepsilon, \phi_0 + \varepsilon]$ and $I_r = [\bar{r} - \varepsilon, \bar{r} + \varepsilon]$, then the set $F(I_\phi \times I_r \times I_r)$ is a neighborhood of $F(\phi_0, \bar{r}, \bar{r})$. Indeed, due to our choice of ϕ_1 the set

$$\{F(\phi_0, r, r) \,:\, r \in [\bar{r} - \varepsilon/2, \bar{r} + \varepsilon/2]\} \subset \mathbb{R}^2 \tag{5.5.46}$$

[3] Here the angle ϕ_0 has to be intended modulo $2\mathbb{S}^\circ$.

is a curve that is not parallel to the x-axis. Moreover, for every small δ such that $|\delta| < \psi_1 - \phi_0 = \phi_0 - \psi_0$ and every $r \in [\bar{r} - \varepsilon/2, \bar{r} + \varepsilon/2]$, the equalities in (5.5.44) imply the following relation:

$$\begin{aligned} F(\phi_0 + \delta, r - \delta, r - \delta) &= (\sin_{\Omega^\circ}(\phi_0 + r) - \sin_{\Omega^\circ}(\phi_0 + \delta), \cos_{\Omega^\circ}(\phi_0 + \delta) - \cos_{\Omega^\circ}(\phi_0 + r)) \\ &= (\sin_{\Omega^\circ}(\phi_0 + r) - \sin_{\Omega^\circ}(\phi_0) - \delta/\bar{x}, \bar{x} - \cos_{\Omega^\circ}(\phi_0 + r)) \\ &= (-\delta/\bar{x}, 0) + F(\phi_0, r, r). \end{aligned}$$

This shows that $F(I_\phi \times I_r \times I_r)$ contains all the sufficiently small horizontal translation of the set in (5.5.46) (see Figure 5.9), so it is a neighborhood of $F(\phi_0, \bar{r}, \bar{r})$. In particular $\mathscr{L}^2(F(I_\phi \times I_r \times I_r)) > 0$.

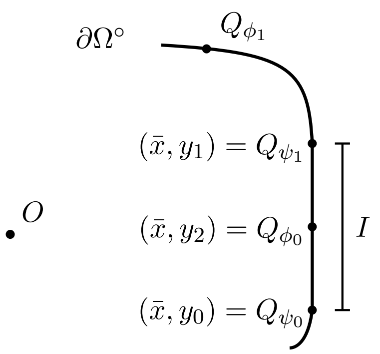

Figure 5.8: The flat part of $\partial\Omega^\circ$.

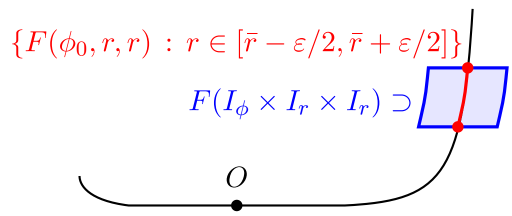

Figure 5.9: Estimate of the set $F(I_\phi \times I_r \times I_r)$.

Now we claim that, for every point $(\tilde{x}, \tilde{y}, \tilde{z}) = G(\tilde{\psi}, \tilde{\omega}, \tilde{r}; 1)$ with $\tilde{\psi} \in I_\phi$, $\tilde{\omega} \in I_r$ and $\tilde{r} \in I_r$, there exists an interval $J_z \ni \tilde{z}$ (depending on $\tilde{x}$ and $\tilde{y}$) such that

$$\{(\tilde{x}, \tilde{y}, z) \,:\, z \in J_z\} \subset G([\tilde{\psi} - \varepsilon, \tilde{\psi} + \varepsilon], [\tilde{\omega} - \varepsilon, \tilde{\omega} + \varepsilon], [\tilde{r} - \varepsilon, \tilde{r} + \varepsilon]; 1). \tag{5.5.47}$$

This is enough to prove (5.5.45), indeed, on the one hand, (5.5.47) implies that

$$\{(\tilde{x}, \tilde{y}, z) \,:\, z \in J_z\} \subset G(I'_\psi \times I'_r \times I'_r; 1), \tag{5.5.48}$$

where $I'_\psi = [\phi_0 - 2\varepsilon, \phi_0 + 2\varepsilon]$, and $I'_r = [\bar{r} - 2\varepsilon, \bar{r} + 2\varepsilon]$. On the other hand, since (5.5.48) holds for every point $(\tilde{x}, \tilde{y}) \in F(I_\phi \times I_r \times I_r)$, we deduce that

$$\mathscr{L}^3(G(I'_\psi \times I'_r \times I'_r; 1)) \geq \int_{F(I_\phi \times I_r \times I_r)} \mathscr{L}^1(J_z(\tilde{x}, \tilde{y}))\, \mathrm{d}\tilde{x}\, \mathrm{d}\tilde{y} > 0.$$

which implies (5.5.45) with $\mathscr{A} = I'_\psi \times I'_r \times I'_r$.

We proceed to the proof of claim (5.5.47): let $(\tilde{x}, \tilde{y}, \tilde{z}) = G(\tilde{\psi}, \tilde{\omega}, \tilde{r}; 1)$ with $\tilde{\psi} \in I_\phi$, $\tilde{\omega} \in I_r$ and $\tilde{r} \in I_r$ and consider the family of parallel lines

$$\{l(s) = \{y = s + kx\} \,:\, s \in \mathbb{R}\} \tag{5.5.49}$$

in $\mathbb{R}^2$, following the direction identified by the vector $(\tilde{x}, \tilde{y})$, see Figure 5.10. Call $S' \subset \mathbb{R}^2$ the sphere $\partial\Omega^\circ$ dilated by $\frac{\tilde{r}}{\tilde{w}}$ and rotated by $-\frac{\pi}{2}$. Then, there exists $\bar{s} \in \mathbb{R}$ such that $l(\bar{s})$ intersects S' in the points

$$\frac{\tilde{r}}{\tilde{w}}(\sin_{\Omega^\circ}(\tilde{\psi}), -\cos_{\Omega^\circ}(\tilde{\psi})) \qquad \text{and} \qquad \frac{\tilde{r}}{\tilde{w}}(\sin_{\Omega^\circ}(\tilde{\psi} + \tilde{r}), -\cos_{\Omega^\circ}(\tilde{\psi} + \tilde{r})).$$

Let $a(s)$ be the function that associates to s the area inside S' and below $l(s)$ and let $d(s)$ be the function that associates to s the (Euclidean) distance between the two intersections of

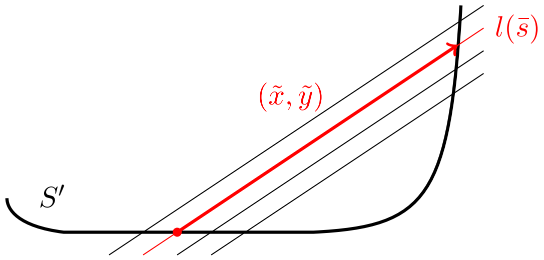

Figure 5.10: The line $l(\bar{s})$ identifies $(\tilde{x},\tilde{y})$.

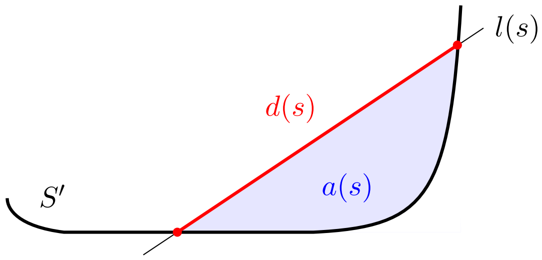

Figure 5.11: Definition of $d(s)$ and $a(s)$.

$l(s)$ with S' (see Figure 5.11). In particular, by our choice of $\bar{s}$, we have $d(\bar{s}) = \|(\tilde{x},\tilde{y})\|_{eu}$ and, according to Proposition 5.5.12, $a(\bar{s}) = \tilde{z}$. Moreover, note that, by Lemma 5.5.28, the function

$$s \mapsto \frac{a(s)}{d(s)^2} \text{ is strictly increasing.} \tag{5.5.50}$$

Now, for every s close enough to $\bar{s}$, the line $l(s)$ intersects S' in the points

$$\frac{\tilde{r}}{\tilde{\omega}}(\sin_{\Omega^\circ}(\psi(s)), -\cos_{\Omega^\circ}(\psi(s))) \qquad \text{and} \qquad \frac{\tilde{r}}{\tilde{\omega}}(\sin_{\Omega^\circ}(\psi(s)+r(s)), -\cos_{\Omega^\circ}(\psi(s)+r(s))),$$

with $\psi(s) \in [\tilde{\psi}-\varepsilon, \tilde{\psi}+\varepsilon]$ and $r(s) \in [\tilde{r}-\varepsilon/2, \tilde{r}+\varepsilon/2]$. By Proposition 5.5.12 and our choice of parallel lines in (5.5.49), we deduce that

$$G\left(\psi(s), r(s), r(s)\frac{\|(\tilde{x},\tilde{y})\|_{eu}}{d(s)}; 1\right) = \left(\tilde{x}, \tilde{y}, \frac{\|(\tilde{x},\tilde{y})\|_{eu}^2}{d(s)^2}\cdot a(s)\right).$$

Observe that, since d is a continuous function and $d(\bar{s}) = \|(\tilde{x},\tilde{y})\|_{eu}$, for every s sufficiently close to $\bar{s}$ we have

$$r(s) \in [\tilde{r}-\varepsilon, \tilde{r}+\varepsilon] \subset I'_r \qquad \text{and} \qquad r(s)\frac{d(s)}{\|(\tilde{x},\tilde{y})\|_{eu}} \in [\tilde{r}-\varepsilon, \tilde{r}+\varepsilon] \subset I'_r.$$

Then, (5.5.50) is sufficient to conclude the existence of an interval $J_z \subset \mathbb{R}$ as in (5.5.47). This concludes the proof of claim (5.5.45) with the choice $\mathscr{A} = I'_\psi \times I'_r \times I'_r$.

Finally, we are ready to disprove the measure contraction property $\mathsf{MCP}(K,N)$, taking as marginals

$$\mu_0 := \delta_{\mathrm{e}} \qquad \text{and} \qquad \mu_1 := \frac{1}{\mathscr{L}^3(G(\mathscr{A};1))}\mathscr{L}^3|_{G(\mathscr{A};1)}.$$

Note that, thanks to our construction of the set $\mathscr{A}$, the curve $t \mapsto G(\lambda;t)$, with $\lambda \in \mathscr{A}$, is the unique geodesic joining the origin and $G(\lambda;1)$ (cf. Theorem 5.5.9). Therefore, according to Remark 5.2.6, it is enough to contradict (5.2.4) with $A' = A = G(\mathscr{A};1)$. In particular, we prove that there exists $t_0 \in (0,1)$ such that

$$M_t(\{\mathrm{e}\}, A) \subset \{y=0, z=0\}, \qquad \forall\, t < t_0. \tag{5.5.51}$$

To this aim, fix any $(\phi,\omega,r) \in \mathscr{A}$ and note that, for every $t < \frac{\psi_1-\phi}{\omega}$, (5.5.44) implies that

$$\cos_{\Omega^\circ}(\phi+\omega t) = \bar{x} \qquad \text{and} \qquad \sin_{\Omega^\circ}(\phi+\omega t) = \sin_{\Omega^\circ}(\phi) + \frac{\omega t}{\bar{x}}.$$

From these relations, it follows immediately that

$$y(\phi,\omega,r;t) = 0 \qquad \text{and} \qquad z(\phi,\omega,r;t) = 0,$$

for every $t < \frac{\psi_1-\phi}{\omega}$. Observe that, by our choice of ε small enough, $\frac{\psi_1-\phi}{\omega}$ is bounded from below by a positive constant uniformly as $\phi \in I'_\phi$ and $\omega \in I'_r$, thus ensuring the existence of a constant $t_0 \in (0,1)$ for which (5.5.51) holds. □

Remark 5.5.27. In the last step of the proof of the preceding theorem, we established the existence of a family of branching geodesics: namely those corresponding to a flat part of $\partial\Omega^\circ$. In particular, when $\mathbb{H}$ is equipped with a strictly convex and singular norm, geodesics can branch, although they are unique. This is remarkable as examples of branching spaces usually occur when geodesics are not unique.

Lemma 5.5.28. *Let $f : \mathbb{R} \to \mathbb{R}$ be a concave and C^1 function. Assume that there exist $\alpha_0 < \beta_0$ such that*

$$f(\alpha_0) = f(\beta_0) = 0 \qquad \text{and} \qquad f > 0 \ \text{ on } \ (\alpha_0, \beta_0). \tag{5.5.52}$$

For every $s \in [0, \max f)$, define $\alpha(s) < \beta(s)$ such that

$$\{y = s\} \cap \operatorname{Graph}(f) = \{(\alpha(s), s) \,;\, (\beta(s), s)\}.$$

Denote by $a(s)$ the area enclosed by the line $\{y = s\}$ and the graph of f, and by $d(s) := \beta(s) - \alpha(s)$ (see Figure 5.12). Then,

$$[0, \max f) \ni s \mapsto \frac{a(s)}{d^2(s)} \qquad \text{is strictly decreasing.}$$

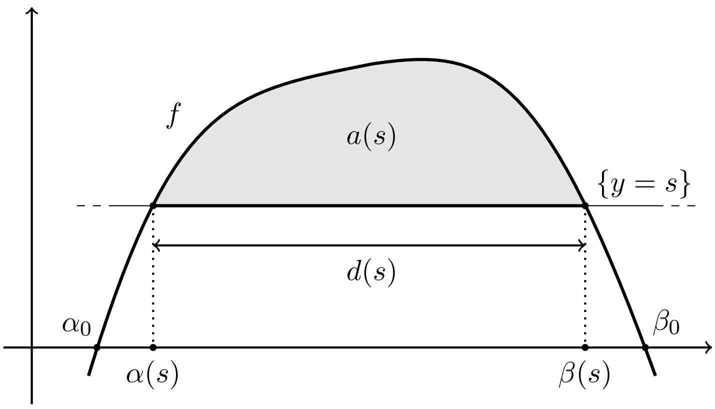

Figure 5.12: Representation of the quantities $\alpha(s)$, $\beta(s)$, $a(s)$ and $d(s)$.

Proof. Fix $0 \le s_1 < s_2 < \max f$, then it is sufficient to prove that

$$A_1 := a(s_1) > \frac{d^2(s_1)}{d^2(s_2)} a(s_2) =: A_2. \tag{5.5.53}$$

Observe that, by definition, $a(s) = \displaystyle\int_{\alpha(s)}^{\beta(s)} (f(t) - s)\,\mathrm{d}t$, therefore:

$$A_1 = \int_{\alpha(s_1)}^{\beta(s_1)} (f(t) - s_1)\,\mathrm{d}t, \qquad A_2 = \frac{d(s_1)}{d(s_2)} \int_{\alpha(s_1)}^{\beta(s_1)} \left[f\left(\alpha(s_2) + \frac{d(s_2)}{d(s_1)}(t - \alpha(s_1)) \right) - s_2 \right] \mathrm{d}t,$$

where, for A_2, we used the change of variables $t \mapsto \alpha(s_1) + \frac{d(s_1)}{d(s_2)}(t - \alpha(s_2))$. For ease of notation, set g to be the integrand of A_2, namely:

$$g(t) := \frac{d(s_1)}{d(s_2)} \left[f\left(\alpha(s_2) + \frac{d(s_2)}{d(s_1)}(t - \alpha(s_1)) \right) - s_2 \right], \qquad \forall\, t \in \mathbb{R}.$$

Now, let $\tilde{t} \in [\alpha(s_1), \beta(s_1)]$ be such that

$$\tilde{t} = \alpha(s_2) + \frac{d(s_2)}{d(s_1)}(\tilde{t} - \alpha(s_1)).$$

Note that, by linearity, $t \leq \tilde{t}$ if and only if $t \leq \alpha(s_2) + \frac{d(s_2)}{d(s_1)}(t - \alpha(s_1))$. Thus, for every $t \leq \tilde{t}$, the concavity of f yields that

$$g'(t) = f'\left(\alpha(s_2) + \frac{d(s_2)}{d(s_1)}(t - \alpha(s_1))\right) \leq f'(t). \tag{5.5.54}$$

Therefore, observing that $g(\alpha(s_1)) = 0$, we deduce that, for every $t \leq \tilde{t}$,

$$g(t) = \int_{\alpha(s_1)}^{t} g'(r)\,\mathrm{d}r \leq \int_{\alpha(s_1)}^{t} f'(r)\,\mathrm{d}r = f(t) - s_1. \tag{5.5.55}$$

The same inequality can be proved for every $t \geq \tilde{t}$, proceeding in a symmetric way. Thus, integrating both sides of (5.5.55), we obtain $A_1 \geq A_2$. Finally, observe that if $A_1 = A_2$ then also (5.5.54) is an equality, for every $t < \tilde{t}$. By concavity, this implies that $f'(t) \equiv c_1$ for every $t < \tilde{t}$. Analogously, $f'(t) \equiv c_2$ for every $t > \tilde{t}$ and (5.5.52) implies that we must have $c_1 \neq c_2$. Thus, f is linear on $(\alpha(s_1)), \tilde{t})$ and $(\tilde{t}, \beta(s_1)))$ and not differentiable at $\tilde{t}$. But, this contradicts that $f \in C^1(\mathbb{R})$, proving claim (5.5.53). □

Paper 6

Convergence of metric measure spaces satisfying the CD condition for negative values of the dimension parameter

with Chiara Rigoni and Gerardo Sosa

We study the problem of whether the curvature-dimension condition with negative values of the generalized dimension parameter is stable under a suitable notion of convergence. To this purpose, first of all we propose an appropriate setting to introduce the $\mathsf{CD}(K,N)$*-condition for* $N<0$*, allowing metric measure structures in which the reference measure is quasi-Radon. Then in this class of spaces we define the distance* $\mathsf{d}_{\mathsf{iKRW}}$*, which extends the already existing notions of distance between metric measure spaces. Finally, we prove that if a sequence of metric measure spaces satisfying the* $\mathsf{CD}(K,N)$*-condition with* $N<0$ *is converging with respect to the distance* $\mathsf{d}_{\mathsf{iKRW}}$ *to some metric measure space, then this limit structure is still a* $\mathsf{CD}(K,N)$ *space.*

All authors of this paper contributed equally to all results.

6.1 Introduction

In the last years, the class of metric measure spaces satisfying the synthetic curvature-dimension condition has been a central object of investigation. These spaces, in which a lower bound on the curvature formulated in terms of optimal transport holds, have been introduced by Sturm in [Stu06a, Stu06b] and independently by Lott and Villani in [LV09]. For a metric measure space $(\mathsf{X},\mathsf{d},\mathfrak{m})$, the curvature-dimension condition $\mathsf{CD}(K,N)$ depends on two parameters $K\in\mathbb{R}$ and $N\in[1,\infty]$ and it relies on a suitable convexity property of the entropy functional defined on the space of probability measures on X: the $\mathsf{CD}(K,N)$-condition for finite N is an appropriate reformulation of the $\mathsf{CD}(K,\infty)$ one introduced as the K-convexity of the relative entropy with respect to $\mathfrak{m}$. Spaces satisfying the curvature-dimension condition are Riemannian manifolds [Stu06a, Stu06b], Finsler spaces [Oht09] and Alexandrov spaces [Pet11, ZZ10]. In particular, in the case of a weighted Riemannian manifold, namely a Riemannian manifold (M,g) equipped with a weighted measure $\mathfrak{m}=e^{-\psi}\mathrm{vol}_{\mathrm{g}}$ which leads to a weighted Ricci curvature tensor Ric_N, being a $\mathsf{CD}(K,N)$ space is equivalent to the condition $\mathrm{Ric}_N\geq K$ that can be regarded as the combination of a lower bound by K on the curvature and an upper bound by N on the dimension. Moreover, in the setting of Riemannian manifolds, it turns out that for $N>0$ it is possible to characterize the $\mathsf{CD}(K,N)$-condition in terms of a property of the relative entropy: the required property is the (K,N)-convexity introduced in [EKS15]. This notion reinforces the one of K-convexity and can be generalized to the case of metric measure spaces.

In the Euclidean setting a direct application of the results in [BL76] ensures that given any convex measure μ, with full dimensional convex support and C^2 density Ψ, the space $(\mathbb{R}^n, \mathsf{d}_{\mathrm{Eucl.}}, \mu)$ satisfies the $\mathsf{CD}(0,N)$-condition for $1/N \in (-\infty, 1/n]$ (i.e. $N \in (-\infty,0) \cup [n,\infty]$), in the sense that $\mathrm{Ric}_N \geq 0$. This class of measures, introduced by Borell in [Bor75], extends the set of the so-called log-concave ones and it has been largely studied for example in [Bob07, BL09, Kol14]. In particular, following the terminology adopted by Bobkov, the case $N \in (-\infty, 0)$ corresponds to the "heavy-tailed measures" (see also [BCR05]), identified by the condition that $1/\Psi^{1/(n-N)}$ is convex. An explicit example of these measures is given by the family of Cauchy probability measures on $(\mathbb{R}^n, \mathsf{d}_{\mathrm{Eucl.}})$

$$\mu^{n,\alpha} = \frac{c_{n,\alpha}}{(1+|x|^2)^{\frac{n+\alpha}{2}}}\,\mathrm{d}x, \quad \alpha > 0, \tag{6.1.1}$$

where $c_{n,\alpha} > 0$ is a normalization constant. It then follows that $(\mathbb{R}^n, \mathsf{d}_{\mathrm{Eucl.}}, \mu^{n,\alpha})$ is a $\mathsf{CD}(0,-\alpha)$ space.

Admitting $N < 0$ may sound strange if one thinks of N as an upper bound on the dimension; however, as explained in [OT11] and [OT13], in the case of weighted Riemannian manifolds, it is useful to consider a generalization of the entropy, called m-relative entropy $H_m(\cdot|\nu)$, $m \in \mathbb{R}\setminus\{1\}$, stemming from the Bregman divergence in information geometry, which is closely related to the Rényi entropies in statistical mechanics. More precisely, in these papers Ohta and Takatsu prove that if (M,ω) is a weighted Riemannian manifold and $\nu = \exp_m(\Psi)\omega$ is a conformal deformation of ω in terms of the m-exponential function, then the fact that $H_m(\cdot|\nu) \geq K$ in the Wasserstein space $(\mathscr{P}_2(\mathsf{M}), W_2)$ is equivalent to the fact that $\mathrm{Hess}\Psi \geq K$ and $\mathrm{Ric}_N \geq 0$ with $N = 1/(1-m)$, where Ric_N is the weighted Ricci curvature tensor associated with (M,ω). In this setting, depending on the choice of the particular entropy, i.e., the value of m, the value of the dimension N can be negative. Hence they show that the bounds $\mathrm{Hess}\Psi \geq K$ and $\mathrm{Ric}_N \geq 0$ imply appropriate variants of the Talagrand, HWI, logarithmic Sobolev and the global Poincaré inequalities as well as the concentration of measures. Moreover, using similar techniques as in [JKO98, Vil03, OS09], they prove that the gradient flow of $H_m(\cdot|\nu)$ produces a weak solution to the porous medium equation (for $m > 1$) or the fast diffusion equation (for $m < 1$) of the form

$$\frac{\partial \rho}{\partial t} = \frac{1}{m}\Delta^{\omega}(\rho^m) + \mathrm{div}_{\omega}(\rho \nabla \Psi), \tag{6.1.2}$$

Δ^{ω} and div_{ω} being the Laplacian and the divergence associated with the measure ω. This result was demonstrated also by Otto [Ott01] in the case in which the reference measure ν in the m-relative entropy $H_m(\cdot|\nu)$ is given by the family of m-Gaussian measures, which is in turn closely related to the Barenblatt solution to (6.1.2) without drift (see [OW10, Tak12]).

In [Oht16], the author extends the range of admissible "dimension parameters" to negative values of N in the theories of (K,N)-convex functions, of tensors Ric_N and of the $\mathsf{CD}(K,N)$-condition in the more general setting of metric measure spaces. In particular, it is proved that the (K,N)-convexity for $N < 0$ is weaker than the K-convexity, thus it covers a wider class of functions. This means that the class of metric measure spaces satisfying the $\mathsf{CD}(K,N)$-condition for negative values of N includes all $\mathsf{CD}(K,\infty)$ ones; in particular, since a metric measure space which satisfies the $\mathsf{CD}(K,N)$-condition for some $N > 0$ is also a $\mathsf{CD}(K,\infty)$ space, it follows that:

$(\mathsf{X},\mathsf{d},\mathfrak{m})$ is a $\mathsf{CD}(K,N)$ space, for some $N > 0$.	$\Rightarrow$	$(\mathsf{X},\mathsf{d},\mathfrak{m})$ is a $\mathsf{CD}(K,\infty)$ space	$\Rightarrow$	$(\mathsf{X},\mathsf{d},\mathfrak{m})$ is a $\mathsf{CD}(K,N)$ space, for any $N < 0$.

The curvature-dimension condition for negative values of the dimension has not been largely studied up to now. In the setting of metric measure spaces, the only paper devoted to the study of this notion is the aforementioned work by Ohta [Oht16]. Therein, many direct consequences are extracted from the definitions, as in the case of the standard curvature-dimension bounds theory and a number of results valid in the case of $N > 0$ are generalized to these spaces, including the Brunn-Minkowski inequality and some other functional ones.

Nevertheless, most of the results on this topic are obtained in the case of weighted Riemannian manifolds. A first example of a model space is provided in [Mil17b]: it is therein proved that the n-dimensional unit sphere equipped with the harmonic measure, namely the hitting distribution by the Brownian motion started at $x \in \mathbb{S}^n$, $|x| < 1$ (which can be equivalently described as the probability measure whose density is proportional to $\mathbb{S}^n \ni y \to 1/|y-x|^{n+1}$) is a $\mathsf{CD}(n-1-(n+1)/4, -1)$ space. More generally, Milman provides an equivalent to the family (6.1.1) of Cauchy measures in $\mathbb{R}^{n+1}$, showing that the family of probability measures on the n-dimensional unit sphere having density proportional to

$$\mathbb{S}^n \ni y \mapsto \frac{1}{|y-x|^{n+\alpha}}$$

satisfies the curvature-dimension condition $\mathsf{CD}(n-1-\frac{n+\alpha}{4}, -\alpha)$ for all $|x| < 1$, $\alpha \geq -n$ and $n \geq 2$. In [Mil17a], the author studies the isoperimetric, functional and concentration properties of n-dimensional weighted Riemannian manifolds satisfying a uniform bound from below on the tensor Ric_N, when $N \in (-\infty, 1)$, providing a new one-dimensional model-space under an additional diameter upper bound (namely, a positively curved sphere of possibly negative dimension). In this setting, many other rigidity results have been obtained (see for example [Sak19, Sak20]). As for the Lorentzian splitting theorem in this setting, we would cite Wylie-Woolgar's paper [WW18]. Other interesting geometric results have been proved when the tensor Ric_N for $N \in (-\infty, 0]$ is uniformly bounded from below: for example, in the paper [KM18], Kolesnikov and Milman prove various Poincaré-type inequalities on the manifolds and their boundaries (making use of the Bochner's inequality and of the Reilly formula, when the boundary is nonempty).

Finally, let us underline that Bochner's inequality, generalized to the setting of weighted Riemannian manifolds satisfying the $\mathsf{CD}(K,N)$-condition for $N < 0$ in [Oht16] and in [KM18], independently, does not yet have a corresponding in the nonsmooth setting of metric measure spaces. We recall that for $N > 0$, this important inequality has been extended to the setting of singular spaces in a series of works, precisely in [OS14] for Finsler manifolds, in [GKO13] and [ZZ10] for Alexandrov spaces and in [AGS15] for $\mathsf{RCD}(K,\infty)$ spaces.

Despite the progress made in [Oht16], some fundamental questions remain open. The objective of this paper is to address the question of whether the curvature-dimension condition with negative value of generalized dimension is stable under convergence in a suitable topology. Special attention has to be payed to establishing an appropriate setting. In fact, inspired by some of the results found in [Oht16], we prove that for any $N < -1$ the interval $I := [-\pi/2, \pi/2]$ equipped with the Euclidean distance and the weighted measure $\mathrm{d}\mathfrak{m}(x) := \cos^N(x)\mathrm{d}\mathcal{L}^1(x)$, $\mathcal{L}^1$ being the 1-dimensional Lebesgue measure on I, is a $\mathsf{CD}(N,N)$ space. This fundamental example shows that the natural setting to introduce this curvature-dimension condition cannot be the one of complete and separable metric (Polish, in short) spaces equipped with Radon measures as in the case of $\mathsf{CD}(K,N)$ spaces with $N > 0$, but rather the one of Polish spaces endowed with quasi-Radon measures, i.e., measures which are Radon outside a negligible set. In fact, roughly speaking, the information that the weighted measure $\cos^N(x)\mathrm{d}\mathcal{L}^1(x)$ is the right one to consider in order to have a space with negative dimension comes from the theory of (K,N)-convex functions (see [Oht16, Section 2]). However, despite the fact that the "natural" domain for the function $\cos^N(x)$ with $N < 0$ would be the open interval $(-\pi/2, \pi/2)$, the theory of optimal transport forces us to consider the underlying metric space to be complete and separable, in order to ensure that also the Wasserstein space $(\mathscr{P}_2(\mathsf{X}), W_2)$ enjoys the same properties. Furthermore, we prove that also the space obtained by gluing together n-copies of the interval $(I, \mathsf{d}_{\mathrm{Eucl.}}, \mathfrak{m})$ introduced above still satisfies the $\mathsf{CD}(N,N)$-condition: this in particular shows that the negligible set of points in which the reference measure explodes is not just appearing in the "boundary" of our space, but also in the interior of it.

In this new and more general setting, a sequence of spaces satisfying the curvature-dimension condition for negative dimension parameter may fail to be stable under the standard measured Gromov-Hausdorff convergence of metric measure spaces. For example, it can be the case that a well-defined limit of a sequence of $\mathsf{CD}(K,N)$ spaces does not exist due to failure of convergence of the metric or the measure. We present some examples of this kind of behavior for metric measure spaces whose reference measures are quasi-Radon:

1) (*σ-finiteness lost in limit*) Consider the sequence of compact metric measure spaces given by $\{([0,1], \mathsf{d}_{\mathrm{Eucl.}}, \mathfrak{m}_n := x^{-n}dx)\}_{n\in\mathbb{N}}$. Since these measures are unbounded, it is not clear a priori in which way we want the measures to converge. One possibility, however, is the following: note that for every neighborhood U of 0 the measure $\mathfrak{m}_n|_{[0,1]\setminus U}$ is finite, therefore, up to being cautious with boundaries, one could ask for the weak*-convergence of the restricted finite measures $\mathfrak{m}_n|_{X\setminus U}$ to some measure $\mathfrak{m}_\infty^U$, for every neighborhood U of the origin. Then using an extension theorem we would obtain a unique measure $\mathfrak{m}_\infty$ defined on the whole interval $[0,1]$. However this construction leads to a measure $\mathfrak{m}_\infty$ which is infinite for every measurable subset A of $[0,1]$ with $\mathscr{L}^1(A) > 0$, losing thus any regularity.

2) (*Bounded measures to unbounded measures*) Consider now the sequence of metric measure spaces given by $\{([2^{-n}, +\infty), |\cdot|, x^N\mathscr{L}^1)\}_{n\in\mathbb{N}}$ for some fixed $N < -1$. For each $n \in \mathbb{N}$, the reference measure of the space is Radon, but the limit measure is such that every neighborhood of 0 has infinite mass.

We recall that in the setting of metric measure spaces, a suitable notion of convergence, called measured Gromov-Hausdorff convergence, was introduced by Fukaya in [Fuk87] as a natural variant of the purely metric Gromov-Hausdorff one. Then the stability of the $\mathsf{CD}(K,N)$-condition for $N \in [1,\infty)$ was proved following these two approaches:

- Lott and Villani proved that the $\mathsf{CD}(K,N)$-condition is stable under pointed measured Gromov-Hausdorff convergence in the class of proper pointed metric measure spaces. Roughly speaking, this means that for any $R > 0$ there is a measured Gromov-Hausdorff convergence of balls of radius R around the given points of the spaces;

- Sturm worked in the setting of Polish spaces equipped with probability measures with finite second moment as reference measures. In this class of spaces he defined a distance $\mathbb{D}$ by putting
$$\mathbb{D}\Big((\mathsf{X}_1, \mathsf{d}_1, \mathfrak{m}_1), (\mathsf{X}_2, \mathsf{d}_2, \mathfrak{m}_2)\Big) := \inf W_2((\iota_1)_\sharp\mathfrak{m}_1, (\iota_2)_\sharp\mathfrak{m}_2),$$
the infimum being taken among all complete and separable metric spaces (X, d) and all the isometric embeddings $\iota_i\colon (\mathrm{supp}(\mathfrak{m}_i), \mathsf{d}_i) \to (\mathsf{X}, \mathsf{d})$, $i = 1, 2$. He then showed that the curvature-dimension condition is stable with respect to this $\mathbb{D}$-convergence.

In particular, these two techniques produce the same convergence in the case of compact and doubling metric measure spaces. Then, in [GMS15] Gigli, Mondino and Savaré introduce a notion of convergence of metric measure spaces, called pointed measured Gromov convergence, which works without any compactness assumptions on the metric structure and for more general Radon measures which are finite on bounded sets. Moreover, they prove that lower Ricci bounds are stable with respect to this convergence.

As the first achievement of this paper we propose a suitable setting to introduce the curvature-dimension condition for negative values of the dimension parameter, extending and complementing the work by Ohta [Oht16]. We then propose an appropriate notion of distance, that we call intrinsic pointed Kantorovich-Rubinstein-Wasserstein distance $\mathsf{d}_{\mathsf{iKRW}}$, and we prove that the curvature-dimension bounds with negative values of the dimension are stable with respect to the $\mathsf{d}_{\mathsf{iKRW}}$-convergence. In particular, this distance extends the one introduced in [GMS15] to the set of equivalence classes of metric measure spaces with more general σ-finite measures, allowing us to analyze sequences of metric measure spaces in which the reference measures may "explode" in some points and are not necessarily finite on bounded sets (we underline that also in this setting we do not require the local compactness assumption on the metric structure).

More specifically, the structures we work with are isomorphism classes of *pointed generalized metric measure spaces* $(\mathsf{X}, \mathsf{d}, \mathfrak{m}, \mathcal{C}, p)$ where:

- (X, d) is a complete separable metric space,
- $\mathfrak{m} \in \mathscr{M}^{qR}(\mathsf{X})$ is a quasi-Radon measure, $\mathfrak{m} \neq 0$,
- $\mathcal{C} \subset \mathsf{X}$ is a closed set with empty interior and $\mathfrak{m}(\mathcal{C}) = 0$,
- $p \in \mathrm{supp}(\mathfrak{m}) \subset \mathsf{X}$ is a distinguished point,

and $(\mathsf{X}_1, \mathsf{d}_1, \mathfrak{m}_1, \mathcal{C}_1, p_1)$ is said to be isomorphic to $(\mathsf{X}_2, \mathsf{d}_2, \mathfrak{m}_2, \mathcal{C}_2, p_2)$ if there exists

an isometric embedding $i\colon \mathrm{supp}(\mathfrak{m}_1) \to \mathsf{X}_2$ such that $i(\mathcal{C}_1) = \mathcal{C}_2, i_\sharp \mathfrak{m}_1 = \mathfrak{m}_2$ and $i(p_1) = p_2$.

Intuitively, here for "quasi-Radon measure" $\mathfrak{m}$ on (X, d) (following the terminology introduced in [Fre06]) we mean a complete σ-finite measure with the following properties:

- there exists a closed negligible set with empty interior $\mathcal{S}_\mathfrak{m} \subset \mathsf{X}$ such that $\mathfrak{m}(U) = \infty$ for every open neighborhood U of $x \in \mathcal{S}_\mathfrak{m}$
- the restricted measure $\mathfrak{m}|_{\mathsf{X}\setminus\mathcal{S}_\mathfrak{m}}$ is Radon on the open set $\mathsf{X} \setminus \mathcal{S}_\mathfrak{m}$.

In this class of spaces we then introduce the intrinsic distance $\mathsf{d}_{\mathsf{iKRW}}$. This is constructed by taking partitions of the space: each element of the partition has finite measure and can be then renormalized; hence, we measure the intrinsic Kantorovich-Rubinstein-Wasserstein distance between these renormalized elements. In doing so, we take inspiration from the ideas behind the construction of the distance $\mathrm{p}\mathbb{G}_W$ in [GMS15]. However, in contrast to their setting, the lack of regularity of the measure becomes an obstacle to find a canonical and appropriate manner to partition the metric measure space. In particular, it turns out that a control on the Hausdorff distance of the singular sets in the definition of the $\mathsf{d}_{\mathsf{iKRW}}$-distance is actually necessary in order to provide an extrinsic realization of the distance given as an intrinsic one.

Then in this setting the $\mathsf{CD}(K, N)$-condition for negative values of N is introduced requiring a suitable convexity property of the extended Rényi entropy functional defined on the space of probability measures on X, as in the case $N > 0$.

We prove that this notion is stable with respect to the $\mathsf{d}_{\mathsf{iKRW}}$-distance: our main result shows that if a sequence of pointed generalized metric measure spaces $\{(\mathsf{X}_n, \mathsf{d}_n, \mathfrak{m}_n, \mathcal{C}_n, p_n)\}_{n\in\mathbb{N}}$ satisfying the $\mathsf{CD}(K, N)$-condition for some $N < 0$ (and some other technical assumptions) is converging in the $\mathsf{d}_{\mathsf{iKRW}}$-distance to some generalized metric measure space $(\mathsf{X}_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, \mathcal{C}_\infty, p_\infty)$, then this limit structure is still a $\mathsf{CD}(K, N)$ space.

This result in the case $N > 0$ strongly relies on the fact that the (standard) Rényi entropy functional is lower semicontinuous with respect to the weak topology in $\mathscr{P}_2(\mathsf{X})$. Unfortunately, the same property does not hold for the extended Rényi entropy functional $S_{N,\mathfrak{m}}$ when the reference measure $\mathfrak{m}$ is quasi-Radon. Therefore we provide a new argument to prove this stability, which extends the proofs of Lott-Villani and Sturm when $N > 0$ and the one of Gigli-Mondino-Savaré when $N = \infty$ (in all these classes of spaces the reference measure $\mathfrak{m}$ is Radon, namely $\mathcal{S}_{\mathfrak{m}} = \emptyset$). We show that $S_{N,\mathfrak{m}}$ is weakly lower semicontinuous on the space

$$\mathscr{P}^{\mathcal{S}_{\mathfrak{m}}}(\mathsf{X}) := \{\mu \in \mathscr{P}_2(\mathsf{X}) \,:\, \mu(\mathcal{S}_{\mathfrak{m}}) = 0\}$$

and that this will be enough to prove the desired stability result, provided that each one of the spaces in the converging sequence $\{(\mathsf{X}_n, \mathsf{d}_n, \mathfrak{m}_n, \mathcal{C}_n, p_n)\}_{n\in\mathbb{N}}$ is not accumulating "too much mass" around any of the points in $\mathcal{S}_{\mathfrak{m}_n}$ in a uniform way.

Finally, in Theorem 6.4.2 we manage to adapt this *Stability Result* also in the case in which we only have at our disposal a distance which is not explicitly dependent on the behavior of the $\mathfrak{m}$-singular sets. Intuitively, one of the examples we would like to include in our theory consists in approximating the $\mathsf{CD}(0, N+1)$ space $([0,\infty), |\cdot|, x^N\mathscr{L}^1)$, where $N < -1$, making use of the sequence of metric measure spaces $([2^{-n}, +\infty), |\cdot|, x^N\mathscr{L}^1)$: clearly each space in this sequence is still a $\mathsf{CD}(0, N+1)$ space for $N < -1$ but now the singularity of the measure is ruled out from the domain, meaning that each metric space $([2^{-n}, +\infty), |\cdot|)$ is actually equipped with a Radon measure. Hence, we rely on an extrinsic approach to convergence which does not require any control on the Hausdorff distance between $\mathfrak{m}$-singular sets in the definition of the $\mathsf{d}_{\mathsf{iKRW}}$-distance.

6.2 Metric spaces equipped with quasi-Radon measures

6.2.1 Measure theory background

Quasi-Radon measures

We begin by introducing some notation and concepts from measure theory. Let X be a set, $\mathcal{T}$, Σ be, respectively, a topology and a σ-algebra on X, and $\mathfrak{m}$ be a positive measure defined on Σ such that $\mathfrak{m}(\mathsf{X}) \neq 0$. Whenever $\mathcal{T} \subseteq \Sigma$, we will call the quadruple $(\mathsf{X}, \mathcal{T}, \Sigma, \mathfrak{m})$ a *topological measure space*.

Definition 6.2.1. Let $(\mathsf{X}, \mathcal{T}, \Sigma, \mathfrak{m})$ be a topological measure space. We say that the measure $\mathfrak{m}$ is:

i) *locally finite* if for every $x \in \mathsf{X}$ there exists a neighborhood $U \in \mathcal{T}$ with $\mathfrak{m}(U) < \infty$;

ii) *effectively locally finite* if for every $A \in \Sigma$ with $\mathfrak{m}(A) > 0$, there exists an open set $U \in \mathcal{T}$ with finite measure such that $\mathfrak{m}(A \cap U) > 0$;

iii) *σ-finite* if there exists $\{A_i\}_{i\in\mathbb{N}} \subset \Sigma$, with $\mathfrak{m}(A_i) < \infty$ for any $i \in \mathbb{N}$, such that $\mathsf{X} = \cup_{i\in\mathbb{N}} A_i$;

iv) *inner regular with respect to a family of sets $\mathcal{F} \subset \Sigma$* if for any $E \in \Sigma$ it holds

$$\mathfrak{m}(E) = \sup\{\mathfrak{m}(A) : A \in \mathcal{F} \text{ and } A \subset E\}.$$

In particular, when $\mathcal{F}$ is the family of compact sets of X, this property is called *tightness*.

We denote by $\mathcal{B}(\mathsf{X})$ the *Borel σ-algebra* of $(\mathsf{X}, \mathcal{T})$, namely the smallest σ-algebra containing all open sets of a topological space $(\mathsf{X}, \mathcal{T})$. A measure defined on $\mathcal{B}(\mathsf{X})$ is referred to as a *Borel measure*. In the following we will always consider measures $\mathfrak{m}$ on X which are Borel.

For the purpose of this paper we can restrict our study to the case in which the Borel measures are defined on a complete and separable metric space (X, d), in short, on a *Polish* space. In particular, it is useful to recall that every metric space is Hausdorff, which ensures that every compact subset is closed and, in particular, in $\mathcal{B}(\mathsf{X})$.

It can be proven that every effectively locally finite Borel measure, which is defined on a metric space, is actually inner regular with respect to closed sets (see [Fre06, Theorem 412E]). Moreover, every finite Borel measure on a Polish space is tight (see [Bog07, Volume 2, Theorem 7.1.7]). In this regard, we recall the following useful characterization of tight measures valid in the setting of Polish spaces (see [Fre06, Corollary 412B]):

Proposition 6.2.2. *Let (X,d) be a Polish space and $\mathfrak{m}$ be a Borel measure on X which is effectively locally finite. Then the following are equivalent:*

i) the measure $\mathfrak{m}$ is tight,

ii) for every $A \in \mathcal{B}(\mathsf{X})$ with $\mathfrak{m}(A) > 0$ there exist a measurable compact set $K \subset A$ with the property that $\mathfrak{m}(K) > 0$.

This result in particular allows us to prove an important property of effectively locally finite measures defined on a Polish space:

Lemma 6.2.3. *Let (X,d) be a Polish space and $\mathfrak{m}$ be a Borel measure on X which is effectively locally finite. Then $\mathfrak{m}$ is tight.*

Proof. Let us fix any subset $A \in \mathcal{B}(\mathsf{X})$ with $\mathfrak{m}(A) > 0$. Since $\mathfrak{m}$ is effectively locally finite, there exists an open set $U \subset A$ such that $0 < \mathfrak{m}(U) < \infty$. Moreover, being $\mathfrak{m}$ defined on a metric space, the inner regularity with respect to closed sets guarantees the existence of a closed set $C \subset U$ with the property that $0 < \mathfrak{m}(C) < \infty$. At this point the elementary observation that every closed subset of a Polish space is still a Polish space together with the fact that a finite Borel measure on a Polish space is tight ensure the existence of a measurable compact set $K \subset A$ with the property that $\mathfrak{m}(K) > 0$. Thanks to Proposition 6.2.2 we can conclude that $\mathfrak{m}$ is tight. □

We can now introduce the following classes of Borel measures, which are of central interest to us. We borrow the terminology proposed in [Fre06, Definitions 411H(a), (b)], where the classes of Radon and quasi Radon measures are defined in the more general setting of topological measure spaces, specializing these characterizations to the setting of Polish spaces.

Definition 6.2.4 (Radon and quasi-Radon measures)**.** Let (X,d) be a Polish space. We say that a complete Borel measure $\mathfrak{m}$ is

i) *Radon* if it is locally finite;

ii) *quasi-Radon* if it is effectively locally finite.

Remark 6.2.5 (Assumptions on inner regularity)**.** We remark that if the metric space (X,d) is just separable but not complete, an additional assumption on the inner regularity of $\mathfrak{m}$ is needed: in fact in this case a Radon measure has to be also inner regular with respect to compact sets, while a quasi-Radon measure is required to be inner regular with respect to closed sets. However, in our setting of Polish spaces both locally finite and effectively locally finite measures are tight, in view of Lemma 6.2.3, and for Hausdorff spaces tight measures are inner regular with respect to closed sets.

Let us now list some properties that these classes of measures satisfy. Before stating and proving these results, we recall that a topological space Y is called *Lindelöf* if for every open cover of Y, there exists a countable sub-cover. Moreover, Y is called *hereditary Lindelöf* if the same property holds for every subset $\mathcal{V} \subset Y$. Now we note that a Polish space (X,d) is second countable, since it is separable, and we recall that second countable topological spaces are Lindelöf. Moreover, since second countability is an hereditary property we have that actually any separable metric space is hereditary Lindelöf.

Proposition 6.2.6. *Let (X, d) be a Polish space equipped with a complete Borel measure $\mathfrak{m}$. Then it holds:*

i) if $\mathfrak{m}$ is a Radon measure, then $\mathfrak{m}$ is a quasi-Radon measure;

ii) conversely, if $\mathfrak{m}$ is an effectively locally finite quasi-Radon measure, then $\mathfrak{m}$ is a Radon measure;

iii) if $\mathfrak{m}$ is a quasi-Radon measure, then $\mathfrak{m}$ is σ-finite;

iv) if $\mathfrak{m}$ is quasi-Radon, then there exists a closed set $\mathcal{S}_\mathfrak{m}$ with empty interior and $\mathfrak{m}(\mathcal{S}_\mathfrak{m}) = 0$ such that $\mathfrak{m}|_{\mathsf{X}\setminus\mathcal{S}_\mathfrak{m}}$ is a Radon measure on the open set $\mathsf{X} \setminus \mathcal{S}_\mathfrak{m}$.

Proof. The first point $i)$ follows from the fact that on a separable metric space (X, d) a locally finite tight measure is essentially locally finite (see [Fre06, 416A] for a proof of this result), while point $ii)$ follows from Lemma 6.2.3.

Let us then prove $iii)$ by showing the existence of a countable collection of open sets $\{U_i\}_{i\in\mathbb{N}}$ such that $\mathfrak{m}(U_i) < \infty$ for every $i \in \mathbb{N}$ and that $\mathfrak{m}(\mathsf{X} \setminus \bigcup_{i\in\mathbb{N}} U_i) = 0$. The effective local finiteness property of the measure $\mathfrak{m}$ ensures the existence of a family $\mathcal{V}$ of open sets $V \subset \mathsf{X}$ with $\mathfrak{m}(V) < \infty$. Using the hereditary Lindelöf property of X, we can extract a countable sub-cover $\{U_i\}_{i\in\mathbb{N}}$ of the family $\mathcal{V}$, still satisfying the property $\mathfrak{m}(U_i) < \infty$ for every $i \in N$. Finally, we observe that $\mathsf{X} \setminus \bigcup_{i\in\mathbb{N}} U_i$ is a closed set with $\mathfrak{m}(\mathsf{X} \setminus \bigcup_{i\in\mathbb{N}} U_i) = 0$, since the intersection of $\mathsf{X} \setminus \bigcup_{i\in\mathbb{N}} U_i$ with each open set of finite measure is empty.

At this point, the proof of $iv)$ is straightforward. In fact, we can take as $\mathcal{S}_\mathfrak{m}$ the set $\mathsf{X}\setminus\bigcup_{i\in\mathbb{N}} U_i$: as already remarked this is a closed set with $\mathfrak{m}(\mathcal{S}_\mathfrak{m}) = 0$. The fact that $\mathcal{S}_\mathfrak{m}$ has empty interior is guaranteed by the fact that $\mathfrak{m}$ is effectively locally finite, while to conclude that $\mathfrak{m}|_{\mathsf{X}\setminus\mathcal{S}_\mathfrak{m}}$ is a Radon measure follows from the fact that it is locally finite. □

Finally, we show the validity of the Radon-Nikodym Theorem for quasi-Radon measures. With this aim, we first introduce another concept which is a strengthening of absolute continuity between measures.

Definition 6.2.7. Let $(X, \Sigma, \mathfrak{m})$ be a measurable space and μ be a measure on Σ. We say that a measure μ is truly continuous with respect to $\mathfrak{m}$ if:

i) μ is absolutely continuous with respect to $\mathfrak{m}$

ii) for any $E \in \Sigma$ with $\mu(E) > 0$ there is $F \in \Sigma$ such that $\mathfrak{m}(F) < \infty$ and $\mu(E \cap F) > 0$.

We refer to [Fre03, Section 232] for a proof of the following result:

Theorem 6.2.8 (Radon-Nikodym Theorem on measurable spaces)**.** *Let $(\mathsf{X}, \Sigma, \mathfrak{m})$ be a measurable space equipped with a quasi-Radon measure, and μ be a measure on X which is truly continuous with respect to $\mathfrak{m}$. Then there exists a measurable function f on X such that for any $B \in \mathcal{B}(\mathsf{X})$ it holds*

$$\mu(B) = \int_B f \,\mathrm{d}\mathfrak{m}.$$

Lemma 6.2.9. *Let $(\mathsf{X}, \Sigma, \mathfrak{m})$ be a measurable space equipped with a quasi-Radon measure which is σ-finite. Then μ is truly continuous with respect to $\mathfrak{m}$ if and only if it is absolutely continuous.*

Proof. Directly from the definition, we have that a measure μ which is truly continuous with respect to $\mathfrak{m}$ is also absolutely continuous with respect to $\mathfrak{m}$. In order to get the conclusion, we have just to show that if μ is an absolutely continuous measure with respect to a σ-finite measure $\mathfrak{m}$, then point *ii)* in Definition 6.2.7 is automatically satisfied. To show this, let $\{X_n\}_{n\in\mathbb{N}}$ be a non-decreasing sequence of sets of finite measure covering X, and μ absolutely continuous with respect to $\mathfrak{m}$. For any $E \in \Sigma$ such that $\mu(E) > 0$, we have that $\lim_{n\to\infty} \mu(E \cap X_n) > 0$, which means that there exists a $\bar{n} \in \mathbb{N}$ with $\mu(E \cap X_{\bar{n}}) > 0$. ☐

Theorem 6.2.10 (Radon-Nikodym Theorem on Polish spaces). *Let $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ be a Polish space equipped with a quasi-Radon measure, and μ be a measure on X which is absolutely continuous with respect to $\mathfrak{m}$. There exists a measurable function f on X such that for any $B \in \mathcal{B}(\mathsf{X})$ it holds*

$$\mu(B) = \int_B f \, \mathrm{d}\mathfrak{m}.$$

Proof. Since in this setting Proposition 6.2.6 ensures that the measure $\mathfrak{m}$ is σ-finite, we can conclude just applying Lemma 6.2.9 and Theorem 6.2.8. ☐

Convergence of quasi-Radon measures

Let (X, d) be a Polish space and let us define

$$\mathscr{P}(\mathsf{X}) := \{\mathfrak{m} : \mathfrak{m} \text{ is a probability measure on } \mathsf{X}\};$$
$$\mathscr{P}_2(\mathsf{X}) := \Big\{\mathfrak{m} \in \mathscr{P}(\mathsf{X}) : \int \mathsf{d}^2(x, x_0) \, \mathrm{d}\mathfrak{m}(x) < \infty \text{ for some, and thus any, } x_0 \in \mathsf{X}\Big\}.$$

On the space $\mathscr{P}_2(\mathsf{X})$ we introduce the 2-Wasserstein distance

$$W_2^2(\mu, \nu) := \inf_{\gamma \in \mathsf{Adm}(\mu,\nu)} \int_{\mathsf{X}\times\mathsf{X}} \mathsf{d}(x, y)^2 \, \mathrm{d}\gamma(x, y), \tag{6.2.1}$$

where $\mathsf{Adm}(\mu, \nu) := \{\gamma \in \mathscr{P}(\mathsf{X} \times \mathsf{X}) \,|\, \pi^1_\sharp \gamma = \mu \text{ and } \pi^2_\sharp \gamma = \nu\}$, $\pi^{1,2} \colon \mathsf{X} \times \mathsf{X} \to \mathsf{X}$ being the natural projection onto the first and the second coordinate respectively.

It is important to recall that the infimum in (6.2.1) is always realized and the plans $\gamma \in \mathsf{Adm}(\mu, \nu)$ such that $\int \mathsf{d}(x, y)^2 \, \mathrm{d}\gamma(x, y) = W_2^2(\mu, \nu)$ are called optimal couplings, or optimal transport plans. The set that contains them all is denoted by $\mathsf{Opt}(\mu, \nu)$. It is well known that W_2 is a complete and separable distance on $\mathscr{P}_2(\mathsf{X})$.

Now let $S \subset \mathsf{X}$. We say that $U \in \mathcal{B}(\mathsf{X})$ is a neighborhood of S, if there exists an open $V \in \mathcal{B}(\mathsf{X})$, such that $S \subset V \subset U$ and we write $\mathcal{N}_S$ for the set of all the neighborhoods of S in X. Let us fix a closed set with empty interior $S \subset \mathsf{X}$ to introduce the following classes of measures:

$$\mathscr{M}(\mathsf{X}) := \{\mathfrak{m} : \mathfrak{m} \text{ is a finite measure on } \mathsf{X}\};$$
$$\mathscr{M}^R_{loc}(\mathsf{X}) := \{\mathfrak{m} : \mathfrak{m} \text{ is a Radon measure on } \mathsf{X} \text{ s.t. } \mathfrak{m}(B) < \infty, \forall\, B \subset \mathsf{X} \text{ bounded}\};$$
$$\mathscr{M}_S(\mathsf{X}) := \{\mathfrak{m} : \mathfrak{m} \text{ is a quasi-Radon measure on } \mathsf{X},\ S \text{ is an } \mathfrak{m}\text{-null set and } \mathfrak{m}|_{X\setminus U} \in \mathscr{M}^R_{loc}(\mathsf{X}), \text{ for every } U \in \mathcal{N}_S\}.$$

The next class of measures is of central importance in our work,

$$\mathscr{M}^{qR}(\mathsf{X}) := \{\mathfrak{m} : \mathfrak{m} \text{ is a quasi-Radon measure on } \mathsf{X} \text{ for which there exists } S \subset \mathsf{X} \text{ closed with the property that } \mathfrak{m}(S) = 0 \text{ and } \mathfrak{m} \in \mathscr{M}_S(\mathsf{X})\}.$$

Notice that we have the following chain of inclusions: $\mathscr{P}(\mathsf{X}) \subset \mathscr{M}(\mathsf{X}) \subset \mathscr{M}^R_{loc}(\mathsf{X}) \subset \mathscr{M}^{qR}(\mathsf{X})$.

The adequate study of quasi-Radon measures will require us to monitor their singularities. Intuitively said, given a closed set $S \subset \mathsf{X}$ with empty interior, in the definition above we isolate the set of singular points of a quasi-Radon measure inside S. Thus one should regard $\mathscr{M}_S(\mathsf{X})$ as the set of quasi-Radon measures which are locally finite and concentrated in $X \setminus S$. Recall that the effective local finiteness implies that all singular sets S of quasi-Radon measures have empty interior, that is, $\mathscr{M}_S(\mathsf{X}) = \emptyset$ if $\mathrm{int}(S) \neq \emptyset$. The local finiteness guarantees as well that S is nowhere dense. Moreover, Proposition 6.2.6 proves that for every $\mathfrak{m} \in \mathscr{M}^{qR}(\mathsf{X})$ there exists a singular set $S_{\mathfrak{m}} \subset \mathsf{X}$, closed with empty interior, providing that $\mathfrak{m} \in \mathscr{M}_{S_{\mathfrak{m}}}(\mathsf{X})$. Finally, note that, in particular, $\mathscr{M}^R_{loc}(\mathsf{X}) \subset \mathscr{M}^{qR}(\mathsf{X}) \cap \mathscr{M}_\emptyset(\mathsf{X})$.

Let us now introduce the following sets of functions

$$\begin{aligned} C_{bs}(\mathsf{X}) &:= \{\text{bounded continuous functions with bounded support on } \mathsf{X}\}, \\ C_b(\mathsf{X}) &:= \{\text{bounded continuous functions on } \mathsf{X}\}, \\ C_S(\mathsf{X}) &:= \{\text{continuous functions on } \mathsf{X} \text{ which vanish on some neighborhood of } S\}, \end{aligned}$$

where S is a closed set with empty interior, and proceed to define a convergence on $\mathscr{M}_S(\mathsf{X})$ in duality with functions in $C_{bs}(\mathsf{X}) \cap C_S(\mathsf{X})$. In detail, we say that

Definition 6.2.11 (Weak convergence for quasi-Radon measures)**.** We say that a sequence of measures $\{\mathfrak{m}_n\}_{n\in\mathbb{N}} \subset \mathscr{M}_S(\mathsf{X})$ converges weakly to $\mathfrak{m}_\infty \in \mathscr{M}_S(\mathsf{X})$, and we write $\mathfrak{m}_n \rightharpoonup \mathfrak{m}_\infty$, if

$$\lim_{n\to\infty} \int f \, \mathrm{d}\mathfrak{m}_n = \int f \, \mathrm{d}\mathfrak{m}_\infty \qquad \text{for every } f \in C_{bs}(\mathsf{X}) \cap C_S(\mathsf{X}).$$

We wish to emphasize that many useful properties enjoyed by Radon measures are not necessarily valid in the setting of quasi-Radon measures. For example, it is well known that in a complete and separable metric space (X, d) equipped with a Radon measure $\mathfrak{m}$, the set $C_b(\mathsf{X}) \cap L^1(\mathfrak{m})$ is dense in $L^1(\mathfrak{m})$ (see [Rud87, Theorem 3.14]), while this result is no more true when $\mathfrak{m}$ is a quasi- Radon measure.

The following proposition substantiates our choice of convergence.

Proposition 6.2.12. *Let (X, d) be a Polish space, $S \subset \mathsf{X}$ be a closed set with empty interior, and $\mathfrak{m}, \mathfrak{n} \in \mathscr{M}_S(\mathsf{X})$ two quasi-Radon measures on X such that $\int f \, \mathrm{d}\mathfrak{m} = \int f \, \mathrm{d}\mathfrak{n}$ for every function $f \in C_{bs}(\mathsf{X}) \cap C_S(\mathsf{X})$. Then $\mathfrak{m} = \mathfrak{n}$.*

Proof. According to [Fre06, Proposition 415I], if $\mathfrak{m}, \mathfrak{n} \in \mathscr{M}^{qR}(\mathsf{X})$ are such that $\int f \, \mathrm{d}\mathfrak{m} = \int f \, \mathrm{d}\mathfrak{n}$, for every function $f \in C_b(\mathsf{X}) \cap L^1(\mathfrak{m}) \cap L^1(\mathfrak{n})$, then $\mathfrak{m} = \mathfrak{n}$. In particular this is valid for measures $\mathfrak{m}, \mathfrak{n} \in \mathscr{M}_S(\mathsf{X}) \subset \mathscr{M}^{qR}(\mathsf{X})$. The conclusion is attained using an approximating argument.

Let $x_0 \in \mathsf{X} \setminus S$ and, for any $n \in \mathbb{N}$, consider a sequence of Lipschitz functions $g_n \colon \mathsf{X} \to [0,1]$ with the property that

$$g_n = \begin{cases} 1 & \text{on } B_{2^n}(x_0) \cap \{x \in \mathsf{X} \,:\, \mathsf{d}(x,S) \geq 2^{-n}\}, \\ 0 & \text{on } \mathsf{X} \setminus B_{2^{n+1}}(x_0) \cap \{x \in \mathsf{X} \,:\, \mathsf{d}(x,S) \leq 2^{-(n+1)}\}. \end{cases}$$

Now, for every $f \in C_b(\mathsf{X}) \cap L^1(\mathfrak{m}) \cap L^1(\mathfrak{n})$, the sequence $f_n := g_n f$ is such that

$$\{f_n\}_{n\in\mathbb{N}} \subset C_{bs}(\mathsf{X}) \cap C_S(\mathsf{X}), \quad \lim_{n\to\infty} f_n = f \quad \mathfrak{m}, \mathfrak{n}\text{-a.e., and } |f_n| \leq |f|,$$

since $\mathfrak{m}(S) = \mathfrak{n}(S) = 0$. We can then conclude applying the dominated convergence theorem. □

Our definition of weak convergence for quasi-Radon measures turns out to be well-fitted for our purposes. Indeed, we have tailored it precisely with this goal. So let us then conclude this Subsection giving some observations regarding the corresponding topology.

Remark 6.2.13. i) For our purposes, we would like to have at our disposal a notion of convergence for quasi-Radon measures without making any a priori assumption on the uniformity of singular sets. However, this seems out of reach: indeed, without having any control on the singular sets of a given sequence, we would be able to generate an unfavorable limiting singular set and thus, for instance, obtain that $C_{S_\infty}(\mathsf{X}) = \{f \equiv 0\}$. In this case, the weak convergence is trivial. As an example, consider a dense and countable collection of points $P = \{p_m\}_{m\in\mathbb{N}} \subset \mathsf{X}$ in a complete and separable space, and non-atomic measures $\nu_n \in \mathscr{M}^{qR}(\mathsf{X})$, $\forall n \in \mathbb{N}$, such that for any neighborhood $U^n \subset \mathsf{X}$ of the set of the first n-points $P_n := \{p_1, \dots, p_n\}$, $\nu_n(U^n) = \infty$ while $\nu_n(\mathsf{X} \setminus U^n) < \infty$. Letting $n \to \infty$, we would expect a limit measure having P as a singular set but, for the reason given above, convergence defined against any meaningful subclass of continuous functions turns out to be trivial. Furthermore, note that such a limit measure would fall outside the realm of quasi-Radon measures.

ii) *Consistency.* Let us underline that by considering $S = \emptyset$ and by restricting the topology to $\mathscr{M}^R_{loc}(\mathsf{X})$ the above definition coincides with the weak* topology (induced in duality with $C_{bs}(\mathsf{X})$); by further restricting the topology to $\mathscr{M}(X)$, the weak topology agrees with the narrow topology (defined in duality with $C_b(\mathsf{X})$).

6.2.2 Pointed generalized metric measure spaces and their convergence

Metric spaces equipped with quasi-Radon measures

In the following we say that $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ is a *metric measure space* if (X, d) is a Polish space equipped with a quasi-Radon measure $\mathfrak{m}$. We will refer to a *generalized metric measure space* meaning a structure $(\mathsf{X}, \mathsf{d}, \mathfrak{m}, \mathcal{C})$ where:

- (X, d) is a complete separable metric space,
- $\mathfrak{m} \in \mathscr{M}^{qR}(\mathsf{X})$ is a quasi-Radon measure, $\mathfrak{m} \neq 0$,
- $\mathcal{C} \subset \mathsf{X}$ is a closed set with empty interior and $\mathfrak{m}(\mathcal{C}) = 0$.

A *pointed generalized metric measure space* is then the structure $(\mathsf{X}, \mathsf{d}, \mathfrak{m}, \mathcal{C}, p)$ consisting of a generalized metric measure space with a distinguished point $p \in \mathrm{supp}(\mathfrak{m}) \subset \mathsf{X}$.
Two generalized metric measure spaces $(\mathsf{X}_i, \mathsf{d}_i, \mathfrak{m}_i, \mathcal{C}_i)$, $i = 1, 2$ are called *isomorphic* if there exists

$$\text{an isometric embedding } i\colon \mathrm{supp}(\mathfrak{m}_1) \to \mathsf{X}_2 \text{ such that } i(\mathcal{C}_1) = \mathcal{C}_2 \text{ and } i_\sharp \mathfrak{m}_1 = \mathfrak{m}_2$$

and, in the case of pointed metric measure spaces $(\mathsf{X}_i, \mathsf{d}_i, \mathfrak{m}_i, \mathcal{C}_i, p_i)$, $i = 1, 2$, we further require that $i(p_1) = p_2$. Any such i is called an *isomorphism* from X_1 to X_2.
We denote by $\mathbb{X} := [\mathsf{X}, \mathrm{d}, \mathfrak{m}, \mathcal{C}, p]$ the equivalence class of the given pointed generalized metric measure space $(\mathsf{X}, \mathrm{d}, \mathfrak{m}, \mathcal{C}, p)$ and by $\mathcal{M}^{qR}$ the collection of all equivalence classes of pointed generalized metric measure spaces.
In particular, the portion of the space outside the support of the measure can be neglected since $(\mathsf{X}, \mathrm{d}, \mathfrak{m}, \mathcal{C})$ (resp. $(\mathsf{X}, \mathrm{d}, \mathfrak{m}, \mathcal{C}, p)$) is isomorphic to $(\mathrm{supp}(\mathfrak{m}), \mathrm{d}, \mathfrak{m}, \mathcal{C})$ (resp. $(\mathrm{supp}(\mathfrak{m}), \mathrm{d}, \mathfrak{m}, \mathcal{C}, p)$). Hence, we will assume that $\mathrm{supp}(\mathfrak{m}) = \mathsf{X}$, except when considering the associated *k-th cuts,* $\mathbb{X}^k$, of a metric measure space, which we now turn to define.

For a quasi-Radon measure $\mathfrak{m} \in \mathscr{M}^{qR}(X)$, let $\mathcal{S}_\mathfrak{m} \subset \mathsf{X}$ be the $\mathfrak{m}$-*singular set*, or singular set in short, namely the set of all points in X for which every open neighborhood has infinite measure

$$\mathcal{S}_\mathfrak{m} := \{x \in \mathsf{X} \,:\, \mathfrak{m}(U) = \infty \text{ for every open neighborhood } U \text{ of } x\}.$$

Recall that from Proposition 6.2.6 we have that $\mathcal{S}_\mathfrak{m}$ is a closed set with $\mathfrak{m}(\mathcal{S}_\mathfrak{m}) = 0$. Moreover $\mathcal{S}_\mathfrak{m} = \emptyset$ if and only if the measure $\mathfrak{m}$ is Radon. In particular, to any metric measure space

$(\mathsf{X},\mathsf{d},\mathfrak{m})$ we can associate a generalized metric measure space in a canonical way by considering $(\mathsf{X},\mathsf{d},\mathfrak{m},\mathcal{S}_\mathfrak{m})$. Now we fix once and for all a cut-off Lipschitz function $f_{\mathsf{cut}}\colon [0,\infty)\to[0,1]$ such that

$$\begin{cases} f_{\mathsf{cut}}(x)=1 & \text{for } 0\le x\le 1,\\ f_{\mathsf{cut}}(x)\in(0,1) & \text{for } 1<x<2,\\ f_{\mathsf{cut}}(x)=0 & \text{for } 2\le x\end{cases}$$

and for $k\in\mathbb{N}$ we define the k-th cut of $\mathbb{X}$ as the generalized metric measure space $\mathbb{X}^k := (\mathsf{X},\mathsf{d},\mathfrak{m}^k,\mathcal{C},p)$ where the measure is given by

$$\mathfrak{m}^k := f^k\,\mathfrak{m},\ \text{ where }\ f^k(x):=\begin{cases} f_{\mathsf{cut}}(\mathsf{d}(x,p)2^{-k})(1-f_{\mathsf{cut}}(\mathsf{d}(x,\mathcal{S}_\mathfrak{m})2^k)) & \text{if } \mathcal{S}_\mathfrak{m}\neq\emptyset,\\ f_{\mathsf{cut}}(\mathsf{d}(x,p)2^{-k}) & \text{if } \mathcal{S}_\mathfrak{m}=\emptyset.\end{cases}$$

Intuitively, the k-th cut $(\mathsf{X},\mathsf{d},\mathfrak{m}^k,\mathcal{C},p)$ *resembles* more $\mathbb{X}$ as k grows (see Remark 6.2.17 below).

Remark 6.2.14 (Regularity of the measure $\mathfrak{m}$)**.** We point out that since we are considering metric measure spaces $(\mathsf{X},\mathsf{d},\mathfrak{m})$ endowed with measures $\mathfrak{m}\in\mathscr{M}^{qR}(\mathsf{X})$, it holds that

- $\mathfrak{m}^k(\mathsf{X})<\infty$ for any $k\in\mathbb{N}$, and that
- there exists a $\tilde{k}\in\mathbb{N}$ such that for any $k\ge\tilde{k}$ it holds $\mathfrak{m}^k(\mathsf{X})>0$.

We say that $(\mathsf{X},\mathsf{d},\mathfrak{m})$ is a metric measure space with $\mathfrak{m}$-regularity parameter $\tilde{k}$ if the aforementioned condition is satisfied for $\tilde{k}\in\mathbb{N}$.

Finally, for a metric measure space $(\mathsf{X},\mathsf{d},\mathfrak{m})$, we define its k-th $\mathfrak{m}$-regular set, or *k-regular set* in short, as

$$\mathcal{R}^k := B_{2^{k+1}}(p)\setminus\mathcal{N}_{2^{-k}}(\mathcal{S}_\mathfrak{m})\ \text{ for any } k\in\mathbb{N}, \tag{6.2.4}$$

where $\mathcal{N}_{2^{-k}}(\mathcal{S}_\mathfrak{m}) := \cup_{x\in\mathcal{S}_\mathfrak{m}} B_{2^{-k}}(x)$. Observe that $\mathfrak{m}|_{\mathcal{R}^k}$ is a finite measure and that $\operatorname{supp}(\mathfrak{m}^k)=\mathcal{R}^k$.

Convergence of pointed metric measure spaces

First of all, we recall what is the intrinsic Kantorovich-Rubinstein-Wasserstein (iKRW, in short) distance between two metric measure spaces of finite mass. For this aim, we start fixing a cost function c, that is,

$$\mathsf{c}\in C([0,\infty)) \text{ is non-constant and concave with } \mathsf{c}(0)=0,\ c(d)>0 \text{ for } d>0 \text{ and } \lim_{d\to\infty}\mathsf{c}(d)<\infty \tag{6.2.5}$$

(e.g., $\mathsf{c}(d)=\tanh(d)$ or $\mathsf{c}(d)=d\wedge 1$). Then the iKRW-distance between two probability measures $\mathfrak{m},\mathfrak{n}\in\mathscr{P}(\mathsf{X})$ on a complete and separable metric space (X,d) is given by

$$W_\mathsf{c}(\mathfrak{m},\mathfrak{n}) := \inf_{\gamma\in\mathsf{Adm}(\mathfrak{m},\mathfrak{n})}\int_{\mathsf{X}\times\mathsf{X}} \mathsf{c}(\mathsf{d}(x,y))\,\mathrm{d}\gamma(x,y).$$

Observe that the distance W_c allows us to deal with all measures in $\mathscr{P}(\mathsf{X})$, rather than with the ones in the restricted set $\mathscr{P}_2(\mathsf{X})$. Moreover, regardless of the choice of c as in (6.2.5), $(\mathscr{P}(\mathsf{X}),W_\mathsf{c})$ is a complete and separable metric space and the convergence with respect to the weak topology of probability measures is equivalent to the convergence provided by the W_c-distance (see [Vil09, Chapter 6]); the last claim is a consequence of the fact that $c\circ\mathsf{d}$ defines a bounded complete distance on X, whose induced topology coincides with the one induced by d.

In the same spirit as Sturm's $\mathbb{D}$ distance, the iKRW-distance is used to define an intrinsic complete separable distance $\mathsf{d}^{fm}_{\mathsf{iKRW}}$ between pointed metric measure spaces with finite mass [GMS15]. Let $\mathbb{X}_1 := (\mathsf{X}_1, \mathsf{d}_1, \mathfrak{m}_1, \mathcal{C}_1, p_1)$, $\mathbb{X}_2 := (\mathsf{X}_2, \mathsf{d}_2, \mathfrak{m}_2, \mathcal{C}_2, p_2) \in \mathcal{M}^{qR}$ be generalized metric measure spaces with finite mass, then we set

$$\mathsf{d}^{fm}_{\mathsf{iKRW}}(\mathbb{X}_1, \mathbb{X}_2) :=$$
$$\left|\log\left(\frac{\mathfrak{m}_1(\mathsf{X}_1)}{\mathfrak{m}_2(\mathsf{X}_2)}\right)\right| + \inf\Big\{\mathsf{d}(i_1(p_1), i_2(p_2)) + \mathsf{d}_H(i_1(\mathcal{C}_1), i_2(\mathcal{C}_2)) + W_{\mathsf{c}}((i_1)_\sharp\bar{\mathfrak{m}}_1, (i_2)_\sharp\bar{\mathfrak{m}}_2)\Big\},$$

where the infimum is taken over all isometric embeddings $i_j\colon (\mathsf{X}_j, \mathsf{d}_j) \to (\mathsf{X}, \mathsf{d})$ into a complete separable metric space, $\bar{\mathfrak{m}}_j := \frac{\mathfrak{m}_j}{\mathfrak{m}_j(\mathsf{X}_j)}$ is a normalization of the measure $\mathfrak{m}_j$, for $j \in \{1,2\}$ and d_H is the Hausdorff distance between the two closed sets $i_1(\mathcal{C}_1)$ and $i_2(\mathcal{C}_2)$. In the following we set $\mathrm{d}_H(\emptyset, A) := +\infty$ if $A \neq \emptyset$ while $\mathrm{d}_H(\emptyset,\emptyset) := 0$.

Notice that the distance $\mathsf{d}^{fm}_{\mathsf{iKRW}}$ is defined only in the case in which the total mass of the two measures $\mathfrak{m}_1$ and $\mathfrak{m}_2$ is finite (and strictly positive). Therefore, in order to define a distance between two generalized metric measure spaces in $\mathcal{M}^{qR}$, we cover the spaces making use of the k-cuts and we sum up the contributions given by the $\mathsf{d}^{fm}_{\mathsf{iKRW}}$-distance between them.

In particular, we need the mass of the k-cuts to be strictly positive: for that purpose, given any $\bar{k} \in \mathbb{N}$, we introduce the following class of spaces

$$\mathcal{M}^{qR}_{\bar{k}} := \Big\{(\mathsf{X}, \mathsf{d}, \mathfrak{m}, \mathcal{C}, p) \in \mathcal{M}^{qR} : \mathfrak{m}^{\bar{k}}(\mathsf{X}) > 0\Big\}$$

Let us observe that for any finite family of generalized metric measure spaces in $\mathcal{M}^{qR}$, there exists a $\bar{k} \in \mathbb{N}$ such that the whole family is contained in $\mathcal{M}^{qR}_{\bar{k}}$ (in particular, it is sufficient to take $\bar{k} := \max \tilde{k}_i$, where $\tilde{k}_i$ is the regularity parameter of the i-th space). Nevertheless, for a sequence in $\mathcal{M}^{qR}$ it is necessary to assume the existence of a common regularity parameter in order to introduce a meaningful distance. Hence, in the following, we will restrict ourselves to the class $\mathcal{M}^{qR}_{\bar{k}}$ for some $\bar{k} \in \mathbb{N}$.

Definition 6.2.15 (Intrinsic pointed Kantorovich-Rubinstein-Wasserstein distance)**.** For any couple of metric measure spaces $\mathbb{X}_i := (\mathsf{X}_i, \mathsf{d}_i, \mathfrak{m}_i, \mathcal{C}_i, p_i) \in \mathcal{M}^{qR}_{\bar{k}}$, $i \in \{1,2\}$, $\bar{k} \in \mathbb{N}$, we define the pointed iKRW-distance as

$$\mathsf{d}_{\mathsf{iKRW}}(\mathbb{X}_1, \mathbb{X}_2) := \sum_{k \geq \bar{k}} \frac{1}{2^k} \min\Big\{1, \mathsf{d}^{fm}_{\mathsf{iKRW}}(\mathbb{X}_1^k, \mathbb{X}_2^k)\Big\},$$

where $\mathbb{X}_i^k = (\mathsf{X}_i, \mathsf{d}_i, \mathfrak{m}_i^k, \mathcal{C}_i, p_i)$ is the k-th cut of $\mathbb{X}_i$, for $i \in \{1,2\}$.

Notice that the distance $\mathsf{d}_{\mathsf{iKRW}}$ depends on the common regularity parameter $\bar{k}$, but we drop this dependence, since it will be clear from the context.

Definition 6.2.16 (Converging sequence of pointed generalized metric measure spaces)**.** We say that a sequence of pointed generalized metric measure spaces $\{\mathbb{X}_n\}_{n\in\mathbb{N}} \subset \mathcal{M}^{qR}_{\bar{k}}$, for some $\bar{k} \in \mathbb{N}$, is iKRW-converging to $\mathbb{X}_\infty \in \mathcal{M}^{qR}_{\bar{k}}$ if

$$\lim_{n\to\infty} \mathsf{d}_{\mathsf{iKRW}}(\mathbb{X}_n, \mathbb{X}_\infty) = 0.$$

Observe that the fact that $\mathsf{d}^{fm}_{\mathsf{iKRW}}$ is a distance function guarantees that also $\mathsf{d}_{\mathsf{iKRW}}\colon \mathcal{M}^{qR}_{\bar{k}} \to \mathbb{R}^+ \cup \{0\}$ defines a finite distance function.

Remark 6.2.17. Directly from the definitions of $\mathsf{d}_{\mathsf{iKRW}}$ and $\mathsf{d}^{fm}_{\mathsf{iKRW}}$, it follows that

$$\lim_{n\to\infty} \mathsf{d}_{\mathsf{iKRW}}(\mathbb{X}_n,\mathbb{X}_\infty)=0 \quad \text{if and only if} \quad \lim_{n\to\infty} \mathsf{d}^{fm}_{\mathsf{iKRW}}(\mathbb{X}^k_n,\mathbb{X}^k_\infty)=0 \text{ for every } k\geq \bar{k}, \tag{6.2.6}$$

where $\bar{k}$ is the common regularity parameter associated to the converging sequence.

In the next result we prove an *extrinsic approach to convergence.* From now on we assume that the generalized metric measure space $(\mathsf{X},\mathsf{d},\mathfrak{m},\mathcal{C})$ is the canonical one associated to $(\mathsf{X},\mathsf{d},\mathfrak{m})$, namely $\mathcal{C}=\mathcal{S}_{\mathfrak{m}}$ is the $\mathfrak{m}$-singular set.

Proposition 6.2.18. *Let $\{\mathbb{X}_n\}_{n\in\mathbb{N}\cup\{\infty\}}\subset \mathcal{M}^{qR}_{\bar{k}}$, $\mathbb{X}_n=(\mathsf{X}_n,\mathsf{d}_n,\mathfrak{m}_n,\mathcal{S}_{\mathfrak{m}_n},p_n)$ be a sequence of pointed generalized metric measure spaces, $\bar{k}\in\mathbb{N}$. Then the following statements are equivalent:*

i) $\lim_{n\to\infty}\mathsf{d}_{\mathsf{iKRW}}(\mathbb{X}_n,\mathbb{X}_\infty)=0$,

ii) there exist a complete and separable metric space (Z,d_Z) and a sequence of isometric embeddings $\{i_n\colon \mathsf{X}_n\to Z\}_{n\in\mathbb{N}}$, for which

$$\begin{aligned}\left|\log\left(\frac{\mathfrak{m}^k_n(\mathsf{X}_n)}{\mathfrak{m}^k_\infty(\mathsf{X}_\infty)}\right)\right| + \mathsf{d}_Z(i_n(p_n),i_\infty(p_\infty)) + (\mathsf{d}_Z)_H(i_n(\mathcal{S}_{\mathfrak{m}_n}),i_\infty(\mathcal{S}_{\mathfrak{m}_\infty}))\\ + W_c((i_n)_\sharp\bar{\mathfrak{m}}^k_n,(i_\infty)_\sharp\bar{\mathfrak{m}}^k_\infty)\overset{n\to\infty}{\to} 0,\end{aligned} \tag{6.2.7}$$

for any $k\geq\bar{k}$.

We refer to $\big((Z,\mathsf{d}_Z),\{i_n\}_{n\in\mathbb{N}}\big)$ *as an* effective realization for the convergence *of $\{\mathbb{X}_n\}_{n\in\mathbb{N}}$ to $\mathbb{X}_\infty$.*

Proof. *i)* $\Rightarrow$ *ii)* We start assuming that $\mathsf{d}_{\mathsf{iKRW}}(\mathbb{X}_n,\mathbb{X}_\infty)\to 0$. In this case, the metric space (Z,d_Z) as well as the isometric embeddings $\{i_n\}_{n\in\mathbb{N}}$ are constructed relying on a twofold gluing argument. Roughly speaking, the strategy is the following: for any fixed $k\geq\bar{k}$ we use a "gluing" procedure to construct a common space Z^k equipped with the metric that makes all the k-th cuts $\{\mathbb{X}^k_n\}_{n\in\mathbb{N}\cup\{\infty\}}$ be isometrically embedded. Next, we show that a certain compatibility condition holds between the spaces $\{Z^k\}_{k\in\mathbb{N}}$: this allows us to "glue" one more time, and obtain the desired common complete and separable metric space (Z,d_Z) in which we will embed the sequence $\{\mathbb{X}_n\}_{n\in\mathbb{N}\cup\{\infty\}}$. In the following we present the detailed argument, which is a suitable adaptation of [GMS15, Theorem 3.15].
For every fixed $k\geq\bar{k}$, (6.2.6) ensures the existence of a sequence of complete and separable metric spaces $\{(Z^k_n,\mathsf{d}_{Z^k_n})\}_{n\in\mathbb{N}}$, and of two sequences of isometric embeddings $\{i^k_n\colon\mathcal{R}^k_n\to Z^k_n\}_{n\in\mathbb{N}}$ and $\{i^k_{\infty,n}\colon\mathcal{R}^k_\infty\to Z^k_n\}_{n\in\mathbb{N}}$, where $\mathcal{R}^k_n=\operatorname{supp}(\mathfrak{m}^k_n)$ and $\mathcal{R}^k_\infty=\operatorname{supp}(\mathfrak{m}^k_\infty)$, with the property that

$$\begin{aligned}\left|\log\left(\frac{\mathfrak{m}^k_n(\mathsf{X}_n)}{\mathfrak{m}^k_\infty(\mathsf{X}_\infty)}\right)\right| + \mathsf{d}_{Z^k_n}\left(i^k_n(p_n),i^k_{\infty,n}(p_\infty)\right) + \left(\mathsf{d}_{Z^k_n}\right)_H\left(i^k_n(\mathcal{S}_{\mathfrak{m}_n}),i^k_{\infty,n}(\mathcal{S}_{\mathfrak{m}_\infty})\right)\\ + W_c\left((i^k_n)_\sharp\bar{\mathfrak{m}}^k_n,(i^k_{\infty,n})_\sharp\bar{\mathfrak{m}}^k_\infty\right)\overset{n\to\infty}{\to} 0.\end{aligned} \tag{6.2.8}$$

We then define the set $Z^k=\sqcup_{n\in\mathbb{N}}Z^k_n$ and the function $\mathsf{d}_{Z^k}\colon Z^k\times Z^k\to[0,\infty)$ by setting

$$\mathsf{d}_{Z^k}(x,y):=\begin{cases}\mathsf{d}_{Z^k_n}(x,y) & \text{if } (x,y)\in Z^k_n\times Z^k_n,\ \exists n\in\mathbb{N},\\ \inf\limits_{w\in X^k_\infty}\mathsf{d}_{Z^k_n}(x,i^k_{\infty,n}(w))+\mathsf{d}_{Z^k_m}(i^k_{\infty,m}(w),y) & \text{if } (x,y)\in Z^k_n\times Z^k_m,\ \exists n\neq m.\end{cases}$$

Thus, we can define an equivalence relation $\sim$ on Z^k saying that $v\sim w$ if and only if $\mathsf{d}_{Z^k}(v,w)=0$, for $v,w\in Z^k$: we take the quotient of Z^k by this relation and then its completion. We denote by

$\tilde{Z}^k$ the resulting space. Note that d_{Z^k} canonically induces a distance function on $\tilde{Z}^k \times \tilde{Z}^k$, which we still denote by d_{Z^k}, and that the operations made so far preserve the separability of the space. Thus, the pair $(\tilde{Z}^k, \mathsf{d}_{Z^k})$ is a complete and separable metric space. By construction, for $n \in \mathbb{N}$, the composition

$$\mathsf{i}_n^k := p^k \circ j_n^k \circ i_n^k : \mathcal{R}_n^k \to \tilde{Z}^k \tag{6.2.10}$$

is an isometric embedding, where $j_n^k \colon Z_n^k \to Z^k$ is the canonical inclusion and $p^k \colon Z^k \to \tilde{Z}^k$ the projection map. Moreover, the fact that for every $m, n \in \mathbb{N}$ the set $j_n^k(i_{\infty,n}^k(\mathcal{R}_\infty^k))$ is identified under the equivalence relation with $j_m^k(i_{\infty,m}^k(\mathcal{R}_\infty^k))$ implies that the maps

$$p^k \circ j_n^k \circ i_{\infty,n}^k \colon \mathcal{R}_\infty^k \to \tilde{Z}^k \qquad \text{and} \qquad p^k \circ j_m^k \circ i_{\infty,m}^k \colon \mathcal{R}_\infty^k \to \tilde{Z}^k$$

coincide for every $n, m \in \mathbb{N}$. In this manner, we see that also $p^k \circ j_m^k \circ i_{\infty,m}^k : \mathcal{R}_\infty^k \to \tilde{Z}^k$ is an isometric embedding, which is independent of m. Let us denote it by i_∞^k. The convergence in (6.2.8) yields

$$\left|\log\left(\frac{\mathfrak{m}_n^k(\mathsf{X}_n)}{\mathfrak{m}_\infty^k(\mathsf{X}_\infty)}\right)\right| + \mathsf{d}_{Z^k}\left(\mathsf{i}_n^k(p_n), \mathsf{i}_\infty^k(p_\infty)\right) + (\mathsf{d}_{Z^k})_H\left(\mathsf{i}_n^k(\mathcal{S}_{\mathfrak{m}_n}), \mathsf{i}_\infty^k(\mathcal{S}_{\mathfrak{m}_\infty})\right) \overset{n\to\infty}{\to} 0. \tag{6.2.11}$$

To finish the first step of the argument we note that the pushforward of a coupling under the map $(p^k \circ j_n^k) \times (p^k \circ j_n^k) : Z_n^k \times Z_n^k \to \tilde{Z}^k \times \tilde{Z}^k$, is again a coupling between the pushforward of the original marginal measures, namely

$$\text{if } \pi \in \mathsf{Adm}\big((i_n^k)_\sharp \bar{\mathfrak{m}}_n^k, (i_{\infty,n}^k)_\sharp \bar{\mathfrak{m}}_\infty^k\big), \text{then } \tilde{\pi} := ((p^k \circ j_n^k)^2)_\sharp \pi \in \mathsf{Adm}\big((\mathsf{i}_n^k)_\sharp \bar{\mathfrak{m}}_n^k, (\mathsf{i}_\infty^k)_\sharp \bar{\mathfrak{m}}_\infty^k\big).$$

Therefore, if we choose $\pi \in \mathsf{Opt}((i_n^k)_\sharp \bar{\mathfrak{m}}_n^k, (i_{\infty,n}^k)_\sharp \bar{\mathfrak{m}}_\infty^k)$, we get

$$W_c^{\tilde{Z}^k}\Big((\mathsf{i}_n^k)_\sharp \bar{\mathfrak{m}}_n^k, (\mathsf{i}_\infty^k)_\sharp \bar{\mathfrak{m}}_\infty^k\Big) \le W_c^{Z_n^k}\Big((i_n^k)_\sharp \bar{\mathfrak{m}}_n^k, (i_{\infty,n}^k)_\sharp \bar{\mathfrak{m}}_\infty^k\Big),$$

since $p^k \circ j_n^k : Z_n^k \to \tilde{Z}^k$ is an isometry. Jointly with the last term in (6.2.8), this inequality implies the convergence $(\mathsf{i}_n^k)_\sharp \bar{\mathfrak{m}}_n^k \to (\mathsf{i}_\infty^k)_\sharp \bar{\mathfrak{m}}_\infty^k$ in $(\mathscr{P}(\tilde{Z}^k), W_c^{\tilde{Z}^k})$. We have hereby shown the existence of a complete and separable metric space $(\tilde{Z}^k, \mathsf{d}_{Z^k})$ and a sequence of isometric embeddings $\{i_n^k \colon \mathcal{R}_n^k \to \tilde{Z}^k\}_{n\in\mathbb{N}\cup\{\infty\}}$ which provide a realization of the convergence $\mathbb{X}_n^k \to \mathbb{X}_\infty^k$ for any $k \ge \bar{k}$.

For the second part of the argument, first of all we prove that for any $\bar{k} \le j < k-1$, the embeddings $i_n^k \colon \mathcal{R}_n^k \to \tilde{Z}^k$ serve as an effective realization for the convergence $\mathbb{X}_n^j \to \mathbb{X}_\infty^j$. To that purpose, let us consider the function

$$\begin{aligned} g_n^{j,k} \colon \tilde{Z}^k &\to [0,1] \\ y &\mapsto \begin{cases} f_{\mathsf{cut}}\left(\mathsf{d}_{Z^k}\left(y, i_n^k(p_n)\right) 2^{-(j+1)}\right)\left(1 - f_{\mathsf{cut}}\left(\mathsf{d}_{Z^k}\left(y, i_n^k\left(\mathcal{S}_{\mathfrak{m}_n}\right)\right) 2^{j+1}\right)\right) & \text{if } \mathcal{S}_{\mathfrak{m}_n} \neq \emptyset, \\ f_{\mathsf{cut}}\left(\mathsf{d}_{Z^k}\left(y, i_n^k(p_n)\right) 2^{-(j+1)}\right) & \text{if } \mathcal{S}_{\mathfrak{m}_n} = \emptyset. \end{cases} \end{aligned}$$

The Lipschitz continuity of the cut-off function f_{cut}, together with the convergence of $\{i_n^k(p_n)\}_{n\in\mathbb{N}}$ to $i_\infty^k(p_\infty)$ by (6.2.8), ensures that the sequence $f_{\mathsf{cut}}\left(\mathsf{d}_{Z^k}\left(y, i_n^k(p_n)\right) 2^{-(j+1)}\right)$ is uniformly converging to $f_{\mathsf{cut}}\left(\mathsf{d}_{Z^k}(y, i_\infty^k(p_\infty)) 2^{-(j+1)}\right)$ as $n \to \infty$. In the same way, the triangular inequality ensures that

$$\left|\mathsf{d}_{Z^k}\left(y, i_n^k\left(\mathcal{S}_{\mathfrak{m}_n}\right)\right) - \mathsf{d}_{Z^k}\left(y, i_\infty^k\left(\mathcal{S}_{\mathfrak{m}_\infty}\right)\right)\right| \le (\mathsf{d}_{Z^k})_H\left(i_n^k\left(\mathcal{S}_{\mathfrak{m}_n}\right), i_\infty^k\left(\mathcal{S}_{\mathfrak{m}_\infty}\right)\right).$$

and the convergence (6.2.8) guarantees that the sequence $f_{\mathsf{cut}}\left(\mathsf{d}_{Z^k}\left(y, i_n^k\left(\mathcal{S}_{\mathfrak{m}_n}\right)\right)2^{j+1}\right)$ uniformly converges to $f_{\mathsf{cut}}\left(\mathsf{d}_{Z^k}\left(y, i_\infty^k\left(\mathcal{S}_{\mathfrak{m}_\infty}\right)\right)2^{j+1}\right)$ as $n\to\infty$. Hence, for every $y\in\tilde{Z}^k$ the sequence $\{g_n^{j,k}(y)\}_{n\in\mathbb{N}}$ uniformly converges as $n\to\infty$ to

$$g^{j,k}(y) := \begin{cases} f_{\mathsf{cut}}\left(\mathsf{d}_{Z^k}(y, i_\infty^k(p_\infty))\,2^{-(j+1)}\right)\left(1 - f_{\mathsf{cut}}\left(\mathsf{d}_{Z^k}\left(y, i_\infty^k\left(\mathcal{S}_{\mathfrak{m}_\infty}\right)\right)2^{j+1}\right)\right) & \text{if } \mathcal{S}_{\mathfrak{m}_\infty}\neq\emptyset,\\ f_{\mathsf{cut}}\left(\mathsf{d}_{Z^k}(y, i_\infty^k(p_\infty))\,2^{-(j+1)}\right) & \text{if } \mathcal{S}_{\mathfrak{m}_\infty}=\emptyset. \end{cases}$$

This in particular implies the weak convergence of the sequence of measures

$$(i_n^k)_\sharp\bar{\mathfrak{m}}_n^j = (g_n^{j,k}\circ i_n^k)_\sharp\bar{\mathfrak{m}}_n^k \overset{n\to\infty}{\rightharpoonup} (i_\infty^k)_\sharp\bar{\mathfrak{m}}^j = (g^{j,k}\circ i_\infty^k)_\sharp\bar{\mathfrak{m}}^k$$

(note that we ask for $\bar{k}\le j<k-1$). The former convergence, together with (6.2.11) and the fact that $\left|\log\left(\frac{\mathfrak{m}_n^j(X_n)}{\mathfrak{m}_\infty^j(X_\infty)}\right)\right|\to 0$, shows that the embeddings $i_n^k\colon\mathcal{R}_n^k\to\tilde{Z}^k$ realize the convergence of the sequence of j-cuts, for every $\bar{k}\le j<k-1$.

At this point an analogous "gluing" argument can be applied to the sequence $(\tilde{Z}^k,\{i_n^k\}_{n\in\mathbb{N}\cup\{\infty\}})$ when $k\ge\bar{k}$: we can in fact construct a common space $Z:=\sqcup_{k\ge\bar{k}}\tilde{Z}^k$, which, endowed with the distance d_Z defined analogously as in (6.2.9), is a complete and separable metric space, and a sequence of embeddings $i_n\colon\mathsf{X}_n\to Z$ for $n\in\mathbb{N}\cup\{\infty\}$ as in (6.2.10). The pair $\left((Z,\mathsf{d}_Z),\{i_n\}_{n\in\mathbb{N}\cup\{\infty\}}\right)$ is the desired effective realization of the iKRW-convergence $\mathbb{X}_n\to\mathbb{X}_\infty$.

ii) $\Rightarrow$ *i)* Note that the existence of an effective realization of the convergence $\mathbb{X}_n\overset{n\to\infty}{\to}\mathbb{X}_\infty$ implies that $\mathsf{d}_{\mathsf{iKRW}}^{fm}(\mathbb{X}_n^k,\mathbb{X}_\infty^k)\overset{n\to\infty}{\to}0$, for all $k\ge\bar{k}$. Then, we can conclude by using (6.2.6). □

In some situations, it would be practical to have at our disposal a metric which is not explicitly dependent on the behavior of the $\mathfrak{m}$-singular sets. For instance, we could gain flexibility by not asking for a control on the Hausdorff distance between $\mathfrak{m}$-singular sets in the definition of the $\mathsf{d}_{\mathsf{iKRW}}$-distance. However, as we just saw, this term is necessary to provide an extrinsic realization of the distance given as an intrinsic one. Therefore, the following definition turns out to be useful.

Definition 6.2.19 (Extrinsic convergence)**.** We say that a sequence of pointed generalized metric measure spaces $\{\mathbb{X}_n\}_{n\in\mathbb{N}}\subset\mathcal{M}_{\bar{k}}^{qR}$ *converges extrinsically* to $\mathbb{X}_\infty\in\mathcal{M}_{\bar{k}}^{qR}$, $\bar{k}\in\mathbb{N}$, if there exist a complete and separable metric space (Z,d_Z) and a sequence of isometric embeddings $\{i_n\colon\mathsf{X}_n\to Z\}_{n\in\mathbb{N}}$ for which

$$\left|\log\left(\frac{\mathfrak{m}_n^k(\mathsf{X}_n)}{\mathfrak{m}_\infty^k(\mathsf{X}_\infty)}\right)\right| + \mathsf{d}_Z(i_n(p_n), i_\infty(p_\infty)) + W_c((i_n)_\sharp\bar{\mathfrak{m}}_n^k,(i_\infty)_\sharp\bar{\mathfrak{m}}_\infty^k)\overset{n\to\infty}{\to}0, \tag{6.2.12}$$

for any $k\ge\bar{k}$.

Note that we dropped the assumption on the Hausdorff distance between singular sets at the cost of presenting ourselves a space providing the extrinsic realization. Furthermore, by Proposition 6.2.18 we know that a iKRW-converging sequence converges also in the extrinsic manner.

We finish the Section with some remarks.

Remark 6.2.20. We observe that the convergence with respect to the Wasserstein distance have the following characterization (cf. [AGS08, Section 7.1]):

$$\mu_n\overset{W_2}{\longrightarrow}\mu \quad\iff\quad \mu_n\rightharpoonup\mu \text{ and } \int\mathsf{d}(x_0,x)^2\,\mathrm{d}\mu_n\to\int\mathsf{d}(x_0,x)^2\,\mathrm{d}\mu \quad\forall x_0\in\mathsf{X}.$$

This description shows in particular that for a sequence of probability measures with uniformly bounded support, the W_2-convergence is equivalent to the weak one and, consequently, to the W_c-convergence. Hence, in this case, (6.2.7) in Proposition 6.2.18 and (6.2.12) in Definition 6.2.19 remain valid when we replace the W_c-distance with the W_2-one.

Remark 6.2.21. *Connection to Gigli-Mondino-Savaré's* $p\mathbb{G}_W$ *distance.* In [GMS15] the authors define a distance between $\mathbb{X}_1$ and $\mathbb{X}_2$ metric measure spaces endowed with Radon measures giving finite mass to bounded sets. This is what inspired us to propose the definition of $\mathsf{d}_{\mathsf{iKRW}}$: in fact, in this case the $\mathfrak{m}$-singular set of a metric measure space in such a family is the empty set and thus our definition coincides with theirs.

Remark 6.2.22. We recall that in [GMS15, Theorem 3.17] the authors prove that the class of all metric measure spaces equipped with Radon measures is complete with respect to the $p\mathbb{G}_W$ distance. It is worth to underline that in our context we cannot hope for a similar completeness result. The main reason is that the set of all closed sets with empty interior is not closed for the Hausdorff distance. Hence, intuitively, we cannot prevent a sequence of quasi-Radon measures from converging to a measure which is not quasi-Radon.

6.3 CD-condition for negative generalized dimension

6.3.1 Basic definitions and properties

We introduce the *Rényi entropy* $S_{N,\mathfrak{m}}$ for $N < 0$ with respect to the reference measure $\mathfrak{m}$ as the functional defined on $\mathscr{P}(\mathsf{X})$ by

$$S_{N,\mathfrak{m}}(\mu) := \begin{cases} \int_{\mathsf{X}} \rho(x)^{\frac{N-1}{N}} \, \mathrm{d}\mathfrak{m}(x) & \text{if } \mu \ll \mathfrak{m},\ \mu = \rho\mathfrak{m}, \\ +\infty & \text{otherwise,} \end{cases}$$

where $\rho = \mathrm{d}\mu/\mathrm{d}\mathfrak{m}$ is the Radon-Nikodym derivative of μ with respect to $\mathfrak{m}$, whose existence is guaranteed by Theorem 6.2.10. In the following we will denote by $\mathscr{P}^{ac}(\mathsf{X}, \mathfrak{m})$ the set of probability measures in $\mathscr{P}_2(\mathsf{X})$ that are absolutely continuous with respect to the reference measure $\mathfrak{m}$.

Remark 6.3.1. We point out that for $N \geq 1$ the "standard" Rényi entropy is defined as $S_{N,\mathfrak{m}}(\rho\mathfrak{m}) := -\int_{\mathsf{X}} \rho(x)^{\frac{N-1}{N}} \, \mathrm{d}\mathfrak{m}(x)$. For $N < 0$, the minus sign is omitted to impose the convexity of the function $h(s) = s^{(N-1)/N}$. Note that, for $N \geq 1$, it suffices to define the "standard" Rényi entropy on Polish spaces equipped with Radon reference measures. In this case, under a volume growth condition on the reference measure $\mathfrak{m}$, the functional $S_{N,\mathfrak{m}}(\cdot)$ is lower semicontinuous with respect to the weak topology and, in particular, it is lower semicontinuous with respect to the 2-Wasserstein convergence in $\mathscr{P}_2(\mathsf{X})$. Unfortunately, the same property is not necessarily true for negative values of $N < 0$ and quasi-Radon reference measures $\mathfrak{m}$. However, we prove in Proposition 6.4.8 that $S_{N,\mathfrak{m}}(\cdot)$ is lower semicontinuous with respect to the weak convergence, on the subspace

$$\mathscr{P}^{\mathcal{S}_\mathfrak{m}}(\mathsf{X}) := \{\mu \in \mathscr{P}_2(\mathsf{X}) \, : \, \mu(\mathcal{S}_\mathfrak{m}) = 0 \text{ where } \mathcal{S}_\mathfrak{m} \text{ is the } \mathfrak{m}\text{-singular set}\}.$$

In fact, we show a more general result stating that the Rényi entropy functional $S_{N,\mathfrak{n}}(\nu)$ is a lower semicontinuous function of $(\mathfrak{n}, \nu) \in \mathscr{M}_S(\mathsf{X}) \times \mathscr{P}^{\mathcal{S}}(\mathsf{X})$, where the convergence of the first coordinate is intended to be the weak convergence of quasi-Radon measures.

In order to give the definition of curvature-dimension bounds, we need also to introduce the following distortion coefficients for $K \in \mathbb{R}$ and $N < 0$:

$$\sigma_{K,N}^{(t)}(\theta) := \begin{cases} \infty, & \text{if } K\theta^2 \le N\pi^2, \\ \dfrac{\sin(t\theta\sqrt{K/N})}{\sin(\theta\sqrt{K/N})} & \text{if } N\pi^2 < K\theta^2 < 0, \\ t & \text{if } K\theta^2 = 0, \\ \dfrac{\sinh(t\theta\sqrt{-K/N})}{\sinh(\theta\sqrt{-K/N})} & \text{if } K\theta^2 > 0 \end{cases}$$

and

$$\tau_{K,N}^{(t)}(\theta) := t^{1/N}\sigma_{K,N-1}^{(t)}(\theta)^{(N-1)/N}, \tag{6.3.1}$$

for every $\theta \in [0,\infty)$ and $t \in [0,1]$.

Definition 6.3.2. For any couple of measures $\mu_0, \mu_1 \in \mathscr{P}^{ac}(\mathsf{X},\mathfrak{m})$, $\mu_i = \rho_i\mathfrak{m}$, we denote by $\pi \in \mathscr{P}(\mathsf{X}\times\mathsf{X})$ a coupling between them, and by $T_{K,N}^{t}(\pi|\mathfrak{m})$ the functional defined by

$$T_{K,N}^{(t)}(\pi|\mathfrak{m}) := \int_{\mathsf{X}\times\mathsf{X}} \left[\tau_{K,N}^{(1-t)}(\mathsf{d}(x,y))\rho_0(x)^{-\frac{1}{N}} + \tau_{K,N}^{(t)}(\mathsf{d}(x,y))\rho_1(y)^{-\frac{1}{N}}\right] \mathrm{d}\pi(x,y).$$

We are ready to introduce the definition of metric measure spaces satisfying a curvature-dimension condition for negative values of the dimensional parameter.

Definition 6.3.3 (CD-condition)**.** For fixed $K \in \mathbb{R}, N \in (-\infty,0)$, we say that a metric measure space $(\mathsf{X},\mathsf{d},\mathfrak{m})$ satisfies the $\mathsf{CD}(K,N)$-*condition* if, for each pair $\mu_0 = \rho_0\mathfrak{m}$, $\mu_1 = \rho_1\mathfrak{m} \in \mathscr{P}^{ac}(\mathsf{X},\mathfrak{m})$, there exists an optimal coupling $\pi \in \mathsf{Opt}(\mu_0,\mu_1)$ and a W_2-geodesic $\{\mu_t\}_{t\in[0,1]} \subset \mathscr{P}_2(\mathsf{X})$ such that

$$S_{N',\mathfrak{m}}(\mu_t) \le T_{K,N'}^{(t)}(\pi|\mathfrak{m}) \tag{6.3.2}$$

holds, for every $t \in [0,1]$, and every $N' \in [N,0)$, provided that $S_{N',\mathfrak{m}}(\mu_0)$, $S_{N',\mathfrak{m}}(\mu_1) < \infty$.

We mention that a notion of $\mathsf{CD}(K,N)$-condition with $N = 0$ was also introduced by Ohta in [Oht18].

Remark 6.3.4. Note that the CD-inequality becomes trivial when $K < 0$ and

$$\pi\Big(\{(x,y) \in \mathsf{X}\times\mathsf{X} : \mathrm{d}(x,y) \ge \pi\sqrt{(N'-1)/K}\}\Big) > 0.$$

Furthermore, observe that, if $K \ge 0$ or if $\mathrm{diam}(\mathsf{X}) < \sqrt{\pi(N-1)/K}$ when $K < 0$, the coefficients $\tau_{K,N}^{(t)}(\cdot)$ are bounded. Now notice that Jensen's inequality shows that $S_{N,\mathfrak{m}}(\mu_0)$, $S_{N,\mathfrak{m}}(\mu_1)$ are finite, if the entropies $S_{N',\mathfrak{m}}(\mu_0)$, $S_{N',\mathfrak{m}}(\mu_1)$ are finite for some $N' \in [N,0)$. Therefore, observe that in this case the $\mathsf{CD}(K,N)$-condition guarantees that the Wasserstein geodesics along which the inequality (6.3.2) holds are absolutely continuous with respect to $\mathfrak{m}$.

Remark 6.3.5. Take notice that Definition 6.3.3 restricts the domain in which it is required to verify inequality (6.3.2) to $\mathcal{D}(S_{N',\mathfrak{m}}) := \{\mu : S_{N',\mathfrak{m}}(\mu) < \infty\}$. This is a common assumption made when dealing with convexity-like properties of functions which take values in the extended real numbers. Nevertheless, we wish to comment in more detail with regard to this.

It turns out that it is not necessary to make this restriction in the classical theory of curvature-dimension bounds for $N \ge 1$ since the Rényi Entropy is bounded for absolutely continuous measures in $\mathscr{P}_2(\mathsf{X})$, as a consequence of the fact that $\mathsf{CD}(K,N)$ spaces possess reference measures with a controlled volume growth. A proof of the finiteness of some entropy functionals, in particular of the Rényi entropy, under volume growth assumptions can be found for example in

[LV09, Proposition E.17]. However, this is not necessarily the case when dealing with negative parameters N and, therefore, we require the finiteness of the entropies at the marginal measures.

We remark that for terminal marginals with bounded supports, Definition 6.3.3 coincides with the one introduced by Ohta in [Oht16]. Indeed, if the supports of μ_0 and μ_1 are bounded in (X,d) the coefficients $\tau_{K,N}^{(1-t)}(\cdot)$ are bounded below away from 0 on the support of any coupling $\pi \in \mathsf{Opt}(\mu_0,\mu_1)$, for fixed $0<t<1$. Thus, if for some $N' \in [N,0)$ one of the terminal measures has unbounded entropy $S_{N',\mathfrak{m}}$, then $T_{K,N'}^{(t)}(\pi|\mathfrak{m}) = \infty$, for any $t \in (0,1)$, and inequality (6.3.2) is always satisfied.

Lastly, we emphasize as well that the assumption on the entropy in Definition 6.3.3 is consistent with the standard definitions of curvature-dimension conditions for positive values of N, in which the requirement is not explicitly made.

We underline that, as in the case $N \geq 1$, the definition of curvature-dimension condition is invariant under standard transformations of metric measure structures. Precisely, the CD-condition is stable under isomorphisms, scalings, and restrictions to convex subsets of metric measure spaces (this can be proved using the same techniques as in [Stu06a, Propositions 4.12, 4.13 and 4.15]) and in [Stu06b, Proposition 1.4]. We also point out, that the "hierarchy property" of $\mathsf{CD}(K,N)$ spaces, with $N<0$, remains valid. Specifically,

Proposition 6.3.6. *If* $(\mathsf{X},\mathsf{d},\mathfrak{m})$ *satisfies the curvature-dimension condition* $\mathsf{CD}(K,N)$ *for some* $K \in \mathbb{R}, N<0$*, then it also satisfies the curvature-dimension condition* $\mathsf{CD}(K',N')$ *for any* $K' \leq K$ *and* $N' \in [N,0)$.

Proof. The monotonicity in N follows directly from Definition 6.3.3, while the monotonicity in K follows from the fact that the coefficient $\sigma_\kappa^{(t)}(\theta)$ is non-decreasing in κ once t and θ are fixed (see [BS10, Remark 2.2]). □

Let us conclude by recalling that the $\mathsf{CD}(K,N)$-condition is weaker than the $\mathsf{CD}(K,\infty)$-condition (see [Oht16]) and, therefore, it follows that $\mathsf{CD}(K,\infty)$ spaces are $\mathsf{CD}(K,N)$ space for every $N<0$.

6.3.2 Examples

In this section we present some examples of negative dimensional CD spaces, referring to [Mil17b, Mil17a, KM18] for other model spaces satisfying the $\mathsf{CD}(K,N)$-condition with $N<0$. Moreover, we show that singular points of the reference measure in negative dimensional CD spaces can appear as inner points of geodesics. This fact motivates us to present the definitions of *approximate* CD*-condition* and *ω-uniform convexity*, objects of Section 6.3.3, which will enable us to deal with this kind of behavior in the proof of our Stability Theorem.

A fundamental notion in the presentation of the examples is the one of (K,N)-convexity of a function on a metric space, for a negative value of N. This definition is the natural counterpart of the one with positive N, and it was introduced by Ohta in [Oht16].

Definition 6.3.7 ((K,N)-convexity)**.** In a metric space (X,d), for every fixed $K \in \mathbb{R}$ and $N \in (-\infty,0)$, a function $f:\mathsf{X} \to \bar{\mathbb{R}}$ is said to be (K,N)-convex if for every $x_0,x_1 \in \{f<+\infty\}$, with $d := \mathsf{d}(x_0,x_1) < \pi\sqrt{N/K}$ when $K<0$, there exists a constant speed geodesic γ connecting x_0 and x_1, such that

$$f_N(\gamma_t) \leq \sigma_{K,N}^{(1-t)}(d) f_N(x_0) + \sigma_{K,N}^{(t)}(d) f_N(x_1) \qquad \forall t \in [0,1], \tag{6.3.3}$$

where $f_N(x) = e^{-f(x)/N}$.

The following result ([Oht16, Corollary 4.12]) is used to produce examples of $\mathsf{CD}(K,N)$ spaces with negative values of the generalized dimension.

Proposition 6.3.8. *Let M be a n-dimensional complete Riemannian manifold with Riemannian distance d_g and Riemannian volume vol_g. Let us then consider a weighted volume measure $\mathfrak{m} = e^{-\psi} vol_g$, for some function $\psi\colon M \to \mathbb{R}$, and let numbers $K_1, K_2 \in \mathbb{R}$, $N_2 \geq n$ and $N_1 < -N_2$ be given.*

Then if $(M, \mathrm{d}_g, \mathfrak{m})$ satisfies the $\mathsf{CD}(K_2, N_2)$-condition, the weighted space $(M, \mathrm{d}_g, e^{-\Psi}\mathfrak{m})$ satisfies the $\mathsf{CD}(K_1 + K_2, N_1 + N_2)$-condition provided that $\Psi \in C^2(M)$ is (K_1, N_1)-convex.

Example 6.3.9 (1-dimensional models)**.** In the following we will denote by $|\cdot|$ the Euclidean distance and by $\mathscr{L}^1$ the 1-dimensional Lebesgue measure.
(i) For any pair of real numbers $K > 0, N < -1$ the weighted space $(\mathbb{R}, |\cdot|, V\mathscr{L}^1)$ with

$$V(x) = \cosh\left(x\sqrt{-\frac{K}{N}}\right)^N$$

satisfies the curvature-dimension condition $\mathsf{CD}(K, N+1)$ with no singular set, i.e. $\mathcal{S}_{V\mathscr{L}^1} = \emptyset$.
(ii) For any pair of real numbers $K > 0, N < -1$ also the weighted space $([0,\infty), |\cdot|, V\mathscr{L}^1)$ with

$$V(x) = \sinh\left(x\sqrt{-\frac{K}{N}}\right)^N$$

satisfies the curvature-dimension condition $\mathsf{CD}(K, N+1)$ with singular set $\mathcal{S}_{V\mathscr{L}^1} = \{0\}$.
(iii) For any $N < -1$ the space $([0,\infty), |\cdot|, x^N\mathscr{L}^1)$ is a $\mathsf{CD}(0, N+1)$ space with singular set $\mathcal{S}_{x^N\mathscr{L}^1} = \{0\}$.
(iv) For any pair of real numbers $K < 0, N < -1$ the weighted space

$$\left(\left[-\frac{\pi}{2}\sqrt{\frac{N}{K}}, \frac{\pi}{2}\sqrt{\frac{N}{K}}\right], |\cdot|, \cos\left(x\sqrt{\frac{K}{N}}\right)^N \mathscr{L}^1\right)$$

satisfies the curvature-dimension condition $\mathsf{CD}(K, N+1)$ with singular set given by

$$\mathcal{S}_{\cos\left(x\sqrt{K/N}\right)^N \mathscr{L}^1} = \left\{-\frac{\pi}{2}\sqrt{\frac{N}{K}}, \frac{\pi}{2}\sqrt{\frac{N}{K}}\right\}.$$

Example 6.3.9 provides negative dimensional CD spaces, whose set of singular points is a subset of their topological boundary. Unfortunately, this is not a general behavior and we proceed now to show this. With this goal in mind, we will rely on a modification of Proposition 6.3.8, whose proof needs a preliminary result.

Lemma 6.3.10. *Let $f : I \to \bar{\mathbb{R}}$ be a function on the interval $I := [a,b] \subset \mathbb{R}$. Assume that there exists $c \in (a,b) \cap \{f < +\infty\}$ such that $f|_{[a,c]}$ and $f|_{[c,b]}$ are (K,N)-convex and for every $x_0 \in [a,c)$, $x_1 \in (c,b]$ it holds that*

$$f_N(c) \leq \sigma_{K,N}^{\left(\frac{x_1 - c}{x_1 - x_0}\right)}(x_1 - x_0) f_N(x_0) + \sigma_{K,N}^{\left(\frac{c - x_0}{x_1 - x_0}\right)}(x_1 - x_0) f_N(x_1). \tag{6.3.4}$$

Then f is (K,N)-convex.

Proof. We have to prove the convexity inequality (6.3.3) for every $x_0, x_1 \in I$ in the domain $\{f < \infty\}$. However, this holds by hypothesis, if $x_0, x_1 \in [a,c]$ or $x_0, x_1 \in [c,b]$, thus it is sufficient to consider the case where $x_0 \in [a,c)$ and $x_1 \in (c,b]$. Without loss of generality, we can assume that $x_t \in [a,c)$, then the (K,N)-convexity of $f|_{[a,c]}$ yields that

$$f_N(x_t) \le \sigma_{K,N}^{\left(\frac{c-x_t}{c-x_0}\right)}(c-x_0) f_N(x_0) + \sigma_{K,N}^{\left(\frac{x_t-x_0}{c-x_0}\right)}(c-x_0) f_N(c).$$

Combining this last inequality with (6.3.4) we obtain

$$\begin{aligned} f_N(x_t) \le \Big[\sigma_{K,N}^{\left(\frac{c-x_t}{c-x_0}\right)}(c-x_0) + \sigma_{K,N}^{\left(\frac{x_t-x_0}{c-x_0}\right)}(c-x_0)\, \sigma_{K,N}^{\left(\frac{x_1-c}{x_1-x_0}\right)}(x_1-x_0)\Big] f_N(x_0) \\ + \sigma_{K,N}^{\left(\frac{x_t-x_0}{c-x_0}\right)}(c-x_0)\, \sigma_{K,N}^{\left(\frac{c-x_0}{x_1-x_0}\right)}(x_1-x_0) f_N(x_1). \end{aligned} \tag{6.3.5}$$

On the other hand it is easy to realize that

$$\sigma_{K,N}^{\left(\frac{x_t-x_0}{c-x_0}\right)}(c-x_0)\, \sigma_{K,N}^{\left(\frac{c-x_0}{x_1-x_0}\right)}(x_1-x_0) = \sigma_{K,N}^{\left(\frac{x_t-x_0}{x_1-x_0}\right)}(x_1-x_0).$$

As an example, we consider $K<0$: it holds that

$$\begin{gathered} \sigma_{K,N}^{\left(\frac{x_t-x_0}{c-x_0}\right)}(c-x_0)\, \sigma_{K,N}^{\left(\frac{c-x_0}{x_1-x_0}\right)}(x_1-x_0) = \\ \frac{\sin(\sqrt{K/N}(x_t-x_0))}{\sin(\sqrt{K/N}(c-x_0))} \cdot \frac{\sin(\sqrt{K/N}(c-x_0))}{\sin(\sqrt{K/N}(x_1-x_0))} = \frac{\sin(\sqrt{K/N}(x_t-x_0))}{\sin(\sqrt{K/N}(x_1-x_0))} = \\ \sigma_{K,N}^{\left(\frac{x_t-x_0}{x_1-x_0}\right)}(x_1-x_0). \end{gathered}$$

Moreover, with an explicit computation, using the sum-to-product trigonometric formulas, it is also possible to prove that

$$\sigma_{K,N}^{\left(\frac{c-x_t}{c-x_0}\right)}(c-x_0) + \sigma_{K,N}^{\left(\frac{x_t-x_0}{c-x_0}\right)}(c-x_0)\, \sigma_{K,N}^{\left(\frac{x_1-c}{x_1-x_0}\right)}(x_1-x_0) = \sigma_{K,N}^{\left(\frac{x_1-x_t}{x_1-x_0}\right)}(x_1-x_0).$$

Combining the previous trigonometric identities with inequality (6.3.5) we obtain the (K,N)-convexity inequality. ☐

An immediate corollary follows from the fact that $f_N(c)=0$ if $f(c)=-\infty$.

Corollary 6.3.11. *Let $f: I \to \bar{\mathbb{R}}$ be a function on the interval $I := [a,b]$. Assume that there exists $c \in (a,b)$ such that $f|_{[a,c]}$ and $f|_{[c,b]}$ are (K,N)-convex and that $f(c) = -\infty$, then f is (K,N)-convex.*

Now, we present an alternative version of Proposition 6.3.8, in which we do not need to assume regularity of the weight function ψ at the price of restricting to the case $M = \mathbb{R}$.

Proposition 6.3.12. *Let $\psi : \mathbb{R} \to [-\infty, \infty)$ be (K,N)-convex with $N < -1$, such that $\mathcal{L}^1(\{\psi = -\infty\}) = 0$. Then the metric measure space $(\mathbb{R}, |\cdot|, e^{-\psi}\mathcal{L}^1)$ is a $\mathsf{CD}(K, N+1)$ space.*

Proof. In this proof we denote with $\mathfrak{m}$ the reference measure $e^{-\psi}\mathcal{L}^1$. In order to prove the CD-condition we fix two absolutely continuous measures $\mu_0 = \rho_0\mathfrak{m}, \mu_1 = \rho_1\mathfrak{m} \in \mathscr{P}_2(\mathbb{R})$. Notice that the assumption $\mathscr{L}^1(\{\psi = -\infty\}) = 0$ ensures that $\mu_0, \mu_1 \ll \mathscr{L}^1$, we are going to call $\tilde\rho_0$ and $\tilde\rho_1$ (respectively) their densities, that is $\mu_0 = \tilde\rho_0\mathscr{L}^1$ and $\mu_1 = \tilde\rho_1\mathscr{L}^1$. Now, Brenier's theorem ensures that there exists a unique optimal transport plan between μ_0 and μ_1, and it is induced by a map T, which is differentiable μ_0-almost everywhere. It is also well known that the map T is increasing, thus $T'(x)$ will be non-negative when defined. Moreover, the unique Wasserstein geodesic connecting μ_0 and μ_1 is given by $\mu_t = (T_t)_\#\mu_0$, where $T_t = (1-t)\mathrm{id} + tT$. Then, calling

$\tilde{\rho}_t$ the density of μ_t with respect to the Lebesgue measure $\mathscr{L}^1$, the Jacobi equation holds and gives that

$$\tilde{\rho}_0(x) = \tilde{\rho}_t(T_t(x))T_t'(x) = \tilde{\rho}_t(T_t(x))(1+t(T'(x)-1)),$$

for μ_0-almost every x. On the one hand it is obvious that $\tilde{\rho}_t = e^{-\psi}\rho_t$ for every $t \in [0,1]$, therefore

$$e^{-\psi(x)}\rho_0(x) = e^{-\psi(T_t(x))}\rho_t(T_t(x))(1+t(T'(x)-1)) \tag{6.3.6}$$

for every $t \in [0,1]$ and μ_0-almost every x. On the other hand notice that for every $N' < -1$

$$\begin{aligned} S_{N'+1}(\mu_t) &= \int \rho_t^{-\frac{1}{N'+1}} \, \mathrm{d}\mu_t = \int \rho_t^{-\frac{1}{N'+1}} \, \mathrm{d}[(T_t)_\# \mu_0] = \int \rho_t(T_t(x))^{-\frac{1}{N'+1}} \, \mathrm{d}\mu_0(x) \\ &= \int \left(e^{-\psi(x)}\rho_0(x)\right)^{-\frac{1}{N'+1}} \left(e^{-\psi(T_t(x))}(1+t(T'(x)-1))\right)^{\frac{1}{N'+1}} \mathrm{d}\mu_0(x). \end{aligned} \tag{6.3.7}$$

We can then prove the convexity pointwise, using the (K,N)-convexity of ψ. In particular, ψ is (K,N')-convex for every $N' \in [N,0)$ (cf. [Oht16, Lemma 2.9]). Therefore, calling $A(x) = T'(x)-1$ in order to ease the notation, for every $N' \in [N,-1)$, it holds that

$$\begin{aligned} \left(e^{-\psi(T_t(x))}(1+tA(x))\right)^{\frac{1}{N'+1}} &= \left[e^{-\psi(T_t(x))/N'}\right]^{\frac{N'}{N'+1}} (1+tA(x))^{\frac{1}{N'+1}} \\ &\leq (1+tA(x))^{\frac{1}{N'+1}} \left[\sigma_{K,N'}^{(1-t)}(|T(x)-x|)e^{-\psi(x)/N'} + \sigma_{K,N'}^{(t)}(|T(x)-x|)e^{-\psi(T(x))/N'}\right]^{\frac{N'}{N'+1}}. \end{aligned}$$

Then, by rewriting the last term, we obtain

$$\begin{aligned} &\left[(1+tA(x))^{\frac{1}{N'}}\sigma_{K,N'}^{(1-t)}(|T(x)-x|)e^{-\psi(x)/N'} + (1+tA(x))^{\frac{1}{N'}}\sigma_{K,N'}^{(t)}(|T(x)-x|)e^{-\psi(T(x))/N'}\right]^{\frac{N'}{N'+1}} \\ &\leq \tau_{K,(N'+1)}^{(1-t)}(|T(x)-x|)e^{-\psi(x)/(N'+1)} + \tau_{K,(N'+1)}^{(t)}(|T(x)-x|)(1+A(x))^{\frac{1}{N'+1}}e^{-\psi(T(x))/(N'+1)}, \end{aligned}$$

where the last inequality is a consequence of Jensen's inequality and of the definition of $\tau_{K,N'}^{(t)}$ (see (6.3.1)). Substituting the above inequality into (6.3.7) and using (6.3.6) with $t=1$, we obtain that

$$\begin{aligned} S_{N'+1}(\mu_t) \leq &\int \tau_{K,(N'+1)}^{(1-t)}(|T(x)-x|)\rho_0(x)^{-\frac{1}{N'+1}} \, \mathrm{d}\mu_0(x) \\ &+ \int \tau_{K,(N'+1)}^{(t)}(|T(x)-x|)(1+A(x))^{\frac{1}{N'+1}}\left(e^{-\psi(x)}\rho_0(x)\right)^{-\frac{1}{N'+1}}e^{-\psi(T(x))/(N'+1)} \, \mathrm{d}\mu_0(x) \\ = &\int \tau_{K,(N'+1)}^{(1-t)}(|T(x)-x|)\rho_0(x)^{-\frac{1}{N'+1}} \, \mathrm{d}\mu_0(x) + \int \tau_{K,(N'+1)}^{(t)}(|T(x)-x|)\rho_1(T(x))^{-\frac{1}{N'+1}} \, \mathrm{d}\mu_0(x) \\ = &\int \left[\tau_{K,(N'+1)}^{(1-t)}(|y-x|)\rho_0(x)^{-\frac{1}{N'+1}} + \tau_{K,(N'+1)}^{(t)}(|y-x|)\rho_1(y)^{-\frac{1}{N'+1}}\right] \mathrm{d}[(\mathrm{id} \times T)_\# \mu_0](x,y), \end{aligned}$$

for every $N' \in [N,-1)$, which is the desired inequality. □

A direct application of the previous result leads to the following refinement of Example 6.3.9:

Example 6.3.13. **(ii')** For any pair of real numbers $K > 0, N < -1$, the weighted space

$$\left(\mathbb{R}, |\cdot|, V\mathscr{L}^1\right) \quad \text{with } V(x) = \sinh\left(x\sqrt{-\frac{K}{N}}\right)^N,$$

obtained gluing two copies of the half-line space in Example 6.3.9-(ii), satisfies the curvature-dimension condition CD$(K, N+1)$ with singular set $\mathcal{S}_{V\mathscr{L}^1} = \{0\}$.

(iii') Similarly, for any $N<-1$ the space $(\mathbb{R},|\cdot|,|x|^N\mathscr{L}^1)$ is a $\mathsf{CD}(0,N+1)$ space with singular set $\mathcal{S}_{|x|^N\mathscr{L}^1}=\{0\}$.
(iv') For any pair of real numbers $K<0, N<-1$ the space which is obtained gluing J-copies of the interval in Example 6.3.9-(iv), for example by considering $\left(\bigcup_{j=1}^J I_j,|\cdot|,V\mathscr{L}^1\right)$ with

$$I_j:=\left[\frac{(2j-1)\pi}{2}\sqrt{\frac{N}{K}},\frac{(2j+1)\pi}{2}\sqrt{\frac{N}{K}}\right]\ \text{and}\ V:=\sum_{j=1}^J \mathbb{1}_{I_j}\cdot\cos\left((x-x_j)\sqrt{\frac{K}{N}}\right)^N,\ x_j:=j\pi\sqrt{\frac{N}{K}},$$

satisfies the curvature-dimension condition $\mathsf{CD}(K,N+1)$ with singular set given by

$$\mathcal{S}_{V\mathscr{L}^1}=\left\{\frac{(2j-1)\pi}{2}\sqrt{\frac{N}{K}}:j=1,\dots,J+1\right\}.$$

We end this section by pointing out that there exist unbounded CD spaces with negative dimension, for every value of the curvature. In particular, unlike to what happens for positive dimensional CD spaces, it is never possible to obtain a bound on the diameter of the space. Actually, this not only happens for singular spaces, as in Example 6.3.13 (iv'), since for example, also the hyperbolic plane satisfies the $\mathsf{CD}(-1,N)$-condition for any $N<0$ (recall that every $\mathsf{CD}(K,N)$ space with $N\geq 1$ is automatically a $\mathsf{CD}(K,N)$ space for any $N<0$). Therefore, there exists no counterpart of the generalized Bonnet-Myers theorem [Stu06b, Corollary 2.6] for negative dimensional CD spaces. This is not completely surprising, since there exist also $\mathsf{CD}(K,\infty)$ spaces with $K>0$ (such as Gaussian spaces), having infinite diameter.

6.3.3 Approximate CD-condition and regularity assumptions

Examples 6.3.9 and 6.3.13 exhibit that $\mathfrak{m}$-singular sets associated to $\mathsf{CD}(K,N)$ spaces are not necessarily empty sets. Moreover, as already noticed, the geometric behaviour in these examples can be extremely different: in opposition to Examples 6.3.9, where points in $\mathcal{S}_\mathfrak{m}$ appear only as terminal points of geodesics in the space, singular points in Examples 6.3.13 occur as inner points of geodesics. This observation shows that, in particular, the k-th regular set $\mathcal{R}^k$ of a space X, which was introduced in (6.2.4), is not necessarily geodesically convex. This turns out to produce major difficulties in the proof of our main result. Indeed, we wish to approximate metric measure spaces by considering their k-th regular sets but the CD-condition is precisely a condition made on geodesics. We point out that this type of issues do not show up for positive values of the dimensional parameter and, in fact, to overcome the problems arising from the non-geodesic convexity will be the main challenge to overcome in the proof of the Stability Theorem 6.4.1. In order to do so we present the next two definitions.

Definition 6.3.14 (Approximate CD-condition)**.** We say that a metric measure space $(\mathsf{X},\mathsf{d},\mathfrak{m})$ satisfies the *approximate curvature-dimension condition* $\mathsf{CD}^a(K,N)$ if the CD-condition in Definition 6.3.3 is satisfied by further requiring that the supports of the measures μ_0, $\mu_1\in\mathscr{P}^{ac}(\mathsf{X},\mathfrak{m})$ satisfy $\mathrm{supp}(\mu_0),\mathrm{supp}(\mu_1)\subset\mathcal{R}^k$, for some $k\in\mathbb{N}$.

Note that in the definition above k is not fixed.

As discussed above, we need to carefully approach the topic of the non-geodesic convexity of $\mathfrak{m}$-singular sets. The following concept directs us in this direction by quantifying in terms of masses - and thus, controlling - to what extent convexity of the k-th cuts is unsatisfied.

Definition 6.3.15 (ω-uniform convexity)**.** A metric measure space $(\mathsf{X},\mathsf{d},\mathfrak{m})$ is *ω-uniformly convex* if there exists a function $\omega\colon\mathbb{N}\times\mathbb{N}\times\mathbb{R}^+\to[0,1]$ with the following properties:

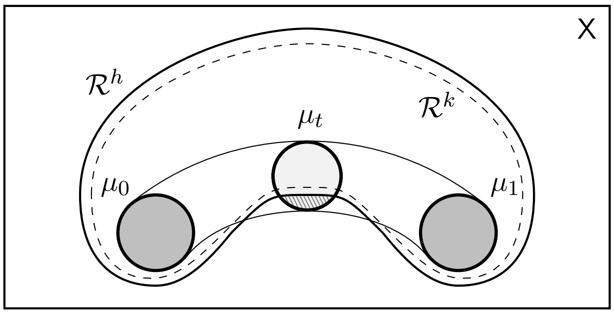

Figure 6.1: A visual representation of the property of ω-uniform convexity. In particular the shaded set has μ_t-mass bounded above by $\omega(k, h, M)$, if $S_{N,\mathfrak{m}}(\mu_0), S_{N,\mathfrak{m}}(\mu_1) \leq M$.

- for any μ_0, $\mu_1 \in \mathscr{P}_2(\mathsf{X})$, with $S_{N,\mathfrak{m}}(\mu_0), S_{N,\mathfrak{m}}(\mu_1) \leq M$ and $\text{supp}(\mu_0), \text{supp}(\mu_1) \subseteq \mathcal{R}^k$, every t-middle point of any geodesic $\{\mu_t\}_{t\in[0,1]} \subset \mathscr{P}_2(\mathsf{X})$ between μ_0 and μ_1 satisfies
$$\mu_t(\mathcal{R}^h) \geq 1 - \omega(k, h, M) \text{ for any } h \in \mathbb{N};$$
- for any $k \in \mathbb{N}$, $M \in \mathbb{R}^+$
$$\lim_{h\to\infty} \omega(k, h, M) = 0. \tag{6.3.8}$$

Figure 6.1 shows a schematic representation of the ω-uniform convexity.

Remark 6.3.16. We illustrate with some examples the concept of ω-uniform convexity.

* If the k-th regular set $\mathcal{R}^k$ is geodesically convex, then we can choose $\omega(\ell, h, M) = 0$, for all entropy bounds $M > 0$ and $\ell \leq k \leq h$. This is the case, for example, of metric measure spaces with empty $\mathfrak{m}$-singular set and the spaces presented in Example 6.3.9.
* Conversely, if μ_0, $\mu_1 \in \mathscr{P}_2(\mathsf{X})$ are supported in $\mathcal{R}^k$ with bounded entropies and the support of μ_t, a geodesic joining μ_0 and μ_1 evaluated at time t, is contained in the complement of $\mathcal{R}^h$, for some $t \in (0, 1)$ and $h \in \mathbb{N}$, then $\omega(k, h, M) = 1$.
In particular, the metric measure space $([-1, 1], |\cdot|, \mathfrak{m})$, with $\mathrm{d}\mathfrak{m} = \delta_{-1} + \delta_1 + 1/x^2 \mathrm{d}\mathcal{L}^1$ serves as an example of a metric measure space which is not ω-uniformly convex. Indeed, at time $t = 1/2$, the support of the unique 2-Wasserstein geodesic $(\delta_{2t-1})_{t\in[0,1]}$ is contained inside $X \setminus \mathcal{R}^h$ for any $h \in \mathbb{N}$, since $\mathcal{S}_\mathfrak{m} = \{0\}$, while its terminal points have entropy equal to 1.
* Lastly, a more interesting behaviour occurs in a convex subset of Example 6.3.13, given by
$$\left([0, \pi], |\cdot|, \mathbb{1}_{[0,\frac{\pi}{2}]}\cos(x)^{-2}\mathcal{L}^1 + \mathbb{1}_{[\frac{\pi}{2},\pi]}\cos(x-\pi)^{-2}\mathcal{L}^1\right),$$
whose singular set is $S_\mathfrak{m} = \{\frac{\pi}{2}\}$. This is a CD$(-2, -1)$ space as well as an ω-uniformly convex space for a non-trivial function $\omega(k, h, M)$. Note that since there exist Wasserstein geodesics which are, at some time t, entirely contained in the complement of $\mathcal{R}^k$, there are actually some values $k, h \in \mathbb{N}$, $M \in \mathbb{R}^+$ for which $\omega(k, h, M) = 1$. Also, for fixed values of $k \in \mathbb{N}$ and $M \in \mathbb{R}^+$, $\omega(k, h, M) \to 0$ as $h \to \infty$, since W_2-geodesics are absolutely continuous and $V\mathcal{L}^1(X \setminus \mathcal{R}^h) \to 0$ as $h \to \infty$. Indeed, the key observation here is that we can not force arbitrarily large amounts of mass to transit through $X \setminus \mathcal{R}^h$, at a given time, without losing the upper bound on the entropy of the terminal points. Intuitively, to produce such geodesics, we would have to consider measures with arbitrarily small supports or which accumulate arbitrarily large masses around a point. However, these type of measures have large entropy.

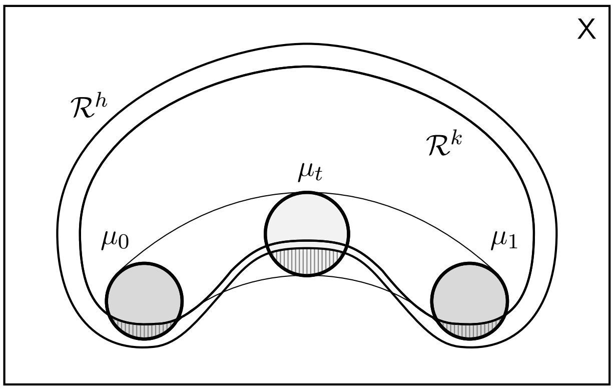

Figure 6.2: A visual representation of the property provided by the function Ω, that is i) in Proposition 6.3.17. In particular the shaded set in the center has μ_t-mass bounded above by $\Omega(k, h, M, \delta)$, if the shaded set on the left and the one on the right have μ_0-mass and μ_1-mass (respectively) less than δ and $S_{N,\mathfrak{m}}(\mu_0), S_{N,\mathfrak{m}}(\mu_1) \leq M$.

A concrete and useful property which ω-uniformly convex metric measure spaces enjoy is that we are able to quantify interpolated mass outside the set $\mathcal{R}^h$, even if the marginals are not necessarily supported on $\mathcal{R}^k$, granted they supply sufficient mass to the k-th regular sets. In the following, we will denote by $\mathrm{Geo}(\mathsf{X})$ the set of all the constant speed geodesics in the Polish space (X, d).

Proposition 6.3.17. *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *be an* ω*-uniformly convex space. Then there exists a function* $\Omega \colon \mathbb{N} \times \mathbb{N} \times \mathbb{R}^+ \times [0,1] \to \mathbb{R}$ *such that:*

i) for any μ_0, $\mu_1 \in \mathscr{P}_2(\mathsf{X})$ *with* $S_{N,\mathfrak{m}}(\mu_0), S_{N,\mathfrak{m}}(\mu_1) \leq M$ *and* $\mu_0(\mathcal{R}^k), \mu_1(\mathcal{R}^k) \geq 1 - \delta$*, any* t*-middle point of the geodesic* $\{\mu_t\}_{t\in[0,1]}$ *satisfies* $\mu_t(\mathcal{R}^h) \geq 1 - \Omega(k, h, M, \delta)$*,*

ii) for every $0 \leq \delta < \frac{1}{4}$ *it holds that*

$$\limsup_{h\to\infty} \Omega(k, h, M, \delta) \leq 2\delta, \tag{6.3.9}$$

for every fixed $k \in \mathbb{N}$ *and* $M \in \mathbb{R}^+$*.*

Proof. Notice that we can limit ourselves to the case when $0 \leq \delta < \frac{1}{4}$, because we can simply put $\Omega(k, h, M, \delta) = 1$ if $\delta \geq \frac{1}{4}$. Fixed μ_0, $\mu_1 \in \mathscr{P}_2(\mathsf{X})$ such that $S_{N,\mathfrak{m}}(\mu_0), S_{N,\mathfrak{m}}(\mu_1) \leq M$ and $\mu_0(\mathcal{R}^k), \mu_1(\mathcal{R}^k) \geq 1 - \delta$, consider a t-middle point of a geodesic $\{\mu_t\}_{t\in[0,1]}$, connecting μ_0 and μ_1. Let $\eta \in \mathscr{P}(\mathrm{Geo}(\mathsf{X}))$ be a representation of $\{\mu_t\}_{t\in[0,1]}$ and define

$$\tilde{\eta} := \frac{1}{\eta(G)} \cdot \eta|_G \in \mathscr{P}(\mathrm{Geo}(\mathsf{X})),$$

where

$$G := \{\gamma \in \mathrm{Geo}(\mathsf{X}) \,:\, \gamma(0), \gamma(1) \in \mathcal{R}^k\}.$$

Notice that $\tilde{\eta}$ is actually well-defined, since our condition on μ_0 and μ_1 ensures that $\eta(G) \geq 1 - 2\delta > 0$. Moreover,

$$\eta = \eta(G) \cdot \tilde{\eta} + \bar{\eta} \quad \text{for some } \bar{\eta} \in \mathscr{M}(\mathrm{Geo}(\mathsf{X})) \text{ with } \bar{\eta}(\mathrm{Geo}(\mathsf{X})) \leq 2\delta. \tag{6.3.10}$$

Observe that $\{\tilde{\mu}_t = (e_t)_\#\tilde{\eta}\}_{t\in[0,1]}$ is a Wasserstein geodesic connecting two measures $\tilde{\mu}_0$ and $\tilde{\mu}_1$, which are supported on $\mathcal{R}^k$ and satisfy

$$\max\{S_{N,\mathfrak{m}}(\tilde{\mu}_0), S_{N,\mathfrak{m}}(\tilde{\mu}_1)\} \leq \left[\frac{1}{\eta(G)}\right]^{1-\frac{1}{N}} \max\{S_{N,\mathfrak{m}}(\mu_0), S_{N,\mathfrak{m}}(\mu_1)\}$$
$$\leq \left[\frac{1}{1-2\delta}\right]^{1-\frac{1}{N}} M \leq 2^{1-\frac{1}{N}} M.$$

Then, the ω-uniform convexity of $(\mathsf{X},\mathsf{d},\mathfrak{m})$ ensures that, for every h,

$$\tilde{\mu}_t(\mathcal{R}^h) \geq 1 - \omega(k,h,2^{1-\frac{1}{N}}M).$$

Moreover, taking into account (6.3.10), we can conclude that

$$\mu_t(\mathcal{R}^h) \geq (1-2\delta)\cdot\tilde{\mu}_t(\mathcal{R}^h) \geq 1 - \omega(k,h,2^{1-\frac{1}{N}}M) - 2\delta.$$

Therefore, to satisfy i), we can set

$$\Omega(k,h,M,\delta) := \omega(k,h,2^{1-\frac{1}{N}}M) + 2\delta.$$

With this definition, ii) is a straightforward consequence of the condition (6.3.8) on $\omega(k,h,M)$. □

6.4 Stability of CD-condition

The aim of the last section is to prove the main result,

Theorem 6.4.1 (Stability). *Let $K \in \mathbb{R}$, $N \in (-\infty, 0)$, and $\{(\mathsf{X}_n,\mathsf{d}_n,\mathfrak{m}_n,\mathcal{S}_{\mathfrak{m}_n},p_n)\}_{n\in\mathbb{N}} \subset \mathcal{M}^{qR}_{\bar{k}}$ be a sequence of pointed generalized metric measure spaces converging to $(\mathsf{X}_\infty,\mathsf{d}_\infty,\mathfrak{m}_\infty,\mathcal{S}_{\mathfrak{m}_\infty},p_\infty) \in \mathcal{M}^{qR}_{\bar{k}}$ in the* iKRW*-distance, for some $\bar{k} \in \mathbb{N}$. Assume further that:*

(i) $(\mathsf{X}_n,\mathsf{d}_n,\mathfrak{m}_n)$ is a $\mathsf{CD}(K,N)$ space for every $n \in \mathbb{N}$;

(ii) there exists $\omega : \mathbb{N}\times\mathbb{N}\times\mathbb{R}^+ \to [0,1]$, for which $(\mathsf{X}_n,\mathsf{d}_n,\mathfrak{m}_n)$ is ω-uniformly convex, for every $n \in \mathbb{N}$;

(iii) $\sup_{n\in\mathbb{N}\cup\{\infty\}} \operatorname{diam}(\mathsf{X}_n,\mathsf{d}_n) < \pi\sqrt{\frac{1}{|K|}}$, if $K < 0$.

Then $(\mathsf{X}_\infty,\mathsf{d}_\infty,\mathfrak{m}_\infty)$ is a $\mathsf{CD}(K,N)$ space.

As a matter of fact, Theorem 6.4.1 is concluded from the slightly more general statement below, since Proposition 6.2.18 provides an effective realization for an iKRW-converging sequence of metric measure spaces. Recall that the extrinsic convergence of metric measure spaces is presented in Definition 6.2.19.

Theorem 6.4.2 (Extrinsic Stability). *Let $K \in \mathbb{R}$, $N \in (-\infty, 0)$. Then the $\mathsf{CD}(K,N)$-condition is stable under the extrinsic convergence of metric measure spaces, granted conditions* (i)-(iii) *from Theorem 6.4.1 are satisfied by the converging sequence.*

The following is an immediate result of Theorem 6.4.2.

Corollary 6.4.3. *Let $K \in \mathbb{R}$, $N \in (-\infty, 0)$, and $\{(\mathsf{X}_n,\mathsf{d}_n,\mathfrak{m}_n,\mathcal{S}_{\mathfrak{m}_n},p_n)\}_{n\in\mathbb{N}} \subset \mathcal{M}^{qR}_{\bar{k}}$ be a sequence converging to $(\mathsf{X}_\infty,\mathsf{d}_\infty,\mathfrak{m}_\infty,\mathcal{S}_{\mathfrak{m}_\infty},p_\infty) \in \mathcal{M}^{qR}_{\bar{k}}$, in the extrinsic or intrinsic manner. Assume that the regular sets $\mathcal{R}^k_n$ are geodesically convex, for all $k,n \in \mathbb{N}$. If for every $n \in \mathbb{N}$ the space $(\mathsf{X}_n,\mathsf{d}_n,\mathfrak{m}_n)$ satisfies the $\mathsf{CD}(K,N)$ condition (with $\sup_{n\in\mathbb{N}\cup\{\infty\}} \operatorname{diam}(\mathsf{X}_n,\mathsf{d}_n) < \pi|K|^{-1/2}$, if $K < 0$), then also $(\mathsf{X}_\infty,\mathsf{d}_\infty,\mathfrak{m}_\infty)$ is a $\mathsf{CD}(K,N)$ space.*

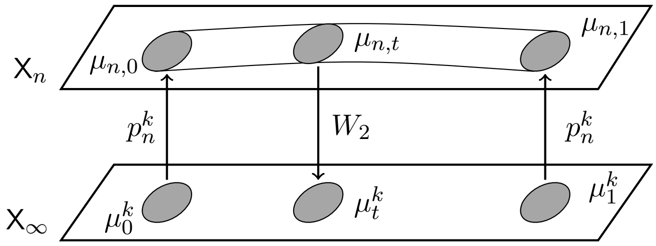

Figure 6.3: Approximation procedure for the midpoints

When compared with *Stability Theorem* 6.4.1, the advantage of *Extrinsic Stability Theorem* 6.4.2, is that no assumptions have to be made, regarding the limiting behaviour of singular sets along the sequence. This contrasts with the *Stability Theorem*, which is stated in terms of the intrinsic $\mathsf{d}_{\mathsf{iKRW}}$-convergence, since the $\mathsf{d}_{\mathsf{iKRW}}$-distance controls the Hausdorff distance between singular sets. Therefore, with the latter Theorem one gains some flexibility to study the aforementioned sets; nevertheless, there is a price to pay in exchange. Namely, it is necessary to be in possession of an effective realization for the convergence. In this sense, we find that both results complement very well each other.

We fix some notation prior to outlining the argument in the proof of Theorem 6.4.2.

First of all, since by assumption we have a realization of the convergence in a complete and separable metric space (Z, d_Z), for simplicity we will identify all objects with their embedded version. In particular, for every $n \in \mathbb{N} \cup \{\infty\}$ we will call X_n the embedded set $i_n(\mathsf{X}_n)$, $\mathfrak{m}_n$ the push-forward measure $(i_n)_\# \mathfrak{m}_n$ (and the same for its restricted and normalized versions), p_n the reference point $i_n(p_n)$. Moreover, since the embeddings are isometries, it will suffice to work with the distance d_Z, which from now on will be denoted by d, for sake of simplicity. With this identification, our extrinsically convergent sequence of pointed generalized metric measure spaces $\{\mathbb{X}_n\}_{n\in\mathbb{N}} \subset \mathcal{M}_{\bar{k}}^{qR}$, which converges to $\mathbb{X}_\infty \in \mathcal{M}_{\bar{k}}^{qR}$, satisfies

$$\left|\log\left(\frac{\mathfrak{m}_n^k(\mathsf{X}_n)}{\mathfrak{m}_\infty^k(\mathsf{X}_\infty)}\right)\right| + \mathsf{d}(p_n, p_\infty) + W_2(\bar{\mathfrak{m}}_n^k, \bar{\mathfrak{m}}_\infty^k) \overset{n\to\infty}{\to} 0, \tag{6.4.1}$$

for any $k \geq \bar{k}$. Notice that it was possible to put the Wasserstein distance W_2 in (6.4.1), according to Remark 6.2.20.

In the remainder, we use the adjective *horizontal* to refer to the approximations we construct inside a fixed space $\mathbb{X}_n$, for $n \in \mathbb{N}$. Respectively, we denote as *vertical* approximations those approximations made over the sequence $\mathbb{X}_n \to \mathbb{X}_\infty$, when we let $n \to \infty$. Our objective is, naturally, to demonstrate, for every pair of measures $\mu_0, \mu_1 \in \mathscr{P}^{ac}(\mathsf{X}_\infty, \mathfrak{m}_\infty)$, the existence of a 2-Wasserstein geodesic $\{\mu_t\}_{t\in[0,1]} \subset \mathscr{P}_2(\mathsf{X}_\infty)$ and an optimal plan $q \in \mathsf{Opt}(\mu_0, \mu_1)$, for which the curvature-dimension inequality (6.3.2) is satisfied. We accomplish this by following the next steps.

1. We assume that $\mathrm{supp}(\mu_i) \subseteq \mathcal{R}_\infty^k$, for $i \in \{0, 1\}$ and fixed $k \in \mathbb{N}$, and construct a geodesic $(\mu_t)_{t\in[0,1]} \subset \mathscr{P}_2(\mathsf{X}_\infty)$ between μ_0 and μ_1 and an optimal plan $q \in \mathsf{Opt}(\mu_0, \mu_1)$, for which the CD-inequality (6.3.2) is fulfilled, relying on the following vertical approximation argument. (Above, and in the following, we write $\mathcal{R}_n^k \subset \mathsf{X}_n$ to denote the k-th $\mathfrak{m}_n$-regular set of X_n, k-regular set in short, for $n \in \mathbb{N} \cup \{\infty\}$.)

   The assumption on the supports allows us to approximate vertically the marginal measures μ_0 and μ_1, by employing a canonical map between Wasserstein spaces $P_n^k : \mathscr{P}^{ac}(\mathsf{X}_\infty, \mathfrak{m}_\infty^k) \to$

$\mathscr{P}^{ac}(\mathsf{X}_n, \mathfrak{m}_n^k)$, induced via an optimal coupling of the normalized reference measures $p_n^k \in \mathsf{Opt}(\bar{\mathfrak{m}}_\infty^k, \bar{\mathfrak{m}}_n^k)$. Let us denote these approximations by $(\mu_{n,i})_{n\in\mathbb{N}}$, for $i \in \{0,1\}$.

At this point we construct the pair $(\mu_t,\, q)$ as the vertical limits of a sequence of geodesics $(\mu_{n,t})_{t\in[0,1]} \subset \mathscr{P}_2(\mathsf{X}_n)$, between $\mu_{n,0}$ and $\mu_{n,1}$, and a sequence of optimal plans $q_n \in \mathsf{Opt}(\mu_{n,0}, \mu_{n,1})$, both indexed by $n \in \mathbb{N}$. Furthermore, we provide these sequences using the CD-hypothesis on $(\mathsf{X}_n, \mathsf{d}_n, \mathfrak{m}_n)$, so in particular we can guarantee that, for $n \in \mathbb{N}$, each pair $(\mu_{n,t},\, q_n)$ satisfies the CD-inequality, for every $t \in [0,1]$.

After demonstrating the lower semicontinuity $S_{N',\mathfrak{m}_\infty}(\mu_t) \le \liminf_{n\to\infty} S_{N',\mathfrak{m}_n}(\mu_{n,t})$ and the upper semicontinuity $\limsup_{n\to\infty} T^{(t)}_{K,N'}(q_n|\mathfrak{m}_n) \le T^{(t)}_{K,N'}(q|\mathfrak{m}_\infty)$, along our sequences as $n \to \infty$, we conclude the validity of the CD-inequality (6.3.2) for $(\mu_t,\, q)$, for every $t \in [0,1]$.

(Look at Figure 6.3 for a schematic representation)

2. Additionally, we produce, for $i \in \{0,1\}$, favorable horizontal approximations $(\mu_i^k)_{k\in\mathbb{N}} \subset \mathscr{P}^{ac}(\mathsf{X}_\infty, \mathfrak{m}_\infty)$, W_2-converging to μ_i, whose supports satisfy $\mathrm{supp}(\mu_i^k) \subseteq \mathcal{R}_\infty^k$, for every $k \in \mathbb{N}$. Subsequently, by approximating with the pairs constructed in Step 1, we show the existence of the sought geodesic $\{\mu_t\}_{t\in[0,1]} \subset \mathscr{P}_2(\mathsf{X}_\infty)$ and optimal plan $q \in \mathsf{Opt}(\mu_0, \mu_1)$. After showing that the appropriate semicontinuity of the functionals $S_{N',\mathfrak{m}_\infty}(\cdot)$ and $T^{(t)}_{K,N'}(\cdot|\mathfrak{m}_\infty)$ hold, we are able to verify the CD-condition and conclude.

Let us now indicate where the complications arise.
By now the rough idea behind a proof of geometric stability in Wasserstein spaces is well known. More precisely, for well-behaved measures, as a first step one shows that $\mathscr{P}(\mathsf{X}_\infty)$ inherits the CD-convexity from the stability of the geometry of $\mathscr{P}(\mathsf{X}_n)$ under vertical approximations. Afterwards, one can conclude the same property for more general measures, by approximating them horizontally with these well-behaved measures. On the way, the two main challenges that one encounters are, of course, semicontinuity and precompactness.

Inspired by techniques used in [LV09], we are able to provide a Legendre-type representation formula for the entropy, which handles one of the functionals in question. At this point, inspired by arguments of Sturm in [Stu06b], we conclude the upper semicontinuity of $T^{(t)}_{K,N'}(\cdot|\mathfrak{m}_\infty)$.

The very challenging obstacles appear then when approaching the problem of the existence of limits and of the convergence of inner points of geodesics. The general class of metric measure structures which we consider is not even locally compact, while the "wildness" of quasi-Radon measures prohibit us to control the reference measures in any uniform way, preventing us to recover any tightness results from them. Thus, to overcome these problems original arguments have to be provided here.
The crucial ingredient to get back into track will be the control of the mass given by Wasserstein geodesics to $\mathfrak{m}$-singular sets when taking the limits and this can be extracted from the ω-uniform convexity.

We advance to the presentation of some auxiliary results in the next Section. The vertical approximation argument is presented afterwards in Section 6.4.2, while Step 2. above is discussed in the final Section 6.4.3.

6.4.1 Auxiliary Results

We collect in this section the preliminary results needed to prove Theorem 6.4.1. In particular, in the first part we present the tools which turn out to be useful in approximating t-midpoints of geodesics, while in the consecutive subsection we deal with the required semicontinuity results.

Approximation and compactness results

We start by exhibiting the existence of well-behaved horizontal approximations to measures. Recall that for a reference measure $\mathfrak{m} \in \mathscr{M}^{qR}(\mathsf{X})$, $\mathfrak{m}^k$ is its k-cut defined by (6.2.3).

Lemma 6.4.4. *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space,* $\mathfrak{m} \in \mathscr{M}^{qR}(\mathsf{X})$*, and* $\mu \in \mathscr{P}^{ac}(\mathsf{X}, \mathfrak{m})$*. Then:*

1) *The sequence of measures* $\{\mathfrak{m}^k\}_{k\in\mathbb{N}}$ *approximates* $\mathfrak{m}$*, in the sense of quasi-Radon measures:*

$$\mathfrak{m}^k \rightharpoonup \mathfrak{m}.$$

2) *There exists a sequence of measures* $\{\mu^k\}_{k\in\mathbb{N}} \subset \mathscr{P}^{ac}(\mathsf{X}, \mathfrak{m})$*,* $\mu^k \ll \mathfrak{m}^k$ *for any* $k \in \mathbb{N}$*, converging to* μ *in the* W_2*-distance. In particular, for every* $k \in \mathbb{N}$*, we have that* $\mathrm{supp}(\mu^k) \subseteq \mathcal{R}^k := B_{2^{k+1}}(p) \setminus \mathcal{N}_{2^{-(k+1)}}(\mathcal{S}_\mathfrak{m})$*, thus these measures have bounded support.*

Proof. We start noticing that 1) follows directly from the definition of weak convergence because $\mathrm{supp}(f) \subseteq \mathcal{R}^k$ holds eventually, for any function $f \in C_{bs}(X) \cap C_{\mathcal{S}_\mathfrak{m}}(X)$.

As for 2), let us consider $\mu^k := c^k f^k \mu$, where f^k is the cut-off function defined in (6.2.3), c^k is the normalization constant providing $\mu^k(X) = 1$, and k is a sufficiently large number, the estimate of which will be determined along the proof. Clearly, $\mu^k \ll \mathfrak{m}^k$. At this point we recall that $\mathrm{supp}(f^k) \subset \mathcal{R}^k$ with $0 \le f^k \le 1$ for any $k \in \mathbb{N}$, and that $f^k \to 1$ pointwise $\mathfrak{m}$-almost everywhere as $k \to \infty$. As a consequence, $f^k \to 1$ pointwise μ-almost everywhere, and $\mathrm{supp}(\mu^k)$ is bounded since $\mathrm{supp}(\mu^k) \subset \mathrm{supp}(\mathfrak{m}^k) \subset \mathcal{R}^k$.
By choosing k_0 sufficiently large, we can assume that $\mathrm{supp}(\mu) \cap \mathcal{R}^k \neq \emptyset$, for all $k \ge k_0$. (Although the particular choice of k_0 does depend on μ, there is no loss of generality, since such a bound exists for every measure μ and we are interested exclusively in the limit behavior of μ^k.) Let $c^k := (\int_X f^k \,\mathrm{d}\mu)^{-1}$: in view of the previous remarks, c^k is well-defined and monotone decreasing in $k \in \mathbb{N}$ and $\lim_{k\to\infty} c^k = 1$.
We can then conclude using the dominated convergence theorem, recalling that a sequence of measures is W_2-convergent if and only if it is weakly convergent and the sequence of its second moments is also convergent. ☐

Take two complete and separable metric measure spaces $(X, \mathsf{d}_X, \mathfrak{m}_X)$ and $(Y, \mathsf{d}_Y, \mathfrak{m}_Y)$ embedded in Z, such that $\mathfrak{m}_X$ and $\mathfrak{m}_Y$ are probability measures. Recall that, given a coupling $p \in \mathsf{Adm}(\mathfrak{m}_X, \mathfrak{m}_Y)$, we can consider a canonical map between their Wasserstein spaces $P : \mathscr{P}^{ac}(X, \mathfrak{m}_X) \to \mathscr{P}^{ac}(Y, \mathfrak{m}_Y)$, which is induced by pushing forward weighted versions of the coupling p. We refer to these maps as *Weighted Marginalizations* and we will use them to produce vertical approximations.

In detail, for each $n, k \in \mathbb{N}$ we consider (and fix) an optimal coupling $p_n^k \in \mathsf{Opt}(\bar{\mathfrak{m}}_\infty^k, \bar{\mathfrak{m}}_n^k)$. Here $\bar{\mathfrak{n}} = \mathfrak{n}(\mathsf{Y})^{-1}\mathfrak{n}$ denotes the normalization of a finite measure $\mathfrak{n} \in \mathscr{M}(\mathsf{Y})$. We then write $\{P_{n,k}(x)\}_{x\in\mathsf{X}_\infty} \subset \mathscr{P}(\{x\} \times \mathsf{X}_n) \approx \mathscr{P}(\mathsf{X}_n)$ and $\{P'_{n,k}(y)\}_{y\in\mathsf{X}_n} \subset \mathscr{P}(\mathsf{X}_\infty \times \{y\}) \approx \mathscr{P}(\mathsf{X}_\infty)$ the disintegration kernels of the coupling p_n^k with respect to the projection maps $\mathsf{p}_1 : \mathsf{X}_\infty \times \mathsf{X}_n \to \mathsf{X}_\infty$ and $\mathsf{p}_2 : \mathsf{X}_\infty \times \mathsf{X}_n \to \mathsf{X}_n$, respectively. More precisely, for $\bar{\mathfrak{m}}_\infty^k$-a.e. $x \in \mathsf{X}_\infty$, and $\bar{\mathfrak{m}}_n^k$-a.e. $y \in \mathsf{X}_n$, we let $P_{n,k}(x)$ and $P'_{n,k}(y)$ be the measures given by the Disintegration Theorem, which are characterized by

$$p_n^k(A) = \int_{\mathsf{X}_\infty} P_{n,k}(x)(A_x)\,\mathrm{d}\bar{\mathfrak{m}}_\infty^k(x) = \int_{\mathsf{X}_n} P'_{n,k}(y)(A_y)\,\mathrm{d}\bar{\mathfrak{m}}_n^k(y).$$

for every measurable $A \subset X_\infty \times X_n$, where $A_x = \{y : (x, y) \in A\}$ and $A_y = \{x : (x, y) \in A\}$. Furthermore, we can define the Weighted Marginalization maps between Wasserstein spaces via the push forward along the coordinate projections of the weighted couplings $\rho\, p_n^k$. With a slight abuse of notation, we denote again these maps as $P'_{n,k}$ and $P_{n,k}$. Specifically, let

$$\begin{aligned} P'_{n,k}\colon \mathscr{P}^{ac}(\mathsf{X}_\infty, \mathfrak{m}_\infty^k) &\to \mathscr{P}^{ac}(\mathsf{X}_n, \mathfrak{m}_n^k) \\ \mu = \rho\, \bar{\mathfrak{m}}_\infty^k &\mapsto P'_{n,k}(\mu) := (\mathsf{p}_2)_\sharp\, \rho\, p_n^k = \rho'\, \bar{\mathfrak{m}}_n^k, \\ &\quad \text{with } \rho'(y) = \int_{X_\infty} \rho(x) P'_{n,k}(y)(\mathrm{d}x). \end{aligned}$$

The map $P_{n,k}\colon \mathscr{P}^{ac}(\mathsf{X}_n, \mathfrak{m}_n^k) \to \mathscr{P}^{ac}(\mathsf{X}_\infty, \mathfrak{m}_\infty^k)$ is defined in an analogous manner. Note that, in particular, $\rho\, p_n^k \in \mathsf{Adm}(\mu, P'_{n,k}(\mu))$. The following lemma shows that the well-known properties of the Weighted Marginalization map $P'_{n,k}$ are still valid in our framework.

Lemma 6.4.5. *Let $\mu = \rho \mathfrak{m}_\infty^k \in \mathscr{P}_2(\mathsf{X}_\infty)$, then $P'_{n,k}$ satisfies the following properties:*

(i) For every $N < 0$ the functional $S_{N,\cdot}(\cdot)$ satisfies the contraction property:

$$\begin{aligned} S_{N,\mathfrak{m}_n^k}(P'_{n,k}(\mu)) &= \mathfrak{m}_n^k(\mathsf{X}_n)^{-\frac{1}{N}} S_{N,\bar{\mathfrak{m}}_n^k}(P'_{n,k}(\mu)) \\ &\le \mathfrak{m}_n^k(\mathsf{X}_n)^{-\frac{1}{N}} S_{N,\bar{\mathfrak{m}}_\infty^k}(\mu) = \left[\frac{\mathfrak{m}_n^k(\mathsf{X}_n)}{\mathfrak{m}_\infty^k(\mathsf{X}_\infty)}\right]^{-\frac{1}{N}} S_{N,\mathfrak{m}_\infty^k}(\mu). \end{aligned} \tag{6.4.2}$$

(ii) If the density ρ of μ is bounded, then the Wasserstein convergence holds:

$$W_2^2(\mu, P'_{n,k}(\mu)) \le \int \mathsf{d}^2(x,y)\tilde{\rho}(x)\, \mathrm{d}p_n^k(x,y) \to 0, \quad \text{as } n \to \infty,$$

where $\tilde{\rho} = \mathfrak{m}_\infty^k(\mathsf{X}_\infty)\, \rho$ is the density of μ with respect to the normalized measure $\bar{\mathfrak{m}}_\infty^k$.

Proof. Observe that the two equalities in (6.4.2) are obvious, then we just have to prove the inequality. Consequently (i) follows directly from Jensen's inequality applied to the convex function $\psi(r) := r^{1-\frac{1}{N}}$. Indeed,

$$\begin{aligned} S_{N,\bar{\mathfrak{m}}_n^k}(P'_{n,k}(\mu)) &= \int_{\mathsf{X}_n} \left[\int_{\mathsf{X}_\infty} \tilde{\rho}(x) P'_{n,k}(y)(\mathrm{d}x)\right]^{1-\frac{1}{N}} \mathrm{d}\bar{\mathfrak{m}}_n^k(y) \\ &\le \int_{\mathsf{X}_n}\int_{\mathsf{X}_\infty} \tilde{\rho}(x)^{1-\frac{1}{N}} P'_{n,k}(y)(\mathrm{d}x)\, \mathrm{d}\bar{\mathfrak{m}}_n^k(y) = S_{N,\bar{\mathfrak{m}}_\infty^k}(\mu). \end{aligned}$$

Regarding (ii), notice that, since ρ is bounded, the same holds for $\tilde{\rho}$. Moreover, we have that $\tilde{\rho}\, p_n^k \in \mathsf{Adm}(\mu, P'_{n,k}(\mu))$, and consequently

$$W_2^2(\mu, P'_{n,k}(\mu)) \le \int \mathsf{d}^2(x,y)\tilde{\rho}(x)\, \mathrm{d}p_n^k(x,y) \le ||\tilde{\rho}||_{L^\infty(\mathfrak{m}_\infty^k)} W_2^2(\bar{\mathfrak{m}}_\infty^k, \bar{\mathfrak{m}}_n^k) \to 0.$$

□

The last result we are going to prove in this subsection is useful to conclude tightness for a sequence of measures, provided that we have a uniform bound on their Rényi entropies, and a tightness condition on the reference measures. The analogous result stated for the relative entropy functional was proven in [GMS15, Proposition 4.1], and this proof can be easily adapted.

Lemma 6.4.6. *Let $\{\mathfrak{n}_n\}_{n\in\mathbb{N}}, \{\mu_n\}_{n\in\mathbb{N}} \subset \mathscr{P}(Z)$ be two sequences of measures such that $\{\mathfrak{n}_n\}_{n\in\mathbb{N}}$ is tight and $\sup_{n\in\mathbb{N}} S_{N,\mathfrak{n}_n}(\mu_n) < \infty$. Then $\{\mu_n\}_{n\in\mathbb{N}}$ is tight.*

Proof. First of all we observe that, being the entropy bounded, we can write $\mu_n = \rho_n \mathfrak{n}_n$. Thus, a direct application of Jensen's inequality gives that, for every $n \in \mathbb{N}$ and for every measurable set $E \subset Z$,

$$\frac{\mu_n(E)^{1-\frac{1}{N}}}{\mathfrak{n}_n(E)^{1-\frac{1}{N}}} \leq \frac{1}{\mathfrak{n}_n(E)} \int_E \rho_n^{1-\frac{1}{N}} d\mathfrak{n}_n \leq \frac{S_{N,\mathfrak{n}_n}(\mu_n)}{\mathfrak{n}_n(E)}.$$

The tightness of $\{\mathfrak{n}_n\}_{n\in\mathbb{N}}$ assures the existence of a sequence of compact sets $\{D_l\}_{l\in\mathbb{N}}$ such that $\sup_{n\in\mathbb{N}} \mathfrak{n}_n(Z\setminus D_l) \to 0$ as $l \to \infty$. We write $E_l = Z\setminus D_l$ and we conclude from the above inequality that $\{\mu_n\}_{n\in\mathbb{N}}$ is tight, since

$$\sup_{n\in\mathbb{N}} \mu_n(E_l)^{1-\frac{1}{N}} \leq \sup_{n\in\mathbb{N}} \mathfrak{n}_n(E_l)^{-\frac{1}{N}} \sup_{n\in\mathbb{N}} S_{N,\mathfrak{n}_n}(\mu_n) \overset{l\to\infty}{\to} 0.$$

□

This result can be applied to our extrinsic converging sequence $\{\mathbb{X}_n\}_{n\in\mathbb{N}\cup\{\infty\}} \subset \mathcal{M}_k^{qR}$, with a straightforward normalization argument by recalling that $\mathfrak{m}_n^k(\mathsf{X}_n)$ approaches $\mathfrak{m}_\infty^k(\mathsf{X}_\infty)$ as $n \to \infty$, for every suitable k.

Corollary 6.4.7. *Given a fixed $k \geq \bar{k}$, and $\{\mu_n\}_{n\in\mathbb{N}} \subset \mathscr{P}(Z)$ a sequence of probability measures, such that $\sup_{n\in\mathbb{N}} S_{N,\mathfrak{m}_n^k}(\mu_n) < \infty$, then $\{\mu_n\}_{n\in\mathbb{N}}$ is tight.*

Semicontinuity properties

We present semicontinuity properties of $S_{N,\cdot}(\cdot)$ and $T_{K,N}^{(t)}(\cdot|\cdot)$ conditioned to their domain of definition. We start by proving the lower semicontinuity of $S_{N,\cdot}(\cdot)$ that we anticipated in Section 6.3.1. This property is well known in classical frameworks, that is, for positive values of N and well-behaved reference measures. For example, lower semicontinuity for a big class of functionals, in which the Rényi entropy is included, was proved in [LV09] for locally compact spaces endowed with reference measures having uniformly bounded volume growth. Inspired by the techniques used in [LV09], we provide below a Legendre-type representation formula for the entropy to attain our result. With this aim, let us write

$$\mathscr{P}^{\mathcal{S}}(\mathsf{X}) := \{\mu \in \mathscr{P}_2(\mathsf{X}) \,:\, \mu(\mathcal{S}) = 0\}.$$

Proposition 6.4.8. *Let $(\mathsf{X}, \mathsf{d}, p)$ be a pointed Polish space and $\mathcal{S} \subset \mathsf{X}$ a closed subset with empty interior. Then the Rényi entropy functional $S_{N,\mathfrak{n}}(\nu)$ is a lower semicontinuous function of $(\mathfrak{n}, \nu) \in \mathscr{M}_{\mathcal{S}}(\mathsf{X}) \times \mathscr{P}^{\mathcal{S}}(\mathsf{X})$. Specifically, for sequences*

$$(\mathfrak{n}_n)_{n\in\mathbb{N}\cup\{\infty\}} \subset \mathscr{M}_{\mathcal{S}}(\mathsf{X}) \text{ and } (\nu_n)_{n\in\mathbb{N}\cup\{\infty\}} \subset \mathscr{P}^{\mathcal{S}}(\mathsf{X}),$$

such that $\mathfrak{n}_n \overset{*}{\rightharpoonup} \mathfrak{n}_\infty$ as quasi-Radon measures and $\nu_n \rightharpoonup \nu_\infty$, we have that,

$$S_{N,\mathfrak{n}_\infty}(\nu_\infty) \leq \liminf_{n\to\infty} S_{N,\mathfrak{n}_n}(\nu_n).$$

In particular, the conclusion remains valid under W_2-convergence in the second argument.

Proof. The semicontinuity of the Rényi entropy functional is verified by exhibiting $S_{N,\mathfrak{m}}$ as the supremum of a set of continuous functions on $\mathscr{M}_{\mathcal{S}}(\mathsf{X}) \times \mathscr{P}^{\mathcal{S}}(\mathsf{X})$ endowed with the corresponding product convergence. In particular we define $\mathcal{R}^k := B_{2^{k+1}}(p) \setminus \mathcal{N}_{2^{-(k+1)}}(\mathcal{S})$ and we show that, for every pair $(\mathfrak{m}, \mu) \in \mathscr{M}_{\mathcal{S}}(\mathsf{X}) \times \mathscr{P}^{\mathcal{S}}(\mathsf{X})$,

$$S_{N,\mathfrak{m}}(\mu) = \sup\left\{ \int F \,\mathrm{d}\mu \quad - \int f^*(F)\,\mathrm{d}\mathfrak{m} \,:\, F \in C_b(\mathsf{X}) \text{ supported in } \mathcal{R}^k, \text{ for some } k \in \mathbb{N}\right\}, \tag{6.4.3}$$

where f^* is the convex conjugate function of $f(x) = |x|^{1-\frac{1}{N}}$, for $x \in \mathbb{R}$. That is,

$$\begin{aligned} f^* : \mathbb{R} &\to [0,\infty) \\ y &\mapsto f^*(y) := \sup_{x\in\mathbb{R}}(y\,x - f(x)) = -\frac{1}{N}\left(\frac{N}{N-1}\right)^{1-N} |y|^{1-N}. \end{aligned}$$

Proving (6.4.3) will be enough to deduce the lower semicontinuity of the functional $S_{N,\cdot}(\cdot)$, in fact, for every $F \in C_b(\mathsf{X})$ supported in some $\mathcal{R}^k$, the functional

$$(\mu, \mathfrak{m}) \mapsto \int F \,\mathrm{d}\mu - \int f^*(F)\,\mathrm{d}\mathfrak{m}$$

is indeed (weakly) continuous in $\mathscr{M}_{\mathcal{S}}(\mathsf{X}) \times \mathscr{P}^{\mathcal{S}}(\mathsf{X})$, since $f^*(F) \in C_{bs}(\mathsf{X}) \cap C_{\mathcal{S}}(\mathsf{X})$. As a result, the functional $S_{N,\cdot}(\cdot)$ will be the supremum of (weakly) continuous functionals, and thus it will be (weakly) lower semicontinuous. For simplicity, let us denote by $\tilde{\mathtt{S}}_{N,\mathfrak{m}}(\mu)$ the expression on the right-hand side of (6.4.3).

We first verify that $S_{N,\mathfrak{m}}(\mu) \geq \tilde{\mathtt{S}}_{N,\mathfrak{m}}(\mu)$, for every $(\mathfrak{m},\mu) \in \mathscr{M}_S(\mathsf{X}) \times \mathscr{P}^{\mathcal{S}}(\mathsf{X})$. We assume that $\mu = \rho\mathfrak{m}$ since the aforementioned inequality is trivially satisfied when $\mu \not\ll \mathfrak{m}$. We get the desired result, integrating the expression $f(z) \geq z\,y^* - f^*(y^*)$ with respect to $\mathfrak{m}$ which, by definition of f^*, holds for any $z, y^* \in \mathbb{R}$, after replacing $z = \rho(x)$ and $y^* = F(x)$, for $F \in C_b(\mathsf{X})$ with support inside some $\mathcal{R}^k$.

Before proceeding with the converse inequality let us point out that $\tilde{\mathtt{S}}_{N,\mathfrak{m}}(\mu) = \infty$ granted $\mu \not\ll \mathfrak{m}$. Indeed, in this case, there exists a Borel set $A \subset \mathsf{X}$ with $\mathfrak{m}(A) = 0$ and $\mu(A) > 0$. At this point, recall that every Borel finite measure in a Polish spaces is inner regular with respect to compact sets and outer regular with respect to open sets (see for instance [Bog07, Theorem 7.1.7]). For this reason, since $\mu \in \mathscr{P}^{\mathcal{S}}(\mathsf{X})$ and A is a compact set, it is possible to assume that $A \cap S = \emptyset$. Observe also that compactness grants the existence of $k \in \mathbb{N}$ for which $A \subset \mathcal{R}^k$. Since $\mathfrak{m}$ and μ restricted to $\mathcal{R}^{k+1}$ are finite measures, there exist a sequence of compact sets $(K_n)_{n\in\mathbb{N}}$ and one of open sets $(A_n)_{n\in\mathbb{N}}$ such that $K_n \subset A \subset A_n \subset \mathcal{R}^{k+1}$ and $(\mu + \mathfrak{m})(A_n \setminus K_n) < 1/n$, for any $n \in \mathbb{N}$. Then for $M > 0$ Tietze's Theorem ensures the existence of a sequence of approximating functions $\left(F_n^M\right)_{n\in\mathbb{N}} \subset C_b(X)$ satisfying: $0 \leq F_n^M \leq M$, $F_n^M = M$ on K_n, and $F_n^M = 0$ on $X \setminus A_n$. Therefore, as $n \to \infty$, the functions F_n^M converge, in $L^1(\mu + \mathfrak{m})$, to the scaled characteristic function $M \cdot \mathbb{1}_A$. On the other hand, since for every n, F_n^M is an admissible function for the supremum in $\tilde{\mathtt{S}}_{N,\mathfrak{m}}$, we have that

$$\int F_n^M \,\mathrm{d}\mu - \int f^*(F_n^M)\,\mathrm{d}\mathfrak{m} \leq \tilde{\mathtt{S}}_{N,\mathfrak{m}}(\mu), \tag{6.4.4}$$

for any $n \in \mathbb{N}$. Passing now to the limit as n goes to infinity in (6.4.4), we obtain that $M \cdot \mu(A) \leq \tilde{\mathtt{S}}_{N,\mathfrak{m}}(\mu)$. The arbitrariness of M implies then that $\tilde{\mathtt{S}}_{N,\mathfrak{m}}(\mu) = \infty$.

We proceed now to prove that $S_{N,\mathfrak{m}}(\mu) \leq \tilde{\mathtt{S}}_{N,\mathfrak{m}}(\mu)$ and we will assume that $\tilde{\mathtt{S}}_{N,\mathfrak{m}}(\mu) < +\infty$, since otherwise there is nothing to prove. The paragraph above enables us to write $\mu = \rho\mathfrak{m}$. We then have the following expression for $\tilde{\mathtt{S}}_{N,\mathfrak{m}}(\mu)$:

$$\tilde{\mathtt{S}}_{N,\mathfrak{m}}(\mu) = \sup\left\{\int F\rho - f^*(F)\,\mathrm{d}\mathfrak{m} \,:\, F \in C_b(\mathsf{X}) \text{ supported in } \mathcal{R}^k, \text{ for some } k \in \mathbb{N}\right\}.$$

And, recalling that $f(x) = (f^*)^*(x) = \sup_{y\in\mathbb{R}}\{x\,y - f^*(y)\}$ because f is finite, convex and continuous, it follows that

$$S_{N,\mathfrak{m}}(\mu) = \int \sup_{s^*\in\mathbb{Q}} \{\rho(x)s^* - f^*(s^*)\}\,\mathrm{d}\mathfrak{m}(x).$$

By fixing $\mathbb{Q} = \{q_n\}_{n\in\mathbb{N}}$, an enumeration of rational numbers with $q_0 = 0$, we introduce the family of approximating functionals

$$\left\{ S^h_{N,\mathfrak{m}}(\mu) := \int \sup_{s^*\in\{q_0,\dots,q_h\}} \{\rho(x)s^* - f^*(s^*)\}\,\mathrm{d}\mathfrak{m}(x) \right\}_{h\in\mathbb{N}}.$$

Observe that the integrands are monotone increasing in h and that $0 = \rho(x)q_0 - f^*(q_0)$. In particular, Beppo Levi's Theorem ensures that $S^h_{N,\mathfrak{m}}(\mu) \to S_{N,\mathfrak{m}}(\mu)$, as $h \to \infty$. Therefore, it suffices to show that $S^h_{N,\mathfrak{m}} \leq \tilde{\mathsf{S}}_{N,\mathfrak{m}}$, for any fixed $h \in \mathbb{N}$. To this aim, one confirms directly that

$$S^h_{N,\mathfrak{m}}(\mu) = \sup\left\{ \int (\rho F - f^*(F))\,\mathrm{d}\mathfrak{m} \,:\, F \text{ is a step function with values in } \{q_0,\dots,q_h\} \right\}.$$

Note that the fact that S is an $\mathfrak{m}$-null set guarantees that we can further require that the aforementioned functions are supported in $\mathcal{R}^k$ for some $k \in \mathbb{N}$ without modifying the supremum, as an approximation argument using the Monotone Convergence Theorem shows. Finally, since $\mathfrak{m}$ and μ are finite measures when restricted to $\mathcal{R}^{k+1}$, every step function with support in $\mathcal{R}^k$ can be obtained as the $L^1(\mu+\mathfrak{m})$-limit of continuous and uniformly bounded functions implying that $S^h_{N,\mathfrak{m}} \leq \tilde{\mathsf{S}}_{N,\mathfrak{m}}$, which concludes the proof. □

Applying this proposition to our extrinsic converging sequence $\{\mathbb{X}_n\}_{n\in\mathbb{N}\cup\{\infty\}} \subset \mathcal{M}^{qR}_{\bar{k}}$, we can extract a useful corollary with a couple of observations. From (6.4.1) we easily deduce that, for any $k \geq \bar{k}$, the sequence $(\mathfrak{m}^k_n)_{n\in\mathbb{N}}$ converges to $\mathfrak{m}^k_\infty \in \mathscr{M}(Z) \subset \mathscr{M}_\emptyset(Z)$ in the weak convergence. Moreover, Remark 6.2.13 states that, granted we restrict ourselves to the set $\mathscr{M}(Z)$, then weak convergence in the sense of quasi-Radon measures coincides with the usual weak one.

Corollary 6.4.9. *Given a fixed $k \in \mathbb{N}$ and $\{\mu_n\}_{n\in\mathbb{N}} \subset \mathscr{P}_2(Z)$ a sequence converging weakly to $\mu \in \mathscr{P}_2(Z)$, it holds that*

$$S_{N,\mathfrak{m}^k_\infty}(\mu) \leq \liminf_{n\to\infty} S_{N,\mathfrak{m}^k_n}(\mu_n).$$

We conclude the section with a corresponding continuity result for the functional $T^{(t)}_{K,N}$. We stress that although, it would be sufficient for the proof of Theorem 6.4.1 to have the upper semicontinuity of $T^{(t)}_{K,N}$, we prefer to present a more general statement.

Proposition 6.4.10. *Let $K \geq 0$ and $N < 0$ and $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be a metric measure space. Furthermore, set $\mu_0 = \rho_0\mathfrak{m}, \mu_1 = \rho_1\mathfrak{m} \in \mathscr{P}(\mathsf{X})$ to be absolutely continuous with respect to the quasi-Radon reference measure $\mathfrak{m}$, with $S_{N,\mathfrak{m}}(\mu_0), S_{N,\mathfrak{m}}(\mu_1) < \infty$. Consider a sequence $(\pi_n)_{n\in\mathbb{N}} \subset \mathscr{P}(\mathsf{X}\times\mathsf{X})$, weakly converging to $\pi \in \mathscr{P}(\mathsf{X}\times\mathsf{X})$ and such that*

$$(\mathsf{p}_1)_\#\pi_n = \mu_0 \quad \textit{and} \quad (\mathsf{p}_2)_\#\pi_n = \mu_1 \qquad \textit{for every } n \in \mathbb{N}.$$

Then, for $t \in [0,1]$, it holds that

$$\lim_{n\to\infty} T^{(t)}_{K,N}(\pi_n|\mathfrak{m}) = T^{(t)}_{K,N}(\pi|\mathfrak{m}).$$

Additionally, the conclusion remains valid for $K < 0$, granted $\operatorname{diam}(\mathsf{X}) < \pi\sqrt{\frac{N-1}{K}}$.

Proof. Let us fix any $t \in (0,1)$, since the statement is clearly true for the remaining values. We want to prove that

$$\lim_{n\to\infty} \int_{X\times X} \tau^{(1-t)}_{K,N}(\mathsf{d}(x,y))\rho_0(x)^{-\frac{1}{N}}\,\mathrm{d}\pi_n(x,y) = \int_{X\times X} \tau^{(1-t)}_{K,N}(\mathsf{d}(x,y))\rho_0(x)^{-\frac{1}{N}}\,\mathrm{d}\pi(x,y), \quad (6.4.5)$$

since the other term of $T^{(t)}_{K,N}(\cdot|\mathfrak{m})$ can be treated analogously. Notice that being $S_{N,\mathfrak{m}}(\mu_0)<\infty$, $\rho_0(x)^{-1/N}\in L^1(\mu_0)$ thus by the density of $C_b(\mathsf{X})\cap L^1(\mathfrak{m})$ in $L^1(\mathfrak{m})$, for every fixed $\varepsilon>0$ there exists $f^\varepsilon\in C_b(\mathsf{X})$ such that $\left\|\rho_0^{-1/N}-f^\varepsilon\right\|_{L^1(\mu_0)}<\varepsilon$. Moreover, notice that the coefficients $\tau^{(1-t)}_{K,N}(\cdot)$ are bounded and continuous. Indeed, this is always the case for $K\geq 0$, and since $\operatorname{diam}(X)<\pi\sqrt{\frac{N-1}{K}}$ is bounded by our assumptions, this holds as well for $K<0$. Therefore,

$$\tau^{(1-t)}_{K,N}(\mathsf{d}(x,y))f^\varepsilon(x)$$

is itself bounded and continuous. Consequently, the weak convergence $(\pi_n)_n\rightharpoonup\pi$ shows that,

$$\lim_{n\to\infty}\int_{X\times X}\tau^{(1-t)}_{K,N}(\mathsf{d}(x,y))f^\varepsilon(x)\,\mathrm{d}\pi_n(x,y)=\int_{X\times X}\tau^{(1-t)}_{K,N}(\mathsf{d}(x,y))f^\varepsilon(x)\,\mathrm{d}\pi(x,y).$$

Furthermore, the boundedness of $\tau^{(1-t)}_{K,N}$ allows to deduce the following estimate

$$\begin{aligned}\limsup_{n\to\infty}\int_{X\times X}&\tau^{(1-t)}_{K,N}(\mathsf{d}(x,y))\rho_0(x)^{-\frac{1}{N}}\,\mathrm{d}\pi_n(x,y)\\&\leq\lim_{n\to\infty}\int_{X\times X}\tau^{(1-t)}_{K,N}(\mathsf{d}(x,y))f^\varepsilon(x)\,\mathrm{d}\pi_n(x,y)+\varepsilon\left\|\tau^{(1-t)}_{K,N}\right\|_{L^\infty}\\&=\int_{X\times X}\tau^{(1-t)}_{K,N}(\mathsf{d}(x,y))f^\varepsilon(x)\,\mathrm{d}\pi(x,y)+\varepsilon\left\|\tau^{(1-t)}_{K,N}\right\|_{L^\infty}\\&\leq\int_{X\times X}\tau^{(1-t)}_{K,N}(\mathsf{d}(x,y))\rho_0(x)^{-\frac{1}{N}}\,\mathrm{d}\pi(x,y)+2\varepsilon\left\|\tau^{(1-t)}_{K,N}\right\|_{L^\infty}.\end{aligned}$$

Analogously, it follows that

$$\liminf_{n\to\infty}\int_{X\times X}\tau^{(1-t)}_{K,N}(\mathsf{d}(x,y))\rho_0(x)^{-\frac{1}{N}}\,\mathrm{d}\pi_n(x,y)\geq\int_{X\times X}\tau^{(1-t)}_{K,N}(\mathsf{d}(x,y))\rho_0(x)^{-\frac{1}{N}}\,\mathrm{d}\pi(x,y)-2\varepsilon\left\|\tau^{(1-t)}_{K,N}\right\|_{L^\infty},$$

and since $\varepsilon>0$ can be chosen arbitrarily, equation (6.4.5) holds true. We conclude by recalling the arbitrariness of t.

□

6.4.2 Proof of the Approximate CD Condition

The objective of this section is to prove the next partial result

Theorem 6.4.11. *Let $K\in\mathbb{R}$, $N\in(-\infty,0)$, and $\{(\mathsf{X}_n,\mathsf{d}_n,\mathfrak{m}_n,\mathcal{S}_{\mathfrak{m}_n},p_n)\}_{n\in\mathbb{N}\cup\{\infty\}}\subset\mathcal{M}^{qR}_{\bar{k}}$ be a sequence of metric measure spaces satisfying the assumptions of the Stability Theorem 6.4.1, for some $\bar{k}\in\mathbb{N}$. Then $(\mathsf{X}_\infty,\mathsf{d}_\infty,\mathfrak{m}_\infty)$ is a $\mathsf{CD}^a(K,N)$ space.*

As discussed in the previous section we can prove this result by referring directly to a realization in a metric space. We recall the discussion before equation (6.4.1) and we remark that in the following we will directly work in the realization space (Z,d) with the embedded versions of our converging pointed metric measure spaces.

We follow the plan explained at the beginning of the section and argue, in steps, using vertical approximations. Specifically, Steps 1 and 2 serve the purpose of constructing useful approximations of the marginals μ_0 and μ_1. Next, we follow the argumentation of Sturm in [Stu06b] in Steps 3 to 6, to exhibit the upper semicontinuity of $T^{(t)}_{K,N}$ along a sequence of optimal couplings, provided by the curvature-dimension assumption. Additionally, we demonstrate the existence of a favorable limiting optimal coupling. Step 7 focuses on proving the convergence of inner points of a vertical sequence of Wasserstein geodesics, as well as, the lower semicontinuity of the Rényi entropy along this sequence.

Let us fix first the notation. Set $k \geq \bar{k}$ and assume that $\mu_0, \mu_1 \in \mathscr{P}^{ac}(\mathsf{X}_\infty, \mathfrak{m}_\infty)$ have supports satisfying $\mathrm{supp}(\mu_0)$, $\mathrm{supp}(\mu_1) \subset \mathcal{R}_\infty^{k-1}$. We denote by ρ_i the density of μ_i with respect to $\mathfrak{m}_\infty$, for $i = \{0, 1\}$. Define the set,

$$I := \{N' \in [N, 0) \, : \, S_{N',\mathfrak{m}_\infty}(\mu_0), S_{N',\mathfrak{m}_\infty}(\mu_1) < \infty\},$$

and observe that I is an interval, as a consequence of Jensen's inequality. Then, surely, we are able to assume that (and set)

$$(\, M := \,) \, \max \{S_{N,\mathfrak{m}_\infty}(\mu_0), \, S_{N,\mathfrak{m}_\infty}(\mu_1)\} < \infty,$$

and that, for every $q \in \mathsf{Opt}(\mu_0, \mu_1)$,

$$T^{(t)}_{K,N'}(q|\mathfrak{m}_\infty) < \infty, \tag{6.4.6}$$

since the CD-condition is trivial in failure of any of these inequalities. Therefore, the arbitrariness of k and initial measures shows that, in order to demonstrate Theorem 6.4.11, we are required to validate the CD-inequality (6.3.2), for every $N' \in I$, and every $t \in [0, 1]$.

Additionally, we fix as before an optimal coupling $p_n \in \mathsf{Opt}\left(\bar{\mathfrak{m}}^k_\infty, \bar{\mathfrak{m}}^k_n\right)$ between the normalized k-cuts of the reference measures, for every $n \in \mathbb{N}$. And, as defined in Section 6.4.1, we consider $\{P_n(x)\}_{x \in \mathsf{X}_\infty} \subset \mathscr{P}(\mathsf{X}_n)$ and $\{P'_n(y)\}_{y \in X_n} \subset \mathscr{P}(\mathsf{X}_\infty)$ the disintegration kernels of p_n with respect to the projections $\mathtt{p}_1$ and $\mathtt{p}_2$ respectively, and consider the map $P'_n \colon \mathscr{P}^{ac}(\mathsf{X}_\infty, \mathfrak{m}^k_\infty) \to \mathscr{P}^{ac}(\mathsf{X}_n, \mathfrak{m}^k_n)$. Note that, in contrast to Section 6.4.1, here we have omitted the dependence on the number k, since it is fixed for now.

STEP 1: Horizontal approximation with bounded densities

During the argument it proves useful to work with bounded-density measures. Therefore we construct here a horizontal approximation of μ_0 and μ_1, for which its elements enjoy this property. For the construction, we fix an arbitrary optimal coupling $\tilde{q} \in \mathsf{Opt}(\mu_0, \mu_1)$ and define, for every $r > 0$,

$$E_r := \{(x_0, x_1) \in X_\infty \times X_\infty \, : \, \rho_0(x_0) < r, \rho_1(x_1) < r\}$$

and consequently, for sufficiently large r,

$$\tilde{q}^{(r)} := \alpha_r^{-1} \tilde{q}(\, . \cap E_r),$$

where $\alpha_r := \tilde{q}(E_r)$. The measure $\tilde{q}^{(r)} \in \mathscr{P}(\mathsf{X}_\infty \times \mathsf{X}_\infty)$ has marginals given by

$$\mu_0^{(r)} := (\mathtt{p}_1)_\sharp \, \tilde{q}^{(r)} \quad \text{ and } \quad \mu_1^{(r)} := (\mathtt{p}_2)_\sharp \, \tilde{q}^{(r)}.$$

Notice that both $\mu_0^{(r)}$ and $\mu_1^{(r)}$ have bounded densities and that $\mu_i^{(r)}$ converges to μ_i in $(\mathscr{P}^2(\mathsf{X}_\infty), W_2)$, for $i = 0, 1$. Moreover, notice that $S_{N,\mathfrak{m}_\infty}(\mu_i^{(r)}) \to S_{N,\mathfrak{m}_\infty}(\mu_i)$ as $r \to \infty$, for $i = 0, 1$. Then we fix $\varepsilon > 0$ and find $r = r(\varepsilon)$ such that $\alpha_r \geq 1 - \varepsilon$ and that the following estimates hold:

$$\max_{i \in \{0,1\}} W_2(\mu_i, \mu_i^{(r)}) \leq \varepsilon \quad \text{ and } \quad \max_{i \in \{0,1\}} S_{N,\mathfrak{m}^k_\infty}(\mu_i^{(r)}) = \max_{i \in \{0,1\}} S_{N,\mathfrak{m}_\infty}(\mu_i^{(r)}) \leq M + \frac{1}{2}. \tag{6.4.7}$$

We point out that the parameter r depends on ε, but we won't be explicit on this dependence for the sake of the presentation.

STEP 2: Vertical approximation

Once we have identified the horizontal approximations $\mu_0^{(r)}$ and $\mu_1^{(r)}$, we may proceed to their vertical approximation. First of all, observe that $\mu_0^{(r)}$ and $\mu_1^{(r)}$ are absolutely continuous with respect to the normalized reference measure $\bar{\mathfrak{m}}_\infty^k$, so we denote by $\tilde{\rho}_0^{(r)}$ and $\tilde{\rho}_1^{(r)}$ their bounded densities. Then, for every $n \in \mathbb{N}$, we define $\mu_{0,n}$, $\mu_{1,n} \in \mathscr{P}_2(\mathsf{X}_n, \mathsf{d}_n, \mathfrak{m}_n^k)$ as

$$\mu_{i,n} := P_n'(\mu_i^{(r)}) = \rho_{i,n}\bar{\mathfrak{m}}_n^k,$$

where $\rho_{i,n}(y) = \int \tilde{\rho}_i^{(r)}(x) P_n'(y)(\mathrm{d}x)$. Notice that $\mu_{0,n}$ and $\mu_{1,n}$ depend on r (and ultimately on ε), but, once again, we prefer not to make this dependence explicit, in order to maintain an easy notation in the following. Anyway, we invite the reader to keep in mind that every object we are going to define depends only on ε. Now, since $\mathfrak{m}_n^k(\mathsf{X}_n) \to \mathfrak{m}_\infty^k(\mathsf{X}_\infty)$, observe that Lemma 6.4.5 guarantees the existence of an $\bar{n} \in \mathbb{N}$, such that if $n \geq \bar{n}$ it holds that

$$\max_{i\in\{0,1\}} W_2(\mu_i^{(r)}, \mu_{i,n}) \leq \varepsilon.$$

and that

$$\max_{i\in\{0,1\}} S_{N,\mathfrak{m}_n}(\mu_{i,n}) \leq \max_{i\in\{0,1\}} S_{N,\mathfrak{m}_n^k}(\mu_{i,n}) \leq M+1. \tag{6.4.8}$$

Moreover, according to Lemma 6.4.5, for every $N' \in I$, it holds that

$$S_{N',\mathfrak{m}_n}(\mu_{i,n}) \leq S_{N',\mathfrak{m}_n^k}(\mu_{i,n}) \leq \left[\frac{\mathfrak{m}_n^k(\mathsf{X}_n)}{\mathfrak{m}_\infty^k(\mathsf{X}_\infty)}\right]^{-\frac{1}{N'}} S_{N',\mathfrak{m}_\infty^k}(\mu_i^{(r)}) < \infty.$$

Therefore, for every $n \in \mathbb{N}$ large enough, since $(\mathsf{X}_n, \mathsf{d}_n, \mathfrak{m}_n)$ is a $\mathsf{CD}(K,N)$ space, there exist an optimal plan $\pi_n \in \mathsf{Opt}(\mu_{0,n}, \mu_{1,n})$ and a 2-Wasserstein geodesic $(\mu_{t,n})_{t\in[0,1]} \subset \mathscr{P}_2(\mathsf{X}_n)$ connecting $\mu_{0,n}$ and $\mu_{1,n}$, for which,

$$S_{N',\mathfrak{m}_n}(\mu_{t,n}) \leq T_{K,N'}^{(t)}(\pi_n|\mathfrak{m}_n) \tag{6.4.9}$$

holds, for every $t \in [0,1]$ and every $N' \in I$. Note that Remark 6.3.4 together with the assumption that $\sup_{i\in\mathbb{N}\cup\{\infty\}} \operatorname{diam}(\mathsf{X}_i, \mathsf{d}_i) < \pi\sqrt{\frac{1}{|K|}}$, if $K < 0$, assures that the geodesic $\mu_{t,n}$ is absolutely continuous with respect to $\mathfrak{m}_n$.

STEP 3: Estimate for $T_{K,N'}^{(t)}$

In this step we start the proof of the upper semicontinuity of the functional $T_{K,N'}^{(t)}$. In particular, we fix $N' \in [N, 0)$ and a time $t \in [0,1]$ and we call Q_n and Q_n' be the disintegrations of π_n with respect to $\mu_{0,n}$ and $\mu_{1,n}$ respectively. Then we define the following two functions

$$v_0(y_0) = \int_{\mathsf{X}_n} \tau_{K,N'}^{(1-t)}(\mathsf{d}(y_0,y_1)) Q_n(y_0, dy_1)$$

and

$$v_1(y_1) = \int_{\mathsf{X}_n} \tau_{K,N'}^{(t)}(\mathsf{d}(y_0,y_1)) Q_n'(y_1, dy_0).$$

A direct application of Jensen's theorem leads to

$$
\begin{aligned}
T^{(t)}_{K,N'}(\pi_n|\bar{\mathfrak{m}}^k_n) &= \sum_{i=0}^{1}\int_{\mathsf{X}_n} \rho_{i,n}(y_i)^{1-1/N'}\cdot v_i(y_i)\,\mathrm{d}\bar{\mathfrak{m}}^k_n(y_i)\\
&= \sum_{i=0}^{1}\int_{\mathsf{X}_n}\left[\int_{\mathsf{X}_\infty}\tilde{\rho}^{(r)}_i(x_i)P'_n(y_i,\mathrm{d}x_i)\right]^{1-1/N'}\cdot v_i(y_i)\,\mathrm{d}\bar{\mathfrak{m}}^k_n(y_i)\\
&\leq \sum_{i=0}^{1}\int_{\mathsf{X}_n}\int_{\mathsf{X}_\infty}\tilde{\rho}^{(r)}_i(x_i)^{1-1/N'}P'_n(y_i,\mathrm{d}x_i)\cdot v_i(y_i)\,\mathrm{d}\bar{\mathfrak{m}}^k_n(y_i)\\
&= \sum_{i=0}^{1}\int_{\mathsf{X}_\infty}\tilde{\rho}^{(r)}_i(x_i)^{1-1/N'}\left[\int_{\mathsf{X}_n} v_i(y_i)P_n(x_i,\mathrm{d}y_i)\right]\mathrm{d}\bar{\mathfrak{m}}^k_\infty(x_i).
\end{aligned}
$$

At this point we see that

$$
\begin{aligned}
\int_{\mathsf{X}_n} v_0(y_0)P_n(x_0,\mathrm{d}y_0) &= \int_{\mathsf{X}_n\times\mathsf{X}_n}\tau^{(1-t)}_{K,N'}(\mathsf{d}(y_0,y_1))Q_n(y_0,\mathrm{d}y_1)P_n(x_0,\mathrm{d}y_0)\\
&= \int_{\mathsf{X}_n\times\mathsf{X}_n\times\mathsf{X}_\infty}\tau^{(1-t)}_{K,N'}(\mathsf{d}(y_0,y_1))\frac{\tilde{\rho}^{(r)}_1(x_1)}{\rho_{1,n}(y_1)}P'_n(y_1,\mathrm{d}x_1)Q_n(y_0,\mathrm{d}y_1)P_n(x_0,\mathrm{d}y_0)\\
&\leq \int_{\mathsf{X}_n\times\mathsf{X}_n\times\mathsf{X}_\infty}\left[\tau^{(1-t)}_{K,N'}(\mathsf{d}(x_0,x_1)) + C\cdot|\mathsf{d}(y_0,y_1)-\mathsf{d}(x_0,x_1)|\right]\\
&\qquad\qquad \frac{\tilde{\rho}^{(r)}_1(x_1)}{\rho_{1,n}(y_1)}P'_n(y_1,\mathrm{d}x_1)Q_n(y_0,\mathrm{d}y_1)P_n(x_0,\mathrm{d}y_0)\\
&\leq \int_{\mathsf{X}_n\times\mathsf{X}_n\times\mathsf{X}_\infty}\left[\tau^{(1-t)}_{K,N'}(\mathsf{d}(x_0,x_1)) + C\cdot(\mathsf{d}(x_0,y_0)+\mathsf{d}(x_1,y_1))\right]\\
&\qquad\qquad \frac{\tilde{\rho}^{(r)}_1(x_1)}{\rho_{1,n}(y_1)}P'_n(y_1,\mathrm{d}x_1)Q_n(y_0,\mathrm{d}y_1)P_n(x_0,\mathrm{d}y_0),
\end{aligned}
$$

and analogously that

$$
\begin{aligned}
\int_{\mathsf{X}_n} v_1(y_1)P_n(x_1,\mathrm{d}y_1) \leq \int_{\mathsf{X}_n\times\mathsf{X}_n\times\mathsf{X}_\infty}&\left[\tau^{(t)}_{K,N'}(\mathsf{d}(x_0,x_1)) + C\cdot(\mathsf{d}(x_0,y_0)+\mathsf{d}(x_1,y_1))\right]\\
&\qquad \frac{\tilde{\rho}^{(r)}_0(x_0)}{\rho_{0,n}(y_0)}P'_n(y_0,\mathrm{d}x_0)Q'_n(y_1,\mathrm{d}y_0)P_n(x_1,\mathrm{d}y_1),
\end{aligned}
$$

where $C := \max_{\theta\in[0,\Theta],s\in[0,1]}\frac{\partial}{\partial\theta}\tau^{(s)}_{K,N'}(\theta)$ and Θ is the maximum between $\mathsf{d}(x_0,x_1)$ and $\mathsf{d}(y_0,y_1)$. Observe that the constant C is indeed finite because we know that $\sup_{n\in\mathbb{N}\cup\{\infty\}}\mathrm{diam}(\mathsf{X}_n,\mathsf{d}_n) < \pi\sqrt{\frac{1}{|K|}}$, if $K<0$, from assumption (iii) in Theorem 6.4.1.
Moreover we notice that

$$
\begin{aligned}
\int_{\mathsf{X}_\infty}\tilde{\rho}^{(r)}_0(x_0)^{1-1/N'}\int_{\mathsf{X}_n\times\mathsf{X}_n\times\mathsf{X}_\infty}&\mathsf{d}(x_0,y_0)\frac{\tilde{\rho}^{(r)}_1(x_1)}{\rho_{1,n}(y_1)}P'_n(y_1,\mathrm{d}x_1)Q_n(y_0,\mathrm{d}y_1)P_n(x_0,\mathrm{d}y_0)\,\mathrm{d}\bar{\mathfrak{m}}^k_\infty(x_0)\\
&= \int_{\mathsf{X}_\infty}\tilde{\rho}^{(r)}_0(x_0)^{1-1/N'}\int_{\mathsf{X}_n}\mathsf{d}(x_0,y_0)P_n(x_0,\mathrm{d}y_0)\,\mathrm{d}\bar{\mathfrak{m}}^k_\infty(x_0)\\
&\leq r^{1-1/N'}\int_{\mathsf{X}_n\times\mathsf{X}_\infty}\mathsf{d}(x_0,y_0)\,\mathrm{d}p_n(x_0,y_0) \leq r^{1-1/N'}W_2(\bar{\mathfrak{m}}^k_n,\bar{\mathfrak{m}}^k_\infty)
\end{aligned}
\tag{6.4.10}
$$

and

$$\begin{aligned}
\int_{\mathsf{X}_\infty} \tilde\rho_0^{(r)}(x_0)^{1-1/N'} &\int_{\mathsf{X}_n\times\mathsf{X}_n\times\mathsf{X}_\infty} \mathsf{d}(x_1,y_1)\frac{\tilde\rho_1^{(r)}(x_1)}{\rho_{1,n}(y_1)}P_n'(y_1,\mathrm{d}x_1)Q_n(y_0,\mathrm{d}y_1)P_n(x_0,\mathrm{d}y_0)\,\mathrm{d}\bar{\mathfrak{m}}_\infty^k(x_0)\\
&\le r^{-1/N'}\int_{\mathsf{X}_n\times\mathsf{X}_n\times\mathsf{X}_\infty\times\mathsf{X}_\infty} \mathsf{d}(x_1,y_1)\frac{\tilde\rho_1^{(r)}(x_1)}{\rho_{1,n}(y_1)}P_n'(y_1,\mathrm{d}x_1)\\
&\qquad\qquad Q_n(y_0,\mathrm{d}y_1)P_n(x_0,\mathrm{d}y_0)\tilde\rho_0^{(r)}(x_0)\,\mathrm{d}\bar{\mathfrak{m}}_\infty^k(x_0)\\
&= r^{-1/N'}\int_{\mathsf{X}_n\times\mathsf{X}_n\times\mathsf{X}_\infty} \mathsf{d}(x_1,y_1)\frac{\tilde\rho_1^{(r)}(x_1)}{\rho_{1,n}(y_1)}P_n'(y_1,\mathrm{d}x_1)Q_n(y_0,\mathrm{d}y_1)\mu_{0,n}(\,\mathrm{d}y_0)\\
&= r^{-1/N'}\int_{\mathsf{X}_n\times\mathsf{X}_\infty} \mathsf{d}(x_1,y_1)\tilde\rho_1^{(r)}(x_1)P_n'(y_1,\mathrm{d}x_1)\,\mathrm{d}\bar{\mathfrak{m}}_n^k(y_1)\\
&\le r^{1-1/N'}\int_{\mathsf{X}_n\times\mathsf{X}_\infty} \mathsf{d}(x_1,y_1)\,\mathrm{d}p_n(x_1,y_1) \le r^{1-1/N'}W_2(\bar{\mathfrak{m}}_n^k,\bar{\mathfrak{m}}_\infty^k),
\end{aligned} \tag{6.4.11}$$

where the last inequality in both chains follows by the Jensen's inequality. Consequently, for every $n\in\mathbb{N}$, we define a – not necessarily optimal – coupling $\bar q_n^{(r)}\in\mathsf{Adm}(\mu_0^{(r)},\mu_1^{(r)})$ by imposing that

$$\begin{aligned}
\mathrm{d}\bar q_n^{(r)}(x_0,x_1) &= \int_{\mathsf{X}_n\times\mathsf{X}_n}\frac{\tilde\rho_0^{(r)}(x_0)\tilde\rho_1^{(r)}(x_1)}{\rho_{0,n}(y_0)\rho_{1,n}(y_1)}P_n'(y_1,\mathrm{d}x_1)P_n'(y_0,\mathrm{d}x_0)\,\mathrm{d}\pi_n(y_0,y_1)\\
&= \int_{\mathsf{X}_n\times\mathsf{X}_n}\frac{\tilde\rho_0^{(r)}(x_0)\tilde\rho_1^{(r)}(x_1)}{\rho_{1,n}(y_1)}P_n'(y_1,\mathrm{d}x_1)Q_n(y_0,\mathrm{d}y_1)P_n(x_0,\mathrm{d}y_0)\,\mathrm{d}\bar{\mathfrak{m}}_\infty^k(x_0)\\
&= \int_{\mathsf{X}_n\times\mathsf{X}_n}\frac{\tilde\rho_0^{(r)}(x_0)\tilde\rho_1^{(r)}(x_1)}{\rho_{0,n}(y_0)}P_n'(y_0,\mathrm{d}x_0)Q_n'(y_1,\mathrm{d}y_0)P_n(x_1,\mathrm{d}y_1)\,\mathrm{d}\bar{\mathfrak{m}}_\infty^k(x_1).
\end{aligned}$$

With this definition of $\bar q_n^{(r)}$ and keeping in mind (6.4.10) and (6.4.11), we end up with

$$T_{K,N'}^{(t)}(\pi_n|\bar{\mathfrak{m}}_n^k)\le T_{K,N'}^{(t)}(\bar q_n^{(r)}|\bar{\mathfrak{m}}_\infty^k)+4Cr^{1-1/N'}W_2(\bar{\mathfrak{m}}_\infty^k,\bar{\mathfrak{m}}_n^k).$$

Now, up to taking a greater $\bar n$, we can require that for every $n\ge\bar n$ it holds that

$$W_2(\bar{\mathfrak{m}}_\infty^k,\bar{\mathfrak{m}}_n^k)\le\frac{\varepsilon}{4Cr^{\frac{N'-1}{N'}}},$$

for every $N'\in[N,\varepsilon)$. As a consequence we obtain that

$$T_{K,N'}^{(t)}(\pi_n|\bar{\mathfrak{m}}_n^k)\le T_{K,N'}^{(t)}(\bar q_n^{(r)}|\bar{\mathfrak{m}}_\infty^k)+\varepsilon,$$

for every $n\ge\bar n$ and every $N'\in[N,\varepsilon)$.

STEP 4: $\bar q_n^{(r)}$ converges to an optimal plan
The objective now is to prove that

$$\int \mathsf{d}^2(x_0,x_1)\,\mathrm{d}\bar q_n^{(r)}(x_0,x_1)\to W_2^2(\mu_0^{(r)},\mu_1^{(r)})\qquad\text{as } n\to\infty. \tag{6.4.12}$$

First of all notice that, since every $\bar q_n^{(r)}$ is an admissible plan between $\mu_0^{(r)}$ and $\mu_1^{(r)}$, then for every $n\in\mathbb{N}$ it holds that

$$\int \mathsf{d}^2(x_0,x_1)\,\mathrm{d}\bar q_n^{(r)}(x_0,x_1)\ge W_2^2(\mu_0^{(r)},\mu_1^{(r)}). \tag{6.4.13}$$

On the other hand the triangular inequality ensures that

$$\mathsf{d}(x_0,x_1) \leq \mathsf{d}(x_0,y_0) + \mathsf{d}(y_0,y_1) + \mathsf{d}(x_1,y_1)$$

and consequently, since $\mathsf{d}(y_0,y_1) < \operatorname{diam}(\mathcal{R}_n^k) \leq 2^{k+2}$ for π_n-almost every pair (y_0,y_1), we have that

$$\mathsf{d}^2(x_0,x_1) - \mathsf{d}^2(y_0,y_1) \leq 2\mathsf{d}^2(x_0,y_0) + 2\mathsf{d}^2(x_1,y_1) + 2^{k+3}\mathsf{d}(x_0,y_0) + 2^{k+3}\mathsf{d}(x_1,y_1)$$

for π_n-almost every pair (y_0,y_1). It is then possible to perform the following estimate

$$\begin{aligned}
&\int_{\mathsf{X}_\infty\times\mathsf{X}_\infty} \mathsf{d}^2(x_0,x_1)\,\mathrm{d}\bar{q}_n^{(r)}(x_0,x_1)\\
&\quad= \int_{\mathsf{X}_\infty\times\mathsf{X}_\infty} \mathsf{d}^2(x_0,x_1) \int_{\mathsf{X}_n\times\mathsf{X}_n} \frac{\tilde{\rho}_0^{(r)}(x_0)\tilde{\rho}_1^{(r)}(x_1)}{\rho_{0,n}(y_0)\rho_{1,n}(y_1)} P_n'(y_1,\mathrm{d}x_1)P_n'(y_0,\mathrm{d}x_0)\,\mathrm{d}\pi_n(y_0,y_1)\\
&\quad\leq \int_{\mathsf{X}_n\times\mathsf{X}_n} \mathsf{d}^2(y_0,y_1)\,\mathrm{d}\pi_n(y_0,y_1)\\
&\qquad+ \int_{\mathsf{X}_\infty}\int_{\mathsf{X}_n\times\mathsf{X}_n} 2\mathsf{d}^2(x_0,y_0)\frac{\tilde{\rho}_0^{(r)}(x_0)}{\rho_{0,n}(y_0)}P_n'(y_0,\mathrm{d}x_0)\,\mathrm{d}\pi_n(y_0,y_1)\\
&\qquad+ \int_{\mathsf{X}_\infty}\int_{\mathsf{X}_n\times\mathsf{X}_n} 2^{k+3}\mathsf{d}(x_0,y_0)\frac{\tilde{\rho}_0^{(r)}(x_0)}{\rho_{0,n}(y_0)}P_n'(y_0,\mathrm{d}x_0)\,\mathrm{d}\pi_n(y_0,y_1)\\
&\qquad+ \int_{\mathsf{X}_\infty}\int_{\mathsf{X}_n\times\mathsf{X}_n} 2\mathsf{d}^2(x_1,y_1)\frac{\tilde{\rho}_1^{(r)}(x_1)}{\rho_{1,n}(y_1)}P_n'(y_1,\mathrm{d}x_1)\,\mathrm{d}\pi_n(y_0,y_1)\\
&\qquad+ \int_{\mathsf{X}_\infty}\int_{\mathsf{X}_n\times\mathsf{X}_n} 2^{k+3}\mathsf{d}(x_1,y_1)\frac{\tilde{\rho}_1^{(r)}(x_1)}{\rho_{1,n}(y_1)}P_n'(y_1,\mathrm{d}x_1)\,\mathrm{d}\pi_n(y_0,y_1)
\end{aligned}$$

We can now consider one term at a time and start by noticing that, according to Lemma 6.4.5,

$$\begin{aligned}
&\int_{\mathsf{X}_\infty}\int_{\mathsf{X}_n\times\mathsf{X}_n} 2\mathsf{d}^2(x_0,y_0)\frac{\tilde{\rho}_0^{(r)}(x_0)}{\rho_{0,n}(y_0)}P_n'(y_0,\mathrm{d}x_0)\,\mathrm{d}\pi_n(y_0,y_1)\\
&\qquad= \int_{\mathsf{X}_\infty}\int_{\mathsf{X}_n} 2\mathsf{d}^2(x_0,y_0)\tilde{\rho}_0^{(r)}(x_0)P_n'(y_0,\mathrm{d}x_0)\,\mathrm{d}\bar{\mathfrak{m}}_n^k(y_0)\\
&\qquad= \int_{\mathsf{X}_\infty}\int_{\mathsf{X}_n} 2\mathsf{d}^2(x_0,y_0)\tilde{\rho}_0^{(r)}(x_0)\,\mathrm{d}p_n(x_0,y_0) \to 0,
\end{aligned}$$

and similarly

$$\int_{\mathsf{X}_\infty}\int_{\mathsf{X}_n\times\mathsf{X}_n} 2\mathsf{d}^2(x_1,y_1)\frac{\tilde{\rho}_1^{(r)}(x_1)}{\rho_{1,n}(y_1)}P_n'(y_1,\mathrm{d}x_1)\,\mathrm{d}\pi_n(y_0,y_1) \to 0.$$

Moreover Hölder's inequality ensures that

$$\begin{aligned}
&\int_{\mathsf{X}_\infty}\int_{\mathsf{X}_n\times\mathsf{X}_n} 2^{k+3}\mathsf{d}(x_0,y_0)\frac{\tilde{\rho}_0^{(r)}(x_0)}{\rho_{0,n}(y_0)}P_n'(y_0,\mathrm{d}x_0)\,\mathrm{d}\pi_n(y_0,y_1)\\
&\qquad\leq 2^{k+3}\left[\int_{\mathsf{X}_\infty}\int_{\mathsf{X}_n\times\mathsf{X}_n} \mathsf{d}^2(x_0,y_0)\frac{\tilde{\rho}_0^{(r)}(x_0)}{\rho_{0,n}(y_0)}P_n'(y_0,\mathrm{d}x_0)\,\mathrm{d}\pi_n(y_0,y_1)\right]^{\frac{1}{2}} \to 0,
\end{aligned}$$

and analogously that

$$\int_{\mathsf{X}_\infty}\int_{\mathsf{X}_n\times\mathsf{X}_n} 2^{k+3}\mathsf{d}(x_1,y_1)\frac{\tilde{\rho}_1^{(r)}(x_1)}{\rho_{1,n}(y_1)}P_n'(y_1,\mathrm{d}x_1)\,\mathrm{d}\pi_n(y_0,y_1) \to 0.$$

Therefore, putting together the estimates on every term, we can conclude that

$$\begin{aligned}\limsup_{n\to\infty}\int \mathsf{d}^2(x_0,x_1)\,\mathrm{d}\bar{q}_n^{(r)}(x_0,x_1) &\leq \limsup_{n\to\infty}\int_{\mathsf{X}_n\times\mathsf{X}_n}\mathsf{d}^2(y_0,y_1)\,\mathrm{d}\pi_n(y_0,y_1)\\ &= \limsup_{n\to\infty} W_2^2(\mu_{0,n},\mu_{1,n}) = W_2^2(\mu_0^{(r)},\mu_1^{(r)}),\end{aligned}$$

where we used that π_n is an optimal plan and that $\mu_{0,n}\xrightarrow{W_2}\mu_0^{(r)}$, $\mu_{1,n}\xrightarrow{W_2}\mu_1^{(r)}$. This last inequality, combined with (6.4.13), allows us to conclude (6.4.12).

STEP 5: Definition of approximating plan with fixed marginals
We have shown the existence of $n(\varepsilon)\geq\bar{n}$ such that

$$\left|\int \mathsf{d}^2(x_0,x_1)\,\mathrm{d}\bar{q}_{n(\varepsilon)}^{(r)}(x_0,x_1) - W_2^2(\mu_0^{(r)},\mu_1^{(r)})\right| < \varepsilon. \tag{6.4.14}$$

Recalling the properties of $\bar{n}$ proven in the previous steps, we also know that

$$T_{K,N'}^{(t)}(\pi_n|\bar{\mathfrak{m}}_n^k) \leq T_{K,N'}^{(t)}(\bar{q}_n^{(r)}|\bar{\mathfrak{m}}_\infty^k)+\varepsilon, \tag{6.4.15}$$

for every $n\geq\bar{n}$ and every $N'\in[N,\varepsilon)$. At this point, using $\bar{q}_{n(\varepsilon)}^{(r)}$, we define a coupling q^ε between μ_0 and μ_1 by

$$q^\varepsilon(\cdot) := \alpha_r\bar{q}_{n(\varepsilon)}^{(r)} + \tilde{q}(\cdot\cap(\mathsf{X}_\infty^2\setminus E_r)).$$

First of all, notice that

$$\left|\int \mathsf{d}^2(x_0,x_1)\,\mathrm{d}\bar{q}_{n(\varepsilon)}^{(r)}(x_0,x_1) - \int \mathsf{d}^2(x_0,x_1)\,\mathrm{d}q^\varepsilon(x_0,x_1)\right| \leq 2(1-\alpha_r)\mathrm{diam}(\mathcal{R}_\infty^{k-1})^2 \leq \varepsilon 2^{2k+3}.$$

Consequently, putting together this last estimate with (6.4.7) and (6.4.14), we can conclude that

$$\int \mathsf{d}^2(x_0,x_1)\,\mathrm{d}q^\varepsilon(x_0,x_1) = W_2^2(\mu_0,\mu_1)+O(\varepsilon). \tag{6.4.16}$$

On the other hand, it is immediate from the definition of q^ε that

$$(1-\varepsilon)^{1-1/N'}T_{K,N'}^{(t)}(\bar{q}_{n(\varepsilon)}^{(r)}|\bar{\mathfrak{m}}_\infty^k) \leq \alpha_r^{1-1/N'}T_{K,N'}^{(t)}(\bar{q}_{n(\varepsilon)}^{(r)}|\bar{\mathfrak{m}}_\infty^k) \leq T_{K,N'}^{(t)}(q^\varepsilon|\bar{\mathfrak{m}}_\infty^k). \tag{6.4.17}$$

STEP 6: Convergence of plans
We turn now to prove the weak convergence of the plans q^ε introduced in the previous step, as $\varepsilon\to 0$, and consequently the upper semicontinuity of $T_{K,N'}^{(t)}$.
We first note that, since for every $\varepsilon>0$, it holds that $q^\varepsilon\in\mathsf{Adm}(\mu_0,\mu_1)$, then the family $(q^\varepsilon)_{\varepsilon>0}$ is tight and Prokhorov Theorem ensures the existence of a sequence $(\varepsilon_m)_{m\in\mathbb{N}}$ converging to 0 such that $q^{\varepsilon_m}\rightharpoonup q\in\mathsf{Adm}(\mu_0,\mu_1)$. Equation (6.4.16) ensures the optimality of $q\in\mathsf{Opt}(\mu_0,\mu_1)$. Furthermore, putting together the estimates (6.4.15) (that holds definitely for every $N'\in[N,0)$) and (6.4.17), we conclude that, for every $N'\in[N,0)$ and $t\in[0,1]$,

$$\begin{aligned}
\limsup_{m\to\infty} T^{(t)}_{K,N'}(\pi_{n(\varepsilon_m)}|\mathfrak{m}_{n(\varepsilon_m)}) &\leq \limsup_{m\to\infty} T^{(t)}_{K,N'}(\pi_{n(\varepsilon_m)}|\mathfrak{m}^k_{n(\varepsilon_m)}) \\
&= \limsup_{m\to\infty} \frac{1}{\mathfrak{m}^k_{n(\varepsilon_m)}(\mathsf{X}_{n(\varepsilon_m)})^{-1/N'}} \cdot T^{(t)}_{K,N'}(\pi_{n(\varepsilon_m)}|\bar{\mathfrak{m}}^k_{n(\varepsilon_m)}) \\
&= \frac{1}{\mathfrak{m}^k_\infty(\mathsf{X}_\infty)^{-1/N'}} \limsup_{m\to\infty} T^{(t)}_{K,N'}(\pi_{n(\varepsilon_m)}|\bar{\mathfrak{m}}^k_{n(\varepsilon_m)}) \\
&\leq \frac{1}{\mathfrak{m}^k_\infty(\mathsf{X}_\infty)^{-1/N'}} \limsup_{m\to\infty} T^{(t)}_{K,N'}(\bar{q}^{(r)}_{n(\varepsilon_m)}|\bar{\mathfrak{m}}^k_\infty) + \varepsilon_m \\
&= \frac{1}{\mathfrak{m}^k_\infty(\mathsf{X}_\infty)^{-1/N'}} \limsup_{m\to\infty} T^{(t)}_{K,N'}(\bar{q}^{(r)}_{n(\varepsilon_m)}|\bar{\mathfrak{m}}^k_\infty) \\
&\leq \frac{1}{\mathfrak{m}^k_\infty(\mathsf{X}_\infty)^{-1/N'}} \limsup_{m\to\infty} \frac{1}{(1-\varepsilon_m)^{1-1/N'}} T^{(t)}_{K,N'}(q^{\varepsilon_m}|\bar{\mathfrak{m}}^k_\infty) \\
&= \limsup_{m\to\infty} T^{(t)}_{K,N'}(q^{\varepsilon_m}|\mathfrak{m}^k_\infty).
\end{aligned}$$

Now, notice that every q^{ε_m} has as marginals μ_0 and μ_1, which are supported in $\mathcal{R}^{k-1}_\infty$ and therefore

$$T^{(t)}_{K,N'}(q^{\varepsilon_m}|\mathfrak{m}^k_\infty) = T^{(t)}_{K,N'}(q^{\varepsilon_m}|\mathfrak{m}_\infty).$$

Thus we can apply Proposition 6.4.10 to $T^{(t)}_{K,N'}(q^{\varepsilon_m}|\mathfrak{m}_\infty)$, for $N' \in I$, which together with the above estimate guarantees that

$$\limsup_{m\to\infty} T^{(t)}_{K,N'}(\pi_{n(\varepsilon_m)}|\mathfrak{m}_{n(\varepsilon_m)}) \leq \limsup_{m\to\infty} T^{(t)}_{K,N'}(q^{\varepsilon_m}|\mathfrak{m}_\infty) = T^{(t)}_{K,N'}(q|\mathfrak{m}_\infty) \tag{6.4.18}$$

holds for every $N' \in I$ and $t \in [0,1]$.

STEP 7: Convergence of midpoints

The goal of this step is to show the existence of a limit geodesic $\{\mu_t\}_{t\in[0,1]}$, such that for any $t \in [0,1]$, $\mu_{t,n(\varepsilon_m)}$ W_2-converges (up to subsequences) to μ_t, as $m \to \infty$. Furthermore, we are going to prove a suitable lower semicontinuity of the Rényi entropies that will allow us to pass to the limit of the CD inequality. In order to ease the notation we will denote the Rényi entropy $S_{N,\mathfrak{m}_{n(\varepsilon_m)}}$ by $S_{N,n(\varepsilon_m)}$.

Claim 1. *For every fixed $t \in [0,1]$, the sequence $(\mu_{t,n(\varepsilon_m)})_{m\in\mathbb{N}}$ converges (up to subsequences) to a measure $\mu_t \in \mathscr{P}(\mathsf{X}_\infty)$.*

First of all, notice that estimate (6.4.18), the CD-condition (6.4.9) and assumption (6.4.6) together ensure that the entropies $S_{N,n(\varepsilon_m)}(\mu_{t,n(\varepsilon_m)})$ are uniformly bounded above by a constant M', for $m \in \mathbb{N}$. Moreover, for every $n(\varepsilon_m)$, the approximation Lemma 6.4.4 provides the existence of a sequence $(\mu^l_{t,n(\varepsilon_m)})_{l\in\mathbb{N}}$, that W_2-converges to $\mu_{t,n(\varepsilon_m)}$, as $l \to \infty$, and such that $\mathrm{supp}(\mu^l_{t,n(\varepsilon)}) \subseteq \mathcal{R}^l_{n(\varepsilon)}$. From the proof of Lemma 6.4.4 we recall that $\mu^l_{t,n(\varepsilon_m)} = c^l f^l \mu_{t,n(\varepsilon_m)}$ so, we can easily notice that,

$$(c^l)^{-1} = \int f^l \,\mathrm{d}\mu_{t,n(\varepsilon_m)} \geq \mu_{t,n(\varepsilon_m)}(\mathcal{R}^{l-1}_{n(\varepsilon_m)}) \geq 1 - \omega(k, l-1, M+1).$$

Consequently, for sufficiently large l it holds that, as measures,

$$\mu^l_{t,n(\varepsilon_m)} \leq \frac{1}{1-\omega(k,l-1,M+1)} \cdot \mu_{t,n(\varepsilon_m)}. \tag{6.4.19}$$

Notice that we took into account that $\operatorname{supp}(\mu_{0,n(\varepsilon_m)}), \operatorname{supp}(\mu_{1,n(\varepsilon_m)}) \subseteq \mathcal{R}^k_{n(\varepsilon_m)}$ and we have used the ω-uniform convexity assumption, keeping in mind that $M+1$ bounds from above the terminal entropies (6.4.8). In turn, inequality (6.4.19) implies that,

$$S_{N,n(\varepsilon_m)}(\mu^l_{t,n(\varepsilon_m)}) \leq \frac{1}{(1-\omega(k,l-1,M+1))^{1-1/N}} S_{N,n(\varepsilon_m)}(\mu_{t,n(\varepsilon_m)}). \tag{6.4.20}$$

Moreover, the fact that $\operatorname{supp}(\mu_{0,n(\varepsilon_m)}), \operatorname{supp}(\mu_{1,n(\varepsilon_m)}) \subseteq \mathcal{R}^k_{n(\varepsilon_m)}$ shows that the measure $\mu_{t,n(\varepsilon_m)}$ has bounded support. In particular, for every $l \in \mathbb{N}$ (and every $m \in \mathbb{N}$)

$$\operatorname{supp}(\mu^l_{t,n(\varepsilon_m)}) \subseteq \operatorname{supp}(\mu_{t,n(\varepsilon_m)}) \subseteq B(p_{n(\varepsilon_m)}, 2^{k+2}).$$

As a consequence of this bound, it is easy to deduce that,

$$W_2^2(\mu^l_{t,n(\varepsilon_m)}, \mu_{t,n(\varepsilon_m)}) \leq (2\cdot 2^{k+2})^2\, \omega(k,l-1,M+1), \tag{6.4.21}$$

because $\mu_{t,n(\varepsilon_m)} \leq \mu^l_{t,n(\varepsilon_m)}$ when restricted to $\mathcal{R}^{l-1}_{n(\varepsilon_m)}$, and $\mu_{t,n(\varepsilon_m)}(X\setminus\mathcal{R}^{l-1}_{n(\varepsilon_m)}) \leq \omega(k,l-1,M+1)$, by ω-uniform convexity. Now, for every fixed l sufficiently large, such that $\omega(k,l-1,M+1)<1$, observe that, according to (6.4.20), the entropies $S_{N,n(\varepsilon_m)}(\mu^l_{t,n(\varepsilon_m)})$ are uniformly bounded above by the constant

$$\frac{1}{(1-\omega(k,l-1,M+1))^{1-1/N}} M',$$

for all $m \in \mathbb{N}$. Notice also that, since $\mu^l_{t,n(\varepsilon_m)}$ is supported in $\mathcal{R}^l_{n(\varepsilon_m)}$, it holds that

$$S_{N,\mathfrak{m}^{l+1}_{n(\varepsilon_m)}}(\mu^l_{t,n(\varepsilon_m)}) = S_{N,n(\varepsilon_m)}(\mu^l_{t,n(\varepsilon_m)}).$$

Therefore, Corollary 6.4.7 shows that $\mu^l_{t,n(\varepsilon_m)}$ weakly converges to some $\mu^l_t \in \mathscr{P}(X_\infty)$ as $m\to\infty$, for some choice of a subsequence. Moreover, we extract the bound $S_{N,\mathfrak{m}^{l+1}_\infty}(\mu^l_t) < \infty$ from Corollary 6.4.9, which guarantees the lower semicontinuity of $S_{N,\cdot}(\cdot)$ along our sequence. Consequently, this implies that the support of μ^l_t is contained in $\mathcal{R}^{l+1}_\infty$. Finally, note that then the sequence of measures $(\mu^l_{t,n(\varepsilon_m)})_{m\in\mathbb{N}}$ is supported in a uniformly bounded set, since $\operatorname{supp}(\mu^l_{t,n(\varepsilon_m)}) \subseteq B\left(p_{n(\varepsilon_m)}, 2^{k+2}\right)$ and $p_{n(\varepsilon_m)} \to p_\infty$, as $m\to\infty$. Thus, we are able to conclude, up to picking again subsequence, that for every sufficiently large $l\in\mathbb{N}$

$$\mu^l_{t,n(\varepsilon_m)} \xrightarrow{W_2} \mu^l_t \quad \text{as} \quad m\to\infty.$$

As a matter of fact, we can show using inequality (6.4.21) that,

$$W_2^2(\mu^i_t,\mu^j_t) \leq 2^{2k+7}\big[\omega(k,i-1,M+1)+\omega(k,j-1,M+1)\big],$$

for every (large enough) $i,j\in\mathbb{N}$. Then, our assumption on ω ensures that $(\mu^l_t)_{l\in\mathbb{N}}$ is a Cauchy sequence, which therefore W_2-converges to $\mu_t \in \mathscr{P}(\mathsf{X}_\infty)$. We conclude by noting that the uniform estimate (6.4.21) guarantees that $\mu_{t,n(\varepsilon_m)} \to \mu_t$.

Claim 2. *For every $t\in[0,1]$ the measure μ_t does not give mass to the set S of singular points.*

For every $m\in\mathbb{N}$ and every $l\in\mathbb{N}$ sufficiently large, let us introduce the measures

$$\tilde\mu^l_{t,n(\varepsilon_m)} = [1-\omega(k,l-1,M+1)]\mu^l_{t,n(\varepsilon_m)},$$

and notice that, for every $l\in\mathbb{N}$ sufficiently large,

$$\tilde\mu^l_{t,n(\varepsilon_m)} \rightharpoonup \tilde\mu^l_t := [1-\omega(k,l-1,M+1)]\mu^l_t.$$

Observe also that all measures $\tilde{\mu}^l_{t,n(\varepsilon_m)}$ have total mass equal to $[1-\omega(k,l-1,M+1)]$, as m varies. Thus, $\tilde{\mu}^l_t$ also has total mass equal to $[1-\omega(k,l-1,M+1)]$. On the other hand it follows from the uniform convexity properties (and in particular from (6.4.19)) that for every $m \in \mathbb{N}$ and every $l \in \mathbb{N}$ sufficiently large, there exists a positive measure $\bar{\mu}^l_{t,n(\varepsilon_m)}$ such that

$$\mu_{t,n(\varepsilon_m)} = \tilde{\mu}^l_{t,n(\varepsilon_m)} + \bar{\mu}^l_{t,n(\varepsilon_m)}.$$

Notice that, since the sequences $\mu_{t,n(\varepsilon_m)}$ and $(\tilde{\mu}^l_{t,n(\varepsilon_m)})_{m\in\mathbb{N}}$ are weakly converging, the sequence $\bar{\mu}^l_{t,n(\varepsilon_m)}$ is also weakly converging to a (positive) measure $\bar{\mu}^l_t$, such that

$$\mu_t = \tilde{\mu}^l_t + \bar{\mu}^l_t.$$

As pointed out before μ^l_t is supported in $\mathcal{R}^{l+1}_\infty$, thus the same holds for $\tilde{\mu}^l_t$, and therefore

$$\mu_t(\mathcal{R}^{l+1}_\infty) \geq 1-\omega(k,l-1,M+1).$$

Finally observe that this is sufficient to prove the claim, because of the arbitrariness of l.

Claim 3. *The lower semicontinuity of the Rényi entropies holds, that is for every $N' \in [N,0)$*

$$S_{N',\mathfrak{m}_\infty}(\mu_t) \leq \liminf_{m\to\infty} S_{N',n(\varepsilon_m)}(\mu_{t,n(\varepsilon_m)}).$$

First of all notice that, the result of Claim 2 combined with Proposition 6.4.8 yields that

$$S_{N',\mathfrak{m}_\infty}(\mu_t) \leq \liminf_{l\to\infty} S_{N',\mathfrak{m}_\infty}(\mu^l_t). \tag{6.4.22}$$

On the other hand Corollary 6.4.9 ensures that for every $l \in \mathbb{N}$ large enough

$$S_{N',\mathfrak{m}_\infty}(\mu^l_t) = S_{N',\mathfrak{m}^{l+2}_\infty}(\mu^l_t) \leq \liminf_{m\to\infty} S_{N',\mathfrak{m}^{l+2}_{n(\varepsilon_m)}}(\mu^l_{t,n(\varepsilon_m)}) = \liminf_{m\to\infty} S_{N',n(\varepsilon_m)}(\mu^l_{t,n(\varepsilon_m)}). \tag{6.4.23}$$

Moreover, we deduce as in Claim 1 the following estimate for every $N' \in [N,0)$

$$S_{N',n(\varepsilon_m)}(\mu^l_{t,n(\varepsilon_m)}) \leq \frac{1}{(1-\omega(k,l-1,M+1))^{1-1/N'}} S_{N',n(\varepsilon_m)}(\mu_{t,n(\varepsilon_m)})$$

and consequently for every $l \in \mathbb{N}$

$$\begin{aligned}\liminf_{m\to\infty} S_{N',n(\varepsilon_m)}(\mu_{t,n(\varepsilon_m)}) &\geq \liminf_{m\to\infty}(1-\omega(k,l-1,M+1))^{1-1/N'} S_{N',n(\varepsilon_m)}(\mu^l_{t,n(\varepsilon_m)}) \\ &\geq (1-\omega(k,l-1,M+1))^{1-1/N'} S_{N',\mathfrak{m}_\infty}(\mu^l_t),\end{aligned}$$

where the last passage follows from (6.4.23). Then, since this last inequality holds for every $l \in \mathbb{N}$, we can conclude that

$$\begin{aligned}\liminf_{m\to\infty} S_{N',n(\varepsilon_m)}(\mu_{t,n(\varepsilon_m)}) &\geq \liminf_{l\to\infty}(1-\omega(k,l-1,M+1))^{1-1/N'} S_{N',\mathfrak{m}_\infty}(\mu^l_t) \\ &\geq S_{N',\mathfrak{m}_\infty}(\mu_t),\end{aligned}$$

where we used (6.4.22). This is exactly what we wanted to prove.

CONCLUSION

So far we were able to prove that for every fixed $t \in [0,1]$, the sequence $(\mu_{t,n(\varepsilon_m)})_{m\in\mathbb{N}}$ converges (up to subsequences) to a measure $\mu_t \in \mathscr{P}_2(\mathsf{X}_\infty)$ and

$$S_{N',\mathfrak{m}_\infty}(\mu_t) \leq \liminf_{m\to\infty} S_{N',n(\varepsilon_m)}(\mu_{t,n(\varepsilon_m)}), \tag{6.4.24}$$

for every $N' \in [N, 0)$. Now, a diagonal argument ensures that, by selecting a suitable subsequence (that we do not rename for sake of simplicity),

$$(\mu_{t,n(\varepsilon_m)}) \xrightarrow{W_2} \mu_t,$$

and that estimate (6.4.24) holds for every $t \in [0,1] \cap \mathbb{Q}$. Our approximation ensures also that $\mu_{0,n(\varepsilon_m)} \xrightarrow{W_2} \mu_0$ and $\mu_{1,n(\varepsilon_m)} \xrightarrow{W_2} \mu_1$ therefore, since $\mu_{t,n(\varepsilon_m)}$ is a t-midpoint of $\mu_{0,n(\varepsilon_m)}$ and $\mu_{1,n(\varepsilon_m)}$, for every $t \in [0,1] \cap \mathbb{Q}$ the limit point μ_t is a t-midpoint of μ_0 and μ_1. Now it is easy to realize that we can extend by continuity μ_t to a Wasserstein geodesic (connecting μ_0 and μ_1) on the whole interval $[0,1]$, obtaining also that

$$(\mu_{t,n(\varepsilon_m)}) \xrightarrow{W_2} \mu_t, \qquad \text{for every } t \in [0,1].$$

This argument will be explained with more details in Lemma 6.4.12 below. Moreover, we know from the proof of Claim 2, that for every $l \in \mathbb{N}$

$$\mu_t(\mathcal{R}_\infty^{l+1}) \geq 1 - \omega(k, l-1, M+1),$$

for every $t \in [0,1] \cap \mathbb{Q}$. Then by continuity we can conclude the same inequality for every $t \in [0,1]$, and consequently we know that μ_t gives no mass to the set of singular points.
Finally, inequality (6.4.24), combined with (6.4.18), allows to pass to the limit as $m \to \infty$ of the inequality (6.4.9) at every rational time and obtaining that

$$S_{N',\mathfrak{m}_\infty}(\mu_t) \leq T^{(t)}_{K,N'}(q|\mathfrak{m}_\infty)$$

holds for every $t \in [0,1] \cap \mathbb{Q}$ and every $N' \in I$. Finally, the lower semicontinuity of the entropy (ensured by the fact that μ_t gives no mass to the set of singular points) and the continuity of $T^{(t)}_{K,N'}(q|\mathfrak{m}_\infty)$ in t (which is a straightforward consequence of the dominated convergence theorem), allow to extend this last inequality to every $t \in [0,1]$, concluding the proof of the approximate CD-condition.

6.4.3 Proof of the CD Condition

This final section is dedicated to the proof of our main result, that is Theorem 6.4.2. As already mentioned, the proof of the approximate CD-condition and the approximation argument that made it possible are the foundation to prove the CD-condition. As the reader will notice, we are going to use basically the same techniques, but refining them a little bit to achieve the more general result. We specify that we could prove the CD-condition directly, but we preferred to divide the proof in order to be clearer.

Before going on, we prove a preliminary lemma, that will help us in the following. Notice that a result of this type is now needed because the marginals may not have bounded support.

Lemma 6.4.12. *Given a metric space (X, d), for every $n \in \mathbb{N}$ let $(\nu_t^n)_{t\in[0,1]} \subset \mathscr{P}_2(X)$ be a Wasserstein geodesic. Assume that for every $t \in [0,1]$ the family $(\nu_t^n)_{n\in\mathbb{N}}$ is tight and that there exist $\nu_0, \nu_1 \in \mathscr{P}_2(X)$ such that*

$$\nu_0^n \xrightarrow{W_2} \nu_0 \quad \text{and} \quad \nu_1^n \xrightarrow{W_2} \nu_1 \qquad \text{as } n \to \infty.$$

Then there exists a Wasserstein geodesic $(\nu_t)_{t\in[0,1]} \subset \mathscr{P}_2(X)$ connecting ν_0 and ν_1 such that, up to subsequences,

$$\nu_t^n \rightharpoonup \nu_t \qquad \text{as } n \to \infty, \text{ for every } t \in [0,1] \cap \mathbb{Q}.$$

Proof. First of all, notice that applying Prokhorov theorem we deduce that, for every fixed t, the sequence $(\nu_t^n)_{n\in\mathbb{N}}$ is weakly convergent, up to subsequences. Thus the diagonal argument ensures that, up to taking a suitable subsequence which we do not recall for simplicity, for every $t \in [0,1]\cap\mathbb{Q}$ there exists $\nu_t \in \mathscr{P}_2(X)$ such that

$$\nu_t^n \rightharpoonup \nu_t \qquad \text{as } n\to\infty, \text{ for every } t\in[0,1]\cap\mathbb{Q}.$$

It is well-known that the Wasserstein distance is lower semicontinuous with respect to the weak convergence (see for example Proposition 7.1.3 in [AGS08]), then

$$W_2(\nu_0,\nu_t) \leq \liminf_{n\to\infty} W_2(\nu_0^n,\nu_t^n) = \liminf_{n\to\infty} t\cdot W_2(\nu_0^n,\nu_1^n) = t\cdot W_2(\nu_0,\nu_1)$$

and analogously

$$W_2(\nu_t,\nu_1) \leq (1-t)\cdot W_2(\nu_0,\nu_1).$$

Combining this two inequalities with the triangular inequality we deduce that

$$W_2(\nu_0,\nu_t) = t\cdot W_2(\nu_0,\nu_1) \quad \text{and} \quad W_2(\nu_t,\nu_1) = (1-t)\cdot W_2(\nu_0,\nu_1),$$

which means that ν_t is a t-midpoint of ν_0 and ν_1. The lower semicontinuity of the Wasserstein distance also ensures that for every $s,t\in[0,1]\cap\mathbb{Q}$ it holds that

$$W_2(\nu_t,\nu_s) \leq \liminf_{n\to\infty} W_2(\nu_t^n,\nu_s^n) = |t-s|\cdot\liminf_{n\to\infty} W_2(\nu_0^n,\nu_1^n) = |t-s|\cdot W_2(\nu_0,\nu_1).$$

Finally, since for every $r\in[0,1]\cap\mathbb{Q}$ ν_r is an r-midpoint of ν_0 and ν_1, the triangular inequality allow us conclude that

$$W_2(\nu_t,\nu_s) = |t-s|\cdot W_2(\nu_0,\nu_1), \qquad \text{for every } s,t\in[0,1]\cap\mathbb{Q},$$

then we can extend ν_t to the whole interval $[0,1]$, finding a Wasserstein geodesic $(\nu_t)_{t\in[0,1]}$ connecting ν_0 and ν_1. ☐

Now that we have this last result at our disposal we can proceed to the proof of Theorem 6.4.2. To this aim, we fix $\mu_0,\mu_1\in\mathscr{P}^{ac}(\mathsf{X}_\infty,\mathfrak{m}_\infty)$. In analogy with the previous section, we can assume that $S_{N,\mathfrak{m}_\infty}(\mu_0), S_{N,\mathfrak{m}_\infty}(\mu_1)<\infty$ and introduce the constant

$$M := \max\{S_{N,\mathfrak{m}_\infty}(\mu_0), S_{N,\mathfrak{m}_\infty}(\mu_1)\}.$$

We can also define the interval

$$I := \{N'\in[N,0) \,:\, S_{N',\mathfrak{m}_\infty}(\mu_0), S_{N',\mathfrak{m}_\infty}(\mu_1)<\infty\},$$

in particular we will need to prove (6.3.2) for every $N'\in I$ and every $t\in[0,1]$. Now, according to Lemma 6.4.4 there exist two sequences $(\mu_0^l)_{l\in\mathbb{N}}$ and $(\mu_1^l)_{l\in\mathbb{N}}$, W_2-converging to μ_0 and μ_1 respectively and such that

$$\operatorname{supp}(\mu_0^l), \operatorname{supp}(\mu_1^l) \subseteq \mathcal{R}_\infty^{l-1} \qquad \text{for every } l\in\mathbb{N}.$$

Moreover, keeping in mind the definition of μ_0^l and μ_1^l (see Lemma 6.4.4), it is easy to realize that for l sufficiently large

$$S_{N',\mathfrak{m}_\infty}(\mu_0^l), S_{N',\mathfrak{m}_\infty}(\mu_1^l)<\infty \quad \text{for every } N'\in I$$

and that the dominated convergence theorem ensures that

$$\lim_{l\to\infty} S_{N,\mathfrak{m}_\infty}(\mu_0^l) = S_{N,\mathfrak{m}_\infty}(\mu_0) \qquad \text{and} \qquad \lim_{l\to\infty} S_{N,\mathfrak{m}_\infty}(\mu_1^l) = S_{N,\mathfrak{m}_\infty}(\mu_1).$$

Thus, for every l large enough

$$S_{N,\mathfrak{m}_\infty}(\mu_0^l), S_{N,\mathfrak{m}_\infty}(\mu_1^l) \leq \max\left\{S_{N,\mathfrak{m}_\infty}(\mu_0), S_{N,\mathfrak{m}_\infty}(\mu_1)\right\} + 1 = M + 1$$

and then we can apply the argument presented in the last section and deduce the existence of an optimal plan $q^l \in \mathsf{Opt}(\mu_0^l, \mu_1^l)$ and of a Wasserstein geodesic $(\mu_t^l)_{t\in[0,1]}$ connecting μ_0^l and μ_1^l, such that

$$S_{N',\mathfrak{m}_\infty}(\mu_t^l) \leq T^{(t)}_{K,N'}(q^l|\mathfrak{m}_\infty) \tag{6.4.25}$$

holds for every $t \in [0,1]$ and every $N' \in I$. Now, we divide the proof into two steps, the first dedicated to the convergence of the plans $(q^l)_{l\in\mathbb{N}}$ and to the upper semicontinuity of $T^{(t)}_{K,N'}$, the second dedicated to the convergence of the measures $(\mu_t^l)_{l\in\mathbb{N}}$ and the lower semicontinuity of $S_{N',\mathfrak{m}_\infty}$.

Step 1: Upper semicontinuity for $T^{(t)}_{K,N'}$

Notice that $(q^l)_{l\in\mathbb{N}}$ is a sequence of probability measures having as marginals two sequences of converging, and thus tight, probability measures. As a consequence the sequence $(q^l)_{l\in\mathbb{N}}$ is itself tight, then up to subsequences it weakly converges to a plan $q \in \mathscr{P}(\mathsf{X}_\infty \times \mathsf{X}_\infty)$. It is well known and easy to prove that $q \in \mathsf{Opt}(\mu_0, \mu_1)$. We are now going to prove that

$$\limsup_{l\to\infty} T^{(t)}_{K,N'}(q^l|\mathfrak{m}_\infty) \leq T^{(t)}_{K,N'}(q|\mathfrak{m}_\infty) \tag{6.4.26}$$

for every $t \in [0,1]$ and every $N' \in I$. The argument we are going to use is essentially the same as the one explained in the proof of Proposition 6.4.10, nevertheless we briefly recall it for the sake of completeness, avoiding to repeat all the details.
In particular, for every $l \in \mathbb{N}$ let us call ρ_0^l and ρ_1^l the densities of μ_0^l and μ_1^l with respect to the reference measure $\mathfrak{m}_\infty$, we just need to prove that

$$\limsup_{l\to\infty} \int \tau^{(1-t)}_{K,N'}(\mathsf{d}(x,y))\rho_0^l(x)^{-\frac{1}{N'}}\,\mathrm{d}q^l \leq \int \tau^{(1-t)}_{K,N'}(\mathsf{d}(x,y))\rho_0(x)^{-\frac{1}{N'}}\,\mathrm{d}q.$$

Notice that, the particular definition of μ_0^l (check Lemma 6.4.4), ensures that the density ρ_0^l is a suitable renormalization of $f^l\rho_0$, then for a fixed $\varepsilon > 0$ we can find $\bar{l} \in \mathbb{N}$ such that

$$\left\|(\rho_0^l)^{-1/N'} - \rho_0^{-1/N'}\right\|_{L^1(\mu_0^l)} < \varepsilon \quad \text{for every } l \geq \bar{l}.$$

Furthermore, recalling that $C_b(\mathsf{X}) \cap L^1(\mathfrak{m})$ is dense in $L^1(\mathfrak{m})$, for the same reason (up to possibly changing $\bar{l}$) we can find $f^\varepsilon \in C_b(X)$ such that

$$\left\|\rho_0^{-1/N'} - f^\varepsilon\right\|_{L^1(\mu_0)} < \varepsilon \qquad \text{and} \qquad \left\|\rho_0^{-1/N'} - f^\varepsilon\right\|_{L^1(\mu_0^l)} < \varepsilon \quad \text{for every } l \geq \bar{l}.$$

Putting together this last two estimates we end up proving that

$$\left\|(\rho_0^l)^{-1/N'} - f^\varepsilon\right\|_{L^1(\mu_0^l)} < 2\varepsilon \quad \text{for every } l \geq \bar{l}.$$

On the other hand, since the function

$$\tau^{(1-t)}_{K,N}(\mathsf{d}(x,y))f^\varepsilon(x)$$

is bounded and continuous, the weak convergence of $(q^l)_l$ to q yields that

$$\lim_{l\to\infty} \int \tau^{(1-t)}_{K,N'}(\mathsf{d}(x,y))f^\varepsilon(x)\,\mathrm{d}q^l = \int \tau^{(1-t)}_{K,N'}(\mathsf{d}(x,y))f^\varepsilon(x)\,\mathrm{d}q.$$

Then, since definitely $l \geq \bar{l}$, we can deduce the following estimate

$$\begin{aligned}\limsup_{l\to\infty} \int \tau_{K,N'}^{(1-t)}(\mathsf{d}(x,y))\rho_0^l(x)^{-\frac{1}{N'}}\,\mathrm{d}q^l &\leq \lim_{l\to\infty}\int \tau_{K,N'}^{(1-t)}(\mathsf{d}(x,y))f^\varepsilon(x)\,\mathrm{d}q^l + 2\varepsilon\left\|\tau_{K,N'}^{(1-t)}\right\|_{L^\infty}\\ &= \int \tau_{K,N'}^{(1-t)}(\mathsf{d}(x,y))f^\varepsilon(x)\,\mathrm{d}q + 2\varepsilon\left\|\tau_{K,N'}^{(1-t)}\right\|_{L^\infty}\\ &\leq \int \tau_{K,N'}^{(1-t)}(\mathsf{d}(x,y))\rho_0(x)^{-\frac{1}{N'}}\,\mathrm{d}q + 3\varepsilon\left\|\tau_{K,N'}^{(1-t)}\right\|_{L^\infty}.\end{aligned}$$

and since $\varepsilon > 0$ can be chosen arbitrarily, (6.4.26) holds true.

Step 2: Lower semicontinuity for $S_{N',\mathfrak{m}_\infty}$

In this second step we prove an additional property on μ_t^l, which is fundamental to prove the CD-condition. Let us start with a preliminary lemma.

Lemma 6.4.13. *Fix $k \leq h \in \mathbb{N}$ and let $\nu \in \mathscr{P}^{ac}(\mathsf{X}_\infty, \mathfrak{m}_\infty)$ with bounded density be such that $\operatorname{supp}(\nu) \subseteq \mathcal{R}_\infty^{k-1}$. Then, for every $\epsilon > 0$, there exists $\tilde{n} \in \mathbb{N}$ large enough such that, $P'_{n,h}(\nu)(\mathcal{R}_n^{k+1}) \geq 1-\epsilon$ for every $n \geq \tilde{n}$.*

Proof. Notice that, according to Lemma 6.4.5, both the sequences $(P'_{n,k}(\nu))_{n\in\mathbb{N}}$ and $(P'_{n,h}(\nu))_{n\in\mathbb{N}}$ W_2-converge to ν. Assume that $P'_{m,h}(\nu)(\mathcal{R}_m^{k+1}) < 1-\epsilon$ for some arbitrarily large $m \in \mathbb{N}$. Observe that

$$\inf\left\{\mathsf{d}(x,y) \,:\, x \in \mathcal{R}_m^k,\ y \in (\mathcal{R}_m^{k+1})^c\right\} \geq 2^{-(k+2)},$$

as a consequence, since $P'_{m,k}(\nu)$ is supported in $\mathcal{R}_m^k$, we obtain that

$$W_2^2(P'_{m,k}(\nu), P'_{m,h}(\nu)) \geq \epsilon\cdot 2^{-(2k+4)}. \tag{6.4.27}$$

On the other hand, since the sequences $(P'_{n,k}(\nu))_{n\in\mathbb{N}}$ and $(P'_{n,h}(\nu))_{n\in\mathbb{N}}$ have the same limit, it holds that

$$W_2^2(P'_{n,k}(\nu), P'_{n,h}(\nu)) \to 0 \qquad \text{as } n\to\infty.$$

Then definitely (6.4.27) cannot hold, proving the desired result. □

Fix $\epsilon > 0$ and take $k(\epsilon) \in \mathbb{N}$ such that $\mu_0(\mathcal{R}_\infty^{k(\epsilon)-1}), \mu_1(\mathcal{R}_\infty^{k(\epsilon)-1}) > 1-\frac{\epsilon}{2}$. Then we take $l > k(\epsilon)$ and repeat the argument of the previous section on μ_0^l and μ_1^l. We are also going to use the same notation, forgetting for the moment the dependence on l. It is easy to realize that there exist two measures $\nu_0^{k(\epsilon)}$ and $\nu_1^{k(\epsilon)}$ with $\operatorname{supp}(\nu_0^{k(\epsilon)}), \operatorname{supp}(\nu_1^{k(\epsilon)}) \subseteq \mathcal{R}_\infty^{k(\epsilon)-1}$ and $\nu_0^{k(\epsilon)}(\mathsf{X}_\infty), \nu_1^{k(\epsilon)}(\mathsf{X}_\infty) > 1-\epsilon$, such that for r sufficiently large (and thus for ϵ sufficiently small) $\mu_0^{(r)} \geq \nu_0^{k(\epsilon)}$ and $\mu_1^{(r)} \geq \nu_1^{k(\epsilon)}$ (in particular this tells us that $\nu_0^{k(\epsilon)}$ and $\nu_1^{k(\epsilon)}$ have bounded density). Then we can apply Lemma 6.4.13 to the probability measures

$$\frac{1}{\nu_0^{k(\epsilon)}(\mathsf{X}_\infty)}\nu_0^{k(\epsilon)} \quad \text{and} \quad \frac{1}{\nu_1^{k(\epsilon)}(\mathsf{X}_\infty)}\nu_1^{k(\epsilon)},$$

obtaining that for m sufficiently large (in particular such that $n(\varepsilon_m) \geq \tilde{n}$) it holds that

$$\mu_{0,n(\varepsilon_m)}(\mathcal{R}_{n(\varepsilon_m)}^{k(\epsilon)+1}), \mu_{1,n(\varepsilon_m)}(\mathcal{R}_{n(\varepsilon_m)}^{k(\epsilon)+1}) \geq (1-\epsilon)^2 \geq 1-2\epsilon.$$

Consequently our uniform convexity assumption ensures that

$$\mu_{t,n(\varepsilon_m)}(\mathcal{R}_{n(\varepsilon_m)}^h) \geq 1 - \Omega(k(\epsilon)+1, h, M+2, 2\epsilon),$$

for every $t \in [0,1]$ and every $h \in \mathbb{N}$. Proceeding as in Step 7 of the previous section (see in particular Claim 2), we can actually conclude that

$$\mu_t^l(\mathcal{R}_\infty^{h+1}) \geq 1 - \Omega(k(\epsilon)+1, h-1, M+2, 2\epsilon), \tag{6.4.28}$$

for every $t \in [0,1]$ and every $h \in \mathbb{N}$ sufficiently large.

Claim 4. *For a fixed $t > 0$, the family $(\mu_t^l)_{l\in\mathbb{N}}$ is tight.*

Given a fixed $\delta > 0$, we have to find a compact set K_δ, such that $\mu_t^l(K_\delta) \geq 1-\delta$ for every $l \in \mathbb{N}$. To this aim we take suitable ϵ and h such that (6.4.28) ensures that

$$\mu_t^l(\mathcal{R}_\infty^{h+1}) \geq 1 - \frac{\delta}{2}. \tag{6.4.29}$$

Moreover, combining the result of Step 1 (that is (6.4.26)) with (6.4.25), we conclude that $S_{N,\mathfrak{m}_\infty}(\mu_t^l)$ is definitely bounded. Then, since $\mathfrak{m}_\infty|_{\mathcal{R}_\infty^{h+1}}$ is a finite Radon measure, we can argue as in the proof of Lemma 6.4.6 and prove the tightness of the family of measures $(\mu_t^l|_{\mathcal{R}_\infty^{h+1}})_{l\in\mathbb{N}}$. As a consequence, keeping in mind (6.4.29), there exists a compact set K_δ such that

$$\mu_t^l(K_\delta) \geq \mu_t^l|_{\mathcal{R}_\infty^{h+1}}(K_\delta) \geq 1-\delta,$$

proving the claim.

Now, we can apply Lemma 6.4.12 and find a Wasserstein geodesic $(\mu_t)_{t\in[0,1]} \subset \mathscr{P}(\mathsf{X}_\infty)$ connecting μ_0 and μ_1 such that (up to subsequences)

$$\mu_t^l \rightharpoonup \mu_t \in \mathscr{P}(\mathsf{X}_\infty) \qquad \text{as } l\to\infty \text{ for every } t\in[0,1]\cap\mathbb{Q}$$

Then, since the bound (6.4.28) is uniform in l, we can conclude that

$$\mu_t(\mathcal{R}_\infty^{h+1}) \geq 1 - \Omega(k(\epsilon)+1, h-1, M+2, 2\epsilon), \tag{6.4.30}$$

for every $t \in [0,1]\cap\mathbb{Q}$ and every $h \in \mathbb{N}$ sufficiently large. Moreover, by continuity we can deduce (6.4.30) for every time $t \in [0,1]$ (and every $h \in \mathbb{N}$ sufficiently large). This is sufficient to conclude that μ_t gives no mass to the set $\mathcal{S}$ of singular points, for every $t \in [0,1]$. In fact, assume by contradiction that $\mu_t(\mathcal{S}) = \delta > 0$, then condition (6.3.9) on Ω ensures that there exist ϵ and $h \in \mathbb{N}$ such that

$$\Omega(k(\epsilon)+1, h-1, M+2, 2\epsilon) < \delta,$$

and consequently

$$\mu_t(\mathcal{R}_\infty^{h+1}) \geq 1-\delta,$$

which contradicts $\mu_t(\mathcal{S}) = \delta$. At this point we know from Proposition 6.4.8 that

$$\liminf_{l\to\infty} S_{N',\mathfrak{m}_\infty}(\mu_t^l) \geq S_{N',\mathfrak{m}_\infty}(\mu_t), \tag{6.4.31}$$

for every $t \in [0,1]\cap\mathbb{Q}$ and $N' \in [N,0)$.

Finally, we can use (6.4.31) and (6.4.26) to pass to the limit as $l\to\infty$ of the inequality (6.4.25) and deduce that

$$S_{N',\mathfrak{m}_\infty}(\mu_t) \leq T_{K,N'}^{(t)}(q|\mathfrak{m}_\infty) \tag{6.4.32}$$

holds for every $t \in [0,1]\cap\mathbb{Q}$ and every $N' \in I$. Then the lower semicontinuity of $S_{N',\mathfrak{m}_\infty}$ (granted by (6.4.30)) and the continuity of $T_{K,N'}^{(t)}(q|\mathfrak{m}_\infty)$ in t (which is a straightforward consequence of the dominated convergence theorem), allow to conclude (6.4.32) for every $t \in [0,1]$, finishing the proof.

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Paper 7

Optimal maps and local-to-global property in negative dimensional spaces with Ricci curvature bounded from below

with Chiara Rigoni

In this paper we investigate two important properties of metric measure spaces satisfying the reduced curvature-dimension condition for negative values of the dimension parameter: the existence of a transport map between two suitable marginals and the so-called local-to-global property.

All authors of this paper contributed equally to all results.

7.1 Introduction

The class of metric measure spaces satisfying the $\mathsf{CD}(K,N)$-condition for $N<0$ was first introduced by Ohta in [Oht16], where the range of admissible "dimension parameters" in the theories of (K,N)-convex functions and of $\mathsf{CD}(K,N)$-spaces was extended to negative values of N. A further study of this family of spaces has been carried out in the recent work [MRS23], where a new setting to introduce the curvature-dimension condition for negative values of N was proposed, extending and complementing the work by Ohta. The structures on which the study is focused are complete and separable metric spaces endowed with quasi-Radon measures, i.e., measures which are Radon outside a negligible set of singular points in which the measure can explode. In particular, this setting is more general than the classical one where the notion of curvature-dimension bounds is introduced for nonsmooth structures (see [Stu06a], [Stu06b], [LV09]): in fact, in the case in which the dimensional parameter is positive, the CD-condition is introduced in complete and separable metric spaces equipped with Radon measures. In this regard, it is important to recall that the class of metric measure spaces satisfying the $\mathsf{CD}(K,N)$-condition for some $N<0$ includes all $\mathsf{CD}(K,\infty)$ spaces and so also all the $\mathsf{CD}(K,N)$ ones for any $N>0$. Furthermore, in [Oht16] the definition of reduced curvature-dimension condition was extended to the setting of negative values of the dimensional parameter. This condition, denoted by $\mathsf{CD}^*(K,N)$ and first introduced by Bacher-Sturm in [BS10] for $N\geq 1$, is obtained from the $\mathsf{CD}(K,N)$-condition by replacing the volume distortion coefficients $\tau^{(t)}_{K,N}(\cdot)$ by the slightly smaller coefficients $\sigma^{(t)}_{K,N}(\cdot)$. Analogously to the result for $N\geq 1$ (see [BS10, Proposition 2.5(i)]), also in the case when $N<0$ the original curvature-condition $\mathsf{CD}(K,N)$ implies the reduced one $\mathsf{CD}^*(K,N)$, as proven in [Oht16, Proposition 4.7].

In this paper we prove two important properties of metric measure spaces satisfying the reduced curvature-dimension condition for negative values of the dimension parameter: the existence of a transport map between two suitable marginals and the so-called local-to-global property. Their validity for positive dimensional CD spaces has been established in a series of works that we are going to recall in the following. It is important to underline that in this setting the proofs of these two properties rely heavily on the lower semicontinuity of the characterizing entropy functional, which allows to perform some nice approximation arguments. Unfortunately in the context of quasi-Radon measures the lower semicontinuity of the entropy functional does not hold on the whole $\mathscr{P}_2(\mathsf{X})$: instead our proofs are based on suitable extensions of these classical arguments, in which we have to pay particular attention to the set of singular points of the reference measure.

The problem of addressing existence and/or uniqueness of optimal transport maps between two given marginals has a long history, since it represents the original formulation of the optimal transport problem by Monge. The first positive answers were given by Brenier [Bre87] in the Euclidean setting, by McCann [McC01] in the setting of Riemannian manifolds, by Ambrosio-Rigot [AR04] and Figalli-Rifford [FR10] in the context of sub-Riemannian manifolds and by Bertrand [Ber08] for Alexandrov spaces. In the context of metric measure spaces, most of the results are proven under a non-branching assumption and a metric curvature bound. In particular, we recall the works by Gigli [Gig12], Rajala-Sturm [RS14], Gigli-Rajala-Sturm [GRS16] and Cavalletti-Mondino [CM17a]. Some results can also be obtained once the space is assumed to be non-branching and a quantitative property on the reference measure, related to the shrinking of sets to points, holds true (see the work by Cavalletti-Huesmann [CH15] and the one by Kell [Kel17]). As shown by Rajala in [Raj16], the non-branching assumption is necessary to prove the uniqueness of the transport map. Nevertheless, some existence results can be proven also without this assumption, see for example the paper by Shultz [Sch18] and by the first author [Mag22b].

In this paper we focus on the case of non-branching spaces satisfying the $\mathsf{CD}^*(K,N)$-condition for some $N<0$ and we prove the existence and uniqueness of the optimal transport map when the marginals have finite entropy and bounded support. We then use this uniqueness result in order to show the existence of a transport map between two general absolutely continuous marginals with possibly unbounded support.

Also the local-to-global property of the reduced curvature-dimension condition is a very important and fundamental feature. In fact, it shows that the metric measure space version of curvature-dimension bounds is a local requirement, as it happens in the case of Ricci curvature bounds in Riemannian manifolds. In the positive dimensional case, the local-to-global property was proven by Sturm [Stu06b, Theorem 4.17] for the $\mathsf{CD}(K,\infty)$-condition, by Villani [Vil09] for the $\mathsf{CD}(0,N)$-condition and then by Bacher-Sturm [BS10, Theorem 5.1] for the general $\mathsf{CD}^*(K,N)$ one. Thereafter, the globalization of the $\mathsf{CD}(K,N)$ condition was proven by Cavalletti-Milman [CM21] with a much more sophisticated argument, which allows them to demonstrate other remarkable properties of the CD condition. We remark that a result similar to the one in [CM21] seems out of reach in the context of $\mathsf{CD}(K,N)$ spaces with $N<0$, due to the pathologies of the reference measure: what we prove in the last section of this paper is the equivalence of the the local version of $\mathsf{CD}^*(K-,N)$ to a global condition $\mathsf{CD}^*(K-,N)$, provided that the metric measure space $(\mathsf{X},\mathsf{d},\mathfrak{m})$ is locally compact.

7.2 Setting and preliminary results

7.2.1 Metric spaces and Wasserstein distance

In this paper, (X, d) will always denote a complete and separable metric space. The set $\mathscr{P}_2(\mathsf{X})$ is the set of probability measures with finite second moment, that we endow with the Wasserstein distance W_2, defined by

$$W_2^2(\mu, \nu) := \inf_{\pi \in \mathsf{Adm}(\mu,\nu)} \int \mathsf{d}^2(x, y) \, \mathrm{d}\pi(x, y), \tag{7.2.1}$$

where $\mathsf{Adm}(\mu, \nu)$ is the set of all the admissible transport plans between μ and ν, namely all the measures in $\mathscr{P}(\mathsf{X}^2)$ such that $(\mathsf{p}_1)_\sharp \pi = \mu$ and $(\mathsf{p}_2)_\sharp \pi = \nu$. The set of optimal transport plans between μ and ν, that is the set of the admissible plans which realize the infimum in (7.2.1), is denoted by $\mathrm{OptPlans}(\mu, \nu)$.

In the following $C([0, 1], \mathsf{X})$ will stand for the space of continuous curves from $[0, 1]$ to X, endowed with the sup-norm, and we recall that this is a complete and separable space. Hence, for any $t \in [0, 1]$ we define the evaluation map $e_t : C([0, 1], \mathsf{X}) \to \mathsf{X}$ by setting $e_t(\gamma) := \gamma_t$ and the stretching/restriction operator restr_r^s in $C([0, 1], \mathsf{X})$, defined, for all $0 \leq r < s \leq 1$, by

$$[\mathrm{restr}_r^s(\gamma)]_t := \gamma_{r+t(s-r)}, \qquad t \in [0, 1].$$

A curve $\gamma : [0, 1] \to \mathsf{X}$ is a (minimizing constant speed) geodesic if

$$\mathsf{d}(\gamma_s, \gamma_t) = |t - s| \mathsf{d}(\gamma_0, \gamma_1) \quad \text{for every } s, t \in [0, 1],$$

we indicate by $\mathrm{Geo}(\mathsf{X})$ the space of geodesics on X, endowed with the sup-norm. Observe that $\mathrm{Geo}(\mathsf{X})$ is complete and separable as soon as (X, d) satisfies the same properties.

In this terminology, if $\mu, \nu \in \mathscr{P}_2(\mathsf{X})$ are joined by a geodesic, then their W_2-distance can be equivalently characterized as

$$W_2^2(\mu, \nu) = \min_{\boldsymbol{\pi}} \int_0^1 |\dot{\gamma}_t|^2 \, \mathrm{d}t \, \mathrm{d}\boldsymbol{\pi}(\gamma),$$

the minimum being taken among all $\boldsymbol{\pi} \in \mathscr{P}(C([0, 1], \mathsf{X}))$ such that $(e_0)_\sharp \boldsymbol{\pi} = \mu$ and $(e_1)_\sharp \boldsymbol{\pi} = \nu$. The set of minimizers will be denoted by $\mathrm{OptGeo}(\mu, \nu)$ and its elements will be called optimal geodesic plans. We underline that optimal geodesic plans are always supported in the set $\mathrm{Geo}(\mathsf{X})$ and that a curve $(\mu_t)_{t \in [0,1]}$ is a geodesic connecting μ and ν if and only if there exists $\boldsymbol{\pi} \in \mathrm{OptGeo}(\mu, \nu)$ such that

$$\mu_t = (e_t)_\sharp \boldsymbol{\pi}.$$

7.2.2 Metric spaces equipped with quasi-Radon measures

In this section we present the setting we are going to work in. In order to motivate it, we recall that in [MRS23] some new 1-dimensional model space have been provided, such as:

1. the weighted space

$$\left(\mathbb{R}, |\cdot|, V \mathscr{L}^1\right) \quad \text{with } V(x) = \sinh\left(x\sqrt{-\frac{K}{N}}\right)^N,$$

   which satisfies the the curvature-dimension condition $\mathsf{CD}(K, N + 1)$ for any $K > 0$ and $N < -1$;

2. the space $(\mathbb{R}, |\cdot|, |x|^N \mathscr{L}^1)$, which is a $\mathsf{CD}(0, N + 1)$-space for any $N < -1$;

3. the weighted space

$$\left(\left[-\frac{\pi}{2}\sqrt{\frac{K}{N}},\frac{\pi}{2}\sqrt{\frac{K}{N}}\right],|\cdot|,\cos\left(x\sqrt{\frac{K}{N}}\right)^{N}\mathscr{L}^1\right)$$

which is a $\mathsf{CD}(K,N+1)$-space for any $K<0$ and $N<-1$.

These examples show in particular that, in negative dimensional CD spaces, the reference measure does not need to be locally finite. Therefore we introduce the class of quasi-Radon measures, specializing to the case of measures defined on a complete and separable metric space. We refer to [MRS23, Section 2.1] for the precise definitions and constructions, presented in the more general setting of topological measure spaces.

Definition 7.2.1 (Quasi-Radon measures)**.** Let $\mathfrak{m}$ be a Borel measure defined on a complete and separable metric space (X,d). We say that $\mathfrak{m}$ is a quasi-Radon measure if it is complete and effectively locally finite, meaning that for every Borel $A\subset\mathsf{X}$ with $\mathfrak{m}(A)>0$, there exists an open set $U\subset\mathsf{X}$ with finite measure such that $\mathfrak{m}(A\cap U)>0$.

In particular, in [MRS23, Proposition 2.7], it is proven that any quasi-Radon measure $\mathfrak{m}$ defined on a complete and separable metric space (X,d) is σ-finite and has the following property:

$$\begin{gathered}\text{there exists a closed set } \mathcal{S}_{\mathfrak{m}}\subset\mathsf{X} \text{ with empty interior and } \mathfrak{m}(\mathcal{S}_{\mathfrak{m}})=0\\ \text{such that } \mathfrak{m}|_{\mathsf{X}\setminus\mathcal{S}_{\mathfrak{m}}} \text{ is a Radon measure on } \mathsf{X}\setminus\mathcal{S}_{\mathfrak{m}}.\end{gathered}$$

Intuitively, the set $\mathcal{S}_{\mathfrak{m}}$ consists of all points x such that $\mathfrak{m}(U)=\infty$ for any open neighborhood U of x in X. In the following, we will refer to $\mathcal{S}_{\mathfrak{m}}$ as the singular set associated to $\mathfrak{m}$.

We recall that in this setting it is still possible to speak about absolutely continuity of a measure μ with respect to $\mathfrak{m}$ (see [MRS23, Definition 2.8, Proposition 2.9]) and that, in this case, a suitable extension of the Radon-Nikodym Theorem holds (see [MRS23, Theorem 2.10]). In fact, if a measure μ is absolutely continuous with respect to the quasi-Radon reference measure $\mathfrak{m}$, then there exists a measurable function f on X such that for any $B\in\mathcal{B}(\mathsf{X})$ it holds

$$\mu(B)=\int_B f\,\mathrm{d}\mathfrak{m}.$$

In the following we will always refer to structures $(\mathsf{X},\mathsf{d},\mathfrak{m})$ where (X,d) is a complete and separable metric space and $\mathfrak{m}$ is a quasi-Radon measure on X, $\mathfrak{m}\neq 0$. Moreover, up to consider the metric measure space $(\mathrm{supp}(\mathfrak{m}),\mathsf{d},\mathfrak{m})$, we can assume, without losing generality, that $\mathfrak{m}$ has full support, that is $\mathrm{supp}(\mathfrak{m})=\mathsf{X}$.

7.2.3 Reduced Curvature-Dimension Condition $\mathsf{CD}^*(K,N)$ for $N<0$

Let $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be a metric measure space and $N<0$. We introduce the *Rényi entropy* $S_{N,\mathfrak{m}}$ with respect to the reference measure $\mathfrak{m}$ as the functional defined on $\mathscr{P}(\mathsf{X})$ by

$$S_{N,\mathfrak{m}}(\mu):=\begin{cases}\displaystyle\int_{\mathsf{X}}\rho(x)^{\frac{N-1}{N}}d\mathfrak{m}(x) & \text{if } \mu\ll\mathfrak{m},\ \mu=\rho\mathfrak{m},\\ +\infty & \text{otherwise.}\end{cases}$$

As already pointed out in the introduction, if $\mathfrak{m}$ is a Radon measure, the functional $S_{N,\mathfrak{m}}$ is lower semicontinuous with respect to the weak topology and thus it is lower semicontinuous with respect to the Wasserstein convergence in $\mathscr{P}_2(\mathsf{X})$. Unfortunately, the same property does not hold for quasi-Radon reference measures $\mathfrak{m}$, but it is possible to prove the following nice result (see [MRS23, Proposition 4.8]):

Proposition 7.2.2. *In a metric measure space* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *the Rényi entropy functional* $S_{N,\mathfrak{m}}$ *is weakly lower semicontinuous on the space*

$$\mathscr{P}^{\mathcal{S}}(\mathsf{X}, \mathfrak{m}) := \{\mu \in \mathscr{P}_2(\mathsf{X}) \,:\, \mu(\mathcal{S}_\mathfrak{m}) = 0\},$$

where $\mathcal{S}_\mathfrak{m}$ *is the singular set associated to the measure* $\mathfrak{m}$.

Before going on, we define some subsets of $\mathscr{P}(\mathsf{X})$ that will be relevant later on:

$$\begin{aligned}
&\mathscr{P}_{ac}(\mathsf{X}, \mathfrak{m}) := \{\mu \in \mathscr{P}_2(\mathsf{X}) : \mu \ll \mathfrak{m}\},\\
&\mathscr{P}_{\infty}(\mathsf{X}, \mathfrak{m}) := \{\mu \in \mathscr{P}_{ac}(\mathsf{X}, \mathfrak{m}) : \mu \text{ has bounded support}\},\\
&\mathscr{P}^*_N(\mathsf{X}, \mathfrak{m}) := \{\mu \in \mathscr{P}_{\infty}(\mathsf{X}, \mathfrak{m}) : S_{N,\mathfrak{m}}(\mu) < \infty\}.
\end{aligned}$$

Observe that, since $\mathfrak{m}(\mathcal{S}_\mathfrak{m}) = 0$, it holds that $\mathscr{P}_{ac}(\mathsf{X}, \mathfrak{m}) \subset \mathscr{P}^{\mathcal{S}}(\mathsf{X}, \mathfrak{m})$, thus in particular the entropy functional $S_{N,\mathfrak{m}}$ is weakly lower semicontinuous on $\mathscr{P}_{ac}(\mathsf{X}, \mathfrak{m})$.

In order to give the definition of reduced curvature-dimension bounds for negative values of the dimensional parameter, we need also to introduce the following distortion coefficients for $N < 0$:

$$\sigma^{(t)}_{K,N}(\theta) := \begin{cases} \infty, & \text{if } K\theta^2 \leq N\pi^2,\\ \dfrac{\sin(t\theta\sqrt{K/N})}{\sin(\theta\sqrt{K/N})} & \text{if } N\pi^2 < K\theta^2 < 0,\\ t & \text{if } K\theta^2 = 0,\\ \dfrac{\sinh(t\theta\sqrt{-K/N})}{\sinh(\theta\sqrt{-K/N})} & \text{if } K\theta^2 > 0. \end{cases}$$

Notice that, for every $N < 0$ and every $t \in [0, 1]$, the map

$$\theta \mapsto \sigma^{(t)}_{K,N}(\theta) \text{ is increasing if } K \leq 0 \text{ and decreasing if } K \geq 0. \tag{7.2.2}$$

We are then ready to define the notion of reduced curvature-dimension condition for negative values of the dimensional parameter N, which was introduced for the first time by Ohta in [Oht16].

Definition 7.2.3. For any couple of measures $\mu_0, \mu_1 \in \mathscr{P}_{ac}(\mathsf{X}, \mathfrak{m})$, $\mu_i = \rho_i\mathfrak{m}$ and any coupling $\pi \in \mathscr{P}(\mathsf{X} \times \mathsf{X})$ between them, we denote by $R^{(t)}_{K,N}(\pi|\mathfrak{m})$ the functional defined by

$$R^{(t)}_{K,N}(\pi|\mathfrak{m}) := \int_{\mathsf{X}\times\mathsf{X}} \left[\sigma^{(1-t)}_{K,N}(\mathsf{d}(x,y))\rho_0(x)^{-\frac{1}{N}} + \sigma^{(t)}_{K,N}(\mathsf{d}(x,y))\rho_1(y)^{-\frac{1}{N}}\right] \mathrm{d}\pi(x,y).$$

Definition 7.2.4 (CD^* condition)**.** For fixed $K \in \mathbb{R}, N \in (-\infty, 0)$, we say that a metric measure space $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ satisfies the $\mathsf{CD}^*(K, N)$-*condition* if for each pair of measures $\mu_0 = \rho_0\mathfrak{m}$, $\mu_1 = \rho_1\mathfrak{m} \in \mathscr{P}^*_N(\mathsf{X}, \mathfrak{m})$ there exists an optimal coupling $\pi \in \mathrm{OptPlans}(\mu_0, \mu_1)$ and a W_2-geodesic $\{\mu_t\}_{t\in[0,1]} \subset \mathscr{P}^*_N(\mathsf{X}, \mathfrak{m})$ such that

$$S_{N',\mathfrak{m}}(\mu_t) \leq R^{(t)}_{K,N'}(\pi|\mathfrak{m}) \tag{7.2.3}$$

holds for every $t \in [0, 1]$ and every $N' \in [N, 0)$.

Remark 7.2.5. We point out that Definition 7.2.4 is not exactly the same as the one adopted by Ohta in [Oht16], where the condition is required to hold for every pair of absolutely continuous marginals. The main difference is basically that we ask the space $\mathscr{P}^*_N(\mathsf{X}, \mathfrak{m})$ to be geodesically convex even when inequality (7.2.3) does not ensure it, as it may happen when $K < 0$, due to the pathologies of the distortion coefficients. This is a technical detail but it is fundamental in this paper. On the other hand, let us point out that Definition 7.2.4 is consistent to the first definition of reduced CD-condition, that was introduced by Bacher and Sturm in [BS10] in the positive dimensional case.

Finally we invite the reader to compare this reduced curvature-dimension condition $\mathsf{CD}^*(K,N)$ with the original $\mathsf{CD}(K,N)$ one (see [Oht16] and [MRS23]).

7.2.4 Essentially Non-Branching Metric Measure Spaces

In this section we briefly introduce the notion of essentially non-branching condition on a metric measure space, that was pioneered by Rajala and Sturm in [RS14], proving also a result that will be fundamental in the following. This property is a weakening of the classical non-branching condition and it fits better the context of metric measure spaces, since it takes into account also the reference measure. An example of this is the result by Rajala and Sturm, which ensures that every strong $\mathsf{CD}(K,\infty)$ space is essentially non-branching.

Definition 7.2.6. In a metric space (X,d), a subset $G\subset \mathrm{Geo}(\mathsf{X})$ is called non-branching if for any pair of geodesics $\gamma_1,\gamma_2\in G$ such that $\gamma_1\neq\gamma_2$, it holds that

$$\mathrm{restr}_0^t\gamma_1\neq \mathrm{restr}_0^t\gamma_1 \qquad \text{for every } t\in(0,1).$$

A metric measure space $(\mathsf{X},\mathsf{d},\mathfrak{m})$ is said to be essentially non-branching if for every absolutely continuous measures $\mu_0,\mu_1\in\mathscr{P}_{ac}(\mathsf{X},\mathfrak{m})$, every optimal geodesic plan η connecting them is concentrated on a non-branching set of geodesics.

The non-branching assumption for (X,d) can be equivalently characterized by requiring that the map $(e_0,e_t)\colon \mathrm{Geo}(\mathsf{X})\to\mathsf{X}^2$ is s injective for some, and thus for any, $t\in(0,1)$.

As one can intuitively realize, the essentially non-branching condition turns out to be useful in order to prevent mass overlap of two Wasserstein geodesics at any intermediate time. The following proposition provides an interesting result in this direction; similar statements can be find in ([BS10, Lemma 2.8] [EKS15, Lemma 3.11]), but, in our setting, we need a slightly more general result. The proof relies on a clever mixing argument, that was used for a different purpose in [RS14]. The same mixing procedure will be very useful in the proof of Proposition 7.3.1.

Proposition 7.2.7. *Let $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be an essentially non-branching metric measure space. Given the probability measures $\mu_0,\nu_0,\mu_1,\nu_1\in\mathscr{P}_{ac}(\mathsf{X},\mathfrak{m})$, consider two optimal geodesic plans $\boldsymbol{\pi}^\mu\in\mathrm{OptGeo}(\mu_0,\mu_1)$ and $\boldsymbol{\pi}^\nu\in\mathrm{OptGeo}(\nu_0,\nu_1)$, and their geodesic representation $(\mu_t)_{t\in[0,1]}=(e_t)_\#\boldsymbol{\pi}^\mu$ and $(\nu_t)_{t\in[0,1]}=(e_t)_\#\boldsymbol{\pi}^\nu$. Assume that $\boldsymbol{\pi}^\mu\perp\boldsymbol{\pi}^\nu$ and that there is an optimal transport plan $\pi\in\mathrm{OptPlans}$ such that*

$$(e_0,e_1)_\#\boldsymbol{\pi}^\mu,(e_0,e_1)_\#\boldsymbol{\pi}^\nu\ll\pi.$$

For every $t\in(0,1)$ if $\mu_t,\nu_t\ll\mathfrak{m}$, then $\mu_t\perp\nu_t$.

Proof. The proof will be done by contradiction, thus assume there exists $t\in(0,1)$ such that $\mu_t\not\perp\nu_t$. Consider the measures

$$\boldsymbol{\pi}^{\mathrm{left}}=\frac{1}{2}\left((\mathrm{restr}_0^t)_\#\boldsymbol{\pi}^\mu+(\mathrm{restr}_0^t)_\#\boldsymbol{\pi}^\nu\right),\qquad \boldsymbol{\pi}^{\mathrm{right}}=\frac{1}{2}\left((\mathrm{restr}_t^1)_\#\boldsymbol{\pi}^\mu+(\mathrm{restr}_t^1)_\#\boldsymbol{\pi}^\nu\right),$$

then let $\{\boldsymbol{\pi}_x^{\mathrm{left}}\}_{x\in\mathsf{X}}$ be the disintegration of $\boldsymbol{\pi}^{\mathrm{left}}$ with respect to e_1 and let $\{\boldsymbol{\pi}_x^{\mathrm{right}}\}_{x\in\mathsf{X}}$ be the disintegration of $\boldsymbol{\pi}^{\mathrm{right}}$ with respect to e_0. Hence, define the measure

$$\eta:=(e_0)_\#\boldsymbol{\pi}^{\mathrm{right}}=(e_1)_\#\boldsymbol{\pi}^{\mathrm{left}}=\frac{\mu_t+\nu_t}{2}=(e_t)_\#\left(\frac{\boldsymbol{\pi}^\mu+\boldsymbol{\pi}^\nu}{2}\right).$$

Consider the splitting map

$$\begin{aligned}\mathrm{Sp}:C([0,1];\mathsf{X})&\to\left\{(\gamma^1,\gamma^2)\in C([0,1];\mathsf{X})\times C([0,1];\mathsf{X})\,:\,\gamma_1^1=\gamma_0^2\right\}\\ \gamma&\mapsto(\mathrm{restr}_0^t\gamma,\mathrm{restr}_t^1\gamma),\end{aligned}$$

and notice that the map Sp is bi-Lipschitz, ensuring the existence of its (measurable) inverse

$$\mathrm{Sp}^{-1} : \Big\{(\gamma^1, \gamma^2) \in C([0,1];\mathsf{X}) \times C([0,1];\mathsf{X}) \,:\, \gamma^1_1 = \gamma^2_0\Big\} \to C([0,1];\mathsf{X}).$$

Now define the collection of measures $\{\boldsymbol{\pi}_x\}_{x\in\mathsf{X}} \subset \mathscr{P}(C([0,1];\mathsf{X}))$ as

$$\boldsymbol{\pi}_x := (\mathrm{Sp}^{-1})_\#(\boldsymbol{\pi}_x^{\mathrm{left}} \times \boldsymbol{\pi}_x^{\mathrm{right}})$$

and finally introduce the "mixed measure" $\boldsymbol{\pi}^{\mathrm{mix}} \in \mathscr{P}(C([0,1];\mathsf{X}))$ as

$$\boldsymbol{\pi}^{\mathrm{mix}}(\mathrm{d}\gamma) = \boldsymbol{\pi}_x(\mathrm{d}\gamma)\eta(\mathrm{d}x).$$

The next few passages of the proof are designed to prove that $\boldsymbol{\pi}^{\mathrm{mix}}$ is an optimal geodesic plan. First of all notice that, since $\pi \in \mathrm{OptPlans}$, there exists a cyclically monotone set $\Gamma \subset \mathsf{X}\times\mathsf{X}$ with $\pi(\Gamma) = 1$. Consider then the set

$$\tilde\Gamma = \{\gamma \in \mathrm{Geo}(\mathsf{X}) \,:\, (\gamma_0, \gamma_1) \in \Gamma\} = (e_0, e_1)^{-1}(\Gamma).$$

Since $(e_0, e_1)_\#\boldsymbol{\pi}^\mu \ll \pi$, it holds

$$\boldsymbol{\pi}^\mu(\tilde\Gamma) = \boldsymbol{\pi}^\mu((e_0, e_1)^{-1}(\Gamma)) = (e_0, e_1)_\#(\boldsymbol{\pi}^\mu)(\Gamma) = 1,$$

and similarly $\boldsymbol{\pi}^\nu(\tilde\Gamma) = 1$. Hence for every pair of curves $\gamma_1, \gamma_2 \in \tilde\Gamma$ with $\gamma^1_t = \gamma^2_t$, the cyclical monotonicity, together with the triangular inequality, gives that

$$\begin{aligned}
\mathsf{d}^2(\gamma^1_0, \gamma^1_1) + \mathsf{d}^2(\gamma^2_0, \gamma^2_1) &\le \mathsf{d}^2(\gamma^1_0, \gamma^2_1) + \mathsf{d}^2(\gamma^2_0, \gamma^1_1)\\
&\le \Big(tl(\gamma^1) + (1-t)l(\gamma^2)\Big)^2 + \Big(tl(\gamma^2) + (1-t)l(\gamma^1)\Big)^2\\
&= l(\gamma^1)^2 + l(\gamma^2)^2 - 2t(1-t)\Big(l(\gamma^1) - l(\gamma^2)\Big)^2\\
&\le l(\gamma^1)^2 + l(\gamma^2)^2 = \mathsf{d}^2(\gamma^1_0, \gamma^1_1) + \mathsf{d}^2(\gamma^2_0, \gamma^2_1),
\end{aligned} \tag{7.2.4}$$

and so all the inequalities in the above chain (7.2.4) are equalities, in particular $l(\gamma^1) = l(\gamma^2)$. Thus, for every $x \in e_t(\tilde\Gamma) =: \tilde\Gamma_t$, there exists l_x such that $l(\gamma) = l_x$, for every $\gamma \in \tilde\Gamma$ with $\gamma_t = x$. On the other hand, notice that

$$\eta(\tilde\Gamma_t) = (e_t)_\#\Big(\frac{\boldsymbol{\pi}^\mu + \boldsymbol{\pi}^\nu}{2}\Big)(e_t(\tilde\Gamma)) \ge \frac{\boldsymbol{\pi}^\mu + \boldsymbol{\pi}^\nu}{2}(\tilde\Gamma) = 1.$$

Moreover, using once again that $\boldsymbol{\pi}^\mu(\tilde\Gamma) = \boldsymbol{\pi}^\nu(\tilde\Gamma) = 1$, follows that, for η-almost every $x \in \mathsf{X}$, $\boldsymbol{\pi}_x^{\mathrm{left}}$ is concentrated on geodesics of type $\mathrm{restr}_0^t\gamma$ with $\gamma \in \tilde\Gamma$ and $\boldsymbol{\pi}_x^{\mathrm{right}}$ is concentrated on geodesics of type $\mathrm{restr}_0^t\gamma$ with $\gamma \in \tilde\Gamma$. Therefore the measure $\boldsymbol{\pi}^{\mathrm{mix}}$ is concentrated on the set

$$\Big\{\gamma \in C([0,1];\mathsf{X}) \,:\, \text{there exist } \gamma^1, \gamma^2 \in \tilde\Gamma \text{ s.t. } \mathrm{restr}_0^t\gamma = \mathrm{restr}_0^t\gamma^1 \text{ and } \mathrm{restr}_t^1\gamma = \mathrm{restr}_t^1\gamma^2\Big\},$$

which, by the equalities in (7.2.4), is a subset of $\mathrm{Geo}(\mathsf{X})$.
Furthermore, since $\boldsymbol{\pi}^\mu$, $\boldsymbol{\pi}^\nu$ and $\boldsymbol{\pi}^{\mathrm{mix}}$ are concentrated in $\tilde\Gamma$, it holds that

$$\begin{aligned}
\int_{\mathrm{Geo}(\mathsf{X})} \mathsf{d}^2(\gamma_0, \gamma_1)\,\mathrm{d}\Big(\frac{\boldsymbol{\pi}^1 + \boldsymbol{\pi}^2}{2}\Big)(\gamma) &= \int_{\mathsf{X}} l_x^2\,\mathrm{d}\left[(e_t)_\#\Big(\frac{\boldsymbol{\pi}^1 + \boldsymbol{\pi}^2}{2}\Big)\right](x)\\
&= \int_{\mathsf{X}} l_x^2\,\mathrm{d}\eta(x)\\
&= \int_{\mathsf{X}} l_x^2\,\mathrm{d}((e_t)_\#\boldsymbol{\pi}^{\mathrm{mix}})(x) = \int_{\mathrm{Geo}(\mathsf{X})} \mathsf{d}^2(\gamma_0, \gamma_1)\,\mathrm{d}\boldsymbol{\pi}^{\mathrm{mix}}(\gamma).
\end{aligned}$$

On the other hand $(e_0, e_1)_\# \left(\frac{1}{2}(\boldsymbol{\pi}^\mu + \boldsymbol{\pi}^\nu)\right)$ is an optimal transport plan, moreover $(e_i)_\# \boldsymbol{\pi}^{\text{mix}} = (e_i)_\# \left(\frac{1}{2}(\boldsymbol{\pi}^\mu + \boldsymbol{\pi}^\nu)\right)$ for $i = 0, 1$, thus the measure $\boldsymbol{\pi}^{\text{mix}}$ is an optimal geodesic plan.

Now, call ρ_t^μ, ρ_t^ν the densities with respect to $\mathfrak{m}$ of μ_t and ν_t, respectively. Since $\mu_t \not\perp \nu_t$ there exists a set $E \subset \mathsf{X}$ of positive $\mathfrak{m}$-measure, where both ρ_t^μ and ρ_t^ν are strictly positive. Notice moreover that, for $\mathfrak{m}$-almost every $x \in E$, at least one of the measures $\boldsymbol{\pi}_x^{\text{left}}$ and $\boldsymbol{\pi}_x^{\text{right}}$ is not a Dirac mass, because $\boldsymbol{\pi}^\mu \perp \boldsymbol{\pi}^\nu$. Suppose in the first instance that $\boldsymbol{\pi}_x^{\text{right}}$ is not a Dirac measure for a $\mathfrak{m}$-positive set $E' \subset E$. Observe that, since both ρ_t^μ and ρ_t^ν are strictly positive in E, we have that $\eta(E') > 0$. Let $A \subseteq \text{Geo}(\mathsf{X})$ with full $\boldsymbol{\pi}^{\text{mix}}$-measure, that is

$$1 = \boldsymbol{\pi}^{\text{mix}}(A) = \int_A \mathrm{d}\boldsymbol{\pi}^{\text{mix}}(\gamma) = \int_{\mathsf{X}} \int_A \boldsymbol{\pi}_x(\mathrm{d}\gamma)\eta(\mathrm{d}x),$$

then it must hold that $\int_A \boldsymbol{\pi}_x(\mathrm{d}\gamma) = 1$ for η-almost every x. On the other hand, we have

$$\begin{aligned}\int_A \boldsymbol{\pi}_x(\mathrm{d}\gamma) &= \int_A \left[(\text{Sp}^{-1})_\#(\boldsymbol{\pi}_x^{\text{left}} \times \boldsymbol{\pi}_x^{\text{right}})\right](\mathrm{d}\gamma) = \int_{\text{Sp}(A)} \mathrm{d}[\boldsymbol{\pi}_x^{\text{left}} \times \boldsymbol{\pi}_x^{\text{right}}](\gamma_1, \gamma_2) \\ &= \int_{\text{Sp}_2(A)} \int_{A(\gamma_2)} \mathrm{d}\boldsymbol{\pi}_x^{\text{left}}(\gamma_1)\, \mathrm{d}\boldsymbol{\pi}_x^{\text{right}}(\gamma_2) \leq \int_{\text{Sp}_2(A)} \mathrm{d}\boldsymbol{\pi}_x^{\text{right}}(\gamma_2) \leq 1\end{aligned}$$

where $\text{Sp}_2(A) := \{\text{restr}_t^1 \gamma : \gamma \in A\}$ and $A(\gamma_2) := \{\gamma_1 \in C([0,1], \mathsf{X}) : \text{Sp}^{-1}(\gamma_1, \gamma_2) \in A\}$. In particular, for η-almost every x, the two inequalities in the last equation must be equalities, consequently $\boldsymbol{\pi}_x^{\text{right}}(\text{Sp}_2(A)) = 1$ and $\boldsymbol{\pi}_x^{\text{left}}(A(\gamma_2)) = 1$ for $\boldsymbol{\pi}_x^{\text{right}}$-almost every γ_2. Now, since $\boldsymbol{\pi}_x^{\text{right}}$ is not a Dirac measure for a η-positive set of x and it is concentrated in $G_x := \{\gamma \in \text{Geo}(\mathsf{X}) : \gamma(0) = x\}$, for some $x \in \mathsf{X}$ we can find $\bar{\gamma}, \tilde{\gamma} \in G_x \cap \text{Sp}_2(A)$ such that $\boldsymbol{\pi}_x^{\text{left}}(A(\bar{\gamma})), \boldsymbol{\pi}_x^{\text{left}}(A(\tilde{\gamma})) = 1$. In particular there exists $\gamma \in \text{Geo}(\mathsf{X})$ such that $\text{Sp}^{-1}(\gamma, \bar{\gamma}), \text{Sp}^{-1}(\gamma, \tilde{\gamma}) \in A$. This shows that A cannot be a non-branching set of geodesics, contradicting the hypothesis of essentially non-branching. If instead $\boldsymbol{\pi}_x^{\text{left}}$ is not a Dirac measure for a $\mathfrak{m}$-positive set, we can argue in the exact same way, considering the time-inverse geodesic transport plan $I_\# \boldsymbol{\pi}^{\text{mix}}$, defined via the mapping

$$I : \text{Geo}(\mathsf{X}) \to \text{Geo}(\mathsf{X}) \qquad \gamma \mapsto (\gamma' : [0,1] \to \mathsf{X} \quad t \mapsto \gamma_{1-t}).$$ □

7.3 Solution to Monge Problem

In this section we investigate existence and/or uniqueness of an optimal transport map in essentially non-branching $\mathsf{CD}^*(K,N)$ spaces with $N < 0$. To this aim, we follow the approach proposed by Gigli in [Gig12], developing some technical results in order to apply it to our more general context. In particular, we start by stating and proving a uniqueness result when the marginals have finite entropy and bounded support, while at the end of the section we show an existence result for general marginals. For these results the essentially non-branching assumption and Proposition 7.2.7 play a central role, but we need to add another preliminary proposition which is the extension of a result by Ambrosio and Gigli [AG13, Proposition 2.16]. Notice that the proof of Proposition 7.3.1 is similar to the one of Proposition 7.2.7, as it relies on the same mixing argument.

Proposition 7.3.1. *Let $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ be an essentially non-branching metric measure space and let $(\mu_t)_{t \in [0,1]} \subset \mathscr{P}_2(\mathsf{X})$ be a constant speed Wasserstein geodesic, such that $\mu_0, \mu_1 \in \mathscr{P}_{ac}(\mathsf{X}, \mathfrak{m})$. Then for every $t \in (0,1)$ there exists only one optimal geodesic plan $\boldsymbol{\pi}^0 \in \text{OptGeo}(\mu_0, \mu_t)$ and only one optimal geodesic plan $\boldsymbol{\pi}^1 \in \text{OptGeo}(\mu_t, \mu_1)$ and they are both induced by a map from μ_t.*

Proof. Pick $\boldsymbol{\pi}_0 \in \text{OptGeo}(\mu_0, \mu_t)$ and $\boldsymbol{\pi}_1 \in \text{OptGeo}(\mu_t, \mu_1)$, then let $\{\boldsymbol{\pi}_x^0\}_{x \in \mathsf{X}}$ be the disintegration of $\boldsymbol{\pi}^0$ with respect to e_1 and let $\{\boldsymbol{\pi}_x^1\}_{x \in \mathsf{X}}$ be the disintegration of $\boldsymbol{\pi}^1$ with respect to e_0.

Furthermore (similarly to what was done in the proof of Proposition 7.2.7), consider the splitting map

$$\begin{aligned}\mathrm{Sp} : C([0,1];\mathsf{X}) &\to \left\{(\gamma^1,\gamma^2) \in C([0,1];\mathsf{X}) \times C([0,1];\mathsf{X}) \,:\, \gamma^1_1 = \gamma^2_0\right\}\\ \gamma &\mapsto (\mathrm{restr}_0^t\gamma, \mathrm{restr}_t^1\gamma).\end{aligned}$$

and its (measurable) inverse

$$\mathrm{Sp}^{-1} : \left\{(\gamma^1,\gamma^2) \in C([0,1];\mathsf{X}) \times C([0,1];\mathsf{X}) \,:\, \gamma^1_1 = \gamma^2_0\right\} \to C([0,1];\mathsf{X}).$$

Now define the collection of measures $\{\boldsymbol{\pi}_x\}_{x\in\mathsf{X}} \subset \mathscr{P}(C([0,1];\mathsf{X}))$ as

$$\boldsymbol{\pi}_x := (\mathrm{Sp}^{-1})_\#(\boldsymbol{\pi}^0_x \times \boldsymbol{\pi}^1_x),$$

and the measure $\boldsymbol{\pi} \in \mathscr{P}(C([0,1];\mathsf{X}))$ as

$$\boldsymbol{\pi}(\mathrm{d}\gamma) = \boldsymbol{\pi}_x(\mathrm{d}\gamma)\mu_t(\mathrm{d}x).$$

In particular observe that, since $(e_1)_\#\boldsymbol{\pi}^0 = (e_0)_\#\boldsymbol{\pi}^1 = \mu_t$, it holds that

$$(\mathrm{restr}_0^t)_\#\boldsymbol{\pi} = \boldsymbol{\pi}^0 \qquad \text{and} \qquad (\mathrm{restr}_t^1)_\#\boldsymbol{\pi} = \boldsymbol{\pi}^1. \tag{7.3.1}$$

Take $\alpha := (e_0, e_t, e_1)_\#\boldsymbol{\pi} \in \mathscr{P}(\mathsf{X}^3)$, then using (7.3.1) follows that

$$\begin{aligned}\|\mathsf{d}(\mathsf{p}_1,\mathsf{p}_3)\|_{L^2(\alpha)} &\leq \|\mathsf{d}(\mathsf{p}_1,\mathsf{p}_2) + \mathsf{d}(\mathsf{p}_2,\mathsf{p}_3)\|_{L^2(\alpha)} \leq \|\mathsf{d}(\mathsf{p}_1,\mathsf{p}_2)\|_{L^2(\alpha)} + \|\mathsf{d}(\mathsf{p}_2,\mathsf{p}_3)\|_{L^2(\alpha)}\\ &= \|\mathsf{d}(e_0,e_1)\|_{L^2(\boldsymbol{\pi}^0)} + \|\mathsf{d}(e_0,e_1)\|_{L^2(\boldsymbol{\pi}^1)}\\ &= W_2(\mu_0,\mu_t) + W_2(\mu_t,\mu_1) = W_2(\mu_0,\mu_1),\end{aligned} \tag{7.3.2}$$

thus $(\mathsf{p}_1,\mathsf{p}_3)_\#\alpha = (e_0,e_1)_\#\boldsymbol{\pi}$ is an optimal transport plan. In particular, the first inequality in the chain (7.3.2) must be an equality: this ensures that $\mathsf{d}(x,z) = \mathsf{d}(x,y) + \mathsf{d}(y,z)$ for α-almost every (x,y,z), which means that x,y,z lie along a geodesic. Furthermore, since also the second inequality has to be an equality, the functions $(x,y,z) \mapsto \mathsf{d}(x,y)$ and $(x,y,z) \mapsto \mathsf{d}(y,z)$ are each a positive multiple of the other, for α-almost every (x,y,z). Thus, it follows that for α-almost every (x,y,z) it holds

$$\mathsf{d}(x,y) = t\mathsf{d}(x,z) \qquad \text{and} \qquad \mathsf{d}(y,z) = (1-t)\mathsf{d}(x,z).$$

This last information, together with (7.3.1), ensures that $\boldsymbol{\pi}$ is concentrated in $\mathrm{Geo}(\mathsf{X})$. Moreover $(e_0,e_1)_\#\boldsymbol{\pi}$ is an optimal transport plan and so $\boldsymbol{\pi}$ is an optimal geodesic plan.

We are now going to prove that $\boldsymbol{\pi}^1$ is induced by a map, the proof for $\boldsymbol{\pi}^0$ is totally analogous. Notice that the essentially non-branching assumption guarantees that $\boldsymbol{\pi}^1_x$ is a Dirac mass for μ_t-almost every $x \in \mathsf{X}$, since otherwise $\boldsymbol{\pi} \in \mathrm{OptGeo}(\mu_0,\mu_1)$ will not be concentrated in a non-branching set of geodesic. In addition, the map $x \mapsto \boldsymbol{\pi}^1_x$ is Borel measurable, therefore there exists a Borel measurable map $T : \mathsf{X} \to \mathrm{Geo}(\mathsf{X})$ such that $\boldsymbol{\pi}^1_x = \delta_{T(x)}$ for μ_t-almost every $x \in \mathsf{X}$. In particular $\boldsymbol{\pi}^1 = T_\#\mu_t$ and this is sufficient to conclude the proof. □

Let us then present the following corollary which is an easy consequence of Proposition 7.3.1.

Corollary 7.3.2. *Let $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be an essentially non-branching $\mathsf{CD}^*(K,N)$ space, for some $K \in \mathbb{R}$ and $N < 0$, and let $\boldsymbol{\pi} \in \mathrm{OptGeo}(\mu_0,\mu_1)$ be such that*

$$\mu_t = \rho_t\mathfrak{m} := (e_t)_\#\boldsymbol{\pi} \in \mathscr{P}^*_N(\mathsf{X},\mathfrak{m}) \text{ for every } t \in [0,1].$$

Then it holds that

$$\lim_{s\to 0} S_{N,\mathfrak{m}}(\mu_s) = S_{N,\mathfrak{m}}(\mu_0).$$

Proof. Observe that Proposition 7.3.1 guarantees that, for a fixed $r \in (0,1)$, there exists a unique optimal geodesic plan between μ_0 and μ_r, which is then $(\mathrm{restr}_0^r)_\#\boldsymbol{\pi}$. Moreover notice that when $K<0$, since μ_0 and μ_1 have bounded support, we can find $r \in (0,1)$ such that

$$\sup_{x\in\mathrm{supp}(\mu_0),\, y\in\mathrm{supp}(\mu_r)} \mathsf{d}(x,y) < \pi\sqrt{\frac{N}{K}}.$$

As a consequence of this uniqueness, (7.2.3) must be true along $(\mathrm{restr}_0^r)_\#\boldsymbol{\pi}$, therefore for a suitable $q \in \mathsf{Opt}(\mu_0,\mu_r)$ it holds

$$S_{N,\mathfrak{m}}(\mu_{tr}) \leq \int \left[\sigma_{K,N}^{(1-t)}(\mathsf{d}(x,y))\rho_0(x)^{-\frac{1}{N}} + \sigma_{K,N}^{(t)}(\mathsf{d}(x,y))\rho_r(y)^{-\frac{1}{N}}\right] \mathrm{d}q(x,y).$$

Now, letting $t \to 0$, the following pointwise convergences holds

$$\sigma_{K,N}^{(1-t)}(\mathsf{d}(x,y)) \to 1 \quad \text{and} \quad \sigma_{K,N}^{(t)}(\mathsf{d}(x,y)) \to 0,$$

and applying the dominated convergence theorem we conclude that $\limsup_{s\to 0} S_{N,\mathfrak{m}}(\mu_s) \leq S_{N,\mathfrak{m}}(\mu_0)$. On the other hand, since μ_0 is in particular absolutely continuous with respect to $\mathfrak{m}$, the lower semicontinuity of the entropy functional $S_{N,\mathfrak{m}}$, proven in Proposition 7.2.2, allows to deduce that $S_{N,\mathfrak{m}}(\mu_s) \to S_{N,\mathfrak{m}}(\mu_0)$ as $s \to 0$. □

Notice that, refining the proof of Corollary 7.3.2, it is possible to prove that for an essentially non-branching $\mathsf{CD}^*(K,N)$ space $(\mathsf{X},\mathsf{d},\mathfrak{m})$, the entropy functional is continuous along every Wasserstein geodesic with domain in $\mathscr{P}_N^*(\mathsf{X},\mathfrak{m})$. Another nice consequence of this last result is stated in the following lemma, which will represent a key element for the proof of the main theorem (Theorem 7.3.4).

Lemma 7.3.3. *Let $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be an essentially non-branching $\mathsf{CD}^*(K,N)$ space for some $K \in \mathbb{R}$ and $N<0$ and assume $\boldsymbol{\pi} \in \mathrm{OptGeo}(\mu_0,\mu_1)$ to be such that*

$$\mu_t := (e_t)_\#\boldsymbol{\pi} \in \mathscr{P}_N^*(\mathsf{X},\mathfrak{m}) \text{ for every } t \in [0,1].$$

Assume also that there exists a set $B \subset \mathsf{X}\setminus S$, with $\mathfrak{m}(B)<\infty$, such that

$$\mathrm{supp}(\mu_t) \subset B \qquad \text{for every } t \in [0,1].$$

Call ρ_t the density of μ_t with respect to the reference measure $\mathfrak{m}$, that is $\mu_t = \rho_t\mathfrak{m}$, then

$$\mathfrak{m}(\{\rho_0>0\}) \leq \liminf_{t\to 0} \mathfrak{m}(\{\rho_t>0\}).$$

Proof. Observe that, since $\mu_0 \ll \mathfrak{m}$, the set $\{\rho_0>0\}$ has positive $\mathfrak{m}$ measure. Moreover $\{\rho_0 > 0\} \subset B$ up to a $\mathfrak{m}$-null set, thus $0<\mathfrak{m}(\{\rho_0>0\})<\infty$. So fix $\epsilon>0$ sufficiently small and take a Borel set $A_\epsilon \subset \{\rho_0>0\}$ such that $\mathfrak{m}(\{\rho_0>0\}) - \mathfrak{m}(A_\epsilon) < \epsilon$ and $c<\rho_0(x)<C$ for every $x \in A_\epsilon$ and suitable constants c and C. Now define the set $\mathcal{A}_\epsilon := (e_0)^{-1}(A_\epsilon)$ and consequently the measures $\boldsymbol{\pi}',\boldsymbol{\pi}'' \in \mathscr{P}(\mathrm{Geo}(\mathsf{X}))$ as

$$\boldsymbol{\pi}' := \frac{1}{\boldsymbol{\pi}(\mathcal{A}_\epsilon)}\boldsymbol{\pi}|_{\mathcal{A}_\epsilon} \quad \text{and} \quad \mathrm{d}\boldsymbol{\pi}''(\gamma) = \frac{1}{\mathfrak{m}(A_\epsilon)\rho_0(\gamma_0)}\mathrm{d}\boldsymbol{\pi}'(\gamma).$$

By construction $\boldsymbol{\pi}'' \ll \boldsymbol{\pi}$ with bounded density, thus if follows that $\tilde{\mu}_t = \tilde{\rho}_t\mathfrak{m} := (e_t)_\#\boldsymbol{\pi}'' \in \mathscr{P}_N^*(\mathsf{X})$ for every $t \in [0,1]$ and $\boldsymbol{\pi}'' \in \mathrm{OptGeo}(\tilde{\mu}_0,\tilde{\mu}_1)$. Furthermore, it is easy to realize that $\tilde{\mu}_0 = \mathfrak{m}(A_\epsilon)^{-1}\mathfrak{m}|_{A_\epsilon}$ and $\mathfrak{m}(\{\tilde{\rho}_t>0\}) \leq \mathfrak{m}(\{\rho_t>0\})$ for every $t\in[0,1]$. Once again $\{\tilde{\rho}_t>0\} \subset B$

up to a $\mathfrak{m}$-null set, therefore $\mathfrak{m}(\{\tilde{\rho}_t > 0\}) < \infty$ and consequently applying Jensen's inequality we can deduce that that

$$S_{N,\mathfrak{m}}(\tilde{\mu}_t) = \int \tilde{\rho}_t^{1-\frac{1}{N}} \,\mathrm{d}\mathfrak{m} = \int_{\{\tilde{\rho}_t>0\}} \tilde{\rho}_t^{1-\frac{1}{N}} \,\mathrm{d}\mathfrak{m} = \mathfrak{m}(\{\tilde{\rho}_t > 0\}) \int \tilde{\rho}_t^{1-\frac{1}{N}} \,\mathrm{d}\Big[\frac{\mathfrak{m}|_{\{\tilde{\rho}_t>0\}}}{\mathfrak{m}(\{\tilde{\rho}_t > 0\})}\Big]$$
$$\geq \mathfrak{m}(\{\tilde{\rho}_t > 0\}) \left(\int \tilde{\rho}_t \,\mathrm{d}\Big[\frac{\mathfrak{m}|_{\{\tilde{\rho}_t>0\}}}{\mathfrak{m}(\{\tilde{\rho}_t > 0\})}\Big]\right)^{1-\frac{1}{N}} = \mathfrak{m}(\{\tilde{\rho}_t > 0\})^{\frac{1}{N}}.$$

It is then possible to apply Corollary 7.3.2 to $\boldsymbol{\pi}''$ and deduce that

$$\lim_{t\to 0} S_{N,\mathfrak{m}}(\tilde{\mu}_t) = S_{N,\mathfrak{m}}(\tilde{\mu}_0) = (\mathfrak{m}(A_\epsilon))^{\frac{1}{N}}.$$

Consequently it holds that

$$\mathfrak{m}(\{\rho_0 > 0\}) - \epsilon < \mathfrak{m}(A_\epsilon) \leq \liminf_{t\to 0} \mathfrak{m}(\{\tilde{\rho}_t > 0\}) \leq \liminf_{t\to 0} \mathfrak{m}(\{\rho_t > 0\}).$$

Then the thesis follows from the arbitrariness of ϵ. □

We are now ready to proof the main result of the section, which provides the uniqueness of Wasserstein geodesics in $\mathscr{P}_N^*(\mathsf{X}, \mathfrak{m})$.

Theorem 7.3.4. *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *be an essentially non-branching* $\mathsf{CD}^*(K, N)$ *space, for some* $K \in \mathbb{R}$ *and* $N < 0$. *Then, for every* $\mu_0, \mu_1 \in \mathscr{P}_N^*(\mathsf{X}, \mathfrak{m})$ *there exists a unique* $\boldsymbol{\pi} \in \mathrm{OptGeo}(\mu_0, \mu_1)$ *which satisfies*

$$\mu_t := (e_t)_\# \boldsymbol{\pi} \in \mathscr{P}_N^*(\mathsf{X}, \mathfrak{m}) \text{ for every } t \in [0, 1] \tag{7.3.3}$$

and it is induced by a map.

Proof. First of all notice that, since the functional $S_{N,\mathfrak{m}}$ is convex with respect to linear interpolation, it is sufficient to prove that every $\boldsymbol{\pi} \in \mathrm{OptGeo}(\mu_0, \mu_1)$ satisfying (7.3.3) is induced by a map. So assume by contradiction that there are two measures $\mu_0 = \rho_0\mathfrak{m}, \mu_1 = \rho_1\mathfrak{m} \in \mathscr{P}_N^*(\mathsf{X}, \mathfrak{m})$ and $\boldsymbol{\pi} \in \mathrm{OptGeo}(\mu_0, \mu_1)$ satisfying (7.3.3) which is not induced by a map. Call $\{\boldsymbol{\pi}_x\}_{x\in\mathsf{X}} \subset \mathscr{P}(\mathrm{Geo}(\mathsf{X}))$ be the disintegration kernel of $\boldsymbol{\pi}$ with respect to the map e_0, then there exists a μ-positive set K, such that ρ_0 is positive $\mathfrak{m}$-almost everywhere on K and $\boldsymbol{\pi}_x$ is not a delta measure for every $x \in K$. As a consequence, the measure

$$\eta = \int_K \boldsymbol{\pi}_x \times \boldsymbol{\pi}_x \,\mathrm{d}\mu(x)$$

is not concentrated in in the diagonal $D : \{(\gamma, \gamma) : \gamma \in \mathrm{Geo}(\mathsf{X})\}$. Therefore there exists a point $(\gamma_1, \gamma_2) \in \mathrm{supp}(\eta) \subset \mathrm{Geo}(\mathsf{X}) \times \mathrm{Geo}(\mathsf{X})$ with $\gamma_1 \neq \gamma_2$. Take $\varepsilon > 0$ small enough such that $B(\gamma_1, \varepsilon) \cap B(\gamma_2, \varepsilon) = \emptyset$, then

$$\eta(B(\gamma_1, \varepsilon) \times B(\gamma_2, \varepsilon)) > 0$$

and consequently, up to restricting K, we can assume that

$$\boldsymbol{\pi}_x(B(\gamma_1, \varepsilon)), \boldsymbol{\pi}_x(B(\gamma_1, \varepsilon)) > 0 \tag{7.3.4}$$

for every $x \in K$.

On the other hand, since $\mathfrak{m}(S) = 0$ and μ is absolutely continuous with respect to $\mathfrak{m}$, it holds that $\mu(S) = 0$. For every $\delta > 0$ define the open set

$$S^\delta = \{x \in \mathsf{X} : \inf_{s\in S} \mathsf{d}(x, s) < \delta\}$$

and notice that, since S is closed,

$$\bigcap_{\delta>0} S^\delta = S,$$

and consequently $\mu(S^\delta) \to 0$ as $\delta \to 0$. Therefore for a suitably small $\tilde{\delta} > 0$ it holds that

$$\mu(K \setminus S^{\bar{\delta}}) > 0,$$

thus, up to further restrict K, we can assume that $K \subset \left(S^{\bar{\delta}}\right)^c$. Then, since μ is a Radon measure measure and therefore it is inner regular, we can also assume that K is compact and consequently $\mathfrak{m}(K) < \infty$. Moreover, the reference measure $\mathfrak{m}$ is locally finite when restricted to $\mathsf{X} \setminus S$, thus for every δ small enough $\mathfrak{m}(K^\delta) < \infty$, where

$$K^\delta = \{x \in \mathsf{X} : \inf_{k\in K} \mathsf{d}(x,k) < \delta\}.$$

Furthermore, since K is closed, there exists $\tilde{\delta} > 0$ such that $\mathfrak{m}(K^{\tilde{\delta}}) < \frac{3}{2}\mathfrak{m}(K)$.
Now, introduce the sets $\Gamma_1, \Gamma_2 \in \mathrm{Geo}(\mathsf{X})$ as

$$\Gamma_1 := B(\gamma_1, \varepsilon) \cap e_0^{-1}(K) \quad \text{and} \quad \Gamma_2 := B(\gamma_2, \varepsilon) \cap e_0^{-1}(K),$$

and notice that (7.3.4) ensures that $\boldsymbol{\pi}(\Gamma_1), \boldsymbol{\pi}(\Gamma_2) > 0$. Consequently define $\boldsymbol{\pi}^1, \boldsymbol{\pi}^2 \in \mathscr{P}(\mathrm{Geo}(\mathsf{X}))$ as

$$\boldsymbol{\pi}^1 = \frac{\boldsymbol{\pi}|_{\Gamma_1}}{\boldsymbol{\pi}(\Gamma_1)} \quad \text{and} \quad \boldsymbol{\pi}^2 = \frac{\boldsymbol{\pi}|_{\Gamma_2}}{\boldsymbol{\pi}(\Gamma_2)}.$$

Observe that $\boldsymbol{\pi}^1, \boldsymbol{\pi}^2 \ll \boldsymbol{\pi}$ with bounded density, thus $\mu_t^1 := (e_t)_\#\boldsymbol{\pi}^1 \in \mathscr{P}_N^*(\mathsf{X}, \mathfrak{m})$ and $\mu_t^2 := (e_t)_\#\boldsymbol{\pi}^2 \in \mathscr{P}_N^*(\mathsf{X}, \mathfrak{m})$ for every $t \in [0,1]$. Moreover, since $B(\gamma_1, \varepsilon) \cap B(\gamma_2, \varepsilon) = \emptyset$ it is clear that $\boldsymbol{\pi}^1 \perp \boldsymbol{\pi}^2$, then, after noticing that $(e_0, e_1)_\#\boldsymbol{\pi}^1, (e_0, e_1)_\#\boldsymbol{\pi}^2 \ll (e_0, e_1)_\#\boldsymbol{\pi} \in \mathrm{OptPlans}$, it is possible to apply Proposition 7.2.7 and conclude that $\mu_t^1 \perp \mu_t^2$ for every $t \in (0,1)$. On the other hand the sets $e_1(B(\gamma_1, \varepsilon))$ and $e_1(B(\gamma_2, \varepsilon))$ are bounded, therefore there exists a time $\bar{t}$ such that $\mathrm{supp}(\mu_t^1), \mathrm{supp}(\mu_t^2) \subset K^{\tilde{\delta}}$ for every $t \in [0, \bar{t}]$. It is then possible to apply Lemma 7.3.3 to the measures $\mathrm{restr}_{0\#}^{\bar{t}}\boldsymbol{\pi}^1$ and $\mathrm{restr}_{0\#}^{\bar{t}}\boldsymbol{\pi}^2$ (with $B = K^{\tilde{\delta}}$) and deduce that

$$\mathfrak{m}(K) = \mathfrak{m}(\{\rho_0^1 > 0\}) \leq \liminf_{t\to 0} \mathfrak{m}(\{\rho_t^1 > 0\})$$

and

$$\mathfrak{m}(K) = \mathfrak{m}(\{\rho_0^2 > 0\}) \leq \liminf_{t\to 0} \mathfrak{m}(\{\rho_t^2 > 0\}),$$

where ρ_t^1 and ρ_t^2 denote the density of μ_t^1 and μ_t^2 with respect to the reference measure $\mathfrak{m}$. In particular, for $t < \bar{t}$ sufficiently small, using that $\mu_t^1 \perp \mu_t^2$ we can conclude that

$$\frac{3}{2}\mathfrak{m}(K) < \mathfrak{m}(\{\rho_t^1 > 0\}) + \mathfrak{m}(\{\rho_t^2 > 0\}) \leq \mathfrak{m}(K^{\tilde{\delta}}) \leq \frac{3}{2}\mathfrak{m}(K),$$

obtaining the desired contradiction. □

This uniqueness result can be used in order to show the existence of a transport map between two general absolutely continuous marginals, with possibly unbounded support. We point out that the existence of a transport map is a global property and in general it cannot be studied locally and then globalized. Anyway the subsequent proof needs to be done by approximation and this is possible only thanks to the uniqueness provided by Theorem 7.3.4.

Corollary 7.3.5. *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *be an essentially non-branching* $\mathsf{CD}^*(K,N)$ *space, for some* $K \in \mathbb{R}$ *and* $N < 0$*. Then, for every* $\mu_0, \mu_1 \in \mathscr{P}_2(\mathsf{X})$ *which are absolutely continuous with respect to* $\mathfrak{m}$ *there exists* $\boldsymbol{\pi} \in \mathrm{OptGeo}(\mu_0, \mu_1)$ *which is induced by a map.*

Proof. First of all assume that $\mu_0 \in \mathscr{P}^*_N(\mathsf{X},\mathfrak{m})$ and fix $\boldsymbol{\pi} \in \mathrm{OptGeo}(\mu_0,\mu_1)$. Consider a countable and measurable family of disjoint bounded sets $(F_n)_{n\in\mathbb{N}^+}$, covering μ_1-almost all X (that is $\mu_1(\mathsf{X}\setminus\bigcup_n F_n)=0$), such that $\mu_1(F_n)>0$ for every n and

$$S_{N,\mathfrak{m}}\left(\frac{\mu_1|_{F_n}}{\mu_1(F_n)}\right)<\infty \qquad \text{for every } n\in\mathbb{N}.$$

We are going to define inductively a sequence $(\boldsymbol{\pi}_n)_{n\in\mathbb{N}}\subset\mathscr{P}(\mathrm{Geo}(\mathsf{X}))$, with the following properties (where $E_n=\bigcup_{i=1}^n F_n$)

(i) for every $n\in\mathbb{N}$, we have $\boldsymbol{\pi}_n\in\mathrm{OptGeo}(\mu_0,\mu_1)$ thus in particular $(e_0)_\#\boldsymbol{\pi}_n=\mu_0$ and $(e_1)_\#\boldsymbol{\pi}_n=\mu_1$,

(ii) for every $n\in\mathbb{N}^+$ and every $m\in\mathbb{N}^+$ such that $m<n$, it holds that $\boldsymbol{\pi}_n|_{e_1^{-1}(E_m)}=\boldsymbol{\pi}_m|_{e_1^{-1}(E_m)}$,

(iii) for every $n\in\mathbb{N}^+$ the optimal geodesic plan $\frac{1}{\mu_1(E_n)}\boldsymbol{\pi}_n|_{e_1^{-1}(E_n)}$ is induced by a map and constitutes a Wasserstein geodesic contained in $\mathscr{P}^*_N(\mathsf{X},\mathfrak{m})$.

First of all put $\boldsymbol{\pi}_0=\boldsymbol{\pi}$, which obviously satisfies all the required properties. Then, given $\boldsymbol{\pi}_{n-1}$, call μ_0^n and μ_1^n the marginals at time 0 and 1 (respectively) of $\frac{1}{\mu_1(F_n)}\boldsymbol{\pi}_{n-1}|_{e_1^{-1}(F_n)}$. Then consider the optimal geodesic plan $\tilde{\boldsymbol{\pi}}_n\in\mathrm{OptGeo}(\mu_0^n,\mu_1^n)$ provided by Theorem 7.3.4. Notice that $\mu_1^n\in\mathscr{P}^*_N(\mathsf{X},\mathfrak{m})$, because of the definition of F_n, and $\mu_0^n\in\mathscr{P}^*_N(\mathsf{X},\mathfrak{m})$, since $\mu_0^n\ll\mu_0$ with bounded density. Consequently define

$$\boldsymbol{\pi}_n=\boldsymbol{\pi}_{n-1}-\boldsymbol{\pi}_{n-1}|_{e_1^{-1}(F_n)}+\mu_1(F_n)\tilde{\boldsymbol{\pi}}_n, \tag{7.3.5}$$

it is easy to realize that $\boldsymbol{\pi}_n$ satisfies (i). Using the inductive assumption it is also clear that property (ii) holds, in fact from (7.3.5) follows that $\boldsymbol{\pi}_n$ and $\boldsymbol{\pi}_{n-1}$ coincide on $e_1^{-1}(F_n^c)$. Moreover it holds that

$$\frac{1}{\mu_1(E_n)}\boldsymbol{\pi}_n|_{e_1^{-1}(E_n)}=\frac{1}{\mu_1(E_n)}\boldsymbol{\pi}_{n-1}|_{e_1^{-1}(E_{n-1})}+\frac{\mu_1(F_n)}{\mu_1(E_n)}\tilde{\boldsymbol{\pi}}_n$$

and therefore $\frac{1}{\mu_1(E_n)}\boldsymbol{\pi}_n|_{e_1^{-1}(E_n)}$ constitutes a Wasserstein geodesic contained in $\mathscr{P}^*_N(\mathsf{X},\mathfrak{m})$. As a consequence, Theorem 7.3.4 ensures that it is induced by a map, proving (iii). Now the combination of properties (i), (ii) and (iii) implies that $(\boldsymbol{\pi}_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to the total variation norm and thus it converges to $\tilde{\boldsymbol{\pi}}\in\mathrm{OptGeo}(\mu_0,\mu_1)$ such that $\tilde{\boldsymbol{\pi}}|_{e_1^{-1}(E_n)}=\boldsymbol{\pi}_n|_{e_1^{-1}(E_n)}$ for every $n\in\mathbb{N}^+$. We are now going to prove that $\boldsymbol{\pi}$ is induced by a map. Assume by contradiction this is not true, calling $\{\tilde{\boldsymbol{\pi}}_x\}_{x\in\mathsf{X}}\subset\mathscr{P}(\mathrm{Geo}(\mathsf{X}))$ the disintegration of $\tilde{\boldsymbol{\pi}}$ with respect to the map e_0, $\tilde{\boldsymbol{\pi}}_x$ is not a delta measure for a μ-positive set of x. Then, since $\mu_1(\mathsf{X}\setminus\bigcup_n F_n)=0$, there exists $\bar{n}\in\mathbb{N}$ such that $\tilde{\boldsymbol{\pi}}_x|_{e_1^{-1}(E_{\bar{n}})}$ is not a delta measure for a μ-positive set of x, contradicting the fact that $\tilde{\boldsymbol{\pi}}|_{e_1^{-1}(E_{\bar{n}})}$ is induced by a map.

We can now explain the proof of the general case, assuming only the absolute continuity on the first marginal. This proof can be done using an approximation procedure, very similar the one showed in the first part of the proof, for this reason we will not explain all the passages. As before, consider a countable and measurable family of disjoint bounded sets $(F_n)_{n\in\mathbb{N}^+}$, covering μ_0-almost all X, such that $\mu_0(F_n)>0$ for every n and

$$S_{N,\mathfrak{m}}\left(\frac{\mu_0|_{F_n}}{\mu_0(F_n)}\right)<\infty \qquad \text{for every } n\in\mathbb{N}.$$

Then it is possible to proceed as before, using an inductive procedure (and the first part of the proof) in order to obtain $\tilde{\boldsymbol{\pi}}\in\mathrm{OptGeo}(\mu_0,\mu_1)$ such that $\tilde{\boldsymbol{\pi}}|_{e_0^{-1}(E_n)}$ (where $E_n=\bigcup_{i=1}^n F_n$) is induced by a map for every $n\in\mathbb{N}^+$. In this case, this is sufficient to conclude that $\tilde{\boldsymbol{\pi}}$ is induced by a map. $\square$

7.4 Local-to-global Property

In this last section we prove a local-to-global result for the reduced CD condition with a negative dimensional parameter. For this purpose, in this section we assume the metric measure space $(\mathsf{X},\mathsf{d},\mathfrak{m})$ to be locally compact.

Definition 7.4.1. For fixed $K\in\mathbb{R}, N\in(-\infty,0)$, we say that a metric measure space $(\mathsf{X},\mathsf{d},\mathfrak{m})$ satisfies the condition $\mathsf{CD}^*(K-,N)$ if for every $\mu_0=\rho_0\mathfrak{m}$, $\mu_1=\rho_1\mathfrak{m}\in\mathscr{P}^*_N(\mathsf{X},\mathfrak{m})$ and every $K'<K$ there exists an optimal coupling $\pi\in\mathrm{OptPlans}(\mu,\nu)$ and a W_2-geodesic $\{\mu_t\}_{t\in[0,1]}\subset\mathscr{P}^*_N(\mathsf{X},\mathfrak{m})$ connecting μ_0 and μ_1 such that

$$S_{N',\mathfrak{m}}(\mu_t)\le R^{(t)}_{K',N'}(\pi|\mathfrak{m}) \tag{7.4.1}$$

holds for every $t\in[0,1]$ and every $N'\in[N,0)$.

We point out that, unlike to what happens in the positive dimensional case, the $\mathsf{CD}^*(K-,N)$ condition is not equivalent to the $\mathsf{CD}^*(K,N)$ one in general. This is basically due to the pathologies of the distortion coefficients $\sigma^{(t)}_{K,N}$ when $K<0$, on the other hand, if the curvature parameter is non-negative, we are able to prove the equivalence. The proof in the positive dimensional case relies on the lower semicontinuity of the entropy functionals, which does not hold in our context. Anyway we can overcome this difficulty using the uniqueness results obtained in section 7.3.

Proposition 7.4.2. *For fixed $K\ge 0, N\in(-\infty,0)$, a metric measure space $(\mathsf{X},\mathsf{d},\mathfrak{m})$ satisfies the condition $\mathsf{CD}^*(K,N)$ if and only if it satisfies the $\mathsf{CD}^*(K-,N)$ one.*

Proof. The “only if” part of the statement is obviously true. In order to prove the “if” part, we start by noticing that Theorem 7.3.4 ensure that every pair of marginals μ_0, $\mu_1\in\mathscr{P}^*_N(\mathsf{X},\mathfrak{m})$ is connected by a unique geodesic $(\mu_t)_{t\in[0,1]}$ with domain in $\mathscr{P}^*_N(\mathsf{X},\mathfrak{m})$. In particular we can take a sequence $(K_n)_{n\in\mathbb{N}}$ such that $K_n\nearrow K$, and find for every n an optimal plan $\pi_n\in\mathrm{OptPlans}(\mu_0,\mu_1)$ such that

$$S_{N,\mathfrak{m}}(\mu_t)\le R^{(t)}_{K_n,N}(\pi_n|\mathfrak{m}),\quad \text{for every } t\in[0,1]. \tag{7.4.2}$$

The sequence $(\pi_n)_{n\in\mathbb{N}}$ is obviously tight, since every measure has the same marginals, thus up to taking a suitable subsequence there exists $\pi\in\mathrm{OptPlans}(\mu_0,\mu_1)$ such that $\pi_n\rightharpoonup\pi$. In order to conclude the proof it is sufficient to show that, for every $t\in[0,1]$,

$$\lim_{n\to\infty}R^{(t)}_{K_n,N}(\pi_n|\mathfrak{m})=R^{(t)}_{K,N}(\pi|\mathfrak{m}), \tag{7.4.3}$$

in fact this will allow to pass (7.4.2) at the limit as $n\to\infty$. To this aim we just need to prove that

$$\lim_{n\to\infty}\int\sigma^{(1-t)}_{K_n,N}(\mathsf{d}(x,y))\rho_0(x)^{-\frac{1}{N}}\,\mathrm{d}\pi_n=\int\sigma^{(1-t)}_{K,N}(\mathsf{d}(x,y))\rho_0(x)^{-\frac{1}{N}}\,\mathrm{d}\pi,$$

the other term of $R^{(t)}_{K,N}$ can be treated analogously. Notice that, since $S_{N,\mathfrak{m}}(\mu_0)<\infty$, then $\rho_0(x)^{-1/N}\in L^1(\mu_0)$ thus, according to [MRS23, Lemma 2.12], for every fixed $\varepsilon>0$ there exists $f^\varepsilon\in C_b(\mathsf{X})$ such that $\left\|\rho_0^{-1/N}-f^\varepsilon\right\|_{L^1(\mu_0)}<\varepsilon$. Moreover, notice that the coefficients $\sigma^{(1-t)}_{K_n,N}$ and $\sigma^{(1-t)}_{K,N}$ are uniformly bounded above by 1 and continuous. Moreover, it is easy to realize that

$$C_b(\mathsf{X}\times\mathsf{X})\ni\sigma^{(1-t)}_{K_n,N}(\mathsf{d}(x,y))f^\varepsilon(x)\to\sigma^{(1-t)}_{K,N}(\mathsf{d}(x,y))f^\varepsilon(x)\in C_b(\mathsf{X}\times\mathsf{X})$$

uniformly. As a consequence, the weak convergence $(\pi_n)_n\rightharpoonup\pi$ ensures that,

$$\lim_{n\to\infty}\int\sigma^{(1-t)}_{K_n,N}(\mathsf{d}(x,y))f^\varepsilon(x)\,\mathrm{d}\pi_n=\int_{\mathsf{X}\times\mathsf{X}}\sigma^{(1-t)}_{K,N}(\mathsf{d}(x,y))f^\varepsilon(x)\,\mathrm{d}\pi.$$

Furthermore, the uniform bound on $\sigma^{(1-t)}_{K_n,N}$ and $\sigma^{(1-t)}_{K,N}$ allows to deduce the following estimate

$$\begin{aligned}\limsup_{n\to\infty}\int \sigma^{(1-t)}_{K_n,N}(\mathsf{d}(x,y))\rho_0(x)^{-\frac{1}{N}}\,\mathrm{d}\pi_n &\leq \lim_{n\to\infty}\int \sigma^{(1-t)}_{K_n,N}(\mathsf{d}(x,y))f^{\varepsilon}(x)\,\mathrm{d}\pi_n+\varepsilon\\ &= \int \sigma^{(1-t)}_{K,N}(\mathsf{d}(x,y))f^{\varepsilon}(x)\,\mathrm{d}\pi+\varepsilon\\ &\leq \int \sigma^{(1-t)}_{K,N}(\mathsf{d}(x,y))\rho_0(x)^{-\frac{1}{N}}\,\mathrm{d}\pi+2\varepsilon.\end{aligned}$$

Analogously, it can be proven that

$$\liminf_{n\to\infty}\int \sigma^{(1-t)}_{K_n,N}(\mathsf{d}(x,y))\rho_0(x)^{-\frac{1}{N}}\,\mathrm{d}\pi_n \geq \int \sigma^{(1-t)}_{K,N}(\mathsf{d}(x,y))\rho_0(x)^{-\frac{1}{N}}\,\mathrm{d}\pi-2\varepsilon,$$

and since $\varepsilon>0$ and can be chosen arbitrarily, equation (7.4.3) holds true.

□

In order to prove the local-to-global property we need a preliminary proposition, which states an equivalent characterization of the $\mathsf{CD}^*(K-,N)$ condition. The analogous result for the $\mathsf{CD}^*(K,N)$ condition for positive N is proven in [BS10, Proposition 2.8], but the approximation argument used relies on the lower semicontinuity of the entropy functionals. For these reason, we need to work with the $\mathsf{CD}^*(K-,N)$ condition and to proceed in a different way.

Proposition 7.4.3 (Equivalent characterizations)**.** *Given a proper essentially non-branching metric measure space* $(\mathsf{X},\mathsf{d},\mathfrak{m})$*, the following statements are equivalent*:

(*i*) $(\mathsf{X},\mathsf{d},\mathfrak{m})$ *satisfies the condition* $\mathsf{CD}^*(K-,N)$*.*

(*ii*) *For every* $K'<K$ *and every pair of marginals* $\mu_0,\mu_1\in\mathscr{P}^*_N(\mathsf{X},\mathfrak{m})$ *there exists a geodesic* $\mu:[0,1]\to\mathscr{P}^*_N(\mathsf{X},\mathfrak{m})$ *connecting* μ_0 *and* μ_1 *such that for all* $t\in[0,1]$ *and all* $N'\in[N,0)$*, it holds that*

$$S_{N'}(\mu_t)\leq \sigma^{(1-t)}_{K',N'}(\theta)S_{N'}(\mu_0)+\sigma^{(t)}_{K',N'}(\theta)S_{N'}(\mu_1),$$

where

$$\theta:=\begin{cases}\inf_{x_0\in\mathcal{S}_0,x_1\in\mathcal{S}_1}\mathsf{d}(x_0,x_1), & \text{if } K\geq 0,\\ \sup_{x_0\in\mathcal{S}_0,x_1\in\mathcal{S}_1}\mathsf{d}(x_0,x_1), & \text{if } K<0,\end{cases}\tag{7.4.4}$$

denoting by $\mathcal{S}_0$ *and* $\mathcal{S}_1$ *the supports of* μ_0 *and* μ_1*, respectively.*

Proof. (i) ⇒ (ii): This implication easily follows from the monotonicity property of the coefficient $\sigma^{(t)}_{K,N'}$, see (7.2.2).

(ii) ⇒ (i): We prove this implication only for $K>0$, our argument applies without any major modification also when $K\leq 0$. Notice that it is sufficient to prove condition (7.4.1) for $0<K'<K$, because of the monotonicity properties of the coefficients $\sigma^{(t)}_{K,N}$. Thus we fix $0<K'<K$ and two measures $\mu_0,\mu_1\in\mathscr{P}^*_N(\mathsf{X},\mathfrak{m})$, then there exist $o\in\mathsf{X}$ and $R>0$ such that $\operatorname{supp}(\mu_0),\operatorname{supp}(\mu_1)\subseteq B_R(o)$. We also fix $K'<\tilde K<K$ and consider an arbitrary coupling $\tilde{\mathsf{q}}\in\mathsf{Opt}(\mu_0,\mu_1)$. Now for every $n\in\mathbb{N}$ let $\mathcal{C}^n=\{C^n_1,\dots,C^n_{m_n}\}$ be a (finite) Borel partition of $B_R(o)$, that is

$$\bigcup_{i=1}^{m_n} C^n_i=B_R(o)\quad\text{ and }\quad C^n_i\cap C^n_j=\emptyset \text{ for every } i\neq j,$$

such that $\operatorname{diam}(C^n_i)\leq\frac{1}{2^{n+1}}$ for $i=1,\dots,m_n$. Moreover we assume that, for every $n\in\mathbb{N}$, $\mathcal{C}_{n+1}$ is consistent with $\mathcal{C}_n$, meaning that for every i we have that $C^n_i=C^{n+1}_{i_1}\bigcup\cdots\bigcup C^{n+1}_{i_k}$ for a suitable choice of indices $i_1, \dots i_k$. Furthermore, for every n we take

$$\delta_n := \left[2^n\left(1-\sqrt{\frac{K'}{\tilde{K}}}\right)\right]^{-1},$$

and we define

$$\tilde{I}_n := \left\{(i,j) \in \{1,\dots,m_n\}^2 \,:\, \inf_{x\in C_i^n, y\in C_j^n} \mathsf{d}(x,y) > \delta_n\right\}.$$

Calling then

$$E_n := \bigcup_{(i,j)\in\tilde{I}_n} C_i^n \times C_j^n \subset B_R(o)\times B_R(o),$$

notice that $E_n \subseteq E_{n+1}$. Consequently we introduce for every $n \in \mathbb{N}$ the set

$$\bar{I}_n = \left\{(i,j) \in \tilde{I}_n \,:\, (C_i^n \times C_j^n) \cap E_{n-1} = \emptyset\right\},$$

where we assume $E_{-1} = \emptyset$. Given these definitions it is easy to realize that, calling $D := \{(x,x) : x \in \mathsf{X}\}$ the diagonal of X, it holds

$$\bigcup_{n=1}^{\infty} \bigcup_{(i,j)\in\bar{I}_n} C_i^n \times C_j^n = B_R(o)\times B_R(o) \setminus D, \tag{7.4.5}$$

moreover $(C_i^n \times C_j^n) \cap (C_k^m \times C_l^m) = \emptyset$ for every $n,m \in \mathbb{N}$ and $(i,j) \in I_n$, $(k,l) \in I_m$. Now, for every $n \in \mathbb{N}$, we define the set of indices

$$I_n := \left\{(i,j) \in \bar{I}_n \,:\, \tilde{\mathsf{q}}(C_i^n \times C_j^n) > 0\right\}$$

and for every $(i,j) \in I_n$ the associated probability measures $\mu_0^{n,ij}$ and $\mu_1^{n,ij}$ by

$$\mu_0^{n,ij}(A) := \frac{1}{\alpha_{ij}^n}\tilde{\mathsf{q}}((A\cap C_i^n)\times C_j^n) \quad \text{and} \quad \mu_1^{n,ij}(A) := \frac{1}{\alpha_{ij}^n}\tilde{\mathsf{q}}(C_i^n \times (A \cap C_j^n)),$$

where each $\alpha_{ij}^n := \tilde{\mathsf{q}}(C_i \times C_j) \neq 0$ is the suitable normalization constant. Then we call

$$\mathcal{S}_0^{n,ij} := \mathsf{supp}(\mu_0^{n,ij}) \subseteq \overline{C_i^n} \quad \text{and} \quad \mathcal{S}_1^{n,ij} =: \mathsf{supp}(\mu_1^{n,ij}) \subseteq \overline{C_j^n}$$

and accordingly to (7.4.4), we introduce

$$\theta^{n,ij} := \inf_{x_0\in\mathcal{S}_0^{ij}, x_1\in\mathcal{S}_1^{ij}} \mathsf{d}(x_0,x_1),$$

observe that, since $\operatorname{diam}(C_i^n) \leq \frac{1}{2^{n+1}}$ for every $i = 1,\dots,m_n$, it holds that $\mathsf{d}(x,y) - \frac{1}{2^n} \leq \theta^{ij}$ for every $x \in \mathcal{S}_0^{ij}$ and $y \in \mathcal{S}_1^{ij}$. By the assumption (ii), for every $n \in \mathbb{N}$ and $(i,j) \in I_n$ there exist $\mathsf{q}^{n,ij} \in \mathsf{Opt}(\mu_0^{n,ij}, \mu_1^{n,ij})$ and a Wasserstein geodesic $\mu^{n,ij} : [0,1] \to \mathscr{P}_\infty(\mathsf{X},\mathfrak{m})$, connecting $\mu_0^{n,ij} = \rho_0^{n,ij}\mathfrak{m}$ to $\mu_1^{n,ij} = \rho_1^{n,ij}\mathfrak{m}$ and satisfying

$$S_{N',\mathfrak{m}}(\mu_t^{n,ij}) \leq \sigma_{\tilde{K},N'}^{(1-t)}(\theta^{n,ij}) S_{N',\mathfrak{m}}(\mu_0^{n,ij}) + \sigma_{\tilde{K},N'}^{(t)}(\theta^{n,ij}) S_{N',\mathfrak{m}}(\mu_1^{n,ij}).$$

Observe in particular that, since $\mathsf{q}^{n,ij} \in \mathsf{Opt}(\mu_0^{n,ij}, \mu_1^{n,ij})$, we have

$$\alpha_{ij}^n \int \mathsf{d}^2 \, \mathrm{d}\mathsf{q}^{n,ij} = \int \mathsf{d}^2 \, \mathrm{d}[\tilde{\mathsf{q}}|_{C_i\times C_j}]. \tag{7.4.6}$$

Then, since $\mathsf{d}(x,y)-\frac{1}{2^n}\leq\theta^{n,ij}$ for every $x\in\mathcal{S}_0^{n,ij}$ and $y\in\mathcal{S}_1^{n,ij}$, it holds that

$$\begin{aligned}
&S_{N',\mathfrak{m}}(\mu_t^{n,ij})\\
&\leq\int\Big[\sigma^{(1-t)}_{\tilde K,N'}(\mathsf{d}(x_0,x_1)-2^{-n})\rho_0^{n,ij}(x_0)^{-1/N'}+\sigma^{(t)}_{\tilde K,N'}(\mathsf{d}(x_0,x_1)-2^{-n})\rho_1^{n,ij}(x_1)^{-1/N'}\Big]\,\mathrm{d}\mathsf{q}^{n,ij}(x_0,x_1)\\
&\leq\int\Big[\sigma^{(1-t)}_{K',N'}(\mathsf{d}(x_0,x_1))\rho_0^{n,ij}(x_0)^{-1/N'}+\sigma^{(t)}_{K',N'}(\mathsf{d}(x_0,x_1))\rho_1^{n,ij}(x_1)^{-1/N'}\Big]\,\mathrm{d}\mathsf{q}^{n,ij}(x_0,x_1)\\
&=R^{(t)}_{K',N'}(\mathsf{q}^{n,ij}|\mathfrak{m})
\end{aligned}\tag{7.4.7}$$

for every $t\in[0,1]$ and every $N'\in[N,0)$, where the second inequality is a consequence of

$$\sqrt{\tilde K}\cdot(\mathsf{d}(x,y)-2^{-n})\geq\sqrt{K'}\cdot\mathsf{d}(x,y),$$

which holds for $\mathsf{q}^{n,ij}$-almost everywhere because of the definition of δ_n and $\tilde I_n$. On the other hand, if $\alpha^D:=\tilde{\mathsf{q}}(D)>0$ we call

$$\rho_0^D\mathfrak{m}=\mu_0^D:=\frac{1}{\alpha^D}(\mathsf{p}_1)_\#[\tilde{\mathsf{q}}|_D]=\frac{1}{\alpha^D}(\mathsf{p}_2)_\#[\tilde{\mathsf{q}}|_D]=:\mu_1^D=\rho_1^D\mathfrak{m}.$$

Then, posing $\mu_t^D\equiv\mu_0^D=\mu_1^D$ and $\mathsf{q}^D=\frac{1}{\alpha^D}\tilde{\mathsf{q}}|_D$, it obviously holds that

$$S_{N',\mathfrak{m}}(\mu_t^D)\leq R^{(t)}_{K',N'}(\mathsf{q}^D|\mathfrak{m}),$$

for every $t\in[0,1]$ and every $N'\in[N,0)$. As a consequence of (7.4.5) it is also possible to conclude that

$$\mu_\iota=\alpha^D\mu_\iota^D+\sum_{n=1}^\infty\sum_{(i,j)\in I_n}\alpha^n_{ij}\mu_\iota^{n,ij}\qquad\text{for }\iota=0,1.$$

We can now define

$$\mathsf{q}:=\alpha^D\mathsf{q}^D+\sum_{n=1}^\infty\sum_{(i,j)\in I_n}\alpha^n_{ij}\mathsf{q}^{n,ij}\quad\text{and}\quad\mu_t:=\alpha^D\mu_t^D+\sum_{n=1}^\infty\sum_{(i,j)\in I_n}\alpha^n_{ij}\mu_t^{n,ij}\quad\text{for every }t\in[0,1],$$

where both the series converge in the total variation norm. As a consequence of (7.4.6) notice that

$$\int\mathsf{d}^2\,\mathrm{d}\mathsf{q}=\int\mathsf{d}^2\,\mathrm{d}\tilde{\mathsf{q}}=W_2^2(\mu_0,\mu_1),$$

thus q is an optimal coupling of μ_0 and μ_1, while μ_t defines a geodesic connecting them. It is also easy to realize that we can apply of Proposition 7.2.7 and deduce that for every $t\in[0,1]$

$$\mu_t^{n,ij}\perp\mu_t^{m,kl}\ \text{ if }n\neq m\text{ or }n=m\text{ and }(i,j)\neq(k,l),$$

moreover

$$\mu_t^{n,ij}\perp\mu_t^D\ \text{ for every }n\in\mathbb{N}\text{ and }(i,j)\in I_n.$$

As a consequence, for every $t\in[0,1]$ and every $N'\in[N,0)$ it holds that

$$S_{N',\mathfrak{m}}(\mu_t)=[\alpha^D]^{1-1/N'}S_{N',\mathfrak{m}}(\mu_t^D)+\sum_{n=1}^\infty\sum_{(i,j)\in I_n}[\alpha^n_{ij}]^{1-1/N'}S_{N',\mathfrak{m}}(\mu_t^{n,ij}).$$

On the other hand, keeping in mind the definitions of q, $\alpha^{n,ij}$ and $\mu_\iota^{n,ij}$ for $\iota = 0, 1$, we can easily conclude that

$$\begin{aligned}
&[\alpha^D]^{1-1/N'} R^{(t)}_{K',N'}(\mathsf{q}^D|\mathfrak{m}) + \sum_{n=1}^{\infty} \sum_{(i,j)\in I_n} [\alpha^n_{ij}]^{1-1/N'} R^{(t)}_{K',N'}(\mathsf{q}^{n,ij}|\mathfrak{m}) \\
&\quad = \int \sigma^{(1-t)}_{K',N'}(\mathsf{d}(x_0,x_1))[\alpha^D \rho_0^D(x_0)]^{-1/N'} + \sigma^{(t)}_{K',N'}(\mathsf{d}(x_0,x_1))[\alpha^D \rho_1^D(x_1)]^{-1/N'} \Big] \,\mathrm{d}[\alpha^D \mathsf{q}^D](x_0,x_1) \\
&\qquad + \sum_{n=1}^{\infty} \sum_{(i,j)\in I_n} \int \Big[\sigma^{(1-t)}_{K',N'}(\mathsf{d}(x_0,x_1))[\alpha^n_{ij}\rho_0^{n,ij}(x_0)]^{-1/N'} + \sigma^{(t)}_{K',N'}(\mathsf{d}(x_0,x_1))[\alpha^n_{ij}\rho_1^{n,ij}(x_1)]^{-1/N'}\Big] \\
&\hspace{30em} \mathrm{d}[\alpha^n_{ij}\mathsf{q}^{n,ij}](x_0,x_1) \\
&\quad \leq \int \Big[\sigma^{(1-t)}_{K',N'}(\mathsf{d}(x_0,x_1))\rho_0(x_0)^{-1/N'} + \sigma^{(t)}_{K',N'}(\mathsf{d}(x_0,x_1))\rho_1(x_1)^{-1/N'}\Big] \,\mathrm{d}\mathsf{q}(x_0,x_1) = R^{(t)}_{K',N'}(\mathsf{q}|\mathfrak{m}).
\end{aligned}$$

Combining this last two relations with (7.4.7) we obtain that

$$S_{N',\mathfrak{m}}(\mu_t) \leq R^{(t)}_{K',N'}(\mathsf{q}|\mathfrak{m}),$$

concluding the proof. □

We are now ready to state the main result of this section:

Theorem 7.4.4. *Let $K, N \in \mathbb{R}$ with $N < 0$ and let $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ be a locally compact, essentially non-branching metric measure space such that $\mathscr{P}^*_N(\mathsf{X}, \mathfrak{m})$ is a geodesic space. If $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ satisfies the condition $\mathsf{CD}^*(K-, N)$ locally, then it satisfies the condition $\mathsf{CD}^*(K-, N)$ globally.*

The proof of this theorem relies on Proposition 7.4.3, so our goal is to demonstrate (*ii*); to this aim we fix $K' < K$. Before presenting the proof, we introduce the basic construction which allows to show its validity.

Let us fix a metric measure space $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ as in the hypothesis of Theorem 7.4.4, satisfying $\mathsf{CD}^*(K-, N)$ locally for some $K \in \mathbb{R}$ and $N < 0$. Observe that, since $\mathscr{P}^*_N(\mathsf{X}, \mathfrak{m})$ is a geodesic space, then (X, d) is a length space. Therefore the metric version of the Hopf-Rinow theorem (see for example [Bal95, Theorem 2.4]) ensures that (X, d) is proper, being locally compact. Finally, we fix a metric ball $B_R(o)$ and for $k \in \mathbb{N} \cup \{0\}$, we introduce the following property, that we denote by $\mathsf{C}(k)$. We remark that this property is similar in spirit to the one proposed in [BS10], where it is formulated in terms of midpoints of geodesics. However, in our situation it is more straightforward to consider directly the whole geodesic, thanks to the uniqueness results proven in the previous section.

$\mathsf{C}(k)$: For each geodesic $\Gamma\colon [0, 1] \to \mathscr{P}^*_N(\mathsf{X}, \mathfrak{m})$ such that $\mathrm{supp}(\mu_0), \mathrm{supp}(\mu_1) \subseteq B_R(o)$ and for each pair of times $s, t \in [0, 1]$, such that $t - s = 2^{-k}$ the (restricted and reparameterized) geodesic Γ between $\Gamma(s)$ and $\Gamma(t)$, satisfies the inequality

$$S_{N',\mathfrak{m}}(\Gamma(s + r(t-s))) \leq \sigma^{(1-r)}_{K',N'}(\theta^k) S_{N',\mathfrak{m}}(\Gamma(s)) + \sigma^{(r)}_{K',N'}(\theta^k) S_{N',\mathfrak{m}}(\Gamma(t)),$$

for all $r \in [0, 1]$ and $N' \in [N, 0)$, where

$$\theta^0 := \inf_{\gamma \in \mathrm{supp}(\Gamma)} \mathsf{d}(\gamma(0), \gamma(1)) \quad \text{and} \quad \theta^k := \frac{\theta^0}{2^k}.$$

In the following we treat the case $K > 0$, the general one follows by analogous computations.

Lemma 7.4.5. *If* $\mathsf{C}(k)$ *is satisfied for some* $k \in \mathbb{N}$*, then also* $\mathsf{C}(k-1)$ *holds true.*

Proof. Let $k \in \mathbb{N}$ be such that $\mathsf{C}(k)$ is satisfied and let Γ be a geodesic in $\mathscr{P}^*_N(\mathsf{X},\mathfrak{m})$ such that $\operatorname{supp}(\mu_0), \operatorname{supp}(\mu_1) \subseteq B_R(o)$, moreover we fix $s,t \in [0,1]$ with $t-s = 2^{1-k}$. First of all, let us observe that property $\mathsf{C}(k)$ ensures that it holds

$$S_{N',\mathfrak{m}}\Big(\Gamma(s + r\cdot 2^{-k})\Big) \leq \sigma^{(1-r)}_{K',N'}(\theta^k) S_{N',\mathfrak{m}}(\Gamma(s)) + \sigma^{(r)}_{K',N'}(\theta^k) S_{N',\mathfrak{m}}\Big(\Gamma(s+2^{-k})\Big), \tag{7.4.8}$$

as well as

$$S_{N',\mathfrak{m}}\Big(\Gamma(s + 2^{-k} + r\cdot 2^{-k})\Big) \leq \sigma^{(1-r)}_{K',N'}(\theta^k) S_{N',\mathfrak{m}}\Big(\Gamma(s+2^{-k})\Big) + \sigma^{(r)}_{K',N'}(\theta^k) S_{N',\mathfrak{m}}(\Gamma(t)), \tag{7.4.9}$$

for all $N' \in [N,0)$. Now, applying property $\mathsf{C}(k)$ between the times $s+2^{-(k+1)}$ and $s+3\cdot 2^{-(k+1)}$, which are at distance 2^{-k}, we can also obtain the following chain of inequalities

$$\begin{aligned} S_{N',\mathfrak{m}}\Big(\Gamma(s+2^{-k})\Big) &\leq \sigma^{(1/2)}_{K',N'}(\theta^k) S_{N',\mathfrak{m}}\Big(\Gamma(s+2^{-(k+1)})\Big) + \sigma^{(1/2)}_{K',N'}(\theta^k) S_{N',\mathfrak{m}}\Big(\Gamma(s+3\cdot 2^{-(k+1)})\Big) \\ &\leq \sigma^{(1/2)}_{K',N'}(\theta^k)\Big[\sigma^{(1/2)}_{K',N'}(\theta^k) S_{N',\mathfrak{m}}(\Gamma(s)) + \sigma^{(1/2)}_{K',N'}(\theta^k) S_{N',\mathfrak{m}}\Big(\Gamma(s+2^{-k})\Big)\Big] \\ &\quad + \sigma^{(1/2)}_{K',N'}(\theta^k)\Big[\sigma^{(1/2)}_{K',N'}(\theta^k) S_{N',\mathfrak{m}}\Big(\Gamma(s+2^{-k})\Big) + \sigma^{(1/2)}_{K',N'}(\theta^k) S_{N',\mathfrak{m}}(\Gamma(t))\Big], \end{aligned}$$

that in particular leads to

$$S_{N',\mathfrak{m}}\Big(\Gamma(s+2^{-k})\Big) \leq \frac{\big(\sigma^{(1/2)}_{K',N'}(\theta^k)\big)^2}{1-2\big(\sigma^{(1/2)}_{K',N'}(\theta^k)\big)^2}\big[S_{N',\mathfrak{m}}(\Gamma(s)) + S_{N',\mathfrak{m}}(\Gamma(t))\big]. \tag{7.4.10}$$

Hence, let us observe that

$$\begin{aligned} \frac{\Big(\sigma^{(1/2)}_{K',N'}(\theta^k)\Big)^2}{1-2\Big(\sigma^{(1/2)}_{K',N'}(\theta^k)\Big)^2} &= \frac{\sinh^2\Big(\frac{\theta^k}{2}\sqrt{-K'/N'}\Big)}{\sinh^2\big(\theta^k\sqrt{-K'/N'}\big)} \cdot \frac{\sinh^2\big(\theta^k\sqrt{-K'/N'}\big)}{\sinh^2\big(\theta^k\sqrt{-K'/N'}\big) - 2\sinh^2\Big(\frac{\theta^k}{2}\sqrt{-K'/N'}\Big)} \\ &= \frac{\sinh^2\Big(\frac{\theta^k}{2}\sqrt{-K'/N'}\Big)}{\cosh^2\big(\theta^k\sqrt{-K'/N'}\big) - \cosh\big(\theta^k\sqrt{-K'/N'}\big)} \\ &= \frac{1}{2}\frac{1}{\cosh\big(\theta^k\sqrt{-K'/N'}\big)} = \sigma^{(1/2)}_{K',N'}(2\theta^k). \end{aligned}$$

Moreover, since $\theta^k = \frac{1}{2}\theta^{k-1}$, it holds that $\sigma^{(1/2)}_{K',N'}(2\theta^k) = \sigma^{(1/2)}_{K',N'}(\theta^{k-1})$ and we can rewrite inequality (7.4.10) as

$$S_{N',\mathfrak{m}}(\Gamma(s+2^{-k})) \leq \sigma^{(1/2)}_{K',N'}(\theta^{k-1}) S_{N',\mathfrak{m}}(\Gamma(s)) + \sigma^{(1/2)}_{K',N'}(\theta^{k-1}) S_{N',\mathfrak{m}}(\Gamma(t)). \tag{7.4.11}$$

Thus, let us consider the geodesic Γ restricted (and reparametrized) between the times s and t by considering the curve $[0,1] \ni r \to \Gamma(s + r\cdot 2^{1-k})$. If $r \in [0,1/2]$, inequality (7.4.8) ensures that

$$\begin{aligned} &S_{N',\mathfrak{m}}(\Gamma(s + r\cdot 2^{1-k})) \leq \sigma^{(1-2r)}_{K',N'}(\theta^k) S_{N',\mathfrak{m}}(\Gamma(s)) + \sigma^{(2r)}_{K',N'}(\theta^k) S_{N',\mathfrak{m}}(\Gamma(s+2^{-k})) \\ &\overset{(7.4.11)}{\leq} \Big(\sigma^{(1-2r)}_{K',N'}(\theta^k) + \sigma^{(2r)}_{K',N'}(\theta^k)\sigma^{(1/2)}_{K',N}(\theta^{k-1})\Big) S_{N',\mathfrak{m}}(\Gamma(s)) + \sigma^{(2r)}_{K',N'}(\theta^k)\sigma^{(1/2)}_{K',N}(\theta^{k-1}) S_{N',\mathfrak{m}}(\Gamma(t)). \end{aligned}$$

A direct computation shows that

$$\sigma^{\big((x_r-s)/2^{-k}\big)}_{K',N'}(\theta^k)\,\sigma^{(1/2)}_{K',N'}(\theta^{k-1}) = \sigma^{\big((x_r-s)/2^{-(k+1)}\big)}_{K',N'}(\theta^{k-1}), \tag{7.4.12}$$

while, using the sum-to-product trigonometric formulas, it is also possible to prove that

$$\sigma_{K',N'}^{\left((s+2^{-k}-x_r)/2^{-k}\right)}(\theta^k)+\sigma_{K',N'}^{\left((x_r-s)/2^{-k}\right)}(\theta^k)\,\sigma_{K',N'}^{(1/2)}(\theta^{k-1})=\sigma_{K',N'}^{\left((s+2^{-(k+1)}-x_r)/2^{-(k+1)}\right)}(\theta^{k-1}), \quad (7.4.13)$$

where x_r is any time between s and $s+2^{-(k+1)}$. Making use of these expressions, we can then write the bound on $S_{N',\mathfrak{m}}(\Gamma(s+r\cdot 2^{1-k}))$ as

$$S_{N',\mathfrak{m}}(\Gamma(s+r\cdot 2^{1-k}))\le\sigma_{K',N'}^{(1-r)}(\theta^{k-1})S_{N',\mathfrak{m}}(\Gamma(s))+\sigma_{K',N'}^{(r)}(\theta^{k-1})S_{N',\mathfrak{m}}(\Gamma(t)),$$

when $r\in[0,1/2]$. In the case in which $r\in[1/2,1]$, we can apply (7.4.9) to obtain that

$$\begin{aligned}S_{N',\mathfrak{m}}(\Gamma(s+r\cdot 2^{1-k}))&\le\sigma_{K,N'}^{(2-2r)}(\theta^k)S_{N',\mathfrak{m}}(\Gamma(s+2^{-k}))+\sigma_{K',N'}^{(2r-1)}(\theta^k)S_{N',\mathfrak{m}}(\Gamma(t))\\&\overset{(7.4.11)}{\le}\sigma_{K',N'}^{(2-2r)}(\theta^k)\sigma_{K',N'}^{(1/2)}(\theta^{k-1})S_{N',\mathfrak{m}}(\Gamma(s))+\Big(\sigma_{K',N'}^{(2-2r)}(\theta^k)\sigma_{K',N'}^{(1/2)}(\theta^{k-1})+\sigma_{K',N'}^{(2r-1)}(\theta^k)\Big)S_{N',\mathfrak{m}}(\Gamma(t)).\end{aligned}$$

Using again the identities in (7.4.12) and (7.4.13), we get the bound

$$S_{N',\mathfrak{m}}(\Gamma(s+r\cdot 2^{1-k}))\le\sigma_{K',N'}^{(1-r)}(\theta^{k-1})S_{N',\mathfrak{m}}(\Gamma(s))+\sigma_{K',N'}^{(r)}(\theta^{k-1})S_{N',\mathfrak{m}}(\Gamma(t)),$$

also when $r\in[1/2,1]$, which shows the validity of property $\mathsf{C}(k-1)$. □

Notice that if Γ is a geodesic in $\mathscr{P}_N^*(\mathsf{X},\mathfrak{m})$ such that $\operatorname{supp}(\mu_0),\operatorname{supp}(\mu_1)\subseteq B_R(o)$, then $\operatorname{supp}(\Gamma(t))\subseteq\bar B_{2R}(o)$ for every $t\in[0,1]$. The compactness of $\bar B_{2R}(o)$ implies the existence of a constant $\lambda>0$, of finitely many disjoint sets $L_1,\dots,L_n$ covering $\bar B_{2R}(o)$ and closed sets $X_1,\dots,X_n$ with $B_\lambda(L_j)\subset X_j$ for $j=1,\dots,n$, that realize the local validity of the $\mathsf{CD}^*(K',N)$ condition. In particular we ask that, for $j=1,\dots,n$, every pair of marginals $\mu_0,\mu_1\in\mathscr{P}_N^*(\mathsf{X},\mathfrak{m})$ such that $\operatorname{supp}(\mu_0),\operatorname{supp}(\mu_1)\subseteq X_j$ can be joined by a geodesic in $\mathscr{P}_N^*(\mathsf{X},\mathfrak{m})$ satisfying (7.2.3). Hence, we choose a $\kappa\in\mathbb{N}$ such that

$$2^{-\kappa}\operatorname{diam}(\bar B_{2R}(o))\le 2^{2-\kappa}R\le\lambda.$$

Lemma 7.4.6. *Under the assumptions of Theorem* 7.4.4, *property* $\mathsf{C}(\kappa)$ *holds true.*

Proof. We fix a geodesic $\Gamma\colon[0,1]\to\mathscr{P}_N^*(\mathsf{X},\mathfrak{m})$ such that $\operatorname{supp}(\mu_0),\operatorname{supp}(\mu_1)\subseteq B_R(o)$ and a pair of times $s,t\in[0,1]$, such that $t-s=2^{-\kappa}$. We consider

$$\hat{\mathsf{q}}=(e_s,e_t)_\#\Gamma\in\mathscr{P}(\mathsf{X}\times\mathsf{X}),$$

it is easy to realize that

$$\mathsf{d}(x,y)\le 2^{-\kappa}\operatorname{diam}(\bar B_{2R}(o))\le\lambda, \quad (7.4.14)$$

for $\hat{\mathsf{q}}$-almost every (x,y). Then, for $j=1,\dots,n$, we define the probability measures $\Gamma_j(s)$ and $\Gamma_j(t)$ by

$$\Gamma_j(s):=\frac{1}{\alpha_j}(\mathsf{p}_1)_\#[\hat{\mathsf{q}}|_{L_j\times\mathsf{X}}]\quad\text{and}\quad\Gamma_j(t):=\frac{1}{\alpha_j}(\mathsf{p}_2)_\#[\hat{\mathsf{q}}|_{L_j\times\mathsf{X}}],$$

provided that $\alpha_j:=\hat{\mathsf{q}}(L_j\times\mathsf{X})>0$ (otherwise we can define $\Gamma_j(s)$ and $\Gamma_j(t)$ arbitrarily). In the last formula $\mathsf{p}_1,\mathsf{p}_2:\mathsf{X}\times\mathsf{X}\to\mathsf{X}$ denote the projection maps on the first and on the second factor, respectively. Then $\operatorname{supp}(\Gamma_j(s))\subseteq\overline{L_j}$ and this, together with (7.4.14), ensures that

$$\operatorname{supp}(\Gamma_j(s))\cup\operatorname{supp}(\Gamma_j(t))\subseteq\overline{B_\lambda(L_j)}\subseteq X_j.$$

Moreover, notice that $\Gamma_j(s),\Gamma_j(t)\in\mathscr{P}_N^*(\mathsf{X},\mathfrak{m})$ for every j, thus, according to our choice of the sets X_j there exists a geodesic $\hat\Gamma_j$ between $\Gamma_j(s)$ and $\Gamma_j(t)\in\mathscr{P}_N^*(\mathsf{X},\mathfrak{m})$ such that

$$S_{N',\mathfrak{m}}(\hat\Gamma_j(r))\le\sigma_{K',N'}^{(1-r)}(\theta_j)S_{N',\mathfrak{m}}(\Gamma_j(s))+\sigma_{K',N'}^{(r)}(\theta_j)S_{N',\mathfrak{m}}(\Gamma_j(t)), \quad (7.4.15)$$

for all $r \in [0,1]$ and $N' \in [N,0)$, where

$$\theta_j := \inf_{\gamma \in \mathrm{supp}(\hat{\Gamma}_j)} \mathsf{d}(\gamma(s), \gamma(t)).$$

Define then the curve $\hat{\Gamma}\colon [0,1] \to \mathscr{P}^*_N(\mathsf{X}, \mathfrak{m})$ by setting for any $r \in [0,1]$

$$\hat{\Gamma}(r) := \sum_{j=1}^n \alpha_j \hat{\Gamma}_j(r).$$

We observe that $\{\hat{\Gamma}(r)\}_{r\in[0,1]}$ is a geodesic between $\Gamma(s) := \sum_{j=1}^n \alpha_j \Gamma_j(s)$ and $\Gamma(t) := \sum_{j=1}^n \alpha_j \Gamma_j(t)$ and so, as a consequence of Proposition 7.3.1 we can actually conclude that

$$\hat{\Gamma}(r) = \Gamma(s + r(t-s)) \quad \forall r \in [0,1].$$

It is easy to realize that this is particular implies that $\theta_j \geq \theta^\kappa$ for every j, and then, keeping in mind (7.4.15), we conclude that

$$S_{N',\mathfrak{m}}(\hat{\Gamma}_j(r)) \leq \sigma^{(1-r)}_{K',N'}(\theta^\kappa) S_{N',\mathfrak{m}}(\Gamma_j(s)) + \sigma^{(r)}_{K',N'}(\theta^\kappa) S_{N',\mathfrak{m}}(\Gamma_j(t)), \tag{7.4.16}$$

for every j. On the other hand, since $\Gamma_j(s)$ are mutually singular for $j = 1, \dots, n$, it is possible to apply Proposition 7.2.7 and conclude that $\hat{\Gamma}(r)$ are mutually singular for $r \in [0,1)$ and $j = 1, \dots, n$. In particular, the fact that $\Gamma_j(s)$ and $\hat{\Gamma}_j(r)$ are mutually singular ensures that for every $N' \in [N,0)$ it holds

$$S_{N',\mathfrak{m}}(\hat{\Gamma}(r)) = \sum_{j=1}^n \alpha_j^{1-\frac{1}{N'}} S_{N',\mathfrak{m}}(\hat{\Gamma}_j(r)) \quad \forall r \in [0,1) \tag{7.4.17}$$

and

$$S_{N',\mathfrak{m}}(\Gamma(s)) = \sum_{j=1}^n \alpha_j^{1-\frac{1}{N'}} S_{N',\mathfrak{m}}(\Gamma_j(s)). \tag{7.4.18}$$

On the other hand, the $\Gamma_j(t)$ are not necessarily mutually singular for $j = 1, \dots, n$, and so

$$S_{N',\mathfrak{m}}(\Gamma(t)) \geq \sum_{j=1}^n \alpha_j^{1-\frac{1}{N'}} S_{N',\mathfrak{m}}(\Gamma_j(t)). \tag{7.4.19}$$

At this point, summing up for $j = 1, \dots, n$ the inequality (7.4.16) and making use of (7.4.17), (7.4.18) and (7.4.19), we obtain

$$\begin{aligned} S_{N',\mathfrak{m}}(\Gamma(s + r(t-s))) = S_{N',\mathfrak{m}}(\hat{\Gamma}(r)) &= \sum_{j=1}^n \alpha_j^{1-\frac{1}{N'}} S_{N',\mathfrak{m}}(\hat{\Gamma}_j(r)) \\ &\leq \sigma^{(1-r)}_{K',N'}(\theta^\kappa) \sum_{j=1}^n \alpha_j^{1-\frac{1}{N'}} S_{N',\mathfrak{m}}(\Gamma_j(s)) + \sigma^{(r)}_{K',N'}(\theta^\kappa) \sum_{j=1}^n \alpha_j^{1-\frac{1}{N'}} S_{N',\mathfrak{m}}(\Gamma_j(t)) \\ &\leq \sigma^{(1-r)}_{K',N'}(\theta^\kappa) S_{N',\mathfrak{m}}(\Gamma(s)) + \sigma^{(r)}_{K',N'}(\theta^\kappa) S_{N',\mathfrak{m}}(\Gamma(t)) \end{aligned}$$

for all $N' \in [N,0)$, proving property $\mathsf{C}(\kappa)$. □

Using these results, the proof of Theorem 7.4.4 is quite straightforward.

Proof of Theorem 7.4.4. Let us fix two probability measures $\mu_0, \mu_1 \in \mathscr{P}_N^*(\mathsf{X}, \mathfrak{m})$ such that

$$\operatorname{supp}(\mu_0), \operatorname{supp}(\mu_1) \subseteq B_R(o).$$

By assumption, there exists a geodesic Γ with domain in $\mathscr{P}_N^*(\mathsf{X}, \mathfrak{m})$ connecting them, i.e. $\Gamma(0) = \mu_0$ and $\Gamma(1) = \mu_1$. Now, by Lemma 7.4.6, property $\mathsf{C}(\kappa)$ is satisfied, while Lemma 7.4.5 ensures that this implies that $\mathsf{C}(k)$ holds also for all $k = \kappa - 1, \kappa - 2, \dots, 0$. In particular, property $\mathsf{C}(0)$ states that the geodesic Γ is such that

$$S_{N',\mathfrak{m}}(\Gamma(t)) \leq \sigma_{K,N'}^{(1-t)}(\theta^0) S_{N',\mathfrak{m}}(\mu_0) + \sigma_{K,N'}^{(t)}(\theta^0) S_{N',\mathfrak{m}}(\mu_1)$$

for all $N' \in [N, 0)$, where

$$\theta^0 = \inf_{\gamma \in \operatorname{supp}(\Gamma)} \mathsf{d}(\gamma(0), \gamma(1)).$$

On the other hand, it is obvious that

$$\theta^0 \geq \theta = \inf_{x_0 \in \mathcal{S}_0, x_1 \in \mathcal{S}_1} \mathsf{d}(x_0, x_1),$$

where $\mathcal{S}_0 = \operatorname{supp}(\mu_0)$ and $\mathcal{S}_1 = \operatorname{supp}(\mu_1)$. As a consequence we can conclude that

$$S_{N',\mathfrak{m}}(\Gamma(t)) \leq \sigma_{K,N'}^{(1-t)}(\theta) S_{N',\mathfrak{m}}(\mu_0) + \sigma_{K,N'}^{(t)}(\theta) S_{N',\mathfrak{m}}(\mu_1).$$

Thanks to the arbitrariness of K' and of the metric ball $B_R(o)$, it is possible to apply Proposition 7.4.3 and show the validity of the condition $\mathsf{CD}^*(K-, N)$ globally in $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$. ☐

We conclude by noticing that combining Theorem 7.4.4 with 7.4.2, we obtain the local-to-global property for the $\mathsf{CD}^*(K, N)$ condition, when the curvature parameter K is non-negative.

Corollary 7.4.7. *Let $K, N \in \mathbb{R}$ with $N < 0 \leq K$ and let $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ be a locally compact, essentially non-branching metric measure space such that $\mathscr{P}_N^*(\mathsf{X}, \mathfrak{m})$ is a geodesic space. If $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ satisfies the condition $\mathsf{CD}^*(K, N)$ locally, then it satisfies the condition $\mathsf{CD}^*(K, N)$ globally.*

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