Study of random Schrödinger operators away from the standard Anderson model: correlations and non-locality

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Dedication

Quisiera dedicar esta tesis a varias personas que me ayudaron de una u otra forma a finalizar esta etapa en mi vida. Quisiera dedicar este trabajo a mi familia, en particular a mi abuelo, Pedro. Gracias por sus enseñanzas... siempre recordaré que: "Si uno tiene que morir, uno muere de frente". También quisiera aprovechar esta instancia para dedicar este trabajo a Florence, Carole y Frauke por ayudarme a mejorar mis habilidades comunicativas e interpersonales. Siempre estaré en deuda de su apoyo incondicional.

Abstract

This thesis focuses on the investigation of two distinct variations of the Anderson model, where we relax the standard assumptions. In the first half of our study, we investigate a version of the Anderson model on the one-dimensional lattice strip or ladder of width 2, denoted as $\Gamma = \mathscr{D}_2 := \mathbb{Z} \times \{0,1\}$ with $\mathrm{d} = 1$, wherein the free operator is the graph Laplacian and the random variables $\omega = (\omega_x)_{x \in \mathscr{D}_2}$ are identically distributed but not always independent. We adapt the supersymmetric approach presented by Klein, Martinelli and Perez in 1986 from the uni-dimensional lattice $\mathbb{Z}$ to $\mathscr{D}_2$, which is not one-dimensional. This adaptation allows us to obtain a representation of the square modulus of the 4-points average of the Green's functions corresponding to two layers of $\mathscr{D}_2$. Although we succeeded in expressing this squared average in terms of transfer operators, the analytic estimate poses technical challenges which we have not been able to overcome. Hence, we are still unable to show localization in our specific context. Nonetheless, our extension of Klein, Martinelli and Perez's approach is interesting on its own and could serve as a starting point for future investigations.

In the second half of our study, we examine a specific instance of the Anderson model on $\Gamma = \mathbb{Z}^{\mathrm{d}}$ with $\mathrm{d} \in \mathbb{N}$ known as the *fractional Anderson model*. In this model the random variables $(\omega_x)_{x \in \mathbb{Z}^{\mathrm{d}}}$ are iid and the free operator is the fractional Laplacian, which is not a local operator and exhibits a slow rate in the decay of its matrix elements. Adapting Schenker's arguments in 2015, we relate the fractional moments of the Green's function to the two-point correlation function of a self-avoiding walk with polynomial long-range jumps. This together with the use of known probabilistic techniques yield sharper bounds for the fractional moments of the Green's function at strong disorder, surpassing the previous bounds in the literature. Notably, we expand the range of the disorder parameter λ where spectral localization happens. Furthermore, for $\mathrm{d} = 1$, we prove polynomial decay of the eigenfunctions almost-surely, assuming some regularity of the probability distribution of ω_x with $x \in \mathbb{Z}$.

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Chapter 1

Introduction

1.1 Physical Motivation

The state of an electron confined to a space Γ at a given time is described by a normalized wave function ψ_0 in a convenient Hilbert space $\mathcal{H}$. The time-evolution of ψ_0 is determined by the Schrödinger equation

$$\begin{cases} i\frac{\partial}{\partial t}\psi(x,t) = \mathrm{H}\psi(x,t), & (x,t) \in \Gamma \times \mathbb{R}, \\ \psi(x,0) = \psi_0(x), & x \in \Gamma, \end{cases} \tag{1.1.1}$$

where H is a linear self-adjoint operator on $\mathcal{H}$, which represents the energy of the particle. Due to the spectral theorem for self-adjoint operators, the solution to Eq. 1.1.1 can be expressed as

$$\psi(x,t) = e^{-it\mathrm{H}}\psi_0(x), \qquad (x,t) \in \Gamma \times \mathbb{R}. \tag{1.1.2}$$

The propagation of the electron corresponds to an extended wave function such as $\psi_0(x) \sim e^{ix}$. This behavior indicates that the material is a conductor. By contrast, the absence of propagation of the electron corresponds to a localized wave function, for example, $\psi_0(x) \sim e^{-x^2}$. In this situation, this behavior indicates that the material is an insulator.

In 1958, Anderson proposed a model to provide an explanation of the absence of quantum wave propagation in disordered lattices [And58]. This physical phenomenon is known nowadays as Anderson localization. He realized that, if some conditions are met, then impurities in the material can refrain the electron from propagating and thus the material behaves as an insulator. To investigate this phenomenon, Anderson focused on a specific type of Random Schrödinger Operator, wherein impurities are modelled through realizations of a random potential in a suitable probability space Ω as follows: let $\Gamma = \mathbb{Z}^{\mathrm{d}}$ with $\mathrm{d} \in \mathbb{N}$ and

$$\mathrm{H}_{\omega,\lambda} = -\Delta + \lambda \mathrm{V}_{\omega}, \qquad \omega \in \Omega, \tag{1.1.3}$$

acting on $\mathcal{H} = \ell^2(\mathbb{Z}^{\mathrm{d}})$, where:

(A) $-\Delta$ is the (negative) discrete Laplacian representing the kinetic energy of the electron,

(B) V_{ω} is a random multiplication operator acting diagonally on the canonical basis by a sequence $\omega = (\omega_x)_{x \in \mathbb{Z}^{\mathrm{d}}}$ of iid random variables with common uniform distribution, each ω_x represents the interaction between the electron and the atomic constitution of the material at $x \in \mathbb{Z}^{\mathrm{d}}$,

(C) $\lambda > 0$ is a parameter representing the intensity of the disorder.

1.2 Anderson Model and types of localization

Let us start by introducing the general mathematical framework of the discrete Anderson model. Let Γ be a discrete set and $\mathcal{H}$ be the Hilbert space defined as

$$\mathcal{H} = \ell^2(\Gamma) := \left\{ \psi : \Gamma \longrightarrow \mathbb{C} \,\middle|\, \sum_{x \in \Gamma} |\psi(x)|^2 < \infty \right\} \tag{1.2.1}$$

with inner product $\langle \phi, \psi \rangle = \sum_{x \in \Gamma} \overline{\phi(x)} \psi(x)$, for all $\phi, \psi \in \ell^2(\Gamma)$, and induced norm $\|\psi\| = \sqrt{\langle \psi, \psi \rangle}$, for all $\psi \in \ell^2(\Gamma)$. The Anderson model acting on $\ell^2(\Gamma)$ is defined as

$$\mathrm{H}_{\omega,\lambda} = \mathrm{H}_0 + \lambda \mathrm{V}_{\omega}, \qquad \omega \in \Omega, \tag{1.2.2}$$

where H_0 is a bounded self-adjoint operator on $\ell^2(\Gamma)$, known as *free operator*, and V_{ω} is a multiplication operator given by

$$\mathrm{V}_{\omega} \psi(x) = \omega_x \psi(x), \qquad \forall \psi \in \ell^2(\Gamma), \, \forall x \in \Gamma,$$

and $\omega := (\omega_x)_{x \in \Gamma}$ is a family of bounded random variables with joint distribution $\mathbb{P}$ on the measurable space $\mathbb{R}^{\Gamma}$ with the $\sigma-$algebra generated by the open cylinder. Here, $\mathbb{E}[\cdot]$ denotes the corresponding expectation.

Consequently, $\mathrm{H}_{\omega,\lambda}$ is self-adjoint and its spectrum $\sigma(\mathrm{H}_{\omega,\lambda})$ is contained in the real line, for all $\omega \in \Omega$. The *standard Anderson model* corresponds to the case where $\Gamma = \mathbb{Z}^{\mathrm{d}}$, $\mathrm{H}_0 = -\Delta$ and $\omega = (\omega_x)_{x \in \Gamma}$ are iid with joint distribution $\mathbb{P} = \bigotimes_{\mathbb{Z}^{\mathrm{d}}} \mathrm{P}_0$. The specific model studied in [And58] corresponds to $\mathrm{P}_0(y) = \frac{1}{2} \mathbb{1}_{[-1,1]}(y)$, $y \in \mathbb{R}$, see Eq. 1.1.3

We can now define the notions of *localization* and *delocalization* within this mathematical framework. From now on, let $\Gamma \subset \mathbb{Z}^{\mathrm{d}}$ with $\mathrm{d} \in \mathbb{N}$ and I be an interval. We say that $\{\mathrm{H}_{\omega,\lambda}\}_{\omega \in \Omega}$ exhibits *spectral localization* in I, if $\sigma(\mathrm{H}_{\omega,\lambda}) \cap I$ is pure-point, for $\omega \in \Omega$

$\mathbb{P}-$almost-surely (a.s.). We say that $\{\mathrm{H}_{\omega,\lambda}\}_{\omega\in\Omega}$ exhibits *Anderson localization* in I, if $\sigma(\mathrm{H}_{\omega,\lambda})\cap I$ is pure-point and the eigenfunctions have exponential decay, for $\omega\in\Omega$ $\mathbb{P}-$a.s. Specifically, there exist constants $0<C_k(\omega),\gamma_k(\omega)<\infty$ and $x_k(\omega)\in\Gamma$ such that the eigenfunctions $\{\psi_k(\cdot,\omega)\}_{k\in\mathbb{N}}\subset\ell^2(\mathbb{Z}^{\mathrm{d}})$ of $\mathrm{H}_{\omega,\lambda}$ satisfy

$$|\psi(x,\omega)|\leq C_k(\omega)e^{-\gamma_k(\omega)|x-x_k(\omega)|},\qquad \forall k\in\mathbb{N}, \tag{1.2.3}$$

where $|\cdot|$ denotes the Euclidean metric on Γ.

There is a stronger notion of localization. It is said that $\{\mathrm{H}_{\omega,\lambda}\}_{\omega\in\Omega}$ exhibits *dynamical localization* in I, if there are constants $0<C,\gamma<\infty$ and $0<\zeta\leq 1$ such that

$$\mathbb{E}\Big(\sup_{t\in\mathbb{R}}\big|\langle\delta_x,e^{-it\mathrm{H}_{\omega,\lambda}}\chi_I(\mathrm{H}_{\omega,\lambda})\delta_y\rangle\big|\Big)\leq Ce^{-\gamma|x-x_k(\omega)|^{\zeta}},\qquad \forall x,y\in\Gamma, \tag{1.2.4}$$

where $\{\delta_x\}_{x\in\Gamma}$ corresponds to the canonical basis of $\ell^2(\Gamma)$ and $\chi_I(\mathrm{H}_{\omega,\lambda})$ is the spectral projection of $\mathrm{H}_{\omega,\lambda}$ associated to the interval I. Observe that dynamical localization implies Anderson localization, a proof of this can be found in [CFKS09, Thm. 9.22]. However, the converse does not hold in general, as shown in [dRJLS95, Appendix 2]. In addition, dynamical localization implies that, for an initial localized ψ_0, its time-evolution $e^{-it\mathrm{H}_{\omega,\lambda}}\psi_0$ remains uniformly localized, for all time t, for $\omega\in\Omega$ $\mathbb{P}-$a.s. To be more precise, for all $\mathrm{p}\geq 0$, for all $\psi\in\mathcal{H}$, it holds

$$\sup_{t\in\mathbb{R}}\big\||X|^{\mathrm{p}}e^{-it\mathrm{H}_{\omega,\lambda}}\chi_I(\mathrm{H}_{\omega,\lambda})\psi\big\|<\infty,\qquad \omega\in\Omega\,\mathbb{P}-\text{a.s.}, \tag{1.2.5}$$

where $|X|$ is the multiplication operator on $\mathcal{H}$ given by $|X|\psi(x)=\|x\|\psi(x)$, for all $\psi\in\mathcal{H}$, for all $x\in\Gamma$. In fact, Condition 1.2.5 is equivalent to dynamical localization, a proof of this can be found in [GK04, Thm. 4.2].

By contrast, we say that the Anderson model $\{\mathrm{H}_{\omega,\lambda}\}_{\omega\in\Omega}$ exhibits *delocalization* in I, if dynamical localization does not hold. Note that if $\sigma(\mathrm{H}_{\omega,\lambda})$ is absolutely continuous $\mathbb{P}-$a.s., then there is delocalization. However, the converse is not true in general. In fact, the Anderson model mentioned in [dRJLS95, Appendix 2] exhibits delocalization, yet it possesses pure-point spectrum $\mathbb{P}-$a.s.

1.3 State of the art

Anderson's seminal work [And58] served as a catalyst for the mathematical investigations of the Anderson model and its localization properties since the late 70s. Indeed, the earliest result can be found in [GMP77].

For $\mathrm{d}=1$, it is expected that $\mathrm{H}_{\omega,\lambda}$ exhibits dynamical localization in the whole real line, for all $\lambda>0$ (e.g., [KS80], for the standard Anderson model). In the unidimensional case, there are several available methods to prove dynamical localization such as Transfer Operators, Supersymmetry (SUSY) and Kunz-Souillard (see [KMP86] and [Dam11]).

For $d > 1$, it is expected that $H_{\omega,\lambda}$ exhibits dynamical localization either in the whole real line, for sufficiently large λ, or at the spectral band edges, for specific values of λ. The first proof of dynamical localization under high disorder λ can be found in [FMSS85], which proof is based on the *Multiscale Analysis* (MSA). However, they did not show the existence of $\lambda_0 > 0$ such that $H_{\omega,\lambda}$ exhibits dynamical localization, for all $\lambda > \lambda_0$. Afterwards, a new approach called *Fractional Moment Method* (FMM) in [AM93] was developed to prove dynamical localization, providing an explicit localization threshold. However, that critical value was larger than the one computed by Anderson in [And58], see [Sch15, Table I] for a numerical comparison between both thresholds. In addition, for fixed values of λ, there is still dynamical localization at energies $|E| \gg 1$. Note that this is meaningful only in the case of unbounded potential; otherwise, the spectrum is bounded and $E \notin \sigma(H_{\omega,\lambda})$, for $|E| \gg 1$, for $\omega \in \Omega$ $\mathbb{P}-$a.s. In general, for arbitrary d, MSA and FMM are the only known methods to prove dynamical localization. A comprehensive exposition of MSA and FMM can be found in [PF92], [Sto01], [CL12] and [AW15]. Recently, *self-avoiding walks* (SAW) have been employed as a complementary tool to prove localization under strong disorder, see [Tau11], [Suz13] and [Sch15].

In contrast to dynamical localization, only a limited number of models have been rigorously established to exhibit delocalization, particularly in cases involving decaying randomness (see [KKO00], [Sim82], [DSS85] and [Kis96]). Delocalization in the case where $\omega = (\omega_x)_{x\in\Gamma}$ iid has only been proved on the Bethe lattice $\Gamma = \mathbb{B}$ (see [Kle94], [Kle98], [ASW06], [FHS07] and [AW13]). However, it remains unknown whether or not delocalization occurs on $\mathbb{Z}^d$ with $d > 1$.

These investigations have given rise to two open problems regarding the standard Anderson model:

1. For $d = 2$, it is conjectured that the standard Anderson model shows localization, similar to the uni-dimensional case, in the complete spectrum, for all $\lambda > 0$.
2. For $d > 2$, it is conjectured that the standard Anderson model experiences a transition from exhibiting extended states within the bulk of the spectrum to localized states at the spectral band edges, which is known as "Anderson metal-insulator transition".

This transition can be regarded as a competition between the two components of the standard Anderson model to dominate the situation. If the free operator $-\Delta$ establishes dominance, then its absolutely continuous spectrum $\sigma(-\Delta) = [0, 4d]$ with associated extended states prevail. Conversely, if the random potential V_ω takes control, then its pure point spectrum and associated localized eigenfunctions prevail almost-surely.

The study of the *long-range* Anderson model, wherein H_0 is a long-range operator, has recently gained growing attention, see [Han19], [PKL+20], [GRM20], [JL21], [Liu23] and [Shi23]. This model, particularly when H_0 exhibits power-law jumps, is relevant in physical phenomena such as the quantum Kepler model (see [AL97]) or nuclear spins in solid-state systems (see [ÁSK15]). In this scenario, instead of exponential decay of the eigenfunctions and dynamical bounds, a *polynomial* decay is obtained, see [Shi23, Corollary 2.3].

For the interested reader who might be interested in learning more about the Anderson model on the discrete setting and its localization properties, [Kir07] and [RM17] offer a clear and self-contained explanation of the subject and its state of the art.

1.4 Our results

The rest of this thesis focuses on the investigation of two different instances of the Anderson model, as given by Eq. 1.2.2.

In Chapter 2, we examine a version of the Anderson model over the one-dimensional lattice strip or ladder of width 2, denoted as $\Gamma = \mathscr{D}_2 := \mathbb{Z} \times \{0, 1\}$. In this case, H_0 is a local operator but $\omega = (\omega_x)_{x\in\mathscr{D}_2}$ are not necessarily independent, which introduces greater complexity to the analysis from a probabilistic point of view. Specifically, we consider

$$H_{\omega,\lambda} = -\Delta + \lambda V_\omega, \qquad \omega \in \Omega, \tag{1.4.1}$$

acting on $\mathscr{D}_2$, where $H_0 = -\Delta$ is the lattice Laplacian and $\omega = (\omega_x)_{x\in\mathscr{D}_2}$ is a sequence of identically distributed random variables, which are not necessarily independent.

In [KMP86], Klein, Martinelli and Perez investigated the case when the underlying lattice $\Gamma = \mathbb{Z}$ and $\omega = (\omega_x)_{x\in\mathbb{Z}}$ are iid, which is a realization of the model considered in [And58] for $d = 1$. They established dynamical localization (see Ineq. 1.2.4) in the whole real line, for all $\lambda > 0$. To accomplish this, they employed a combination of SUSY and Transfer Operators. More precisely, they used a SUSY representation of $\mathbb{E}[|G_z(x_0, x_1)|^2]$, with $x_0, x_1 \in \mathbb{Z}$. Then, they explicitly carried out the derivation over the fermionic variables to extract a suitable transfer operator from the resulting expression. This approach heavily relied on the fact that the underlying lattice $\mathbb{Z}$ is one-dimensional. For the reader's convenience, a comprehensive exposition of SUSY formalism is provided in Section 2.3 below.

Our aim is to adapt this strategy to our model on the lattice $\mathscr{D}2$, which is not one-dimensional. However, instead of studying $\mathbb{E}[|G_z(x_0, x_1)|^2]$, with $x_0, x_1 \in \mathscr{D}_2$, it is more natural in our setting to consider the square of the 4-points average of the Green's functions associated to two layers of $\mathscr{D}_2$. Although we manage to use the SUSY approach to find a representation via transfer operator of the mentioned squared average, the examination of its corresponding properties presents challenges, which we are not able to solve. A more detailed description of this issue can be found in Section 2.5.

In Chapter 3, we explore an instance of the Anderson model on $\mathbb{Z}^d$ with $d \in \mathbb{N}$, where $\{\omega_x\}_{x\in\mathbb{Z}^d}$ are iid but H_0 is a non-local operator. To be explicit, the *fractional Anderson model* is defined as

$$H_{\omega,\lambda,\alpha} = (-\Delta)^\alpha + \lambda V_\omega, \qquad \omega \in \Omega, \tag{1.4.2}$$

acting on $\Gamma = \mathbb{Z}^d$, where $H_0 = (-\Delta)^\alpha$, for $0 < \alpha < 1$, and $\{\omega_x\}_{x\in\mathbb{Z}^d}$ are iid. The operator $(-\Delta)^\alpha$ is the discrete fractional (negative) Laplacian, which is defined via functional calculus. Unlike the standard Laplacian, the operator $(-\Delta)^\alpha$ is not local with strictly negative off-diagonal matrix elements (see Remark 3.2.2). However, this alone does not pose a problem, when it comes to prove dynamical localization. In this case, dynamical localization is established using [AM93, Lemma 3.2]. However, in the fractional Anderson model, the decay of $(-\Delta)^\alpha$ is not fast enough because its off-diagonal matrix elements decay as $|(-\Delta)^\alpha(x,y)| \sim \frac{1}{|x-y|^{d+2\alpha}}$, see [GRM20, Thm. 2.2 (iii)]. In fact, when polynomial decay is involved, additional complexities arise, making the computations more challenging. This is further supported by the recent work of [Shi23] on the MSA for long-range models.

In [Sch15], Schenker studied the Anderson model corresponding to $\Gamma = \mathbb{Z}^d$, $H_0 = -\Delta$ and $\{\omega_x\}_{x\in\mathbb{Z}^d}$ iid with common uniform distribution supported on $[-1,1]$. This is the model originally considered by Anderson in [And58]. He proved that dynamical localization in the whole real line above a localization threshold $\lambda_{\mathrm{And}} > 0$. In particular, the constant λ_{And} matches exactly the critical value proposed in [And58]. To establish that result, the author employed the FMM, that is, he estimated the fractional moments of the Green's function. Initially, the depleted resolvent identity was employed to find an upper bound on $\mathbb{E}[|G_z(x,x_0)|^s]$. This bound was subsequently reformulated in terms of the two-point correlation function of the nearest-neighbours SAW induced by the standard Laplacian up to a constant.

However, when we replicated Schenker's approach in our setting, we instead obtained that our SAW has long jumps with polynomial decay, which reflects the fact that the fractional Laplacian is a non-local operator with a polynomial slow decay. Our main result is that we relate $\mathbb{E}[|G_z(x,x_0)|^s]$, with $x_0, x_1 \in \mathbb{Z}^d$, $0 < s < 1$, to the two-point correlation function of the long-range self-avoiding walk (SAW) induced by $(-\Delta)^\alpha$ (see Thm. 3.5.1). This generalizes [Sch15, Thm. 1] to the Anderson model with a fractional Laplacian perturbed by a random potential. The challenging part revolves around determining the convergence conditions of the two-point correlation function of the SAW induced by $(-\Delta)^\alpha$. In particular, its two-point correlation function decays polynomially, as shown in Lemma [CS15, Lemma 2.4]. Unfortunately, this polynomial decay was insufficient to establish dynamical localization since exponential decay of the two-point correlation function was required. Nevertheless, we managed to prove that there is a constant $\lambda_0 > 0$ such that, for all $\lambda > \lambda_0$, spectral localization occurs. In addition, in the case of $d = 1$ and assuming some conditions on the one-site probability distribution, the eigenfunctions decay polynomially, for $\omega \in \Omega$ $\mathbb{P}-$almost-surely as detailed in Thm. 3.5.5 below. Furthermore, our critical value $\lambda_0 > 0$ is smaller than previously found values in the literature (see Thm. 3.3.2 and 3.3.11) and our estimation of the decay of eigenfunctions is sharper than a prior result found in [AM93, Lemma 3.2] (compare Estimate 3.3.7 with 3.5.24).

Chapter 2

Random Schrödinger Operator with dependent random variables

2.1 Introduction

In this chapter we consider the Anderson model of the form

$$\mathrm{H}_{\omega,\lambda} = -\Delta + \lambda \mathrm{V}_\omega, \qquad \omega \in \Omega, \tag{2.1.1}$$

acting on $\Gamma = \mathscr{D}_2 = \mathbb{Z} \times \{0,1\}$, where $-\Delta$ is the graph Laplacian and V_ω is a random multiplication operator acting diagonally on the canonical basis by a sequence $\omega = (\omega_x)_{x\in\mathbb{Z}}$ of real, bounded and identically distributed random variables defined in a suitable probability space Ω. These random variables are not necessarily independent. When the underlying graph is $\Gamma = \mathbb{Z}$ with $\mathrm{d} = 1$ and $\omega = (\omega_x)_{x\in\mathbb{Z}}$ are iid, [KMP86] established Anderson localization, see Corollary 3 therein, under some additional conditions on the one-site probability measure, see Assumption 2.5.3 below. Their proof used a representation of the second moment of the Green function through suitable transfer operators.

The main objective of this chapter is to extend the strategy presented in [KMP86] to the case when the graph is the one-dimensional lattice strip or ladder of width 2, $\mathscr{D}_2 := \mathbb{Z}\times\{0,1\}$, and $\omega = (\omega_x)_{x\in\mathbb{Z}}$ are not necessarily independent. Instead of studying $\mathbb{E}[|\mathrm{G}_z(x_0,x_1)|^2]$, with $x_0, x_1 \in \mathscr{D}_2$, it is more natural in our setting to consider the second moment of the 4-point average of the Green's functions corresponding the j_0-th and j_1-th layers of the ladder $\mathscr{D}_2$ with $j_0, j_1 \in \mathbb{Z}$. Although we arrive to a representation of the square of the mentioned 4-point average via transfer operators, the analysis of the properties of the corresponding second moment of the 4-point average poses problems, which we cannot still solve.

Now, we provide an overview of the remaining sections. In Section 2.1, we rigorously define the Anderson model on $\mathscr{D}_2$ and state Thm. 2.2.1, which is the result of adapting the strategy shown in [KMP86] to the lattice $\mathscr{D}_2$, which does not have dimension 1. Then, Section 2.3, we give a concise and self-contained exposition of Supersymmetry, which might be of interest to the reader. Next, in Section 2.4, we present the proof of Thm. 2.2.1 by introducing a representation of the square of the average version of the Green's functions corresponding two layers of $\mathscr{D}_2$ in terms of transfer operators. Finally, in Section 2.5, we explain why we are not able to continue with the Klein, Martinelli and Perez's approach in our case. To be more precise, in Subsection 2.5.1, we outline the strategy given in [KMP86]. Then, in Subsection 2.5, we compare their approach to the proof of Thm. 2.2.1 and why we cannot continue with their method to prove localization in our setting. In addition, we provide some paths of research which might be helpful in the future to surmount the obstacle, which we came across.

We set the notation of the rest of the chapter. Let $(\delta_x)_{x\in\mathscr{D}_2}$ be the canonical orthonormal basis of $\ell^2(\mathscr{D}_2)$. For an operator A acting on $\ell^2(\mathscr{D}_2)$, we denote the matrix elements of A by $A(x_0,x) := \langle\delta_{x_0}, A\,\delta_x\rangle$ with $x, x_0 \in \mathscr{D}_2$. We write $\langle\delta_x,\cdot\rangle\delta_x$ for the projection onto the subspace generated by δ_x. Any element of $\mathscr{D}_2$ can be written as $x = \begin{pmatrix} j & \sigma\end{pmatrix}^T$, where $j \in \mathbb{Z}$ and $\sigma \in \{0,1\}$. Setting $e = \begin{pmatrix} 0 & 1\end{pmatrix}^T$, we formally write $x = j + \sigma e$, to make our computations more concise. In addition, we denote the ℓ^{p}-distance in $\mathscr{D}_2$ by $|x_0 - x_1|_{\mathrm{p}} = (|j_0 - j_1|^{\mathrm{p}} + |\sigma_0 - \sigma_1|^{\mathrm{p}})^{\frac{1}{\mathrm{p}}}$, for $1 \le \mathrm{p} < \infty$, and $|x_0 - x_1|_\infty = \max\{|j_0 - j_1|, |\sigma_0 - \sigma_1|\}$, for $\mathrm{p} = \infty$, for all $x_k = j_k + \sigma_k \in \mathscr{D}_2$, $k \in \{0,1\}$. In the case $\mathrm{p} = 2$, we use the short hand notation $|\cdot|$. Observe that we wrote "ℓ^{p}-distance in $\mathscr{D}_2$" and *not* "ℓ^{p}-norm in $\mathscr{D}_2$" because $\mathscr{D}_2$ equipped with the standard addition and scalar multiplication is not a vector space. Finally, let $f \in \mathrm{L}^1(\mathbb{R}^{\mathrm{d}})$ with $\mathrm{d} \in \mathbb{N}$, the Fourier transform of f is given by $[\mathscr{F}f](k) = \hat{f}(k) = \int_{\mathbb{R}^{\mathrm{d}}} \mathrm{d}y e^{-ik\cdot y} f(y)$, for all $k \in \mathbb{R}^{\mathrm{d}}$.

2.2 Model

We consider the discrete Anderson model on the strip $\mathscr{D}_2$ given by

$$\mathrm{H}_{\omega,\lambda} := -\mathrm{P} + \lambda \mathrm{V}_\omega, \tag{2.2.1}$$

where $\mathrm{P} = \mathbb{1}_{|x-y|_1 = 1}$ is the adjacency matrix and V_ω is a multiplication operator defined as

$$\mathrm{V}_\omega u(x) = \omega_x u(x), \qquad \forall u \in \ell^2(\mathscr{D}_2),\ \forall x \in \mathscr{D}_2,$$

where we assume that $\omega_j = \omega_{j+e}$, for all $j \in \mathbb{Z}$, and $\omega := (\omega_j)_{j\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ being a family of bounded iid random variables defined in the probability space $\Omega := \mathbb{R}^{\mathbb{Z}}$ equipped with the Borel probability measure $\mathbb{P} := \bigotimes_{\mathbb{Z}} \mathrm{P}_0$ with expectation $\mathbb{E}[\cdot]$. The Fourier transform of P_0, denoted by $\hat{\mathrm{P}}_0$, is defined as $\hat{\mathrm{P}}_0(x) := \mathbb{E}\big[e^{-i\omega_j x}\big]$, for all $x \in \mathbb{R}$.

To relate this model to the the usual representation of the Anderson Model, we express Eq. 2.2.1 as

$$\mathrm{H}_{\omega,\lambda} := -\Delta + (\lambda \mathrm{V}_\omega - 3), \tag{2.2.2}$$

where $-\Delta$ represents the (negative) standard discrete Laplacian on $\ell^2(\mathscr{D}_2)$ defined by $(-\Delta)_{ij} = \delta_{ij}\mathrm{d}_j - \mathrm{P}_{ij}$, $\mathrm{d}_j := \sum_{\substack{k\in\mathscr{D}_2\\ |k-j|_1=1}} 1 = 3$. Consequently, $\{\mathrm{H}_{\omega,\lambda}\}_{\omega\in\Omega}$ is a family of bounded ergodic self-adjoint operators. Let $z \in \mathbb{C}\setminus\mathbb{R}$ and $\mathrm{G}_{z,\omega,\lambda} := (\mathrm{H}_{\omega,\lambda} - z)^{-1}$. Let $L \in \mathbb{N}$ and $\Lambda_L := \{-L, \dots, L\} \times \{0, 1\}$. Let $\mathrm{H}^{\Lambda_L}_{\omega,\lambda} = \mathbb{1}_{\Lambda_L}\mathrm{H}_{\omega,\lambda}\mathbb{1}_{\Lambda_L}$ denote the restriction of $\mathrm{H}_{\omega,\lambda}$ to $\ell^2(\Lambda_L)$ with Dirichlet boundary conditions. Finally, let $\mathrm{G}^{\Lambda_L}_{z,\omega,\lambda} = (\mathrm{H}^{\Lambda_L}_{\omega,\lambda} - z)^{-1}$ restricted to $\ell^2(\Lambda_L)$.

By employing supersymmetric formalism, we obtain a transfer operator representation of the square of the 4-points average of the Green's function between two layers of $\mathscr{D}_2$, which is the content of Thm. 2.2.1 below.

Theorem 2.2.1. *Let $\varepsilon \in \{+,-\}$, $j \in \mathbb{Z}$, $\alpha_{\varepsilon,\omega_j} = \varepsilon i(-\omega_j + \mathrm{E} + i\eta - 1)$,*

$$\Gamma_{\varepsilon,\omega_j} = \begin{pmatrix} g_{0,\varepsilon,\omega_j} & 0 & 0 & 0 \\ g_{1,\varepsilon,\omega_j} & g_{0,\varepsilon,\omega_j} & 0 & 0 \\ g_{2,\varepsilon,\omega_j} & 0 & g_{0,\varepsilon,\omega_j} & 0 \\ g_{3,\varepsilon,\omega_j} & 2g_{1,\varepsilon,\omega_j} & 2g_{2,\varepsilon,\omega_j} & g_{0,\varepsilon,\omega_j} \end{pmatrix}, \quad \tilde{\Gamma}_{\varepsilon,\omega_j} = g_{0,\varepsilon,\omega_j}\begin{pmatrix} 1 & 0 \\ \alpha_{\varepsilon,\omega_j} & 1 \end{pmatrix}$$

$$\mathcal{T}_{\varepsilon,\omega_j} = \frac{1}{\pi^2}\mathscr{F}_2\begin{pmatrix} g_{3,\varepsilon,\omega_j} & 2g_{1,\varepsilon,\omega_j} & 2g_{2,\varepsilon,\omega_j} & g_{0,\varepsilon,\omega_j} \\ g_{1,\varepsilon,\omega_j} & g_{0,\varepsilon,\omega_j} & 0 & 0 \\ -g_{2,\varepsilon,\omega_j} & 0 & -g_{0,\varepsilon,\omega_j} & 0 \\ g_{0,\varepsilon,\omega_j} & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{\mathcal{T}}_{\varepsilon,\omega_j} = \frac{\varepsilon i g_{0,\varepsilon,\omega_j}}{\pi^2}\mathscr{F}_{2\varepsilon}\begin{pmatrix} \alpha_{\varepsilon,\omega_j} & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.2.3}$$

where $\mathscr{F}_{2\varepsilon}$ denotes the Fourier transform after scaling by 2ε and

$$\begin{aligned} g_{0,\varepsilon,\omega_j}(x_1+x_2,y) &= e^{-\varepsilon i\omega_j(x_1+x_2)}e^{\varepsilon i(\mathrm{E}+i\eta)(x_1+x_2)}e^{-\varepsilon 2iy}, \\ g_{1,\varepsilon,\omega_j}(x_1+x_2,y) &= \frac{\partial g_{0,\varepsilon,\omega_j}(x_1+x_2,y)}{\partial x_1} = \frac{\partial g_{0,\varepsilon,\omega_j}(x_1+x_2,y)}{\partial x_2} \\ &= \varepsilon i[-\omega_j + \mathrm{E} + i\eta]g_{0,\varepsilon,\omega_j}(x_1+x_2,y), \\ g_{2,\varepsilon,\omega_j}(x_1+x_2,y) &= \frac{\partial g_{0,\varepsilon,\omega_j}(x_1+x_2,y)}{\partial y} = -2\varepsilon i g_{0,\varepsilon,\omega_j}(x_1+x_2,y), \\ g_{3,\varepsilon,\omega_j}(x_1+x_2,y) &= \left[\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} - \frac{1}{4}\frac{\partial^2}{\partial y^2}\right]g_{0,\varepsilon,\omega_j}(x_1+x_2,y) \\ &= (1-[-\omega_j+\mathrm{E}+i\eta]^2)g_{0,\varepsilon,\omega_j}(x_1+x_2,y), \quad \forall x_1,x_2 \geq 0, \forall y, \omega_j \in \mathbb{R}. \end{aligned} \tag{2.2.4}$$

Then,

$$\frac{1}{4}\sum_{\sigma_0,\sigma_1=0}^{1} \mathrm{G}^{\Lambda_L}_{z,\omega,\lambda}(j_0+\sigma_0 e, j_1+\sigma_1 e) = \frac{-i}{2\pi^2}\int_{\mathbb{R}^2\times\mathbb{R}^2} \mathrm{d}^2\mathrm{X}^+_{j_0}\mathrm{d}^2\mathrm{X}^+_{j_0+e}$$
$$\Big[\mathrm{R}_{1,+}\tilde{\mathrm{R}}^{j_0,j_1}_{0,+} + \mathrm{R}_{0,+}\tilde{\mathrm{R}}^{j_0,j_1}_{1,+}\Big]\Big(\mathrm{X}^+_{j_0},\mathrm{X}^+_{j_0+e}\Big),$$
$$\frac{1}{4}\sum_{\sigma_0,\sigma_1=0}^{1} \overline{\mathrm{G}^{\Lambda_L}_{z,\omega,\lambda}(j_0+\sigma_0 e, j_1+\sigma_1 e)} = \frac{i}{2\pi^2}\int_{\mathbb{R}^2\times\mathbb{R}^2} \mathrm{d}^2\mathrm{X}^-_{j_0}\mathrm{d}^2\mathrm{X}^-_{j_0+e}$$
$$\Big[\mathrm{R}_{1,-}\tilde{\mathrm{R}}^{j_0,j_1}_{0,-} + \mathrm{R}_{0,-}\tilde{\mathrm{R}}^{j_0,j_1}_{1,+}\Big]\Big(\mathrm{X}^-_{j_0},\mathrm{X}^-_{j_0+e}\Big), \tag{2.2.5}$$

where $\mathrm{R}_{k,\varepsilon} = \mathrm{R}_{k,\varepsilon,\omega_{j_0},\ldots,\omega_{-L}}$, $\tilde{\mathrm{R}}^{j_0,j_1}_{k,\varepsilon} = \tilde{\mathrm{R}}^{j_0,j_1}_{k,\varepsilon,\omega_{j_0+1},\ldots,\omega_L}$, *for* $k\in\{0,1\}$, *and*

$$\mathrm{R}_{0,\varepsilon}(\mathrm{X}_{j_0},\mathrm{X}_{j_0+e}) = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}^T \left[\Gamma_{\varepsilon,\omega_{j_0}}\prod_{j=j_0-1}^{-L}\mathcal{T}_{\varepsilon,\omega_j}\right]\begin{pmatrix}1\\0\\0\\0\end{pmatrix},$$
$$\mathrm{R}_{1,\varepsilon}(\mathrm{X}_{j_0},\mathrm{X}_{j_0+e}) = \begin{pmatrix}0\\1\\-1\\0\end{pmatrix}^T \left[\Gamma_{\varepsilon,\omega_{j_0}}\prod_{j=j_0-1}^{-L}\mathcal{T}_{\varepsilon,\omega_j}\right]\begin{pmatrix}1\\0\\0\\0\end{pmatrix},$$
$$\tilde{\mathrm{R}}^{j_0,j_1}_{0,\varepsilon}(\mathrm{X}_{j_0},\mathrm{X}_{j_0+e}) = \frac{\varepsilon i}{\pi^2}\begin{pmatrix}1\\0\end{pmatrix}^T\left[\prod_{j=j_0+1}^{j_1-1}\mathcal{T}_{\varepsilon,\omega_j}\right]\mathscr{F}_2\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\mathrm{R}_{0,\varepsilon,\omega_{j_1},\ldots,\omega_L}\\ \mathrm{R}_{1,\varepsilon,\omega_{j_1},\ldots,\omega_L}\end{pmatrix},$$
$$\tilde{\mathrm{R}}^{j_0,j_1}_{1,\varepsilon}(\mathrm{X}_{j_0},\mathrm{X}_{j_0+e}) = \frac{\varepsilon i}{\pi^2}\begin{pmatrix}0\\1\end{pmatrix}^T\left[\prod_{j=j_0+1}^{j_1-1}\mathcal{T}_{\varepsilon,\omega_j}\right]\mathscr{F}_2\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\mathrm{R}_{0,\varepsilon,\omega_{j_1},\ldots,\omega_L}\\ \mathrm{R}_{1,\varepsilon,\omega_{j_1},\ldots,\omega_L}\end{pmatrix}. \tag{2.2.6}$$

Furthermore, if we assume that P_0 *has moments up to at least order 4, then*

$$\mathbb{E}\left[\left|\frac{1}{4}\sum_{\sigma_0,\sigma_1=0}^{1}\mathrm{G}^{\Lambda_L}_{z,\omega,\lambda}(j_0+\sigma_0 e,j_1+\sigma_1 e)\right|^2\right] = \frac{1}{4\pi^4}\int_{\mathbb{R}^2\times\mathbb{R}^2}\int_{\mathbb{R}^2\times\mathbb{R}^2}\mathrm{d}^2\mathrm{X}^+_{j_0}\mathrm{d}^2\mathrm{X}^+_{j_0+e}\mathrm{d}^2\mathrm{X}^-_{j_0}\mathrm{d}^2\mathrm{X}^-_{j_0+e}$$
$$\mathbb{E}\Big[\Big\{\mathrm{R}_{1,+}\tilde{\mathrm{R}}^{j_0,j_1}_{0,+}+\mathrm{R}_{0,+}\tilde{\mathrm{R}}^{j_0,j_1}_{1,+}\Big\}\Big(\mathrm{X}^+_{j_0},\mathrm{X}^+_{j_0+e}\Big)\Big\{\mathrm{R}_{1,-}\tilde{\mathrm{R}}^{j_0,j_1}_{0,-}+\mathrm{R}_{0,-}\tilde{\mathrm{R}}^{j_0,j_1}_{1,-}\Big\}\Big(\mathrm{X}^-_{j_0},\mathrm{X}^-_{j_0+e}\Big)\Big]. \tag{2.2.7}$$

2.3 Supersymmetric formalism

2.3.1 Grassmann algebra

Let $N\in\mathbb{N}$. Let V be a real $N-$dimensional vector space with basis $\mathscr{B}=\{\rho_j\}_{j=1}^N$. The antisymmetric tensor product is defined as

$$\begin{aligned}\wedge : \mathrm{V}\times\mathrm{V} &\longrightarrow \mathrm{V}\otimes_{as}\mathrm{V}\\ (v,w)&\longmapsto v\wedge w,\end{aligned} \tag{2.3.1}$$

where $\mathrm{V} \otimes_{as} \mathrm{V}$ denotes the antisymmetric tensor product of V with itself (see [Abd04]). To lighten the notation, $v \wedge w$ will be written vw. This binary operator has the property that

$$vw = -wv, \qquad \forall v, w \in \mathrm{V}. \tag{2.3.2}$$

It follows that

$$v^2 = 0, \qquad \forall v \in \mathrm{V}. \tag{2.3.3}$$

The Grassmann algebra (also known as exterior or graded algebra) on $\mathbb{R}$ generated by $\mathscr{B}$ is the associative algebra with unity given by

$$\mathcal{A}[\mathscr{B}] := \bigoplus_{k \geq 0} \mathrm{V}^k, \tag{2.3.4}$$

where $\mathrm{V}^k := \underbrace{\mathrm{V} \otimes_{as} \ldots \otimes_{as} \mathrm{V}}_{(k-1) \text{ times}}$, for all $k \geq 1$, and $\mathrm{V}^0 := \mathbb{R}$. Observe that $\mathrm{V}^k \cap \mathrm{V}^j = \{0\}$, for all $k \neq j$. In addition, $\mathrm{V}^p = \{0\}$, for all $p > N$, because some of the generators would appear more than once. The antisymmetric tensor product given in Eq. 2.3.1 defines a product operation on $\mathcal{A}[\mathscr{B}]$. By abuse of notation, we will also denote it as $\otimes_{as}$. This binary operation has the following property: Let $p, q \in \mathbb{N}$. If $v \in \mathrm{V}^p$ and $w \in \mathrm{V}^q$, then $vw \in \mathrm{V}^4$ and

$$vw = (-1)^{pq} wv. \tag{2.3.5}$$

An immediate consequence of Eq. 2.3.5 is that if $v \in \mathrm{V}^p$ and p is odd, then $v^2 = 0$. Indeed, $v^2 = (-1)^{p^2} v^2 = -v^2$.

For any $v \in \mathcal{A}[\mathscr{B}]$, there are scalars $\{v_{i_1,\ldots,i_k}\}_{i_1,\ldots,i_k=1}^N$ such that

$$v = v_0 + \sum_{k \geq 1} \sum_{i_1,\ldots,i_k=1}^{N} v_{i_1 \ldots i_k}\, \rho_{i_1} \cdots \rho_{i_k}. \tag{2.3.6}$$

The above decomposition becomes unique, if the coefficients $v_{i_1,\ldots,i_k}$ are antisymmetric under any interchange of pair of indices.

Bosonic and Fermionic variables

By Eq. 2.3.5, for all $v \in \mathrm{V}^p$ and $w \in \mathrm{V}^q$, $vw = (-1)^{pq} wv$. Hence, $vw = -wv$, if p and q are odd. By contrast, if p is even, then $vw = wv$, for all q. This motivates the following decomposition $\mathcal{A}[\mathscr{B}] = \mathcal{A}_0[\mathscr{B}] \oplus \mathcal{A}_1[\mathscr{B}]$, where

$$\mathcal{A}_0[\mathscr{B}] = \bigoplus_{k \geq 0} \mathrm{V}^{2k}, \qquad \mathcal{A}_1[\mathscr{B}] = \bigoplus_{k \geq 0} \mathrm{V}^{2k+1}. \tag{2.3.7}$$

Since $\mathrm{V}^p = \{0\}$, for all $p > N$, the above direct sums are finite. It holds that

$$v \in \mathcal{A}_0[\mathscr{B}],\ w \in \mathcal{A}_0[\mathscr{B}] \cup \mathcal{A}_1[\mathscr{B}] \Rightarrow vw = wv, \tag{2.3.8}$$

$$v, w \in \mathcal{A}_1[\mathscr{B}] \Rightarrow vw = -wv. \tag{2.3.9}$$

Then the elements of $\mathcal{A}_1[\mathscr{B}]$ anticommute with each other. On the other hand, the elements of $\mathcal{A}_0[\mathscr{B}]$ commute with all the other elements of $\mathcal{A}[\mathscr{B}]$ and hence it is a subset of the center of $\mathcal{A}[\mathscr{B}]$. Moreover $\mathcal{A}_0[\mathscr{B}]$ is a subalgebra since it is closed under multiplication, addition and scalar multiplication.

Note that $\mathcal{A}_0[\mathscr{B}]$ is a strict subset of the center unless the number of generators is even. This is the content of the next lemma.

Lemma 2.3.1. *Let $N \in \mathbb{N}$ and $\mathscr{B} = \{\rho_i\}_{i=1}^N$ be the basis of a real vector space* V*. Let $\mathcal{A}[\mathscr{B}]$ be the Grassmann algebra generated by $\mathscr{B}$ and $\mathrm{Z}[\mathcal{A}[\mathscr{B}]]$ be the center of it. If N is even, then $\mathrm{Z}[\mathcal{A}[\mathscr{B}]] = \mathcal{A}_0[\mathscr{B}]$. By contrast, if N is odd, then $\mathrm{Z}[\mathcal{A}[\mathscr{B}]] = \mathcal{A}_0[\mathscr{B}] \oplus \mathrm{V}^N$. In particular, for $N = 1$, $\mathcal{A}[\mathscr{B}]$ is a commutative algebra.*

Proof. Let $v \in \mathrm{Z}[\mathcal{A}[\mathscr{B}]]$. Since $v \in \mathcal{A}[\mathscr{B}]$, v can be uniquely written as $v = v_0 + v_1$, where $v_0 \in \mathcal{A}_0[\mathscr{B}]$ and $v_1 \in \mathcal{A}_1[\mathscr{B}]$. This yields, for all $i \in \{1, \dots, N\}$,

$$v_0\rho_i + v_1\rho_i = v\rho_i = \rho_i v = \rho_i v_0 + \rho_i v_1 = v_0\rho_i - v_1\rho_i, \tag{2.3.10}$$

where we used that v belongs to the center, $\rho_i, v_1 \in \mathcal{A}_1[\mathscr{B}]$ and $v_0 \in \mathcal{A}_0[\mathscr{B}]$. It follows

$$v_1\rho_i = 0, \qquad \forall i \in \{1, \dots, N\}. \tag{2.3.11}$$

As a result of this, $v_1 = \lambda\rho_1 \cdot \ldots \cdot \rho_N$, for some $\lambda \in \mathbb{R}$. Indeed, v_1 can be written as $v_1 = w_0 + w_1\rho_1$, where $w_0, w_1 \in \mathcal{A}[\mathscr{B}]$ are independent of ρ_1. Hence,

$$0 = v_1\rho_1 = w_0\rho_1 + w_1\rho_1^2 = w_0\rho_1. \tag{2.3.12}$$

This implies that $w_0 = 0$ since w_0 is independent of ρ_1, and therefore $v_1 = w_1\rho_1$. Now we can repeat the same argument on w_1 with the generator ρ_2. By a recursive argument, we can conclude that $v_1 \in \mathrm{V}^N$. If N is odd, then $\mathrm{V}^N \subset \mathcal{A}_1[\mathscr{B}]$ and $\mathrm{Z}[\mathcal{A}[\mathscr{B}]] = \mathcal{A}_0[\mathscr{B}] \oplus \mathrm{V}^N$. By contrast, if N is even, then $\mathrm{V}^N \subset \mathcal{A}_0[\mathscr{B}]$ and $v_1 \in \mathcal{A}_0[\mathscr{B}] \cap \mathcal{A}_1[\mathscr{B}] = \{0\}$. Hence, $\mathrm{Z}[\mathcal{A}[\mathscr{B}]] = \mathcal{A}_0[\mathscr{B}]$. $\square$

From now on, we call the elements of $\mathcal{A}_0[\mathscr{B}]$ and $\mathcal{A}_1[\mathscr{B}]$ as **bosonic** (or **even**) and **fermionic** (or **odd**) **variables**, respectively. In addition, we usually denote the bosonic variables by Latin letters and the fermionic variables by Greek letters.

Remark 2.3.2. *For all fermionic variable ψ, it holds that $\psi^2 = 0$, due to Eq. 2.3.5.*

Any bosonic variable X can be uniquely written as

$$\mathrm{X} = x(\mathrm{X}) + n(\mathrm{X}), \text{ where } x(\mathrm{X}) \in \mathbb{R} \text{ and } n(\mathrm{X}) \in \bigoplus_{k \geq 1} \mathrm{V}^{2k}. \tag{2.3.13}$$

We say that $x(\mathrm{X})$ and $n(\mathrm{X})$ are, respectively, the body and soul of X. Note that $n(\mathrm{X})$ is nilpotent since $n(\mathrm{X})^{\lceil \frac{N}{2} \rceil} = 0$, where $\lceil \cdot \rceil$ is the ceiling function.

Inverse of a Grassmann variable

Let $v \in \mathcal{A}[\mathscr{B}]$. We say that v has an inverse, if there is a $v^{-1} \in \mathcal{A}[\mathscr{B}]$ such that $vv^{-1} = v^{-1}v = 1$. If v^{-1} exists, then it is unique. Indeed, for the sake of contradiction, assume that $w_1, w_2 \in \mathcal{A}[\mathscr{B}]$ are inverses of v but $w_1 \neq w_2$. Then, $w_1 = w_1(vw_2) = (w_1v)w_2 = w_2$.

For any bosonic variable $\mathrm{X} = x+n$ with nonzero body (i.e., $x \neq 0$), X has an inverse and this is given by

$$\mathrm{X}^{-1} = \frac{1}{x} + \frac{1}{x}\sum_{j\geq 1}\left(\frac{-n}{x}\right)^j, \tag{2.3.14}$$

which can be checked by direct computation. Due to the fact that n is nilpotent, the above sum is finite.

Remark 2.3.3. *By contrast, if* ψ *is a fermionic variable, then it has no inverse. For the sake of contradiction, assume that* ψ^{-1} *exists. As result of this,* $1 = \psi(\psi^{-1}\psi)\psi^{-1} = \psi(\psi\psi^{-1})\psi^{-1} = 0$*, where we used that* $\psi\psi^{-1} = \psi^{-1}\psi = 1$ *and* $\psi^2 = 0$*.*

Function of Grassmann variables

Any function $f \in \mathcal{C}^\infty(\mathbb{R};\mathbb{R})$ can be extended to a mapping $f : \mathcal{A}_0[\mathscr{B}] \longrightarrow \mathcal{A}_0[\mathscr{B}]$ [1] in the following way:

$$\begin{aligned} f : \mathcal{A}_0[\mathscr{B}] &\longrightarrow \mathcal{A}_0[\mathscr{B}], \\ \mathrm{X} = x(\mathrm{X}) + n(\mathrm{X}) &\longmapsto f(\mathrm{X}) := f(x(\mathrm{X})) + \sum_{k\geq 1}\frac{f^{(k)}(x(\mathrm{X}))}{k!}n(\mathrm{X})^k. \end{aligned} \tag{2.3.15}$$

Note that the above sum is always finite since $n(\mathrm{X})$ is nilpotent. The same construction can be used to extend $f \in C^\infty(U;\mathbb{R})$ with $U \subset \mathbb{R}$ open to $f : U \oplus \bigoplus_{k\geq 1}\mathrm{V}^{2k} \longrightarrow \mathcal{A}_0[\mathscr{B}]$, by requiring that $x(\mathrm{X}) \in U$. Similarly, any function $f \in \mathcal{C}^\infty(\mathbb{R}^{p\times q};\mathbb{R})$ with $p, q \in \mathbb{N}$ can be extended to a mapping $f : \mathcal{A}_0^{p\times q}[\mathscr{B}] \longrightarrow \mathcal{A}_0[\mathscr{B}]$ as follows

$$\begin{aligned} f : \mathcal{A}_0^{p\times q}[\mathscr{B}] &\longrightarrow \mathcal{A}_0[\mathscr{B}], \\ \mathrm{X} = (x_{ij}(\mathrm{X}) + n_{ij}(\mathrm{X}))_{\substack{1\leq i\leq p,\\ 1\leq j\leq q}} &\longmapsto f(\mathrm{X}) := \sum_{k\in\mathbb{N}_0^{p\times q}}\frac{1}{k!}\partial_x^k f(x(\mathrm{X}))n^k(\mathrm{X}), \end{aligned} \tag{2.3.16}$$

where we used the multi-index notation, for all $k \in \mathbb{N}_0^{p\times q}$,

$$\partial_x^k := \prod_{\substack{1\leq i\leq p\\ 1\leq j\leq q}}\frac{\partial^{k_{ij}}}{\partial x_{ij}^{k_{ij}}}, \qquad k! := \prod_{\substack{1\leq i\leq p\\ 1\leq j\leq q}} k_{ij}!, \qquad n^k(\mathrm{X}) := \prod_{\substack{1\leq i\leq p\\ 1\leq j\leq q}} n_{ij}^{k_{ij}}(\mathrm{X}). \tag{2.3.17}$$

Once again, since all terms $n_{ij}(\mathrm{X})$ are nilpotent, the above sum is finite. Any $f \in \mathcal{C}^\infty(U;\mathbb{C})$ can be decomposed as $f = \operatorname{Re} f + i\operatorname{Im} f$, where $\operatorname{Re} f, \operatorname{Im} f \in \mathcal{C}^\infty(U;\mathbb{R})$.

[1] we will call the new function again f, by abuse of notation.

Thus, f can be extended to a mapping $f : U \oplus \bigoplus_{k\geq 1} \mathrm{V}^{2k} \to \mathcal{A}_0[\mathscr{B}] \oplus i\mathcal{A}_0[\mathscr{B}]$ defined by

$$f(\mathrm{X}) = \operatorname{Re} f(\mathrm{X}) + i \operatorname{Im} f(\mathrm{X}). \tag{2.3.18}$$

Finally, we can also replace the range of f in the above construction by a real finite-dimensional vector space W. In this case, the corresponding extension of f is defined componentwise.

In the following we will need the extension of the inverse, exponential and logarithm.

(i) Inverse. Let $U = \mathbb{R}\backslash\{0\}$ and consider $f(x) = \frac{1}{x}$. This function is smooth in U. For any bosonic variable $\mathrm{X} = x + n$ with nonzero body (i.e., $x \neq 0$), the function $f(\mathrm{X}) = (x+n)^{-1}$ coincides with Eq. 2.3.14. We can extend this construction to suitable square matrices with bosonic entries as follows: let $p \in \mathbb{N}$ and $a = (a_{ij})_{1\leq i,j\leq p} \in \mathcal{A}_0^{p\times p}[\mathscr{B}]$. If $x(a)$ is invertible, then a is invertible and its inverse is given by

$$a^{-1} = x(a)^{-1} + x(a)^{-1} \sum_{j\geq 1} (-1)^j \left[x(a)^{-1} n(a)\right]^j, \tag{2.3.19}$$

by the same argument as Eq. 2.3.14.

(ii) Exponential and Logarithm. We can define $\exp : \mathcal{A}_0[\mathscr{B}] \to \mathcal{A}_0[\mathscr{B}]$ as

$$e^{\mathrm{X}} = e^{x(\mathrm{X})} + \sum_{k\geq 1} \frac{e^{x(\mathrm{X})}}{k!} n(\mathrm{X})^k. \tag{2.3.20}$$

Just like the standard exponential, this function satisfies

$$e^{\mathrm{X}_1+\mathrm{X}_2} = e^{\mathrm{X}_1} e^{\mathrm{X}_2}, \qquad \forall \mathrm{X}_1, \mathrm{X}_2 \in \mathcal{A}_0. \tag{2.3.21}$$

Let $U = \mathbb{R}^+$ and consider $f(x) = \ln x$. This function satisfies $f \in \mathcal{C}^\infty(U; \mathbb{R})$. For all $\mathrm{X} \in \mathcal{A}_0[\mathscr{B}]$ with $x(\mathrm{X}) > 0$, we have

$$\ln \mathrm{X} = \ln(x(\mathrm{X}) + n(\mathrm{X})) := \ln(x(\mathrm{X})) - \sum_{j\geq 1} \frac{(-1)^j}{j} \left(\frac{n(\mathrm{X})}{x(\mathrm{X})}\right)^j. \tag{2.3.22}$$

We can again extend the above constructions to suitable square matrices with bosonic entries as follows: let $p \in \mathbb{N}$ and $a = (a_{ij})_{1\leq i,j\leq p} \in \mathcal{A}_0^{p\times p}[\mathscr{B}]$. Then e^a can be defined as

$$e^a = e^{x(a)+n(a)} := e^{x(a)} \sum_{j\geq 0} \frac{1}{j!} n(a)^j. \tag{2.3.23}$$

Moreover, if $\ln x(a)$ exists, then $\ln a$ can be defined as

$$\ln a = \ln(x(a) + n(a)) := \ln x(a) - \sum_{j\geq 1} \frac{(-1)^j}{j} \left[x^{-1}(a) n(a)\right]^j. \tag{2.3.24}$$

Remark 2.3.4. *Let $p \in \mathbb{N}$ and $a = (a_{ij})_{1\leq i,j\leq p} \in \mathcal{A}_0^{p\times p}[\mathscr{B}]$. If $x(a)$ is invertible and $\ln x(a)$ exists, then*

$$\operatorname{tr} \ln a = \ln \det a. \tag{2.3.25}$$

Indeed, let $f(x) = \operatorname{tr} \ln x$ and $g(x) = \ln \det x$. Then, $f(x) = g(x)$, for all $x \in \mathbb{R}^{p\times p}$ invertible such that $\ln x$ exists. Since f and g are smooth functions, their corresponding extensions to $\mathcal{A}_0^{p\times p}[\mathscr{B}]$ also coincide when we extend them.

Derivation of Grassmann variables

To define a derivative operator, note that every $v \in \mathcal{A}[\mathscr{B}]$ can be seen as a polynomial whose degree is at most 1 in each ρ_i. Hence, it can be uniquely decomposed as

$$v(\mathscr{B}) = v_1 + \rho_i v_2^{\mathrm{L}} = v_1 + v_2^{\mathrm{R}} \rho_i, \tag{2.3.26}$$

where v_1, v_2^{L} and v_2^{R} do not depend on ρ_i. We define the **left** and **right derivative** with respect to ρ_i as

$$\frac{\overrightarrow{\partial}}{\partial \rho_i} v := v_2^{\mathrm{L}}, \qquad\qquad v\frac{\overleftarrow{\partial}}{\partial \rho_i} := v_2^{\mathrm{R}}. \tag{2.3.27}$$

Note that when v does not depend on ρ_i, $\frac{\overrightarrow{\partial}}{\partial \rho_i} v = \frac{\overleftarrow{\partial}}{\partial \rho_i} v = 0$. However, in general, $\frac{\overrightarrow{\partial}}{\partial \rho_i} v \neq v\frac{\overleftarrow{\partial}}{\partial \rho_i}$. Indeed, let $v = \rho_i \rho_j$, for $j \neq i$. Then, $\frac{\overrightarrow{\partial}}{\partial \rho_i} v = \rho_j$ and $\frac{\overleftarrow{\partial}}{\partial \rho_i} v = -\rho_j$.

From now on, we will make use only of the left derivative in our computations. To lighten the notation, we will call it simply **derivative** and $\frac{\partial}{\partial \rho_i} := \frac{\overrightarrow{\partial}}{\partial \rho_i}$. Note that the derivative behaves as a Grassmann variable in the following sense:

$$\frac{\partial}{\partial \rho_i}\frac{\partial}{\partial \rho_j} v = -\frac{\partial}{\partial \rho_j}\frac{\partial}{\partial \rho_i} v, \qquad \forall i \neq j, \tag{2.3.28}$$

$$\frac{\partial}{\partial \rho_i}\frac{\partial}{\partial \rho_i} v = 0. \tag{2.3.29}$$

The following special case will be useful later.

Lemma 2.3.5 (**Derivation of the exponential function.**). *Let* V *be the* $2N$*-dimensional vector space with base* $\mathscr{B} := \left\{\psi_j, \overline{\psi}_j\right\}_{j=1}^N$. *We consider* $\mathcal{A}[\mathscr{B}]$. *Then,*

$$\prod_{j=1}^N \frac{\partial}{\partial \overline{\psi}_j}\frac{\partial}{\partial \psi_j} e^{-\sum_{i,j=1}^N \overline{\psi}_i A_{ij} \psi_j} = \det A, \quad \forall A \in \mathbb{C}^{N\times N}, \tag{2.3.30}$$

$$\prod_{j=1}^N \frac{\partial}{\partial \overline{\psi}_j}\frac{\partial}{\partial \psi_j} \psi_k \overline{\psi}_l e^{-\sum_{i,j=1}^N \overline{\psi}_i A_{ij} \psi_j} = \det A_{lk}, \quad \forall A \in \mathbb{C}^{N\times N},\ \forall k,l \in \{1,\dots,N\}, \tag{2.3.31}$$

where $\det A_{lk}$ *corresponds to the minor of* A *after deleting the* l*-th row and* $k-$*th column. In particular, if* A *is invertible, then*

$$\prod_{j=1}^N \frac{\partial}{\partial \overline{\psi}_j}\frac{\partial}{\partial \psi_j} \psi_k \overline{\psi}_l e^{-\sum_{i,j=1}^N \overline{\psi}_i A_{ij} \psi_j} = (\det A) A^{-1}_{kl}, \quad \forall k,l \in \{1,\dots,N\}. \tag{2.3.32}$$

Proof. Let $n = \sum_{i,j=1}^N \overline{\psi}_i A_{ij} \psi_j$. Clearly, $n \in \bigoplus_{k\geq 1} \mathrm{V}^{2k} \subset \mathcal{A}_0[\mathscr{B}]$. Then, n is nilpotent and e^{-n} is well-defined. Hence,

$$\prod_{j=1}^N \frac{\partial}{\partial \overline{\psi}_j}\frac{\partial}{\partial \psi_j} e^{-\sum_{i,j=1}^N \overline{\psi}_i A_{ij} \psi_j} = \sum_{k=0}^N \frac{(-1)^k}{k!} \prod_{j=1}^N \frac{\partial}{\partial \overline{\psi}_j}\frac{\partial}{\partial \psi_j} n^k \tag{2.3.33}$$

By explicit computation,

$$n^k = \sum_{\substack{i_1,\dots,i_k \\ j_1,\dots,j_k}} \left(\overline{\psi}_{i_1} A_{i_1 j_1} \psi_{j_1}\right) \cdot \ldots \cdot \left(\overline{\psi}_{i_k} A_{i_k j_k} \psi_{j_k}\right) \tag{2.3.34}$$

Observe that the only term which possesses all the generators of $\mathcal{A}[\mathscr{B}]$ is n^N and they appear exactly once in the product. Hence, in Eq. 2.3.33, n^k vanishes after deriving, for $k < N$ and we only need to focus on the case $k = N$. Let S_N be the group of permutations of $\{1, \dots, N\}$. Recall that $|S_N| = N!$. Given $\tau \in S_N$, let $\varepsilon(\tau)$ be the sign of τ.

$$\begin{aligned}
n^N &= \sum_{\sigma,\tau \in S_N} \left(\overline{\psi}_{\sigma_1} \psi_{\tau_1}\right) \cdot \ldots \cdot \left(\overline{\psi}_{\sigma_N} \psi_{\tau_N}\right) A_{i_{\sigma_1} j_{\tau_1}} \cdot \ldots \cdot A_{i_{\sigma_N} j_{\tau_N}} \\
&= N! \sum_{\tau \in S_N} \left(\overline{\psi}_1 \psi_{\tau_1}\right) \cdot \ldots \cdot \left(\overline{\psi}_N \psi_{\tau_N}\right) A_{1 j_{\tau_1}} \cdot \ldots \cdot A_{N j_{\tau_N}} \\
&= N! \left(\overline{\psi}_1 \psi_1\right) \cdot \ldots \cdot \left(\overline{\psi}_N \psi_N\right) \sum_{\tau \in S_N} \varepsilon(\tau) A_{1 j_{\tau_1}} \cdot \ldots \cdot A_{N j_{\tau_N}} &(2.3.35) \\
&= N! \left(\overline{\psi}_1 \psi_1\right) \cdot \ldots \cdot \left(\overline{\psi}_q \psi_q\right) \det A. &(2.3.36)
\end{aligned}$$

As a result of this,

$$\prod_{j=1}^{N} \frac{\partial}{\partial \overline{\psi}_j} \frac{\partial}{\partial \psi_j} e^{-\sum_{i,j=1}^{N} \overline{\psi}_i A_{ij} \psi_j} = (-1)^N \det A \prod_{j=1}^{N} \frac{\partial}{\partial \overline{\psi}_j} \frac{\partial}{\partial \psi_j} \overline{\psi}_j \psi_j = \det A, \tag{2.3.37}$$

where we used $\frac{\partial}{\partial \overline{\psi}_j} \frac{\partial}{\partial \psi_j} \overline{\psi}_j \psi_j = -1$. This proves Eq. 2.3.30. Moreover, Eq. 2.3.31 is proved in the same way. Finally, Eq. 2.3.32 is a direct consequence of Eq. 2.3.31 and Cramer's rule.

□

2.3.2 Supervectors

Let $p \in \mathbb{N}$ and $q \in \mathbb{N}$. Let $\mathrm{X}_1, \dots, \mathrm{X}_p \in \mathcal{A}_0[\mathscr{B}]$ and $\Psi_1, \dots, \Psi_q \in \mathcal{A}_1[\mathscr{B}]$. A supervector Φ is given by

$$\Phi = \begin{pmatrix} \mathrm{X} \\ \Psi \end{pmatrix}, \text{ where } \mathrm{X} = \begin{pmatrix} \mathrm{X}_1 \\ \vdots \\ \mathrm{X}_p \end{pmatrix} \text{ and } \Psi = \begin{pmatrix} \Psi_1 \\ \vdots \\ \Psi_q \end{pmatrix}. \tag{2.3.38}$$

We only need to consider the case $p = q = 2$. In the following, we set

$$\mathrm{X} = \begin{pmatrix} \mathrm{X}_1 \\ \mathrm{X}_2 \end{pmatrix}, \qquad \Psi = \begin{pmatrix} \overline{\psi} \\ \psi \end{pmatrix}. \tag{2.3.39}$$

We can also define functions on supervectors. Given $\psi, \overline{\psi} \in \mathcal{A}_1[\mathscr{B}]$. We define

$$\begin{aligned}
&F : \mathcal{A}_0[\mathscr{B}] \longrightarrow \mathcal{A}[\mathscr{B}], \\
&F(\Phi) = F_0(\mathrm{X}) + F_1(\mathrm{X})\overline{\psi} + F_2(\mathrm{X})\psi + F_3(\mathrm{X})\overline{\psi}\psi, &(2.3.40)
\end{aligned}$$

where $F_j : \mathcal{A}_0[\mathscr{B}] \longrightarrow \mathcal{A}[\mathscr{B}]$ such that F_j is independent of $\overline{\psi}$ and ψ, for all $j \in \{0, 1, 2, 3\}$. A simple example is $F(\Phi) = \overline{\psi}$, where $F_0 = F_2 = F_3 = 0$ and $F_1 = 1$. We will always consider functions $\Phi \mapsto F(\Phi) \in \mathcal{A}_0[\Phi]$. In this case, $F_0, F_3 : \mathcal{A}_0[\mathscr{B}] \longrightarrow \mathcal{A}_0[\mathscr{B}]$ and $F_1, F_2 : \mathcal{A}_0[\mathscr{B}] \longrightarrow \mathcal{A}_1[\mathscr{B}]$.

If F_3 is integrable, then the integral of F is defined as

$$\int \mathrm{d}\Phi\, F(\Phi) := \int_{\mathbb{R}^2} \frac{\mathrm{d}^2\mathrm{X}}{\pi} \frac{\partial}{\partial\overline{\psi}} \frac{\partial}{\partial\psi} F(\Phi). \tag{2.3.41}$$

Similarly, we can consider $F(\Phi_1, \dots, \Phi_n)$, for $n \in \mathbb{N}$. The integral of F is defined via Fubini

$$\int \prod_{j=1}^{n} \mathrm{d}\Phi_j\, F(\Phi_1, \dots, \Phi_n) := \int_{\mathbb{R}^{2n}} \prod_{j=1}^{n} \frac{\mathrm{d}^2\mathrm{X}_j}{\pi} \frac{\partial}{\partial\overline{\psi}_j} \frac{\partial}{\partial\psi_j} F(\Phi_1, \dots, \Phi_n) \tag{2.3.42}$$

We define a scalar product of two supervectors as

$$\Phi \cdot \Phi' := \mathrm{X} \cdot \mathrm{X}' + \frac{1}{2}\big(\overline{\psi}\psi' + \overline{\psi}'\psi\big) \in \mathcal{A}_0[\mathscr{B}], \qquad \forall \Phi, \Phi' \in \mathcal{A}_0^2[\mathscr{B}] \times \mathcal{A}_1^2[\mathscr{B}], \tag{2.3.43}$$

where $\mathrm{X} \cdot \mathrm{X}' := \mathrm{X}_1\mathrm{X}_1' + \mathrm{X}_2\mathrm{X}_2'$. In particular,

$$\Phi^2 := \Phi \cdot \Phi = \mathrm{X} \cdot \mathrm{X} + \overline{\psi}\psi. \tag{2.3.44}$$

We can rewrite the scalar product in terms of matrices,

$$\Phi \cdot \Phi' = \Phi^T \Sigma \Phi', \text{ where } \Phi^T := \begin{pmatrix} \mathrm{X}^{\mathrm{T}} & \Psi^{\mathrm{T}} \end{pmatrix},\ \Sigma := \begin{pmatrix} \mathrm{I}_2 & 0 \\ 0 & l \end{pmatrix}, \text{ and } l := \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.3.45}$$

Theorem 2.3.6. *Let $N \in \mathbb{N}$. Let $M \in \mathbb{C}^{N \times N}$. If $M = M^{\mathrm{T}}$ and $\operatorname{Re} M := \frac{A+A^*}{2} > 0$, where A^* is the adjoint of M. Then,*

$$M^{-1}_{x,y} = \int \mathrm{d}\Phi\, \psi_x \overline{\psi}_y e^{-\Phi \cdot M\Phi},$$

where $\Phi_j = \begin{pmatrix} \mathrm{X}_j \\ \Psi_j \end{pmatrix}$, $\mathrm{X}_j \in \mathbb{R}^2$, $\Psi_j = \begin{pmatrix} \overline{\psi}_j \\ \psi_j \end{pmatrix}$, for all $j \in \{1, \dots, N\}$, and $\Phi \cdot M\Phi = \sum_{j,k=1}^{N} \Phi_j \cdot M_{jk}\Phi_k$.

Proof. We compute

$$\begin{aligned} \Phi \cdot M\Phi &= \sum_{j,k=1}^{N} M_{jk}\Phi_j \cdot \Phi_k = \sum_{j,k=1}^{N} M_{jk}\mathrm{X}_j \cdot \mathrm{X}_k + \sum_{j,k=1}^{N} M_{jk} \frac{\overline{\psi}_j\psi_k + \overline{\psi}_k\psi_j}{2} \\ &= \sum_{j,k=1}^{N} M_{jk}\mathrm{X}_j \cdot \mathrm{X}_k + \sum_{x,y=1}^{N} \overline{\psi}_j M_{jk}\psi_k, \end{aligned} \tag{2.3.46}$$

where in the second line we used that $M_{xy} = M_{yx}$, for all $x, y \in \{1, \ldots, N\}$. Hence,

$$\begin{aligned}
\int \mathrm{d}\Phi\, \psi_x \overline{\psi}_y e^{-\Phi\cdot M\Phi} &= \int_{\mathbb{R}^{2N}} \prod_{j=1}^{n} \frac{\mathrm{d}^2 \mathrm{X}_j}{\pi} e^{-\sum_{j,k=1}^{N} M_{jk} \mathrm{X}_j \cdot \mathrm{X}_k} \prod_{j=1}^{N} \frac{\partial}{\partial \overline{\psi}_j} \frac{\partial}{\partial \psi_j} \psi_x \overline{\psi}_y e^{-\sum_{x,y=1}^{N} \overline{\psi}_j M_{jk} \psi_k} \\
&= (\det M) M_{xy}^{-1} \int_{\mathbb{R}^{2N}} \prod_{j=1}^{n} \frac{\mathrm{d}^2 \mathrm{X}_j}{\pi} e^{-\sum_{j,k=1}^{N} M_{jk} \mathrm{X}_j \cdot \mathrm{X}_k} \\
&= (\det M) M_{xy}^{-1} \left(\int_{\mathbb{R}^N} \prod_{j=1}^{n} \frac{\mathrm{d}\mathrm{Z}_j}{\sqrt{\pi}} e^{-\sum_{j,k=1}^{N} \mathrm{Z}_j M_{jk} \mathrm{Z}_k} \right)^2 = (\det M) M_{xy}^{-1} \left(\frac{1}{\sqrt{\det M}} \right)^2 = M_{xy}^{-1},
\end{aligned} \tag{2.3.47}$$

where in the second line we used invertibility of M together with Lemma 2.3.5 and in the last line we used $\operatorname{Re} M > 0$ to compute the Gaussian integral. $\square$

The above construction can be extended to p and q arbitrary but for the purpose of our computations the case $p = q = 2$ is sufficient.

2.3.3 Supermatrices

A linear transformation $M : \mathcal{A}_0^2[\mathscr{B}] \times \mathcal{A}_1^2[\mathscr{B}] \longrightarrow \mathcal{A}_0^2[\mathscr{B}] \times \mathcal{A}_1^2[\mathscr{B}]$ must have a specific block-matrix representation

$$M = \begin{pmatrix} a & \sigma \\ \chi & b \end{pmatrix}, \tag{2.3.48}$$

where $a \in \mathcal{A}_0^{2\times 2}[\mathscr{B}]$, $b \in \mathcal{A}_0^{2\times 2}[\mathscr{B}]$, $\sigma \in \mathcal{A}_1^{2\times 2}[\mathscr{B}]$ and $\chi \in \mathcal{A}_1^{2\times 2}[\mathscr{B}]$. M is called a supermatrix.

Let $M, N : \mathcal{A}_0^2[\mathscr{B}] \times \mathcal{A}_1^2[\mathscr{B}] \longrightarrow \mathcal{A}_0^2[\mathscr{B}] \times \mathcal{A}_1^2[\mathscr{B}]$ be supermatrices. The sum, product and scalar multiplication of supermatrices are defined as usual:

$$(M+N)_{ij} = M_{ij} + N_{ij}, \qquad \forall i, j \in \{1, 2, 3, 4\}, \tag{2.3.49}$$

$$(MN)_{ij} = \sum_{k=1}^{4} M_{ik} N_{kj}, \qquad \forall i, j \in \{1, 2, 3, 4\}, \tag{2.3.50}$$

$$(\lambda M)_{ij} = \lambda M_{ij}, \qquad \forall i, j \in \{1, 2, 3, 4\},\ \forall \lambda \in \mathbb{R}. \tag{2.3.51}$$

By a direct computation, $M + N$, MN and λM are also supermatrices.

In general, we can take $a \in \mathcal{A}_0^{p\times p}[\mathscr{B}]$, $b \in \mathcal{A}_0^{q\times q}[\mathscr{B}]$, $\sigma \in \mathcal{A}_1^{p\times q}[\mathscr{B}]$ and $\rho \in \mathcal{A}_1^{q\times p}[\mathscr{B}]$ with p, q arbitrary. Just like before, we only need the case $p = q = 2$.

Supertrace

In the case of supermatrices, the standard definition of a trace is not invariant under cyclic permutations. Indeed, let $p, q \in \mathbb{N}$ and $M_i : \mathcal{A}_0^2[\mathscr{B}] \times \mathcal{A}_1^2[\mathscr{B}] \longrightarrow \mathcal{A}_0^2[\mathscr{B}] \times \mathcal{A}_1^2[\mathscr{B}]$

be a supermatrix with $M_i = \begin{pmatrix} a_i & \sigma_i \\ \chi_i & b_i \end{pmatrix}$, for $i \in \{1,2\}$. After an explicit computation,

$$\operatorname{tr}(M_1M_2) = \operatorname{tr}(a_1a_2) + \operatorname{tr}(\sigma_1\chi_2) + \operatorname{tr}(\chi_1\sigma_2) + \operatorname{tr}(b_1b_2), \tag{2.3.52}$$

$$\begin{aligned}\operatorname{tr}(M_2M_1) &= \operatorname{tr}(a_2a_1) + \operatorname{tr}(\sigma_2\chi_1) + \operatorname{tr}(\chi_2\sigma_1) + \operatorname{tr}(b_2b_1) \\ &= \operatorname{tr}(a_1a_2) - \operatorname{tr}(\sigma_1\chi_2) - \operatorname{tr}(\chi_1\sigma_2) + \operatorname{tr}(b_1b_2) \neq \operatorname{tr}(M_1M_2),\end{aligned} \tag{2.3.53}$$

where in the last step we used the fact that σ_i and χ_i are matrices whose coefficients are fermionic variables, for $i \in \{1,2\}$, and

$$\operatorname{tr}(\sigma\chi) = -\operatorname{tr}(\chi\sigma), \qquad \forall\sigma \in \mathcal{A}_1^{2\times 2}[\mathscr{B}], \forall\chi \in \mathcal{A}_1^{2\times 2}[\mathscr{B}]. \tag{2.3.54}$$

To circumvent this problem, we modify the definition of the standard trace as follows:

$$\operatorname{Str}\begin{pmatrix} a & \sigma \\ \chi & b \end{pmatrix} = \operatorname{tr} a - \operatorname{tr} b, \tag{2.3.55}$$

which is called the **supertrace** of a supermatrix. By definition, $\operatorname{Str}M \in \mathcal{A}_0[\mathscr{B}]$, for all supermatrix M. In addition, the supertrace has the desired property:

$$\operatorname{Str}(M_1M_2) = \operatorname{Str}(M_2M_1), \qquad \text{for arbitrary supermatrices } M_1 \text{ and } M_2 \text{ as long as they have the same size.} \tag{2.3.56}$$

Indeed,

$$\begin{aligned}\operatorname{Str}(M_1M_2) &= \operatorname{tr}(a_1a_2) + \operatorname{tr}(\sigma_1\chi_2) - \operatorname{tr}(\chi_1\sigma_2) - \operatorname{tr}(b_1b_2), \\ &= \operatorname{tr}(a_2a_1) + \operatorname{tr}(\sigma_2\chi_1) - \operatorname{tr}(\chi_2\sigma_1) - \operatorname{tr}(b_2b_1) = \operatorname{Str}(M_2M_1).\end{aligned}$$

Remark 2.3.7.

- *If $a = b$ in Eq. 2.3.48, then $\operatorname{Str}(M) = 0$. In particular,* $\operatorname{Str}\begin{pmatrix} \mathrm{I}_2 & 0 \\ 0 & \mathrm{I}_2 \end{pmatrix} = 0$.
- *The supertrace is a linear operator*

$$\operatorname{Str}(\lambda_1M_1 + \lambda_2M_2) = \lambda_1\operatorname{Str}(M_1) + \lambda_2\operatorname{Str}(M_2), \qquad \forall\lambda_1, \lambda_2 \in \mathbb{R}, \tag{2.3.57}$$

Supertranspose of a Supermatrix

Let $M = \begin{pmatrix} a & \sigma \\ \chi & b \end{pmatrix}$ be a supermatrix. We define the supertranspose of M as

$$M^{\mathrm{T}} := \begin{pmatrix} a^{\mathrm{T}} & \chi^{\mathrm{T}} \\ -\sigma^{\mathrm{T}} & b^{\mathrm{T}} \end{pmatrix}. \tag{2.3.58}$$

Thus, the following equations hold

$$M\Phi\cdot\Phi' = (M\Phi)^T\Sigma\Phi' = \Phi^{\mathrm{T}}M^{\mathrm{T}}\Sigma\Phi', \qquad \forall\Phi, \Phi' \in \mathcal{A}_0^2[\mathscr{B}] \times \mathcal{A}_1^2[\mathscr{B}]. \tag{2.3.59}$$

Remark 2.3.8.

- *The supertranspose is linear,*

$$(\lambda_1 M_1 + \lambda_2 M_2)^{\mathrm{T}} = \lambda_1 M_1^{\mathrm{T}} + \lambda_2 M_2^{\mathrm{T}}, \qquad \forall \lambda_1, \lambda_2 \in \mathbb{R}, \tag{2.3.60}$$

- *The supertrace is invariant under the supertranspose,*

$$\mathrm{Str}(M) = \mathrm{Str}(M^{\mathrm{T}}). \tag{2.3.61}$$

- *Unlike the usual transpose, the supertranspose is not an involution. To be more explicit,*

$$\left(M^{\mathrm{T}}\right)^{\mathrm{T}} \neq M, \tag{2.3.62}$$

unless $\sigma = \chi = 0$.

- $$(M_1 M_2)^{\mathrm{T}} = M_2^{\mathrm{T}} M_1^{\mathrm{T}}, \text{ for all supermatrices } M_1 \text{ and } M_2 \text{ of the same size.} \tag{2.3.63}$$

Indeed, let $M_i = \begin{pmatrix} a_i & \sigma_i \\ \chi_i & b_i \end{pmatrix}$ *be a supermatrix, for* $i \in \{1, 2\}$*. Then,*

$$\begin{aligned} M_2^{\mathrm{T}} M_1^{\mathrm{T}} &= \begin{pmatrix} a_2^{\mathrm{T}} & \chi_2^{\mathrm{T}} \\ -\sigma_2^{\mathrm{T}} & b_2^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} a_1^{\mathrm{T}} & \chi_1^{\mathrm{T}} \\ -\sigma_1^{\mathrm{T}} & b_1^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} a_2^{\mathrm{T}} a_1^{\mathrm{T}} - \chi_2^{\mathrm{T}} \sigma_1^{\mathrm{T}} & a_2^{\mathrm{T}} \chi_1^{\mathrm{T}} + \chi_2^{\mathrm{T}} b_1^{\mathrm{T}} \\ -(\sigma_2^{\mathrm{T}} a_1^{\mathrm{T}} + b_2^{\mathrm{T}} \sigma_1^{\mathrm{T}}) & -\sigma_2^{\mathrm{T}} \chi_1^{\mathrm{T}} + b_2^{\mathrm{T}} b_1^{\mathrm{T}} \end{pmatrix} \\ &= \begin{pmatrix} a_1 a_2 + \sigma_1 \chi_2 & a_1 \sigma_2 + \sigma_1 b_2 \\ \chi_1 a_2 + b_1 \chi_2 & \chi_1 \sigma_2 + b_1 b_2 \end{pmatrix}^{\mathrm{T}} = \left(\begin{pmatrix} a_1 & \sigma_1 \\ \chi_1 & b_1 \end{pmatrix} \begin{pmatrix} a_2 & \sigma_2 \\ \chi_2 & b_2 \end{pmatrix} \right)^{\mathrm{T}} = (M_1 M_2)^{\mathrm{T}}. \end{aligned} \tag{2.3.64}$$

Superdeterminant and Logarithm

In the theory of conventional square matrices, the determinant has the following properties:

(a) $\det(AB) = \det A \det B, \qquad \forall A, B \in \mathbb{C}^{n \times n}, \forall n \in \mathbb{N},$

(b) $\operatorname{tr} \ln A = \ln \det A$, whenever $\ln A$ is well-defined.

These equations remain true when A and B have entries in $\mathcal{A}_0[\mathscr{B}]$. Indeed, let $f(x, y) = \det(xy)$ and $g(x, y) = \det x \det y$, for all $x, y \in \mathbb{R}^{n \times n}$. This functions are polynomials and hence thet are smooth. Therefore, their extensions on $\mathcal{A}_0^{n \times n}[\mathscr{B}]$ defined via Taylor expansion coincide. This proves Property (a). On the other hand, Property (b) was already proved in Eq. 2.3.25 via an analogue argument.

We want to define the analog of the determinant for a supermatrix such that its relationship with the supertrace mimics Properties (a) and (b). More precisely, we look for a definition of superdeterminant such that:

(A) $\mathrm{Sdet}(M_1 M_2) = \mathrm{Sdet}(M_1)\mathrm{Sdet}(M_2)$, for arbitrary supermatrices M_1 and M_2 as long as they have the same size.

(B) $\operatorname{Str}\ln M = \ln \operatorname{Sdet} M$.

For (B) to be well-defined, we first need to define $\ln M$, for M supermatrix. We distinguish three cases:

Case 1: Assume that M is a block-diagonal matrix, $M = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, such that a, b are invertible and $\ln x(a), \ln x(b)$ are well-defined (cf. Eq. 2.3.24). In this case, we define

$$\ln \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} := \begin{pmatrix} \ln a & 0 \\ 0 & \ln b \end{pmatrix}. \tag{2.3.65}$$

Case 2: Assume $a = \mathrm{I}_2$ and $b = \mathrm{I}_2$, then $M = \begin{pmatrix} \mathrm{I}_2 & \sigma \\ \chi & \mathrm{I}_2 \end{pmatrix}$. In this case, we define

$$\ln \left(\mathrm{I} + \begin{pmatrix} 0 & \sigma \\ \chi & 0 \end{pmatrix} \right) := - \sum_{j \geq 1} \frac{(-1)^j}{j} \begin{pmatrix} 0 & \sigma \\ \chi & 0 \end{pmatrix}^j, \tag{2.3.66}$$

where the sum is finite since $\begin{pmatrix} 0 & \sigma \\ \chi & 0 \end{pmatrix}$ is nilpotent. Observe

$$\begin{pmatrix} 0 & \sigma \\ \chi & 0 \end{pmatrix}^{2k} = \begin{pmatrix} (\sigma\chi)^k & 0 \\ 0 & (\chi\sigma)^k \end{pmatrix}, \quad \begin{pmatrix} 0 & \sigma \\ \chi & 0 \end{pmatrix}^{2k-1} = \begin{pmatrix} 0 & (\sigma\chi)^{k-1}\sigma \\ (\chi\sigma)^{k-1}\chi & 0 \end{pmatrix}, \qquad \forall k \in \mathbb{N}. \tag{2.3.67}$$

If we plug both equations into Eq. 2.3.66, we obtain

$$\begin{aligned} \ln \left(\mathrm{I} + \begin{pmatrix} 0 & \sigma \\ \chi & 0 \end{pmatrix} \right) &= - \sum_{k \geq 1} \frac{(-1)^{2k}}{2k} \begin{pmatrix} 0 & \sigma \\ \chi & 0 \end{pmatrix}^{2k} - \sum_{k \geq 1} \frac{(-1)^{2k-1}}{2k-1} \begin{pmatrix} 0 & \sigma \\ \chi & 0 \end{pmatrix}^{2k-1} \\ &= - \sum_{k \geq 1} \frac{1}{2k} \begin{pmatrix} (\sigma\chi)^k & 0 \\ 0 & (\chi\sigma)^k \end{pmatrix} + \sum_{k \geq 1} \frac{1}{2k-1} \begin{pmatrix} 0 & (\sigma\chi)^{k-1}\sigma \\ (\chi\sigma)^{k-1}\chi & 0 \end{pmatrix}, \end{aligned} \tag{2.3.68}$$

where in the first step we used the fact that the sum in Eq. 2.3.66 is finite, hence it can be reordered into a sum over even and odd powers.

Case 3: In the general case, let $M = \begin{pmatrix} a & \sigma \\ \chi & b \end{pmatrix}$ be a supermatrix such that a, b are invertible and $\ln x(a), \ln x(b)$ are well-defined. We define

$$\begin{aligned} \ln M = \ln \left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} \mathrm{I}_2 & a^{-1}\sigma \\ b^{-1}\chi & \mathrm{I}_2 \end{pmatrix} \right) &:= \ln \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} + \ln \left(\mathrm{I} + \begin{pmatrix} 0 & a^{-1}\sigma \\ b^{-1}\chi & 0 \end{pmatrix} \right), \\ &= \begin{pmatrix} \ln a - \sum_{k \geq 1} \frac{1}{2k} (a^{-1}\sigma b^{-1}\chi)^k & \sum_{k \geq 1} \frac{1}{2k-1} (a^{-1}\sigma b^{-1}\chi)^{k-1} a^{-1}\sigma \\ \sum_{k \geq 1} \frac{1}{2k-1} (b^{-1}\chi a^{-1}\sigma)^{k-1} b^{-1}\chi & \ln b - \sum_{k \geq 1} \frac{1}{2k} (b^{-1}\chi a^{-1}\sigma)^k \end{pmatrix}. \end{aligned} \tag{2.3.69}$$

Now we are able to define the superdeterminant. As a first step, let M be a block-diagonal supermatrix, $M = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, such that a, b are invertible and $\ln x(a), \ln x(b)$ are well-defined. We compute

$$\operatorname{Str} \ln M = \operatorname{Str} \begin{pmatrix} \ln a & 0 \\ 0 & \ln b \end{pmatrix} = \operatorname{tr} \ln a - \operatorname{tr} \ln b = \ln \det a - \ln \det b = \ln \left(\frac{\det a}{\det b} \right), \tag{2.3.70}$$

where we used the definition of Supertrace and Property (b). In order to satisfy Property (B), we need

$$\ln \mathrm{Sdet} M = \mathrm{Str} \ln M = \ln \left(\frac{\det a}{\det b} \right). \tag{2.3.71}$$

Hence, we define

$$\mathrm{Sdet} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} := \frac{\det a}{\det b}. \tag{2.3.72}$$

This definition ensures that the superdeterminant of block-diagonal supermatrices satisfies Property (A) too. Indeed,

$$\begin{aligned} \mathrm{Sdet}\left(\begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix} \begin{pmatrix} a_2 & 0 \\ 0 & b_2 \end{pmatrix} \right) &= \mathrm{Sdet} \begin{pmatrix} a_1 a_2 & 0 \\ 0 & b_1 b_2 \end{pmatrix} = \frac{\det(a_1 a_2)}{\det(b_1 b_2)} \\ &= \frac{\det a_1}{\det b_1} \frac{\det a_2}{\det b_2} = \mathrm{Sdet} \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix} \mathrm{Sdet} \begin{pmatrix} a_2 & 0 \\ 0 & b_2 \end{pmatrix}, \end{aligned} \tag{2.3.73}$$

where in the second line we used the multiplicativity of the usual determinant.

For the general case we have the following proposition

Proposition 2.3.9. *Suppose that a, b are invertible and $\ln a, \ln b$ are well-defined. Assume that Properties (A) and (B) hold. Then,*

$$\mathrm{Sdet} \begin{pmatrix} a & \sigma \\ \chi & b \end{pmatrix} = \frac{\det a}{\det b} \det\left(\mathrm{I}_2 - a^{-1} \sigma b^{-1} \chi\right). \tag{2.3.74}$$

In particular, $\mathrm{Sdet} \begin{pmatrix} a & \sigma \\ \chi & b \end{pmatrix} \in \mathcal{A}_0[\mathscr{B}]$.

Proof. To begin with, assume that $a = \mathrm{I}_2$ and $b = \mathrm{I}_2$. In this case, we need to prove

$$\mathrm{Sdet}\left(\mathrm{I} + \begin{pmatrix} 0 & \sigma \\ \chi & 0 \end{pmatrix} \right) = \det(\mathrm{I}_2 - \sigma\chi). \tag{2.3.75}$$

Using Property (B) and Eq. 2.3.68, we obtain

$$\begin{aligned} \ln \mathrm{Sdet}\left(\mathrm{I} + \begin{pmatrix} 0 & \sigma \\ \chi & 0 \end{pmatrix} \right) &= \mathrm{Str} \ln \left(\mathrm{I} + \begin{pmatrix} 0 & \sigma \\ \chi & 0 \end{pmatrix} \right) \\ &= \mathrm{Str} \left(- \sum_{k \geq 1} \frac{1}{2k} \begin{pmatrix} (\sigma\chi)^k & 0 \\ 0 & (\chi\sigma)^k \end{pmatrix} + \sum_{k \geq 1} \frac{1}{2k-1} \begin{pmatrix} 0 & (\sigma\chi)^{k-1}\sigma \\ (\chi\sigma)^{k-1}\chi & 0 \end{pmatrix} \right) \\ &= - \sum_{k \geq 1} \frac{1}{2k} \mathrm{Str} \begin{pmatrix} (\sigma\chi)^k & 0 \\ 0 & (\chi\sigma)^k \end{pmatrix} + \sum_{k \geq 1} \frac{1}{2k-1} \mathrm{Str} \begin{pmatrix} 0 & (\sigma\chi)^{k-1}\sigma \\ (\chi\sigma)^{k-1}\chi & 0 \end{pmatrix}, \end{aligned} \tag{2.3.76}$$

where in the last step we used the linearity of the supertrace for finite sums. We compute each term of the above sum

$$\begin{aligned} &\mathrm{Str} \begin{pmatrix} (\sigma\chi)^k & 0 \\ 0 & (\chi\sigma)^k \end{pmatrix} = \mathrm{tr}\left((\sigma\chi)^k\right) - \mathrm{tr}\left((\chi\sigma)^k\right) = 2\,\mathrm{tr}\left((\sigma\chi)^k\right), \\ &\mathrm{Str} \begin{pmatrix} 0 & (\sigma\chi)^{k-1}\sigma \\ (\chi\sigma)^{k-1}\chi & 0 \end{pmatrix} = 0, \qquad \forall k \in \mathbb{N}, \end{aligned} \tag{2.3.77}$$

where in the first line we used Eq. 2.3.54 since σ, χ are matrices with fermionic entries. By plugging the above results into Eq. 2.3.76, we obtain

$$\begin{aligned}\mathrm{Str}\ln\left(\mathrm{I}+\begin{pmatrix}0 & \sigma\\ \chi & 0\end{pmatrix}\right) &= -\sum_{k\geq 1}\frac{1}{k}\,\mathrm{tr}\left((\sigma\chi)^k\right) = -\,\mathrm{tr}\left(\sum_{k\geq 1}\frac{1}{k}(\sigma\chi)^k\right)\\ &= \mathrm{tr}\ln(\mathrm{I}_2-\sigma\chi) = \ln\det(\mathrm{I}_2-\sigma\chi),\end{aligned} \tag{2.3.78}$$

where in the second line we used the definition of logarithm of a matrix with bosonic entries (cf. 2.3.24) and Property (b). Hence, Eq. 2.3.74 holds.

For the general case, assume that a, b are invertible and $\ln a, \ln b$ are well-defined. Using Property A and Eq. 2.3.72, 2.3.75, we get

$$\begin{aligned}\mathrm{Sdet}\begin{pmatrix}a & \sigma\\ \chi & b\end{pmatrix} &= \mathrm{Sdet}\left(\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}\left\{\mathrm{I}+\begin{pmatrix}0 & a^{-1}\sigma\\ b^{-1}\chi & 0\end{pmatrix}\right\}\right)\\ &= \mathrm{Sdet}\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}\mathrm{Sdet}\left(\mathrm{I}+\begin{pmatrix}0 & a^{-1}\sigma\\ b^{-1}\chi & 0\end{pmatrix}\right) = \frac{\det a}{\det b}\det\left(\mathrm{I}_2-a^{-1}\sigma b^{-1}\chi\right).\end{aligned} \tag{2.3.79}$$

This concludes the proof of the proposition. □

2.3.4 Superrotation & SUSY Invariant Functions

A superrotation is a linear transformation $\Phi \mapsto \mathrm{R}\Phi$ such that

$$\mathrm{R}\Phi\cdot\mathrm{R}\Phi' = \Phi\cdot\Phi', \qquad \forall\Phi,\Phi'\in\mathcal{A}_0^2[\mathscr{B}]\times\mathcal{A}_1^2[\mathscr{B}]. \tag{2.3.80}$$

By Eq. 2.3.59, the LHS of the above equation is equivalent to

$$\mathrm{R}\Phi\cdot\mathrm{R}\Phi' = (\mathrm{R}\Phi)^{\mathrm{T}}\Sigma\mathrm{R}\Phi' = \Phi^{\mathrm{T}}\mathrm{R}^{\mathrm{T}}\Sigma\mathrm{R}\Phi', \quad \forall\Phi,\Phi'\in\mathcal{A}_0^2[\mathscr{B}]\times\mathcal{A}_1^2[\mathscr{B}], \tag{2.3.81}$$

where R^{T} is defined in Eq. 2.3.58 and Σ is given in Eq. 2.3.45. Therefore, R is a superrotation iff

$$\mathrm{R}^{\mathrm{T}}\Sigma\mathrm{R} = \Sigma. \tag{2.3.82}$$

A simple example is the case when $\mathrm{R} = \begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}$ with $a\in\mathbb{R}^{2\times 2}$ a standard rotation (i.e., $a^{\mathrm{T}} = a^{-1}$) and $b\in\mathbb{R}^{2\times 2}$ be such that $b^{\mathrm{T}}lb = l$. Then, R is a superrotation. Indeed,

$$\mathrm{R}\Phi\cdot\mathrm{R}\Phi' = (\mathrm{R}\Phi)^{\mathrm{T}}\Sigma(\mathrm{R}\Phi') = \mathrm{X}^{\mathrm{T}}a^{\mathrm{T}}a\mathrm{X}' + \Psi^{\mathrm{T}}b^{\mathrm{T}}lb\Psi' = \mathrm{X}^{\mathrm{T}}a^{\mathrm{T}}a\mathrm{X}' + \Psi^{\mathrm{T}}l\Psi' = \Phi\cdot\Phi'.$$

The next result provides conditions to guarantee that R is a superrotation.

Proposition 2.3.10. *Let* $M = \begin{pmatrix}a & \sigma\\ \chi & b\end{pmatrix}$ *be a supermatrix,* $t\in\mathbb{R}$ *and* $\mathrm{R}_t = e^{tM}$*. Then,* $\mathrm{R}_t = e^{tM}$ *is a superrotation for all* $t\in\mathbb{R}$ *iff* $M^{\mathrm{T}}\Sigma = -\Sigma M$*. Equivalently,* R_t *is a superrotation for all* $t\in\mathbb{R}$ *iff*

$$a^{\mathrm{T}} = -a, \quad b^{\mathrm{T}}l = -lb, \quad \sigma = -\chi^{\mathrm{T}}l, \quad \sigma^{\mathrm{T}} = l\chi. \tag{2.3.83}$$

In particular, $\mathrm{Str}M = 0$*, and therefore* $\mathrm{Sdet}\mathrm{R}_t = 1$*, for all* $t\in\mathbb{R}$*.*

Proof. Assume that $M^{\mathrm{T}}\Sigma = -\Sigma M$. Hence, $M^{\mathrm{T}} = -\Sigma M \Sigma^{-1}$. Using also Eq. 2.3.58 and 2.3.63, we compute

$$\begin{aligned}
\mathrm{R}_t^{\mathrm{T}} &= \left(\sum_{k\geq 0}\frac{t^k}{k!}M^k\right)^{\mathrm{T}} = \sum_{k\geq 0}\frac{t^k}{k!}(M^k)^{\mathrm{T}} = \sum_{k\geq 0}\frac{t^k}{k!}(M^{\mathrm{T}})^k = \sum_{k\geq 0}\frac{t^k}{k!}(-\Sigma M\Sigma^{-1})^k \\
&= \Sigma\left(\sum_{k\geq 0}\frac{(-t)^k}{k!}M^k\right)\Sigma^{-1} = \Sigma e^{-tM}\Sigma^{-1} = \Sigma \mathrm{R}_t^{-1}\Sigma^{-1}. \qquad (2.3.84)
\end{aligned}$$

Therefore,

$$\mathrm{R}_t^{\mathrm{T}}\Sigma\mathrm{R}_t = \Sigma\mathrm{R}_t^{-1}\Sigma^{-1}\Sigma\mathrm{R}_t = \Sigma, \qquad (2.3.85)$$

which is the desired result.

Suppose now that $\mathrm{R}_t = e^{tM}$ is a superrotation for all $t \in \mathbb{R}$. Expanding in powers of t, we obtain

$$\mathrm{R}_t\Phi\cdot\mathrm{R}_t\Phi' = \Phi\cdot\Phi' + t(M\Phi\cdot\Phi' + \Phi\cdot M\Phi') + o(t), \quad \forall\Phi,\Phi'\in\mathcal{A}_0^2[\mathscr{B}]\times\mathcal{A}_1^2[\mathscr{B}]. \qquad (2.3.86)$$

Since we suppose that R_t is a superrotation, the linear term in t on the RHS of the above equation must be zero, that is,

$$0 = M\Phi\cdot\Phi' + \Phi\cdot M\Phi' = (M\Phi)^{\mathrm{T}}\Sigma\Phi' + \Phi^{\mathrm{T}}\Sigma M\Phi' = \Phi^{\mathrm{T}}M^{\mathrm{T}}\Sigma\Phi' + \Phi^{\mathrm{T}}\Sigma M\Phi'. \qquad (2.3.87)$$

Hence, we must have $M^{\mathrm{T}}\Sigma = -\Sigma M$.

By a direct computation, $M^{\mathrm{T}}\Sigma = -\Sigma M$ is equivalent to $a^{\mathrm{T}} = -a$, $b^{\mathrm{T}}l = -lb$, $\sigma = -\chi^{\mathrm{T}}l$ and $\sigma^{\mathrm{T}} = l\chi$. It follows that a is antisymmetric and in particular, $\operatorname{tr} a = 0$. On the other hand, $b = -lb^{\mathrm{T}}l^{-1}$. Hence,

$$\operatorname{tr} b = -\operatorname{tr}\left(lb^{\mathrm{T}}l^{-1}\right) = \operatorname{tr}\left(b^{\mathrm{T}}l^{-1}l\right) = -\operatorname{tr} b^{\mathrm{T}} = -\operatorname{tr} b, \qquad (2.3.88)$$

where we used the invariance under cyclic permutations of the usual trace and its invariance under the transpose. Consequently, we obtain $\operatorname{tr} b = 0$.

Finally, using $\mathrm{Str}M = \operatorname{tr} a - \operatorname{tr} b = 0$, we have

$$\mathrm{Sdet}\mathrm{R}_t = e^{\ln\mathrm{Sdet}\mathrm{R}_t} = e^{t\mathrm{Str}M} = e^0 = 1. \qquad (2.3.89)$$

This concludes the proof. □

Our next goal is to define functions invariant under superrotations. Recall that we have defined functions on supervectors (cf. Eq. 2.3.40). Let $\Phi \mapsto F\Phi \in \mathcal{A}_0[\mathscr{B}]$ be a function. F is called **SUSY invariant** (or **supersymmetric invariant**), if

$$F(\mathrm{R}\Phi) = F(\Phi), \qquad \forall\mathrm{R} = e^{tM}\ \text{superrotation}. \qquad (2.3.90)$$

A SUSY invariant function has a simplified form. This is the content of the next lemma.

Lemma 2.3.11. *Let $\Phi \mapsto F\Phi$ be SUSY invariant. Then, there is a function $f \in \mathcal{C}^\infty(\mathbb{R}^+) \cap \mathcal{C}^1([0,\infty))$ such that*

$$F(\Phi) = f(\Phi^2). \tag{2.3.91}$$

Proof. Let M be a supermatrix of the form

$$M = \begin{pmatrix} 0 & \sigma \\ l^{-1}\sigma & 0 \end{pmatrix}, \text{ where } \sigma = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}, \; l^{-1}\sigma = \begin{pmatrix} 0 & 2\alpha_2 \\ -2\alpha_1 & 0 \end{pmatrix}. \tag{2.3.92}$$

Then, $\mathrm{R} = e^{tM}$ is a superrotation, for all $t \in \mathbb{R}$, due to Proposition 2.3.10. Remember $\Phi = \begin{pmatrix} \mathrm{X} \\ \Psi \end{pmatrix}$, where $\mathrm{X} = \begin{pmatrix} \mathrm{X}_1 \\ \mathrm{X}_2 \end{pmatrix} \in \mathcal{A}_0^2[\mathscr{B}]$ and $\Psi = \begin{pmatrix} \overline{\psi} \\ \psi \end{pmatrix} \in \mathcal{A}_1^2[\mathscr{B}]$. It suffices to prove Eq. 2.3.91 for $\mathrm{X} \in \mathbb{R}^2$. Indeed, the general case is obtained by Taylor expansion on both sides. For t small, we expand

$$\Phi \mapsto \mathrm{R}\Phi = \Phi + tM\Phi + o(t), \text{ where } M\Phi = \begin{pmatrix} n \\ \rho \end{pmatrix}, \; n = \begin{pmatrix} \alpha_1\overline{\psi} \\ \alpha_2\psi \end{pmatrix}, \text{ and } \rho = 2\begin{pmatrix} \alpha_2\mathrm{X}_2 \\ -\alpha_1\mathrm{X}_1 \end{pmatrix}. \tag{2.3.93}$$

Since $F(\mathrm{R}_t\Phi) = F(\Phi)$ and F is smooth, we obtain

$$F(\Phi + tM\Phi) - F(\Phi) = o(t). \tag{2.3.94}$$

We compute

$$\begin{aligned} F(\Phi + tM\Phi) = F_0(\mathrm{X} + tn) &+ F_1(\mathrm{X}+tn)\big(\overline{\psi} + t\rho_1\big) + F_2(\mathrm{X}+tn)(\psi + t\rho_2) \\ &+ F_3(\mathrm{X}+tn)\big(\overline{\psi} + t\rho_1\big)(\psi + t\rho_2) \end{aligned} \tag{2.3.95}$$

Expanding F_j in tn, we get

$$F_j(\mathrm{X}+tn) = F_j(\mathrm{X}) + \frac{\partial F_j(\mathrm{X})}{\partial \mathrm{X}_1} tn_1 + \frac{\partial F_j(\mathrm{X})}{\partial \mathrm{X}_2} tn_2 + o(t). \tag{2.3.96}$$

Hence,

$$\begin{aligned} F(\Phi + tM\Phi) - F(\Phi) = t\sum_{j=1}^{2} &\left[\frac{\partial F_0(\mathrm{X})}{\partial \mathrm{X}_j} + \frac{\partial F_1(\mathrm{X})}{\partial \mathrm{X}_j}\overline{\psi} + \frac{\partial F_2(\mathrm{X})}{\partial \mathrm{X}_j}\psi\right] n_j \\ &+ t\big[F_1(\mathrm{X})\rho_1 + F_2(\mathrm{X})\rho_2 + F_3(\mathrm{X})(\overline{\psi}\rho_2 + \rho_1\psi)\big] + o(t). \end{aligned} \tag{2.3.97}$$

Observe

$$\overline{\psi}n_1 = 0, \qquad \overline{\psi}n_2 = -\alpha_2\overline{\psi}\psi, \qquad \overline{\psi}\rho_2 = -2\alpha_1\mathrm{X}_1\overline{\psi}, \tag{2.3.98}$$

$$\psi n_1 = \alpha_1\overline{\psi}\psi, \qquad \psi_1 n_2 = 0, \qquad \rho_1\psi = 2\alpha_2\mathrm{X}_2\psi. \tag{2.3.99}$$

It follows

$$\begin{aligned} F(\Phi + tM\Phi) - F(\Phi) = t\alpha_1 &\left[\frac{\partial F_0(\mathrm{X})}{\partial \mathrm{X}_1}\overline{\psi} - 2F_2(\mathrm{X})\mathrm{X}_1 + \frac{\partial F_2(\mathrm{X})}{\partial \mathrm{X}_1}\overline{\psi}\psi - 2F_3(\mathrm{X})\mathrm{X}_1\overline{\psi}\right] \\ + t\alpha_2 &\left[\frac{\partial F_0(\mathrm{X})}{\partial \mathrm{X}_2}\psi + 2F_1(\mathrm{X})\mathrm{X}_2 - \frac{\partial F_1(\mathrm{X})}{\partial \mathrm{X}_2}\overline{\psi}\psi - 2F_3(\mathrm{X})\mathrm{X}_2\psi\right] + o(t). \end{aligned} \tag{2.3.100}$$

Setting $\psi = \overline{\psi} = 0$, we get $F_1 = F_2 = 0$. Hence,

$$F(\Phi + tM\Phi) - F(\Phi) = t\alpha_1\left[\frac{\partial F_0(\mathrm{X})}{\partial \mathrm{X}_1} - 2F_3(\mathrm{X})\mathrm{X}_1\right]\overline{\psi} + t\alpha_2\left[\frac{\partial F_0(\mathrm{X})}{\partial \mathrm{X}_2} - 2F_3(\mathrm{X})\mathrm{X}_2\right]\psi + o(t),$$

which implies

$$\nabla F_0(\mathrm{X}) = 2F_3(\mathrm{X})\begin{pmatrix}\mathrm{X}_1\\ \mathrm{X}_2\end{pmatrix}. \tag{2.3.101}$$

Changing to polar coordinates,

$$\hat{e}_1 = \cos\theta\hat{e}_r - \sin\theta\hat{e}_\theta, \qquad \hat{e}_2 = \sin\theta\hat{e}_r + \cos\theta\hat{e}_\theta, \tag{2.3.102}$$

we obtain

$$\begin{aligned}\nabla F(\mathrm{X}_1, \mathrm{X}_2) &= \frac{\partial F_0(r,\theta)}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial F_0(r,\theta)}{\partial \theta}\hat{e}_\theta,\\ \begin{pmatrix}\mathrm{X}_1\\ \mathrm{X}_2\end{pmatrix} &= [r\cos\theta(\cos\theta\hat{e}_r - \sin\theta\hat{e}_\theta) + r\sin\theta(\sin\theta\hat{e}_r + \cos\theta\hat{e}_\theta)]\\ &= r(\cos^2\theta + \sin^2\theta)\hat{e}_r = r\hat{e}_r.\end{aligned} \tag{2.3.103}$$

In this way, Eq. 2.3.101 becomes

$$\frac{\partial F_0(r,\theta)}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial F_0(r,\theta)}{\partial \theta}\hat{e}_\theta = 2F_3(r,\theta)r\hat{e}_r. \tag{2.3.104}$$

It follows $\frac{\partial F_0(r,\theta)}{\partial \theta} = 0$, hence F_0 is radial and there is a function $f \in C^\infty(\mathbb{R})$ such that

$$F_0(r) = f(r^2). \tag{2.3.105}$$

Hence, the RHS of the above equation does not depend on θ. In particular, F_3 is also radial and

$$F_3(r) = \left.\frac{\mathrm{d}f(u)}{\mathrm{d}u}\right|_{u=r^2}. \tag{2.3.106}$$

□

2.4 Transfer Operator representation

We give a proof of Thm. 2.2.1 below.

Proof. Let $z \in \mathbb{C}$ with $\operatorname{Im} z > 0$. Observe

$$\begin{aligned}&\left|\frac{\sum_{\sigma_0,\sigma_1=0}^{1} \mathrm{G}_z^{\Lambda_L}(j_0 + \sigma_0 e, j_1 + \sigma_1 e)}{4}\right|^2\\ &= \left(\frac{\sum_{\sigma_0,\sigma_1=0}^{1} \mathrm{G}_z^{\Lambda_L}(j_0 + \sigma_0 e, j_1 + \sigma_1 e)}{4}\right)\overline{\left(\frac{\sum_{\sigma_0',\sigma_1'=0}^{1} \mathrm{G}_z^{\Lambda_L}(j_0 + \sigma_0' e, j_1 + \sigma_1' e)}{4}\right)}\\ &= \sum_{\sigma_0,\sigma_1,\sigma_0',\sigma_1'=0}^{1} \frac{\mathrm{G}_z^{\Lambda_L}(j_0 + \sigma_0 e, j_1 + \sigma_1 e)\overline{\mathrm{G}_z^{\Lambda_L}(j_0 + \sigma_0' e, j_1 + \sigma_1' e)}}{16}.\end{aligned} \tag{2.4.1}$$

We represent each $\mathrm{G}_z^{\Lambda_L}(j_0+\sigma_0 e, j+\sigma e)\overline{\mathrm{G}_z^{\Lambda_L}(j_0+\sigma_0' e, j+\sigma' e)}$ in terms of a Gaussian integral involving real and Grassmann variables as in Section 2.3. Let $\mathfrak{G} := \mathcal{A}[\mathscr{B}]$ be the real Grassmann algebra generated by

$$\mathscr{B} := \left\{\psi_x^+, \overline{\psi}_x^+, \psi_x^-, \overline{\psi}_x^-\right\}_{x\in\Lambda_L} \tag{2.4.2}$$

Remember that $\mathfrak{G} = \mathfrak{G}_0 \oplus \mathfrak{G}_1$, see Eq. 2.3.7. Recall that for a given function $F(\Phi^+)$ or $F(\Phi^-)$ the corresponding integral is defined by

$$\int \mathrm{d}\Phi^\pm \mathbf{F}(\Phi^\pm) := \int_{\mathbb{R}^2} \frac{\mathrm{d}^2 \mathrm{X}^\pm}{\pi} \frac{\partial}{\partial \overline{\psi}^\pm} \frac{\partial}{\partial \psi^\pm} \mathbf{F}(\Phi^\pm),$$
$$\text{where } \Phi^\pm = \begin{pmatrix} \mathrm{X}^\pm \\ \Psi^\pm \end{pmatrix},\ \mathrm{X}^\pm \in \mathbb{R}^2,\ \Psi^\pm = \begin{pmatrix} \overline{\psi}^\pm \\ \psi^\pm \end{pmatrix} \in \mathfrak{G}_1,$$

see Eq. 2.3.41. Using Thm. 2.3.6, we can write

$$\mathrm{G}_z^{\Lambda_L}(j_0+\sigma_0 e, j_1+\sigma_1 e) = i\int \mathrm{d}\Phi^+\ \psi^+_{j_0+\sigma_0 e}\overline{\psi}^+_{j_1+\sigma_1 e} e^{i\Phi^+\cdot\left[z-\mathrm{H}^{\Lambda_L}\right]\Phi^+}, \tag{2.4.3}$$

$$\begin{aligned}\overline{\mathrm{G}_z^{\Lambda_L}(j_0+\sigma_0' e, j_1+\sigma_1' e)} &= (\mathrm{H}^{\Lambda_L}-\overline{z})^{-1}_{j_1+\sigma_1' e, j_0+\sigma_0' e} \\ &= -i\int \mathrm{d}\Phi^-\ \psi^-_{j_1+\sigma_1' e}\overline{\psi}^-_{j_0+\sigma_0' e} e^{-i\Phi^-\cdot\left[\overline{z}-\mathrm{H}^{\Lambda_L}\right]\Phi^-},\end{aligned} \tag{2.4.4}$$

where we used that H^{Λ_L} is self-adjoint and $\operatorname{Re} iz < 0$.

For simplicity, we drop the superindex of Φ^+ in the representation 2.4.3. Expanding the argument of the exponential, we obtain

$$\begin{aligned} i\Phi\cdot\left[z-\mathrm{H}^{\Lambda_L}\right]\Phi = &-\eta\sum_{j=-L}^{L}\sum_{\sigma=0}^{1}\Phi_{j+\sigma e}^2 + \sum_{j=-L}^{L} i(\mathrm{E}-\omega_j)\sum_{\sigma=0}^{1}\Phi^2_{j+\sigma e} \\ &- 2i\sum_{j=-L}^{L-1}\sum_{\sigma=0}^{1}\Phi_{j+\sigma e}\cdot\Phi_{j+1+\sigma e} - 2i\sum_{j=-L}^{L}\Phi_j\cdot\Phi_{j+e}. \end{aligned} \tag{2.4.5}$$

Hence,

$$\begin{aligned} &\frac{\sum_{\sigma_0,\sigma_1=0}^{1}\mathrm{G}_z^{\Lambda_L}(j_0+\sigma_0 e, j_1+\sigma_1 e)}{4} \\ &= i\sum_{\sigma_0,\sigma_1,\sigma_0',\sigma_1'=0}^{1}\int\prod_{j=-L}^{L}\mathrm{d}\Phi_j\,\mathrm{d}\Phi_{j+e}\frac{\psi_{j_0+\sigma_0 e}\overline{\psi}_{j_1+\sigma_1 e}}{4}\prod_{j=-L}^{L-1}e^{-2i\Phi_j\cdot\Phi_{j+1}}e^{-2i\Phi_{j+e}\cdot\Phi_{j+1+e}} \\ &\quad\times\prod_{j=-L}^{L}e^{-\eta(\Phi_j^2+\Phi_{j+e}^2)}e^{i\mathrm{E}(\Phi_j^2+\Phi_{j+e}^2)}e^{-i\omega_j(\Phi_j^2+\Phi_{j+e}^2)}e^{-2i\Phi_j\cdot\Phi_{j+e}} \\ &= i\int\prod_{j=-L}^{L}\mathrm{d}\Phi_j\,\mathrm{d}\Phi_{j+e}\left(\frac{\psi_{j_0}+\psi_{j_0+e}}{2}\right)\left(\frac{\overline{\psi}_{j_1}+\overline{\psi}_{j_1+e}}{2}\right)\prod_{j=-L}^{L-1}e^{-2i\Phi_j\cdot\Phi_{j+1}}e^{-2i\Phi_{j+e}\cdot\Phi_{j+1+e}} \\ &\quad\times\prod_{j=-L}^{L}e^{-\eta(\Phi_j^2+\Phi_{j+e}^2)}e^{i\mathrm{E}(\Phi_j^2+\Phi_{j+e}^2)}e^{-i\omega_j(\Phi_j^2+\Phi_{j+e}^2)}e^{-2i\Phi_j\cdot\Phi_{j+e}}, \end{aligned} \tag{2.4.6}$$

where in the second line we used the linearity of the integral and the following equation

$$\sum_{\sigma_0,\sigma_1=0}^{1} \frac{\psi_{j_0+\sigma_0 e}\overline{\psi}_{j_1+\sigma_1 e}}{4} = \left(\frac{\psi_{j_0}+\psi_{j_0+e}}{2}\right)\left(\frac{\overline{\psi}_{j_1}+\overline{\psi}_{j_1+e}}{2}\right). \tag{2.4.7}$$

To introduce a transfer operator representation, we define

$$\begin{aligned}[]
[\mathbf{T}_+\mathrm{F}](\Phi,\Phi') &:= \int d\Theta\, d\Theta'\, e^{-2i\Phi\cdot\Theta} e^{-2i\Phi'\cdot\Theta'}\mathrm{F}(\Theta,\Theta'),\\
\Gamma_{+,\omega_j}(\Phi,\Phi') &:= e^{-\eta\left(\Phi^2+\Phi'^2\right)} e^{i\mathrm{E}\Phi\cdot\Phi} e^{i\mathrm{E}\Phi'\cdot\Phi'} e^{-i\omega_j(\Phi^2+\Phi'^2)} e^{-2i\Phi\cdot\Phi'},
\end{aligned} \tag{2.4.8}$$

where Θ and Θ' are two supervectors. The relevant observables above are

$$\mathscr{O}(\Phi,\Phi') := \frac{\psi+\psi'}{2}, \qquad \overset{*}{\mathscr{O}}(\Phi,\Phi') := \frac{\overline{\psi}+\overline{\psi}'}{2} = -\overline{\mathscr{O}(\Phi,\Phi')}.$$

By hypothesis, the observables are located on the j_0-th and j_1-th layer of the ladder $\mathscr{D}_2$ with $j_0 < j_1$. To make the notation more succinct, let

$$\mathscr{o}\mathrm{F}(\Phi,\Phi') := \mathscr{O}(\Phi,\Phi')\mathrm{F}(\Phi,\Phi'), \qquad \overset{*}{\mathscr{o}}\mathrm{F}(\Phi,\Phi') := \overset{*}{\mathscr{O}}(\Phi,\Phi')\mathrm{F}(\Phi,\Phi'). \tag{2.4.9}$$

Finally, we introduce the functions $\mathrm{F}_r(\Phi,\Phi') = \mathrm{F}_l(\Phi,\Phi') = 1$. With these definitions, we can rewrite Eq. 2.4.1 as

$$\begin{aligned}
&\frac{\sum_{\sigma_0,\sigma_1=0}^{1} \mathrm{G}_z^{\omega,\Lambda_L}(j_0+\sigma_0 e, j_1+\sigma_1 e)}{4}\\
&= i\int \prod_{j=-L}^{L} d\Phi_j\, d\Phi_{j+e}\left(\frac{\psi_{j_0}+\psi_{j_0+e}}{2}\right)\prod_{j=-L}^{j_0-1} e^{-2i\Phi_j\cdot\Phi_{j+1}} e^{-2i\Phi_{j+e}\cdot\Phi_{j+1+e}}\Gamma_{+,\omega_j}(\Phi_j,\Phi_{j+e})\\
&\quad\times \Gamma_{+,\omega_{j_0}}(\Phi_{j_0},\Phi_{j_0+e})\prod_{j=j_0}^{j_1-1} e^{-2i\Phi_j\cdot\Phi_{j+1}} e^{-2i\Phi_{j+e}\cdot\Phi_{j+1+e}}\Gamma_{+,\omega_j}(\Phi_{j+1},\Phi_{j+1+e})\\
&\quad\times \left(\frac{\overline{\psi}_{j_1}+\overline{\psi}_{j_1+e}}{2}\right)\prod_{j=j_1}^{L-1} e^{-2i\Phi_j\cdot\Phi_{j+1}} e^{-2i\Phi_{j+e}\cdot\Phi_{j+1+e}}\Gamma_{+,\omega_j}(\Phi_{j+1},\Phi_{j+1+e})\\
&= i\int d\Phi_{j_0}\, d\Phi_{j_0+e}\left[\mathscr{o}\Gamma_{+,\omega_{j_0}}\mathbf{T}_+\Gamma_{+,\omega_{j_0-1}}\cdots\mathbf{T}_+\Gamma_{+,\omega_{-L}}\mathrm{F}_l\right](\Phi_{j_0},\Phi_{j_0+e})\\
&\quad\times\left[\mathbf{T}_+\Gamma_{+,\omega_{j_0+1}}\cdots\mathbf{T}_+\mathscr{o}^*\Gamma_{+,\omega_{j_1}}\mathbf{T}_+\Gamma_{+,\omega_{j_1+1}}\cdots\mathbf{T}_+\Gamma_{+,\omega_L}\mathrm{F}_r\right](\Phi_{j_0},\Phi_{j_0+e}).
\end{aligned}$$

Note that Γ_{+,ω_j} can be written as

$$\begin{aligned}
\Gamma_{+,\omega_j}(\Phi,\Phi') &= g_{0,+,\omega_j}(|\mathrm{X}|^2+|\mathrm{X}'|^2,\mathrm{X}\cdot\mathrm{X}') + g_{1,+,\omega_j}(|\mathrm{X}|^2+|\mathrm{X}'|^2,\mathrm{X}\cdot\mathrm{X}')\{\overline{\psi}\psi+\overline{\psi}'\psi'\}\\
&+ g_{2,+,\omega_j}(|\mathrm{X}|^2+|\mathrm{X}'|^2,\mathrm{X}\cdot\mathrm{X}')\{\overline{\psi}'\psi+\overline{\psi}\psi'\} + g_{3,+,\omega_j}(|\mathrm{X}|^2+|\mathrm{X}'|^2,\mathrm{X}\cdot\mathrm{X}')\overline{\psi}\psi\overline{\psi}'\psi'
\end{aligned} \tag{2.4.10}$$

for the complex-valued smooth functions $g_{0,+,\omega_j}$, $g_{1,+,\omega_j}$, $g_{2,+,\omega_j}$ and $g_{3,+,\omega_j}$ on $[0,\infty)\times\mathbb{R}$ given in Eq. 2.2.4.

Let F be a function of the form

$$\mathrm{F}(\Phi,\Phi') = f_0(\mathrm{X},\mathrm{X}') + f_1(\mathrm{X},\mathrm{X}')\{\overline{\psi}\psi+\overline{\psi}'\psi'\} + f_2(\mathrm{X},\mathrm{X}')\{\overline{\psi}'\psi+\overline{\psi}\psi'\} + f_3(\mathrm{X},\mathrm{X}')\overline{\psi}\psi\overline{\psi}'\psi'. \tag{2.4.11}$$

By Lemma 2.4.1 below, $[\mathbf{T}_+\mathrm{F}](\Phi,\Phi')$ and $\left[\Gamma_{+,\omega_j}\mathrm{F}\right](\Phi,\Phi')$ have the same representation as Eq. 2.4.11. Moreover, in terms of matrices, $\mathbf{T}_+\Gamma_{+,\omega_j} = \mathcal{T}_{+,\omega_j}$, see Eq. 2.2.3. Indeed,

$$
\begin{aligned}
\begin{pmatrix} f_0 \\ f_1 \\ f_3 \\ f_3 \end{pmatrix} \longmapsto \left[\mathbf{T}_+\Gamma_{+,\omega_j}\right] \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix}
&= \frac{1}{\pi^2}\mathscr{F}_2 \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} g_{0,+,\omega_j} & 0 & 0 & 0 \\ g_{1,+,\omega_j} & g_{0,+,\omega_j} & 0 & 0 \\ g_{2,+,\omega_j} & 0 & g_{0,+,\omega_j} & 0 \\ g_{3,+,\omega_j} & 2g_{1,+,\omega_j} & 2g_{2,+,\omega_j} & g_{0,+,\omega_j} \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix} \\
&= \frac{1}{\pi^2}\mathscr{F}_2 \begin{pmatrix} g_{3,+,\omega_j} & 2g_{1,+,\omega_j} & 2g_{2,+,\omega_j} & g_{0,+,\omega_j} \\ g_{1,+,\omega_j} & g_{0,+,\omega_j} & 0 & 0 \\ -g_{2,+,\omega_j} & 0 & -g_{0,+,\omega_j} & 0 \\ g_{0,+,\omega_j} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix} = \mathcal{T}_{+,\omega_j} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix}.
\end{aligned} \tag{2.4.12}
$$

Recall that $\mathrm{F}_l(\Phi,\Phi') = 1$. If we repeatedly use the above argument, then the term $\left[\Gamma_{+,\omega_{j_0}}\mathbf{T}_+\Gamma_{+,\omega_{j_0-1}}\cdots\mathbf{T}_+\Gamma_{+,\omega_{-L}}\mathrm{F}_l\right](\Phi_{j_0},\Phi_{j_0+e})$ has also the same representation as 2.4.11. Furthermore, it can be written as a product of matrices. Indeed,

$$
\left[\Gamma_{+,\omega_{j_0}}\prod_{j=j_0-1}^{-L}\mathbf{T}_+\Gamma_{+,\omega_j}\right]\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \left[\Gamma_{+,\omega_{j_0}}\prod_{j=j_0-1}^{-L}\mathcal{T}_{+,\omega_j}\right]\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{2.4.13}
$$

Now we apply the observable at j_0 to the above expression. By Lemma 2.4.2 below, we obtain

$$
\begin{aligned}
&\left[\mathscr{O}\Gamma_{+,\omega_{j_0}}\mathbf{T}_+\Gamma_{+,\omega_{j_0-1}}\cdots\mathbf{T}_+\Gamma_{+,\omega_{-L}}\mathrm{F}_l\right](\Phi_{j_0},\Phi_{j_0+e}) = \mathrm{R}_{0,+,\omega_{j_0},\ldots,\omega_{-L}}(\mathrm{X}_{j_0},\mathrm{X}_{j_0+e})\frac{\psi_{j_0}+\psi_{j_0+e}}{2} \\
&+ \mathrm{R}_{1,+,\omega_{j_0},\ldots,\omega_{-L}}(\mathrm{X}_{j_0},\mathrm{X}_{j_0+e})\frac{\psi_{j_0+e}\overline{\psi}_{j_0}\psi_{j_0}+\psi_{j_0}\overline{\psi}_{j_0+e}\psi_{j_0+e}}{2},
\end{aligned} \tag{2.4.14}
$$

where $\mathrm{R}_{0,+,\omega_{j_0},\ldots,\omega_{-L}}$ and $\mathrm{R}_{1,+,\omega_{j_0},\ldots,\omega_{-L}}$ are complex-valued functions on $\mathbb{R}^2\times\mathbb{R}^2$ given by Eq. 2.2.6. Likewise, we can use the previous line of reasoning to compute the term $[\mathscr{O}^*\Gamma_{+,\omega_{j_1}}\mathbf{T}_+\Gamma_{+,\omega_{j_1+1}}\cdots\mathbf{T}_+\Gamma_{+,\omega_L}\mathrm{F}_r](\Phi_{j_1},\Phi_{j_1+e})$. Thus, we get

$$
\begin{aligned}
&\left[\mathscr{O}^*\Gamma_{+,\omega_{j_1}}\mathbf{T}_+\Gamma_{+,\omega_{j_1+1}}\cdots\mathbf{T}_+\Gamma_{+,\omega_L}\mathrm{F}_r\right](\Phi_{j_1},\Phi_{j_1+e}) = \mathrm{R}_{0,+,\omega_{j_1},\ldots,\omega_L}(\mathrm{X}_{j_1},\mathrm{X}_{j_1+e})\frac{\overline{\psi}_{j_1}+\overline{\psi}_{j_1+e}}{2} \\
&\qquad + \mathrm{R}_{1,+,\omega_{j_1},\ldots,\omega_L}(\mathrm{X}_{j_1},\mathrm{X}_{j_1+e})\frac{\overline{\psi}_{j_1+e}\overline{\psi}_{j_1}\psi_{j_1}+\overline{\psi}_{j_1}\overline{\psi}_{j_1+e}\psi_{j_1+e}}{2},
\end{aligned} \tag{2.4.15}
$$

where $\mathrm{R}_{0,+,\omega_{j_1},\ldots,\omega_L}(\mathrm{X}_{j_1},\mathrm{X}_{j_1+e})$ and $\mathrm{R}_{1,+,\omega_{j_1},\ldots,\omega_L}(\mathrm{X}_{j_1},\mathrm{X}_{j_1+e})$ are given in Eq. 2.2.6. Let $\mathrm{R}(\Phi,\Phi')$ be a function of the form

$$
\mathrm{R}(\Phi,\Phi') = r_0(\mathrm{X},\mathrm{X}')\frac{\overline{\psi}+\overline{\psi}'}{2} + r_1(\mathrm{X},\mathrm{X}')\frac{\overline{\psi'}\,\overline{\psi}\psi+\overline{\psi}\,\overline{\psi'}\psi'}{2}. \tag{2.4.16}
$$

Due to Lemma 2.4.2, $[\mathbf{T}_+\mathrm{R}](\Phi,\Phi')$ and $\left[\Gamma_{+,\omega_j}\mathrm{R}\right](\Phi,\Phi')$ have also the same structure as Eq. 2.4.11. Hence, the action of its composition $\mathbf{T}_+\Gamma_{+,\omega_j}$ also preserve the representation. Furthermore, in terms of matrices, $\mathbf{T}_+\Gamma_{+,\omega_j}=\tilde{\mathcal{T}}_{+,\omega_j}$. Indeed,

$$\begin{aligned}\begin{pmatrix}r_0\\r_1\end{pmatrix}\mapsto\mathbf{T}_+\Gamma_{+,\omega_j}\begin{pmatrix}r_0\\r_1\end{pmatrix}&=\frac{ig_{0,+,\omega_j}}{\pi^2}\mathscr{F}_2\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&0\\\alpha_{+,\omega_j}&1\end{pmatrix}\begin{pmatrix}r_0\\r_1\end{pmatrix}\\&=\frac{ig_{0,+,\omega_j}}{\pi^2}\mathscr{F}_2\begin{pmatrix}\alpha_{+,\omega_j}&1\\1&0\end{pmatrix}\begin{pmatrix}r_0\\r_1\end{pmatrix}=\tilde{\mathcal{T}}_{+,\omega_j}\begin{pmatrix}r_0\\r_1\end{pmatrix}.\end{aligned}\tag{2.4.17}$$

Hence,

$$\begin{aligned}&\left[\mathbf{T}_+\Gamma_{+,\omega_{j_0+1}}\cdots\mathbf{T}_+\mathcal{O}^*\Gamma_{+,\omega_{j_1}}\mathbf{T}_+\Gamma_{+,\omega_{j_1+1}}\cdots\mathbf{T}_+\Gamma_{+,\omega_L}\mathrm{F}_r\right](\Phi_{j_0},\Phi_{j_0+e})\\&=\tilde{\mathrm{R}}^{j_0,j_1}_{0,+,\omega_{j_0+1},\dots,\omega_L}(\mathrm{X}_{j_0},\mathrm{X}_{j_0+e})\frac{\overline{\psi}_{j_0}+\overline{\psi}_{j_0+e}}{2}\\&\quad+\tilde{\mathrm{R}}^{j_0,j_1}_{1,+,\omega_{j_0+1},\dots,\omega_L}(\mathrm{X}_{j_0},\mathrm{X}_{j_0+e})\frac{\overline{\psi}_{j_0+e}\overline{\psi}_{j_0}\psi_{j_0}+\overline{\psi}_{j_0}\overline{\psi}_{j_0+e}\psi_{j_0+e}}{2},\end{aligned}\tag{2.4.18}$$

where $\tilde{\mathrm{R}}^{j_0,j_1}_{0,+,\omega_{j_0+1},\dots,\omega_L}$ and $\tilde{\mathrm{R}}^{j_0,j_1}_{1,+,\omega_{j_0+1},\dots,\omega_L}$ are given by Eq. 2.2.6. Therefore,

$$\begin{aligned}&\frac{\sum_{\sigma_0,\sigma_1=0}^1\mathrm{G}_z^{\Lambda_L}(j_0+\sigma_0e,j_1+\sigma_1e)}{4}=i\int_{\mathbb{R}^2\times\mathbb{R}^2}\frac{d^2\mathrm{X}_{j_0}d^2\mathrm{X}_{j_0+e}}{\pi^2}\frac{\partial}{\partial\overline{\psi}_{j_0}}\frac{\partial}{\partial\psi_{j_0}}\frac{\partial}{\partial\overline{\psi}_{j_0+e}}\frac{\partial}{\partial\psi_{j_0+e}}\\&\times\left\{\mathrm{R}_{0,+}(\mathrm{X}_{j_0},\mathrm{X}_{j_0+e})\frac{\psi_{j_0}+\psi_{j_0+e}}{2}+\mathrm{R}_{1,+}(\mathrm{X}_{j_0},\mathrm{X}_{j_0+e})\frac{\psi_{j_0+e}\overline{\psi}_{j_0}\psi_{j_0}+\psi_{j_0}\overline{\psi}_{j_0+e}\psi_{j_0+e}}{2}\right\}\\&\times\left\{\tilde{\mathrm{R}}^{j_0,j_1}_{0,+}(\mathrm{X}_{j_0},\mathrm{X}_{j_0+e})\frac{\overline{\psi}_{j_0}+\overline{\psi}_{j_0+e}}{2}+\tilde{\mathrm{R}}^{j_0,j_1}_{1,+}(\mathrm{X}_{j_0},\mathrm{X}_{j_0+e})\frac{\overline{\psi}_{j_0+e}\overline{\psi}_{j_0}\psi_{j_0}+\overline{\psi}_{j_0}\overline{\psi}_{j_0+e}\psi_{j_0+e}}{2}\right\}\\&=\frac{-i}{2\pi^2}\int_{\mathbb{R}^2\times\mathbb{R}^2}d^2\mathrm{X}_{j_0}d^2\mathrm{X}_{j_0+e}\left[\mathrm{R}_{1,+}\tilde{\mathrm{R}}^{j_0,j_1}_{0,+}+\mathrm{R}_{0,+}\tilde{\mathrm{R}}^{j_0,j_1}_{1,+}\right](\mathrm{X}_{j_0},\mathrm{X}_{j_0+e}),\end{aligned}\tag{2.4.19}$$

where in the second equality we used the anticommutativity of the fermionic variables and the fact that the only products which have all the generators at j_0 and j_0+e are $\frac{\psi_{j_0}+\psi_{j_0+e}}{2}\cdot\frac{\overline{\psi}_{j_0+e}\overline{\psi}_{j_0}\psi_{j_0}+\overline{\psi}_{j_0}\overline{\psi}_{j_0+e}\psi_{j_0+e}}{2}$ and $\frac{\psi_{j_0+e}\overline{\psi}_{j_0}\psi_{j_0}+\psi_{j_0}\overline{\psi}_{j_0+e}\psi_{j_0+e}}{2}\cdot\frac{\overline{\psi}_{j_0}+\overline{\psi}_{j_0+e}}{2}$. Indeed,

$$\begin{aligned}\frac{\psi_{j_0}+\psi_{j_0+e}}{2}\cdot\frac{\overline{\psi}_{j_0+e}\overline{\psi}_{j_0}\psi_{j_0}+\overline{\psi}_{j_0}\overline{\psi}_{j_0+e}\psi_{j_0+e}}{2}&=\frac{\psi_{j_0+e}\overline{\psi}_{j_0+e}\overline{\psi}_{j_0}\psi_{j_0}+\psi_{j_0}\overline{\psi}_{j_0}\overline{\psi}_{j_0+e}\psi_{j_0+e}}{4}\\&=\frac{(-2)\overline{\psi}_{j_0}\psi_{j_0}\overline{\psi}_{j_0+e}\psi_{j_0+e}}{4}=\frac{-\overline{\psi}_{j_0}\psi_{j_0}\overline{\psi}_{j_0+e}\psi_{j_0+e}}{2}.\end{aligned}\tag{2.4.20}$$

By the same token,

$$\frac{\psi_{j_0+e}\overline{\psi}_{j_0}\psi_{j_0}+\psi_{j_0}\overline{\psi}_{j_0+e}\psi_{j_0+e}}{2}\cdot\frac{\overline{\psi}_{j_0}+\overline{\psi}_{j_0+e}}{2}=\frac{-\overline{\psi}_{j_0}\psi_{j_0}\overline{\psi}_{j_0+e}\psi_{j_0+e}}{2}.$$

We take the expectation on the average of the second moment of the Green's function,

$$\begin{aligned}&\mathbb{E}\left[\left|\frac{1}{4}\sum_{\sigma_0,\sigma_1=0}^1\mathrm{G}^{\Lambda_L}_{z,\omega,\lambda}(j_0+\sigma_0e,j_1+\sigma_1e)\right|^2\right]\\&=\frac{1}{4\pi^4}\int_\Omega d\mathbb{P}(\omega)\int_{\mathbb{R}^2\times\mathbb{R}^2}\int_{\mathbb{R}^2\times\mathbb{R}^2}d^2\mathrm{X}^+_{j_0}d^2\mathrm{X}^+_{j_0+e}d^2\mathrm{X}^-_{j_0}d^2\mathrm{X}^-_{j_0+e}\\&\quad\times\left\{\mathrm{R}_{1,+}\tilde{\mathrm{R}}^{j_0,j_1}_{0,+}+\mathrm{R}_{0,+}\tilde{\mathrm{R}}^{j_0,j_1}_{1,+}\right\}\left(\mathrm{X}^+_{j_0},\mathrm{X}^+_{j_0+e}\right)\left\{\mathrm{R}_{1,-}\tilde{\mathrm{R}}^{j_0,j_1}_{0,-}+\mathrm{R}_{0,-}\tilde{\mathrm{R}}^{j_0,j_1}_{1,-}\right\}\left(\mathrm{X}^-_{j_0},\mathrm{X}^-_{j_0+e}\right).\end{aligned}\tag{2.4.21}$$

Remember that the terms $\mathrm{R}_{k,-}$ and $\tilde{\mathrm{R}}_{k,-}^{j_0,j_1}$, for $k \in \{0,1\}$, are written in terms of a product of the matrices $\Gamma_{\varepsilon,\omega_j}$, $\tilde{\Gamma}_{\varepsilon,\omega_j}$, $\mathcal{T}_{\varepsilon,\omega_j}$ and $\tilde{\mathcal{T}}_{\varepsilon,\omega_j}$. The matrix elements of these matrices are of the form

$$g_{k,\varepsilon,\omega_j}(\mathrm{X},\mathrm{X}') = e^{-\eta(|\mathrm{X}|^2+|\mathrm{X}'|^2)} e^{\varepsilon i \mathrm{E}(|\mathrm{X}|^2+|\mathrm{X}'|^2)} e^{-\varepsilon i \mathrm{X}\cdot\mathrm{X}'} (a_{k,\varepsilon}\omega_j^2 + b_{k,\varepsilon}\omega_j + c_{k,\varepsilon}) e^{-i\varepsilon\omega_j(|\mathrm{X}|^2+|\mathrm{X}'|^2)}, \tag{2.4.22}$$

where the coefficients $a_{k,\varepsilon}, b_{k,\varepsilon}$ and $c_{k,\varepsilon}$ are complex constants determined by Eq. 2.2.4. Hence, there are complex polynomials $p_\varepsilon\Big(\{\mathrm{X}_j\}_{j=-L}^{L}, \big\{\mathrm{X}_j'\big\}_{j=-L}^{L}\Big)$ of order 2 in $(\mathrm{X}_j)_l$, $(\mathrm{X}_j')_l$, for $l \in \{1,2\}$, and $q_\varepsilon\Big(\{\omega_j\}_{j=-L}^{L}\Big)$ of order 2 in each ω_j such that

$$\begin{aligned}
&\mathbb{E}\left[\left|\frac{1}{4}\sum_{\sigma_0,\sigma_1=0}^{1} \mathrm{G}_{z,\omega,\lambda}^{\Lambda_L}(j_0+\sigma_0 e, j_1+\sigma_1 e)\right|^2\right] \\
&= \frac{1}{4\pi^4}\int_\Omega \prod_{j=-L}^{L} \mathrm{dP}_0(\omega_j) \int_{(\mathbb{R}^2\times\mathbb{R}^2)^{2L+1}} \prod_{j=-L}^{L} \mathrm{dX}_j^+\,\mathrm{dX}_j^-\,\mathrm{dX}_{j+e}^+\,\mathrm{dX}_{j+e}^- \\
&\times e^{-\eta\sum_{j=-L}^{L}|\mathrm{X}_j^+|^2+|\mathrm{X}_{j+e}^+|^2+|\mathrm{X}_j^-|^2+|\mathrm{X}_{j+e}^-|^2} \\
&\times e^{ip_+\left(\{\mathrm{X}_j^+\}_{j=-L}^{L},\{\mathrm{X}_{j+e}^+\}_{j=-L}^{L}\right)} e^{-ip_-\left(\{\mathrm{X}_j^-\}_{j=-L}^{L},\{\mathrm{X}_{j+e}^-\}_{j=-L}^{L}\right)} \\
&\times q_+\Big(\{\omega_j\}_{j=-L}^{L}\Big) q_-\Big(\{\omega_j\}_{j=-L}^{L}\Big) e^{-i\sum_{j=-L}^{L}\omega_j\left(|\mathrm{X}_j^+|^2+|\mathrm{X}_{j+e}^+|^2-|\mathrm{X}_j^-|^2-|\mathrm{X}_{j+e}^-|^2\right)}
\end{aligned} \tag{2.4.23}$$

To interchange the order of the integrals in the RHS of above equation, it is sufficient to prove that the absolute value of the integrand is integrable. By Fubini-Tonelli, we obtain

$$\begin{aligned}
&\int_\Omega \prod_{j=-L}^{L} \mathrm{dP}_0(\omega_j) \int_{(\mathbb{R}^2\times\mathbb{R}^2)^{2L+1}} \prod_{j=-L}^{L} \mathrm{dX}_j^+\,\mathrm{dX}_j^-\,\mathrm{dX}_{j+e}^+\,\mathrm{dX}_{j+e}^- \\
&\quad\times e^{-\eta\sum_{j=-L}^{L}|\mathrm{X}_j^+|^2+|\mathrm{X}_{j+e}^+|^2+|\mathrm{X}_j^-|^2+|\mathrm{X}_{j+e}^-|^2} \\
&\quad\times \left| q_+\Big(\{\omega_j\}_{j=-L}^{L}\Big) q_-\Big(\{\omega_j\}_{j=-L}^{L}\Big) e^{-i\sum_{j=-L}^{L}\omega_j\left(|\mathrm{X}_j^+|^2+|\mathrm{X}_{j+e}^+|^2-|\mathrm{X}_j^-|^2-|\mathrm{X}_{j+e}^-|^2\right)}\right| \\
&\le \int_{(\mathbb{R}^2\times\mathbb{R}^2)^{2L+1}} \prod_{j=-L}^{L} \mathrm{dX}_j^+\,\mathrm{dX}_j^-\,\mathrm{dX}_{j+e}^+\,\mathrm{dX}_{j+e}^-\, e^{-\eta\sum_{j=-L}^{L}|\mathrm{X}_j^+|^2+|\mathrm{X}_{j+e}^+|^2+|\mathrm{X}_j^-|^2+|\mathrm{X}_{j+e}^-|^2} \\
&\quad\times \mathbb{E}\left[\left| q_+\Big(\{\omega_j\}_{j=-L}^{L}\Big) q_-\Big(\{\omega_j\}_{j=-L}^{L}\Big) e^{-i\sum_{j=-L}^{L}\omega_j\left(|\mathrm{X}_j^+|^2+|\mathrm{X}_{j+e}^+|^2-|\mathrm{X}_j^-|^2-|\mathrm{X}_{j+e}^-|^2\right)}\right|\right].
\end{aligned} \tag{2.4.24}$$

Note that a sufficient condition to assure that the above integral is convergent is that there is a constant $K > 0$ such that

$$\mathbb{E}\left[\left| q_+\Big(\{\omega_j\}_{j=-L}^{L}\Big) q_-\Big(\{\omega_j\}_{j=-L}^{L}\Big) e^{-i\sum_{j=-L}^{L}\omega_j x_j}\right|\right] \le K, \qquad \forall x_j \in \mathbb{R}. \tag{2.4.25}$$

Observe that there are complex constants $\{c_\alpha\}_{\alpha\in\{0,1,2,3,4\}^{2L+1}}$ such that

$$q_+\Big(\{\omega_j\}_{j=-L}^{L}\Big) q_-\Big(\{\omega_j\}_{j=-L}^{L}\Big) e^{-i\sum_{j=-L}^{L}\omega_j x_j} = \sum_{\alpha\in\{0,1,2,3,4\}^{2L+1}} c_\alpha \prod_{j=-L}^{L} \omega_j^{\alpha_j} e^{-i\omega_j}, \tag{2.4.26}$$

where the constants c_α can be written in terms of η, E and i as they depend on the constants $a_{k,\varepsilon}$, $b_{k,\varepsilon}$ and $c_{k,\varepsilon}$, see Eq. 2.4.27. Due to the linearity of the expectation and the independence of $\{\omega_j\}_{j=-L}^{L}$, we obtain

$$\mathbb{E}\Big[q_+\big(\{\omega_j\}_{j=-L}^{L}\big)q_-\big(\{\omega_j\}_{j=-L}^{L}\big)e^{-i\sum_{j=-L}^{L}\omega_j x_j}\Big] = \sum_{\alpha\in\{0,1,2,3,4\}^{2L+1}} c_\alpha \mathbb{E}\left[\prod_{j=-L}^{L} \omega_j^{\alpha_j} e^{-i\omega_j x_j}\right]$$

$$= \sum_{\alpha\in\{0,1,2,3,4\}^{2L+1}} c_\alpha \prod_{j=-L}^{L} \int_{\mathbb{R}} \mathrm{dP}_0(\omega_j)\Big[\omega_j^{\alpha_j} e^{-i\omega_j x_j}\Big]. \tag{2.4.27}$$

Since we assume that P_0 has moments up to order 4, we get

$$\mathbb{E}\Big[q_+\big(\{\omega_j\}_{j=-L}^{L}\big)q_-\big(\{\omega_j\}_{j=-L}^{L}\big)e^{-i\sum_{j=-L}^{L}\omega_j x_j}\Big]$$

$$= \sum_{\alpha\in\{0,1,2,3,4\}^{2L+1}} c_\alpha \prod_{j=-L}^{L} \int_{\mathbb{R}} \mathrm{dP}_0(\omega_j)\omega_j^{\alpha_j} e^{-i\omega_j x_j} = \sum_{\alpha\in\{0,1,2,3,4\}^{2L+1}} c_\alpha \prod_{j=-L}^{L} i^{\alpha_j}\frac{\mathrm{d}^{\alpha_j}}{\mathrm{d}x_j^{\alpha_j}}\hat{\mathrm{P}}_0(x_j). \tag{2.4.28}$$

Note that

$$\left|\frac{\mathrm{d}^{\alpha_j}}{\mathrm{d}x_j^{\alpha_j}}\hat{\mathrm{P}}_0(x_j)\right| = \left|\int_{\mathbb{R}} \mathrm{dP}_0(\omega_j)\omega_j^{\alpha_j} e^{-i\omega_j x_j}\right| \le \max_{\omega_j\in\mathrm{suppP}_0} |\omega_j|^{\alpha_j}, \qquad \forall x_j \in \mathbb{R}. \tag{2.4.29}$$

Therefore, Ineq. 2.4.25 is valid, for $K = \sum_{\alpha\in\{0,1,2,3,4\}^{2L+1}} |c_\alpha| \prod_{j=-L}^{L} \max_{\omega_j\in\mathrm{suppP}_0} |\omega_j|^{\alpha_j}$, by Eq. 2.4.28 and 2.4.29. $\square$

Below we state and provide a proof of some lemmas which were used in the proof of Thm. 2.2.1.

Lemma 2.4.1. *Let* F *be a function of the form*

$$\mathrm{F}(\Phi,\Phi') = f_0(\mathrm{X},\mathrm{X}') + f_1(\mathrm{X},\mathrm{X}')\{\overline{\psi}\psi + \overline{\psi'}\psi'\} + f_2(\mathrm{X},\mathrm{X}')\{\overline{\psi'}\psi + \overline{\psi}\psi'\} + f_3(\mathrm{X},\mathrm{X}')\overline{\psi}\psi\overline{\psi'}\psi', \tag{2.4.30}$$

where $f_0, f_1, f_2, f_3 \in \mathrm{L}^1(\mathbb{R}^2\times\mathbb{R}^2)$. *Then,* $\Gamma_{\varepsilon,\omega_j}\mathrm{F}$ *and* $\mathbf{T}_\varepsilon\mathrm{F}$ *preserve the representation of* F. *More precisely,*

$$\begin{aligned}\left[\Gamma_{\varepsilon,\omega_j}\mathrm{F}\right](\Phi,\Phi') =& g_{0,\varepsilon,\omega_j} f_0 + \left[g_{0,\varepsilon,\omega_j} f_1 + g_{,\varepsilon,\omega_j} f_0\right]\{\overline{\psi}\psi + \overline{\psi'}\psi'\}\\ &+ \left[g_{0,\varepsilon,\omega_j} f_2 + g_{2,\varepsilon,\omega_j} f_0\right]\{\overline{\psi'}\psi + \overline{\psi}\psi'\}\\ &+ \left[g_{3,\varepsilon,\omega_j} f_0 + 2g_{1,\varepsilon,\omega_j} f_1 + 2g_{2,\varepsilon,\omega_j} f_2 + g_{0,\varepsilon,\omega_j} f_3\right]\overline{\psi}\psi\overline{\psi'}\psi', \end{aligned}\tag{2.4.31}$$

$$\begin{aligned}\left[\mathbf{T}_\varepsilon\mathrm{F}\right](\Phi,\Phi') =& \frac{1}{\pi^2}\left[\hat{f}_3(2\varepsilon\mathrm{X}, 2\varepsilon\mathrm{X}') + \hat{f}_2(2\varepsilon\mathrm{X}, 2\varepsilon\mathrm{X}')\{\overline{\psi}\psi + \overline{\psi'}\psi'\}\right]\\ &+ \frac{1}{\pi^2}\left[-\hat{f}_1(2\varepsilon\mathrm{X}, 2\varepsilon\mathrm{X}')\{\overline{\psi'}\psi + \overline{\psi}\psi'\} + \hat{f}_0(2\varepsilon\mathrm{X}, 2\varepsilon\mathrm{X}')\overline{\psi}\psi\overline{\psi'}\psi'\right],\end{aligned} \tag{2.4.32}$$

where $\hat{f}_j$ *denotes the Fourier transform given by*

$$\hat{f}_j(\mathrm{X},\mathrm{X}') = \int_{\mathbb{R}^2\times\mathbb{R}^2} \mathrm{dY}\,\mathrm{dY}'\, e^{-i\mathrm{X}\cdot\mathrm{Y}} e^{-i\mathrm{X}'\cdot\mathrm{Y}'} f_j(\mathrm{Y},\mathrm{Y}'), \qquad \forall \mathrm{X},\mathrm{X}' \in \mathbb{R}^2. \tag{2.4.33}$$

Equivalently, we can also represent the action of Γ_{+,ω_j} and $\mathbf{T}$ on F via matrices as follows:

$$\begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix} \longmapsto \Gamma_{\varepsilon,\omega_j} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} g_{0,\varepsilon,\omega_j} f_0 \\ g_{0,\varepsilon,\omega_j} f_1 + g_{1,\varepsilon,\omega_j} f_0 \\ g_{0,\varepsilon,\omega_j} f_2 + g_{2,\varepsilon,\omega_j} f_0 \\ g_{3,\varepsilon,\omega_j} f_0 + 2g_{1,\varepsilon,\omega_j} f_1 + 2g_{2,\varepsilon,\omega_j} f_2 + g_{0,\varepsilon,\omega_j} f_3 \end{pmatrix}$$

$$= \begin{pmatrix} g_{0,\varepsilon,\omega_j} & 0 & 0 & 0 \\ g_{1,\varepsilon,\omega_j} & g_{0,\varepsilon,\omega_j} & 0 & 0 \\ g_{2,\varepsilon,\omega_j} & 0 & g_{0,\varepsilon,\omega_j} & 0 \\ g_{3,\varepsilon,\omega_j} & 2g_{1,\varepsilon,\omega_j} & 2g_{2,\varepsilon,\omega_j} & g_{0,\varepsilon,\omega_j} \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix}, \tag{2.4.34}$$

$$\begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix} \longmapsto \mathbf{T}_\varepsilon \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix} = \frac{1}{\pi^2}\mathscr{F}_{2\varepsilon} \begin{pmatrix} f_3 \\ f_1 \\ -f_2 \\ f_0 \end{pmatrix} = \frac{1}{\pi^2}\mathscr{F}_{2\varepsilon} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix}, \tag{2.4.35}$$

where $\mathscr{F}_{2\varepsilon}$ denotes the Fourier transform after scaling by 2ε.

Proof. Let F be of the form 2.4.30. By a direct computation, $\Gamma_{+,\omega_j}\mathrm{F}$ has the desired representation. Hence, we only focus on the expression $[\mathbf{T}_\varepsilon \mathrm{F}](\Phi, \Phi')$, for $\varepsilon \in \{-1, 1\}$. Let $\Theta = \begin{pmatrix} \mathrm{Y} \\ \overline{\chi} \\ \chi \end{pmatrix}$ and $\Theta' = \begin{pmatrix} \mathrm{Y}' \\ \overline{\chi}' \\ \chi' \end{pmatrix}$ be supervectors. Using the representation of F, we get

$$[\mathbf{T}_\varepsilon \mathrm{F}](\Phi, \Phi') = \int \mathrm{d}\Theta \mathrm{d}\Theta' e^{-2\varepsilon i \Phi\cdot\Theta} e^{-2\varepsilon i \Phi'\cdot\Theta'} \mathrm{F}(\Theta, \Theta')$$
$$= \int \mathrm{d}\Theta \mathrm{d}\Theta' e^{-2\varepsilon i \Phi\cdot\Theta} e^{-2\varepsilon i \Phi'\cdot\Theta'} \big(f_0 + f_1\{\overline{\chi}\chi + \overline{\chi}'\chi'\} + f_2\{\overline{\chi}'\chi + \overline{\chi}\chi'\} + f_3\overline{\chi}\chi\overline{\chi}'\chi'\big).$$

By linearity of the integral, we can separately compute each addend. To make the computations below more succinct, note that the following equations are valid

$$\frac{\partial}{\partial\overline{\chi}}\frac{\partial}{\partial\chi} e^{-\varepsilon i(\overline{\psi}\chi + \overline{\chi}\psi)} = \frac{\partial}{\partial\overline{\chi}}\frac{\partial}{\partial\chi}\{1 - \varepsilon i(\overline{\psi}\chi + \overline{\chi}\psi) - \overline{\psi}\chi\overline{\chi}\psi\} = -\overline{\psi}\psi, \tag{2.4.36}$$

$$\frac{\partial}{\partial\overline{\chi}}\frac{\partial}{\partial\chi} e^{-\varepsilon i(\overline{\psi}\chi + \overline{\chi}\psi)}\overline{\chi} = \frac{\partial}{\partial\overline{\chi}}\frac{\partial}{\partial\chi}\{1 - \varepsilon i(\overline{\psi}\chi + \overline{\chi}\psi) - \overline{\psi}\chi\overline{\chi}\psi\}\overline{\chi} = -\varepsilon i\overline{\psi}, \tag{2.4.37}$$

$$\frac{\partial}{\partial\overline{\chi}}\frac{\partial}{\partial\chi} e^{-\varepsilon i(\overline{\psi}\chi + \overline{\chi}\psi)}\overline{\chi}\chi = \frac{\partial}{\partial\overline{\chi}}\frac{\partial}{\partial\chi}\{1 - \varepsilon i(\overline{\psi}\chi + \overline{\chi}\psi) - \overline{\psi}\chi\overline{\chi}\psi\}\overline{\chi}\chi = -1, \tag{2.4.38}$$

By linearity of the integral, we can separately compute each addend. To begin with,

$$\int \mathrm{d}\Theta \mathrm{d}\Theta' e^{-2\varepsilon i \Phi\cdot\Theta} e^{-2\varepsilon i \Phi'\cdot\Theta'} f_0(\mathrm{Y}, \mathrm{Y}')$$
$$= \int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{\mathrm{d}^2\mathrm{Y}\mathrm{d}^2\mathrm{Y}'}{\pi^2} e^{-2\varepsilon i \mathrm{X}\cdot\mathrm{Y}} e^{-2\varepsilon i \mathrm{X}'\cdot\mathrm{Y}'} f_0(\mathrm{Y}, \mathrm{Y}') \frac{\partial}{\partial\overline{\chi}}\frac{\partial}{\partial\chi} e^{-\varepsilon i(\overline{\psi}\chi + \overline{\chi}\psi)} \frac{\partial}{\partial\overline{\chi}'}\frac{\partial}{\partial\chi'} e^{-\varepsilon i(\overline{\psi}'\chi' + \overline{\chi}'\psi')}$$
$$= \int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{\mathrm{d}^2\mathrm{Y}\mathrm{d}^2\mathrm{Y}'}{\pi^2} e^{-2\varepsilon i \mathrm{X}\cdot\mathrm{Y}} e^{-2\varepsilon i \mathrm{X}'\cdot\mathrm{Y}'} f_0(\mathrm{Y}, \mathrm{Y}')\{-\overline{\psi}\psi\}\{-\overline{\psi}'\psi'\}$$
$$= \frac{1}{\pi^2}\hat{f}_0(2\varepsilon\mathrm{X}, 2\varepsilon\mathrm{X}')\overline{\psi}\psi\overline{\psi}'\psi', \tag{2.4.39}$$

where we used Eq. 2.4.36 twice in the second line. Next,

$$\begin{aligned}
&\int \mathrm{d}\Theta\mathrm{d}\Theta' e^{-2\varepsilon i\Phi\cdot\Theta} e^{-2\varepsilon i\Phi'\cdot\Theta'} f_1(\mathrm{Y},\mathrm{Y}')\overline{\chi}\chi \\
&= \int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{\mathrm{d}^2\mathrm{Y}\mathrm{d}^2\mathrm{Y}'}{\pi^2} e^{-2\varepsilon i\mathrm{X}\cdot\mathrm{Y}} e^{-2\varepsilon i\mathrm{X}'\cdot\mathrm{Y}'} f_1(\mathrm{Y},\mathrm{Y}')\frac{\partial}{\partial\overline{\chi}}\frac{\partial}{\partial\chi} e^{-\varepsilon i(\overline{\psi}\chi+\overline{\chi}\psi)}\overline{\chi}\chi\frac{\partial}{\partial\overline{\chi}'}\frac{\partial}{\partial\chi'} e^{-\varepsilon i(\overline{\psi}'\chi'+\overline{\chi}'\psi')} \\
&= \int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{\mathrm{d}^2\mathrm{Y}\mathrm{d}^2\mathrm{Y}'}{\pi^2} e^{-2\varepsilon i\mathrm{X}\cdot\mathrm{Y}} e^{-2\varepsilon i\mathrm{X}'\cdot\mathrm{Y}'} f_1(\mathrm{Y},\mathrm{Y}')(-1)^2\overline{\psi}'\psi' = \frac{1}{\pi^2}\hat{f}_1(2\varepsilon\mathrm{X},2\varepsilon\mathrm{X}')\overline{\psi}'\psi'.
\end{aligned} \tag{2.4.40}$$

where we used Eq. 2.4.36 and 2.4.38 in the second line. By the same token,

$$\int \mathrm{d}\Theta\mathrm{d}\Theta' e^{-2\varepsilon i\Phi\cdot\Theta} e^{-2\varepsilon i\Phi'\cdot\Theta'} f_1(\mathrm{Y},\mathrm{Y}')\overline{\chi}'\chi' = \frac{1}{\pi^2}\hat{f}_1(2\varepsilon\mathrm{X},2\varepsilon\mathrm{X}')\overline{\psi}\psi. \tag{2.4.41}$$

Therefore,

$$\int \mathrm{d}\Theta\mathrm{d}\Theta' e^{-2\varepsilon i\Phi\cdot\Theta} e^{-2\varepsilon i\Phi'\cdot\Theta'} f_1(\mathrm{Y},\mathrm{Y}')\{\overline{\chi}\chi+\overline{\chi}'\chi'\} = \frac{1}{\pi^2}\hat{f}_1(2\varepsilon\mathrm{X},2\varepsilon\mathrm{X}')\{\overline{\psi}\psi+\overline{\psi}'\psi'\}. \tag{2.4.42}$$

Next,

$$\begin{aligned}
&\int \mathrm{d}\Theta\mathrm{d}\Theta' e^{-2\varepsilon i\Phi\cdot\Theta} e^{-2\varepsilon i\Phi'\cdot\Theta'} f_2(\mathrm{Y},\mathrm{Y}')\overline{\chi}\chi' \\
&= \int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{\mathrm{d}^2\mathrm{Y}\mathrm{d}^2\mathrm{Y}'}{\pi^2} e^{-2\varepsilon i\mathrm{X}\cdot\mathrm{Y}} e^{-2\varepsilon i\mathrm{X}'\cdot\mathrm{Y}'} f_2(\mathrm{Y},\mathrm{Y}')\frac{\partial}{\partial\overline{\chi}}\frac{\partial}{\partial\chi} e^{-\varepsilon i(\overline{\psi}\chi+\overline{\chi}\psi)}\overline{\chi}\frac{\partial}{\partial\overline{\chi}'}\frac{\partial}{\partial\chi'} e^{-\varepsilon i(\overline{\psi}'\chi'+\overline{\chi}'\psi')}\chi' \\
&= \int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{\mathrm{d}^2\mathrm{Y}\mathrm{d}^2\mathrm{Y}'}{\pi^2} e^{-2\varepsilon i\mathrm{X}\cdot\mathrm{Y}} e^{-2\varepsilon i\mathrm{X}'\cdot\mathrm{Y}'} f_2(\mathrm{Y},\mathrm{Y}')(-\varepsilon i\overline{\psi})(-\varepsilon i\psi') = \frac{-1}{\pi^2}\hat{f}_2(2\varepsilon\mathrm{X},2\varepsilon\mathrm{X}')\overline{\psi}\psi',
\end{aligned} \tag{2.4.43}$$

where we used Eq. 2.4.37 twice in the second line. Likewise,

$$\int \mathrm{d}\Theta\mathrm{d}\Theta' e^{-2\varepsilon i\Phi\cdot\Theta} e^{-2\varepsilon i\Phi'\cdot\Theta'} f_2(\mathrm{Y},\mathrm{Y}')\overline{\chi}'\chi = \frac{-1}{\pi^2}\hat{f}_2(2\varepsilon\mathrm{X},2\varepsilon\mathrm{X}')\overline{\psi}'\psi. \tag{2.4.44}$$

Hence,

$$\int \mathrm{d}\Theta\mathrm{d}\Theta' e^{-2\varepsilon i\Phi\cdot\Theta} e^{-2\varepsilon i\Phi'\cdot\Theta'} f_2(\mathrm{Y},\mathrm{Y}')\{\overline{\chi}\chi'+\overline{\chi}'\chi\} = \frac{-1}{\pi^2}\hat{f}_2(2\varepsilon\mathrm{X},2\varepsilon\mathrm{X}')\{\overline{\psi}\psi'+\overline{\psi}'\psi\}. \tag{2.4.45}$$

Finally, the last addend can be computed as follows

$$\begin{aligned}
&\int \mathrm{d}\Theta\mathrm{d}\Theta' e^{-2\varepsilon i\Phi\cdot\Theta} e^{-2\varepsilon i\Phi'\cdot\Theta'} f_3(\mathrm{Y},\mathrm{Y}')\overline{\chi}\chi\overline{\chi}'\chi' \\
&= \int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{\mathrm{d}^2\mathrm{Y}\mathrm{d}^2\mathrm{Y}'}{\pi^2} e^{-2\varepsilon i\mathrm{X}\cdot\mathrm{Y}} e^{-2\varepsilon i\mathrm{X}'\cdot\mathrm{Y}'} f_3(\mathrm{Y},\mathrm{Y}')\frac{\partial}{\partial\overline{\chi}}\frac{\partial}{\partial\chi} e^{-\varepsilon i(\overline{\psi}\chi+\overline{\chi}\psi)}\overline{\chi}\chi\frac{\partial}{\partial\overline{\chi}'}\frac{\partial}{\partial\chi'} e^{-\varepsilon i(\overline{\psi}'\chi'+\overline{\chi}'\psi')}\overline{\chi}'\chi' \\
&= \int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{\mathrm{d}^2\mathrm{Y}\mathrm{d}^2\mathrm{Y}'}{\pi^2} e^{-2\varepsilon i\mathrm{X}\cdot\mathrm{Y}} e^{-2\varepsilon i\mathrm{X}'\cdot\mathrm{Y}'} f_3(\mathrm{Y},\mathrm{Y}')(-1)^2 = \frac{1}{\pi^2}\hat{f}_3(2\varepsilon\mathrm{X},2\varepsilon\mathrm{X}'),
\end{aligned} \tag{2.4.46}$$

where we used Eq. 2.4.38 twice in the second line. □

Lemma 2.4.2. *Let* F *of the form* (2.4.30), *then*

$$\frac{\overline{\psi}+\overline{\psi}'}{2}\mathrm{F}(\Phi,\Phi')=f_0(\mathrm{X},\mathrm{X}')\frac{\overline{\psi}+\overline{\psi}'}{2}+(f_1(\mathrm{X},\mathrm{X}')-f_2(\mathrm{X},\mathrm{X}'))\frac{\overline{\psi}'\,\overline{\psi}\psi+\overline{\psi}\,\overline{\psi}'\psi'}{2}. \tag{2.4.47}$$

Let R *be a function of the form*

$$\mathrm{R}(\Phi,\Phi')=r_0(\mathrm{X},\mathrm{X}')\frac{\overline{\psi}+\overline{\psi}'}{2}+r_1(\mathrm{X},\mathrm{X}')\frac{\overline{\psi}'\,\overline{\psi}\psi+\overline{\psi}\,\overline{\psi}'\psi'}{2}, \tag{2.4.48}$$

where $r_0,r_1\in \mathrm{L}^1(\mathbb{R}^2\times\mathbb{R}^2)$. *Let* $\alpha_{\omega_j}=i(-\omega_j+\mathrm{E}+i\eta-1)$. *Then,*

$$\Gamma_{+,\omega_j}\mathrm{R}(\Phi,\Phi')=g_{0,\varepsilon,\omega_j}(\mathrm{X},\mathrm{X}')\left[r_0(\mathrm{X},\mathrm{X}')\frac{\overline{\psi}+\overline{\psi}'}{2}+(\alpha_{\omega_j}r_0(\mathrm{X},\mathrm{X}')+r_1(\mathrm{X},\mathrm{X}'))\frac{\overline{\psi}'\,\overline{\psi}\psi+\overline{\psi}\,\overline{\psi}'\psi'}{2}\right], \tag{2.4.49}$$

$$\mathbf{T}_\varepsilon\mathrm{R}(\Phi,\Phi')=\frac{\varepsilon i}{\pi^2}\left[\hat{r}_1(2\varepsilon\mathrm{X},2\varepsilon\mathrm{X}')\frac{\overline{\psi}+\overline{\psi}'}{2}+\hat{r}_0(2\varepsilon\mathrm{X},2\varepsilon\mathrm{X}')\frac{\overline{\psi}'\,\overline{\psi}\psi+\overline{\psi}\,\overline{\psi}'\psi'}{2}\right]. \tag{2.4.50}$$

Equivalently, we can also represent the action of Γ_{+,ω_j} *and* $\mathbf{T}_\varepsilon$ *on* R *via matrices as follows:*

$$\begin{pmatrix}r_0\\r_1\end{pmatrix}\longmapsto\Gamma_{+,\omega_j}\begin{pmatrix}r_0\\r_1\end{pmatrix}=\begin{pmatrix}g_{0,\varepsilon,\omega_j}r_0\\g_{0,\varepsilon,\omega_j}(\alpha r_0+r_1)\end{pmatrix}=g_{0,\varepsilon,\omega_j}\begin{pmatrix}1&0\\\alpha&1\end{pmatrix}\begin{pmatrix}r_0\\r_1\end{pmatrix}, \tag{2.4.51}$$

$$\begin{pmatrix}r_0\\r_1\end{pmatrix}\longmapsto\mathbf{T}_\varepsilon\begin{pmatrix}r_0\\r_1\end{pmatrix}=\frac{\varepsilon i}{\pi^2}\mathscr{F}_{2\varepsilon}\begin{pmatrix}r_1\\r_0\end{pmatrix}=\frac{\varepsilon i}{\pi^2}\mathscr{F}_{2\varepsilon}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}r_0\\r_1\end{pmatrix}, \tag{2.4.52}$$

where $\mathscr{F}_{2\varepsilon}$ *denotes the Fourier transform after scaling by* 2ε.

Proof. Let F and R be functions of the form 2.4.30 and 2.4.48, respectively. By direct computations, $\frac{\overline{\psi}+\overline{\psi}'}{2}\mathrm{F}(\Phi,\Phi')$ and $\Gamma_{+,\omega_j}\mathrm{R}(\Phi,\Phi')$ have the desired representations. Hence, we only focus on the expression $\mathbf{T}_\varepsilon\mathrm{R}(\Phi,\Phi')$. Note

$$\begin{aligned}\mathbf{T}_\varepsilon\mathrm{R}(\Phi,\Phi')&=\int \mathrm{d}\Theta\mathrm{d}\Theta' e^{-2\varepsilon i\Phi\cdot\Theta}e^{-2\varepsilon i\Phi'\cdot\Theta'}\mathrm{R}(\Theta,\Theta')\\&=\int \mathrm{d}\Theta\mathrm{d}\Theta' e^{-2\varepsilon i\Phi\cdot\Theta}e^{-2\varepsilon i\Phi'\cdot\Theta'}\left\{r_0(\mathrm{Y},\mathrm{Y}')\frac{\overline{\chi}+\overline{\chi}'}{2}+r_1(\mathrm{Y},\mathrm{Y}')\frac{\overline{\chi}'\,\overline{\chi}\chi+\overline{\chi}\,\overline{\chi}'\chi'}{2}\right\}.\end{aligned} \tag{2.4.53}$$

Due to the linearity of the integral, we can separately compute each addend. On the one hand,

$$\begin{aligned}&\int \mathrm{d}\Theta\mathrm{d}\Theta' e^{-2i\Phi\cdot\Theta}e^{-2i\Phi'\cdot\Theta'}r_0(\mathrm{Y},\mathrm{Y}')\frac{\overline{\chi}}{2}\\&=\int_{\mathbb{R}^2\times\mathbb{R}^2}\frac{\mathrm{d}^2\mathrm{Y}\mathrm{d}^2\mathrm{Y}'}{\pi^2}e^{-2\varepsilon i\mathrm{X}\cdot\mathrm{Y}}e^{-2\varepsilon i\mathrm{X}'\cdot\mathrm{Y}'}r_0(\mathrm{Y},\mathrm{Y}')\frac{\partial}{\partial\overline{\chi}}\frac{\partial}{\partial\chi}e^{-\varepsilon i(\overline{\psi}\chi+\overline{\chi}\psi)}\frac{\overline{\chi}}{2}\frac{\partial}{\partial\overline{\chi}'}\frac{\partial}{\partial\chi'}e^{-\varepsilon i(\overline{\psi}'\chi'+\overline{\chi}'\psi')}\\&=\int_{\mathbb{R}^2\times\mathbb{R}^2}\frac{\mathrm{d}^2\mathrm{Y}\mathrm{d}^2\mathrm{Y}'}{\pi^2}e^{-2\varepsilon i\mathrm{X}\cdot\mathrm{Y}}e^{-2\varepsilon i\mathrm{X}'\cdot\mathrm{Y}'}r_0(\mathrm{Y},\mathrm{Y}')\frac{(-\varepsilon i\overline{\psi})}{2}(-\overline{\psi}'\psi')=\frac{\varepsilon i}{\pi^2}\hat{r}_0(2\varepsilon\mathrm{X},2\varepsilon\mathrm{X}')\frac{\overline{\psi}\,\overline{\psi}'\psi'}{2},\end{aligned}$$

where we used Eq. 2.4.36 and 2.4.37 in the second step. By the same token,

$$\int \mathrm{d}\Theta\mathrm{d}\Theta' e^{-2i\Phi\cdot\Theta} e^{-2i\Phi'\cdot\Theta'} r_0(\mathrm{Y},\mathrm{Y}')\frac{\overline{\chi'}}{2} = \frac{\varepsilon i}{\pi^2}\hat{r}_0(2\varepsilon\mathrm{X},2\varepsilon\mathrm{X}')\frac{\overline{\psi'}\,\overline{\psi}\psi}{2}.$$

Hence,

$$\int \mathrm{d}\Theta\mathrm{d}\Theta' e^{-2\varepsilon i\Phi\cdot\Theta} e^{-2\varepsilon i\Phi'\cdot\Theta'} r_0(\mathrm{Y},\mathrm{Y}')\frac{\overline{\chi}+\overline{\chi'}}{2} = \frac{\varepsilon i}{\pi^2}\hat{r}_0(2\varepsilon\mathrm{X},2\varepsilon\mathrm{X}')\left(\frac{\overline{\psi'}\,\overline{\psi}\psi}{2}+\frac{\overline{\psi}\,\overline{\psi'}\psi'}{2}\right). \tag{2.4.54}$$

On the other hand,

$$\begin{aligned}
&\int \mathrm{d}\Theta\mathrm{d}\Theta' e^{-2\varepsilon i\Phi\cdot\Theta} e^{-2\varepsilon i\Phi'\cdot\Theta'} r_1(\mathrm{Y},\mathrm{Y}')\frac{\overline{\chi'}\,\overline{\chi}\chi}{2}\\
&=\int_{\mathbb{R}^2\times\mathbb{R}^2}\frac{\mathrm{d}^2\mathrm{Y}\mathrm{d}^2\mathrm{Y}'}{\pi^2}e^{-2\varepsilon i\mathrm{X}\cdot\mathrm{Y}}e^{-2\varepsilon i\mathrm{X}'\cdot\mathrm{Y}'}r_1(\mathrm{Y},\mathrm{Y}')\frac{\partial}{\partial\overline{\chi}}\frac{\partial}{\partial\chi}e^{-\varepsilon i\left(\overline{\psi}\chi+\overline{\chi}\psi\right)}\frac{\overline{\chi}\chi}{2}\frac{\partial}{\partial\overline{\chi'}}\frac{\partial}{\partial\chi'}e^{-\varepsilon i\left(\overline{\psi'}\chi'+\overline{\chi'}\psi'\right)}\overline{\chi'}\\
&=\int_{\mathbb{R}^2\times\mathbb{R}^2}\frac{\mathrm{d}^2\mathrm{Y}\mathrm{d}^2\mathrm{Y}'}{\pi^2}e^{-2\varepsilon i\mathrm{X}\cdot\mathrm{Y}}e^{-2\varepsilon i\mathrm{X}'\cdot\mathrm{Y}'}r_1(\mathrm{Y},\mathrm{Y}')\frac{(-1)}{2}(-\varepsilon i\overline{\psi'}) = \frac{\varepsilon i}{\pi^2}\hat{r}_1(2\varepsilon\mathrm{X},2\varepsilon\mathrm{X}')\frac{\overline{\psi'}}{2},
\end{aligned} \tag{2.4.55}$$

where we used Eq. 2.4.37 and 2.4.38 in the third line. Analogously,

$$\int \mathrm{d}\Theta\mathrm{d}\Theta' e^{-2\varepsilon i\Phi\cdot\Theta} e^{-2\varepsilon i\Phi'\cdot\Theta'} r_1(\mathrm{Y},\mathrm{Y}')\frac{\overline{\chi}\,\overline{\chi'}\chi'}{2} = \frac{\varepsilon i}{\pi^2}\hat{r}_1(2\varepsilon\mathrm{X},2\varepsilon\mathrm{X}')\frac{\overline{\psi}}{2}. \tag{2.4.56}$$

It follows

$$\int \mathrm{d}\Theta\mathrm{d}\Theta' e^{-2\varepsilon i\Phi\cdot\Theta} e^{-2\varepsilon i\Phi'\cdot\Theta'} r_1(\mathrm{Y},\mathrm{Y}')\frac{\overline{\chi'}\,\overline{\chi}\chi+\overline{\chi}\,\overline{\chi'}\chi'}{2} = \frac{\varepsilon i}{\pi^2}\hat{r}_1(2\varepsilon\mathrm{X},2\varepsilon\mathrm{X}')\frac{\overline{\psi}+\overline{\psi'}}{2}. \tag{2.4.57}$$

□

2.5 Discussion

As we pointed out in Section 2.1, we did not manage to fully replicate the strategy laid out in [KMP86] to prove localization in our model because challenges arose when we tried to implement it.

2.5.1 Klein, Martinelli and Perez's strategy

Klein, Martinelli and Perez worked on an instance of the Anderson model with $\mathrm{d} = 1$. Let $\lambda > 0$,

$$\mathrm{H}_{\omega,\lambda} = -\Delta + \lambda\mathrm{V}_\omega, \quad \omega\in\Omega \tag{2.5.1}$$

acting on $\ell^2(\mathbb{Z})$, where $-\Delta$ is the usual (negative) discrete Laplacian and V_ω is a multiplication operator given by

$$\mathrm{V}_\omega u(x) = \omega_x u(x), \qquad \forall u\in\ell^2(\mathbb{Z}),\ \forall x\in\mathbb{Z},$$

where $\omega := (\omega_j)_{j\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ being a family of bounded iid random variables defined in the probability space $\Omega := \mathbb{R}^{\mathbb{Z}}$ equipped with the Borel probability measure $\mathbb{P}_1 := \bigotimes_{\mathbb{Z}} \mathrm{P}_1$ with expectation $\mathbb{E}_1[\cdot]$. Recall that the Fourier transform of P_1 is defined as $\hat{\mathrm{P}}_1(x) := \mathbb{E}_1\left[e^{-i\omega_j x}\right]$, for all $x \in \mathbb{R}$. We assume that P_1 has at least moments up to order 2 at least.

Let $\tilde{\Lambda}_L := \{-L, \dots, L\}$, for $L \in \mathbb{N}$. Under some assumptions of the integrability of $\hat{\mathrm{P}}_1$, they proved that there are some constants $K > 0$, $r > 0$ such that $\forall L \in \mathbb{N}$ there is $\eta_0 > 0$

$$\mathbb{E}_1\left[\left|\mathrm{G}^{\tilde{\Lambda}_L}_{\mathrm{E}+i\eta,\lambda}(x_0,x)\right|^2\right] \leq \frac{K}{\eta^2}\left(1 - \frac{\ln 2}{|\ln\eta|^r} + \mathcal{O}\left(\frac{1}{|\ln\eta|^{2r}}\right)\right)^{|x-x_0|}, \quad \forall L \in \mathbb{N},\ \forall x, x_0 \in \tilde{\Lambda}_L,\ \forall \eta \in (0,\eta_0], \tag{2.5.2}$$

which is the content of [KMP86, Main Thm.]. This result together with the assumption that *for every* $\eta > 0$, $\beta > 0$ *and* $\mathrm{E}_0 > 0$, *there is* $\delta > 0$ *and* $\mathrm{L}_0 > 0$ *such that*

$$\mathbb{P}_1\left(\mathrm{dist}\left(E, \sigma\left(\mathrm{H}^{\tilde{\Lambda}_L}_{\lambda}\right)\right) < e^{-\alpha L^\beta}\right) \leq e^{-\delta L^\beta},\ \forall E \in [-\mathrm{E}_0, \mathrm{E}_0], \forall L \geq \mathrm{L}_0 \tag{2.5.3}$$

yield Anderson localization, see [KMP86, Corollary 3].

The proof of Main Thm. in [KMP86] can be decomposed into two parts: in the first half, the author used the SUSY representation of $\mathbb{E}_1\left[\left|\mathrm{G}^{\tilde{\Lambda}_L}_{\mathrm{E}+i\eta,\lambda}(x_0,x)\right|^2\right]$ to obtain

$$\mathbb{E}_1\left[\left|\mathrm{G}^{\tilde{\Lambda}_L}_{\mathrm{E}+i\eta,\lambda}(x_0,x)\right|^2\right] = \int \mathrm{d}\Phi^+_{x_0}\mathrm{d}\Phi^-_{x_0}[\tilde{\sigma}(\tilde{T}\tilde{\Gamma})^{L-x_0}F_l](\Phi^+_{x_0},\Phi^-_{x_0})e^{-\frac{\eta}{2}({\Phi^+_{x_0}}^2+{\Phi^-_{x_0}}^2)} \times \left[\tilde{\Gamma}(\tilde{T}\tilde{\Gamma})^{x-x_0}\tilde{\sigma}^*(\tilde{T}\tilde{\Gamma})^{L-x}F_r\right](\Phi^+_{x_0},\Phi^-_{x_0}), \tag{2.5.4}$$

where

$$\begin{aligned}
[\tilde{T}\mathrm{F}](\Phi,\Phi') &= \int \mathrm{d}\Theta\mathrm{d}\Theta' e^{-i\Phi\cdot\Theta}e^{i\Phi'\cdot\Theta'}\mathrm{F}(\Theta,\Theta'),\\
[\tilde{\Gamma}\mathrm{F}](\Phi,\Phi') &= e^{-\frac{\eta}{2}(\Phi^2+\Phi'^2)}\hat{\mathrm{P}}_1(\Phi^2-\Phi'^2)e^{i\mathrm{E}(\Phi^2-\Phi'^2)}F(\Phi,\Phi'),\\
[\tilde{\sigma}F](\Phi^+,\Phi^-) &= \psi^+\overline{\psi}^-F(\Phi^+,\Phi^-), \qquad [\tilde{\sigma}^*F](\Phi^+,\Phi^-) = \psi^+\overline{\psi}^-F(\Phi^+,\Phi^-),\\
F_l(\Phi_+^2,\Phi_-^2) &= F_r(\Phi_+^2,\Phi_-^2) = e^{-\frac{\eta}{2}(\Phi_+^2+\Phi_-^2)}.
\end{aligned} \tag{2.5.5}$$

To obtain Eq. 2.5.4, Klein, Martinelli and Perez had to interchange the integral with respect to the supervectors with the expectation. As we did in the proof of Thm. 2.2.1, we need to proof that the term $\hat{\mathrm{P}}_1(\Phi^2-\Phi'^2)$ is well-defined. Moreover, $\mathbb{E}_1\left[e^{-i\omega_j(\Phi^2-\Phi'^2)}\right]$ is also well-defined and

$$\hat{\mathrm{P}}_1(\Phi^2-\Phi'^2) = \mathbb{E}\left[e^{-i\omega_j(\Phi^2-\Phi'^2)}\right]. \tag{2.5.6}$$

Indeed, let $\Phi^2 - \Phi'^2 = |\mathrm{X}|^2 - |\mathrm{X}'|^2 + \overline{\psi}\psi - \overline{\psi'}\psi' := x + n$, where $x \in \mathbb{R}$ and $n \in \mathfrak{G}_0$ with $n^3 = 0$. Then

$$\mathbb{E}_1\Big[e^{-i\omega_j(x+n)}\Big] = \int_{\mathbb{R}} \mathrm{dP}_1(\omega_j)\, e^{-i\omega_j(x+n)} = \int_{\mathbb{R}} \mathrm{dP}_1(\omega_j)\, e^{-i\omega_j x}\left[1 + \sum_{l\geq 1} \frac{(-i\omega_j)^l}{l!} n^l\right]$$
$$= \hat{\mathrm{P}}_1(x) + n \int_{\mathbb{R}} \mathrm{dP}_1(\omega_j)\,(-i\omega_j)e^{-i\omega_j x} + \frac{n^2}{2}\int_{\mathbb{R}} \mathrm{dP}_1(\omega_j)\,(-i\omega_j)^2 e^{-i\omega_j x}$$
$$= \hat{\mathrm{P}}_1(x) + \left(\frac{\mathrm{d}^l}{\mathrm{d}x^l}\hat{\mathrm{P}}_1(x)\right)n + \frac{1}{2}\left(\frac{\mathrm{d}^l}{\mathrm{d}x^l}\hat{\mathrm{P}}_1(x)\right)n^2 = \hat{\mathrm{P}}_1(x+n), \tag{2.5.7}$$

where in the third step, we used that $n^3 = 0$, and in the fourth step, the fact that P_1 has finite moments up to order 2.

To simplify the derivation over the fermionic variable on the RHS of Eq. 2.5.4, the authors used the fact that F_r is *separately SUSY invariant*, which means that

$$F_r(\Phi^+, \Phi^-) = F_r(\mathrm{R}^+\Phi^+, \mathrm{R}^-\Phi^-), \qquad \forall \mathrm{R}^+, \mathrm{R}^- \text{ superrotations.} \tag{2.5.8}$$

The use of Lemma 2.3.11 twice yields that there is a function $f \in \mathcal{C}^\infty((0,\infty)^2) \cap \mathcal{C}^2([0,\infty)^2)$ such that

$$F(\Phi, \Phi') = f(\Phi^2, \Phi'^2) \tag{2.5.9}$$

In addition, if F is separately SUSY invariant, then $\tilde{F}$ and $\tilde{\Gamma}$ are also separately SUSY invariant. After they performed the explicit derivation over the fermionic variables, they rewrote the second moment of the Green's function as the inner product of some functions given in terms of the above transfer operators on a certain complex Hilbert space. Indeed,

$$\mathbb{E}_1\left[\left|\mathrm{G}^{\tilde{\Lambda}_L}_{z,\lambda}(x_0,x)\right|^2\right] = \langle \overline{(\tilde{T}\tilde{\Gamma})^{L-x_0}F_l}, e^{-\frac{\eta}{2}(r_+^2+r_-^2)}\tilde{\Gamma}(\tilde{T}\tilde{\Gamma})^{x-x_0}(\tilde{T}\tilde{\Gamma})^{L-x}F_r\rangle, \tag{2.5.10}$$

where $\langle,\rangle$ is the inner product in the Hilbert space

$$\mathcal{H} := \left\{ f : [0,\infty)^2 \longrightarrow \mathbb{C} \,\middle|\, \|f\|^2 = 4\int_{[0,\infty)^2} \mathrm{d}r_+\,\mathrm{d}r_-\; r_+ r_- \left|f(r_+^2, r_-^2)\right|^2 < \infty \right\}, \tag{2.5.11}$$

and the operator $\tilde{T}$ on $\mathcal{H}$ takes the form

$$[\tilde{T}f](r_+^2, r_-^2) = \int_{[0,\infty)^2} \mathrm{d}s_+\,\mathrm{d}s_-\; s_+ s_- J_0(r_+s_+)J_0(r_-s_-)f(s_+^2, s_-^2), \tag{2.5.12}$$

with $J_0(s) = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\theta e^{-is\cos\theta}$ the Bessel function of order 0.

In the second half of the proof of [KMP86, Main Thm.], the authors exploit the properties of the transfer operators $\tilde{T}$ and $\tilde{\Gamma}$ to obtain the estimate given in Ineq. 2.5.2. To be precise, Cauchy-Schwarz inequality together with the fact that $\tilde{T}$ and $\tilde{\Gamma}$ are bounded operators on $\mathcal{H}$ yield

$$\mathbb{E}_1\left[\left|\mathrm{G}^{\tilde{\Lambda}_L}_{z,\lambda}(x_0,x)\right|^2\right] \leq \left\|\overline{(\tilde{T}\tilde{\Gamma})^{L-x_0}F_l}\right\|_{\mathcal{H}} \cdot \left\|\tilde{\Gamma}(\tilde{T}\tilde{\Gamma})^{x-x_0}\right\|_{op} \cdot \left\|(\tilde{T}\tilde{\Gamma})^{L-x}F_r\right\|_{\mathcal{H}}. \tag{2.5.13}$$

On the one hand, we have the following estimate

$$\left\|(\tilde{T}\tilde{\Gamma})^n F_r\right\|_{\mathcal{H}} = \left\|\overline{(\tilde{T}\tilde{\Gamma})^n F_l}\right\|_{\mathcal{H}} \leq \frac{1}{\eta}, \qquad \forall \eta > 0,\ \forall \mathrm{E} \in \mathbb{R},\ \forall n \in \mathbb{N}, \tag{2.5.14}$$

which is the content of Lemma 3.1 in [KMP86]. On the other hand, there is a constant $\eta_0 > 0$ such that

$$\left\|\tilde{\Gamma}(\tilde{T}\tilde{\Gamma})\right\|_{op} \leq \left(1 - \frac{\ln 2}{|\ln(\eta)|^r} + \mathcal{O}\left(\frac{1}{|\ln(\eta)|^{2r}}\right)\right), \quad \forall \mathrm{E} \in \mathbb{R},\ \forall \eta \in (0, \eta_0], \tag{2.5.15}$$

which is the content of Lemma 3.2 in [KMP86]. To determine the above estimate, the authors use the double Hankel transform of order 0 given by

$$[H_0 g](r_+, r_-) = \int_{[0,\infty)^2} \mathrm{d}s_+\,\mathrm{d}s_-\ \sqrt{r_+ s_+} J_0(r_+ s_+) \sqrt{r_- s_-} J_0(r_- s_-) g(s_+, s_-). \tag{2.5.16}$$

H_0 is unitary operator on $\mathrm{L}^2(\mathbb{R}^2, \mathrm{d}r_+\mathrm{d}r_-)$ and $H_0 : \mathrm{L}^1(\mathbb{R}^2, \mathrm{d}r_+\mathrm{d}r_-) \longrightarrow \mathrm{L}^\infty(\mathbb{R}^2, \mathrm{d}r_+\mathrm{d}r_-)$ with $\|H_0 g\|_{\mathrm{L}^\infty(\mathbb{R}^2, \mathrm{d}r_+\mathrm{d}r_-)} \leq \|g\|_{\mathrm{L}^1(\mathbb{R}^2, \mathrm{d}r_+\mathrm{d}r_-)}$, for all $g \in \mathrm{L}^1(\mathbb{R}^2, \mathrm{d}r_+\mathrm{d}r_-)$. Due to Riesz-Thorin Interpolation Theorem (see [Gra14, Thm. 1.3.4]), we obtain

$$\|H_0 g\|_{\mathrm{L}^q(\mathbb{R}^2, \mathrm{d}r_+\mathrm{d}r_-)} \leq \|g\|_{\mathrm{L}^{q'}(\mathbb{R}^2, \mathrm{d}r_+\mathrm{d}r_-)}, \quad \forall g \in \mathrm{L}^{q'}(\mathbb{R}^2, \mathrm{d}r_+\mathrm{d}r_-),\ \forall q' \in [1, 2] \text{ with } \frac{1}{q} + \frac{1}{q'} = 1. \tag{2.5.17}$$

Note that we can write $\tilde{T}$ in terms of H_0 as

$$\sqrt{r_+ r_-}[\tilde{T} f](r_+^2, r_-^2) = -H_0(\sqrt{s_+ s_-} f(s_+^2, s_-^2))(r_+, r_-). \tag{2.5.18}$$

Hence, $\tilde{T}$ inherit all the above properties of H_0. Indeed, $\tilde{T}$ is a unitary operator on $\mathcal{H}$ and

$$\left\|\sqrt{r_+ r_-}[\tilde{T} f](r_+^2, r_-^2)\right\|_{\mathrm{L}^q(\mathbb{R}^2, \mathrm{d}r_+\mathrm{d}r_-)} \leq \left\|\sqrt{r_+ r_-} f(r_+^2, r_-^2)\right\|_{\mathrm{L}^{q'}(\mathbb{R}^2, \mathrm{d}r_+\mathrm{d}r_-)},$$
$$\forall f : [0,\infty)^2 \to \mathbb{C} \text{ with } \int_{[0,\infty)^2} \mathrm{d}r_+\mathrm{d}r_- \left|\sqrt{r_+ r_-} f(r_+^2, r_-^2)\right|^{q'} < \infty,\ \forall q' \in [1, 2] \text{ with } \frac{1}{q} + \frac{1}{q'} = 1. \tag{2.5.19}$$

For the sake of simplicity, assume that P_1 is absolutely continuous with respect to the Lebesgue measure. We have that

$$\left\|\tilde{\Gamma}\tilde{T}\tilde{\Gamma}\right\|_{op} \leq \left\|\Gamma(r_+^2, r_-^2)\right\|^2_{\mathrm{L}^p(\mathbb{R}^2, \mathrm{d}r_+\mathrm{d}r_-)} \leq \left\|\hat{\mathrm{P}}_1(r_+^2 - r_-^2) e^{\eta(r_+^2 + r_-^2)}\right\|^2_{\mathrm{L}^p(\mathbb{R}^2, \mathrm{d}r_+\mathrm{d}r_-)}, \quad \forall p \geq 2. \tag{2.5.20}$$

Indeed, let $f \in \mathcal{H}$, $p \geq 2$ and $q \in [1, 2]$ with $\frac{2}{p} + \frac{2}{q} = 1$. By Cauchy-Schwarz inequality and Ineq. 2.5.19, we obtain

$$\left\|\tilde{\Gamma}\tilde{T}\tilde{\Gamma}f\right\|_{\mathcal{H}} = \left(\int_{[0,\infty)^2} \mathrm{d}r_+\,\mathrm{d}r_-\left|\sqrt{r_+r_-}\tilde{\Gamma}\tilde{T}\tilde{\Gamma}f(r_+^2,r_-^2)\right|^2\right)^{\frac{1}{2}}$$
$$\leq \left(\int_{[0,\infty)^2} \mathrm{d}r_+\,\mathrm{d}r_-\left|\tilde{\Gamma}(r_+^2,r_-^2)\right|^p\right)^{\frac{1}{p}}\left(\int_{[0,\infty)^2} \mathrm{d}r_+\,\mathrm{d}r_-\left|\sqrt{r_+r_-}\tilde{T}\tilde{\Gamma}f(r_+^2,r_-^2)\right|^q\right)^{\frac{1}{q}}$$
$$= \left\|\Gamma(r_+^2,r_-^2)\right\|_{\mathrm{L}^p(\mathbb{R}^2,\mathrm{d}r_+\mathrm{d}r_-)}\left\|\sqrt{r_+r_-}[\tilde{T}\tilde{\Gamma}f](r_+^2,r_-^2)\right\|_{\mathrm{L}^q(\mathbb{R}^2,\mathrm{d}r_+\mathrm{d}r_-)}$$
$$\leq \left\|\Gamma(r_+^2,r_-^2)\right\|_{\mathrm{L}^p(\mathbb{R}^2,\mathrm{d}r_+\mathrm{d}r_-)}\left\|\sqrt{r_+r_-}[\tilde{\Gamma}f](r_+^2,r_-^2)\right\|_{\mathrm{L}^{q'}(\mathbb{R}^2,\mathrm{d}r_+\mathrm{d}r_-)}$$
$$\leq \left\|\Gamma(r_+^2,r_-^2)\right\|^2_{\mathrm{L}^p(\mathbb{R}^2,\mathrm{d}r_+\mathrm{d}r_-)}\left\|\sqrt{r_+r_-}f(r_+^2,r_-^2)\right\|_{\mathrm{L}^2(\mathbb{R}^2,\mathrm{d}r_+\mathrm{d}r_-)} = \left\|\Gamma(r_+^2,r_-^2)\right\|^2_{\mathrm{L}^p(\mathbb{R}^2,\mathrm{d}r_+\mathrm{d}r_-)}\|f\|_{\mathcal{H}}, \tag{2.5.21}$$

where in the third line we used Cauchy-Schwarz inequality again because $1 = \frac{q'}{p} + \frac{q'}{2}$. By direct computations and assumptions on the integrability of powers of the Fourier transform of P_1, there are constants $p_0 > 2$ and $C_1 > 0$ such that

$$\left\|\hat{\mathrm{P}}_1(r_+^2 - r_-^2)e^{\eta(r_+^2+r_-^2)}\right\|^2_{\mathrm{L}^p(\mathbb{R}^2,\mathrm{d}r_+\mathrm{d}r_-)} \leq \left(\frac{C_1}{4}\frac{|\ln\eta|}{p^{\frac{1}{r}}} + \mathrm{R}(p)\right)^2, \qquad \forall p > p_0, \tag{2.5.22}$$

where $\lim_{p\to\infty}\mathrm{R}(p) = 0$. Let $p(\eta) = C_1^r|\ln\eta|^r$. Then, there is $\eta_0 > 0$ such that

$$\left\|\tilde{\Gamma}\tilde{T}\tilde{\Gamma}\right\|_{op} \leq \left(\frac{1}{2}\right)^{\frac{1}{p(\eta)}}, \qquad \forall \mathrm{E}\in\mathbb{R},\ \forall\eta\in(0,\eta_0], \tag{2.5.23}$$

By Taylor's expansion, $\left(\frac{1}{2}\right)^{\frac{1}{p(\eta)}} = 1 - \frac{\ln 2}{p(\eta)} + \mathcal{O}\left(\left(\frac{1}{p(\eta)}\right)^2\right)$ and Ineq. 2.5.15 follows.

2.5.2 Challenges arising in the Anderson model on $\mathscr{D}_2$

We could partially implement the first half of the strategy in [KMP86] for the Anderson model on $\mathscr{D}_2$ after some minor changes. More explicitly, for fixed $\omega\in\Omega$ and $z\in\mathbb{C}\setminus\mathbb{R}$, we could write $\frac{1}{4}\sum_{\sigma_0,\sigma_1=0}^1 \mathrm{G}^{\Lambda_L}_{z,\omega,\lambda}(j_0+\sigma_0 e, j_1+\sigma_1 e)$ and $\frac{1}{4}\sum_{\sigma_0,\sigma_1=0}^1 \mathrm{G}^{\Lambda_L}_{z,\omega,\lambda}(j_0+\sigma_0 e, j_1+\sigma_1 e)$ in terms of bosonic and fermionic variables, see Eq. 2.4.6. Then, we carried out the derivation over the fermionic variables to obtain transfer operator representation, see Eq. 2.2.5. As a result of this,

$$\left|\frac{1}{4}\sum_{\sigma_0,\sigma_1=0}^1 \mathrm{G}^{\Lambda_L}_{z,\omega,\lambda}(j_0+\sigma_0 e, j_1+\sigma_1 e)\right|^2 = \frac{1}{4\pi^4}\int_{\mathbb{R}^2\times\mathbb{R}^2}\int_{\mathbb{R}^2\times\mathbb{R}^2} \mathrm{d}^2\mathrm{X}^+_{j_0}\mathrm{d}^2\mathrm{X}^+_{j_0+e}\mathrm{d}^2\mathrm{X}^-_{j_0}\mathrm{d}^2\mathrm{X}^-_{j_0+e}$$
$$\left[\left\{\mathrm{R}_{1,+}\tilde{\mathrm{R}}^{j_0,j_1}_{0,+} + \mathrm{R}_{0,+}\tilde{\mathrm{R}}^{j_0,j_1}_{1,+}\right\}\left(\mathrm{X}^+_{j_0},\mathrm{X}^+_{j_0+e}\right)\left\{\mathrm{R}_{1,-}\tilde{\mathrm{R}}^{j_0,j_1}_{0,-} + \mathrm{R}_{0,-}\tilde{\mathrm{R}}^{j_0,j_1}_{1,-}\right\}\left(\mathrm{X}^-_{j_0},\mathrm{X}^-_{j_0+e}\right)\right], \quad \forall\omega\in\Omega, \tag{2.5.24}$$

where $\mathrm{R}_{k,\varepsilon} = \mathrm{R}_{k,\varepsilon,\omega_{j_0},\dots,\omega_{-L}}$ and $\tilde{\mathrm{R}}^{j_0,j_1}_{k,\varepsilon} = \tilde{\mathrm{R}}^{j_0,j_1}_{k,\varepsilon,\omega_{j_0+1},\dots,\omega_L}$, for $k\in\{0,1\}$, are given by Eq. 2.4.8. However, when we tried to compute the second moment of the Green function by taking the expectation in the above expression, we could not manage to maintain the transfer operator representation as we could not find an explicit formula of the constants c_α involved and there are interactions between points on the same layer. More precisely, at the end of the proof of Thm. 2.2.1 in Section 2.4, we proved that we can interchange the expectation with the integral on $((\mathbb{R}^2\times\mathbb{R}^2)\times(\mathbb{R}^2\times\mathbb{R}^2))^{(2L+1)}$ in order to prove Eq. 2.2.7. This result together Eq. 2.4.23 and 2.4.28 yield

$$\begin{aligned}
&\mathbb{E}\left[\left|\frac{1}{4}\sum_{\sigma_0,\sigma_1=0}^{1} \mathrm{G}^{\Lambda_L}_{z,\omega,\lambda}(j_0+\sigma_0 e, j_1+\sigma_1 e)\right|^2\right]\\
&=\frac{1}{4\pi^4}\int_{(\mathbb{R}^2\times\mathbb{R}^2)^{2(2L+1)}}\prod_{j=-L}^{L} \mathrm{dX}^+_j\,\mathrm{dX}^-_j\,\mathrm{dX}^+_{j+e}\,\mathrm{dX}^-_{j+e}\\
&\times e^{-\eta\sum_{j=-L}^{L}|\mathrm{X}^+_j|^2+|\mathrm{X}^+_{j+e}|^2+|\mathrm{X}^-_j|^2+|\mathrm{X}^-_{j+e}|^2}\\
&\times e^{ip_+\left(\{\mathrm{X}^+_j\}_{j=-L}^{L},\{\mathrm{X}^+_{j+e}\}_{j=-L}^{L}\right)}e^{-ip_-\left(\{\mathrm{X}^-_j\}_{j=-L}^{L},\{\mathrm{X}^-_{j+e}\}_{j=-L}^{L}\right)}\\
&\times\mathbb{E}\left[q_+\left(\{\omega_j\}_{j=-L}^{L}\right)q_-\left(\{\omega_j\}_{j=-L}^{L}\right)e^{-i\sum_{j=-L}^{L}\omega_j\left(|\mathrm{X}^+_j|^2+|\mathrm{X}^+_{j+e}|^2-|\mathrm{X}^-_j|^2-|\mathrm{X}^-_{j+e}|^2\right)}\right]\\
&=\frac{1}{4\pi^4}\int_{(\mathbb{R}^2\times\mathbb{R}^2)^{2L+1}}\prod_{j=-L}^{L} \mathrm{dX}^+_j\,\mathrm{dX}^-_j\,\mathrm{dX}^+_{j+e}\,\mathrm{dX}^-_{j+e} \qquad (2.5.25)\\
&\times e^{-\eta\sum_{j=-L}^{L}|\mathrm{X}^+_j|^2+|\mathrm{X}^+_{j+e}|^2+|\mathrm{X}^-_j|^2+|\mathrm{X}^-_{j+e}|^2}\\
&\times e^{ip_+\left(\{\mathrm{X}^+_j\}_{j=-L}^{L},\{\mathrm{X}^+_{j+e}\}_{j=-L}^{L}\right)}e^{-ip_-\left(\{\mathrm{X}^-_j\}_{j=-L}^{L},\{\mathrm{X}^-_{j+e}\}_{j=-L}^{L}\right)}\\
&\times\sum_{\alpha\in\{0,1,2,3,4\}^{2L+1}}c_\alpha\prod_{j=-L}^{L} i^{\alpha_j}\hat{\mathrm{P}}_0^{(\alpha_j)}\left(\left|\mathrm{X}^+_j\right|^2+\left|\mathrm{X}^+_{j+e}\right|^2-\left|\mathrm{X}^-_j\right|^2-\left|\mathrm{X}^-_{j+e}\right|^2\right).
\end{aligned}$$

Recall that the constants c_α can be written in terms of η and E as they depend on the constants $a_{k,\varepsilon}$, $b_{k,\varepsilon}$ and $c_{k,\varepsilon}$, see Eq. 2.4.27. Although we do not have an explicit formula for the c_α, we might try to estimate it. Suppose, for the sake of simplicity, we were able to find an upper bound for $|c_\alpha|$, for all $\alpha\in\{0,1,2,3\}$. Following the spirit of Subsection 2.5.1, we want to find an estimate similar to Ineq. 2.5.15. However, in our case, after taking expectation, there is an interaction between the two points in the layer, which is coded in the term

$$K_l(\mathrm{X},\mathrm{X}',\mathrm{Y},\mathrm{Y}') := e^{-2i(\mathrm{X}\cdot\mathrm{Y}-\mathrm{X}'\cdot\mathrm{Y}')}\hat{\mathrm{P}}_0^{(l)}\left(|\mathrm{X}|^2+\left|\mathrm{X}'\right|^2-|\mathrm{Y}|^2-\left|\mathrm{Y}'\right|^2\right),\quad l\in\{0,1,2,3,4\}. \tag{2.5.26}$$

Let

$$\begin{aligned}
&[\mathbf{S}_l f](\mathrm{X},\mathrm{X}',\mathrm{Y},\mathrm{Y}') := \mathscr{F}[K_l f](2\mathrm{X},2\mathrm{X}',-2\mathrm{Y},-2\mathrm{Y}'),\\
&[\mathbf{G}f](\mathrm{X},\mathrm{X}',\mathrm{Y},\mathrm{Y}') := [\mathbf{G}_+](\mathrm{X},\mathrm{X}')[\mathbf{G}_-](\mathrm{Y},\mathrm{Y}')f(\mathrm{X},\mathrm{X}',\mathrm{Y},\mathrm{Y}'),\quad\text{where}\\
&\qquad[\mathbf{G}_\varepsilon](\mathrm{X},\mathrm{X}') := e^{-\eta\left(|\mathrm{X}|^2+|\mathrm{X}'|^2\right)}e^{\varepsilon i\mathrm{E}(|\mathrm{X}|^2+|\mathrm{X}'|^2)},\quad\text{for } l\in\{0,1,2,3,4\},\ \varepsilon\in\{+,-\}.
\end{aligned}\tag{2.5.27}$$

Hence, we need to find a convenient upper bound of $\|\mathbf{S}_l\mathrm{G}\|_{op}$ but we could not replicate the argument using Hausdorff-Young inequality of the Fourier transform.

Thm. 2.2.1 provides a representation of the second moments of the Green's function using product random matrices. This suggests a potential path of investigation for understanding the localization of the Anderson model on $\mathscr{D}_2$ through the study of Lyapunov exponents, as explored in [Dam11].

Chapter 3

Decay of the Green's function of the fractional Anderson model and connection to long-range SAW

3.1 Introduction

This chapter is based upon [DMERM23] and shows the contribution of the PhD candidate to it in the form of Thm. 3.5.1 and 3.5.5 below. For the reader's convenience, a copy of [DMERM23] is provided in Appendix A located at the end of this thesis.

Let us recall the definition of the fractional Anderson model. Let $\mathrm{d} \in \mathbb{N}$, $0 < \alpha < 1$, and $\lambda > 0$. The fractional Anderson model $\mathrm{H}_{\omega,\lambda,\alpha}$ is given by

$$\mathrm{H}_{\omega,\lambda,\alpha} = (-\Delta)^{\alpha} + \lambda \mathrm{V}_{\omega}, \qquad \omega \in \Omega, \tag{3.1.1}$$

acting on $\ell^2(\mathbb{Z}^{\mathrm{d}})$, where $(-\Delta)^{\alpha}$ is the fractional Laplacian (which is defined in Section 3.2 below) and V_{ω} is a random multiplication operator acting diagonally on the canonical basis by a sequence $\omega = (\omega_x)_{x\in\mathbb{Z}^{\mathrm{d}}}$ of real, iid and bounded random variables with common distribution P_0 and defined in a suitable probability space Ω.

When $\alpha = 1$ and P_0 is uniform, [Sch15] proved dynamical localization for disorder parameter $\lambda > \lambda_{\mathrm{And}}$, where λ_{And} is exactly the disorder threshold proposed in [And58]. His proof makes an exhaustive use of the depleted resolvent identity and self-avoiding walks (abbreviated SAW).

This chapter has two main objectives:

(i) Relate the fractional moments of the Green's function to the SAW induced by $(-\Delta)^{\alpha}$, thus extending [Sch15, Thm. 1] to the fractional Laplacian case, see Thm. 3.5.1 below. This enables us to show, under some conditions, pure-point spectrum and polynomial decay of eigenfunctions at large disorder λ a.s. in the fractional Anderson model, as outlined in Thm. 3.5.5.

(ii) Compare different estimates of the fractional moments of the Green's function available in the literature, leading us to conclude that our estimate provides the sharpest result. Furthermore, the localization threshold $\lambda_0(s)$ in Thm 3.5.5 is shown to be smaller than the thresholds found in previously known results. Notably, we expand the range of values of λ where spectral localization happens a.s.

The remaining sections of this chapter are outlined below:

In Section 3.2, we define the fractional Laplacian and we show some properties which will be useful in subsequent analysis. More precisely, Subsection 3.2.1 establishes that, similar to the usual Laplacian case, the fractional Laplacian exhibits translation invariance, and the matrix elements of the diagonal can be computed as the sum of the off-diagonal terms in a row (cf. Eq. 3.2.1). In addition, Eq. 3.2.2 expresses the off-diagonal matrix elements as an integral in terms of Bessel functions, following [Kwa17, Thm. 1.1]. Then, in Subsection 3.2.2, we provide upper and lower bounds for the sum of the fractional powers of the off-diagonal terms of the matrix elements of $(-\Delta)^{\alpha}$. Next, in Subsection 3.2.3, for $0 < \alpha < \frac{\mathrm{d}}{2}$, we define the inverse of the fractional Laplacian as $(-\Delta)^{-\alpha}(x_0, x) := \lim_{\mathrm{m}\downarrow 0}[(-\Delta)^{-\alpha} + \mathrm{m}^2]^{-1}(x_0, x)$, for $x, x_0 \in \mathbb{Z}^{\mathrm{d}}$. We establish in Thm. 3.2.7 that the limit is well-defined by using the Fourier transform and the Riesz potential (see Def. 3.2.8 below). Moreover, $(-\Delta)^{-\alpha}$ also possesses translation-invariance and its off-diagonal matrix elements decay polynomially as $|(-\Delta)^{-\alpha}(x_0, x)| \lesssim \frac{1}{|x-x_0|^{\mathrm{d}-2\alpha}}$, for $x, x_0 \in \mathbb{Z}^{\mathrm{d}}$, $x_0 \neq x$. Moving forward, in Subsection 3.2.4, we prove some known properties of the Riesz potential used in the proof of Thm. 3.2.7, following [LL01].

In Section 3.3, we focus on the fractional Anderson model and present our findings. Specifically, in Subsection 3.3.1, we rigorously define the fractional Anderson model. Then, in Subsection 3.3.2, we present some known results regarding the decay of fractional moments of the Green's functions, as shown in Ineq. 3.3.35 and 3.3.50. Additionally, Thm. 3.3.2 establishes conditions to guarantee pure-point spectrum and polynomial decay of the eigenfunctions a.s. Next, in Subsection 3.4, we introduce the self-avoiding walk X induced by $(-\Delta)^{\alpha}$ and its associated two-point correlation function. Moving forward, in Section 3.5, we address the achievement of Objectives (i) and (ii). More precisely, in Subsection 3.5.1, we show the relationship between the fractional moments of the Green's function and the two-point correlation function, which is the content of Thm. 3.5.1 below. This result together with [CS15, Lemma 2.4] yield our estimate of the fractional moments of the Green's function, see Ineq. 3.5.23. A direct consequence of Thm. 3.5.1 is Corollary 3.5.2, where we express the correlation function as an averaged weighted sum of the arrivals at x by the self-avoiding walk X starting at x_0 and staying within the volume $\Lambda \subset \mathbb{Z}^{\mathrm{d}}$ until it lands outside of it. This extends [FV17, Lemma 8.13] to the fractional Laplacian case (see Remark 3.5.3). Moreover, we determine spectral localization at strong disorder. Additionally, for $\mathrm{d} = 1$ and assuming some regularity on the common distribution P_0, we find that the eigenfunctions decay polynomially. After comparing our decay of the eigenfunctions with the one found in Thm. 3.3.2, we conclude that our estimate is sharper. Finally, in Subsection 3.5.2, we contrast the different regimes of decay of the fractional moments of the Green's function explored in this chapter. Based on this analysis, we reach the conclusion that our estimate is the sharpest and that the localization region, in terms of the disorder parameter λ, is larger.

We set the notation of the rest of the chapter. Let $\mathrm{d} \in \mathbb{N}$ and $(\delta_x)_{x\in\mathbb{Z}^{\mathrm{d}}}$ be the canonical orthonormal base of $\ell^2(\mathbb{Z}^{\mathrm{d}})$. For an operator A acting on $\ell^2(\mathbb{Z}^{\mathrm{d}})$, we denote the matrix elements of A by $A(x_0, x) := \langle \delta_{x_0}, A\,\delta_x\rangle$ with $x, x_0 \in \mathbb{Z}^{\mathrm{d}}$. We write $\langle \delta_x, \cdot\rangle\delta_x$ for the projection onto the subspace generated by δ_x. In addition, we denote the ℓ^p-norm in $\mathbb{Z}^{\mathrm{d}}$ by $|x|_{\mathrm{p}} = \left(\sum_{i=1}^{\mathrm{d}} |x_i|^{\mathrm{p}}\right)^{\frac{1}{\mathrm{p}}}$, for $1 \leq \mathrm{p} < \infty$, and $|x|_\infty = \max_{1\leq i\leq \mathrm{d}} |x_i|$, for $\mathrm{p} = \infty$. In the case $\mathrm{p} = 2$, we use the short hand notation $|\cdot|$.

3.2 Fractional Laplacian

Let $0 < \alpha < 1$. By functional calculus, the discrete fractional Laplacian is defined as the linear bounded operator $(-\Delta)^{\alpha}$ acting on $\ell^2(\mathbb{Z}^{\mathrm{d}})$, where $-\Delta$ denotes the discrete (negative) Laplacian on $\ell^2(\mathbb{Z}^{\mathrm{d}})$ given by $(-\Delta\varphi)(x_0) := \sum_{|x-x_0|_1=1} \left(\varphi(x_0) - \varphi(x)\right)$, for all $\varphi \in \ell^2(\mathbb{Z}^{\mathrm{d}})$, for all $x \in \mathbb{Z}^{\mathrm{d}}$.

3.2.1 Relationship between the coefficients of the fractional Laplacian

The discrete fractional Laplacian shares some properties with the discrete Laplacian such as the invariance under translations and the fact that the matrix elements of the diagonal can be written as the sum of the off-diagonal elements in a row, which are the content of Proposition 3.2.1 below, where, in addition, we write the matrix elements of the fractional Laplacian as a certain integral. This proposition and its corresponding proof are extensions from [CRS+18, Thm. 1.2] to the case $\mathrm{d} \geq 1$. Recall that the gamma function Γ has simple poles in the set of the non-positive integers so $\frac{1}{\Gamma(x)} := 0$ for all $x \in \{0, -1, -2, \ldots\}$.

Proposition 3.2.1. *Let $0 < \alpha < 1$ and $\mathrm{d} \in \mathbb{N}$. The following equation for the fractional Laplacian holds:*

$$(-\Delta)^{\alpha}(x, x) = - \sum_{y \in \mathbb{Z}^{\mathrm{d}} \setminus \{x\}} (-\Delta)^{\alpha}(x, y), \qquad x \in \mathbb{Z}^{\mathrm{d}}. \tag{3.2.1}$$

In addition, the terms $(-\Delta)^{\alpha}(x, y)$ can be computed as

$$(-\Delta)^{\alpha}(x, y) = \frac{-1}{|\Gamma(-\alpha)|} \int_0^{\infty} \frac{\mathrm{d}t}{t^{1+\alpha}} e^{-2\mathrm{d}t} \prod_{j=1}^{\mathrm{d}} \mathrm{I}_{(x_j - y_j)}(2t) \quad \forall x, y \in \mathbb{Z}^{\mathrm{d}} \text{ with } x \neq y, \tag{3.2.2}$$

where I_{p} is the modified Bessel function of order $p \in \mathbb{Z}$ which is defined as

$$\mathrm{I}_{\mathrm{p}}(t) := \sum_{\mathrm{q} \geq 0} \frac{1}{\mathrm{q}! \Gamma(\mathrm{p} + \mathrm{q} + 1)} \left(\frac{t}{2}\right)^{2\mathrm{q}+\mathrm{p}}. \tag{3.2.3}$$

Remark 3.2.2. *In particular, Eq. 3.2.2 says that $(-\Delta)^{\alpha}(x, y) < 0$ for all $x, y \in \mathbb{Z}^{\mathrm{d}}$ with $x \neq y$. This, together with Eq. 3.2.1, implies that $(-\Delta)^{\alpha}(x, x) > 0$ for all $x \in \mathbb{Z}^{\mathrm{d}}$. Moreover, it follows that $(-\Delta)^{\alpha}$ is invariant under translations.*

Proof. In order to compute the scalar $(-\Delta)^{\alpha}(x, y)$, we will use of the following representation of the fractional Laplacian:

$$(-\Delta)^{\alpha} = \frac{-1}{|\Gamma(-\alpha)|} \int_0^{\infty} \frac{\mathrm{d}t}{t^{1+\alpha}} (e^{t\Delta} - \mathrm{I}), \tag{3.2.4}$$

where I is the identity operator and the integral converges under the operator norm, see [Kwa17, Theorem 1.1 (c)]. Hence,

$$(-\Delta)^{\alpha}(x, y) = \frac{-1}{|\Gamma(-\alpha)|} \int_0^{\infty} \frac{\mathrm{d}t}{t^{1+\alpha}} (e^{t\Delta}(x, y) - \langle \delta_x, \delta_y \rangle). \tag{3.2.5}$$

In the proof of [GRM20, Thm. 2.2], it was shown that

$$e^{t\Delta}(x, y) = e^{-2\mathrm{d}t} \prod_{j=1}^{\mathrm{d}} \mathrm{I}_{(x_j - y_j)}(2t). \tag{3.2.6}$$

We proceed to prove that

$$(-\Delta)^{\alpha}(x, x) = - \sum_{y \in \mathbb{Z}^{\mathrm{d}} \setminus \{x\}} (-\Delta)^{\alpha}(x, y) \tag{3.2.7}$$

We already know that

$$(-\Delta)^{\alpha}(x, x) = \frac{-1}{|\Gamma(-\alpha)|} \int_0^{\infty} \frac{\mathrm{d}t}{t^{1+\alpha}} (e^{t\Delta}(x, x) - 1) \tag{3.2.8}$$

Claim 3.2.3.

$$-\sum_{y\in\mathbb{Z}^{\mathrm{d}}\setminus\{x\}} e^{t\Delta}(x,y) = e^{t\Delta}(x,x) - 1. \tag{3.2.9}$$

Therefore, due to Claim 3.2.3 and Fubini-Tonelli, we obtain that

$$\begin{aligned}(-\Delta)^{\alpha}(x,x) &= -\frac{-1}{|\Gamma(-\alpha)|}\int_0^\infty \frac{\mathrm{d}t}{t^{1+\alpha}} \sum_{y\in\mathbb{Z}^{\mathrm{d}}\setminus\{x\}} e^{t\Delta}(x,y) = -\frac{-1}{|\Gamma(-\alpha)|}\sum_{y\in\mathbb{Z}^{\mathrm{d}}\setminus\{x\}}\int_0^\infty \frac{\mathrm{d}t}{t^{1+\alpha}} e^{t\Delta}(x,y) \\ &= -\sum_{y\in\mathbb{Z}^{\mathrm{d}}\setminus\{x\}} (-\Delta)^{\alpha}(x,y). \end{aligned} \tag{3.2.10}$$

It only remains to prove Claim 3.2.3.

$$\begin{aligned}\sum_{y\in\mathbb{Z}^{\mathrm{d}}} e^{t\Delta}(y,0) &= e^{-2\mathrm{d}t}\sum_{y=(y_1,\dots,y_{\mathrm{d}})\in\mathbb{Z}^{\mathrm{d}}} \mathrm{I}_{y_1}(2t)\cdot\ldots\cdot \mathrm{I}_{y_{\mathrm{d}}}(2t) \\ &= e^{-2\mathrm{d}t}\prod_{j=1}^{\mathrm{d}}\sum_{y_j\in\mathbb{Z}} \mathrm{I}_{y_j}(2t) = \left(e^{-2t}\sum_{p\in\mathbb{Z}} \mathrm{I}_p(2t)\right)^{\mathrm{d}}.\end{aligned}$$

Thus, it is sufficient to prove that $e^{-2t}\sum_{p\in\mathbb{Z}} \mathrm{I}_p(2t) = 1$.

$$e^{-2t}\sum_{p\in\mathbb{Z}} \mathrm{I}_p(2t) = e^{-2t}\sum_{p\in\mathbb{Z}}\sum_{q\geq 0}\frac{1}{q!\Gamma(p+q+1)}t^{2q+p} = e^{-2t}\sum_{q\geq 0}\sum_{p\in\mathbb{Z}}\frac{1}{q!\Gamma(p+q+1)}t^{2q+p}, \tag{3.2.11}$$

where in the last equality we could interchange the order of series because we are only adding up non-negative terms. In addition, Γ has poles at each of the non-positive integers, whereas $\Gamma(n) = (n-1)!$ for $n\in\mathbb{N}$. Hence,

$$\begin{aligned} e^{-2t}\sum_{p\in\mathbb{Z}} \mathrm{I}_p(2t) &= e^{-2t}\sum_{q\geq 0}\sum_{p\geq -q}\frac{1}{q!\Gamma(p+q+1)}t^{2q+p} \\ &= e^{-2t}\sum_{q\geq 0}\frac{t^q}{q!}\sum_{p\geq -q}\frac{t^{p+q}}{(p+q)!} = e^{-t}\sum_{q\geq 0}\frac{t^q}{q!} = 1. \end{aligned}$$

□

3.2.2 Estimation of the sum of the fractional powers of the off-diagonal terms of the Fractional Laplacian

The off-diagonal matrix elements of the fractional Laplacian decay polynomially, as shown in [GRM20, Thm. 2.2(iii)]. Indeed, there are constants $0 < c_{\alpha,\mathrm{d}} < \mathcal{C}_{\alpha,\mathrm{d}}$ such that

$$\frac{c_{\alpha,\mathrm{d}}}{|x-x_0|^{\mathrm{d}+2\alpha}} \leq -(-\Delta)^{\alpha}(x_0,x) \leq \frac{\mathcal{C}_{\alpha,\mathrm{d}}}{|x-x_0|^{\mathrm{d}+2\alpha}}, \qquad \forall x,x_0\in\mathbb{Z}^{\mathrm{d}},\ x\neq x_0. \tag{3.2.12}$$

A consequence of the above inequality is that it provides a sufficient and necessary condition for convergence of the series $\sum_{x\in\mathbb{Z}^{\mathrm{d}}\setminus\{0\}} |\triangle^{\alpha}(0,x)|^s$, for all $\frac{\mathrm{d}}{\mathrm{d}+2\alpha} < s \leq 1$, see Corollary 3.2.4 below. Moreover, when it converges, we find an upper and lower bound of its value.

Corollary 3.2.4. *Let $0 < \alpha < 1$ and $\mathrm{d} \in \mathbb{N}$. Let $\mathcal{C}_{\alpha,\mathrm{d}}$ and $c_{\alpha,\mathrm{d}}$ be as in Ineq. 3.2.12. Then, there is $N_\mathrm{d} \in \mathbb{N} \cup \{0\}$ such that*

$$\frac{c_{\alpha,\mathrm{d}}}{\sqrt{\mathrm{d}^{\mathrm{d}+2\alpha}}} \frac{1}{\alpha} \leq \sum_{x \in \mathbb{Z}^\mathrm{d} \setminus \{0\}} \triangle^\alpha(0,x) \leq 2^{N_\mathrm{d}} \mathcal{C}_{\alpha,\mathrm{d}} \Big(2 + \frac{1}{\alpha}\Big). \tag{3.2.13}$$

Furthermore, let $\frac{\mathrm{d}}{\mathrm{d}+2\alpha} < s \leq 1$ and let α_s be the constant satisfying

$$\mathrm{d} + 2\alpha_s = s(\mathrm{d} + 2\alpha). \tag{3.2.14}$$

Then,

$$\frac{c^s_{\alpha,\mathrm{d}}}{\mathcal{C}_{\alpha_s,\mathrm{d}}} \Delta^{\alpha_s}(0,x) \leq |\Delta^\alpha(0,x)|^s \leq \frac{\mathcal{C}^s_{\alpha,\mathrm{d}}}{c_{\alpha_s,\mathrm{d}}} \Delta^{\alpha_s}(0,x) \qquad \forall y \in \mathbb{Z}^\mathrm{d} \setminus \{0\}. \tag{3.2.15}$$

In particular, for all $s \in \Big(\frac{\mathrm{d}}{\mathrm{d}+2\alpha}, 1\Big)$, we obtain

$$\sum_{x \in \mathbb{Z}^\mathrm{d} \setminus \{0\}} |\triangle^\alpha(0,x)|^s < \infty \Longleftrightarrow \sum_{x \in \mathbb{Z}^\mathrm{d} \setminus \{0\}} \triangle^{\alpha_s}(0,x) < \infty. \tag{3.2.16}$$

Remark 3.2.5. *Note that $\alpha_s = s\big(\alpha + \frac{\mathrm{d}}{2}\big) - \frac{\mathrm{d}}{2}$. Hence, α_s is a strictly-increasing function in s and $0 < \alpha_s < \alpha$, for all $s \in \Big(\frac{\mathrm{d}}{\mathrm{d}+2\alpha}, 1\Big)$*

Proof of Corollary 3.2.4. By Ineq. 3.2.12, there are constants $0 < c_{\alpha,\mathrm{d}} < \mathcal{C}_{\alpha,\mathrm{d}}$ such that

$$\sum_{x \in \mathbb{Z}^\mathrm{d} \setminus \{0\}} \frac{c_{\alpha,\mathrm{d}}}{|x|^{\mathrm{d}+2\alpha}} \leq \sum_{x \in \mathbb{Z}^\mathrm{d} \setminus \{0\}} \triangle^\alpha(0,x) \leq \sum_{x \in \mathbb{Z}^\mathrm{d} \setminus \{0\}} \frac{\mathcal{C}_{\alpha,\mathrm{d}}}{|x|^{\mathrm{d}+2\alpha}} \tag{3.2.17}$$

Thus, $\sum_{x \in \mathbb{Z}^\mathrm{d} \setminus \{0\}} \triangle^\alpha(0,x)$ is convergent iff $\sum_{x \in \mathbb{Z}^\mathrm{d} \setminus \{0\}} \frac{1}{|x|^{\mathrm{d}+2\alpha}}$ is convergent. So our problem boils down to computing the latter series.

For the case $\mathrm{d} = 1$, we can make use of the Integral test. Indeed, let $f(y) = \frac{1}{|y|^{1+2\alpha}}$ for $y \in \mathbb{R} \setminus \{0\}$. Clearly, f is positive, monotone decreasing and

$$\sum_{x \in \mathbb{Z} \setminus \{0\}} \frac{1}{|x|^{1+2\alpha}} = 2 \sum_{n \geq 1} f(n). \tag{3.2.18}$$

The Integral test asserts that $\sum_{n \geq 1} f(n)$ converges if and only if $\int_1^\infty \mathrm{d}y f(y)$ converges. To be more precise, the test gives us the following bounds:

$$\int_1^\infty \frac{\mathrm{d}y}{y^{1+2\alpha}} \leq \sum_{n \geq 1} \frac{1}{n^{1+2\alpha}} \leq f(1) + \int_1^\infty \frac{\mathrm{d}y}{y^{1+2\alpha}}, \tag{3.2.19}$$

A direct computation shows that the above series converges iff $\alpha > 0$. Furthermore, if $\alpha > 0$, then

$$\frac{1}{2\alpha} \leq \sum_{n \geq 1} \frac{1}{n^{1+2\alpha}} \leq 1 + \frac{1}{2\alpha}. \tag{3.2.20}$$

Therefore, for $\mathrm{d}=1$, we have proven

$$c_{\alpha,1}\frac{1}{\alpha}\leq\sum_{x\in\mathbb{Z}\setminus\{0\}}\triangle^{\alpha}(0,x)\leq\mathcal{C}_{\alpha,1}\Big(2+\frac{1}{\alpha}\Big). \tag{3.2.21}$$

From now on, let $\mathrm{d}\geq 2$. Note that $|x|_\infty\leq|x|\leq\sqrt{\mathrm{d}}|x|_\infty$, for all $x\in\mathbb{Z}^{\mathrm{d}}$. Hence

$$\frac{1}{\mathrm{d}^{\frac{\mathrm{d}+2\alpha}{2}}}\sum_{x\in\mathbb{Z}^{\mathrm{d}}\setminus\{0\}}\frac{1}{|x|_\infty^{\mathrm{d}+2\alpha}}\leq\sum_{x\in\mathbb{Z}^{\mathrm{d}}\setminus\{0\}}\frac{1}{|x|^{\mathrm{d}+2\alpha}}\leq\sum_{x\in\mathbb{Z}^{\mathrm{d}}\setminus\{0\}}\frac{1}{|x|_\infty^{\mathrm{d}+2\alpha}}. \tag{3.2.22}$$

Given $n\in\mathbb{N}$, let $B^{\infty,\mathrm{d}}(0,n):=\big\{x\in\mathbb{Z}^{\mathrm{d}}\mid|x|_\infty<n\big\}$ be the open ball in the metric space $(\mathbb{Z}^{\mathrm{d}},|\cdot|_\infty)$ centered at 0 and with radius n. Similarly, its corresponding sphere is defined as $\partial B^{\infty,\mathrm{d}}(0,n):=\big\{x\in\mathbb{Z}^{\mathrm{d}}\mid|x|_\infty=n\big\}$. In addition, $\big\{\partial B^{\infty,\mathrm{d}}(0,n)\big\}_{n\in\mathbb{N}}$ forms a partition of $\mathbb{Z}^{\mathrm{d}}\setminus\{0\}$. Hence,

$$\sum_{x\in\mathbb{Z}^{\mathrm{d}}\setminus\{0\}}\frac{1}{|x|_\infty^{\mathrm{d}+2\alpha}}=\sum_{n\geq1}\sum_{x\in\partial B^{\infty,\mathrm{d}}(0,n)}\frac{1}{|x|_\infty^{\mathrm{d}+2\alpha}}=\sum_{n\geq1}\frac{\#\partial B^{\infty,\mathrm{d}}(0,n)}{n^{\mathrm{d}+2\alpha}}, \tag{3.2.23}$$

where $\#\partial B^{\infty,\mathrm{d}}(0,n)$ is the cardinality of $\partial B^{\infty,\mathrm{d}}(0,n)$, which can be computed as follows:

$$\#\partial B^{\infty,\mathrm{d}}(0,n)=\#B^{\infty,\mathrm{d}}(0,n+1)-\#B^{\infty,\mathrm{d}}(0,n). \tag{3.2.24}$$

Since the open balls with the metric $|\cdot|_\infty$ are hypercubes, it follows that $\#B^{\infty,\mathrm{d}}(0,n)=(2n-1)^{\mathrm{d}}$. Additionally, $a^{\mathrm{d}}-b^{\mathrm{d}}=(a-b)\big(a^{\mathrm{d}-1}+a^{\mathrm{d}-2}b+\ldots+b^{\mathrm{d}-1}\big)$ for $a,b\in\mathbb{R}$. Therefore

$$\#\partial B^{\infty,\mathrm{d}}(0,n)=(2n+1)^{\mathrm{d}}-(2n-1)^{\mathrm{d}}=2\sum_{j=0}^{\mathrm{d}-1}(2n+1)^{\mathrm{d}-1-j}(2n-1)^{j}. \tag{3.2.25}$$

As a result of this, there is a constant $N_{\mathrm{d}}\in\mathbb{N}\cup\{0\}$ such that

$$2n^{\mathrm{d}-1}\leq\#\partial B^{\infty,\mathrm{d}}(0,n)\leq2^{N_{\mathrm{d}}+1}n^{\mathrm{d}-1}\qquad\forall n\in\mathbb{N}. \tag{3.2.26}$$

Consequently,

$$2\sum_{n\geq1}\frac{1}{n^{1+2\alpha}}\leq\sum_{n\geq1}\frac{\#\partial B^{\infty,\mathrm{d}}(0,n)}{n^{\mathrm{d}+2\alpha}}\leq2^{N_{\mathrm{d}}+1}\sum_{n\geq1}\frac{1}{n^{1+2\alpha}}. \tag{3.2.27}$$

By Ineq. 3.2.20, 3.2.23 and 3.2.27, we establish

$$\frac{1}{\alpha}\leq\sum_{x\in\mathbb{Z}^{\mathrm{d}}\setminus\{0\}}\frac{1}{|x|_\infty^{\mathrm{d}+2\alpha}}\leq2^{N_{\mathrm{d}}}\Big(2+\frac{1}{\alpha}\Big). \tag{3.2.28}$$

This together with Ineq. 3.2.22 yield

$$\frac{1}{\mathrm{d}^{\frac{\mathrm{d}+2\alpha}{2}}}\frac{1}{\alpha}\leq\sum_{x\in\mathbb{Z}^{\mathrm{d}}\setminus\{0\}}\frac{1}{|x|^{\mathrm{d}+2\alpha}}\leq2^{N_{\mathrm{d}}}\Big(2+\frac{1}{\alpha}\Big). \tag{3.2.29}$$

By Ineq. 3.2.17 and 3.2.29, Ineq. 3.2.13 follows.

Let $\frac{\mathrm{d}}{\mathrm{d}+2\alpha} < s < 1$ and α_s be as in Eq. 3.2.14. Then $0 < \alpha_s < \alpha < 1$. Furthermore, due to Ineq. 3.2.12, we establish

$$|\triangle^{\alpha}(x,y)|^s \leq \frac{\mathcal{C}^s_{\alpha,\mathrm{d}}}{|x-y|^{s(\mathrm{d}+2\alpha)}} = \frac{\mathcal{C}^s_{\alpha,\mathrm{d}}}{|x-y|^{\mathrm{d}+2\alpha_s}} \leq \frac{\mathcal{C}^s_{\alpha,\mathrm{d}}}{c_{\alpha_s,\mathrm{d}}} \triangle^{\alpha_s}(x,y). \tag{3.2.30}$$

By the same token, the remaining part of Ineq. 3.2.15 follows. □

Incidentally, the above proof yields a sufficient and necessary condition for convergence of the series $\sum_{x\in\mathbb{Z}^{\mathrm{d}}\setminus\{0\}} \frac{1}{|x|^{\mathrm{d}+\mathrm{r}}}$, for $\mathrm{r} \in \mathbb{R}$, see Corollary 3.2.6 below. Furthermore, when it converges, we find an upper and lower bound of its value.

Corollary 3.2.6. *Let $\mathrm{r} \in \mathbb{R}$, then the series $\sum_{x\in\mathbb{Z}^{\mathrm{d}}\setminus\{0\}} \frac{1}{|x|^{\mathrm{d}+\mathrm{r}}}$ converges iff $\mathrm{r} > 0$. Moreover, let $\mathrm{r} > 0$ and $N_{\mathrm{d}} \in \mathbb{N}\cup\{0\}$ be as in Corollary 3.2.4, then*

$$\frac{4}{\mathrm{d}^{\frac{\mathrm{d}+2\alpha}{2}}\mathrm{r}} \leq \sum_{x\in\mathbb{Z}^{\mathrm{d}}\setminus\{0\}} \frac{1}{|x|^{\mathrm{d}+\mathrm{r}}} \leq 2^{N_{\mathrm{d}}+1}\Big(1+\frac{1}{\mathrm{r}}\Big). \tag{3.2.31}$$

3.2.3 Inverse of Fractional Laplacian

Let $0 < \alpha < \frac{\mathrm{d}}{2}$. It can be determined that the inverse $(-\Delta)^{\alpha}$ is well-defined and Ineq. 3.2.33 is valid, see [Sla18, Sect. 2] and references therein. Here, we provide proof that, we believe, is new in this context. It follows the arguments in [GRM20, Lemma A.1]. In Thm. 3.2.7 below, we show that this object is well-defined and we provide a pointwise estimate.

Theorem 3.2.7. *Let $0 < \alpha < \frac{\mathrm{d}}{2}$ and $\mathrm{m} > 0$ (so that $-\mathrm{m}^2 \notin \sigma((-\Delta)^{\alpha}) = [0,(4\mathrm{d})^{\alpha}]$). Then,*

$$(-\Delta)^{-\alpha}(x_0,x) := \lim_{\mathrm{m}\downarrow 0}\left[(-\Delta)^{\alpha}+\mathrm{m}^2\right]^{-1}(x_0,x), \qquad \forall x,x_0 \in \mathbb{Z}^{\mathrm{d}}, \tag{3.2.32}$$

is well-defined. Moreover, $(-\Delta)^{-\alpha}$ is invariant under translations and

$$\left|(-\Delta)^{-\alpha}(x_0,x)\right| \leq \frac{\tilde{\mathcal{C}}_{\alpha,\mathrm{d}}}{|x-x_0|^{\mathrm{d}-2\alpha}}, \qquad \forall x,x_0 \in \mathbb{Z}^{\mathrm{d}},\ x \neq x_0, \tag{3.2.33}$$

where $\tilde{\mathcal{C}}_{\alpha,\mathrm{d}} > 0$ is a constant.

Proof. We divide our proof into two parts. In the first half, we prove the existence of the inverse of the fractional Laplacian, see Eq. 3.2.46 below. In the second half, we estimate the decaying of the operator, see Eq. 3.2.56 below.

Let $x_0, x \in \mathbb{Z}^{\mathrm{d}}$ and $\mathrm{m} > 0$. As a first step to prove the existence of $(-\Delta)^{-\alpha}(x_0,x)$, we will rewrite $\left[(-\Delta)^{\alpha}+\mathrm{m}^2\right]^{-1}(x_0,x)$ in terms of the the Fourier transform (see Eq. 3.2.37 below). Then, we will let $\mathrm{m} \downarrow 0$ and prove that the limit is well-defined. Let $\mathcal{F} : \ell^2(\mathbb{Z}^{\mathrm{d}}) \longrightarrow \mathrm{L}^2([-\pi,\pi]^{\mathrm{d}})$ be the discrete Fourier transform

$$[\mathcal{F}u](k) = \frac{1}{(2\pi)^{\frac{\mathrm{d}}{2}}} \sum_{x\in\mathbb{Z}^{\mathrm{d}}} e^{-iy\cdot k}u(x), \quad \forall u \in \ell^2(\mathbb{Z}^{\mathrm{d}}),\ \forall k \in [-\pi,\pi]^{\mathrm{d}} \tag{3.2.34}$$

and let $\mathcal{F}^{-1} : \mathrm{L}^2([-\pi,\pi]^{\mathrm{d}}) \longrightarrow \ell^2(\mathbb{Z}^{\mathrm{d}})$ be the inverse of $\mathcal{F}$

$$\left[\mathcal{F}^{-1}g\right](x) = \frac{1}{(2\pi)^{\frac{\mathrm{d}}{2}}} \int_{[-\pi,\pi]^{\mathrm{d}}} \mathrm{d}k\, e^{ix\cdot k} g(k), \quad \forall g \in \mathrm{L}^2([-\pi,\pi]^{\mathrm{d}}),\ \forall x \in \mathbb{Z}^{\mathrm{d}}. \tag{3.2.35}$$

The discrete Laplacian is diagonalized by the discrete Fourier transform, i.e.,

$$\left[\mathcal{F}(-\Delta)\mathcal{F}^{-1}g\right](k) = \left(\sum_{j=1}^{\mathrm{d}} (2 - 2\cos k_j)\right) g(k) := \mathrm{f}(k) g(k), \qquad \forall g \in \mathrm{L}^2([-\pi,\pi]^{\mathrm{d}}),\ \forall k \in [-\pi,\pi]^{\mathrm{d}}. \tag{3.2.36}$$

Then,

$$\left[(-\Delta)^{\alpha} + \mathrm{m}^2\right]^{-1}(x_0, x) = \frac{1}{(2\pi)^{\mathrm{d}}} \int_{[-\pi,\pi]^{\mathrm{d}}} \mathrm{d}k \frac{e^{i(x-x_0)\cdot k}}{\mathrm{f}^{\alpha} + \mathrm{m}^2} = \frac{1}{(2\pi)^{\frac{\mathrm{d}}{2}}} \mathcal{F}^{-1}\left[\frac{1}{\mathrm{f}^{\alpha} + \mathrm{m}^2}\right](x - x_0). \tag{3.2.37}$$

Hence, the operator $\left[(-\Delta)^{\alpha} + \mathrm{m}^2\right]^{-1}(x_0, x)$ only depends on the difference $x - x_0$. In particular, the operator is invariant under translations. Without loss of generality, we can assume $x_0 = 0$ from now on. Observe that $\mathrm{f}(0) = 0$ (see Eq. 3.2.36). To circumvent this problem in the denominator on the RHS of Eq. 3.2.37 as $\mathrm{m} \downarrow 0$, we will introduce a suitable smooth function with compact support on $\mathbb{R}^{\mathrm{d}}$. Let $\psi \in C^{\infty}(\mathbb{R}^{\mathrm{d}})$ such that:

$$\mathrm{supp}\psi \subseteq \partial B(0,1), \qquad 0 \le \psi \le 1, \qquad \psi(k) = 1, \quad \forall k \in B\Big(0, \frac{1}{2}\Big). \tag{3.2.38}$$

Hence,

$$\left[(-\Delta)^{\alpha} + \mathrm{m}^2\right]^{-1}(0, x) = \frac{1}{(2\pi)^{\frac{\mathrm{d}}{2}}} \mathcal{F}^{-1}\left[\frac{\psi}{\mathrm{f}^{\alpha} + \mathrm{m}^2}\right](x) + \frac{1}{(2\pi)^{\frac{\mathrm{d}}{2}}} \mathcal{F}^{-1}\left[\frac{1-\psi}{\mathrm{f}^{\alpha} + \mathrm{m}^2}\right](x). \tag{3.2.39}$$

On the one hand, note $(1 - \psi(k)) = 0$, for all $k \in [-\frac{1}{2}, \frac{1}{2}]$, then the pointwise-limit of the second term on the RHS as $\mathrm{m} \downarrow 0$ is well-defined. Let $\frac{1-\psi}{\mathrm{f}^{\alpha}} := \lim_{\mathrm{m}\downarrow 0} \frac{1-\psi}{\mathrm{f}^{\alpha}+\mathrm{m}^2}$. Hence,

$$\left|\frac{1}{\mathrm{f}^{\alpha}(k) + \mathrm{m}^2}(1 - \psi(k))\right| \le \frac{1 - \psi(k)}{|\mathrm{f}^{\alpha}(k)|} \in \mathrm{L}^1([-\pi,\pi]^{\mathrm{d}}). \tag{3.2.40}$$

By Dominated Convergence Theorem, we obtain

$$\lim_{\mathrm{m}\downarrow 0} \mathcal{F}^{-1}\left[\frac{1-\psi}{\mathrm{f}^{\alpha} + \mathrm{m}}\right](x) = \frac{1}{(2\pi)^{\frac{\mathrm{d}}{2}}} \int_{[-\pi,\pi]^{\mathrm{d}}} \mathrm{d}k\, e^{ix\cdot k} \frac{1 - \psi(k)}{\mathrm{f}^{\alpha}(k)}. \tag{3.2.41}$$

On the other hand,

$$\left|\frac{1}{\mathrm{f}^{\alpha}(k) + \mathrm{m}^2} \psi(k)\right| \le \frac{1}{|k|^{2\alpha}} |\Phi_{\alpha,\psi}(k)|, \tag{3.2.42}$$

where $\Phi_{\alpha,\psi}$ be a real-valued function given by

$$\Phi_{\alpha,\psi}(k) = \begin{cases} \left(\frac{|k|^2}{\mathrm{f}(k)}\right)^{\alpha} \psi(k), & k \neq 0, \\ 1, & k = 0. \end{cases} \tag{3.2.43}$$

Observe that $\Phi_{\alpha,\psi}$ is a smooth function with compact support. In fact, let us assume for the moment that $\mathrm{d}=1$,

$$\frac{\mathrm{f}(k)}{|k|^2} = \frac{2-2\cos k}{|k|^2} = 2\sum_{l\geq 0}\frac{(-1)^l}{(2l+2)!}k^{2l} > 0 \qquad \forall k\in\left[-\frac{1}{2},\frac{1}{2}\right], \tag{3.2.44}$$

where we made use of the Taylor's expansion of the cosine function. Note that the function $\mathbb{R}\setminus\{0\}\ni x\in\mathbb{R}\mapsto x^{-\alpha}$ is $C^\infty(\mathbb{R})$. So after composing and multiplying smooth functions, $\Phi_{\alpha,\psi}(k)\in C^\infty(\mathbb{R})$ and in particular $\lim_{|k|\to 0}\Phi_{\alpha,\psi}(k)=1$. Since ψ has compact support, $\Phi_{\alpha,\psi}(k)\in C_c^\infty(\mathbb{R})$. The general case goes along the same lines. Moreover, $\frac{1}{|k|^{2\alpha}}\Phi_{\alpha,\psi}(k)\in L^1(\mathbb{R}^\mathrm{d})$. Indeed,

$$\int_{B(0,1)}\mathrm{d}k\frac{1}{|k|^{2\alpha}}|\Phi_{\alpha,\psi}(k)| \leq \|\Phi_{\alpha,\psi}\|_\infty\int_0^1\mathrm{d}r\frac{r^{\mathrm{d}-1}}{r^{2\alpha}}\int_{B(0,1)}\mathrm{d}s\,(\theta) = \frac{|\partial B(0,1)|}{\mathrm{d}-2\alpha}\|\Phi_{\alpha,\psi}\|_\infty. \tag{3.2.45}$$

After applying Dominated Convergence Theorem as $\mathrm{m}\downarrow 0$, we obtain

$$\lim_{\mathrm{m}\downarrow 0}\mathcal{F}^{-1}\left[\frac{\psi}{\mathrm{f}^\alpha+\mathrm{m}^2}\right](x) = \frac{1}{(2\pi)^{\frac{\mathrm{d}}{2}}}\int_{[-\pi,\pi]^\mathrm{d}}\mathrm{d}k\,e^{ix\cdot k}\frac{1}{|k|^{2\alpha}}\Phi_{\alpha,\psi}(k).$$

Therefore,

$$(-\Delta)^{-\alpha}(0,x) = \frac{1}{(2\pi)^\mathrm{d}}\int_{[-\pi,\pi]^\mathrm{d}}\mathrm{d}k\,e^{ix\cdot k}\frac{\Phi_{\alpha,\psi}(k)}{|k|^{2\alpha}} + \frac{1}{(2\pi)^\mathrm{d}}\int_{[-\pi,\pi]^\mathrm{d}}\mathrm{d}k\,e^{ix\cdot k}\frac{1-\psi(k)}{\mathrm{f}^\alpha(k)}. \tag{3.2.46}$$

This proves the fact that the inverse of the fractional Laplacian is well-defined.

Now, we prove Ineq. 3.2.33. By Eq. 3.2.46, we get

$$\begin{aligned}&\lim_{|x|\to\infty}|x|^{\mathrm{d}-2\alpha}(-\Delta)^{-\alpha}(0,x)\\ &\quad=\lim_{|x|\to\infty}\frac{|x|^{\mathrm{d}-2\alpha}}{(2\pi)^\mathrm{d}}\int_{[-\pi,\pi]^\mathrm{d}}\mathrm{d}k\,e^{ix\cdot k}\frac{1}{|k|^{2\alpha}}\Phi_{\alpha,\psi}(k) + \frac{|x|^{\mathrm{d}-2\alpha}}{(2\pi)^\mathrm{d}}\int_{[-\pi,\pi]^\mathrm{d}}\mathrm{d}k\,e^{ix\cdot k}\frac{1-\psi(k)}{\mathrm{f}^\alpha(k)}\end{aligned} \tag{3.2.47}$$

On the one hand, the function $\frac{1-\psi}{\mathrm{f}^\alpha}$ is $\mathbb{C}^\infty([-\pi,\pi]^\mathrm{d})$, hence its Fourier coefficients decay faster than any polynomial $p(x)$ as $|x|\to\infty$ (see [Gra14, Theorem 3.2.9]). In particular,

$$\lim_{|x|\to\infty}\frac{|x|^{\mathrm{d}-2\alpha}}{(2\pi)^\mathrm{d}}\int_{[-\pi,\pi]^\mathrm{d}}\mathrm{d}k\,e^{ix\cdot k}\frac{1-\psi(k)}{\mathrm{f}^\alpha(k)} = \lim_{|x|\to\infty}\frac{|x|^{\mathrm{d}-2\alpha}}{(2\pi)^{\frac{\mathrm{d}}{2}}}\mathcal{F}^{-1}\left[\frac{1-\psi}{\mathrm{f}^\alpha}\right](x) = 0. \tag{3.2.48}$$

Therefore, it remains to compute the limit of the first term on the RHS of Eq. 3.2.47. To do that, we have to extend our analysis to $\mathbb{R}^\mathrm{d}$, where the limit is well-studied. This computation follows the same strategy as the proof provided in [GRM20, Lemma A.1]. Let $\mathcal{F}_c:\mathrm{L}^2(\mathbb{R}^\mathrm{d})\longrightarrow\mathrm{L}^2(\mathbb{R}^\mathrm{d})$ be the continuous Fourier transform

$$[\mathcal{F}_c g](k) = \frac{1}{(2\pi)^{\frac{\mathrm{d}}{2}}}\int_{\mathbb{R}^\mathrm{d}}\mathrm{d}y\,e^{-iy\cdot k}g(y), \qquad \forall g\in\mathrm{L}^2(\mathbb{R}^\mathrm{d})\;\forall k\in\mathbb{R}^\mathrm{d}. \tag{3.2.49}$$

and $\mathcal{F}_c^{-1} : \mathrm{L}^2(\mathbb{R}^{\mathrm{d}}) \longrightarrow \mathrm{L}^2(\mathbb{R}^{\mathrm{d}})$ be its inverse

$$\left[\mathcal{F}_c^{-1} g\right](y) = \frac{1}{(2\pi)^{\frac{\mathrm{d}}{2}}} \int_{\mathbb{R}^{\mathrm{d}}} \mathrm{d}k\, e^{iy\cdot k} g(k), \quad \forall g \in \mathrm{L}^2(\mathbb{R}^{\mathrm{d}})\, \forall y \in \mathbb{R}^{\mathrm{d}}. \tag{3.2.50}$$

By abuse of notation, we also denote by $\Phi_{\alpha,\psi}$ the natural extension of $\Phi_{\alpha,\psi}$ into $\mathbb{R}^{\mathrm{d}}$. Then, $\Phi_{\alpha,\psi} \in C_c(\mathbb{R}^{\mathrm{d}}) \subseteq \mathcal{S}(\mathbb{R}^{\mathrm{d}})$. Since the Fourier transform is an invertible mapping which maps $\mathcal{S}(\mathbb{R}^{\mathrm{d}})$ onto itself (see [Gra14, Corollary 2.2.15]), $\mathcal{F}_c^{-1}\Phi_{\alpha,\psi} \in \mathcal{S}(\mathbb{R}^{\mathrm{d}})$ and Eq. 3.2.47 can be rewritten as

$$\lim_{|x|\to\infty} |x|^{\mathrm{d}-2\alpha}(-\Delta)^{-\alpha}(0,x) = \lim_{|y|\to\infty} \frac{|y|^{\mathrm{d}-2\alpha}}{(2\pi)^{\mathrm{d}}} \int_{\mathbb{R}^{\mathrm{d}}} \mathrm{d}k\, e^{iy\cdot k} \frac{1}{|k|^{2\alpha}} \mathcal{F}_c[\mathcal{F}_c^{-1}\Phi_{\alpha,\psi}](k) \tag{3.2.51}$$

In order to compute the above limit, we make use of two propositions, which will be proven later. The first one can be found in [LL01, Section 5.9] (the constants are slightly different because of our choice of definition of Fourier transform). The second one goes along the same lines of the proof of [GRM20, Lemma A.1]. But before stating such propositions, we will define *the Riesz potential*, which will help us to make the computations clearer.

Definition 3.2.8 (The Riesz potential)**.** *Let* $0 < \alpha < \frac{\mathrm{d}}{2}$ *and* $\mathfrak{c}_\alpha = \frac{\Gamma(\frac{\mathrm{d}}{2}-\alpha)}{\Gamma(\alpha)} 2^{\frac{\mathrm{d}}{2}-2\alpha}$. *The Riesz potential of order* α *is defined as*

$$\left[\mathrm{I}_\alpha \varphi\right](y) = \mathfrak{c}_\alpha \int_{\mathbb{R}^{\mathrm{d}}} \frac{\mathrm{d}w}{|y-w|^{\mathrm{d}-2\alpha}} \varphi(w), \qquad \forall \varphi \in \mathcal{S}(\mathbb{R}^{\mathrm{d}}),\, \forall y \in \mathbb{R}^{\mathrm{d}}. \tag{3.2.52}$$

Proposition 3.2.9 below allows us to rewrite the RHS of Eq. 3.2.51 in terms of the Riesz potential.

Proposition 3.2.9. *[LL01, Thm. 5.9] Let* $\alpha \in (0, \frac{\mathrm{d}}{2})$ *and let* $\varphi \in \mathcal{S}(\mathbb{R}^{\mathrm{d}})$, *then*

$$\left[\mathrm{I}_\alpha \varphi\right](y) = \frac{1}{(2\pi)^{\frac{\mathrm{d}}{2}}} \int_{\mathbb{R}^{\mathrm{d}}} \mathrm{d}k e^{ik\cdot y} \frac{1}{|k|^{2\alpha}} \left[\mathcal{F}_c \varphi\right](k). \tag{3.2.53}$$

Remark 3.2.10. *Let* $\varphi \in \mathcal{S}(\mathbb{R}^{\mathrm{d}})$. *Then,* $[\mathcal{F}_c\varphi] \in \mathcal{S}(\mathbb{R}^{\mathrm{d}})$ *(see [Gra14, Corollary 2.2.15]). Hence* $\frac{1}{|k|^{2\alpha}}[\mathcal{F}_c\varphi](k)$ *is* $L^1(\mathbb{R}^{\mathrm{d}})-$*integrable and its Fourier transform is well-defined. Prop. 3.2.9 is equivalent to saying that* $\mathcal{F}_c[\mathrm{I}_\alpha\varphi](k) = |k|^{-(2\alpha)}[\mathcal{F}_c\varphi](k)$ *holds in the sense that*

$$\langle I_\alpha \phi, \psi\rangle_{L^2(\mathbb{R}^{\mathrm{d}})} = \langle \mathcal{F}_c[I_\alpha\phi], \mathcal{F}_c\psi\rangle_{L^2(\mathbb{R}^{\mathrm{d}})} \qquad \forall \phi, \psi \in \mathcal{S}(\mathbb{R}^{\mathrm{d}}). \tag{3.2.54}$$

Proposition 3.2.11. *Let* $\alpha \in (0, \frac{\mathrm{d}}{2})$ *and* $\varphi \in \mathcal{S}(\mathbb{R}^{\mathrm{d}})$, *then*

$$\lim_{|y|\to\infty} |y|^{\mathrm{d}-2\alpha}[\mathrm{I}_\alpha\varphi](y) = (2\pi)^{\frac{\mathrm{d}}{2}} \mathfrak{c}_\alpha [\mathcal{F}\varphi](0). \tag{3.2.55}$$

Applying Prop. 3.2.9 and Prop. 3.2.11 to Eq. 3.2.51 to get that

$$\begin{aligned} \lim_{|x|\to\infty} |x|^{\mathrm{d}-2\alpha}(-\Delta)^{-\alpha}(0,x) &= \lim_{|y|\to\infty} \frac{|y|^{\mathrm{d}-2\alpha}}{(2\pi)^{\mathrm{d}}} \int_{\mathbb{R}^{\mathrm{d}}} \mathrm{d}k\, e^{iy\cdot k} \frac{1}{|k|^{2\alpha}} \mathcal{F}_c[\mathcal{F}_c^{-1}\Phi_{\alpha,\psi}](k) \\ &= \frac{1}{(2\pi)^{\frac{\mathrm{d}}{2}}} \lim_{|y|\to\infty} |y|^{\mathrm{d}-2\alpha}[\mathrm{I}_\alpha(\mathcal{F}_c^{-1}\Phi_{\alpha,\psi})](y) = \mathfrak{c}_\alpha \mathcal{F}_c[\mathcal{F}_c^{-1}\Phi_{\alpha,\psi}](0) = \mathfrak{c}_\alpha, \end{aligned} \tag{3.2.56}$$

where, in the last step, we used $\Phi_{\alpha,\psi}(0) = 1$ (see definition of $\Phi_{\alpha,\psi}$ Eq. 3.2.43). □

3.2.4 Riesz Potential

As we previously pointed it out, Prop. 3.2.9 can be found in [LL01, Thm. 5.9] and Prop. 3.2.11 is based on an argument in [GRM20, Lemma A.1] but for the sake of self-containment, we provide a proof for each one of them.

Proof of Prop. 3.2.9. To begin with, we use the following identity,

$$\frac{1}{\rho^{\alpha}} = \frac{1}{2^{\alpha}\Gamma(\alpha)}\int_0^{\infty} \mathrm{d}t\, t^{\alpha-1}e^{-\frac{t}{2}\rho}. \tag{3.2.57}$$

Indeed, by the definition of the gamma function and a change of variables, we obtain

$$\Gamma(\alpha) := \int_0^{\infty} \mathrm{d}y\, x^{\alpha-1}e^{-x} = \frac{\rho^{\alpha}}{2^{\alpha}}\int_0^{\infty} \mathrm{d}t\, t^{\alpha-1}e^{-\frac{t}{2}\rho}.$$

Therefore,

$$\frac{1}{|k|^{2\alpha}} = \frac{1}{2^{\alpha}\Gamma(\alpha)}\int_0^{\infty} \mathrm{d}t\, t^{\alpha-1}e^{-\frac{t}{2}|k|^2}, \qquad \forall k \in \mathbb{R}^{\mathrm{d}} \setminus \{0\}. \tag{3.2.58}$$

and the RSH of Eq. 3.2.53 can be written as

$$\frac{1}{(2\pi)^{\frac{\mathrm{d}}{2}}}\int_{\mathbb{R}^{\mathrm{d}}} \mathrm{d}k e^{ik\cdot x}\frac{1}{|k|^{2\alpha}}[\mathcal{F}_c\varphi](k) = \frac{1}{(2\pi)^{\frac{\mathrm{d}}{2}}}\int_{\mathbb{R}^{\mathrm{d}}} \mathrm{d}k\, e^{ik\cdot x}[\mathcal{F}_c\varphi](k)\left\{\frac{1}{2^{\alpha}\Gamma(\alpha)}\int_0^{\infty}\mathrm{d}t\, t^{\alpha-1}e^{-\frac{t}{2}|k|^2}\right\}.$$

Let $\mathrm{g}(y) = e^{-\frac{1}{2}|y|^2}$, $y \in \mathbb{R}^{\mathrm{d}}$ be the Gaussian function. Given $\mathrm{a} > 0$, let $\mathrm{g}_{\mathrm{a}}(y) := \mathrm{g}(\mathrm{a}y)$. By our choice of Fourier transform, $[\mathcal{F}_c\mathrm{g}_{\mathrm{a}}](k) = \mathrm{a}^{-\mathrm{d}}e^{-\frac{|k|^2}{2\mathrm{a}^2}}$.

We apply Fubini-Tonelli and the Fourier transform of a Gaussian function to the RHS of the above equation, which yields

$$\frac{1}{(2\pi)^{\frac{\mathrm{d}}{2}}}\int_{\mathbb{R}^{\mathrm{d}}} \mathrm{d}k e^{ik\cdot y}\frac{1}{|k|^{2\alpha}}[\mathcal{F}_c\varphi](k) = \frac{1}{2^{\alpha}\Gamma(\alpha)}\int_0^{\infty}\mathrm{d}t\, t^{\alpha-1}\left\{\frac{1}{(2\pi)^{\frac{\mathrm{d}}{2}}}\int_{\mathbb{R}^{\mathrm{d}}}\mathrm{d}k\, e^{ik\cdot y}\mathrm{g}_{\sqrt{t}}(k)[\mathcal{F}_c\varphi](k)\right\}$$
$$= \frac{1}{2^{\alpha}\Gamma(\alpha)}\int_0^{\infty}\mathrm{d}t\, t^{\alpha-1}\left\{\frac{1}{(2\pi)^{\frac{\mathrm{d}}{2}}}\int_{\mathbb{R}^{\mathrm{d}}}\mathrm{d}k\, e^{ik\cdot y}t^{-\frac{\mathrm{d}}{2}}\left[\mathcal{F}_c\mathrm{g}_{\frac{1}{\sqrt{t}}}\right](k)[\mathcal{F}_c\varphi](k)\right\}$$
$$= \frac{1}{2^{\alpha}\Gamma(\alpha)}\int_0^{\infty}\mathrm{d}t\, t^{(\alpha-\frac{\mathrm{d}}{2})-1}\left\{\mathcal{F}_c^{-1}\left(\left[\mathcal{F}_c\mathrm{g}_{\frac{1}{\sqrt{t}}}\right][\mathcal{F}_c\varphi]\right)\right\}(y)$$

We know $\mathcal{F}_c^{-1}[f_1 * f_2](k) = \mathcal{F}_c^{-1}[f_1](k)\mathcal{F}_c^{-1}[f_2](k)$, for all $f_1, f_2 \in L^2(\mathbb{R}^{\mathrm{d}})$. Hence, after using this identity and Fubini-Tonelli again, we obtain that

$$\frac{1}{(2\pi)^{\frac{\mathrm{d}}{2}}}\int_{\mathbb{R}^{\mathrm{d}}} \mathrm{d}k e^{ik\cdot y}\frac{1}{|k|^{2\alpha}}[\mathcal{F}_c\varphi](k) = \frac{1}{2^{\alpha}\Gamma(\alpha)}\int_0^{\infty}\mathrm{d}t\, t^{(\alpha-\frac{\mathrm{d}}{2})-1}\left\{\mathcal{F}_c^{-1}\left[\mathcal{F}_c\left(\mathrm{g}_{\frac{1}{\sqrt{t}}} * \varphi\right)\right]\right\}(y)$$
$$= \frac{1}{2^{\alpha}\Gamma(\alpha)}\int_0^{\infty}\mathrm{d}t\, t^{(\alpha-\frac{\mathrm{d}}{2})-1}\int_{\mathbb{R}^{\mathrm{d}}}\mathrm{d}w\, e^{-\frac{1}{2t}|y-w|^2}\varphi(w)$$

We use Fubini-Tonelli and Eq. 3.2.58 to get

$$\frac{1}{(2\pi)^{\frac{\mathrm{d}}{2}}}\int_{\mathbb{R}^{\mathrm{d}}} \mathrm{d}k e^{ik\cdot y}\frac{1}{|k|^{2\alpha}}[\mathcal{F}_c\varphi](k) = \frac{1}{2^{\alpha}\Gamma(\alpha)}\int_{\mathbb{R}^{\mathrm{d}}}\mathrm{d}w\,\varphi(w)\int_0^{\infty}\mathrm{d}t\, t^{(\alpha-\frac{\mathrm{d}}{2})-1}e^{-\frac{1}{2t}|y-w|^2}$$
$$= \frac{1}{2^{\alpha}\Gamma(\alpha)}\int_{\mathbb{R}^{\mathrm{d}}}\mathrm{d}y\,\varphi(w)\int_0^{\infty}\mathrm{d}\tilde{t}\,\tilde{t}^{(\frac{\mathrm{d}}{2}-\alpha)-1}e^{-\frac{\tilde{t}}{2}|y-w|^2} = \frac{\Gamma(\frac{\mathrm{d}}{2}-\alpha)}{\Gamma(\alpha)}2^{\frac{\mathrm{d}}{2}-2\alpha}\int_{\mathbb{R}^{\mathrm{d}}}\frac{\mathrm{d}w}{|y-w|^{\mathrm{d}-2\alpha}}\varphi(w)$$
$$= [\mathrm{I}_{\alpha}\varphi](y).$$

$\square$

Proof of Prop. 3.2.11.

$$\int_{\mathbb{R}^{\mathrm{d}}} \frac{\mathrm{d}w}{|y-w|^{\mathrm{d}-2\alpha}}\varphi(w) = \int_{B\left(y,\frac{|y|}{2}\right)} \frac{\mathrm{d}w}{|y-w|^{\mathrm{d}-2\alpha}}\varphi(w) + \int_{B\left(y,\frac{|y|}{2}\right)^C} \frac{\mathrm{d}w}{|y-w|^{\mathrm{d}-2\alpha}}\varphi(w)$$

Note

$$\lim_{|y|\to\infty} |y|^{\mathrm{d}-2\alpha} \int_{B\left(y,\frac{|y|}{2}\right)} \frac{\mathrm{d}w}{|y-w|^{\mathrm{d}-2\alpha}}\varphi(w) = \int_{\mathbb{R}^{\mathrm{d}}} \mathrm{d}w\, \varphi(w) = (2\pi)^{\frac{\mathrm{d}}{2}}[\mathcal{F}_c\varphi](0). \tag{3.2.59}$$

Indeed, given $y \in \mathbb{R}^{\mathrm{d}}$

$$\left| \frac{|y|^{\mathrm{d}-2\alpha}}{|y-w|^{\mathrm{d}-2\alpha}}\varphi(w)\mathbb{1}_{B\left(y,\frac{|y|}{2}\right)^C}(w) \right| \leq 2^{\mathrm{d}-2\alpha}|\varphi(w)| \in L^1(\mathbb{R}^{\mathrm{d}}). \tag{3.2.60}$$

In addition,

$$\lim_{|y|\to\infty} \frac{|y|^{\mathrm{d}-2\alpha}}{|y-w|^{\mathrm{d}-2\alpha}}\varphi(w)\mathbb{1}_{B\left(y,\frac{|y|}{2}\right)^C}(w) = \varphi(w) \qquad \forall w \in \mathbb{R}^{\mathrm{d}}. \tag{3.2.61}$$

Consequently, by Dominated Convergence Theorem, the desired result follows. Therefore, it remains to show that

$$\lim_{|y|\to\infty} |y|^{\mathrm{d}-2\alpha} \int_{B\left(y,\frac{|y|}{2}\right)^C} \frac{\mathrm{d}w}{|y-w|^{\mathrm{d}-2\alpha}}\varphi(w) = 0. \tag{3.2.62}$$

To this end, we will decompose the term into two addends and prove that each one of them tends to zero as $|y| \to \infty$. To be more explicit,

$$\begin{aligned} &|y|^{\mathrm{d}-2\alpha} \int_{B\left(y,\frac{|y|}{2}\right)^C} \frac{\mathrm{d}w}{|y-w|^{\mathrm{d}-2\alpha}}\varphi(w) \\ &= |y|^{\mathrm{d}-2\alpha} \int_{B\left(y,\frac{|y|}{2}\right)} \mathrm{d}w \frac{\varphi(w)-\varphi(y)}{|y-w|^{\mathrm{d}-2\alpha}} + |y|^{\mathrm{d}-2\alpha}\varphi(y) \int_{B\left(y,\frac{|y|}{2}\right)} \frac{\mathrm{d}w}{|y-w|^{\mathrm{d}-2\alpha}} =: \mathcal{I}_1(y) + \mathcal{I}_2(y) \end{aligned} \tag{3.2.63}$$

On the one hand,

$$\begin{aligned} \mathcal{I}_2(y) &= |y|^{\mathrm{d}-2\alpha}\varphi(x) \int_{B\left(y,\frac{|y|}{2}\right)} \frac{\mathrm{d}w}{|y-w|^{\mathrm{d}-2\alpha}} = |y|^{\mathrm{d}-2\alpha}\varphi(x) \int_0^{\frac{|y|}{2}} \mathrm{d}r \frac{r^{\mathrm{d}-1}}{r^{\mathrm{d}-2\alpha}} \int_{\partial B(0,1)} \mathrm{d}s\,(\theta) \\ &= |\partial B(0,1)||y|^{\mathrm{d}-2\alpha}\varphi(y) \int_0^{\frac{|y|}{2}} \mathrm{d}r\, r^{2\alpha-1} = \frac{|\partial B(0,1)|}{2\alpha}|y|^{\mathrm{d}}\varphi(y). \end{aligned}$$

As a result of the above computation and the fact that $\varphi \in \mathcal{S}(\mathbb{R})$, we get that $\lim_{|y|\to\infty} |\mathcal{I}_2(y)| = 0$.

On the other hand,

$$\mathcal{I}_1(y) = |y|^{\mathrm{d}-2\alpha} \int_0^{\frac{|y|}{2}} \mathrm{d}r \frac{r^{\mathrm{d}-1}}{r^{\mathrm{d}-2\alpha}} \int_{\partial B(0,1)} \mathrm{d}s\,(\theta)[\varphi(y+r\theta)-\varphi(y)]. \tag{3.2.64}$$

Let $h(r) = \int_{\partial B(0,1)} \mathrm{d}s\,(\theta)\varphi(y + r\theta)$, $r > 0$. Then,

$$\mathcal{I}_1(y) = |y|^{\mathrm{d}-2\alpha} \int_0^{\frac{|y|}{2}} \mathrm{d}r\, r^{2\alpha-1}[h(r) - h(0)] = |y|^{\mathrm{d}-2\alpha} \int_0^{\frac{|y|}{2}} \mathrm{d}r\, r^{2\alpha-1} \int_0^r \mathrm{d}s\ h'(s). \tag{3.2.65}$$

Let us compute $h'(s)$,

$$\begin{aligned} h'(s) &= \int_{\partial B(0,1)} \mathrm{d}s\,(\theta)\, \nabla\varphi(y + s\theta) \cdot w = \frac{1}{s^{\mathrm{d}-1}} \int_{\partial B(y,s)} \mathrm{d}s\,(v)\, \nabla\varphi(v) \cdot \frac{v-y}{s} \\ &= \frac{1}{s^{\mathrm{d}-1}} \int_{B(y,s)} \mathrm{d}y\, \Delta\varphi(y), \end{aligned} \tag{3.2.66}$$

the last equality is justified by the Divergence Theorem. It follows that

$$\mathcal{I}_1(y) = |y|^{\mathrm{d}-2\alpha} \int_0^{\frac{|y|}{2}} \mathrm{d}r\, r^{2\alpha-1} \int_0^r \mathrm{d}s\, \frac{1}{s^{\mathrm{d}-1}} \int_{B(y,s)} \mathrm{d}y\, \Delta\varphi(y). \tag{3.2.67}$$

Let $\mathcal{M} := \sup_{w \in B\left(y, \frac{|y|}{2}\right)} |\Delta\varphi(w)|$, then

$$\begin{aligned} |\mathcal{I}_1(y)| &\le \mathcal{M}|y|^{\mathrm{d}-2\alpha} \int_0^{\frac{|y|}{2}} \mathrm{d}r\, r^{2\alpha-1} \int_0^r \mathrm{d}s\, \frac{1}{s^{\mathrm{d}-1}} |B(0,1)| s^{\mathrm{d}} \\ &= \frac{|B(0,1)|}{2} \mathcal{M}|y|^{\mathrm{d}-2\alpha} \int_0^{\frac{|y|}{2}} \mathrm{d}r\, r^{2\alpha+1} = \frac{|B(0,1)|}{4(\alpha+1)} \mathcal{M}|y|^{\mathrm{d}+2} \end{aligned}$$

As a result of this, we obtain that $\lim_{|x|\to\infty} |\mathcal{I}_1(y)| = 0$. □

3.3 Fractional Anderson model

3.3.1 Definition of the model

Let $0 < \alpha \le 1$ and $\lambda > 0$. We consider the discrete fractional Anderson model of the form

$$\mathrm{H}_{\omega,\lambda,\alpha} := (-\Delta)^{\alpha} + \lambda \mathrm{V}_{\omega}, \qquad \omega \in \Omega, \tag{3.3.1}$$

acting on the Hilbert space $\ell^2(\mathbb{Z}^{\mathrm{d}})$, where $(-\Delta)^{\alpha}$ is defined in Section 3.2 and V_{ω} is the random potential given by $\mathrm{V}_{\omega} := \sum_{x\in\mathbb{Z}^{\mathrm{d}}} \omega_x \langle \delta_x, \cdot \rangle \delta_x$ with $\omega := (\omega_x)_{x\in\mathbb{Z}^{\mathrm{d}}} \in \mathbb{R}^{\mathbb{Z}^d}$ being a family of bounded iid random variables with common distribution P_0 defined in the probability space $\Omega := \mathbb{R}^{\mathbb{Z}^{\mathrm{d}}}$ endowed with the Borel probability measure $\mathbb{P} := \bigotimes_{\mathbb{Z}^{\mathrm{d}}} \mathrm{P}_0$ with expectation $\mathbb{E}[\cdot]$. By the boundedness of the random variables ω_x, the support of P_0 (denoted by $\mathrm{supp}\mathrm{P}_0$) is compact. We assume that P_0 is non-trivial and absolutely continuous with respect to the Lebesgue measure. Furthermore, given $\tau \in (0,1]$, we suppose that P_0 is τ-regular in the sense of Definition 3.3.1 below.

Definition 3.3.1 ($\tau-$regularity)**.** *Let $\tau \in (0,1]$. We say that a probability measure μ is $\tau-$regular if there is a $\mathrm{C} > 0$ such that $\mu([\mathrm{v}-\delta, \mathrm{v}+\delta]) \le \mathrm{C}\delta^{\tau}$, $\forall \mathrm{v} \in \mathbb{R}$, $\forall \delta > 0$. If μ is $\tau-$regular, we define*

$$\mathrm{M}_{\tau}(\mu) := \inf\{\mathrm{C} > 0 \mid \mu([\mathrm{v}-\delta, \mathrm{v}+\delta]) \le \mathrm{C}\delta^{\tau},\ \forall \mathrm{v} \in \mathbb{R},\ \forall \delta > 0\}. \tag{3.3.2}$$

Let $\sigma(\mathrm{H}_{\omega,\lambda,\alpha})$ be the spectrum of $\mathrm{H}_{\omega,\lambda,\alpha}$. As a result of the translation invariance of the free operator $(-\Delta)^{\alpha}$ and the fact $\omega := (\omega_x)_{x\in\mathbb{Z}^{\mathrm{d}}} \in \mathbb{R}^{\mathbb{Z}^d}$ being a family of bounded iid random variables, $\{\mathrm{H}_{\omega,\lambda,\alpha}\}_{\omega\in\Omega}$ is a family of ergodic operators in the sense of [PF92]. Therefore, $\sigma(\mathrm{H}_{\omega,\lambda,\alpha})$ is deterministic, for $\mathbb{P}-$a.s $\omega \in \Omega$. Moreover, it holds that $\sigma(\mathrm{H}_{\omega,\lambda,\alpha}) = [0,(4\mathrm{d})^{\alpha}] + \lambda \mathrm{supp}\mathrm{P}_0$, for $\mathbb{P}-$a.s $\omega \in \Omega$, see [PF92]. For $z \notin [0,(4\mathrm{d})^{\alpha}] + \lambda\mathrm{supp}\mathrm{P}_0$, the operator $\mathrm{G}_{z,\omega,\lambda,\alpha} = (\mathrm{H}_{\omega,\lambda,\alpha} - z)^{-1}$ is well-defined, for $\mathbb{P}-$a.s $\omega \in \Omega$. From now on, we will regularly omit the dependence on ω from $\mathrm{H}_{\omega,\lambda,\alpha}$ and $\mathrm{G}_{z,\omega,\lambda,\alpha}$ to lighten the notation.

3.3.2 Known results

In the subsequent discussion, we present three established results that offer precise rates of decay for the fractional moments of the Green's function. More precisely, they allow us to quantify the magnitude of λ required to achieve spectral localization and polynomial decay of the eigenfunctions, based on one of these estimates.

Thm. 3.3.2 below is a consequence of both Ineq. 3.2.15 and an adaptation of [AM93, Thm. 3.1] to the fractional Anderson model. From now on, we set $\triangle^{\alpha}(x_0,x) = -(-\Delta)^{\alpha}(x_0,x)$, for all $x, x_0 \in \mathbb{Z}^{\mathrm{d}}$.

Theorem 3.3.2. *Let* $\frac{\mathrm{d}}{\mathrm{d}+2\alpha} < s < \tau \leq 1$. *Let* $\alpha_s = s\big(\alpha + \frac{\mathrm{d}}{2}\big) - \frac{\mathrm{d}}{2}$, *the constant in Eq. 3.2.14. Then,*

$$\mathrm{B}_s(t) = \sum_{y\in\mathbb{Z}^{\mathrm{d}}\setminus\{0\}} \triangle^{\alpha_s}(y,0)(1+|y|)^t < \infty, \qquad \forall t \in (0, 2\alpha_s), \tag{3.3.3}$$

and there is a unique constant $\beta_{\lambda,s} \in (0, 2\alpha_s)$ *satisfying the following equation*

$$\mathrm{B}_s(\beta_{\lambda,s}) = \lambda^s \left(\frac{\tau - s}{2\tau \mathrm{M}_{\tau}(\mathrm{P}_0)^{\frac{1}{\tau}}} \right)^s. \tag{3.3.4}$$

Moreover, the constant $\beta_{\lambda,s}$ *can be computed as*

$$\beta_{\lambda,s} = 2\alpha_s - \mathcal{O}\Big(\frac{1}{\lambda^s}\Big). \tag{3.3.5}$$

Let $A_s = A_{\mathrm{P}_0,\alpha,\mathrm{d},\tau,s} := 2^{1-\frac{1}{s}} \mathcal{C}_{\alpha,\mathrm{d}} \Big(\frac{1}{c_{\alpha_s,\mathrm{d}}}\Big)^{\frac{1}{s}} \frac{\tau \mathrm{M}_{\tau}(\mathrm{P}_0)^{\frac{1}{\tau}}}{\tau - s}$. *If* $\lambda > A_s[2\,(-\Delta)^{\alpha}(0,0)]^{\frac{1}{s}} := \lambda_{\mathrm{AG}}(s)$, *then*

$$\mathbb{E}\left[\sum_{x\in\mathbb{Z}^{\mathrm{d}}} |\mathrm{G}_{z,\lambda,\alpha}(x_0,x)|^s |x - x_0|^t \right] \leq \frac{1}{2[\mathrm{B}_s(\beta_{\lambda,s}) - \mathrm{B}_s(t)]}, \quad \forall t \in (0, \beta_{\lambda,s}),\ \forall x_0 \in \mathbb{Z}^{\mathrm{d}}$$

and $\sigma(\mathrm{H}_{\lambda,\alpha})$ *is pure-point a.s. Furthermore, let* g *be the density of* P_0 *(i.e.,* $\mathrm{dP}_0\,(v) = \mathrm{d}v\, g(v)$*) and let* $I \subset \mathbb{R}$ *be an open and bounded Borel set. If there are constants* $M > 0$ *and* $0 < \kappa < 1$ *such that*

$$\sup_{v\in\mathbb{R}} (1+|v|)^{\kappa} g(v) \leq M, \tag{3.3.6}$$

then, for $\mathbb{P}-$a.s. $\omega \in \Omega$, for all $\mathrm{E} \in \sigma(\mathrm{H}_{\lambda,\alpha}) \cap I$, there is a localization center $\mathrm{x_E}(\omega) \in \mathbb{Z}^{\mathrm{d}}$ such that the corresponding normalized eigenfunction $\varphi_{\mathrm{E}}(\cdot,\omega)$ satisfies

$$|\varphi_{\mathrm{E}}(y,\omega)|^2 \leq 4\mathrm{A}_{I,t}(\omega)\left[\frac{(-\Delta)^{\alpha}(0,0)}{\Delta^{\alpha}(0,\mathrm{x_E}(\omega))}\right]^2 \frac{1}{(1+|y-\mathrm{x_E}(\omega)|)^t}, \forall t \in (0,\beta_{\lambda,s}), \forall y \in \mathbb{Z}^{\mathrm{d}}, \tag{3.3.7}$$

where $\mathrm{A}_{I,t}$ is an integrable random variable given in Eq. 3.3.14 below.

Remark 3.3.3.

- *In [AM93], $\mathrm{P}_0(v) = \frac{1}{2}\mathbb{1}[-1,1](v)$, which satisfies Ineq. 3.3.6.*
- *Ineq. 3.3.6 implies that the density function g of P_0 is bounded and $\tau = 1$.*
- *We obtain a meaningful upper bound for the eigenfunction only for the case $\mathrm{d} = 1$. Indeed, by Corollary 3.2.6, $\sum_{y\in\mathbb{Z}^{\mathrm{d}}} \frac{1}{(1+|y-\mathrm{x_E}(\omega)|)^t} < \infty$ iff $\mathrm{d} = 1$, α, t, s and λ are such that $\lambda > \lambda_{\mathrm{AG}}$ and $1 < t < \beta_{\lambda,s} < 2\alpha_s < 2\alpha < 2$.*
- *Let $\frac{\mathrm{d}}{\mathrm{d}+2\alpha} < s < \tau \leq 1$. By Ineq. 3.3.30, if $\lambda > \lambda_{\mathrm{AG}}$, then*

$$\mathbb{E}[|\mathrm{G}_{z,\lambda,\alpha}(x_0,x)|^s] \leq \frac{1}{2[\mathrm{B}_s(\beta_{\lambda,s})-\mathrm{B}_s(t)]}\frac{1}{(1+|x-x_0|)^t}, \quad \forall t \in (0,\beta_{\lambda,s}),\ \forall x, x_0 \in \mathbb{Z}^{\mathrm{d}},\ x \neq x_0,\ \textit{unif. in } z \in \mathbb{C}\setminus\mathbb{R}. \tag{3.3.8}$$

Prior to presenting the proof of Thm. 3.3.2, we introduce several technical lemmas. These lemmas will be instrumental in deriving the upper bounds mentioned in Ineq. 3.3.7. In particular, we will use the Decoupling Lemma (Lemma 3.3.4), which is presented below.

Lemma 3.3.4 (Decoupling lemma, [AG98])**.** *Let $0 < s < \tau \leq 1$ and μ be a $\tau-$regular probability measure. Let $\theta_{\mu,s} := \left(\frac{2\tau}{\tau-s}\right)\mathrm{M}_{\tau}(\mu)^{\frac{1}{\tau}}$. Then*

$$\left(\frac{1}{\theta_{\mu,s}}\right)^s \int_{\mathbb{R}} \mathrm{d}\mu(\mathrm{v})\frac{1}{|\mathrm{v}-\beta|^s} \leq \int_{\mathbb{R}} \mathrm{d}\mu(\mathrm{v})\frac{|\mathrm{v}-\alpha|^s}{|\mathrm{v}-\beta|^s}, \qquad \forall \beta, \alpha \in \mathbb{C}, \tag{3.3.9}$$

$$\int \mathrm{d}\mu(\mathrm{v})\frac{1}{|\mathrm{v}-\alpha|^s} \leq \frac{1}{2}\theta_{\mu,s}^s, \qquad \forall \alpha \in \mathbb{C} \tag{3.3.10}$$

On the other hand, to establish spectral localization and the polynomial decay of the eigenfunctions, we rely on Lemma 3.3.5 provided below.

Lemma 3.3.5. *Let $I \subset \mathbb{R}$ be an open Borel set. The eigenfunction correlator Q is defined as*

$$Q(x,y,\omega;I) := \sup_{\substack{F\in C(\mathbb{R})\\ \|F\|_\infty \leq 1}} |\langle \delta_x, P_I(\mathrm{H}_{\omega,\lambda,\alpha})F(\mathrm{H}_{\omega,\lambda,\alpha})\delta_y\rangle|, \qquad \forall x,y \in \mathbb{Z}^{\mathrm{d}},\ \forall \omega \in \Omega, \tag{3.3.11}$$

where $P_I(\mathrm{H}_{\omega,\lambda,\alpha})$ *is the spectral projection operator. Let* $\mathbb{C}^+ := \{z \in \mathbb{C} \mid \mathrm{Im}\{z\} > 0\}$. *Assume that there are constants* $K > 0$, $s \in (0,1)$ *and* $t > 0$ *such that*

$$\mathbb{E}\left[\sum_{y\in\mathbb{Z}^{\mathrm{d}}} |\mathrm{G}_{z,\lambda,\alpha}(x,y)|^s(1+|x-y|)^t\right] \leq K, \quad \forall x \in \mathbb{Z}^{\mathrm{d}}, \forall z \in \mathbb{C}^+. \tag{3.3.12}$$

Then, $\sigma(\mathrm{H}_{\lambda,\alpha})$ *is pure-point a.s. Furthermore, let* I *be bounded. If there are constants* $0 < \kappa < 1$ *and* C *such that Ineq. 3.3.6 holds, then, for a.s.* $\omega \in \Omega$ *and for each of the eigenvalues* $\mathrm{E} \in I$ *of* $\mathrm{H}_{\omega,\lambda,\alpha}$, *there is a localization center* $\mathrm{x_E}(\omega) \in \mathbb{Z}^{\mathrm{d}}$ *such that the corresponding normalized eigenfuction* $\varphi_{\mathrm{E}}(\cdot,\omega)$ *satisfies*

$$|\varphi_{\mathrm{E}}(y,\omega)|^2 \leq 4\mathrm{A}_{I,t}(\omega)\left[\frac{(-\Delta)^\alpha(0,0)}{|\triangle^\alpha(0,\mathrm{x_E}(\omega))|}\right]^2 \frac{1}{(1+|y-\mathrm{x_E}(\omega)|)^t}, \qquad \forall y \in \mathbb{Z}^{\mathrm{d}}, \tag{3.3.13}$$

where $\mathrm{A}_{I,t}$ *is an integrable random variable given by*

$$\mathrm{A}_{I,t}(\omega) := \sum_{x\in\mathbb{Z}^{\mathrm{d}}} \frac{|\triangle^\alpha(0,x)|}{2(-\Delta)^\alpha(0,0)} \sum_{y\in\mathbb{Z}^{\mathrm{d}}}(1+|x-y|)^t Q(x,y,\omega;I)^2, \quad \omega \in \Omega. \tag{3.3.14}$$

Proof. To prove that $\sigma(\mathrm{H}_{\lambda,\alpha})$ is pure-point a.s., we can use the Simon-Wolff Criterion [AW15, Thm. 5.7], which we state below as Thm. 3.3.6 in our setting.

Theorem 3.3.6 (Simon-Wolff Criterion, [AW15]). *If for all* $x \in \mathbb{Z}^{\mathrm{d}}$, *for Lebesgue-a.e.* $\mathrm{E} \in \mathbb{R}$ *and for a.s.* $\omega \in \Omega$,

$$\lim_{\eta\downarrow 0}\sum_{y\in\mathbb{Z}^{\mathrm{d}}} |\mathrm{G}_{\mathrm{E}+i\eta,\omega,\lambda,\alpha}(x,y)|^2 < \infty, \tag{3.3.15}$$

then $\sigma(\mathrm{H}_{\omega,\lambda,\alpha})$ *is pure-point for* $\omega \in \Omega$ $\mathbb{P}-$*a.s.*

Let $K > 0$, $s \in (0,1)$ and $t > 0$ be such that Ineq. 3.3.12 holds. By Fatou's lemma and the fact that $w \mapsto w^{\frac{s}{2}}$ is continuous and concave, we get

$$\begin{aligned}
&\mathbb{E}\left[\left(\lim_{\eta\downarrow 0}\sum_{y\in\mathbb{Z}^{\mathrm{d}}} |\mathrm{G}_{\mathrm{E}+i\eta,\omega,\lambda,\alpha}(x,y)|^2\right)^{\frac{s}{2}}\right] \leq \varliminf_{\eta\downarrow 0}\mathbb{E}\left[\left(\sum_{y\in\mathbb{Z}^{\mathrm{d}}} |\mathrm{G}_{\mathrm{E}+i\eta,\omega,\lambda,\alpha}(x,y)|^2\right)^{\frac{s}{2}}\right] \\
&\leq \varliminf_{\eta\downarrow 0}\mathbb{E}\left[\sum_{y\in\mathbb{Z}^{\mathrm{d}}} |\mathrm{G}_{\mathrm{E}+i\eta,\omega,\lambda,\alpha}(x,y)|^s\right] \leq \varliminf_{\eta\downarrow 0}\mathbb{E}\left[\sum_{y\in\mathbb{Z}^{\mathrm{d}}} |\mathrm{G}_{\mathrm{E}+i\eta,\omega,\lambda,\alpha}(x,y)|^s(1+|x-y|)^t\right] \\
&\leq K, \qquad \forall x \in \mathbb{Z}^{\mathrm{d}},
\end{aligned} \tag{3.3.16}$$

which implies that $\sigma(\mathrm{H}_{\omega,\lambda,\alpha})$ is pure-point for $\omega \in \Omega$ $\mathbb{P}$-a.s., by the Simon-Wolff criterion (Thm. 3.3.6).

To show that the eigenvectors have the decay as Ineq. 3.5.24, we will use [AW15, Thm. 7.4], which we also state below in our setting. To do so, we take the function $g(x) = \frac{2(-\Delta)^\alpha(0,0)}{\triangle^\alpha(0,x)}$, for all $x \in \mathbb{Z}^{\mathrm{d}}$, therein.

Theorem 3.3.7 (Eigenfunction localization, [AW15]). *Let $I \subset \mathbb{R}$ be a Borel set. Let $t > 0$ and $\mathrm{A}_{I,t}$ be as in Lemma 3.3.5. If $\mathrm{A}_{I,t} \in \mathrm{L}^1(\Omega)$, then, for a.s. ω and for each of the simple eigenvalues $\mathrm{E} \in \sigma(\mathrm{H}_{\lambda,\alpha}) \cap I$, there is a localization center $\mathrm{x_E}(\omega) \in \mathbb{Z}^{\mathrm{d}}$ such that the corresponding normalized eigenfuction $\varphi_{\mathrm{E}}(\cdot, \omega)$ satisfies Ineq. 3.5.24.*

By [AW15, Thm. 5.8], the pure-point part of the spectrum of $\mathrm{H}_{\lambda,\alpha}$ is simple a.s. because $\{\omega(x)\}_{x\in\mathbb{Z}^{\mathrm{d}}}$ are iid and $\omega(x)$ has a τ-regular distribution P_0. Hence, we can use Thm. 3.3.7 to show that all the eigenvectors have the desired decay.

After using Fubini-Tonelli, we need to guarantee

$$\mathbb{E}[\mathrm{A}_{I,t}] = \sum_{x\in\mathbb{Z}^{\mathrm{d}}} \frac{|\triangle^{\alpha}(0,x)|}{2(-\Delta)^{\alpha}(0,0)} \sum_{y\in\mathbb{Z}^{\mathrm{d}}} (1+|x-y|)^t \mathbb{E}\big[Q(x,y;I)^2\big] < \infty, \tag{3.3.17}$$

where I is an open Borel set.

Note that $Q(x,y,\omega;I)^2 \le Q(x,y,\omega;I) \le 1$, for all $x,y \in \mathbb{Z}^{\mathrm{d}}$, for all $\omega \in \Omega$. Hence, it is sufficient to obtain an estimate for $\mathbb{E}[Q(x,y;I)]$. Let I be bounded. Due to our assumption that Ineq. 3.3.6 holds together with the fact that $\{\omega(x)\}_{x\in\mathbb{Z}^{\mathrm{d}}}$ are iid with a common τ-regular compactly support distribution P_0, we can use [AW15, Thm. 7.7]

$$\mathbb{E}[Q(x,y;I)] \le k_s \lim_{\eta\downarrow 0} \int_I \mathrm{dE}\ \mathbb{E}[|\mathrm{G}_{\mathrm{E}+i\eta,\omega,\lambda,\alpha}(x,y)|^s],\ \forall x,y\in\mathbb{Z}^{\mathrm{d}}, \tag{3.3.18}$$

where k_s is a finite constant.

By the above argument, Fubini-Tonelli, Fatou's lemma, Assumption 3.3.12 and Eq. 3.2.1, it holds

$$\begin{aligned}
\mathbb{E}[\mathrm{A}_{I,t}] &\le \sum_{x\in\mathbb{Z}^{\mathrm{d}}} \frac{|\triangle^{\alpha}(0,x)|}{2(-\Delta)^{\alpha}(0,0)} \sum_{y\in\mathbb{Z}^{\mathrm{d}}} (1+|x-y|)^t \mathbb{E}[Q(x,y;I)] \\
&\le k_s \sum_{x\in\mathbb{Z}^{\mathrm{d}}} \frac{|\triangle^{\alpha}(0,x)|}{2(-\Delta)^{\alpha}(0,0)} \sum_{y\in\mathbb{Z}^{\mathrm{d}}} (1+|x-y|)^t \lim_{\eta\downarrow 0} \int_I \mathrm{dE}\ \mathbb{E}[|\mathrm{G}_{\mathrm{E}+i\eta,\omega,\lambda,\alpha}(x,y)|^s] \\
&\le k_s \varliminf_{\eta\downarrow 0} \sum_{x\in\mathbb{Z}^{\mathrm{d}}} \frac{|\triangle^{\alpha}(0,x)|}{2(-\Delta)^{\alpha}(0,0)} \sum_{y\in\mathbb{Z}^{\mathrm{d}}} (1+|x-y|)^t \int_I \mathrm{dE}\ \mathbb{E}[|\mathrm{G}_{\mathrm{E}+i\eta,\omega,\lambda,\alpha}(x,y)|^s] \\
&\le k_s \varliminf_{\eta\downarrow 0} \sum_{x\in\mathbb{Z}^{\mathrm{d}}} \frac{|\triangle^{\alpha}(0,x)|}{2(-\Delta)^{\alpha}(0,0)} \int_I \mathrm{dE}\ \mathbb{E}\left[\sum_{y\in\mathbb{Z}^{\mathrm{d}}} (1+|x-y|)^t |\mathrm{G}_{\mathrm{E}+i\eta,\omega,\lambda,\alpha}(x,y)|^s\right] \\
&\le k_s K|I| \sum_{x\in\mathbb{Z}^{\mathrm{d}}} \frac{|\triangle^{\alpha}(0,x)|}{2(-\Delta)^{\alpha}(0,0)} = k_s K|I| < \infty,
\end{aligned} \tag{3.3.19}$$

where $|I|$ is the Lebesgue measure of I, which is finite since we assume that I is bounded. $\square$

Remark 3.3.8. *In the previous proof, we used [AW15, Thm. 7.4] (Thm. 3.3.7) with the choice $g(x) = \frac{2(-\Delta)^{\alpha}(0,0)}{\triangle^{\alpha}(0,x)}$, for all $x \in \mathbb{Z}^{\mathrm{d}}$. However, any strictly-positive-valued function $\tilde{g}$ on $\mathbb{Z}^{\mathrm{d}}$ such that $\sum_{x\in\mathbb{Z}^{\mathrm{d}}} \tilde{g}(x)^{-1} = 1$ could have been used instead of g. We specifically chose g because it is intrinsic to the fractional Anderson model and depends on α.*

Proof of Thm 3.3.2. Let $\frac{\mathrm{d}}{\mathrm{d}+2\alpha} < s < \tau \leq 1$. Due to Corollary 3.2.4, $\mathrm{B}_s(t) < \infty$ iff $\sum_{y\in\mathbb{Z}^{\mathrm{d}}\setminus\{x_0\}} |y|^{t-(\mathrm{d}+2\alpha_s)} < \infty$. By Corollary 3.2.6, the latter series converges, if $0 < t < 2\alpha_s$. Thus, it remains to find the suitable $t \in (0, 2\alpha)$ such that $\mathrm{B}_s(t) = \left(\frac{\lambda}{\theta_{s,\mathrm{P}_0}}\right)^s$. Note that B_s is a strictly-increasing and continuous in t. In addition, we have

$$\lim_{t\downarrow 0} \mathrm{B}_s(t) = \left(\frac{\lambda_{\mathrm{AG}}(s)}{\theta_{\mathrm{P}_0,s}}\right)^s, \qquad \lim_{t\uparrow 2\alpha} \mathrm{B}_s(t) = \infty. \tag{3.3.20}$$

Hence, we can establish the existence of $\beta_{\lambda,s} \in (0, 2\alpha_s)$ such that $\left(\frac{\lambda}{\theta_{\mathrm{P}_0,s}}\right)^s = \mathrm{B}_s(\beta_{\lambda,s})$. Moreover, we can estimate $\beta_{\lambda,s}$ by using Ineq. 3.2.12 and Corollary 3.2.6. In fact,

$$\begin{aligned}
\left(\frac{\lambda}{\theta_{\mathrm{P}_0,s}}\right)^s &= \mathrm{B}_s(\beta_{\lambda,s}) = \sum_{x\in\mathbb{Z}^{\mathrm{d}}\setminus\{0\}} \triangle^{\alpha_s}(x_0,x)(1+|x-x_0|)^{\beta_{\lambda,s}} \\
&\propto \sum_{x\in\mathbb{Z}^{\mathrm{d}}\setminus\{0\}} \frac{1}{(1+|x-x_0|)^{\mathrm{d}+2\alpha_s-\beta_{\lambda,s}}} \propto \frac{1}{2\alpha_s-\beta_{\lambda,s}} \\
&\Rightarrow \beta_{\lambda,s} = 2\alpha_s - \mathcal{O}\left(\frac{1}{\lambda^s}\right).
\end{aligned} \tag{3.3.21}$$

To prove Ineq. 3.3.12, we employ the following strategy: in the first place, we derive a condition similar to Ineq. 3.3.12 for finite boxes (see Ineq. 3.3.27). Subsequently, we extend our analysis to $\mathbb{Z}^{\mathrm{d}}$, by choosing a specific value for t.

Let $L \in \mathbb{N}$, $\Lambda_L := \{-L, \dots, L\}^{\mathrm{d}}$, we define $\mathrm{H}^{\Lambda_L}_{\omega,\lambda,\alpha} := \mathbb{1}_{\Lambda_L}\mathrm{H}_{\omega,\lambda,\alpha}\mathbb{1}_{\Lambda_L}$. Let $z \in \mathbb{C}\setminus\mathbb{R}$ and $\mathrm{G}^{\Lambda_L}_{z,\omega,\lambda,\alpha}$ be the operator on $\ell^2(\mathbb{Z}^{\mathrm{d}})$ given by

$$\mathrm{G}^{\Lambda_L}_{z,\omega,\lambda,\alpha}(x,y) = \begin{cases} \left(\mathrm{H}^{\Lambda_L}_{\omega,\lambda,\alpha} - z\right)^{-1}(x,y), & \forall x,y \in \Lambda_L, \\ 0, & \text{otherwise.} \end{cases} \tag{3.3.22}$$

Once again, to lighten the notation, we will frequently omit the dependence on ω in the above functions. Let

$$\mathrm{F}_{s,L}(t) := \sum_{x\in\Lambda_L} \mathbb{E}\left[\left|\mathrm{G}^{\Lambda_L}_{z,\lambda,\alpha}(x_0,x)\right|^s\right](1+|x-x_0|)^t, \tag{3.3.23}$$

which is finite since Λ_L is a finite set.

For now, let us assume that for $\frac{\mathrm{d}}{\mathrm{d}+2\alpha} < s < \tau$, the following inequality holds

$$\begin{aligned}
\left(\frac{\lambda}{\theta_{\mathrm{P}_0,s}}\right)^s \mathbb{E}\left[\left|\mathrm{G}^{\Lambda_L}_{z,\lambda,\alpha}(x_0,x)\right|^s\right] \leq &\frac{\mathcal{C}^s_{\alpha,\mathrm{d}}}{c_{\alpha_s,\mathrm{d}}} \sum_{y\in\mathbb{Z}^{\mathrm{d}}\setminus\{x\}} \triangle^{\alpha_s}(y,x)\mathbb{E}\left[\left|\mathrm{G}^{\Lambda_L}_{z,\lambda,\alpha}(x_0,y)\right|^s\right], \\
&\forall x \in \Lambda_L\setminus\{x_0\},\ \forall\lambda > \lambda_{\mathrm{AG}}(s).
\end{aligned} \tag{3.3.24}$$

We will provide a proof of this inequality later.

Due to the Decoupling Lemma (Lemma 3.3.4), Ineq. 3.3.24, Ineq. 3.3.44 and definition of the function $\mathrm{B}_s(t)$ (see Ineq. 3.3.45), we obtain that

$$\begin{aligned}
\mathrm{F}_{s,L}(t) &\leq \frac{1}{2}\left(\frac{\theta_{\mathrm{P}_0,s}}{\lambda}\right)^s + \left(\frac{\theta_{\mathrm{P}_0,s}}{\lambda}\right)^s \frac{\mathcal{C}^s_{\alpha,\mathrm{d}}}{c_{\alpha,\mathrm{d}}} \sum_{x\in\Lambda_L\setminus\{x_0\}} (1+|x-x_0|)^t \\
&\qquad \times \sum_{y\in\Lambda_L\setminus\{x\}} \triangle^{\alpha_s}(y,x)\mathbb{E}\left[\left|\mathrm{G}^{\Lambda_L}_{z,\lambda,\alpha}(x_0,y)\right|^s\right] \\
&\leq \frac{1}{2}\left(\frac{\theta_{\mathrm{P}_0,s}}{\lambda}\right)^s + \left(\frac{\theta_{\mathrm{P}_0,s}}{\lambda}\right)^s \sum_{y\in\Lambda_L} \mathbb{E}\left[\left|\mathrm{G}^{\Lambda_L}_{z,\lambda,\alpha}(x_0,y)\right|^s\right](1+|y-x_0|)^t \\
&\qquad \times \frac{\mathcal{C}^s_{\alpha,\mathrm{d}}}{c_{\alpha,\mathrm{d}}} \sum_{x\in\Lambda_L\setminus\{x_0,y\}} \triangle^{\alpha_s}(y,x)(1+|y-x|)^t \\
&\leq \frac{1}{2}\left(\frac{\theta_{\mathrm{P}_0,s}}{\lambda}\right)^s + \left(\frac{\theta_{\mathrm{P}_0,s}}{\lambda}\right)^s \mathrm{F}_{s,L}(t)\mathrm{B}_s(t).
\end{aligned} \tag{3.3.25}$$

If $0 < t < \beta_{\lambda,s}$, then

$$\mathrm{F}_{s,L}(t) \leq \frac{1}{2\left[\left(\frac{\lambda}{\theta_{\mathrm{P}_0,s}}\right)^s - \mathrm{B}_s(t)\right]} = \frac{1}{2[\mathrm{B}_s(\beta_{\lambda,s}) - \mathrm{B}_s(t)]}, \qquad \forall L \in \mathbb{N}. \tag{3.3.26}$$

Due to Fatou's Lemma, Ineq. 3.3.26 and Fubini-Tonelli, we get

$$\begin{aligned}
&\mathbb{E}\left[\varliminf_{L\to\infty} \sum_{x\in\mathbb{Z}^\mathrm{d}} \left|\mathrm{G}^{\Lambda_L}_{z,\lambda,\alpha}(x_0,x)\right|^s (1+|x-x_0|)^t\right] \\
&\leq \varliminf_{L\to\infty} \mathbb{E}\left[\sum_{x\in\mathbb{Z}^\mathrm{d}} \left|\mathrm{G}^{\Lambda_L}_{z,\lambda,\alpha}(x_0,x)\right|^s (1+|x-x_0|)^t\right] = \varliminf_{L\to\infty} \mathrm{F}_{s,L}(t) \leq \frac{1}{2[\mathrm{B}_s(\beta_{\lambda,s}) - \mathrm{B}_s(t)]}.
\end{aligned} \tag{3.3.27}$$

The above result together with the definition of lim inf imply that there is a sequence $\{L_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ such that, for a.s. $\omega \in \Omega$,

$$\varliminf_{L\to\infty} \sum_{x\in\mathbb{Z}^\mathrm{d}} \left|\mathrm{G}^{\omega,\Lambda_L}_z(x_0,x)\right|^s (1+|x-x_0|)^t = \lim_{k\to\infty} \sum_{x\in\mathbb{Z}^\mathrm{d}} \left|\mathrm{G}^{\omega,\Lambda_{L_k}}_z(x_0,x)\right|^s (1+|x-x_0|)^t < \infty, \tag{3.3.28}$$

This result combined with an argument similar to Ineq. 3.3.16 yield that the sequence of functions $\left\{\mathrm{G}^{\omega,(\Lambda_{L_k}(x_0))}_{\mathrm{E}+i\eta,\omega,\lambda,\alpha}(x_0,\cdot)\right\}$ are uniformly bounded in $\ell^2(\mathbb{Z}^\mathrm{d})$, for a.s. $\omega \in \Omega$. Hence, due to Banach-Alaouglu Theorem, there is a subsequence $\{L_{k_j}\}_{j\in\mathbb{N}}$ such that, for all $x \in \mathbb{Z}^\mathrm{d}$, for a.s. $\omega \in \Omega$,

$$\lim_{j\to\infty} \mathrm{G}^{\omega,\Lambda_{L_{k_j}}}_z(x_0,x) = \mathrm{G}^\omega_z(x_0,x). \tag{3.3.29}$$

By the application of Fatou's lemma twice, we get

$$\begin{aligned}
&\mathbb{E}\left[\sum_{x\in\mathbb{Z}^\mathrm{d}} |\mathrm{G}_{z,\lambda,\alpha}(x_0,x)|^s|x-x_0|^t\right] \leq \varliminf_{j\to\infty} \mathbb{E}\left[\sum_{x\in\mathbb{Z}^\mathrm{d}} \left|\mathrm{G}^{\Lambda_{L_{k_j}}}_z(x_0,x)\right|^s (1+|x-x_0|)^t\right] \\
&\leq \frac{1}{2[\mathrm{B}_s(\beta_{\lambda,s}) - \mathrm{B}_s(t)]}, \quad 0 < t < \beta_{\alpha,s}.
\end{aligned} \tag{3.3.30}$$

Hence, by Lemma 3.3.5, $\sigma(\mathrm{H}_{\lambda,\alpha})$ is pure-point a.s. Moreover, if Ineq. 3.3.18 holds, then the eigenfunctions have the stated decay in Thm. 3.3.2.

We now proceed to prove Inequality 3.3.24. As a first step, note

$$0 = \left\langle \delta_{x_0}, \mathrm{G}^{\Lambda_L}_{z,\lambda,\alpha}\left[\mathrm{H}^{\Lambda_L}_{\lambda,\alpha} - z\right]\delta_x \right\rangle = \sum_{y\in\mathbb{Z}^{\mathrm{d}}} \left\langle \delta_{x_0}, \mathrm{G}^{\Lambda_L}_{z,\lambda,\alpha}\delta_y \right\rangle \left\langle \delta_y, \left[\mathrm{H}^{\Lambda_L}_{\lambda,\alpha} - z\right]\delta_x \right\rangle$$
$$= \left(- \sum_{y\in\mathbb{Z}^{\mathrm{d}}\setminus\{x\}} \triangle^{\alpha}(y,x)\mathrm{G}^{\Lambda_L}_{z,\lambda,\alpha}(x_0,y) \right) + [\lambda\omega(x) - \triangle^{\alpha}(x,x) - z]\mathrm{G}^{\Lambda_L}_{z,\lambda,\alpha}(x_0,x).$$

Given $s\in(0,1)$, we use the concavity of the function $w \mapsto w^s$, for $w\in\mathbb{R}$, and then we take expectations on both sides to get

$$\mathbb{E}\Big[|\lambda\omega(x) - \triangle^{\alpha}(x,x) - z|^s \Big|\mathrm{G}^{\Lambda_L}_{z,\lambda,\alpha}(x_0,x)\Big|^s\Big] \le \sum_{y\in\mathbb{Z}^{\mathrm{d}}\setminus\{x\}} |\triangle^{\alpha}(y,x)|^s \mathbb{E}\Big[\Big|\mathrm{G}^{\Lambda_L}_{z,\lambda,\alpha}(x_0,y)\Big|^s\Big]. \tag{3.3.31}$$

On the one hand, by Corollary 3.2.4, we get

$$\mathbb{E}\Big[|\lambda\omega(x) - \triangle^{\alpha}(x,x) - z|^s \Big|\mathrm{G}^{\Lambda_L}_{z,\lambda,\alpha}(x_0,x)\Big|^s\Big] \le \frac{\mathcal{C}^s_{\alpha,\mathrm{d}}}{c_{\alpha_s,\mathrm{d}}} \sum_{y\in\mathbb{Z}^{\mathrm{d}}\setminus\{x\}} \triangle^{\alpha_s}(y,x) \mathbb{E}\Big[\Big|\mathrm{G}^{\Lambda_L}_{z,\lambda,\alpha}(x_0,y)\Big|^s\Big]. \tag{3.3.32}$$

On the other hand, due to Krein formula, there exists a pair of complex-valued functions ξ and Φ which depend on three elements λ, z and $\mathrm{V}_{\{x\}^c} := \{\omega(y)\}_{y\in\mathbb{Z}^{\mathrm{d}}\setminus\{x\}}$ such that

$$\mathrm{G}^{\Lambda_L}_{z,\lambda,\alpha}(x_0,x) = \frac{\xi}{\lambda\omega(x) - \Phi}. \tag{3.3.33}$$

Then, the LHS of Ineq. 3.3.32 can be rewritten as

$$\begin{aligned}
&\mathbb{E}\Big[|\lambda\omega(x) - \triangle^{\alpha}(x,x) - z|^s \Big|\mathrm{G}^{\Lambda_L}_{z,\lambda,\alpha}(x_0,x)\Big|^s\Big] \\
&= \int d\mathrm{V}_{\{x\}^c} \int_{\mathbb{R}} d\omega(x) |\lambda\omega(x) - \triangle^{\alpha}(x,x) - z|^s \frac{|\xi|^s}{|\lambda\omega(x) - \Phi|^s} \\
&\ge \left(\frac{\lambda}{\theta_{\mathrm{P}_0,s}}\right)^s \int d\mathrm{V}_{\{x\}^c} \int_{\mathbb{R}} d\omega(x) \frac{|\xi|^s}{|\lambda\omega(x) - \Phi|^s} = \left(\frac{\lambda}{\theta_{\mathrm{P}_0,s}}\right)^s \mathbb{E}\Big[\Big|\mathrm{G}^{\Lambda_L}_{z,\lambda,\alpha}(x_0,x)\Big|^s\Big],
\end{aligned}$$

where we have used Decoupling Lemma (Lemma 3.3.4). This proves Ineq. 3.3.40. □

Remark 3.3.9. *In the proof, we could have avoided the step from Eq. 3.3.31 to Eq. 3.3.32, and then, following the same line of reasoning, we would have ended up with the condition that*

$$\lambda > \theta_{\mathrm{P}_0,s} \left(\sum_{y\in\mathbb{Z}^{\mathrm{d}}\setminus\{x_0\}} |\triangle^{\alpha}(x_0,y)|^s \right)^{\frac{1}{s}} := \tilde{\lambda}_{\mathrm{AG}}(s). \tag{3.3.34}$$

We do not know if $\tilde{\lambda}_{\mathrm{AG}}(s)$ yields a sharper condition than $\lambda_0(s)$ under some conditions on α, d and/or s because we do not know the explicit value of $\mathcal{C}_{\alpha,\mathrm{d}}$ and $c_{\alpha_s,\mathrm{d}}$ given by Eq. 3.2.12.

By adapting the arguments presented in [AG98, Thm.1'] to our specific context, we can make a slight improvement to Ineq. 3.3.8

$$\mathbb{E}[|\mathrm{G}_{z,\lambda,\alpha}(x_0,x)|^s] \leq 2^{1-\frac{1}{s}}\left(\frac{\mathrm{K}_{\mathrm{P}_0,\tau,s}}{\lambda}\right)^s \frac{1}{(1+|x-x_0|)^{\beta_{\lambda,s}}},,$$
$$\forall x, x_0 \in \mathbb{Z}^{\mathrm{d}},\ x \neq x_0,\ \text{unif. in } z \in \mathbb{C}\setminus\mathbb{R}, \tag{3.3.35}$$

where $\mathrm{K}_{\mathrm{P}_0,\tau,s} = \frac{2\tau \mathrm{M}_\tau(\mathrm{P}_0)^{\frac{1}{\tau}}}{2^{\frac{1}{s}}(\tau - s)}$.

Proof of Ineq. 3.3.35. Let $x_0 \in \mathbb{Z}^{\mathrm{d}}$ and

$$\phi(x) = \mathbb{E}[|\mathrm{G}_{z,\lambda,\alpha}(x_0,x)|^s] - \frac{\mathbb{E}[|\mathrm{G}_{z,\lambda,\alpha}(x_0,x_0)|^s]}{(1+|x-x_0|)^t}, \qquad \forall x \in \mathbb{Z}^{\mathrm{d}}, \tag{3.3.36}$$

where t is a positive constant to be defined later.

By Functional Calculus, $\mathbb{E}[|\mathrm{G}_{z,\lambda,\alpha}(x,y)|^s] \leq \frac{1}{|\mathrm{Im}\, z|^s}$, for all $x, y \in \mathbb{Z}^{\mathrm{d}}$, and thus $\phi \in \ell^\infty(\mathbb{Z}^{\mathrm{d}})$. Additionally, $\phi(x_0) = 0$. If we are able to prove that $\phi(x) \leq 0$, for all $x \in \mathbb{Z}^{\mathrm{d}} \setminus \{x_0\}$, for a suitable choice of t and λ, then

$$\mathbb{E}[|\mathrm{G}_{z,\lambda,\alpha}(x_0,x)|^s] \leq \frac{\mathbb{E}[|\mathrm{G}_{z,\lambda,\alpha}(x_0,x_0)|^s]}{(1+|x-x_0|)^t} \leq \frac{1}{2}\left(\frac{\theta_{\mathrm{P}_0,s}}{\lambda}\right)^s \frac{1}{(1+|x-x_0|)^t},\ \forall x \in \mathbb{Z}^{\mathrm{d}} \setminus \{x_0\}, \tag{3.3.37}$$

where, in the last step, we apply the Decoupling lemma (Lemma 3.3.4). This inequality resembles Ineq. 3.3.35.

Now we provide a criterion for $\phi(x) \leq 0$, for all $x \in \mathbb{Z}^{\mathrm{d}} \setminus \{x_0\}$.

Claim 3.3.10. *If $\lambda > \lambda_{\mathrm{AG}}(s)$ and*

$$\left(\frac{\lambda}{\theta_{\mathrm{P}_0,s}}\right)^s \phi(x) \leq \frac{\mathcal{C}^s_{\alpha,\mathrm{d}}}{c_{\alpha_s,\mathrm{d}}} \sum_{y \in \mathbb{Z}^{\mathrm{d}}\setminus\{x\}} \triangle^{\alpha_s}(y,x)\phi(y), \qquad \forall x \in \mathbb{Z}^{\mathrm{d}} \setminus \{x_0\}, \tag{3.3.38}$$

then $\phi(x) \leq 0$, for all $x \in \mathbb{Z}^{\mathrm{d}} \setminus \{x_0\}$.

For the sake of contradiction, assume that $\phi(x^*) > 0$ for some $x^* \in \mathbb{Z}^{\mathrm{d}} \setminus \{x_0\}$. Hence, $\mathfrak{M} := \phi(x) > 0$. If $\lambda > \lambda_{\mathrm{AG}}(s)$, then

$$\left(\frac{\lambda}{\theta_{\mathrm{P}_0,s}}\right)^s \mathfrak{M} \leq \mathfrak{M}\frac{\mathcal{C}^s_{\alpha,\mathrm{d}}}{c_{\alpha_s,\mathrm{d}}} \sum_{y \in \mathbb{Z}^{\mathrm{d}}\setminus\{x\}} \triangle^{\alpha_s}(y,x) = \mathfrak{M}\frac{\mathcal{C}^s_{\alpha,\mathrm{d}}}{c_{\alpha_s,\mathrm{d}}}(-\Delta)^{\alpha_s}(x,x)$$
$$= \mathfrak{M}\frac{\mathcal{C}^s_{\alpha,\mathrm{d}}}{c_{\alpha_s,\mathrm{d}}}(-\Delta)^{\alpha_s}(0,0),$$

where we used Eq. 3.2.1 and the invariance under translation of the fractional Laplacian. The above inequality implies that $\lambda \leq \lambda_{\mathrm{AG}}(s)$.

Using the definition of $\phi(x)$, we rewrite Ineq. 3.3.38 as

$$\left(\frac{\lambda}{\theta_{P_0,s}}\right)^s\left(\mathbb{E}[|G_{z,\lambda,\alpha}(x_0,x)|^s] - \frac{\mathbb{E}[|G_{z,\lambda,\alpha}(x_0,x_0)|^s]}{(1+|x-x_0|)^t}\right)$$
$$\leq \frac{\mathcal{C}^s_{\alpha,d}}{c_{\alpha_s,d}} \sum_{y\in\mathbb{Z}^d\setminus\{x\}} \triangle^{\alpha_s}(y,x)\left(\mathbb{E}[|G_{z,\lambda,\alpha}(x_0,y)|^s] - \frac{\mathbb{E}[|G_{z,\lambda,\alpha}(x_0,x_0)|^s]}{(1+|y-x_0|)^t}\right), \quad \forall x\in\mathbb{Z}^d\setminus\{x_0\}. \tag{3.3.39}$$

To prove that Ineq. 3.3.39 holds, it is enough to verify the validity of Ineq. 3.3.40 and Ineq. 3.3.42 below, for all $\lambda > \lambda_{AG}(s)$,

$$\left(\frac{\lambda}{\theta_{P_0,s}}\right)^s \mathbb{E}[|G_{z,\lambda,\alpha}(x_0,x)|^s] \leq \frac{\mathcal{C}^s_{\alpha,d}}{c_{\alpha_s,d}} \sum_{y\in\mathbb{Z}^d\setminus\{x\}} \triangle^{\alpha_s}(y,x)\mathbb{E}[|G_{z,\lambda,\alpha}(x_0,y)|^s], \tag{3.3.40}$$

$$\frac{\mathcal{C}^s_{\alpha,d}}{c_{\alpha_s,d}} \sum_{y\in\mathbb{Z}^d\setminus\{x\}} \frac{\triangle^{\alpha_s}(y,x)\mathbb{E}[|G_{z,\lambda,\alpha}(x_0,x_0)|^s]}{(1+|y-x_0|)^t} \leq \left(\frac{\lambda}{\theta_{P_0,s}}\right)^s \frac{\mathbb{E}[|G_{z,\lambda,\alpha}(x_0,x_0)|^s]}{(1+|x-x_0|)^t}, \tag{3.3.41}$$

The proof of Ineq. 3.3.40 follows the same line of reasoning of the proof of Ineq. 3.3.24 above. On the other hand, with respect to Ineq. 3.3.41, WLOG we can assume $\mathbb{E}[|G_{z,\lambda,\alpha}(x_0,x_0)|^s] \neq 0$. Hence, after dividing such a term on both sides of Ineq. 3.3.41, it is enough to prove

$$\frac{\mathcal{C}^s_{\alpha,d}}{c_{\alpha_s,d}} \sum_{y\in\mathbb{Z}^d\setminus\{x\}} \frac{\triangle^{\alpha_s}(y,x)}{(1+|y-x_0|)^t} \leq \left(\frac{\lambda}{\theta_{P_0,s}}\right)^s \frac{1}{(1+|x-x_0|)^t}. \tag{3.3.42}$$

Note

$$\sum_{y\in\mathbb{Z}^d\setminus\{x\}} \frac{\triangle^{\alpha_s}(y,x)}{(1+|y-x_0|)^t} \leq \left(\sum_{y\in\mathbb{Z}^d\setminus\{x\}} \triangle^{\alpha_s}(y,x)(1+|y-x|)^t\right)\frac{1}{(1+|x-x_0|)^t}, \tag{3.3.43}$$

where we use the fact that the function $w \mapsto w^t$ is strictly-increasing for $t>0$ and

$$\frac{1}{1+|y-x_0|} \leq \frac{1+|x-y|}{1+|x-x_0|} \qquad \forall x,y\in\mathbb{Z}^d, x\neq x_0\; x\neq y. \tag{3.3.44}$$

Indeed, due to the triangular inequality,

$$1+|x-x_0| \leq 1+|x-y|+|y-x_0|+|y-x_0| \leq (1+|x-y|)(1+|y-x_0|).$$

By the translation under invariance of the fractional Laplacian, Ineq. 3.3.43 can be rewritten as

$$\sum_{y\in\mathbb{Z}^d\setminus\{x\}} \frac{\triangle^{\alpha_s}(y,x)}{(1+|y-x_0|)^t} \leq \underbrace{\left(\sum_{y\in\mathbb{Z}^d\setminus\{0\}} \triangle^{\alpha_s}(y,0)(1+|y|)^t\right)}_{=B_s(t)}\frac{1}{(1+|x-x_0|)^t}. \tag{3.3.45}$$

If we choose $t=\beta_{\lambda,s}$, then $B_s(\beta_{\lambda,s}) = \left(\frac{\lambda}{\theta_{P_0,s}}\right)^s$, by construction. This proves Ineq. 3.3.42 and ends the proof. □

Another known result is the Combes-Thomas estimate, [AW15, Thm. 10.5]. For the reader's convenience, we state it for our model. To do so, it is enough to take the metric $\tilde{d}(x_0, x) = \ln(1 + |x - x_0|)$, for all $x, x_0 \in \mathbb{Z}^{\mathrm{d}}$ therein. Here, we stress out the dependence on ω since the bounds Ineq. 3.3.47 and 3.3.48 hold, for a.s. $\omega \in \Omega$.

Theorem 3.3.11. *[Combes-Thomas estimate] Let* $0 < \mathcal{N} < 2\alpha$. *Then*

$$\mathrm{S}_{\mathcal{N}}(\alpha, \mathrm{d}) := \sum_{x \in \mathbb{Z}^{\mathrm{d}} \setminus \{0\}} \triangle^{\alpha}(x_0, x)\Big(1 - \frac{1}{(1 + |x - x_0|)^{\mathcal{N}}}\Big) < \infty. \tag{3.3.46}$$

Let $z \notin [0, (4\mathrm{d})^{\alpha}] + \mathrm{suppP}_0$ *and* $\mathrm{D}_z := \mathrm{dist}(z, [0, (4\mathrm{d})^{\alpha}] + \mathrm{suppP}_0)$. *Then*

$$|\mathrm{G}_z^{\omega}(x_0, x)| \leq \frac{1}{\mathrm{D}_z - \mathrm{S}_{\nu}(\alpha, \mathrm{d})} \frac{1}{(1 + |x - x_0|)^{\nu}} \qquad a.s.\ \forall \nu \leq \mathcal{N} \text{ with } \mathrm{S}_{\nu}(\alpha, \mathrm{d}) < \mathrm{D}_z. \tag{3.3.47}$$

In particular:

1. *Let* $\tilde{\eta}_{z,\alpha,\mathrm{d}} := \frac{\mathcal{N} \mathrm{D}_z}{4\alpha S_{\mathcal{N}}(\alpha, \mathrm{d})}$. *If* $\mathrm{D}_z \leq 2 S_{\mathcal{N}}(\alpha, \mathrm{d})$, *then* $0 < \tilde{\eta}_{z,\alpha,\mathrm{d}} < 1$ *and*

$$|\mathrm{G}_z^{\omega}(x_0, x)| \leq \frac{2}{\mathrm{D}_z} \frac{1}{(1 + |x - x_0|)^{2\alpha \tilde{\eta}_{z,\alpha,\mathrm{d}}}} \qquad a.s. \tag{3.3.48}$$

2. *Let* $\eta_{\mathrm{m},\alpha,\mathrm{d}} := \frac{\mathcal{N} \mathrm{m}}{2\alpha \sqrt{2 S_{\mathcal{N}}(\alpha, \mathrm{d})}}$. *If* $\mathrm{m}^2 \leq S_{\mathcal{N}}(\alpha, \mathrm{d})$, *then* $0 < \eta_{\mathrm{m},\alpha,\mathrm{d}} < 1$ *and*

$$\Big| \big[(-\Delta)^{\alpha} + \mathrm{m}^2\big]^{-1}(x_0, x) \Big| \leq \frac{2}{\mathrm{m}^2} \frac{1}{(1 + |x - x_0|)^{2\alpha \eta_{\mathrm{m},\alpha,\mathrm{d}}}} \tag{3.3.49}$$

As a result of Thm. 3.3.11, we get

$$\mathbb{E}[|\mathrm{G}_{z,\lambda,\alpha}(x_0, x)|^s] \leq \frac{2}{\mathrm{D}_z} \frac{1}{(1 + |x - x_0|)^{2s\alpha\tilde{\eta}_{z,\alpha,\mathrm{d}}}}. \tag{3.3.50}$$

Remark 3.3.12. *Observe that Ineq. 3.5.23, 3.3.35 and 3.5.21 are uniform in* z. *By contrast, the term on the RHS of Ineq. 3.3.50 is not and diverges as* z *approaches to* $\sigma(\mathrm{H}_{\lambda,\alpha})$.

3.4 Self-avoiding walks with long jumps

In this Section, we explicitly construct the long-range self-avoiding random walk generated by the fractional Laplacian $(-\Delta)^{\alpha}$ and its corresponding two-points correlation function.

Definition 3.4.1 (Random walk generated by the Fractional Laplacian)**.** *Let*

$$T_{\alpha,\mathrm{d}}(x_0, x) = \begin{cases} \frac{\triangle^{\alpha}(x_0, x)}{\sum_{y \in \mathbb{Z}^{\mathrm{d}} \setminus \{x_0\}} \triangle^{\alpha}(x_0, y)}, & \textit{if } x_0 \neq x, \\ 0, & \textit{otherwise.} \end{cases} \tag{3.4.1}$$

Let $\mathrm{X} = \{\mathrm{X}_n\}_{n\geq 0}$ *be defined as the long-range random walk in* $\mathbb{Z}^{\mathrm{d}}$ *whose jumps or transition probabilities are given by the numbers* $\{T_{\alpha,\mathrm{d}}(x_0,x)\}_{x_0,x\in\mathbb{Z}^{\mathrm{d}}}$*. We say that* X *is the random walk generated by the Fractional Laplacian* $(-\Delta)^{\alpha}$*. We denote by* $\mathrm{Q}^{\alpha}_{x_0} = \mathrm{Q}_{x_0,\alpha,\mathrm{d}}$ *the distribution of such a walk starting at* $x_0 \in \mathbb{Z}^{\mathrm{d}}$*. In other words,*

$$\begin{aligned} \mathrm{Q}^{\alpha}_{x_0}(\{\mathrm{X}_n = x_n \mid \mathrm{X}_{n-1} = x_{n-1}\}) &= T_{\alpha,\mathrm{d}}(x_n, x_{n-1}), \qquad \forall n \in \mathbb{N}, \\ \mathrm{Q}^{\alpha}_{x_0}(\{\mathrm{X}_0 = x_0\}) &= 1. \end{aligned} \tag{3.4.2}$$

Finally, $\mathbb{E}^{\alpha}_{x_0}[\cdot]$ *denotes the corresponding expectation.*

Remark 3.4.2. *By Eq. 3.2.1 and invariance under translation of* $(-\Delta)^{\alpha}$*, we get that*

$$\sum_{y\in\mathbb{Z}^{\mathrm{d}}\setminus\{x_0\}} \triangle^{\alpha}(x_0,y) = (-\Delta)^{\alpha}(x_0,x_0) = (-\Delta)^{\alpha}(0,0). \tag{3.4.3}$$

Hence,

$$T_{\alpha,\mathrm{d}}(x_0,x) = \frac{\triangle^{\alpha}(x_0,x)}{(-\Delta)^{\alpha}(0,0)} \mathbb{1}_{\{x_0\neq x\}}. \tag{3.4.4}$$

Definition 3.4.3 (Two-points correlation function)**.** *Let* $n \in \mathbb{N}$*. Let* $x, x_0 \in \mathbb{Z}^{\mathrm{d}}$ *and* $\mathcal{W}_n(x_0,x) := \left\{ w = (w_j)_{j=0}^{n} \subset \mathbb{Z}^{n\mathrm{d}} \mid w_0 = x_0,\ w_n = x \right\}$*. We say that* $w \in \mathcal{W}_n(x_0,x)$ *is a self-avoiding walk of length* n (SAW) *if* $w_k \neq w_l$ *for all* $k \neq l$ *with* $k,l \leq n$*. We set* $\triangle^{\alpha}(x_0,x) = -(-\Delta)^{\alpha}(x_0,x)$*, for all* $x, x_0 \in \mathbb{Z}^{\mathrm{d}}$*. Following [Sch15], we define*

$$\mathrm{c}^{\alpha,\mathrm{RW}}_n(x) := \sum_{w\in\mathcal{W}_n(0,x)} \prod_{j=0}^{n-1} \triangle^{\alpha}(w_j,w_{j+1}) \mathbb{1}_{\{w_j\neq w_{j+1}\}}, \tag{3.4.5}$$

$$\mathrm{c}^{\alpha,\mathrm{SAW}}_n(x) := \sum_{w\in\mathcal{W}_n(0,x)} \prod_{j=0}^{n-1} \triangle^{\alpha}(w_j,w_{j+1}) \mathbb{1}_{\{w \text{ } SAW\}}, \qquad \forall n \in \mathbb{N}. \tag{3.4.6}$$

Let $\bigstar \in \{\mathrm{RW}, \mathrm{SAW}\}$ *and* $\gamma \geq 0$*, the* $\bigstar$*-two-point correlation function* $\mathrm{C}^{\alpha,\bigstar}_{\gamma}$ *of the random walk* X *generated by* $(-\Delta)^{\alpha}$ *between 0 and* x *is defined as*

$$\mathrm{C}^{\alpha,\bigstar}_{\gamma}(x) := \sum_{n\geq 1} \mathrm{c}^{\alpha,\mathrm{RW}}_n(x)\gamma^n, \qquad \mathrm{C}^{\alpha,\bigstar}_{\gamma}(0) := 1, \quad \forall x \in \mathbb{Z}^{\mathrm{d}} \setminus \{0\}. \tag{3.4.7}$$

with $\mathrm{R}_{\mathrm{C}^{\alpha,\bigstar}(x)}$ *as the radius of convergence.*

Our definition of SAW-two-point correlation function matches the definition in [Sch15]. Indeed, let $\gamma > 0$,

$$\begin{aligned} \mathrm{C}^{1,\mathrm{SAW}}_{\gamma}(x) := \sum_{n\geq 1} \mathrm{c}^{1,\mathrm{SAW}}_n(x)\gamma^n &= \sum_{n\geq 1} \gamma^n \sum_{w\in\mathcal{W}_n(x_0,x)} \prod_{j=1}^{n} \triangle(w_{j-1},w_j) \mathbb{1}_{\{w \text{ SAW}\}} \\ &= \sum_{n\geq 1} \gamma^n \#\mathcal{S}_n(x,0), \end{aligned} \tag{3.4.8}$$

where $\mathcal{S}_n(x,0) = \left\{ w \in \mathcal{W}_n(0,x) \mid w \text{ SAW},\ |w_{j-1} - w_j|_1 = 1 \quad \forall j \in \{1,\ldots,n\} \right\}$.

Remark 3.4.4. *In [CS15], the correlation function is defined in a slightly different way. To recover their definition, we make the following change of variable*

$$\gamma \mapsto \tilde{\gamma} := \frac{\gamma}{(-\Delta)^{\alpha}(0,0)}. \tag{3.4.9}$$

We will rewrite the $\bigstar$-two-points correlation function in terms of X and its distribution, for $\bigstar \in \{\mathrm{RW}, \mathrm{SAW}\}$, see Eq. 3.4.16 and Eq. 3.4.19. To make the computations easier, we will define a new operator. Let $T_{\alpha,\mathrm{d}}$ be the linear operator on $\ell^2(\mathbb{Z}^{\mathrm{d}})$ given by

$$[T_{\alpha,\mathrm{d}}u](x_0) = \sum_{x_1 \in \mathbb{Z}^{\mathrm{d}}} T_{\alpha,\mathrm{d}}(x_0, x_1) u(x_1), \qquad \forall u \in \ell^2(\mathbb{Z}^{\mathrm{d}}),\ \forall x_0 \in \mathbb{Z}^{\mathrm{d}}. \tag{3.4.10}$$

Note that $T_{\alpha,\mathrm{d}}$ is bounded. In fact,

$$\sup_{x \in \mathbb{Z}^{\mathrm{d}}} \sum_{y \in \mathbb{Z}^{\mathrm{d}} \setminus \{x\}} |T_{\alpha,\mathrm{d}}(x,y)| = \sup_{y \in \mathbb{Z}^{\mathrm{d}}} \sum_{x \in \mathbb{Z}^{\mathrm{d}} \setminus \{y\}} |T_{\alpha,\mathrm{d}}(x,y)| = 1. \tag{3.4.11}$$

Hence, due to Schur's bound test, $T_{\alpha,\mathrm{d}}$ is bounded and

$$\|T_{\alpha,\mathrm{d}}\| \leq 1. \tag{3.4.12}$$

Let $x \in \mathbb{Z}^{\mathrm{d}}$, we can rewrite $\mathrm{c}_n^{\alpha,\mathrm{RW}}(x)$ and $\mathrm{C}_n^{\alpha,\mathrm{RW}}(x)$, for $n \in \mathbb{N}$, in terms of the random walk X and its distribution Q_0^{α}. Indeed, by the invariance under translations of $(-\Delta)^{\alpha}$, we have

$$\begin{aligned}
\mathrm{c}_n^{\alpha,\mathrm{RW}}(x) &= [(-\Delta)^{\alpha}(0,0)]^n \sum_{w \in \mathcal{W}_n(0,x)} \prod_{j=1}^{n} \frac{\triangle^{\alpha}(w_{j-1}, w_j)}{(-\Delta)^{\alpha}(0,0)} \mathbb{1}_{\{w_j \neq w_{j+1}\}} \\
&= [(-\Delta)^{\alpha}(0,0)]^n \sum_{w_1,\dots,w_{j-1} \in \mathbb{Z}^{\mathrm{d}}} T_{\alpha,\mathrm{d}}(0, w_1) \cdot \ldots \cdot T_{\alpha,\mathrm{d}}(w_{j-1}, x) \\
&= [(-\Delta)^{\alpha}(0,0)]^n T_{\alpha,\mathrm{d}}^n(0,x),
\end{aligned} \tag{3.4.13}$$

where $T_{\alpha,\mathrm{d}}^n$ denotes the $n-$th power of $T_{\alpha,\mathrm{d}}$. Observe

$$T_{\alpha,\mathrm{d}}^n(0,x) = \mathrm{Q}_0^{\alpha}(\{\mathrm{X}_n = x\}), \qquad \forall n \in \mathbb{N}. \tag{3.4.14}$$

In fact, for $n = 1$, $T_{\alpha,\mathrm{d}}(0,x) = \mathrm{Q}_0^{\alpha}(\{X_1 = x\})$, by definition. For $n = 2$, we have

$$\begin{aligned}
T_{\alpha,\mathrm{d}}^2(0,x) &= \sum_{w_1 \in \mathbb{Z}^{\mathrm{d}}} T_{\alpha,\mathrm{d}}(0,w_1) T_{\alpha,\mathrm{d}}(w_1,x) = \sum_{w_1 \in \mathbb{Z}^{\mathrm{d}}} \mathrm{Q}_0^{\alpha}(\{X_1 = w_1\}) \mathrm{Q}_0^{\alpha}(\{\mathrm{X}_2 = x | X_1 = w_1\}) \\
&= \mathrm{Q}_0^{\alpha}(\{\mathrm{X}_2 = x\}).
\end{aligned}$$

The general case holds, due to an inductive argument. Hence, Eq. 3.4.13 and Eq. 3.4.14 yield

$$\mathrm{c}_n^{\alpha,\mathrm{RW}}(x) = [(-\Delta)^{\alpha}(0,0)]^n \mathrm{Q}_0^{\alpha}(\{\mathrm{X}_n = x\}), \qquad \forall n \in \mathbb{Z}^{\mathrm{d}}. \tag{3.4.15}$$

Let $\gamma \geq 0$. Then Eq. 3.4.15 yields in turn

$$\mathrm{C}_{\gamma}^{\alpha,\mathrm{RW}}(x) = \sum_{n \geq 1} \gamma^n \mathrm{c}_n^{\alpha,\mathrm{RW}}(x) = \sum_{n \geq 1} \gamma^n [(-\Delta)^{\alpha}(0,0)]^n \mathrm{Q}_0^{\alpha}(\{\mathrm{X}_n = x\}), \qquad \forall x \in \mathbb{Z}^{\mathrm{d}}. \tag{3.4.16}$$

By a similar argument, $\mathrm{c}_n^{\alpha,\mathrm{SAW}}(x)$ can be rewritten as

$$\mathrm{c}_n^{\alpha,\mathrm{SAW}}(x) = [(-\Delta)^{\alpha}(0,0)]^n \mathrm{Q}_0^{\alpha}(\{\mathrm{X}_n = x, \mathrm{X} \text{ SAW up to } n\}), \tag{3.4.17}$$

where the event $\{\mathrm{X} \text{ SAW up to } n\}$ is defined as

$$\{\mathrm{X} \text{ SAW up to } n\} := \bigcup_{\substack{w \in \mathcal{W}_n(0,x) \\ w \text{ SAW}}} \{\mathrm{X}_0 = 0, \mathrm{X}_1 = w_1, \ldots, \mathrm{X}_n = x\}, \quad \forall n \in \mathbb{N}. \tag{3.4.18}$$

Additionally, for $\gamma \geq 0$, it also holds that

$$\mathrm{C}_{\gamma}^{\alpha,\mathrm{SAW}}(x) = \sum_{n \geq 1} \gamma^n [(-\Delta)^{\alpha}(0,0)]^n \mathrm{Q}_0^{\alpha}(\{\mathrm{X}_n = x, \mathrm{X} \text{ SAW up to } n\}). \tag{3.4.19}$$

Definition 3.4.5 (Susceptibility)**.** *Let* $\bigstar \in \{\mathrm{RW}, \mathrm{SAW}\}$ *and* $\gamma \geq 0$. *We define the susceptibility* $\chi^{\alpha,\bigstar}(\gamma)$ *of the random walk* X *generated by* $(-\Delta)^{\alpha}$ *as*

$$\chi^{\alpha,\bigstar}(\gamma) := \sum_{x \in \mathbb{Z}^{\mathrm{d}}} \mathrm{C}_{\gamma}^{\alpha,\bigstar}(x) = 1 + \sum_{x \in \mathbb{Z}^{\mathrm{d}} \setminus \{0\}} \sum_{n \geq 1} \mathrm{c}_n^{\alpha,\bigstar}(x) \gamma^n, \tag{3.4.20}$$

with $\mathrm{R}_{\chi^{\alpha,\bigstar}}$ *as the radius of convergence.*

The fact that $\mathrm{R}_{\chi^{\alpha,\mathrm{SAW}}} > 0$ is known, see [CS15]. However, we prove it for the sake of self-containment, see Proposition 3.4.6 below. In fact, we show that $\mathrm{R}_{\chi^{\alpha,\mathrm{SAW}}} \geq \frac{1}{(-\Delta)^{\alpha}(0,0)}$.

Proposition 3.4.6. *Let* $x \in \mathbb{Z}^{\mathrm{d}} \setminus \{0\}$. *Then,*

$$\frac{1}{(-\Delta)^{\alpha}(0,0)} \leq \mathrm{R}_{\chi^{\alpha,\mathrm{SAW}}} \leq \mathrm{R}_{\mathrm{C}^{\alpha,\mathrm{SAW}}(x)}. \tag{3.4.21}$$

Proof. Observe

$$\begin{aligned} \mathrm{R}_{\mathrm{C}^{\alpha,\mathrm{SAW}}(x)} &= \frac{1}{\limsup_{n\to\infty} \left(\mathrm{c}_n^{\alpha,\mathrm{SAW}}(x)\right)^{\frac{1}{n}}} \\ &\geq \frac{1}{\limsup_{n\to\infty} \left(\sum_{x \in \mathbb{Z}^{\mathrm{d}} \setminus \{x_0\}} \mathrm{c}_n^{\alpha,\mathrm{SAW}}(x)\right)^{\frac{1}{n}}} = \mathrm{R}_{\chi^{\alpha,\mathrm{SAW}}}. \end{aligned} \tag{3.4.22}$$

Hence, it remains to prove $\mathrm{R}_{\chi^{\alpha,\mathrm{SAW}}} \geq \frac{1}{(-\Delta)^{\alpha}(0,0)}$. Let $0 \leq \gamma \leq \frac{\varepsilon}{(-\Delta)^{\alpha}(0,0)}$ with $0 < \varepsilon < 1$. Using Eq. 3.4.19 and Fubini-Tonelli, we obtain

$$\begin{aligned} &\chi^{\alpha,\mathrm{SAW}}(\gamma) = 1 + \sum_{x \in \mathbb{Z}^{\mathrm{d}} \setminus \{0\}} \mathrm{C}_n^{\alpha,\mathrm{SAW}}(x) \\ &= 1 + \sum_{x \in \mathbb{Z}^{\mathrm{d}} \setminus \{0\}} \sum_{n \geq 1} \gamma^n [(-\Delta)^{\alpha}(0,0)]^n \mathrm{Q}_0^{\alpha}(\mathrm{X}_n = x, \mathrm{X} \text{ SAW up to } n) \\ &= 1 + \sum_{n \geq 1} \gamma^n [(-\Delta)^{\alpha}(0,0)]^n \sum_{x \in \mathbb{Z}^{\mathrm{d}} \setminus \{x_0\}} \mathrm{Q}_0^{\alpha}(\mathrm{X}_n = x, \mathrm{X} \text{ SAW up to } n) \\ &= 1 + \sum_{n \geq 1} \gamma^n [(-\Delta)^{\alpha}(0,0)]^n \mathrm{Q}_0^{\alpha}(\mathrm{X} \text{ SAW up to } n) \leq 1 + \sum_{n \geq 1} \varepsilon^n = \frac{1}{1-\varepsilon} < \infty, \end{aligned} \tag{3.4.23}$$

where, in the fourth equality, we used the σ−subadditivity and the fact that

$$\{X_n = x, X \text{ SAW up to } n\} \cap \{X_n = \tilde{x}, X \text{ SAW up to } n\} = \varnothing \\ \forall x, \tilde{x} \in \mathbb{Z}^d,\ x \neq \tilde{x},\ \forall n \in \mathbb{N}. \tag{3.4.24}$$

This shows that $R_{\chi^{\alpha,SAW}} \geq \frac{1}{(-\Delta)^\alpha(0,0)}$. □

Note that if $0 < \gamma < R_{C^{\alpha,RW}(x)}$, then

$$C^{\alpha,SAW}_\gamma(x) < C^{\alpha,RW}_\gamma(x), \tag{3.4.25}$$

because, by construction, we have that $c^{\alpha,SAW}_n(x) = c^{\alpha,RW}_n(x)$, $\forall n \leq 2$ and $c^{\alpha,SAW}_n(x) < c^{\alpha,RW}_n(x)$, $\forall n \geq 3$.

In Thm. 3.4.7 below, we obtain an estimate on the correlation functions in terms of the inverse of the fractional Laplacian defined as in Subsection 3.2.3, which produces, in turn, an upper bound on the decay of the Green's functions, see Ineq. 3.5.20.

Theorem 3.4.7. *Let $0 < \alpha < \frac{d}{2}$ and $x \in \mathbb{Z}^d \setminus \{0\}$. Let $0 < \gamma \leq \frac{1}{(-\Delta)^\alpha(0,0)}$. Then,*

$$C^{\alpha,SAW}_\gamma(x) < C^{\alpha,RW}_\gamma(x) \leq C^{\alpha,RW}_{\frac{1}{(-\Delta)^\alpha(0,0)}}(x) = (-\Delta)^\alpha(0,0) \cdot (-\Delta)^{-\alpha}(0,x). \tag{3.4.26}$$

In particular, $C^{\alpha,RW}_\gamma(x) < \infty$ and $(-\Delta)^{-\alpha}(0,x) > 0$, for all $x \in \mathbb{Z}^d \setminus \{0\}$.

Proof. Let $x \in \mathbb{Z}^d \setminus \{0\}$. In Prop. 3.4.6, it was proven that $R_{C^{\alpha,RW}(x)} \geq \frac{1}{(-\Delta)^\alpha(0,0)}$. In addition, it was also shown that $C^{\alpha,SAW}_\gamma(x) < C^{\alpha,RW}_\gamma(x)$, for all $0 < \gamma < \frac{1}{(-\Delta)^\alpha(0,0)}$, see Ineq. 3.4.25. Since $C^{\alpha,RW}_\gamma(x)$ is increasing in γ, it only remains to prove that $C^{\alpha,RW}_{\frac{1}{(-\Delta)^\alpha(0,0)}}(x) < \infty$ and $C^{\alpha,RW}_{\frac{1}{(-\Delta)^\alpha(0,0)}}(x) = (-\Delta)^\alpha(0,0) \cdot (-\Delta)^{-\alpha}(0,x)$.

Let $0 < \alpha < \frac{d}{2}$. By Thm. 3.2.7 , $(-\Delta)^{-\alpha}(0,x)$ is well-defined in the sense that

$$(-\Delta)^{-\alpha}(0,x) := \lim_{m \downarrow 0} \left[(-\Delta)^\alpha + m^2\right]^{-1}(0,x).$$

Let us write $(-\Delta)^\alpha + m^2 = D - B$, where D is a diagonal operator and the diagonal entries of B are zero. Due to the invariance under translations, it holds that $(-\Delta)^\alpha(y,y) = (-\Delta)^\alpha(0,0)$, for all $y \in \mathbb{Z}^d$, and

$$D = \left[(-\Delta)^\alpha(0,0) + m^2\right] I, \tag{3.4.27}$$

where I is the identity. On the other hand, from Eq. 3.4.10, it holds that

$$B = (-\Delta)^\alpha(0,0) T_{\alpha,d}. \tag{3.4.28}$$

Hence,

$$(-\Delta)^\alpha + m^2 = \left[(-\Delta)^\alpha(0,0) + m^2\right]\left[I - \frac{(-\Delta)^\alpha(0,0)}{(-\Delta)^\alpha(0,0) + m^2} T_{\alpha,d}\right]. \tag{3.4.29}$$

Since $\|T\|_{\alpha,\mathrm{d}} \leq 1$ (see Ineq. 3.4.12), it holds that $\left\|\frac{(-\Delta)^{\alpha}(0,0)}{(-\Delta)^{\alpha}(0,0)+\mathrm{m}^2}T_{\alpha,\mathrm{d}}\right\| < 1$, $\forall \mathrm{m} > 0$. Thus, we can use Neumann series, to get that

$$\left[(-\Delta)^{\alpha}+\mathrm{m}^2\right]^{-1} = \frac{1}{(-\Delta)^{\alpha}(0,0)+\mathrm{m}^2}\sum_{n\geq 0}\left[\frac{(-\Delta)^{\alpha}(0,0)}{(-\Delta)^{\alpha}(0,0)+\mathrm{m}^2}\right]^n T^n_{\alpha,\mathrm{d}}, \tag{3.4.30}$$

where the above limit converges in operator-norm. Hence, the following series converges

$$\left[(-\Delta)^{\alpha}+\mathrm{m}^2\right]^{-1}(0,x) = \frac{1}{(-\Delta)^{\alpha}(0,0)+\mathrm{m}^2}\sum_{n\geq 1}\left[\frac{(-\Delta)^{\alpha}(0,0)}{(-\Delta)^{\alpha}(0,0)+\mathrm{m}^2}\right]^n \mathrm{Q}^{\alpha}_0(\mathrm{X}_n = x), \tag{3.4.31}$$

where we used the fact that $T^n(0,x) = \mathrm{Q}^{\alpha}_0(\mathrm{X}_n = x)$, see Eq. 3.4.14. Note that the series starts at $n = 1$ and not at $n = 0$ because $T^0(0,x) = I(0,x) = 0$, for $x \neq 0$. In addition, all the terms of the series on the RHS are nonnegative and they increase as $\mathrm{m} \downarrow 0$. By the Monotone Convergence Theorem, it follows that

$$\begin{aligned}
(-\Delta)^{-\alpha}(0,x) &= \lim_{\mathrm{m}\downarrow 0}\left[(-\Delta)^{\alpha}+\mathrm{m}^2\right]^{-1}(0,x)\\
&= \lim_{\mathrm{m}\downarrow 0}\frac{1}{(-\Delta)^{\alpha}(0,0)+\mathrm{m}^2}\sum_{n\geq 1}\left(\frac{(-\Delta)^{\alpha}(0,0)}{(-\Delta)^{\alpha}(0,0)+\mathrm{m}^2}\right)^n \mathrm{Q}^{\alpha}_0(\mathrm{X}_n = x)\\
&= \frac{1}{(-\Delta)^{\alpha}(0,0)}\sum_{n\geq 1}\mathrm{Q}^{\alpha}_0(\mathrm{X}_n = x) = \frac{1}{(-\Delta)^{\alpha}(0,0)}\mathrm{C}^{\alpha,\mathrm{RW}}_{\frac{1}{(-\Delta)^{\alpha}(0,0)}}(x),
\end{aligned} \tag{3.4.32}$$

where, in the last step, we used Eq. 3.4.19. Since $(-\Delta)^{\alpha}(0,x)$ is well-defined, for $0 < \alpha < \frac{\mathrm{d}}{2}$, the series $\mathrm{C}^{\alpha,\mathrm{RW}}_{\frac{1}{(-\Delta)^{\alpha}(0,0)}}(x)$ is convergent and the result follows at once. □

3.5 Decay of Green's function

This sections is structured in the following way: In the first part, we establish the aforementioned connection between the fractional moments of the Green's function and the two-point correlation function introduced earlier. This result is stated in Theorem 3.5.1 and serves as a generalization of [Sch15, Thm. 1]. By combining this result with Lemma 3.3.5, we can conclude spectral localization and exponential decay of the eigenfunctions, subject to certain conditions. Additionally, we also extend [FV17, Lemma 8.13] to the fractional Anderson model, see Corollary 3.5.2. In the second part, we make a comparison between different estimates of the Green's function of the fractional Laplacian, exploring their strengths and limitations.

3.5.1 Decay of Green's function in terms of SAW

We now state our key estimate which relates the fractional moments of the Green's function to the SAW-two-points-correlation function introduced in the previous section. Let $\Lambda \subset \mathbb{Z}^{\mathrm{d}}$, recall that $\mathrm{G}^{\Lambda}_{z,\omega,\lambda,\alpha} = \mathbb{1}_{\Lambda}\mathrm{H}^{\alpha,\Lambda}\mathbb{1}_{\Lambda}$.

Theorem 3.5.1. *Let* $\frac{\mathrm{d}}{\mathrm{d}+2\alpha} < s < \tau \le 1$. *Let* $\alpha_s = s\big(\alpha + \frac{\mathrm{d}}{2}\big) - \frac{\mathrm{d}}{2}$, *the constant in Eq. 3.2.14, and* A_s *be as in Thm. 3.3.2. If* $\lambda > \frac{A_s}{\left(\mathrm{R}_{\chi^{\alpha_s},\mathrm{SAW}}\right)^{\frac{1}{s}}} := \lambda_0(s)$, *then*

$$\sup_{\Lambda\subset\mathbb{Z}^{\mathrm{d}}} \mathbb{E}\Big[\big|\mathrm{G}_z^{\Lambda}(x_0,x)\big|^s\Big] \le \left(\frac{\mathrm{K}_{\mathrm{P}_0,\tau,s}}{\lambda}\right)^s \mathrm{C}^{\alpha_s,\mathrm{SAW}}_{\left(\frac{A_s}{\lambda}\right)^s}(x-x_0),$$
$$\forall x,x_0\in\mathbb{Z}^{\mathrm{d}},\ x\neq x_0,\ \textit{unif. in } z\in\mathbb{C}\setminus\mathbb{R}, \tag{3.5.1}$$

where $\mathrm{K}_{\mathrm{P}_0,\tau,s} = \frac{2\tau \mathrm{M}_\tau(\mathrm{P}_0)^{\frac{1}{\tau}}}{2^{\frac{1}{s}}(\tau-s)}$ *as in Ineq. 3.3.35.*

Proof. Let $\Lambda\subset\mathbb{Z}^{\mathrm{d}}$, $z\notin\sigma(\mathrm{H}_{\lambda,\alpha})$. Without loss of generality, let $x_0,x\in\Lambda$, $x\neq x_0$, and $s\in(0,1)$. As a first step towards Ineq. 3.5.1, we will prove that

$$\mathbb{E}\Big[\big|\mathrm{G}_z^{\Lambda}(x_0,x)\big|^s\Big] \le \frac{1}{2}\left(\frac{\theta_{\mathrm{P}_0,s}}{\lambda}\right)^s \sum_{w_1\in\Lambda\setminus\{x_0\}} |\triangle^{\alpha}(x_0,w_1)|^s \mathbb{E}\Big[\Big|\mathrm{G}_z^{\Lambda\setminus\{x_0\}}(w_1,x)\Big|^s\Big], \tag{3.5.2}$$

where we recall from Decoupling Lemma (Lemma 3.3.4) that $\theta_{\mathrm{P}_0,s} = \left(\frac{2\tau}{\tau-s}\right)\mathrm{M}_\tau(\mu)^{\frac{1}{\tau}}$. Indeed, the resolvent identity and then the concavity of the function $y\mapsto y^s$ yield that

$$\big|\mathrm{G}_z^{\Lambda}(x_0,x)\big|^s \le \big|\mathrm{G}_z^{\Lambda}(x_0,x_0)\big|^s \sum_{w_1\in\Lambda\setminus\{x_0\}} |\triangle^{\alpha}(x_0,w_1)|^s \Big|\mathrm{G}_z^{\Lambda\setminus\{x_0\}}(w_1,x)\Big|^s.$$

On the one hand, note that $\sum_{w_1\in\Lambda_L\setminus\{x_0\}} |\triangle^{\alpha}(x_0,w_1)|^s \Big|\mathrm{G}_z^{\Lambda_L\setminus\{x_0\}}(w_1,x)\Big|^s$ depends only on $\mathrm{V}_{\{x_0\}^c} := \{\omega(y)\}_{y\in\mathbb{Z}^{\mathrm{d}}\setminus\{x_0\}}$. On the other hand, Krein formula yields that

$$\mathrm{G}_z^{\Lambda}(x_0,x_0) = \frac{1}{\lambda\omega(x_0)-\Phi}, \tag{3.5.3}$$

where Φ is a complex-valued functions which depend on λ, z and $\mathrm{V}_{\{x_0\}^c}$. After taking expectation on both sides, we get that

$$\begin{aligned}
&\mathbb{E}\Big[\big|\mathrm{G}_z^{\Lambda}(x_0,x)\big|^s\Big] \\
&\le \int d\mathrm{V}_{\{x_0\}^c} \sum_{w_1\in\Lambda\setminus\{x_0\}} |\triangle^{\alpha}(x_0,w_1)|^s \Big|\mathrm{G}_z^{\Lambda\setminus\{x_0\}}(w_1,x)\Big|^s \int \frac{d\omega(x_0)}{|\lambda\omega(x_0)-\Phi|^s} \\
&\le \frac{1}{2}\left(\frac{\theta_{\mathrm{P}_0,s}}{\lambda}\right)^s \sum_{w_1\in\Lambda\setminus\{x_0\}} |\triangle^{\alpha}(x_0,w_1)|^s \mathbb{E}\Big[\Big|\mathrm{G}_z^{\Lambda\setminus\{x_0\}}(w_1,x)\Big|^s\Big],
\end{aligned}$$

where in the last step we use the Decoupling Lemma (Lemma 3.3.4). This proves 3.5.2.

Recall that $\mathcal{W}_n(x_0,x) := \Big\{w := (w_j)_{j=0}^n \subset \mathbb{Z}^{(n+1)\mathrm{d}} | w_0 = x_0,\ w_n = x\Big\}$. We introduce the subset $\mathcal{W}_n^{\Lambda}(x_0,x) := \Big\{w := (w_j)_{j=0}^n \subset \Lambda^{n+1} | w_0 = x_0,\ w_n = x\Big\}$. As a second step, for all $N\in\mathbb{N}$, we will prove that

$$\begin{aligned}
\mathbb{E}\Big[\big|\mathrm{G}_z^{\Lambda}(x_0,x)\big|^s\Big] \le \frac{1}{2}\left(\frac{\theta_{\mathrm{P}_0,s}}{\lambda}\right)^s \sum_{n=1}^{n}\left[\frac{1}{2}\left(\frac{\theta_{\mathrm{P}_0,s}}{\lambda}\right)^s\right]^n \sum_{\substack{w\in\mathcal{W}_n^{\Lambda}(x_0,x)\\ w \text{ SAW}}} \prod_{j=1}^{n} |\triangle^{\alpha}(w_{j-1},w_j)|^s \\
+ \frac{1}{|\mathrm{Im}\, z|}\left[\frac{1}{2}\left(\frac{\theta_{\mathrm{P}_0,s}}{\lambda}\right)^s\right]^N \sum_{\substack{w\in\mathcal{W}_N^{\Lambda}(x_0,w_N)\\ w \text{ SAW}\\ w_N\neq x}} \prod_{j=1}^{n} |\triangle^{\alpha}(w_{j-1},w_j)|^s.
\end{aligned} \tag{3.5.4}$$

In fact, Ineq. 3.5.2 and the Decoupling Lemma applied to $\mathbb{E}\left[\left|\mathrm{G}_z^{\Lambda\setminus\{x_0\}}(x,x)\right|^s\right]$ yield that

$$\mathbb{E}\left[\left|\mathrm{G}_z^{\Lambda}(x_0,x)\right|^s\right] \leq \frac{1}{2}\left(\frac{\theta_{\mathrm{P}_0,s}}{\lambda}\right)^{2s}|\triangle^{\alpha}(x_0,x)|^s + \frac{1}{2}\left(\frac{\theta_{\mathrm{P}_0,s}}{\lambda}\right)^{s}\sum_{w_1\in\Lambda\setminus\{x_0,x\}}|\triangle^{\alpha}(x_0,w_1)|^s\mathbb{E}\left[\left|\mathrm{G}_z^{\Lambda\setminus\{x_0\}}(w_1,x)\right|^s\right]. \quad (3.5.5)$$

By the same token, for all $w_1\in\Lambda_L\setminus\{0,x\}$, we obtain that

$$\mathbb{E}\left[\left|\mathrm{G}_z^{\Lambda\setminus\{x_0\}}(w_1,x)\right|^s\right] \leq \frac{1}{2^2}\left(\frac{\theta_{\mathrm{P}_0,s}}{\lambda}\right)^{2s}|\triangle^{\alpha}(w_1,x)|^s + \frac{1}{2}\left(\frac{\theta_{\mathrm{P}_0,s}}{\lambda}\right)^{s}\sum_{w_2\in\Lambda\setminus\{0,w_1,x\}}|\triangle^{\alpha}(w_1,w_2)|^s\mathbb{E}\left[\left|\mathrm{G}_z^{\Lambda\setminus\{x_0,w_1\}}(w_2,x)\right|^s\right]. \quad (3.5.6)$$

If we insert Ineq. 3.5.6 into Ineq. 3.5.5, then we obtain

$$\begin{aligned}&\mathbb{E}\left[\left|\mathrm{G}_z^{\Lambda}(x_0,x)\right|^s\right]\\ &\leq \sum_{n=1}^{2}\left[\frac{1}{2}\left(\frac{\theta_{\mathrm{P}_0,s}}{\lambda}\right)^{s}\right]^{n+1}\sum_{\substack{w\in\mathcal{W}_n^{\Lambda}(x_0,x)\\ w\ \mathrm{SAW}}}\prod_{j=1}^{n}|\triangle^{\alpha}(w_{j-1},w_j)|^s\\ &+\left[\frac{1}{2}\left(\frac{\theta_{\mathrm{P}_0,s}}{\lambda}\right)^{s}\right]^{2}\sum_{\substack{w\in\mathcal{W}_2^{\Lambda}(x_0,w_2)\\ w\ \mathrm{SAW}\\ w_2\neq x}}\mathbb{E}\left[\left|\mathrm{G}_z^{\Lambda\setminus\{x_0,w_1\}}(w_2,x)\right|^s\right]\prod_{j=1}^{2}|\triangle^{\alpha}(w_{j-1},w_j)|^s. \quad (3.5.7)\end{aligned}$$

By an inductive argument and the fact that $\mathbb{E}\left[\left|\mathrm{G}_z^{\Lambda\setminus\{w_k\}_{k=0}^{N-1}}(w_N,x)\right|^s\right]\leq\frac{1}{|\mathrm{Im}\,z|^s}$, we get that Ineq. 3.5.4 holds for $N\in\mathbb{N}$.

Let α_s be as in Eq. 3.2.14. Next, we bound the RHS of Ineq. 3.5.4, up to a constant, by the correlation function $\mathrm{C}_{\gamma}^{\alpha_s,\mathrm{SAW}}(x-x_0)$, for $\gamma>0$ to be defined later. Let $\mathcal{C}_{\alpha,\mathrm{d}}$ and $c_{\alpha_s,\mathrm{d}}$ be as in Ineq. 3.2.12. As a result of Ineq. 3.2.15 and the invariance under translations of the fractional Laplacian, we get that

$$\begin{aligned}&\mathbb{E}\left[\left|\mathrm{G}_z^{\Lambda}(x_0,x)\right|^s\right]\\ &\leq\frac{1}{2}\left(\frac{\theta_{\mathrm{P}_0,s}}{\lambda}\right)^{s}\sum_{n=1}^{n}\left[\frac{1}{2c_{\alpha_s,\mathrm{d}}}\left(\frac{\theta_{\mathrm{P}_0,s}\mathcal{C}_{\alpha,\mathrm{d}}}{\lambda}\right)^{s}\right]^{n}\sum_{\substack{w'\in\mathcal{W}_n^{\Lambda}(0,x-x_0)\\ w'\ \mathrm{SAW}}}\prod_{j=1}^{n}\triangle^{\alpha_s}(w'_{j-1},w'_j)\\ &+\frac{1}{|\mathrm{Im}\,z|}\left[\frac{1}{2c_{\alpha_s,\mathrm{d}}}\left(\frac{\theta_{\mathrm{P}_0,s}\mathcal{C}_{\alpha,\mathrm{d}}}{\lambda}\right)^{s}\right]^{N}\sum_{\substack{w'\in\mathcal{W}_N^{\Lambda}(0,y-x_0)\\ w'\ \mathrm{SAW}\\ y\neq x}}\prod_{j=1}^{n}\triangle^{\alpha_s}(w'_{j-1},w'_j). \quad (3.5.8)\end{aligned}$$

By the definition of $\mathrm{c}_n^{\alpha,\mathrm{SAW}}$ and the fact that $A_s := \left(\frac{1}{2c_{\alpha_s,\mathrm{d}}}\right)^{\frac{1}{s}} \theta_{\mathrm{P}_0,s}\mathcal{C}_{\alpha,\mathrm{d}}$, we get that

$$\mathbb{E}\Big[\big|\mathrm{G}_z^{\Lambda}(x_0,x)\big|^s\Big] < \frac{1}{2}\left(\frac{\theta_{\mathrm{P}_0,s}}{\lambda}\right)^s \sum_{n=1}^{n}\left(\frac{A_s}{\lambda}\right)^{sn} \mathrm{c}_n^{\alpha_s,\mathrm{SAW}}(x-x_0) + \frac{1}{|\mathrm{Im}\, z|}\left(\frac{A_s}{\lambda}\right)^{sN} \sum_{y\in\mathbb{Z}^{\mathrm{d}}\setminus\{x_0\}} \mathrm{c}_N^{\alpha_s,\mathrm{SAW}}(y-x_0). \tag{3.5.9}$$

Now, we compute the limit as $N\to\infty$ on the RHS of the above inequality. On the one hand,

$$\lim_{N\to\infty}\left(\frac{A_s}{\lambda}\right)^{sN}\sum_{y\in\mathbb{Z}^{\mathrm{d}}\setminus\{x_0\}} \mathrm{c}_N^{\alpha_s,\mathrm{SAW}}(y-x_0) = 0, \qquad \forall \lambda > \lambda_0(s) = \frac{A_s}{\left(\mathrm{R}_{\chi^{\alpha,\mathrm{SAW}}}\right)^{\frac{1}{s}}}. \tag{3.5.10}$$

Indeed, $\left(\frac{A_s}{\lambda}\right)^{sN}\sum_{y\in\mathbb{Z}^{\mathrm{d}}\setminus\{x_0\}} \mathrm{c}_N^{\alpha_s,\mathrm{SAW}}(y-x_0)$ corresponds to the N-th term of the series $\chi^{\alpha_s,\mathrm{SAW}}\left(\left(\frac{A_s}{\lambda}\right)^s\right)$, which is convergent for $\lambda>\lambda_0(s)$. On the other hand,

$$\sum_{n=1}^{\infty}\left(\frac{A_s}{\lambda}\right)^{sn}\mathrm{c}_n^{\alpha_s,\mathrm{SAW}}(x-x_0) = \mathrm{C}^{\alpha_s,\mathrm{SAW}}_{\left(\frac{A_s}{\lambda}\right)^s}(x-x_0) < \infty, \qquad \forall\lambda>\lambda_0(s),$$

since $\left(\frac{A_s}{\lambda}\right)^s < \mathrm{R}_{\chi^{\alpha,\mathrm{SAW}}} \le \mathrm{R}_{\mathrm{C}^{\alpha,\mathrm{SAW}}(x-x_0)}$ (see Prop. 3.4.6 below). Therefore,

$$\mathbb{E}\Big[\big|\mathrm{G}_z^{\Lambda}(x_0,x)\big|^s\Big] \le \frac{1}{2}\left(\frac{\theta_{\mathrm{P}_0,s}}{\lambda}\right)^s \mathrm{C}^{\alpha_s,\mathrm{SAW}}_{\left(\frac{A_s}{\lambda}\right)^s}(x-x_0), \quad \forall\lambda>\lambda_0(s),\ \forall x\in\mathbb{Z}^{\mathrm{d}}\setminus\{x_0\}. \tag{3.5.11}$$

This proves Ineq. 3.5.1. □

Corollary 3.5.2 below restates Thm. 3.5.1 using the expectation $\mathbb{E}_0^{\alpha_s}[\cdot]$ and the associated random walk X. This corollary serves as an extension of [FV17, Lemma 8.13] within the framework of the fractional Anderson model.

Corollary 3.5.2. *Let $\frac{\mathrm{d}}{\mathrm{d}+2\alpha} < s < \tau \le 1$. Let α_s be as in Eq. 3.2.14. Let $x_0\in\mathbb{Z}^{\mathrm{d}}$ and $\mathbb{E}_{x_0}^{\alpha_s}[\cdot]$ be as in Def. 3.4.1. Let A_s be as in Thm. 3.5.1. Let $L\in\mathbb{N}$, $\Lambda_L := \{-L,\dots,L\}^{\mathrm{d}}$ and $\mathrm{G}_z^{\Lambda_L}$ be as in Eq. 3.3.22, and $\mathrm{T}_{\Lambda^c} := \inf_{n\in\mathbb{N}}\{\mathrm{X}_n\notin\Lambda_L\}$; that is, $\mathrm{T}_{\Lambda_L^c}$ is a random variable which indicates the first time that the random walk lands outside of Λ. If $\lambda>0$, then*

$$\mathbb{E}\Big[\big|\mathrm{G}_z^{\Lambda_L}(x_0,x)\big|^s\Big] \le \frac{1}{2}\left(\frac{\theta_{\mathrm{P}_0,s}}{\lambda}\right)^s \mathbb{E}_{x_0}^{\alpha_s}\left[\sum_{n=1}^{\mathrm{T}_{\Lambda_L^c}-1}\left(\frac{A_s}{\lambda}\right)^{sn}[(-\Delta)^{\alpha_s}(0,0)]^n \mathbb{1}_{\{\mathrm{X}_n=x,\mathrm{X}\ \mathit{SAW\ up\ to}\ n\}}\right], \quad \forall x\in\Lambda\setminus\{x_0\}. \tag{3.5.12}$$

Furthermore, if $\left(\frac{A_s}{\lambda}\right)^s < \mathrm{R}_{\chi^{\alpha,\mathrm{SAW}}}$, then

$$\mathbb{E}[|\mathrm{G}_z(x_0,x)|^s] \le \frac{1}{2}\left(\frac{\theta_{\mathrm{P}_0,s}}{\lambda}\right)^s \mathbb{E}_{x_0}^{\alpha_s}\left[\sum_{n=1}^{\infty}\left(\frac{A_s}{\lambda}\right)^{sn}[(-\Delta)^{\alpha_s}(0,0)]^n \mathbb{1}_{\{\mathrm{X}_n=x,\mathrm{X}\ \mathit{SAW\ up\ to}\ n\}}\right] < \infty, \quad \forall x\in\mathbb{Z}^{\mathrm{d}}\setminus\{x_0\}. \tag{3.5.13}$$

Proof. Let $L \in \mathbb{N}$ and $\lambda > 0$. Using Ineq. 3.5.4 from the proof of Thm. 3.5.1 for $N = L$, and then applying Eq. 3.4.17 and Fubini-Tonelli, we get that

$$\begin{aligned}
&\mathbb{E}\left[\left|\mathrm{G}_z^{\Lambda}(x_0, x)\right|^s\right] \\
&\leq \frac{1}{2}\left(\frac{\theta_{\mathrm{P}_0,s}}{\lambda}\right)^s \sum_{n=1}^{L}\left(\frac{A_s}{\lambda}\right)^{sn} \sum_{\substack{w \in \mathcal{W}_n^{\Lambda}(x_0,x) \\ w \text{ SAW}}} \prod_{j=1}^{n} \triangle^{\alpha_s}(w_{j-1}, w_j) \\
&= \frac{1}{2}\left(\frac{\theta_{\mathrm{P}_0,s}}{\lambda}\right)^s \sum_{n=1}^{L}\left(\frac{A_s}{\lambda}\right)^{sn} [(-\Delta)^{\alpha_s}(0,0)]^n \mathrm{Q}_{x_0}^{\alpha_s}(\mathrm{X}_n = x, \mathrm{X} \text{ SAW up to } n, \mathrm{T}_{\Lambda^c} > n) \\
&= \frac{1}{2}\left(\frac{\theta_{\mathrm{P}_0,s}}{\lambda}\right)^s \mathbb{E}_{x_0}^{\alpha_s}\left[\sum_{n=1}^{\mathrm{T}_{\Lambda^c}-1}\left(\frac{A_s}{\lambda}\right)^{sn} [(-\Delta)^{\alpha_s}(0,0)]^n \mathbb{1}_{\{\mathrm{X}_n = x, \mathrm{X} \text{ SAW up to } n\}}\right],
\end{aligned} \tag{3.5.14}$$

which is the desired inequality.

Now, if $\left(\frac{A_s}{\lambda}\right)^s < \mathrm{R}_{\chi^{\alpha,\mathrm{SAW}}}$, then $\mathrm{C}^{\alpha,\mathrm{SAW}}_{\left(\frac{A_s}{\lambda}\right)^s}(x - x_0) < \infty$. In addition, by Thm. 3.5.1, Eq. 3.4.19 and Fubini-Tonelli, we get that

$$\begin{aligned}
&\mathbb{E}[|\mathrm{G}_z(x_0, x)|^s] \\
&\leq \frac{1}{2}\left(\frac{\theta_{\mathrm{P}_0,s}}{\lambda}\right)^s \mathrm{C}^{\alpha_s,\mathrm{SAW}}_{\left(\frac{A_s}{\lambda}\right)^s}(x - x_0) \\
&= \frac{1}{2}\left(\frac{\theta_{\mathrm{P}_0,s}}{\lambda}\right)^s \sum_{n \geq 1}\left(\frac{A_s}{\lambda}\right)^{sn} [(-\Delta)^{\alpha_s}(0,0)]^n \mathrm{Q}_{x_0}^{\alpha_s}(\mathrm{X}_n = x, \mathrm{X} \text{ SAW up to } n) \\
&= \frac{1}{2}\left(\frac{\theta_{\mathrm{P}_0,s}}{\lambda}\right)^s \mathbb{E}_{x_0}^{\alpha_s}\left[\sum_{n=1}^{\infty}\left(\frac{A_s}{\lambda}\right)^{sn} [(-\Delta)^{\alpha_s}(0,0)]^n \mathbb{1}_{\{\mathrm{X}_n = x, \mathrm{X} \text{ SAW up to } n\}}\right].
\end{aligned} \tag{3.5.15}$$

□

Remark 3.5.3. *The case of the usual discrete Laplacian is well-studied (see [FV17]). In such scenario, we instead obtain a simple symmetric random walk on $\mathbb{Z}^{\mathrm{d}}$. To be more precise,*

$$-\frac{1}{2\mathrm{d}}\Delta = \mathrm{I} - \mathcal{T}, \tag{3.5.16}$$

where I is the identity operator on $\ell^2(\mathbb{Z}^{\mathrm{d}})$ and $\mathcal{T}$ is the operator given by

$$\mathcal{T}(x_0, x) = \begin{cases} \frac{1}{2\mathrm{d}}, & |x_0 - x|_1 = 1, \\ 0, & \textit{otherwise}. \end{cases} \tag{3.5.17}$$

We can define the simple symmetric random walk $\mathcal{X} = \{\mathcal{X}\}_{k \geq 0}$ on $\mathbb{Z}^{\mathrm{d}}$ as the random walk which has transition probabilities given by the entries of $\mathcal{T}$. Let $\mathcal{Q}_{x_0}$ be the distribution of the simple symmetric random walk which begins at 0; i.e.,

$$\begin{aligned}
\mathcal{Q}_{x_0}(\mathcal{X}_n = w_n \mid \mathcal{X}_{n-1} = w_{n-1}) &= \mathcal{T}(w_n, w_{n-1}), \\
\mathcal{Q}_{x_0}(\mathcal{X}_0 = x_0) &= 1.
\end{aligned} \tag{3.5.18}$$

Let $\tilde{E}_{x_0}$ be the expectation given by $\mathcal{Q}_{x_0}$. Let Λ_L be as in Corollary 3.5.2 and $\tilde{\mathrm{T}}_{\Lambda^c} := \inf_{n\in\mathbb{N}}\{\mathcal{X}_n \notin \Lambda_L\}$. Then

$$\left(-\frac{1}{2\mathrm{d}}\Delta^{\Lambda}\right)^{-1}(x_0, x) = \tilde{E}_{x_0}\left[\sum_{n=0}^{\tilde{\mathrm{T}}_{\Lambda^c}-1} \mathbb{1}_{\{\mathcal{X}_n = x\}}\right], \tag{3.5.19}$$

see [FV17, Lemma 8.13]. That is, the Green's function of the usual discrete Laplacian restricted to Λ_L at (x_0, x) corresponds to the average of arrivals at x by a simple symmetric random walk starting at x_0, before it lands outside of Λ_L.

Let A_s be as in Thm. 3.3.2 and let $\lambda_1(s) := A_s|(-\Delta)^{\alpha}(0,0)|^{\frac{1}{s}}$. By Prop. 3.4.6, it holds that $\lambda_1(s) \geq \lambda_0(s)$, for $s > 0$. Therefore, applying Thm. 3.5.1 and 3.4.7, we obtain that

$$\begin{aligned}\mathbb{E}[|\mathrm{G}_z(x_0, x)|^s] \leq &\left(\frac{(-\Delta)^{\alpha}(0,0)\,\mathrm{K}_{\mathrm{P}_0,\tau,s}}{\lambda}\right)^s \Delta^{-\alpha_s}(x_0, x),\\ &\forall\lambda \geq \lambda_1(s),\ \forall s \in \left(\frac{\mathrm{d}}{\mathrm{d}+2\alpha}, \min\left\{\tau, \frac{2\mathrm{d}}{\mathrm{d}+2\alpha}\right\}\right).\end{aligned} \tag{3.5.20}$$

Together with Thm. 3.2.7, we obtain the estimate

$$\begin{aligned}\mathbb{E}[|\mathrm{G}_z(x_0, x)|^s] \leq &\left(\frac{(-\Delta)^{\alpha}(0,0)\,\mathrm{K}_{\mathrm{P}_0,\tau,s}}{\lambda}\right)^s \frac{1}{|x - x_0|^{\mathrm{d}-2\alpha_s}},\\ &\forall\lambda \geq \lambda_1(s),\ \forall s \in \left(\frac{\mathrm{d}}{\mathrm{d}+2\alpha}, \min\left\{\tau, \frac{2\mathrm{d}}{\mathrm{d}+2\alpha}\right\}\right),\end{aligned} \tag{3.5.21}$$

which gives a slower decay than Ineq. 3.5.23.

Remark 3.5.4. *We require that $s \in \left(\frac{\mathrm{d}}{\mathrm{d}+2\alpha}, \frac{2\mathrm{d}}{\mathrm{d}+2\alpha}\right)$ to ensure that $0 < \alpha_s < \frac{\mathrm{d}}{2}$. For $\mathrm{d} = 1$ and $\frac{1}{2} < \alpha < 1$, it holds that $\frac{1}{2} \leq \alpha_s < \alpha$ for $s \in \left[\frac{2}{1+2\alpha}, 1\right)$.*

We recall [CS15, Lemma 2.4]: Given $0 < \gamma < \mathrm{R}_{\chi^{\alpha,\mathrm{SAW}}}$, then

$$\mathrm{C}_{\gamma}^{\alpha,\mathrm{SAW}}(x - x_0) \leq \frac{\tilde{\mathrm{K}}_{\mathrm{P}_0,\tau,s,\gamma}}{|x - x_0|^{\mathrm{d}+2\alpha}}, \tag{3.5.22}$$

where $\tilde{\mathrm{K}}_{\mathrm{P}_0,\tau,s,\gamma} = \chi_{\gamma}^{\alpha,\mathrm{SAW}}\left(6^{\mathrm{d}+2\alpha}2\gamma\chi_{\gamma}^{\alpha,\mathrm{SAW}}\mathcal{C}_{\alpha,\mathrm{d}} + \tilde{\ell}_{\alpha,\mathrm{d},\gamma}^{\mathrm{d}+2\alpha}\right)$, the constant $\tilde{\ell}_{\alpha,\mathrm{d},\gamma} > 0$ is given in the proof of [CS15, Lemma 2.4]. This combined with Thm. 3.5.1 yield our main result.

Theorem 3.5.5. *Let $\frac{\mathrm{d}}{\mathrm{d}+2\alpha} < s < \tau \leq 1$. Let $\alpha_s = s\left(\alpha + \frac{\mathrm{d}}{2}\right) - \frac{\mathrm{d}}{2}$, the constant in Eq. 3.2.14. Let A_s be as in Thm. 3.3.2. Let $\lambda_0(s) = \frac{A_s}{(\mathrm{R}_{\chi^{\alpha_s,\mathrm{SAW}}})^{\frac{1}{s}}}$ be as in Thm. 3.5.1. If $\lambda > \lambda_0(s)$, then*

$$\mathbb{E}[|\mathrm{G}_z(x_0, x)|^s] \leq \left(\frac{\mathcal{K}_{\mathrm{P}_0,\tau,s,\lambda}}{\lambda}\right)^s \frac{1}{|x - x_0|^{\mathrm{d}+2\alpha_s}},\ \forall x, x_0 \in \mathbb{Z}^{\mathrm{d}},\ x \neq x_0,\ \textit{unif. in } z \in \mathbb{C}\setminus\mathbb{R}, \tag{3.5.23}$$

where $\mathcal{K}_{\mathrm{P}_0,\tau,s,\lambda} = \mathrm{K}_{\mathrm{P}_0,\tau,s}\left(\tilde{\mathrm{K}}_{\mathrm{P}_0,\tau,s,(\frac{A_s}{\lambda})^s}\right)^{\frac{1}{s}}$, *whose factors are given, respectively, in Thm. 3.5.1 and Ineq. 3.5.22. Moreover,* $\sigma(\mathrm{H}_\alpha)$ *is pure-point, for* $\mathbb{P}-a.s$ $\omega \in \Omega$ *. Furthermore, let* g *be the density of* P_0*, (i.e.,* $\mathrm{dP}_0(v) = \mathrm{d}v\, g(v)$*) and let* $I \subset \mathbb{R}$ *be an open and bounded Borel set. If there are constants* $M > 0$ *and* $0 < \kappa < 1$ *such that Ineq. 3.3.6 holds, then, for* $\mathbb{P}-a.s.$ $\omega \in \Omega$*, for all* $\mathrm{E} \in \sigma(\mathrm{H}_\alpha) \cap I$*, there is a localization center* $\mathrm{x}_\mathrm{E}(\omega) \in \mathbb{Z}^\mathrm{d}$ *such that the corresponding normalized eigenfunction* $\varphi_\mathrm{E}(\cdot,\omega)$ *satisfies*

$$|\varphi_\mathrm{E}(y,\omega)|^2 \leq 4\mathrm{A}_{I,t}(\omega)\left[\frac{(-\Delta)^\alpha(0,0)}{\triangle^\alpha(0,\mathrm{x}_\mathrm{E}(\omega))}\right]^2 \frac{1}{(1+|y-\mathrm{x}_\mathrm{E}(\omega)|)^t},\ \forall\, t \in (0, 2\alpha_s),\ \forall y \in \mathbb{Z}^\mathrm{d}, \tag{3.5.24}$$

where $\mathrm{A}_{I,t}$ *is an integrable random variable given in Eq. 3.3.14 above.*

Remark 3.5.6. *Let* $\mathrm{d} = \tau = 1$*,* $\frac{1}{2} < \alpha < 1$*,* $\frac{1}{1+2\alpha} < s < 1$ *and* $\lambda > \lambda_0(s)$*. If we assume that* $\sup_{v\in\mathbb{R}}(1+|v|)^\kappa g(v) \leq M$*, for some constants* $M > 0$ *and* $0 < \kappa < 1$*, then we establish pure-point spectrum and polynomial decaying of the eigenfunctions. Explicitely, let* $\varphi \in \ell^2(\mathbb{Z})$ *be an eigenfunction, then, for* $|x|$ *large,* $|\varphi(x)|^2 \lesssim \frac{1}{|x|^t}$*, where* $1 < t < s(1+2\alpha) - 1 < 2$*.*

Proof. Let $\frac{\mathrm{d}}{\mathrm{d}+2\alpha} < s < \tau \leq 1$ and $\lambda > \lambda_0(s)$. As mentioned in Section 3.3.1, we note that Ineq. 3.5.23 can be derived from both Ineq. 3.5.1 and 3.5.22. To complete the proof, we need to establish spectral localization. Furthermore, if we can find a constant $M > 0$ satisfying Ineq. 3.3.6, then the eigenfunctions decay polynomially, as stated in Ineq. 3.5.24. To accomplish this, we use Lemma 3.3.5. Thus, it is enough to find a pair of constants $t > 0$ and $K > 0$ such that Ineq. 3.3.12 holds. In fact, the Decoupling Lemma (Lemma 3.3.4), Thm. 3.5.1 and Corollary 3.2.6 yield that

$$\begin{aligned}
&\mathbb{E}\left[\sum_{x\in\mathbb{Z}^\mathrm{d}} |\mathrm{G}_{z,\omega,\lambda,\alpha}(x_0,x)|^s(1+|x-x_0|)^t\right] \leq \frac{1}{2}\left(\frac{\theta_{\mathrm{P}_0,s}}{\lambda}\right)^s + \sum_{x\in\mathbb{Z}^\mathrm{d}\setminus\{x_0\}}\left(\frac{\mathcal{K}_{\mathrm{P}_0,\tau,s,\lambda}}{\lambda}\right)^s \frac{(1+|x-x_0|)^t}{|x-x_0|^{\mathrm{d}+2\alpha_s}}\\
&\leq \frac{1}{2}\left(\frac{\theta_{\mathrm{P}_0,s}}{\lambda}\right)^s + \sum_{x\in\mathbb{Z}^\mathrm{d}\setminus\{x_0\}}\left(\frac{\mathcal{K}_{\mathrm{P}_0,\tau,s,\lambda}}{\lambda}\right)^s \frac{2^t}{|x-x_0|^{\mathrm{d}+2\alpha_s-t}}\\
&\leq \frac{1}{2}\left(\frac{\theta_{\mathrm{P}_0,s}}{\lambda}\right)^s + 2^{N_\mathrm{d}+1+t}\left(\frac{\mathcal{K}_{\mathrm{P}_0,\tau,s,\lambda}}{\lambda}\right)^s\left(1+\frac{1}{2\alpha_s - t}\right) < \infty,\ \forall x_0 \in \mathbb{Z}^\mathrm{d}, \forall z \in \mathbb{C}^+, \forall t \in (0, 2\alpha_s).
\end{aligned} \tag{3.5.25}$$

This ends the proof of Thm. 3.5.5. □

3.5.2 Comparison of decaying rates

We examine and compare the various decay regimes of the fractional moments of the Green's function, as indicated by Ineq. 3.3.35, 3.3.50, 3.5.21, and 3.5.23. Then, we establish a hierarchy based on the corresponding exponents: $\beta_{\lambda,s}$, $2s\alpha\tilde{\eta}z$, α, d, $\mathrm{d}-2\alpha s$, and $\mathrm{d}+2\alpha_s$. The summarized discussion can be found in Table 3.1 and 3.2 below. After this comparison, we conclude that our main result yields the sharpest decay.

By Prop. 3.4.6, it holds that $(-\Delta)^{\alpha_s}(0,0) \geq \frac{1}{\mathrm{R}_{\chi^{\alpha_s},\mathrm{SAW}}}$. Then, $\lambda_{\mathrm{AG}}(s) > \lambda_0(s)$ and Ineq. 3.5.23 yields a faster decay than 3.3.35, $\forall\lambda > \lambda_{\mathrm{AG}}(s)$, $\forall s \in \left(\frac{\mathrm{d}}{\mathrm{d}+2\alpha}, \tau\right)$. In addition, since $\beta_{\lambda,s} < 2\alpha_s$, the eigenfunctions have a better decay in Thm. 3.5.1 than in 3.3.2 a.s.

However, if we compare Ineq. 3.3.35 with Ineq. 3.5.21, we obtain that

$$\beta_{\lambda,s} < \mathrm{d} - 2\alpha_s, \quad \forall\lambda > \lambda_{AG}(s),\ \forall s \in \left(\frac{\mathrm{d}}{\mathrm{d}+2\alpha}, \tau\right) \cap \left(\frac{\mathrm{d}}{\mathrm{d}+2\alpha}, \frac{3\mathrm{d}}{2(\mathrm{d}+2\alpha)}\right], \tag{3.5.26}$$

where we note that the interval for s is always non-empty and $\lambda_{AG}(s) > \lambda_1(s)$, for all $s > 0$. If $\tau \leq \frac{3\mathrm{d}}{2(\mathrm{d}+2\alpha)}$ (e.g., for $\mathrm{d} \geq 4$ or $0 < \alpha \leq \frac{\mathrm{d}}{4}$), then Ineq. 3.5.21 provides a better decay than 3.3.35, $\forall\lambda > \lambda_{AG}(s)$, $\forall s \in \left(\frac{\mathrm{d}}{\mathrm{d}+2\alpha}, \tau\right)$. Otherwise, if $\frac{3\mathrm{d}}{2(\mathrm{d}+2\alpha)} < \tau$, then Ineq. 3.5.21 yields a better decay than 3.3.35, $\forall\lambda > \lambda_{AG}(s)$, $\forall s \in \left(\frac{\mathrm{d}}{\mathrm{d}+2\alpha}, \frac{3\mathrm{d}}{2(\mathrm{d}+2\alpha)}\right]$. Alternatively, for all $s \in \left(\frac{3\mathrm{d}}{2(\mathrm{d}+2\alpha)}, \min\left\{\tau, \frac{2\mathrm{d}}{\mathrm{d}+2\alpha}\right\}\right)$, there is $\lambda^*(s) > \lambda_{AG}(s)$ such that

$$\mathrm{d} - 2\alpha_s < \beta_{\lambda,s} \qquad \forall\lambda \geq \lambda^*(s), \tag{3.5.27}$$

and Ineq. 3.3.35 has a better decay than 3.5.21, $\forall s \in \left(\frac{3\mathrm{d}}{2(\mathrm{d}+2\alpha)}, \min\left\{\tau, \frac{2\mathrm{d}}{\mathrm{d}+2\alpha}, \right\}\right)$, $\lambda \geq \lambda^*(s)$.

Remark 3.5.7. *Note that* $\mathrm{d} - 2\alpha_s = 2\mathrm{d} - s(2\alpha + \mathrm{d})$ *is strictly-decreasing in* s *and* $\max\{\mathrm{d} - 2\alpha, 0\} < \mathrm{d} - 2\alpha_s < \mathrm{d}$, *for all* $s \in \left(\frac{\mathrm{d}}{\mathrm{d}+2\alpha}, \min\left\{1, \frac{2\mathrm{d}}{\mathrm{d}+2\alpha}\right\}\right)$. *In addition,*

$$\mathrm{d} - 2\alpha_{s^*} = 2\alpha_{s^*} = \sup_{\lambda > \lambda_0(s^*)} \beta_{\lambda,s^*}, \qquad \textit{for } s^* = \frac{3\mathrm{d}}{2(\mathrm{d}+2\alpha)}. \tag{3.5.28}$$

The above discussion is summed up by Table 3.1 below.

τ	s	λ	Decay
$\frac{\mathrm{d}}{\mathrm{d}+2\alpha} < \tau \leq \frac{3\mathrm{d}}{2(\mathrm{d}+2\alpha)}$	$\frac{\mathrm{d}}{\mathrm{d}+2\alpha} < s < \tau$	$\lambda > \lambda_{AG}(s)$	$\beta_{\lambda,s} < \mathrm{d} - 2\alpha_s$
$\frac{3\mathrm{d}}{2(\mathrm{d}+2\alpha)} < \tau \leq 1$	$\frac{\mathrm{d}}{\mathrm{d}+2\alpha} < s \leq \frac{3\mathrm{d}}{2(\mathrm{d}+2\alpha)}$	$\lambda > \lambda_{AG}(s)$	
$\frac{3\mathrm{d}}{2(\mathrm{d}+2\alpha)} < \tau \leq 1$	$\frac{3\mathrm{d}}{2(\mathrm{d}+2\alpha)} < s < \min\left\{\tau, \frac{2\mathrm{d}}{\mathrm{d}+2\alpha}\right\}$	$\lambda > \lambda^*(s)$	$\mathrm{d} - 2\alpha_s < \beta_{\lambda,s}$

Table 3.1: Thm. 3.3.35 vs. Thm. 3.2.7

From Ineq. 3.3.49 and Ineq. 3.5.21, we get that

$$2s\alpha\eta < \mathrm{d} - 2\alpha_s, \quad \forall s \in \left(\frac{\mathrm{d}}{\mathrm{d}+2\alpha}, \min\left\{\tau, \frac{2\mathrm{d}}{\mathrm{d}+2\alpha(1+\eta)}\right\}\right), \forall\eta(0,1). \tag{3.5.29}$$

If $\tau \leq \frac{2\mathrm{d}}{\mathrm{d}+2\alpha(1+\eta)}$ (e.g., for $\mathrm{d} = 4$ or $0 < \alpha \leq \frac{\mathrm{d}}{4}$), then Ineq. 3.5.21 provides a better decay than 3.3.49, $\forall s \in \left(\frac{\mathrm{d}}{\mathrm{d}+2\alpha}, \tau\right)$. Otherwise, if $\tau > \frac{2\mathrm{d}}{\mathrm{d}+2\alpha(1+\eta)}$, then Ineq. 3.5.21 yields a faster decay than 3.3.49, $\forall s \in \left(\frac{\mathrm{d}}{\mathrm{d}+2\alpha}, \frac{2\mathrm{d}}{\mathrm{d}+2\alpha(1+\eta)}\right)$. Alternatively, for all $s \in \left[\frac{2\mathrm{d}}{\mathrm{d}+2\alpha(1+\eta)}, \min\left\{\tau, \frac{2\mathrm{d}}{\mathrm{d}+2\alpha}\right\}\right)$, it holds that

$$\mathrm{d} - 2\alpha_s \leq 2s\alpha\eta \tag{3.5.30}$$

and Ineq. 3.3.49 yields a sharper decay than 3.5.21 $\forall s \in \left[\frac{2\mathrm{d}}{\mathrm{d}+2\alpha(1+\eta)}, \min\left\{\tau, \frac{2\mathrm{d}}{\mathrm{d}+2\alpha}\right\}\right)$.

Remark 3.5.8. *Like in Remark 3.5.7, we have that* $(-\Delta)^{-\alpha_s}$ *is well-defined since* $\frac{\mathrm{d}}{\mathrm{d}+2\alpha} < s < \frac{2\mathrm{d}}{\mathrm{d}+2\alpha(1+\eta)} < \frac{2\mathrm{d}}{\mathrm{d}+2\alpha}$ $\forall \eta \in (0,1)$. *In addition, given* $\eta \in (0,1)$, $2s\alpha\eta$ *is strictly-increasing in* s, $\mathrm{d}-2\alpha_s$ *is strictly-decreasing and they are equal at* $s = \frac{2\mathrm{d}}{\mathrm{d}+2\alpha(1+\eta)}$.

The gist of the above discussion is shown in Table 3.2 below.

τ	s	λ	Decay
$\frac{\mathrm{d}}{\mathrm{d}+2\alpha} < \tau \leq \frac{2\mathrm{d}}{\mathrm{d}+2\alpha(1+\eta)}$ $\frac{2\mathrm{d}}{\mathrm{d}+2\alpha(1+\eta)} < \tau \leq 1$	$\frac{\mathrm{d}}{\mathrm{d}+2\alpha} < s < \tau$ $\frac{\mathrm{d}}{\mathrm{d}+2\alpha} < s < \frac{2\mathrm{d}}{\mathrm{d}+2\alpha(1+\eta)}$	$\lambda > \lambda_1(s)$ $\lambda > \lambda_1(s)$	$2s\alpha\eta < \mathrm{d} - 2\alpha_s$
$\frac{2\mathrm{d}}{\mathrm{d}+2\alpha(1+\eta)} < \tau \leq 1$	$\frac{2\mathrm{d}}{\mathrm{d}+2\alpha(1+\eta)} \leq s < \min\left\{\tau, \frac{2\mathrm{d}}{\mathrm{d}+2\alpha}\right\}$	$\lambda > \lambda^*(s)$	$\mathrm{d} - 2\alpha_s < 2s\alpha\eta$

Table 3.2: Thm. 3.2.7 vs. Thm. 3.3.11 [Massive vs. Massless case]

Since $0 < \tilde{\eta}_{z,\alpha,\mathrm{d}} < 1$, Ineq. 3.3.48 yields a slower decay than Ineq. 3.5.23. Instead, if we make the comparison between Ineq. 3.3.48 and Ineq. 3.3.35, it is unclear which one is better than the other since $2s\alpha\tilde{\eta}$ and $\mathrm{d} - \alpha_s$ decay slower than $2s\alpha$. Finally, we provide an estimation of $\beta_{\lambda,s}$ and $\tilde{\eta}_{z,\alpha,\mathrm{d}}$ as α tends to 0 or 1.

Theorem 3.5.9. *(a) Let* $\frac{\mathrm{d}}{\mathrm{d}+2\alpha} < \tau \leq 1$ *and* $\beta_{\lambda,s}$ *be as in Thm. 3.3.2. Then,*

$$\lim_{\alpha \uparrow 1} \sup_{\frac{\mathrm{d}}{\mathrm{d}+2\alpha} < s < \tau} \left| \beta_{\lambda,s} + \mathcal{O}\left(\frac{1}{\lambda^s}\right) \right| = 0 \tag{3.5.31}$$

Furthermore, if $\tau = 1$, *then*

$$\lim_{\alpha \downarrow 0} \sup_{\frac{\mathrm{d}}{\mathrm{d}+2\alpha} < s < 1} \beta_{\lambda,s} = 0. \tag{3.5.32}$$

(b) Let $\mathrm{d} = 1$, $0 < \mathcal{N} < 2\alpha$, $\mathcal{R}_\alpha := \{z \in \mathbb{C} \mid 0 < \mathrm{D}_z \leq 2S_{\mathcal{N}}(\alpha, 1)\}$ *and* $\tilde{\eta}_{z,\alpha,\mathrm{d}}$ *be as in Thm. 3.3.11. Then,*

$$\lim_{\alpha \downarrow 0} \sup_{\substack{0<s<1 \\ 0<\mathcal{N}<2\alpha \\ z \in \mathcal{R}_\alpha}} 2s\alpha\tilde{\eta}_z = 0. \tag{3.5.33}$$

Moreover, let $0 < \mathcal{N} < 2$ *and* $0 < \varepsilon < 1$, *then* $\inf_{\varepsilon \leq \alpha < 1} S_{\mathcal{N}}(\alpha, 1) > 0$ *and*

$$\lim_{\alpha \uparrow 1} \sup_{\substack{0<s<1 \\ S_{\mathcal{N}}(\alpha,1)<\infty}} \left| 2s\alpha\tilde{\eta}_{z,\alpha,1} - \frac{\mathcal{N}\mathrm{D}_z}{2\left(1 - \frac{1}{2^{\mathcal{N}}}\right)} \right| = 0, \quad \forall\, z \in \bigcap_{\varepsilon<\alpha<1} \mathcal{R}_\alpha. \tag{3.5.34}$$

Let $z \notin \sigma(\mathrm{H}_{\lambda,\alpha})$. Although $\tilde{\eta}_{z,\alpha,\mathrm{d}}$ has a limit in the sense of Eq. 3.5.33 as $\alpha \downarrow 0$, the RHS of Eq. 3.3.48 diverges because $\mathrm{D}_z \leq 2S_{\mathcal{N}}(\alpha, 1) \downarrow 0$.

The proof of Thm. 3.5.9 below is presented in two parts. In the first half, we evaluate the limits of $\beta_{\lambda,s}$ as α approaches 1 and as α approaches 0. In the second half, we aim to apply the same limits to $\tilde{\eta}_{z,\alpha,1}$. However, in this case, the computations involve the Γ function. It is important to recall that Γ is a meromorphic function defined on $\mathbb{C} \setminus \{0, -1, -2, \ldots\}$, and it possesses two useful properties: $\Gamma(n) = (n-1)!$ for all $n \in \mathbb{N}$, and Γ has simple poles at non-positive integers.

Proof.

(a) Let $\frac{\mathrm{d}}{\mathrm{d}+2\alpha} < s < \tau$ and α_s be as in Eq. 3.2.14. Due to Thm 3.3.2, we have $\beta_{\lambda,s} = 2\alpha_s - \mathcal{O}\big(\frac{1}{\lambda^s}\big) = s(2\alpha+\mathrm{d}) - \mathrm{d} - \mathcal{O}\big(\frac{1}{\lambda^s}\big) > 0$. Hence,

$$\lim_{\alpha\uparrow 1} \sup_{\frac{\mathrm{d}}{\mathrm{d}+2\alpha}<s<\tau} \left|\beta_{\lambda,s} + \mathcal{O}\Big(\frac{1}{\lambda^s}\Big)\right| = \tau(2+\mathrm{d}) - \mathrm{d}. \tag{3.5.35}$$

On the other hand, we know $0 < \beta_{\lambda,s} < 2\alpha_s < 2\alpha$. If $\tau = 1$, then

$$\lim_{\alpha\downarrow 0} \sup_{\frac{\mathrm{d}}{\mathrm{d}+2\alpha}<s<1} \beta_{\lambda,s} = 0. \tag{3.5.36}$$

(b) Let $\mathrm{d} = 1$, $0 < s < 1$, $0 < \mathcal{N} < 2\alpha$ and $0 < \mathrm{D}_z \leq 2S_{\mathcal{N}}(\alpha,1)$. To compute the limits of $\tilde{\eta}_{z,\alpha,1} = \frac{\mathcal{N}\mathrm{D}_z}{4\alpha S_{\mathcal{N}}(\alpha,1)}$, we need to estimate $S_{\mathcal{N}}(\alpha,1)$. Recall that

$$\mathrm{S}_{\mathcal{N}}(\alpha,1) := \sum_{x\in\mathbb{Z}\setminus\{0\}} \triangle^\alpha(0,x)\Big(1 - \frac{1}{(1+|x|)^{\mathcal{N}}}\Big). \tag{3.5.37}$$

[CRS+18, Thm. 1.2 (b)] provides an explicit value of $\triangle^\alpha(0,x)$, which is given by

$$\triangle^\alpha(0,x) = \frac{4^\alpha}{\sqrt{\pi}} \frac{\Gamma\big(\frac{1}{2}+\alpha\big)}{|\Gamma(-\alpha)|} \frac{\Gamma(|x|-\alpha)}{\Gamma(|x|+\alpha+1)}, \quad x\in\mathbb{Z}\setminus\{0\}. \tag{3.5.38}$$

Replacing Eq. 3.5.38 into Eq. 3.5.37, we obtain that

$$\begin{aligned}\mathrm{S}_{\mathcal{N}}(\alpha,1) =& 2\frac{4^\alpha}{\sqrt{\pi}} \frac{\Gamma\big(\frac{1}{2}+\alpha\big)}{|\Gamma(-\alpha)|} \frac{\Gamma(1-\alpha)}{\Gamma(2+\alpha)}\Big(1 - \frac{1}{2^{\mathcal{N}}}\Big)\\ &+ 2\frac{4^\alpha}{\sqrt{\pi}} \frac{\Gamma\big(\frac{1}{2}+\alpha\big)}{|\Gamma(-\alpha)|} \sum_{n\geq 2} \frac{\Gamma(n-\alpha)}{\Gamma(n+\alpha+1)}\Big(1 - \frac{1}{(1+n)^{\mathcal{N}}}\Big).\end{aligned} \tag{3.5.39}$$

On the one hand, the Gamma function has the property that $\alpha|\Gamma(-\alpha)| = \Gamma(1-\alpha)$. Hence, the first term on the RHS of the above equation can be rewritten as

$$2\frac{4^\alpha}{\sqrt{\pi}} \frac{\Gamma\big(\frac{1}{2}+\alpha\big)}{|\Gamma(-\alpha)|} \frac{\Gamma(1-\alpha)}{\Gamma(2+\alpha)}\Big(1 - \frac{1}{2^{\mathcal{N}}}\Big) = 2\alpha\frac{4^\alpha}{\sqrt{\pi}} \frac{\Gamma\big(\frac{1}{2}+\alpha\big)}{\Gamma(2+\alpha)}\Big(1 - \frac{1}{2^{\mathcal{N}}}\Big). \tag{3.5.40}$$

On the other hand, to estimate the second term, we need to know how it decays the general term of the series. The Stirling's formula for the Gamma function is

$$\Gamma(y) \sim \sqrt{\frac{2\pi}{y}}\Big(\frac{y}{e}\Big)^y, \qquad y > 0. \tag{3.5.41}$$

Note that for $y = n \in \mathbb{N}$, the above formula gives $\Gamma(n) = (n-1)! \sim \sqrt{\frac{2\pi}{n}}\big(\frac{x}{e}\big)^n$. By multiplying by n on both sides, we recover the usual Stirling's formula. Hence,

$$\begin{aligned}&\frac{\Gamma(n-\alpha)}{\Gamma(n+\alpha+1)} \sim \sqrt{\frac{n+\alpha+1}{n-\alpha}}\Big(\frac{n-\alpha}{e}\Big)^{n-\alpha}\Big(\frac{e}{n+\alpha+1}\Big)^{n+\alpha+1}\\ &\sim \Big(\frac{e}{n}\Big)^{1+2\alpha}\Big(1-\frac{\alpha}{n}\Big)^{n-\alpha}\left(\frac{1}{1+\frac{\alpha+1}{n}}\right)^{n+\alpha+1} \sim \Big(\frac{e}{n}\Big)^{1+2\alpha} e^{-\alpha}e^{-1-\alpha} = \frac{1}{n^{1+2\alpha}}.\end{aligned}$$

Consequently,

$$
\begin{aligned}
&\mathrm{S}_{\mathcal{N}}(\alpha,1)\\
&\sim 2\frac{4^{\alpha}}{\sqrt{\pi}}\Gamma\Big(\frac{1}{2}+\alpha\Big)\left[\frac{\alpha}{\Gamma(2+\alpha)}\Big(1-\frac{1}{2^{\mathcal{N}}}\Big)+\frac{1}{|\Gamma(-\alpha)|}\sum_{n\geq 2}\frac{1}{n^{1+2\alpha}}\Big(1-\frac{1}{(1+n)^{\mathcal{N}}}\Big)\right].
\end{aligned}
\tag{3.5.42}
$$

By the continuity of $\Gamma(y)$, for $y>0$, and the facts that $\Gamma(\frac{1}{2})=\frac{\sqrt{\pi}}{2}$, $\Gamma(n)=(n-1)!$, for $n\in\mathbb{N}$, and $\lim_{y\to 0}\frac{1}{\Gamma(y)}=\lim_{y\to -1}\frac{1}{\Gamma(y)}=0$, it holds that

$$
\lim_{\alpha\uparrow 1}\mathrm{S}_{\mathcal{N}}(\alpha,1)=2\Big(1-\frac{1}{2^{\mathcal{N}}}\Big),\qquad\qquad \lim_{\alpha\downarrow 0}\mathrm{S}_{\mathcal{N}}(\alpha,1)=0. \tag{3.5.43}
$$

Let $\mathcal{R}_{\alpha}:=\{z\in\mathbb{C}\,|\,0<\mathrm{D}_z\leq 2S_{\mathcal{N}}(\alpha,1)\}$. Recall that $\tilde{\eta}_{z,\alpha,1}=\frac{\mathcal{N}\mathrm{D}_z}{4\alpha S_{\mathcal{N}}(\alpha,1)}$. Hence,

$$
0\leq 2s\alpha\tilde{\eta}_z\leq\frac{s\mathcal{N}\mathrm{D}_z}{2S_{\mathcal{N}}(\alpha,1)}\leq s\mathcal{N},\qquad \forall z\in\mathcal{R}_{\alpha}. \tag{3.5.44}
$$

Therefore,

$$
\lim_{\alpha\downarrow 0}\sup_{\substack{0<s<1\\ 0<\mathcal{N}<2\alpha\\ z\in\mathcal{R}_{\alpha}}} 2s\alpha\tilde{\eta}_z=0. \tag{3.5.45}
$$

Moreover, let $0<\varepsilon<1$, then $\inf_{\varepsilon\leq\alpha<1}S_{\mathcal{N}}(\alpha,1)>0$ due to Eq. 3.5.43. By Ineq. 3.5.43, it follows that

$$
\lim_{\alpha\uparrow 1}\sup_{\substack{0<s<1\\ S_{\mathcal{N}}(\alpha,1)<\infty}}\left|2s\alpha\tilde{\eta}_{z,\alpha,1}-\frac{\mathcal{N}\mathrm{D}_z}{2\big(1-\frac{1}{2^{\mathcal{N}}}\big)}\right|=0,\quad \forall\, z\in\bigcap_{\varepsilon<\alpha<1}\mathcal{R}_{\alpha}. \tag{3.5.46}
$$

□

Remark 3.5.10.

- *Note that in Eq. 3.5.32, we need to take* $\tau=1$ *because* $\lim_{\alpha\downarrow 0}\frac{\mathrm{d}}{\mathrm{d}+2\alpha}=1$.
- *Observe that the value of* λ *was improved in [DMERM23], giving a larger interval for* λ *such that pp spectrum with polynomially decaying eigenfunctions holds, compared to [AM93] and [AG98].*

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Appendix A

Submitted paper

Decay of the Green's function of the fractional Anderson model and connection to long-range SAW

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Abstract

We prove a connection between the Green's function of the fractional Anderson model and the two point function of a self-avoiding random walk with long range jumps, adapting a strategy proposed by Schenker in 2015. This connection allows us to exploit results from the theory of self-avoiding random walks to improve previous bounds known for the fractional Anderson model at strong disorder. In particular, we enlarge the range of the disorder parameter where spectral localization occurs. Moreover we prove that the decay of Green's function at strong disorder for any $0 < \alpha < 1$ is arbitrarily close to the decay of the massive resolvent of the corresponding fractional Laplacian, in agreement with the case of the standard Anderson model $\alpha = 1$. We also derive upper and lower bounds for the resolvent of the discrete fractional Laplacian with arbitrary mass $m \geq 0$, that are of independent interest.

Keywords: fractional Laplacian, random Schödinger operator, self-avoiding random walk, Anderson localization

MSC: 82B44, 82B41, 35R11 (primary), 47B80, 81Q10 (secondary)

1 Introduction

Transport phenomena in disordered environment are often described via random Schrödinger operators. On the lattice $\mathbb{Z}^d$, $d \geq 1$, they take the form of an infinite random matrix $H_\omega = T + \lambda V_\omega \in \mathbb{R}^{\mathbb{Z}^d \times \mathbb{Z}^d}_{sym}$ where T is a deterministic matrix (the kinetic part) and V_ω is a diagonal matrix with random entries. In its most standard formulation T is the negative discrete Laplacian $-\Delta$ defined via $-\Delta(x,y) := -\delta_{|x-y|=1} + 2d\,\delta_{|x-y|=0}$, where $|\cdot|$ denotes the ℓ^2 norm. This defines a self-adjoint bounded operator $-\Delta\colon \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$ with absolutely continuum spectrum and delocalized generalized eigenfunctions. More generally, T can be a symmetric matrix with decaying off-diagonal terms.
Operators of the form $H = T + V$ where V is a, possibly random, multiplication operator and T is long-range have attracted increasing interest in recent years [Han19, PKL+20, GRM20, SS21, JL21, Liu23, Shi23]. In particular, the usual exponential decay of eigenfunctions and dynamical bounds is replaced in this case by a polynomial decay.

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In this paper we consider the case when T is the discrete fractional Laplacian $(-\Delta)^\alpha$ with $0<\alpha<1$, which is obtained from $-\Delta$ via functional calculus. This operator has been subject to increasing interest in recent years. Just as the standard discrete Laplacian, it is bounded and translation invariant $(-\Delta)^\alpha(x,y)=(-\Delta)^\alpha(0,y-x)$, non-negative as a quadratic form and satisfies (see [GRM20, Thm. 2.2])

$$(-\Delta)^\alpha(x,x)>0, \qquad (-\Delta)^\alpha(x,y)\le 0 \quad \forall x\neq y.$$

Its off-diagonal matrix elements decay polynomially. See [GRM20, Thm. 2.2(iii)], [Sla18, Lemma 2.1] or, for the one dimensional case, [CRS+18, Thm. 1.1]. More precisely there are constants $0<c_{\alpha,d}<\mathcal{C}_{\alpha,d}$ such that

$$\frac{c_{\alpha,d}}{|x-y|^{d+2\alpha}}\le -(-\Delta)^\alpha(x,y)\le \frac{\mathcal{C}_{\alpha,d}}{|x-y|^{d+2\alpha}}, \qquad \forall x,y\in\mathbb{Z}^d,\ x\neq y. \tag{1.1}$$

In particular $(-\Delta)^\alpha$ has a summable kernel $\sum_{x\in\mathbb{Z}^d}|(-\Delta)^\alpha(0,x)|<\infty$ $\forall 0<\alpha\le 1$ and

$$(-\Delta)^\alpha(0,0)=-\sum_{x\neq 0}(-\Delta)^\alpha(0,x). \tag{1.2}$$

The long range nature of $(-\Delta)^\alpha$ for $\alpha<1$ changes drastically the behavior of the resolvent $((-\Delta)^\alpha+m^2)^{-1}$ with $m>0$. While for $\alpha=1$ the corresponding kernel decays exponentially, we only have polynomial decay for $\alpha<1$. Precisely, for $\alpha=1$

$$c\,e^{-m|x-y|}\le(-\Delta+m^2)^{-1}(x,y)\le C\,e^{-m|x-y|} \qquad \forall x\neq y, \tag{1.3}$$

for some constants $c,C>0$, depending on d,m, while for $\alpha<1$

$$\frac{c_1}{|x-y|^{d+2\alpha}}\ \le\ ((-\Delta)^\alpha+m^2)^{-1}(x,y)\ \le\ \frac{C_1}{|x-y|^{d+2\alpha}} \qquad \forall\ x\neq y, \tag{1.4}$$

for some constants $c_1,C_1>0$, depending on d,m,α. See [Sla18, Lemma 3.2] or Thm. 7 in Section 5 below. The limit $m\downarrow 0$ is well defined for $d>2\alpha$, $0<\alpha\le 1$ and behaves polynomially both for $\alpha=1$ and for $\alpha<1$. See [Sla18, Sect. 2] and references therein, Thm. 8 in Section 5 below, or, for the one-dimensional case, [CRS+18, Thm. 1.3]. Precisely

$$\frac{c_2}{|x-y|^{d-2\alpha}}\le((-\Delta)^\alpha)^{-1}(x,y)\le\frac{C_2}{|x-y|^{d-2\alpha}} \qquad \forall x\neq y. \tag{1.5}$$

for some constants $c_2,C_2>0$, depending on d,α.

In this paper we consider the so-called fractional Anderson model, which is obtained by perturbing the fractional Laplacian with a random diagonal matrix as follows

$$\mathrm{H}_{\alpha,\omega}=(-\Delta)^\alpha+\lambda\mathrm{V}_\omega\in\mathbb{R}^{\mathbb{Z}^d\times\mathbb{Z}^d} \tag{1.6}$$

where $\lambda>0$ is the disorder parameter, $\mathrm{V}_\omega(x,y):=\delta_{|x-y|=0}\,\omega_x$ and $\omega:=(\omega_x)_{x\in\mathbb{Z}^d}\in\mathbb{R}^{\mathbb{Z}^d}$ is a family of i.i.d. real random variables endowed with the Borel probability measure $\mathbb{P}:=\bigotimes_{\mathbb{Z}^d}\mathrm{P}_0$, with compactly supported one site probability measure P_0. With these assumptions the operator $\mathrm{H}_{\alpha,\omega}$ is bounded and self-adjoint. By translation invariance it is also ergodic and hence the spectrum $\sigma(\mathrm{H}_{\alpha,\omega})$ is a.s. a deterministic bounded interval of $\mathbb{R}$.

The fractional Anderson model in the discrete setting is known to exhibit pure point spectrum with eigenfunctions decaying at least polynomially at strong disorder, see [AM93, Thm. 3.2], and to exhibit fractional Lifshitz tails [GRM20]. Localization for this operator is not yet available in the continuous setting, therefore it is important to understand better the mechanism causing pure point spectrum.
In this paper we contribute to these efforts by giving an alternative proof of spectral localization that exploits a connection to self-avoiding random walks (SAW), following [Sch15]. This allows us to improve known results, such as the polynomial decay rate of the Green's function and eigenfunctions at strong disorder. In particular we enlarge the range of the disorder parameter where spectral localization occurs. Moreover we prove that the decay of Green's function at strong disorder for any $0 < \alpha < 1$ is arbitrarily close to the decay of the massive resolvent (1.4) of the corresponding fractional Laplacian. We conjecture that this is the optimal decay rate one can obtain by the Fractional Moment Method. Note that the same kind of result holds in the case of the standard Anderson model $\alpha = 1$.

Organization of the paper. In Section 2 we state our main results and discuss connections with the existing literature. In Section 3 we introduce the regularity assumption we need on the random potential and the basic definitions to introduce self-avoiding random walks, including a known result on the decay of the two-point correlation function of a SAW with long jumps. In Section 4 we establish a comparison between the decay of the averaged fractional resolvent of our model and the two-point correlation function of a particular SAW with long jumps, proving our main result. Finally, in Section 5 we complement our results by studying properties of the discrete fractional Laplacian. This section might be of independent interest.

2 Main results and discussion

In the following we assume P_0 is absolutely continuous with respect to Lebesgue and $\tau-$regular for some $\tau \in (\frac{d}{d+2\alpha}, 1)$ with $\tau-$constant $M_\tau(\mathrm{P}_0)$ (cf. Def. 3 below). Note that the decay of $(-\Delta)^\alpha$, Eq. (1.1), ensures that

$$\sum_{x\in\mathbb{Z}^d} |(-\Delta)^\alpha(0,x)|^s < \infty \qquad \text{holds } \forall s \in \left(\frac{d}{d+2\alpha}, 1\right] \tag{2.1}$$

since in that interval we have $s(d+2\alpha) > d$. In particular this holds for all s in the non empty interval $(\frac{d}{d+2\alpha}, \tau)$ and hence, by the Fractional Moment Method [AM93, Thm 3.2], the spectrum of $\mathrm{H}_{\alpha,\omega}$ for large disorder $\lambda > \lambda_{AM}(s)$, with

$$\lambda_{AM}(s) := M_\tau(\mathrm{P}_0)^{\frac{1}{\tau}} \left(\frac{2\tau}{\tau - s} \sum_{x\in\mathbb{Z}^d} |(-\Delta)^\alpha(0,x)|^s \right)^{\frac{1}{s}} \tag{2.2}$$

consists only of pure point spectrum, with random square summable eigenvectors. In this sense the fractional Anderson model undergoes the same localization phenomenon at large disorder as the standard non fractional one, but contrary to the case $\alpha = 1$, the operator $\mathrm{H}_{\alpha,\omega}$ is expected to undergo a phase transition in $d = 1$ for $\alpha < 1/2$ and in $d = 2$ for all $0 < \alpha < 1$ between complete pure point spectrum at large disorder and coexistence of absolutely continuous and pure point spectrum at weak disorder [JM99].

Indeed, in dimension $d = 1$, the random walk with long jumps generated by $-(-\Delta)^{\alpha}$ is transient in the case $0 < \alpha < 1/2$, and recurrent in the case $1/2 \leq \alpha \leq 1$, while in dimension $d = 2$ and above, it is transient for all $0 < \alpha < 1$, see e.g. [CFG09, Appendix B.1].
The situation changes drastically when considering the spatial decay of the corresponding eigenvectors. While these are exponentially localized around some random point for $\alpha = 1$, polynomial decay is expected for $\alpha < 1$ in any dimension due to the long range nature of $(-\Delta)^{\alpha}$. Upper and lower polynomial bounds have been proved, for example, in the case of the fractional Laplacian perturbed by a negative potential vanishing at infinity, see [CMS90, Prop. IV.1 and IV.3]

The key observable giving information on the spectral properties of $\mathrm{H}_{\alpha,\omega}$ is the fractional average Green's function $\mathbb{E}[|\mathrm{G}_z(x_0, x)|^s]$ with $0 < s < 1$, where

$$\mathrm{G}_z = (\mathrm{H}_{\alpha,\omega} - z)^{-1}$$

is a well defined bounded operator for all $z \in \mathbb{C} \setminus \mathbb{R}$. In this article we adapt a strategy developed in [Sch15] to bound $\mathbb{E}[|\mathrm{G}_z(x, y)|^s]$ by the two point function of a self-avoiding walk (SAW) generated by $D(x, y) := |(-\Delta)^{\alpha}(x, y)|^s$. We use this bound to enlarge the set of values for λ where pure spectrum occurs, and to derive improved decay estimates on the corresponding eigenvectors.

To formulate our main result we need some notions to describe a SAW with long jumps generated by D. These are collected in Section 3. In particular we denote by $\chi^{\alpha,SAW}$ the susceptibility, with radius of convergence $\mathrm{R}_{\chi^{\alpha,SAW}}$, and by $\mathrm{C}_{\gamma}^{D,\mathrm{SAW}}(x)$ the two-point correlation function with parameter $\gamma > 0$, see (3.5) and (3.6) below. With this notation, our main result is summarized in the following theorem.

Theorem 1. *Assume the one site probability measure* P_0 *is* $\tau-$*regular, for some* $\tau \in \left(\frac{d}{d+2\alpha}, 1\right)$ *with* $\tau-$*constant* $M_\tau(\mathrm{P}_0)$. *For* $s \in \left(\frac{d}{d+2\alpha}, \tau\right)$ *we consider the self-avoiding walk generated by* $D(x, y) := |(-\Delta)^{\alpha}(x, y)|^s$, *with* $x \neq y$. *Set*

$$\theta_s := \frac{\tau}{\tau - s} M_\tau(\mathrm{P}_0)^{\frac{s}{\tau}} \tag{2.3}$$

and

$$\lambda_0(s) := \left(\frac{\theta_s}{\mathrm{R}_{\chi^{\alpha,SAW}}}\right)^{\frac{1}{s}}. \tag{2.4}$$

Then for all $\lambda > \lambda_0(s)$ *and* $\forall x \neq x_0 \in \mathbb{Z}^d$ *it holds*

$$\mathbb{E}[|\mathrm{G}_z(x_0, x)|^s] \leq \ \gamma_0(\lambda, s)\ \mathrm{C}_{\gamma_0(\lambda,s)}^{D,\mathrm{SAW}}(x - x_0), \tag{2.5}$$

uniformly in $z \in \mathbb{C} \setminus \mathbb{R}$, *where* $\gamma_0(\lambda, s) := \frac{\theta_s}{\lambda^s}$.

The proof is in Section 4. Note that the kernel D is summable by (2.1). Moreover, translation invariance of $(-\Delta)^{\alpha}$ implies that D is translation invariant too. The condition $\lambda > \lambda_0(s)$ ensures that the two point function is finite $\mathrm{C}_{\gamma_0(\lambda,s)}^{D,\mathrm{SAW}}(x) < \infty\ \forall x \in \mathbb{Z}^d$ (cf. Section 3 below). The next result uses the bound (2.5) to prove existence of pure point spectrum and decay estimates on the eigenvectors.

Theorem 2. *For $s \in \left(\frac{d}{d+2\alpha}, \tau\right)$ we define*

$$2\alpha_s := s(d+2\alpha) - d, \tag{2.6}$$

which satisfies, by the constraints on s, the inequality

$$0 < \alpha_s < \alpha < 1.$$

Remember the definitions of $\lambda_0(s)$ (2.4) and θ_s (2.3) above.

(i) *For all $\lambda > \lambda_0(s)$ it holds*

$$\mathbb{E}[|G_z(x,y)|^s] \le \frac{K_0\theta_s}{\lambda^s}\, \frac{1}{|x-y|^{d+2\alpha_s}} \qquad \forall x, y \in \mathbb{Z}^d,\ x \neq y, \tag{2.7}$$

uniformly in $z \in \mathbb{C} \setminus \mathbb{R}$. The constant $K_0 = K_0(\lambda, s) > 0$ is defined in (3.9) below.

(ii) *For all $\lambda > \lambda_0(s)$ the spectrum of $H_{\alpha,\omega}$ consists only of pure point spectrum.*

(iii) *Assume the density of P_0 is bounded (in particular $\tau = 1$). Then, for a.s. $\omega \in \Omega$ and for all $E \in \sigma(H_\alpha)$, there is a localization center $x_E(\omega) \in \mathbb{Z}^d$ such that the corresponding normalized eigenfunction $\varphi_E(\cdot, \omega)$ satisfies $\forall y \in \mathbb{Z}^d$*

$$|\varphi_E(y,\omega)|^2 \le 4A_t(\omega)\left[\frac{(-\Delta)^\alpha(0,0)}{-(-\Delta)^\alpha(0,x_E(\omega))}\right]^2 \frac{1}{(1+|y-x_E(\omega)|)^t}, \quad \forall 0 < t < 2\alpha_s, \tag{2.8}$$

where A_t is an integrable random variable.

Proof. The first statement follows directly from the bound (2.5) together with the decay (3.8). To prove the last two statements note that the bound (2.7) ensures that for all $\lambda > \lambda_0(s)$,

$$\sum_{x \in \mathbb{Z}^d} \mathbb{E}\left[|G_z(0,x)|^s |x|^t\right] < \infty \qquad \forall 0 \le t < 2\alpha_s \tag{2.9}$$

holds uniformly in $z \in \mathbb{C} \setminus \mathbb{R}$. Note that $\lim_{\eta \downarrow 0} \sum_{y \in \mathbb{Z}^d} |G_{E+i\eta}(x,y)|^2$ always exists (but may be infinite) since $\sum_{y \in \mathbb{Z}^d} |G_{E+i\eta}(x,y)|^2 = [(E - H_{\alpha,\omega})^2 + \eta^2]^{-1}(x,x)$ which is a monotone function in η. We argue

$$\begin{aligned}
\mathbb{E}\left[\left(\lim_{\eta\downarrow 0} \sum_{y\in\mathbb{Z}^d} |G_{E+i\eta}(x,y)|^2\right)^{\frac{s}{2}}\right] &= \mathbb{E}\left[\lim_{\eta\downarrow 0}\left(\sum_{y\in\mathbb{Z}^d} |G_{E+i\eta}(x,y)|^2\right)^{\frac{s}{2}}\right] \\
&\le \mathbb{E}\left[\lim_{\eta\downarrow 0}\left(\sum_{y\in\mathbb{Z}^d} |G_{E+i\eta}(x,y)|^s\right)\right] \le \liminf_{\eta\downarrow 0} \sum_{y\in\mathbb{Z}^d} \mathbb{E}[|G_{E+i\eta}(x,y)|^s] < \infty,
\end{aligned}$$

where in the first two steps we used that the function $x \mapsto x^{\frac{s}{2}}$ is monotone and $(\sum_n a_n)^s \le \sum_n a_n^s$ for all $a_n \ge 0$ and $0 < s < 1$. The last two inequalites follow by Fatou and (2.9) with $t = 0$. Since we assumed the one site probability measure P_0 has a density, this bound implies by Simon-Wolff criterion [AW15, Thm. 5.7] that the spectrum is pure point only. Finally, since P_0 has compact support and bounded density we have

$$\sup_{v\in\mathbb{R}} (1+|v|)^s P_0(v) < \infty.$$

The eigenfunction decay follows from this bound and (2.9) by standard arguments (cf. [AW15, Thm. 7.4]).

□

Discussion of the results. Note that, assuming again $\frac{d}{d+2\alpha} < s < \tau$, a direct application of [AG98, Thm.1'] with $|x-y|$ replaced by $\ln(1+|x-y|)$ and $K(x,y)$ replaced by $(-\Delta)^{\alpha}(x,y)$ gives the following estimate $\forall x, x_0 \in \mathbb{Z}^d$

$$\mathbb{E}[|\mathrm{G}_z(x_0,x)|^s] \;\le\; M_\tau(\mathrm{P}_0)^{\frac{s}{\tau}} \tfrac{\tau}{\tau-s} \frac{1}{(1+|x-x_0|)^{\beta}}, \qquad \forall 0<\beta<2\alpha_s \tag{2.10}$$

which holds uniformly in $z \in \mathbb{C}\setminus\mathbb{R}$, as long as λ satisfies

$$\lambda > \lambda_{AG}(\beta,s) := M_\tau(\mathrm{P}_0)^{\frac{1}{\tau}} \left(\frac{2\tau}{\tau-s} \sum_{x\in\mathbb{Z}^d} |(-\Delta)^{\alpha}(0,x)|^s (1+|x|)^{\beta} \right)^{\frac{1}{s}}.$$

While the bound (2.7) in Theorem 2 is summable for all $\alpha > 0$, the bound (2.10) is never summable when $\alpha < \frac{d}{2}$, hence Simon-Wolff criterion cannot be applied directly. Instead one proves the inequality [AM93]

$$\sup_{\eta>0} \mathbb{E}\left[\sum_{y\in\mathbb{Z}^d} |\mathrm{G}_{\mathrm{E}+i\eta}(x,y)|^s (1+|x-y|)^{\beta} \right] < \infty,$$

for $\lambda > \lambda_{AG}(\beta,s)$. This ensures the existence of pure point spectrum and the decay (2.8) for the eigenfunctions. Note that, using (2.4), (3.7) and (2.2) we obtain

$$\begin{aligned}
\lambda_0(s) &= M_\tau(\mathrm{P}_0)^{\frac{1}{\tau}} \left(\frac{\tau}{\tau-s} \frac{1}{\mathrm{R}_{\chi^{\alpha},SAW}} \right)^{\frac{1}{s}} \le M_\tau(\mathrm{P}_0)^{\frac{1}{\tau}} \left(\frac{\tau}{\tau-s} \sum_{x\neq 0} D(0,x) \right)^{\frac{1}{s}} \\
&= M_\tau(\mathrm{P}_0)^{\frac{1}{\tau}} \left(\frac{\tau}{\tau-s} \sum_{x\neq 0} |(-\Delta)^{\alpha}(0,x)|^s \right)^{\frac{1}{s}} \\
&< M_\tau(\mathrm{P}_0)^{\frac{1}{\tau}} \left(\frac{2\tau}{\tau-s} \sum_{x\neq 0} |(-\Delta)^{\alpha}(0,x)|^s \right)^{\frac{1}{s}} = \lambda_{AM}(s) \\
&< M_\tau(\mathrm{P}_0)^{\frac{1}{\tau}} \left(\frac{2\tau}{\tau-s} \sum_{x\in\mathbb{Z}^d} |(-\Delta)^{\alpha}(0,x)|^s (1+|x|)^{\beta} \right)^{\frac{1}{s}} = \lambda_{AG}(s,\beta).
\end{aligned}$$

Hence $\lambda_0(s) < \lambda_{AM}(s) < \lambda_{AG}(s,\beta)$ for all $0<\beta<2\alpha_s$ and $s\in(0,1)$.
Our bound (2.7) ensures one can get arbitrarily close to the decay $t = \tau(d+2\alpha)$. Remember that we need $1 \ge \tau > \frac{d}{d+2\alpha}$ so τ must be near one for d large or α small. In the case $\tau = 1$ our results imply we can get arbitrarily close to $(d+2\alpha)$ which is the decay of the massive resolvent $((-\Delta)^{\alpha}+m^2)^{-1}$, but also of $(-\Delta)^{\alpha}$. On the other hand, the best decay we can obtain via (2.10) is $2\alpha_s = s(d+2\alpha)-d$, and this at the cost of an infinitely large disorder since $\lim_{\beta\uparrow 2\alpha_s} \lambda_{AG}(\beta,s) = \infty$. For s near 1 this approximates $d+2\alpha-d = 2\alpha$ so the bound misses the optimal decay by a factor d.

3 Preliminary definitions and results

$\tau-$regularity and apriori bound.

Definition 3 ($\tau-$regularity)**.** *Let $\tau \in (0,1]$. We say that a probability measure μ is $\tau-$regular if there is a $C > 0$ such that $\mu([v-\delta, v+\delta]) \leq C\delta^\tau$, $\forall v \in \mathbb{R}$, $\forall \delta > 0$. If μ is $\tau-$regular, the corresponding $\tau-$constant is defined by*

$$M_\tau(\mu) := \inf\{C > 0 \mid \mu([v-\delta, v+\delta]) \leq C\delta^\tau,\ \forall v \in \mathbb{R},\ \forall \delta > 0\}. \tag{3.1}$$

The $\tau-$regularity of P_0 enters in the bounds for G_z via the so-called *a priori bound*

$$\mathbb{E}\Big[|G_z(x,x)|^s|\omega_{\mathbb{Z}^d\setminus\{x\}}\Big] \leq \frac{\theta_s}{\lambda^s}, \qquad \forall x \in \mathbb{Z}^d,\ 0 < s < \tau, \tag{3.2}$$

uniformly in $\omega_{\mathbb{Z}^d\setminus\{x\}}$, where $\theta_s = \frac{\tau}{\tau-s}M_\tau(P_0)^{\frac{s}{\tau}}$ (cf. equation (2.3)). This bound is obtained remarking that $|G_z(x,x)| = (\lambda|\omega_x + \eta(\omega_{\mathbb{Z}^d\setminus\{x\}})|)^{-1}$ where the random complex number $\eta(\omega_{\mathbb{Z}^d\setminus\{x\}})$ is independent of ω_x. Since ω_x and $\omega_{\mathbb{Z}^d\setminus\{x\}}$ are independent, the problem reduces to the following estimate

$$\int_{\mathbb{R}} \frac{1}{|v-\eta|^s} dP_0(v) \leq \theta_s, \qquad \forall \eta \in \mathbb{C} \tag{3.3}$$

which holds $\forall 0 < s < \tau$ (cf.[AG98, App. B]).

Self-avoiding walks with long-range jumps. Let $D \in [0,\infty)^{\mathbb{Z}^d\times\mathbb{Z}^d}$ be an infinite matrix. Assume D is translation invariant and

$$0 < \sum_{x\neq 0} D(0,x) < \infty.$$

We consider the random walk on $\mathbb{Z}^d$ with transition probability from x to $y \neq x$

$$p(x,y) = \frac{D(x,y)}{\sum_{z\neq x} D(x,z)}.$$

For $x, x_0 \in \mathbb{Z}^d$ we consider for $n \geq 1$

$$\mathcal{W}_n(x_0,x) := \Big\{w = (w_j)_{j=0}^n \subset \mathbb{Z}^{nd} \mid w_0 = x_0,\ w_n = x\Big\}$$

the set of paths in $\mathbb{Z}^d$ going from x_0 to x in $n \geq 1$ steps. For $x = x_0$ we may also have paths of length zero $\mathcal{W}_0(x_0,x_0) := \{x_0\}$. We say that $w \in \mathcal{W}_n(x_0,x)$ is a self-avoiding walk (SAW) of length n if $w_k \neq w_l$ for all $k \neq l$ with $k,l \leq n$. The set of self-avoiding paths in $\mathbb{Z}^d$ going from x_0 to x in $n \geq 0$ steps is denoted by $\mathcal{W}_n^{SAW}(x_0,x)$. Note that $\mathcal{W}_0^{SAW}(x_0,x_0) = \mathcal{W}_0(x_0,x_0)$ and

$$\mathcal{W}_n^{SAW}(x_0,x_0) = \emptyset = \mathcal{W}_0^{SAW}(x_0,x) \qquad \forall x_0 \neq x, n \geq 1.$$

Following [Sch15], we define, for $n \geq 0$, and $x \in \mathbb{Z}^d$

$$c_n^D(x) = c_n^{D,\mathrm{SAW}}(x) := \sum_{w\in\mathcal{W}_n^{SAW}(0,x)} \prod_{j=0}^{n-1} D(w_j, w_{j+1}), \tag{3.4}$$

where we took the convention that the sum over an empty set equals 0 and the product over an empty set equals 1. In particular $\mathrm{c}_0^D(x) = \delta_{x,0}$ and $c_n^D(0) = 0\ \forall n \geq 1$. The function $\mathrm{c}_n^D(x)$ for $n \geq 1$, is proportional to the probability that a self-avoiding random walk goes from 0 to x in n steps. The corresponding two-point correlation function is defined as

$$\mathrm{C}_\gamma^D(x) = \mathrm{C}_\gamma^{D,\mathrm{SAW}}(x) := \sum_{n\geq 0} \mathrm{c}_n^D(x)\gamma^n, \quad \forall x \in \mathbb{Z}^d. \tag{3.5}$$

The sum starts at $n = 1$ when $x \neq 0$, while for $x = 0$ we have $\mathrm{C}_\gamma^\alpha(0) = \gamma^0 = 1$. The corresponding radius of convergence is denoted by $\mathrm{R}_{\mathrm{C}^{D,SAW}(x)}$. Note that in [CS15], the correlation function is defined in a slightly different way, which can be recovered from our definition performing the change of variable

$$\gamma \mapsto \tilde{\gamma} := \frac{\gamma}{\sum_{z\neq x} D(x,z)}.$$

Summing over x we obtain the susceptibility

$$\chi_\gamma^D = \chi_\gamma^{D,SAW} := \sum_{x\in\mathbb{Z}^d} \mathrm{C}_\gamma^D(x) = \sum_{n\geq 0}\gamma^n \sum_{x\in\mathbb{Z}^d} \sum_{w\in\mathcal{W}_n^{SAW}(0,x)} \prod_{j=0}^{n-1} D(w_j, w_{j+1}), \tag{3.6}$$

The corresponding radius of convergence is denoted by $\mathrm{R}_{\chi^{D,SAW}}$. For $n = 2$ we have

$$\sum_{x\in\mathbb{Z}^d} \sum_{w\in\mathcal{W}_2^{SAW}(0,x)} \prod_{j=0}^{n-1} D(w_j, w_{j+1}) = \sum_{x\neq 0}\sum_{x_1\neq 0,x} D(0,x_1)D(x_1,x)$$

$$= \sum_{x_1\neq 0} D(0,x_1) \sum_{x\neq x_1,0} D(x_1,x) \leq \sum_{x_1\neq 0} D(0,x_1) \sum_{x\neq x_1} D(x_1,x) = \left(\sum_{z\neq 0} D(0,z)\right)^2.$$

Repeating this argument for general $n \geq 1$ we obtain

$$\sum_{x\in\mathbb{Z}^d} \sum_{w\in\mathcal{W}_n^{SAW}(0,x)} \prod_{j=0}^{n-1} D(w_j, w_{j+1}) \leq \left(\sum_{z\neq 0} D(0,z)\right)^n,$$

and hence

$$\mathrm{R}_{\chi^{D,SAW}} \geq \frac{1}{\sum_{z\neq 0} D(0,z)} > 0. \tag{3.7}$$

The decay of the two point function of the SAW introduced above has been estimated in [CS15, Lemma 2.4]. We recall the result here, together with a sketch of the proof, translated into our language.

Lemma 4 (Decay of the SAW two point function)**.** *Assume that $D(0,x) \leq \mathcal{C}\frac{1}{|x|^{d+a}}$ holds for some $\mathcal{C}, a > 0$ and for all $x \neq 0$. Then the two point function of the SAW generated by D is bounded by*

$$\mathrm{C}_\gamma^{D,\mathrm{SAW}}(x) \leq K_0 \frac{1}{|x|^{d+a}}, \tag{3.8}$$

for all $\gamma < \mathrm{R}_{\chi^{D,SAW}}$. The constant $K_0 = K_0(d,a) > 0$ can be explicitely written in terms of the susceptibility χ_γ^D as follows:

$$K_0 = \tilde{\ell}^{d+a}\chi_\gamma^D + 2(\chi_\gamma^D)^2\gamma\mathcal{C} \tag{3.9}$$

where $\tilde{\ell} = \tilde{\ell}(d,a) > 0$ *is the minimal distance such that*

$$c(x) := \sum_{|u| \le \frac{|x|}{3} \le |v|} \mathrm{C}_\gamma^D(u)\ \gamma D(u,v) \le \frac{1}{2} 2^{-(d+a)} \qquad \forall |x| \ge \tilde{\ell}.$$

Note that $\tilde{\ell}$ is well defined since $\lim_{|x|\to\infty} c(x) = 0$ (see below).

Proof. The assumption $\gamma < \mathrm{R}_{\chi^{\alpha,SAW}}$ ensures $\mathrm{C}_\gamma^D(x) < \infty\ \forall x \in \mathbb{Z}^d$. The key ingredient of the proof is the following inequality, which holds for any $0 < \ell < |x|$

$$\mathrm{C}_\gamma^D(x) \le \sum_{\substack{u,v \in \mathbb{Z}^d \\ |u| \le \ell < |v|}} \mathrm{C}_\gamma^D(u)\ \gamma D(u,v)\ \mathrm{C}_\gamma^D(x-v) \tag{3.10}$$

To prove it, remember that

$$\mathrm{C}_\gamma^D(x) = \sum_{n \ge 0} \sum_{w \in \mathcal{W}_n^{SAW}(0,x)} \gamma^n \prod_{j=0}^{n-1} D(w_j, w_{j+1}).$$

For a given path $w \in \mathcal{W}_n^{SAW}(0,x)$ we define $u := w_{j_m}$ and $v := w_{j_m+1}$ where

$$j_m := \max\{j \in \{0,\dots,n\} |\ |w_j| < \ell\}.$$

Since $0 < \ell < |x|$ this set is non-empty and $0 \le j_m < n$. With this definitions the sum above can be reorganized as

$$\mathrm{C}_\gamma^D(x) = \sum_{\substack{u,v \in \mathbb{Z}^d \\ |u| \le \ell < |v|}} \sum_{n,m \ge 0} \sum_{w \in \mathcal{W}_n^{SAW}(0,u)} \sum_{w' \in \mathcal{W}_m^{SAW}(v,x)} [\gamma^n \prod_{j=0}^{n-1} D(w_j, w_{j+1})] \gamma D(u,v) \cdot$$
$$[\gamma^m \prod_{j=0}^{m-1} D(w'_j, w'_{j+1})]\ \mathbb{1}_{\{w \cup w' \text{ is SAW}\}} \ \le \sum_{\substack{u,v \in \mathbb{Z}^d \\ |u| \le \ell < |v|}} \mathrm{C}_\gamma^D(u)\ \gamma D(u,v)\ \mathrm{C}_\gamma^D(x-v),$$

where in the last step we applied $\mathbb{1}_{\{w \cup w' \text{ is SAW}\}} \le 1$ and the translation invariance of D. Set now $\ell = \frac{|x|}{3}$. The sum on the RHS of (3.10) can be reorganized as follows

$$\sum_{|u| \le \frac{|x|}{3} < |v|} = \sum_{|u| \le \frac{|x|}{3},\ \frac{|x|}{2} < |v|} + \sum_{|u| \le \frac{|x|}{3} < |v| \le \frac{|x|}{2}}.$$

We estimate the first sum as follows:

$$\sum_{|u| \le \frac{|x|}{3},\ \frac{|x|}{2} < |v|} \mathrm{C}_\gamma^D(u)\ \gamma D(u,v)\ \mathrm{C}_\gamma^D(x-v) \le (\chi_\gamma^D)^2 \gamma \mathcal{C} \frac{1}{|x|^{d+a}} \tag{3.11}$$

where we used (3.6), $|u-v| \ge |x|/6$ and $D(u,v) \le \frac{\mathcal{C}}{|u-v|^{d+a}}$. The second sum is bounded by

$$\sum_{|u| \le \frac{|x|}{3} < |v| \le \frac{|x|}{2}} \mathrm{C}_\gamma^D(u)\ \gamma D(u,v)\ \mathrm{C}_\gamma^D(x-v) \le c(x) \sup_{|v| \le \frac{|x|}{2}} \mathrm{C}_\gamma^D(x-v) = c(x) \sup_{|v| \ge \frac{|x|}{2}} \mathrm{C}_\gamma^D(v),$$

where we defined $c(x) := \sum_{|u|\le\frac{|x|}{3}<|v|} \mathrm{C}^D_\gamma(u)\ \gamma D(u,v)$. Putting all this together we obtain

$$\mathrm{C}^D_\gamma(x) \le c(x) \sup_{|v|\ge\frac{|x|}{2}} \mathrm{C}^D_\gamma(v) + (\chi^D_\gamma)^2\gamma\mathcal{C}\frac{1}{|x|^{d+a}}. \tag{3.12}$$

Using the spatial decay of D we argue

$$c(x) \le \sum_{|u|<\frac{|x|}{5},\ \frac{|x|}{3}<|v|} \mathrm{C}^D_\gamma(u)\ \gamma D(u,v) + \sum_{\frac{|x|}{5}\le|u|\le\frac{|x|}{3}<|v|} \mathrm{C}^D_\gamma(u)\ \gamma D(u,v)$$

$$\le \chi^D_\gamma \sum_{|v|>\frac{|x|}{5}} \gamma\mathcal{C}\frac{1}{|v|^{d+a}} + \sum_{|u|\ge\frac{|x|}{5}} \mathrm{C}^D_\gamma(u) \sum_{|v|\ge 1} \gamma\mathcal{C}\frac{1}{|v|^{d+a}} \le \mathcal{C}'\left[\frac{1}{|x|^a} + \sum_{|u|\ge\frac{|x|}{5}} \mathrm{C}^D_\gamma(u)\right],$$

for some constant $\mathcal{C}'$. Since $\lim_{|x|\to\infty}\sum_{|u|\ge|x|} \mathrm{C}^D_\gamma(u) = 0$ we obtain $\lim_{|x|\to\infty} c(x) = 0$. Hence there is a $\tilde\ell = \tilde\ell(d,a)$ such that

$$c(x) \le \frac{1}{2}2^{-(d+a)} \qquad \forall |x| \ge \tilde\ell.$$

For $2^{n-1}\tilde\ell \le |x| < 2^n\tilde\ell$ with $n\ge 1$ we apply n times the inequality (3.12) and obtain

$$\mathrm{C}^D_\gamma(x) \le \frac{1}{2^{n(d+a+1)}} \sup_{|v|\ge\frac{|x|}{2}} \mathrm{C}^D_\gamma(v) + \Big(\sum_{j=0}^n \frac{1}{2^j}\Big)\frac{(\chi^D_\gamma)^2\gamma\mathcal{C}}{|x|^{d+a}} \le \frac{\tilde\ell^{d+a}\chi^D_\gamma + 2(\chi^D_\gamma)^2\gamma\mathcal{C}}{|x|^{d+a}}.$$

When $|x| < \tilde\ell$ we apply the simple bound $\mathrm{C}^D_\gamma(x) \le \chi^D_\gamma \le \tilde\ell^{d+a}\chi^D_\gamma/|x|^{d+a}$. □

4 Comparison with a long range SAW.

The proof of Theorem 1 adapts the strategy of [Sch15, Thm. 1] to the fractional Anderson model. In particular, this requires to work with Green's functions defined on different volumes. Therefore, for any $\Lambda\subset\mathbb{Z}^d$ subset of $\mathbb{Z}^d$ (finite or infinite) we introduce the restricted Green's function G^Λ

$$\mathrm{G}^\Lambda_z(x,y) := \begin{cases} (\mathrm{H}^\Lambda_{\alpha,\omega} - z)^{-1}(x,y), & \forall x,y\in\Lambda, \\ 0, & \text{otherwise.} \end{cases} \tag{4.1}$$

where $\mathrm{H}^\Lambda_{\alpha,\omega} - z \in \mathbb{C}^{\Lambda\times\Lambda}$ is the matrix (finite or infinite) obtained by restricting $\mathrm{H}_{\alpha,\omega} - z$ to Λ. This matrix is invertible for all $z\in\mathbb{C}\setminus\mathbb{R}$. In particular $\mathrm{G}^{\mathbb{Z}^d}_z = \mathrm{G}_z$. In the following $\Lambda = \mathbb{Z}^d$, but we leave the notation Λ through the proof below to stress the fact that the same result holds for any volume.
To simplify the notation we also set[1]

$$\triangle^\alpha(x_0,x) := -(-\Delta)^\alpha(x_0,x), \qquad \forall x,x_0\in\mathbb{Z}^d.$$

Note that with this convention $\triangle^\alpha(x,y) > 0\ \forall x\ne y$.

[1] Note that this definition differs from the corresponding operator defined via functional calculus by a phase.

Proof of Theorem 1. By the resolvent identity we have, for all $x \neq x_0 \in \Lambda$

$$\begin{aligned}\mathrm{G}_z^{\Lambda}(x_0,x) &= \mathrm{G}_z^{\Lambda}(x_0,x_0) \triangle^{\alpha} (x_0,x)\mathrm{G}_z^{\Lambda\setminus\{x_0\}}(x,x) \\ &\quad + \mathrm{G}_z^{\Lambda}(x_0,x_0) \sum_{w_1\in\Lambda\setminus\{x_0,x\}} \triangle^{\alpha}(x_0,w_1)\mathrm{G}_z^{\Lambda\setminus\{x_0\}}(w_1,x).\end{aligned}$$

Repeating the procedure N times we obtain

$$\begin{aligned}\mathrm{G}_z^{\Lambda}(x_0,x) = \mathrm{G}_z^{\Lambda}(x_0,x_0)\Big[&\sum_{n=1}^{N} \sum_{w\in\mathcal{W}_n^{SAW}(x_0,x)} \prod_{j=0}^{n-1} \triangle^{\alpha}(w_j,w_{j+1}) \prod_{j=1}^{n} \mathrm{G}_z^{\Lambda\setminus\{w_0,\dots,w_{j-1}\}}(w_j,w_j) \\ &+ \sum_{w\in\mathcal{W}_{N+1}^{SAW}(x_0,x)} \prod_{j=0}^{N-1} \triangle^{\alpha}(w_j,w_{j+1}) \prod_{j=1}^{N-1} \mathrm{G}_z^{\Lambda\setminus\{w_0,\dots,w_{j-1}\}}(w_j,w_j)\mathrm{G}_z^{\Lambda\setminus\{w_0,\dots,w_N\}}(w_N,x))\Big]\end{aligned}$$

Taking the average and using the concavity of the function $y \mapsto y^s$ we have

$$\begin{aligned}\mathbb{E}\left[|\mathrm{G}_z^{\Lambda}(x_0,x)|^s\right] \leq &\sum_{n=1}^{N} \sum_{w\in\mathcal{W}_n^{SAW}(x_0,x)} \prod_{j=0}^{n-1} \triangle^{\alpha}(w_j,w_{j+1})^s \cdot \\ &\cdot \mathbb{E}\left[|\mathrm{G}_z^{\Lambda}(x_0,x_0)|^s \prod_{j=1}^{n} |\mathrm{G}_z^{\Lambda\setminus\{w_0,\dots,w_{j-1}\}}(w_j,w_j)|^s\right] + \sum_{w\in\mathcal{W}_{N+1}^{SAW}(x_0,x)} \prod_{j=0}^{N-1} \triangle^{\alpha}(w_j,w_{j+1})^s \cdot \\ &\cdot \mathbb{E}\left[|\mathrm{G}_z^{\Lambda}(x_0,x_0)|^s \prod_{j=1}^{N-1} |\mathrm{G}_z^{\Lambda\setminus\{w_0,\dots,w_{j-1}\}}(w_j,w_j)|^s \ |\mathrm{G}_z^{\Lambda\setminus\{w_0,\dots,w_N\}}(w_N,x))|^s\right].\end{aligned}$$

The resolvent $\mathrm{G}_z^{\Lambda\setminus\{w_0,\dots,w_{j-1}\}}(y,y')$ does not depend on the random variables $\omega_{w_i}, i = 0,\dots j-1$, hence recursive applications of the apriori bound (3.2), which holds since we assume $s < \tau$, yield

$$\mathbb{E}\left[|\mathrm{G}_z^{\Lambda}(x_0,x_0)|^s \prod_{j=1}^{n} |\mathrm{G}_z^{\Lambda\setminus\{w_0,\dots,w_{j-1}\}}(w_j,w_j)|^s\right] \leq \left(\frac{\theta_s}{\lambda^s}\right)^{n+1}$$

$$\mathbb{E}\left[|\mathrm{G}_z^{\Lambda}(x_0,x_0)|^s \prod_{j=1}^{N-1} |\mathrm{G}_z^{\Lambda\setminus\{w_0,\dots,w_{j-1}\}}(w_j,w_j)|^s \ |\mathrm{G}_z^{\Lambda\setminus\{w_0,\dots,w_N\}}(w_N,x))|^s\right] \leq \left(\frac{\theta_s}{\lambda^s}\right)^{N} \frac{1}{|\operatorname{Re} z|^s}$$

where we also applied the inequality $|\mathrm{G}_z^{\Lambda\setminus\{w_0,\dots,w_N\}}(w_N,x)| \leq \frac{1}{|\operatorname{Re} z|}$. Inserting these estimates in the sums above and using the translation invariance of $\triangle^{\alpha}$ we get

$$\begin{aligned}\mathbb{E}\left[|\mathrm{G}_z^{\Lambda}(x_0,x)|^s\right] &\leq \frac{\theta_s}{\lambda^s} \sum_{n=1}^{N} \left(\frac{\theta_s}{\lambda^s}\right)^{n} \sum_{w\in\mathcal{W}_n^{SAW}(0,x-x_0)} \prod_{j=0}^{n-1} \triangle^{\alpha}(w_j,w_{j+1})^s \\ &\quad + \left(\frac{\theta_s}{\lambda^s}\right)^{N} \frac{1}{|\operatorname{Re} z|^s} \sum_{w\in\mathcal{W}_{N+1}^{SAW}(0,x-x_0)} \prod_{j=0}^{N-1} \triangle^{\alpha}(w_j,w_{j+1})^s \\ &\leq \frac{\theta_s}{\lambda^s} C_{\gamma}^{D}(x-x_0) + \frac{1}{|\operatorname{Re} z|^s} Err(N)\end{aligned}$$

where $C_\gamma^D(x-x_0)$ is the two point function of the SAW generated by $D(x,y) := \triangle^\alpha(x,y)^s = |(-\Delta)^\alpha(x,y)|^s$ with $\gamma = \frac{\theta_s}{\lambda^s}$. The error term $Err(N)$ satisfies

$$
\begin{aligned}
Err(N) &:= \left(\frac{\theta_s}{\lambda^s}\right)^N \sum_{w \in \mathcal{W}_{N+1}^{SAW}(0,x-x_0)} \prod_{j=0}^{N-1} \triangle^\alpha(w_j, w_{j+1})^s \\
&\leq \left(\frac{\theta_s}{\lambda^s}\right)^N \sum_{y \in Z^d} \sum_{w \in \mathcal{W}_N^{SAW}(0,y)} \prod_{j=0}^{N-1} \triangle^\alpha(w_j, w_{j+1})^s.
\end{aligned}
$$

Up to now, the above sums may be infinite. The generating kernel for the SAW satisfies

$$
\sum_{y\neq 0} D(y) = \sum_{y \neq 0} \triangle^\alpha(0,y)^s = \sum_{y\neq 0} |(-\Delta)^\alpha(0,y)|^s \leq C \sum_{y \neq 0} \frac{1}{|x-y|^{(d+2\alpha)s}}.
$$

The sum above is finite for all $s > \frac{d}{d+2\alpha}$. Finally, the assumption $\lambda > \lambda_0$ ensures that $\gamma < \mathrm{R}_{\chi^{\alpha,SAW}}$ holds and hence $\lim_{N\to\infty} Err(N) = 0$ since the susceptibility is finite. This completes the proof of the theorem. $\square$

5 Properties of the fractional Laplacian.

5.1 Matrix elements

In this section we collect some properties of the fractional Laplacian. The matrix elements of $(-\Delta)^\alpha$ admit the following explicit representation for all $x, y \in \mathbb{Z}^d$

$$
(-\Delta)^\alpha(x,y) = \frac{-1}{|\Gamma(-\alpha)|} \int_0^\infty \frac{\mathrm{d}t}{t^{1+\alpha}} \left[e^{-2dt} \prod_{j=1}^d \mathrm{I}_{(x_j-y_j)}(2t) - \delta_{xy} \right] \tag{5.1}
$$

where $\mathrm{I_p}$ is the modified Bessel function of order $p \in \mathbb{Z}$, which is defined as

$$
\mathrm{I_p}(t) := \sum_{\mathrm{q}\geq 0} \frac{1}{\mathrm{q}!\Gamma(\mathrm{p}+\mathrm{q}+1)} \left(\frac{t}{2}\right)^{2\mathrm{q}+\mathrm{p}}. \tag{5.2}
$$

This follows from the representation

$$
(-\Delta)^\alpha = \frac{-1}{|\Gamma(-\alpha)|} \int_0^\infty \frac{\mathrm{d}t}{t^{1+\alpha}} (e^{t\Delta} - \mathrm{Id}), \tag{5.3}
$$

where the integral converges under the operator norm, see [Kwa17, Theorem 1.1 (c)], together with the relation (cf. equations (5.3)-(5.4) in [GRM20])

$$
e^{t\Delta}(x,y) = e^{-2dt} \prod_{j=1}^d \mathrm{I}_{(x_j-y_j)}(2t) \quad \forall x, y \in \mathbb{Z}^d.
$$

Note that, since Γ has simple poles in the set of the non-positive integers, the equality

$$
\mathrm{I_p}(2t) = \sum_{\mathrm{q}\geq 0, p+q \geq 0} \frac{1}{\mathrm{q}!(\mathrm{p}+\mathrm{q})!} t^{2\mathrm{q}+\mathrm{p}}
$$

holds for all $p \in \mathbb{Z}$. In particular this implies $I_p(2t) > 0$ $\forall t > 0$ and $p \in \mathbb{Z}$ and

$$\sum_{p\in\mathbb{Z}} \mathrm{I_p}(2t) = e^{2t}. \tag{5.4}$$

Note that $I_p = I_{-p}$ $\forall p > 0$, hence, using also $\Gamma(n+1) = n!$ for all $n \geq 0$ we obtain

$$\mathrm{I_p}(t) = \sum_{\mathrm{q}\geq 0} \frac{1}{\mathrm{q}!(|\mathrm{p}|+\mathrm{q})!}\left(\frac{t}{2}\right)^{2\mathrm{q}+|\mathrm{p}|} \qquad \forall p \in \mathbb{Z}. \tag{5.5}$$

Proposition 5. *The matrix elements of $(-\Delta)^\alpha$ satisfy $(-\Delta)^\alpha(x,y) < 0$ for all $x \neq y$ and*

$$(-\Delta)^\alpha(x,x) = -\sum_{y\in\mathbb{Z}^d\setminus\{x\}} (-\Delta)^\alpha(x,y), \qquad \forall x \in \mathbb{Z}^d. \tag{5.6}$$

In particular $(-\Delta)^\alpha(x,x) = (-\Delta)^\alpha(0,0) > 0$.

Proof. The first statement follows from the fact that $I_p(t) > 0$ holds $\forall t > 0$, $p \in \mathbb{Z}$. To prove (5.6) we argue

$$\begin{aligned} -\sum_{y\in\mathbb{Z}^d\setminus\{x\}} (-\Delta)^\alpha(x,y) &= \frac{1}{|\Gamma(-\alpha)|}\sum_{y\in\mathbb{Z}^d\setminus\{x\}}\int_0^\infty \frac{\mathrm{d}t}{t^{1+\alpha}} e^{-2dt}\prod_{j=1}^d \mathrm{I}_{(x_j-y_j)}(2t) \\ &= \frac{1}{|\Gamma(-\alpha)|}\int_0^\infty \frac{\mathrm{d}t}{t^{1+\alpha}} e^{-2dt}\sum_{y\in\mathbb{Z}^d\setminus\{x\}}\prod_{j=1}^d \mathrm{I}_{(x_j-y_j)}(2t). \end{aligned}$$

Using $I_p = I_{-p}$ and (5.4) we compute

$$\begin{aligned} \sum_{y\in\mathbb{Z}^d\setminus\{x\}}\prod_{j=1}^d \mathrm{I}_{(x_j-y_j)}(2t) &= \sum_{y\neq 0}\prod_{j=1}^d \mathrm{I}_{y_j}(2t) \\ &= \sum_{y\in\mathbb{Z}^d}\prod_{j=1}^d \mathrm{I}_{y_j}(2t) - \mathrm{I}_0(2t)^d = e^{2dt} - \mathrm{I}_0(2t)^d, \end{aligned}$$

and hence

$$-\sum_{y\in\mathbb{Z}^d\setminus\{x\}} (-\Delta)^\alpha(x,y) = \frac{1}{|\Gamma(-\alpha)|}\int_0^\infty \frac{\mathrm{d}t}{t^{1+\alpha}}\left[1 - e^{-2dt}\mathrm{I}_0(2t)^d\right] = (-\Delta)^\alpha(x,x).$$

This concludes the proof. ∎

The limits $\alpha \to 0$ and $\alpha \to 1$ can be controlled. This is the content of the next proposition. The proof extends the strategy of [CRS+18, Thm. 1.2] to the case $d \in \mathbb{N}$.

Theorem 6. *The matrix elements of $(-\Delta)^\alpha$ satify,*

$$\lim_{\alpha\to 1}\sup_{|x|>1} |(-\Delta)^\alpha(x,0)| = 0, \quad \lim_{\alpha\to 0}\sup_{|x|>1} |(-\Delta)^\alpha(x,0)| = 0 \qquad \text{if } |x| > 1 \tag{5.7}$$

$$\lim_{\alpha\to 1} |(-\Delta)^\alpha(x,0)| = 1, \quad \lim_{\alpha\to 0} |(-\Delta)^\alpha(x,0)| = 0 \qquad \text{if } |x| = 1 \tag{5.8}$$

$$\lim_{\alpha\to 1} |(-\Delta)^\alpha(0,0)| = 2d, \quad \lim_{\alpha\to 0} |(-\Delta)^\alpha(0,0)| = 1 \qquad \text{if } |x| = 0. \tag{5.9}$$

Note that this result implies, in particular, using (5.6),

$$\lim_{\alpha\to1}\sum_{x\neq0}|(-\Delta)^{\alpha}(x,0)|=2d,\quad \lim_{\alpha\to0}\sum_{x\neq0}|(-\Delta)^{\alpha}(x,0)|=1.$$

Proof. Remember that for $x=(x_1,\ldots,x_d)$ we defined $|x|=|x|_2=(\sum_{j=1}^d x_j^2)^{\frac{1}{2}}$. Set also $|x|_1:=\sum_{j=1}^d|x_j|$.
Using (5.1) and (5.5), we have

$$|\Gamma(-\alpha)|\,|(-\Delta)^{\alpha}(x,0)|=\int_0^\infty\frac{\mathrm{d}t}{t^{1+\alpha}}\left[e^{-2dt}\prod_{j=1}^{d}\mathrm{I}_{x_j}(2t)-\delta_{0x}\right]$$

$$=\int_0^\infty\frac{\mathrm{d}t}{t^{1+\alpha}}\left[e^{-2dt}\prod_{i=1}^{d}\frac{1}{|x_i|!}\,t^{|x|_1}-\delta_{0x}\right]+\sum_{|q|_1\geq1}\prod_{i=1}^{d}\frac{1}{q_i!(q_i+|x_i|)!}\int_0^\infty\mathrm{d}t\,t^{2|q|_1+|x|_1-1-\alpha}e^{-2dt}$$

$$=S_1(\alpha,|x|_1)+S_2(\alpha,|x|_1),$$

where

$$S_1(\alpha,0)=\int_0^\infty\mathrm{d}t\,t^{-1-\alpha}\left(1-e^{-2dt}\right),\tag{5.10}$$

$$S_1(\alpha,|x|_1)=\prod_{i=1}^{d}\frac{1}{|x_i|!}\,\frac{1}{(2d)^{|x|_1-\alpha}}\Gamma(|x|_1-\alpha)\quad\text{for }|x|_1>0,\tag{5.11}$$

and for all $|x|_1\geq0$

$$S_2(\alpha,|x|_1)=\sum_{|q|_1\geq1}\prod_{i=1}^{d}\frac{1}{q_i!(q_i+|x_i|)!}\,\frac{1}{(2d)^{2|q|_1+|x|_1-\alpha}}\Gamma(2|q|_1+|x|_1-\alpha).\tag{5.12}$$

We claim that, $\forall|x|_1\geq0$,

$$0\leq\frac{S_2(\alpha,|x|_1)}{|\Gamma(-\alpha)|}\leq\frac{(2d)^{\alpha}(1-\alpha)\Gamma(1-\alpha)}{|\Gamma(-\alpha)|}C'_d=(2d)^{\alpha}\alpha(1-\alpha)C'_d,\tag{5.13}$$

for some constant $C'_d>0$ independent of α and x. This implies

$$\lim_{\alpha\to0}\frac{S_2(\alpha,|x|_1)}{|\Gamma(-\alpha)|}=0=\lim_{\alpha\to1}\frac{S_2(\alpha,|x|_1)}{|\Gamma(-\alpha)|}\qquad\forall|x|_1\geq0.\tag{5.14}$$

To prove the claim, note that, since $|q|_1\geq1$ we have $2|q|_1+|x|_1\geq2$ and hence, using

$$\Gamma(z+n)=z(z+1)\cdots(z+n-1)\Gamma(z)=z\Gamma(z)\prod_{l=1}^{n-1}(z+l),$$

and $0<\alpha<1$, we get

$$\Gamma(2|q|_1+|x|_1-\alpha)=(1-\alpha)\Gamma(1-\alpha)\prod_{l=2}^{2|q|_1+|x|_1-1}(l-\alpha)$$
$$\leq(1-\alpha)\Gamma(1-\alpha)(2|q|_1+|x|_1-1)!=\frac{(1-\alpha)\Gamma(1-\alpha)}{2|q|_1+|x|_1}(2|q|_1+|x|_1)!\,.$$

Inserting this bound in $S_2(\alpha, |x|_1)$ we obtain

$$S_2(\alpha, |x|_1) \leq (2d)^\alpha (1-\alpha)\Gamma(1-\alpha) \sum_{|q|_1 \geq 1} \frac{1}{2|q|_1 + |x|_1} \frac{(2|q|_1 + |x|_1)!}{\prod_{i=1}^d q_i!(q_i + |x_i|)!} \frac{1}{(2d)^{2|q|_1 + |x|_1}}. \tag{5.15}$$

In the case $d = 1$ we have

$$S_2(\alpha, |x|_1) \leq (2d)^\alpha (1-\alpha)\Gamma(1-\alpha) \sum_{q \geq 1} \frac{1}{2q + |x|} \frac{1}{2^{2q+|x|}} \frac{(2q + |x|)!}{q!(q + |x|)!}.$$

Note that the binomial coefficient $\frac{n!}{n_1! n_2!}$ is maximal at $n_1 = n_2$, hence, together with Stirling's formula, we get

$$\frac{n!}{n_1! n_2!} \leq \frac{n!}{(\lfloor \frac{n}{2} \rfloor)!^2} \leq C_1 \frac{2^n}{n^{\frac{1}{2}}},$$

for some constant $C_1 > 0$. Inserting this bound above we obtain

$$\sum_{q \geq 1} \frac{1}{2q + |x|} \frac{1}{2^{2q+|x|}} \frac{(2q + |x|)!}{q!(q + |x|)!} \leq C_1 \sum_{q \geq 1} \frac{1}{2q + |x|} \frac{1}{2^{2q+|x|}} \frac{2^{2q+|x|}}{\sqrt{q}} \leq \frac{C_1}{2} \sum_{q \geq 1} \frac{1}{q^{\frac{3}{2}}} =: C_1' < \infty,$$

which proves (5.13) for $d = 1$.

In the case $d \geq 2$ we write

$$S_2(\alpha, |x|_1) = (2d)^\alpha (1-\alpha)\Gamma(1-\alpha) \sum_{n \geq 1} \frac{1}{2n + |x|_1} \frac{1}{(2d)^{2n+|x|_1}} \sum_{|q|_1 = n} \frac{(2n + |x|_1)!}{\prod_{i=1}^d q_i!(q_i + |x_i|)!}$$

We develop the binomial coefficient as follows

$$\frac{(2n + |x|_1)!}{\prod_{i=1}^d q_i!(q_i + |x_i|)!} = \frac{(2n + |x|_1)!}{n!\,(n + |x|_1)!} \frac{(n + |x|_1)!}{\prod_{i=1}^d (q_i + |x_i|)!} \frac{n!}{\prod_{i=1}^d q_i!}. \tag{5.16}$$

Using $\frac{n!}{q!(n-q)!} \leq 2^n$ for all $0 \leq q \leq n$, and the fact that the multinomial coefficient $\frac{n!}{\prod_{i=1}^d q_j!}$ is maximal when all q_j are equal, together with Stirling's formula, we get

$$\frac{(2n + |x|_1)!}{n!\,(n + |x|_1)!} \leq 2^{2n+|x|_1} \quad \text{and} \quad \frac{n!}{\prod_{i=1}^d q_j!} \leq \frac{n!}{(\lfloor \frac{n}{d} \rfloor)!^d} \leq C_d \frac{d^n}{n^{\frac{d-1}{2}}},$$

for some constant $C_d > 0$. Inserting these bounds in (5.16) we obtain

$$\frac{(2n + |x|_1)!}{\prod_{i=1}^d q_i!(q_i + |x_i|)!} \leq 2^{2n+|x|_1} \frac{(n + |x|_1)!}{\prod_{i=1}^d (q_i + |x_i|)!} C_d \frac{d^n}{n^{\frac{d-1}{2}}}.$$

Using

$$\sum_{|q|_1 = n} \frac{(n + |x|_1)!}{\prod_{i=1}^d (q_i + |x_i|)!} \leq d^{n+|x|_1}$$

and inserting all these bounds in $S_2(\alpha, |x|_1)$ we obtain

$$\begin{aligned} S_2(\alpha, |x|_1) &\leq (2d)^\alpha (1-\alpha)\Gamma(1-\alpha) C_d \sum_{n \geq 1} \frac{1}{2n + |x|_1} \frac{1}{(2d)^{2n+|x|_1}} 2^{2n+|x|_1} d^{n+|x|_1} \frac{d^n}{n^{\frac{d-1}{2}}} \\ &\leq (2d)^\alpha (1-\alpha)\Gamma(1-\alpha) C_d \sum_{n \geq 1} \frac{1}{2n^{1+\frac{d-1}{2}}} = (2d)^\alpha (1-\alpha)\Gamma(1-\alpha) C_d' \end{aligned}$$

where $0 < C'_d < \infty$. This proves (5.13) for $d > 1$.

We study now the term $S_1(\alpha, |x|_1)$. We distinguish three cases.

Case 1: $|x| > 1$. In this case $|x|_1 \geq 2$ and therefore, using $0 < \alpha < 1$,

$$\Gamma(|x|_1 - \alpha) \leq (1-\alpha)\Gamma(1-\alpha)(|x|_1 - 1)!\ .$$

It follows, using (5.11),

$$\frac{S_1(\alpha, |x|_1)}{|\Gamma(-\alpha)|} \leq \frac{(1-\alpha)\Gamma(1-\alpha)}{|\Gamma(-\alpha)|} \frac{(|x|_1 - 1)!}{\prod_{i=1}^d |x_i|!} \frac{1}{(2d)^{|x|_1 - \alpha}} \leq \alpha(1-\alpha)(2d)^\alpha.$$

Together with (5.14) this yields (5.7).

Case 2: $|x|_2 = 1 = |x|_1$. In this case

$$\frac{S_1(\alpha, 1)}{|\Gamma(-\alpha)|} = \frac{1}{(2d)^{1-\alpha}} \frac{\Gamma(1-\alpha)}{|\Gamma(-\alpha)|} = \frac{\alpha}{(2d)^{1-\alpha}}.$$

Together with (5.14) this yields (5.8).

Case 3: $|x|_2 = 0 = |x|_1$. Inserting $1 - e^{-2dt} = \int_0^1 2dte^{-2dts} ds$ in (5.10) we obtain

$$\frac{S_1(\alpha, 0)}{|\Gamma(-\alpha)|} = (2d)^\alpha \frac{\Gamma(1-\alpha)}{|\Gamma(-\alpha)|} \int_0^1 \mathrm{d}s\ s^{-1+\alpha} = (2d)^\alpha \frac{\Gamma(1-\alpha)}{\alpha|\Gamma(-\alpha)|} = (2d)^\alpha.$$

Hence $\lim_{\alpha\to 1} \frac{S_1(\alpha,0)}{|\Gamma(-\alpha)|} = 2d$ and $\lim_{\alpha\to 1} \frac{S_1(\alpha,0)}{|\Gamma(-\alpha)|} = 1$. Together with (5.14) this yields (5.9) and concludes the proof of the theorem. □

5.2 Resolvent decay

In this section we consider the operator $[(-\Delta)^\alpha + m^2]^{-1}$ with $m > 0$. This operator is well defined and bounded since $-m^2 \notin \sigma((-\Delta)^\alpha) = [0, (4d)^\alpha]$, for all $m > 0$.
Recall that (see [GRM20, Thm. 2.2] or [Sla18, Lemma 2.1])

$$c_{\alpha,d} = \lim_{|x-y|\to\infty} |x-y|^{d+2\alpha}(-(-\Delta)^\alpha(x,y)) > 0. \tag{5.17}$$

Theorem 7. *Set $m > 0$ and $0 < \alpha < 1$. The matrix elements of the resolvent satisfy*

$$\inf_{m>0} ((-\Delta)^\alpha + m^2)^{-1}(x,y) > 0 \qquad \forall x, y \in \mathbb{Z}^d, \tag{5.18}$$

and

$$\lim_{|x-y|\to\infty} |x-y|^{d+2\alpha}((-\Delta)^\alpha + m^2)^{-1}(x,y) = \frac{c_{\alpha,d}}{m^4} \tag{5.19}$$

where $c_{\alpha,d}$ is the constant introduced in (5.17). Moreover there are constants $C_1 = C_1(m, \alpha, d) > 0$ and $c_1 = c_1(\alpha, d) > 0$ such that

$$\frac{c_1}{|x-y|^{d+2\alpha}} \leq ((-\Delta)^\alpha + m^2)^{-1}(x,y) \leq \frac{C_1}{|x-y|^{d+2\alpha}} \qquad \forall x \neq y. \tag{5.20}$$

where the constant c_1 is independent of the mass m.

Note that the asymptotic behavior (5.19) is compatible with the upper bound obtained in [Sla18, Lemma 3.2]) with other techniques.

Proof. To prove the lower bound note that

$$(-\Delta)^\alpha + m^2 = m_\alpha^2 \mathrm{Id} - P,$$

where $P(x,y) := -(-\Delta)^\alpha(x,y) = |(-\Delta)^\alpha(x,y)| > 0$ for $x \neq y$, $P(x,x) := 0$ and $m_\alpha^2 = m^2 + (-\Delta)^\alpha(0,0)$. For $m > 0$ the Neumann series

$$((-\Delta)^\alpha + m^2)^{-1}(x,y) = \frac{1}{m_\alpha^2}\mathrm{Id} + \sum_{n\geq 1} \frac{1}{m_\alpha^2}\left(P\frac{1}{m_\alpha^2}\right)^n (x,y)$$

is a sum of positive terms and converges for all $x, y \in \mathbb{Z}^d$. Bounding the sum below by the first non-zero term we obtain for $x = y$

$$((-\Delta)^\alpha + m^2)^{-1}(x,x) \geq \frac{1}{m_\alpha^2} \geq \frac{1}{(-\Delta)^\alpha(0,0)} > 0 \tag{5.21}$$

uniformly in $m > 0$. In the case $x \neq y$, using also (1.1), we obtain

$$\begin{aligned}((-\Delta)^\alpha + m^2)^{-1}(x,y) &\geq \frac{1}{m_\alpha^4}P(x,y) = \frac{1}{m_\alpha^4}|(-\Delta)^\alpha(x,y)| \geq \frac{c_{\alpha,d}}{m_\alpha^4}\frac{1}{|x-y|^{d+2\alpha}} \\ &\geq \frac{c_{\alpha,d}}{(-\Delta)^\alpha(0,0)^2}\frac{1}{|x-y|^{d+2\alpha}}.\end{aligned} \tag{5.22}$$

This concludes the proof of (5.18) and the lower bound in (5.20).

To prove (5.19) and the upper bound in (5.20), note that $\left[(-\Delta)^\alpha + m^2\right]^{-1}$ is defined via discrete Fourier transform as follows:

$$\left[(-\Delta)^\alpha + m^2\right]^{-1}(x,y) = \int_{[-\pi,\pi]^d} \frac{\mathrm{d}k}{(2\pi)^d}\frac{e^{i(x-y)\cdot k}}{\mathrm{f}(k)^\alpha + m^2}, \tag{5.23}$$

where

$$\mathrm{f}(k) := \sum_{j=1}^d 2(1-\cos k_j). \tag{5.24}$$

This operator is invariant under translations, hence it suffices to consider the case $y = 0$. Applying $N \geq 2$ times the identity

$$\frac{1}{\mathrm{f}(k)^\alpha + m^2} = \frac{1}{m^2} - \frac{\mathrm{f}(k)^\alpha}{m^2(\mathrm{f}(k)^\alpha + m^2)}$$

we obtain

$$\frac{1}{\mathrm{f}(k)^\alpha + m^2} = \sum_{j=0}^{N-1}\frac{(-1)^j}{m^{2(j+1)}}\mathrm{f}(k)^{\alpha j} + \frac{(-1)^j}{m^{2N}}\frac{\mathrm{f}(k)^{N\alpha}}{\mathrm{f}(k)^\alpha + m^2}.$$

Inserting this decomposition in the integral and using $\int_{[-\pi,\pi]^d} e^{ix\cdot k}\mathrm{d}k = 0$ for all $x \neq 0$, we obtain

$$\begin{aligned}\left[(-\Delta)^\alpha + m^2\right]^{-1}(x,0) &= \frac{1}{m^4}(-(-\Delta)^\alpha(x,y)) + \sum_{j=2}^{N-1}\tfrac{(-1)^j}{m^{2(j+1)}}(-\Delta)^{j\alpha}(x,0) \\ &\quad + \tfrac{(-1)^N}{m^{2N}(2\pi)^d}\int_{[-\pi,\pi]^d}\mathrm{d}k\ F(\mathrm{f}(k)^\alpha)\ e^{ix\cdot k},\end{aligned}$$

where the function $F\colon [0,\infty) \to [0,\infty)$ is defined by

$$F(x) := \frac{x^N}{x+m^2}.$$

Since $0 < \alpha < 1$ there is a $N_\alpha \geq 2$ such that $\alpha(N_\alpha - 1) \leq 1$ and $\alpha N_\alpha > 1$. Setting $N = N_\alpha$ and using (1.1) (for $\alpha(N_\alpha - 1) = 1$ we obtain $(-\Delta)(x,y)$ which is a finite range kernel)

$$\left| \sum_{j=2}^{N_\alpha - 1} \frac{(-1)^j}{m^{2(j+1)}} [(-\Delta)^{j\alpha}(x,y)] \right| \leq \sum_{j=2}^{N_\alpha - 1} \frac{C_{j\alpha,d}}{m^{2(j+1)}|x-y|^{d+2j\alpha}} < \frac{C_1^{(1)}}{|x-y|^{d+4\alpha}},$$

where

$$C_1^{(1)} := N_\alpha \max_{j=2\ldots N_\alpha} \frac{C_{j\alpha,d}}{m^{2(j+1)}}.$$

The limit (5.19) and the upper bound in (5.20) now follow from the following estimate

$$\left| \int_{[-\pi,\pi]^d} \mathrm{d}k \; F(\mathrm{f}(k)^\alpha) \; e^{ix\cdot k} \right| \leq \frac{C_1^{(2)}}{|x|^{d+2}} \quad \forall x \neq 0, \tag{5.25}$$

for some constant $C_1^{(2)} > 0$. To prove it, note that $F \in C^\infty([0,\infty))$ with $F(x) = O(x^N)$ as $x \to 0$. On the contrary, the function $k \mapsto \mathrm{f}(k)^\alpha$ is in $C^\infty([-\pi,\pi]^d \setminus \{0\})$. The first derivative equals

$$\partial_{k_j} \mathrm{f}(k)^\alpha = \alpha \mathrm{f}(k)^\alpha \; \frac{2\sin k_j}{\mathrm{f}(k)},$$

and hence $|\partial_{k_j} \mathrm{f}(k)^\alpha| \leq O(|k|^{2\alpha-1})$. Any additional derivative brings an additional $|k|^{-1}$ divergence factor. Therefore near $k = 0$ we have

$$|\partial_k^\beta F(\mathrm{f}(k)^\alpha)| \leq C_{|\beta|,\alpha,N} O(|k|^{2\alpha N_\alpha - |\beta|}). \tag{5.26}$$

This implies that $\partial_k^\beta F(\mathrm{f}(k)^\alpha) \in L^1([-\pi,\pi]^d)$ for all $|\beta| \leq d+2$. In addition $\mathrm{f}(k)$ is periodic with period 2π in all variables. Since $\mathrm{f}(k)$ is even, we can assume without loss of generality $x_j \geq 0 \; \forall j = 1,\ldots,d$, so that $|x|_1 = \sum_j x_j$. We argue

$$\begin{aligned} -ix_j \int_{[-\pi,\pi]^d} \mathrm{d}k \; \partial_k^\beta F(\mathrm{f}(k)^\alpha)) \; e^{ix\cdot k} &= -\int_{[-\pi,\pi]^d} \mathrm{d}k \; \partial_k^\beta F(\mathrm{f}(k)^\alpha)) \; \partial_{k_j} e^{ix\cdot k} \\ &= -\lim_{\varepsilon\to 0} \int_{[-\pi,\pi]^d \setminus B_\varepsilon(0)} \mathrm{d}k \; \partial_k^\beta F(\mathrm{f}(k)^\alpha)) \; \partial_{k_j} e^{ix\cdot k} \end{aligned}$$

Performing partial integration we obtain

$$\begin{aligned} \int_{[-\pi,\pi]^d \setminus B_\varepsilon(0)} \mathrm{d}k \; \partial_k^\beta F(\mathrm{f}(k)^\alpha)) \; \partial_{k_j} e^{ix\cdot k} &= \int_{\partial B_\varepsilon(0)} \mathrm{d}\mathcal{H}^{d-1} \partial_k^\beta F(\mathrm{f}(k)^\alpha)) \; \nu_j(k) e^{ix\cdot k} \\ &\quad - \int_{[-\pi,\pi]^d \setminus B_\varepsilon(0)} \mathrm{d}k \; \partial_{k_j} \partial_k^\beta F(\mathrm{f}(k)^\alpha)) \; e^{ix\cdot k} \end{aligned}$$

where $\nu(k) := \frac{1}{|k|} k$ and the periodicity of $\mathrm{f}(k)$ garantees there is no contribution from the boundary of $[-\pi,\pi]^d$. For $|\beta| \leq d+1$ and $\alpha N_\alpha > 1$, using (5.26)

$$\limsup_{\varepsilon\to 0} \left| \int_{\partial B_\varepsilon(0)} \mathrm{d}\mathcal{H}^{d-1} \partial_k^\beta F(\mathrm{f}(k)^\alpha)) \; \nu_j(k) e^{ix\cdot k} \right| \leq C \limsup_{\varepsilon\to 0} \varepsilon^{2\alpha N_\alpha - |\beta|} \varepsilon^{d-1} = 0,$$

and hence

$$-ix_j \int_{[-\pi,\pi]^d} \mathrm{d}k\ \partial_k^\beta F(\mathrm{f}(k)^\alpha))\ e^{ix\cdot k} = \int_{[-\pi,\pi]^d} \mathrm{d}k\ \partial_{k_j}\partial_k^\beta F(\mathrm{f}(k)^\alpha))\ e^{ix\cdot k},$$

where the last integral is well defined since $|\beta| + 1 \leq d + 2$. The integrability of the derivative ensures we can repeat the procedure above inductively until $|\beta| + 1 = d + 2$. This concludes the proof of (5.25) and of the theorem. $\square$

5.3 Inverse

Theorem 8. *Let $0 < \alpha < \frac{d}{2}$. The inverse $(-\Delta)^{-\alpha}(x,y) := \lim_{m\downarrow 0}\left[(-\Delta)^\alpha + m^2\right]^{-1}(x,y)$ is well-defined and admits the represention via discrete Fourier transform*

$$(-\Delta)^{-\alpha}(x,y) = \int_{[-\pi,\pi]^d} \frac{\mathrm{d}k}{(2\pi)^d} \frac{e^{i(x-y)\cdot k}}{\mathrm{f}(k)^\alpha}, \tag{5.27}$$

This operator is invariant under translations, its matrix elements satisfy $(-\Delta)^{-\alpha}(x,y) > 0\ \forall x,y \in \mathbb{Z}^d$, and

$$\lim_{|x-y|\to\infty} |x-y|^{d-2\alpha}(-\Delta)^{-\alpha}(x,y) = \mathfrak{c}_\alpha \tag{5.28}$$

where $\mathfrak{c}_\alpha$ is the constant introduced in (5.33). Moreover there are constants $C_2 = C_2(\alpha,d) > 0$ and $c_2 = c_2(\alpha,d) > 0$ such that

$$c_2 \frac{1}{|x-y|^{d-2\alpha}} \leq (-\Delta)^{-\alpha}(x,y) \leq C_2 \frac{1}{|x-y|^{d-2\alpha}} \qquad \forall x \neq y. \tag{5.29}$$

The fact that $(-\Delta)^{-\alpha}$ is well-defined and Ineq. 5.29 holds are known (see e.g. [Sla18, Sect. 2] and references therein). Here, we provide an alternative, more analytical proof, which we believe is new in this context. It uses the discrete Fourier transform and is based on arguments in [GRM20, Lemma A.1].

Proof.
By (5.18) we have $(-\Delta)^{-\alpha}(x,y) := \lim_{m\downarrow 0}\left[(-\Delta)^\alpha + m^2\right]^{-1}(x,y) > 0\ \forall x,y \in \mathbb{Z}^d$. Moreover, remember that, for all $m > 0$, (cf. (5.23))

$$\left[(-\Delta)^\alpha + m^2\right]^{-1}(x,y) = \int_{[-\pi,\pi]^d} \frac{\mathrm{d}k}{(2\pi)^d} \frac{e^{i(x-y)\cdot k}}{\mathrm{f}(k)^\alpha + m^2},$$

where $\mathrm{f}(k)$ is defined in (5.24). Note that $[\mathrm{f}(k)^\alpha + m^2]^{-1} \leq \mathrm{f}(k)^{-\alpha}$ which is an unbounded integrable function. Indeed this function behaves near $k = 0$ as $\frac{1}{|k|^{2\alpha}}$ which integrably divergent as long as $\alpha < \frac{d}{2}$. Therefore, by dominated convergence, the limit $\varepsilon \to 0$ is well defined and formula (5.27) holds.

To prove (5.29) we approximate the discrete Laplacian $-\Delta$ on $\mathbb{Z}^d$, with eigenvalues $\mathrm{f}(k)$, by the continuous Laplacian $-\Delta_c$ on $\mathbb{R}^d$, with eigenvalues $|k|^2$, and use the known decay $|(-\Delta_c)^{-1}(x,y)| \leq \frac{C}{|x-y|^{d-2\alpha}}$, which holds in distributional sense for some $C > 0$ (cf. Proposition 9 below for a precise statement). By translation invariance it suffices to consider the case $y = 0$, $x \neq 0$.

The two functions $\mathrm{f}(k)$ and $|k|^2$ coincide only near $k=0$, therefore we introduce a smooth cut-off function $\psi \in C_c^\infty(\mathbb{R}^d,[0,1])$ such that $\operatorname{supp}\psi \subset B_1(0)$ and $\psi(k)=1$ $\forall k \in B_{\frac{1}{2}}(0)$. Hence $(-\Delta)^{-\alpha}(0,x) = I_1(x)+I_2(x)$ where

$$I_1(x) := \int_{[-\pi,\pi]^d} \frac{\mathrm{d}k}{(2\pi)^d} \frac{\psi(k)}{\mathrm{f}(k)^\alpha} e^{ix\cdot k}, \qquad I_2(x) := \int_{[-\pi,\pi]^d} \frac{\mathrm{d}k}{(2\pi)^d} \frac{1-\psi(k)}{\mathrm{f}(k)^\alpha} e^{ix\cdot k}. \tag{5.30}$$

Note that $k \mapsto \frac{1-\psi(k)}{\mathrm{f}(k)^\alpha} \in C^\infty([-\pi,\pi]^d)$. Moreover, along the boundary of $[-\pi,\pi]^d$ we have $\frac{1-\psi(k)}{\mathrm{f}(k)^\alpha} = \frac{1}{\mathrm{f}(k)^\alpha}$, which is a periodic function with period 2π in all variables. Therefore, by partial integration, we obtain

$$|x_j^N I_2(x)| \le \int_{[-\pi,\pi]^d} \frac{\mathrm{d}k}{(2\pi)^d} \left| \partial_{k_j}^N \frac{1-\psi(k)}{\mathrm{f}(k)^\alpha} \right| \le C_N,$$

where no boundary contribution appears by periodicity. It follows, for all $N \ge 1$

$$|I_2(x)| \le \frac{C_N}{|x|^N} \qquad \forall x \neq 0,$$

for some constant $C_N > 0$. Therefore we only need to study the first integral $I_1(x)$. We write

$$I_1(x) = \int_{[-\pi,\pi]^d} \frac{\mathrm{d}k}{(2\pi)^d} \frac{1}{|k|^{2\alpha}} \Phi(k)\ e^{ix\cdot k} = \int_{\mathbb{R}^d} \frac{\mathrm{d}k}{(2\pi)^d} \frac{1}{|k|^{2\alpha}} \Phi(k)\ e^{ix\cdot k}$$

where we defined

$$\Phi(k) := \begin{cases} \left(\frac{|k|^2}{\mathrm{f}(k)}\right)^\alpha \psi(k), & k \neq 0, \\ 1, & k = 0. \end{cases} \tag{5.31}$$

and in the last step we used the fact the ψ has support inside $B_1(0)$ to extend the integral from $[-\pi,\pi]^d$ to $\mathbb{R}^d$. The function Φ is smooth $\Phi(k) \in C_c^\infty(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$, and hence it is the continuous Fourier transform of a function $\varphi \in \mathcal{S}(\mathbb{R}^d)$ (see [Gra14, Corollary 2.2.15])

$$\Phi(k) = \hat{\varphi}(k) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \mathrm{d}y\, e^{-iy\cdot k} \varphi(y).$$

It follows, by Proposition 9 below,

$$I_1(x) = \int_{\mathbb{R}^d} \frac{\mathrm{d}k}{(2\pi)^d} \frac{1}{|k|^{2\alpha}} \Phi(k)\ e^{ix\cdot k} = \mathfrak{c}_\alpha \int_{\mathbb{R}^d} \mathrm{d}y\ \frac{1}{|x-y|^{d-2\alpha}} \varphi(y)$$

where the constant $\mathfrak{c}_\alpha$ is given in (5.33). The integral above is well-defined since $\varphi \in \mathcal{S}(\mathbb{R}^d)$[2]. Using this result and Proposition 10 below, we argue

$$\begin{aligned} \lim_{|x|\to\infty} |x|^{d-2\alpha} (-\Delta)^{-\alpha}(0,x) &= \mathfrak{c}_\alpha \lim_{|x|\to\infty} \int_{\mathbb{R}^d} \mathrm{d}y\ \frac{|x|^{d-2\alpha}}{|x-y|^{d-2\alpha}} \varphi(y) \\ &= \mathfrak{c}_\alpha \int_{\mathbb{R}^d} \mathrm{d}y\ \varphi(y) = \mathfrak{c}_\alpha \hat{\varphi}(0) = \mathfrak{c}_\alpha \Phi(0) = \mathfrak{c}_\alpha > 0, \end{aligned}$$

where in the last step we used $\Phi(0) = 1$. The limit (5.28), as well as the upper and lower bounds in (5.29) now follow. □

[2]The function $\mathrm{I}_\alpha[\varphi](y) := \mathfrak{c}_\alpha \int_{\mathbb{R}^d} \mathrm{d}y\ \frac{1}{|x-y|^{d-2\alpha}} \varphi(y)$ is called Riesz potential.

We collect finally two techical results that are necessary for the proof above. The first proposition can be found in [LL01, Section 5.9] (the constants are slightly different because of our choice of definition of Fourier transform), see also [Ste70, Chapter 5]. For completeness we give here a sketch of the proof. The second proposition is based on the same arguments as the proof of [GRM20, Lemma A.1].

Proposition 9. *[LL01, Thm. 5.9] Let* $\alpha \in (0, \frac{d}{2})$ *and let* $\varphi \in \mathcal{S}(\mathbb{R}^d)$*, then*

$$\int_{\mathbb{R}^d} \frac{\mathrm{d}k}{(2\pi)^{\frac{d}{2}}} \frac{1}{|k|^{2\alpha}} \hat{\varphi}(k) e^{+ik\cdot x} = \mathfrak{c}_\alpha \int_{\mathbb{R}^d} \mathrm{d}y \, \frac{1}{|x-y|^{d-2\alpha}} \varphi(y) \tag{5.32}$$

where

$$\mathfrak{c}_\alpha := \frac{\Gamma(\frac{d}{2}-\alpha)}{\Gamma(\alpha)} 2^{\frac{d}{2}-2\alpha}. \tag{5.33}$$

Proof. Using the identity, which holds for all $\rho > 0$ and $0 < \alpha < 1$,

$$\frac{1}{\rho^\alpha} = \frac{1}{2^\alpha \Gamma(\alpha)} \int_0^\infty \mathrm{d}t \, t^{\alpha-1} e^{-\frac{t}{2}\rho},$$

we can write

$$\frac{1}{|k|^{2\alpha}} = \frac{1}{2^\alpha \Gamma(\alpha)} \int_0^\infty \mathrm{d}t \, t^{\alpha-1} e^{-\frac{t}{2}|k|^2}, \qquad \forall k \in \mathbb{R}^d \setminus \{0\}. \tag{5.34}$$

It follows, using Fubini and the Fourier transform of a product,

$$\begin{aligned}
\int_{\mathbb{R}^d} \frac{\mathrm{d}k}{(2\pi)^{\frac{d}{2}}} \frac{1}{|k|^{2\alpha}} \hat{\varphi}(k) e^{+ik\cdot x} &= \frac{1}{2^\alpha \Gamma(\alpha)} \int_0^\infty \mathrm{d}t \, t^{\alpha-1} \int_{\mathbb{R}^d} \frac{\mathrm{d}k}{(2\pi)^{\frac{d}{2}}} e^{-\frac{t}{2}|k|^2} \hat{\varphi}(k) e^{+ik\cdot x} \\
&= \frac{1}{2^\alpha \Gamma(\alpha)} \int_0^\infty \mathrm{d}t \, t^{\alpha-1} \frac{1}{t^{\frac{d}{2}}} \int_{\mathbb{R}^d} \mathrm{d}y e^{-\frac{1}{2t}|x-y|^2} \varphi(y) \\
&= \frac{1}{2^\alpha \Gamma(\alpha)} \int_{\mathbb{R}^d} \mathrm{d}y \varphi(y) \int_0^\infty \mathrm{d}t \, t^{\alpha-1} \frac{1}{t^{\frac{d}{2}}} e^{-\frac{1}{2t}|x-y|^2} = \mathfrak{c}_\alpha \int_{\mathbb{R}^d} \mathrm{d}y \, \frac{1}{|x-y|^{d-2\alpha}} \varphi(y).
\end{aligned}$$

This concludes the proof. ☐

Proposition 10. *Let* $\alpha \in (0, \frac{d}{2})$ *and let* $\varphi \in \mathcal{S}(\mathbb{R}^d)$*, then*

$$\lim_{|x|\to\infty} \int_{\mathbb{R}^d} \mathrm{d}y \, \frac{|x|^{d-2\alpha}}{|x-y|^{d-2\alpha}} \varphi(y) = \int_{\mathbb{R}^d} \mathrm{d}y \, \varphi(y). \tag{5.35}$$

Proof. We decompose the integral as follows:

$$\int_{\mathbb{R}^d} \mathrm{d}y \, \frac{|x|^{d-2\alpha}}{|x-y|^{d-2\alpha}} \varphi(y) = \int_{B_{\frac{|x|}{2}}(x)} \mathrm{d}y \, \frac{|x|^{d-2\alpha}}{|x-y|^{d-2\alpha}} \varphi(y) + \int_{B^c_{\frac{|x|}{2}}(x)} \mathrm{d}y \, \frac{|x|^{d-2\alpha}}{|x-y|^{d-2\alpha}} \varphi(y).$$

The first integral can be reorganized as

$$\int_{B_{\frac{|x|}{2}}(x)} \mathrm{d}y \, \frac{|x|^{d-2\alpha}}{|x-y|^{d-2\alpha}} \varphi(y) = \int_{B_{\frac{|x|}{2}}(0)} \mathrm{d}y \, \frac{|x|^{d-2\alpha}}{|y|^{d-2\alpha}} \varphi(x+y).$$

Since $\varphi \in \mathcal{S}(\mathbb{R}^d)$ we have $|\varphi(y)| \le \frac{C_N}{|y|^N}$ for $y \neq 0$ for all $N \ge 1$. Therefore, since $|x+y| \ge \frac{|x|}{2}$ inside the ball $B_{\frac{|x|}{2}}(0)$ we have

$$|\varphi(x+y)| \le \frac{C_N}{|x+y|^N} \le \frac{2^N C_N}{|x|^N}.$$

Inserting this bound in the integral and fixing $N > d$ we obtain

$$\begin{aligned}\limsup_{|x|\to\infty} \int_{B_{\frac{|x|}{2}}(0)} \mathrm{d}y \, \frac{|x|^{d-2\alpha}}{|y|^{d-2\alpha}} |\varphi(y+x)| &\le \limsup_{|x|\to\infty} 2^N C_N \frac{|x|^{d-2\alpha}}{|x|^N} \int_{B_{\frac{|x|}{2}}(0)} \mathrm{d}y \, \frac{1}{|y|^{d-2\alpha}} \\ &= \frac{2^N C_N |\mathcal{S}^{d-1}|}{2\alpha \, 2^{2\alpha}} \limsup_{|x|\to\infty} \frac{|x|^d}{|x|^N} = 0,\end{aligned}$$

where $|\mathcal{S}^{d-1}|$ is the surface volume of the unit sphere in $\mathbb{R}^d$. Hence

$$\lim_{|x|\to\infty} \int_{B_{\frac{|x|}{2}}(x)} \mathrm{d}y \, \frac{|x|^{d-2\alpha}}{|x-y|^{d-2\alpha}} \varphi(y) = 0.$$

We consider now the first integral. Note that, since, the center of the ball $B_{\frac{|x|}{2}}(x)$ escapes at infinity as $|x| \to \infty$ it holds

$$\lim_{|x|\to\infty} \frac{|x|^{d-2\alpha}}{|x-y|^{d-2\alpha}} \mathbf{1}_{B^c_{\frac{|x|}{2}}(x)}(y) = 1 \qquad \forall y \in \mathbb{Z}^d.$$

Since in addition

$$\frac{|x|^{d-2\alpha}}{|x-y|^{d-2\alpha}} \mathbf{1}_{B^c_{\frac{|x|}{2}}(x)}(y) |\varphi(y)| \le 2^{d-2\alpha} |\varphi(y)| \in L^1(\mathbb{R}^d)$$

holds, we obtain, by dominated convergence,

$$\lim_{|x|\to\infty} \int_{B^c_{\frac{|x|}{2}}(x)} \mathrm{d}y \, \frac{|x|^{d-2\alpha}}{|x-y|^{d-2\alpha}} \varphi(y) = \int_{\mathbb{R}^d} \mathrm{d}y \, \varphi(y),$$

which concludes the proof of the proposition. □

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