Improved Approximation Algorithms for Weighted k-Set Packing

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Chapter 1

Introduction

In this thesis, we study the weighted k-Set Packing problem. It is defined as follows: The input consists of a family $\mathcal{S}$ of non-empty sets, each of cardinality at most k. Every set $S \in \mathcal{S}$ is equipped with a positive weight $w(S)$. The task is to compute a disjoint subfamily of $\mathcal{S}$, i.e., a family consisting of pairwise disjoint sets, of maximum total weight. See Fig. 1.1 for an example.

The weighted k-Set Packing problem constitutes a fundamental problem in combinatorial optimization. In particular, the case $k = 2$ is equivalent to the well-studied *Maximum Weight Matching problem*. Note that this implies that for $k = 1, 2$, the weighted k-Set Packing problem can be solved in polynomial time [22]. In contrast, for $k \geq 3$, even the special case where all sets have a weight of 1, the *unweighted k-Set Packing problem*, is NP-hard since it generalizes the 3-Dimensional Matching problem [35]. In the following chapters, we will study *approximation algorithms* for the weighted k-Set Packing problem for constant $k \geq 3$.

The remainder of this chapter is organized as follows: In Section 1.1, we discuss previous works that our results build upon. Next, we provide an overview of those results in Section 1.2. After that, we review further related works in Section 1.3. Finally, Section 1.4 provides an outline of the remainder of the thesis.

1.1 Previous works

Berman and Karpinski [9] have shown that already the unweighted 3-Set Packing problem is NP-hard to approximate within a factor of less than $\frac{98}{97}$. In addition, Kann [34] has proven that the unweighted 3-Set Packing problem is APX-hard[1]. Moreover, Hazan, Safra and Schwartz [32] have established that unless P = NP, the unweighted k-Set Packing problem does not admit a polynomial-time $o(\frac{k}{\log k})$-approximation.

[1]Note that MAX-SNP-hardness implies APX-hardness, see [36].

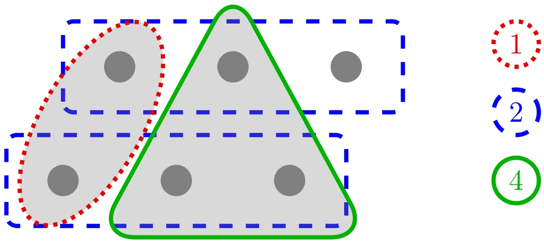


Figure 1.1: An instance of the weighted 3-Set Packing problem, where filled circles represent set elements, and frames indicate the sets. The color and pattern of a frame encodes the weight of the respective set as shown on the right. An optimum solution is highlighted in gray.

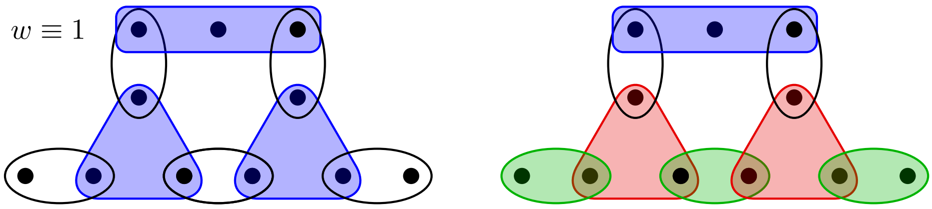
A simple greedy approach that traverses the sets in order of decreasing weight and selects every set that does not intersect an already chosen one is known to yield an approximation guarantee of k (see, e.g., [15]). The best approximation guarantee that is known for the unweighted k-Set Packing problem is $\frac{k+1+\epsilon}{3}$ [16, 28], where $\epsilon > 0$ can be chosen to be arbitrarily small, but constant. Prior to our work, the state-of-the-art for general weights has been Berman's $\frac{k+1+\epsilon}{2}$-approximation algorithm [7].

The technique that has proven most successful in designing approximation algorithms for both the weighted and the unweighted k-Set Packing problem is *local search*. The basic idea can be described as follows: Start with an arbitrary solution, e.g., the empty one. As long as there exists a *local improvement* from a certain class, apply it to improve the solution maintained by the algorithm. Once no more local improvement can be found, return the current solution.

In the context of (weighted) k-Set Packing, a local improvement of a feasible solution A is a disjoint collection X of sets such that

$$w(X) > w(\{a \in A : \exists x \in X : a \cap x \neq \emptyset\}),$$

that is, the weight of X exceeds the weight of the collection of sets in A that are intersected by sets in X (including the sets in $A \cap X$). Note that these are exactly those sets that we need to remove from A in order to be able to add the sets in X and maintain a feasible solution. See Fig. 1.2 for an illustration. We call $|X|$ the *size* of the local improvement X. Note that whenever the current solution A is suboptimal, every optimum solution B constitutes a local improvement of A. However, if we aim at a polynomial running time per iteration, it is of course infeasible to check every possible subset of the set family $\mathcal{S}$. Even more, unless P = NP, for $k \geq 3$, we certainly cannot check for the existence of a (general) local improvement in polynomial time because if we could, this would imply a polynomial-time



(a) An instance of the unweighted 3-Set Packing problem. A feasible solution A is marked in blue.

(b) A local improvement X of A of size 3 is highlighted in green. The sets from A that sets from X intersect are marked in red.

Figure 1.2: Example of a local improvement.

algorithm for the NP-hard unweighted k-Set Packing problem. To this end, observe that for unit weights, any local search algorithm terminates after at most $|\mathcal{S}| + 1$ iterations since the cardinality of the solution strictly increases with each local improvement.

As a consequence, in order to be able to search for a local improvement in polynomial time, several previous works restrict themselves to local improvements of constant size [4, 5, 7, 10, 15, 33], which can be searched for via exhaustive enumeration. In addition, the more recent works on the unweighted k-Set Packing problem also allow certain well-structured local improvements of up to logarithmic size, which they search for using techniques from fixed-parameter tractability such as path decompositions or color-coding [16, 28, 48].

We point out that for general weights, in contrast to the unit weight setting, a polynomial upper bound on the number of local improvement steps we need to perform is not immediate. In fact, by combining reductions from [21] and [20], it follows that a weighted version of the 3-Dimensional Matching problem is tight PLS-complete with respect to certain types of swaps of constant size. Thus, it appears unlikely that locally optimum solutions can be computed in polynomial time. Instead, this problem is usually circumvented by scaling and truncating the weight function, which allows for a polynomial bound on the number of iterations at the cost of an arbitrarily small loss in the approximation guarantee [7, 15]. See Section 2.3 for more details.

In the following, we first give an overview of local search based approximation algorithms for the unweighted k-Set Packing problem, including the state-of-the-art works on this problem. Then, we discuss previous results on local search for general weights.

1.1.1 The unit weight case

In 1989, Hurkens and Schrijver [33] have shown that local improvements of arbitrarily large, but constant size yield approximation guarantees arbitrarily close to $\frac{k}{2}$ for the unweighted k-Set Packing problem. In their paper, they provide exact formulas for the approximation guarantee as a function of k and the maximum improvement size p, and give matching lower bound examples. By taking into account local improvements of up to logarithmic size, at the cost of a quasi-polynomial running time, Halldórsson [31] managed to obtain an improved approximation guarantee of $\frac{k+2}{3}$. Still with a quasi-polynomial runing time, Cygan, Grandoni and Mastrolilli [17] could improve on this, achieving a guarantee of $\frac{k+1+\epsilon}{3}$. The first polynomial-time improvement over the result of Hurkens and Schrijver [33] was achieved by Sviridenko and Ward [48]. They observed that in order to obtain an approximation guarantee of $\frac{k+2}{3}$, it suffices to search for certain, well-structured local improvements of up to logarithmic size, which they call *canonical*, as well as local improvements of size 1. In addition, Sviridenko and Ward explain how to apply the color-coding technique in order to search for canonical improvements in polynomial time. By considering local improvements of *bounded pathwidth* (in a certain auxiliary graph), Cygan [16] managed to obtain an approximation guarantee of $\frac{k+1+\epsilon}{3}$, where the running time is polynomial (for any fixed $\epsilon > 0$), but depends on ϵ^{-1} in a doubly exponential way. The state-of-the-art algorithm for the unweighted k-Set Packing problem in terms of approximation guarantee and running time is due to Fürer and Yu [28]. They enhance the notion of canonical improvements considered in [48] by studying *canonical improvements with tail changes*. In doing so, they obtain a polynomial-time $\frac{k+1+\epsilon}{3}$-approximation algorithm for the unweighted k-Set Packing problem, whose running time is singly exponential in ϵ^{-2}. In addition, Fürer and Yu provide instances with locality gap $\frac{k+1}{3}$ for any algorithm considering local improvements of size $\mathcal{O}(|\mathcal{S}|^{\frac{1}{5}})$, i.e., instances for which there exists a feasible solution that does not admit any local improvement of size $\mathcal{O}(|\mathcal{S}|^{\frac{1}{5}})$, but that is by a factor of $\frac{k+1}{3}$ smaller than the optimum. Hence, new techniques will be required to get below the threshold guarantee of $\frac{k+1}{3}$ for unit weights.

1.1.2 General weights

When it comes to general weights, the situation appears to be even more challenging. For instance, while in the unit weight case, local improvements of constant size were sufficient to achieve approximation guarantees arbitrarily close to $\frac{k}{2}$ [33], Arkin and Hassin [4] have shown that for general weights, applying local improvements of constant size t in an arbitrary manner until no more exist only results in an approximation guarantee of $k - 1 + \frac{1}{t}$. Contrasting this result, Chandra and Halldórsson [15] have found that if in each iteration, among a certain class of local improvements of constant size, one of maximum *payoff factor*, the ratio between the total weights of sets added to and removed from the solution is chosen, then this results in an approximation guarantee of $\frac{2\cdot(k+1)}{3}$. By scaling and truncating the weight function, they obtain a polynomial-time $\frac{2\cdot(k+1)+\epsilon}{3}$-approximation algorithm for the weighted k-Set Packing problem.

Prior to our work, the state-of-the-art for the weighted k-Set Packing problem has been Berman's algorithm *SquareImp* [7]. As the name indicates, SquareImp performs a *misdirected local search* in that it does not optimize the original, but the *squared weight* w^2 of the solution (meaning that the weight of each set is squared individually). For the case where the size of the local improvement is bounded by 2, the optimum misdirection (among those of the form w^α) is known [10]. In contrast, the type of local improvement that SquareImp considers can be of size up to k. In [7], Berman shows that the weight of a solution that is locally optimum w.r.t. this type of improvement is by a factor of at most $\frac{k+1}{2}$ smaller than the optimum, and that this is tight. In particular, Berman's result implies a polynomial-time $\frac{k+1+\epsilon}{2}$-approximation for the weighted k-Set Packing problem.

The first improvement over this guarantee, which has been unchallenged for twenty years, was obtained in the course of my Master's thesis [39]. By considering local improvements of size up to $\mathcal{O}(k^2)$, I managed to obtain a (slightly) improved approximation guarantee of $\frac{k+1-\frac{1}{31,850,496\cdot k}+\epsilon}{2}$. After the completion of the thesis, this result was further refined to obtain an improved guarantee of $\frac{k+1+\epsilon}{2}-\frac{1}{63,700,992}$ [40]. Very recently and independent of our work, Thiery and Ward [49] have shown that by taking into account local improvements of size up to $\mathcal{O}(k^3)$, one can obtain approximation guarantees of $\frac{k+1-\tau_k}{2}$, where $\tau_k \geq 0.428$ and $\lim_{k\to\infty}\tau_k = \frac{2}{3}$. In particular, their result implies a 1.786-approximation for $k = 3$.

1.2 Our results

In this thesis, we first refine and simplify the ideas used in [40]. More precisely, in Chapter 3, we show that it suffices to consider the type of local improvement of size $\leq k$ SquareImp searches for, as well as local improvements of size 3, to obtain a polynomial-time $\left(\frac{k+1}{2}-\frac{1}{1000}\right)$-approximation algorithm for the weighted k-Set Packing problem for $k \geq 3$. Even though in terms of the approximation ratio, this result is now dominated by the guarantees in [49], we believe that it is still worthwhile to present it as part of this thesis for the following reasons: First of all, it shows that Berman's algorithm SquareImp can be improved upon without the (significant) increase in the size of the improvements considered that is inherent to [39, 40, 49] and the results in Chapters 4 and 5. Second, the presentation of this comparably simple way to obtain an improvement may serve as a warm-up for the reader, providing some intuition for the types of arguments that we will use in our more involved analyses. Finally, in the course of the analysis, we develop and showcase a structural result that will play a crucial role when deriving the subsequent two results.

For the first one of these results, we explore the possibilities of local improvements of up to logarithmic size in the weighted setting. Recall that even for unit weights, local improvements of constant size cannot result in a better guarantee than $\frac{k}{2}$ [33], whereas taking into account local improvements of up to logarithmic size allows to bring the approximation guarantee for the unweighted k-Set Packing problem down to $\frac{k+1+\epsilon}{3}$. Thus, it appears natural to ask whether a similar behavior can be observed for general weights. Surprisingly, this turns out not to be the case. More precisely, in Chapter 4, we show that by considering local improvements of logarithmically bounded size with respect to a fixed additive local search objective (e.g., $w^2(A) = \sum_{S\in A} w^2(S)$ as in Berman's algorithm [7]), one cannot obtain a better guarantee than $\frac{k}{2}$. Complementing this negative result, we derive a suitable notion of a well-structured local improvement of logarithmic size for the weighted setting, and use it to obtain improved approximation guarantees of $\frac{k+1-\lambda_k}{2}$, where $\lim_{k\to\infty} \lambda_k = 1$. Note that in light of our lower bound, these guarantees are asymptotically best possible. The results presented in Chapter 4 have been published in the form of an extended abstract in the proceedings of IPCO 2022 [41].[2] A full version of the paper has recently been accepted for publication in Mathematical Programming.

At first sight, the previous results seem to conclude the story of local search for the weighted k-Set Packing problem, given that well-structured local improvements of logarithmically bounded size lie on the border of what is still tractable via enumeration based approaches all of the prior works rely on. However, it turns out that by employing a *black box algorithm* for the *unweighted k-Set Packing problem* to carefully chosen subinstances, we can in fact circumvent the logarithmic size bound, and more excitingly, pass the threshold guarantee of $\frac{k}{2}$. We point out that this approach is motivated by the fact that instances on which Berman's analysis [7] (which we build upon) is close to being tight are "close to being unweighted in a certain sense". See Chapters 2 and 3 for the details. Formalizing the above ideas results in a polynomial-time $(0.4999 \cdot k + 0.501)$-approximation algorithm for the weighted k-Set Packing problem, which we present in Chapter 5. A version of this result where we obtain slightly improved constants at the cost of a (much) more tedious analysis has been published in the proceedings of SODA 2023 [42]. Even though the absolute improvement in the approximation guarantee we obtain is quite small and only dominates our previous results, let alone the recent improvements by Thiery and Ward [49] for (very) large

[2] The lower bound result in [41] is slightly weaker than the one presented in this thesis in that it only considers local search objectives of the form w^α with $\alpha \in \mathbb{R}$.

values of k, we believe that as a proof of concept, our result is exciting nonetheless.

In this spirit, in Chapter 6, we show that the techniques from Chapter 5 allow us to establish a general link between the approximation ratios for the weighted and the unweighted k-Set Packing problem. More precisely, we show that for every $\sigma \in (0,1)$, there exists a constant $\tau \in (0,1)$ with the property that for every $k \geq 3$, a polynomial-time $(\tau \cdot k)$-approximation algorithm for the unweighted k-Set Packing problem immediately gives rise to a polynomial-time $(\sigma \cdot k)$-approximation algorithm for general weights. In terms of lower bounds, this result tells us that it cannot happen that for unit weights, the current lower bound of $\Omega(\frac{k}{\log k})$ can be attained, whereas for general weights, obtaining an $o(k)$-approximation is hard. This does not seem to be clear a priori. Moreover, the lines of argumentation in Chapters 5 and 6 may serve as a recipe to translate improved guarantees for the unweighted k-Set Packing problem into improved guarantees for general weights.

Complementing the rather abstract result in Chapter 6, in Chapter 7, we study the *hereditary* 2-3-*Set Packing problem*, a (very) special case of weighted 3-Set Packing, and provide a rather simple $\frac{4}{3}$-approximation, improving on the previous best known guarantee of $\frac{7}{5}$ [25]. The hereditary 2-3-Set Packing problem arises as a subtask in an approximation algorithm for the Maximum Leaf Spanning Arborescence problem (MLSA) in acyclic digraphs (dags) (see Chapter 7) by Fernandes and Lintzmayer [25]. They show that any polynomial-time α-approximation for the hereditary 2-3-Set Packing problem gives rise to a polynomial-time $\max\{\alpha, \frac{4}{3}\}$-approximation for the MLSA in dags. In particular, the $\frac{4}{3}$-approximation we provide allows us to tap the full potential of the approach by Fernandes and Lintzmayer. In addition, our algorithm and our analysis are arguably simpler than the $\frac{7}{5}$-approximation in [25].

1.3 Further related works

1.3.1 Linear and semidefinite programming relaxations for weighted Set Packing

The weighted Set Packing problem is defined analogously to the weighted k-Set Packing problem, but allows sets to be of arbitrary large (positive) cardinality. Unless P = NP, even the unit weight case cannot be approximated within a factor of $n^{1-\varepsilon}$ for any constant $\varepsilon > 0$, where n denotes the number of sets contained in the instance. This follows from [50]. In this section, we review works that investigate the strength of linear and semidefinite programming relaxations for the weighted Set Packing problem and relate them to the sizes of the sets that occur, as well as certain structural properties of the instances.

For a set family $\mathcal{S}$, denote the underlying universe by $U(\mathcal{S}) := \bigcup \mathcal{S}$. Given a collection $\mathcal{S}$ of non-empty sets and $w\colon \mathcal{S} \to \mathbb{R}_{>0}$, we can model the weighted Set Packing problem on $(\mathcal{S}, w)$ by the following integer program:

$$\max \sum_{S \in \mathcal{S}} w(S) \cdot x_S \qquad \text{(Set-Packing-IP}(\mathcal{S}, w))$$

$$\text{subject to} \sum_{S: u \in S} x_S \leq 1 \qquad \text{for all } u \in U(\mathcal{S}) \tag{1.1}$$

$$x_S \in \{0, 1\} \qquad \text{for all } S \in \mathcal{S} \tag{1.2}$$

If we relax the integrality constraints in (1.2) to $x_S \in [0, 1]$ for each $S \in \mathcal{S}$, we obtain the *standard linear programming relaxation for* $(\mathcal{S}, w)$:

$$\max \sum_{S \in \mathcal{S}} w(S) \cdot x_S \qquad \text{(Set-Packing-LP}(\mathcal{S}, w))$$

$$\text{subject to} \sum_{S: u \in S} x_S \leq 1 \qquad \text{for all } u \in U(\mathcal{S}) \tag{1.3}$$

$$x_S \in [0, 1] \qquad \text{for all } S \in \mathcal{S} \tag{1.4}$$

Denote the optimum value of (Set-Packing-LP$(\mathcal{S}, w)$) by $\nu^*(\mathcal{S}, w)$. The Füredi-Kahn-Seymour-Conjecture [26] states that there exists a feasible solution A to $(\mathcal{S}, w)$ such that

$$\sum_{S \in A} \left(|S| - 1 + \frac{1}{|S|} \right) \cdot w(S) \geq \nu^*(\mathcal{S}, w). \tag{1.5}$$

Füredi [27] has verified the conjecture for the special case where all sets in $\mathcal{S}$ have the same cardinality and the same weight. Note that for an instance of the weighted k-Set Packing problem, we can assume without loss of generality that all sets are of cardinality *exactly* k. In case $\mathcal{S}$ is k*-partite*, i.e., $U(\mathcal{S})$ can be partitioned into $U_1, \ldots, U_k$ such that $|S \cap U_i| = 1$ holds for all $S \in \mathcal{S}$, and again, all sets have the same weight, the coefficient of $w(S)$ in (1.5) can be decreased to $|S| - 1 = k - 1$ [27]. In [26], the authors show that the conjecture holds if one of the following three conditions is satisfied: all sets in $\mathcal{S}$ have the same cardinality, $\mathcal{S}$ is *intersecting*, meaning that each two sets in $\mathcal{S}$ have a non-empty intersection, or w is constant (i.e., each set receives the same weight). In particular, for instances $(\mathcal{S}, w)$ of the weighted k-Set Packing problem, the integrality gap of (Set-Packing-LP$(\mathcal{S}, w)$) is bounded by $k - 1 + \frac{1}{k}$. For projective planes, this bound is tight, as noted in [26]. For general set families and weight functions, the Füredi-Kahn-Seymour-Conjecture remains open and has been subject to recent research [3, 6].

For set families in which all sets have the same cardinality k, Chan and Lau [14] provide an algorithmic proof of (1.5), obtaining an LP-relative $(k - 1 + \frac{1}{k})$-approximation for the weighted k-Set Packing problem. For k-partite instances, they even obtain an LP-relative $(k - 1)$-approximation, which, for $k = 3$, improves upon Berman's approximation guarantee of $2 + \epsilon$ (with respect to the value of an optimum solution). Parekh and Pritchard managed to generalize the aforementioned results from [14] to the b-matching problem on k-uniform, respectively k-uniform and bipartite hypergraphs [43].

Chan and Lau [14] further investigate how to strengthen the standard LP relaxation by adding further constraints, and study the integrality gaps of the resulting formulations. For a uniform set size of $k = 3$ and unit weights, they prove that the Fano LP has an integrality gap of 2. In addition, for $k \geq 3$, Chan and Lau construct families $\mathcal{S}$ of sets of cardinality k, in which the integrality gap of (Set-Packing-LP($\mathcal{S}, w$)) remains at least $k - 2$ for $\Omega\left(\frac{|U(\mathcal{S})|}{k^3}\right)$ rounds of Sherali-Adams relaxations, even for unit weights. On the other hand, they show that for constant k, there exists a polynomial-sized LP relaxation for the unweighted k-Set Packing problem with integrality gap at most $\frac{k+1}{2}$. Finally, they provide a polynomial-sized semidefinite programming relaxation of the unweighted Set Packing problem with integrality gap at most $\frac{\max_{S \in \mathcal{S}} |S|+1}{2}$.

For the unweighted k-Set Packing problem, Singh and Talwar [46] have shown that $\mathcal{O}(k^2)$ rounds of the Chvátal-Gomory-truncation suffice to reduce the integrality gap of (Set-Packing-LP($\mathcal{S}, w$)) to $\frac{k+1}{2}$.

1.3.2 The Maximum Weight Independent Set problem in $(k+1)$-claw free graphs

Several approximation algorithms for the (weighted) k-Set Packing problem actually tackle a more general task, the *Maximum Weight Independent Set problem (MWIS) in* $(k+1)$*-claw free graphs*. An undirected graph $G = (V, E)$ is $(k+1)$*-claw free* if for any vertex $v \in V$ and any independent set $I \subseteq V$, v has at most k neighbors in I. There exists an approximation-preserving reduction from the weighted k-Set Packing problem to the MWIS in $(k+1)$-claw free graphs. See Chapter 2 for further details. We refer to the unit weight version of the MWIS as MIS.

The approximation algorithms from [4, 7, 15, 33, 39, 40, 49] and Chapter 3 generalize to the M(W)IS in $(k+1)$-claw free graphs (in the sense that algorithms for the unweighted k-Set Packing problem generalize to algorithms for the MIS, and the algorithms for general weights can be interpreted in terms of the MWIS). In addition, the arguments in Chapter 6 yield an analogous result for the MWIS in $(k+1)$-claw free graphs.

The algorithms in [16, 17, 28, 31, 48] and the ones presented in Chapters 4 and 5, as well as their analyses, generalize to the MWIS in $(k+1)$-claw free graphs in a straightforward way. In particular, the generalized analyses yield the same approximation ratios as for the weighted k-Set Packing problem. However, for the MWIS in $(k+1)$-claw free graphs, only a quasi-polynomial running time can be guaranteed. (Note that the algorithms in [17] and [31] even require a quasi-polynomial running time for the weighted k-Set Packing problem.)

While the best lower bound on the approximability of the weighted k-Set Packing problem is $\Omega(\frac{k}{\log k})$ [32], stronger inapproximability results have been recently obtained for the MWIS in $(k+1)$-claw free graphs: Unless NP = BPP, there is no polynomial-time $\left(\frac{k}{3+2\sqrt{2}} - \Omega(k)\right)$-approximation for the MWIS in $(k+1)$-claw free graphs [37]. When additionally assuming the Unique Games Conjecture, the inapproximability can be strengthened to $\frac{k}{4} - \Omega(k)$ [37].

Another very recent paper [13] provides lower bounds concerning the integrality gap of the clique constrained stable set polytope after an (almost) linear (w.r.t. the number of vertices) number of rounds of the Sherali-Adams hierarchy. These lower bounds are stronger than the ones proven in [14] for the special case of weighted k-Set Packing.

1.4 Organization of the thesis

The remainder of this thesis is organized as follows: In Chapter 2, we first shed some more light on the relation between the weighted k-Set Packing problem and the MWIS in $(k+1)$-claw free graphs. Then, we discuss Berman's algorithm SquareImp and its analysis, which constitutes the starting point for our improvements. In particular, we investigate the structural properties of instances where Berman's analysis is tight. These will motivate the route we take towards better approximation guarantees in the following chapters.

In Chapter 3, we present a polynomial-time $\left(\frac{k+1}{2} - \frac{1}{1000}\right)$-approximation algorithm for the MWIS in $(k+1)$-claw free graphs. The analysis of the algorithm is comparably simple and allows us to introduce one of the main tools used in the analyses of our more involved algorithms, which are presented in the following two chapters.

Chapter 4 investigates the power of local improvements of logarithmically bounded size for weighted k-Set Packing. On the one hand, we present a polynomial-time $\frac{k+1-\lambda_k}{2}$-approximation algorithm, where $\lim_{k\to\infty} \lambda_k = 1$. On the other hand, we show that this result is asymptotically best possible: We establish a lower bound of $\frac{k}{2}$ on the approximation guarantees of local search algorithms that only consider local improvements of logarithmically bounded size (w.r.t. a fixed additive local search objective).

In Chapter 5, we explain how to breach the $\frac{k}{2}$-barrier (at least for large enough values of k) by employing a black box algorithm for the unweighted k-Set Packing problem to generate local improvements. Building upon this approach, we manage to more generally link the approximation guarantees achievable for the weighted k-Set Packing problem to those that can be obtained for unit weights in Chapter 6.

In Chapter 7, we shift our focus to a more concrete problem again and study a special case of weighted 3-Set Packing, the hereditary 2-3-Set Packing problem. We provide a simple $\frac{4}{3}$-approximation algorithm for this problem. Using the connection to the Maximum Leaf Spanning Arborescence problem in acyclic digraphs established by Fernandes and Lintzmayer [25], this immediately yields a $\frac{4}{3}$-approximation for the latter problem.

Finally, we conclude the thesis in Chapter 8.

Chapter 2

Preliminaries

In this chapter, we formally introduce the weighted k-Set Packing problem and the Maximum Weight Independent Set problem (MWIS) in $(k+1)$-claw free graphs in Section 2.1. We further establish that the MWIS in $(k+1)$-claw free graphs constitutes a (strict) generalization of the weighted k-Set Packing problem.

In Section 2.2, we then present Berman's algorithm SquareImp, which yields a $\frac{k+1}{2}$-approximation for the MWIS in $(k+1)$-claw free graphs [7]. Prior to our work [40], Berman's result has been the state-of-the-art for both the MWIS in $(k+1)$-claw free graphs and the weighted k-Set Packing problem. Thus, it constitutes a natural starting point for our improvements.

In Section 2.3, we show how to guarantee that SquareImp runs in polynomial time. Following [15], we outline how to scale and truncate the weight function in order to obtain integral weights and a polynomial bound on the weight of an optimum solution, while incurring only an arbitrarily small loss in the approximation guarantee. We will re-employ this result several times in the running time analyses of our algorithms.

In Section 2.4, we bound the approximation ratio that SquareImp attains by $\frac{k+1}{2}$. The analysis is based on [7] and [40] and phrased in a way that allows us to reuse certain results when bounding the approximation guarantees achieved by our (improved) algorithms.

We conclude this chapter in Section 2.5 by discussing an example instance provided by Berman [7] proving his analysis to be best possible. We observe that this instance features unit weights and argue that in fact, all instances where SquareImp does not perform better than a $\frac{k+1}{2}$-approximation are unweighted (in a certain sense) and highly structured. In particular, all of them admit local improvements of size 3. This observation will inspire the algorithm presented in Chapter 3.

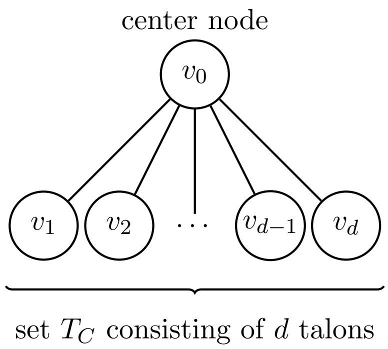


Figure 2.1: Illustration of a d-claw C.

2.1 The weighted k-Set Packing problem and the Maximum Weight Independent Set problem in $(k+1)$-claw free graphs

In this section, we lay out the well-known connection between the weighted k-Set Packing problem and the MWIS in $(k+1)$-claw free graphs. We first provide formal definitions of the two problems. The weighted k-Set Packing problem is defined as follows:

Definition 2.1 (weighted k-Set Packing problem)**.**

Input: a collection $\mathcal{S}$ of non-empty sets, each of cardinality at most k, positive set weights $w\colon \mathcal{S} \to \mathbb{R}_{>0}$

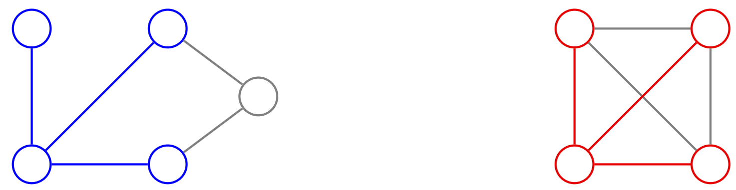
Task: Compute a *disjoint subcollection* $A \subseteq \mathcal{S}$ (meaning that we require the sets in A to be pairwise disjoint) of maximum total weight.

Next, we introduce the notions of an independent set and a d-claw free graph.

Definition 2.2 (independent set)**.** Let $G = (V, E)$ be a graph. A vertex set $I \subseteq V$ is called *independent* if the vertices in I are pairwise non-adjacent.

Definition 2.3 (d-claw)**.** Let $d \in \mathbb{Z}_{\geq 1}$. A *$d$-claw* is a star on $d+1$ vertices, i.e., a graph of the form $C = (\{v_0, v_1, \ldots, v_d\}, \{\{v_0, v_i\}, i = 1, \ldots, d\})$. We call v_0 the *center node* and $v_1, \ldots, v_d$ the *talons* of C, and we denote the set of talons by T_C. See Fig. 2.1 for an illustration.

We remark that for $d = 1$, either of the two vertices can be regarded as the center or the unique talon, respectively. In the following, it will always be clear from the context which vertex is to be considered the center node.



(a) The blue edges and vertices form a 3-claw in the displayed graph.

(b) The red edges and vertices do *not* form a 3-claw in the displayed graph. Even more, the graph is 3-claw free.

Figure 2.2: Illustration of Definition 2.4.

Definition 2.4 (d-claw free graph)**.** Let $d \in \mathbb{Z}_{\geq 1}$. We call an induced subgraph C of G that constitutes a d-claw a *d-claw in G*. We say that a graph G is *d-claw free* if there is no d-claw in G. See Fig. 2.2 for an illustration.

A graph is 1-claw free if and only if it does not contain any edge. A 2-claw free graph is a disjoint union of complete graphs. We remark that the notion of d-claw freeness from Definition 2.4 agrees with the definition that we gave in the introduction.

Proposition 2.5. Let $d \in \mathbb{Z}_{\geq 1}$. A graph G is d-claw free if and only if for any vertex $v \in V(G)$ and any independent set $I \subseteq V(G)$, v has at most $d-1$ neighbors in I.

Proof. Assume that G is d-claw free, let $v \in V(G)$ and let T be the set of neighbors of v in an independent set I. If T is non-empty, then $C := G[\{v\} \cup T]$ constitutes a $|T|$-claw in G. In particular, $|T| \leq d-1$ because otherwise, C would contain an induced subgraph that forms a d-claw, a contradiction. Now, assume that G is not d-claw free. Then G contains a d-claw C, and its center node has d neighbors in the independent set T_C formed by its talons. $\square$

Definition 2.6 (Maximum Weight Independent Set problem (MWIS))**.**

Input: a graph $G = (V, E)$, $w \colon V \to \mathbb{R}_{>0}$

Task: Find an independent set $A \subseteq V$ such that $w(A)$ is maximum.

The special case where $w \equiv 1$ is called the *Maximum Cardinality Independent Set problem (MIS)*. The *M(W)IS in* $(k+1)$*-claw free graphs* is the restriction of the M(W)IS to instances where the input graph G is $(k+1)$-claw free.

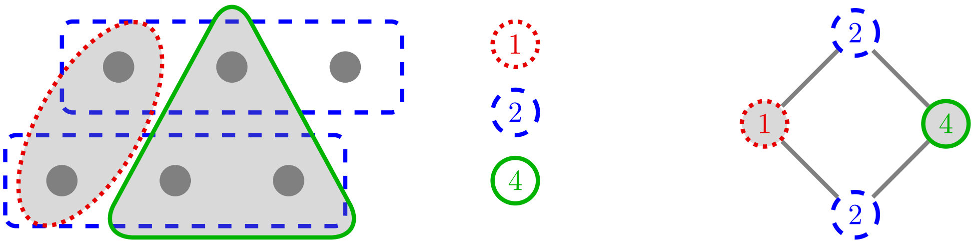


Figure 2.3: An instance of weighted 3-Set Packing (left) and the corresponding conflict graph (right), equipped with the respective vertex weights. An optimum solution is highlighted. Colors and patterns indicate the weights.

We remark that for both the weighted (or unweighted) k-Set Packing problem and the M(W)IS in $(k+1)$-claw free graphs, the positive integer k is part of the problem definition and not of the input. Thus, when discussing polynomial-time (approximation) algorithms, we will consider k to be a fixed constant. In particular, a running time of $\mathcal{O}(n^k)$, where n denotes the number of sets/vertices will be regarded as polynomial.

To reduce the weighted k-Set Packing problem to the MWIS in $(k+1)$-claw free graphs, we introduce the notion of the *conflict graph* [7, 15].

Definition 2.7 (conflict graph)**.** Let $\mathcal{S}$ be a family of sets. The *conflict graph* $G_{\mathcal{S}}$ is defined as follows:

- The vertices of $G_{\mathcal{S}}$ correspond to the sets in $\mathcal{S}$, i.e., $V(G_{\mathcal{S}}) = \mathcal{S}$.
- The edges of $G_{\mathcal{S}}$ model non-empty set intersections, i.e., $E(G_{\mathcal{S}}) = \{\{s_1, s_2\} : s_1, s_2 \in \mathcal{S}, s_1 \neq s_2, s_1 \cap s_2 \neq \emptyset\}$.

See Fig. 2.3 for an illustration.

By definition of the conflict graph, $A \subseteq \mathcal{S}$ consists of pairwise disjoint sets if and only if A constitutes an independent set in $G_{\mathcal{S}}$. This yields an approximation factor-preserving polynomial-time reduction from the (weighted) k-Set Packing problem to the M(W)IS. On its own, this observation is, however, not particularly helpful since for every constant $\varepsilon > 0$, it is known to be NP-hard to even approximate the Maximum Cardinality Independent Set problem within a factor of $n^{1-\varepsilon}$, where n denotes the number of vertices [50]. Fortunately, it turns out that the conflict graphs of instances of the weighted k-Set Packing problem are $(k+1)$-claw free, which provides additional structure that we can exploit towards (much) stronger approximation guarantees than for the general MWIS.

Proposition 2.8 (see, e.g., [15])**.** Let $k \in \mathbb{Z}_{\geq 1}$ and let $\mathcal{S}$ be a family of sets, each of cardinality at most k. Then $G_{\mathcal{S}}$ is $(k+1)$-claw free.

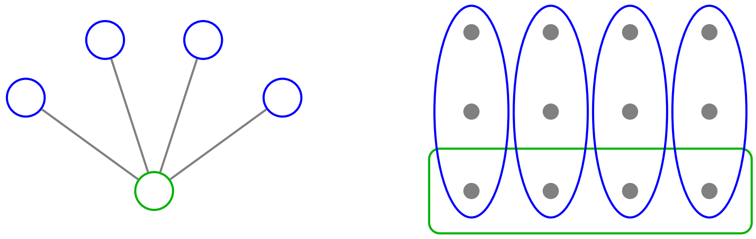


Figure 2.4: If there is a $(k+1)$-claw in $G_{\mathcal{S}}$, then the set corresponding to the center has to intersect the $k+1$ pairwise disjoint sets corresponding to the talons.

Proof. Assume towards a contradiction that there were a $(k+1)$-claw C in $G_{\mathcal{S}}$. Then the $k+1$ sets corresponding to the talons must be pairwise disjoint, and each of them has to intersect the set that constitutes the center of the claw. But this contradicts the fact that each set contains at most k elements. See Fig. 2.4 for an illustration. □

We can even show the following, slightly stronger statement:

Proposition 2.9. Let $k \in \mathbb{Z}_{\geq 1}$ and let $\mathcal{S}$ be a family of sets, each of cardinality at most k. Let $S \in \mathcal{S}$ and let U be the neighborhood of S in $G_{\mathcal{S}}$, i.e., the collection of sets in $\mathcal{S} \setminus \{S\}$ that have a non-empty intersection with S. Then there exist $s := |S|$ set collections $U_1, \ldots, U_s$ such that U_i is a (possibly empty) clique in $G_{\mathcal{S}}$ for every $i = 1, \ldots, s$, and $U = \bigcup_{i=1}^{s} U_i$.

Proof. Let $S = \{a_1, \ldots, a_s\}$ and define $U_i := \{S' \in U : a_i \in S'\}$. As every set in U intersects S, we have $U = \bigcup_{i=1}^{s} U_i$. Moreover, U_i constitutes a clique in $G_{\mathcal{S}}$ for every $i = 1, \ldots, s$ since all sets in U_i intersect in a_i. □

We remark that for $k \geq 2$, not every $(k+1)$-claw free graph arises as the conflict graph of a family of sets that each have a cardinality of at most k.

Proposition 2.10. Let $k \geq 2$. There exists a $(k+1)$-claw free graph G with the following property: There is no family $\mathcal{S}$ consisting of sets of cardinality at most k such that $G_{\mathcal{S}}$ is isomorphic to G.

Proof. Let $k \geq 2$ and consider a graph G that consists of a cycle C of length $2k+1$ and a vertex v^* connected to each vertex of the cycle (see Fig. 2.5). It is not hard to see that G is $(k+1)$-claw free: First of all, there is no $(k+1)$-claw in G centered at v^* because the maximum cardinality of an independent set in C is k. Moreover, every vertex of C has $3 \leq k+1$ neighbors in G, and one of them, v^*, is adjacent to both other neighbors.

However, we cannot cover the neighborhood of v^* by at most k cliques. The maximum size of a clique in C is 2, and thus, at most k cliques do not

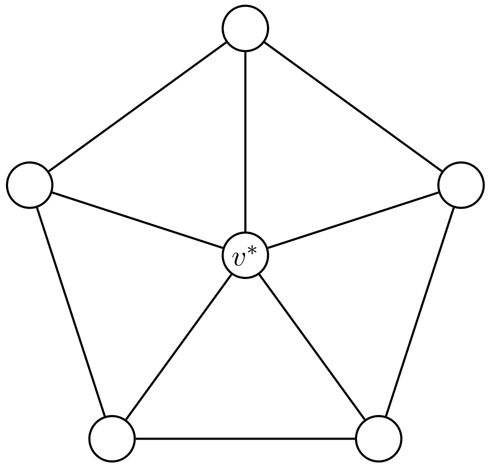


Figure 2.5: A 3-claw free graph that does not arise as the conflict graph of a set family consisting of sets of cardinality at most 2.

suffice to cover the $2k+1$ vertices of C. Thus, Proposition 2.9 tells us that G is not isomorphic to the conflict graph of any set family consisting of sets of cardinality at most k. $\square$

Hence, the M(W)IS in $(k+1)$-claw free graphs constitutes a strict generalization of the (weighted) k-Set Packing problem.

2.2 Berman's algorithm SquareImp

In this section, we discuss Berman's algorithm *SquareImp* [7], which has been the state-of-the-art for both the weighted k-Set Packing problem and the MWIS in $(k+1)$-claw free graphs for twenty years, and constitutes the starting point for our improvements. SquareImp is a local search algorithm that starts with the empty solution and iteratively applies local improvements w.r.t. the *squared weight function* from a certain class until no more exist. We call the type of local improvement that SquareImp considers *claw-shaped*. In order to formally define the notion of a claw-shaped improvement, we require the following notation, which is based on [7].

Notation 2.11. Let $U \subseteq V$ be sets and let $w\colon V \to \mathbb{R}_{>0}$. We write $w^2(U) \coloneqq \sum_{u\in U} w^2(u)$.

In particular, we have $w^2(U) \neq (w(U))^2$ in general.

Definition 2.12 (neighborhood [7])**.** Let $G = (V, E)$ be a graph and let $U, W \subseteq V$. We call $N(U,W) \coloneqq \{w \in W : \exists u \in U : u = w \vee \{u,w\} \in E\}$ the *neighborhood of U in W*. For $v \in V$, we write $N(v,W) \coloneqq N(\{v\},W)$.

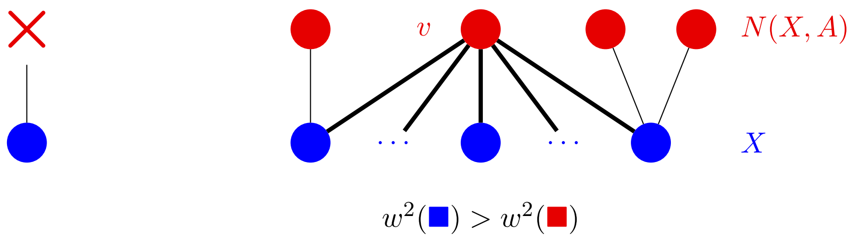


Figure 2.6: A claw-shaped improvement consists of a single vertex without any neighbor in A, or of the set of talons of a claw centered at a vertex in A.

Algorithm 1: Berman's algorithm SquareImp [7]

Input: a $(k+1)$-claw free graph $G = (V, E)$, $w\colon V \to \mathbb{R}_{>0}$
Output: an independent set $A \subseteq V$

$A \leftarrow \emptyset$
while *there exists a claw-shaped improvement* X *of* A **do**
  $A \leftarrow (A \setminus N(X, A)) \cup X$
end
return A

Note that Definition 2.12 differs from the more standard definition of the neighborhood in that we include all vertices from U that are contained in W, treating them as "adjacent to themselves".

Definition 2.13 (claw-shaped improvement)**.** Let $G = (V, E)$ be a graph, $w\colon V \to \mathbb{R}_{>0}$ and let A and X be independent sets in G. We say that X is a *local improvement* of A (w.r.t. w^2) if $w^2(X) > w^2(N(X, A))$, and we call $|X|$ the *size* of the local improvement X.

We further call a local improvement X *claw-shaped* if $|X| = 1$ and $N(X, A) = \emptyset$ or if there is $v \in A$ such that $\{v\} \cup X$ induces a $|X|$-claw in G centered at v. We say that *no claw improves* A to state that there is no claw-shaped improvement of A. See Fig. 2.6 for an illustration.

Observe that if G is $(k+1)$-claw free, then a claw-shaped improvement is of size at most k. Using Definition 2.13, we can formulate SquareImp as shown in Algorithm 1. One iteration of SquareImp can be implemented to run in time $\mathcal{O}(|V|^k \cdot (|V| + |E|))$ by enumerating all subsets of V of cardinality at most k and, for each such subset X, checking in linear time whether it constitutes a claw-shaped improvement. As $w^2(A)$ strictly increases in each iteration, it is also clear that SquareImp terminates. However, for general weights, it is *not* clear how to obtain a polynomial bound on the number of iterations that SquareImp performs. This issue is resolved by pre-processing the instance in such a way that the weights become integral and the weight of an optimum solution can be polynomially bounded, while incurring only an arbitrarily small increase in the approximation ratio [7, 15]. We provide the details in Section 2.3.

2.3 A polynomial running time

In this section, we explain how to achieve a polynomial running time at the cost of an arbitrarily small loss in the approximation guarantee. To this end, following [7], we first observe that in case the weight function w is integral and $\mathrm{OPT}(G, w)$, the maximum weight of an independent set in (G, w), is polynomially bounded in the number of vertices, then the number of iterations of SquareImp is polynomially bounded. Indeed, if the weights are integral, then $w^2(A)$ increases by at least 1 in each iteration. As

$$0 \leq w^2(A) \leq (w(A))^2 \leq (\mathrm{OPT}(G, w))^2,$$

the number of iterations is at most $(\mathrm{OPT}(G, w))^2$, which is polynomially bounded. We have already discussed in the previous section how to perform one iteration of SquareImp in polynomial time.

Hence, it remains to pre-process the instance in an appropriate way to obtain integral weights and a polynomial bound on the maximum weight of an independent set, while only incurring a small error in terms of the approximation guarantee. This is taken care of by the following lemma:

Lemma 2.14. Let $N \in \mathbb{Z}_{\geq 2}$ and let $(G = (V, E), w)$ be an instance of the MWIS in $(k + 1)$-claw free graphs. Then we can, in polynomial time, compute $U \subseteq V$ and a weight function $w' \colon U \to \mathbb{Z}_{>0}$ with the following properties:

(i) Let $\rho \geq 1$ and let $A \subseteq U$ be a ρ-approximate solution to the MWIS in $(G[U], w')$. Then A constitutes an $\frac{N}{N-1} \cdot \rho$-approximate solution to the MWIS in (G, w).

(ii) $\mathrm{OPT}(G[U], w') \leq k \cdot N \cdot |V|$.

We remark that as an induced subgraph of G, $G[U]$ is $(k + 1)$-claw free again, and in particular, we can apply SquareImp to $(G[U], w')$. Moreover, if $G = G_{\mathcal{S}}$ is the conflict graph of a set family $\mathcal{S}$, then $G[U] = G_U$ is the conflict graph of the subfamily $U \subseteq V(G) = \mathcal{S}$. The proof of Lemma 2.14 is based on [7] and [15].

Proof of Lemma 2.14. If $V = \emptyset$, there is nothing to show, so assume that $V \neq \emptyset$. We obtain U and w' as follows:

1. Apply the greedy algorithm to compute a k-approximate solution A_0 to the MWIS in (G, w) (see, e.g., [15]). Note that $w(A_0) > 0$ since all weights are strictly positive and $V \neq \emptyset$.
2. Define a scaled weight function w_s by $w_s(v) := \frac{N \cdot |V|}{w(A_0)} \cdot w(v)$ for all $v \in V$.
3. Let $U := \{v \in V : \lfloor w_s(v) \rfloor > 0\}$ and set $w'(u) := \lfloor w_s(u) \rfloor$ for all $u \in U$.

To prove (i), let B be an optimum solution to (G, w). Then $B \cap U$ is a feasible solution to $(G[U], w')$ of value at least

$$w'(B \cap U) = \lfloor w_s \rfloor(B) \geq w_s(B) - |V| = \frac{N \cdot |V|}{w(A_0)} \cdot \left(w(B) - \frac{w(A_0)}{N} \right)$$
$$\geq \frac{N \cdot |V|}{w(A_0)} \cdot \frac{N-1}{N} \cdot w(B).$$

Let further A be a ρ-approximate solution to $(G[U], w')$. Then

$$\rho \cdot w(A) = \rho \cdot \frac{w(A_0)}{N \cdot |V|} \cdot w_s(A) \geq \frac{w(A_0)}{N \cdot |V|} \cdot \rho \cdot w'(A)$$
$$\geq \frac{w(A_0)}{N \cdot |V|} \cdot w'(B \cap U) \geq \frac{N-1}{N} \cdot w(B).$$

Hence, A constitutes a $\frac{N}{N-1} \cdot \rho$-approximate solution to the MWIS in (G, w).

For (ii), let B' be an optimum solution to $(G[U], w')$. Then

$$w'(B') \leq w_s(B') = N \cdot |V| \cdot \frac{w(B')}{w(A_0)} \leq k \cdot N \cdot |V|$$

by our choice of A_0. This concludes the proof. $\square$

2.4 Analysis of SquareImp

In this section, we show that SquareImp achieves an approximation guarantee of $\frac{k+1}{2}$ for the MWIS in $(k+1)$-claw free graphs. The analysis we present is based on [7], but rephrased in a way that allows us to reuse certain results in the following chapters.

Theorem 2.15 ([7])**.** Let $k \in \mathbb{Z}_{\geq 1}$, let (G, w) be an instance of the MWIS in $(k+1)$-claw free graphs, let B be an independent set in G and let A be an independent set in G with the property that no claw improves A. Then $w(B) \leq \frac{k+1}{2} \cdot w(A)$.

Applying Theorem 2.15 in the situation where A denotes the solution returned by SquareImp and B is an optimum solution yields the desired approximation guarantee.

For the remainder of this section, fix $k \in \mathbb{Z}_{\geq 1}$ and an instance (G, w) of the MWIS in $(k+1)$-claw free graphs. In addition, let B be an optimum solution to (G, w) and let A be a feasible solution to (G, w) such that no claw improves A.

The main idea of Berman's analysis [7] is to charge the vertices in A for preventing adjacent vertices in B from being included into A. (Observe that by positivity of the weight function, A is an inclusion-wise maximal independent set.) More precisely, Berman shows how to spread the weight of the vertices in B among their neighbors in A in such a way that no vertex in A receives more than $\frac{k+1}{2}$ times its own weight. The suggested weight distribution proceeds in two steps:

- First, each vertex $u \in B$ invokes costs of $\frac{w(v)}{2}$ at each $v \in N(u, A)$.
- In the second step, each vertex $u \in B$ sends the remaining amount of $w(u) - \frac{1}{2} \cdot w(N(u, A))$ to a heaviest neighbor it possesses in A, which is captured by the following definition of *charges*.

Definition 2.16 (charges [7])**.** For each $u \in B$, pick a vertex $v \in N(u, A)$ of maximum weight and call it $n(u)$. Define a map charge : $B \times A \to \mathbb{R}$ via

$$\text{charge}(u, v) := \begin{cases} w(u) - \frac{1}{2} \cdot w(N(u, A)) & , \text{ if } v = n(u) \\ 0 & , \text{ otherwise} \end{cases}.$$

The suggested weight distribution provides the following bound on $w(B)$.

$$w(B) \leq \sum_{v \in A} \left(\underbrace{\frac{|N(v, B)|}{2} \cdot w(v)}_{\text{paid in step 1}} + \underbrace{\sum_{u \in B : v = n(u)} \text{charge}(u, v)}_{\text{paid in step 2}} \right) \tag{2.1}$$

To bound the amount a vertex $v \in A$ has to pay in the first step, we employ Proposition 2.17.

Proposition 2.17 ([7])**.** Let $k \in \mathbb{Z}_{\geq 1}$, let $G = (V, E)$ be $(k+1)$-claw free, let $v \in V$ and let $Z \subseteq V$ be independent. Then $|N(v, Z)| \leq k$.

Proof. If $v \in Z$, then $N(v, Z) = \{v\}$ since Z is independent. Otherwise, Proposition 2.5 yields the desired statement. $\square$

In particular, the total amount a vertex $v \in A$ has to pay in the first step is bounded by $\frac{k}{2} \cdot w(v)$. Together with (2.1), this results in the following lemma, which we explicitly state here for better reusability.

Lemma 2.18 ([7])**.** $w(B) \leq \frac{k}{2} \cdot w(A) + \sum_{u \in B} \text{charge}(u, n(u))$ $\square$

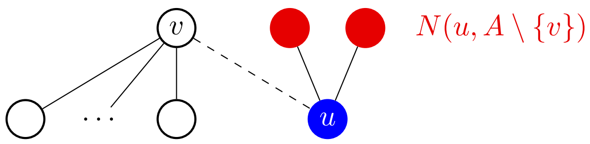


Figure 2.7: Motivation of Definition 2.19.

In order to bound the total charges a vertex from A has to pay in the second step, we need to exploit the fact that when the algorithm terminates, there is no claw-shaped improvement. To this end, let $v \in A$. Our goal is to construct an improving claw C centered at v with $T_C \subseteq B$. Consider adding a vertex $u \in N(v, B)$ to the set of talons T_C. On the bright side, this of course increases $w^2(T_C)$ by $w^2(u)$. On the negative side, $w^2(N(T_C, A))$ may also increase by up to $w^2(N(u, A) \setminus \{v\})$. (If we want our claw to be centered at v, we will have to pay $w^2(v)$ anyway.) Thus, if $w^2(u) > w^2(N(u, A) \setminus \{v\})$, we certainly want to add u to T_C, whereas in case $w^2(u) \leq w^2(N(u, A) \setminus \{v\})$, we may choose not to. This is captured by the notion of the *contribution* (Definition 2.19). It measures how much adding (or not adding) a vertex u to the set of talons of a claw centered at v contributes (at least) towards making that claw improving. See Fig. 2.7 for an illustration.

Definition 2.19. For $u \in V$ and $v \in A$, define

$$\operatorname{contr}(u, v) := \begin{cases} \max\left\{0, \frac{w^2(u) - w^2(N(u,A)\setminus\{v\})}{w(v)}\right\} & , \text{ if } v \in N(u, A) \\ 0 & , \text{ else} \end{cases}.$$

The main reason that we divide by $w(v)$ is to normalize the contribution to make it comparable to the notion of charges (cf. Corollary 2.22).

The fact that there is no claw-shaped improvement yields an upper bound on the total contribution to a vertex $v \in A$ (see Proposition 2.20). Corollary 2.22 further bounds the charges a vertex $v \in A$ has to pay to a vertex $u \in B$ in terms of the contribution of u to v. This allows us to bound the total charges v has to pay in the second step of the weight distribution.

Proposition 2.20 ([7])**.** For each $v \in A$, we have $\sum_{u \in B} \operatorname{contr}(u, v) \leq w(v)$.

Proof. If $v \in B$, the statement is true because $N(v, B) = N(v, A) = \{v\}$ and $\operatorname{contr}(v, v) = w(v)$ in this case. So let $v \in A \setminus B$ and assume towards a contradiction that

$$\sum_{u \in B} \operatorname{contr}(u, v) > w(v).$$

Define $T := \{u \in N(v, B) : \operatorname{contr}(u, v) > 0\}$. Then for $u \in T$, we have

$$\operatorname{contr}(u, v) = \frac{w^2(u) - w^2(N(u, A) \setminus \{v\})}{w(v)}.$$

By non-negativity of the contribution, this yields

$$\sum_{u \in T} \frac{w^2(u) - w^2(N(u,A) \setminus \{v\})}{w(v)} = \sum_{u \in T} \operatorname{contr}(u,v) = \sum_{u \in B} \operatorname{contr}(u,v) > w(v).$$

Multiplying by $w(v) > 0$ allows us to conclude that

$$w^2(T) = \sum_{u \in T} w^2(u) > w^2(v) + \sum_{u \in T} w^2(N(u,A) \setminus \{v\}) \geq w^2(N(T,A)).$$

As B is independent, T constitutes the set of talons of a claw centered at v, and, hence, a claw-shaped improvement. This contradicts our assumptions on A and concludes the proof. □

The following lemma provides a lower bound on the difference between contribution and charges that will be useful in the following chapter. Moreover, it will allow us to derive Corollary 2.22.

Lemma 2.21 ([7])**.** Let $u \in B$ and $v \in N(u,A)$. Then

$$\begin{aligned}&(\operatorname{contr}(u,v) - 2 \cdot \operatorname{charge}(u, n(u))) \cdot w(v)\\ &\geq (w(u) - w(v))^2 + \sum_{x \in N(u,A)} (w(v) - w(x)) \cdot w(x).\end{aligned}$$

Proof. Using $\operatorname{contr}(u,v) \cdot w(v) \geq w^2(u) - w^2(N(u,A) \setminus \{v\})$ by Definition 2.19 and $2 \cdot \operatorname{charge}(u, n(u)) = 2 \cdot w(u) - w(N(u,A))$ by Definition 2.16, we calculate

$$\begin{aligned}&(\operatorname{contr}(u,v) - 2 \cdot \operatorname{charge}(u, n(u))) \cdot w(v)\\ &\geq w^2(u) - w^2(N(u,A) \setminus \{v\}) - (2 \cdot w(u) - w(N(u,A))) \cdot w(v)\\ &= w^2(u) - 2 \cdot w(u) \cdot w(v) + w^2(v) + w(N(u,A)) \cdot w(v) - w^2(N(u,A))\\ &= (w(u) - w(v))^2 + \sum_{x \in N(u,A)} (w(v) - w(x)) \cdot w(x).\end{aligned}$$

□

Corollary 2.22 ([7])**.** Let $u \in B$ and $v \in A$. Then

$$2 \cdot \operatorname{charge}(u,v) \leq \operatorname{contr}(u,v).$$

Proof. If $v \neq n(u)$, then $2 \cdot \operatorname{charge}(u,v) = 0 \leq \operatorname{contr}(u,v)$. Thus, we may assume that $v = n(u)$. In particular,

$$v = n(u) \in \operatorname{argmax}\{w(x) : x \in N(u,A)\}. \tag{2.2}$$

By Lemma 2.21, this yields

$$\begin{aligned}&(\mathrm{contr}(u,v) - 2\cdot \mathrm{charge}(u,v))\cdot w(v)\\ &\geq (w(u)-w(v))^2 + \sum_{x\in N(u,A)} (w(v)-w(x))\cdot w(x) \geq 0\end{aligned} \tag{2.3}$$

since real squares are non-negative, $w(v) - w(x) \geq 0$ for all $x \in N(u, A)$ and $w(x) > 0$ for all $x \in N(u, A)$. □

Combining Proposition 2.17, Proposition 2.20 and Corollary 2.22 results in Theorem 2.23, which, in particular, implies Theorem 2.15. For better reusability, we explicitly restate our set of assumptions here.

Theorem 2.23 ([7])**.** Let $k \in \mathbb{Z}_{\geq 1}$ and let (G, w) be an instance of the MWIS in $(k+1)$-claw free graphs. Let further B be an independent set in G and let A be independent in G with the property that no claw improves A. Then

$$\begin{aligned}w(B) &\leq \frac{k+1}{2}\cdot w(A) - \frac{1}{2}\cdot\sum_{v\in A}(k - |N(v,B)|)\cdot w(v)\\ &\quad - \frac{1}{2}\cdot\sum_{u\in B}\left(\sum_{v\in A}\mathrm{contr}(u,v) - 2\cdot\mathrm{charge}(u,n(u))\right)\\ &\leq \frac{k+1}{2}\cdot w(A).\end{aligned}$$

Proof. We would like to apply (2.1). To this end, we rewrite

$$\sum_{v\in A}\frac{|N(v,B)|}{2}\cdot w(v) = \frac{k}{2}\cdot w(A) - \frac{1}{2}\cdot\sum_{v\in A}(k-|N(v,B)|)\cdot w(v). \tag{2.4}$$

In addition, using Proposition 2.20, we bound

$$\begin{aligned}&\sum_{v\in A}\sum_{u\in B: v=n(u)}\mathrm{charge}(u,v)\\ &= \frac{1}{2}\cdot\sum_{v\in A}\sum_{u\in B}\mathrm{contr}(u,v) - \frac{1}{2}\cdot\sum_{u\in B}\left(\sum_{v\in A}\mathrm{contr}(u,v) - 2\cdot\mathrm{charge}(u,n(u))\right)\\ &\leq \frac{w(A)}{2} - \frac{1}{2}\cdot\sum_{u\in B}\left(\sum_{v\in A}\mathrm{contr}(u,v) - 2\cdot\mathrm{charge}(u,n(u))\right).\end{aligned} \tag{2.5}$$

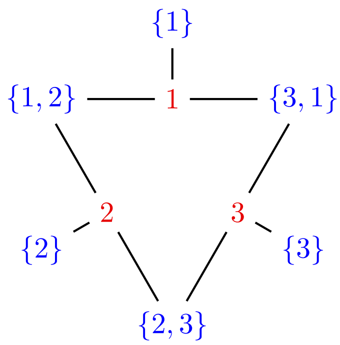


Figure 2.8: The tight example provided by Berman for $k = 3$.
$A = \{1, 2, 3\}$ is depicted in red, $B = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{3, 1\}\}$ is drawn in blue.

Plugging (2.4) and (2.5) into (2.1) results in

$$\begin{aligned} w(B) \leq \frac{k+1}{2} \cdot w(A) - \frac{1}{2} \cdot \sum_{v \in A} (k - |N(v, B)|) \cdot w(v) \\ - \frac{1}{2} \cdot \sum_{u \in B} \left(\sum_{v \in A} \mathrm{contr}(u, v) - 2 \cdot \mathrm{charge}(u, n(u)) \right) \\ \leq \frac{k+1}{2} \cdot w(A), \end{aligned} \tag{2.6}$$

where the last inequality follows from Proposition 2.17, Corollary 2.22 and the fact that the contribution is non-negative. □

2.5 A tight example

For every $k \geq 1$, Berman [7] provides an instance G of the MIS (i.e., all weights are equal to 1) in $(k+1)$-claw free graphs for which his analysis is tight. The vertex set of G can be partitioned into a locally optimum solution $A := \{1, \dots, k\}$ and an optimum solution $B := \binom{A}{1} \cup \binom{A}{2}$. Two vertices $v \in A$ and $u \in B$ share an edge if and only if $v \in u$. See Fig. 2.8 for an example. We remark that G is isomorphic to the conflict graph of the k-Set Packing instance that we obtain from $V(G)$ by replacing each vertex by its set of incident edges. Note that the degree of each vertex in G is bounded by k.

By construction, both A and B constitute independent sets in G, and moreover,

$$w(B) = |B| = \binom{k}{1} + \binom{k}{2} = \frac{k \cdot (k+1)}{2} = \frac{k+1}{2} \cdot |A| = \frac{k+1}{2} \cdot w(A).$$

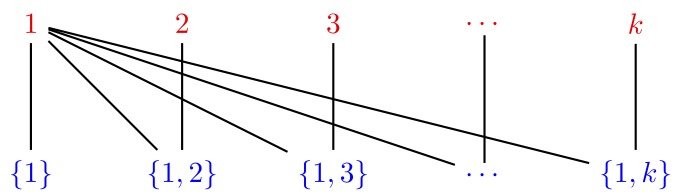


Figure 2.9: There is no improving claw centered at 1.

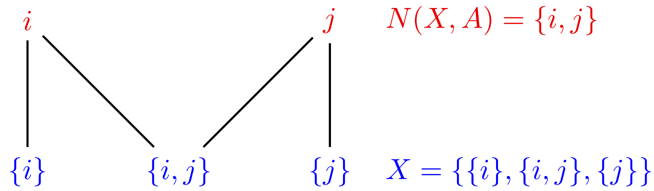


Figure 2.10: For $1 \leq i < j \leq k$, $X = \{\{i\}, \{i, j\}, \{j\}\}$ constitutes a local improvement of size 3.

We further observe that no claw improves A. First of all, A constitutes a maximal independent set. Due to the symmetry of the construction, it suffices to see that there is no claw centered at $1 \in A$ that improves A. This follows from the fact that for each $u \in N(1, B) = \{\{1\}, \{1, 2\}, \{1, 3\}, \ldots, \{1, k\}\}$, $(\max u) \in A$ is a neighbor of u and the vertices $(\max u), u \in N(1, B)$ are pairwise distinct. Thus, $w^2(N(T, A)) = |N(T, A)| \geq |T| = w^2(T)$ for every $T \subseteq N(1, B)$. See Fig. 2.9 for an illustration.

However, as Berman's instance features unit weights, the result by Hurkens and Schrijver [33] implies that for $k \geq 3$, local improvements of constant size must exist[1]. Indeed, these local improvements are not hard to find: In fact, for every $1 \leq i < j \leq k$, $\{\{i\}, \{i, j\}, \{j\}\}$ constitutes a local improvement of size 3 (see Fig. 2.10).

This is not a coincidence. In fact, we will see in the remainder of this section that *every* instance where Berman's analysis is tight bears the same local structure as the example we have just discussed, and, in particular, allows for a local improvement of size 3. Extending this result to instances where Berman's analysis is only *close to tight*, i.e., where $w(B) \geq \frac{k+1-\varepsilon}{2} \cdot w(A)$ for some small, but constant $\varepsilon > 0$ will yield our first improvement over Berman's result in the next chapter.

For the remainder of this section, fix again $k \in \mathbb{Z}_{\geq 1}$ and an instance (G, w) of the MWIS in $(k+1)$-claw free graphs. In addition, let B be an

[1] Recall that Hurkens and Schrijver have shown in [33] that in the unit weight case, local improvements of constant size are sufficient to obtain approximation guarantees arbitrarily close to $\frac{k}{2}$ for $k \geq 3$.

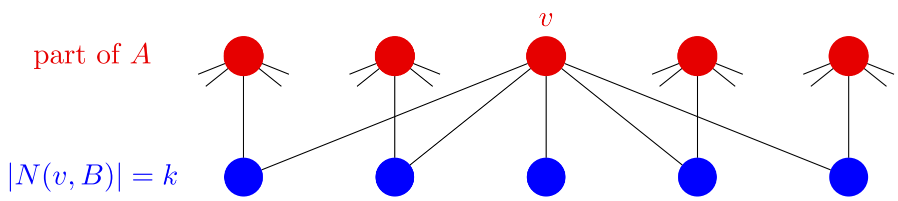


Figure 2.11: Local structure of tight instances.

optimum solution to (G, w) and let A be a feasible solution to (G, w) such that no claw improves A, and assume that $w(B) = \frac{k+1}{2} \cdot w(A)$.

We start by proving that every tight instance is (essentially) unweighted.

Lemma 2.24. Let $u \in B$. For every $v \in N(u, A)$, we have $w(u) = w(v)$. In particular, weights are equal in each connected component of $G[A \cup B]$.

Proof. If Theorem 2.23 is tight, then in particular, the inequality in (2.6) is tight, which in turn implies that Corollary 2.22 must be tight for u and $n(u)$. This tells us that

$$(w(u) - w(n(u)))^2 + \sum_{x \in N(u,A)} (w(n(u)) - w(x)) \cdot w(x) = 0$$

(cf. (2.3)). As all weights are positive and $n(u)$ is of maximum weight in $N(u, A)$, this allows us to conclude that both

$$(w(u) - w(n(u)))^2 = 0 \quad \text{and} \quad \sum_{x \in N(u,A)} (w(n(u)) - w(x)) \cdot w(x) = 0.$$

The first equation yields $w(u) = w(n(u))$. The second one, by positivity of weights, implies that $w(x) = w(n(u))$ for all $x \in N(u, A)$. $\square$

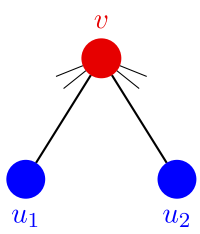
Note that the fact that no claw improves A implies that for every connected component C of $G[A \cup B]$, no claw in C improves $A \cap V(C)$. In particular,

$$w(B \cap V(C)) \leq \frac{k+1}{2} \cdot w(A \cap V(C)),$$

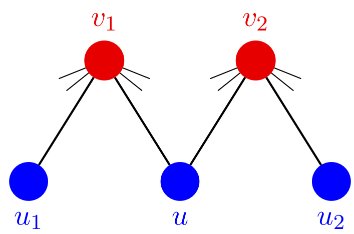
and the fact that $w(B) = \frac{k+1}{2} \cdot w(A)$ implies that we must have equality for every connected component. By Lemma 2.24, this tells us that $|B \cap V(C)| = \frac{k+1}{2} \cdot |A \cap V(C)|$ for every connected component C of $G[A \cup B]$.

Lemma 2.25. Let $v \in A$. Then $N(v, B) = \{u_0, u_1, \ldots, u_{k-1}\}$ with $N(u_0, A) = \{v\}$ and $|N(u_i, A)| = 2$ for $i = 1, \ldots, k-1$.

See Fig. 2.11 for an illustration.



(a) No $v \in A$ can have two neighbors in B_1.



(b) Tight instances feature local improvements of size 3.

Figure 2.12: Forbidden situations and local improvements in tight instances.

Proof. We can assume without loss of generality that $G[A \cup B]$ is connected; if not, we consider one connected component instead. As no claw improves A, we know that $N(u, A) \neq \emptyset$ for every $u \in B$. Partition B into the sets B_1, B_2 and $B_{\geq 3}$ consisting of all vertices $u \in B$ with $|N(u, A)| = 1$, $|N(u, A)| = 2$ and $|N(u, A)| \geq 3$, respectively. By Proposition 2.17, we know that

$$k \cdot |A| \geq |B_1| + 2 \cdot |B_2| + 3 \cdot |B_{\geq 3}|, \tag{2.7}$$

and if this is tight, then $|N(v, B)| = k$ for every $v \in A$. Moreover, for every $v \in A$, there can be at most one $u \in B$ with $N(u, A) = \{v\}$ because if there were two such vertices u_1 and u_2, then $w(u_1) = w(u_2) = w(v)$ by Lemma 2.24 would imply that $\{u_1, u_2\}$ constitutes a claw-shaped improvement (see Fig. 2.12a). As a consequence,

$$|A| \geq |B_1| \tag{2.8}$$

and this is tight if and only if for every $v \in A$, there is exactly one $u \in B$ with $N(u, A) = \{v\}$. Adding (2.7) and (2.8) yields

$$(k+1) \cdot |A| \geq (k+1) \cdot |A| - |B_{\geq 3}| \geq 2 \cdot (|B_1| + |B_2| + |B_{\geq 3}|) = 2 \cdot |B|.$$

Hence, $\frac{k+1}{2} \cdot |A| = |B|$ implies that $B_{\geq 3} = \emptyset$ and that (2.7) and (2.8) are tight. By the previous considerations, this yields the statement of the lemma. □

Corollary 2.26. If $k \geq 2$ and $B \neq \emptyset$, there exists a local improvement $X \subseteq B$ of size 3.

Proof. $B \neq \emptyset$ implies $A \neq \emptyset$ by positivity of weights. Hence, let $v_1 \in A$. By Lemma 2.25, there is $u \in N(v_1, B)$ with $|N(u, A)| = 2$. Let $N(u, A) = \{v_1, v_2\}$. Again by Lemma 2.25, there are $u_1, u_2 \in B$ with $N(u_i, A) = \{v_i\}$ for $i = 1, 2$. By Lemma 2.24, $\{u_1, u, u_2\}$ constitutes a local improvement of size 3 (see Fig. 2.12b). □

Chapter 3

Simple improvement

In this chapter, we leverage our observations regarding the structure of tight instances for SquareImp (cf. Section 2.5) towards a first improvement over the approximation guarantee of $\frac{k+1}{2}$. We study the algorithm *SimpleImp* (Algorithm 2) that iteratively searches for claw-shaped local improvements and local improvements of size 3 until no more exist. As in the previous chapter, local improvements are defined with respect to the squared weight function (see Definition 2.13). Our main result for this chapter is Theorem 3.1, which tells us that SimpleImp yields a better-than-$\frac{k+1}{2}$-approximation for the MWIS in $(k+1)$-claw free graphs for $k \geq 3$. Note that for $k \leq 2$, there exists a polynomial-time exact algorithm, see [38].

Theorem 3.1. Let $k \in \mathbb{Z}_{\geq 3}$ and let (G, w) be an instance of the MWIS in $(k+1)$-claw free graphs. Let further B be an independent set in G and let A be an independent set with the property that no claw improves A and there is no local improvement of A of size 3 (w.r.t. w^2).

Then $w(B) \leq \left(\frac{k+1}{2} - 0.00123\right) \cdot w(A)$.

Recall that we have already seen in the previous chapter that we can search for a claw-shaped improvement in polynomial time. By simply iterating over $\binom{V}{3}$, we can further check whether there is a local improvement of size 3 in polynomial time. Together with Lemma 2.14, this yields the following corollary:

Corollary 3.2. Let $k \in \mathbb{Z}_{\geq 3}$. There exists a polynomial-time $\left(\frac{k+1}{2} - \frac{1}{1000}\right)$-approximation algorithm for the MWIS in $(k+1)$-claw free graphs. □

Let ε be a constant satisfying the following inequalities that will pop up

Algorithm 2: SimpleImp

Input: a $(k+1)$-claw free graph $G=(V,E)$, $w\colon V \to \mathbb{R}_{>0}$
Output: an independent set $A \subseteq V$

$A \leftarrow \emptyset$
while $\exists$ *local improvement* X *s.t.* X *is claw-shaped or* $|X| = 3$ **do**
  $A \leftarrow (A \setminus N(X,A)) \cup X$
end
return A

during our analysis:

$$0 < \varepsilon < \frac{1}{4} \tag{3.1}$$

$$\frac{\varepsilon^2}{(1+\varepsilon)\cdot(2+\varepsilon^2)} \le \frac{\varepsilon^2}{2} \le 1 - 3\varepsilon - \varepsilon^2 - \varepsilon^3 \tag{3.2}$$

$$2 + (3\varepsilon+\varepsilon^2+\varepsilon^3)^2 + 2\varepsilon^2 < 3\cdot(1-\varepsilon)^2 \tag{3.3}$$

$$1 + 2\varepsilon^2 < 2\cdot(1-\varepsilon)^2 \tag{3.4}$$

We remark that (3.1) and (3.3) imply (3.4).

Proposition 3.3. Let $\varepsilon \in (0, 0.138]$. Then ε satisfies (3.1)-(3.4).

Proof. The inequalities in (3.1) and the first inequality in (3.2) are clear. For all other inequalities, we observe that the right-hand side minus the left-hand side is a monotonically decreasing function in $\varepsilon \in (0,1)$, and for $\varepsilon = 0.138$, all inequalities hold. □

For the analysis of SimpleImp, we set the constant to $\varepsilon := 0.138$, but in the following two chapters, we will choose smaller values of ε. We use ε as a threshold to decide whether the neighborhood $N(v,B)$ of a vertex $v \in A$ closely aligns with the structure that we have derived for tight instances in Section 2.5 (see Fig. 2.11). If this is the case, we refer to v as a *regular vertex*, otherwise, we call v *irregular*. The precise definitions are introduced in Section 3.1. Next, in Section 3.2, we show that the irregular vertices must make up a large fraction of $w(A)$ because otherwise, we can find a local improvement of size 3, similar as in Fig. 2.12b. Finally, in Section 3.3, we discuss how irregular vertices keep Berman's analysis from being tight. More precisely, we observe that each irregular vertex improves the bound $w(B) \le \frac{k+1}{2}\cdot w(A)$ by a constant fraction of its weight. Combining this with the results from Section 3.2 allows us to derive Theorem 3.1.

For the remainder of this chapter, fix $k \in \mathbb{Z}_{\ge 3}$ and an instance (G,w) of the MWIS in $(k+1)$-claw free graphs, and write $G=(V,E)$. In accordance with Theorem 3.1, let B be an independent set in G and let A be an independent set with the property that no claw improves A and there is no local

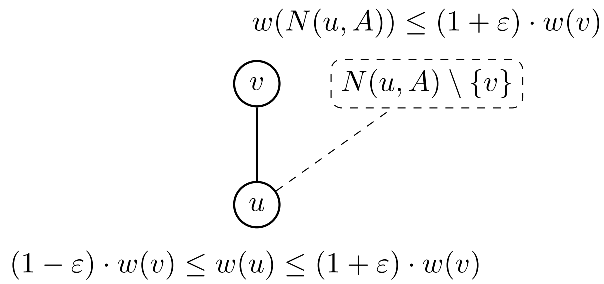


Figure 3.1: u is a single neighbor of $v = n(u)$.

improvement of A of size 3 w.r.t. w^2. In addition, we fix a map $n\colon V \to A$ mapping $u \in V$ to a vertex in $N(u, A)$ of maximum weight. Recall that $N(u, A) \neq \emptyset$ for all $u \in V$ since the fact that no claw improves A implies that A constitutes a maximal independent set. Moreover, note that $n \restriction B$ complies with the requirements of Definition 2.16. We further fix a map $n_2\colon \{u \in V : |N(u, A)| \geq 2\} \to A$ mapping u to a vertex in $N(u, A) \setminus \{n(u)\}$ of maximum weight.

3.1 Regular and irregular vertices

In this section, we introduce the notion of a *regular vertex.* Regular vertices can be thought of as vertices from A with the property that their neighborhood in B (almost) obeys the structure outlined in Lemma 2.25. To specify what we mean by this, we introduce the notions of *single* and *double vertices* in B. They play the role of vertices from B_1 and B_2 in a tight instance (cf. proof of Lemma 2.25).

Definition 3.4 (single vertex)**.** We call a vertex $u \in V$ *single* if

$$(1 - \varepsilon) \cdot w(n(u)) \leq w(u) \leq (1 + \varepsilon) \cdot w(n(u)) \text{ and} \tag{3.5}$$
$$w(N(u, A)) \leq (1 + \varepsilon) \cdot w(n(u)). \tag{3.6}$$

See Fig. 3.1 for an illustration.

Definition 3.5 (double vertex)**.** We call a vertex $u \in V$ with $|N(u, A)| \geq 2$ *double* if

$$(1 - \varepsilon) \cdot w(n(u)) \leq w(n_2(u)) \leq w(n(u)), \tag{3.7}$$
$$(1 - \varepsilon) \cdot w(n(u)) \leq w(u) \leq (1 + \varepsilon) \cdot w(n(u)), \text{ and} \tag{3.8}$$
$$w(N(u, A)) \leq (2 + \varepsilon^2) \cdot w(u). \tag{3.9}$$

See Fig. 3.2 for an illustration.

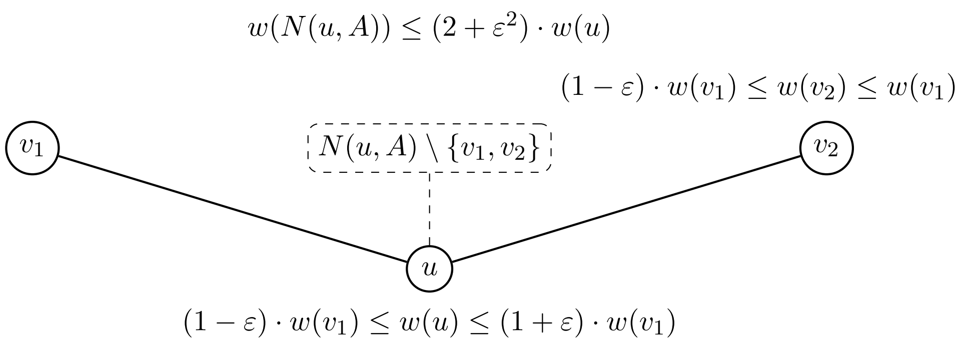


Figure 3.2: u is a double neighbor of $v_1 = n(u)$ and $v_2 = n_2(u)$.

We point out that Definition 3.4 implies that for a single vertex u, we have $w(N(u,A) \setminus \{n(u)\}) = \mathcal{O}(\varepsilon) \cdot w(u)$. Moreover, Definition 3.5 yields $w(N(u,A) \setminus \{n(u), n_2(u)\}) = \mathcal{O}(\varepsilon) \cdot w(u)$ for a double vertex u.

We further remark that the notions of being single or double mutually exclude each other.

Proposition 3.6. No $u \in V$ is both single and double.

Proof. Assume that $u \in V$ were both single and double. Then

$$(1-\varepsilon) \cdot w(n(u)) \overset{(3.7)}{\leq} w(n_2(u)) \leq w(N(u,A) \setminus \{n(u)\}) \overset{(3.6)}{\leq} \varepsilon \cdot w(n(u)),$$

a contradiction to $w(n(u)) > 0$ and (3.1). □

We would like to consider a single vertex u' as only being adjacent to $n(u')$ in A, and to view $n(u'')$ and $n_2(u'')$ as the only neighbors a double vertex u'' features in A. This idea is captured by the following definition.

Definition 3.7 (regular/irregular neighbors)**.** Let $v \in A$ and $u \in N(v,V)$. We say that u and v are *regular neighbors* if u is single and $v = n(u)$, or if u is double and $v \in \{n(u), n_2(u)\}$. Otherwise, we say that they are *irregular neighbors*.

For $v \in A$, denote the set of regular neighbors that v has in B by $\mathrm{reg}_B(v)$, and let $\mathrm{irreg}_B(v) := N(v,B) \setminus \mathrm{reg}_B(v)$ be the set of irregular neighbors of v in B.

Analogously, for $u \in V$, let $\mathrm{reg}_A(u)$ ($\mathrm{irreg}_A(u)$) denote the set of regular (irregular) neighbors of u in A.

We remark that for $x, y \in A$ with $y \in N(x,V)$, independence of A implies $x = y$. In particular, the notion of x and y being regular/irregular neighbors is well-defined in the sense that it does not depend on the way we choose v among x and y since they are equal.

The previous definitions now allow for a natural generalization of the structural property described in Lemma 2.25. Namely, we replace the condition that $v \in A$ features one neighbor in B with degree 1 to A and $k-1$ neighbors in B with degree 2 to A by the requirement that v has one *regular single neighbor* and $k-1$ *regular double neighbors* in B. Observe that by Proposition 2.17, these neighbors already make up all of $N(v, B)$.

Definition 3.8 (regular/irregular vertices)**.** We call a vertex $v \in A$ *regular* if $N(v, B)$ consists of one regular neighbor that is single, and $k-1$ regular neighbors that are double. Otherwise, we call v *irregular*. We denote the set of irregular vertices by I.

We observe that no vertex in A can feature two (or more) regular single neighbors in B because this immediately implies a local improvement.

Lemma 3.9. Let $v \in A$ and assume that $u_1, u_2 \in N(v, B)$ are two distinct regular single neighbors of v. Then $X = \{u_1, u_2\}$ constitutes a claw-shaped improvement.

Proof. As $N(v, B)$ contains two distinct vertices, we must have $v \in A \setminus B$. In particular, X constitutes the set of talons of a claw centered at v. Thus, it remains to see that $w^2(N(X, A)) < w^2(X)$. We calculate

$$\begin{aligned}
w^2(N(X, A)) &\leq w^2(v) + w^2(N(u_1, A) \setminus \{v\}) + w^2(N(u_2, A) \setminus \{v\}) \\
&\leq w^2(v) + (w(N(u_1, A) \setminus \{v\}))^2 + (w(N(u_2, A) \setminus \{v\}))^2 \\
&\overset{(3.6)}{\leq} (1 + 2 \cdot \varepsilon^2) \cdot w^2(v) \\
&\overset{(3.4)}{<} 2 \cdot (1 - \varepsilon)^2 \cdot w^2(v) \\
&\overset{(3.5)}{\leq} w^2(u_1) + w^2(u_2) = w^2(X).
\end{aligned}$$

□

3.2 There are many irregular vertices

Our main result for this section is Lemma 3.10, which tells us that the irregular vertices constitute a constant fraction of A in terms of weight.

Lemma 3.10. We have

$$w(I) \geq \frac{(1 - \varepsilon)^2 \cdot (k-1)}{(1 - \varepsilon)^2 \cdot (k-1) + (1 + \varepsilon) \cdot k} \cdot w(A).$$

The proof of Lemma 3.10 proceeds in two steps. First, we show that there is no double vertex $u \in B$ with the property that both $n(u)$ and $n_2(u)$ are regular (cf. Lemma 3.11). To this end, we observe that if for a

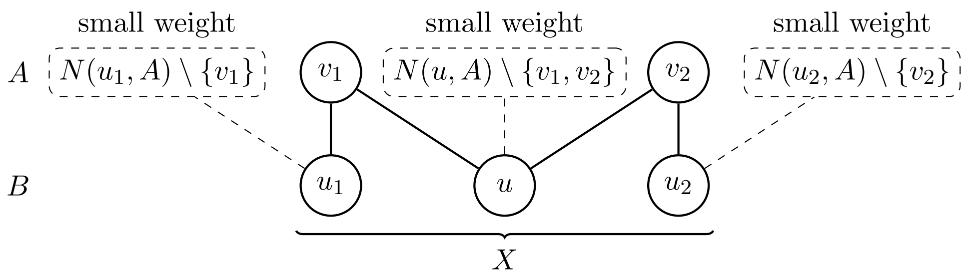


Figure 3.3: A double vertex $u \in B$ such that $v_1 := n(u)$ and $v_2 := n_2(u)$ are regular implies a local improvement of size 3.

double vertex $u \in B$, both $v_1 := n(u)$ and $v_2 := n_2(u)$ are regular, then $X := \{u, u_1, u_2\}$ constitutes a local improvement of size 3, where u_i denotes the regular single neighbor of v_i in B for $i = 1, 2$. Intuitively, this follows from the facts that the weights of v_1, v_2, u_1, u_2 and u are roughly the same and moreover, if ε is small enough, the total weight of $N(u_i, A) \setminus \{v_i\}$ for $i = 1, 2$ and of $N(u, A) \setminus \{v_1, v_2\}$ is almost negligible. See Fig. 3.3 for an illustration. As a consequence of Lemma 3.11, we can obtain a bipartite graph H with bipartitions I and $A \setminus I$ by adding an edge $\{n(u), n_2(u)\}$ for each double vertex $u \in B$ with $\{n(u), n_2(u)\} \cap (A \setminus I) \neq \emptyset$. By Definition 3.8, each vertex in $A \setminus I$ has degree $k - 1$ in H, whereas Proposition 2.17 implies that every vertex in I has degree at most k. Moreover, (3.7) tells us that the weights of adjacent vertices are roughly the same. This allows us to derive Lemma 3.10. The remainder of this section provides formal proofs of Lemma 3.11 and Lemma 3.10.

Lemma 3.11. There is no double vertex $u \in B$ such that both $n(u)$ and $n_2(u)$ are regular vertices.

Proof. Assume towards a contradiction that $u \in B$ were a double vertex such that $v_1 := n(u)$ and $v_2 := n_2(u)$ are regular. By Definition 3.8, let $u_i \in N(v_i, B)$ be the regular single neighbor of v_i in B for $i = 1, 2$. Note that $n(u_i) = v_i$ for $i = 1, 2$ implies $u_1 \neq u_2$. Moreover, Proposition 3.6 tells us that $u \notin \{u_1, u_2\}$. Let $X := \{u_1, u, u_2\}$. Then $|X| = 3$. We claim that X is a local improvement. To verify this, we need to show that $w^2(X) > w^2(N(X, A))$. We first note that (3.6) tells us that

$$w^2(N(u_i, A) \setminus \{v_i\}) \leq (w(N(u_i, A) \setminus \{v_i\}))^2 \leq (\varepsilon \cdot w(v_i))^2 = \varepsilon^2 \cdot w^2(v_i) \quad (3.10)$$

for $i = 1, 2$. Moreover, we compute

$$\begin{aligned} w^2(N(u, A) \setminus \{v_1, v_2\}) &\leq (w(N(u, A) \setminus \{v_1, v_2\}))^2 \\ &\overset{(3.9)}{\leq} ((2 + \varepsilon^2) \cdot w(u) - w(v_1) - w(v_2))^2 \end{aligned}$$

$$\overset{(3.7)}{\underset{(3.8)}{\leq}} ((2+\varepsilon^2)\cdot(1+\varepsilon)\cdot w(v_1) - w(v_1) - (1-\varepsilon)\cdot w(v_1))^2$$
$$= (3\varepsilon+\varepsilon^2+\varepsilon^3)^2\cdot w^2(v_1). \tag{3.11}$$

Thus, we obtain

$$\begin{aligned} &w^2(N(X,A)) \\ &\quad \leq w^2(v_1)+w^2(v_2)+w^2(N(u,A)\setminus\{v_1,v_2\}) \\ &\qquad + w^2(N(u_1,A)\setminus\{v_1\}) + w^2(N(u_2,A)\setminus\{v_2\}) \\ &\quad \overset{(3.10)}{\underset{(3.11)}{\leq}} \left(1+(3\varepsilon+\varepsilon^2+\varepsilon^3)^2+\varepsilon^2\right)\cdot w^2(v_1) + (1+\varepsilon^2)\cdot w^2(v_2). \end{aligned} \tag{3.12}$$

Moreover, we get

$$\begin{aligned} w^2(X) &= w^2(u_1)+w^2(u)+w^2(u_2) \\ &\overset{(3.5)}{\underset{(3.8)}{\geq}} 2\cdot(1-\varepsilon)^2\cdot w^2(v_1) + (1-\varepsilon)^2\cdot w^2(v_2). \end{aligned} \tag{3.13}$$

Combining (3.12) and (3.13) yields

$$\begin{aligned} &w^2(X) - w^2(N(X,A)) \\ &\quad \geq 2\cdot(1-\varepsilon)^2\cdot w^2(v_1) + (1-\varepsilon)^2\cdot w^2(v_2) \\ &\qquad - \left(1+(3\varepsilon+\varepsilon^2+\varepsilon^3)^2+\varepsilon^2\right)\cdot w^2(v_1) - (1+\varepsilon^2)\cdot w^2(v_2) \\ &\quad = \left(2\cdot(1-\varepsilon)^2 - \left(1+(3\varepsilon+\varepsilon^2+\varepsilon^3)^2+\varepsilon^2\right)\right)\cdot w^2(v_1) \\ &\qquad - \left((1+\varepsilon^2)-(1-\varepsilon)^2\right)\cdot w^2(v_2) \\ &\quad \overset{(3.7)}{\geq} \left(3\cdot(1-\varepsilon)^2 - \left(2+(3\varepsilon+\varepsilon^2+\varepsilon^3)^2+2\varepsilon^2\right)\right)\cdot w^2(v_1) \\ &\quad \overset{(3.3)}{>} 0. \end{aligned}$$

This concludes the proof. □

Proof of Lemma 3.10. Let Y be the set of double vertices $u \in B$ such that $n(u)$ or $n_2(u)$ is a regular vertex (i.e., contained in $A \setminus I$). We provide two estimates for $w(Y)$. By Definition 3.8 and Proposition 3.6, for every $v \in A \setminus I$, we have $|N(v,Y)| = k-1$, and moreover, every $u \in N(v,Y)$ is a regular double neighbor of v. In particular, every $u \in N(v,Y)$ satisfies

$$w(u) \overset{(3.8)}{\geq} (1-\varepsilon)\cdot w(n(u)) \overset{(3.7)}{=} (1-\varepsilon)\cdot\max\{w(n(u)), w(n_2(u))\} \geq (1-\varepsilon)\cdot w(v),$$

where the last inequality follows from Definition 3.7. Hence, we obtain

$$w(N(v,Y)) \geq (1-\varepsilon)\cdot(k-1)\cdot w(v) \text{ for every } v \in A\setminus I. \tag{3.14}$$

Lemma 3.11 further tells us that the sets $N(v, Y), v \in A \setminus I$ are pairwise disjoint. This implies

$$\begin{aligned} w(Y) = \sum_{v \in A \setminus I} w(N(v, Y)) &\overset{(3.14)}{\geq} (1-\varepsilon) \cdot (k-1) \cdot w(A \setminus I) \\ &= (1-\varepsilon) \cdot (k-1) \cdot (w(A) - w(I)). \end{aligned} \tag{3.15}$$

On the other hand, Lemma 3.11 tells us that for every $u \in Y$, at least one of $n(u)$ and $n_2(u)$ is contained in I and as further $\{n(u), n_2(u)\} \cap A \setminus I \neq \emptyset$, exactly one of them is. In addition, for $u \in Y$ and $v \in \{n(u), n_2(u)\}$, we have

$$\begin{aligned} w(u) &\overset{(3.8)}{\leq} (1+\varepsilon) \cdot w(n(u)) \overset{(3.7)}{\leq} \frac{1+\varepsilon}{1-\varepsilon} \cdot \min\{w(n(u)), w(n_2(u))\} \\ &\leq \frac{1+\varepsilon}{1-\varepsilon} \cdot w(v). \end{aligned} \tag{3.16}$$

This yields

$$\begin{aligned} w(Y) &= \sum_{v \in I} \sum_{\substack{u \in Y: \\ v \in \{n(u), n_2(u)\}}} w(u) \\ &\overset{(3.16)}{\leq} \sum_{v \in I} \sum_{\substack{u \in Y: \\ v \in \{n(u), n_2(u)\}}} \frac{1+\varepsilon}{1-\varepsilon} \cdot w(v) \\ &\leq \sum_{v \in I} |N(v, Y)| \cdot \frac{1+\varepsilon}{1-\varepsilon} \cdot w(v) \\ &\leq \frac{1+\varepsilon}{1-\varepsilon} \cdot k \cdot w(I), \end{aligned} \tag{3.17}$$

where the last inequality follows from Proposition 2.17. Combining (3.15) and (3.17) implies

$$(1-\varepsilon)^2 \cdot (k-1) \cdot (w(A) - w(I)) \leq (1-\varepsilon) \cdot w(Y) \leq (1+\varepsilon) \cdot k \cdot w(I),$$

which results in

$$w(I) \geq \frac{(1-\varepsilon)^2 \cdot (k-1)}{(1-\varepsilon)^2 \cdot (k-1) + (1+\varepsilon) \cdot k} \cdot w(A). \qquad \square$$

3.3 Proof of the approximation guarantee

In this section, we conclude the proof of Theorem 3.1. For this purpose, we show that each irregular vertex improves the bound $w(B) \leq \frac{k+1}{2} \cdot w(A)$ that we get from Berman's analysis of SquareImp [7] by a constant fraction of its weight. To obtain this result, we distinguish two types of irregular vertices, which we handle differently. Namely, we let D denote the set of irregular vertices $v \in I$ with the property that $N(v, B)$ consists of k regular double neighbors of v, and consider D and $I \setminus D$ separately. We observe that in contrast to the vertices in D, every vertex in $I \setminus D$ has at most $k-1$ regular neighbors in B.

Proposition 3.12. Let $v \in I \setminus D$. Then $|\mathrm{reg}_B(v)| \leq k - 1$.

Proof. Assume towards a contradiction that $v \in I \setminus D$ with $|\mathrm{reg}_B(v)| \geq k$. Then Proposition 2.17 tells us that $|\mathrm{reg}_B(v)| = k$ and $\mathrm{reg}_B(v) = N(v, B)$. As $v \notin D$, $\mathrm{reg}_B(v)$ cannot exclusively consist of double vertices, so at least one of the vertices is single. As v is irregular, at least two of the vertices in $\mathrm{reg}_B(v)$ have to be single. But this is impossible by Lemma 3.9. □

The remainder of this section is organized as follows: In Section 3.3.1, we prove Lemma 3.13, which improves our bound on $w(B)$ by a constant fraction of $w(D)$. In contrast, Lemma 3.15, which we establish in Section 3.3.2, yields an improvement proportional to $w(I \setminus D)$. Finally, we balance the two bounds in Section 3.3.3 and prove Theorem 3.1.

3.3.1 Vertices in D improve the analysis

Lemma 3.13. We have

$$w(B) \leq \frac{k+1}{2} \cdot w(A) - \frac{1-\varepsilon}{4-2\varepsilon} \cdot w(D).$$

The key ingredient towards Lemma 3.13 is Lemma 3.14, which tells us that for a double vertex from B, its total contribution amounts to almost four times the charges it invokes. A similar result also appears in [40]. Note that for general vertices from B, Corollary 2.22 only guarantees a factor of 2, and for a single vertex $u \in B$ with $w(u) = w(n(u))$ and $N(u, A) = \{n(u)\}$, this is also best possible.

As the charges that vertices in D have to pay are invoked by double vertices exclusively, we can count these charges almost four times instead of just twice when establishing a lower bound on the total contribution $\sum_{u \in B} \sum_{v \in A} \mathrm{contr}(u, v)$, which is in turn upper bounded by $w(A)$ (cf. Proposition 2.20). As a consequence, if the total charges paid by D amount to the maximum possible value of $\frac{1}{2} \cdot w(D)$, we gain almost the same amount in our bound on the total charges paid by A. On the other hand, if the the vertices in D pay zero charges, we can upper bound the total charges invoked by $\frac{1}{2} \cdot w(A \setminus D)$. The proof of Lemma 3.13 essentially balances these two extreme cases.

Lemma 3.14. Let $u \in B$ be double. Then

$$\operatorname{contr}(u, n(u)) + \operatorname{contr}(u, n_2(u)) \geq 2 \cdot (2 - \varepsilon) \cdot \operatorname{charge}(u, n(u)).$$

Before diving into the proof, we provide some intuition why Lemma 3.14 holds. Observe that in case $w(n(u)) = w(n_2(u))$, it is completely arbitrary which one of the two vertices we define to be $n(u)$. In particular, we have $\operatorname{contr}(u, n_2(u)) = \operatorname{contr}(u, n(u))$ in this case, which even yields

$$\operatorname{contr}(u, n(u)) + \operatorname{contr}(u, n_2(u)) \geq 4 \cdot \operatorname{charge}(u, n(u))$$

by Corollary 2.22. If $w(n_2(u)) < w(n(u))$, then $\operatorname{contr}(u, n_2(u))$ can become smaller than $\operatorname{contr}(u, n(u))$. However, we can use the "slack terms" on the right-hand side of Lemma 2.21 to counterbalance this effect.

After having outlined the intuition behind Lemma 3.14, we provide a formal proof. Then, we conclude this section with the proof of Lemma 3.13.

Proof of Lemma 3.14. By (3.1), we have $2 \cdot (2 - \varepsilon) > 0$. Thus, in case $\operatorname{charge}(u, n(u)) \leq 0$, the statement of the lemma follows by non-negativity of the contribution. Hence, we may assume that

$$\operatorname{charge}(u, n(u)) > 0. \tag{3.18}$$

Let $v_1 \coloneqq n(u)$ and $v_2 \coloneqq n_2(u)$. Then Lemma 2.21 tells us that for $i = 1, 2$, we have

$$\begin{aligned}
&\operatorname{contr}(u, v_i) \cdot w(v_i) - 2 \cdot \operatorname{charge}(u, v_1) \cdot w(v_i) \\
&\geq (w(u) - w(v_i))^2 + (w(v_i) - w(v_{3-i})) \cdot w(v_{3-i}) \\
&\quad + \sum_{x \in N(u,A) \setminus \{v_1, v_2\}} (w(v_i) - w(x)) \cdot w(x) \\
&\geq (w(u) - w(v_i))^2 + (w(v_i) - w(v_{3-i})) \cdot w(v_{3-i}),
\end{aligned} \tag{3.19}$$

where the last inequality follows by definition of $n(u) = v_1$ and $n_2(u) = v_2$. We distinguish two cases:

Case 1: $w(v_1) \geq w(u)$. Then Definition 2.16 implies

$$\begin{aligned}
&2 \cdot \operatorname{charge}(u, v_1) \\
&= 2 \cdot w(u) - w(N(u, A)) \leq 2 \cdot w(v_1) - w(N(u, A)) \\
&\leq 2 \cdot w(v_1) - w(v_1) - w(v_2) = w(v_1) - w(v_2).
\end{aligned} \tag{3.20}$$

Applying (3.19) for $i = 1$ yields

$$\begin{aligned}
&\operatorname{contr}(u, v_1) \cdot w(v_1) - 2 \cdot \operatorname{charge}(u, v_1) \cdot w(v_1) \\
&\geq (w(u) - w(v_1))^2 + (w(v_1) - w(v_2)) \cdot w(v_2) \\
&\geq (w(v_1) - w(v_2)) \cdot w(v_2) \\
&\overset{(3.7)}{\geq} (1 - \varepsilon) \cdot (w(v_1) - w(v_2)) \cdot w(v_1) \\
&\overset{(3.20)}{\geq} (1 - \varepsilon) \cdot 2 \cdot \operatorname{charge}(u, v_1) \cdot w(v_1).
\end{aligned}$$

Rearranging the inequality and dividing by $w(v_1) > 0$ results in

$$\operatorname{contr}(u, v_1) \geq (2 - \varepsilon) \cdot 2 \cdot \operatorname{charge}(u, v_1).$$

Using $\operatorname{contr}(u, v_2) \geq 0$ by Definition 2.19 completes the proof.

Case 2: $w(v_1) < w(u)$. By (3.7) and positivity of weights, we obtain $0 < w(v_2) \leq w(v_1) < w(u)$, and, thus,

$$(w(u) - w(v_2))^2 > (w(v_1) - w(v_2))^2. \tag{3.21}$$

Now, (3.19) yields

$$\begin{aligned}
&\operatorname{contr}(u, v_1) \cdot w(v_1) - 2 \cdot \operatorname{charge}(u, v_1) \cdot w(v_1) \\
&\quad + \operatorname{contr}(u, v_2) \cdot w(v_2) - 2 \cdot \operatorname{charge}(u, v_1) \cdot w(v_2) \\
&\geq (w(u) - w(v_1))^2 + (w(v_1) - w(v_2)) \cdot w(v_2) \\
&\quad + (w(u) - w(v_2))^2 + (w(v_2) - w(v_1)) \cdot w(v_1) \\
&= (w(u) - w(v_1))^2 + (w(u) - w(v_2))^2 - (w(v_1) - w(v_2))^2 \\
&\overset{(3.21)}{>} 0.
\end{aligned}$$

By non-negativity of the contribution and since $\operatorname{charge}(u, v_1) > 0$ by (3.18), we obtain

$$\begin{aligned}
&(\operatorname{contr}(u, v_1) + \operatorname{contr}(u, v_2)) \cdot w(v_1) \\
&\overset{(3.7)}{\geq} \operatorname{contr}(u, v_1) \cdot w(v_1) + \operatorname{contr}(u, v_2) \cdot w(v_2) \\
&\geq 2 \cdot \operatorname{charge}(u, v_1) \cdot (w(v_1) + w(v_2)) \\
&\overset{(3.7)}{\geq} (2 - \varepsilon) \cdot 2 \cdot \operatorname{charge}(u, v_1) \cdot w(v_1).
\end{aligned}$$

Division by $w(v_1) > 0$ completes the proof. □

Proof of Lemma 3.13. By Lemma 2.18, it suffices to show that

$$\sum_{u \in B} \operatorname{charge}(u, n(u)) \leq \frac{1}{2} \cdot w(A) - \frac{1 - \varepsilon}{4 - 2\varepsilon} \cdot w(D).$$

To this end, using Proposition 2.20, Corollary 2.22, Lemma 3.14 and non-negativity of the contribution, we calculate

$$\begin{aligned}
&\sum_{u\in B} 2\cdot \text{charge}(u,n(u)) + (1-\varepsilon)\cdot \sum_{v\in D}\sum_{\substack{u\in B:\\ n(u)=v}} 2\cdot\text{charge}(u,n(u))\\
&= \sum_{v\in A\setminus D}\sum_{\substack{u\in B:\\ n(u)=v}} 2\cdot\text{charge}(u,n(u)) + \sum_{v\in D}\sum_{\substack{u\in B:\\ n(u)=v}} (2-\varepsilon)\cdot 2\cdot\text{charge}(u,n(u))\\
&\le \sum_{v\in A\setminus D}\sum_{\substack{u\in B:\\ n(u)=v}} \text{contr}(u,n(u)) + \sum_{v\in D}\sum_{\substack{u\in B:\\ n(u)=v}} \big(\text{contr}(u,n(u)) + \text{contr}(u,n_2(u))\big)\\
&\le \sum_{u\in B}\sum_{v\in A}\text{contr}(u,v) \le w(A).
\end{aligned}$$

This yields

$$\sum_{u\in B} 2\cdot\text{charge}(u,n(u)) \le w(A) - (1-\varepsilon)\cdot\sum_{v\in D}\sum_{\substack{u\in B:\\ n(u)=v}} 2\cdot\text{charge}(u,n(u)). \tag{3.22}$$

On the other hand, using again Proposition 2.20, Corollary 2.22 and non-negativity of the contribution, we compute

$$\begin{aligned}
&\sum_{u\in B} 2\cdot\text{charge}(u,n(u))\\
&= \sum_{v\in A\setminus D}\sum_{\substack{u\in B:\\ n(u)=v}} 2\cdot\text{charge}(u,n(u)) + \sum_{v\in D}\sum_{\substack{u\in B:\\ n(u)=v}} 2\cdot\text{charge}(u,n(u))\\
&\le w(A\setminus D) + \sum_{v\in D}\sum_{\substack{u\in B:\\ n(u)=v}} 2\cdot\text{charge}(u,n(u)).
\end{aligned}\tag{3.23}$$

Adding (3.22) and $(1-\varepsilon)\cdot$ (3.23) results in

$$(4-2\varepsilon)\cdot\sum_{u\in B}\text{charge}(u,n(u)) \le (2-\varepsilon)\cdot w(A) - (1-\varepsilon)\cdot w(D),$$

and division by $4-2\varepsilon>0$ yields the claim. □

3.3.2 Vertices in $I\setminus D$ improve the analysis

In this section, we prove Lemma 3.15, which tells us that we can improve upon the bound of $w(B)\le \frac{k+1}{2}\cdot w(A)$ by a constant fraction of $w(I\setminus D)$.

Lemma 3.15. We have

$$w(B) \le \frac{k+1}{2}\cdot w(A) - \frac{\varepsilon^2}{2\cdot(1+\varepsilon)\cdot(2+\varepsilon^2)}\cdot w(I\setminus D).$$

The main idea for the proof of Lemma 3.15 can be described as follows: By Proposition 3.12, we know that every vertex $v \in I \setminus D$ has at most $k-1$ regular neighbors in B. In case v has at most $k-1$ neighbors in B, i.e., $|N(v,B)| \leq k-1$, Theorem 2.23 tells us that we gain $\frac{1}{2} \cdot w(v)$ in our bound on $w(B)$. On the other hand, if $|N(v,B)| = k$, then v needs to have at least one irregular neighbor $u \in B$. We show that this neighbor u can reimburse v, and, in fact, each one of its irregular neighbors in A, by a constant fraction of its weight by distributing the slack in the inequality

$$\sum_{\tilde{v} \in A} \operatorname{contr}(u, \tilde{v}) \geq 2 \cdot \operatorname{charge}(u, n(u))$$

among them (cf. Theorem 2.23). Definition 3.16 and Lemma 3.17 formalize this idea.

Definition 3.16 (slack)**.** For $u \in B$, we define

$$\operatorname{slack}(u) \coloneqq \sum_{v \in A} \operatorname{contr}(u, v) - 2 \cdot \operatorname{charge}(u, n(u)).$$

Note that non-negativity of the contribution implies

$$\operatorname{slack}(u) \geq \operatorname{contr}(u, n(u)) - 2 \cdot \operatorname{charge}(u, n(u)). \tag{3.24}$$

Lemma 3.17. Let $u \in B$. Then

$$\operatorname{slack}(u) \geq \frac{\varepsilon^2}{(1+\varepsilon) \cdot (2+\varepsilon^2)} \cdot w(\operatorname{irreg}_A(u)).$$

The intuition behind Lemma 3.17 is that each irregular neighbor of u makes the inequality $\sum_{v \in A} \operatorname{contr}(u, v) \geq 2 \cdot \operatorname{charge}(u, n(u))$ less tight.

For better readability, we split the proof of Lemma 3.17 into two parts. Lemma 3.18 takes care of the cases where u is single or double, respectively. Lemma 3.19 deals with the remaining vertices that are neither single nor double.

Finally, we conclude the section with the proof of Lemma 3.15.

Lemma 3.18. Let $u \in B$ be single or double. Then

$$\operatorname{slack}(u) \geq (1 - 3\varepsilon - \varepsilon^2 - \varepsilon^3) \cdot w(\operatorname{irreg}_A(u)) \geq \frac{\varepsilon^2}{(1+\varepsilon) \cdot (2+\varepsilon^2)} \cdot w(\operatorname{irreg}_A(u)).$$

Proof. By Lemma 2.21 and (3.24), we obtain

$$
\begin{aligned}
&\operatorname{slack}(u) \cdot w(n(u)) \\
&\geq (w(u) - w(n(u)))^2 + \sum_{x \in N(u,A)} (w(n(u)) - w(x)) \cdot w(x) \\
&\geq \sum_{x \in \operatorname{irreg}_A(u)} (w(n(u)) - w(x)) \cdot w(x) \\
&\geq (w(n(u)) - w(\operatorname{irreg}_A(u))) \cdot w(\operatorname{irreg}_A(u)). \qquad (3.25)
\end{aligned}
$$

Here, the second inequality follows from the fact that $n(u)$ is of maximum weight in $N(u,A) \supseteq \operatorname{irreg}_A(u)$. The last inequality is implied by $w(x) \leq w(\operatorname{irreg}_A(u))$ for all $x \in \operatorname{irreg}_A(u)$.

If u is single, then $\operatorname{irreg}_A(u) = N(u,A) \setminus \{n(u)\}$ and we compute

$$
\begin{aligned}
&w(n(u)) - w(\operatorname{irreg}_A(u)) = 2 \cdot w(n(u)) - w(N(u,A)) \\
&\overset{(3.6)}{\geq} (1-\varepsilon) \cdot w(n(u)) \geq (1 - 3\varepsilon - \varepsilon^2 - \varepsilon^3) \cdot w(n(u)). \qquad (3.26)
\end{aligned}
$$

If u is double, we obtain $\operatorname{irreg}_A(u) = N(u,A) \setminus \{n(u), n_2(u)\}$. Hence,

$$
\begin{aligned}
&w(n(u)) - w(\operatorname{irreg}_A(u)) = 2 \cdot w(n(u)) + w(n_2(u)) - w(N(u,A)) \\
&\underset{(3.9)}{\overset{(3.7)}{\geq}} 2 \cdot w(n(u)) + (1-\varepsilon) \cdot w(n(u)) - (2+\varepsilon^2) \cdot w(u) \\
&\overset{(3.8)}{\geq} (3-\varepsilon) \cdot w(n(u)) - (2+\varepsilon^2) \cdot (1+\varepsilon) \cdot w(n(u)) \\
&= (1 - 3\varepsilon - \varepsilon^2 - \varepsilon^3) \cdot w(n(u)). \qquad (3.27)
\end{aligned}
$$

Plugging (3.26) or (3.27), respectively, into (3.25) and dividing by the positive value $w(n(u))$ yields the first inequality. The second one follows by positivity of weights and (3.2). □

Lemma 3.19. Let $u \in B$ be neither single nor double. Then

$$\operatorname{slack}(u) \geq \frac{\varepsilon^2}{(1+\varepsilon) \cdot (2+\varepsilon^2)} \cdot w(\operatorname{irreg}_A(u)).$$

Proof. We first observe that $\operatorname{irreg}_A(u) = N(u,A)$. In case we have

$$w(N(u,A)) > (2+\varepsilon^2) \cdot w(u),$$

(3.24) and non-negativity of the contribution tell us that

$$
\begin{aligned}
\operatorname{slack}(u) &\geq -2 \cdot \operatorname{charge}(u, n(u)) = w(N(u,A)) - 2 \cdot w(u) \\
&> \left(1 - \frac{2}{2+\varepsilon^2}\right) \cdot w(N(u,A)) \\
&= \frac{\varepsilon^2}{2+\varepsilon^2} \cdot w(N(u,A)).
\end{aligned}
$$

Thus, we may assume that

$$w(N(u,A)) \leq (2+\varepsilon^2) \cdot w(u) \tag{3.28}$$

in the following. By (3.24), Lemma 2.21 and since $n(u)$ is of maximum weight in $N(u,A)$, we can infer that

$$\text{slack}(u) \cdot w(n(u)) \geq (w(u) - w(n(u)))^2 \text{ and} \tag{3.29}$$

$$\text{slack}(u) \cdot w(n(u)) \geq \sum_{x \in N(u,A) \setminus \{n(u)\}} (w(n(u)) - w(x)) \cdot w(x). \tag{3.30}$$

We distinguish two cases:

Case 1: $|w(u) - w(n(u))| > \varepsilon \cdot w(n(u))$. Then

$$\frac{\max\{w(n(u)), w(u)\}}{\min\{w(n(u)), w(u)\}} > 1 + \varepsilon. \tag{3.31}$$

Thus,

$$\begin{aligned}
&\text{slack}(u) \cdot w(n(u)) \overset{(3.29)}{\geq} (w(u) - w(n(u))^2 \\
&\quad = (\max\{w(u), w(n(u))\} - \min\{w(u), w(n(u))\})^2 \\
&\quad \overset{(3.31)}{\geq} \left(1 - \frac{1}{1+\varepsilon}\right) \cdot \max\{w(u), w(n(u))\} \cdot (1+\varepsilon-1) \cdot \min\{w(u), w(n(u))\} \\
&\quad = \left(1 - \frac{1}{1+\varepsilon}\right) \cdot (1+\varepsilon-1) \cdot w(u) \cdot w(n(u)) \\
&\quad = \frac{\varepsilon^2}{1+\varepsilon} \cdot w(u) \cdot w(n(u)) \overset{(3.28)}{\geq} \frac{\varepsilon^2}{(1+\varepsilon)\cdot(2+\varepsilon^2)} \cdot w(N(u,A)) \cdot w(n(u)).
\end{aligned}$$

Division by $w(n(u)) > 0$ yields the desired statement.

Case 2: $|w(u) - w(n(u))| \leq \varepsilon \cdot w(n(u))$. (3.32)

Then the fact that u is not single implies

$$w(N(u,A) \setminus \{n(u)\}) > \varepsilon \cdot w(n(u)). \tag{3.33}$$

In particular, $N(u,A) \setminus \{n(u)\} \neq \emptyset$, and, thus, $|N(u,A)| \geq 2$. By (3.32) and (3.28), the fact that u is not double yields

$$w(n_2(u)) < (1-\varepsilon) \cdot w(n(u)). \tag{3.34}$$

But as $n_2(u)$ is a vertex of maximum weight in $N(u,A) \setminus \{n(u)\}$, this allows us to conclude that

$$\begin{aligned}
\operatorname{slack}(u) \cdot w(n(u)) &\overset{(3.30)}{\geq} \sum_{x \in N(u,A) \setminus \{n(u)\}} (w(n(u)) - w(x)) \cdot w(x) \\
&\geq \sum_{x \in N(u,A) \setminus \{n(u)\}} (w(n(u)) - w(n_2(u))) \cdot w(x) \\
&= (w(n(u)) - w(n_2(u))) \cdot w(N(u,A) \setminus \{n(u)\}) \\
&\underset{(3.34)}{\overset{(3.33)}{>}} \varepsilon^2 \cdot w^2(n(u)) \\
&\overset{(3.32)}{\geq} \frac{\varepsilon^2}{1+\varepsilon} \cdot w(u) \cdot w(n(u)) \\
&\overset{(3.28)}{\geq} \frac{\varepsilon^2}{(1+\varepsilon) \cdot (2+\varepsilon^2)} \cdot w(N(u,A)) \cdot w(n(u)).
\end{aligned}$$

Division by $w(n(u)) > 0$ concludes the proof. □

From this, we may derive the following bound.

Lemma 3.20. We have

$$w(B) \leq \frac{k+1}{2} \cdot w(A) - \frac{1}{2} \cdot \frac{\varepsilon^2}{(1+\varepsilon) \cdot (2+\varepsilon^2)} \cdot \sum_{v \in A} (k - |\operatorname{reg}_B(v)|) \cdot w(v).$$

Proof. By Theorem 2.23, it suffices to show that

$$\begin{aligned}
&\sum_{v \in A} (k - |N(v,B)|) \cdot w(v) + \sum_{u \in B} \left(\sum_{v \in A} \operatorname{contr}(u,v) - 2 \cdot \operatorname{charge}(u, n(u)) \right) \\
&\geq \frac{\varepsilon^2}{(1+\varepsilon) \cdot (2+\varepsilon^2)} \cdot \sum_{v \in A} (k - |\operatorname{reg}_B(v)|) \cdot w(v). \qquad (3.35)
\end{aligned}$$

We have

$$\begin{aligned}
&\sum_{v \in A} (k - |N(v,B)|) \cdot w(v) \\
&\overset{(3.1)}{\geq} \frac{\varepsilon^2}{(1+\varepsilon) \cdot (2+\varepsilon^2)} \cdot \sum_{v \in A} (k - |N(v,B)|) \cdot w(v). \qquad (3.36)
\end{aligned}$$

Moreover, Definition 3.16 and Lemma 3.17 yield

$$\begin{aligned}
&\sum_{u\in B}\left(\sum_{v\in A}\mathrm{contr}(u,v) - 2\cdot \mathrm{charge}(u,n(u))\right) &(3.37)\\
&= \sum_{u\in B}\mathrm{slack}(u)\\
&\geq \frac{\varepsilon^2}{(1+\varepsilon)\cdot(2+\varepsilon^2)}\cdot\sum_{u\in B} w(\mathrm{irreg}_A(u))\\
&= \frac{\varepsilon^2}{(1+\varepsilon)\cdot(2+\varepsilon^2)}\cdot\sum_{v\in A} |\mathrm{irreg}_B(v)|\cdot w(v). &(3.38)
\end{aligned}$$

Combining (3.36) and (3.38) and using $|\mathrm{reg}_B(v)| = |N(v,B)| - |\mathrm{irreg}_B(v)|$ by Definition 3.7 results in (3.35). □

Now, we have accumulated all ingredients that we need in order to prove Lemma 3.15.

Proof of Lemma 3.15. By Proposition 3.12 and Proposition 2.17, we know that $\sum_{v\in A}(k - |\mathrm{reg}_B(v)|)\cdot w(v) \geq w(I\setminus D)$. Now, Lemma 3.20 concludes the proof. □

3.3.3 Balancing the bounds

Now, we combine Lemma 3.13 and Lemma 3.15 to obtain Theorem 3.21.

Theorem 3.21.

$$w(B) \leq \frac{k+1}{2}\cdot w(A) - \frac{\beta\gamma}{\beta+\gamma}\cdot\alpha\cdot w(A),$$

where

$$\alpha := \frac{(1-\varepsilon)^2\cdot(k-1)}{(1-\varepsilon)^2\cdot(k-1)+(1+\varepsilon)\cdot k},\quad \beta := \frac{1-\varepsilon}{4-2\varepsilon}\quad\text{and}$$
$$\gamma := \frac{\varepsilon^2}{2\cdot(1+\varepsilon)\cdot(2+\varepsilon^2)}.$$

Proof. First, note that (3.1) implies $\alpha,\beta,\gamma\in(0,1)$. By Lemma 3.10, we know that $w(I)\geq\alpha\cdot w(A)$. In case $w(D)\geq\frac{\gamma}{\beta+\gamma}\cdot w(I)$, we use Lemma 3.13 to conclude that

$$w(B) \leq \frac{k+1}{2}\cdot w(A) - \beta\cdot w(D) \leq \frac{k+1}{2} - \frac{\beta\gamma}{\beta+\gamma}\cdot\alpha\cdot w(A).$$

Otherwise, we have

$$w(I\setminus D) > \frac{\beta}{\beta+\gamma}\cdot w(I),$$

and Lemma 3.15 yields

$$w(B) \leq \frac{k+1}{2} - \gamma \cdot w(I \setminus D) < \frac{k+1}{2} - \frac{\beta\gamma}{\beta+\gamma} \cdot \alpha \cdot w(A)$$

as desired. □

Finally, we prove Theorem 3.1.

Proof of Theorem 3.1. We observe that

$$\begin{aligned}\alpha &= \frac{(1-\varepsilon)^2 \cdot (k-1)}{(1-\varepsilon)^2 \cdot (k-1) + (1+\varepsilon) \cdot k} = \frac{(1-\varepsilon)^2}{(1-\varepsilon)^2 + \frac{k}{k-1} \cdot (1+\varepsilon)} \\ &\geq \frac{(1-\varepsilon)^2}{(1-\varepsilon)^2 + \frac{3}{2} \cdot (1+\varepsilon)}.\end{aligned}$$

Plugging this bound and our choice of $\varepsilon = 0.138$ into Theorem 3.21 yields the desired statement. □

Chapter 4

Local improvements of logarithmic size

In the previous chapter, we have seen how to obtain a first improvement over Berman's algorithm SquareImp by not only considering claw-shaped improvements, but also local improvements of size 3. This of course raises the question whether taking into account an even broader class of local improvements allows us to further decrease the approximation guarantee we obtain, and if yes, by how much. One natural idea would be to consider even larger improvements of constant size. In fact, a recent paper by Thiery and Ward [49] shows that local improvements of size $\mathcal{O}(k^3)$ suffice to achieve approximation guarantees of $\frac{k+1-\tau_k}{2}$ for the MWIS in $(k+1)$-claw free graphs, where $\tau_k \geq 0.428$ for $k \geq 3$ and $\lim_{k\to\infty} \tau_k = \frac{2}{3}$.

However, the lower bound result by Hurkens and Schrijver [33] also tells us that even for the unweighted k-Set Packing problem, local improvements of constant size cannot yield an approximation guarantee below $\frac{k}{2}$. In contrast, recall that searching for well-structured local improvements of up to logarithmic size can decrease the approximation guarantee for the unweighted k-Set packing problem by a factor of almost $\frac{2}{3}$ to $\frac{k+1+\epsilon}{3}$ [16, 28]. This motivates the question whether considering an appropriate class of local improvements of logarithmically bounded size can also give rise to (significant) improvements for the weighted k-Set Packing problem. Note that structured local improvements of logarithmically bounded size lie on the border of what can be searched for via enumeration/dynamic programming based approaches (e.g., using the color-coding technique as in [28]). This makes it even more interesting to determine the limits of local search, restricted to improvements of logarithmically bounded size, in the weighted setting.

In this chapter, we provide an asymptotically tight answer. On the algorithmic side, we manage to transfer the notion of local improvement that lies at the core of the results in [28] and [48] to the weighted setting.

In doing so, we obtain improved approximation guarantees for the weighted k-Set Packing problem, which get arbitrarily close to $\frac{k}{2}$ as k tends to infinity:

Theorem 4.1. There exists a polynomial-time $\frac{k+1-\lambda_k}{2}$-approximation algorithm for the weighted k-Set Packing problem, where $\lim_{k\to\infty}\lambda_k = 1$.

At the cost of a quasi-polynomial running time, this result generalizes to the MWIS in $(k+1)$-claw free graphs.

In terms of lower bounds, we can show that the approximation ratios we obtain in Theorem 4.1 are asymptotically best possible if we limit ourselves to local improvements of logarithmically bounded size, with respect to a fixed additive local search objective (see Definition 4.2 and Theorem 4.3). In particular, there is an asymptotic multiplicative gap of $\frac{3}{2}$ compared to the (matching) guarantees and lower bounds established in [28] for unit weights.

Definition 4.2. Let $f\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ and let $k \in \mathbb{Z}_{\geq 3}$. Let further $(\mathcal{S}, w)$ be an instance of the weighted k-Set Packing problem and let $A \subseteq \mathcal{S}$ be a feasible solution. We call a disjoint subcollection $X \subseteq \mathcal{S}$ a *local improvement of A with respect to $f \circ w$* if

$$(f \circ w)(X) := \sum_{x\in X} f(w(x)) > \sum_{a\in A} f(w(a)) =: (f \circ w)(A).$$

We call $|X|$ the *size* of the local improvement X.

Theorem 4.3. Let $k \in \mathbb{Z}_{\geq 3}$, $f\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$, $\varepsilon \in (0,1)$ and $C > 0$. Then there exist

- an instance $(\mathcal{S}, w)$ of the weighted k-Set Packing problem and
- a feasible solution $A \subseteq \mathcal{S}$

with the following properties:

- There is no local improvement of A with respect to $f \circ w$ of size at most $C \cdot \log(|\mathcal{S}|)$.
- $\left(\frac{k}{2} - \varepsilon\right) \cdot w(A) \leq \mathrm{OPT}(\mathcal{S}, w)$, where $\mathrm{OPT}(\mathcal{S}, w)$ denotes the value of an optimum solution to $(\mathcal{S}, w)$.

Observe that in the setting of Theorem 4.3, a local search algorithm that searches for local improvements w.r.t. $f \circ w$ of size bounded by $C \cdot \log(|\mathcal{S}|)$ until no more exist may just pick the locally optimum solution A set by set and then terminate. In particular, we can reach A via local improvements of infinite payoff factor[1]. This shows that a similar approach as in [15] cannot be used to obtain approximation ratios below $\frac{k}{2}$.

[1] Recall that in [15], the payoff factor of a local improvement is defined as the ratio between the total weight of the sets added to and the total weight of the sets removed from the current solution.

The remainder of this chapter is organized as follows: In Section 4.1, following [48], we explain how to obtain an approximation guarantee of $\frac{k+2+\epsilon}{3}$ for the unweighted k-Set Packing problem to showcase the way local improvements of logarithmically bounded size are used in the unweighted setting. In Section 4.2, we discuss how to generalize ideas from the unweighted to the weighted case. In doing so, we develop a new type of local improvement, which we call *circular improvement*. Section 4.3 presents a criterion that allows us to derive the existence of a circular improvement. This criterion will guide the analysis of our new algorithm *LogImp*, which we introduce and study in Section 4.4. In particular, we prove Theorem 4.1 in this section. Finally, we conclude this chapter with a proof of Theorem 4.3 in Section 4.5.

The algorithms in Sections 4.1 and 4.4, as well as the analyses of their approximation guarantees, easily generalize to the M(W)IS in $(k+1)$-claw free graphs. The only point where we actually need the underlying structure of an instance of the k-Set Packing problem is to get down to a polynomial (instead of quasi-polynomial) running time. Thus, we will phrase (most of) the following results in terms of the more general M(W)IS in $(k+1)$-claw free graphs. This is also more convenient notation-wise.

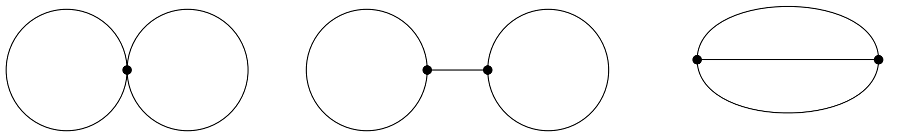
4.1 A $\frac{k+2+\epsilon}{3}$-approximation for unweighted k-Set Packing

In this section, we explain how to obtain a quasi-polynomial-time $\frac{k+2+\epsilon}{3}$-approximation algorithm for the MIS in $(k+1)$-claw free graphs via local improvements of logarithmically bounded size. For conflict graphs of instances of the unweighted k-Set Packing problem, a polynomial running time can be achieved [48]. The algorithm and the analysis we present are based on the work by Sviridenko and Ward [48][2].

To provide some intuition about where we are heading, first assume that we are given a $(k+1)$-claw free graph G and a *maximal* independent set A. We would like to compare $|A|$ to the size of a *maximum* independent set B. For this purpose, we partition B into the three sets B_1, B_2 and $B_{\geq 3}$ containing those vertices $u \in B$ with $|N(u, A)| = 1$, $|N(u, A)| = 2$ and $|N(u, A)| \geq 3$, respectively. As A is maximal, there is no $u \in B$ with $N(u, A) = \emptyset$. By double-counting adjacencies between A and B, i.e., ordered pairs $(u, v) \in B \times A$ with $u \in N(v, B)$ resp. $v \in N(u, A)$, we obtain

$$|B_1| + 2|B_2| + 3|B_{\geq 3}| \leq \sum_{u \in B} |N(u, A)| = \sum_{v \in A} |N(v, B)| \leq k \cdot |A|, \qquad (4.1)$$

[2]For simplicity, we omit one step from the analysis in [48] that would allow us to drop the ϵ-term in the approximation guarantee.



(a) two cycles joined in a single node (a path of length 0)

(b) two cycles joined by a path of positive length

(c) two nodes joined by three paths

Figure 4.1: The structure of a minimal binocular.

where the last inequality follows from Proposition 2.17. In particular, we can observe that if B were to solely contain vertices from $B_{\geq 3}$, we would even obtain a guarantee of $\frac{k}{3}$. In contrast, if B consists exclusively of vertices from $B_1 \cup B_2$, the best we can hope for is a guarantee of $\frac{k}{2}$, which is larger than what we are aiming for (at least for $k \geq 5$). Thus, we need to establish a bound on $|B_1| + |B_2|$. This is where local search comes into play: The strategy in [48] is to encode vertices from $B_1 \cup B_2$ as edges in an auxiliary graph G_A on the vertex set A, and to observe that there is a one-to-one correspondence between subgraphs of G_A featuring more edges than vertices, and local improvements $X \subseteq B_1 \cup B_2$. One can show that in case

$$|B_1| + |B_2| = |E(G_A)| \geq (1 + \epsilon) \cdot |V(G_A)| = |A|$$

for some $\epsilon > 0$, there exists such a subgraph of size $\mathcal{O}(\epsilon \cdot \log(|A|))$ [8, 48]. This subgraph gives rise to a well-structured local improvement which we can search for in polynomial time, provided the input graph is the conflict graph of an instance of the unweighted k-Set Packing problem [48].

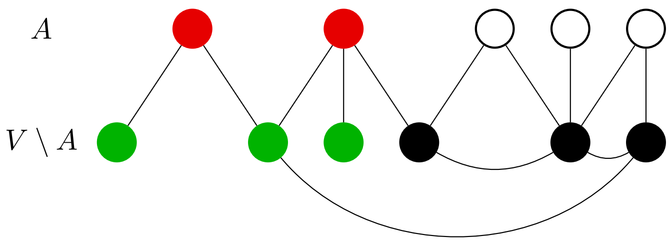
To formalize this idea, we require the following definition:

Definition 4.4 (binocular [8])**.** A *binocular* is a (multi)graph with more edges than vertices, where we allow parallel edges and loops. The *size* of a binocular is its number of edges.

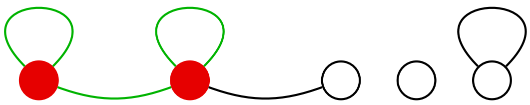
A *minimal binocular* is a binocular J such that no proper subgraph of J constitutes a binocular.

One can show that a minimal binocular always consists of two edge-disjoint cycles connected by a path (which may have length zero), or of three edge- and internally vertex-disjoint paths between a pair of vertices (see, e.g., [48]). This structural result explains the choice of the name “binocular”. See Fig. 4.1 for an illustration.

Berman and Fürer [8] have shown that sufficiently dense graphs contain binoculars of logarithmically bounded size.



(a) The graph G, with vertices from A drawn in the top, and vertices from $V \setminus A$ drawn in the bottom row. A binocular improvement $X \subseteq V \setminus A$ is depicted in green, $N(X, A)$ is marked in red.



(b) The auxiliary graph H. The edges corresponding to the binocular improvement X are drawn in green, $N(X, A)$ is again marked in red.

Figure 4.2: Illustration of the auxiliary graph H and Definition 4.7.

Lemma 4.5 ([8])**.** Let $s \in \mathbb{N}$ and let $G = (V, E)$ be a non-empty (multi)graph (where we allow parallel edges and loops) such that $|E| \geq \frac{s+1}{s} \cdot |V|$. Then G contains a binocular of size at most $2 \cdot s \cdot (2 \cdot \lfloor \log(|V|) \rfloor + 1)$.[3]

In the spirit of the above outline, we would now ideally like to consider an algorithm that, in each iteration, first greedily extends the current solution A to a maximal independent set, and then, if exists, applies a local improvement corresponding to the edge set of a binocular in G_A of logarithmically bounded size. The issue with this approach is of course that the optimum solution B, and, thus, the graph G_A, is unknown to us. Instead, we need to consider a larger[4] auxiliary graph H, where *every* vertex $u \in V \setminus A$ with $|N(u, A)| \leq 2$ induces a loop on its neighbor in A (if $|N(u, A)| = 1$) or an edge connecting its neighbors in A (if $|N(u, A)| = 2$). We will use e_u to denote the loop/edge corresponding to u.

Other than for G_A, not every binocular J in H will correspond to a local improvement, but only those with the property that $E(J)$ corresponds to an *independent set*. Searching for such a binocular (of logarithmically bounded size) in polynomial time is possible for conflict graphs of k-Set Packing instances by employing the *color-coding technique* (coloring the underlying

[3] We remark that in [8], a bound of $4 \cdot s \cdot \log(|V|)$ is claimed, but this is not entirely correct, e.g., for $|V| = 1$. We provide a proof of the corrected statement in Section 4.3.

[4] Technically speaking, H is not always a super-graph of G_A because we lose loops that vertices from $A \cap B$ induce on themselves. However, these vertices do not have any further incident edges and can be ignored when searching for local improvements.

Algorithm 3: $\frac{k+2+\epsilon}{3}$-approximation for the MIS in $(k+1)$-claw free graphs based on [48]

Input: a $(k+1)$-claw free graph $G = (V, E)$, a parameter $\epsilon > 0$
Output: an independent set $A \subseteq V$

$A \leftarrow \emptyset$
$\tau \leftarrow 2 \cdot \lceil 2\epsilon^{-1} \rceil \cdot (2 \cdot \lfloor \log(|V|) \rfloor + 1)$
`improvement_found` $\leftarrow$ `true`
while `improvement_found` **do**
    if $\exists$ *local improvement* X *with* $|X| = 1$ **then**
        $A \leftarrow A \cup X$ `//` $N(X, A) = \emptyset$ `in this case`
    else if $\exists$ *binocular improvement* X *with* $|X| \leq \tau$ **then**
        $A \leftarrow (A \setminus N(X, A)) \cup X$
    else
        `improvement_found` $\leftarrow$ `false`
    end
end
return A

universe) [48]. All in all, we arrive at the notion of a *binocular improvement* (Definition 4.7). It corresponds to the notion of a *canonical improvement* in [48]. To formally define it, we employ the following notation:

Notation 4.6 (multiset of edges)**.** Let V be a non-empty finite set, let I be a finite index set and let $(S_i)_{i \in I}$ be a sequence of subsets of V, each of cardinality 1 or 2. We write $\{e_i = S_i : i \in I\}$ to denote the multiset of edges that contains one copy of the edge S_i for each $i \in I$. If $|S_i| = 1$, this edge is a loop, if $|S_i| = 2$, it constitutes a two-vertex edge.

Definition 4.7. Let $G = (V, E)$ be a graph and let $A \subseteq V$ be independent. We call an independent set $X \subseteq \{u \in V \setminus A : 1 \leq |N(u, A)| \leq 2\}$ a *binocular improvement of* A if the multigraph

$$H_X \coloneqq (N(X, A), \{e_u = N(u, A) : u \in X\})$$

constitutes a binocular. We call $|X|$ the *size* of the binocular improvement X. See Fig. 4.2 for an illustration.

Using Definition 4.7, the algorithm that we would like to analyze can be formulated as in Algorithm 3. It is almost identical to the algorithm studied in [48]. We observe that Algorithm 3 terminates after at most $|V| + 1$ iterations because the cardinality of A increases in each iteration except for the last one. Moreover, considering $\epsilon > 0$ to be a fixed constant, each iteration can be implemented to run in quasi-polynomial, and, for conflict graphs of instances of the unweighted k-Set Packing problem, even in polynomial time, see [48]. For the analysis of the approximation guarantee, we denote the solution returned by Algorithm 3 by A and pick an optimum solution B. Then A is a maximal independent set and in particular, we know that (4.1) holds. In addition, we observe that every binocular of size at most τ in G_A yields a binocular improvement because B is independent and after removing connected components consisting of an isolated vertex and one incident loop that correspond to vertices in $A \cap B$, G_A becomes a subgraph of H. By the termination criterion of our algorithm, we may, hence, conclude that G_A contains no binocular of size at most τ. By Lemma 4.5, we can infer that

$$|B_1| + |B_2| = |E(G_A)| \leq \left(1 + \frac{\epsilon}{2}\right) \cdot |V(G_A)| = \left(1 + \frac{\epsilon}{2}\right) \cdot |A|. \qquad (4.2)$$

Adding $2 \cdot (4.2)$ to (4.1) yields

$$3 \cdot |B| \leq 3|B_1| + 4|B_2| + 3|B_{\geq 3}| \leq (k + 2 + \epsilon) \cdot |A|,$$

and, thus,

$$|B| \leq \frac{k + 2 + \epsilon}{3} \cdot |A|$$

as desired.

4.2 Extending the notion of binocular improvements: circular improvements

The construction of the auxiliary graph H reminds us of the notions of single and double vertices (Definitions 3.4 and 3.5) that we have introduced in the previous chapter. Thus, it appears natural to define a similar auxiliary graph in which a single vertex induces a loop on its regular neighbor, and a double vertex corresponds to an edge connecting its regular neighbors. For simplicity, we again denote the resulting graph by H. Recall that Lemma 3.20 tells us that if $\sum_{v \in A} w(v) \cdot \deg_H(v)$, the weighted sum of degrees in H, is smaller than, say $\frac{k}{2} \cdot w(A)$, we obtain an improvement of $\Omega(k) \cdot w(A)$ in our approximation guarantee. This is more than sufficient to prove Theorem 4.1.

However, for the above approach to be successful, we still need to take care of the following two points: First of all, in the spirit of Definition 4.7, we have to come up with the notion of a certain type of subgraph of H that gives rise to a local improvement of logarithmically bounded size (and can be searched for efficiently). Second, we have to make sure that in case the weighted sum of degrees in H is large (think for example of $\frac{k}{2} \cdot w(A)$), then one of said subgraphs has to exist. In this section, we deal with the first point. The second one is handled in Section 4.3.

Binoculars do not suffice in the weighted case

Given our construction of the auxiliary graph H, the first idea one might have is to again look at independent sets $X \subseteq V \setminus A$ that induce the edges of a binocular in H. Unfortunately, these sets do not necessarily correspond to local improvements. To see this, consider a scenario where $A \cap B = \emptyset$ and all vertices in A feature a weight of $1 + \varepsilon$, whereas vertices in $V \setminus A$ only have a weight of 1. Now, if we want $X \subseteq V \setminus A$ to constitute a local improvement (w.r.t. w^2), we need $|X|$ to be by a *constant factor* of (more than) $(1+\varepsilon)^2$ larger than $|N(X, A)|$, and not just by an additive constant of 1. However, we will see in Section 4.5 that even if every vertex in H assumes its maximum possible degree of k, there does not need to be such a set X of logarithmically bounded size.

Contributions to the rescue

One might argue that in the above example where the weights of the vertices in A are by a factor of $1+\varepsilon$ larger than the weights of the vertices in B, we do not need to worry about whether or not we can find a local improvement after all because we have already achieved the desired approximation guarantee. However, we can modify the example and, instead, look at a scenario where again $A \cap B = \emptyset$, H is bipartite with bipartitions A_+ and A_- of equal size, and where vertices in $A_\pm$ feature a weight of $1 \pm \varepsilon$, whereas vertices from $V \setminus A \supseteq B$ once more receive a weight of 1. Now, the average weight is the same in A and B, respectively. Nonetheless, the *squared weight* of a double vertex from B remains by a constant factor smaller than the average squared weight of its neighbors in A, which amounts to $\frac{1}{2} \cdot ((1-\varepsilon)^2 + (1+\varepsilon)^2) = 1+\varepsilon^2$. As a consequence, we run into the same problem as before. Also, note that Lemma 3.20 crucially relies on the fact that we conduct local search with respect to the squared weight function, so getting rid of this assumption does not appear to be a viable solution.

Instead, our idea is to augment candidate improvements $X \subseteq B$ we get from H by adding further vertices from B with a *positive contribution* to the vertices in $N_H(X, A)$. To this end, for $v \in A$, let

$$T_v := \{u \in B : n(u) = v \text{ and } \operatorname{contr}(u, v) > 0\}.$$

Note that as $T_v \subseteq \{u \in B : n(u) = v\}$, the sets $T_v, v \in A$ are pairwise disjoint. Moreover, we define

$$A' := \left\{ v \in A : \sum_{u \in T_v} \operatorname{contr}(u, v) > \frac{1 - \lambda_k}{2} \cdot w(v) \right\},$$

where $\frac{k+1-\lambda_k}{2}$ is the approximation guarantee we are targeting. Now, we make the following observations: First, Corollary 2.22 tells us that

$$\sum_{u \in B} \mathrm{charge}(u, v) \leq \frac{1-\lambda_k}{4} \cdot w(v) \text{ for all } v \in A \setminus A'.$$

Second, we may assume that $w(A') \geq \frac{1-\lambda_k}{2}$ holds: Otherwise, Corollary 2.22 and Proposition 2.20 yield

$$\begin{aligned}
&\sum_{u \in B} \mathrm{charge}(u, n(u)) = \sum_{v \in A} \sum_{u \in B} \mathrm{charge}(u, v) \\
&= \sum_{v \in A'} \sum_{u \in B} \mathrm{charge}(u, v) + \sum_{v \in A \setminus A'} \sum_{u \in B} \mathrm{charge}(u, v) \\
&\leq \frac{w(A')}{2} + \frac{1-\lambda_k}{4} \cdot w(A \setminus A') \\
&\leq \frac{1-\lambda_k}{4} \cdot w(A) + \frac{1-\lambda_k}{4} \cdot w(A) = \frac{1-\lambda_k}{2} \cdot w(A),
\end{aligned}$$

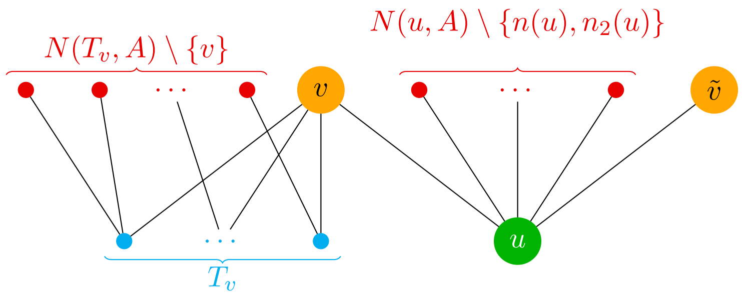
and Lemma 2.18 allows us to derive the desired approximation guarantee.

We would like to make sure that for every candidate improvement $X \subseteq B$ we get from H, $N_H(X, A)$ contains many vertices from A' since for these, we can add the vertices in T_v to augment X. Thus, we simply restrict H to contain only edges with at least one endpoint in A'. Note that as $w(A')$ makes up a constant fraction of $w(A)$, we can still employ Lemma 3.20 to conclude that either we have reached the desired approximation guarantee, or the weighted sum of degrees in H amounts to $\Omega(k) \cdot w(A)$. Now, instead of a binocular, we consider a *cycle* C in H of logarithmically bounded size. For a vertex $u \in V \setminus A$ that induces an edge $e_u = \{n(u), n_2(u)\}$, we use $w^2(u)$ and half of the contributions from the sets T_v with $v \in \{n(u), n_2(u)\} \cap A'$ to cover for $\frac{1}{2} \cdot w^2(\{n(u), n_2(u)\})$ (the other half will be taken care of by the other incident edges in the cycle) as well as $w^2(N(u, A \setminus \{n(u), n_2(u)\}))$. See Fig. 4.3 for an illustration.

In principle, for this argument to work, it suffices to choose the threshold $\varepsilon = \varepsilon_k$ (see Definitions 3.4 and 3.5) small enough compared to $1 - \lambda_k$. However, note that we have to be a bit careful because the vertex u might be contained in one of the sets T_v with $v \in \{n(u), n_2(u)\}$. Luckily, if u is double, then its contribution will be negligible compared to the total contribution from T_v. While it is indeed necessary to exclude cycles consisting of only one loop corresponding to a single vertex for the above reason, this does not harm our bound on the weighted sum of degrees because no vertex from A can have more than one regular single neighbor in B (cf. Lemma 3.9).

Circular improvements

The previous considerations finally lead us to the notion of a *circular improvement*.



$$w^2(u) + \tfrac{1}{2} \cdot \sum_{z \in T_v} \underbrace{w^2(z) - w^2(N(z, A) \setminus \{v\})}_{= \mathrm{contr}(z,v) \cdot w(v)}$$
$$> \tfrac{1}{2} \cdot w^2(\{n(u), n_2(u)\}) + w^2(N(u, A) \setminus \{n(u), n_2(u)\})$$

Figure 4.3: The double vertex u induces an edge $e_u = \{n(u), n_2(u)\} =: \{v, \tilde{v}\}$, where $v \in A'$. We use $w^2(u)$ and half of the contributions from T_v to v to pay for half of $w^2(\{n(u), n_2(u)\})$, as well as $w^2(N(u, A) \setminus \{n(u), n_2(u)\})$.

Definition 4.8 (circular improvement)**.** Let $k \in \mathbb{Z}_{\geq 3}$, let $G = (V, E)$ be a $(k+1)$-claw free graph, let $w \colon V \to \mathbb{R}_{>0}$ and let $A \subseteq V$ be independent.

Let further two maps

- $n \colon \{u \in V : |N(u, A)| \geq 1\} \to A$ mapping u to an element of $N(u, A)$ of maximum weight, and
- $n_2 \colon \{u \in V : |N(u, A)| \geq 2\} \to A$ mapping u to an element of $N(u, A) \setminus \{n(u)\}$ of maximum weight be given.

We call an independent set $X \subseteq \{u \in V \setminus A : |N(u, A)| \geq 1\}$ a *circular improvement of A of length ℓ* if there is $U \subseteq X$ with the following properties:

(4.8.1) $|N(u, A)| \geq 2$ for each $u \in U$ and

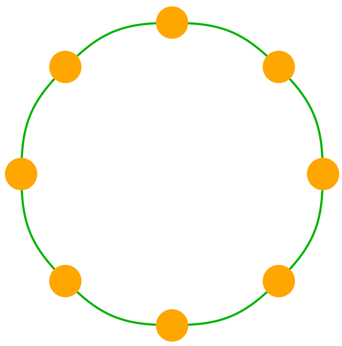
$$C := \left(\bigcup_{u \in U} \{n(u), n_2(u)\}, \{e_u = \{n(u), n_2(u)\} : u \in U\} \right)$$

is a cycle of length $|U| = |E(C)| = \ell$.

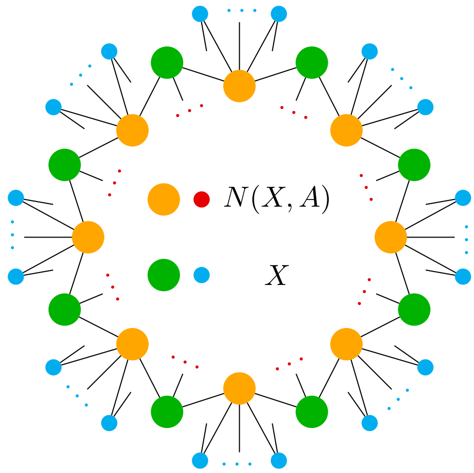
(4.8.2) If we let $Y_v := \{x \in X \setminus U : n(x) = v\}$, then $X = U \cup \bigcup_{v \in V(C)} Y_v$.

(4.8.3) For every $u \in U$, we have

$$w^2(u) + \frac{1}{2} \cdot \sum_{v \in \{n(u), n_2(u)\}} \sum_{z \in Y_v} w^2(z) - w^2(N(z, A) \setminus \{v\})$$
$$> \frac{w^2(n(u)) + w^2(n_2(u))}{2} + w^2(N(u, A) \setminus \{n(u), n_2(u)\}).$$



(a) The cycle C.



(b) A circular improvement.

Figure 4.4: Illustration of a circular improvement. Vertices from U are indicated by big green circles, vertices from $\bigcup_{v \in V(C)} Y_v$ are drawn as small cyan circles. Big circles drawn in orange represent the vertices in $V(C)$, and red dots indicate vertices from $N(X, A) \setminus V(C)$.

See Fig. 4.4 for an illustration.

We remark that the sets Y_v play the role of the sets T_v (minus the vertices from U appearing in them). We first establish the following bound on the size of a circular improvement.

Proposition 4.9. If X is a circular improvement and C and $(Y_v)_{v \in V(C)}$ are as in Definition 4.8, then we have $|Y_v| \leq k$ for all $v \in V(C)$. In particular, we obtain $|X| \leq (k+1) \cdot |E(C)|$.

Proof. Let further U be as in Definition 4.8. As X is independent and $Y_v \subseteq N(v, X)$ by definition of Y_v and the map n, Proposition 2.17 tells us that $|Y_v| \leq k$ for all $v \in V(C)$. In particular, we obtain

$$|X| = |U| + \sum_{v \in V(C)} |Y_v| \leq |U| + k \cdot |V(C)| = (k+1) \cdot |E(C)|,$$

where $|U| = |V(C)|$ follows by definition of C. □

We further observe that a circular improvement indeed constitutes a local improvement.

Proposition 4.10. Let X be a circular improvement of an independent set A. Then X constitutes a local improvement of A (w.r.t. w^2).

Proof. Let X be a circular improvement of A and let U, C and $(Y_v)_{v\in V(C)}$ be as in Definition 4.8. As X is independent, it remains to check that we have $w^2(X) > w^2(N(X,A))$. We compute

$$\begin{aligned}
w^2(X) &= w^2(U) + \sum_{v\in V(C)} w^2(Y_v)\\
&\overset{(*)}{=} \sum_{u\in U}\left[w^2(u) + \frac{1}{2}\cdot\sum_{v\in\{n(u),n_2(u)\}}\sum_{z\in Y_v} w^2(z)\right]\\
&\overset{(4.8.3)}{>} \sum_{u\in U}\left[\frac{w^2(n(u))+w^2(n_2(u))}{2} + w^2(N(u,A)\setminus\{n(u),n_2(u)\})\right.\\
&\qquad\qquad \left. + \frac{1}{2}\cdot\sum_{v\in\{n(u),n_2(u)\}}\sum_{z\in Y_v} w^2(N(z,A)\setminus\{v\})\right]\\
&\overset{(*)}{=} \sum_{v\in V(C)}\left[w^2(v) + \sum_{z\in Y_v} w^2(N(z,A)\setminus\{v\})\right]\\
&\quad + \sum_{u\in U} w^2(N(u,A)\setminus\{n(u),n_2(u)\}))\\
&\geq w^2(V(C)) + \sum_{v\in V(C)} w^2(N(Y_v,A)\setminus V(C)) + \sum_{u\in U} w^2(N(u,A)\setminus V(C))\\
&\geq w^2(N(X,A)).
\end{aligned}$$

The equations labeled $(*)$ follow from the fact that

$$C = \left(\bigcup_{u\in U}\{n(u),n_2(u)\}, \{e_u = \{n(u),n_2(u)\} : u\in U\}\right)$$

forms a cycle, meaning that each $v\in V(C)$ occurs exactly twice among all of the sets $\{n(u),n_2(u)\}, u\in U$. □

4.3 Existence of circular improvements

In this section, we prove Lemma 4.11. Applying it to the auxiliary graph H from the previous section will allow us to derive the existence of a circular improvement, provided that the weighted sum of degrees of vertices in A' is sufficiently large. Note that by construction of H and (3.7), we may choose $\alpha = (1-\varepsilon)^{-1}$.

Lemma 4.11. Let $G=(V,E)$ be a non-empty (multi)graph (where we allow parallel edges and loops) equipped with positive vertex weights $w\colon V\to\mathbb{R}_{>0}$.

Let further $\alpha\geq 1$ such that for every edge $\{u,v\}\in E$, we have

$$\alpha^{-1}\cdot w(u)\leq w(v)\leq \alpha\cdot w(u),$$

and let $s \in \mathbb{N}$ such that

$$\sum_{v \in V} w(v) \cdot |\deg_G(v)| \geq \frac{s+1}{s} \cdot 2 \cdot \alpha \cdot \sum_{v \in V} w(v),$$

where $\deg_G(v)$ denotes the degree of v in G. Then G contains a cycle of length at most $s \cdot (2 \cdot \lfloor \log(|V|) \rfloor + 1)$.

Our strategy to prove Lemma 4.11 is to show that G contains a subgraph G' with $|E(G')| \geq \frac{s+1}{s} \cdot |V(G')|$, and to then apply Lemma 4.12, which follows from the proof of Lemma 4.5 presented in [8].

Lemma 4.12 ([8])**.** Let $s \in \mathbb{N}_{>0}$ and let $G = (V, E)$ be a non-empty (multi)graph (where we allow parallel edges and loops) with $|E| \geq \frac{s+1}{s} \cdot |V|$. Then G contains a cycle of size at most $s \cdot (2 \cdot \lfloor \log(|V|) \rfloor + 1)$.

Note that Lemma 4.12 is the special case of Lemma 4.11 where $w \equiv 1$ and $\alpha = 1$.

In the following, we first prove two technical lemmata that allow us to convert weighted into unweighted sums, and vice versa. We need one of them to derive the existence of the subgraph G' mentioned above; the other one will be used in the next chapter. However, as the statements and proofs are rather similar, we prefer to present the two lemmata together.

Next, we show how to derive Lemma 4.11 from Lemma 4.12 using one of the two technical lemmata.

Finally, we prove Lemma 4.12 and Lemma 4.5.

4.3.1 Two technical lemmata

Lemma 4.13. Let S be a finite set, $\varphi \colon S \to \mathbb{R}_{\geq 0}$, $\mu \colon S \to \mathbb{R}_{\geq 0}$ and $\eta > 0$ such that

$$\sum_{s \in S} \varphi(s) \cdot \mu(s) > \eta \cdot \varphi(S).$$

Let further $\lambda > 0$. Then there exists $x \in \mathbb{R}_{>0}$ such that

$$\sum_{s \in S : \lambda \cdot \varphi(s) \geq x} \mu(s) > \lambda \cdot \eta \cdot |\{s \in S : \varphi(s) \geq x\}|.$$

Proof. Assume towards a contradiction that there was no $x \in \mathbb{R}_{>0}$ with the desired property. We get

$$\sum_{s \in S} \varphi(s) \cdot \mu(s) = \sum_{s \in S} \lambda^{-1} \cdot \lambda \cdot \varphi(s) \cdot \mu(s) = \sum_{s \in S} \lambda^{-1} \cdot \int_0^{\lambda \cdot \varphi(s)} \mu(s) \; dx$$
$$= \sum_{s \in S} \lambda^{-1} \cdot \int_0^{\infty} \mu(s) \cdot \mathbb{1}_{\lambda \cdot \varphi(s) \geq x} \; dx = \lambda^{-1} \cdot \int_0^{\infty} \sum_{s \in S} \mu(s) \cdot \mathbb{1}_{\lambda \cdot \varphi(s) \geq x} \; dx$$

$$
\begin{aligned}
&= \lambda^{-1} \cdot \int_0^\infty \sum_{s \in S : \lambda \cdot \varphi(s) \geq x} \mu(s) \ dx \leq \lambda^{-1} \cdot \int_0^\infty \lambda \cdot \eta \cdot |\{s \in S : \varphi(s) \geq x\}| \ dx \\
&= \eta \cdot \int_0^\infty |\{s \in S : \varphi(s) \geq x\}| \ dx = \eta \cdot \int_0^\infty \sum_{s \in S} \mathbb{1}_{\varphi(s) \geq x} \ dx \\
&= \eta \cdot \sum_{s \in S} \int_0^\infty \mathbb{1}_{\varphi(s) \geq x} \ dx = \eta \cdot \sum_{s \in S} \int_0^{\varphi(s)} 1 \ dx \\
&= \eta \cdot \varphi(S) < \sum_{s \in S} \varphi(s) \cdot \mu(s),
\end{aligned}
$$

a contradiction. Hence, there is $x \in \mathbb{R}_{>0}$ such that

$$\sum_{s \in S : \lambda \cdot \varphi(s) \geq x} \mu(s) > \lambda \cdot \eta \cdot |\{s \in S : \varphi(s) \geq x\}|.$$

□

Lemma 4.14. Let S_1 and S_2 be finite sets, $\varphi \colon S_1 \cup S_2 \to \mathbb{R}_{>0}$ and $\eta > 0$ such that

$$|S_1| > \eta \cdot |S_2|.$$

Let further $\lambda > 0$. Then there exists $x \in \mathbb{R}_{>0}$ such that

$$\sum_{s \in S_1 : \varphi(s) \leq x} \varphi(s) > \lambda \cdot \eta \cdot \sum_{s \in S_2 : \varphi(s) \leq \lambda^{-1} \cdot x} \varphi(s).$$

We remark that we can generalize the statement of the lemma by considering two different maps φ_1 and φ_2 for S_1 and S_2, respectively. In this setting, by rescaling φ_2, we can further assume without loss of generality that $\lambda = 1$. However, we decided to formulate the lemma in a way that is closest possible to the way it is applied in Chapter 5.

Proof. Let $\Phi \coloneqq \{\varphi(s) : s \in S_1\} \cup \{\lambda \cdot \varphi(s) : s \in S_2\}$. As S_1 and S_2 are finite and $S_1 \neq \emptyset$ (since $|S_1| > \eta \cdot |S_2| \geq 0$), Φ is a finite, non-empty set. Let

$$x_0 \coloneqq \min\{x \in \Phi : |\{s \in S_1 : \varphi(s) \leq x\}| > \eta \cdot |\{s \in S_2 : \varphi(s) \leq \lambda^{-1} \cdot x\}|\} > 0.$$

Note that we take the minimum over a non-empty set of values since for $x = \max \Phi$, the sets we obtain are S_1 and S_2, respectively, which satisfy $|S_1| > \eta \cdot |S_2|$. To simplify notation, let

$$S_1' \coloneqq \{s \in S_1 : \varphi(s) \leq x_0\} \text{ and } S_2' \coloneqq \{s \in S_2 : \varphi(s) \leq \lambda^{-1} \cdot x_0\}.$$

By definition, we have $|S_1'| > \eta \cdot |S_2'|$. Further observe that the two sets $\{s \in S_1 : \varphi(s) \leq x\}$ and $\{s \in S_2 : \varphi(s) \leq \lambda^{-1} \cdot x\}$ are empty if $x < \min \Phi$, and only depend on $\max\{y \in \Phi : y \leq x\}$ otherwise. Thus, for $0 < x < x_0$, we have

$$\begin{aligned} |\{s \in S_1' : \varphi(s) \leq x\}| = |\{s \in S_1 : \varphi(s) \leq x\}| &\leq \eta \cdot |\{s \in S_2 : \varphi(s) \leq \lambda^{-1} \cdot x\}| \\ &= \eta \cdot |\{s \in S_2' : \varphi(s) \leq \lambda^{-1} \cdot x\}| \qquad (4.3) \end{aligned}$$

by minimality of x_0. We compute

$$\begin{aligned} x_0 \cdot |S_1'| &= \sum_{s \in S_1'} \varphi(s) + x_0 - \varphi(s) = \varphi(S_1') + \sum_{s \in S_1'} x_0 - \varphi(s) \\ &= \varphi(S_1') + \sum_{s \in S_1'} \int_{\varphi(s)}^{x_0} 1 \; dx = \varphi(S_1') + \sum_{s \in S_1'} \int_0^{x_0} \mathbb{1}_{x \geq \varphi(s)} \; dx \\ &= \varphi(S_1') + \int_0^{x_0} \sum_{s \in S_1'} \mathbb{1}_{x \geq \varphi(s)} \; dx \\ &= \varphi(S_1') + \int_0^{x_0} |\{s \in S_1' : \varphi(s) \leq x\}| \; dx \\ &\overset{(4.3)}{\leq} \varphi(S_1') + \eta \cdot \int_0^{x_0} |\{s \in S_2' : \varphi(s) \leq \lambda^{-1} \cdot x\}| \; dx \\ &= \varphi(S_1') + \eta \cdot \int_0^{x_0} \sum_{s \in S_2'} \mathbb{1}_{\varphi(s) \leq \lambda^{-1} \cdot x} \; dx \\ &= \varphi(S_1') + \eta \cdot \sum_{s \in S_2'} \int_0^{x_0} \mathbb{1}_{\varphi(s) \leq \lambda^{-1} \cdot x} \; dx \\ &= \varphi(S_1') + \eta \cdot \sum_{s \in S_2'} \int_0^{x_0} \mathbb{1}_{\lambda \cdot \varphi(s) \leq x} \; dx \\ &= \varphi(S_1') + \eta \cdot \sum_{s \in S_2'} \int_{\lambda \cdot \varphi(s)}^{x_0} 1 \; dx = \varphi(S_1') + \eta \cdot \sum_{s \in S_2'} x_0 - \lambda \cdot \varphi(s) \\ &= \varphi(S_1') + x_0 \cdot \eta \cdot |S_2'| - \lambda \cdot \eta \cdot \varphi(S_2'). \end{aligned}$$

This results in

$$\varphi(S_1') \geq \lambda \cdot \eta \cdot \varphi(S_2') + x_0 \cdot (|S_1'| - \eta \cdot |S_2'|) > \lambda \cdot \eta \cdot \varphi(S_2'),$$

where the last inequality follows since $|S_1'| > \eta \cdot |S_2'|$ and $x_0 > 0$. This finishes the proof. □

4.3.2 Proving Lemma 4.11

Proof of Lemma 4.11, assuming Lemma 4.12. If G contains a loop, we are done, so assume that this is not the case. By Lemma 4.12, it suffices to show that G contains a non-empty subgraph G' with $|E(G')| \geq \frac{s+1}{s} \cdot |V(G')|$. As G

has only finitely many subgraphs, it suffices to prove that for each $\varepsilon \in (0,1)$, there is a non-empty subgraph G' with $|E(G')| \geq \frac{s+1}{s} \cdot (1-\varepsilon) \cdot |V(G')|$. Fix $\varepsilon \in (0,1)$. As G is non-empty and $w > 0$, we have

$$\sum_{v \in V} w(v) \cdot |\deg_G(v)| > \frac{s+1}{s} \cdot 2 \cdot \alpha \cdot (1-\varepsilon) \cdot \sum_{v \in V} w(v).$$

We apply Lemma 4.13 with $S := V$, $\varphi := w$, $\mu := \deg_G$, $\lambda := \alpha^{-1}$ and $\eta := \frac{s+1}{s} \cdot 2 \cdot \alpha \cdot (1-\varepsilon)$ to obtain $x \in \mathbb{R}_{>0}$ with

$$\sum_{\substack{v \in V: \\ w(v) \geq \alpha \cdot x}} \deg_G(v) > \frac{s+1}{s} \cdot 2 \cdot (1-\varepsilon) \cdot |\{v \in V : w(v) \geq x\}|. \tag{4.4}$$

Let $V' := \{v \in V : w(v) \geq x\} \supseteq \{v \in V : w(v) \geq \alpha \cdot x\} \neq \emptyset$ and define $G' := G[V']$. By our assumption that the weights of neighboring vertices differ by a factor of at most α, we know that for every $v \in V$ with $w(v) \geq \alpha \cdot x \geq x$, all neighbors of v are contained in V'. In particular, (4.4) allows us to conclude that

$$|E(G')| \geq \frac{1}{2} \cdot \sum_{\substack{v \in V: \\ w(v) \geq \alpha \cdot x}} \deg_G(v) > \frac{s+1}{s} \cdot (1-\varepsilon) \cdot |V'| = \frac{s+1}{s} \cdot (1-\varepsilon) \cdot |V(G')|.$$

□

4.3.3 Proving Lemma 4.12 and Lemma 4.5

The following proof is based on [8].

Proof. For this proof, we count loops twice towards the degree of a vertex, i.e., the degree of a vertex equals twice its number of incident loops plus its number of incident two-vertex edges. Fix $s \in \mathbb{N}_{>0}$ and let $G = (V, E)$ be a (multi)graph with $|E| \geq \frac{s+1}{s} \cdot |V|$ and $V \neq \emptyset$. In particular, $|E| > |V|$. We show that G contains a cycle of length at most $s \cdot (2\lfloor \log(|V|) \rfloor + 1)$ and a binocular of size at most $2 \cdot s \cdot (2\lfloor \log(|V|) \rfloor + 1)$. To this end, we first perform the following two reduction steps, until none of them applies anymore:

(i) If there is a vertex of degree at most 1, delete it and its incident edge, if exists.

(ii) If there is a walk P in G that contains at least $s+1$ edges and such that all inner vertices of P have degree 2 in G, delete the edges of P and all inner vertices of P from G.

First of all, as each reduction step reduces the number of vertices, this process terminates. Next, we observe that both (i) and (ii) preserve the inequality $|E| \geq \frac{s+1}{s} \cdot |V|$ since the number of edges we remove is by a factor of at most $\frac{s+1}{s}$ larger than the number of vertices we delete. Moreover, both (i) and (ii) preserve the property that $V \neq \emptyset$. For (ii), this is clear since the endpoint(s) of P survive. For (i), we note that $|E| - |V|$ may only increase. Thus, if we have $V \neq \emptyset$ before (i), then $|E| \geq \frac{s+1}{s} \cdot |V|$ implies that $|E| - |V| > 0$ holds before (i). Hence, $E \neq \emptyset$, and, thus, also $V \neq \emptyset$, holds after (i).

After the reduction steps, $G = (V, E)$ has the following properties:

(a) $|E| \geq \frac{s+1}{s} \cdot |V|$ and $V \neq \emptyset$.

(b) Every vertex in G has degree at least 2.

(c) Every walk in G on which all inner vertices have degree 2 in G has length at most s.

We may further assume that G is connected since we can just pick a connected component C of G with $|E(C)| \geq \frac{s+1}{s} \cdot |V(C)|$ otherwise. Under this assumption, no vertex of degree 2 in G can have an incident loop because then connectivity would imply that G consists of exactly one vertex and one edge, contradicting (a). Let $\mathcal{P}$ be the family of maximal walks P in G with the property that all inner vertices of P have degree exactly 2 in G.

Claim 4.15. For every $P \in \mathcal{P}$, the endpoint(s) of P have degree at least 3 in G.

Proof. Let $P \in \mathcal{P}$ and let v be an endpoint of P. By (b), v has degree at least 2 in G. Assume towards a contradiction that the degree of v is exactly 2. Then v does not have an incident loop. Moreover, if both incident edges of v were contained in P, then P would constitute a cycle and a connected component of G. Again, this contradicts (a), combined with our assumption that G is connected. Thus, one of the incident edges of v is not contained in P. But this contradicts the maximality of P. □

Claim 4.16. The walks in $\mathcal{P}$ are internally vertex- and pairwise edge-disjoint.

Proof. By Claim 4.15, for a vertex v that appears as an inner vertex of a walk $P \in \mathcal{P}$, we can recover P by traversing G from v until we hit vertices of degree 3. This shows that the walks in $\mathcal{P}$ are internally vertex-disjoint. In particular, no edge incident to a vertex of degree 2 in G can appear on more than one walk. But for each edge e both endpoints of which have degree ≥ 3 in G, the only walk from $\mathcal{P}$ containing e is the walk that does not contain any edge other than e. By (b), this concludes the proof. □

For each $P \in \mathcal{P}$ consisting of more than one edge, we delete all edges and inner vertices of P and add a new edge between the endpoints of P. In case P is a closed walk, this edge will be a loop. Call the resulting graph $G' = (V', E')$. Then $|E'| - |V'| = |E| - |V| > 0$. In particular, G' is non-empty. Moreover, every vertex in G' has degree at least 3 (where again, we count loops twice towards the degree). Finally, by (c), each cycle/binocular H' in G' corresponds to a cycle/binocular H in G with $|E(H)| \leq s \cdot |E(H')|$. Thus, it suffices to see that G' contains a cycle of length at most $2 \cdot \lfloor \log(|V'|) \rfloor + 1$ and a binocular of length at most $2 \cdot (2 \cdot \lfloor \log(|V'|) \rfloor + 1)$. In case $|V'| = 1$, the unique vertex must have at least 2 incident loops and we are done. To deal with the case $|V'| \geq 2$, we show the following claim:

Claim 4.17 ([8])**.** Let Γ be a connected (multi)graph with $|E(\Gamma)| \geq |V(\Gamma)|$, and let $r \in V(\Gamma)$ such that every vertex in $V(\Gamma) \setminus \{r\}$ has degree at least 3. Then Γ contains a connected subgraph J with the properties that $r \in V(J)$, $|E(J)| \geq |V(J)|$ and $|E(J)| \leq 2 \cdot \lfloor \log(|V(\Gamma)|) \rfloor + 1$.

Proof. If $|V(\Gamma)| = 1$, then r must have an incident loop and we are done. Next, assume $|V(\Gamma)| \geq 2$. Pick a BFS-tree T for Γ rooted in r. Let V_i be the set of vertices at distance i from r in T. Let $m := \lfloor \log(|V(\Gamma)|) \rfloor$. If there is $v \in \bigcup_{i=1}^{m} V_i$ such that v has an incident loop or two parallel incident edges, then together with the r-v-path in T, we obtain the desired subgraph. Hence, we may assume that no $v \in \bigcup_{i=1}^{m} V_i$ is incident to a loop or to two parallel edges. We distinguish two cases:

Case 1: r has exactly one child in T. Denote this child by s. If every $v \in V_i$ has at least two children in T for $i = 1, \ldots, m$, then we obtain

$$|V(T)| \geq \left| \{r\} \,\dot\cup\, \dot{\bigcup_{i=1}^{m+1}} V_i \right| \geq 1 + \sum_{i=1}^{m+1} 2^{i-1} = 2^{m+1} > |V(\Gamma)|,$$

a contradiction. Thus, pick $1 \leq i \leq m$ and $v \in V_i$ such that v has at most one child in T. As v has degree at least 3 in Γ, v is adjacent (in Γ) to a vertex $w \in \bigcup_{\ell=0}^{i+1} V_\ell$ that is neither the parent nor a child of v. Then the subgraph J consisting of the r-v-path $T[r, v]$ in T, the r-w-path $T[r, w]$ in T, and the edge $\{v, w\}$ has the desired properties since $T[r, v]$ is of length at most $\lfloor \log(|V(\Gamma)|) \rfloor$, $T[r, w]$ is of length at most $\lfloor \log(|V(\Gamma)|) \rfloor + 1$, and $T[r, v]$ and $T[r, w]$ share at least one edge, namely $\{r, s\}$.

Case 2: r has at least two children in T. If there exist $i \in \{1, \ldots, m-1\}$ and $v \in V_i$ with the property that v has at most one child in T, then again, the fact that v has degree at least 3 in Γ implies that there is $w \in \bigcup_{\ell=1}^{i+1} V_\ell$ that is neither the parent nor a child of v, but adjacent to v in Γ. Then the subgraph consisting of the r-v-path in T, the r-w-path in T, and the edge $\{v, w\}$

has the desired properties. Next, assume that for every $i = 1, \dots, m-1$, every $v \in V_i$ has at least two children in T. Then

$$|V(T)| \geq \left|\{r\} \,\dot\cup\, \dot{\bigcup}_{i=1}^{m} V_i\right| \geq 1 + \sum_{i=1}^{m} 2^i = 2^{m+1} - 1 \geq |V(\Gamma)|,$$

so $V(\Gamma) = \{r\} \cup \bigcup_{i=1}^{m} V_i$. Let $v \in V_m$. As v has degree at least 3 in Γ, v has a neighbor w in Γ that is not the parent of v. Then the subgraph consisting of the r-v-path in T, the r-w-path in T and the edge $\{v, w\}$ has the desired properties. $\square$

We pick $r \in V'$ arbitrarily and apply the claim to obtain a connected subgraph J with $r \in V(J)$, $|E(J)| \leq 2 \cdot \lfloor \log(|V'|) \rfloor + 1$ and $|E(J)| \geq |V(J)|$. In particular, J contains a cycle, which concludes the proof of Lemma 4.12. If further $|E(J)| > |V(J)|$, then J even yields the desired binocular. Otherwise, we contract J (where contracting a loop means removing it). We define the vertex resulting from the contraction of $V(J) \ni r$ to be the new r. All vertices in $V(\Gamma) \setminus V(J)$ keep a degree of at least 3. Moreover, as $|V(J)| = |E(J)|$, the contraction reduces $|E'| - |V'|$ only by one, and thus, after the contraction, we still have $|E'| \geq |V'|$. Hence, we may apply the claim again to obtain another subgraph J_2. Glueing together J and J_2 yields a binocular. $\square$

4.4 Our algorithm LogImp

We have now accumulated all of the ingredients that we need to introduce our algorithm *LogImp*, which we will show to yield the guarantees promised in Theorem 4.1. As before, we will phrase both the algorithm and its analysis in terms of the more general MWIS in $(k+1)$-claw free graphs, and only exploit the structure of the underlying Set Packing instance to obtain a polynomial (instead of quasi-polynomial) running time. Like SquareImp (Algorithm 1) and SimpleImp (Algorithm 2), LogImp (Algorithm 4) is a local search algorithm that iteratively applies local improvement w.r.t. the *squared* weight function until no more exist. The types of local improvement it considers are local improvements of size 3, claw-shaped improvements and circular ones. We can easily observe that LogImp terminates after a finite number of iterations because $w^2(A)$ strictly increases in each iteration (except for the last one), so no solution $A \subseteq V$ can be attained twice. A polynomial number of iterations can be achieved at the cost of an arbitrary small loss in the approximation guarantee by applying Lemma 2.14. By simple enumeration, each single iteration can be implemented to run in quasi-polynomial time by Proposition 4.9, assuming κ to be a fixed constant. Finally, the definition of a local improvement implies that LogImp is correct in the sense that it returns an independent set.

Algorithm 4: LogImp

Input: a $(k+1)$-claw free graph $G=(V,E)$, $w\colon V \to \mathbb{R}_{>0}$, a parameter $\kappa \in \mathbb{N}_{>0}$

Output: an independent set $A \subseteq V$

$\tau \leftarrow \kappa \cdot (2 \cdot \lfloor \log(|V|) \rfloor + 1)$
$A \leftarrow \emptyset$
`improvement_found` $\leftarrow$ `true`
while `improvement_found` **do**
  if $\exists$ *local improvement* X *of size* 3 **then**
    $A \leftarrow (A \setminus N(X,A)) \cup X$
  end
  else if $\exists$ *claw-shaped improvement* X **then**
    $A \leftarrow (A \setminus N(X,A)) \cup X$
  end
  else if $\exists$ *circular improvement* X *of length at most* τ **then**
    $A \leftarrow (A \setminus N(X,A)) \cup X$
  end
  else
    `improvement_found` $\leftarrow$ `false`
  end
end
return A

The remainder of this section is organized as follows: In Section 4.4.1, we analyze the approximation ratio that LogImp attains. Then, in Section 4.4.2, we show how to implement a single iteration of LogImp in polynomial time, provided the input graph constitutes the conflict graph of an instance of the weighted k-Set Packing problem. In Section 4.4.3, we combine these two results with Lemma 2.14 to obtain Theorem 4.1.

4.4.1 Analysis

Our main result for this section is the following theorem:

Theorem 4.18. Let $k \in \mathbb{Z}_{\geq 3}$. Setting $\kappa := 10$, LogImp achieves an approximation guarantee of $\frac{k+1-\theta_k}{2}$ for the MWIS in $(k+1)$-claw free graphs, where

$$\theta_k := \begin{cases} 0.00246, & k = 3,4 \\ \frac{k-3.358}{14.48k+462}, & 5 \leq k \leq 7999 \\ \frac{k-3.316}{k+48.77 \cdot k^{\frac{2}{3}}}, & k \geq 8000 \end{cases}.$$

First, we observe that $\lim_{k\to\infty} \theta_k = 1$. In addition, we remark that the cases $k=3$ and $k=4$ follow directly from Theorem 3.1.

Next, we point out that our choice of the value of κ is, ultimately, arbitrary. As we will see in the proof of Theorem 4.18, larger values of κ result in (slightly) improved approximation guarantees due to a factor of $\frac{\kappa+1}{\kappa}$ that appears in the calculations of the constants. However, the running time of our algorithm depends exponentially on κ (see Section 4.4.2). Thus, we decided for a "moderate" choice of κ for which the approximation guarantee we achieve is quite close to the limit for $\kappa \to \infty$, and that gives the desired asymptotic behavior we are interested in.

Finally, we remark that we deliberately use θ_k instead of λ_k (as in Theorem 4.1) here because in order to guarantee a polynomial running time, we will lose another factor of $\frac{N}{N-1}$ (with N arbitrarily large, but constant) in the approximation ratio.

Fix $k \in \mathbb{Z}_{\geq 5}$, a $(k+1)$-claw free graph $G = (V, E)$, and a weight function $w\colon V \to \mathbb{R}_{>0}$. Let A be the solution returned by LogImp with input (G, w) and let B be an optimum solution to the MWIS in (G, w). Moreover, fix two maps n and n_2 as in Definition 4.8. Note that A is a maximal independent set, so the domain of n is V.

Our goal is to show that

$$w(B) \leq \frac{k+1-\theta_k}{2} \cdot w(A).$$

To prove this, we would like to re-use some of the results from Chapter 3 (mainly the notion of regular neighbors, as well as Lemma 3.20). To do so, let ε be a constant subject to (3.1)-(3.4). Moreover, let $\delta \in (0, 1)$ satisfy

$$1-(1-\varepsilon)^2+((2+\varepsilon^2)\cdot(1+\varepsilon)-(2-\varepsilon))^2 \leq \frac{1}{2}\cdot(\delta-8\varepsilon)\cdot(1-\varepsilon)^2. \tag{4.5}$$

In the proof of Theorem 4.18, we will pick the constants ε and δ depending on k, and verify that they meet the respective constraints. The constant δ can be thought of as a threshold for the total contribution to a vertex in A to be large. We use it to define the set A' that was introduced in Section 4.2.

Definition 4.19 (T_v and A')**.** For $v \in A$, we let

$$T_v := \{u \in B : n(u) = v \text{ and } \mathrm{contr}(u, v) > 0\}.$$

We further define

$$A' := \left\{ v \in A : \sum_{u \in T_v} \mathrm{contr}(u, v) > \delta \cdot w(v) \right\}.$$

As outlined in Section 4.2, we introduce an auxiliary graph H, and show that cycles in H of logarithmically bounded size give rise to circular improvements (see Lemma 4.21). From this, we derive an upper bound on the weighted sum of the numbers of regular neighbors the vertices in A' feature (Corollary 4.23). In case $w(A')$ makes up a large enough fraction of $w(A)$, this allows us to improve our bound $w(B) \leq \frac{k+1}{2} \cdot w(A)$ (cf. Lemma 4.24). On the other hand, if $w(A')$ is significantly smaller than $w(A)$, then we obtain an improved guarantee due to the fact that the total charges a vertex $v \in A \setminus A'$ receives can be bounded by $\frac{\delta}{2} \cdot w(v)$ instead of just $\frac{1}{2} \cdot w(v)$, which is the bound used in the Berman analysis. This observation is captured by Lemma 4.25. Combining Lemma 4.24 and Lemma 4.25 allows us to prove Theorem 4.18.

Definition 4.20 (H)**.** The multigraph H on the vertex set $V(H) := A$ is given as follows: Every double vertex $u \in B$ induces an edge between its two regular neighbors, provided at least one of them is contained in A', i.e.,

$$E(H) := \{e_u := \mathrm{reg}_A(u) : u \in B \text{ is double and } \mathrm{reg}_A(u) \cap A' \neq \emptyset\}.$$

Recall that for a double vertex $u \in B$, we have $\mathrm{reg}_A(u) = \{n(u), n_2(u)\}$, so e_u corresponds to the edge that u would induce in a circular improvement.

Lemma 4.21. Assume that H contains a cycle C. Then there exists a circular improvement of A of length $|E(C)|$.

Proof. Let C be a cycle in H. To obtain a circular improvement, denote the set of vertices from B that induce the edges of C by U, and define

$$X := U \cup \bigcup_{v \in V(C) \cap A'} T_v.$$

Our goal is to show that X constitutes a circular improvement. First of all, X is independent as a subset of B.

Claim 4.22. $X \subseteq V \setminus A$.

Proof. Note that $V(C) \subseteq A \setminus B$ since vertices in $A \cap B$ are only adjacent to themselves in B, and, hence, isolated in H. As every vertex in X is adjacent to $V(C) \subseteq A \setminus B$, we can conclude that $X \cap A \cap B = \emptyset$ because no vertex in $A \cap B$ can be adjacent to $A \setminus B$ by independence of A. Consequently, $X \subseteq B \setminus A \subseteq V \setminus A$. □

By maximality of A, we obtain $X \subseteq \{u \in V \setminus A : |N(u, A)| \geq 1\}$. It remains to check that (4.8.1)-(4.8.3) from Definition 4.8 are satisfied. (4.8.1) follows from our choice of U. For (4.8.2), we observe that

$$Y_v = \begin{cases} T_v \setminus U, & v \in V(C) \cap A' \\ \emptyset, & v \in V(C) \setminus A' \end{cases}.$$

It remains to check (4.8.3). Let $u \in U$ and let $\{v, z\} := \mathrm{reg}_A(u)$ such that $v \in A'$ (and we do not make any assumptions on whether or not $z \in A'$). This is possible by construction of H. We observe that

$$\sum_{x \in Y_z} w^2(x) - w^2(N(x, A) \setminus \{z\}) \geq 0,$$

which is clear if $z \notin A'$, and follows from Definition 2.19 and Definition 4.19 if $z \in A'$. As a consequence, to verify (4.8.3), it is enough to prove (4.6).

$$\begin{aligned}
&\frac{1}{2} \cdot \left(\sum_{x \in T_v \setminus U} w^2(x) - w^2(N(x, A) \setminus \{v\}) \right) \\
&> \frac{w^2(\mathrm{reg}_A(u))}{2} - w^2(u) + w^2(N(u, A) \setminus \mathrm{reg}_A(u)). \qquad (4.6)
\end{aligned}$$

We first derive a lower bound on the left-hand side: Definition 2.19 and Definition 4.19 yield

$$\sum_{x \in T_v} w^2(x) - w^2(N(x, A) \setminus \{v\}) > \delta \cdot w^2(v). \qquad (4.7)$$

In order to obtain an upper bound on

$$\sum_{x \in T_v \cap U} w^2(x) - w^2(N(x, A) \setminus \{v\}),$$

we notice that every vertex $x \in T_v \cap U$ satisfies $n(x) = v$ by Definition 4.19, and induces the edge $e_x = \mathrm{reg}_A(x) \ni n(x) = v$ in $E(C)$, which is incident to v. As v has exactly two incident edges in C, $|T_v \cap U| \leq 2$. Finally, every $x \in T_v \cap U$ is a regular double neighbor of $v = n(x)$, which yields

$$\begin{aligned}
w^2(x) - w^2(N(x, A) \setminus \{v\}) &\leq w^2(x) - w^2(n_2(x)) \\
&\underset{(3.8)}{\overset{(3.7)}{\leq}} \left((1+\varepsilon)^2 - (1-\varepsilon)^2\right) \cdot w^2(v) = 4\varepsilon \cdot w^2(v).
\end{aligned}$$

Together with (4.7), this implies

$$\begin{aligned}
&\sum_{x \in T_v \setminus U} w^2(x) - w^2(N(x, A) \setminus \{v\}) \\
&> (\delta - 8\varepsilon) \cdot w^2(v) \\
&\overset{(3.7)}{\geq} (\delta - 8\varepsilon) \cdot (1-\varepsilon)^2 \cdot w^2(n(u)) \qquad (4.8)
\end{aligned}$$

since $v \in \mathrm{reg}_A(u) = \{n(u), n_2(u)\}$. Next, we prove an upper bound on the right-hand side of (4.6). As u is a double vertex, we obtain

$$\begin{aligned}
&|w(N(u,A)) - w(\mathrm{reg}_A(u))| = |w(N(u,A)) - w(n(u)) - w(n_2(u))| \\
&\quad = w(N(u,A)) - w(n(u)) - w(n_2(u)) \\
&\quad \overset{(3.9)}{\leq} (2+\varepsilon^2) \cdot w(u) - w(n(u)) - w(n_2(u)) \\
&\quad \underset{(3.8)}{\overset{(3.7)}{\leq}} \big((2+\varepsilon^2)\cdot(1+\varepsilon) - (2-\varepsilon)\big) \cdot w(n(u)).
\end{aligned} \tag{4.9}$$

Hence, we get

$$\begin{aligned}
&\frac{w^2(\mathrm{reg}_A(u))}{2} - w^2(u) + w^2(N(u,A) \setminus \mathrm{reg}_A(u)) \\
&\underset{(3.8)}{\overset{(3.7)}{\leq}} w^2(n(u)) - (1-\varepsilon)^2 \cdot w^2(n(u)) + w^2(N(u,A) \setminus \mathrm{reg}_A(u)) \\
&\leq w^2(n(u)) - (1-\varepsilon)^2 \cdot w^2(n(u)) + (w(N(u,A)) - w(\mathrm{reg}_A(u)))^2 \\
&\overset{(4.9)}{\leq} \big(1 - (1-\varepsilon)^2 + ((2+\varepsilon^2)\cdot(1+\varepsilon) - (2-\varepsilon))^2\big) \cdot w^2(n(u)).
\end{aligned} \tag{4.10}$$

Combining (4.8), (4.10) and (4.5) completes the proof. □

We remark that a (slightly) weaker version of (4.5) would be sufficient if we, for example, did not use *both* the lower *and* the upper bound on $w(n_2(u))$ and $w(u)$ provided by (3.7) and (3.8), respectively, in our estimates, but, instead, viewed our bounds as a function in $w(n_2(u))$ and $w(u)$ and calculated where they become tightest. However, as this increases the complexity of the calculations without providing further insights or changing the order of magnitude of our improvements, we decided in favor of the simpler version.

Next, we exploit that there is no circular improvement of length at most $\kappa \cdot (2 \cdot \lfloor \log(|V|) \rfloor + 1)$ to derive an upper bound on $\sum_{v \in A'} w(v) \cdot |\mathrm{reg}_B(v)|$.

Corollary 4.23. We have

$$\sum_{v \in A'} w(v) \cdot (|\mathrm{reg}_B(v)| - 1) \leq \frac{2}{1-\varepsilon} \cdot \frac{\kappa+1}{\kappa} \cdot w(A).$$

Proof. Assume towards a contradiction that

$$\sum_{v \in A'} w(v) \cdot (|\mathrm{reg}_B(v)| - 1) > \frac{2}{1-\varepsilon} \cdot \frac{\kappa+1}{\kappa} \cdot w(A).$$

By the termination criterion of LogImp, it suffices to show that there is a circular improvement of length at most $\kappa \cdot (2 \cdot \lfloor \log(|V|) \rfloor + 1)$. By Lemma 3.9, every vertex in A can have at most one regular single neighbor in B. Thus, every $v \in A'$ has at least $|\mathrm{reg}_B(v)| - 1$ many regular double neighbors in B. This yields

$$\sum_{v \in A} w(v) \cdot \deg_H(v) \geq \sum_{v \in A'} w(v) \cdot (|\mathrm{reg}_B(v)| - 1) > \frac{2}{1-\varepsilon} \cdot \frac{\kappa+1}{\kappa} \cdot w(A). \quad (4.11)$$

We further observe that by (3.7), the weights of adjacent vertices in H differ by a factor of at most $(1-\varepsilon)^{-1}$ multiplicatively. Thus, we can apply Lemma 4.11 and Lemma 4.21 to derive the desired contradiction. □

We remark that we could obtain a slightly stronger statement using the following observations:

1. For vertices $v \in A'$ that do not feature a regular single neighbor in B, we can omit the -1 on the left-hand side. In particular, we obtain an improved bound unless they only make up a very small fraction of $w(A')$. In this case, the weighted sum over their degrees in H is also small by Proposition 2.17.
2. By the same argument as in the proof of Lemma 3.11, we may observe that no two vertices in A' with a regular single neighbor in B can share an edge in H. In particular, whenever $v \in A'$ has a regular single neighbor in B, all of its incident edges in H go to vertices from A' without a regular single neighbor in B, or to $A \setminus A'$. By (3.7) and the first observation, we can assume that a large fraction of the sum $\sum_{v \in A'} w(v) \cdot \deg_H(v)$ comes from edges of the second type.
3. Currently, for edges connecting A' to $A \setminus A'$, in (4.11), we only account for the endpoint from A', even though (3.7) tells us that the second endpoint contributes a term almost as large.

However, the improvement we can obtain by formally carrying out these arguments is limited to, at best, getting rid of the -1 on the left-hand side, and unnecessarily increases the proof complexity without changing the qualitative statement we aim at. Thus, we omit the details.

Now, we prove bounds on the approximation guarantee of LogImp for the cases where $w(A')$ is "large" (Lemma 4.24) or "small" (Lemma 4.25), respectively.

Lemma 4.24. Let $\mu \in [0, 1]$. If $w(A') \geq \mu \cdot w(A)$, then

$$\begin{aligned} w(B) \leq & \frac{k+1}{2} \cdot w(A) \\ & - \frac{1}{2} \cdot \frac{\varepsilon^2}{(1+\varepsilon)\cdot(2+\varepsilon^2)} \cdot \left((k-1)\cdot\mu - \frac{2}{1-\varepsilon} \cdot \frac{\kappa+1}{\kappa}\right) \cdot w(A). \end{aligned}$$

Observe that this only results in an improved guarantee if we have $\mu > \frac{1}{k-1} \cdot \frac{2}{1-\varepsilon} \cdot \frac{\kappa+1}{\kappa}$.

Proof. Using Proposition 2.17 and Corollary 4.23, we calculate

$$\begin{aligned}\sum_{v\in A}(k - |\mathrm{reg}_B(v)|) \cdot w(v) &\geq \sum_{v\in A'}(k - |\mathrm{reg}_B(v)|) \cdot w(v)\\ &= (k-1)\cdot w(A') - \sum_{v\in A'}(|\mathrm{reg}_B(v)| - 1)\cdot w(v)\\ &\geq (k-1)\cdot w(A') - \frac{2}{1-\varepsilon}\cdot\frac{\kappa+1}{\kappa}\cdot w(A)\\ &\geq \left((k-1)\cdot\mu - \frac{2}{1-\varepsilon}\cdot\frac{\kappa+1}{\kappa}\right)\cdot w(A).\end{aligned}$$

Now, Lemma 3.20 yields the desired bound. □

Lemma 4.25. Let $\mu \in [0,1]$. If $w(A') \leq \mu \cdot w(A)$, then

$$w(B) \leq \frac{k+1}{2}\cdot w(A) - \frac{(1-\delta)\cdot(1-\mu)}{2}\cdot w(A).$$

Proof. We first observe that by Corollary 2.22, we get

$$\sum_{\substack{u\in B:\\ n(u)=v}} \mathrm{charge}(u,v) \leq \frac{\delta}{2}\cdot w(v) \text{ for all } v \in A\setminus A'.$$

Thus, we may apply Lemma 2.18, Proposition 2.20 and Corollary 2.22 to obtain

$$\begin{aligned}w(B) &\leq \frac{k}{2}\cdot w(A) + \sum_{u\in B}\mathrm{charge}(u,n(u))\\ &\leq \frac{k}{2}\cdot w(A) + \sum_{v\in A'}\sum_{\substack{u\in B:\\ n(u)=v}}\mathrm{charge}(u,v) + \sum_{v\in A\setminus A'}\sum_{\substack{u\in B:\\ n(u)=v}}\mathrm{charge}(u,v)\\ &\leq \frac{k}{2}\cdot w(A) + \frac{1}{2}\cdot w(A') + \frac{\delta}{2}\cdot w(A\setminus A')\\ &= \frac{k+1}{2}\cdot w(A) - \frac{1-\delta}{2}\cdot w(A\setminus A')\\ &\leq \frac{k+1}{2}\cdot w(A) - \frac{(1-\delta)\cdot(1-\mu)}{2}\cdot w(A).\end{aligned}$$

□

Now, we are ready to prove Theorem 4.18.

Proof of Theorem 4.18. The cases $k = 3$ and $k = 4$ are a direct consequence of Theorem 3.1. To deal with the case $k \geq 5$, we define

$$\mu^* := \frac{1 - \delta + \frac{\varepsilon^2}{(1+\varepsilon)\cdot(2+\varepsilon^2)} \cdot \frac{2}{1-\varepsilon} \cdot \frac{\kappa+1}{\kappa}}{1 - \delta + \frac{\varepsilon^2}{(1+\varepsilon)\cdot(2+\varepsilon^2)} \cdot (k-1)}.$$

Note that $\mu^* \in [0, 1]$ because $\delta \in (0, 1)$ and

$$\frac{2}{1-\varepsilon} \cdot \frac{\kappa+1}{\kappa} \underset{\kappa=10}{\overset{(3.1)}{<}} \frac{2}{0.75} \cdot 1.1 < 3 < k - 1.$$

Moreover, μ^* balances the bounds from Lemma 4.24 and Lemma 4.25 in that both yield a guarantee of

$$w(B) \leq \left(\frac{k+1}{2} - \frac{1}{2} \cdot \frac{k - 1 - \frac{2}{1-\varepsilon} \cdot \frac{\kappa+1}{\kappa}}{\frac{k-1}{1-\delta} + \frac{(1+\varepsilon)\cdot(2+\varepsilon^2)}{\varepsilon^2}} \right) \cdot w(A). \tag{4.12}$$

Indeed, we calculate

$$\begin{aligned}
&\frac{\varepsilon^2}{(1+\varepsilon)\cdot(2+\varepsilon^2)} \cdot \left((k-1) \cdot \mu^* - \frac{2}{1-\varepsilon} \cdot \frac{\kappa+1}{\kappa} \right) \\
&= \left(\frac{\varepsilon^2}{(1+\varepsilon)\cdot(2+\varepsilon^2)} \cdot (k-1) + (1-\delta) \right) \cdot \mu^* - (1-\delta) \cdot \mu^* \\
&\quad - \frac{\varepsilon^2}{(1+\varepsilon)\cdot(2+\varepsilon^2)} \cdot \frac{2}{1-\varepsilon} \cdot \frac{\kappa+1}{\kappa} \\
&= 1 - \delta + \frac{\varepsilon^2}{(1+\varepsilon)\cdot(2+\varepsilon^2)} \cdot \frac{2}{1-\varepsilon} \cdot \frac{\kappa+1}{\kappa} - (1-\delta) \cdot \mu^* \\
&\quad - \frac{\varepsilon^2}{(1+\varepsilon)\cdot(2+\varepsilon^2)} \cdot \frac{2}{1-\varepsilon} \cdot \frac{\kappa+1}{\kappa} \\
&= (1-\delta) \cdot (1-\mu^*)
\end{aligned}$$

and

$$\begin{aligned}
&(1-\delta) \cdot (1-\mu^*) \\
&= (1-\delta) \cdot \frac{\frac{\varepsilon^2}{(1+\varepsilon)\cdot(2+\varepsilon^2)} \cdot (k-1) - \frac{\varepsilon^2}{(1+\varepsilon)\cdot(2+\varepsilon^2)} \cdot \frac{2}{1-\varepsilon} \cdot \frac{\kappa+1}{\kappa}}{1 - \delta + \frac{\varepsilon^2}{(1+\varepsilon)\cdot(2+\varepsilon^2)} \cdot (k-1)} \\
&= (1-\delta) \cdot \frac{\varepsilon^2}{(1+\varepsilon)\cdot(2+\varepsilon^2)} \cdot \frac{k - 1 - \frac{2}{1-\varepsilon} \cdot \frac{\kappa+1}{\kappa}}{1 - \delta + \frac{\varepsilon^2}{(1+\varepsilon)\cdot(2+\varepsilon^2)} \cdot (k-1)} \\
&= \frac{k - 1 - \frac{2}{1-\varepsilon} \cdot \frac{\kappa+1}{\kappa}}{\frac{k-1}{1-\delta} + \frac{(1+\varepsilon)\cdot(2+\varepsilon^2)}{\varepsilon^2}}.
\end{aligned}$$

Now, to prove the bound for $5 \leq k \leq 7999$, we pick $\varepsilon := 0.067$ and $\delta := 0.9309$. Proposition 3.3 tells us that ε meets (3.1)-(3.4) and one can easily verify (4.5) by plugging in the two values. Moreover, plugging these constants and $\kappa = 10$ into (4.12) yields the desired bound.

Finally, for $k \geq 8000 = 20^3$, we pick $\varepsilon := k^{-\frac{1}{3}}$ and $\delta := 14\varepsilon$. Then Proposition 3.3 tells us that ε satisfies (3.1)-(3.4). As far as (4.5) is concerned, we calculate

$$\begin{aligned}
& 1 - (1-\varepsilon)^2 + ((2+\varepsilon^2)\cdot(1+\varepsilon) - (2-\varepsilon))^2 \leq \frac{1}{2}\cdot(14\varepsilon - 8\varepsilon)\cdot(1-\varepsilon)^2 \\
\Leftrightarrow\ & 1 - (1-\varepsilon)^2 + ((2+\varepsilon^2)\cdot(1+\varepsilon) - (2-\varepsilon))^2 \leq 3\varepsilon\cdot(1-\varepsilon)^2 \\
\Leftrightarrow\ & 14\varepsilon^2 + 3\varepsilon^3 + 7\varepsilon^4 + 2\varepsilon^5 + \varepsilon^6 \leq \varepsilon \quad | \quad \varepsilon > 0 \\
\Leftrightarrow\ & 14\varepsilon + 3\varepsilon^2 + 7\varepsilon^3 + 2\varepsilon^4 + \varepsilon^5 \leq 1.
\end{aligned}$$

The last inequality is true since $\varepsilon \leq \frac{1}{20}$. Using $\varepsilon = k^{-\frac{1}{3}} \leq \frac{1}{20}$ and $\delta = 14\varepsilon$, we first calculate

$$\frac{1}{1-\delta} = 1 + \frac{\delta}{1-\delta} \leq 1 + \frac{\delta}{1 - 14\cdot\frac{1}{20}} = 1 + \frac{10}{3}\cdot\delta = 1 + \frac{140}{3}\cdot\varepsilon. \tag{4.13}$$

Further using $\kappa = 10$, we obtain

$$\begin{aligned}
\frac{k - 1 - \frac{2}{1-\varepsilon}\cdot\frac{\kappa+1}{\kappa}}{\frac{k-1}{1-\delta} + \frac{(1+\varepsilon)\cdot(2+\varepsilon^2)}{\varepsilon^2}} &> \frac{k - 1 - \frac{2}{1-\varepsilon}\cdot\frac{\kappa+1}{\kappa}}{\frac{k}{1-\delta} + \frac{(1+\varepsilon)\cdot(2+\varepsilon^2)}{\varepsilon^2}} \\
&\overset{(4.13)}{\geq} \frac{k - 1 - \frac{2}{1-\frac{1}{20}}\cdot\frac{11}{10}}{k\cdot(1+\frac{140}{3}\cdot\varepsilon) + \frac{(1+\frac{1}{20})\cdot(2+\frac{1}{20^2})}{\varepsilon^2}} \\
&= \frac{k - 1 - \frac{2}{1-\frac{1}{20}}\cdot\frac{11}{10}}{k\cdot(1+\frac{140}{3}\cdot k^{-\frac{1}{3}}) + \frac{(1+\frac{1}{20})\cdot(2+\frac{1}{20^2})}{k^{-\frac{2}{3}}}} \\
&= \frac{k - \frac{63}{19}}{k + \frac{1170463}{24000}\cdot k^{\frac{2}{3}}} \\
&\geq \frac{k - 3.316}{k + 48.77\cdot k^{\frac{2}{3}}}.
\end{aligned}$$

Hence, (4.12) yields

$$w(B) \leq \left(\frac{k+1}{2} - \frac{1}{2}\cdot\frac{k-3.316}{k+48.77\cdot k^{\frac{2}{3}}}\right)\cdot w(A)$$

as desired. □

We point out that the constants for the case $5 \leq k \leq 7999$ are chosen to (approximately) optimize the guarantee we obtain for $k = 5$, whereas the constants for $k \geq 8000$ are chosen to highlight the asymptotic behavior of the approximation guarantees.

We further remark that the same arguments we use for $k \geq 5$ would also work for $k = 4$, but result in a worse constant than the simpler analysis from Chapter 3. For $k = 3$, we would require a stronger version of Corollary 4.23. Note that for $k = 3$, Corollary 4.23 actually follows from Proposition 2.17. While we have already discussed how such a stronger version could be obtained, we omit this here because we are more interested in the asymptotic behavior of the approximation guarantees.

4.4.2 Running time

Let $\kappa \in \mathbb{N}$ be a fixed constant. Our main result for this section is given by the following theorem:

Theorem 4.26. Let $k \in \mathbb{Z}_{\geq 3}$ be a fixed constant. Let $(\mathcal{S}, w)$ be an instance of the weighted k-Set Packing problem and let $G \coloneqq (V, E) \coloneqq G_{\mathcal{S}}$. Let further $A \subseteq V$ be a maximal independent set. We can, in polynomial time, either return a circular improvement X of A of length at most $\kappa \cdot (2 \cdot \lfloor \log(|V|) \rfloor + 1)$, or decide that none exists.

Fix $k \in \mathbb{Z}_{\geq 3}$ and an instance $(\mathcal{S}, w)$ of the weighted k-Set Packing problem and let $G \coloneqq (V, E) \coloneqq G_{\mathcal{S}}$. Let $A \subseteq V$ be a maximal independent set. In accordance with Definition 4.8, we compute two maps

- $n\colon \{u \in V : |N(u, A)| \geq 1\} \to A$ mapping u to an element of $N(u, A)$ of maximum weight, and
- $n_2\colon \{u \in V : |N(u, A)| \geq 2\} \to A$ mapping u to an element of $N(u, A) \setminus \{n(u)\}$ of maximum weight.

These maps can easily be computed in polynomial time. As A is maximal, the domain of n is V. We define an auxiliary multigraph H as follows:

Definition 4.27. The vertices of H are pairs (v, Y), where

(4.27.1) $v \in A$ and

(4.27.2) $Y \subseteq \{u \in V \setminus A : n(u) = v\}$ is an independent set.

The multiset $E(H)$ consists of the edges $e(u, Y_1, Y_2) \coloneqq \{(n(u), Y_1), (n_2(u), Y_2)\}$, where

(4.27.3) $u \in V \setminus A$ with $|N(u, A)| \geq 2$ and $(n(u), Y_1), (n_2(u), Y_2) \in V(H)$,

(4.27.4) $\{u\}$, Y_1 and Y_2 are pairwise disjoint, and

(4.27.5) We have

$$\begin{aligned} w^2(u) + \frac{1}{2} \cdot \Bigg(&\sum_{x \in Y_1} w^2(x) - w^2(N(x,A) \setminus \{n(u)\}) \\ &+ \sum_{x \in Y_2} w^2(x) - w^2(N(x,A) \setminus \{n_2(u)\}) \Bigg) \\ > \frac{w^2(n(u)) + w^2(n_2(u))}{2} &+ w^2(N(u,A) \setminus \{n(u), n_2(u)\}). \end{aligned}$$

We say that the edge $e(u, Y_1, Y_2)$ is *induced by* u and define $\mu \colon E(H) \to V \setminus A$, $e(u, Y_1, Y_2) \mapsto u$.

Proposition 4.28. For every $(v, Y) \in V(H)$, we have $|Y| \leq k$.

Proof. $G = G_{\mathcal{S}}$ is $(k+1)$-claw free by Proposition 2.8 and Y constitutes the set of talons of a claw centered at v, provided it is non-empty. □

Proposition 4.29. H can be constructed in polynomial time.

Proof. We first consider the set of vertices. By Proposition 4.28, for each $v \in A$, there are at most

$$\sum_{i=0}^{k} |V|^i = \frac{|V|^{k+1} - 1}{|V| - 1} \in \mathcal{O}(|V|^k)$$

many possible choices for Y, and we can check in polynomial time whether one of them meets (4.27.2).

As far as the edges are concerned, there are at most $|V|$ many possible choices for u and $\mathcal{O}(|V|^k)$ possibilities to choose each of Y_1 and Y_2. For given choices of u, Y_1 and Y_2, we can check (4.27.3)-(4.27.5) in polynomial time. □

We claim that there is a one-to-one correspondence between circular improvements and certain cycles $\bar{C}$ in H, which we call *proper*.

Definition 4.30. We call a cycle $\bar{C}$ in H *proper* if the following conditions hold:

(4.30.1) For every $v \in A$, there is at most one set Y with $(v, Y) \in V(\bar{C})$.

(4.30.2) $X(\bar{C}) := \{\mu(e), e \in E(\bar{C})\} \cup \bigcup_{(v,Y) \in V(\bar{C})} Y$ constitutes an independent set and the union is disjoint.

We remark that requirement that the union is disjoint can be omitted since it follows from (4.30.1), (4.27.2) and (4.27.4). However, adding this requirement shortens the subsequent arguments.

Proposition 4.31. A set $X \subseteq V$ constitutes a circular improvement of length ℓ if and only if there is a proper cycle $\bar{C}$ in H of length ℓ such that $X = X(\bar{C})$.

Proof. Let X constitute a circular improvement of length ℓ. Let U, C and $(Y_v)_{v\in V(C)}$ be as in Definition 4.8. By definition of the sets $(Y_v)_{v\in V(C)}$ and (4.8.3), we have

$$\bar{e}_u := e(u, Y_{n(u)}, Y_{n_2(u)}) = \{(n(u), Y_{n(u)}), (n_2(u), Y_{n_2(u)})\} \in E(H)$$

for every $u \in U$, and $\mu(\bar{e}_u) = u$. In particular,

$$\bar{C} := (\{(v, Y_v), v \in V(C)\}, \{\bar{e}_u, u \in U\})$$

yields a cycle in H of length ℓ by (4.8.1). (4.30.2) follows since X is circular and by definition of the sets Y_v, (4.30.1) follows by construction.

Now, let $\bar{C}$ be a proper cycle of length ℓ. By (4.30.2), (4.27.2) and (4.27.3), $X = X(\bar{C}) \subseteq V \setminus A$ constitutes an independent set.

Define $U := \{\mu(e) : e \in E(\bar{C})\}$. Then (4.30.1) tells us that for every $u \in U$ and every $v \in \{n(u), n_2(u)\}$, there is at most one set Y with $(v, Y) \in V(\bar{C})$. In particular, u induces at most one edge of $\bar{C}$. Hence, $|U| = |E(\bar{C})| = \ell$. Additionally,

$$C := \left(\bigcup_{u\in U} \{n(u), n_2(u)\}, \{e_u = \{n(u), n_2(u)\}, u \in U\} \right)$$

is a closed edge sequence because it arises from $\bar{C}$ by projecting every vertex $(v, Y) \in V(\bar{C})$ to its first component. By (4.30.1), C is a cycle. Thus, (4.8.1) holds. We further observe that (4.27.2), (4.30.1) and (4.30.2) imply $Y_v = Y$ for every $(v, Y) \in V(\bar{C})$, where Y_v is defined according to (4.8.2). Now, (4.8.2) and (4.8.3) follow from (4.30.2) and (4.27.5). □

It remains to see how to, in polynomial time, find a proper cycle in H of length at most $\kappa \cdot (2 \cdot \lfloor \log(|V|) \rfloor + 1)$, or decide that none exists.

To this end, we apply the *color-coding technique* and introduce the following terminology:

Definition 4.32 (t-perfect family of hash functions, [2])**.** For $t, m \in \mathbb{N}$ with $t \leq m$, a family $\mathcal{F} \subseteq {}^{\{1,\dots,m\}}\{1, \dots, t\}$ of functions mapping $\{1, \dots, m\}$ to $\{1, \dots, t\}$ is called a *t-perfect family of hash functions* if for all $I \subseteq \{1, \dots, m\}$ of size at most t, there is $f \in \mathcal{F}$ with $f \restriction I$ injective.

Theorem 4.33 (stated in [2] referring to [44])**.** For $t, m \in \mathbb{N}$ with $t \leq m$, a t-perfect family $\mathcal{F}$ of hash functions of cardinality $2^{\mathcal{O}(t)} \cdot (\log(m))^2$, where each function is encoded using $\mathcal{O}(t) + 2 \log \log m$ many bits, can be explicitly constructed such that the query time is constant.

In order to find a proper cycle $\bar{C}$ in H meeting the desired size bound, we color our current solution A, as well as the underlying universe $\bigcup \mathcal{S}$ of our k-Set Packing instance. Then, we strengthen our requirements on $\bar{C}$ by demanding that the vertices v with $(v, Y) \in V(\bar{C})$ receive pairwise distinct colors, and that moreover, the sets of colors corresponding to the elements of $X(\bar{C})$ (when interpreted as sets in $\mathcal{S}$) are pairwise disjoint. This leads to the notion of a *colorful cycle* (see Definition 4.35).

In the following, we first show that we can pick our colorings from appropriately chosen families of perfect hash functions (of polynomial size) in such a way that for each proper cycle $\bar{C}$ in H of length at most $\kappa \cdot (2 \cdot \lfloor \log(|V|) \rfloor + 1)$, there exist colorings from our families for which $\bar{C}$ is colorful. The families we consider only require a logarithmic number of different colors.

Next, we explain how to, for fixed colorings from our families, use dynamic programming to check for the existence of a colorful cycle in polynomial time.

In particular, by looping over all colorings from our families and checking for a colorful cycle, we will find a circular improvement, if one exists.

By Theorem 4.33, let $\mathcal{F}$ be a $t_1 := \kappa \cdot (2 \cdot \lfloor \log(|V|) \rfloor + 1)$-perfect family of hash functions with domain $A \subseteq V$ of size

$$2^{\mathcal{O}(\log(|V|))} \cdot (\log(|V|))^2 = |V|^{\mathcal{O}(1)}.$$

In addition, we pick a $t_2 := (k+1) \cdot k \cdot \kappa \cdot (2 \cdot \lfloor \log(|V|) \rfloor + 1)$-perfect family of hash functions $\mathcal{G}$ with domain $\bigcup \mathcal{S}$, the underlying universe of the k-Set Packing instance. Clearly, $|\bigcup \mathcal{S}| \leq k \cdot |\mathcal{S}| = k \cdot |V|$. Thus, by Theorem 4.33, we can choose $\mathcal{G}$ to be of size

$$2^{\mathcal{O}(k^2 \cdot \log(|V|))} \cdot (\log(k \cdot |V|))^2 = |V|^{\mathcal{O}(k^2)},$$

which is polynomial.

We explain how we use the hash functions to color the vertices and edges of H. In doing so, we employ the term "k-set" to refer to elements of $\mathcal{S}$, whereas the term "set" may also refer to a set of vertices, for example.

Definition 4.34 (colors)**.** Let $f \in \mathcal{F}$ and $g \in \mathcal{G}$.

For every $v = (z, Y) \in V(H)$, we set $\mathrm{col}^f(v) := f(z)$, meaning that we color v with the color assigned to $z \in A$ by f. Additionally, we define

$$\mathrm{col}^g(v) := g\left(\bigcup Y\right) = \{g(x) : \exists u \in Y : x \in u\},$$

i.e., we color v with all of the colors occurring among the elements of the k-sets represented by the vertices in Y.

Finally, we assign to an edge $e = \{(v_1, Y_1), (v_2, Y_2)\} \in E(H)$ the set of colors

$$\mathrm{col}^g(e) := g(\mu(e)) = \{g(x), x \in \mu(e)\},$$

where we interpret $\mu(e)$ as the corresponding k-set.

Definition 4.35 (colorful path/cycle). Let $f \in \mathcal{F}$ and $g \in \mathcal{G}$. We call a path or cycle P in H *colorful for f and g* if the following conditions are satisfied:

(4.35.1) The colors $\mathrm{col}^f(v), v \in V(P)$ are pairwise distinct.

(4.35.2) The color sets $\mathrm{col}^g(e), e \in E(P)$ and $\mathrm{col}^g(v), v \in V(P)$ are pairwise disjoint.

Lemma 4.36. Let $\bar{C}$ be a cycle in H of length at most $\kappa \cdot (2 \cdot \lfloor \log(|V|) \rfloor + 1)$. Then the following are equivalent:

- There exist $f \in \mathcal{F}$ and $g \in \mathcal{G}$ for which $\bar{C}$ is colorful.
- $\bar{C}$ is a proper cycle.

Proof. Let $\bar{C}$ be colorful for f and g. (4.35.1) implies (4.30.1). By (4.35.2), the (unions of) k-sets in the list $(\bigcup Y)_{(v,Y)\in V(\bar{C})}$, $\mu(e)_{e\in E(\bar{C})}$ are pairwise disjoint. In particular, the union in (4.30.2) is disjoint since all sets in $\mathcal{S}$ are non-empty. Moreover, (4.27.2) ensures that for each $(v, Y) \in V(H)$, the k-sets in Y are pairwise disjoint. Thus, $X(\bar{C})$ constitutes a disjoint collection of k-sets, or, equivalently, an independent set in G. Hence, (4.30.2) holds.

Now, let $\bar{C}$ be a proper cycle of length $\leq \kappa \cdot (2 \cdot \lfloor \log(|V|) \rfloor + 1) = t_1$. By (4.30.1), there is $f \in \mathcal{F}$ that assigns a different color to each one of the vertices in $V(\bar{C})$. By (4.30.2), the k-sets contained in the sets $Y, (v, Y) \in V(H)$ and the k-sets $\mu(e), e \in E(\bar{C})$ are pairwise distinct and pairwise disjoint. In addition, as $|Y| \leq k$ for every $(v, Y) \in V(H)$ by Proposition 4.28, the total number of elements from $\bigcup \mathcal{S}$ contained in

$$\bigcup_{(v,Y)\in V(\bar{C})} \left(\bigcup Y\right) \cup \bigcup_{e\in E(\bar{C})} \mu(e)$$

can be bounded by

$$k^2 \cdot |V(\bar{C})| + k \cdot |E(\bar{C})| \leq (k+1) \cdot k \cdot \kappa \cdot (2 \cdot \lfloor \log(|V|) \rfloor + 1) = t_2.$$

Hence, there exists $g \in \mathcal{G}$ assigning distinct colors to all of these elements. Thus, $\bar{C}$ is colorful for f and g. □

As the sizes of $\mathcal{F}$ and $\mathcal{G}$ are polynomially bounded, Lemma 4.37 concludes the proof of Theorem 4.26.

Lemma 4.37. Let $f \in \mathcal{F}$ and $g \in \mathcal{G}$. We can, in polynomial time, either find a cycle $\bar{C}$ of length at most $\kappa \cdot (2 \cdot \lfloor \log(|V|) \rfloor + 1)$ in H that is colorful for f and g, or decide that none exists.

Proof. There is a colorful cycle $\bar{C}$ of length at most $\kappa \cdot (2 \cdot \lfloor \log(|V|) \rfloor + 1)$ if and only if there exist an edge $e_0 = \{s, t\} \in E(H)$ and a colorful s-t-path P of length at most $\kappa \cdot (2 \cdot \lfloor \log(|V|) \rfloor + 1) - 1$ such that $\mathrm{col}^g(e_0)$ and $\bigcup_{e \in E(P)} \mathrm{col}^g(e) \cup \bigcup_{v \in V(P)} \mathrm{col}^g(v)$ are disjoint. Note that e_0 cannot appear on the s-t-path P since the k-set $\mu(e_0)$ and, thus, the color set $\mathrm{col}^g(e_0)$ is non-empty. For

- $s, t \in V(H)$,
- $C^f \subseteq \{1, \ldots, t_1\}$,
- $C^g \subseteq \{1, \ldots, t_2\}$, and
- $i \in \{0, \ldots, \kappa \cdot (2 \cdot \lfloor \log(|V|) \rfloor + 1) - 1\}$,

we define the Boolean value $\mathrm{Path}(s, t, C^f, C^g, i)$ to be true if and only if there exists a colorful s-t-path P of length i with

$$\{\mathrm{col}^f(v), v \in V(P)\} = C^f \text{ and } \bigcup_{e \in E(P)} \mathrm{col}^g(e) \cup \bigcup_{v \in V(P)} \mathrm{col}^g(v) = C^g.$$

We apply dynamic programming to compute these values in order of non-decreasing i, and we use backlinks to be able to retrace a corresponding path for each value that is set to true. The total number of values we have to compute is bounded by

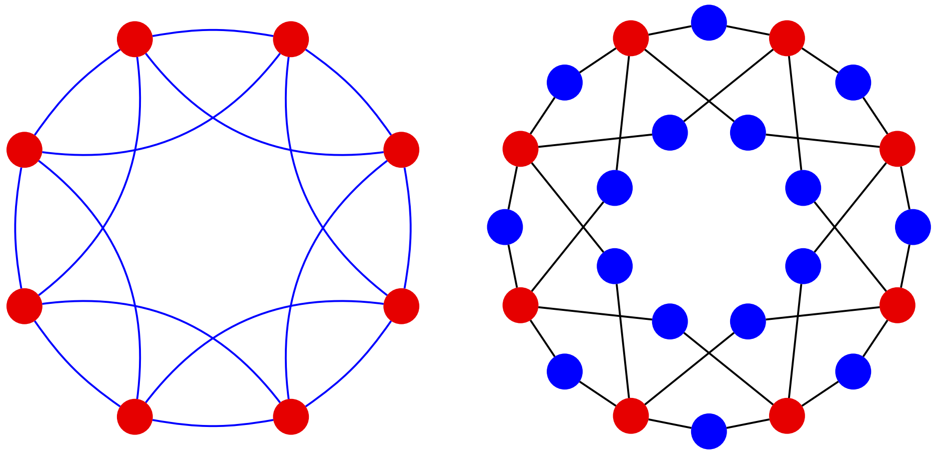
$$\begin{aligned}&|V(H)|^2 \cdot 2^{t_1} \cdot 2^{t_2} \cdot \kappa \cdot (2 \cdot \lfloor \log(|V|) \rfloor + 1)\\ &= |V|^{\mathcal{O}(k)} \cdot |V|^{\mathcal{O}(1)} \cdot |V|^{\mathcal{O}(k^2)} \cdot \log(|V|) = |V|^{\mathcal{O}(k^2)},\end{aligned}$$

which is polynomially bounded. Hence, it suffices to show how to, in polynomial time, determine $\mathrm{Path}(s, t, C^f, C^g, i)$, provided that all of the values $\mathrm{Path}(s', t', C^{f,'}, C^{g,'}, i')$ with $i' < i$ are already known to us. We have $\mathrm{Path}(s, t, C^f, C^g, 0) = \text{true}$ if and only if $s = t$, $C^f = \{\mathrm{col}^f(s)\}$ and $C^g = \mathrm{col}^g(s)$. For $i \geq 1$, we observe that $\mathrm{Path}(s, t, C^f, C^g, i) = \text{true}$ if and only if $\mathrm{col}^f(s) \in C^f$ and there exists an edge $e = \{s, v\} \in E(H)$ such that

- $\mathrm{col}^g(s) \cap \mathrm{col}^g(e) = \emptyset$,
- $\mathrm{col}^g(s) \cup \mathrm{col}^g(e) \subseteq C^g$ and
- $\mathrm{Path}(v, t, C^f \setminus \{\mathrm{col}^f(s)\}, C^g \setminus (\mathrm{col}^g(s) \cup \mathrm{col}^g(e)), i - 1) = \text{true}$.

We can check this in polynomial time. Observe that we indeed compute a path (as opposed to an edge sequence, potentially visiting vertices multiple times) since every vertex receives a color via f and in particular, we cannot encounter a vertex twice. $\square$

We remark that similar arguments are used in [28] and [48] to obtain a polynomial running time.



(a) A 4-regular graph H with girth 3.

(b) The graph G with vertex set $V(G) = V(H) \cup E(H)$ arises by connecting each $e \in E(H)$ (blue) to its endpoints in H (red).

Figure 4.5: Illustration of the construction of the graph G.

4.4.3 Proof of Theorem 4.1

For $k = 1, 2$, we can solve the weighted k-Set Packing problem exactly in polynomial time [22]. Next, fix $k \in \mathbb{Z}_{\geq 3}$ and consider the following algorithm: Given an instance $(\mathcal{S}, w)$ of the weighted k-Set Packing problem, we first apply Lemma 2.14 with $N = (k+1) \cdot k + 1$ to obtain a new instance $(\mathcal{S}', w')$ with $\mathcal{S}' \subseteq \mathcal{S}$. Then, we run LogImp on $(G_{\mathcal{S}'}, w')$. By Lemma 2.14 (ii), we have

$$w'^{\,2}(A) \leq (w'(A))^2 \leq (k \cdot N \cdot |\mathcal{S}|)^2 \in \mathcal{O}(k^6 \cdot |\mathcal{S}|^2)$$

for any feasible solution A to $(G_{\mathcal{S}'}, w')$. As w' is integral, LogImp terminates on $(G_{\mathcal{S}'}, w')$ after $\mathcal{O}(k^6 \cdot |\mathcal{S}|^2)$ iterations. Moreover, by Theorem 4.26, each iteration can be implemented to run in polynomial time. Finally, Theorem 4.18 and Lemma 2.14 (i) allow us to conclude that our algorithm achieves an approximation guarantee of

$$\frac{N}{N-1} \cdot \frac{k+1-\theta_k}{2} \leq \frac{k+1+\frac{1}{k}-\theta_k}{2} = \frac{k+1-\lambda_k}{2},$$

where $\lambda_k := \theta_k - \frac{1}{k}$. Now, $\lim_{k\to\infty} \theta_k = 1$ implies $\lim_{k\to\infty} \lambda_k = 1$, which concludes the proof.

4.5 Lower bound

In this section, we show that Theorem 4.1 is asymptotically best possible in the sense of Theorem 4.3. We first establish the following auxiliary statement.

Lemma 4.38. Let $f\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$. Then one of the following holds:

(i) f is constant.

(ii) For every $\delta \in (0,1)$, there exist $x, y \in \mathbb{R}_{>0}$ such that $x > (1-\delta)\cdot y$ and $f(x) < f(y)$.

Proof. Assume that (ii) does not hold. Then there exists $\delta \in (0,1)$ with the following property:

$$\text{For every } y \in \mathbb{R}_{>0} \text{ and every } x > (1-\delta)\cdot y, \text{ we have } f(x) \geq f(y). \tag{4.14}$$

Claim 4.39. Let $\alpha \in \mathbb{R}_{>0}$. Then f is constant on the interval $((1-\delta)\cdot\alpha, \alpha)$.

Proof. Let $r, z \in ((1-\delta)\cdot\alpha, \alpha)$ with $r \leq z$. We need to show that $f(r) = f(z)$. Then (4.14) with $y = r$ and $x = z$ yields $f(z) \geq f(r)$. On the other hand, we have

$$r > (1-\delta)\cdot\alpha > (1-\delta)\cdot z,$$

so we can also apply (4.14) with $y = z$ and $x = r$ to conclude $f(r) \geq f(z)$. Hence, $f(r) = f(z)$. $\square$

Claim 4.40. Let $n \in \mathbb{N}$. Then f is constant on the interval $\left((1-\delta)^{\frac{n}{2}}, (1-\delta)^{-\frac{n}{2}}\right)$.

Proof. Induction on n. The base case $n = 1$ follows from Claim 4.39 with $\alpha = (1-\delta)^{-\frac{1}{2}}$. Now, assume that the statement of the claim is true for n. Applying Claim 4.39 tells us that f is constant on the intervals $\left((1-\delta)^{\frac{n+1}{2}}, (1-\delta)^{\frac{n-1}{2}}\right)$ and $\left((1-\delta)^{-\frac{n-1}{2}}, (1-\delta)^{-\frac{n+1}{2}}\right)$, which both have a non-empty intersection with the interval $\left((1-\delta)^{\frac{n}{2}}, (1-\delta)^{-\frac{n}{2}}\right)$. Hence, f is also constant on the interval

$$\begin{aligned}
&\left((1-\delta)^{\frac{n+1}{2}}, (1-\delta)^{-\frac{n+1}{2}}\right) \\
&= \left((1-\delta)^{\frac{n+1}{2}}, (1-\delta)^{\frac{n-1}{2}}\right) \cup \left((1-\delta)^{\frac{n}{2}}, (1-\delta)^{-\frac{n}{2}}\right) \\
&\quad \cup \left((1-\delta)^{-\frac{n-1}{2}}, (1-\delta)^{-\frac{n+1}{2}}\right).
\end{aligned}$$

$\square$

Claim 4.41. (i) holds.

Proof. Let $x, y \in \mathbb{R}_{>0}$. We need to show that $f(x) = f(y)$. Pick $n \in \mathbb{N}$ such that $x, y \in \left((1-\delta)^{\frac{n}{2}}, (1-\delta)^{-\frac{n}{2}}\right)$. Claim 4.40 yields the desired statement. $\square$

□

Now, we are ready to prove Theorem 4.3.

Proof of Theorem 4.3. Let $k \in \mathbb{Z}_{\geq 3}$, $f\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$, $\varepsilon \in (0,1)$ and $C > 0$. We distinguish two cases, depending on which one of the statements (i) or (ii) from Lemma 4.38 holds. We first deal with the easier case where f is constant.

Case 1: f is constant. Let $\mathcal{S} \coloneqq \{\{1,2\},\{1,3\}\}$ and define $w(\{1,2\}) \coloneqq 1$ and $w(\{1,3\}) \coloneqq \frac{k}{2}$. Finally, let $A \coloneqq \{\{1,2\}\}$. Then there is no local improvement of A w.r.t. $f \circ w$ because $\{1,2\}$ and $\{1,3\}$ intersect and $f(w(\{1,2\})) = f(w(\{1,3\}))$ since f is constant. On the other hand, we have

$$\mathrm{OPT}(\mathcal{S}, w) = w(\{1,3\}) = \frac{k}{2} > \left(\frac{k}{2} - \varepsilon\right) \cdot w(\{1,2\}) = \left(\frac{k}{2} - \varepsilon\right) \cdot w(A).$$

This concludes the proof for the case where f is constant. Next, we deal with the more interesting case where (ii) from Lemma 4.38 holds true.

Case 2: (ii) holds. We apply (ii) with $\delta \coloneqq \frac{\varepsilon}{k}$ to obtain $x, y \in \mathbb{R}_{>0}$ with the following property:

$$x > \left(1 - \frac{\varepsilon}{k}\right) \cdot y \text{ and } f(x) < f(y). \tag{4.15}$$

Let $s \in \mathbb{N}$ such that

$$\frac{s+1}{s} \cdot f(x) < f(y). \tag{4.16}$$

Pick $n_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}_{\geq n_0}$, we have

$$s \cdot \left(2 \cdot \left\lfloor \log\left(C \cdot \log\left(\frac{k+2}{2} \cdot n\right)\right)\right\rfloor + 1\right) < \frac{\log(n)}{\log(k-1)} - 1. \tag{4.17}$$

This is possible since the left-hand side grows asymptotically slower than the right-hand side.

Now, we employ a result by Erdős and Sachs [23] telling us that there exists a k-regular graph H on $|V(H)| \geq n_0$ vertices such that

$$\mathrm{girth}(H) \geq \frac{\log(|V(H)|)}{\log(k-1)} - 1, \tag{4.18}$$

where $\mathrm{girth}(H)$ denotes the girth of H, i.e., the minimum length of a cycle in H. Consider the graph G with vertex set $V(G) := V(H) \cup E(H)$ and edge set $E(G) := \{\{v, e\} : v \in e \in E(H)\}$, i.e., each edge of H is connected via edges in G to both of its endpoints. See Fig. 4.5 for an illustration. We define $\mathcal{S} := \{\delta_G(x), x \in V(G)\}$, where $\delta_G(x)$ is the set of incident edges of x in G. By k-regularity of H, $|\delta_G(v)| = k \geq 3$ for $v \in V(H)$, and $|\delta_G(e)| = 2$ for $e \in E(H)$, so each element of $\mathcal{S}$ has cardinality at most k. By definition, G is simple, so no two vertices share more than one edge. As all degrees are at least two, the sets $\delta_G(x), x \in V(G)$ are pairwise distinct. Finally, $V(H)$ and $E(H)$ constitute independent sets in G, implying that $A := \{\delta_G(v), v \in V(H)\}$ and $B := \{\delta_G(e), e \in E(H)\}$ each consist of pairwise disjoint sets. We define positive weights on $\mathcal{S}$ by setting $w(a) := y$ for $a \in A$ and $w(b) := x$ for $b \in B$. By k-regularity of H, we have

$$|B| = |E(H)| = \frac{k}{2} \cdot |V(H)| = \frac{k}{2} \cdot |A|. \tag{4.19}$$

Hence, we obtain

$$\begin{aligned}\mathrm{OPT}(\mathcal{S}, w) \geq w(B) = x \cdot |B| &\overset{(4.19)}{=} x \cdot \frac{k}{2} \cdot |A| \overset{(4.15)}{>} \left(1 - \frac{\varepsilon}{k}\right) \cdot \frac{k}{2} \cdot y \cdot |A| \\ &= \frac{k - \varepsilon}{2} \cdot w(A).\end{aligned}$$

It remains to see that there is no local improvement X of A w.r.t. $f \circ w$ of size

$$|X| \leq C \cdot \log(|\mathcal{S}|) = C \cdot \log(|V(G)|) \overset{(4.19)}{=} C \cdot \log\left(\frac{k+2}{2} \cdot |V(H)|\right). \tag{4.20}$$

As X constitutes a local improvement of A w.r.t. $f \circ w$ if and only if $X \setminus A$ does, we may restrict ourselves to the case where $X \subseteq \mathcal{S} \setminus A = B$. Assume towards a contradiction that $X \subseteq B$ is a local improvement of A w.r.t. $f \circ w$ meeting the size bound in (4.20). Then our choice of weights implies

$$\begin{aligned}f(x) \cdot |X| = (f \circ w)(X) &> (f \circ w)(\{a \in A : \exists b \in X : a \cap b \neq \emptyset\}) \\ &= f(y) \cdot |\{a \in A : \exists b \in X : a \cap b \neq \emptyset\}|.\end{aligned}$$

In particular, (4.16) implies

$$|X| > \frac{s+1}{s} \cdot |\{a \in A : \exists b \in X : a \cap b \neq \emptyset\}|. \tag{4.21}$$

But the sets from A that X intersects are precisely the sets $\delta_G(v)$ for those vertices $v \in V(H)$ that are endpoints of edges $e \in E(H)$ with $\delta_G(e) \in X$. Let J be the subgraph of H containing precisely the edges e with $\delta_G(e) \in X$ and their endpoints. Then

$$C \cdot \log\left(\frac{k+2}{2} \cdot |V(H)|\right) \overset{(4.20)}{\geq} |X| = |E(J)| \overset{(4.21)}{>} \frac{s+1}{s} \cdot |V(J)|.$$

By Lemma 4.12, this implies the existence of a cycle in H of length at most

$$s \cdot \left(2 \cdot \left\lfloor \log\left(C \cdot \log\left(\frac{k+2}{2} \cdot |V(H)|\right)\right)\right\rfloor + 1\right).$$

But by (4.17) and since we chose H with $|V(H)| \geq n_0$, this contradicts our lower bound on the girth of H from (4.18). Hence, there is no local improvement X of A wr.t. $f \circ w$ meeting the size bound in (4.20). $\square$

We remark that a similar construction was also used by Bafna, Narayanan and Ravi [5] to show that local improvements of constant size do not yield an approximation guarantee better than $\frac{k}{2}$ for the MIS in $(k+1)$-claw free graphs. We further point out that Theorem 4.3 easily generalizes to all functions $f : \mathbb{R}_{>0} \to \mathbb{R}$: If there is $x \in \mathbb{R}_{>0}$ with $f(x) \leq 0$, we may consider the instance given by $\mathcal{S} = \{\{1\}\}$ and $w(\{1\}) = x$. Then $\emptyset$ is locally optimum w.r.t. $f \circ w$, but has a total weight of 0, whereas the optimum solution $\mathcal{S}$ features a positive weight of x.

Chapter 5

A better-than-$k/2$-approximation

In the previous chapter, we have seen that if we restrict ourselves to local improvements of logarithmically bounded size (w.r.t. a fixed additive local search objective), we cannot get better than a $\frac{k}{2}$-approximation for the weighted k-Set Packing problem (cf. Theorem 4.3). Moreover, for $k \to \infty$, we can asymptotically meet these lower bound guarantees (Theorem 4.1). At first sight, this seems to conclude the story of local search for the weighted k-Set Packing problem, given that well-structured local improvements of logarithmically bounded size lie on the border of what we can still search for via enumeration/dynamic programming based approaches. Note that the previous works studying polynomial-time local search algorithms for (weighted) k-Set Packing rely on these paradigms [7, 15, 16, 17, 28, 31, 33, 48]. As a consequence, new ideas are required in order to breach the barrier of $\frac{k}{2}$ in the weighted case.

In this chapter, we present a polynomial-time local search algorithm for the weighted k-Set Packing problem that beats the approximation threshold of $\frac{k}{2}$, at least for sufficiently large values of k. Once more, we conduct local search with respect to the squared weight function and base our analysis on Lemma 3.20, telling us that either "most of" our instance is "close to being unweighted", or we obtain an $\Omega(k)$-improvement in the approximation guarantee. However, in contrast to the previous two chapters, we do not deal with the first scenario by trying to generalize techniques from the unweighted setting, limiting ourselves to local improvements of logarithmically bounded size. Instead, we directly apply a *black box algorithm* for the unweighted k-Set Packing problem to carefully chosen subinstances in order to generate candidate improvements. In doing so, we are able to also consider potential improvements of super-logarithmic size. This allows us to escape the local optima from the lower bound instances constructed in the proof of Theorem 4.3.

Our main result for this chapter is given by the following theorem:

Theorem 5.1. Let $k \in \mathbb{Z}_{\geq 3}$.
There is a polynomial-time $\min\{0.5 \cdot k + 0.499, 0.4999 \cdot k + 0.501\}$-approximation algorithm for the weighted k-Set Packing problem.

For $k \leq 20$, the first term attains the minimum, for $k \geq 20$, the second term does. For $k \geq 5011$, we obtain a guarantee below $\frac{k}{2}$.

We remark that by conducting a more involved analysis, including a stronger (but much more tedious to prove) version of Lemma 3.20, and using circular improvements to boost the approximation ratio, we can in fact obtain an asymptotically slightly better guarantee of $0.4986 \cdot k + 0.5194$, see [42] for the details.

We further point out that, at the cost of a quasi-polynomial running time, Theorem 5.1 generalizes to the MWIS in $(k+1)$-claw free graphs.[1] In particular, we will, once more, phrase our algorithm and our analysis in terms of the more general MWIS in $(k+1)$-claw free graphs, which we also consider to be more convenient notation-wise. The additional structure an instance of the weighted k-Set Packing problem provides is only needed in order to obtain a polynomial running time: We exploit it to make sure that the subroutine we use to approximate the MIS can be implemented to run in polynomial time. Note that this is the case for induced subgraphs of the conflict graph of a k-Set Packing instance because they correspond to k-Set Packing subinstances.

The remainder of this chapter is organized as follows: In Section 5.1, we provide some intuition on how running a black box algorithm for the unweighted k-Set Packing problem/the MIS in $(k+1)$-claw free graphs may help us to generate local improvements and obtain an approximation guarantee below $\frac{k}{2}$ for the weighted case. This will motivate the algorithm that we introduce and analyze in Section 5.2. In particular, we prove Theorem 5.1 in this section.

5.1 Motivation

In order to obtain a (quasi-polynomial-time) better-than-$\frac{k}{2}$-approximation for the MWIS in $(k+1)$-claw free graphs (for sufficiently large k), we once more rely on Lemma 3.20. It tells us that either we obtain an approximation guarantee of $\frac{k+1}{2} - \Omega(k)$ (which is sufficient for our purposes), or the instance at hand is highly structured in that a large fraction (in terms of weight) of the vertices in the solution we find feature many regular neighbors in an optimum solution. In order to be able to apply Lemma 3.20, we again

[1] Note that the state-of-the-art $\frac{k+1+\epsilon}{3}$-approximation for the unweighted k-Set Packing problem by Fürer and Yu [28] also, at the cost of a quasi-polynomial running time, generalizes to the MIS in $(k+1)$-claw free graphs.

study an algorithm that iteratively searches for certain local improvements w.r.t. the squared weight function, including improvements of size 3 and claw-shaped ones (Definition 2.13). Denote the solution that our algorithm returns by A, and let B be an optimum solution. Let further ε be a constant subject to (3.1)-(3.4), which is used to define the notions of single and double vertices and regular neighbors (Definitions 3.4, 3.5 and 3.7). Lemma 3.20 tells us that either, say,

$$\sum_{v \in A} w(v) \cdot |\mathrm{reg}_B(v)| \geq 0.99 \cdot k \cdot w(A), \tag{5.1}$$

meaning that "on average" (in a weighted sense), every $v \in A$ has at least $0.99 \cdot k$ many regular neighbors in B, or we obtain an improved approximation guarantee of

$$w(B) \leq \left(\frac{k+1}{2} - \frac{\varepsilon^2}{(1+\varepsilon) \cdot (2+\varepsilon^2)} \cdot \frac{k}{200}\right) \cdot w(A).$$

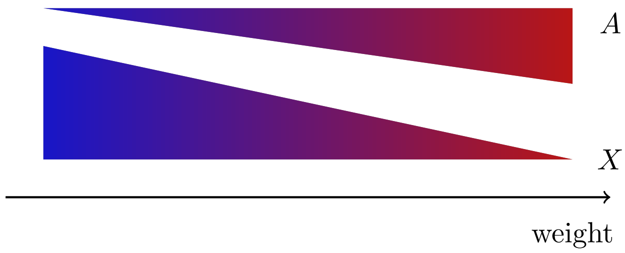
Note that for k sufficiently large, this yields a guarantee below $\frac{k}{2}$. Hence, we may focus on the case where (5.1) holds true. We consider an auxiliary graph H^* whose vertex set is the disjoint union of A and the set of vertices from B that are single or double. Moreover, $v \in A$ and $u \in B$ are connected if and only if they are regular neighbors. There are no edges in H^* within A or B.

H^* is "locally unweighted" in the sense that the weights of adjacent vertices can only differ by a factor of $1 \pm \mathcal{O}(\varepsilon)$ (cf. Definitions 3.4, 3.5 and 3.7). Thus, H^* can be thought of as an "unweighted approximation" of our instance. This approximation has the nice property that it almost preserves the notion of a local improvement: For a single or double vertex $u \in B$, the total weight of its irregular neighbors only amounts to an $\mathcal{O}(\varepsilon)$-fraction of $w(u)$. As a consequence, we can infer that a set $X \subseteq V(H^*) \cap B$ with $w^2(N_{H^*}(X, A)) < (1 - \mathcal{O}(\varepsilon^2)) \cdot w^2(X)$ constitutes a local improvement of A.

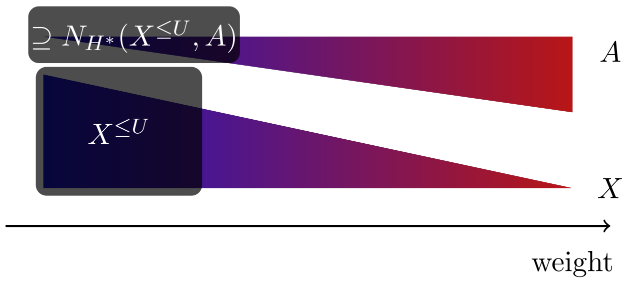
Finally, (5.1) tells us that the average weighted degree that vertices from A have in H^* amounts to $0.99 \cdot k$, whereas each vertex from B can have a degree of at most 2 in H^* since it can have at most two regular neighbors in A. Using the fact that the weights of neighboring vertices only differ by a factor of $1 \pm \mathcal{O}(\varepsilon)$, we can apply Lemma 4.13 to obtain a weight threshold L with the property that if we consider the induced subgraph $H^*_{\geq L}$ of H^* featuring only vertices of weight at least L, then the average degree in A amounts to at least $(1 - \mathcal{O}(\varepsilon)) \cdot 0.99 \cdot k$. In particular,

$$\frac{|V(H^*_{\geq L}) \cap B|}{|V(H^*_{\geq L}) \cap A|} \geq (1 - \mathcal{O}(\varepsilon)) \cdot 0.99 \cdot \frac{k}{2}.$$

Hence, applying the (quasi-polynomial-time) $\frac{k+1+\epsilon}{3}$-approximation algorithm



(a) Even though the cardinality of X is larger than the cardinality of A, $w^2(A)$ is larger than $w^2(X)$ because most vertices in A have a large weight, whereas most vertices in X feature a small weight.



(b) For a sufficiently small weight threshold U, $X^{\leq U}$ constitutes a local improvement of A w.r.t. w^2. This is because by local similarity of weights, all neighbors of X in A have a weight of at most $(1 + \mathcal{O}(\varepsilon)) \cdot U$, and there are few such vertices.

Figure 5.1: The figure displays weight distributions for A and X in which X has a larger cardinality, but A has a larger (squared) weight. The weights are drawn along the horizontal axis and, in addition, indicated by colors (left, blue = small weight, right, red = large weight). The weight distribution for A is displayed on top, the weight distribution for X is shown at the bottom. The height of the respective triangle at a given coordinate indicates the number of vertices of the respective weight.

for the MIS in $(k+1)$-claw free graphs from [28] to $H^*_{\geq L}$ yields X with

$$|X| \geq \frac{3}{k+1+\epsilon} \cdot |B \cap V(H^*_{\geq L})| \gtrapprox \frac{3}{2} \cdot |A \cap V(H^*_{\geq L})|.$$

We would like to use local similarity of weights to turn X into a local improvement of A. Unfortunately, X itself does not need to constitute a local improvement of A w.r.t. w^2 because it could happen that all of the cardinality of X is accumulated at small weights, whereas most of the vertices in A feature a large weight. However, for this particular weight distribution, taking only the vertices in X that have a small weight does the trick since their neighbors in A will be among those vertices in A that have a small weight as well, of which there are few. See Fig. 5.1 for an illustration. In general, we can use Lemma 4.14 to show that there is a weight threshold U such that

Algorithm 5: Better-than-$k/2$-approximation

Input: a $(k+1)$-claw free graph $G=(V,E)$, $w\colon V \to \mathbb{R}_{>0}$
Output: an independent set $A \subseteq V$

$A \leftarrow \emptyset$
`continue` $\leftarrow$ **true**
while `continue` **do**
  `continue` $\leftarrow$ `RunIteration`(G, A, w) `// see Algorithm 6`
end
return A

the set $X^{\leq U} := \{x \in X : w(x) \leq U\}$ constitutes a local improvement of A.

The above ideas give rise to an improved (quasi-polynomial-time) approximation algorithm for the MWIS in $(k+1)$-claw free graphs (that can be implemented to run in polynomial-time for conflict graphs of k-Set Packing instances), except for the problem that we of course do not know the optimum solution B, and, thus, cannot construct the graph H^* as outlined above. However, this issue can be fixed rather easily by looking at a graph where we include *all* single and double vertices from $V \setminus A$ and all edges between them, as well as all edges from vertices in A to their regular neighbors.

5.2 Algorithm and analysis

Let $\epsilon := 0.01$ and let $\rho := \frac{k+1+\epsilon}{3}$. Denote the quasi-polynomial-time ρ-approximation algorithm for the MIS in $(k+1)$-claw free graphs from [28] by `MIS`.[2] Moreover, let `MIS`(I) be the solution that `MIS` returns when we apply it to the instance I. Recall that `MIS` can be implemented to run in polynomial-time for conflict graphs of k-Set Packing instances.

We further let $\varepsilon := 0.06984$. By Proposition 3.3, our choice of ε satisfies (3.1)-(3.4), and, thus, allows us to reuse the results from Chapter 3. In particular, we employ the notions of single and double vertices (Definitions 3.4 and 3.5), as well as the notion of regular neighbors (Definition 3.7), and we make use of Lemma 3.20. Implementing the ideas outlined in the previous section, we arrive at Algorithm 5. Our main result for this section is given by Theorem 5.2.

Theorem 5.2. Algorithm 5 is a $\min\{0.5 \cdot k + 0.49877, 0.49985 \cdot k + 0.501\}$-approximation for the MWIS in $(k+1)$-claw free graphs for $k \in \mathbb{Z}_{\geq 3}$.

[2] Our choice of ϵ is, to some extent, arbitrary because we can choose $\epsilon > 0$ as small as we like and the smaller it is, the better our approximation guarantee becomes. However, the improvements we could obtain by choosing an even smaller value of ϵ are subsumed in the rounding of the constants in our approximation guarantee.

Algorithm 6: `RunIteration`(G, A, w)

Input: a $(k+1)$-claw free graph $G = (V, E)$, an independent set $A \subseteq V$, $w\colon V \to \mathbb{R}_{>0}$
Output: `true` if a local improvement was found, `false` otherwise

if $\exists$ *local improvement* X *such that* $|X| = 3$ **then**
  $A \leftarrow (A \setminus N(X, A)) \cup X$
  return `true`
end
if $\exists$ *claw-shaped improvement* X **then**
  $A \leftarrow (A \setminus N(X, A)) \cup X$
  return `true`
end
$V_{reg} \leftarrow \{u \in V \setminus A : \ u \text{ is single or double}\}$
for $L \in \{w(v) : v \in V\}$ **do**
  $A_{\geq L} \leftarrow \{v \in A : w(v) \geq L\}$
  $V_{\geq L} \leftarrow A_{\geq L} \cup \{u \in V_{reg} : w(u) \geq L \text{ and } \operatorname{reg}_A(u) \subseteq A_{\geq L}\}$
  $Y \leftarrow \mathrm{MIS}(G[V_{\geq L}])$, $X \leftarrow Y \setminus A$
  for $U \in \{w(v) : v \in V\}$ **do**
    $X^{\leq U} \leftarrow \{x \in X : w(x) \leq U\}$
    if $w^2(X^{\leq U}) > w^2(N(X^{\leq U}, A))$ **then**
      $A \leftarrow (A \setminus N(X^{\leq U}, A)) \cup X^{\leq U}$
      return `true`
    end
  end
end
return `false`

We point out that Theorem 5.2 implies Theorem 5.1.

Proof of Theorem 5.1, assuming Theorem 5.2. Fix $k \in \mathbb{Z}_{\geq 3}$ and pick $N \in \mathbb{N}$ such that

$$\begin{aligned}\frac{N}{N-1} \cdot \min\{0.5 \cdot k + 0.49877, 0.49985 \cdot k + 0.501\} \\ \leq \min\{0.5 \cdot k + 0.499, 0.4999 \cdot k + 0.501\}. \qquad (5.2)\end{aligned}$$

We consider the following algorithm: Given a $(k+1)$-claw free graph $G = (V, E)$ and $w\colon V \to \mathbb{R}_{>0}$, we apply Lemma 2.14 to, in polynomial time, compute $U \subseteq V$ and $w' : U \to \mathbb{Z}_{>0}$ subject to Lemma 2.14 (i) and (ii). Then, we apply Algorithm 5 to $(G[U], w')$. By Theorem 5.2, (5.2) and (i), this algorithm yields the desired approximation guarantee. Moreover, (ii) tells us that Algorithm 5 terminates on $(G[U], w')$ after a polynomial number of iterations.

As far as a single iteration is concerned, we can search for a local improvement of size 3, as well as for a claw-shaped local improvement, in polynomial time via brute-force enumeration. Moreover, the two for-loops over the weights in our instance only need a polynomial number of iterations, and except for line 13, each of the lines 9-22 of Algorithm 6 can be executed in polynomial time. The call to `MIS` in line 13 can be performed in quasi-polynomial time since as an induced subgraph of the input graph, $G[V_{\geq L}]$ is $(k+1)$-claw free. Moreover, if G is the conflict graph of a k-Set Packing instance, then $G[V_{\geq L}]$ is the conflict graph of the k-Set Packing subinstance given by the sets in $V_{\geq L}$. In particular, `MIS` can be implemented to run in polynomial time in this case. This concludes the proof of Theorem 5.1. $\square$

The remainder of this chapter is dedicated to the proof of Theorem 5.2. To this end, fix $k \in \mathbb{Z}_{\geq 3}$, a $(k+1)$-claw free graph $G = (V, E)$ and a weight function $w\colon V \to \mathbb{R}_{>0}$. Let A be the solution returned by Algorithm 5 when applied to (G, w) and denote an optimum solution to the MWIS in (G, w) by B. As in Chapter 3, we fix a map $n\colon V \to A$ that maps each $u \in V$ to an element of $N(u, A)$ of maximum weight. Note that we have $N(u, A) \neq \emptyset$ for every $u \in V$ because no claw improves A. In addition, we fix a map $n_2\colon \{u \in V : |N(u, A)| \geq 2\} \to A$ that maps a vertex u in its domain to an element of $N(u, A) \setminus \{n(u)\}$ of maximum weight.

In order to prove Theorem 5.2, we first show Lemma 5.3, which yields an upper bound on the weighted sum over the number of regular neighbors a vertex in A features in B. Then, we plug this upper bound into Lemma 3.20 to obtain the desired guarantee.

Lemma 5.3. We have

$$\sum_{v \in A} |\mathrm{reg}_B(v)| \cdot w(v) \leq \frac{2\rho \cdot (1-\varepsilon)^{-3}}{1 - (\varepsilon^2 + \frac{3\varepsilon}{1+\varepsilon})^2} \cdot w(A).$$

Note that $1 - (\varepsilon^2 + \frac{3\varepsilon}{1+\varepsilon})^2 > 0$ by (3.1).

5.2.1 Proof of Lemma 5.3

To prove Lemma 5.3, we assume towards a contradiction that

$$\sum_{v \in A} |\mathrm{reg}_B(v)| \cdot w(v) > \frac{2\rho \cdot (1-\varepsilon)^{-3}}{1 - (\varepsilon^2 + \frac{3\varepsilon}{1+\varepsilon})^2} \cdot w(A) \tag{5.3}$$

holds. By the termination criterion of Algorithm 5, A is not modified in the last call to `RunIteration` (Algorithm 6). In particular, (5.3) holds in the beginning of the last call to `RunIteration`. We will derive a contradiction by showing that (5.3) guarantees that we find a local improvement in said call. To this end, we may, first of all, assume that we do not find a local improvement of size 3 in line 1, or a claw-shaped improvement in line 5 because we would be done otherwise. In particular, lines 9-16 are executed.

Compliant with Algorithm 6, we employ the following notation:

- We let $V_{reg} := \{u \in V \setminus A : u \text{ is single or double}\}$. V_{reg} equals the set of vertices from $V \setminus A$ that serve as regular neighbors of vertices in A (see Definition 3.7).
- For $x \in \mathbb{R}_{>0}$, we let $A_{\geq x} = \{v \in A : w(v) \geq x\}$ consist of all vertices in A of weight at least x and define

  $$V_{\geq x} := A_{\geq x} \cup \{u \in V_{reg} : w(u) \geq x \text{ and } \mathrm{reg}_A(u) \subseteq A_{\geq x}\}$$

  to contain all vertices from V_{reg} with the property that these and all of their regular neighbors in A are of weight at least x. The intuition behind this definition is that we want to make sure that for every vertex $u \in V_{\geq x} \setminus A$ that might appear in our candidate local improvement X, $G[V_{\geq x}]$ captures enough information about $N(u, A)$ in that it contains all of u's neighbors in A that make up a significant fraction of $w(N(u, A))$. These are precisely the regular neighbors of u.

Proposition 5.4. Let $x > 0$ and $u \in V_{\geq x}$. Then $\mathrm{reg}_A(u) \subseteq A_{\geq x}$.

Proof. For $u \in V \setminus A$, this follows from the definition of $V_{\geq x}$. In case we have $u \in V_{\geq x} \cap A = A_{\geq x}$, we obtain $\mathrm{reg}_A(u) \subseteq N(u, A) = \{u\} \subseteq A_{\geq x}$ since A is independent. Actually, we even have $\mathrm{reg}_A(u) = N(u, A) = \{u\}$ in this case, but this stronger statement is not needed here. ☐

The proof of Lemma 5.3 consists of two main parts. First, Lemma 5.5 shows that there exists a weight L with the property that $|B \cap V_{\geq L}|$ is sufficiently large compared to $|A_{\geq L}|$. This ensures that `MIS`, when applied to $G[V_{\geq L}]$, outputs a set Y the cardinality of which is by some constant factor larger than the cardinality of $A_{\geq L}$. This factor is chosen in a way that it cuts us enough slack to cover for the following two facts: First, the weights of the vertices in $X := Y \setminus A$ might be by a factor of $1 - \varepsilon$ smaller than the weights of their neighbors in $A_{\geq L}$. Second, there might be some further neighbors in $A \setminus A_{\geq L}$, the total (squared) weight of which is, however, bounded by $\mathcal{O}(\varepsilon^2) \cdot w^2(X)$. Lemma 5.6 tells us that one of the sets $X^{\leq U}$ we consider constitutes a local improvement of A.

Lemma 5.5. Assume that

$$\sum_{v \in A} |\mathrm{reg}_B(v)| \cdot w(v) > \frac{2\rho \cdot (1-\varepsilon)^{-3}}{1 - (\varepsilon^2 + \frac{3\varepsilon}{1+\varepsilon})^2} \cdot w(A).$$

Then there is $L \in \{w(v) : v \in V\}$ such that

$$|B \cap V_{\geq L}| > \frac{\rho \cdot (1-\varepsilon)^{-2}}{1 - (\varepsilon^2 + \frac{3\varepsilon}{1+\varepsilon})^2} \cdot |A_{\geq L}|.$$

In particular, `MIS`, when applied to $G[V_{\geq L}]$, returns an independent set Y of cardinality

$$|Y| > \frac{(1-\varepsilon)^{-2}}{1-(\varepsilon^2+\frac{3\varepsilon}{1+\varepsilon})^2} \cdot |A_{\geq L}|.$$

Lemma 5.6. Let $L \in \{w(v) : v \in V\}$ and let $Y \subseteq V_{\geq L}$ be an independent set such that

$$|Y| > \frac{(1-\varepsilon)^{-2}}{1-(\varepsilon^2+\frac{3\varepsilon}{1+\varepsilon})^2} \cdot |A_{\geq L}|.$$

Then there exists $U \in \{w(v) : v \in V\}$ such that the set $X^{\leq U} := \{x \in Y \setminus A : w(x) \leq U\}$ constitutes a local improvement of A (w.r.t. w^2).

Proof of Lemma 5.5. We first prove the following auxiliary statement.

Claim 5.7. Let $x > 0$. For $v \in A$ with $(1-\varepsilon)^{-1} \cdot x \leq w(v)$, we have $\mathrm{reg}_B(v) \subseteq V_{\geq x}$.

Proof. If $u \in \mathrm{reg}_B(v) \cap A$, then $u \in N(v, A)$ and, thus, $u = v$ since A is independent. In particular, $u = v \in A_{\geq x} \subseteq V_{\geq x}$. Next, assume that $u \in \mathrm{reg}_B(v) \setminus A$. Then $u \in V \setminus A$ and u is single or double, so $u \in V_{reg}$. By Definition 3.4 and Definition 3.5, we know that one of the following applies:

- $\mathrm{reg}_A(u) = \{n(u)\} = \{v\}$ and $x \leq (1-\varepsilon) \cdot w(v) \leq w(u)$.
- $\mathrm{reg}_A(u) = \{n(u), n_2(u)\} \ni v$ and
  $x \leq (1-\varepsilon) \cdot w(v) \leq (1-\varepsilon) \cdot w(n(u)) \leq \min\{w(u), w(n_2(u)), w(n(u))\}$.

In either case, u and all of the vertices in $\mathrm{reg}_A(u)$ bear a weight of at least x. This implies that $u \in V_{\geq x}$. ☐

Next, we prove the existence of L with the property that the sum of regular neighbors the vertices in $A_{\geq L}$ have in $V_{\geq L}$ is large compared to the cardinality of $A_{\geq L}$.

Claim 5.8. There is $L \in \{w(v) : v \in V\}$ such that

$$\sum_{v \in A_{\geq L}} |\mathrm{reg}_B(v) \cap V_{\geq L}| > \frac{2\rho \cdot (1-\varepsilon)^{-2}}{1-(\varepsilon^2+\frac{3\varepsilon}{1+\varepsilon})^2} \cdot |A_{\geq L}|.$$

Proof. We want to apply Lemma 4.13. To this end, set $S := A$, $\mu(v) := |\mathrm{reg}_B(v)|$, $\varphi(v) := w(v)$, $\lambda := 1-\varepsilon$ and $\eta := \frac{2\rho \cdot (1-\varepsilon)^{-3}}{1-(\varepsilon^2+\frac{3\varepsilon}{1+\varepsilon})^2}$. Then Lemma 4.13

tells us that there is $L \in \mathbb{R}_{>0}$ such that

$$\begin{aligned}\sum_{v\in A_{\geq L}} |\mathrm{reg}_B(v)\cap V_{\geq L}| &\overset{\text{Claim 5.7}}{\geq} \sum_{v\in A_{\geq(1-\varepsilon)^{-1}\cdot L}} |\mathrm{reg}_B(v)| \\ &\overset{\text{Lem. 4.13}}{>} (1-\varepsilon)\cdot\frac{2\rho\cdot(1-\varepsilon)^{-3}}{1-(\varepsilon^2+\frac{3\varepsilon}{1+\varepsilon})^2}\cdot|A_{\geq L}| \\ &= \frac{2\rho\cdot(1-\varepsilon)^{-2}}{1-(\varepsilon^2+\frac{3\varepsilon}{1+\varepsilon})^2}\cdot|A_{\geq L}|.\end{aligned}$$

The strict inequality tells us that $A_{\geq L} \neq \emptyset$ (otherwise, the sum on the left-hand side would be empty). As a consequence, we can increase L to $\min\{w(v) : v \in V, w(v) \geq L\}$ without changing $A_{\geq L}$ or $V_{\geq L}$. Thus, we may assume $L \in \{w(v) : v \in V\}$. □

Pick L as in Claim 5.8. We calculate

$$\begin{aligned}2\cdot|B\cap V_{\geq L}| \geq \sum_{u\in B\cap V_{\geq L}} |\mathrm{reg}_A(u)| &= \sum_{v\in A_{\geq L}} |\mathrm{reg}_B(v)\cap V_{\geq L}| \\ &> \frac{2\rho\cdot(1-\varepsilon)^{-2}}{1-(\varepsilon^2+\frac{3\varepsilon}{1+\varepsilon})^2}\cdot|A_{\geq L}|.\end{aligned}$$

Here, the first inequality follows since $|\mathrm{reg}_A(u)| \leq 2$ for all $u \in B$ by definition. The next equation is implied by Proposition 5.4, $\mathrm{reg}_B(v) \subseteq B$ for $v \in A$, and $v \in \mathrm{reg}_A(u) \Leftrightarrow u \in \mathrm{reg}_B(v)$ for $u \in B$ and $v \in A$. Division by 2 yields

$$|B\cap V_{\geq L}| > \frac{\rho\cdot(1-\varepsilon)^{-2}}{1-(\varepsilon^2+\frac{3\varepsilon}{1+\varepsilon})^2}\cdot|A_{\geq L}|.$$

As $B \cap V_{\geq L}$ is independent in G, we can conclude that the algorithm `MIS`, applied to $G[V_{\geq L}]$, finds an independent set Y of size at least

$$|Y| \geq \rho^{-1}\cdot|B\cap V_{\geq L}| > \frac{(1-\varepsilon)^{-2}}{1-(\varepsilon^2+\frac{3\varepsilon}{1+\varepsilon})^2}\cdot|A_{\geq L}|.$$

Observe that as $G[V_{\geq L}]$ is an induced subgraph of G, Y is independent in G as well. □

For the proof of Lemma 5.6, we require the following auxiliary statement:

Proposition 5.9. Let $u \in V_{reg}$. Then

$$w(N(u,A)\setminus \mathrm{reg}_A(u)) \leq \left(\varepsilon^2+\frac{3\varepsilon}{1+\varepsilon}\right)\cdot w(u) < w(u).$$

Proof. By definition of V_{reg}, we know that u is single or double. If u is single, we calculate

$$\begin{aligned} w(N(u,A) \setminus \mathrm{reg}_A(u)) &= w(N(u,A)) - w(n(u)) \overset{(3.6)}{\leq} \varepsilon \cdot w(n(u)) \\ \overset{(3.5)}{\leq} \frac{\varepsilon}{1-\varepsilon} \cdot w(u) &\overset{(3.1)}{<} \frac{3\varepsilon}{1+\varepsilon} \cdot w(u) < \left(\varepsilon^2 + \frac{3\varepsilon}{1+\varepsilon}\right) \cdot w(u). \end{aligned}$$

In case u is double, we obtain

$$\begin{aligned} & w(N(u,A) \setminus \mathrm{reg}_A(u)) = w(N(u,A)) - w(n(u)) - w(n_2(u)) \\ & \overset{(3.9)}{\leq} (2+\varepsilon^2) \cdot w(u) - w(n(u)) - w(n_2(u)) \\ & \overset{(3.7)}{\leq} (2+\varepsilon^2) \cdot w(u) - (2-\varepsilon) \cdot w(n(u)) \\ & \overset{(3.8)}{\leq} \left(2 + \varepsilon^2 - \frac{2-\varepsilon}{1+\varepsilon}\right) \cdot w(u) \\ & = \left(\varepsilon^2 + \frac{3\varepsilon}{1+\varepsilon}\right) \cdot w(u). \end{aligned}$$

This proves the first inequality. The second one follows from (3.1) and $w(u) > 0$. □

Proof of Lemma 5.6. Define $X := Y \setminus A$ and let $X^{\leq x} := \{u \in X : w(u) \leq x\}$ for $x > 0$. We point out that it suffices to prove the existence of any $U > 0$ such that $X^{\leq U}$ constitutes a local improvement because this property will ensure that $X^{\leq U} \neq \emptyset$ and, hence, decreasing U to the maximum weight in $X^{\leq U}$ will preserve the set $X^{\leq U}$, and ensure that $U \in \{w(v) : v \in V\}$.

Our goal is to apply Lemma 4.14. For this purpose, we have to derive a lower bound on the cardinality of X. As $Y \cap A \subseteq V_{\geq L} \cap A = A_{\geq L}$, we get

$$|X| > \frac{(1-\varepsilon)^{-2}}{1-(\varepsilon^2 + \frac{3\varepsilon}{1+\varepsilon})^2} \cdot |A_{\geq L}| - |Y \cap A_{\geq L}| \geq \frac{(1-\varepsilon)^{-2}}{1-(\varepsilon^2 + \frac{3\varepsilon}{1+\varepsilon})^2} \cdot |A_{\geq L} \setminus Y|,$$

where the last inequality follows from the fact that $\frac{(1-\varepsilon)^{-2}}{1-(\varepsilon^2+\frac{3\varepsilon}{1+\varepsilon})^2} > 1$. Moreover, as Y is independent in G, no vertex in X is adjacent to a vertex in $A_{\geq L} \cap Y$, implying that $N(X, A_{\geq L}) \subseteq A_{\geq L} \setminus Y$. Hence, we obtain

$$|X| > \frac{(1-\varepsilon)^{-2}}{1-(\varepsilon^2 + \frac{3\varepsilon}{1+\varepsilon})^2} \cdot |N(X, A_{\geq L})|.$$

Claim 5.10. There is $U \in \mathbb{R}_{>0}$ such that

$$w^2(X^{\leq U}) > \frac{1}{1-(\varepsilon^2 + \frac{3\varepsilon}{1+\varepsilon})^2} \cdot w^2(N(X^{\leq U}, A_{\geq L})).$$

Proof. We apply Lemma 4.14 with $S_1 := X$, $S_2 := N(X, A_{\geq L})$, $\varphi(s) := w^2(s) > 0$ for $s \in S_1 \cup S_2$, $\eta := \frac{(1-\varepsilon)^{-2}}{1-(\varepsilon^2+\frac{3\varepsilon}{1+\varepsilon})^2}$ and $\lambda := (1-\varepsilon)^2$. In this setting, Lemma 4.14 tells us that there is $U \in \mathbb{R}_{>0}$ such that

$$\begin{aligned}
&w^2(X^{\leq U}) \\
&= w^2(\{u \in X : w^2(u) \leq U^2\}) \\
&> \frac{1}{1-(\varepsilon^2+\frac{3\varepsilon}{1+\varepsilon})^2} \cdot w^2(\{v \in N(X, A_{\geq L}) : w^2(v) \leq (1-\varepsilon)^{-2} \cdot U^2\}).
\end{aligned}$$

By definition of $V_{\geq L}$, we know that for $u \in X \subseteq V_{\geq L} \setminus A \subseteq V_{reg}$, any vertex in $N(u, A)$ is of weight at most $(1-\varepsilon)^{-1} \cdot w(u)$: This holds for the vertices in $\text{reg}_A(u)$ by (3.5), (3.7) and (3.8), and moreover, the total weight of $N(u, A) \setminus \text{reg}_A(u)$ is bounded by $w(u)$ by Proposition 5.9.

In particular, for $u \in X^{\leq U}$ and $v \in N(u, A)$, we have

$$w^2(v) \leq (1-\varepsilon)^{-2} \cdot w^2(u) \leq (1-\varepsilon)^{-2} \cdot U^2.$$

Thus, we obtain

$$\begin{aligned}
&w^2(X^{\leq U}) \\
&> \frac{1}{1-(\varepsilon^2+\frac{3\varepsilon}{1+\varepsilon})^2} \cdot w^2(\{v \in N(X, A_{\geq L}) : w^2(v) \leq (1-\varepsilon)^{-2} \cdot U^2\}) \\
&\geq \frac{1}{1-(\varepsilon^2+\frac{3\varepsilon}{1+\varepsilon})^2} \cdot w^2(N(X^{\leq U}, A_{\geq L})).
\end{aligned}$$

□

Pick U as provided by Claim 5.10. To finally see that $X^{\leq U}$ constitutes a local improvement of A w.r.t. w^2, it remains to bound $w^2(N(X^{\leq U}, A \setminus A_{\geq L}))$. As $X^{\leq U} \subseteq V_{\geq L} \setminus A \subseteq V_{reg}$, we have $\text{reg}_A(u) \subseteq A_{\geq L}$ for every $u \in X$. By Proposition 5.9, we can infer that

$$w(N(u, A \setminus A_{\geq L})) \leq w(N(u, A) \setminus \text{reg}_A(u)) \leq \left(\varepsilon^2 + \frac{3\varepsilon}{1+\varepsilon}\right) \cdot w(u),$$

and, hence,

$$w^2(N(u, A \setminus A_{\geq L})) \leq (w(N(u, A \setminus A_{\geq L})))^2 \leq \left(\varepsilon^2 + \frac{3\varepsilon}{1+\varepsilon}\right)^2 \cdot w^2(u)$$

for $u \in X^{\leq U}$. Consequently, we obtain

$$\begin{aligned}
w^2(N(X^{\leq U}, A \setminus A_{\geq L})) &\leq \sum_{u \in X^{\leq U}} w^2(N(u, A \setminus A_{\geq L})) \\
&\leq \left(\varepsilon^2 + \frac{3\varepsilon}{1+\varepsilon}\right)^2 \cdot w^2(X^{\leq U}).
\end{aligned}$$

Combining this with Claim 5.10 finally yields

$$\begin{aligned}
w^2(X^{\leq U}) &= \left(1 - \left(\varepsilon^2 + \frac{3\varepsilon}{1+\varepsilon}\right)^2\right) \cdot w^2(X^{\leq U}) + \left(\varepsilon^2 + \frac{3\varepsilon}{1+\varepsilon}\right)^2 \cdot w^2(X^{\leq U}) \\
&> w^2(N(X^{\leq U}, A_{\geq L})) + w^2(N(X^{\leq U}, A \setminus A_{\geq L})) \\
&= w^2(N(X^{\leq U}, A)).
\end{aligned}$$

As $X^{\leq U} \subseteq Y$ is independent, it constitutes a local improvement of A. $\square$

Combining Lemma 5.5 and Lemma 5.6 yields the desired contradiction to the termination criterion of our algorithm, which proves Lemma 5.3.

5.2.2 Proof of Theorem 5.2

We conclude this chapter by proving Theorem 5.2.

Proof of Theorem 5.2. The inequality

$$w(B) \leq (0.5 \cdot (k+1) - 0.00123) \cdot w(A) = (0.5 \cdot k + 0.49877) \cdot w(A)$$

follows from Theorem 3.1.

Combining the bound on $\sum_{v \in A} |\mathrm{reg}_B(v)| \cdot w(v)$ provided by Lemma 5.3 with Lemma 3.20 yields

$$\begin{aligned}
w(B) &\leq \frac{k+1}{2} \cdot w(A) - \frac{1}{2} \cdot \frac{\varepsilon^2}{(1+\varepsilon)\cdot(2+\varepsilon^2)} \cdot \left(k - \frac{2\rho \cdot (1-\varepsilon)^{-3}}{1 - (\varepsilon^2 + \frac{3\varepsilon}{1+\varepsilon})^2}\right) \cdot w(A) \\
&= \left(k + 1 - \frac{\varepsilon^2}{(1+\varepsilon)\cdot(2+\varepsilon^2)} \cdot \left(k - \frac{2\cdot(k+1+\epsilon)\cdot(1-\varepsilon)^{-3}}{3\cdot(1-(\varepsilon^2+\frac{3\varepsilon}{1+\varepsilon})^2)}\right)\right) \cdot \frac{w(A)}{2} \\
&= \frac{1}{2} \cdot \left(\left(1 - \frac{\varepsilon^2}{(1+\varepsilon)\cdot(2+\varepsilon^2)} \cdot \left(1 - \frac{2\cdot(1-\varepsilon)^{-3}}{3\cdot(1-(\varepsilon^2+\frac{3\varepsilon}{1+\varepsilon})^2)}\right)\right) \cdot k \right. \\
&\qquad \left. + \left(1 + \frac{\varepsilon^2}{(1+\varepsilon)\cdot(2+\varepsilon^2)} \cdot \frac{2\cdot(1+\epsilon)\cdot(1-\varepsilon)^{-3}}{3\cdot(1-(\varepsilon^2+\frac{3\varepsilon}{1+\varepsilon})^2)}\right)\right) \cdot w(A).
\end{aligned}$$

Plugging in our choices of constants $\epsilon = 0.01$ and $\varepsilon = 0.06984$ results in $w(B) \leq (0.49985 \cdot k + 0.501) \cdot w(A)$. $\square$

We remark that our choice of ε, up to rounding, minimizes the coefficient of k in the approximation guarantee, leading to the best asymptotic behavior. However, especially for small values of k, a different choice of ε may result in a better guarantee.

Chapter 6

From weighted to unweighted k-Set Packing

In this chapter, we would like to shed some light on the more general relation between approximation guarantees for the weighted and the unweighted k-Set Packing problem. Our main result for this chapter is given by the following theorem.

Theorem 6.1. For any constant $\sigma \in (0, 1)$, there is a constant $\tau \in (0, 1)$ with the following property: Let $k \in \mathbb{Z}_{\geq 3}$ and assume that there exists a polynomial-time $(\tau \cdot k)$-approximation algorithm for the unweighted k-Set Packing problem (the MIS in $(k+1)$-claw free graphs). Then there exists a polynomial-time $(\sigma \cdot k)$-approximation algorithm for the weighted k-Set Packing problem (the MWIS in $(k+1)$-claw free graphs).

We point out that Theorem 6.1 is not a direct consequence of the analysis presented in the previous chapter. To see this, recall that the analysis in Chapter 5 relies on Lemma 3.20, which in turn requires $\varepsilon < 0.14$ by (3.3). In particular, Lemma 3.20 cannot yield a better guarantee than $\frac{k+1}{2} - \frac{0.14^2 \cdot k}{4}$, which is not sufficient for our purposes.

Instead, we basically repeat a coarse version of the previous analysis and do the following: First we relax the notion of regular neighbors enough to obtain a sufficiently strong version of Lemma 3.20. Then, we calculate how small τ needs to be, compared to σ, such that applying a $(\tau \cdot k)$-approximation algorithm for the unweighted problem results in a local improvement, provided a large fraction of the sets/vertices in the solution found by our algorithm feature many regular neighbors.

A (very) similar version of this result has been published in [42]. For the presentation in this thesis, we have adjusted the notation to align with the previous chapters. In addition, we have reformulated Theorem 6.1 in a (slightly) more concise way and we have simplified some of the subsequent arguments.

The remainder of this chapter is dedicated to the proof of Theorem 6.1. We first observe that we may assume $\mathrm{P} \neq \mathrm{NP}$. To this end, we note that if $\mathrm{P} = \mathrm{NP}$, then there exist polynomial-time algorithms for the decision versions of the weighted k-Set Packing problem and the MWIS in $(k+1)$-claw free graphs. Standard techniques further yield polynomial-time exact algorithms for the optimization versions. In particular, setting $\tau = \sigma$ meets the requirements of Theorem 6.1 in this case. Thus, we will assume $\mathrm{P} \neq \mathrm{NP}$ for the remainder of this chapter. In particular, we can make use of the following result by Berman and Karpinski [9].

Proposition 6.2 ([9])**.** Assume $\mathrm{P} \neq \mathrm{NP}$ and let $k \in \mathbb{Z}_{\geq 3}$. Then there is a constant $C > 1$ such that there is no polynomial-time C-approximation algorithm for the unweighted k-Set Packing problem.

Proof. For $k \geq 3$, the unweighted k-Set Packing problem generalizes the Maximum 3-Dimensional Matching problem. It is shown in [9] that for $\epsilon \in (0, \frac{1}{97})$, there is no polynomial-time $(\frac{98}{97} - \epsilon)$-approximation algorithm for the Maximum 3-Dimensional Matching problem unless $\mathrm{P} = \mathrm{NP}$. Thus, we may choose $C = \frac{98}{97} - \frac{1}{2 \cdot 97} = \frac{195}{194}$, for example. $\square$

Pick C according to Proposition 6.2. Fix $\sigma \in (0,1)$ and let

$$\tilde{\sigma} := \frac{C-1}{C} \cdot \sigma \text{ and } \tau := \frac{\left(1 - \frac{\tilde{\sigma}}{4}\right) \cdot \tilde{\sigma}^9}{2 \cdot 4^8 \cdot \left\lceil \frac{4^3}{\tilde{\sigma}^3} \right\rceil} \in (0,1). \tag{6.1}$$

We show that τ meets the requirements of Theorem 6.1 (for both the weighted k-Set Packing problem and the MWIS in $(k+1)$-claw free graphs). Fix $k \in \mathbb{Z}_{\geq 3}$ and let $\mathcal{A}$ be a polynomial-time $(\tau \cdot k)$-approximation algorithm for the unweighted k-Set Packing problem or the MIS in $(k+1)$-claw free graphs, respectively. We consider a local search algorithm that is very similar to Algorithm 5. We start with the empty solution and iteratively search for a local improvement of one of the following two types: a claw-shaped improvement or an improvement that we obtain via an application of $\mathcal{A}$ to certain subinstances. We will denote the solution that $\mathcal{A}$ returns when being applied to the instance I by $\mathcal{A}(I)$.

To formally define the algorithm we would like to analyze, we need to introduce some notation. As in the previous chapters, we employ the terminology from the more general MWIS. We first introduce the notion of the "i-th largest neighbor in a given vertex set".

Definition 6.3. Let $G = (V, E)$ be a graph, let $A \subseteq V$, and let $w \colon V \to \mathbb{R}_{>0}$. Let further $\preceq$ be a total ordering of V by non-decreasing weight, meaning that for $u, v \in V$, we have $u \preceq v \Rightarrow w(u) \leq w(v)$. For $i \in \{1, \dots, |A|\}$, define

$$n_i \colon \{u \in V : |N(u, A)| \geq i\} \to A$$
$$u \mapsto \text{ the } i\text{-th largest element of } N(u, A) \text{ w.r.t. } \preceq$$

Note that n_1 is a valid choice for the map n considered in the previous chapters. The definition of n_2 is compatible with the way it was defined in the previous chapters.

Next, we redefine the notion of regular neighbors. To this end, we need to introduce three constants $\delta \in \left(0, \frac{1}{4}\right)$, $\alpha > 1$ and $m \in \mathbb{N}$. We choose them to be

$$\delta := \frac{\tilde{\sigma}}{4} \in \left(0, \frac{1}{4}\right) \tag{6.2}$$

$$\alpha := \delta^{-2} \quad \text{and} \tag{6.3}$$

$$m := \lceil \delta^{-3} \rceil. \tag{6.4}$$

Observe that (6.1) implies

$$\tau = \frac{(1-\delta) \cdot \tilde{\sigma}}{2 \cdot m \cdot \alpha^4}. \tag{6.5}$$

The constant α serves as a threshold on the proximity of weights: We will consider the weights of two vertices similar if they differ by a factor of at most α multiplicatively. The constant δ has two purposes: First, we use δ as a threshold to determine whether the weight of the non-regular neighbors is small enough, similar to the constants ε and ε^2 in (3.6) and (3.9), respectively. In addition, the term $\frac{1}{2} - \delta$ plays the role of the constant $\frac{\varepsilon^2}{2 \cdot (1+\varepsilon) \cdot (2+\varepsilon^2)}$ in our previous analyses (cf. Lemma 3.20).

The natural number m tells us how many regular neighbors we can allow a vertex $u \in V$ to have. On the one hand, even if $w(N(u, A)) \leq 2 \cdot w(u)$, $N(u, A)$ may still contain $2 \cdot \alpha$ vertices that are within a weight range of $[\alpha^{-1} \cdot w(u), \alpha \cdot w(u)]$ and we would like u to be a regular neighbor of as many of these as possible in order to be able to guarantee a sizeable reimbursement for the remaining, irregular neighbors in the spirit of Lemma 3.17. On the other hand, we would like to conduct a similar argument as in the proof of Lemma 5.5 during the subsequent analysis. For this purpose, we require an upper bound on the number of regular neighbors vertices can feature in A: It allows us to translate a lower bound on the sum over the numbers of regular neighbors vertices in A have in an optimum solution B into a lower bound on the cardinality of B. Our choice of m provides a trade-off between these two conflicting goals.

Using the above constants, we redefine the notion of regular neighbors (see Definition 6.4). The construction can be described as follows: Given a vertex $u \in V$, we consider the maximal final segment of $N(u, A)$ w.r.t. $\preceq$ that only contains vertices within the weight range $[\alpha^{-1} \cdot w(u), \alpha \cdot w(u)]$. Then, we truncate this final segment at a maximum length of m vertices. Call the resulting set of vertices R. In case $w^2(N(u, A) \setminus R) \leq \delta \cdot w^2(u)$, we declare R to be the set of regular neighbors of u in A. Otherwise, we say that u does not have any regular neighbors in A.

Definition 6.4 (regular neighbor)**.** Let G be a $(k+1)$-claw free graph, $w\colon V \to \mathbb{R}_{>0}$ and $A \subseteq V$ independent. Let further $\preceq$ be a total ordering of V by non-decreasing weight (see Definition 6.3).

For $u \in V$, we define

$$i_0^u \coloneqq \min\{i \in \{1, \ldots, |N(u,A)|\} : w(n_i(u)) \notin [\alpha^{-1} \cdot w(u), \alpha \cdot w(u)]\},$$

where $\min \emptyset := \infty$, and set

$$i_{end}^u \coloneqq \min\{|N(u,A)| + 1, m+1, i_0^u\}.$$

We define the set of regular neighbors of u in A to be

$$\mathrm{reg}_A(u) \coloneqq \begin{cases} \{n_i(u) : 1 \leq i < i_{end}^u\} & , \sum_{i=i_{end}^u}^{|N(u,A)|} w^2(n_i(u)) \leq \delta \cdot w^2(u) \\ \emptyset & , \text{otherwise} \end{cases}.$$

We further let $\mathrm{irreg}_A(u) \coloneqq N(u,A) \setminus \mathrm{reg}_A(u)$.

Now, we are ready to introduce our algorithm *BlackBoxImp* (Algorithm 7). While BlackBoxImp is phrased in terms of the MWIS in $(k+1)$-claw free graphs, as in the previous chapters, we can cast it as an algorithm for the weighted k-Set Packing problem by restricting the input to conflict graphs of instances of weighted k-Set Packing. If the input to BlackBoxImp is the (weighted) conflict graph $(G_{\mathcal{S}}, w)$ of an instance $(\mathcal{S}, w)$ of the weighted k-Set Packing problem, then each graph we apply $\mathcal{A}$ to is an induced subgraph of $G_{\mathcal{S}}$ (cf. line 13). Note that for $\mathcal{S}' \subseteq V(G_{\mathcal{S}}) = \mathcal{S}$, $G_{\mathcal{S}}[\mathcal{S}']$ equals the conflict graph $G_{\mathcal{S}'}$ of $\mathcal{S}'$. In particular, approximating the MIS in $G_{\mathcal{S}}[\mathcal{S}']$ is equivalent to approximating the unweighted k-Set Packing problem on $\mathcal{S}'$.

We further point out that by our choice of $\mathcal{A}$, each iteration of (the while-loop of) BlackBoxImp can be implemented to run in polynomial time. Moreover, $w^2(A)$ strictly increases in each iteration.

The remainder of this chapter is dedicated to the proof of Lemma 6.5, which tells us that BlackBoxImp attains an approximation guarantee of $1 + \frac{\tilde{\sigma}}{2} \cdot k$. By Corollary 6.6, this is sufficient to conclude the proof of Theorem 6.1.

Lemma 6.5. Let $G = (V, E)$ be a $(k+1)$-claw free graph, let $w\colon V \to \mathbb{R}_{>0}$, let B be an optimum solution to the MWIS in (G, w) and let A be the solution returned by BlackBoxImp. Then

$$w(B) \leq \frac{1 + \tilde{\sigma} \cdot k}{2} \cdot w(A).$$

Algorithm 7: BlackBoxImp

Input: a $(k+1)$-claw free graph $G = (V, E)$, $w\colon V \to \mathbb{R}_{>0}$
Output: an independent set $A \subseteq V$

1 Sort V by non-decreasing weight to obtain a total ordering $\preceq$ according to Definitions 6.3 and 6.4.
2 $A \leftarrow \emptyset$
3 `improvement_found` $\leftarrow$ `true`
4 **while** `improvement_found` **do**
5   **if** $\exists$ *claw-shaped improvement* X **then**
6         $A \leftarrow (A \setminus N(X, A)) \cup X$
7   **goto** line 4
8   **end**
9     $V_{reg} \leftarrow \{u \in V \setminus A : \mathrm{reg}_A(u) \neq \emptyset\}$
10   **for** $L \in \{w(v) : v \in V\}$ **do**
11         $A_{\geq L} \leftarrow \{v \in A : w(v) \geq L\}$
12         $V_{\geq L} \leftarrow A_{\geq L} \cup \{u \in V_{reg} : w(u) \geq L \text{ and } \mathrm{reg}_A(u) \subseteq A_{\geq L}\}$
13         $Y \leftarrow \mathcal{A}(G[V_{\geq L}])$, $X \leftarrow Y \setminus A$
14   **for** $U \in \{w(v) : v \in V\}$ **do**
15             $X^{\leq U} \leftarrow \{x \in X : w(x) \leq U\}$
16   **if** $w^2(X^{\leq U}) > w^2(N(X^{\leq U}, A))$ **then**
17                 $A \leftarrow (A \setminus N(X^{\leq U}, A)) \cup X^{\leq U}$
18   **goto** line 4
19   **end**
20   **end**
21   **end**
22   `improvement_found` $\leftarrow$ `false`
23 **end**
24 **return** A

Corollary 6.6. There exists a polynomial-time $(\sigma \cdot k)$-approximation algorithm for the weighted k-Set Packing problem/the MWIS in $(k+1)$-claw free graphs, respectively.

Proof. Combining Lemma 6.5 with Lemma 2.14 yields the existence of a polynomial-time $(1 + \tilde{\sigma} \cdot k)$-approximation algorithm for the weighted k-Set Packing problem/the MWIS in $(k+1)$-claw free graphs when choosing $N := 2$. By our choice of the constant C according to Proposition 6.2 and since both the weighted k-Set Packing problem and the MWIS in $(k+1)$-claw free graphs generalize the unweighted k-Set Packing problem, we get

$$C < 1 + \tilde{\sigma} \cdot k \overset{(6.1)}{=} 1 + \frac{C-1}{C} \cdot \sigma \cdot k \quad \text{and, thus,} \quad 1 < \frac{\sigma \cdot k}{C}.$$

From this, using again (6.1), we compute

$$1 + \tilde{\sigma} \cdot k < \frac{\sigma \cdot k}{C} + \frac{C-1}{C} \cdot \sigma \cdot k = \sigma \cdot k.$$

In particular, the polynomial-time $(1 + \tilde{\sigma} \cdot k)$-approximation algorithm we have devised also constitutes a $(\sigma \cdot k)$-approximation algorithm for the weighted k-Set Packing problem or the MWIS in $(k+1)$-claw free graphs, respectively. □

Hence, it remains to prove Lemma 6.5. To this end, pick $G = (V, E)$, w, A and B as in the statement of the lemma. By the termination criterion of BlackBoxImp, A is not modified anymore in the last iteration. Hence, no claw improves A, which further implies that A constitutes a maximal independent set. In particular, the domain of n_1 (Definition 6.3) is V. Moreover, we may reuse results from the analysis of SquareImp [7] that we presented in Chapter 2. In particular, we will re-employ the notion of charges and the notion of the contribution (Definitions 2.16 and 2.19).

Following the arguments in Chapter 3, our first goal is to obtain a statement similar to Lemma 3.20. To this end, we define a (slightly) weaker notion of the slack (Definition 3.16) in Definition 6.7. Then, in analogy to Lemma 3.17, we prove Lemma 6.8.

Definition 6.7 (slack)**.** Let $u \in B$. We define

$$\text{slack}(u) \coloneqq \text{contr}(u, n_1(u)) - 2 \cdot \text{charge}(u, n_1(u)).$$

Note that Corollary 2.22 implies $\text{slack}(u) \geq 0$ for all $u \in B$.

Lemma 6.8. For $u \in B$, we have

$$\text{slack}(u) \geq (1 - 2\delta) \cdot w(\text{irreg}_A(u)).$$

.

We first show the following auxiliary statement:

Lemma 6.9. Let $u \in B$ such that $w(N(u, A)) \geq \delta^{-1} \cdot w(u)$. Then

$$\text{slack}(u) \geq (1 - 2\delta) \cdot w(\text{irreg}_A(u)).$$

Proof. By non-negativity of the contribution, we obtain

$$\begin{aligned}
\text{slack}(u) &= \text{contr}(u, n_1(u)) - 2 \cdot \text{charge}(u, n_1(u)) \\
&\geq -2 \cdot \text{charge}(u, n_1(u)) \\
&= w(N(u, A)) - 2 \cdot w(u) \\
&= \left(1 - 2 \cdot \frac{w(u)}{w(N(u, A))}\right) \cdot w(N(u, A)) \\
&\geq (1 - 2\delta) \cdot w(N(u, A)) \\
&\geq (1 - 2\delta) \cdot w(\text{irreg}_A(u))
\end{aligned}$$

since $1 - 2\delta > 0$ by (6.2) and $\text{irreg}_A(u) \subseteq N(u, A)$ by Definition 6.4. □

Proof of Lemma 6.8. We distinguish 4 cases.

Case 1: $w(n_1(u)) > \alpha \cdot w(u)$. Then $\delta \in (0, 1)$ by (6.2) implies

$$w(N(u, A)) \geq w(n_1(u)) > \alpha \cdot w(u) \stackrel{(6.3)}{=} \delta^{-2} \cdot w(u) > \delta^{-1} \cdot w(u).$$

Thus, Lemma 6.9 yields the claim.

Case 2: $w(n_1(u)) < \alpha^{-0.5} \cdot w(u)$. As $\alpha > 1$ by (6.3), this implies $u \in B \setminus A$ since $N(u, A) = \{u\}$ and $n_1(u) = u$ otherwise. The fact that there is no claw-shaped improvement of A and, in particular, $\{u\}$ cannot constitute one, yields

$$w^2(u) \leq w^2(N(u, A)) \leq w(n_1(u)) \cdot w(N(u, A)) < \alpha^{-0.5} \cdot w(u) \cdot w(N(u, A)),$$

where for the second inequality, we used that $n_1(u)$ is of maximum weight in $N(u, A)$ by Definition 6.3. Division by $\alpha^{-0.5} \cdot w(u) > 0$ results in

$$w(N(u, A)) > \alpha^{0.5} \cdot w(u) \stackrel{(6.3)}{=} \delta^{-1} \cdot w(u).$$

Again, we can apply Lemma 6.9 to conclude the desired statement.

Case 3: $\alpha^{-0.5} \cdot w(u) \leq w(n_1(u)) \leq \alpha \cdot w(u)$ and there exists an index $j \in \{2, \ldots, \min\{|N(u, A)|, m\}\}$ for which we have $w(n_j(u)) < \alpha^{-1} \cdot w(u)$. Pick j minimum with this property. Then for all $i \in \{1, \ldots, j-1\}$, we have $w(n_i(u)) \in [\alpha^{-1} \cdot w(u), \alpha \cdot w(u)]$.

Case 3.1: $\sum_{i=j}^{|N(u,A)|} w^2(n_i(u)) \leq \delta \cdot w^2(u)$.
Then Definition 6.4 yields

$$\text{reg}_A(u) = \{n_i(u) : 1 \leq i < j\} \text{ and } \text{irreg}_A(u) = \{n_i(u) : j \leq i \leq |N(u, A)|\}.$$

Thus, we calculate

$$\begin{aligned}
&\mathrm{slack}(u) \cdot w(n_1(u)) \\
&= \mathrm{contr}(u, n_1(u)) \cdot w(n_1(u)) - 2 \cdot \mathrm{charge}(u, n_1(u)) \cdot w(n_1(u)) \\
&\geq (w(u) - w(n_1(u)))^2 + \sum_{i=1}^{|N(u,A)|} (w(n_1(u)) - w(n_i(u))) \cdot w(n_i(u)) \\
&\geq \sum_{i=j}^{|N(u,A)|} (w(n_1(u)) - w(n_i(u))) \cdot w(n_i(u)) \\
&> \sum_{i=j}^{|N(u,A)|} (w(n_1(u)) - \alpha^{-1} \cdot w(u)) \cdot w(n_i(u)) \\
&\geq \sum_{i=j}^{|N(u,A)|} (1 - \alpha^{-0.5}) \cdot w(n_1(u)) \cdot w(n_i(u)) \\
&\overset{(6.3)}{=} (1 - \delta) \cdot w(n_1(u)) \cdot w(\{n_i(u) : j \leq i \leq |N(u,A)|\}) \\
&= (1 - \delta) \cdot w(n_1(u)) \cdot w(\mathrm{irreg}_A(u)) \\
&\geq (1 - 2\delta) \cdot w(n_1(u)) \cdot w(\mathrm{irreg}_A(u)).
\end{aligned}$$

The first inequality follows from Lemma 2.21, the second one is implied by the fact that all weights are positive and $n_1(u)$ is of maximum weight in $N(u, A)$. For the third inequality, we used that

$$w(n_i(u)) \leq w(n_j(u)) < \alpha^{-1} \cdot w(u) \text{ for all } j \leq i \leq |N(u, A)|.$$

Our case assumption $\alpha^{-0.5} \cdot w(u) \leq w(n_1(u))$ implies the fourth inequality. Division by $w(n_1(u)) > 0$ yields the claim.

Case 3.2: $\sum_{i=j}^{|N(u,A)|} w^2(n_i(u)) > \delta \cdot w^2(u)$.
As $\alpha^{-1} \cdot w(u) > w(n_i(u))$ for all $j \leq i \leq |N(u, A)|$, we compute

$$\begin{aligned}
&\alpha^{-1} \cdot w(u) \cdot w(\{n_i(u) : j \leq i \leq |N(u,A)|\}) \\
&> w^2(\{n_i(u) : j \leq i \leq |N(u,A)|\}) \\
&> \delta \cdot w^2(u).
\end{aligned}$$

Division by $\alpha^{-1} \cdot w(u) > 0$ leads to

$$w(N(u,A)) \geq w(\{n_i(u) : j \leq i \leq |N(u,A)|\}) > \alpha \cdot \delta \cdot w(u) \overset{(6.3)}{=} \delta^{-1} \cdot w(u).$$

Once more, we can apply Lemma 6.9 to conclude the claim.

Case 4: $\alpha^{-0.5} \cdot w(u) \leq w(n_1(u)) \leq \alpha \cdot w(u)$ and $w(n_j(u)) \geq \alpha^{-1} \cdot w(u)$

holds for all $j \in \{2, \dots, \min\{|N(u,A)|, m\}\}$.
If $|N(u,A)| \leq m$, this implies $\mathrm{reg}_A(u) = N(u,A)$ and $\mathrm{irreg}_A(u) = \emptyset$. In particular, Corollary 2.22 yields the desired statement.

In case $m < |N(u,A)|$, we obtain

$$w(N(u,A)) \geq \sum_{i=1}^{m} w(n_i(u)) \geq m \cdot \alpha^{-1} \cdot w(u) \underset{(6.4)}{\overset{(6.3)}{\geq}} \delta^{-1} \cdot w(u).$$

Applying Lemma 6.9 concludes the proof. □

Next, we prove a statement similar to Lemma 3.20. To this end, we require the following addition to Definition 6.4.

Definition 6.10 (regular neighbors in B)**.** For $v \in A$, define

$$\mathrm{reg}_B(v) \coloneqq \{u \in B : v \in \mathrm{reg}_A(u)\}.$$

Lemma 6.11.

$$w(B) \leq \frac{k+1}{2} \cdot w(A) - \left(\frac{1}{2} - \delta\right) \cdot \sum_{v \in A} (k - |\mathrm{reg}_B(v)|) \cdot w(v).$$

Proof. By Theorem 2.23, we know that

$$\begin{aligned} w(B) &\leq \frac{k+1}{2} \cdot w(A) - \frac{1}{2} \cdot \sum_{v \in A} (k - |N(v,B)|) \cdot w(v) \\ &\quad - \frac{1}{2} \cdot \sum_{u \in B} \left(\sum_{v \in A} \mathrm{contr}(u,v) - 2 \cdot \mathrm{charge}(u, n_1(u)) \right) \\ &\leq \frac{k+1}{2} \cdot w(A) - \frac{1}{2} \cdot \sum_{v \in A} (k - |N(v,B)|) \cdot w(v) \\ &\quad - \frac{1}{2} \cdot \sum_{u \in B} \mathrm{slack}(u), \end{aligned} \tag{6.6}$$

where the last inequality follows from Definition 6.7 and the non-negativity of the contribution. Using Lemma 6.8, we calculate

$$\begin{aligned} \sum_{u \in B} \mathrm{slack}(u) &\geq \sum_{u \in B} (1 - 2\delta) \cdot w(\mathrm{irreg}_A(u)) \\ &= (1 - 2\delta) \cdot \sum_{v \in A} |\{u \in B : v \in \mathrm{irreg}_A(u)\}| \cdot w(v) \\ &= (1 - 2\delta) \cdot \sum_{v \in A} (|N(v,B)| - |\mathrm{reg}_B(v)|) \cdot w(v). \end{aligned}$$

Plugging this into (6.6) and using $\delta > 0$ and $|N(v,B)| \leq k$ for all $v \in A$ by Proposition 2.17, we obtain

$$\begin{aligned} w(B) &\leq \frac{k+1}{2} \cdot w(A) - \frac{1}{2} \cdot \sum_{v \in A} (k - |N(v,B)|) \cdot w(v) \\ &\quad - \left(\frac{1}{2} - \delta\right) \cdot \sum_{v \in A} (|N(v,B)| - |\mathrm{reg}_B(v)|) \cdot w(v) \\ &\leq \frac{k+1}{2} \cdot w(A) - \left(\frac{1}{2} - \delta\right) \cdot \sum_{v \in A} (k - |\mathrm{reg}_B(v)|) \cdot w(v). \end{aligned}$$

□

To conclude the proof of Lemma 6.5, we need to establish an upper bound on $\sum_{v \in A} |\mathrm{reg}_B(v)| \cdot w(v)$. This is taken care of by Lemma 6.12.

Lemma 6.12.

$$\sum_{v \in A} |\mathrm{reg}_B(v)| \cdot w(v) \leq \frac{\tilde{\sigma}}{2} \cdot k \cdot w(A)$$

The proof of Lemma 6.12 proceeds analogously to the proof of Lemma 5.3 from the previous chapter. We assume that

$$\sum_{v \in A} |\mathrm{reg}_B(v)| \cdot w(v) > \frac{\tilde{\sigma}}{2} \cdot k \cdot w(A) \tag{6.7}$$

holds and show that this implies that BlackBoxImp finds a local improvement in the last iteration, a contradiction to its termination criterion. To prove the latter statement, we may assume that BlackBoxImp does not find a claw-shaped improvement in the last iteration since we are done otherwise. In particular, lines 9-16 are executed.

In accordance with BlackBoxImp, we employ the following notation:

- We let $V_{reg} := \{u \in V \setminus A : \mathrm{reg}_A(u) \neq \emptyset\}$.
- For $x \in \mathbb{R}_{>0}$, we let $A_{\geq x} = \{v \in A : w(v) \geq x\}$ consist of all vertices in A of weight at least x and define

  $$V_{\geq x} := A_{\geq x} \cup \{u \in V_{reg} : w(u) \geq x \text{ and } \mathrm{reg}_A(u) \subseteq A_{\geq x}\}$$

  to contain the vertices from $A_{\geq x}$ and all vertices from V_{reg} with the property that these and all of their regular neighbors in A are of weight at least x.

Proposition 6.13. Let $x > 0$ and $u \in V_{\geq x}$. Then $\mathrm{reg}_A(u) \subseteq A_{\geq x}$.

Proof. For $u \in V \setminus A$, this follows from the definition of $V_{\geq x}$. If $u \in V_{\geq x} \cap A = A_{\geq x}$, then Definition 6.4 implies $\text{reg}_A(u) \subseteq N(u, A) = \{u\} \subseteq A_{\geq x}$ since A is independent. We can even show $\text{reg}_A(u) = \{u\}$ in this case, but that is not needed here. □

Like the proof of Lemma 5.3, the proof of Lemma 6.12 consists of two main parts. Lemma 6.14 tells us that if (6.7) holds, then there exists a lower weight threshold L with the property that $|B \cap V_{\geq L}|$ is large compared to $|A_{\geq L}|$. In particular, applying $\mathcal{A}$ to $G[V_{\geq L}]$ produces an independent set Y the cardinality of which is by a constant factor larger than $|A_{\geq L}|$. Lemma 6.15 tells us that for this set Y, one of the sets $X^{\leq U}$ defined in line 15 yields a local improvement.

Lemma 6.14. If we have $\sum_{v \in A} |\text{reg}_B(v)| \cdot w(v) > \frac{\tilde{\sigma}}{2} \cdot k \cdot w(A)$, then there is $L \in \{w(v) : v \in V\}$ such that

$$|B \cap V_{\geq L}| > \frac{\tilde{\sigma} \cdot k}{2 \cdot m \cdot \alpha^2} \cdot |A_{\geq L}|.$$

In particular, $\mathcal{A}$, when applied to $G[V_{\geq L}]$, returns an independent set Y of cardinality

$$|Y| > \frac{\tilde{\sigma}}{2 \cdot \tau \cdot m \cdot \alpha^2} \cdot |A_{\geq L}|.$$

Lemma 6.15. Let $L \in \{w(v) : v \in V\}$ and let $Y \subseteq V_{\geq L}$ be an independent set with

$$|Y| > \frac{\tilde{\sigma}}{2 \cdot \tau \cdot m \cdot \alpha^2} \cdot |A_{\geq L}|.$$

Then there exists $U \in \{w(v) : v \in V\}$ such that the set $X^{\leq U} := \{x \in Y \setminus A : w(x) \leq U\}$ constitutes a local improvement of A (w.r.t. w^2).

Proof of Lemma 6.14. The proof is analogous to the proof of Lemma 5.5:

Claim 6.16. Let $x > 0$ and $v \in A$ with $\alpha^2 \cdot x \leq w(v)$. Then $\text{reg}_B(v) \subseteq V_{\geq x}$.

Proof. If $u \in \text{reg}_B(v) \cap A$, then $u \in N(v, A)$, implying $u = v$ by independence of A. In particular, $u = v \in A_{\geq x} \subseteq V_{\geq x}$. Now, let $u \in \text{reg}_B(v) \setminus A$. Then $v \in \text{reg}_A(u) \neq \emptyset$ by Definition 6.10, so $u \in V_{reg}$. Definition 6.4 yields $w(u) \geq \alpha^{-1} \cdot w(v) \geq \alpha \cdot x > x$. Moreover, again by Definition 6.4, every $z \in \text{reg}_A(u)$ satisfies $w(z) \geq \alpha^{-1} \cdot w(u) \geq x$, so $\text{reg}_A(u) \subseteq A_{\geq x}$. This implies that $u \in V_{\geq x}$. □

Our next goal is to show the following statement:

Claim 6.17. There is $L \in \{w(v) : v \in V\}$ such that

$$\sum_{v \in A_{\geq L}} |\text{reg}_B(v) \cap V_{\geq L}| > \frac{\tilde{\sigma} \cdot k}{2 \cdot \alpha^2} \cdot |A_{\geq L}|.$$

Proof. We want to apply Lemma 4.13. To this end, set $S := A$, $\mu(v) := |\text{reg}_B(v)|$, $\varphi(v) := w(v)$, $\lambda := \alpha^{-2}$ and $\eta := \frac{\tilde{\sigma}\cdot k}{2}$. Then Lemma 4.13 tells us that there is $L \in \mathbb{R}_{>0}$ such that

$$\begin{aligned}\sum_{v\in A_{\geq L}} |\text{reg}_B(v) \cap V_{\geq L}| &\overset{\text{Claim 6.16}}{\geq} \sum_{v\in A_{\geq \alpha^2\cdot L}} |\text{reg}_B(v)| \\ &\overset{\text{Lem. 4.13}}{>} \frac{\tilde{\sigma}\cdot k}{2\cdot\alpha^2}\cdot |A_{\geq L}|.\end{aligned}$$

In particular, the strict inequality implies that $A_{\geq L} \neq \emptyset$, and by increasing L to $\min\{w(v) : v \in V, w(v) \geq L\}$, we may assume $L \in \{w(v) : v \in V\}$. $\square$

Pick L as implied by the claim. By Definition 6.4, we have $|\text{reg}_A(u)| \leq m$ for every $u \in V$. Thus,

$$m \cdot |B \cap V_{\geq L}| \geq \sum_{u\in B\cap V_{\geq L}} |\text{reg}_A(u)| \overset{(*)}{=} \sum_{v\in A_{\geq L}} |\text{reg}_B(v) \cap V_{\geq L}| > \frac{\tilde{\sigma}\cdot k}{2\cdot\alpha^2}\cdot |A_{\geq L}|.$$

The equation marked $(*)$ is implied by Proposition 6.13, $\text{reg}_B(v) \subseteq B$ for $v \in A$, and $u \in \text{reg}_B(v) \Leftrightarrow v \in \text{reg}_A(u)$ for $u \in B$ and $v \in A$. Division by m yields the first part of the desired statement. Moreover, as $B \cap V_{\geq L}$ is an independent set in $G[V_{\geq L}]$, we know that $\mathcal{A}$, applied to $G[V_{\geq L}]$, outputs an independent set Y of cardinality

$$|Y| \geq \frac{|B \cap V_{\geq L}|}{k\cdot\tau} > \frac{\tilde{\sigma}}{2\cdot\tau\cdot m\cdot\alpha^2}\cdot |A_{\geq L}|. \qquad \square$$

For the proof of Lemma 6.15, we first show the following auxiliary statement.

Proposition 6.18. Let $u \in V_{reg}$. Then

(i) $w(v) \leq \alpha \cdot w(u)$ for every $v \in N(u, A)$.

(ii) $w^2(N(u, A) \setminus \text{reg}_A(u)) \leq \delta \cdot w^2(u)$.

Proof. Let $u \in V_{reg}$. Then $\text{reg}_A(u) \neq \emptyset$, which, by Definition 6.4, implies (ii), as well as

$$w(v) \leq \alpha \cdot w(u) \quad \text{for all } v \in \text{reg}_A(u). \tag{6.8}$$

(ii) further yields

$$w(v) \leq \sqrt{\delta}\cdot w(u) \underset{(6.3)}{\overset{(6.2)}{<}} \alpha \cdot w(u) \quad \text{for all } v \in N(u, A) \setminus \text{reg}_A(u). \tag{6.9}$$

Combining (6.8) and (6.9) gives (i). $\square$

The proof of Lemma 6.15 is analogous to the proof of Lemma 5.6.

Proof of Lemma 6.15. Define $X \coloneqq Y \setminus A$. In addition, for $x > 0$, let $X^{\leq x} \coloneqq \{u \in X : w(u) \leq x\}$. As $Y \cap A \subseteq V_{\geq L} \cap A = A_{\geq L}$, we get

$$|X| > \frac{\tilde{\sigma}}{2 \cdot \tau \cdot m \cdot \alpha^2} \cdot |A_{\geq L}| - |Y \cap A| \geq \frac{\tilde{\sigma}}{2 \cdot \tau \cdot m \cdot \alpha^2} \cdot |A_{\geq L} \setminus Y|,$$

where the last inequality follows from the fact that $\frac{\tilde{\sigma}}{2 \cdot \tau \cdot m \cdot \alpha^2} = \frac{\alpha^2}{1-\delta} > 1$ by (6.5), (6.2) and (6.3). As Y is independent in $G[V_{\geq L}]$, no vertex in $X = Y \setminus A$ is adjacent to a vertex in $Y \cap A_{\geq L}$, implying that $N(X, A_{\geq L}) \subseteq A_{\geq L} \setminus Y$. Hence, we obtain

$$|X| > \frac{\tilde{\sigma}}{2 \cdot \tau \cdot m \cdot \alpha^2} \cdot |N(X, A_{\geq L})|.$$

Claim 6.19. There is $U \in \{w(v) : v \in V\}$ such that

$$w^2(X^{\leq U}) > \frac{\tilde{\sigma}}{2 \cdot \tau \cdot m \cdot \alpha^4} \cdot w^2(N(X^{\leq U}, A_{\geq L})).$$

Proof. We want to apply Lemma 4.14. Let $S_1 \coloneqq X$, $S_2 \coloneqq N(X, A_{\geq L})$, $\varphi(s) \coloneqq w^2(s) > 0$ for $s \in S_1 \cup S_2$, $\eta \coloneqq \frac{\tilde{\sigma}}{2 \cdot \tau \cdot m \cdot \alpha^2}$ and $\lambda \coloneqq \alpha^{-2}$. In this setting, Lemma 4.14 tells us that there is $U > 0$ such that

$$w^2(X^{\leq U}) > \frac{\tilde{\sigma}}{2 \cdot \tau \cdot m \cdot \alpha^4} \cdot w^2(\{v \in N(X, A_{\geq L}) : w^2(v) \leq \alpha^2 \cdot U^2\}).$$

As $X \subseteq V_{\geq L} \setminus A \subseteq V_{reg}$, Proposition 6.18 (i) further implies

$$N(X^{\leq U}, A_{\geq L}) \subseteq \{v \in N(X, A_{\geq L}) : w^2(v) \leq \alpha^2 \cdot U^2\}.$$

Hence,

$$w^2(X^{\leq U}) > \frac{\tilde{\sigma}}{2 \cdot \tau \cdot m \cdot \alpha^4} \cdot w^2(N(X^{\leq U}, A_{\geq L})).$$

In particular, the strict inequality tells us that $X^{\leq U} \neq \emptyset$. By decreasing U to the maximum weight among the vertices in $X^{\leq U}$, we can thus achieve $U \in \{w(v) : v \in V\}$. ☐

Let U as implied by the claim. To verify that $X^{\leq U}$ constitutes a local improvement, we have to bound $w^2(N(X^{\leq U}, A \setminus A_{\geq L}))$.

As $X^{\leq U} \subseteq V_{\geq L} \setminus A$, we know that $\mathrm{reg}_A(u) \subseteq A_{\geq L}$ for every $u \in X^{\leq U}$. Using $V_{\geq L} \setminus A \subseteq V_{reg}$, Proposition 6.18 (ii) allows us to conclude that

$$w^2(N(X^{\leq U}, A \setminus A_{\geq L})) \leq \sum_{u \in X^{\leq U}} w^2(N(u, A) \setminus \mathrm{reg}_A(u)) \leq \delta \cdot w^2(X^{\leq U}). \quad (6.10)$$

Combining Claim 6.19 and (6.10) gives

$$\begin{aligned} w^2(N(X^{\leq U}, A)) &= w^2(N(X^{\leq U}, A_{\geq L})) + w^2(N(X^{\leq U}, A \setminus A_{\geq L})) \\ &< \left(\frac{2 \cdot \tau \cdot m \cdot \alpha^4}{\tilde{\sigma}} + \delta\right) \cdot w^2(X^{\leq U}) \\ &\overset{(6.5)}{=} w^2(X^{\leq U}). \end{aligned}$$

As $X^{\leq U} \subseteq Y$ is independent, it constitutes a local improvement of A. $\square$

Combining Lemma 6.14 and Lemma 6.15 proves Lemma 6.12. We have finally assembled all ingredients we need for the proof of Lemma 6.5, which concludes this chapter.

Proof of Lemma 6.5. By combining Lemma 6.11 and Lemma 6.12 and using $\delta \in (0, \frac{1}{2})$ by (6.2), we get

$$\begin{aligned} w(B) &\leq \frac{k+1}{2} \cdot w(A) - \left(\frac{1}{2} - \delta\right) \cdot \sum_{v \in A} (k - |\mathrm{reg}_B(v)|) \cdot w(v) \\ &= \frac{k+1}{2} \cdot w(A) - \left(\frac{1}{2} - \delta\right) \cdot k \cdot w(A) + \left(\frac{1}{2} - \delta\right) \cdot \sum_{v \in A} |\mathrm{reg}_B(v)| \cdot w(v) \\ &\leq \left(\frac{1}{2} + \delta \cdot k\right) \cdot w(A) + \left(\frac{1}{2} - \delta\right) \cdot \frac{\tilde{\sigma}}{2} \cdot k \cdot w(A) \\ &\overset{(6.2)}{=} \left(\frac{1}{2} + \frac{\tilde{\sigma}}{4} \cdot k\right) \cdot w(A) + \left(\frac{1}{2} - \delta\right) \cdot \frac{\tilde{\sigma}}{2} \cdot k \cdot w(A) \\ &< \left(\frac{1}{2} + \frac{\tilde{\sigma}}{4} \cdot k + \frac{\tilde{\sigma}}{4} \cdot k\right) \cdot w(A) \\ &= \frac{1 + \tilde{\sigma} \cdot k}{2} \cdot w(A). \end{aligned}$$

$\square$

Chapter 7

A 4/3-approximation for the Maximum Leaf Spanning Arborescence problem in acyclic digraphs

In this chapter, we study the *hereditary* 2-3-*Set Packing problem* (Definition 7.1), a special case of weighted 3-Set Packing, which arises as a subtask in the state-of-the-art approximation algorithm for the Maximum Leaf Spanning Arborescence problem (MLSA) in acyclic digraphs (dags) by Fernandes and Lintzmayer [25]. They showed that a polynomial-time α-approximation algorithm for the hereditary 2-3-Set Packing problem gives rise to a polynomial-time $\max\{\frac{4}{3}, \alpha\}$-approximation algorithm for the MLSA in dags (see Theorem 7.2). Via this approach, Fernandes and Lintzmayer [25] obtained $\frac{7}{5}$-approximations for both problems, which have been unchallenged so far. In this chapter, we improve upon their result and provide a polynomial-time $\frac{4}{3}$-approximation algorithm for the hereditary 2-3-Set Packing problem. By [25], this implies the same approximation guarantee for the MLSA in dags. Moreover, an approximation ratio of $\frac{4}{3}$ is the best we can achieve via Theorem 7.2.

The remainder of this chapter is organized as follows: In Section 7.1, we formally introduce the MLSA and provide a short overview of previous works. In particular, we define the hereditary 2-3-Set Packing problem (Definition 7.1) and state the result from [25] (Theorem 7.2) that allows us to translate approximation algorithms for the latter problem into approximation algorithms for the MLSA in dags. In Section 7.2, we present our algorithm for the hereditary 2-3-Set Packing problem and show that it attains an approximation ratio of $\frac{4}{3}$ (Theorem 7.7).

7.1 Introduction

The Maximum Leaf Spanning Arborescence problem (MLSA) is defined as follows: The input consists of a directed graph G and a *root* $r \in V(G)$ such that every vertex of G is reachable from r via a directed path. The task is to find a spanning r-arborescence with the maximum number of leaves possible.

The MLSA plays an important role in the context of broadcasting [25]: Given a network consisting of a set of nodes containing one distinguished source, and a set of available arcs, a message needs to be transferred from the source to all other nodes along a subset of the arcs, which forms (the edge set of) an arborescence rooted at the source. As internal nodes do not only need to be able to receive, but also to re-distribute messages, they are more expensive. Hence, it is desirable to have as few of them as possible, or equivalently, to maximize the number of leaves (nodes with out-degree zero).

Already the special case where every arc may be used in both directions, the *Maximum Leaves Spanning Tree problem*, is known to be NP-hard, even if the input graph is 4-regular or planar with maximum degree at most 4 (see [30], problem ND2). It has further been shown to be APX-hard [29][1], even when restricted to cubic graphs [12]. The best that is known for the Maximum Leaves Spanning Tree problem is an approximation guarantee of 2 [47].

In contrast, for general digraphs, the state-of-the-art is a $\min\{\sqrt{\mathrm{OPT}}, 92\}$-approximation [18, 19]. Moreover, there is a line of research focusing on FPT-algorithms for the MLSA [1, 11, 18].

The special case where the graph G is assumed to be a dag (directed acyclic graph) has been proven to be APX-hard by Schwartges, Spoerhase and Wolff [45]. They further provided a 2-approximation, which was then improved to $\frac{3}{2}$ by Fernandes and Lintzmayer [24]. Recently, they managed to enhance their approach to obtain a $\frac{7}{5}$-approximation [25], which constitutes the current state-of-the-art. In this thesis, following the approach by Fernandes and Lintzmayer, we improve on the result in [25] and obtain a $\frac{4}{3}$-approximation for the MLSA in dags.

Fernandes and Lintzmayer [25] tackle the MLSA in dags by reducing it, up to an approximation guarantee of $\frac{4}{3}$, to a special case of the weighted 3-Set Packing problem, which we call the *hereditary* 2-3-*Set Packing problem* (Definition 7.1). Fernandes and Lintzmayer [25] prove it to be NP-hard via a reduction from 3-Dimensional Matching [35].

Definition 7.1 (hereditary 2-3-Set Packing)**.** An instance of the *hereditary* 2-3-*Set Packing problem* is an instance $(\mathcal{S}, w)$ of the weighted 3-Set Packing problem, where

[1]Note that MAX-SNP-hardness implies APX-hardness, see [36].

- $w(s) = |s| - 1$ for all $s \in \mathcal{S}$, and
- for every $s \in \mathcal{S}$ with $|s| = 3$, all two-element subsets of s are contained in $\mathcal{S}$.

Note that by the assumption that weights are strictly positive, an instance $(\mathcal{S}, w)$ of the hereditary 2-3-Set Packing problem features only sets of cardinality 2 or 3.

Theorem 7.2 ([25])**.** Let $\alpha \geq 1$ and assume that there is a polynomial-time α-approximation algorithm for the hereditary 2-3-Set Packing problem. Then there exists a polynomial-time $\max\{\alpha, \frac{4}{3}\}$-approximation for the MLSA in dags.

Theorem 7.3 ([25])**.** There exists a polynomial-time $\frac{7}{5}$-approximation algorithm for the hereditary 2-3-Set Packing problem.

Note that the guarantee of $\frac{7}{5}$ in Theorem 7.3 is better than the state-of-the-art guarantee of 1.786 [49] for general weighted 3-Set Packing.

In order to prove Theorem 7.3, Fernandes and Lintzmayer [25] consider a modified version of Berman's algorithm SquareImp [7] (see Section 2.2), which they call SQUARE$^+$IMP. It differs from SquareImp in the following two aspects:

- Instead of squaring the original weights, SQUARE$^+$IMP conducts local search with respect to the weight function $v \mapsto (w(v) + 1)^2$.
- In addition to claw-shaped improvements, SQUARE$^+$IMP also incorporates another, more involved class of local improvements that are related to alternating paths in a certain auxiliary graph. This makes the analysis more complicated because in addition to charging arguments similar to ours, more intricate considerations regarding the structure of the auxiliary graph are required.

The local search algorithm that we study in the following section considers local improvements consisting of up to 10 sets with respect to the following objective: We lexicographically first maximize the weight of the current solution, and second the number of sets of weight 2 that are contained in it. We show that this algorithm yields a polynomial-time $\frac{4}{3}$-approximation for the hereditary 2-3-Set Packing problem. In particular, this results in a polynomial-time $\frac{4}{3}$-approximation algorithm for the MLSA in dags, tapping the full potential of Theorem 7.2. Our analysis is based on a two-stage charging argument.

7.2 Algorithm and analysis

In this section, we present a polynomial-time $\frac{4}{3}$-approximation for the hereditary 2-3-Set Packing problem. In order to define it, we formally introduce the notion of local improvement that we consider. It aims at lexicographically maximizing first the weight of the solution we find, and second the number of sets of weight 2 contained in it.

To simplify notation, we first define the *neighborhood* $N(U, W)$ of a family of sets U in another set family W. It corresponds to the neighborhood of U in W in the conflict graph of $U \cup W$ (see Definition 2.7).

Definition 7.4 (neighborhood)**.** Let U and W be two set families. We define the *neighborhood* of U in W to be

$$N(U, W) := \{w \in W : \exists u \in U : u \cap w \neq \emptyset\}.$$

Moreover, for a single set u, we write $N(u, W) := N(\{u\}, W)$.

Definition 7.5 (local improvement)**.** Let $(\mathcal{S}, w)$ be an instance of the hereditary 2-3-Set Packing problem and let A be a feasible solution. We call a disjoint set collection $X \subseteq \mathcal{S}$ a *local improvement of A of size* $|X|$ if

- $w(X) > w(N(X, A))$ or
- $w(X) = w(N(X, A))$ and X contains more sets of weight 2 than $N(X, A)$.

We analyze Algorithm 8, which starts with the empty solution and iteratively searches for a local improvement of size at most 10 (and performs the respective swap) until no more exists. We first observe that it runs in polynomial time.

Proposition 7.6. Algorithm 8 can be implemented to run in polynomial time.

Proof. A single iteration can be performed in polynomial time via brute-force enumeration. Thus, it remains to bound the number of iterations. By our definition of a local improvement, $w(A)$ can never decrease throughout the algorithm. Initially, we have $w(A) = 0$, and moreover, $w(A) \leq w(\mathcal{S}) \leq 2 \cdot |\mathcal{S}|$ holds throughout. As all weights are integral, we can infer that there are at most $2 \cdot |\mathcal{S}|$ iterations in which $w(A)$ strictly increases. In between two consecutive such iterations, there can be at most $|\mathcal{S}|$ iterations in which $w(A)$ remains constant since the number of sets of weight 2 in A strictly increases in each such iteration. All in all, we can bound the total number of iterations by $\mathcal{O}(|\mathcal{S}|^2)$. □

The remainder of this section is dedicated to the proof of Theorem 7.7, which implies that Algorithm 8 constitutes a $\frac{4}{3}$-approximation for the hereditary 2-3-Set Packing problem.

Algorithm 8: 4/3-approximation for the hereditary 2-3-Set Packing problem

Input: an instance $(\mathcal{S}, w)$ of the hereditary 2-3-Set Packing problem
Output: a disjoint subcollection of $\mathcal{S}$

$A \leftarrow \emptyset$
while $\exists$ *local improvement* X *of* A *of size at most* 10 **do**
  $A \leftarrow (A \setminus N(X, A)) \cup X$
end
Return A

Theorem 7.7. Let $(\mathcal{S}, w)$ be an instance of the hereditary 2-3-Set Packing problem and let $A \subseteq \mathcal{S}$ be a feasible solution such that there is no local improvement of A of size at most 10. Let further $B \subseteq \mathcal{S}$ be an optimum solution. Then $w(B) \leq \frac{4}{3} \cdot w(A)$.

Let $\mathcal{S}$, w, A and B be as in the statement of the theorem. Our goal is to distribute the weights of the sets in B among the sets in A they intersect in such a way that no set in A receives more than $\frac{4}{3}$ times its own weight. We remark that each set in B must intersect at least one set in A because otherwise, it would constitute a local improvement of size 1.

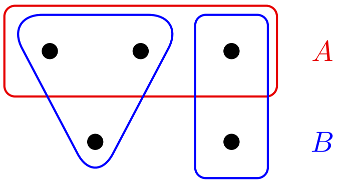
In order to present our weight distribution, we introduce a multigraph version of the *conflict graph*, which allows us to phrase our analysis using graph terminology. This new notion is very similar to Definition 2.7, but differs from it in the following two aspects:

- The vertex set of the conflict graph is now the disjoint union of the set families A and B. In particular, sets in $A \cap B$ will correspond to two vertices, whereas sets from $\mathcal{S} \setminus (A \cup B)$ do not appear anymore.
- We connect two intersecting sets a and b by $|a \cap b|$ *parallel edges* instead of *one single edge*.

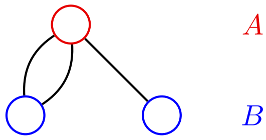
A similar construction is used in [25].

Definition 7.8 (conflict graph)**.** The conflict graph G is defined as follows: Its vertex set is the disjoint union of A and B, i.e., $V(G) = A \dot{\cup} B$. Its edge set is obtained by adding, for each pair $(a, b) \in A \times B$, $|a \cap b|$ parallel edges connecting a to b. See Fig. 7.1 for an illustration.

We remark that for $X \subseteq B$, $N(X, A)$ as defined in Definition 7.4 agrees with the (graph) neighborhood of X in the bipartite graph G (see Definition 2.12). In the following, we will simultaneously interpret sets from $A \dot{\cup} B$ as sets from $\mathcal{S}$ and as the corresponding vertices in G. In particular, we will talk about their degrees, their incident edges and their neighbors. We make the following observation.



(a) The figure displays two collections A (red) and B (blue) consisting of pairwise disjoint sets of cardinality 2 or 3. Black dots represent set elements.



(b) The figure shows the conflict graph of A and B. Vertices from A are drawn in red, vertices from B are drawn in blue.

Figure 7.1: Construction of the conflict graph.

Proposition 7.9. Let $v \in V(G)$ correspond to the set $s \in A \cup B$. Then v has at most $|s|$ incident edges in G.

Proof. As A and B both consist of pairwise disjoint sets, each element of s can induce at most one incident edge of s. □

7.2.1 Step 1 of the weight distribution

Our weight distribution proceeds in two steps. The first step works as follows:

Definition 7.10 (Step 1 of the weight distribution)**.** Let B_1 consist of all sets $u \in B$ with *exactly one neighbor* in A. Each $u \in B_1$ sends its full weight to its unique neighbor in A.

Let further B_2 consist of those $u \in B$ with $w(u) = 2$ and *exactly two incident edges*, with the additional property that they connect to *two distinct vertices* from A. Each $u \in B_2$ sends half of its weight (i.e., 1) along each of its edges to the respective endpoint in A. See Fig. 7.2 for an illustration.

Observe that in the first stage, $v \in A$ receives weight precisely from the sets in $N(v, B_1 \cup B_2)$.

We first prove Lemma 7.11, which tells us that we can represent the total amount of weight a collection $W \subseteq A$ receives in the first step as the weight of a set collection X with $N(X, A) \subseteq W$. We obtain X by first adding all sets from $B_1 \cup B_2$ that send their whole weight to W. Second, for each set $u \in B_2$ that sends only one unit of weight to W, we remove the element in which it intersects its neighboring set in $A \setminus W$, and add the resulting subset of cardinality 2 to X. See Fig. 7.3 for an illustration. This construction will allow us to combine X with subcollections of $B \setminus (B_1 \cup B_2)$ to obtain local improvements.

Lemma 7.11. Let $W \subseteq A$. There is $X \subseteq \mathcal{S}$ with the following properties:

(a) Every set from B_1 sends its whole weight to its unique neighbor in A (to which it may be connected via multiple edges).

(b) Every set from B_2 sends one unit of weight to each of its neighbors in A.

Figure 7.2: The first step of the weight distribution. Vertices from A are drawn in red, vertices from B are drawn in blue. A number inscribed within a vertex indicates its weight, a number next to an edge denotes the amount of weight that is sent along the edge to its endpoint in A. Dashed lines indicate edges that can, but need not exist.

(7.11.1) $N(X, A) \subseteq W$.

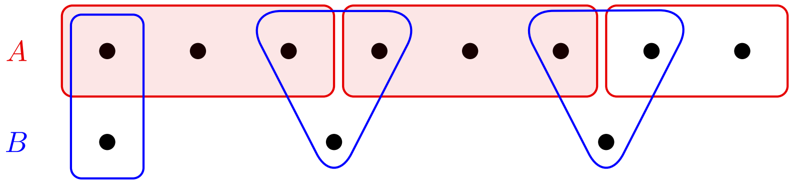
(7.11.2) $w(X)$ equals the total amount of weight that W receives in the first step.

(7.11.3) There is a bijection $N(W, B_1 \cup B_2) \leftrightarrow X$ mapping $u \in B_1 \cup B_2$ to itself or to one of its two-element subsets.

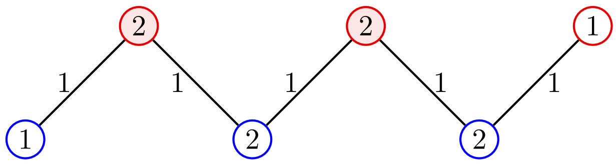
Proof. We obtain X as follows: First, we add those sets in $N(W, B_1 \cup B_2)$ to X that send all of their weight to W (i.e., whose neighborhood in A is contained in W). This includes all sets in $N(W, B_1)$. Second, for each set $u \in B_2$ that has one incident edge to a set $v \in W$ and one incident edge to a set $r \in A \setminus W$, we add its two-element subset $u \setminus r$ to X. Note that B_2 consists of three-element sets only by Definition 7.10. By construction, (7.11.1)-(7.11.3) hold. $\square$

Corollary 7.12. No set in A receives more than its own weight in the first step.

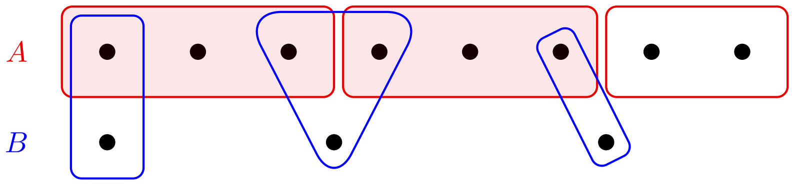
Proof. Assume towards a contradiction that $v \in A$ receives more than $w(v)$ in the first step. Apply Lemma 7.11 with $W = \{v\}$ to obtain a collection $X \subseteq \mathcal{S}$ subject to (7.11.1)-(7.11.3). Then $w(X) > w(v) \geq w(N(X, A))$ by (7.11.1) and (7.11.2). In addition, (7.11.3) and Proposition 7.9 imply that X is a disjoint set family with $|X| \leq |v| \leq 3$. Thus, X constitutes a local improvement of size at most 3, a contradiction. $\square$



(a) The left blue set is contained in B_1 and sends its whole weight to the unique set from A it intersects. The two triangular blue sets are contained in B_2. The left one only intersects sets in A that are contained in W, whereas the right one also intersects a set in $A \setminus W$.



(b) Part of the conflict graph and the weight distribution corresponding to the set configuration in Fig. 7.3a.



(c) The set collection X (blue) we construct in the proof of Lemma 7.11 contains the left and the middle blue set because they send all of their weight to W. For the right triangular set, we remove the element in which it intersects a set from $A \setminus W$. Then, we add the resulting set of cardinality 2 to X.

Figure 7.3: Illustration of the construction in the proof of Lemma 7.11. Fig. 7.3a shows a collection $W \subseteq A$ of sets (red, filled, horizontal), the collection $N(W, B_1 \cup B_2)$ (blue) of sets the sets in W receive weight from in the first step, and further sets from A (red, not filled, horizontal) the sets in $N(W, B_1 \cup B_2)$ send weight to. Fig. 7.3b displays the weight distribution from $N(W, B_1 \cup B_2)$ to A. Fig. 7.3c illustrates the construction of the set collection X.

7.2.2 Removing "covered" sets

Definition 7.13. Let C consist of those sets from A that receive exactly their own weight in the first step.

The intuitive idea behind our analysis is that the sets in C are "covered" by the sets sending weight to them in the sense of Lemma 7.11. Hence, we can "remove" the sets in C from our current solution A and the sets in $B_1 \cup B_2$ from our optimum solution B: If we can find a local improvement in the remaining instance, we will use Lemma 7.11 to transform it into a local improvement in the original instance, leading to a contradiction. See Lemma 7.14 for an example how to apply this reasoning. But under the assumption that no local improvement in the remaining instance exists, we can design the second step of the weight distribution in such a way that overall, no set in A receives more than $\frac{4}{3}$ times its own weight.

7.2.3 Step 2 of the weight distribution

In order to define the second step of the weight distribution, we make the following observations:

Lemma 7.14. There is no $u \in B \setminus (B_1 \cup B_2)$ with $w(N(u, A \setminus C)) < w(u)$.

Proof. Assume towards a contradiction that there is $u \in B \setminus (B_1 \cup B_2)$ with $w(N(u, A \setminus C)) < w(u)$. Apply Lemma 7.11 to $W := N(u, C)$ to obtain X subject to (7.11.1)-(7.11.3). By (7.11.3), $X \dot\cup \{u\}$ consists of pairwise disjoint sets. Proposition 7.9 further yields $|W| = |N(u, C)| \leq |u| \leq 3$. Thus, another application of Proposition 7.9 results in

$$|X| \overset{(7.11.3)}{=} |N(W, B_1 \cup B_2)| \leq \sum_{v \in W} |v| \leq 3 \cdot |W| \leq 9.$$

Finally, we have $w(X) = w(N(u, C))$ by (7.11.2) and Definition 7.13. Hence, using our assumption $w(N(u, A \setminus C)) < w(u)$, we obtain

$$\begin{aligned} w(X \cup \{u\}) = w(X) + w(u) &> w(N(u, C)) + w(N(u, A \setminus C)) \\ &\overset{(7.11.1)}{=} w(N(X \cup \{u\}, A)). \end{aligned}$$

So $X \cup \{u\}$ is a local improvement of A of size at most 10, a contradiction. □

Proposition 7.15. Let $u \in B \setminus (B_1 \cup B_2)$. Then:

(i) u has at least one neighbor in $A \setminus C$.

(ii) If $w(u) = 1$, then u has exactly two neighbors in A.

(iii) If $w(u) = 2$, then u has three incident edges.

Proof. (i) follows from Lemma 7.14. For (ii) and (iii), we remind ourselves that each $u \in B \setminus (B_1 \cup B_2)$ has at most $|u|$ neighbors in A/incident edges by Proposition 7.9, but at least 1 neighbor in A by (i). In particular, (ii) holds since $u \in B_1$ otherwise. For (iii), we observe that in case u has at most 2 incident edges, then either u has only one neighbor in A, or two distinct neighbors to which it is connected by a single edge each. In either case, we have $u \in B_1 \cup B_2$. □

Definition 7.16 (second step of the weight distribution)**.**
Let $u \in B \setminus (B_1 \cup B_2)$ with $w(u) = 1$.

(a) If u has a neighbor in C, then this neighbor receives $\frac{1}{3}$ and the neighbor in $A \setminus C$ receives $\frac{2}{3}$.

(b) Otherwise, both neighbors in $A \setminus C$ receive $\frac{1}{2}$.

Now, let $u \in B \setminus (B_1 \cup B_2)$ with $w(u) = 2$.

(c) If u has degree 1 to $A \setminus C$, then u sends $\frac{1}{3}$ along each edge to C and $\frac{4}{3}$ to the neighbor in $A \setminus C$. Note that this neighbor must have a weight of 2 by Lemma 7.14.

(d) If u has degree 2 to $A \setminus C$, u sends 1 along each edge to a vertex in $A \setminus C$ of weight 2, $\frac{2}{3}$ along each edge to a vertex in $A \setminus C$ of weight 1, and the remaining amount to the neighbor in C.

(e) If all three incident edges of u connect to $A \setminus C$, then u sends $\frac{2}{3}$ along each of these edges.

We denote the set of vertices to which case ℓ with $\ell \in \{a, b, c, d, e\}$ applies by B_ℓ. See Fig. 7.4 for an illustration.

7.2.4 No set in C receives more than 4/3 times its weight

Lemma 7.17. Let $u \in B_d$ and let $v \in N(u, C)$ be the unique neighbor of u in C. If v receives more than $\frac{1}{3}$ from u, then $w(v) = 2$ and v has at most one incident edge to $B \setminus (B_1 \cup B_2)$.

Proof. Denote the (other) endpoints of the edges connecting u to $A \setminus C$ by v_1 and v_2. Assume v receives more than $\frac{1}{3}$ from u. Then $w(v_1) = w(v_2) = 1$. In particular, v_1 and v_2 are distinct by Lemma 7.14. Apply Lemma 7.11 to $W := \{v\}$ to obtain X subject to (7.11.1)-(7.11.3). Then by (7.11.3), $Y := X \dot\cup \{u\}$ is a disjoint collection of sets. Moreover, Proposition 7.9 yields

$$|X| \overset{(7.11.3)}{=} |N(v, B_1 \cup B_2)| \leq |v| \leq 3.$$

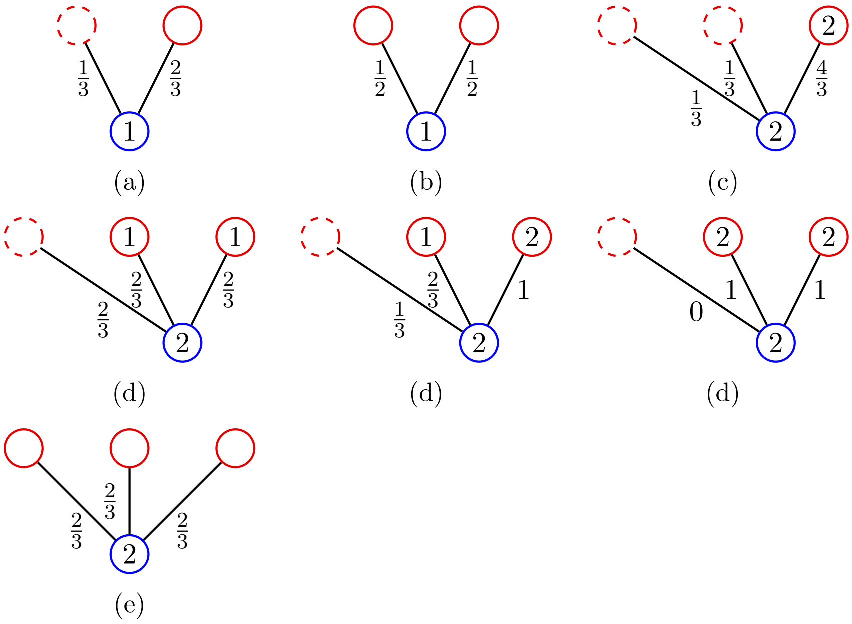


Figure 7.4: Illustration of the second step of the weight distribution. Red circles in the top row indicate sets from A, if they are dashed, the corresponding set is contained in C. Blue circles in the bottom row indicate sets from $B \setminus (B_1 \cup B_2)$. The number within a circle indicates the weight of the corresponding set in case it is relevant. Even though drawn as individual circles, the endpoints in A of the incident edges of a set $u \in B \setminus (B_1 \cup B_2)$ need not be distinct. For example, in (e), two of the sets represented by the red circles may agree, in which case the corresponding set receives $\frac{4}{3}$.

Hence, $|Y| \leq 4$. By (7.11.2) and as $v \in C$, we get $w(v) = w(X)$. Thus, $w(v_1) + w(v_2) = 1 + 1 = 2 = w(u)$ results in

$$w(N(Y, A)) \overset{(7.11.1)}{=} w(v) + w(v_1) + w(v_2) = w(X) + w(u) = w(Y).$$

As Y does not constitute a local improvement, $N(Y, A) = \{v_1, v_2, v\}$ contains at least as many vertices of weight 2 as Y. As $w(v_1) = w(v_2) = 1$, but $w(u) = 2$, this implies that $w(v) = 2$ and that all elements of X have a weight of 1. By (7.11.2) and since $w(v) = 2$, this yields $|X| = 2$, and by (7.11.3), v has degree at least 2 to $B_1 \cup B_2$. Hence, v can have at most one incident edge to $B \setminus (B_1 \cup B_2)$ by Proposition 7.9. □

Lemma 7.18. Each set in C receives at most $\frac{4}{3}$ times its own weight during our weight distribution.

Proof. First, let $v \in C$ with $w(v) = 1$. Then v receives 1 in the first step and has at most one incident edge to $B \setminus (B_1 \cup B_2)$ by Proposition 7.9. By Lemma 7.17, v receives at most $\frac{1}{3}$ via this edge.

Next, let $v \in C$ with $w(v) = 2$. Then v receives 2 in the first step and v has at most two incident edges to $B \setminus (B_1 \cup B_2)$ by Proposition 7.9. If v has two incident edges to $B \setminus (B_1 \cup B_2)$, then Lemma 7.17 implies that v can receive at most $\frac{1}{3}$ via each of the edges to $B \setminus (B_1 \cup B_2)$. Thus, v receives at most $\frac{8}{3} = \frac{4}{3} \cdot w(v)$ in total. If v has one incident edge to $B \setminus (B_1 \cup B_2)$, then the maximum amount v can receive via this edge is $\frac{2}{3}$. Again, v receives at most $\frac{8}{3}$ in total. □

7.2.5 No set in $A \setminus C$ receives more than 4/3 times its weight

In order to make sure that no vertex from $A \setminus C$ receives more than $\frac{4}{3}$ times its weight, we need Lemma 7.19, which essentially states the following:

- If a vertex $v \in A \setminus C$ with $w(v) = 2$ receives $\frac{4}{3}$ from a vertex in B_c, then it does not receive weight from any further vertex in $B_1 \cup B_2 \cup B_c \cup B_d$.
- A vertex $v \in A \setminus C$ with $w(v) = 2$ may, in total, receive at most 2 units of weight from vertices in $B_1 \cup B_2 \cup B_d$.

Lemma 7.19. Let $v \in A \setminus C$ with $w(v) = 2$. Denote the set of vertices $u \in B_d$ that are connected to v by one/two parallel edges by D_1 and D_2, respectively.

Then $|N(v, B_1 \cup B_2)| + 2|N(v, B_c)| + |D_1| + 2|D_2| \leq 2$.

Proof. Assume towards a contradiction that

$$|N(v, B_1 \cup B_2)| + 2|N(v, B_c)| + |D_1| + 2|D_2| \geq 3.$$

Note that $|N(v, B_1 \cup B_2)| \leq 1$ because $v \notin C$ and v receives at least one unit of weight per neighbor in $B_1 \cup B_2$. Pick an inclusion-wise minimal set $\bar{Y} \subseteq N(v, B_c \cup B_d)$ such that

$$|N(v, B_1 \cup B_2)| + 2|\bar{Y} \cap B_c| + |\bar{Y} \cap D_1| + 2|\bar{Y} \cap D_2| \geq 3. \tag{7.1}$$

Then

$$|N(v, B_1 \cup B_2)| + 2|\bar{Y} \cap B_c| + |\bar{Y} \cap D_1| + 2|\bar{Y} \cap D_2| = 3, \text{ or} \tag{7.2}$$
$$\bar{Y} \cap D_1 = \emptyset \text{ and } |N(v, B_1 \cup B_2)| + 2|\bar{Y} \cap B_c| + 2|\bar{Y} \cap D_2| = 4. \tag{7.3}$$

We construct a set collection Y as follows: First, we add all sets contained in $\bar{Y} \cap (B_c \cup D_2)$ to Y. Note that for a set $u \in \bar{Y} \cap (B_c \cup D_2)$, we have $N(u, A \setminus C) = \{v\}$ (see Fig. 7.4). Second, for each $u \in \bar{Y} \cap D_1$, let u' be the set of cardinality 2 containing the element in which u intersects a set from C, and the element in which u intersects v. Add u' to Y. Then Y has the following properties:

$$\begin{aligned} N(Y, A) &\subseteq C \cup \{v\} && (7.4)\\ |Y| &= |\bar{Y} \cap B_c| + |\bar{Y} \cap D_1| + |\bar{Y} \cap D_2| && (7.5)\\ w(Y) &= 2|\bar{Y} \cap B_c| + |\bar{Y} \cap D_1| + 2|\bar{Y} \cap D_2| \\ &\overset{(7.1)}{\geq} 3 - |N(v, B_1 \cup B_2)| && (7.6)\\ |N(Y, C)| &\leq 2|\bar{Y} \cap B_c| + |\bar{Y} \cap D_1| + |\bar{Y} \cap D_2|. && (7.7) \end{aligned}$$

See Fig. 7.4 for (7.7). Let $W := N(Y, C) \cup \{v\}$. Apply Lemma 7.11 to obtain X subject to (7.11.1)-(7.11.3). Then

$$w(X) \geq w(N(Y, C)) + |N(v, B_1 \cup B_2)| \tag{7.8}$$

because each set in $N(Y, C)$ receives its weight in the first step, and v receives at least one per neighbor in $B_1 \cup B_2$. By (7.11.3) and since the sets in Y constitute disjoint subsets of sets in $B \setminus (B_1 \cup B_2)$, $X \dot{\cup} Y$ is a family of pairwise disjoint sets. We would like to show that $X \cup Y$ yields a local improvement of size at most 10. By (7.8) and (7.6), we obtain

$$\begin{aligned} w(X \cup Y) = w(X) + w(Y) &\geq 3 + w(N(Y, C)) \\ &> w(v) + w(N(Y, C)) \geq w(N(X \cup Y, A)), \end{aligned}$$

where $N(X \cup Y, A) \subseteq N(Y, C) \cup \{v\}$ follows from (7.11.1) and (7.4). Thus, it remains to show that $|X \cup Y| \leq 10$. By (7.11.3), we have

$$\begin{aligned} |X| = |N(W, B_1 \cup B_2)| &\leq |N(v, B_1 \cup B_2)| + |N(N(Y, C), B_1 \cup B_2)| \\ &\leq |N(v, B_1 \cup B_2)| + 2|N(Y, C)|. \end{aligned} \tag{7.9}$$

For the last inequality, we used Proposition 7.9, which tells us that each set $z \in N(Y, C)$ has degree at most 3 in G. In addition, z must intersect at least one set from Y, and thus, from $\bar{Y}$. In particular, z has at least one incident edge to $B \setminus (B_1 \cup B_2) \supseteq \bar{Y}$, and, thus, at most two incident edges to $B_1 \cup B_2$. Hence, we obtain

$$\begin{aligned} |Y| + |X| &\overset{(7.9)}{\leq} |Y| + |N(v, B_1 \cup B_2)| + 2|N(Y, C)| \\ &\underset{(7.7)}{\overset{(7.5)}{\leq}} \underbrace{|N(v, B_1 \cup B_2)| + 5|\bar{Y} \cap B_c| + 3|\bar{Y} \cap D_1| + 3|\bar{Y} \cap D_2|}_{=:(*)}. \end{aligned}$$

If (7.2) holds, we can bound $(*)$ by 3 times the left-hand side of (7.2) and deduce an upper bound of 9. In case (7.3) is satisfied, we can bound $(*)$ by $\frac{5}{2}$ times the left-hand side of (7.3) and obtain an upper bound of 10. Thus, we have found a local improvement of size at most 10, a contradiction. □

Lemma 7.20. Each set $v \in A \setminus C$ receives at most $\frac{4}{3}$ times its own weight during our weight distribution.

Proof. If $w(v) = 1$, then v cannot receive any weight in the first step because otherwise, it would receive at least 1 and be contained in C. Moreover, v has at most two incident edges and receives at most $\frac{2}{3}$ via either of them in the second step.

Next, consider the case where $w(v) = 2$. If v receives $\frac{4}{3}$ from a vertex in B_c, then by Lemma 7.19, there is no further vertex in $B_1 \cup B_2 \cup B_c \cup B_d$ from which v receives weight. As v receives at most $\frac{2}{3}$ per edge in all remaining cases, v receives at most $\frac{4}{3} + 2 \cdot \frac{2}{3} = \frac{8}{3} = \frac{4}{3} \cdot w(v)$.

Next, assume that $N(v, B_c) = \emptyset$. In the first step, v can receive at most 1 in total (otherwise, $v \in C$) and this can only happen if v has a neighbor in $B_1 \cup B_2$. The maximum amount v can receive through one edge in the second step is 1, and this can only happen in situation (d). By Lemma 7.19, there are at most 2 edges via which v receives 1 (taking both steps into account). Moreover, v can receive at most $\frac{2}{3}$ via the remaining edges. Again, we obtain an upper bound of $1 + 1 + \frac{2}{3} = \frac{8}{3}$ on the total weight received. □

Combining Lemma 7.18 and Lemma 7.20 proves Theorem 7.7. Together with Proposition 7.6 and Theorem 7.2, we obtain Corollary 7.21.

Corollary 7.21. There is a polynomial-time $\frac{4}{3}$-approximation algorithm for the MLSA in dags. □

Chapter 8

Conclusion

In this thesis, we have studied local search based approximation algorithms for the weighted k-Set Packing problem and managed to improve on Berman's $\frac{k+1+\epsilon}{2}$-approximation algorithm SquareImp [7]. Berman's result [7] has been the state-of-the-art for twenty years. Our improvements are based on a deeper understanding of the structural properties of instances where Berman's analysis [7] is close to being tight: We have shown that such instances are "close to unweighted" in a certain sense, which allowed us to port techniques used in the unweighted setting to general weights.

In Chapter 3, we have seen that enhancing SquareImp by further considering local improvements of size 3 yields an improved approximation guarantee of $\frac{k+1}{2} - \frac{1}{1000}$. In particular, this result shows that the significant blow-up in the improvement size that is inherent to the algorithms in [40] and [49][1], is not needed to obtain an approximation guarantee below $\frac{k+1}{2}$. This raises the question how far one can push the approximation guarantee of local search algorithms when limiting the size of the local improvements considered to $\theta(k)$, the size of improvements searched for by SquareImp. The answer to this question may also be interesting in terms of applicability. Note that misdirected local search with a constant improvement size of t that is independent of k has been studied exemplarily for $t = 2$ in [10].

On the upper end of the size range, we have shown in Chapter 4 that even if we allow local improvements of up to logarithmic size (with respect to a fixed additive local search objective), we cannot obtain a better guarantee than $\frac{k}{2}$. With our algorithm LogImp, we manage to meet this lower bound asymptotically: LogImp obtains approximation ratios of $\frac{k+1-\lambda_k}{2}$ with $\lim_{k\to\infty} \lambda_k = 1$ (cf. Theorem 4.1). As a topic for future research, it would be interesting to see whether it is possible to get arbitrarily close to a guarantee of $\frac{k}{2}$ for every $k \geq 3$, and not just in the limit for $k \to \infty$. We remark that for $k = 3$, this goal cannot be achieved via local search with respect to the

[1]These algorithms consider improvements of size $\theta(k^2)$ and $\theta(k^3)$, respectively, whereas the claw-shaped improvements SquareImp searches for are of size $\theta(k)$.

squared weight function because this approach implies a lower bound of $\sqrt{3}$ ($\sqrt{k}$ in general), see for example [49].

We further point out that for general weights, it is still open whether local improvements of *constant size* suffice to obtain guarantees arbitrarily close to $\frac{k}{2}$: In the unweighted setting, this is true [33], and the lower bound for general weights from [4] only applies to local search with respect to the original weight function. An affirmative answer to the above question would be particularly interesting if the result generalizes to the MWIS in $(k+1)$-claw free graphs. Recall that all previous works employing local improvements of logarithmic size [16, 17, 28, 31, 41, 42, 48] could only extend their algorithms to the MWIS in $(k+1)$-claw free graphs at the cost of a quasi-polynomial running time.

The lower bound result from Chapter 4 limits the scope of approximation ratios that we can hope for via an approach using enumeration or dynamic programming based local search. Note that all previous works on local search for the weighted k-Set Packing problem obey this paradigm [7, 15, 16, 17, 28, 31, 33, 48]. Surprisingly, in Chapter 5, we have seen that by using a black box algorithm for the unweighted k-Set Packing problem in order to generate candidate improvements, we can pass the threshold guarantee of $\frac{k}{2}$ established in Chapter 4, at least for large enough values of k. Once more, it would be interesting to see whether a more refined analysis, e.g., using techniques from [49], can result in more significant improvements also for small values of k.

In Chapter 6, we employ the ideas from Chapter 5 to establish a general link between the approximation guarantees for the weighted and the unweighted k-Set Packing problem (see Theorem 6.1). This result has two main consequences: As far as lower bounds are concerned, it tells us that an $o(k)$-approximation hardness for the weighted k-Set Packing problem would translate to the unit weight case. Given the current gap between the $o(\frac{k}{\log k})$-approximation hardness for the unweighted k-Set Packing problem [32] and the state-of-the-art guarantees in the order of $\theta(k)$ for both the unweighted and the weighted setting [16, 28, 42, 49], Theorem 6.1 could provide an angle for improvements in terms of lower bounds.

On the algorithmic side, the analyses in Chapters 5 and 6 provide a recipe to translate improved approximation ratios for the unweighted k-Set Packing problem (the MIS in $(k+1)$-claw free graphs) into better guarantees for general weights. Thus, our results provide further motivation to study the unit weight case. However, as the lower bound instances in [28] show, new ideas will be required to improve upon the state-of-the-art of $\frac{k+1+\epsilon}{3}$ [16, 28].

Another, potentially more accessible direction for future research is pointed out in Chapter 7. There, we studied the hereditary 2-3-Set Packing problem, a special case of weighted 3-Set Packing, which appears as a subproblem in an algorithm for the Maximum Leaf Spanning Arborescence problem (MLSA) in acyclic digraphs (dags) [25]. Via local improvements of constant size and a simple two-stage charging argument, we managed to obtain a $\frac{4}{3}$ approximation for both the hereditary 2-3-Set Packing problem, and, using the reduction from [25], the MLSA in dags. In doing so, we improve on the previous state-of-the-art guarantee of $\frac{7}{5}$ for both problems [25]. Following up on this result, it would be interesting to identify further special classes of weight functions that arise naturally from other combinatorial optimization problems and bear enough structure to allow for improved approximation guarantees, compared to general weights.

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