



Institut für Numerische Simulation

Rheinische Friedrich-Wilhelms-Universität Bonn

Friedrich-Hirzebruch-Allee 7 • 53115 Bonn • Germany  
phone +49 228 73-69828 • fax +49 228 73-69847  
[www.ins.uni-bonn.de](http://www.ins.uni-bonn.de)

B. Brown, M. Griebel, F. Y. Kuo, I. H. Sloan

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# On the expected uniform error of Brownian motion approximated by the Lévy-Ciesielski construction

Bruce Brown, Michael Griebel, Frances Y. Kuo, and Ian H. Sloan\*<sup>†</sup>

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## Abstract

It is known that the Brownian bridge or Lévy-Ciesielski construction of Brownian paths almost surely converges uniformly to the true Brownian path. In the present article the focus is on the uniform error. In particular, we show constructively that at level  $N$ , at which there are  $d = 2^N$  points evaluated on the Brownian path, the uniform error and its square, and the uniform error of geometric Brownian motion, have upper bounds of order  $\mathcal{O}(\sqrt{\ln d/d})$ , matching the known orders. We apply the results to an option pricing example.

## 1 Introduction

For  $t \in [0, 1]$ , let  $B(t) = B(\omega)(t)$  denote the standard Brownian motion on a probability space  $(\Omega, \mathcal{F}, P)$ . That is, for each  $t \in [0, 1]$ ,  $B(t)$  is a zero-mean Gaussian random variable, and for each pair  $t, s \in [0, 1]$  the covariance is  $\mathbb{E}[B(t)B(s)] = \min(t, s)$ .

In this paper we are concerned with the Lévy-Ciesielski (or Brownian bridge) construction of the Brownian paths. The Lévy-Ciesielski construction expresses the Brownian path  $B(t)$  in terms of a Faber-Schauder basis  $\{\eta_0, \eta_{n,i} : n \in \mathbb{N}, i = 1, \dots, 2^{n-1}\}$  of continuous functions on  $[0, 1]$ , where  $\eta_0(t) := t$  and

$$\eta_{n,i}(t) := \begin{cases} 2^{(n-1)/2} \left( t - \frac{2i-2}{2^n} \right), & t \in \left[ \frac{2i-2}{2^n}, \frac{2i-1}{2^n} \right], \\ 2^{(n-1)/2} \left( \frac{2i}{2^n} - t \right), & t \in \left[ \frac{2i-1}{2^n}, \frac{2i}{2^n} \right], \\ 0 & \text{otherwise.} \end{cases}$$

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\*School of Mathematics and Statistics, UNSW Sydney 2052, Australia ([bruce.brown@unsw.edu.au](mailto:bruce.brown@unsw.edu.au), [f.kuo@unsw.edu.au](mailto:f.kuo@unsw.edu.au), [i.sloan@unsw.edu.au](mailto:i.sloan@unsw.edu.au))

Universität Bonn, Institut für Numerische Simulation, Bonn, Germany, and Fraunhofer Institute SCAI, Schloss Birlinghoven, Sankt Augustin, Germany ([griebel@ins.uni-bonn.de](mailto:griebel@ins.uni-bonn.de))

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For a proof that this is a basis in  $\mathcal{C}[0, 1]$ , see [9, Theorem 2.1(iii)] or [10]. In this construction, the Brownian path corresponding to the sample point  $\omega \in \Omega$  is given by

$$B(t) := X_0(\omega) \eta_0(t) + \sum_{n=1}^{\infty} \sum_{i=1}^{2^{n-1}} X_{n,i}(\omega) \eta_{n,i}(t), \quad (1.1)$$

where  $X_0$  and all the  $X_{n,i}$ ,  $i = 1, \dots, 2^{n-1}$ ,  $n \in \mathbb{N}$  are independent standard normal random variables. For  $N \in \mathbb{N}$  we define the truncated Lévy-Ciesielski expansion by

$$B_N(t) := X_0(\omega) \eta_0(t) + \sum_{n=1}^N \sum_{i=1}^{2^{n-1}} X_{n,i}(\omega) \eta_{n,i}(t). \quad (1.2)$$

Then  $B_N(t)$  is for each  $\omega \in \Omega$  a piecewise-linear function of  $t$  coinciding with  $B(t)$  at special values of  $t$ : we easily see that  $B(0) = B_N(0) = 0$ ,  $B(1) = B_N(1) = X_0$ , and

$$B\left(\frac{2\ell - 1}{2^N}\right) = B_N\left(\frac{2\ell - 1}{2^N}\right), \quad \ell = 1, \dots, 2^{N-1},$$

because the terms in (1.1) with  $n > N$  vanish at these points.

The Lévy-Ciesielski construction has the important property that it converges almost surely to a continuous Brownian path, see the original works by [2, 4] or [8]. The precise statement is that, almost surely,

$$\|B - B_N\|_{\infty} := \sup_{t \in [0,1]} |B(t) - B_N(t)| \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

The convergence rate for the expected uniform error of the Lévy-Ciesielski expansion was obtained in [6, Theorem 2]: in the language of this paper, we have

$$\mathbb{E} [\|B - B_N\|_{\infty}] \sim \sqrt{\frac{\ln d}{2d}}, \quad (1.3)$$

where  $d := 1 + \sum_{n=1}^N 2^{n-1} = 2^N$  is the dimension of the Faber-Schauder basis to level  $N$ .

The meaning of the expected value  $\mathbb{E}$  will be made precise in Section 2. The asymptotic notation  $\alpha(x) \sim \beta(x)$  means that  $\lim_{x \rightarrow \infty} |\alpha(x)/\beta(x)| \rightarrow 1$ . Thus (1.3) gives the precise leading term for the expected uniform error of the Lévy-Ciesielski expansion. Actually, the article [6] shows also that the Lévy-Ciesielski approximation is optimal among all constructions that use information at  $d$  points and Wiener measure. The article [5] contains results for more general problems.

The main result of this paper is Theorem 1.1 below which gives upper bounds of the same order as (1.3), with a slightly worse constant which is larger by a factor of  $2 + \sqrt{2} \approx 3.41421$ . We prove this constructively in Section 3 using a different line of argument to [6], namely extreme value statistics.

**Theorem 1.1** *Let  $B$  be the Lévy-Ciesielski expansion of the standard Brownian motion (1.1), and let  $B_N$  be the corresponding truncated expansion (1.2). Then, with  $d = 2^N$ ,*

$$\begin{aligned} \mathbb{E}[\|B - B_N\|_{\infty}] &\leq (2 + \sqrt{2}) \sqrt{\frac{\ln d}{2d}} \left(1 + \mathcal{O}\left(\frac{1}{\sqrt{\ln d}}\right)\right), \\ \sqrt{\mathbb{E}[\|B - B_N\|_{\infty}^2]} &\leq (2 + \sqrt{2}) \sqrt{\frac{\ln d}{2d}} \left(1 + \mathcal{O}\left(\frac{1}{\sqrt{\ln d}}\right)\right). \end{aligned}$$

Geometric Brownian motion is the solution  $S(t) = S(\omega)(t)$  at time  $t$  of the stochastic differential equation

$$dS(t) = S(t) (r dt + \sigma dB(t)), \quad t \in [0, 1] \quad (1.4)$$

for given initial data  $S(0)$ , where  $r > 0$  is the drift,  $\sigma > 0$  is the volatility, and  $B(t)$  is the standard Brownian motion. The solution to (1.4) is given explicitly by

$$S(t) = S(0) \exp \left( \left( r - \frac{\sigma^2}{2} \right) t + \sigma B(t) \right). \quad (1.5)$$

Let  $S_N$  be the approximation defined by

$$S_N(t) := S(0) \exp \left( \left( r - \frac{\sigma^2}{2} \right) t + \sigma B_N(t) \right), \quad (1.6)$$

where  $B_N$  is the truncated Lévy-Ciesielski approximation of  $B$  given by (1.2). Then we prove in Section 4 the following corollary to Theorem 1.1.

**Corollary 1.2** *Let  $S$  be the geometric Brownian motion (1.5), and let  $S_N$  be the truncated approximation (1.6). Then, with  $d = 2^N$ ,*

$$\mathbb{E} [\|S - S_N\|_\infty] = \mathcal{O} \left( \sqrt{\frac{\ln d}{d}} \right),$$

where the implied constant depends only on  $r$  and  $\sigma$ .

Section 5 gives an application to the problem of pricing an arithmetic Asian option.

## 2 The expected value as an integral over a sequence space

In this section we show that the expected value in Theorem 1.1 can be expressed as an integral over a sequence space. We remark that we will sometimes find it convenient to use interchangeably the language of measure and integration or alternatively that of probability and expectation.

Recall that the Lévy-Ciesielski expansion (1.1) expresses the Brownian path  $B(t)$  in terms of an infinite sequence  $\mathbf{X}(\omega) = (X_0, (X_{n,i})_{n \in \mathbb{N}, i=1, \dots, 2^{n-1}})$  of independent standard normal random variables. In the following we will denote a particular realization of this sequence  $\mathbf{X}$  by

$$\mathbf{x} = (x_0, (x_{n,i})_{n \in \mathbb{N}, i=1, \dots, 2^{n-1}}) = (x_0, x_1, x_2, x_3, \dots) \in \mathbb{R}^\infty,$$

where we will switch freely between the double-index labeling  $(x_0, x_{1,1}, x_{2,1}, x_{2,2}, \dots)$  and a single-index labeling  $(x_0, x_1, x_2, x_3, \dots)$  as appropriate, with the indexing convention that  $x_{n,i}$  becomes  $x_{2^{n-1}+i}$  for  $n \geq 1$  and  $1 \leq i \leq 2^{n-1}$ .

It is clear from (1.1) that, for  $t \in [0, 1]$  and a fixed  $\omega \in \Omega$ ,

$$\begin{aligned} |B_N(t)| &\leq |X_0| + \sum_{n=1}^N \left( \max_{1 \leq i \leq 2^{n-1}} |X_{n,i}| \right) \sum_{i=1}^{2^{n-1}} \eta_{n,i}(t) \\ &\leq |X_0| + \sum_{n=1}^N \max_{1 \leq i \leq 2^{n-1}} |X_{n,i}| 2^{-(n+1)/2}, \end{aligned} \quad (2.1)$$

where in the last step we used the fact that for a given  $n \geq 1$  the disjoint nature of the Faber-Schauder functions ensures that at most one value of  $i$  contributes to the sum over  $i$ , and also that the  $\eta_{n,i}$  for  $i = 1, \dots, 2^{n-1}$  have the same maximum value  $2^{-(n+1)/2}$ .

Motivated by the bound (2.1), and following [3], we define a norm of the sequence  $\mathbf{x} = (x_0, (x_{n,i})_{n \in \mathbb{N}, i=1, \dots, 2^{n-1}})$  by

$$\|\mathbf{x}\|_{\mathcal{X}} := |x_0| + \sum_{n=1}^{\infty} \max_{1 \leq i \leq 2^{n-1}} |x_{n,i}| 2^{-(n+1)/2},$$

and we define a corresponding normed space by  $\mathcal{X} := \{\mathbf{x} \in \mathbb{R}^{\infty} : \|\mathbf{x}\|_{\mathcal{X}} < \infty\}$ . It is easily seen that  $\mathcal{X}$  is a Banach space.

Each choice of  $\mathbf{x} \in \mathcal{X}$  corresponds to a particular  $\omega \in \Omega$  (but not *vice versa*, since there are sample points  $\omega \in \Omega$  corresponding to sequences  $\mathbf{x}$  for which the norm  $\|\mathbf{x}\|_{\mathcal{X}}$  is not finite). Hence to each  $\mathbf{x} \in \mathcal{X}$  there corresponds a particular Brownian path via (1.1), or expressed in terms of  $\mathbf{x}$ ,

$$B(\mathbf{x})(t) = x_0 \eta_0(t) + \sum_{n=1}^{\infty} \sum_{i=1}^{2^{n-1}} x_{n,i} \eta_{n,i}(t), \quad t \in [0, 1]. \quad (2.2)$$

That the resulting path is continuous on  $[0, 1]$  follows from the observation that the path is the pointwise limit of the truncated series

$$B_N(\mathbf{x})(t) = x_0 \eta_0(t) + \sum_{n=1}^N \sum_{i=1}^{2^{n-1}} x_{n,i} \eta_{n,i}(t), \quad t \in [0, 1], \quad (2.3)$$

which is uniformly convergent since

$$\|B_N\|_{\infty} \leq |x_0| + \sum_{n=1}^{\infty} \max_{1 \leq i \leq 2^{n-1}} |x_{n,i}| 2^{-(n+1)/2} = \|\mathbf{x}\|_{\mathcal{X}} < \infty \quad \text{for } \mathbf{x} \in \mathcal{X},$$

so that (2.2) does indeed define a continuous function for  $\mathbf{x} \in \mathcal{X}$ .

We define  $\mathcal{A}_{\mathbb{R}^{\infty}}$  to be the  $\sigma$ -algebra generated by products of Borel sets of  $\mathbb{R}$ , see [1, p. 372]. On the Banach space  $\mathcal{X}$ , we now define a product Gaussian measure  $\rho(d\mathbf{x}) := \otimes_{j=0}^{\infty} \phi(x_j) dx_j$ , see [1, p. 392 and Example 2.35], where  $\phi$  is the standard normal probability density  $\phi(x) := \exp(-x^2/2)/\sqrt{2\pi}$ .

We next show that the space  $\mathcal{X}$  has full Gaussian measure, i.e. that

$$\mathbb{P} \left( |X_0| + \sum_{n=1}^{\infty} \max_{1 \leq i \leq 2^{n-1}} |X_{n,i}| 2^{-(n+1)/2} < \infty \right) = 1.$$

This fact is the basis of the classical proof that the Lévy-Ciesielski construction almost surely converges uniformly to the Brownian path. For a brief explanation, we define

$$H_n(\omega) := \begin{cases} |X_0(\omega)| & \text{for } n = 0, \\ \max_{1 \leq i \leq 2^{n-1}} |X_{n,i}(\omega)| 2^{-(n+1)/2} & \text{for } n \geq 1. \end{cases}$$

As a consequence of the Borel-Cantelli lemma, one can construct a sequence  $(\beta_n)_{n \geq 1}$  of positive numbers such that

$$\sum_{n=1}^{\infty} \beta_n < \infty, \quad \text{and} \quad \mathbb{P}(H_n(\cdot) > \beta_n \text{ infinitely often}) = 0.$$

We now define  $\tilde{\Omega}$  to be the subset of  $\Omega$  consisting of the sample points  $\omega$  for which  $H_n(\omega) > \beta_n$  for only finitely many values of  $n$ . Then  $\tilde{\Omega}$  is of full Gaussian measure, and for each  $\omega \in \tilde{\Omega}$  there exists  $N(\omega) \in \mathbb{N}$  such that  $H_n(\omega) \leq \beta_n$  for  $n > N(\omega)$ , giving

$$\sum_{n=1}^{\infty} H_n(\omega) \leq \sum_{n=1}^{N(\omega)} H_n(\omega) + \sum_{n=N(\omega)+1}^{\infty} \beta_n < \infty \quad \text{for } \omega \in \tilde{\Omega}.$$

Thus  $\mathbb{P}(\sum_{n=0}^{\infty} H_n < \infty) = 1$  as claimed, and  $\mathcal{X}$  is of full Gaussian measure.

We now study integration on the measure space  $(\mathcal{X}, \mathcal{A}_{\mathbb{R}^\infty}, \rho)$ , and we denote the integral, or the expected value, of a measurable function  $f$  by  $\mathbb{E}[f] := \int_{\mathcal{X}} f(\mathbf{x}) \rho(d\mathbf{x})$ .

### 3 Expected uniform error of standard Brownian motion

We devote this section to proving Theorem 1.1. We have from (1.1) and (1.2)

$$|B(t) - B_N(t)| = \left| \sum_{n=N+1}^{\infty} \sum_{i=1}^{2^{n-1}} X_{n,i} \eta_{n,i}(t) \right| \leq \sum_{n=N+1}^{\infty} \left( \max_{1 \leq i \leq 2^{n-1}} |X_{n,i}| \right) \sum_{i=1}^{2^{n-1}} \eta_{n,i}(t).$$

Using the same disjoint support argument as in (2.1), we conclude that

$$\|B - B_N\|_{\infty} \leq \sum_{n=N+1}^{\infty} \max_{1 \leq i \leq 2^{n-1}} |X_{n,i}| 2^{-(n+1)/2} = \sum_{\ell=2^N, 2^{N+1}, 2^{N+2}, \dots} \frac{M_{\ell}}{2\sqrt{\ell}},$$

where we introduced new random variables

$$M_{\ell} := M_{2^{n-1}} := \max_{1 \leq i \leq 2^{n-1}} |X_{n,i}| \quad \text{for } \ell = 2^{n-1} \text{ and } n \geq 1. \quad (3.1)$$

Thus

$$\mathbb{E}[\|B - B_N\|_{\infty}] \leq \sum_{\ell=2^N, 2^{N+1}, 2^{N+2}, \dots} \frac{\mathbb{E}[M_{\ell}]}{2\sqrt{\ell}}, \quad (3.2)$$

and since  $M_\ell$  and  $M_{\ell'}$  are independent random variables for  $\ell \neq \ell'$ ,

$$\begin{aligned} \mathbb{E} [\|B - B_N\|_\infty^2] &\leq \sum_{\ell, \ell' = 2^N, 2^{N+1}, 2^{N+2}, \dots} \frac{\mathbb{E} [M_\ell M_{\ell'}]}{2\sqrt{\ell} \cdot 2\sqrt{\ell'}} \\ &= \sum_{\ell = 2^N, 2^{N+1}, 2^{N+2}, \dots} \frac{\mathbb{E} [M_\ell^2]}{2\sqrt{\ell} \cdot 2\sqrt{\ell}} + \sum_{\substack{\ell, \ell' = 2^N, 2^{N+1}, 2^{N+2}, \dots \\ \ell \neq \ell'}} \frac{\mathbb{E} [M_\ell] \mathbb{E} [M_{\ell'}]}{2\sqrt{\ell} \cdot 2\sqrt{\ell'}}. \end{aligned} \quad (3.3)$$

Now we are in the territory of extreme value statistics. It is known that the distribution function of the maximum of the absolute value of  $\ell$  independent and identical Gaussian random variables converges (after appropriate centering and scaling, as below) to the Gumbel distribution. A first step is to obtain an explicit expression for the distribution function of  $M_\ell$ . Because  $X_{n,1}, X_{n,2}, \dots, X_{n,\ell}$  are  $\mathcal{N}(0, 1)$  random variables, for  $x \in \mathbb{R}^+$  and  $i = 1, \dots, \ell$ , we have

$$\mathbb{P}(X_{n,i} \leq x) = \int_{-\infty}^x \phi(t) dt =: \Phi(x),$$

where  $\phi$  is the standard normal density. Similarly,

$$\mathbb{P}(|X_{n,i}| \leq x) = \int_{-x}^x \phi(t) dt = \Phi(x) - \Phi(-x) = 2\Phi(x) - 1.$$

Therefore (since  $X_{n,1}, X_{n,2}, \dots, X_{n,\ell}$  are independent random variables) we have

$$\mathbb{P}(M_\ell \leq x) = \mathbb{P}(|X_{n,1}| \leq x \text{ and } |X_{n,2}| \leq x \text{ and } \dots \text{ and } |X_{n,\ell}| \leq x) = (2\Phi(x) - 1)^\ell.$$

Thus the distribution function of  $M_\ell$  is

$$\Psi_\ell(x) := (2\Phi(x) - 1)^\ell, \quad x \in \mathbb{R}^+. \quad (3.4)$$

We now define a new random variable  $Y_\ell$  for  $\ell \geq 1$ , which is a recentered and rescaled version of  $M_\ell$ :

$$Y_\ell := \frac{M_\ell - a_\ell}{b_\ell}, \quad \text{or equivalently, } M_\ell = a_\ell + b_\ell Y_\ell, \quad a_\ell > 0, b_\ell > 0. \quad (3.5)$$

It is known (see below) to be appropriate to take  $a_\ell$  and  $b_\ell$  to satisfy

$$a_\ell = \sqrt{2 \ln \ell} + o(1), \quad \text{and } b_\ell = \frac{1}{a_\ell}. \quad (3.6)$$

More precisely, for later convenience we will define  $a_\ell$  to be the unique solution of

$$\frac{1}{\ell} = \sqrt{\frac{2}{\pi}} \frac{e^{-a_\ell^2/2}}{a_\ell} =: g(a_\ell). \quad (3.7)$$

We now show that (3.7) implies (3.6).



**Lemma 3.1** Equation (3.7) for  $\ell \geq 1$  has a unique positive solution of the form  $a_\ell = \sqrt{2 \ln \ell} + o(1)$ . Moreover, for  $\ell \geq 3$  we have  $a_\ell \in (1, \sqrt{2 \ln \ell})$ .

**Proof.** The fact that any solution of (3.7) is positive is immediate. Now observe that  $g$  in (3.7) is monotonically decreasing from  $+\infty$  to 0 on  $\mathbb{R}^+$ . It follows immediately that there is a unique solution  $a_\ell \in (0, \infty)$  for (3.7). Moreover, we have

$$a_\ell > 1 \quad \Leftrightarrow \quad \frac{1}{\ell} < \sqrt{\frac{2}{\pi}} \frac{e^{-1^2/2}}{1} = \sqrt{\frac{2}{\pi e}} = 0.484\dots,$$

which holds if and only if  $\ell \geq 3$ . Now observe that (3.7) is equivalent to

$$a_\ell = \sqrt{2 \left( \ln \ell - \ln \left( \sqrt{\frac{\pi}{2}} a_\ell \right) \right)}. \quad (3.8)$$

For  $\ell \geq 3$  we have  $a_\ell > 1$  and hence  $\ln(\sqrt{\pi/2} a_\ell) > \ln(\sqrt{\pi/2}) > 0$ , so from (3.8) we have  $a_\ell < \sqrt{2 \ln \ell}$ . In turn it follows that

$$a_\ell > \sqrt{2 \ln \ell - 2 \ln \left( \sqrt{\frac{\pi}{2}} \sqrt{2 \ln \ell} \right)}.$$

Thus for  $\ell \geq 3$  we have  $1 \leq a_\ell = \sqrt{2 \ln \ell} + o(1) \leq \sqrt{2 \ln \ell}$ , completing the proof.  $\square$

It is well known that the distribution function of  $Y_\ell$  converges in distribution to a random variable with the Gumbel distribution  $\exp(-e^{-y})$ . For later convenience we state this as a lemma and give a short proof.

**Lemma 3.2** The random variable  $Y_\ell$  defined in (3.5), with  $a_\ell$  defined by (3.7) and  $b_\ell = 1/a_\ell$ , converges in distribution to a random variable  $Y$  with Gumbel distribution function  $\mathbb{P}(Y \leq y) = \exp(-e^{-y})$ .

**Proof.** The proof is based on the asymptotic version of Mill's ratio [7],

$$1 - \Phi(x) \sim \frac{\phi(x)}{x}, \quad x \rightarrow +\infty,$$

where, as in the Introduction,  $\sim$  means that the quotient of the two sides converges to 1. From this it follows that for  $y \in \mathbb{R}$

$$\begin{aligned} \mathbb{P}(Y_\ell \leq y) &= \mathbb{P}(M_\ell \leq a_\ell + b_\ell y) = (2\Phi(a_\ell + b_\ell y) - 1)^\ell = (1 - 2[1 - \Phi(a_\ell + b_\ell y)])^\ell \\ &\sim \left( 1 - \sqrt{\frac{2}{\pi}} \frac{\exp(-\frac{1}{2}(a_\ell + b_\ell y)^2)}{a_\ell + b_\ell y} \right)^\ell \sim \left( 1 - \sqrt{\frac{2}{\pi}} \frac{\exp(-\frac{1}{2}a_\ell^2 - a_\ell b_\ell y)}{a_\ell} \right)^\ell \\ &= (1 - \exp(-y)/\ell)^\ell \sim \exp(-e^{-y}) \quad \text{as } \ell \rightarrow \infty, \end{aligned}$$

where in the second step we dropped a higher order term, and in the second last step we used (3.7) and  $a_\ell b_\ell = 1$ , thus proving the lemma.  $\square$

A deeper result, which we need, is that  $Y_\ell$  converges in expectation to the limit  $Y$ . This is proved in the following lemma.

**Lemma 3.3** *The random variable  $Y_\ell$  defined in (3.5), with  $a_\ell$  defined by (3.7) and  $b_\ell = 1/a_\ell$ , converges in expectation to a random variable  $Y$  with Gumbel distribution  $\exp(-e^{-y})$ , thus*

$$\lim_{\ell \rightarrow \infty} \mathbb{E}[Y_\ell] = \mathbb{E}[Y] = \int_{-\infty}^{\infty} y \exp(-y - e^{-y}) dy = \gamma,$$

where  $\gamma$  is Euler's constant.

**Proof.** For a sequence of real-valued random variables  $Y_1, Y_2, \dots$  converging in distribution to a random variable  $Y$ , it is well known that a sufficient condition for convergence in expectation is uniform integrability of the  $Y_\ell$ . In turn a sufficient condition for uniform integrability is that for sufficiently large  $\ell$

$$\mathbb{P}(Y_\ell \geq y) \leq Q(y) \text{ for } y > 0, \quad \text{and} \quad \mathbb{P}(Y_\ell \leq y) \leq R(y) \text{ for } y < 0,$$

where  $Q(y)$  is integrable on  $\mathbb{R}^+$  and  $R(y)$  is integrable on  $\mathbb{R}^-$ .

First assume  $y > 0$ . We have from (3.4) that

$$\begin{aligned} \mathbb{P}(Y_\ell \geq y) &= \mathbb{P}(M_\ell \geq a_\ell + b_\ell y) = 1 - \mathbb{P}(M_\ell \leq a_\ell + b_\ell y) \\ &= 1 - (2\Phi(a_\ell + b_\ell y) - 1)^\ell = 1 - (1 - 2[1 - \Phi(a_\ell + b_\ell y)])^\ell \\ &\leq 1 - \left(1 - 2 \frac{\phi(a_\ell + b_\ell y)}{a_\ell + b_\ell y}\right)^\ell = 1 - \left(1 - \sqrt{\frac{2}{\pi}} \frac{\exp(-\frac{1}{2}(a_\ell + b_\ell y)^2)}{a_\ell + b_\ell y}\right)^\ell \\ &\leq 1 - \left(1 - \sqrt{\frac{2}{\pi}} \frac{\exp(-\frac{1}{2}a_\ell^2 - a_\ell b_\ell y)}{a_\ell}\right)^\ell, \end{aligned}$$

where we used the upper bound form of Mills' ratio [7],

$$1 - \Phi(x) < \frac{\phi(x)}{x}, \quad x \in \mathbb{R}^+, \quad (3.9)$$

and dropped harmless terms in both the denominator and the exponent in the numerator. Using now (3.7) and also  $a_\ell b_\ell = 1$  we have

$$\mathbb{P}(Y_\ell \geq y) \leq 1 - \left(1 - \frac{\exp(-y)}{\ell}\right)^\ell \leq \exp(-y) =: Q(y), \quad (3.10)$$

where we used the fact that the function  $(1 - c/x)^x$  is increasing on  $[1, \infty)$  for  $c \in [0, 1]$ , and hence takes its minimum at  $x = 1$ . It follows that

$$\int_0^\infty \mathbb{P}(Y_\ell \geq y) dy \leq \int_0^\infty \exp(-y) dy = 1.$$

Now we consider  $y < 0$ . Note first that  $M_\ell = a_\ell + b_\ell Y_\ell$  takes only non-negative values, thus we may restrict  $y$  to  $y \geq -a_\ell/b_\ell$ . We have

$$\mathbb{P}(Y_\ell \leq y) = \mathbb{P}(M_\ell \leq a_\ell + b_\ell y) = (2\Phi(a_\ell + b_\ell y) - 1)^\ell.$$

Now for  $t > 0$  the standard normal distribution  $\Phi$  has negative second derivative,

$$\Phi''(t) = \phi'(t) < 0 \quad \text{for } t > 0,$$

and first derivative  $\Phi'(t) = \phi(t)$ , from which it follows that

$$\Phi(a_\ell + b_\ell y) \leq \Phi(a_\ell) + b_\ell y \phi(a_\ell) \quad \text{for } y \geq -a_\ell/b_\ell.$$

Thus on using  $b_\ell = 1/a_\ell$ , we obtain

$$\begin{aligned} \mathbb{P}(Y_\ell \leq y) &\leq \left( 2\Phi(a_\ell) + 2a_\ell^{-1}y\phi(a_\ell) - 1 \right)^\ell \\ &\leq \left( 1 - 2a_\ell^{-1}\phi(a_\ell)(1 - y - a_\ell^{-2}) \right)^\ell = \left( 1 - \frac{1}{\ell}(1 - y - a_\ell^{-2}) \right)^\ell, \end{aligned}$$

where in the second last step we used the lower bound form of Mills' ratio, see [7, p. 44]

$$1 - \Phi(t) \geq \frac{\phi(t)}{t}(1 - t^{-2}) \quad \text{for } t > 0, \quad (3.11)$$

and in the last step we used (3.7). If we now take  $\ell \geq L$  then we have

$$\begin{aligned} \mathbb{P}(Y_\ell \leq y) &\leq \left( 1 - \frac{1}{\ell}(1 - y - a_L^{-2}) \right)^\ell \\ &\leq \exp(-(1 - y - a_L^{-2})) = \exp(-(1 - a_L^{-2})) \exp(y) =: R(y), \end{aligned} \quad (3.12)$$

since the convergence in the last limit is monotone increasing. The function  $R(y)$  so defined is integrable on  $\mathbb{R}^-$ , completing the proof that  $Y_\ell$  converges in expectation.

It then follows from Lemma 3.2 that the limit of  $\mathbb{E}[Y_\ell]$  is precisely  $\mathbb{E}[Y] = \gamma$ .  $\square$

Since Lemma 3.3 establishes the convergence of  $\mathbb{E}[Y_\ell]$  as  $\ell \rightarrow \infty$ , it can be inferred that there exists a positive constant  $c$  such that

$$\mathbb{E}[Y_\ell] \leq c \quad \text{and hence} \quad \mathbb{E}[M_\ell] \leq a_\ell + b_\ell c \leq a_\ell + c, \quad (3.13)$$

where we used  $b_\ell = a_\ell^{-1} \leq 1$  for  $\ell \geq 3$ . We then conclude from (3.2) that

$$\mathbb{E}[\|B - B_N\|_\infty] \leq \sum_{\ell=2^N, 2^{N+1}, 2^{N+2}, \dots} \frac{a_\ell + c}{2\sqrt{\ell}}. \quad (3.14)$$

It only remains to estimate the sum in (3.14). Using Lemma 3.1 with  $N \geq 2$  (and hence  $\ell \geq 3$ ), we have  $a_\ell < \sqrt{2 \ln \ell}$ , and on setting  $\ell = 2^{N+j}$ ,

$$\begin{aligned} \sum_{\ell=2^N, 2^{N+1}, 2^{N+2}, \dots} \frac{a_\ell}{\sqrt{\ell}} &\leq \sum_{j=0}^{\infty} \frac{\sqrt{2(\ln 2)(N+j)}}{\sqrt{2^{N+j}}} = 2^{-(N-1)/2} \sqrt{\ln 2} \sum_{j=0}^{\infty} \frac{\sqrt{N+j}}{2^{j/2}} \\ &\leq 2^{-(N-1)/2} \sqrt{\ln 2} \sqrt{N} (2 + \sqrt{2}) (1 + \mathcal{O}(N^{-1/2})), \end{aligned}$$

where in the final step we used  $\sqrt{N+j} \leq \sqrt{N} + \sqrt{j}$  and  $\sum_{j=0}^{\infty} 1/2^{j/2} = 2 + \sqrt{2}$ , while noting that  $\sum_{j=0}^{\infty} \sqrt{j}/2^{j/2}$  is finite and independent of  $N$ . Moreover, by a similar argument we conclude that  $\sum_{\ell=2^N, 2^{N+1}, 2^{N+2}, \dots} c/\sqrt{\ell} = \mathcal{O}(2^{-N/2})$ , thus altogether we obtain from (3.14)

$$\begin{aligned} \mathbb{E} [\|B - B_N\|_{\infty}] &\leq \frac{1}{2} \cdot 2^{-(N-1)/2} \sqrt{\ln 2} \sqrt{N} (2 + \sqrt{2}) (1 + \mathcal{O}(N^{-1/2})) \\ &= (2 + \sqrt{2}) \frac{\sqrt{N \ln 2}}{\sqrt{2} \cdot 2^N} (1 + \mathcal{O}(N^{-1/2})) = (2 + \sqrt{2}) \frac{\sqrt{\ln d}}{\sqrt{2d}} \left(1 + \mathcal{O}\left(\frac{1}{\sqrt{\ln d}}\right)\right), \end{aligned}$$

which proves the first bound in Theorem 1.1.

To prove the second bound in Theorem 1.1, we need first to bound  $\mathbb{E} [M_{\ell}^2]$ . With  $Y_{\ell}$  and  $Y$  defined as above, for  $v > 0$  we have

$$\begin{aligned} \mathbb{P}(Y_{\ell}^2 \geq v) &= \mathbb{P}(Y_{\ell} \geq \sqrt{v}) + \mathbb{P}(Y_{\ell} \leq -\sqrt{v}) \\ &\rightarrow \mathbb{P}(Y \geq \sqrt{v}) + \mathbb{P}(Y \leq -\sqrt{v}) = \mathbb{P}(Y^2 \geq v) \quad \text{as } \ell \rightarrow \infty, \end{aligned} \tag{3.15}$$

while for  $v < 0$  we have  $\mathbb{P}(Y_{\ell}^2 \geq v) = \mathbb{P}(Y^2 \geq v) = 1$ . Thus by Lemma 3.2,  $Y_{\ell}^2$  converges in distribution to  $Y^2$ , where  $Y$  is the Gumbel distribution. To prove convergence in expectation, we use (3.15) with (3.10) and (3.12) to give, for  $\ell \geq L$  and  $v > 0$ ,

$$\mathbb{P}(Y_{\ell}^2 \geq v) \leq \exp(-\sqrt{v}) + \exp(-(1 - a_L^{-2})) \exp(-\sqrt{v}),$$

which is integrable on  $\mathbb{R}^+$ , proving  $\mathbb{E} [Y_{\ell}^2] \rightarrow \mathbb{E} [Y^2] < \infty$ . In turn it follows that there exists  $c' > 0$  such that  $\mathbb{E} [Y_{\ell}^2] \leq c'$ , and together with (3.13) we have

$$\mathbb{E} [M_{\ell}^2] \leq a_{\ell}^2 + 2a_{\ell} b_{\ell} c + b_{\ell}^2 c' \leq a_{\ell}^2 + 2a_{\ell} c + c' \leq (a_{\ell} + c'')^2,$$

where we used  $b_{\ell} = a_{\ell}^{-1} \leq 1$  for  $\ell \geq 3$  and introduced  $c'' := \max(c, \sqrt{c'})$ . We now have from (3.3) that

$$\begin{aligned} \mathbb{E} [\|B - B_N\|_{\infty}^2] &\leq \sum_{\ell=2^N, 2^{N+1}, 2^{N+2}, \dots} \frac{(a_{\ell} + c'')^2}{2\sqrt{\ell} \cdot 2\sqrt{\ell}} + \sum_{\substack{\ell, \ell'=2^N, 2^{N+1}, 2^{N+2}, \dots \\ \ell \neq \ell'}} \frac{(a_{\ell} + c'')(a_{\ell'} + c'')}{2\sqrt{\ell} \cdot 2\sqrt{\ell'}} \\ &= \left( \sum_{\ell=2^N, 2^{N+1}, 2^{N+2}, \dots} \frac{a_{\ell} + c''}{2\sqrt{\ell}} \right)^2, \end{aligned}$$

which is the square of the right-hand side of (3.14), with  $c$  replaced by  $c''$ . The second bound in Theorem 1.1 then follows.

## 4 Expected uniform error of geometric Brownian motion

We are now in the position to give a proof of Corollary 1.2. From (1.5) and (1.6) it follows that  $S(t) - S_N(t) = S(0) e^{(r-\sigma^2/2)t} (\exp(\sigma B(t)) - \exp(\sigma B_N(t)))$ , and thus

$$\|S - S_N\|_{\infty} \leq S(0) e^{|r-\sigma^2/2|t} \|\exp(\sigma B) - \exp(\sigma B_N)\|_{\infty}. \tag{4.1}$$

In turn it follows that

$$\begin{aligned} \|S - S_N\|_\infty &\leq S(0) e^{|r-\sigma^2/2|} \left\| \exp(\sigma B_N) (\exp(\sigma(B - B_N)) - 1) \right\|_\infty \\ &\leq S(0) e^{|r-\sigma^2/2|} \exp(\sigma \|B_N\|_\infty) (\exp(\sigma \|B - B_N\|_\infty) - 1). \end{aligned} \quad (4.2)$$

Using  $|\exp(x) - 1| \leq |x| \exp(|x|)$  for  $x \in \mathbb{R}$  and  $\|B_N\|_\infty \leq \|B\|_\infty$  for  $N \in \mathbb{N}$ , we have

$$\begin{aligned} \|S - S_N\|_\infty &\leq S(0) e^{|r-\sigma^2/2|} \exp(\sigma \|B_N\|_\infty) \sigma \|B - B_N\|_\infty \exp(\sigma \|B - B_N\|_\infty) \\ &\leq S(0) e^{|r-\sigma^2/2|} \sigma \exp(3\sigma \|B\|_\infty) \|B - B_N\|_\infty, \end{aligned}$$

where we used  $\|B - B_N\|_\infty \leq \|B\|_\infty + \|B_N\|_\infty \leq 2\|B\|_\infty$ . By the Cauchy-Schwarz inequality we obtain

$$\mathbb{E} [\|S - S_N\|_\infty] \leq S(0) e^{|r-\sigma^2/2|} \sigma \sqrt{\mathbb{E} [\exp(6\sigma \|B\|_\infty)]} \sqrt{\mathbb{E} [\|B - B_N\|_\infty^2]}.$$

It is well known that  $\mathbb{E} [\exp(\alpha \|B\|_\infty)] < \infty$  for every  $\alpha > 0$ . Hence the result now follows from the second bound of Theorem 1.1.

## 5 Application to option pricing

Now we consider a continuous version of a path-dependent call option with strike price  $K$  in a Black-Scholes model with risk-free interest rate  $r > 0$  and constant volatility  $\sigma > 0$ . Recall that the asset price  $S(t)$  at time  $t$  is given explicitly by (1.5). The discounted payoff for the case of a continuous arithmetic Asian option with terminal time  $T = 1$  is therefore

$$\begin{aligned} P &:= e^{-rT} \max \left( \frac{1}{T} \int_0^T S(t) dt - K, 0 \right) \\ &= e^{-r} \max \left( S(0) \int_0^1 \exp \left( \left( r - \frac{\sigma^2}{2} \right) t + \sigma B(t) \right) dt - K, 0 \right). \end{aligned} \quad (5.1)$$

The pricing problem is then to compute the expected value  $\mathbb{E}(P)$ .

We use the Lévy-Ciesielski expansion (2.2) and (2.3) for  $B(t)$  and  $B_N(t)$ , and define

$$P_N := e^{-r} \max \left( S(0) \int_0^1 \exp \left( \left( r - \frac{\sigma^2}{2} \right) t + \sigma B_N(t) \right) dt - K, 0 \right). \quad (5.2)$$

We are interested in estimating how fast  $\mathbb{E} [|P - P_N|]$  converges to 0 as  $N \rightarrow \infty$ .

**Corollary 5.1** *For  $P$  and  $P_N$  defined by (5.1) and (5.2), we have*

$$\mathbb{E} [|P - P_N|] = \mathcal{O} \left( \sqrt{\frac{\ln d}{d}} \right),$$

where  $d = 2^N$  and the implied constant is independent of  $d$ .

**Proof.** It can be easily verified that

$$|\max(\alpha - K, 0) - \max(\beta - K, 0)| \leq |\alpha - \beta|.$$

Thus

$$\begin{aligned} |P - P_N| &\leq \left| e^{-r} S(0) \int_0^1 e^{(r-\sigma^2/2)t} \left( \exp(\sigma B(t)) - \exp(\sigma B_N(t)) \right) dt \right| \\ &\leq e^{-r} S(0) e^{|r-\sigma^2/2|} \int_0^1 \left| \exp(\sigma B(t)) - \exp(\sigma B_N(t)) \right| dt \\ &\leq e^{-r} S(0) e^{|r-\sigma^2/2|} \|\exp(\sigma B) - \exp(\sigma B_N)\|_\infty, \end{aligned}$$

where the last upper bound differs from the upper bound (4.1) on  $\|S - S_N\|_\infty$  only by a factor of  $e^{-r}$ . Hence the result follows from Corollary 1.2.  $\square$

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