Stability of Solitary Waves for the nonlinear Schrödinger Equation

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Abstract

Consider the one-dimensional nonlinear Schrödinger equation $i\partial_t u = -\Delta_x u - |u|^{p-1}\, u$ with initial data $u(0,\cdot) = u_0 \in H^1(\mathbb{R})$ and subcritical/critical exponent $3 \leq p \leq 5$.

This thesis examines a question derived from the so-called soliton resolution conjecture. The NLS admits regular solutions of the form $u(t,x) = e^{i\lambda t} Q_\lambda(x)$, $\lambda > 0$ called solitons.

The soliton resolution conjecture claims that every global solution of the NLS will eventually resolve into a sum of soliton-like solutions and a radiation component which disperses like a linear solution.

We consider the related question of 'asymptotic stability'. For initial data close to a soliton, does the solution resolve into a soliton-like solution and radiation?

Specifically, we examine the linearisation of the NLS around the soliton. Let L denote the Hamiltonian of the resulting linear equation $\partial_t u = Lu$.

We show the following in this thesis.

1. We fully characterise the spectrum of L. Apart from several well-known eigenvalues in 0, iL admits a resonance in ± 1 for $p = 3$, a symmetrical pair of eigenvalues $\pm E \in (-1,1)\backslash\{0\}$ for $3 < p < 5$, as well as two additional generalised eigenvalues in 0 for $p = 5$.

2. Based on the above characterisation of the spectrum of L, we show the existence of a wave operator for $3 < p < 5$, mapping $\partial_t u = Lu$ onto the free Schrödinger equation. This is accomplished by constructing a distorted Fourier transform mapping L onto a multiplication operator.

3. We show that the wave operator acts as a bounded operator $L^q \to L^q$ for every $1 \leq q \leq \infty$. As a consequence, for $3 < p < 5$, the equation $\partial_t u = Lu$ allows for the same dispersive estimates as the free equation.

4. For $3 < p < 5$, we show a local smoothing estimate for $\partial_t u = Lu$. Due to the absence of resonances, this local smoothing estimate allows for significantly stronger local decay than the case of the free equation $i\partial_t u = -\Delta u$.

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Contents

Convention (On the use of constants) *Throughout this thesis, the variable C is used to denote constants without quantifying them. Dependencies are denoted as sub-indices.*

Consider the following example. Given functions $f, g : \mathbb{R}^3 \to \mathbb{R}$, the inequality:

$$\forall x, y, z \in \mathbb{R} : f(x, y, z) \leq Cg(x, y, z) \tag{0.1}$$

is to be read as:

$$\exists C > 0 \ \forall x, y, z \in \mathbb{R} : f(x, y, z) \leq Cg(x, y, z). \tag{0.2}$$

Similarly, the inequality:

$$\forall x, y, z \in \mathbb{R} : f(x, y, z) \leq C_{x,y} g(x, y, z) \tag{0.3}$$

is to be read as:

$$\forall x, y \in \mathbb{R} \ \exists C > 0 \ \forall z \in \mathbb{R} : f(x, y, z) \leq Cg(x, y, z). \tag{0.4}$$

Further, no relationship is implied between the constants of different inequalities, even though the same variable name C is used. Using C as in (0.1) *or* (0.3) *simply implies the existence of a constant in the sense of* (0.2) *or* (0.4)*.*

Consequently, given $f, g : \mathbb{R} \to \mathbb{R}$, the inequality:

$$\forall x \in \mathbb{R} : f(x) \leq Cg(x) \leq C^2 \tag{0.5}$$

implies:

$$\forall x \in \mathbb{R} : f(x) \leq C. \tag{0.6}$$

Clearly, the constant C denotes different numbers in (0.5) *and* (0.6)*.*

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The following table gives an overview over the most important quantities used throughout this thesis. The arrangement is loosely based on the order in which the quantities are introduced.

Quantity	Description
Q	Ground state, see (1.3) on page 12.
L	Linear operator, arises by linearising the NLS around a soliton, see (1.35) on page 22.
L_0	$L_0 = (-\Delta + 1)I$, see definition 10.0.2 on page 225.
ζ_1, ζ_2	Internal Modes. Eigenfunctions of L associated with non-zero eigenvalues, see definition 6.1.2 on page 161.
$\mathcal{H}$	Hilbert space for which iL is self-adjoint, see definition 1.9.1 on page 23.
H	$H = \{w \in \mathcal{H} \| \langle w, \zeta_1\rangle_{\mathcal{H}} = \langle w, \zeta_2\rangle_{\mathcal{H}} = 0\}$, see definition 6.2.1 on page 161.
H_s	Generalisation of H for $s \geq 0$ with $H_1 = H$, see chapter 9.5 on page 218.
H_e, H_o	Subspaces of H, containing only even and odd functions respectively.
$\hat{H}^s$	$\hat{H}^s = \{f \in L^2(\mathbb{R}) \| (1 + \|\cdot\|)^{\frac{s}{2}} f \in L^2(\mathbb{R})\}$. See definition 9.2.22 on page 198.
T	$T = \mathcal{G}F$. Wave operator, unitary transform $H \to H^1(\mathbb{R})^2$ mapping the linearised operator L onto $(-\Delta + 1)I$, see definition 10.2.1 on page 228.
I	$I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, matrix equivalent of the imaginary unit i.
F	Unitary transform, $H \to \hat{H}^1 \times \hat{H}^1$, mapping L onto the multiplication operator $i\,\mathrm{sgn}(\xi)(\xi^2 + 1)$, see definition 9.3.1 on page 201. See also (9.2), (9.3) on page 186.
G	Unitary transform, $\hat{H}^1 \times \hat{H}^1 \to H$, inverse of F. See lemma 9.2.27 on page 200, as well as (9.4), (9.5) on page 186.
$\mathcal{G}$	Unitary transform, $\hat{H}^1 \times \hat{H}^1 \to H^1(\mathbb{R})^2$, mapping the multiplication operator $i\,\mathrm{sgn}(\xi)(\xi^2 + 1)$ onto $(-\Delta + 1)I$, see definition 10.1.1 on page 226.
$W_e(\xi, x)$	Bounded even solution to $Lw = i\,\mathrm{sgn}(\xi)(\xi^2 + 1)w$ wrt. to x. Used as a kernel when constructing F and G. See definition 8.5.1 on page 183.
$W_o(\xi, x)$	Bounded odd solution to $Lw = i\,\mathrm{sgn}(\xi)(\xi^2 + 1)w$ wrt. to x. Used as a kernel when constructing F and G. See definition 8.5.1 on page 183.
U_e, V_e	Component functions of $W_e = (U_e, V_e)$.
U_o, V_o	Component functions of $W_o = (U_o, V_o)$.
c_e, s_e	Coefficient functions used when constructing W_e, see theorem 8.3.5 and definition 8.3.6 on pages 176 and 178.
c_o, s_o	Coefficient functions used when constructing W_o, compare c_e, s_e.
R_e, R_o	Remainder terms, they arise by comparing the asymptotics of W_e, W_o with $c_e(\xi)\cos(\xi x)$ and $s_e(\xi)\sin(\xi x)$, see definition 8.5.6 on page 184. R_e, R_o feature discontinuities in $\xi = 0$ and $x = 0$ in contrast to W_e, W_o.

Quantity	Description
$R_{e,U}, R_{e,V}$	Component functions of $R_e = (R_{e,U}, R_{e,V})$.
$R_{o,U}, R_{o,V}$	Component functions of $R_o = (R_{o,U}, R_{o,V})$.
$\tilde{R}_e, \tilde{R}_o$	Remainder terms of higher order, they decay faster than R_e, R_o with respect to ξ. See definition 8.5.12 on page 185.
ρ_e, ρ_o	Remainder terms, replacing R_e, R_o from chapter 11 onwards. They use a cut-off function to ensure smoothness in $x = 0$ and $\xi = 0$. See definition 11.1.1 on page 229.
$\rho_{e,U}, \rho_{e,V}$	Component functions of $\rho_e = (\rho_{e,U}, \rho_{e,V})$.
$\rho_{o,U}, \rho_{o,V}$	Component functions of $\rho_o = (\rho_{o,U}, \rho_{o,V})$.
$\tilde{\rho}_e, \tilde{\rho}_o$	Remainder terms of higher order. See definition 11.1.6 on page 230.
χ	Cut-off function. Smooth even function $\mathbb{R} \to [0,1]$ fulfilling $\chi(x) = 1$ for $\|\xi\| \geq 2$ and $\chi(x) = 0$ for $\|\xi\| \leq 1$.
$A_{e,k}, A_{o,k}$	$1 \leq k \leq 6$. Operators arising from decomposing the wave operator. Used to show $L^q \to L^q$ estimates for the wave operator in chapter 11. See definition 11.3.1 and 11.3.3 on page 234/235.
$B_{e,k}, B_{o,k}$	$1 \leq k \leq 6$. Operators arising from decomposing the inverse of the wave operator. See definition 11.3.1 and 11.3.3 on page 234/235.
J_t	$J_t = \frac{x}{2} + It\partial_x$, $t \in \mathbb{R}$. Operator related to the Galilean invariance.
$\tilde{\mathcal{F}}$	Matrix Fourier transform. See definition 12.1.1 on page 246.

Part I. Introduction

1. Setting

Consider the nonlinear Schrödinger equation (NLS) in one space dimension

$$i\partial_t u = -\Delta_x u - |u|^{p-1} u, \tag{1.1}$$

for $u : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$, $3 \leq p \leq 5$ and initial data $u(0, \cdot) = u_0$. We mostly consider the cases $u_0 \in H^1(\mathbb{R})$ and $u_0 \in L^2(\mathbb{R})$.

This Cauchy problem is a special case of the d-dimensional nonlinear Schrödinger equation

$$i\partial_t u = -\Delta u + \mu |u|^{p-1} u, \tag{1.2}$$

for $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$, $1 \leq p \leq 1 + \frac{4}{d}$ and $\mu = \pm 1$. The case of $\mu = -1$, which we consider, is called the focusing equation. In contrast, for $\mu = 1$, (1.2) is known as the defocusing equation.

By Bourgain [4], (1.2), given $\mu = -1$ and assuming sufficient regularity s, is locally well-posed in H^s for every $p \geq 1$. In particular, $1 \leq p \leq 1 + \frac{4}{d}$ and $s \geq 0$ ensures local well-posedness in L^2 (see in addition Cazenave and Weissler [6]), while $s \geq \frac{d}{2}$ ensures local well-posedness in H^s for every $p \geq 1$.

By Ginibre and Velo [17], (1.2) is globally well-posed in H^1 for $1 \leq p < 1 + \frac{4}{d}$. For $p = 1 + \frac{4}{d}$ and $\mu = -1$, blow-up solutions are explicitly known, see e.g. [24].

1.1. Solitary Waves

Consider a ground state $Q_\lambda : \mathbb{R}^d \to \mathbb{R}$, $\lambda > 0$ fulfilling

$$\Delta Q_\lambda + |Q_\lambda|^{p-1} Q_\lambda = \lambda Q_\lambda \tag{1.3}$$

and decaying $Q_\lambda(x) \to 0$ for $|x| \to \infty$. Then, $u(t, x) = e^{i\lambda t} Q_\lambda(x)$ fulfils the nonlinear Schrödinger equation (1.2). Such a solution is called a solitary wave or a soliton. Solitary waves are a feature of many nonlinear dispersive equations, notably the Korteweg-de Vries equation, the Sine-Gordon equation and the nonlinear wave equation.

By Strauss [29] and Berestycki and Lions [2], there exist non-trivial ground states solving (1.3) for every dimension $d \geq 1$. By Kwong [22], building on Gidas, Ni and Nirenberg [16], the solution Q_λ of (1.3) is unique up to translation and phase invariance, meaning multiplication with $e^{i\mu}, \mu \in \mathbb{R}$. Further, there is a base solution which is positive and radially symmetric.

In the one-dimensional case $d = 1$, the ground state can be given explicitly as $Q_\lambda(x) = \lambda^{\frac{1}{p-1}} Q(\lambda^{\frac{1}{2}} x)$, whereby

$$Q(x)^{p-1} = \frac{p+1}{2\cosh^2(\frac{p-1}{2}x)} = \frac{2(p+1)e^{(p-1)x}}{(e^{(p-1)x}+1)^2}. \tag{1.4}$$

The explicit formula (1.4) makes the one-dimensional easier to study, and is the reason we only consider $d = 1$ in this thesis.

1.2. Why are Solitary Waves interesting?

The existence of soliton solutions, with solitons being particle-like solutions that persist for all time, can in some ways be seen as surprising. Prior to the 1950's, it was generally expected for nonlinear interactions to lead to "thermalisation". The idea being that any solution would approach a thermal equilibrium through the non-linear interaction acting chaotically on the solution.

This idea would naturally lead to the conclusion that, given enough time, any solution of the NLS would ultimately amount to radiation, i.e. behave like a solution of the free equation $i\partial_t = -\Delta u$. It is to be noted that, given small enough initial data with respect to $||xu_0||_{L^2(\mathbb{R})^2}, ||u_0||_{L^2(\mathbb{R})^2}$, it is well-known that thermalisation indeed occurs. See, e.g. [19] or, alternatively, lemma 13.0.3 in this thesis.

There is a famous experiment by Enrico Fermi, John Pasta, Stanislaw Ulam and Mary Tsingou in 1953-54 on one of the early computers, the Los Alamos MANIAC computer. They used the MANIAC computer to solve a discrete system of nearest-neighbor coupled oscillators, hoping to gain inside into the process of thermalisation. Instead, the system exhibited a complicated quasi-periodic behaviour.

This perplexing behaviour was explained by Zabusky and Kruskal [36] in 1965. They showed the continuum limit of the above system of coupled ODEs to be the Korteweg-de Vries (KdV) equation. The quasi-periodic behaviour being explained by the existence of solition solutions, as well as the surprising fact that solitons apparently can pass through one another without affecting each others asymptotic shapes.

This discovery naturally lead to a great amount of effort being spend on studying soliton mechanics, in hopes of understanding the long term behaviour of non-linear dispersive equations. Perhaps the most powerful tool was developed by Gardner, Greene, Kruskal and Miura [15], that being the inverse scattering transform, an ingenious method for solving the KdV equation.

The inverse scattering transform is to technically complex to be described in any detail here. We give a very basic run-down. Given a solution $u(t, x)$ of the KdV equation, one can fin a Lax pair, two linear operators $L = -\partial_x^2 + u$ and B, such that $L_t = [B, L]$.

Then, if $\psi(t)$ fulfils the time evolution $\partial_t \psi = B\psi$, and $\psi(0)$ is an eigenfunction of L, $L\psi(0) = \lambda\psi(0)$, it follows that $\psi(t)$ is an eigenfunction of L with the same eigenvalue λ.

One now fixes a potential $u \in \mathcal{S}(\mathbb{R})$ and characterises the so-called scattering data, meaning the time evolution of the eigenfunctions associated with each eigenvalue λ, as well as the transmission and reflection coefficient. The two coefficients are given as follows. If $w(x)$ solves $L - \kappa^2$, $\kappa > 0$ and behaves asymptotically as $e^{i\kappa x}$ for $x \to -\infty$, then we also find $w(x) \sim a(\kappa)e^{i\kappa x} + b(\kappa)e^{-i\kappa x}$. a is called transmission coefficient, while b is called reflection coefficient.

Having characterised the scattering data, one can recover the potential u by solving the Gelfand-Levitan-Marchenko integral equation.

The inverse scattering transform was soon extended to many completely integrable systems, most notably for us: the nonlinear Schrödinger equation (1.1), for $p = 3$. Other systems for which the inverse scattering transform can be used include the Sine-Gordon equation, the Kadomtsev–Petviashvili equation, the Ishimori equation and the Dym equation.

A far more comprehensive overview of the development of soliton theory is provided by Palais in [27]. Still, the vastness of the literature is such that even [27] can only give an introduction to the topic.

1.3. Asymptotic Stability

The discovery of solitons has given rise to a new conjecture on the long term behaviour of many nonlinear dispersive equation, replacing thermalisation.

The so-called soliton resolution conjecture predicts that any reasonably regular solution, given enough time, resolves into a finite number of soliton solutions and a radiative term, meaning something which behaves like a solution of the linear equation.

As mentioned, this conjecture is also posed for other nonlinear dispersive equations, notably the nonlinear wave equation and the Korteweg-de Vries equation. Tao [32] gives a more comprehensive overview over solitons and the soliton resolution conjecture for a variety of different dispersive equations.

The topic of this thesis derives from the related question of asymptotic stability. Considering initial data $u_0 = Q + w_0$, whereby we have perturbed a soliton with some small w_0. Asymptotic stability states that a perturbed soliton converges to a possibly different solitary wave and radiation.

Generally speaking, asymptotic stability is a case-by-case problem with no known universal approach or criterium. Instead, there are a number of different ways to approach the problem. The following overview is largely taken from [21].

1. Integrability methods, mainly the inverse scattering transform. Because the in-

verse scattering transform actually solves the problem, it can result in much more information than just asymptotic stability. For systems that are not completely integrable, this Ansatz is not viable, although some spectral information may be obtained from the integrability structure nonetheless.

2. Perturbative Methods, meaning one studies the flow in a neighbourhood of the solitary wave. One usually presupposes suitable assumptions on the initial data, or global in time assumptions on the solution, or restriction of the initial data to a manifold of finite co-dimension. One then decomposes the solution into a wave, described by a set number of geometrical parameters, and a small residue. Asymptotic stability reduces to understanding the long term behaviour of the residue and the evolution of the parameters.

   Ideally, one would like to obtain "scattering", meaning that the residue behaves like a solution of a linear dispersive equation. If scattering can be shown, then controlling the parameters is usually not all to difficult.

   The perturbative approach can in principle be applied to study any soliton problem. However, the linearised PDE satisfied by the residue is often hard to study. The following ideas exist.

   a) Dispersive estimates. The idea is to mimic the small data global Cauchy problem, after first proving dispersive estimates for the linear equation resulting from linearising around a soliton. This usually requires strong spectral information on the resulting Schrödinger operator which can present a problem, especially if the soltion is not given by an explicit formula.

      Dispersive estimates become more difficult to use for low power nonlinearities and/or low space dimension. Also, if the linearised Schrödinger operator features non-trivial eigenvalues, the corresponding eigenfunctions being called internal modes, then further complications usually arise.

   b) Liapunov functionals. Virial type arguments can provide convergence in a weaker sense, under different, sometimes weaker spectral information. This method is often useful for low dimensional problems with low power nonlinearities, since dispersive estimates are not needed.

      A useful tool to be noted in this context are the Morawetz inequalities, a type of estimate that was originally derived by studying and bounding the radial derivative $\frac{x \cdot \nabla u}{|x|}$ of certain nonlinear equations. In recent years so called interaction Morawetz inequalities were introduced, which are not tied to radially symmetric data. As an example, an interaction Morawetz identity can be used to control the correlation quantity

$$\int_{\mathbb{R}} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(t,x)|^2 \, |u(t,y)|^p}{|x-y|} dx dy dt \tag{1.5}$$

      in case of the three-dimensional nonlinear Schrödinger equation. See for example [20].

This thesis considers the question of asymptotic stability by establishing dispersive estimates on the linearised Schrödinger equation as described in point 2a). Using the explicit formula of the ground state (1.4), we fully characterise the spectrum of the linearised operator, which we then use to define a wave operator and establish several dispersive estimate. Using the results of this thesis should make it relatively straightforward to establish asymptotic stability results for the NLS. However, proving asymptotic stability is beyond the scope of this thesis.

1.4. Symmetries

Before we proceed further, let us consider some basic facts about the NLS. We begin by noting that the NLS conserves several quantities. As stated before, initial data $u_0 \in H^1$ suffices to ensure local well-posedness. We consider well-posedness in more detail in chapter 1.7.

Lemma 1.4.1 *Let* $3 \leq p \leq 5$. *Consider a solution* $u \in C^0H^1([0,t_0) \times \mathbb{R})$ *to* (1.1) *with initial data* $u(0,\cdot) = u_0 \in H^1(\mathbb{R})$ *and time of existence* $t_0 \in [0,\infty]$. *The following quantities are conserved for* $0 \leq t < t_0$*:*

1. *Mass:*
$$||u(t,\cdot)||_{L^2} = ||u_0||_{L^2}. \tag{1.6}$$

2. *Energy:*
$$\frac{1}{2}\,||\nabla u(t,\cdot)||_{L^2}^2 - \frac{1}{p+1}\,||u(t,\cdot)||_{L^2}^{p+1} = \frac{1}{2}\,||\nabla u_0||_{L^2}^2 - \frac{1}{p+1}\,||u_0||_{L^2}^{p+1}. \tag{1.7}$$

3. *Momentum:*
$$\operatorname{Im}\left(\int_{\mathbb{R}} \nabla u(t,x)\overline{u(t,x)}dx\right) = \operatorname{Im}\left(\int_{\mathbb{R}} \nabla u_0(x)\overline{u_0(x)}dx\right). \tag{1.8}$$

Proof. Taking the derivative of each quantity with respect to t and then applying (1.1) gives the desired result. ☐

Next, we note the symmetries of (1.1):

1. Translation invariance: If u is a solution, so is $(t,x) \mapsto u(t+t_0, x+x_0)$, $t_0, x_0 \in \mathbb{R}$.
2. Phase invariance: If u is a solution, so is $e^{i\nu}u$, $\nu \in \mathbb{R}$.
3. Scaling invariance: If u is a solution, so is $(t,x) \mapsto \lambda^{\frac{2}{p-1}}u(\lambda^2 t, \lambda x)$, $\lambda > 0$.
4. Galilean invariance: If u is a solution, so is $(t,x) \mapsto u(t, x-\beta t)e^{i\frac{\beta}{2}(x-\frac{\beta}{2}t)}$, $\beta \in \mathbb{R}$.

In the critical case $p = 5$, (1.1) admits one additional symmetry not in H^1, via the so called pseudoconformal transformation:

5. Pseudoconformal invariance: If u is a solution, so is $(t,x) \mapsto \frac{1}{|t|^{\frac{1}{2}}}\overline{u}(\frac{1}{t}, \frac{x}{t})e^{i\frac{|x|^2}{4t}}$.

The pseudoconformal transformation immediately ensures the existence of blow-up solutions. Indeed, applied to a solitary wave, it gives rise to a solution which blows up at time $t = 0$.

We note that not every symmetry corresponds to a conserved quantities. By a well-known theorem of Noether, every conserved quantity should correspond to a symmetry preserving the Hamiltonian.

This is simply due to the fact that neither the scaling, the Galilean nor the pseudo-conformal invariance preserve the Hamiltonian.

In the above case, the translation invariance in space corresponds to the conservation of momentum, while the phase invariance and translation invariance in time correspond to the conservation of mass and energy respectively.

1.5. Criticality

$p = 5$ allows for an additional symmetry. Somehow, (1.1) is fundamentally different for $p < 5$ and $p = 5$. The case of $p = 3$ is also different, as then the system is completely integrable. In this context it is useful to introduce the concept of criticality.

Consider the scaling symmetry for $\lambda > 0$:

$$u(t,x) \mapsto \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x). \tag{1.9}$$

Given a Banach space of initial data, the problem (1.2) is called critical if the norm stays invariant under the scaling operator (1.9).

If the norm diverges as $\lambda \to \infty$ the problem is called subcritical. If the norm instead converges to zero, the problem is called supercritical.

(1.2) is called mass- or L^2-critical for $p = \frac{4}{d} + 1$, related to the conservation of mass:

$$\int_{\mathbb{R}} |u_0|^2 \, dx = \int_{\mathbb{R}} |u(t,x)|^2 \, dx. \tag{1.10}$$

For $p = \frac{4}{d-2} + 1$, it is called energy- or H^1-critical, related to the conservation of energy:

$$\frac{1}{2}\int_{\mathbb{R}} |\nabla u_0|^2 \, dx + \frac{\mu}{p+1}\int_{\mathbb{R}} |u_0|^{p+1} \, dx = \frac{1}{2}\int_{\mathbb{R}} |\nabla u(t,x)|^2 \, dx + \frac{\mu}{p+1}\int_{\mathbb{R}} |u(t,x)|^{p+1} \, dx. \tag{1.11}$$

In this thesis, we consider the one-dimensional focusing equation (1.1) in the mass-subcritical case $3 \leq p < 5$ and the mass-critical case $p = 5$.

1.6. Solution

We have not yet established what it means to be a solution of (1.1).

If sufficient regularity of the non-linear term $|u|^{p-1} u$ is ensured, then (1.1) can be expressed via Duhamel's formula:

$$u(t,x) = e^{it\Delta}u_0(x) + i \int_0^t e^{i(t-s)\Delta} \left(|u(s,x)|^{p-1} u(s,x) \right) ds \tag{1.12}$$

Hereby, $e^{it\Delta}w_0$ denotes the solution $w(t,x)$ to the free equation $i\partial_t w = -\Delta w$ with initial data $w(0,x) = w_0(x)$. (1.12) is better suited than (1.1) to define the notion of a solution.

Lemma 1.6.1 (Dispersive estimate) *Consider the free Schrödinger equation in* $\mathbb{R}^1$*:*

$$i\partial_t u(t,x) = -\Delta u(t,x). \tag{1.13}$$

Let the solution for initial data $u(0,\cdot) = u_0$ *be given by* $e^{it\Delta}u_0$*. Let* $q \in [2,\infty]$ *with dual exponent* $\frac{1}{q} + \frac{1}{q'} = 1$*. Then, for* $t > 0$*:*

$$\left|\left| e^{it\Delta}u_0(t,\cdot) \right|\right|_{L^q} \leq Ct^{-\frac{1}{2}+\frac{1}{q}} \left|\left| u_0 \right|\right|_{L^{q'}}. \tag{1.14}$$

Proof. The case $q = \infty, q' = 1$ follows directly from the explicit solution of the Schrödinger equation:

$$e^{it\Delta}u_0(t,x) = \frac{1}{2\sqrt{it}} \int_{\mathbb{R}} e^{i\frac{(\xi-x)^2}{4t}} u_0(\xi) d\xi. \tag{1.15}$$

As the Schrödinger equation conserves the L^2-norm, the claim holds by interpolation.□

Lemma 1.6.2 (Strichartz estimate in one space dimension) *We consider the free Schrödinger equation* (1.13) *with solution* $e^{it\Delta}u_0$ *for initial data* $u(0,\cdot) = u_0$*.*

Assume $q, r \in [2,\infty]$ *satisfy* $\frac{2}{q} + \frac{1}{r} = \frac{1}{2}$ *and* $\tilde{q}', \tilde{r}' \in [1,2]$ *satisfy* $\frac{2}{\tilde{q}'} + \frac{1}{\tilde{r}'} = \frac{5}{2}$*. Then, the following estimates hold true:*

1. *The homogeneous Strichartz estimate:*

$$\left|\left| e^{it\Delta}u_0 \right|\right|_{L_t^q L_x^r} \leq C \left|\left| u_0 \right|\right|_{L^2}. \tag{1.16}$$

2. *The dual homogeneous Strichartz estimate:*

$$\left|\left| \int_{\mathbb{R}} e^{-is\Delta} F(s,\cdot) ds \right|\right|_{L^2} \leq C \left|\left| F \right|\right|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}. \tag{1.17}$$

3. *The inhomogeneous Strichartz estimate:*

$$\left|\left| \int_0^t e^{i(t-s)\Delta} F(s,\cdot) ds \right|\right|_{L_t^q L_x^r} \leq C \left|\left| F \right|\right|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}. \tag{1.18}$$

Proof. These estimates are originally due to Strichartz [30]. For the version given here, see e.g. Tao [31]. ☐

Definition 1.6.3 (Solution in H^1) *Let initial data $u_0 \in H^1(\mathbb{R})$ be given. Let further $\mathcal{I} \ni 0$ be an interval containing the origin.*

$u \in C_t^0 H_x^1(\mathcal{I} \times \mathbb{R})$ is said to be a (strong) solution to (1.1) *if it satisfies Duhamel's formula* (1.12) *and $u(0, \cdot) = u_0$.*

By the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ and the Strichartz estimates, (1.12) is well-defined for $u \in C_t^0 H_x^1(I \times \mathbb{R})$.

We also define solutions in L^2. In this case we actually do require additional integrability. We choose the space $L_t^6 L_x^6$. On the one hand, the Strichartz estimates then ensure:

$$\left|\left|e^{it\Delta} u_0\right|\right|_{L_t^6 L_x^6} \leq C \left|\left|u_0\right|\right|_{L^2}. \tag{1.19}$$

Recall that we consider $p \leq 5$. Let $\theta \in (0, 1)$ be given by $\frac{1}{p} = \frac{1-\theta}{6} + \frac{\theta}{2}$. Let further $q \in [1, \infty]$ be given by $\frac{3}{4p} = \frac{1-\theta}{6} + \frac{\theta}{q}$. Then, for every $t_0 \geq 0$:

$$\begin{aligned}
\left|\left|u^p\right|\right|_{L_t^{\frac{4}{3}}(0,t_0) L_x^1} &\leq C \left|\left|u\right|\right|^p_{L_t^{\frac{4}{3}p}(0,t_0) L_x^p} \\
&\leq C \left(\left|\left|u\right|\right|^{1-\theta}_{L_t^6(0,t_0)L_x^6} \left|\left|u\right|\right|^{\theta}_{L_t^q(0,t_0)L_x^2} \right)^p \\
&\leq C^2 \left(\left|\left|u\right|\right|_{L_t^6(0,t_0)L_x^6} + t_0^{\frac{1}{q}} \left|\left|u\right|\right|_{L_t^\infty(0,t_0)L_x^2} \right)^p.
\end{aligned} \tag{1.20}$$

We conclude:

$$\begin{aligned}
&\left|\left|\int_0^t e^{i(t-s)\Delta} \left(\left|u(s,x)\right|^{p-1} u(s,x) \right) ds \right|\right|_{L_t^6(0,t_0)L_x^6} \\
&+ \left|\left|\int_0^t e^{i(t-s)\Delta} \left(\left|u(s,x)\right|^{p-1} u(s,x) \right) ds \right|\right|_{L_t^\infty(0,t_0)L_x^2} \\
\leq\ & C \left(\left|\left|u\right|\right|_{L_t^6(0,t_0)L_x^6} + t_0^{\frac{1}{q}} \left|\left|u\right|\right|_{L_t^\infty(0,t_0)L_x^2} \right)^p.
\end{aligned} \tag{1.21}$$

Consequently, given $t_0 \geq 0$ and $u \in C_t^0 L_x^2([0, t_0] \times \mathbb{R}) \cap L_t^6 L_x^6([0, t_0] \times \mathbb{R})$, it follows:

$$\begin{aligned}
&e^{it\Delta} u(0, x) + i \int_0^t e^{i(t-s)\Delta} \left(\left|u(s,x)\right|^{p-1} u(s,x) \right) ds \\
\in\ & C_t^0 L_x^2([0, t_0] \times \mathbb{R}) \cap L_t^6 L_x^6([0, t_0] \times \mathbb{R}).
\end{aligned} \tag{1.22}$$

Definition 1.6.4 (Solution in L^2) *Let initial data $u_0 \in L^2(\mathbb{R})$ be given. Let further $\mathcal{I} \ni 0$ be an interval containing the origin.*

$u \in C_t^0 L_x^2(\mathcal{I} \times \mathbb{R}) \cap L_t^6 L_x^6(\mathcal{I} \times \mathbb{R})$ is said to be a (strong) solution to (1.1) *if it satisfies Duhamel's formula* (1.12) *and $u(0, \cdot) = u_0$.*

1.7. Well-Posedness

Now that a notion of what it means to solve (1.1) is established, we give some basic well-posedness results in H^1.

More general well-posedness results are shown in the literature ([4], [6], [17], compare page 12 in this thesis).

Lemma 1.7.1 (Local well-posedness in H^1) *Consider* (1.1)*. Let $p \geq 1$ and $K > 0$. Then, there exists $T > 0$, such that for every $u_0 \in H^1(\mathbb{R})$ with $||u_0||_{H^1} \leq K$, there is a unique solution $u \in C^0H^1([-T,T] \times \mathbb{R})$. u depends continuously on u_0.*

Proof. Recall Duhamel's formula (1.12). For $w \in C^0H^1([-T,T] \times \mathbb{R})$ with $T > 0$ to be chosen later, we define the operator:

$$Bw := e^{it\Delta}u_0 + i\int_0^t e^{i(t-s)\Delta}\left(|w|^{p-1}\,w\right) ds. \tag{1.23}$$

For $w_1, w_2 \in C^0H^1([-T,T] \times \mathbb{R})$, it follows:

$$\begin{aligned}
||Bw_1 - Bw_2||_{L^\infty H^1} &\leq T \left|\left|\left(|w_1|^{p-1}\,w_1 - |w_2|^{p-1}\,w_2\right)\right|\right|_{L^\infty H^1} \\
&\leq CT\,||w_1 - w_2||_{L^\infty H^1}\left(\left|\left|w_1^{p-1}\right|\right|_{L^\infty L^\infty} + \left|\left|w_2^{p-1}\right|\right|_{L^\infty L^\infty}\right) \\
&\quad + CT\,||w_1 - w_2||_{L^\infty L^\infty}\left(\left|\left|w_1^{p-1}\right|\right|_{L^\infty H^1} + \left|\left|w_2^{p-1}\right|\right|_{L^\infty H^1}\right) \\
&\leq C^2T\,||w_1 - w_2||_{L^\infty H^1}\left(||w_1||_{L^\infty H^1}^{p-1} + ||w_2||_{L^\infty H^1}^{p-1}\right).
\end{aligned} \tag{1.24}$$

In particular, that implies:

$$\begin{aligned}
||Bw_1||_{L^\infty H^1} \leq\ &\left|\left|e^{it\Delta}u_0\right|\right|_{L^\infty H^1} + CT\,||w_1||_{L^\infty H^1}^p \\
\leq\ &K + CT\,||w_1||_{L^\infty H^1}^p.
\end{aligned} \tag{1.25}$$

With $C > 0$ large enough to fulfil both (1.24) and (1.25), consider $T = \min\left(\frac{1}{2^pK^pC}, \frac{1}{2^pK^{p-1}C^2}\right)$. By (1.25), B maps the ball $\{w \in L^\infty H^1([-T,T] \times \mathbb{R}) |\, ||w||_{L^\infty H^1} \leq 2K\}$ onto itself.

The lemma follows from (1.24) and the Banach fixed-point theorem. □

Lemma 1.7.2 (Global well-posedness in H^1) *Assume $3 \leq p < 5$. Given $u_0 \in H^1(\mathbb{R})$,* (1.1) *is globally well-posed. For every $t \in \mathbb{R}$:*

$$||u(t,\cdot)||_{H^1} \leq C\,||u_0||_{H^1} + C_p\,||u_0||_{L^2}^{\frac{p+3}{5-p}}. \tag{1.26}$$

Proof. Let $E(t)$ denote the energy (1.7):

$$E(t) = \frac{1}{2}\,||\nabla u(t,\cdot)||_{L^2}^2 - \frac{1}{p+1}\,||u(t,\cdot)||_{L^2}^{p+1}. \tag{1.27}$$

Consider the Gagliardo-Nirenberg inequality (see [14], [26]):

$$||u||_{L^{p+1}} \leq C\,||\nabla u||_{L^2}^{\frac{1}{2}-\frac{1}{p+1}}\,||u||_{L^2}^{\frac{1}{2}+\frac{1}{p+1}}. \tag{1.28}$$

Consequently, as the L^2-norm and energy are conserved:

$$\begin{aligned} ||\nabla u||_{L^2}^2 &\leq 2E(0) + \frac{2}{p+1}\,||u||_{L^{p+1}}^{p+1} \\ &\leq 2\,||u_0||_{H^1}^2 + C\,||\nabla u||_{L^2}^{\frac{p-1}{2}}\,||u_0||_{L^2}^{\frac{p+3}{2}} \\ &\leq 2\,||u_0||_{H^1}^2 + \frac{1}{2}\,||\nabla u||_{L^2}^2 + C^2\,||u_0||_{L^2}^{\frac{2(p+3)}{5-p}}. \end{aligned} \tag{1.29}$$

We conclude (1.26). As the H^1-norm is uniformly bounded, lemma 1.7.1 implies existence for all times. □

For $p = 5$, (1.1) is not globally well-posed in H^1. Explicit counterexamples can be constructed through use of the pseudoconformal transformation (more general counterexamples can be found in [24]).

We also note the virial identity, a type of Morawetz identity (see [25]), which in some way can be seen as the equivalent of a conserved quantity for the Galilean invariance.

Lemma 1.7.3 (Virial Identiy) *Assume* $3 \leq p < 5$. *Consider* (1.1) *and assume* $xu_0 \in L^2(\mathbb{R})$ *in addition to* $u_0 \in H^1(\mathbb{R})$. *Then, for every* $t \geq 0$*:*

$$\frac{d}{dt}\,||xu||_{L^2}^2 = 4\int_{\mathbb{R}} x\,\mathrm{Im}(\overline{u}\nabla u)dx. \tag{1.30}$$

Further:

$$||xu||_{L^2} \leq ||xu_0||_{L^2} + C_p t\left(||u_0||_{H^1} + ||u_0||_{L^2}^{\frac{p+3}{5-p}}\right). \tag{1.31}$$

Proof. (1.30) follows by direct computation. Consequently:

$$2\,||xu||_{L^2}\,\frac{d}{dt}\,||xu||_{L^2} = \frac{d}{dt}\,||xu||_{L^2}^2 \leq 4\,||\nabla u||_{L^2}\,||xu||_{L^2}. \tag{1.32}$$

It follows $\frac{d}{dt}\,||xu||_{L^2} \leq 2\,||\nabla u||_{L^2}$. Lemma 1.7.2 concludes the proof. □

1.8. Linearisation and Eigenvalue in 0

In order to examine the question of soliton stability for (1.1), we now follow an approach laid out by Weinstein in [34]. We start be linearising (1.1) around the soliton via $u = e^{it}(Q + w)$:

$$i\partial_t w = -\Delta w + w - \frac{p+1}{2}Q^{p-1}w - \frac{p-1}{2}Q^{p-1}\overline{w}. \tag{1.33}$$

Due to $|\cdot|$ not being $\mathbb{C}$-differentiable, (1.33) is only $\mathbb{R}$- and not $\mathbb{C}$-linear. In order to ensure $\mathbb{C}$-linearity, $\mathbb{C}$ is identified with $\mathbb{R}^2 \hookrightarrow \mathbb{C}^2$ and $w = u + iv$ is rewritten as $w = (u, v)$. (1.33) now reads:

$$\partial_t w = -Lw. \tag{1.34}$$

L denotes the following $\mathbb{C}$-linear operator:

$$Lw = \begin{pmatrix} 0 & -L_V \\ L_U & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{1.35}$$

L_U and L_V are given by:

$$L_U u = -\Delta u + u - pQ^{p-1}u, \tag{1.36}$$

$$L_V v = -\Delta v + v - Q^{p-1}v. \tag{1.37}$$

L admits several generalised eigenfunctions, directly connected to the symmetries of (1.1). The eigenmodes are given by

$$L \begin{pmatrix} 0 \\ Q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{1.38}$$

$$L \begin{pmatrix} \partial_x Q \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{1.39}$$

(1.38) and (1.39) relate to the phase invariance and the translation invariance in space respectively. Related to the scaling and Galilean invariance, there also exist two generalised eigenmodes of rank 1 given by:

$$L \begin{pmatrix} \frac{2}{p-1}Q + x\partial_x Q \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -2Q \end{pmatrix}, \tag{1.40}$$

$$L \begin{pmatrix} 0 \\ xQ \end{pmatrix} = \begin{pmatrix} 2\partial_x Q \\ 0 \end{pmatrix}. \tag{1.41}$$

Additionally, the case of $p = 5$ gives rise to two additional generalised eigenmodes:

$$L \begin{pmatrix} 0 \\ x^2 Q \end{pmatrix} = \begin{pmatrix} 4(\frac{1}{2}Q + xQ_x) \\ 0 \end{pmatrix}, \qquad L \begin{pmatrix} \rho \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ x^2 Q \end{pmatrix}. \tag{1.42}$$

The former eigenmode is of course related to the pseudo-conformal invariance, while $\rho \in L^2(\mathbb{R})$ is not given explicitly, and is not connected to any symmetry.

1.9. Hilbert Space

Still following Weinstein [34], we use the eigenfunctions of L to define a Hilbert space, which allows us to treat L as a self-adjoint operator.

Consider the following orthogonality conditions for $u, v \in L^2(\mathbb{R})$:

$$\langle u, Q\rangle_{L^2} = \langle u, xQ\rangle_{L^2} = \langle v, Q_x\rangle_{L^2} = \langle v, \frac{2}{p-1}Q + xQ_x\rangle_{L^2} = 0. \tag{1.43}$$

For $p = 5$, we require two additional orthogonality conditions:

$$\langle u, x^2 Q\rangle_{L^2} = \langle v, \rho\rangle_{L^2} = 0. \tag{1.44}$$

Lemma & Definition 1.9.1 *Let for $p < 5$:*

$$\mathcal{H} := \{w = (u, v) \in H^1(\mathbb{R})^2 | \ (1.43) \ \textit{holds true}\}, \tag{1.45}$$

as well as for $p = 5$:

$$\mathcal{H} := \{w = (u, v) \in H^1(\mathbb{R})^2 | \ (1.43) \ \textit{and} \ (1.44) \ \textit{hold true}\}. \tag{1.46}$$

Further, let $\langle \cdot, \cdot \rangle_{\mathcal{H}} : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ be given by:

$$\begin{aligned} &\langle (u_1, v_1), (u_2, v_2)\rangle_{\mathcal{H}} \\ = &\int_{\mathbb{R}} \Big(\nabla u_1 \nabla \overline{u}_2 + u_1 \overline{u}_2 + \nabla v_1 \nabla \overline{v}_2 + v_1 \overline{v}_2 - pQ^{p-1} u_1 \overline{u}_2 - Q^{p-1} v_1 \overline{v}_2 \Big)\, dx. \end{aligned} \tag{1.47}$$

Then, $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ constitutes a Hilbert space and L maps $\mathcal{H}$ onto itself. Further, iL is self-adjoint with respect to $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. Finally, $||\cdot||_{\mathcal{H}}$ and $||\cdot||_{H^1(\mathbb{R})^2}$ constitute equivalent norms.

Proof. See Weinstein [34]. □

Remark *The $\mathcal{H}$-scalar product is derived from the linearised operator via*

$$\langle (u_1, v_1), (u_2, v_2)\rangle_{\mathcal{H}} = \langle u_1, L_U u_2\rangle_{L^2} + \langle v_1, L_V v_2\rangle_{L^2}. \tag{1.48}$$

Remark *L was derived by complexifying (1.33). This has the effect of mirroring the spectrum. As a simple example, consider the operator $-\Delta + 1$, the spectrum of which is given by $[1, \infty)$. Now consider the matrix equivalent of the imaginary unit i, given by*

$$I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{1.49}$$

The spectrum of the complexification $iI(-\Delta + 1)$ is given by $(-\infty, -1] \cup [1, \infty)$. For the same reason the essential spectrum of iL is given by $(-\infty, -1] \cup [1, \infty)$. This also means that $-iL$ is not positive definite with respect to $\langle \cdot, \cdot \rangle_{\mathcal{H}}$, even though $-IL$ is.

1.10. Spectrum of L

We finally reach the main topic of the thesis.

In order to establish dispersive estimates on the linearised equation $\partial_t w = -Lw$, it is necessary to study the spectrum of L. As compact perturbations leave the essential spectrum unchanged, the essential spectrum of iL is given by $(-\infty, -1] \cup [1, \infty)$. Both eigenvalues and resonances in ± 1 are problematic when establishing estimates.

In the literature the following questions are often considered separately.

1. Are there eigenvalues embedded within the essential spectrum $(-\infty, -1] \cup [1, \infty)$?
2. Do the endpoints ± 1 of the essential spectrum constitute resonances?
3. Apart from 0, are there eigenvalues within the spectral gap $[-1, 1]$?

There are several numerical results on the existence of eigenvalues of L, see [13], [23].

The main result of this thesis is the full characterisation of the spectrum of L in the (sub)critical case $3 \leq p \leq 5$.

Theorem 1.10.1 (Theorem 5.8.7) *For $p = 3$, iL possesses no eigenvalues apart from 0. 1 and -1 constitute resonances.*

For $p \in (3, 5)$, iL admits two simple eigenvalues $E_1, E_2 \in (-1, 1) \backslash \{0\}$ with $E_1 = -E_2$. iL admits no further eigenvalues or resonances apart from 0. The associated eigenfunctions of E_1, E_2 are even. It holds $E_1 \to 1$ for $p \to 3$ and $E_1 \to 0$ for $p \to 5$.

Finally, for $p = 5$, iL admits no eigenvalues or resonances apart from 0.

Remark *The resonances for $p = 3$ are given by:*

$$(L + i) \begin{pmatrix} 1 - Q^2 \\ i \end{pmatrix} = (L - i) \begin{pmatrix} 1 - Q^2 \\ -i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{1.50}$$

The proof of theorem 1.10.1 will make up a majority of this thesis and the entirety of part II, beginning on page 52. For now, we give a brief overview of the strategy of proof.

By using the explicit formulation of the ground state (1.4), we are able to reformulate the eigenvalue equation $Lw = iEw$ as a system of hypergeometric equations. After several transformations we find an equivalent to the eigenvalue equation in (3.16) and (3.17):

$$0 = u_{zz} - \frac{2z}{1 - z^2} u_z + \left(-\frac{4(1 + E)}{(p - 1)^2} \frac{1}{(1 - z^2)^2} + \frac{(p + 1)^2}{(p - 1)^2} \frac{1}{1 - z^2} \right) u + \frac{p + 1}{p - 1} \frac{1}{1 - z^2} v, \tag{1.51}$$

$$0 = v_{zz} - \frac{2z}{1 - z^2} v_z + \left(-\frac{4(1 - E)}{(p - 1)^2} \frac{1}{(1 - z^2)^2} + \frac{(p + 1)^2}{(p - 1)^2} \frac{1}{1 - z^2} \right) v + \frac{p + 1}{p - 1} \frac{1}{1 - z^2} u. \tag{1.52}$$

The symmetries of (1.51), (1.52) allow us to consider even and odd solutions separately. In the even case, we can substitute $\xi = 1 - z^2$ and solve (1.51), (1.52) by calculating the coefficients of

$$u(\xi) = \xi^{\frac{\sqrt{1+E}}{p-1}} \sum_{k=0}^{\infty} a_k \xi^k, \tag{1.53}$$

$$v(\xi) = \xi^{\frac{\sqrt{1+E}}{p-1}} \sum_{k=0}^{\infty} b_k \xi^k. \tag{1.54}$$

The coefficients $(a_k)_k, (b_k)_k$ can be explicitly calculated (lemma 3.5.1) via the recursion $(a_0, b_0) = (1, 0)$,

$$\begin{aligned} a_{k+1} = &\frac{4k^2 + 2k + \frac{8\sqrt{1+E}}{p-1}k + \frac{4(1+E)}{(p-1)^2} + \frac{2\sqrt{1+E}}{p-1} - \frac{(p+1)^2}{(p-1)^2}}{4(k+1)^2 + \frac{8\sqrt{1+E}}{p-1}(k+1)} a_k \\ &- \frac{p+1}{p-1} \frac{1}{4(k+1)^2 + \frac{8\sqrt{1+E}}{p-1}(k+1)} b_k, \end{aligned} \tag{1.55}$$

and

$$\begin{aligned} b_{k+1} = &\frac{4k^2 + 2k + \frac{8\sqrt{1+E}}{p-1}k + \frac{4(1+E)}{(p-1)^2} + \frac{2\sqrt{1+E}}{p-1} - \frac{(p+1)^2}{(p-1)^2}}{4(k+1)^2 + \frac{8\sqrt{1+E}}{p-1}(k+1) + \frac{8E}{(p-1)^2}} b_k \\ &- \frac{p+1}{p-1} \frac{1}{4(k+1)^2 + \frac{8\sqrt{1+E}}{p-1}(k+1) + \frac{8E}{(p-1)^2}} a_k. \end{aligned} \tag{1.56}$$

This idea can be used to construct a solution $w = (u, v) : [0, 1) \to \mathbb{R}^2$ of (1.51), (1.52) with $w(0) = (0, 0)$ for every $E > -1$. The condition $w(0) = (0, 0)$ is equivalent to the condition $\lim_{x \to -\infty} h(x) = (0, 0)$ for the eigenvalue equation $Lh = iEh$.

For $E = -1$, the solution w does not vanish in $\xi = 0$, but is bounded on $[0, 1)$. This solution is therefore relevant when considering resonances, but not when considering eigenvalues.

We follow an almost identical idea when studying odd solutions. Recall (1.51) and (1.52). We instead look for solutions of the form $u = z\tilde{u}$, $v = z\tilde{v}$, whereby $\tilde{u}$ and $\tilde{v}$ are of the form (1.53), (1.54):

$$\tilde{u}(\xi) = \xi^{\frac{\sqrt{1+E}}{p-1}} \sum_{k=0}^{\infty} a_k \xi^k, \tag{1.57}$$

$$\tilde{v}(\xi) = \xi^{\frac{\sqrt{1+E}}{p-1}} \sum_{k=0}^{\infty} b_k \xi^k. \tag{1.58}$$

The coefficients can again be explicitly calculated (lemma 3.6.1) via $(a_0, b_0) = (1, 0)$, as well as

$$
\begin{aligned}
a_{k+1} = & \frac{4k^2 + 6k + \frac{8\sqrt{1+E}}{p-1}k + \frac{4(1+E)}{(p-1)^2} + \frac{6\sqrt{1+E}}{p-1} + 2 - \frac{(p+1)^2}{(p-1)^2}}{4(k+1)^2 + \frac{8\sqrt{1+E}}{p-1}(k+1)} a_k \\
& - \frac{p+1}{p-1} \frac{1}{4(k+1)^2 + \frac{8\sqrt{1+E}}{p-1}(k+1)} b_k,
\end{aligned} \tag{1.59}
$$

and

$$
\begin{aligned}
b_{k+1} = & \frac{4k^2 + 6k + \frac{8\sqrt{1+E}}{p-1}k + \frac{4(1+E)}{(p-1)^2} + \frac{6\sqrt{1+E}}{p-1} + 2 - \frac{(p+1)^2}{(p-1)^2}}{4(k+1)^2 + \frac{8\sqrt{1+E}}{p-1}(k+1) + \frac{8E}{(p-1)^2}} b_k \\
& - \frac{p+1}{p-1} \frac{1}{4(k+1)^2 + \frac{8\sqrt{1+E}}{p-1}(k+1) + \frac{8E}{(p-1)^2}} a_k.
\end{aligned} \tag{1.60}
$$

1.11. Essential Spectrum

In both the even and the odd case, the question of eigenvalues can be resolved by studying the coefficients $(a_k)_k, (b_k)_k$. The argument goes as follows.

We consider the space of potential eigenfunctions

$$M_{E,-} := \{w \in C^2(\mathbb{R}) | Lw = iEw \wedge \lim_{x \to -\infty} w(x) = (0,0)\}, \tag{1.61}$$

$$D_{E,-} := \{(u,v) \in C^2(-1,0)^2 | (1.51), (1.52) \text{ hold}, \lim_{z \to -1} (u(z), v(z)) = (0,0)\}. \tag{1.62}$$

It is quite easy to see (lemma 2.3.21) that $M_{E,-}$ is one-dimensional for $|E| \geq 1$ and two-dimensional for $|E| < 1$. The same holds for $D_{E,-}$, which is simply the equivalent of $M_{E,-}$ for the transformed eigenvalue equation (1.51), (1.52). This makes studying the essential spectrum $|E| \geq 1$ far simpler.

Indeed, given $|E| \geq 1$, consider the solution $w_E = (u_E, v_E)$ given by (1.53), (1.54). If and only if

$$\lim_{z \nearrow 0} \partial_z w_E(1 - z^2) = (0,0), \tag{1.63}$$

then the analytical continuation of $w_E(1-z^2)$ to $z \in [-1,1]$ is even. Due to $M_{E,-}$ being one-dimensional, this is equivalent to the existence of a non-trivial even eigenfunction $Lw = iEw$.

Completely analogously, consider the solution $\tilde{w}_E = (\tilde{u}_E, \tilde{v}_E)$ given by (1.57), (1.58). If and only if

$$\lim_{z \nearrow 0} z \tilde{w}_E(1 - z^2) = (0,0), \tag{1.64}$$

does a non-trivial odd eigenfunction $Lw = iEw$ exist.

Now the coefficients $(a_k)_k, (b_k)_k$ come back into play, as the conditions (1.63), (1.64) can be restated in terms of $(a_k)_k$ and $(b_k)_k$. Let $(a_k)_k$ and $(b_k)_k$ be given by (1.55), (1.56). Then, with the single exception of $(E, p) = (-1, 3)$, we can show (lemma 4.1.2) for every $k \geq 0$:

$$(a_k, b_k) \neq (0, 0). \tag{1.65}$$

The exception of $E = -1$, $p = 3$ is directly related to the existence of a resonance in the cubic case $p = 3$. (1.65) allows us define

$$c_k = \frac{a_k}{b_k} \in \mathbb{R} \cup \{\infty\}. \tag{1.66}$$

We are then able to show $\lim_{k\to\infty} c_k \in \mathbb{R} \cup \{\infty\}$ and construct $(j_1, j_2) \in \mathbb{R}^2 \backslash \{(0, 0)\}$, such that (corollary 4.1.7):

$$\lim_{k\to\infty} c_k = \frac{j_1}{j_2}, \tag{1.67}$$

and (lemma 4.1.10):

$$\lim_{z\nearrow 0} w_E'(1 - z^2) = -(j_1, j_2) \neq (0, 0). \tag{1.68}$$

This is precisely condition (1.63), hence no eigenvalues with even eigenfunctions exist embedded within the essential spectrum $|E| \geq 1$.

The case of odd eigenfunctions is treated analogously. Let $(a_k)_k$ and $(b_k)_k$ be given by (1.59), (1.60). Without exception this time, we find $(a_k, b_k) \neq (0, 0)$ for $k \geq 0$.

We define $c_k = \frac{a_k}{b_k}$ and can construct $(j_1, j_2) \in \mathbb{R}^2 \backslash \{(0, 0)\}$, such that (corollary 4.2.7):

$$\lim_{k\to\infty} c_k = \frac{j_1}{j_2}, \tag{1.69}$$

and (lemma 4.2.10):

$$\lim_{z\nearrow 0} z\tilde{w}_E(z) = -(j_1, j_2) \neq (0, 0). \tag{1.70}$$

This is precisely condition (1.64), consequently eigenvalues with odd eigenfunctions do not exist embedded within the essential spectrum $|E| \geq 1$ either.

1.12. Spectral Gap

For eigenvalues within the spectral gap $|E| < 1$, the situation is more complicated as the space of potential eigenfunctions $M_{E,-}$ is two-dimensional. The same complication arises in the study of resonances for $|E| = 1$, as

$$\tilde{D}_{1,-} := \{w = (u, v) \in C^2(-1, 0)^2 | (1.51), (1.52) \text{ hold}, \lim_{z\to -1} |w(z)| < \infty\} \tag{1.71}$$

is two-dimensional as well. It no longer suffices to simply construct a non-even and non-odd solution $w \in M_{E,-}$. Instead, we need to construct two linearly independent solutions $w, h \in M_{E,-}$ and examine if any possible linear combination is even or odd.

Luckily, there are enough tools at our disposal. As described in chapter 1.11, we can construct, for every $E \in [-1, 1]$, some solution $w_E = (u_E, v_E) : (-1, 0) \to \mathbb{C}^2$ to the eigenvalue equation (1.51) and (1.52) with

$$\lim_{z \to 0} w_E(z) = (j_1(E, p), j_2(E, p)) \neq (0, 0). \tag{1.72}$$

Through symmetry (lemma 2.0.2), we can construct a second solution $\tilde{w}_E = (v_{-E}, u_{-E})$. The asymptotics of these solutions (lemma 4.1.10) for $z \to 0$ ensure that w_E and $\tilde{w}_E$ are linearly independent, meaning $D_{E,-} = \operatorname{span}(w_E, \tilde{w}_E)$.

Consequently, there exists an even linear combination of w_E and $\tilde{w}_E$, if and only if

$$\det \begin{pmatrix} j_1(E, p) & j_2(-E, p) \\ j_2(E, p) & j_1(-E, p) \end{pmatrix} = 0. \tag{1.73}$$

This condition can be expressed using the recursively defined sequence $(c_k(E, p))_k$ which fulfils $\lim_{k \to \infty} c_k = \frac{j_1}{j_2}$. In conclusion, iL admits an eigenvalue or resonance $E \in [-1, 1] \backslash \{0\}$ with even eigenfunction, if and only if

$$\mathcal{C}(E, p) := \lim_{k \to \infty} c_k(E, p) = \lim_{k \to \infty} \frac{1}{c_k(-E, p)} =: \frac{1}{\mathcal{C}(-E, p)}. \tag{1.74}$$

The exact same strategy also works for odd eigenfunctions.

Using (1.74) and the equivalent for odd solutions, we can fully characterise the spectrum of L. We show five distinct lemmata. All five statements are proven using heavy calculations to the point that using a computer algebra system to follow along is recommended.

1. Lemma 5.3.13: Let $p \in (3, 5]$. Then, -1 and 1 are not resonances or eigenvalues with even eigenfunctions of iL, meaning no even bounded solutions to $(iL \pm 1)w = 0$ exist.

2. Lemma 5.4.21: Let $p \in [3, 5]$. Then, -1 and 1 are not resonances or eigenvalues with odd eigenfunctions of iL, meaning no odd bounded solutions to $(iL \pm 1)w = 0$ exist.

3. Lemma 5.5.18: Let $p = 3$. Then, iL admits no eigenvalues with even eigenfunctions within the spectral gap $(-1, 1)$, apart from 0.

4. Lemma 5.6.13: Let $p = 3$. Then, iL admits no eigenvalues or resonances with odd eigenfunctions within the spectral gap $[-1, 1]$, apart from 0.

5. Corollary 5.7.15: There exist $E_1 \in [0, 1)$ and $p_1 \in (3, 5)$, such that for every $p \in (3, p_1]$, there exists exactly one eigenvalue $E \in (E_1, 1)$ of iL.

These five statements suffice to prove theorem 1.10.1. Using lemma 5.3.13, lemma 5.4.21 and corollary 5.7.15, we can prove that the total multiplicity of generalised eigenvalues and resonances remains invariant with respect to $p \in [3, 5]$ (lemma 5.8.6). Lemma 5.5.18 and lemma 5.6.13 imply the number of resonances and eigenvalues to be 6. Theorem 1.10.1 follows from the fact that 0 is an eigenvalue of order 4 for $3 \leq p < 5$ and of order 6 for $p = 5$.

1.13. Wave Operator

Having characterised the spectrum of L, we move on to part III. Part III has two goals. The definition of a wave operator, and establishing several dispersive bounds for the linearised equation $\partial_t w = Lw$.

Let $H \subset \mathcal{H}$ denote the subspace orthogonal to the eigenfunctions of the non-zero eigenvalues of L. We can construct a wave operator, unitarily mapping solutions of the linearised equation onto solutions of a free Schrödinger equation. Such a wave operator $T : H \to H^1(\mathbb{R})^2$ is formally given by

$$Tw = \lim_{t\to\infty} e^{tI(\Delta-1)} e^{tL} w, \tag{1.75}$$

$$I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{1.76}$$

If the limit exists, then any solution w of (1.34) gives rise to a solution Tw of the free Schrödinger equation:

$$I\partial_t w = (-\Delta + 1)w. \tag{1.77}$$

In order to construct T, we spend some time examining the bounded solutions of the eigenvalue equation $Lw = iEw$, $E \in \mathbb{R}\backslash(-1, 1)$.

We are particularly interested in Jost solutions. Let $E = \operatorname{sgn}(\xi)(\xi^2 + 1)$. Heuristically, a Jost solution of $Lw = iEw$ denotes a solution which exhibits asymptotic behaviour:

$$w(x) = e^{\pm I\xi x} + o(1). \tag{1.78}$$

In actuality, the solutions we consider behave closer to

$$w(x) = c_e(|\xi|) \begin{pmatrix} \cos(\xi x) \\ -i\operatorname{sgn}(\xi)\cos(\xi x) \end{pmatrix} + s_e(|\xi|) \begin{pmatrix} \sin(|\xi x|) \\ -i\operatorname{sgn}(\xi)\sin(|\xi x|) \end{pmatrix} + o(1). \tag{1.79}$$

c_e, s_e are well-behaved real-valued coefficient functions. (1.79) describes a solution to $Lw = iEw$ that is even in x. An analogous odd solution exists as well.

These solutions are used as a kernel to define a transform F mirroring the Fourier transform, a so called distorted Fourier transform. This transform maps solutions of (1.34) onto solutions of

$$\partial_t h = -i\operatorname{sgn}(\xi)(\xi^2 + 1)h. \tag{1.80}$$

We then define an operator $\mathcal{G} : L^2(\mathbb{R})^2 \to L^2(\mathbb{R})^2$, which maps (1.80) onto (1.77):

$$\mathcal{G}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \mathcal{F}f_e \\ -i\mathcal{F}(\mathrm{sgn}(\cdot)f_o) \end{pmatrix} - \begin{pmatrix} i\mathcal{F}(\mathrm{sgn}(\cdot)g_e) \\ \mathcal{F}g_o \end{pmatrix}. \tag{1.81}$$

$f = f_e + f_o$ and $g = g_e + g_o$ denote the decomposition into even and odd functions. The wave operator arises as the composite function of the Fourier-like transforms F and $\mathcal{G}$.

This idea of a distorted Fourier transform usually arises in the context of a linear Schrödinger equation with rapidly decaying potential, $i\partial_t u = -\Delta u - Vu$. A brief discussion of this idea can be found in [7].

1.14. Dispersive Estimates

Having defined the wave operator T, we establish several bounds. Most notably, we show that both T and its inverse are bounded $L^q \to L^q$ for every $1 \leq q \leq \infty$.

That allows us to conclude that the classical dispersive estimates for the free Schrödinger equation hold for the linear equation as well. As a direct analogue of lemma 1.6.1 and lemma 1.6.2:

Theorem 1.14.1 (Theorem 12.0.1) *Let $3 < p < 5$. Consider the linearised Schrödinger equation*

$$\partial_t w(t,x) = -Lw(t,x), \quad w : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2 \tag{1.82}$$

Let the solution for initial data $w(0,\cdot) = w_0 \in L^2(\mathbb{R})^2$ be given by $e^{-tL}w_0$. Assume that w_0 fulfils the orthogonality conditions (1.43)*. Let $q \in [2,\infty]$ with dual exponent $q' \in [1,2]$. Then, for every $t > 0$:*

$$\left|\left|e^{-tL}w_0\right|\right|_{L^q_x} \leq Ct^{-\frac{1}{2}+\frac{1}{q}} \left|\left|w_0\right|\right|_{L^{q'}}. \tag{1.83}$$

Hereby, $||(u,v)||_{L^q}$ is used as shorthand for $||(u,v)||_{(L^q)^2} = (||u||^2_{L^q} + ||v||^2_{L^q})^{\frac{1}{2}}$.

Theorem 1.14.2 (Theorem 12.0.2) *Let $3 < p < 5$. Consider the linearised Schrödinger equation* (1.82) *and the solution for initial data $w(0,\cdot) = w_0 \in L^2(\mathbb{R})^2$ be given by $e^{-tL}w_0$. Assume that w_0 fulfils the orthogonality conditions* (1.43)*. Assume $q \in [4,\infty]$ and $r \in [2,\infty]$ satisfy $\frac{2}{q} + \frac{1}{r} = \frac{1}{2}$. Further assume $\tilde{q}' \in [1, \frac{4}{3}]$ and $\tilde{r}' \in [1,2]$ satisfy $\frac{2}{\tilde{q}'} + \frac{1}{\tilde{r}'} = \frac{5}{2}$. Then, the following holds true:*

1. *The homogeneous Strichartz estimates:*

$$\left|\left|e^{-tL}w_0\right|\right|_{L^q_t L^r_x} \leq C\left|\left|w_0\right|\right|_{L^2}. \tag{1.84}$$

2. *The dual homogeneous Strichartz estimates:*

$$\left|\left|\int_{\mathbb{R}} e^{sL}h(s,\cdot)ds\right|\right|_{L^2} \leq C\left|\left|h\right|\right|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x}. \tag{1.85}$$

3. *The inhomogeneous Strichartz estimates:*

$$\left|\left|\int_0^t e^{-(t-s)L}h(s,\cdot)ds\right|\right|_{L_t^q L_x^r} \leq C\,||h||_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}. \tag{1.86}$$

We finally conclude our examination of the linearised operator L by showing a local smoothing estimate. Due to the absence of resonances for $3 < p < 5$, this local smoothing estimate is stronger than the equivalent for the free equation for which 0 is a resonance.

Theorem 1.14.3 (Equivalent of Theorem 12.1.9) *Let* $3 < p < 5$. *Let* $w_0 \in H$ *and assume* $xw_0 \in L^2(\mathbb{R})^2$. *Then:*

$$\begin{aligned}&\left|\left|\frac{t}{\ln(2+t)^3}(1+x^2)^{-1}e^{tL}w_0\right|\right|_{L_t^4(0,\infty)H_x^1} + \left|\left|t(1+x^2)^{-1}e^{tL}w_0\right|\right|_{L_t^\infty(0,\infty)H_x^1}\\ &\leq C\left(||w_0||_{H^1} + ||xw_0||_{L^2}\right).\end{aligned} \tag{1.87}$$

1.15. Outlook: Stability of the nonlinear Schrödinger Equation

At the end of the thesis, we discuss some ideas on how the above results on the spectrum of L and the wave operator T might be used to prove asymptotic stability.

I suspect that the following theorem or a slight variation thereof can be proven. However, this is merely conjecture and we will discuss no more than some basic ideas of proof.

See Buslaev and Sulem [5], for an example of how asymptotic stability in the presence of internal modes can be shown for a class of nonlinear Schrödinger equations, albeit under simpler assumptions than we are afforded. (In particular, they consider nonlinear Schrödinger equations $i\partial_t u = -\Delta u + F(|u|^2)$, whereby the nonlinearity $s \mapsto F(s)$ has a root of order ≥ 4 in $s = 0$.)

Conjecture 1.15.1 *Consider the nonlinear Schrödinger equation*

$$i\partial_t u = -\Delta u - |u|^{p-1}u, \qquad (t,x) \in [0,\infty)\times\mathbb{R}, \tag{1.88}$$

with exponent $p > 3$ *close to* 3. *Assume* $u(0,\cdot) = u_0 \in H^1(\mathbb{R})$ *and let* $w_0 := u_0 - Q$. *Assume:*

1. $\mathrm{Re}\langle w_0, Q\rangle_{L^2} = \mathrm{Re}\langle w_0, xQ\rangle_{L^2} = \mathrm{Im}\langle w_0, Q_x\rangle_{L^2} = \mathrm{Im}\langle w_0, \frac{2}{p-1}Q + xQ_x\rangle_{L^2} = 0.$
2. *For sufficiently small* $\varepsilon_p = \varepsilon > 0$*:* $||w_0||_{H^1}, \left|\left|x^2 w_0\right|\right|_{L^2} < \varepsilon$.

Then, there are functions $\nu, \beta, y : [0,\infty) \to \mathbb{R}$, $\lambda : [0,\infty) \to (0,\infty)$ *and* $w : [0,\infty)\times\mathbb{R} \to \mathbb{C}$, *fulfilling* $\nu(0) = \beta(0) = y(0) = 0$, $\lambda(0) = 1$, *and:*

$$u(t,x) = e^{i\nu + i\lambda^2 t + i\frac{\beta}{2}(x - \frac{\beta}{2}t)}\lambda^{\frac{2}{p-1}}Q(\lambda(x - \beta t + y)) + w(t,x). \tag{1.89}$$

For every $t > 0$, w *admits the dispersive estimate:*

$$||w(t,x)||_{L_x^\infty} \leq C_p(1+t)^{-\frac{1}{2}}\left(||w_0||_{H^1} + \left|\left|x^2 w_0\right|\right|_{L^2}\right). \tag{1.90}$$

Further, for every Strichartz pair $q, r \in [2, \infty]$, $\frac{2}{q} + \frac{1}{r} = \frac{1}{2}$:

$$||w(t,x)||_{L_t^q W_x^{1,r}} \leq C_p \left(||w_0||_{H^1} + \left|\left| x^2 w_0 \right|\right|_{L^2} \right). \tag{1.91}$$

ν, β, y and λ fulfil:

$$\begin{aligned} &\left|\left|(1+t)^2(\nu'(t) + 2\lambda(t)\lambda'(t)t)\right|\right|_{L_t^2}, \left|\left|(1+t)^2\lambda'(t)\right|\right|_{L_t^2}, \\ &\left|\left|(1+t)^2\beta'(t)\right|\right|_{L_t^2}, \left|\left|(1+t)^2(\beta'(t)t - y'(t)\right|\right|_{L_t^2} \\ \leq\ & C_p \left(||w_0||_{H^1} + \left|\left| x^2 w_0 \right|\right|_{L^2} \right). \end{aligned} \tag{1.92}$$

Consequently, for $t \to \infty$, ν, β, y and λ possess limits ν_0, β_0, y_0 and λ_0, fulfilling:

$$|\nu_0|, |\beta_0|, |y_0|, |\lambda_0 - 1| \leq C_p \left(||w_0||_{H^1} + \left|\left| x^2 w_0 \right|\right|_{L^2} \right). \tag{1.93}$$

2. Fundamental System

We start with a purely preliminary chapter. The results of chapter 2 are of technical importance for the rest of the thesis, but are rather basic and unsurprising by themselves.

Using simple fixed-point techniques, we construct several similar fundamental systems for the eigenvalue equation $LW = iEW$, based on slightly different assumptions for $x \to -\infty$.

When characterising the spectrum of L (theorem 1.10.1), all we require from chapter 2 is a simple a priori bound on the dimension of the eigenspaces of L, given by lemma 2.3.21. This lemma is a direct consequence of the decay of the functions making up the fundamental system.

When constructing the distorted Fourier transform resp. the wave operator, we require a bounded solution matrix $W(\xi, x) \in \mathbb{C}^{2\times 2}$ to $LW(\xi, \cdot) = i\,\mathrm{sgn}(\xi)(\xi^2+1)W(\xi, \cdot)$, in other words a Jost solution. This solution matrix will be constructed in chapter 8, based on two additional fundamental systems for $LW = i(\xi^2 + 1)W$, which we define in chapter 2.4. One of these fundamental systems is well-defined for large $\xi >> 0$ and every $x \in \mathbb{R}$, while the other is well-defined for small $x << 0$ and every $\xi \in \mathbb{R}$.

Remark *It is left at the reader's discretion to skip this chapter partially or entirely. In case the chapter is skipped, the relevant results are lemma 2.3.21, which gives the aforementioned a priori bound, as well as theorem 2.4.7 and theorem 2.4.12, which give the two fundamental systems solving* $LW = i(\xi^2 + 1)W$.

Let $W = (U, V) \in H^1(\mathbb{R})^2$ be a solution to $LW = iEW$. As iL is self-adjoint with respect to $\langle \cdot, \cdot \rangle_H$, we only need to consider $E \in \mathbb{R}$. By (1.35):

$$\Delta V - V + Q^{p-1}V = iEU, \tag{2.1}$$

$$-\Delta U + U - pQ^{p-1}U = iEV. \tag{2.2}$$

Substituting $V = i\tilde{V}$ gives:

$$-\Delta\tilde{V} + \tilde{V} + EU = Q^{p-1}\tilde{V}, \tag{2.3}$$

$$-\Delta U + U + E\tilde{V} = pQ^{p-1}U. \tag{2.4}$$

We define $u := U + \tilde{V}$ and $v := U - \tilde{V}$ to reach the system of equations:

$$(-\Delta + 1 + E)u = \frac{p+1}{2}Q^{p-1}u + \frac{p-1}{2}Q^{p-1}v, \tag{2.5}$$

$$(-\Delta + 1 - E)v = \frac{p+1}{2}Q^{p-1}v + \frac{p-1}{2}Q^{p-1}u. \tag{2.6}$$

Lemma 2.0.1 *Consider* (2.5) *and* (2.6) *for some* $E \in \mathbb{R}$. *Let* $x_0 \in \mathbb{R}$. *There exists a unique solution* $(u, v) \in C^2(\mathbb{R})$ *for every quadruple of initial data:*

$$(u(x_0), u'(x_0), v(x_0), v'(x_0)) = y_0 \in \mathbb{C}^4. \tag{2.7}$$

The solution space $M_E := \{(u, v) \in C^2(\mathbb{R}) |$ (2.5), (2.6) *hold true*$\}$ *is 4-dimensional.*

Lemma 2.0.2 *Consider* (2.5) *and* (2.6) *for some* $E = E_0 \in \mathbb{R}$. *If* $(u_0, v_0) \in C^2(\mathbb{R})$ *constitutes a solution, then* $(u_1, v_1) := (v_0, u_0)$ *solves* (2.5) *and* (2.6) *for* $E = -E_0$.

2.1. The Integral Equation

The free Schrödinger equation $-\Delta f + \lambda^2 f = F$, $(f(x_0), f'(x_0)) = y_0 \in \mathbb{C}^2$ for $\lambda \in \mathbb{C}\backslash\{0\}$, $x_0 \in \mathbb{R}$ and $F \in L^1_{\text{loc}}(\mathbb{R})$ has the unique explicit solution via Duhamel's formula:

$$f(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x} + \frac{2}{\lambda} \int_{x_0}^{x} \left(e^{\lambda(x-y)} - e^{\lambda(y-x)} \right) F(y) dy. \tag{2.8}$$

$c_1, c_2 \in \mathbb{C}$ are uniquely determined by the initial data y_0.

(2.8) allows us to write (2.5) and (2.6) as integral equations for $|E| \neq 1$. Lemma 2.0.2 permits us to only consider $E \geq 0$.

Given initial data $(u(x_0), u'(x_0), v(x_0), v'(x_0)) = y_0 \in \mathbb{C}^4$, $x_0 \in \mathbb{R}$, (2.5) and (2.6) are equivalent to:

$$\begin{aligned} u(x) = {} & k_1 e^{\sqrt{1+E}x} + k_2 e^{-\sqrt{1+E}x} \\ & + \frac{1}{2\sqrt{1+E}} \int_{x_0}^{x} e^{\sqrt{1+E}(x-y)} \left(\frac{p+1}{2} Q(y)^{p-1} u(y) + \frac{p-1}{2} Q(y)^{p-1} v(y) \right) dy \\ & - \frac{1}{2\sqrt{1+E}} \int_{x_0}^{x} e^{\sqrt{1+E}(y-x)} \left(\frac{p+1}{2} Q(y)^{p-1} u(y) + \frac{p+1}{2} Q(y)^{p-1} v(y) \right) dy, \end{aligned} \tag{2.9}$$

$$\begin{aligned} v(x) = {} & k_3 e^{\sqrt{1-E}x} + k_4 e^{-\sqrt{1-E}x} \\ & + \frac{1}{2\sqrt{1-E}} \int_{x_0}^{x} e^{\sqrt{1-E}(x-y)} \left(\frac{p+1}{2} Q(y)^{p-1} u(y) + \frac{p-1}{2} Q(y)^{p-1} v(y) \right) dy \\ & - \frac{1}{2\sqrt{1-E}} \int_{x_0}^{x} e^{\sqrt{1-E}(y-x)} \left(\frac{p+1}{2} Q(y)^{p-1} u(y) + \frac{p-1}{2} Q(y)^{p-1} v(y) \right) dy. \end{aligned} \tag{2.10}$$

$k_1, k_2, k_3, k_4 \in \mathbb{C}$ correspond to the initial data via:

$$k_1 e^{\sqrt{1+E}x_0} + k_2 e^{-\sqrt{1+E}x_0} = u(x_0), \tag{2.11}$$

$$\sqrt{1+E} \left(k_1 e^{\sqrt{1+E}x_0} - k_2 e^{-\sqrt{1+E}x_0} \right) = u'(x_0), \tag{2.12}$$

$$k_3 e^{\sqrt{1-E}x_0} + k_4 e^{-\sqrt{1-E}x_0} = v(x_0), \tag{2.13}$$

$$\sqrt{1-E} \left(k_3 e^{\sqrt{1-E}x_0} - k_4 e^{-\sqrt{1-E}x_0} \right) = v'(x_0). \tag{2.14}$$

In case of $|E| = 1$, (2.5) and (2.6) can also be expressed as integral equations. We only consider the case $E = 1$, courtesy of lemma 2.0.2. (2.5) and (2.6) are equivalent to:

$$\begin{aligned} u(x) &= k_1 e^{\sqrt{2}x} + k_2 e^{-\sqrt{2}x} \\ &\quad + \frac{1}{2\sqrt{2}} \int_{x_0}^{x} e^{\sqrt{2}(x-y)} \left(\frac{p+1}{2} Q^{p-1} u + \frac{p-1}{2} Q^{p-1} v \right) dy \\ &\quad - \frac{1}{2\sqrt{2}} \int_{x_0}^{x} e^{\sqrt{2}(y-x)} \left(\frac{p+1}{2} Q^{p-1} u + \frac{p+1}{2} Q^{p-1} v \right) dy, \end{aligned} \tag{2.15}$$

$$v(x) = k_3 + k_4 x + \int_{x_0}^{x} (x-y) \left(\frac{p+1}{2} Q^{p-1} v + \frac{p-1}{2} Q^{p-1} u \right) dy. \tag{2.16}$$

Once again, $k_1, k_2, k_3, k_4 \in \mathbb{C}$ correspond to the initial data via:

$$k_1 e^{\sqrt{2}x_0} + k_2 e^{-\sqrt{2}x_0} = u(x_0), \tag{2.17}$$

$$\sqrt{2} \left(k_1 e^{\sqrt{2}x_0} - k_2 e^{-\sqrt{2}x_0} \right) = u'(x_0), \tag{2.18}$$

$$k_3 + k_4 x_0 = v(x_0), \tag{2.19}$$

$$k_4 = v'(x_0). \tag{2.20}$$

2.2. Fundamental System

We are primarily interested in the decay of solutions of (2.5) and (2.6) as $|x| \to \infty$.

Therefore, we construct a fundamental system (w_1, w_2, w_3, w_4) of (2.5) and (2.6), such that every solution w_k exhibits different asymptotic behaviour as $x \to -\infty$.

Consider again the integral equations (2.9) and (2.10). Heuristically, by substituting $x_0 = -\infty$, we would expect solutions of the form:

$$u(x) \approx c_1 e^{\sqrt{1+E}x} + c_2 e^{-\sqrt{1+E}x}, \quad v(x) \approx c_3 e^{\sqrt{1-E}x} + c_4 e^{-\sqrt{1-E}x}. \tag{2.21}$$

We will construct a fundamental system with precisely this asymptotic behaviour. However, some care must be taken to ensure the respective integrals converge.

We need some basic fixed-point results.

Lemma 2.2.1 (Banach Fixed-Point Theorem) *Let $(X, ||\cdot||_X)$ be a Banach space. Consider any contraction $B : X \to X$ with Lipschitz constant $K < 1$:*

$$\forall f, g \in X : ||B(f) - B(g)||_X \leq K \, ||f - g||_X \, . \tag{2.22}$$

Then, there exists a unique fixed-point $h \in X$:

$$B(h) = h. \tag{2.23}$$

Proof. See [1]. □

Lemma 2.2.2 *Let $(X, ||\cdot||_X)$ be a Banach space and let $B : X \to X$ be a linear contraction with Lipschitz constant $K < 1$. Then, given $f \in X$, the equation:*

$$g = f + Bg, \tag{2.24}$$

admits a unique solution $g \in X$. This solution fulfils:

$$||g||_X \leq \frac{1}{1-K} ||f||_X . \tag{2.25}$$

Proof. Let $G : X \to X$ be given by $G(g) := f + Bg$. By definition, G constitutes a contraction with Lipschitz constant K. Hence, $G(g) = g$ admits a unique solution $g \in X$. It follows:

$$||g||_X - ||f||_X \leq ||g - f||_X = ||G(g) - G(0)||_X \leq K ||g||_X . \tag{2.26}$$

That concludes the proof. □

Our strategy of constructing fundamental systems is based on lemma 2.2.2. As we are interested in the decay of solutions, it is prudent to have this decay reflected in the Banach spaces when applying lemma 2.2.2. Consider the following weighted spaces.

Definition 2.2.3 *Consider an interval $\mathcal{I} \subseteq \mathbb{R}$ and $\nu \in C(\mathcal{I})$ with $\nu(x) \neq 0$ for every $x \in \mathcal{I}$. We define:*

$$L^\infty_\nu(\mathcal{I}) := \{f \in L^\infty_{\text{loc}}(\mathcal{I}) | \nu^{-1} f \in L^\infty(\mathcal{I})\}, \tag{2.27}$$

$$||f||_{L^\infty_\nu(\mathcal{I})} := \left|\left|\nu^{-1} f\right|\right|_{L^\infty(\mathcal{I})} . \tag{2.28}$$

The contractions will be based on the integral equations (2.9), (2.10) and (2.15), (2.16) respectively.

As the dynamics are different for the essential spectrum $|E| > 1$, the spectral boundary $|E| = 1$ and the spectral gap $|E| < 1$, we consider these cases separately.

2.3. Prioritizing Decay

2.3.1. Essential Spectrum: $|E| > 1$

We construct a fundamental system for $E > 1$. By lemma 2.0.2 that also yields a fundamental system for $E < -1$.

We define the following class of operators, slightly generalising (2.9) and (2.10). This definition will only be used in chapter 2.3.1.

Definition 2.3.1 *Let $E > 1$ and $(x_1, x_2, x_3) \in [-\infty, \infty]^3$. Given a suitable function $w = (u, v) : \mathbb{R} \to \mathbb{C}^2$, we define $B_{E,x_1,x_2,x_3} w : \mathbb{R} \to \mathbb{C}^2$ by $B_{E,x_1,x_2,x_3} w = (f, g)$:*

$$\begin{aligned} f(x) = & \frac{1}{2\sqrt{1+E}} \int_{x_1}^x e^{\sqrt{1+E}(x-y)} Q^{p-1} \left(\frac{p+1}{2} u + \frac{p-1}{2} v \right) dy \\ & - \frac{1}{2\sqrt{1+E}} \int_{x_2}^x e^{\sqrt{1+E}(y-x)} Q^{p-1} \left(\frac{p+1}{2} u + \frac{p-1}{2} v \right) dy, \end{aligned} \tag{2.29}$$

$$g(x) = \int_{x_3}^x \frac{\sin(\sqrt{E-1}(x-y))}{\sqrt{E-1}} Q^{p-1} \left(\frac{p+1}{2} u + \frac{p-1}{2} v \right) dy. \tag{2.30}$$

Furthermore, given $\omega \in (-\infty, \infty]$*, we define* $B_{k,\omega} = B_{E,k,\omega}$*,* $k \in \{1,2,3,4\}$ *by:*

$$B_{1,\omega} := B_{E,-\infty,-\infty,-\infty}, \tag{2.31}$$

$$B_{2,\omega} := B_{E,\omega,-\infty,\omega}, \tag{2.32}$$

$$B_{3,\omega} = B_{4,\omega} := B_{E,\omega,-\infty,-\infty}. \tag{2.33}$$

We define the following Banach spaces.

Definition 2.3.2 *Let* $E > 1$ *and* $\omega \in (-\infty, \infty]$*. We define functions* $f_k = f_{E,k} : \mathbb{R} \to \mathbb{R}$ *and Banach spaces* $X_k = X_{E,k} = X_{E,k,\omega}$ *for* $k \in \{1,2,3,4\}$ *by:*

$$f_1(x) := e^{\sqrt{1+E}x}, \tag{2.34}$$

$$f_2(x) := e^{-\sqrt{1+E}x}, \tag{2.35}$$

$$f_3(x) := e^{i\sqrt{E-1}x}, \tag{2.36}$$

$$f_4(x) := e^{-i\sqrt{E-1}x}, \tag{2.37}$$

as well as:

$$X_k := L^\infty_{f_k}(-\infty, \omega)^2. \tag{2.38}$$

In particular, that implies $X_3 = X_4 = L^\infty(-\infty, \omega)^2$*.*

Lemma 2.3.3 *Let* $E > 1$ *and* $\omega \in (-\infty, \infty]$*. Then, for* $x < \omega$*:*

$$\forall w \in X_1 : \ |B_{1,\omega} w(x)| \leq C \frac{1}{\sqrt{E}} \frac{e^{\sqrt{1+E}x}}{1 + e^{-2x}} \, ||w||_{X_1}, \tag{2.39}$$

$$\forall w \in X_2 : \ |B_{2,\omega} w(x)| \leq C \frac{1}{\sqrt{E}} \frac{e^{-\sqrt{1+E}x}}{1 + e^{-\frac{1}{2}x}} \, ||w||_{X_2}, \tag{2.40}$$

$$\forall w \in X_3 : \ |B_{3,\omega} w(x)| \leq C \min\left(\frac{1+|x|}{\sqrt{E}}, \frac{1}{\sqrt{E-1}} \right) \frac{1}{1+e^{-x}} \, ||w||_{X_3}. \tag{2.41}$$

Proof. Note $Q(x)^{p-1} \leq Ce^{-(p-1)|x|} \leq e^{-2|x|}$ and

$$\left| \xi^{-1} \sin(\xi x) \right| \leq 2(1+\xi)^{-1}(1+|x|) \leq 4(1+\xi)^{-1} e^{\frac{1}{2}|x|} \tag{2.42}$$

for $\xi > 0$. Let $w \in X_1$ with $||w||_{X_1} = 1$. It follows for $x < \omega$:

$$\begin{aligned}
\sqrt{E} \, |B_{1,\omega} w(x)| &\leq C \int_{-\infty}^{x} e^{\sqrt{1+E}x} e^{-2|y|} dy + C \int_{-\infty}^{x} e^{\sqrt{1+E}(2y-x)} e^{-2|y|} dy \\
&\quad + C \int_{-\infty}^{x} e^{\frac{1}{2}(x-y)} e^{\sqrt{1+E}y} e^{-2|y|} dy \\
&\leq 2Ce^{\sqrt{1+E}x} \int_{-\infty}^{x} e^{-2|y|} dy \\
&\leq C^2 \frac{e^{\sqrt{1+E}x}}{1+e^{-2x}}.
\end{aligned} \tag{2.43}$$

Let $w \in X_2$ with $||w||_{X_2} = 1$. It follows for $x < \omega$:

$$
\begin{aligned}
\sqrt{E}\,|B_{2,\omega}w(x)| &\leq C\int_x^\omega e^{\sqrt{1+E}(x-2y)}e^{-2|y|}dy + C\int_{-\infty}^x e^{-\sqrt{1+E}x}e^{-2|y|}dy \\
&\quad + C\int_x^\omega e^{\frac{1}{2}(y-x)}e^{-\sqrt{1+E}y}e^{-2|y|}dy \\
&\leq Ce^{\sqrt{1+E}x}\int_x^\omega e^{-2\sqrt{1+E}y-|y|}e^{-|y|}dy + C^2\frac{e^{-\sqrt{1+E}x}}{1+e^{-2x}} \\
&\quad + Ce^{-\frac{1}{2}x}\int_x^\omega e^{-(\sqrt{1+E}-\frac{1}{2})y-\frac{1}{2}|y|}e^{-\frac{3}{2}|y|}dy \\
&\leq Ce^{\sqrt{1+E}x}e^{-2\sqrt{1+E}x-|x|}\int_x^\omega e^{-|y|}dy + C^2\frac{e^{-\sqrt{1+E}x}}{1+e^{-2x}} \\
&\quad + Ce^{-\frac{1}{2}x}e^{-(\sqrt{1+E}-\frac{1}{2})x-\frac{1}{2}|x|}\int_x^\omega e^{-\frac{3}{2}|y|}dy \\
&\leq C^3\frac{e^{-\sqrt{1+E}x}}{1+e^{-\frac{1}{2}x}}. \qquad (2.44)
\end{aligned}
$$

Let $w \in X_3$ with $||w||_{X_3} = 1$. It follows for $x < \omega$:

$$
\begin{aligned}
\sqrt{E}\,|B_{3,\omega}w(x)| &\leq C\int_x^\omega e^{\sqrt{1+E}(x-y)}e^{-2|y|}dy + C\int_{-\infty}^x e^{\sqrt{1+E}(y-x)}e^{-2|y|}dy \\
&\quad + C\int_{-\infty}^x (1+x-y)e^{-2|y|}dy \\
&\leq C^2\frac{1+|x|}{1+e^{-x}}. \qquad (2.45)
\end{aligned}
$$

On the other hand, due to $|\xi^{-1}\sin(\xi x)| \leq \xi^{-1}$:

$$
\begin{aligned}
\sqrt{E}\,|B_{3,\omega}w(x)| &\leq C\int_x^\omega e^{\sqrt{1+E}(x-y)}e^{-2|y|}dy + C\int_{-\infty}^x e^{\sqrt{1+E}(y-x)}e^{-2|y|}dy \\
&\quad + C\frac{\sqrt{E}}{\sqrt{E-1}}\int_{-\infty}^x e^{-2|y|}dy \\
&\leq C^2\frac{\sqrt{E}}{\sqrt{E-1}}\frac{1}{1+e^{-x}}. \qquad (2.46)
\end{aligned}
$$

That concludes the proof. $\square$

Corollary 2.3.4 *Let $E > 1$ and $\omega \in (-\infty, \infty]$. Then:*

$$\forall w \in X_1: \ ||B_{1,\omega}w||_{X_1} \leq \frac{C}{\sqrt{E}}e^{\frac{1}{2}\omega}\,||w||_{X_1}\,, \qquad (2.47)$$

$$\forall w \in X_2: \ ||B_{2,\omega}w||_{X_2} \leq \frac{C}{\sqrt{E}}e^{\frac{1}{2}\omega}\,||w||_{X_2}\,, \qquad (2.48)$$

$$\forall w \in X_3: \ ||B_{3,\omega}w||_{X_3} \leq \frac{C}{\sqrt{E}}e^{\frac{1}{2}\omega}\,||w||_{X_3}\,. \qquad (2.49)$$

Definition 2.3.5 *Let $E > 1$. Let $C > 0$ be chosen large enough to satisfy the conclusion of corollary 2.3.4. We define $\omega < 0$ as an arbitrarily chosen number fulfilling $Ce^{\frac{1}{2}\omega} < \frac{1}{2}$.*
We define $h_k : \mathbb{R} \to \mathbb{R}^2$, $k \in \{1,2,3,4\}$ by:

$$h_1 = (f_1, 0), h_2 = (f_2, 0), h_3 = (0, f_3), h_4 = (0, f_4). \tag{2.50}$$

Definition 2.3.5 is only used in chapter 2.3.1.

Lemma 2.3.6 *Let $E > 1$ and consider (2.5), (2.6). Let $\omega < 0$ be given by definition 2.3.5. Then, there exists a unique fundamental system (w_1, w_2, w_3, w_4), such that $w_k = (u_k, v_k)$, $k \in \{1,2,3,4\}$ solves:*

$$w_k = h_k + B_{k,\omega} w_k \tag{2.51}$$

on $(-\infty, \omega)$. For every $x < \omega$:

$$\left|u_1(x) - e^{\sqrt{1+E}x}\right| + |v_1(x)| \leq \frac{C}{\sqrt{E}} e^{\frac{1}{2}x} e^{\sqrt{1+E}x}, \tag{2.52}$$

$$\left|u_2(x) - e^{-\sqrt{1+E}x}\right| + |v_2(x)| \leq \frac{C}{\sqrt{E}} e^{\frac{1}{2}x} e^{-\sqrt{1+E}x}, \tag{2.53}$$

$$|u_3(x)| + \left|v_3(x) - e^{i\sqrt{E-1}x}\right| \leq \frac{C}{\sqrt{E}} e^{\frac{1}{2}x}, \tag{2.54}$$

$$|u_4(x)| + \left|v_4(x) - e^{-i\sqrt{E-1}x}\right| \leq \frac{C}{\sqrt{E}} e^{\frac{1}{2}x}. \tag{2.55}$$

Proof. For $x < \omega$, we define w_k by (2.51). Lemma 2.2.2 and corollary 2.3.4 imply:

$$||w_k||_{X_k} \leq \frac{1}{1 - \frac{1}{2}} ||h_k||_{X_k} = 2. \tag{2.56}$$

Lemma 2.3.3 implies the estimates (2.52) - (2.55), due to $x < \omega < 0$. □

2.3.2. Spectral Gap: $|E| < 1$

We construct a fundamental system for $0 \leq E < 1$. By lemma 2.0.2 that also yields a fundamental system for $0 \geq E > -1$. We proceed analogously to chapter 2.3.1.

Definition 2.3.7 *Let $0 \leq E < 1$ and $(x_1, x_2, x_3, x_4) \in [-\infty, \infty]^4$. Given a suitable function $w = (u, v) : \mathbb{R} \to \mathbb{C}^2$, we define $B_{E,x_1,x_2,x_3,x_4} w : \mathbb{R} \to \mathbb{C}^2$ by $B_{E,x_1,x_2,x_3,x_4} w = (f, g)$:*

$$\begin{aligned} f(x) = {} & \frac{1}{2\sqrt{1+E}} \int_{x_1}^{x} e^{\sqrt{1+E}(x-y)} Q^{p-1} \left(\frac{p+1}{2} u + \frac{p-1}{2} v\right) dy \\ & - \frac{1}{2\sqrt{1+E}} \int_{x_2}^{x} e^{\sqrt{1+E}(y-x)} Q^{p-1} \left(\frac{p+1}{2} u + \frac{p-1}{2} v\right) dy, \end{aligned} \tag{2.57}$$

$$\begin{aligned} g(x) = {} & \frac{1}{2\sqrt{1-E}} \int_{x_3}^{x} e^{\sqrt{1-E}(x-y)} Q^{p-1} \left(\frac{p+1}{2} u + \frac{p-1}{2} v\right) dy \\ & - \frac{1}{2\sqrt{1-E}} \int_{x_4}^{x} e^{\sqrt{1-E}(y-x)} Q^{p-1} \left(\frac{p+1}{2} u + \frac{p-1}{2} v\right) dy. \end{aligned} \tag{2.58}$$

Furthermore, given $\omega \in (-\infty, \infty]$, *we define* $B_{k,\omega} = B_{E,k,\omega}$, $k \in \{1, 2, 3, 4\}$ *for* $0 \leq E \leq \frac{3}{4}$ *by:*

$$B_{1,\omega} := B_{E,-\infty,-\infty,-\infty,-\infty}, \tag{2.59}$$
$$B_{2,\omega} := B_{E,\omega,-\infty,\omega,-\infty}, \tag{2.60}$$
$$B_{3,\omega} := B_{E,\omega,-\infty,-\infty,-\infty}, \tag{2.61}$$
$$B_{4,\omega} := B_{E,\omega,-\infty,\omega,-\infty}. \tag{2.62}$$

For $\frac{3}{4} < E < 1$, *we define:*

$$B_{1,\omega} := B_{E,-\infty,-\infty,-\infty,-\infty}, \tag{2.63}$$
$$B_{2,\omega} := B_{E,\omega,-\infty,\omega,\omega}, \tag{2.64}$$
$$B_{3,\omega} := B_{E,\omega,-\infty,-\infty,-\infty}, \tag{2.65}$$
$$B_{4,\omega} := B_{E,\omega,-\infty,-\infty,-\infty}. \tag{2.66}$$

We define Banach spaces:

Definition 2.3.8 *Let* $0 \leq E < 1$ *and* $\omega \in (-\infty, \infty]$. *We define functions* $f_k = f_{E,k} : \mathbb{R} \to \mathbb{R}$ *and Banach spaces* $X_k = X_{E,k} = X_{E,k,\omega}$ *for* $k \in \{1, 2, 3, 4\}$ *by:*

$$f_1(x) := e^{\sqrt{1+E}x}, \tag{2.67}$$
$$f_2(x) := e^{-\sqrt{1+E}x}, \tag{2.68}$$
$$f_3(x) := e^{\sqrt{1-E}x}, \tag{2.69}$$
$$f_4(x) := e^{-\sqrt{1-E}x}, \tag{2.70}$$

as well as:

$$X_k := L^\infty_{f_k}(-\infty, \omega)^2. \tag{2.71}$$

Lemma 2.3.9 *Let* $0 \leq E \leq \frac{3}{4}$ *and* $\omega < 0$. *Then, for every* $x < \omega$:

$$\forall w \in X_1 : \ |B_{1,\omega} w(x)| \leq C e^{\sqrt{1+E}x} e^{2x} \left|\left|w\right|\right|_{X_1}, \tag{2.72}$$
$$\forall w \in X_2 : \ |B_{2,\omega} w(x)| \leq C e^{-\sqrt{1+E}x} e^{x} \left|\left|w\right|\right|_{X_2}, \tag{2.73}$$
$$\forall w \in X_3 : \ |B_{3,\omega} w(x)| \leq C e^{\sqrt{1-E}x} e^{x} \left|\left|w\right|\right|_{X_3}, \tag{2.74}$$
$$\forall w \in X_4 : \ |B_{4,\omega} w(x)| \leq C e^{-\sqrt{1-E}x} e^{x} \left|\left|w\right|\right|_{X_4}. \tag{2.75}$$

Proof. Let $w \in X_1$ with $||w||_{X_1} = 1$. It follows for $x < \omega$:

$$|B_{1,\omega} w(x)| \leq C e^{\sqrt{1+E}x} \int_{-\infty}^{x} e^{2y} dy \leq C^2 e^{\sqrt{1+E}x} e^{2x}. \tag{2.76}$$

Let $w \in X_2$ with $||w||_{X_2} = 1$. It follows for $x < \omega < 0$:

$$\begin{aligned}
|B_{2,\omega} w(x)| &\leq C \int_{-\infty}^{x} e^{\sqrt{1-E}(y-x)} e^{-\sqrt{1+E}y} e^{2y} dy \\
&\quad + C \int_{x}^{\omega} e^{\sqrt{1-E}(x-y)} e^{-\sqrt{1+E}y} e^{2y} dy \\
&\leq C^2 e^{-\sqrt{1+E}x} e^{2x} + C^2 (\omega - x) e^{\sqrt{1-E}x} \\
&\leq 2C^2 e^{-\sqrt{1+E}x} e^{x}. \qquad (2.77)
\end{aligned}$$

(2.74) and (2.75) follow completely analogously. ☐

Lemma 2.3.10 *Let $\frac{3}{4} < E < 1$ and $\omega < 0$. Then, for every $x < \omega$:*

$$\forall w \in X_1 : \; |B_{1,\omega} w(x)| \leq C e^{\sqrt{1+E}x} e^{2x} \, ||w||_{X_1}, \qquad (2.78)$$

$$\forall w \in X_2 : \; |B_{2,\omega} w(x)| \leq C e^{-\sqrt{1+E}x} e^{\frac{1}{2}x} \, ||w||_{X_2}, \qquad (2.79)$$

$$\forall w \in X_3 : \; |B_{3,\omega} w(x)| \leq C e^{\sqrt{1-E}x} e^{\frac{1}{2}x} \, ||w||_{X_3}, \qquad (2.80)$$

$$\forall w \in X_4 : \; |B_{4,\omega} w(x)| \leq C e^{-\sqrt{1-E}x} e^{x} \, ||w||_{X_4}. \qquad (2.81)$$

Proof. Note $(1-E)^{-\frac{1}{2}} \left| \sinh(\sqrt{1-E}(x-y)) \right| \leq C \, |x-y| \, e^{\sqrt{1-E}|x-y|}$.

Let $w \in X_1$ with $||w||_{X_1} = 1$. It follows for $x < \omega$:

$$|B_{1,\omega} w(x)| \leq C e^{\sqrt{1+E}x} \int_{-\infty}^{x} e^{2y} dy \leq C^2 e^{\sqrt{1+E}x} e^{2x}. \qquad (2.82)$$

Let $w \in X_2$ with $||w||_{X_2} = 1$. It follows for $x < \omega < 0$:

$$\begin{aligned}
|B_{2,\omega} w(x)| &\leq C \int_{-\infty}^{x} e^{\sqrt{1+E}(y-x)} e^{-\sqrt{1+E}y} e^{2y} dy \\
&\quad + C \int_{x}^{\omega} (y-x) e^{\sqrt{1-E}(y-x)} e^{-\sqrt{1+E}y} e^{2y} dy \\
&\leq C^2 e^{-\sqrt{1+E}x} e^{2x} + C^2 \, |x| \, e^{-\sqrt{1-E}x} \\
&\leq C^3 e^{-\sqrt{1+E}x} e^{\frac{1}{2}x}. \qquad (2.83)
\end{aligned}$$

Let $w \in X_3$ with $||w||_{X_3} = 1$. It follows for $x < \omega < 0$:

$$\begin{aligned}
|B_{3,\omega} w(x)| &\leq C \int_{-\infty}^{x} (x-y) e^{\sqrt{1-E}(x-y)} e^{\sqrt{1-E}y} e^{2y} dy \\
&\quad + C \int_{x}^{\omega} e^{\sqrt{1+E}(x-y)} e^{\sqrt{1-E}y} e^{2y} dy \\
&\leq C^2 (1 + |x|) e^{\sqrt{1-E}x} e^{2x} + C^2 e^{\sqrt{1+E}x} \\
&\leq C^3 e^{\sqrt{1-E}x} e^{\frac{1}{2}x}. \qquad (2.84)
\end{aligned}$$

Let $w \in X_4$ with $||w||_{X_4} = 1$. It follows for $x < \omega < 0$:

$$\begin{aligned}|B_{4,\omega}w(x)| &\leq C\int_{-\infty}^{x}(x-y)e^{\sqrt{1-E}(x-y)}e^{-\sqrt{1-E}y}e^{2y}dy \\ &\quad + C\int_{x}^{\omega}e^{\sqrt{1+E}(x-y)}e^{-\sqrt{1-E}y}e^{2y}dy \\ &\leq C^2(1+|x|)e^{-\sqrt{1-E}x}e^{2x} + Ce^{\sqrt{1+E}x} \\ &\leq C^3e^{-\sqrt{1-E}x}e^{x}. \end{aligned} \tag{2.85}$$

That concludes the proof. $\square$

Definition 2.3.11 *Let $0 \leq E < 1$. We define $\omega < 0$ as an arbitrarily chosen number fulfilling $Ce^{\frac{1}{2}\omega} < \frac{1}{2}$. Hereby, $C > 0$ is chosen large enough to satisfy the conclusions of lemma 2.3.9 and lemma 2.3.10.*

We define $h_k : \mathbb{R} \to \mathbb{R}^2$, $k \in \{1,2,3,4\}$ by:

$$h_1 = (f_1, 0), h_2 = (f_2, 0), h_3 = (0, f_3), h_4 = (0, f_4). \tag{2.86}$$

Definition 2.3.11 is only used in chapter 2.3.2.

Corollary 2.3.12 *Let $0 \leq E < 1$. Then, for $k \in \{1,2,3,4\}$:*

$$\forall w \in X_k : ||B_k w||_{X_k} \leq \frac{1}{2}||w||_{X_k}. \tag{2.87}$$

Proof. Follows from lemma 2.3.9 and lemma 2.3.10. $\square$

Lemma 2.3.13 *Let $0 \leq E < 1$ and consider (2.5), (2.6). Let $\omega < 0$ be given by definition 2.3.11. Then, there exists a unique fundamental system (w_1, w_2, w_3, w_4), such that $w_k = (u_k, v_k)$, $k \in \{1,2,3,4\}$ solves:*

$$w_k = h_k + B_{k,\omega}w_k \tag{2.88}$$

on $(-\infty, \omega)$. For every $x < \omega$:

$$\left|u_1(x) - e^{\sqrt{1+E}x}\right| + |v_1(x)| \leq Ce^{\frac{1}{2}x}e^{\sqrt{1+E}x}, \tag{2.89}$$

$$\left|u_2(x) - e^{-\sqrt{1+E}x}\right| + |v_2(x)| \leq Ce^{\frac{1}{2}x}e^{-\sqrt{1+E}x}, \tag{2.90}$$

$$|u_3(x)| + \left|v_3(x) - e^{\sqrt{1-E}x}\right| \leq Ce^{\frac{1}{2}x}e^{\sqrt{1-E}x}, \tag{2.91}$$

$$|u_4(x)| + \left|v_4(x) - e^{-\sqrt{1-E}x}\right| \leq Ce^{\frac{1}{2}x}e^{-\sqrt{1-E}x}. \tag{2.92}$$

Proof. For $x < \omega$, we define w_k by (2.88). Lemma 2.2.2 and corollary 2.3.12 imply:

$$||w_k||_{X_k} \leq \frac{1}{1-\frac{1}{2}}||h_k||_{X_k} = 2. \tag{2.93}$$

Corollary 2.3.12 implies the estimates (2.89) - (2.92), due to $x < \omega < 0$. $\square$

2.3.3. Spectral Boundary: $|E| = 1$

Last but not least, we construct a fundamental system for $E = 1$. By lemma 2.0.2 that also yields a fundamental system for $E = -1$.

Definition 2.3.14 *Let $(x_1, x_2, x_3) \in [-\infty, \infty]^3$. Given a suitable function $w = (u, v) : \mathbb{R} \to \mathbb{C}^2$, we define $B_{1,x_1,x_2,x_3} w : \mathbb{R} \to \mathbb{C}^2$ by $B_{1,x_1,x_2,x_3} w = (f, g)$:*

$$\begin{aligned} f(x) = {} & \frac{1}{2\sqrt{2}} \int_{x_1}^{x} e^{\sqrt{2}(x-y)} Q^{p-1} \left(\frac{p+1}{2} u + \frac{p-1}{2} v \right) dy \\ & - \frac{1}{2\sqrt{2}} \int_{x_2}^{x} e^{\sqrt{2}(y-x)} Q^{p-1} \left(\frac{p+1}{2} u + \frac{p-1}{2} v \right) dy, \end{aligned} \tag{2.94}$$

$$g(x) = \int_{x_3}^{x} (x-y) Q^{p-1} \left(\frac{p+1}{2} u + \frac{p-1}{2} v \right) dy. \tag{2.95}$$

Furthermore, given $\omega \in (-\infty, \infty]$, we define $B_{k,\omega} = B_{E,k,\omega}$, $k \in \{1, 2, 3, 4\}$ by:

$$B_{1,\omega} := B_{E,-\infty,-\infty,-\infty}, \tag{2.96}$$

$$B_{2,\omega} := B_{E,\omega,-\infty,\omega}, \tag{2.97}$$

$$B_{3,\omega} = B_{4,\omega} := B_{E,\omega,-\infty,-\infty}. \tag{2.98}$$

Definition 2.3.15 *Let $\omega < 0$. We define functions $f_k : \mathbb{R} \to \mathbb{R}$ and Banach spaces $X_k = X_{k,\omega}$ for $k \in \{1, 2, 3, 4\}$ by:*

$$f_1(x) := e^{\sqrt{2}x}, \tag{2.99}$$

$$f_2(x) := e^{-\sqrt{2}x}, \tag{2.100}$$

$$f_3(x) := 1, \tag{2.101}$$

$$f_4(x) := x, \tag{2.102}$$

as well as:

$$X_1 := L^\infty_{f_1}(-\infty, \omega)^2, \tag{2.103}$$

$$X_2 := L^\infty_{f_2}(-\infty, \omega)^2, \tag{2.104}$$

$$X_3 = X_4 := L^\infty_{1+|x|}(-\infty, \omega)^2. \tag{2.105}$$

Lemma 2.3.16 *Let $\omega < 0$. Then, for every $x < \omega$:*

$$\forall w \in X_1 : \ |B_{1,\omega} w(x)| \le C e^{\sqrt{2}x} e^{\frac{1}{2}x} \left|\left| w \right|\right|_{X_1}, \tag{2.106}$$

$$\forall w \in X_2 : \ |B_{2,\omega} w(x)| \le C e^{-\sqrt{2}x} e^{\frac{1}{2}x} \left|\left| w \right|\right|_{X_2}, \tag{2.107}$$

$$\forall w \in X_3 : \ |B_{3,\omega} w(x)| \le C e^{\frac{1}{2}x} \left|\left| w \right|\right|_{X_3}. \tag{2.108}$$

Proof. Follows analogously to lemma 2.3.9. □

Definition 2.3.17 *Let $C > 0$ be chosen large enough to satisfy the conclusions of lemma 2.3.16. We define $\omega < 0$ as an arbitrarily chosen number fulfilling $Ce^{\frac{1}{2}\omega} < \frac{1}{2}$.*
We define $h_k : \mathbb{R} \to \mathbb{R}^2$, $k \in \{1, 2, 3, 4\}$ by:

$$h_1 = (f_1, 0), h_2 = (f_2, 0), h_3 = (0, f_3), h_4 = (0, f_4). \tag{2.109}$$

Definition 2.3.17 is only used in chapter 2.3.3.

Corollary 2.3.18 *For $k \in \{1, 2, 3, 4\}$:*

$$\forall w \in X_k : ||B_{k,\omega} w||_{X_k} \leq \frac{1}{2} ||w||_{X_k} . \tag{2.110}$$

Lemma 2.3.19 *Let $E = 1$ and consider (2.5), (2.6). Let $\omega < 0$ be given by definition 2.3.17. Then, there exists a unique fundamental system (w_1, w_2, w_3, w_4), such that $w_k = (u_k, v_k)$, $k \in \{1, 2, 3, 4\}$ solves:*

$$w_k = h_k + B_{k,\omega} w_k \tag{2.111}$$

on $(-\infty, \omega)$. For every $x < \omega$:

$$\left|u_1(x) - e^{\sqrt{2}x}\right| + |v_1(x)| \leq Ce^{\frac{1}{2}x} e^{\sqrt{2}x}, \tag{2.112}$$

$$\left|u_2(x) - e^{-\sqrt{2}x}\right| + |v_2(x)| \leq Ce^{\frac{1}{2}x} e^{-\sqrt{2}x}, \tag{2.113}$$

$$|u_3(x)| + |v_3(x) - 1| \leq Ce^{\frac{1}{2}x}, \tag{2.114}$$

$$|u_4(x)| + |v_4(x) - x| \leq Ce^{\frac{1}{2}x}. \tag{2.115}$$

Proof. Follows analogously to lemma 2.3.6 and lemma 2.3.13. □

2.3.4. Dimension of the Solution Space

Definition 2.3.20 *Let $E \in \mathbb{R}$. Consider the following subspaces $M_{E,-}, M_{E,+}$ of the solution space (2.5) and (2.6).*

$$M_{E,-} := \{(u, v) \in C^2(\mathbb{R}) | (2.5),\ (2.6) \textit{ hold}, \lim_{x\to-\infty} (u(x), v(x)) = (0, 0)\}, \tag{2.116}$$

$$M_{E,+} := \{(u, v) \in C^2(\mathbb{R}) | (2.5),\ (2.6) \textit{ hold}, \lim_{x\to\infty} (u(x), v(x)) = (0, 0)\}. \tag{2.117}$$

Clearly, $(u, v) \in C^2(\mathbb{R})$ constitutes an H^1-eigenfunction of (2.5) and (2.6) with corresponding eigenvalue $E \in \mathbb{R}$, if and only if $(u, v) \in M_{E,+} \cap M_{E,-}$.

Lemma 2.3.21 *The dimension of $M_{E,+}$ and $M_{E,-}$ is given by:*

$$\dim M_{E,+} = \dim M_{E,-} = \begin{cases} 2, & \textit{if } |E| < 1, \\ 1, & \textit{if } |E| \geq 1. \end{cases}$$

Proof. Lemma 2.0.2 allows us to only consider $E \geq 0$.
For every $w = (u, v) \in M_{E,-}$, we find unique $(c_1, c_2, c_3, c_4) \in \mathbb{C}^4$ with $w = \sum_{k=1}^{4} c_k w_k$.

In case of $E > 1$ and $E = 1$, lemma 2.3.6 and lemma 2.3.19 imply:

$$w \in M_{E,-} \Leftrightarrow c_2 = c_3 = c_4 = 0. \tag{2.118}$$

In case of $E < 1$, lemma 2.3.13 implies:

$$w \in M_{E,-} \Leftrightarrow c_2 = c_4 = 0. \tag{2.119}$$

Symmetry ensures $\dim M_{E,+} = \dim M_{E,-}$. That concludes the proof. □

2.4. Prioritizing Continuity

After proving theorem 1.10.1, we construct a distorted Fourier transform using the bounded solutions to $Lw = iEw$ as a kernel.

If we disregard eigenfunctions, only the essential spectrum $|E| \geq 1$ needs to be considered when it comes to bounded solutions.

By lemma 2.0.2, it suffices to consider $E \geq 1$. We construct a fundamental system exhibiting better continuity then the one defined in chapter 2.3.1.

Consider $\xi = \sqrt{E-1} \in [0,\infty)$ as a new coordinate replacing $E \in [1,\infty)$. Otherwise, we proceed as we did in chapter 2.3.1.

Definition 2.4.1 *Let $\xi \geq 0$ and $(x_1,x_2,x_3) \in [-\infty,\infty]^3$. Given a suitable function $w = (u,v) : \mathbb{R} \to \mathbb{C}^2$, we define $B_{\xi,x_1,x_2,x_3}w : \mathbb{R} \to \mathbb{C}^2$ by $B_{\xi,x_1,x_2,x_3}w = (f,g)$:*

$$\begin{aligned} f(x) = {} & \frac{1}{2\sqrt{2+\xi^2}} \int_{x_1}^{x} e^{\sqrt{2+\xi^2}(x-y)} Q^{p-1}\left(\frac{p+1}{2}u + \frac{p-1}{2}v\right) dy \\ & - \frac{1}{2\sqrt{2+\xi^2}} \int_{x_2}^{x} e^{\sqrt{2+\xi^2}(y-x)} Q^{p-1}\left(\frac{p+1}{2}u + \frac{p-1}{2}v\right) dy, \end{aligned} \tag{2.120}$$

$$g(x) = \int_{x_3}^{x} \frac{\sin(\xi(x-y))}{\xi} Q^{p-1}\left(\frac{p+1}{2}u + \frac{p-1}{2}v\right) dy. \tag{2.121}$$

We understand $\frac{\sin(\xi(x-y))}{\xi}$ to be a holomorphic function, extended to $\xi = 0$ via its limit $\lim_{\xi\to 0} \frac{\sin(\xi(x-y))}{\xi} = x - y$.

Given $\omega \in (-\infty,\infty]$, we define $B_{\xi,k} = B_{\xi,k,\omega}$, $k \in \{1,2,3,4\}$ by:

$$B_{\xi,1} := B_{\xi,-\infty,-\infty,-\infty}, \tag{2.122}$$
$$B_{\xi,2} := B_{\xi,\omega,-\infty,\omega}, \tag{2.123}$$
$$B_{\xi,3} = B_{\xi,4} := B_{\xi,\omega,-\infty,-\infty}. \tag{2.124}$$

Definition 2.4.2 *We define functions $f_k : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ for $k \in \{1,2,3,4\}$ by:*

$$f_1(\xi,x) := e^{\sqrt{2+\xi^2}x}, \tag{2.125}$$
$$f_2(\xi,x) := e^{-\sqrt{2+\xi^2}x}, \tag{2.126}$$
$$f_3(\xi,x) := \cos(\xi x), \tag{2.127}$$
$$f_4(\xi,x) := \frac{\xi+1}{\xi}\sin(\xi x), \tag{2.128}$$

Again, we understand $\frac{\sin(\xi x)}{\xi}$ as a holomorphic function extending to $\xi = 0$.

Given $\omega \in (-\infty,\infty]$ and $\xi \geq 0$, we define Banach spaces $X_{\xi,k} = X_{\xi,k,\omega}$:

$$X_{\xi,1} := L^\infty_{f_1(\xi,\cdot)}(-\infty,\omega)^2, \tag{2.129}$$
$$X_{\xi,2} := L^\infty_{f_2(\xi,\cdot)}(-\infty,\omega)^2, \tag{2.130}$$
$$X_{\xi,3} = X_{\xi,4} := L^\infty_{|\cdot|+1}(-\infty,\omega)^2. \tag{2.131}$$

We further define $h_k : [0,\infty) \times \mathbb{R} \to \mathbb{R}^2$ by:

$$h_1 = (f_1, 0), h_2 = (f_2, 0), h_3 = (0, f_3), h_4 = (0, f_4). \tag{2.132}$$

Remark *Choosing $f_4 = \frac{\xi+1}{\xi}\sin(\xi x)$ ensures $f_4(x) = x$ for $\xi = 0$, which allows to treat both $\xi > 0$ and $\xi = 0$ at once. We maintain suitable decay, as $f_4 \approx \sin(\xi x)$ for large ξ.*

Lemma 2.4.3 *Let $\xi \geq 0$ and $\omega \in (-\infty, \infty]$. Then, for $x < \omega$ and suitable functions w:*

$$|B_{\xi,1}w(x)| \leq \frac{C}{1+\xi}\frac{e^{\sqrt{2+\xi^2}x}}{1+e^{-\frac{1}{2}x}}\,||w||_{X_{\xi,1}}, \tag{2.133}$$

$$|B_{\xi,2}w(x)| \leq \frac{C}{1+\xi}\frac{e^{-\sqrt{2+\xi^2}x}}{1+e^{-\frac{1}{2}x}}\,||w||_{X_{\xi,2}}, \tag{2.134}$$

$$|B_{\xi,3}w(x)| \leq \min\left(\frac{1+|x|}{1+\xi}, \frac{1}{\xi}\right)\frac{C}{1+e^{-x}}\,||w||_{X_{\xi,3}}. \tag{2.135}$$

Proof. Follows completely analogously to lemma 2.3.3. ☐

Corollary 2.4.4 *Let $\xi \geq 0$ and $\omega \in (-\infty, \infty]$. Then:*

$$||B_{\xi,1}w||_{X_{\xi,1}} \leq \frac{C}{1+\xi}\frac{1}{1+e^{-\frac{1}{2}\omega}}\,||w||_{X_{\xi,1}}, \tag{2.136}$$

$$||B_{\xi,2}w||_{X_{\xi,2}} \leq \frac{C}{1+\xi}\frac{1}{1+e^{-\frac{1}{2}\omega}}\,||w||_{X_{\xi,2}}, \tag{2.137}$$

$$||B_{\xi,3}w||_{X_{\xi,3}} \leq \frac{C}{1+\xi}\frac{1}{1+e^{-\frac{1}{2}\omega}}\,||w||_{X_{\xi,3}}. \tag{2.138}$$

Lemma 2.4.5 *Let $\omega \in (-\infty, \infty]$ and $k \in \{1,2,3,4\}$. Then $\xi \mapsto B_{\xi,k,\omega}w$ is infinitely differentiable. For $\xi \geq 0$ and $n \geq 0$:*

$$\left|\left|\frac{d^n}{d\xi^n}B_{\xi,k,\omega}w\right|\right|_{X_{\xi,k}} \leq \frac{C_n}{1+\xi}\frac{1}{1+e^{-\frac{1}{2}\omega}}\,||w||_{X_{\xi,k}}. \tag{2.139}$$

Proof. Follows by direct computation. Note that $\xi \mapsto \xi^{-1}\sin(\xi x)$ is analytical with $\left|\partial_\xi^n(\xi^{-1}\sin(\xi x))\right| \leq C_n\,|x|^{n+1}$ for $x, \xi \in \mathbb{R}$, $n \in \mathbb{N}$. ☐

As before, we can ensure $B_{\xi,k,\omega}$, $k = 1,2,3,4$ to constitute contractions by requiring ω to be small enough. By corollay 2.4.4, restricting ourself to $\xi \geq \xi_0 >> 0$ suffices to ensure a contraction as well.

We will do both: Constructing a fundamental system once for $\omega << 0$ and $\xi_0 = 0$ and once for $\omega = \infty$ and $\xi_0 >> 0$.

Definition 2.4.6 *Let $C > 0$ be chosen large enough to satisfy the conclusions of corollary 2.4.4. We define $\omega < 0$ and $\xi_0 \geq 0$ as arbitrarily chosen numbers fulfilling $Ce^{\frac{1}{4}\omega} < \frac{1}{2}$ and $C(1+\xi_0)^{-1} < \frac{1}{2}$.*

2.4.1. Small x

Theorem 2.4.7 *Let $\xi \geq 0$, $E = 1 + \xi^2$ and consider* (2.5), (2.6)*. Then, there exists a unique fundamental system $(w_1, w_2, w_3, w_4)(\xi, \cdot)$, such that $w_k = (u_k, v_k)$, $k \in \{1, 2, 3, 4\}$ solves:*

$$w_k(\xi, \cdot) = h_k(\xi, \cdot) + B_{\xi,k,\omega} w_k(\xi, \cdot) \tag{2.140}$$

on $(-\infty, \omega)$. For every $x < \omega$:

$$\left| u_1(\xi, x) - e^{\sqrt{2+\xi^2}x} \right| + |v_1(\xi, x)| \leq \frac{C}{1+\xi} e^{\frac{1}{2}x} e^{\sqrt{2+\xi^2}x}, \tag{2.141}$$
$$\left| u_2(\xi, x) - e^{-\sqrt{2+\xi^2}x} \right| + |v_2(\xi, x)| \leq \frac{C}{1+\xi} e^{\frac{1}{2}x} e^{-\sqrt{2+\xi^2}x}, \tag{2.142}$$
$$|u_3(\xi, x)| + |v_3(\xi, x) - \cos(\xi x)| \leq \frac{C}{1+\xi} e^{\frac{1}{2}x}, \tag{2.143}$$
$$|u_4(\xi, x)| + \left| v_4(\xi, x) - \frac{1+\xi}{\xi} \sin(\xi x) \right| \leq \frac{C}{1+\xi} e^{\frac{1}{2}x}. \tag{2.144}$$

Proof. For $x < \omega$, we define w_k by (2.140). Lemma 2.2.2 and corollary 2.4.4 imply:

$$||w_k||_{X_k} \leq \frac{1}{1 - \frac{1}{2}} ||h_k||_{X_k} = 2. \tag{2.145}$$

Lemma 2.4.3 implies the estimates (2.141) - (2.144), due to $x < \omega < 0$. □

Definition 2.4.8 *Motivated by theorem 2.4.7, we define the following remainder terms for $\xi \geq 0$ and $x < \omega$:*

$$r_{u,1}(\xi, x) := u_1(\xi, x) - e^{\sqrt{2+\xi^2}x}, \tag{2.146}$$
$$r_{v,1}(\xi, x) := v_1(\xi, x), \tag{2.147}$$
$$r_{u,2}(\xi, x) := u_2(\xi, x) - e^{-\sqrt{2+\xi^2}x}, \tag{2.148}$$
$$r_{v,2}(\xi, x) := v_2(\xi, x), \tag{2.149}$$
$$r_{u,3}(\xi, x) := u_3(\xi, x), \tag{2.150}$$
$$r_{v,3}(\xi, x) := v_3(\xi, x) - \cos(\xi x), \tag{2.151}$$
$$r_{u,4}(\xi, x) := u_4(\xi, x), \tag{2.152}$$
$$r_{v,4}(\xi, x) := v_4(\xi, x) - (1+\xi)\xi^{-1} \sin(\xi x). \tag{2.153}$$

Lemma 2.4.9 *For every $\xi \geq 0$ and $x < \omega$:*

$$|\partial_x r_{u,1}| + |\partial_x r_{v,1}| \leq C e^{\frac{1}{2}x} e^{\sqrt{2+\xi^2}x}, \tag{2.154}$$
$$|\partial_x r_{u,2}| + |\partial_x r_{v,2}| \leq C e^{\frac{1}{2}x} e^{-\sqrt{2+\xi^2}x}, \tag{2.155}$$
$$|\partial_x r_{u,3}| + |\partial_x r_{v,3}| \leq C e^{\frac{1}{2}x}, \tag{2.156}$$
$$|\partial_x r_{u,4}| + |\partial_x r_{v,4}| \leq C e^{\frac{1}{2}x}. \tag{2.157}$$

Proof. Differentiating $w_k(\xi,x) - h_k(\xi,x) = B_{\xi,k,\omega} w_k(\xi,x)$ with respect to x yields the claim. $\square$

Lemma 2.4.10 *Given $x < \omega$, $w_k(\xi,x)$ is infinitely differentiable with respect to $\xi \geq 0$. For every $\xi \geq 0$, $x < \omega$ and $n \geq 0$:*

$$\left|\partial_\xi^n r_{u,1}\right| + \left|\partial_\xi^n r_{v,1}\right| \leq \frac{C_n}{1+\xi} e^{\frac{1}{2}x} e^{\sqrt{2+\xi^2}x}, \tag{2.158}$$

$$\left|\partial_\xi^n r_{u,2}\right| + \left|\partial_\xi^n r_{v,2}\right| \leq \frac{C_n}{1+\xi} e^{\frac{1}{2}x} e^{-\sqrt{2+\xi^2}x}, \tag{2.159}$$

$$\left|\partial_\xi^n r_{u,3}\right| + \left|\partial_\xi^n r_{v,3}\right| \leq \frac{C_n}{1+\xi} e^{\frac{1}{2}x}, \tag{2.160}$$

$$\left|\partial_\xi^n r_{u,4}\right| + \left|\partial_\xi^n r_{v,4}\right| \leq \frac{C_n}{1+\xi} e^{\frac{1}{2}x}. \tag{2.161}$$

Proof. Let $k \in \{1,2,3,4\}$. Consider for $n \geq 0$:

$$B^{(n)}_{\xi,k,\omega} := \frac{d^n}{d\xi^n} B_{\xi,k,\omega}. \tag{2.162}$$

By lemma 2.4.5, $B^{(n)}_{\xi,k,\omega}$ is well-defined as an operator $X_{\xi,k} \to X_{\xi,k}$. We conclude, by taking the derivative of $w_k(\xi,x) = h_k(\xi,x) + B_{\xi,k,\omega} w_k(\xi,x)$:

$$\begin{aligned}\frac{d^n}{d\xi^n} w_k(\xi,x) &= \frac{d^n}{d\xi^n} h_k(\xi,x) + \sum_{l=0}^{n} \binom{n}{l} B^{(n-l)}_{\xi,k,\omega} \left(\frac{d^l}{d\xi^l} w_k(\xi,\cdot)\right) \\ &=: h_{k,n}(\xi,x) + B_{\xi,k,\omega}\left(\frac{d^n}{d\xi^n} w_k(\xi,\cdot)\right). \end{aligned} \tag{2.163}$$

Clearly, if for every $x < \omega$:

$$|h_{1,n}(\xi,x)| \leq \frac{C}{1+\xi} e^{\sqrt{2+\xi^2}x}, \tag{2.164}$$

then lemma 2.2.2 and corollary 2.4.4 imply:

$$\left|\frac{d^n}{d\xi^n} w_1(\xi,x)\right| \leq \frac{2C}{1+\xi} e^{\sqrt{2+\xi^2}x}. \tag{2.165}$$

(2.164) and (2.165) follow easily by induction. (2.158) follows from (2.165) and lemma 2.4.3.

For $k = 2,3,4$, the lemma is proven analogously. $\square$

Lemma 2.4.11 *Let $w_k = (u_k, v_k)$, $k \in \{1,2,3,4\}$ be as in theorem 2.4.7. $w_k : [0,\infty) \times \mathbb{R} \to \mathbb{R}^2$ is a real analytic function for every k.*

Proof. Analyticity in x is trivial. It follows by bootstrapping, as $w_k(\xi,\cdot)$ solves (2.5), (2.6).

The definition of $B_{\xi,k,\omega}$ as given by 2.4.1 can be extended to $\xi \in \mathbb{C}$. We consider $\zeta \in \mathbb{C}$, $|\zeta| < \frac{1}{2}$. Analogously to theorem 2.4.7, the equation

$$w_k(\xi+\zeta,\cdot) = h_k(\xi+\zeta,\cdot) + B_{\xi+\zeta,k,\omega} w_k(\xi+\zeta,\cdot) \tag{2.166}$$

admits a unique fixed-point $w_k(\xi+\zeta,\cdot)$ for every $\xi \geq 0$ and $k = 1,2,3,4$.

We show that w_k satisfies the Cauchy-Riemann equations, or equivalently $\frac{\partial}{\partial\bar{\xi}} w_k = 0$. $\frac{\partial}{\partial\bar{\xi}}$ is the Wirtinger derivative:

$$\frac{\partial f}{\partial\bar{\xi}} := \frac{1}{2}\partial_\varphi f - \frac{i}{2}\partial_\psi f. \tag{2.167}$$

Hereby, $\varphi = \mathrm{Re}(\xi), \psi = \mathrm{Im}(\xi)$. It follows:

$$\frac{\partial}{\partial\bar{\xi}} w_k = \frac{\partial}{\partial\bar{\xi}} h_k + \frac{\partial}{\partial\bar{\xi}} \left(B_{\xi,k,\omega} w_k\right) = B_{\xi,k,\omega}\left(\frac{\partial}{\partial\bar{\xi}} w_k\right). \tag{2.168}$$

Lemma 2.2.2 implies $\frac{\partial}{\partial\bar{\xi}} w_k = 0$ for every $\xi \geq 0$. That concludes the proof. □

2.4.2. Large ξ

We state the direct analogues to the theorem and lemmata from chapter 2.4.1. We omit proof, as that too is completely analogous to chapter 2.4.1.

Theorem 2.4.12 *Let $\xi \geq \xi_0$, $E = 1+\xi^2$ and consider* (2.5), (2.6). *Then, there exists a unique fundamental system $(\tilde{w}_1, \tilde{w}_2, \tilde{w}_3, \tilde{w}_4)(\xi,\cdot)$, such that $\tilde{w}_k = (\tilde{u}_k, \tilde{v}_k)$, $k \in \{1,2,3,4\}$ solves:*

$$\tilde{w}_k(\xi,\cdot) = h_k(\xi,\cdot) + B_{\xi,k,\infty}\tilde{w}_k(\xi,\cdot). \tag{2.169}$$

For every $x \in \mathbb{R}$:

$$\left|\tilde{u}_1(\xi,x) - e^{\sqrt{2+\xi^2}x}\right| + |\tilde{v}_1(\xi,x)| \leq \frac{C}{1+\xi} e^{\frac{1}{2}x} e^{\sqrt{2+\xi^2}x}, \tag{2.170}$$

$$\left|\tilde{u}_2(\xi,x) - e^{-\sqrt{2+\xi^2}x}\right| + |\tilde{v}_2(\xi,x)| \leq \frac{C}{1+\xi} e^{\frac{1}{2}x} e^{-\sqrt{2+\xi^2}x}, \tag{2.171}$$

$$|\tilde{u}_3(\xi,x)| + |\tilde{v}_3(\xi,x) - \cos(\xi x)| \leq \frac{C}{1+\xi} e^{\frac{1}{2}x}, \tag{2.172}$$

$$|\tilde{u}_4(\xi,x)| + \left|\tilde{v}_4(\xi,x) - \frac{1+\xi}{\xi}\sin(\xi x)\right| \leq \frac{C}{1+\xi} e^{\frac{1}{2}x}. \tag{2.173}$$

Definition 2.4.13 *We define the following remainder terms for $\xi \geq \xi_0$ and $x \in \mathbb{R}$:*

$$\tilde{r}_{u,1}(\xi, x) := \tilde{u}_1(\xi, x) - e^{\sqrt{2+\xi^2}x}, \tag{2.174}$$
$$\tilde{r}_{v,1}(\xi, x) := \tilde{v}_1(\xi, x), \tag{2.175}$$
$$\tilde{r}_{u,2}(\xi, x) := \tilde{u}_2(\xi, x) - e^{-\sqrt{2+\xi^2}x}, \tag{2.176}$$
$$\tilde{r}_{v,2}(\xi, x) := \tilde{v}_2(\xi, x), \tag{2.177}$$
$$\tilde{r}_{u,3}(\xi, x) := \tilde{u}_3(\xi, x), \tag{2.178}$$
$$\tilde{r}_{v,3}(\xi, x) := \tilde{v}_3(\xi, x) - \cos(\xi x), \tag{2.179}$$
$$\tilde{r}_{u,4}(\xi, x) := \tilde{u}_4(\xi, x), \tag{2.180}$$
$$\tilde{r}_{v,4}(\xi, x) := \tilde{v}_4(\xi, x) - (1+\xi)\xi^{-1}\sin(\xi x). \tag{2.181}$$

Lemma 2.4.14 *For $\xi \geq \xi_0$ and $x \in \mathbb{R}$:*

$$|\partial_x \tilde{r}_{u,1}| + |\partial_x \tilde{r}_{v,1}| \leq Ce^{\frac{1}{2}x}e^{\sqrt{2+\xi^2}x}, \tag{2.182}$$
$$|\partial_x \tilde{r}_{u,2}| + |\partial_x \tilde{r}_{v,2}| \leq Ce^{\frac{1}{2}x}e^{-\sqrt{2+\xi^2}x}, \tag{2.183}$$
$$|\partial_x \tilde{r}_{u,3}| + |\partial_x \tilde{r}_{v,3}| \leq Ce^{\frac{1}{2}x}, \tag{2.184}$$
$$|\partial_x \tilde{r}_{u,4}| + |\partial_x \tilde{r}_{v,4}| \leq Ce^{\frac{1}{2}x}. \tag{2.185}$$

Lemma 2.4.15 *Given $x \in \mathbb{R}$, $\tilde{w}_k(\xi, x)$ is infinitely differentiable with respect to $\xi \geq \xi_0$. For every $\xi \geq \xi_0$, $x \in \mathbb{R}$ and $n \geq 0$:*

$$\left|\partial_\xi^n \tilde{r}_{u,1}\right| + \left|\partial_\xi^n \tilde{r}_{v,1}\right| \leq \frac{C_n}{1+\xi}e^{\frac{1}{2}x}e^{\sqrt{2+\xi^2}x}, \tag{2.186}$$
$$\left|\partial_\xi^n \tilde{r}_{u,2}\right| + \left|\partial_\xi^n \tilde{r}_{v,2}\right| \leq \frac{C_n}{1+\xi}e^{\frac{1}{2}x}e^{-\sqrt{2+\xi^2}x}, \tag{2.187}$$
$$\left|\partial_\xi^n \tilde{r}_{u,3}\right| + \left|\partial_\xi^n \tilde{r}_{v,3}\right| \leq \frac{C_n}{1+\xi}e^{\frac{1}{2}x}, \tag{2.188}$$
$$\left|\partial_\xi^n \tilde{r}_{u,4}\right| + \left|\partial_\xi^n \tilde{r}_{v,4}\right| \leq \frac{C_n}{1+\xi}e^{\frac{1}{2}x}. \tag{2.189}$$

Lemma 2.4.16 *Let $\tilde{w}_k = (\tilde{u}_k, \tilde{v}_k)$, $k \in \{1, 2, 3, 4\}$ be as in theorem 2.4.12. Then, $\tilde{w}_k : [\xi_0, \infty) \times \mathbb{R} \to \mathbb{R}^2$ is a real analytic function for every k.*

Part II.
Spectrum of L

3. Meromorphic Differential Equations

We begin with the characterisation of the spectrum of L. In this chapter we transform $Lw = iEw$ into a system of hypergeometric equations, which then can be solved by calculating the coefficients of a power series.

3.1. First Transformation

Recall (2.5) and (2.6). If we write Q out as in (1.4), then (2.5) and (2.6) become:

$$(-\Delta + 1 + E)u = \frac{2(p+1)e^{(p-1)x}}{(e^{(p-1)x}+1)^2}\left(\frac{p+1}{2}u + \frac{p-1}{2}v\right), \tag{3.1}$$

$$(-\Delta + 1 - E)v = \frac{2(p+1)e^{(p-1)x}}{(e^{(p-1)x}+1)^2}\left(\frac{p+1}{2}v + \frac{p-1}{2}u\right). \tag{3.2}$$

Considering the change of coordinates $e^{(p-1)x} = y \in (0,\infty)$ comes natural when trying to simplify (3.1) and (3.2). We calculate:

$$\partial_x = (p-1)e^{(p-1)x}\partial_y = (p-1)y\partial_y, \tag{3.3}$$

$$\partial_x^2 = (p-1)^2 e^{(p-1)x}\partial_y + (p-1)^2 e^{2(p-1)x}\partial_y^2 = (p-1)^2 y\partial_y + (p-1)^2 y^2\partial_y^2. \tag{3.4}$$

Combining this with (3.1) yields:

$$\begin{aligned}&-(p-1)^2y^2u_{yy} - (p-1)^2yu_y + (1+E)u\\ &= (p+1)^2\frac{y}{(y+1)^2}u + (p-1)(p+1)\frac{y}{(1+y)^2}v.\end{aligned} \tag{3.5}$$

It follows:

$$0 = u_{yy} + \frac{1}{y}u_y + \left(-\frac{1+E}{(p-1)^2}\frac{1}{y^2} + \frac{(p+1)^2}{(p-1)^2}\frac{1}{y(1+y)^2}\right)u + \frac{p+1}{p-1}\frac{1}{y(1+y)^2}v. \tag{3.6}$$

Analogously, it follows from (3.2):

$$0 = v_{yy} + \frac{1}{y}v_y + \left(-\frac{1-E}{(p-1)^2}\frac{1}{y^2} + \frac{(p+1)^2}{(p-1)^2}\frac{1}{y(1+y)^2}\right)v + \frac{p+1}{p-1}\frac{1}{y(1+y)^2}u. \tag{3.7}$$

3.2. Hypergeometric Equation

(3.6) and (3.7) are similar to Riemann's differential equation, a generalized form of the hypergeometric differential equation.

Riemann's differential equation in its most general form is given by:

$$
\begin{aligned}
0 = {} & \frac{d^2}{dz^2}w + \left(\frac{1-a_1-a_2}{z-\alpha} + \frac{1-b_1-b_2}{z-\beta} + \frac{1-c_1-c_2}{z-\gamma}\right)\frac{d}{dz}w \\
& + \left(\frac{(\alpha-\beta)(\alpha-\gamma)a_1a_2}{z-\alpha} + \frac{(\beta-\alpha)(\beta-\gamma)b_1b_2}{z-\beta} + \frac{(\gamma-\alpha)(\gamma-\beta)c_1c_2}{z-\gamma}\right)\frac{w}{(z-\alpha)(z-\beta)(z-\gamma)},
\end{aligned}
\tag{3.8}
$$

whereby:

$$a_1 + a_2 + b_1 + b_2 + c_1 + c_2 = 1. \tag{3.9}$$

$\{a_1, a_2\}, \{b_1, b_2\}, \{c_1, c_2\}$ are called the indicial exponent pairs at the regular singular points α, β and γ respectively.

(3.8) assumes $\alpha, \beta, \gamma \neq \infty$. In case α, β or γ is equal infinity, the corresponding limit in (3.8) is considered. In particular, for $\gamma = \infty$ that means:

$$
\begin{aligned}
0 = {} & \frac{d^2}{dz^2}w + \left(\frac{1-a_1-a_2}{z-\alpha} + \frac{1-b_1-b_2}{z-\beta}\right)\frac{d}{dz}w \\
& + \left(\frac{(\alpha-\beta)a_1a_2}{z-\alpha} + \frac{(\beta-\alpha)b_1b_2}{z-\beta} + c_1c_2\right)\frac{w}{(z-\alpha)(z-\beta)}.
\end{aligned}
\tag{3.10}
$$

At the singularity α, assuming $a_1 - a_2 \notin \mathbb{Z}$, one can construct two linear independent local solutions of the form:

$$(z-\alpha)^{a_1}F, (z-\alpha)^{a_2}G. \tag{3.11}$$

F and G are holomorphic in some neighbourhood of α and can be expressed via the hypergeometric function:

$$
\begin{aligned}
& w(z) \\
= {} & \left(\frac{z-\alpha}{z-\beta}\right)^{a_1}\left(\frac{z-\gamma}{z-\beta}\right)^{c_1} {}_2F_1\left(a_1+b_1+c_1, a_1+b_2+c_1, 1+a_1-a_2, \frac{(z-\alpha)(\gamma-\beta)}{(z-\beta)(\gamma-\alpha)}\right).
\end{aligned}
\tag{3.12}
$$

The second solution is obtained by switching a_1 and a_2, b_1 and b_2, c_1 and c_2 respectively.

As it turns out, (3.8) locally behaves similar to Riemann's differential equation. Thus, our Ansatz in solving (3.6), (3.7) mirrors (3.11). We solve the equations explicitly by calculating the coefficients of a power series.

No attempt is made to connect solutions at different singularities.

An overview on the topic of Riemann's differential equation can be found in the NIST Digital Library of Mathematical Functions [8, Chapter 15].

I am not aware of this Ansatz being used before on the NLS. However, it is to be noted that using a coordinate transform to restate dispersive equations in a form akin to (3.6) and (3.7) is not new. Donninger [9], [10], [11] used so-called similarity coordinates to examine the wave equation and study self-similar blow-up solutions. [9] in particular, considers an eigenvalue equation loosely mirroring (3.6) and (3.7).

3.3. Second Transformation

We consider (3.6) and (3.7) and conduct some simplifications.

As (2.5) is invariant under the transformation $x \mapsto -x$, (3.6) is invariant under the transformation $y \to \frac{1}{y}$.

We use another Möbius transform to transpose (3.6) into a more immediately symmetrical form.

We achieve this feat by mapping the two singular points of (3.6) which correspond to the invariance, $y = 0$ and $y = \infty$, onto symmetrical points $z = -1$ and $z = 1$, while at the same time mapping the third singular point of (3.6), $y = -1$, onto $z = \infty$.

Let thus $z = Sy = \frac{y-1}{y+1} \in (-1, 1)$. Also consider the inverse transform $y = S^{-1}z = \frac{1+z}{1-z}$. We calculate:

$$\partial_y = \left(\frac{1}{y+1} - \frac{y-1}{(y+1)^2}\right)\partial_z = \frac{2}{(y+1)^2}\partial_z, \tag{3.13}$$

$$\partial_y^2 = -\frac{4}{(y+1)^3}\partial_z + \frac{4}{(y+1)^4}\partial_z^2. \tag{3.14}$$

Furthermore, note:

$$\frac{1}{y} = \frac{1-z}{1+z}, \quad \frac{1}{y+1} = \frac{1-z}{2}. \tag{3.15}$$

Plugging this into (3.6) yields:

$$\begin{aligned}
0 &= u_{yy} + \frac{1}{y}u_y + \left(-\frac{1+E}{(p-1)^2}\frac{1}{y^2} + \frac{(p+1)^2}{(p-1)^2}\frac{1}{y(1+y)^2}\right)u + \frac{p+1}{p-1}\frac{1}{y(1+y)^2}v \\
&= \frac{(1-z)^4}{4}u_{zz} + \left(\frac{(1-z)^2}{2}\frac{1-z}{1+z} - \frac{(1-z)^3}{2}\right)u_z \\
&\quad + \left(-\frac{1+E}{(p-1)^2}\frac{(1-z)^2}{(1+z)^2} + \frac{(p+1)^2}{4(p-1)^2}\frac{(1-z)^3}{1+z}\right)u + \frac{p+1}{4(p-1)}\frac{(1-z)^3}{1+z}v \\
&= \frac{(1-z)^4}{4}u_{zz} - \frac{z}{2}\frac{(1-z)^3}{(1+z)}u_z + \left(-\frac{1+E}{(p-1)^2}\frac{(1-z)^2}{(1+z)^2} + \frac{(p+1)^2}{4(p-1)^2}\frac{(1-z)^3}{1+z}\right)u \\
&\quad + \frac{p+1}{4(p-1)}\frac{(1-z)^3}{1+z}v \\
&\Leftrightarrow \\
0 &= u_{zz} - \frac{2z}{1-z^2}u_z + \left(-\frac{4(1+E)}{(p-1)^2}\frac{1}{(1-z^2)^2} + \frac{(p+1)^2}{(p-1)^2}\frac{1}{1-z^2}\right)u + \frac{p+1}{p-1}\frac{1}{1-z^2}v.
\end{aligned} \tag{3.16}$$

We conclude completely analogously:

$$0 = v_{zz} - \frac{2z}{1-z^2} v_z + \left(-\frac{4(1-E)}{(p-1)^2} \frac{1}{(1-z^2)^2} + \frac{(p+1)^2}{(p-1)^2} \frac{1}{1-z^2} \right) v + \frac{p+1}{p-1} \frac{1}{1-z^2} u. \tag{3.17}$$

To summarize:

Lemma 3.3.1 *Let $E \in \mathbb{R}$. Consider the coordinate transform $T : C((-1,1), \mathbb{R}^2) \to C(\mathbb{R}, \mathbb{R}^2)$, given by:*

$$Tw = w \left(\frac{e^{(p-1)x} - 1}{e^{(p-1)x} + 1} \right). \tag{3.18}$$

Then, $w : (-1,1) \to \mathbb{R}^2$ solves (3.16) *and* (3.17)*, if and only if Tw solves* (2.5) *and* (2.6)*.*

3.4. Strategy of Proof

3.4.1. Eigenvalues

Recall $M_{E,+}$ and $M_{E,-}$ as given by definition 2.3.20. We define a direct analogue based on (3.16) and (3.17).

Definition 3.4.1 *Let $E \in \mathbb{R}$. Consider:*

$$N_{E,-} := \{(u,v) \in C^2(-1,1)^2 | (3.16),\ (3.17) \textit{ hold}, \lim_{z \to -1} (u(z), v(z)) = (0,0)\}, \tag{3.19}$$

$$N_{E,+} := \{(u,v) \in C^2(-1,1)^2 | (3.16),\ (3.17) \textit{ hold}, \lim_{z \to 1} (u(z), v(z)) = (0,0)\}. \tag{3.20}$$

Lemma 3.4.2 *Let $E \in \mathbb{R}$. Then, iE is an eigenvalue of L, if and only if $N_{E,-} \cap N_{E,+} \neq \{0\}$.*

Proof. By definition, $W \in H^1(\mathbb{R})^2$ solves (2.5) and (2.6) for $E \in \mathbb{R}$, if and only if $T^{-1}W \in N_{E,-} \cap N_{E,+}$. □

Definition 3.4.3 *Let $E \in \mathbb{R}$. Consider:*

$$N_{E,e,-} := \{w \in N_{E,-} | w \textit{ is even}\}, \tag{3.21}$$

$$N_{E,o,-} := \{w \in N_{E,-} | w \textit{ is odd}\}. \tag{3.22}$$

Lemma 3.4.4 *Let $E \in \mathbb{R}$. Then, iE is an eigenvalue of L, if and only if $N_{E,e,-} \neq \{0\}$ or $N_{E,o,-} \neq \{0\}$. $N_{E,e,-} \neq \{0\}$ is equivalent to the existence of an even eigenfunction. Likewise, $N_{E,o,-} \neq \{0\}$ is equivalent to the existence of an odd eigenfunction.*

Proof. Follows from lemma 3.4.2 and $N_{E,e,-} \oplus N_{E,o,-} = N_{E,-} \cap N_{E,+}$. □

Definition 3.4.5 *Let $E \in \mathbb{R}$. Consider:*

$$D_{E,-} := \{(u,v) \in C^2(-1,0)^2 | (3.16),\ (3.17)\ \textit{hold},\ \lim_{z\to -1}(u(z),v(z)) = (0,0)\}, \quad (3.23)$$

$$D_{E,e,-} := \{w \in D_{E,-} | \lim_{z\nearrow 0} w'(z) = (0,0)\}, \quad (3.24)$$

$$D_{E,o,-} := \{w \in D_{E,-} | \lim_{z\nearrow 0} w(z) = (0,0)\}. \quad (3.25)$$

Lemma 3.4.6 *Let $E \in \mathbb{R}$. Then, iE is an eigenvalue of L, if and only if $D_{E,e,-} \neq \{0\}$ or $D_{E,o,-} \neq \{0\}$. $D_{E,e,-} \neq \{0\}$ is equivalent to the existence of an even eigenfunction. Likewise, $D_{E,o,-} \neq \{0\}$ is equivalent to the existence of an odd eigenfunction.*

Proof. Let $w \in N_{E,e,-}$. w is even by definition, which implies $w'(0) = (0,0)$. It follows $w|_{(-1,0)} \in D_{E,e,-}$. We conclude $N_{E,e,-} \neq \{0\} \Rightarrow D_{E,e,-} \neq \{0\}$ and analogously $N_{E,o,-} \neq \{0\} \Rightarrow D_{E,o,-} \neq \{0\}$.

Let now $w \in D_{E,e,-}$. As w solves (3.16) and (3.17) it can be analytically extended to a solution $\tilde{w} : [-1,1] \to \mathbb{C}^2$. $w \in D_{E,e,-}$ implies $\tilde{w}'(0) = (0,0)$ meaning that $\tilde{w}$ is even. That in turn implies $\tilde{w} \in N_{E,e,-}$. We have shown $D_{E,e,-} \neq \{0\} \Rightarrow N_{E,e,-} \neq \{0\}$. $D_{E,o,-} \neq \{0\} \Rightarrow N_{E,o,-} \neq \{0\}$ follows completely analogously.

Lemma 3.4.4 concludes the proof. □

$D_{E,e,-}$ and $D_{E,o,-}$ allow for falsifiable conditions. We focus on solving (3.16) and (3.17) for $-1 < z < 0$. If, for $E \in \mathbb{R}$, we can show

$$\lim_{z\nearrow 0} w'(z) \neq (0,0),\ \lim_{z\nearrow 0} w(z) \neq (0,0), \quad (3.26)$$

for every $w \in D_{E,-}$, then iE is no eigenvalue of L.

3.4.2. Resonances

The exact same strategy also works for resonances.

Definition 3.4.7 *Let $E = -1$ or $E = 1$. Consider:*

$$\tilde{N}_{E,-} := \{w = (u,v) \in C^2(-1,1)^2 | (3.16),\ (3.17)\ \textit{hold},\ \lim_{z\to -1} |w(z)| < \infty\}, \quad (3.27)$$

$$\tilde{N}_{E,+} := \{w = (u,v) \in C^2(-1,1)^2 | (3.16),\ (3.17)\ \textit{hold},\ \lim_{z\to 1} |w(z)| < \infty\}, \quad (3.28)$$

$$\tilde{N}_{E,e,-} := \{w \in \tilde{N}_{E,-} | w\ \textit{is even}\}, \quad (3.29)$$

$$\tilde{N}_{E,o,-} := \{w \in \tilde{N}_{E,-} | w\ \textit{is odd}\}. \quad (3.30)$$

Definition 3.4.8 *Let $E = -1$ or $E = 1$. Consider:*

$$\tilde{D}_{E,-} := \{(u,v) \in C^2(-1,0)^2 | (3.16)\ \textit{and}\ (3.17)\ \textit{hold},\ \lim_{z\to -1} |w(z)| < \infty\}, \quad (3.31)$$

$$\tilde{D}_{E,e,-} := \{w \in \tilde{D}_{E,-} | \lim_{z\nearrow 0} w'(z) = (0,0)\}, \quad (3.32)$$

$$\tilde{D}_{E,o,-} := \{w \in \tilde{D}_{E,-} | \lim_{z\nearrow 0} w(z) = (0,0)\}. \quad (3.33)$$

Lemma 3.4.9 *Let $E = -1$ or $E = 1$. Then, $\tilde{D}_{E,e,-} \neq \{0\}$ is equivalent to the existence of a non-trivial even bounded solution to $Lw = iEw$. Likewise, $\tilde{D}_{E,o,-} \neq \{0\}$ is equivalent to the existence of a non-trivial odd bounded solution to $Lw = iEw$.*

3.5. The Even Case

We introduce the coordinate transform $\xi = 1 - z^2$. This has the effect of mapping the singular points $z = 1$ and $z = -1$ onto $\xi = 0$ and introducing a new singular point in $\xi = 1$, which corresponds to the branch point $z = 0$ of $1 - z^2$.

After transforming (3.16) and (3.17) via $\xi = 1 - z^2$, we solve the resulting equation via a power series in $\xi = 0$. This power series will have convergence radius 1, due to the nearest singularity being $\xi = 1$.

Such a solution $\xi \mapsto w_E(\xi)$ will be constructed for every $E \geq -1$. In chapter 4, we show that w_E does not extend to an even solution with respect to $z \in (-1, 1)$, with the single exception of $E = -1$, $p = 3$. This exception leads to the existence of a resonance for $p = 3$, see the remark on page 68 for an explicit formula.

The fact that w_E is not even is enough to show that no embedded eigenvalues $|E| \geq 1$ with even eigenfunctions can exist. Because the space of potential eigenfunctions $D_{E,-}$ is one-dimensional for $E \geq 1$, it is spanned by w_E, thus even eigenfunctions can not exist.

In chapter 5, we examine the spectral gap $E \in [-1, 1]$ for eigenvalues and resonances. Due to $D_{E,-}$ resp. $\tilde{D}_{E,-}$ being two-dimensional for $E \in (-1, 1)$ resp. $E = \pm 1$, we instead have to consider linear combinations of solutions. This makes checking for eigenvalues far more challenging. Chapter 5 consists almost entirely of very technical calculations.

We return to the issue at hand. Based on $\xi = 1 - z^2$, we calculate:

$$\partial_z = -2z\partial_\xi, \tag{3.34}$$

$$\partial_{zz} = 4z^2\partial_{\xi\xi} - 2\partial_\xi = 4(1-\xi)\partial_{\xi\xi} - 2\partial_\xi. \tag{3.35}$$

Plugging this into (3.16) gives:

$$\begin{aligned}
0 &= 4(1-\xi)u_{\xi\xi} - 2u_\xi + \frac{4z^2}{1-z^2}u_\xi \\
&\quad + \left(-\frac{4(1+E)}{(p-1)^2}\frac{1}{(1-z^2)^2} + \frac{(p+1)^2}{(p-1)^2}\frac{1}{1-z^2}\right)u + \frac{p+1}{p-1}\frac{1}{1-z^2}v \\
&= 4(1-\xi)u_{\xi\xi} - 2u_\xi + \frac{4(1-\xi)}{\xi}u_\xi + \left(-\frac{4(1+E)}{(p-1)^2}\frac{1}{\xi^2} + \frac{(p+1)^2}{(p-1)^2}\frac{1}{\xi}\right)u + \frac{p+1}{p-1}\frac{1}{\xi}v \\
&= 4(1-\xi)u_{\xi\xi} + \left(\frac{4}{\xi} - 6\right)u_\xi + \left(-\frac{4(1+E)}{(p-1)^2}\frac{1}{\xi^2} + \frac{(p+1)^2}{(p-1)^2}\frac{1}{\xi}\right)u + \frac{p+1}{p-1}\frac{1}{\xi}v. \qquad (3.36)
\end{aligned}$$

Analogously:

$$\begin{aligned}0 &= 4(1-\xi)v_{\xi\xi} + \left(\frac{4}{\xi}-6\right)v_\xi + \left(-\frac{4(1-E)}{(p-1)^2}\frac{1}{\xi^2} + \frac{(p+1)^2}{(p-1)^2}\frac{1}{\xi}\right)v + \frac{p+1}{p-1}\frac{1}{\xi}u \\ &= 4(1-\xi)v_{\xi\xi} + \left(\frac{4}{\xi}-6\right)v_\xi + \left(-\frac{4(1+E)}{(p-1)^2}\frac{1}{\xi^2} + \frac{(p+1)^2}{(p-1)^2}\frac{1}{\xi}\right)v + \frac{p+1}{p-1}\frac{1}{\xi}u \\ &\quad + \frac{8E}{(p-1)^2}\frac{1}{\xi^2}v. \end{aligned} \tag{3.37}$$

As expected, we identify the three singular points $\xi = 0$, $\xi = 1$ and $\xi = \infty$. Our goal is to solve (3.36) and (3.37) for $\xi \in (0,1)$. Clearly, any solution $w : (0,1) \to \mathbb{R}$ is real analytic.

We make the Ansatz:

$$u(\xi) = \xi^{\frac{\sqrt{1+E}}{p-1}} U(\xi), \tag{3.38}$$

$$v(\xi) = \xi^{\frac{\sqrt{1+E}}{p-1}} V(\xi). \tag{3.39}$$

Hereby, U and V are assumed to be analytic at $\xi = 0$. (3.38) and (3.39) are based on the asymptotic behaviour of w_1 described in lemma 2.3.6, lemma 2.3.13 and lemma 2.3.19:

$$\begin{aligned}(u_1, v_1) &\approx e^{\sqrt{1+E}x}(1,0) \\ &= \left(e^{(p-1)x}\right)^{\frac{\sqrt{1+E}}{p-1}}(1,0) \\ &= (1+z)^{\frac{\sqrt{1+E}}{p-1}}(1-z)^{-\frac{\sqrt{1+E}}{p-1}}(1,0) \\ &= \xi^{\frac{\sqrt{1+E}}{p-1}}(1+\sqrt{1-\xi})^{-2\frac{\sqrt{1+E}}{p-1}}(1,0). \end{aligned} \tag{3.40}$$

Based on (3.38), (3.39), we calculate:

$$u_\xi = \xi^{\frac{\sqrt{1+E}}{p-1}} U_\xi + \frac{\sqrt{1+E}}{p-1}\xi^{\frac{\sqrt{1+E}}{p-1}-1}U, \tag{3.41}$$

$$u_{\xi\xi} = \xi^{\frac{\sqrt{1+E}}{p-1}} U_{\xi\xi} + \frac{2\sqrt{1+E}}{p-1}\xi^{\frac{\sqrt{1+E}}{p-1}-1}U_\xi + \left(\frac{1+E}{(p-1)^2} - \frac{\sqrt{1+E}}{p-1}\right)\xi^{\frac{\sqrt{1+E}}{p-1}-2}U. \tag{3.42}$$

The same holds true for v and V.

Plugging (3.41) and (3.42) into (3.36) and dividing by $\xi^{\frac{\sqrt{1+E}}{p-1}-2}$ yields:

$$\begin{aligned}
0 &= 4(1-\xi)\left[\xi^2 U_{\xi\xi} + \frac{2\sqrt{1+E}}{p-1}\xi U_\xi + \left(\frac{1+E}{(p-1)^2} - \frac{\sqrt{1+E}}{p-1}\right) U\right] \\
&\quad + \left(\frac{4}{\xi} - 6\right)\left[\xi^2 U_\xi + \frac{\sqrt{1+E}}{p-1}\xi U\right] + \left[-\frac{4(1+E)}{(p-1)^2} + \frac{(p+1)^2}{(p-1)^2}\xi\right] U + \frac{p+1}{p-1}\xi V \\
&= 4(1-\xi)\xi^2 U_{\xi\xi} + \left[4(1-\xi)\frac{2\sqrt{1+E}}{p-1} + 4 - 6\xi\right]\xi U_\xi + \frac{p+1}{p-1}\xi V \\
&\quad + \left[4(1-\xi)\left(\frac{1+E}{(p-1)^2} - \frac{\sqrt{1+E}}{p-1}\right) + (4-6\xi)\frac{\sqrt{1+E}}{p-1} - \frac{4(1+E)}{(p-1)^2} + \frac{(p+1)^2}{(p-1)^2}\xi\right] U \\
&= 4\xi^2 U_{\xi\xi} - 4\xi^3 U_{\xi\xi} + \left[\frac{8\sqrt{1+E}}{p-1} + 4 - \left(6 + \frac{8\sqrt{1+E}}{p-1}\right)\xi\right]\xi U_\xi \\
&\quad + \left[-\frac{4(1+E)}{(p-1)^2}\xi - \frac{2\sqrt{1+E}}{p-1}\xi + \frac{(p+1)^2}{(p-1)^2}\xi\right] U + \frac{p+1}{p-1}\xi V \\
&= \left[4\xi^2 U_{\xi\xi} + \left(\frac{8\sqrt{1+E}}{p-1} + 4\right)\xi U_\xi\right] \\
&\quad - \xi\left[4\xi^2 U_{\xi\xi} + \left(6 + \frac{8\sqrt{1+E}}{p-1}\right)\xi U_\xi + \left(\frac{4(1+E)}{(p-1)^2} + \frac{2\sqrt{1+E}}{p-1} - \frac{(p+1)^2}{(p-1)^2}\right) U - \frac{p+1}{p-1} V\right].
\end{aligned} \tag{3.43}$$

Analogously, it follows:

$$\begin{aligned}
0 &= \left[4\xi^2 V_{\xi\xi} + \left(\frac{8\sqrt{1+E}}{p-1} + 4\right)\xi V_\xi\right] \\
&\quad - \xi\left[4\xi^2 V_{\xi\xi} + \left(6 + \frac{8\sqrt{1+E}}{p-1}\right)\xi V_\xi + \left(\frac{4(1+E)}{(p-1)^2} + \frac{2\sqrt{1+E}}{p-1} - \frac{(p+1)^2}{(p-1)^2}\right) V - \frac{p+1}{p-1} U\right] \\
&\quad + \frac{8E}{(p-1)^2} V \\
&= \left[4\xi^2 V_{\xi\xi} + \left(\frac{8\sqrt{1+E}}{p-1} + 4\right)\xi V_\xi + \frac{8E}{(p-1)^2} V\right] \\
&\quad - \xi\left[4\xi^2 V_{\xi\xi} + \left(6 + \frac{8\sqrt{1+E}}{p-1}\right)\xi V_\xi + \left(\frac{4(1+E)}{(p-1)^2} + \frac{2\sqrt{1+E}}{p-1} - \frac{(p+1)^2}{(p-1)^2}\right) V - \frac{p+1}{p-1} U\right].
\end{aligned} \tag{3.44}$$

Based on our Ansatz that U and V constitute analytic functions, we assume:

$$U(\xi) = \sum_{k=0}^{\infty} a_k \xi^k, \tag{3.45}$$

$$V(\xi) = \sum_{k=0}^{\infty} b_k \xi^k. \tag{3.46}$$

(3.43) now reads:

$$
\begin{aligned}
0 = & \sum_{k=0}^{\infty}\left[4\xi^2 a_k k(k-1)\xi^{k-2}+\left(\frac{8\sqrt{1+E}}{p-1}+4\right)\xi a_k k\xi^{k-1}\right] \\
& -\xi\sum_{k=0}^{\infty}\left[4\xi^2 a_k k(k-1)\xi^{k-2}+\left(6+\frac{8\sqrt{1+E}}{p-1}\right)\xi a_k k\xi^{k-1}\right] \\
& -\xi\sum_{k=0}^{\infty}\left[\left(\frac{4(1+E)}{(p-1)^2}+\frac{2\sqrt{1+E}}{p-1}-\frac{(p+1)^2}{(p-1)^2}\right)a_k\xi^k-\frac{p+1}{p-1}b_k\xi^k\right] \\
= & \sum_{k=1}^{\infty}\left[4a_k k^2\xi^k+\frac{8\sqrt{1+E}}{p-1}a_k k\xi^k\right] \\
& -\sum_{k=0}^{\infty}\left[4a_k k^2\xi^{k+1}+\left(2+\frac{8\sqrt{1+E}}{p-1}\right)a_k k\xi^{k+1}\right] \\
& -\sum_{k=0}^{\infty}\left[\left(\frac{4(1+E)}{(p-1)^2}+\frac{2\sqrt{1+E}}{p-1}-\frac{(p+1)^2}{(p-1)^2}\right)a_k\xi^{k+1}-\frac{p+1}{p-1}b_k\xi^{k+1}\right] \\
= & \sum_{k=0}^{\infty}a_{k+1}\xi^{k+1}\left[4(k+1)^2+\frac{8\sqrt{1+E}}{p-1}(k+1)\right]+\sum_{k=0}^{\infty}b_k\xi^{k+1}\frac{p+1}{p-1} \\
& -\sum_{k=0}^{\infty}a_k\xi^{k+1}\left[4k^2+\left(2+\frac{8\sqrt{1+E}}{p-1}\right)k+\frac{4(1+E)}{(p-1)^2}+\frac{2\sqrt{1+E}}{p-1}-\frac{(p+1)^2}{(p-1)^2}\right].
\end{aligned}
\tag{3.47}
$$

Analogously:

$$
\begin{aligned}
0 = & \sum_{k=0}^{\infty}b_{k+1}\xi^{k+1}\left[4(k+1)^2+\frac{8\sqrt{1+E}}{p-1}(k+1)+\frac{8E}{(p-1)^2}\right]+\sum_{k=0}^{\infty}a_k\xi^{k+1}\frac{p+1}{p-1} \\
& -\sum_{k=0}^{\infty}b_k\xi^{k+1}\left[4k^2+\left(2+\frac{8\sqrt{1+E}}{p-1}\right)k+\frac{4(1+E)}{(p-1)^2}+\frac{2\sqrt{1+E}}{p-1}-\frac{(p+1)^2}{(p-1)^2}\right] \\
& +b_0\frac{8E}{(p-1)^2}.
\end{aligned}
\tag{3.48}
$$

Lemma 3.5.1 *Let $-1 \leq E \neq 0$. Let $(a_k)_{k\in\mathbb{N}}, (b_k)_{k\in\mathbb{N}} \subseteq \mathbb{R}$ be given by $(a_0, b_0) = (1, 0)$, as well as:*

$$
\begin{aligned}
a_{k+1} = & \frac{4k^2+2k+\frac{8\sqrt{1+E}}{p-1}k+\frac{4(1+E)}{(p-1)^2}+\frac{2\sqrt{1+E}}{p-1}-\frac{(p+1)^2}{(p-1)^2}}{4(k+1)^2+\frac{8\sqrt{1+E}}{p-1}(k+1)}a_k \\
& -\frac{p+1}{p-1}\frac{1}{4(k+1)^2+\frac{8\sqrt{1+E}}{p-1}(k+1)}b_k,
\end{aligned}
\tag{3.49}
$$

and:

$$b_{k+1} = \frac{4k^2 + 2k + \frac{8\sqrt{1+E}}{p-1}k + \frac{4(1+E)}{(p-1)^2} + \frac{2\sqrt{1+E}}{p-1} - \frac{(p+1)^2}{(p-1)^2}}{4(k+1)^2 + \frac{8\sqrt{1+E}}{p-1}(k+1) + \frac{8E}{(p-1)^2}} b_k - \frac{p+1}{p-1} \frac{1}{4(k+1)^2 + \frac{8\sqrt{1+E}}{p-1}(k+1) + \frac{8E}{(p-1)^2}} a_k. \tag{3.50}$$

Then, $w = (u, v)$ given by:

$$u(z) = (1-z^2)^{\frac{\sqrt{1+E}}{p-1}} \sum_{k=0}^{\infty} a_k (1-z^2)^k, \tag{3.51}$$

$$v(z) = (1-z^2)^{\frac{\sqrt{1+E}}{p-1}} \sum_{k=0}^{\infty} b_k (1-z^2)^k, \tag{3.52}$$

is well-defined for $-1 < z < 0$ and fulfils (3.16), (3.17)*, as well as $w \in \tilde{D}_{E,-}$. If $E > -1$, then $w \in D_{E,-}$.*

Proof. By comparing coefficients, we find that (3.47) is equivalent to (3.49). Also by comparing coefficients (3.48) is equivalent to

$$\frac{8E}{(p-1)^2} b_0 = 0 \Leftrightarrow b_0 = 0. \tag{3.53}$$

and (3.50). Further, (3.49) and (3.50) imply that $\sum_k a_k \xi^k$ and $\sum_k b_k \xi^k$ have at least convergence radius 1. The same convergence radius can also be derived from the distance of the singularities $\xi = 0$ and $\xi = 1$. That concludes the proof. □

3.6. The Odd Case

Given a solution $(\tilde{u}, \tilde{v}) \in C(-1,0)^2$ of (3.16) and (3.17), consider $(u, v) \in C(-1,0)^2$ as given by:

$$\tilde{u} = zu, \tag{3.54}$$

$$\tilde{v} = zv. \tag{3.55}$$

(3.16) is equivalent to:

$$\begin{aligned} 0 &= zu_{zz} + 2u_z - \frac{2z^2}{1-z^2}u_z - \frac{2z}{1-z^2}u \\ &\quad + \left(-\frac{4(1+E)}{(p-1)^2} \frac{z}{(1-z^2)^2} + \frac{(p+1)^2}{(p-1)^2} \frac{z}{1-z^2} \right) u + \frac{p+1}{p-1} \frac{z}{1-z^2} v \\ \Leftrightarrow 0 &= u_{zz} + \frac{2}{z}u_z - \frac{2z}{1-z^2}u_z - \frac{2}{1-z^2}u \\ &\quad + \left(-\frac{4(1+E)}{(p-1)^2} \frac{1}{(1-z^2)^2} + \frac{(p+1)^2}{(p-1)^2} \frac{1}{1-z^2} \right) u + \frac{p+1}{p-1} \frac{1}{1-z^2} v. \end{aligned} \tag{3.56}$$

Analogously, (3.17) can be restated as:

$$\begin{aligned}0 &= v_{zz} + \frac{2}{z}v_z - \frac{2z}{1-z^2}v_z - \frac{2}{1-z^2}v\\&\quad + \left(-\frac{4(1-E)}{(p-1)^2}\frac{1}{(1-z^2)^2} + \frac{(p+1)^2}{(p-1)^2}\frac{1}{1-z^2}\right)v + \frac{p+1}{p-1}\frac{1}{1-z^2}u.\end{aligned} \tag{3.57}$$

We proceed as in chapter 3.5. We introduce the coordinate transform $\xi = 1 - z^2$, construct a solution $\xi \mapsto w_E(\xi)$ via a power series in $\xi = 0$ with convergence radius 1.

In chapter 4, we show that the corresponding solution of (3.16) and (3.17) is non-odd, which suffices to rule out embedded eigenvalues. In chapter 5, we consider the spectral gap, which proves more complicated due to $D_{E,-}$ being two-dimensional.

Applying (3.34) and (3.35) to (3.56) and (3.57) yields:

$$\begin{aligned}0 &= 4(1-\xi)u_{\xi\xi} - 2u_\xi - 4u_\xi + \frac{4z^2}{1-z^2}u_\xi - \frac{2}{1-z^2}u\\&\quad + \left(-\frac{4(1+E)}{(p-1)^2}\frac{1}{(1-z^2)^2} + \frac{(p+1)^2}{(p-1)^2}\frac{1}{1-z^2}\right)u + \frac{p+1}{p-1}\frac{1}{1-z^2}v\\&= 4(1-\xi)u_{\xi\xi} - 6u_\xi + \frac{4(1-\xi)}{\xi}u_\xi + \left(-\frac{4(1+E)}{(p-1)^2}\frac{1}{\xi^2} + \frac{(p+1)^2}{(p-1)^2}\frac{1}{\xi} - \frac{2}{\xi}\right)u + \frac{p+1}{p-1}\frac{1}{\xi}v\\&= 4(1-\xi)u_{\xi\xi} + \left(\frac{4}{\xi} - 10\right)u_\xi + \left(-\frac{4(1+E)}{(p-1)^2}\frac{1}{\xi^2} + \frac{(p+1)^2}{(p-1)^2}\frac{1}{\xi} - \frac{2}{\xi}\right)u + \frac{p+1}{p-1}\frac{1}{\xi}v.\end{aligned} \tag{3.58}$$

Analogously:

$$\begin{aligned}0 &= 4(1-\xi)v_{\xi\xi} + \left(\frac{4}{\xi} - 10\right)v_\xi + \left(-\frac{4(1-E)}{(p-1)^2}\frac{1}{\xi^2} + \frac{(p+1)^2}{(p-1)^2}\frac{1}{\xi} - \frac{2}{\xi}\right)v + \frac{p+1}{p-1}\frac{1}{\xi}u\\&= 4(1-\xi)v_{\xi\xi} + \left(\frac{4}{\xi} - 10\right)v_\xi + \left(-\frac{4(1+E)}{(p-1)^2}\frac{1}{\xi^2} + \frac{(p+1)^2}{(p-1)^2}\frac{1}{\xi} - \frac{2}{\xi}\right)v + \frac{p+1}{p-1}\frac{1}{\xi}u\\&\quad + \frac{8E}{(p-1)^2}\frac{1}{\xi^2}v.\end{aligned} \tag{3.59}$$

We make the Ansatz:

$$u(\xi) = \xi^{\frac{\sqrt{1+E}}{p-1}}U(\xi), \tag{3.60}$$

$$v(\xi) = \xi^{\frac{\sqrt{1+E}}{p-1}}V(\xi). \tag{3.61}$$

Analogously to (3.41) and (3.42):

$$u_\xi = \xi^{\frac{\sqrt{1+E}}{p-1}}U_\xi + \frac{\sqrt{1+E}}{p-1}\xi^{\frac{\sqrt{1+E}}{p-1}-1}U, \tag{3.62}$$

$$u_{\xi\xi} = \xi^{\frac{\sqrt{1+E}}{p-1}}U_{\xi\xi} + \frac{2\sqrt{1+E}}{p-1}\xi^{\frac{\sqrt{1+E}}{p-1}-1}U_\xi + \left(\frac{1+E}{(p-1)^2} - \frac{\sqrt{1+E}}{p-1}\right)\xi^{\frac{\sqrt{1+E}}{p-1}-2}U. \tag{3.63}$$

The same holds true for v and V. Plugging this into (3.58) and dividing by $\xi^{\frac{\sqrt{1+E}}{p-1}-2}$ yields:

$$
\begin{aligned}
0 &= 4(1-\xi)\left[\xi^2 U_{\xi\xi} + \frac{2\sqrt{1+E}}{p-1}\xi U_\xi + \left(\frac{1+E}{(p-1)^2} - \frac{\sqrt{1+E}}{p-1}\right)U\right] \\
&\quad + \left(\frac{4}{\xi} - 10\right)\left[\xi^2 U_\xi + \frac{\sqrt{1+E}}{p-1}\xi U\right] + \left[-\frac{4(1+E)}{(p-1)^2} + \frac{(p+1)^2}{(p-1)^2}\xi - 2\xi\right]U + \frac{p+1}{p-1}\xi V \\
&= 4(1-\xi)\xi^2 U_{\xi\xi} + \left[4(1-\xi)\frac{2\sqrt{1+E}}{p-1} + 4 - 10\xi\right]\xi U_\xi + \frac{p+1}{p-1}\xi V \\
&\quad + \left[4(1-\xi)\left(\frac{1+E}{(p-1)^2} - \frac{\sqrt{1+E}}{p-1}\right) + (4-10\xi)\frac{\sqrt{1+E}}{p-1} - \frac{4(1+E)}{(p-1)^2} + \frac{(p+1)^2}{(p-1)^2}\xi - 2\xi\right]U \\
&= 4\xi^2 U_{\xi\xi} - 4\xi^3 U_{\xi\xi} + \left[\frac{8\sqrt{1+E}}{p-1} + 4 - \left(10 + \frac{8\sqrt{1+E}}{p-1}\right)\xi\right]\xi U_\xi \\
&\quad + \left[-\frac{4(1+E)}{(p-1)^2}\xi - \frac{6\sqrt{1+E}}{p-1}\xi + \frac{(p+1)^2}{(p-1)^2}\xi - 2\xi\right]U + \frac{p+1}{p-1}\xi V \\
&= \left[4\xi^2 U_{\xi\xi} + \left(\frac{8\sqrt{1+E}}{p-1} + 4\right)\xi U_\xi\right] \\
&\quad - \xi\left[4\xi^2 U_{\xi\xi} + \left(10 + \frac{8\sqrt{1+E}}{p-1}\right)\xi U_\xi + \left(\frac{4(1+E)}{(p-1)^2} + \frac{6\sqrt{1+E}}{p-1} + 2 - \frac{(p+1)^2}{(p-1)^2}\right)U - \frac{p+1}{p-1}V\right].
\end{aligned}
\tag{3.64}
$$

Analogously, it follows:

$$
\begin{aligned}
0 &= \left[4\xi^2 V_{\xi\xi} + \left(\frac{8\sqrt{1+E}}{p-1} + 4\right)\xi V_\xi\right] \\
&\quad - \xi\left[4\xi^2 V_{\xi\xi} + \left(10 + \frac{8\sqrt{1+E}}{p-1}\right)\xi V_\xi + \left(\frac{4(1+E)}{(p-1)^2} + \frac{6\sqrt{1+E}}{p-1} + 2 - \frac{(p+1)^2}{(p-1)^2}\right)V - \frac{p+1}{p-1}U\right] \\
&\quad + \frac{8E}{(p-1)^2}V \\
&= \left[4\xi^2 V_{\xi\xi} + \left(\frac{8\sqrt{1+E}}{p-1} + 4\right)\xi V_\xi + \frac{8E}{(p-1)^2}V\right] \\
&\quad - \xi\left[4\xi^2 V_{\xi\xi} + \left(10 + \frac{8\sqrt{1+E}}{p-1}\right)\xi V_\xi + \left(\frac{4(1+E)}{(p-1)^2} + \frac{6\sqrt{1+E}}{p-1} + 2 - \frac{(p+1)^2}{(p-1)^2}\right)V - \frac{p+1}{p-1}U\right].
\end{aligned}
\tag{3.65}
$$

We assume U and V to be analytic in $\xi = 0$ and write:

$$U(\xi) = \sum_{k=0}^{\infty} a_k \xi^k, \tag{3.66}$$

$$V(\xi) = \sum_{k=0}^{\infty} b_k \xi^k. \tag{3.67}$$

It follows from (3.64):

$$\begin{aligned}
0 = & \sum_{k=0}^{\infty} \left[4\xi^2 a_k k(k-1)\xi^{k-2} + \left(\frac{8\sqrt{1+E}}{p-1} + 4 \right) \xi a_k k \xi^{k-1} \right] \\
& - \xi \sum_{k=0}^{\infty} \left[4\xi^2 a_k k(k-1)\xi^{k-2} + \left(10 + \frac{8\sqrt{1+E}}{p-1} \right) \xi a_k k \xi^{k-1} \right] \\
& - \xi \sum_{k=0}^{\infty} \left[\left(\frac{4(1+E)}{(p-1)^2} + \frac{6\sqrt{1+E}}{p-1} + 2 - \frac{(p+1)^2}{(p-1)^2} \right) a_k \xi^k - \frac{p+1}{p-1} b_k \xi^k \right] \\
= & \sum_{k=1}^{\infty} \left[4a_k k^2 \xi^k + \frac{8\sqrt{1+E}}{p-1} a_k k \xi^k \right] \\
& - \sum_{k=0}^{\infty} \left[4a_k k^2 \xi^{k+1} + \left(6 + \frac{8\sqrt{1+E}}{p-1} \right) a_k k \xi^{k+1} \right] \\
& - \sum_{k=0}^{\infty} \left[\left(\frac{4(1+E)}{(p-1)^2} + \frac{6\sqrt{1+E}}{p-1} + 2 - \frac{(p+1)^2}{(p-1)^2} \right) a_k \xi^{k+1} - \frac{p+1}{p-1} b_k \xi^{k+1} \right] \\
= & \sum_{k=0}^{\infty} a_{k+1} \xi^{k+1} \left[4(k+1)^2 + \frac{8\sqrt{1+E}}{p-1}(k+1) \right] + \sum_{k=0}^{\infty} b_k \xi^{k+1} \frac{p+1}{p-1} \\
& - \sum_{k=0}^{\infty} a_k \xi^{k+1} \left[4k^2 + \left(6 + \frac{8\sqrt{1+E}}{p-1} \right) k + \frac{4(1+E)}{(p-1)^2} + \frac{6\sqrt{1+E}}{p-1} + 2 - \frac{(p+1)^2}{(p-1)^2} \right].
\end{aligned} \tag{3.68}$$

Analogously, from (3.65):

$$\begin{aligned}
0 = & \sum_{k=0}^{\infty} b_{k+1} \xi^{k+1} \left[4(k+1)^2 + \frac{8\sqrt{1+E}}{p-1}(k+1) + \frac{8E}{(p-1)^2} \right] + \sum_{k=0}^{\infty} a_k \xi^{k+1} \frac{p+1}{p-1} \\
& - \sum_{k=0}^{\infty} b_k \xi^{k+1} \left[4k^2 + \left(6 + \frac{8\sqrt{1+E}}{p-1} \right) k + \frac{4(1+E)}{(p-1)^2} + \frac{6\sqrt{1+E}}{p-1} + 2 - \frac{(p+1)^2}{(p-1)^2} \right] \\
& + b_0 \frac{8E}{(p-1)^2}.
\end{aligned} \tag{3.69}$$

Lemma 3.6.1 *Let $-1 \leq E \neq 0$. Let $(a_k)_{k\in\mathbb{N}}, (b_k)_{k\in\mathbb{N}} \subseteq \mathbb{R}$ be given by $(a_0, b_0) = (1, 0)$,* *as well as:*

$$a_{k+1} = \frac{4k^2 + 6k + \frac{8\sqrt{1+E}}{p-1}k + \frac{4(1+E)}{(p-1)^2} + \frac{6\sqrt{1+E}}{p-1} + 2 - \frac{(p+1)^2}{(p-1)^2}}{4(k+1)^2 + \frac{8\sqrt{1+E}}{p-1}(k+1)} a_k - \frac{p+1}{p-1} \frac{1}{4(k+1)^2 + \frac{8\sqrt{1+E}}{p-1}(k+1)} b_k, \tag{3.70}$$

and:

$$b_{k+1} = \frac{4k^2 + 6k + \frac{8\sqrt{1+E}}{p-1}k + \frac{4(1+E)}{(p-1)^2} + \frac{6\sqrt{1+E}}{p-1} + 2 - \frac{(p+1)^2}{(p-1)^2}}{4(k+1)^2 + \frac{8\sqrt{1+E}}{p-1}(k+1) + \frac{8E}{(p-1)^2}} b_k - \frac{p+1}{p-1} \frac{1}{4(k+1)^2 + \frac{8\sqrt{1+E}}{p-1}(k+1) + \frac{8E}{(p-1)^2}} a_k. \tag{3.71}$$

Then, $w = (u, v)$ given by:

$$u(z) = (1 - z^2)^{\frac{\sqrt{1+E}}{p-1}} z \sum_{k=0}^{\infty} a_k (1 - z^2)^k, \tag{3.72}$$

$$v(z) = (1 - z^2)^{\frac{\sqrt{1+E}}{p-1}} z \sum_{k=0}^{\infty} b_k (1 - z^2)^k, \tag{3.73}$$

is well-defined for $-1 < z < 0$ and fulfils (3.16) *and* (3.17)*, as well as $w \in \tilde{D}_{E,-}$. If $E > -1$, then $w \in D_{E,-}$.*

Proof. Follows analogously to lemma 3.5.1. □

4. Absence of Embedded Eigenvalues

We have constructed solutions to (3.16) and (3.17), which is just a reformulation of the eigenvalue equation $Lw = iEw$. In this chapter we examine the coefficients $(a_k)_k$, $(b_k)_k$ given by lemma 3.5.1 and lemma 3.6.1.

We establish a connection between the coefficients $(a_k)_k, (b_k)_k$ and the initial data $w(0)$ and $w'(0)$ for solutions of (3.16) and (3.17). In particular, we show that the constructed solutions are non-even and non-odd respectively.

This then directly implies that no eigenvalues exist within the essential spectrum $|E| \geq 1$.

The entire chapter is based on using the following lemma to compare $(a_k)_k, (b_k)_k$ to the coefficients of well-known power series.

Lemma 4.0.1 *Consider $\alpha, \beta, A, B \in \mathbb{R}$ with $\alpha - \beta = A - B$. Let $c, d \in \mathbb{R}\backslash\{0\}$. Let $(\gamma_k)_{k\in\mathbb{N}}, (\delta)_{k\in\mathbb{N}} \subseteq \mathbb{R}$ and $S > 0$, such that $|\gamma_k|, |\delta_k| \leq S$ for every $k \in \mathbb{N}$.*

Choose $K > 0$ with $k^2 + \alpha k + \gamma_k, k^2 + \beta k + \delta_k, k + A, k + B > 0$ for $k \geq K$, and define:

$$c_K = c, \tag{4.1}$$

$$c_{k+1} = \frac{k^2 + \alpha k + \gamma_k}{k^2 + \beta k + \delta_k} c_k, \tag{4.2}$$

$$d_K = d, \tag{4.3}$$

$$d_{k+1} = \frac{k + A}{k + B} d_k. \tag{4.4}$$

Then, there exists $j \in \mathbb{R}\backslash\{0\}$ and $C > 0$, such that for every $k > K$:

$$|jc_k - d_k| \leq \frac{C}{k} |d_k|. \tag{4.5}$$

Proof. Consider for $k \geq K$:

$$\ln\left(\frac{c_{k+1}}{c_k}\right) = \ln\left(\frac{1 + \frac{\alpha}{k} + \frac{\gamma_k}{k^2}}{1 + \frac{\beta}{k} + \frac{\delta_k}{k^2}}\right) = \ln\left(1 + \frac{\frac{\alpha-\beta}{k} + \frac{\gamma_k - \delta_k}{k^2}}{1 + \frac{\beta}{k} + \frac{\delta_k}{k^2}}\right) = \frac{\alpha - \beta}{k} + \mathcal{O}(k^{-2}). \tag{4.6}$$

Analogously:

$$\ln\left(\frac{d_{k+1}}{d_k}\right) = \frac{A - B}{k} + \mathcal{O}(k^{-2}) = \frac{\alpha - \beta}{k} + \mathcal{O}(k^{-2}). \tag{4.7}$$

We define:

$$\mu_k := \ln\left(\frac{c_{k+1}}{c_k}\right) - \frac{\alpha - \beta}{k} \in \mathcal{O}(k^{-2}), \tag{4.8}$$

$$\nu_k := \ln\left(\frac{d_{k+1}}{d_k}\right) - \frac{\alpha - \beta}{k} \in \mathcal{O}(k^{-2}). \tag{4.9}$$

It follows for $k \geq l \geq K$:

$$\begin{aligned}\left|\ln\left(\frac{c_k}{d_k}\right) - \ln\left(\frac{c_l}{d_l}\right)\right| &= \left|\ln\left(\frac{c_k}{c_l}\right) - \ln\left(\frac{d_k}{d_l}\right)\right| \\ &\leq \sum_{n=l}^{k-1}\left|\ln\left(\frac{c_{n+1}}{c_n}\right) - \ln\left(\frac{d_{n+1}}{d_n}\right)\right| \\ &= \sum_{n=l}^{k-1}|\mu_n - \nu_n| \in \mathcal{O}(l^{-1}). \end{aligned} \tag{4.10}$$

We conclude that $\ln\left(\frac{c_k}{d_k}\right)$ is a Cauchy sequence. Let $j := \lim_{k\to\infty}\ln\left(\frac{c_k}{d_k}\right) \in \mathbb{R}$. We define:

$$\varepsilon_k := j - \ln\left(\frac{c_k}{d_k}\right) = \sum_{n=k}^{\infty}\left(\ln\left(\frac{c_{n+1}}{c_n}\right) - \ln\left(\frac{d_{n+1}}{d_n}\right)\right). \tag{4.11}$$

It follows from (4.10):

$$|\varepsilon_k| \leq \frac{C}{k}. \tag{4.12}$$

We conclude:

$$\frac{c_k}{d_k} = e^{j-\varepsilon_k} \Rightarrow \left|\frac{c_k}{d_k} - e^j\right| = e^j\left|e^{-\varepsilon_k} - 1\right| \leq e^j\left|e^{\frac{C}{k}} - 1\right| \leq \frac{C'e^j}{k}. \tag{4.13}$$

That concludes the proof. $\square$

4.1. The Even Case

Definition 4.1.1 *Let $E \geq -1$ and $k \geq 0$. We define:*

$$A_k = A_k(E) := 4k^2 + 2k + \frac{8\sqrt{1+E}}{p-1}k + \frac{4(1+E)}{(p-1)^2} + \frac{2\sqrt{1+E}}{p-1} - \frac{(p+1)^2}{(p-1)^2}, \tag{4.14}$$

$$B_k = B_k(E) := 4(k+1)^2 + \frac{8\sqrt{1+E}}{p-1}(k+1). \tag{4.15}$$

Recall lemma 3.5.1. (3.49) and (3.50) can be rewritten as:

$$a_{k+1} = \frac{A_k}{B_k}a_k - \frac{p+1}{p-1}\frac{1}{B_k}b_k, \tag{4.16}$$

$$b_{k+1} = \frac{A_k}{B_k + \frac{8E}{(p-1)^2}}b_k - \frac{p+1}{p-1}\frac{1}{B_k + \frac{8E}{(p-1)^2}}a_k. \tag{4.17}$$

Lemma 4.1.2 *Let $E \geq -1$, but exclude the case $E = -1$, $p = 3$. Let $(a_k)_{k\in\mathbb{N}}, (b_k)_{k\in\mathbb{N}} \subseteq \mathbb{R}$ be given by $(a_0, b_0) = (1, 0)$ and (4.16), (4.17). Then, for every $k \geq 0$:*

$$(a_k, b_k) \neq (0, 0). \tag{4.18}$$

Proof. By definition, $a_0 \neq 0$ and $b_1 = -\frac{p+1}{p-1}\frac{a_0}{B_k+\frac{8E}{(p-1)^2}} \neq 0$.

Assume the lemma does not hold. We find $k \geq 1$ with $a_{k+1} = b_{k+1} = 0$ and $(a_k, b_k) \neq (0,0)$. It follows:

$$0 = A_k a_k - \frac{p+1}{p-1} b_k = A_k b_k - \frac{p+1}{p-1} a_k \Rightarrow A_k^2 = \left(\frac{p+1}{p-1}\right)^2. \tag{4.19}$$

For $E > -1$, $k \geq 1$, this is a contradiction to:

$$A_k > 6 - 2^2 = 2 \geq \frac{p+1}{p-1}. \tag{4.20}$$

Similarly, $E = -1$, $p > 3$, $k \geq 1$ implies:

$$A_k \geq 6 - 2^2 = 2 > \frac{p+1}{p-1}. \tag{4.21}$$

That concludes the proof. $\square$

Remark *Consider $E = -1$ and $p = 3$. By direct calculation, $(a_1, b_1) = (-1, -1)$ and $(a_k, b_k) = (0,0)$ for $k \geq 2$. That corresponds to a resonance of L:*

$$(L+i)\begin{pmatrix} 1-Q^2 \\ i \end{pmatrix} = (L-i)\begin{pmatrix} 1-Q^2 \\ -i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{4.22}$$

As we will later on prove, no such resonance exists for $3 < p \leq 5$.

Using lemma 4.0.1, we compare $\sum_k a_k \xi^k$ and $\sum_k b_k \xi^k$ to the power series:

$$z - 1 = \sqrt{1-\xi} - 1 = \sum_{n=1}^{\infty} \prod_{k=1}^{n} \frac{k - \frac{3}{2}}{k} \xi^n = \sum_{n=1}^{\infty} \binom{n - \frac{3}{2}}{n} \xi^n. \tag{4.23}$$

Lemma 4.1.3 *Let $E \geq -1$, but exclude the case $E = -1$, $p = 3$. Let $(a_k)_{k\in\mathbb{N}}, (b_k)_{k\in\mathbb{N}} \subseteq \mathbb{R}$ be given by $(a_0, b_0) = (1, 0)$ and (4.16), (4.17). Then, there are $j_1, j_2 \in \mathbb{R}$ and $C > 0$ with:*

$$\forall n \geq 1 : \left| a_n - j_1 \prod_{k=1}^{n} \frac{k-\frac{3}{2}}{k} \right| + \left| b_n - j_2 \prod_{k=1}^{n} \frac{k-\frac{3}{2}}{k} \right| \leq \frac{C}{n} \left| \prod_{k=1}^{n} \frac{k-\frac{3}{2}}{k} \right|. \tag{4.24}$$

Proof. Let $a := a_1$, $b := b_1 < 0$. We define sequences $(a_{k,+})_{k\geq 1}, (b_{k,+})_{k\geq 1} \subseteq \mathbb{R}$ by

$$(a_{1,+}, b_{1,+}) = (a, 0), \tag{4.25}$$
$$(a_{1,-}, b_{1,-}) = (0, b) \tag{4.26}$$

and mandating that both $(a_{k,+}, b_{k,+})_{k\geq 1}$ and $(a_{k,-})_{k\geq 1}, (b_{k,-})_{k\geq 1}$ fulfil (4.16) and (4.17). Then, for $k \geq 1$:

$$a_k = a_{k,+} + a_{k,-}, \tag{4.27}$$
$$b_k = b_{k,+} + b_{k,-}. \tag{4.28}$$

Considering $a_{1,-} = 0$ and $b_{1,-} = b < 0$, (4.16) and (4.17) imply inductively:

$$\forall k \geq 2 : a_{k,-} > 0 > b_{k,-}. \tag{4.29}$$

We define $c_{k,-} := \left|\frac{a_{k,-}}{b_{k,-}}\right| = -\frac{a_{k,-}}{b_{k,-}} > 0$ for $k \geq 2$. It follows:

$$\begin{aligned}
0 < c_{k+1,-} &= -\frac{B_k + \frac{8E}{(p-1)^2}}{B_k} \frac{A_k a_{k,-} - \frac{p+1}{p-1} b_{k,-}}{A_k b_{k,-} - \frac{p+1}{p-1} a_{k,-}} \\
&= \left(1 + \frac{8E}{(p-1)^2 B_k}\right) \frac{A_k c_{k,-} + \frac{p+1}{p-1}}{A_k + \frac{p+1}{p-1} c_{k,-}} && (4.30) \\
&= \left(1 + \frac{8E}{(p-1)^2 B_k}\right) \left(1 - \frac{(p+1)(c_{k,-} - c_{k,-}^{-1})}{(p-1)(A_k + \frac{p+1}{p-1} c_{k,-})}\right) c_{k,-}. && (4.31)
\end{aligned}$$

We choose some $0 < c < 1$. Then, for $k \geq 2$ with $c_{k,-} < c$:

$$\begin{aligned}
&\left(1 + \frac{8E}{(p-1)^2 B_k}\right) \left(1 - \frac{p+1}{(p-1)(A_k + \frac{p+1}{p-1} c_{k,-})} (c_{k,-} - c_{k,-}^{-1})\right) \\
&> \left(1 - \frac{8}{(p-1)^2 B_k}\right) \left(1 + \frac{p+1}{(p-1)(A_k + \frac{p+1}{p-1} c_{k,-})} (c^{-1} - c)\right). && (4.32)
\end{aligned}$$

Choosing c small enough ensures $c_{k+1,-} > c_{k,-}$. On the other hand, $c_{k,-} \geq c$ implies:

$$\begin{aligned}
c_{k+1,-} &= \left(1 + \frac{8E}{(p-1)^2 B_k}\right) \left(\frac{A_k c_{k,-} + \frac{p+1}{p-1}}{A_k + \frac{p+1}{p-1} c_{k,-}}\right) \\
&\geq \left(1 - \frac{8}{(p-1)^2 B_1}\right) \frac{A_k c + \frac{p+1}{p-1}}{A_k + \frac{p+1}{p-1} c} \\
&\geq c \left(1 - \frac{8}{(p-1)^2 B_1}\right) \\
&\geq \frac{c}{2}. && (4.33)
\end{aligned}$$

Therefore, we find $j > 0$ with:

$$\forall k \geq 2 : c_{k,-} > j. \tag{4.34}$$

Completely analogously, we find $J > 0$ with:

$$\forall k \geq 2 : c_{k,-} < J. \tag{4.35}$$

Plugging $\frac{a_{k,-}}{b_{k,-}} = -c_{k,-}$ into (3.49) and (3.50) implies for $k \geq 2$:

$$a_{k+1,-} = \frac{k^2 + \frac{1}{2}k + \frac{2\sqrt{1+E}}{p-1}k + \frac{1+E}{(p-1)^2} + \frac{\sqrt{1+E}}{2(p-1)} - \frac{(p+1)^2}{4(p-1)^2} + \frac{p+1}{p-1}\frac{c_{k,-}^{-1}}{4}}{k^2 + 2k + \frac{2\sqrt{1+E}}{p-1}k + \frac{2\sqrt{1+E}}{p-1} + 1} a_{k,-}, \tag{4.36}$$

$$b_{k+1,-} = \frac{k^2 + \frac{1}{2}k + \frac{2\sqrt{1+E}}{p-1}k + \frac{1+E}{(p-1)^2} + \frac{\sqrt{1+E}}{2(p-1)} - \frac{(p+1)^2}{4(p-1)^2} + \frac{p+1}{p-1}\frac{c_{k,-}}{4}}{k^2 + 2k + \frac{2\sqrt{1+E}}{p-1}k + \frac{2\sqrt{1+E}}{p-1} + 1 + \frac{2E}{(p-1)^2}} b_{k,-}. \tag{4.37}$$

As both c_k and c_k^{-1} are bounded, lemma 4.0.1 implies the existence of $j_{1,-}, j_{2,-} \in \mathbb{R}\backslash\{0\}$ and $C > 0$ with:

$$\forall n \geq 2 : \left| a_{n,-} - j_{1,-} \prod_{k=1}^{n} \frac{k - \frac{3}{2}}{k} \right|, \left| b_{n,-} - j_{2,-} \prod_{k=1}^{n} \frac{k - \frac{3}{2}}{k} \right| \leq \frac{C}{n} \left| \prod_{k=1}^{n} \frac{k - \frac{3}{2}}{k} \right|. \tag{4.38}$$

By repeating the same argument for $(a_{k,+})_k, (b_{k,+})_k$, we find $j_{1,+}, j_{2,+} \in \mathbb{R}\backslash\{0\}$ and $C > 0$ with:

$$\forall n \geq 2 : \left| a_{n,+} - j_{1,+} \prod_{k=1}^{n} \frac{k - \frac{3}{2}}{k} \right|, \left| b_{n,+} - j_{2,+} \prod_{k=1}^{n} \frac{k - \frac{3}{2}}{k} \right| \leq \frac{C}{n} \left| \prod_{k=1}^{n} \frac{k - \frac{3}{2}}{k} \right|. \tag{4.39}$$

We define $j_1 := j_{1,+} + j_{1,-}$ and $j_2 := j_{2,+} + j_{2,-}$. That concludes the proof. □

Definition 4.1.4 *Let $E \geq -1$, but exclude the case $E = -1$, $p = 3$. Let $(a_k)_{k\in\mathbb{N}}, (b_k)_{k\in\mathbb{N}} \subseteq \mathbb{R}$ be given by $(a_0, b_0) = (1, 0)$ and (4.16), (4.17). We define $c_k \in \mathbb{R} \cup \{\infty\}$ for $k \geq 0$ by:*

$$c_k = \frac{a_k}{b_k}. \tag{4.40}$$

To denote the dependency on p and E, we also write $c_k = c_k(E, p)$. If E or p are clear from context, as will be the case for large parts of chapter 5, we also write $c_k = c_k(E)$ or $c_k = c_k(p)$.

Lemma 4.1.5 *Let $E \geq -1$, but exclude the case $E = -1$, $p = 3$. Consider the Riemann sphere $\mathbb{C} \cup \{\infty\}$. Then, $(c_k)_{k\in\mathbb{N}}$ is given by $c_0 = \infty$ and recursion via the Möbius transformation:*

$$c_{k+1} = \left(1 + \frac{8E}{(p-1)^2 B_k}\right) \frac{A_k c_k - \frac{p+1}{p-1}}{A_k - \frac{p+1}{p-1} c_k}. \tag{4.41}$$

Proof. Follows by definition and lemma 4.1.2. □

Remark *The singularity in $(E, p) = (-1, 3)$ of the multivariable meromorphic function $c_k(E, p)$ for $k \geq 2$ is not removable. In general, $\lim_{E\searrow -1} c_k(E, 3) \neq \lim_{p\searrow 3} c_k(-1, p)$.*

Lemma 4.1.6 *Let $E \geq -1$, but exclude the case $E = -1$, $p = 3$. Let $(a_k)_{k\in\mathbb{N}}, (b_k)_{k\in\mathbb{N}} \subseteq \mathbb{R}$ be given by $(a_0, b_0) = (1, 0)$ and (4.16), (4.17). Then, for $j_1, j_2 \in \mathbb{R}$ as given by lemma 4.1.3:*

$$(j_1, j_2) \neq (0, 0). \tag{4.42}$$

Proof. Assume $E \geq 0$. Then, by definition, $|c_k| \geq 1$ implies $|c_{k+1}| \geq 1$. Now assume $E < 0$. In that case, $|c_k| \leq 1$ implies $|c_{k+1}| \leq 1$.

In any case, we find $K \geq 1$, such that:

$$\forall k \geq K : |c_k| \leq 1 \tag{4.43}$$

or:

$$\forall k \geq K : \left|c_k^{-1}\right| \leq 1. \tag{4.44}$$

We restate (4.16) and (4.17):

$$a_{k+1} = \frac{k^2 + \frac{1}{2}k + \frac{2\sqrt{1+E}}{p-1}k + \frac{1+E}{(p-1)^2} + \frac{\sqrt{1+E}}{2(p-1)} - \frac{(p+1)^2}{4(p-1)^2} + \frac{p+1}{p-1}\frac{c_k^{-1}}{4}}{k^2 + 2k + \frac{2\sqrt{1+E}}{p-1}k + \frac{2\sqrt{1+E}}{p-1} + 1} a_k, \tag{4.45}$$

$$b_{k+1} = \frac{k^2 + \frac{1}{2}k + \frac{2\sqrt{1+E}}{p-1}k + \frac{1+E}{(p-1)^2} + \frac{\sqrt{1+E}}{2(p-1)} - \frac{(p+1)^2}{4(p-1)^2} + \frac{p+1}{p-1}\frac{c_k}{4}}{k^2 + 2k + \frac{2\sqrt{1+E}}{p-1}k + \frac{2\sqrt{1+E}}{p-1} + 1 + \frac{2E}{(p-1)^2}} b_k. \tag{4.46}$$

By lemma 4.0.1, (4.44) and (4.45) imply $j_1 \neq 0$, while (4.43) and (4.46) imply $j_2 \neq 0$. That concludes the proof. □

Corollary 4.1.7 *Let $E \geq -1$, but exclude the case $E = -1$, $p = 3$. Let $(c_k)_{k\in\mathbb{N}}$ be given by definition 4.1.4. Let further $j_1, j_2 \in \mathbb{R}$ be as given by lemma 4.1.3 Then:*

$$\lim_{k\to\infty} c_k = \frac{j_1}{j_2}. \tag{4.47}$$

Proof. Follows from lemma 4.1.3 and definition 4.1.4. □

Definition 4.1.8 *Let $E \geq -1$, but exclude the case $E = -1$, $p = 3$. We define*

$$\mathcal{C}(E, p) = \lim_{k\to\infty} c_k(E, p). \tag{4.48}$$

Lemma 4.1.9 *$(E, p) \mapsto \mathcal{C}(E, p)$ is a real meromorphic function for $(-1, 3) \neq (E, p) \in [-1, \infty) \times [3, 5]$.*

Proof. By corollary 4.1.7, $\mathcal{C}(E, p) = \frac{j_1(E,p)}{j_2(E,p)}$. By definition, j_1 and j_2 are real analytic functions with $(j_1, j_2) \neq (0, 0)$. That already concludes the proof. □

Lemma 4.1.10 *Let $E \geq -1$, but exclude the case $E = -1$, $p = 3$. Let $(a_k)_{k\in\mathbb{N}}, (b_k)_{k\in\mathbb{N}} \subseteq \mathbb{R}$ be given by $(a_0, b_0) = (1, 0)$ and (4.16), (4.17). Then, $w = (u, v)$ as given by:*

$$u(z) = (1-z^2)^{\frac{\sqrt{1+E}}{p-1}} \sum_{k=0}^{\infty} a_k (1-z^2)^k, \tag{4.49}$$

$$v(z) = (1-z^2)^{\frac{\sqrt{1+E}}{p-1}} \sum_{k=0}^{\infty} b_k (1-z^2)^k \tag{4.50}$$

is well-defined and solves (3.16) and (3.17) for $-1 < z < 0$. Given, j_1, j_2 as in lemma 4.1.3, w fulfils:

$$\lim_{z \nearrow 0} w'(z) = -(j_1, j_2) \neq (0, 0). \tag{4.51}$$

Proof. By lemma 3.5.1, it suffices to prove (4.51). We define:

$$U(z) := \sum_{n=0}^{\infty} a_n (1-z^2)^n, \tag{4.52}$$

$$V(z) := \sum_{n=0}^{\infty} b_n (1-z^2)^n. \tag{4.53}$$

Let $\xi = 1 - z^2$. It follows for $-1 < z < 0$:

$$\begin{aligned} U(z) &= a_0 + \underbrace{\sum_{n=1}^{\infty} \left(a_n - j_1 \prod_{k=1}^{n} \frac{k - \frac{3}{2}}{k} \right) \xi^n}_{=:f(\xi)} + j_1 \sum_{n=1}^{\infty} \prod_{k=1}^{n} \frac{k - \frac{3}{2}}{k} \xi^n \\ &= a_0 + f(\xi) + j_1(\sqrt{1-\xi} - 1), \end{aligned} \tag{4.54}$$

$$\begin{aligned} V(z) &= b_0 + \underbrace{\sum_{n=1}^{\infty} \left(b_n - j_2 \prod_{k=1}^{n} \frac{k - \frac{3}{2}}{k} \right) \xi^n}_{=:g(\xi)} + j_2 \sum_{n=1}^{\infty} \prod_{k=1}^{n} \frac{k - \frac{3}{2}}{k} \xi^n \\ &= b_0 + g(\xi) + j_2(\sqrt{1-\xi} - 1). \end{aligned} \tag{4.55}$$

By lemma 4.1.3, for $0 < \xi < 1$:

$$\begin{aligned} |f(\xi)|, |g(\xi)| &\leq C \sum_{n=1}^{\infty} \prod_{k=1}^{n} \left| \frac{k - \frac{3}{2}}{k} \right| \xi^n \\ &= -C \sum_{n=1}^{\infty} \prod_{k=1}^{n} \frac{k - \frac{3}{2}}{k} \xi^n \\ &= C \left(1 - \sqrt{1-\xi} \right) \leq C. \end{aligned} \tag{4.56}$$

Analogously:

$$
\begin{aligned}
|f'(\xi)| &= \left| \sum_{n=1}^{\infty} \left(a_n - j_1 \prod_{k=1}^{n} \frac{k-\frac{3}{2}}{k} \right) n\xi^{n-1} \right| \\
&\leq C \sum_{n=1}^{\infty} \prod_{k=1}^{n} \left| \frac{k-\frac{3}{2}}{k} \right| \xi^{n-1} \\
&\leq C \frac{1-\sqrt{1-\xi}}{\xi} \\
&= \frac{C}{1+\sqrt{1-\xi}} \leq C, && (4.57) \\
|g'(\xi)| &\leq \frac{C}{1+\sqrt{1-\xi}} \leq C. && (4.58)
\end{aligned}
$$

Putting everything together yields:

$$
\begin{aligned}
U(z) &= a_0 + f(1-z^2) + j_1(-z-1), && (4.59) \\
V(z) &= b_0 + g(1-z^2) + j_2(-z-1). && (4.60)
\end{aligned}
$$

It follows:

$$
\begin{aligned}
U'(z) &= -2zf'(1-z^2) - j_1, && (4.61) \\
V'(z) &= -2zg'(1-z^2) - j_2. && (4.62)
\end{aligned}
$$

By definition:

$$
\begin{aligned}
u(z) &= (1-z^2)^{\frac{\sqrt{1+E}}{p-1}} U(z), && (4.63) \\
v(z) &= (1-z^2)^{\frac{\sqrt{1+E}}{p-1}} V(z). && (4.64)
\end{aligned}
$$

It follows:

$$
\begin{aligned}
\lim_{z \nearrow 0} u'(z) &= \lim_{z \nearrow 0} U'(z) = -j_1, && (4.65) \\
\lim_{z \nearrow 0} v'(z) &= \lim_{z \nearrow 0} V'(z) = -j_2. && (4.66)
\end{aligned}
$$

That concludes the proof. $\square$

4.2. The Odd Case

We proceed analogously to the even case.

Definition 4.2.1 *Let $E \geq -1$ and $k \geq 0$. We define:*

$$A_k = A_k(E) := 4k^2 + 6k + \frac{8\sqrt{1+E}}{p-1}k + \frac{4(1+E)}{(p-1)^2} + \frac{6\sqrt{1+E}}{p-1} + 2 - \frac{(p+1)^2}{(p-1)^2}, \tag{4.67}$$

$$B_k = B_k(E) := 4(k+1)^2 + \frac{8\sqrt{1+E}}{p-1}(k+1). \tag{4.68}$$

Recall lemma 3.6.1. (3.70) and (3.73) can be rewritten as:

$$a_{k+1} = \frac{A_k}{B_k}a_k - \frac{p+1}{p-1}\frac{1}{B_k}b_k, \tag{4.69}$$

$$b_{k+1} = \frac{A_k}{B_k + \frac{8E}{(p-1)^2}}b_k - \frac{p+1}{p-1}\frac{1}{B_k + \frac{8E}{(p-1)^2}}a_k. \tag{4.70}$$

Lemma 4.2.2 *Let $E \geq -1$. Let $(a_k)_{k\in\mathbb{N}}, (b_k)_{k\in\mathbb{N}} \subseteq \mathbb{R}$ be given by $(a_0, b_0) = (1, 0)$ and (4.69), (4.70). Then, for every $k \geq 0$:*

$$(a_k, b_k) \neq (0, 0). \tag{4.71}$$

Proof. By definition, $a_0 \neq 0$ and $b_1 = -\frac{p+1}{p-1}\frac{a_0}{B_k+\frac{8E}{(p-1)^2}} \neq 0$.

Assume that the lemma does not hold. We find $k \geq 1$ with $a_{k+1} = b_{k+1} = 0$ and $(a_k, b_k) \neq (0, 0)$. It follows:

$$0 = A_k a_k - \frac{p+1}{p-1}b_k = A_k b_k - \frac{p+1}{p-1}a_k \Rightarrow A_k^2 = \left(\frac{p+1}{p-1}\right)^2. \tag{4.72}$$

This is a contradiction to:

$$A_k \geq 12 - 2^2 > 2 \geq \frac{p+1}{p-1} \tag{4.73}$$

for $k \geq 1$. That concludes the proof. □

Using lemma 4.0.1, we compare $\sum_k a_k\xi^k$ and $\sum_k b_k\xi^k$ to the power series:

$$\frac{1}{z} - 1 = \frac{1}{\sqrt{1-\xi}} - 1 = \sum_{n=1}^{\infty}\prod_{k=1}^{n}\frac{k-\frac{1}{2}}{k}\xi^n = \sum_{n=1}^{\infty}\binom{n-\frac{1}{2}}{n}\xi^n. \tag{4.74}$$

Lemma 4.2.3 *Let $E \geq -1$. Let $(a_k)_{k\in\mathbb{N}}, (b_k)_{k\in\mathbb{N}} \subseteq \mathbb{R}$ be given by $(a_0, b_0) = (1, 0)$ and (4.69), (4.70). Then, there are $j_1, j_2 \in \mathbb{R}$ and $C > 0$ with:*

$$\left|a_n - j_1\prod_{k=1}^{n}\frac{k-\frac{1}{2}}{k}\right|, \left|b_n - j_2\prod_{k=1}^{n}\frac{k-\frac{1}{2}}{k}\right| \leq \frac{C}{n}\prod_{k=1}^{n}\frac{k-\frac{1}{2}}{k}. \tag{4.75}$$

Proof. Follows analogously to lemma 4.1.3. ☐

Lemma 4.2.4 *Let $E \geq -1$. Let $(a_k)_{k\in\mathbb{N}}, (b_k)_{k\in\mathbb{N}} \subseteq \mathbb{R}$ be given by $(a_0, b_0) = (1, 0)$ and* (4.69)*,* (4.70)*. Then, for $j_1, j_2 \in \mathbb{R}$ as given by lemma 4.2.3:*

$$(j_1, j_2) \neq (0, 0). \tag{4.76}$$

Proof. Follows analogously to lemma 4.1.6. ☐

Definition 4.2.5 *Let $E \geq -1$. Let $(a_k)_{k\in\mathbb{N}}, (b_k)_{k\in\mathbb{N}} \subseteq \mathbb{R}$ be given by $(a_0, b_0) = (1, 0)$ and* (4.69)*,* (4.70)*. We define $c_k \in \mathbb{R} \cup \{\infty\}$ for $k \geq 0$ by:*

$$c_k = \frac{a_k}{b_k}. \tag{4.77}$$

To denote the dependency on p and E, we also write $c_k = c_k(E, p)$. If E or p are clear from context, as will be the case for large parts of chapter 5, we also write $c_k = c_k(E)$ or $c_k = c_k(p)$.

Lemma 4.2.6 *Let $E \geq -1$. Consider the Riemann sphere $\mathbb{C} \cup \{\infty\}$. Then, $(c_k)_{k\in\mathbb{N}}$ is given by $c_0 = \infty$ and:*

$$c_{k+1} = \left(1 + \frac{8E}{(p-1)^2 B_k}\right) \frac{A_k c_k - \frac{p+1}{p-1}}{A_k - \frac{p+1}{p-1} c_k}. \tag{4.78}$$

Proof. Follows by definition and lemma 4.2.2. ☐

Corollary 4.2.7 *Let $E \geq -1$. Let $(c_k)_{k\in\mathbb{N}}$ be given by definition 4.2.5. Let further $j_1, j_2 \in \mathbb{R}$ be as given by lemma 4.2.3 Then:*

$$\lim_{k\to\infty} c_k = \frac{j_1}{j_2}. \tag{4.79}$$

Proof. Follows analogously to corollary 4.1.7. ☐

Definition 4.2.8 *Let $E \geq -1$. We define*

$$\mathcal{C}(E, p) = \lim_{k\to\infty} c_k(E, p). \tag{4.80}$$

Lemma 4.2.9 *$(E, p) \mapsto \mathcal{C}(E, p)$ is a real meromorphic function for $(E, p) \in [-1, \infty) \times [3, 5]$.*

Proof. Follows analogously to lemma 4.1.9. ☐

Lemma 4.2.10 *Let $E \geq -1$. Let $(a_k)_{k\in\mathbb{N}}, (b_k)_{k\in\mathbb{N}} \subseteq \mathbb{R}$ be given by $(a_0, b_0) = (1, 0)$ and* (4.69), (4.70)*. Then, $w = (u, v)$ as given by:*

$$u(z) = (1-z^2)^{\frac{\sqrt{1+E}}{p-1}} z \sum_{k=0}^{\infty} a_k (1-z^2)^k, \tag{4.81}$$

$$v(z) = (1-z^2)^{\frac{\sqrt{1+E}}{p-1}} z \sum_{k=0}^{\infty} b_k (1-z^2)^k \tag{4.82}$$

is well-defined and solves (3.16) *and* (3.17) *for $-1 < z < 0$. Given, j_1, j_2 as in lemma 4.2.3, w fulfils:*

$$\lim_{z \nearrow 0} w(z) = -(j_1, j_2) \neq (0, 0). \tag{4.83}$$

Proof. By lemma 3.6.1, it suffices to prove (4.83). We define:

$$U(z) := \sum_{n=0}^{\infty} a_n (1-z^2)^n, \tag{4.84}$$

$$V(z) := \sum_{n=0}^{\infty} b_n (1-z^2)^n. \tag{4.85}$$

Let $\xi = 1 - z^2$. It follows for $-1 < z < 0$:

$$\begin{aligned} U(z) &= a_0 + \underbrace{\sum_{n=1}^{\infty} \left(a_n - j_1 \prod_{k=1}^{n} \frac{k-\frac{1}{2}}{k} \right) \xi^n}_{=:f(\xi)} + j_1 \sum_{n=1}^{\infty} \prod_{k=1}^{n} \frac{k-\frac{1}{2}}{k} \xi^n \\ &= a_0 + f(\xi) + j_1 \left(\frac{1}{\sqrt{1-\xi}} - 1 \right), \end{aligned} \tag{4.86}$$

$$\begin{aligned} V(z) &= b_0 + \underbrace{\sum_{n=1}^{\infty} \left(b_n - j_2 \prod_{k=1}^{n} \frac{k-\frac{3}{2}}{k} \right) \xi^n}_{=:g(\xi)} + j_2 \sum_{n=1}^{\infty} \prod_{k=1}^{n} \frac{k-\frac{1}{2}}{k} \xi^n \\ &= b_0 + g(\xi) + j_2 \left(\frac{1}{\sqrt{1-\xi}} - 1 \right). \end{aligned} \tag{4.87}$$

By lemma 4.2.3, it follows for $0 < \xi < 1$:

$$\begin{aligned}
|f(\xi)|\,, |g(\xi)| &\leq C \sum_{n=1}^{\infty} \prod_{k=1}^{n} \frac{k - \frac{1}{2}}{k} \frac{\xi^n}{n} \\
&\leq \frac{2C}{\xi} \sum_{n=1}^{\infty} \prod_{k=1}^{n} \frac{k - \frac{1}{2}}{k} \frac{\xi^{n+1}}{n+1} \\
&= \frac{2C}{\xi} \left(2 - 2\sqrt{1-\xi} - \xi\right) \\
&= 2C \left(\frac{2 - 2\sqrt{1-\xi}}{\xi} - 1\right) \\
&= 2C \left(\frac{2}{1 + \sqrt{1-\xi}} - 1\right) \leq 2C. \qquad (4.88)
\end{aligned}$$

Putting everything together yields:

$$U(z) = a_0 + f(1 - z^2) + j_1 \left(-\frac{1}{z} - 1\right), \qquad (4.89)$$

$$V(z) = b_0 + g(1 - z^2) + j_2 \left(-\frac{1}{z} - 1\right). \qquad (4.90)$$

By definition:

$$u(z) = (1 - z^2)^{\frac{\sqrt{1+E}}{p-1}} z U(z), \qquad (4.91)$$

$$v(z) = (1 - z^2)^{\frac{\sqrt{1+E}}{p-1}} z V(z) \qquad (4.92)$$

It follows:

$$\lim_{z \nearrow 0} u(z) = -j_1, \qquad (4.93)$$

$$\lim_{z \nearrow 0} v(z) = -j_2. \qquad (4.94)$$

That concludes the proof. □

4.3. Absence of Eigenvalues

A direct consequence of the existence of the solutions given by lemma 4.1.10 and lemma 4.2.10 is that embedded eigenvalues can not exist.

The space of potential eigenfunctions $D_{E,-}$ is one-dimensional, consequently both of the solutions given by lemma 4.1.10 and by lemma 4.2.10 span said space. Because these two solutions are non-even and non-odd respectively, no eigenfunction can exist.

Theorem 4.3.1 *Let $p \in [3, 5]$. Then, iL has no eigenvalues embedded within the essential spectrum $(-\infty, -1] \cup [1, \infty)$.*

Proof. Symmetry allows us to only consider $E \geq 1$. Lemma 2.3.21 implies $\dim D_{E,-} = 1$. It follows $D_{E,e,-} = \{0\}$ from lemma 4.1.10 and $D_{E,o,-} = \{0\}$ from lemma 4.2.10. Lemma 3.4.6 concludes the proof. □

5. Spectrum between -1 and 1

5.1. Overview

We have concluded our investigation of the essential spectrum of iL, which was aided by $D_{E,-}$ being one-dimensional.

When checking for eigenvalues $E \in (-1,1)\backslash\{0\}$ and resonances $E = \pm 1$, $D_{E,-}$ resp. $\tilde{D}_{E,-}$ is two-dimensional, which makes showing $D_{E,e,-} = D_{E,o,-} = \{0\}$ much more difficult.

Let us recount the tools at our disposal. Through lemma 4.1.10, we have constructed, for every $E \in [-1,1]$ and $p \in [3,5]$ some solution $w_{E,p} = (u_{E,p}, v_{E,p}) : (-1,0) \to \mathbb{C}^2$ to the eigenvalue equation (3.16) and (3.17) with

$$\lim_{z\to 0} w_{E,p}(z) = (j_1(E,p), j_2(E,p)) \neq 0. \tag{5.1}$$

Through symmetry (lemma 2.0.2), we can construct a second solution $\tilde{w}_{E,p} = (v_{-E,p}, u_{-E,p})$. The asymptotics of these solutions (compare lemma 4.1.10) for $z \to 0$ ensure that $w_{E,p}$ and $\tilde{w}_{E,p}$ are linearly independent, meaning $D_{E,-} = \operatorname{span}(w_{E,p}, \tilde{w}_{E,p})$.

Consequently, $D_{E,e,-} = \{0\}$ is equivalent to

$$\det \begin{pmatrix} j_1(E,p) & j_2(-E,p) \\ j_2(E,p) & j_1(-E,p) \end{pmatrix} \neq 0. \tag{5.2}$$

Lemma 4.1.5 gives a recursively defined sequence $(c_k(E,p))_k$ which, by corollary 4.1.7 fulfils $\lim_{k\to\infty} c_k = \frac{j_1}{j_2}$.

That gives us a condition for the existence of eigenvalues resp. resonances. iL admits an (even) eigenvalue or resonance $E \in [-1,1]\backslash\{0\}$, if and only if

$$\mathcal{C}(E,p) := \lim_{k\to\infty} c_k(E,p) = \lim_{k\to\infty} \frac{1}{c_k(-E,p)} = \frac{1}{\mathcal{C}(-E,p)}. \tag{5.3}$$

We employ the exact same strategy for odd eigenfunctions.

5.1.1. The Even Case

Definition 5.1.1 *Let $E \in [-1,1]$ and $p \in [3,5]$. We define $w_{E,p} = (u_{E,p}, v_{E,p}) \in D_{E,-}$ as the function given by lemma 4.1.10. Further, lemma 2.0.2 allows us to define $\tilde{w}_{E,p} := (v_{-E,p}, u_{-E,p}) \in D_{E,-}$.*

Definition 5.1.2 *For $E \in [-1,1]$ and $p \in [3,5]$, we define $j_1(E,p), j_2(E,p) \in \mathbb{R}$ as given by lemma 4.1.3.*

Definition 5.1.3 *Let $E \in [-1,1]$ and $p \in [3,5]$, we define $(c_k(E,p))_{k\in\mathbb{N}} \subseteq \mathbb{R} \cup \{\infty\}$ as given by lemma 4.1.5. As before, we need to exclude the case $E = -1$, $p = 3$.*

Definition 5.1.4 (Definition 4.1.8) *For $E \in [-1,1]$ and $p \in [3,5]$, but exclude the case $E=-1$, $p=3$. By corollary 4.1.7, $\lim_{k\to\infty} c_k(E,p)$ is well-defined. We define:*

$$\mathcal{C}(E,p) = \lim_{k\to\infty} c_k(E,p). \tag{5.4}$$

Lemma 5.1.5 (Lemma 4.1.9) *$(E,p) \mapsto \mathcal{C}(E,p)$ is a real meromorphic function for $(-1,3) \neq (E,p) \in [-1,\infty) \times [3,5]$.*

Lemma 5.1.6 *Let $E \in (-1,1)\backslash\{0\}$ and $p \in [3,5]$. Then, $D_{E,-} = \operatorname{span}(w_{E,p}, \tilde{w}_{E,p})$.*

Proof. By lemma 2.3.21, we only need to show that $w_{E,p}$ and $\tilde{w}_{E,p}$ are linear independent. By lemma 4.1.10, the decay of $w_{E,p}$ for $z \to -1$ is given by $\sim (1-z^2)^{\frac{\sqrt{1+E}}{p-1}}$, while the decay of $w_{-E,p}$ and thus $\tilde{w}_{E,p}$ is given by $\sim (1-z^2)^{\frac{\sqrt{1-E}}{p-1}}$.

That concludes the proof. □

Lemma 5.1.7 *Let $E \in (-1,1)\backslash\{0\}$ and $p \in [3,5]$. If and only if*

$$\mathcal{C}(E,p) \neq \frac{1}{\mathcal{C}(-E,p)}, \tag{5.5}$$

then $D_{E,e,-} = \{0\}$.

Proof. Assume $0 \neq w \in D_{E,e,-}$. By lemma 5.1.6, we find $k,l \in \mathbb{R}$ with $w = kw_{E,p} + l\tilde{w}_{E,p}$ and $(k,l) \neq (0,0)$. By lemma 4.1.10 and $w \in D_{E,e,-}$:

$$k\begin{pmatrix} -j_1(E,p) \\ -j_2(E,p) \end{pmatrix} + l\begin{pmatrix} -j_2(-E,p) \\ -j_1(-E,p) \end{pmatrix} = k\lim_{z\nearrow 0} w'_{E,p}(z) + l\lim_{z\nearrow 0} \tilde{w}'_{E,p}(z) = \lim_{z\nearrow 0} w'(z) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5.6}$$

It follows:

$$\frac{j_1(E,p)}{j_2(-E,p)} = -\frac{l}{k} = \frac{j_2(E,p)}{j_1(-E,p)}. \tag{5.7}$$

We reach a contradiction, as corollary 4.1.7 implies:

$$\lim_{k\to\infty} c_k(E,p) = \frac{j_1(E,p)}{j_2(E,p)} = \frac{j_2(-E,p)}{j_1(-E,p)} = \lim_{k\to\infty} \frac{1}{c_k(-E,p)}. \tag{5.8}$$

The inverse implication follows analogously. That concludes the proof. □

$\tilde{D}_{E,e,-}$ can be characterised completely analogously to lemma 5.1.7.

Lemma 5.1.8 *Let $E=1$ or $E=-1$, $p \in (3,5]$. If and only if*

$$\mathcal{C}(E,p) \neq \frac{1}{\mathcal{C}(-E,p)}, \tag{5.9}$$

then $\tilde{D}_{E,e,-} = \{0\}$.

Remark *The reason we require* $p > 3$ *in lemma 5.1.8 is that* $E = -1$, $p = 3$ *constitutes the one exception to definition 4.1.4, due to* $a_2 = b_2 = 0$. *This fact is directly related to the existence of a resonance as noted in the remark on page 68.*

Corollary 5.1.9 *Let* $E \in (0, 1]$ *and* $p \in [3, 5]$ *with* $(E, p) \neq (1, 3)$. *Then* E *is an eigenvalue or resonance of* iL *with even eigenfunction, if and only if*

$$\mathcal{C}(E,p) \neq \frac{1}{\mathcal{C}(-E,p)}. \tag{5.10}$$

5.1.2. The Odd Case

Definition 5.1.10 *Let* $E \in [-1, 1]$ *and* $p \in [3, 5]$. *We define* $w_{E,p} = (u_{E,p}, v_{E,p}) \in D_{E,-}$ *as the function given by lemma 4.2.10. Further, lemma 2.0.2 allows us to define* $\tilde{w}_E := (v_{-E}, u_{-E}) \in D_{E,-}$.

Definition 5.1.11 *For* $E \in [-1, 1]$, *we define* $j_1(E), j_2(E) \in \mathbb{R}$ *as given by lemma 4.2.3.*

Definition 5.1.12 *For* $E \in [-1, 1]$ *and* $p \in [3, 5]$, *we define* $(c_k(E, p))_{k\in\mathbb{N}} \subseteq \mathbb{R} \cup \{\infty\}$ *as given by definition 4.2.5.*

Definition 5.1.13 (Definition 4.2.8) *Let* $E \in [-1, 1]$ *and* $p \in [3, 5]$. *By corollary 4.2.7,* $\lim_{k\to\infty} c_k(E, p)$ *is well-defined. We define:*

$$\mathcal{C}(E,p) = \lim_{k\to\infty} c_k(E,p). \tag{5.11}$$

Lemma 5.1.14 (Lemma 4.2.9) $(E, p) \mapsto \mathcal{C}(E, p)$ *is a real meromorphic function for* $(E, p) \in [-1, \infty) \times [3, 5]$.

Lemma 5.1.15 *Let* $E \in (-1, 1)\backslash\{0\}$ *and* $p \in [3, 5]$. *Then,* $D_{E,-} = \operatorname{span}(w_{E,p}, \tilde{w}_{E,p})$.

Proof. Follows completely analogously to lemma 5.1.6. □

Lemma 5.1.16 *Let* $E \in (-1, 1)\backslash\{0\}$ *and* $p \in [3, 5]$. *If and only if*

$$\mathcal{C}(E,p) \neq \frac{1}{\mathcal{C}(-E,p)}, \tag{5.12}$$

then $D_{E,o,-} = \{0\}$.

Proof. Follows analogously to lemma 5.1.7. □

Lemma 5.1.17 *Let* $E = 1$ *or* $E = -1$. *Let* $p \in [3, 5]$. *If and only if*

$$\mathcal{C}(E,p) \neq \frac{1}{\mathcal{C}(-E,p)}, \tag{5.13}$$

then $\tilde{D}_{E,o,-} = \{0\}$.

Proof. Follows analogously to lemma 5.1.8. □

Corollary 5.1.18 *Let $E \in (0,1]$ and $p \in [3,5]$. Then E is an eigenvalue or resonance of iL with odd eigenfunction, if and only if*

$$\mathcal{C}(E,p) \neq \frac{1}{\mathcal{C}(-E,p)}. \tag{5.14}$$

5.1.3. Five Goals

As described in chapter 1.12, in order to use the condition $\mathcal{C}(E,p) = \frac{1}{\mathcal{C}(-E,p)}$ to characterise the spectrum, we employ the following strategy.

We show five distinct statements, three for the even solution space and two for the odd solution space. All five statements are proven using heavy calculations to the point that using a computer algebra system to follow along is recommended.

1. Lemma 5.3.13: Let $p \in (3,5]$. Then, -1 and 1 are not resonances or eigenvalues with even eigenfunctions of iL, meaning no even bounded solutions to $(iL \pm 1)w = 0$ exist.

2. Lemma 5.4.21: Let $p \in [3,5]$. Then, -1 and 1 are not resonances or eigenvalues with odd eigenfunctions of iL, meaning no odd bounded solutions to $(iL \pm 1)w = 0$ exist.

3. Lemma 5.5.18: Let $p = 3$. Then, iL admits no eigenvalues with even eigenfunctions within the spectral gap $(-1,1)$, apart from 0.

4. Lemma 5.6.13: Let $p = 3$. Then, iL admits no eigenvalues or resonances with odd eigenfunctions within the spectral gap $[-1,1]$, apart from 0.

5. Corollary 5.7.15: There exist $E_1 \in [0,1)$ and $p_1 \in (3,5)$, such that for every $p \in (3, p_1]$, there exists exactly one (even) eigenvalue $E \in (E_1, 1)$ of iL.

In chapter 5.8, we use the above results and basic spectral methods to prove theorem 1.10.1. Chapters 5.3 - 5.7 are dedicated to showing the above five results one at a time.

Chapter 5.2 tries to give some idea how $\mathcal{C}(E,p)$ and $\frac{1}{\mathcal{C}(-E,p)}$ behave, by plotting $c_k(E,p)$ and $\frac{1}{c_k(-E,p)}$ for $k = 1000$ and different values of p and E.

As chapters 5.3 - 5.7 consist entirely of calculations, the inclined read may want to skip ahead to chapter 5.8, only revisiting chapters 5.3 - 5.7 if they are interested in checking pure calculations. In that case using a computer algebra system for help is recommended.

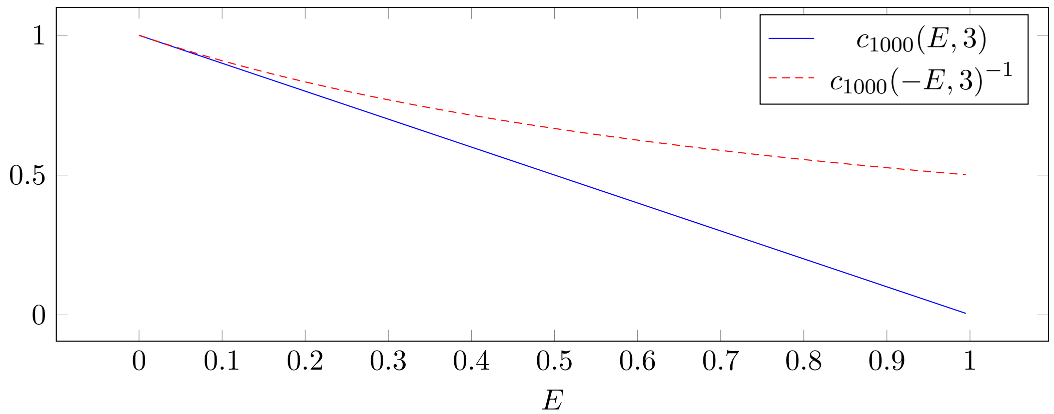
5.2. Graphics

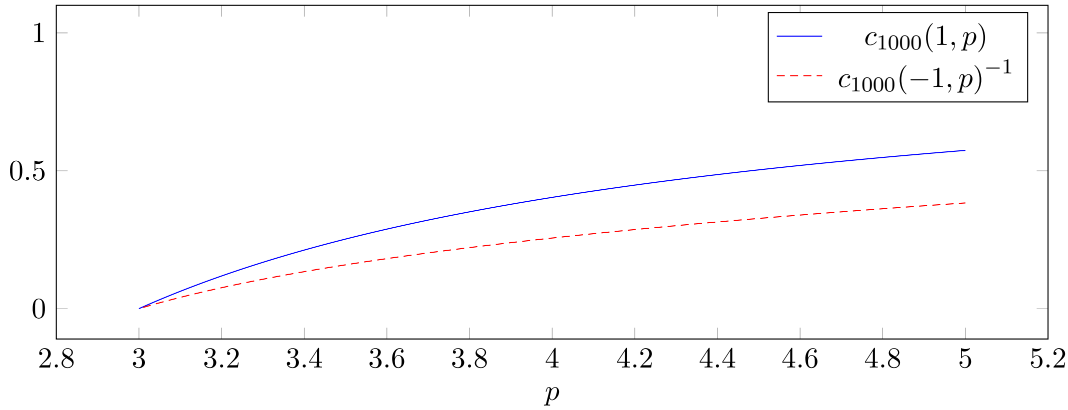
To give an idea of how $\mathcal{C}(E,p)$ and $\mathcal{C}(-E,p)$ behave, we plot $c_k(E,p)$, $c_k(-E,p)^{-1}$ (in the even case) and $c_k(E,p)^{-1}$, $c_k(-E,p)$ (in the odd case), for certain choices of E, p, as well as $k = 1000$.

The existence of an eigenvalue or resonance is indicated by the curves intersecting.

5.2.1. The Even Case

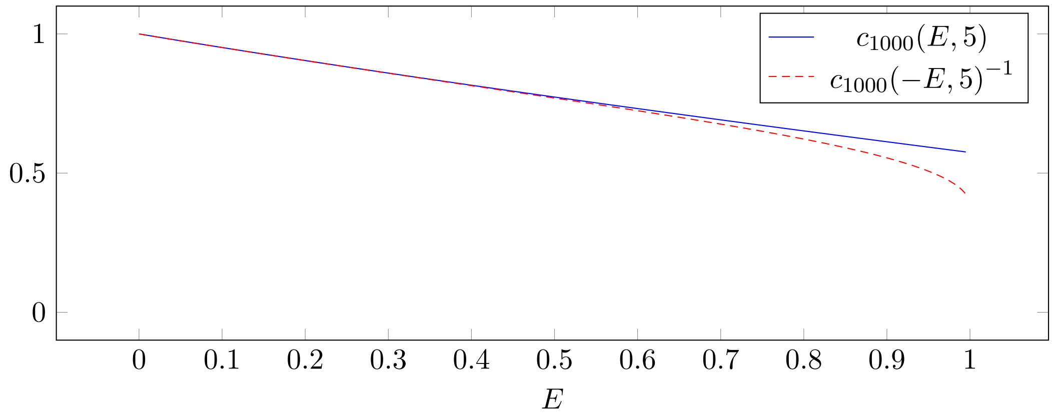
The following graphics show $c_{1000}(E,p)$ and $c_{1000}(-E,p)^{-1}$ for $p = 3$ and $E = 1$ respectively. Note that the discontinuity of $\mathcal{C}(E,p)$ near $(E,p) = (-1,3)$ is clearly visible. The first graphic indicates $\lim_{E\searrow -1} \mathcal{C}(-E,3)^{-1} \approx 0.5$ and the second $\lim_{p\searrow 3} \mathcal{C}(-1,p)^{-1} = 0$.





These graphics imply the behaviour we prove in section 5.3 and 5.5, namely that no non-zero eigenvalues exist for $p = 3$ and that no resonances exist for $p > 3$.

For $p = 5$, the fact that $E = 0$ is an eigenvalue of higher multiplicity (4 instead of 2) can be clearly seen in the corresponding graphic.



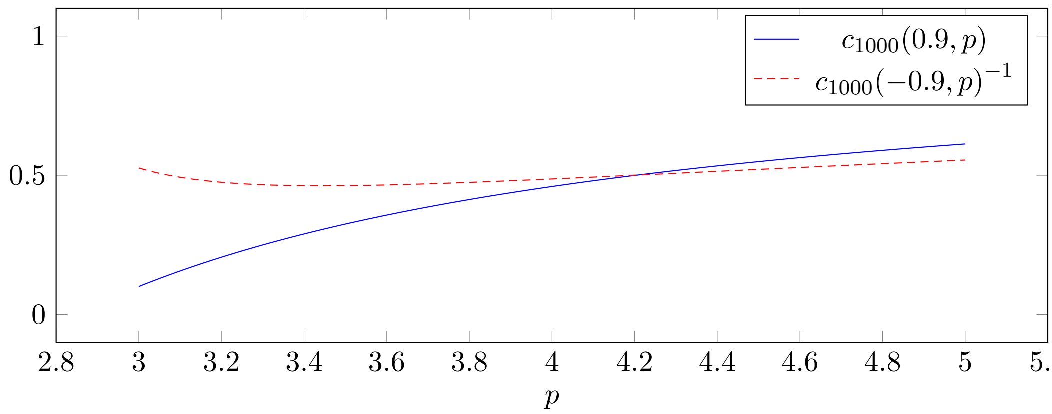
Note that the graphics indicate for $p = 3$:

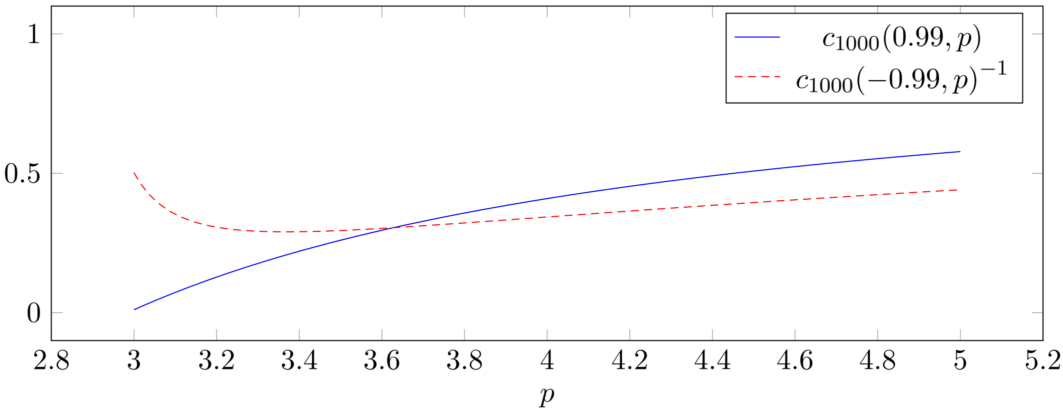
$$\mathcal{C}(E,p) < \frac{1}{\mathcal{C}(-E,p)} \tag{5.15}$$

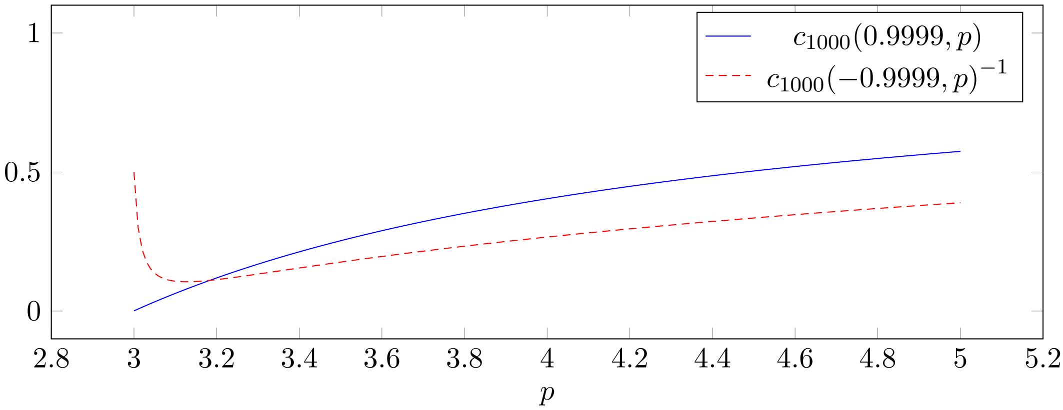
with the sign of the estimate switching for $p = 5$. Thus, by the the intermediate value theorem, for every $E \in (0,1)$ we find some $p \in (3,5)$, such that E constitutes an eigenvalue.

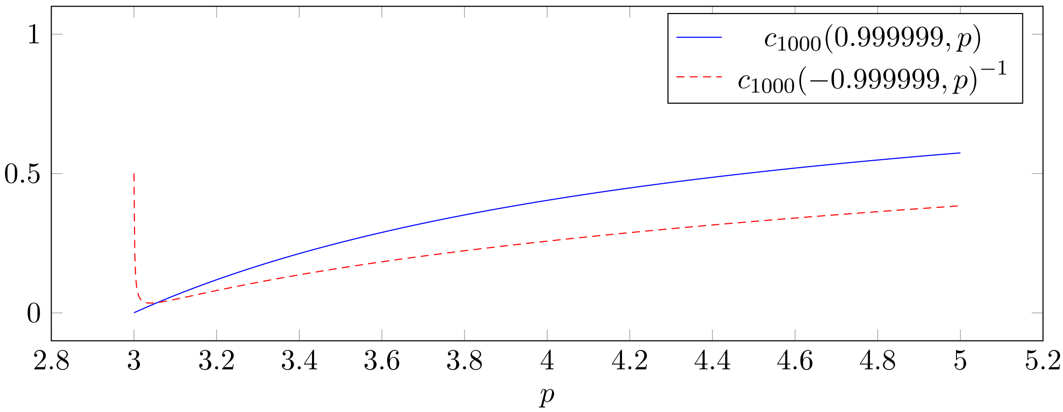
This eigenvalue starts as a resonance in $E = 1$ for $p = 3$ and merges with the eigenvalue $E = 0$ for $p = 5$.

The following four graphics depict the position of the eigenvalue for $E = 0.9$, $E = 0.99$, $E = 0.9999$ and $E = 0.999999$. The intersection point between $\mathcal{C}(E,p) \approx c_{1000}(E,p)$ and $\mathcal{C}(-E,p)^{-1} \approx c_{1000}(-E,p)^{-1}$ indicates the exponent p for which E is an eigenvalue of L.

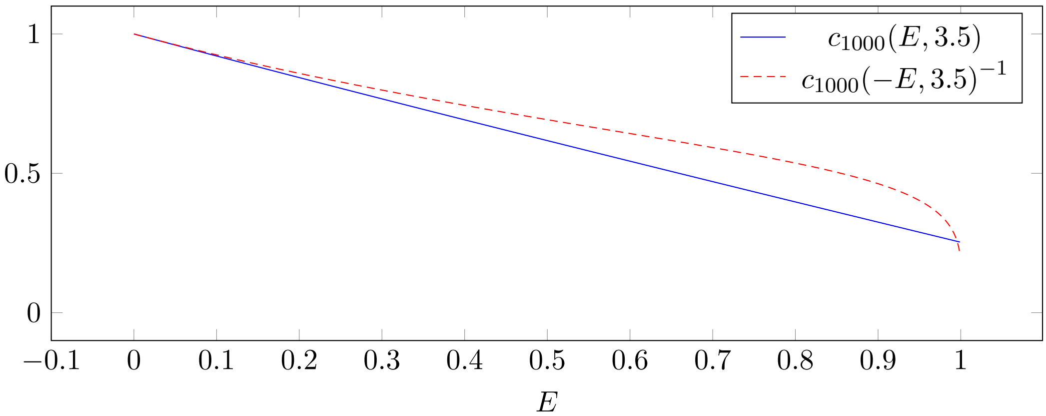




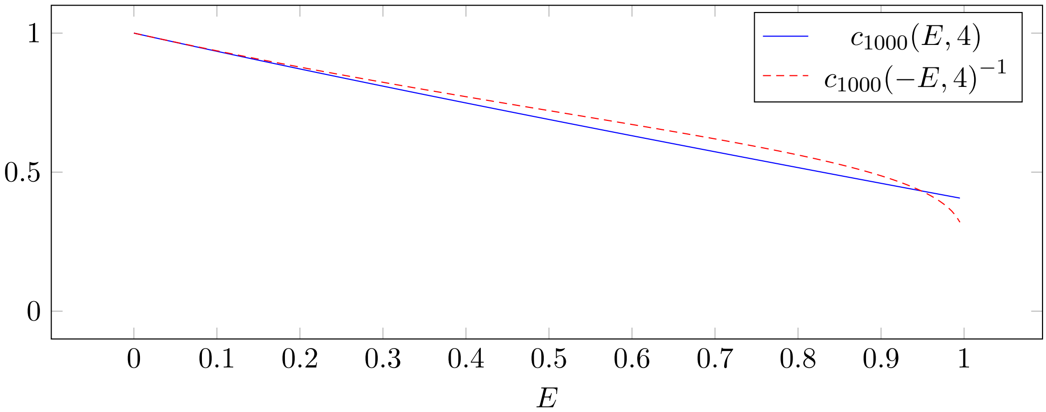




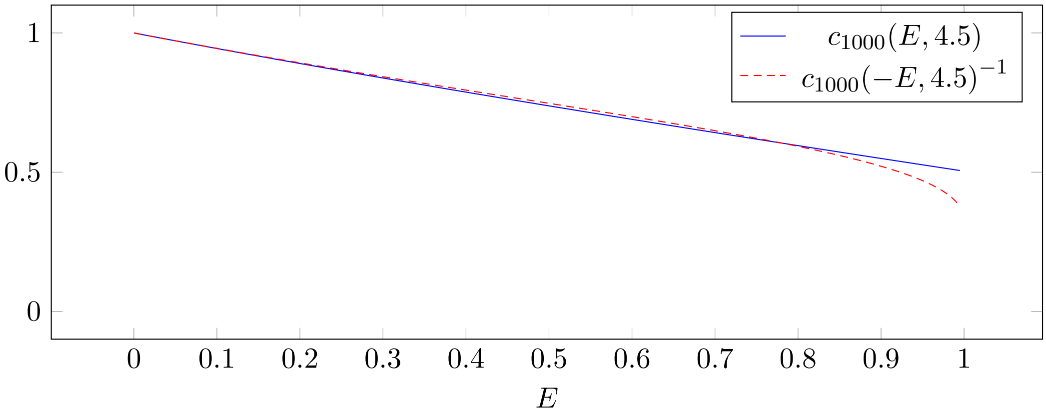
Note that the eigenvalue moves very slowly for exponents close to 3. The eigenvalue $E = 0.99$ corresponds to some $p \geq 3.6$. The reasons behind that become more clear in section 5.7. The following three graphics show the position of the eigenvalue given exponents $p = 3.5$, $p = 4$ and $p = 4.5$.



$c_{1000}(E, 3.5)$
$c_{1000}(-E, 3.5)^{-1}$
1
0.5
0
−0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
E



$c_{1000}(E, 4)$
$c_{1000}(-E, 4)^{-1}$
1
0.5
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
E



$c_{1000}(E, 4.5)$
$c_{1000}(-E, 4.5)^{-1}$
1
0.5
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
E

5.2.2. The Odd Case

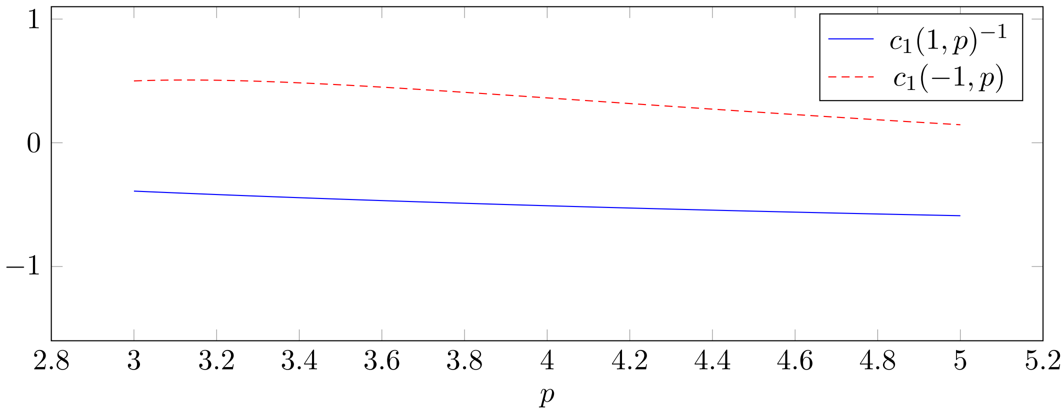
The situation is simpler for odd eigenfunctions. Because non-zero eigenvalues do not emerge, it holds for every $E \in (0, 1]$, $p \in [3, 5]$ and $k \geq 0$:

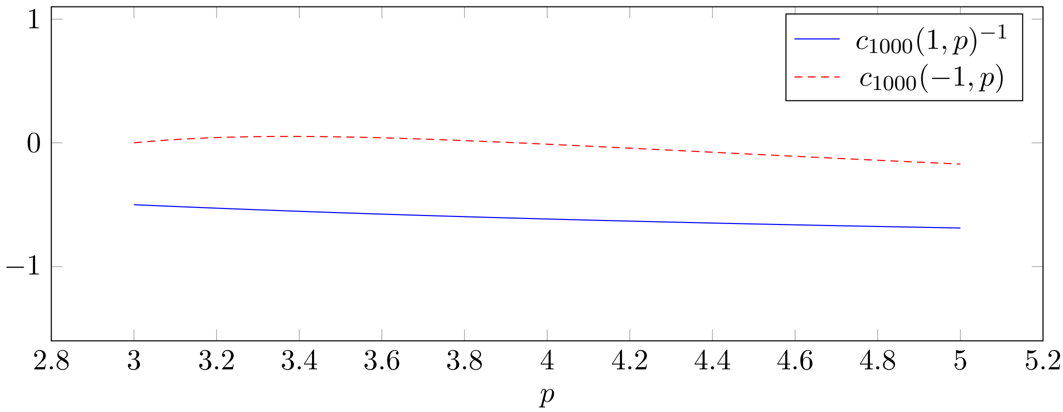
$$c_k(-E, p) > c_k(E, p)^{-1}. \tag{5.16}$$

The curves only intersect for the eigenvalue $E = 0$.

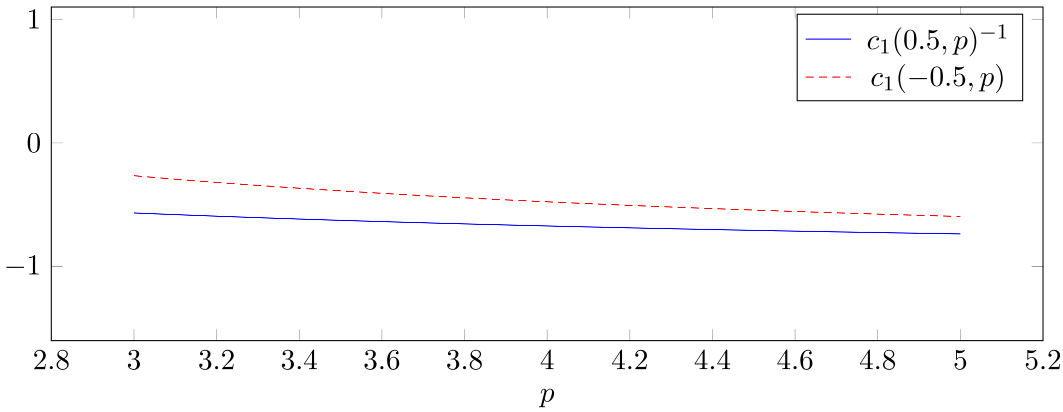
The following graphics depict $c_1(E, p)^{-1}$ and $c_1(-E, p)$, as well as $c_{1000}(E, p)^{-1}$ and $c_{1000}(-E, p)$ for $E = 1$, $E = 0.5$, $p = 3$, $p = 4$ and $p = 5$ respectively.

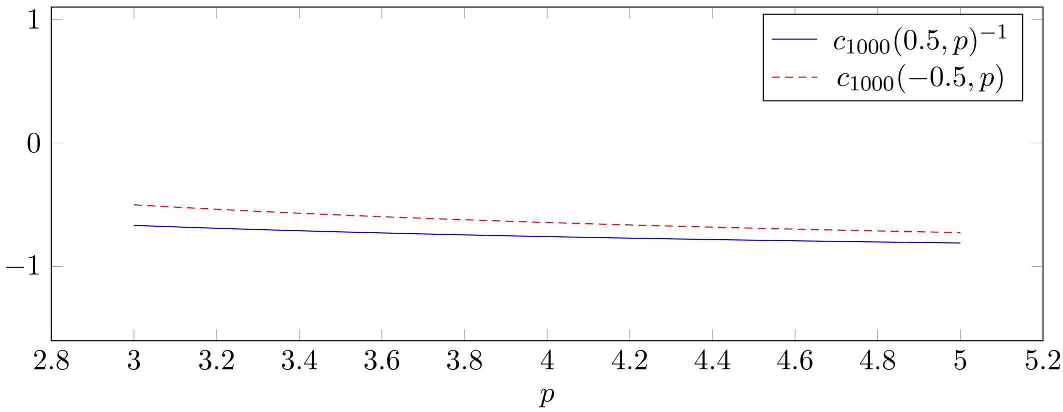
The case of $E = 1$ depicted below corresponds to chapter 5.4. Because the curves do not intersect even for small k, showing the absence of odd resonances via $\frac{1}{\mathcal{C}(1,p)} \neq \mathcal{C}(-1, p)$ is simpler than the case of even resonances.





As mentioned, the curves only intersect for $E = 0$, which is depicted below for $E = 0.5$.



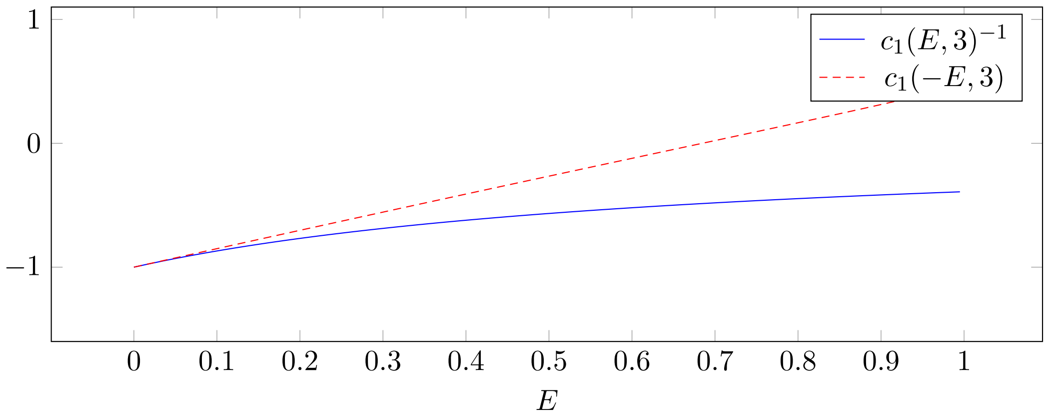


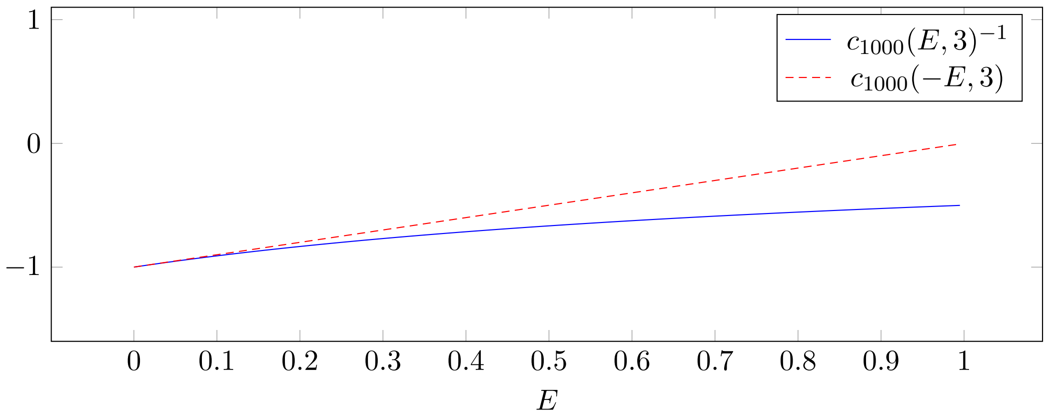
The case of $p = 3$ depicted via the two following graphics corresponds to chapter 5.6. Proving that no eigenvalues exist is somewhat simpler than in the even case depicted on page 82.

This can be explained through the following heuristic. In the odd case, the graphics below seem to indicate

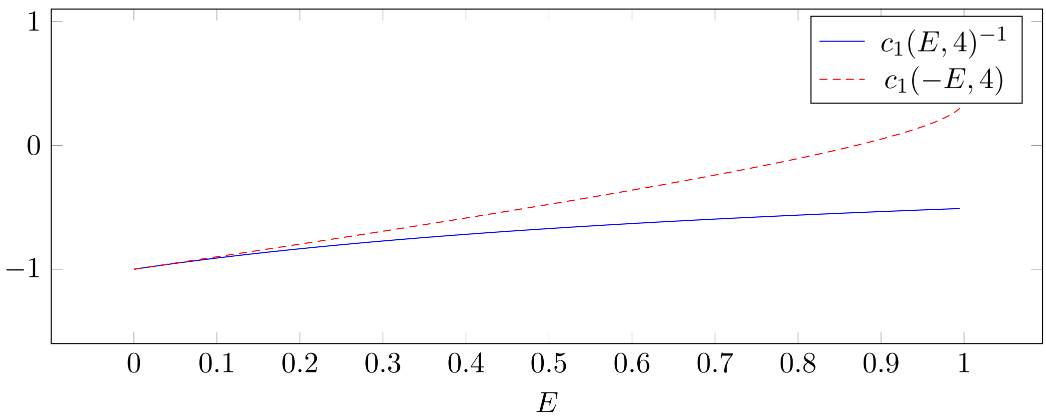
$$\frac{d}{dE}\left(\mathcal{C}(-E, 3) - \frac{1}{\mathcal{C}(E, 3)}\right) > 0. \tag{5.17}$$

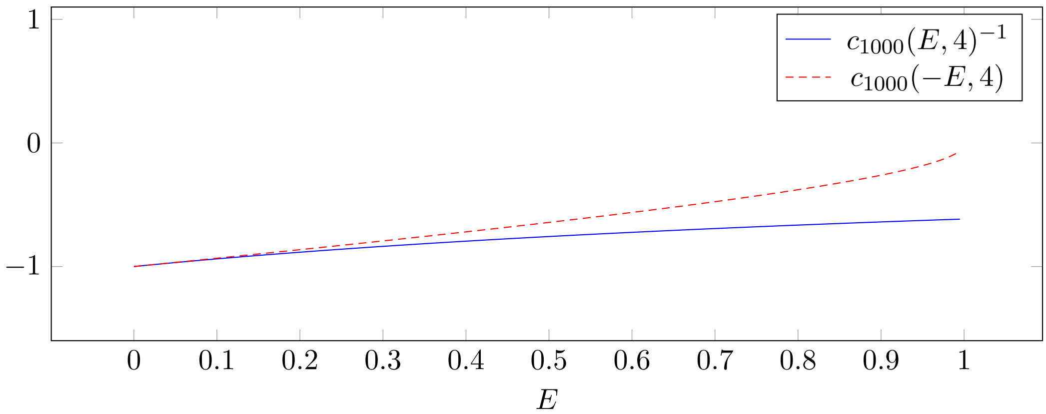
But, if we compare with the graphic on page 82, the even case admits no equivalent to (5.17). Consequently, the absence of eigenvalues in the odd case can be proven by mainly considering E close to 0, while the even case requires an extra step.



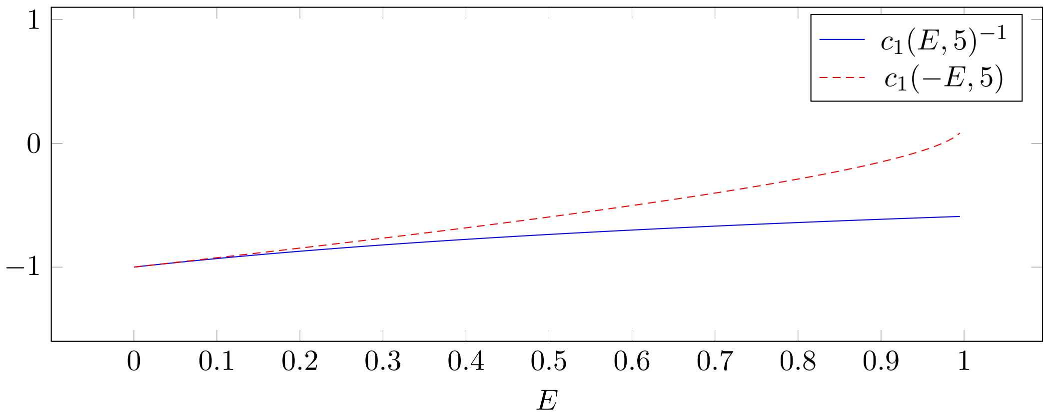


We end with four graphics depicting $c_k(-E,p) > c_k(E,p)^{-1}$ for $p = 4$, $p = 5$ and $k = 1$, $k = 1000$. The fact that $E = 0$ is an eigenvalue of multiplicity > 1, namely multiplicity 2, is visible in all of the below graphics.

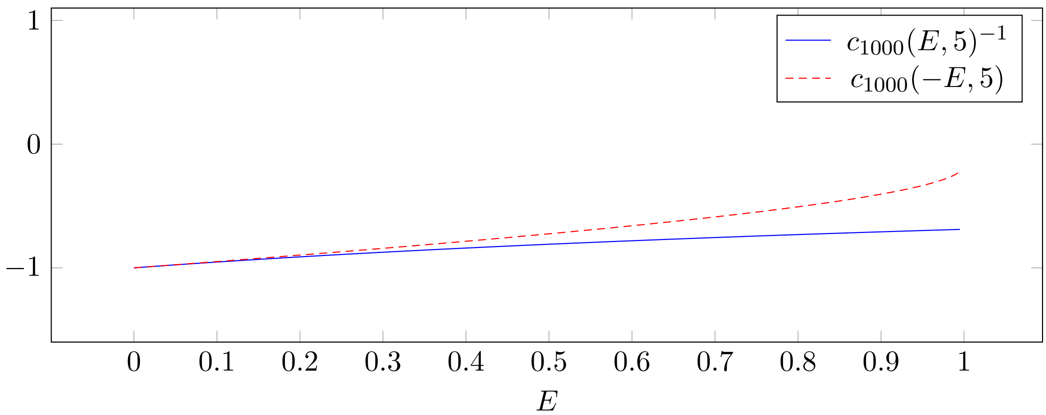




$c_{1000}(E,4)^{-1}$
$c_{1000}(-E,4)$
1
0
−1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
E



$c_1(E,5)^{-1}$
$c_1(-E,5)$
1
0
−1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
E



$c_{1000}(E,5)^{-1}$
$c_{1000}(-E,5)$
1
0
−1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
E

5.3. Even Resonances

In this chapter we show

$$\mathcal{C}(1,p) > \frac{1}{\mathcal{C}(-1,p)}. \tag{5.18}$$

for every $p \in (3,5]$. This implies that no resonances exists for $p > 3$. This chapter involves some heavy calculations, even compared to the rest of chapter 5. Using a computer algebra system for assistance is recommended.

5.3.1. Notation

Definition 5.3.1 *Let $E \in [-1,1]$, $p \in [3,5]$ and $k \geq 0$. We define A_k and B_k as given by definition 4.1.1, meaning:*

$$\begin{aligned} A_k = A_k(E,p) &:= 4k^2 + 2k + \frac{8\sqrt{1+E}}{p-1}k + \frac{4(1+E)}{(p-1)^2} + \frac{2\sqrt{1+E}}{p-1} - \frac{(p+1)^2}{(p-1)^2} \\ &= 4\left(k + \frac{\sqrt{1+E}}{p-1}\right)^2 + 2\left(k + \frac{\sqrt{1+E}}{p-1}\right) - \frac{(p+1)^2}{(p-1)^2}, \end{aligned} \tag{5.19}$$

$$B_k = B_k(E,p) := 4(k+1)^2 + \frac{8\sqrt{1+E}}{p-1}(k+1). \tag{5.20}$$

Definition 5.3.2 *For $E \in [-1,1]$, $p \in [3,5]$, we define $(c_k(E,p))_{k\in\mathbb{N}} \subseteq \mathbb{R} \cup \{\infty\}$ as given by lemma 4.1.5, meaning $c_0 = \infty$ and*

$$c_{k+1} = \left(1 + \frac{8E}{(p-1)^2 B_k}\right) \frac{A_k c_k - \frac{p+1}{p-1}}{A_k - \frac{p+1}{p-1} c_k}. \tag{5.21}$$

As before, we need to exclude the case $E = -1$, $p = 3$.

Notation 5.3.3 *In chapter 5.3, we use the shorthand notation $c_k^+ = c_k^+(p) = c_k(1,p)$ and $c_k^- = c_k^-(p) = c_k(-1,p)$. We also abbreviate $A_k^+ = A_k(1,p)$ and $A_k^- = A_k(-1,p)$.*

The chapter is in essence a proof by induction. We establish bounds on c_2^+ and c_2^- and extend them to c_k^+ and c_k^- via induction. The bounds are inspired by the explicit formulas for $c_k^+(3)$, $\lim_{p\searrow 3} c_k^-(p)$ which will be calculated in chapter 5.3.2.

The results of chapter 5.3.2 are not needed for the proof of (5.18), here they only act as an inspiration for the bounds on $c_k^+(p), c_k^-(p)$. The results of chapter 5.3.2 are however used in in chapter 5.7, when we examine

$$\mathcal{C}(E,p) - \frac{1}{\mathcal{C}(-E,p)}. \tag{5.22}$$

close to the resonance $(E,p) = (1,3)$.

5.3.2. Resonance in the Cubic Case

We calculate $c_k^+(3)$ and $\lim_{p\searrow 3} c_k^-(p)^{-1}$ explicitly.

Lemma 5.3.4 *For $E = 1$, $p = 3$ and $k \in \mathbb{N}$, $c_k(E,p)$ is given by*

$$c_k^+(3) = \frac{k + \frac{1}{\sqrt{2}}}{2(k+\sqrt{2})k}. \tag{5.23}$$

Proof. By lemma 4.1.5:

$$\begin{aligned} c_{k+1}^+ &= \frac{2(k+1)^2 + 2\sqrt{2}(k+1) + 1}{2(k+1)^2 + 2\sqrt{2}(k+1)} \frac{(4k^2 + 2k + 4\sqrt{2}k + \sqrt{2} - 2)c_k^+ - 2}{4k^2 + 2k + 4\sqrt{2}k + \sqrt{2} - 2 - 2c_k^+} \\ &= \frac{(k+1+\frac{1}{\sqrt{2}})^2}{(k+1+\sqrt{2})(k+1)} \frac{(4k^2 + 2k + 4\sqrt{2}k + \sqrt{2} - 2)c_k^+ - 2}{4k^2 + 2k + 4\sqrt{2}k + \sqrt{2} - 2 - 2c_k^+}. \end{aligned} \tag{5.24}$$

We show the claim by induction. For $k = 0$ the claim holds, due to $c_0^+ = \infty$. We calculate:

$$\begin{aligned} & c_{k+1}^+ \frac{2(k+1+\sqrt{2})(k+1)}{k+1+\frac{1}{\sqrt{2}}} \\ &= 2\left(k+1+\frac{1}{\sqrt{2}}\right) \frac{(4k^2 + 2k + 4\sqrt{2}k + \sqrt{2} - 2)(k + \frac{1}{\sqrt{2}}) - 4(k+\sqrt{2})k}{2(4k^2 + 2k + 4\sqrt{2}k + \sqrt{2} - 2)(k+\sqrt{2})k - 2(k + \frac{1}{\sqrt{2}})} \\ &= 2\left(k+1+\frac{1}{\sqrt{2}}\right) \frac{4(k+1+\frac{1}{\sqrt{2}})(k + \frac{\sqrt{2}-1}{2})(k - 1 + \frac{1}{\sqrt{2}})}{8(k+1+\frac{1}{\sqrt{2}})^2(k + \frac{\sqrt{2}-1}{2})(k - 1 + \frac{1}{\sqrt{2}})} \\ &= 1. \end{aligned} \tag{5.25}$$

That concludes the proof. □

Lemma 5.3.5 *For every $k \in \mathbb{N}$:*

$$\lim_{p\searrow 3} c_k^-(p) = \frac{2k^2 - 1}{k}. \tag{5.26}$$

Proof. By lemma 4.1.5, $c_0^-(p) = \infty$ and

$$c_{k+1}^-(p) = \left(1 - \frac{2}{(p-1)^2(k+1)^2}\right) \frac{\left(4k^2 + 2k - \frac{(p+1)^2}{(p-1)^2}\right) c_k^-(p) - \frac{p+1}{p-1}}{4k^2 + 2k - \frac{(p+1)^2}{(p-1)^2} - \frac{p+1}{p-1} c_k^-(p)}. \tag{5.27}$$

We conclude:

$$c_1^-(p) = \left(1 - \frac{2}{(p-1)^2}\right) \frac{p+1}{p-1}. \tag{5.28}$$

Consequently:

$$\lim_{p\searrow 3} c_1^-(p) = \frac{1}{2}\frac{4}{2} = 1. \tag{5.29}$$

Further:

$$\begin{aligned}
\lim_{p\searrow 3} \partial_p c_1^-(p) &= -2\lim_{p\searrow 3} \partial_p \left(\frac{2}{(p-1)^2}\right) + \frac{1}{2}\lim_{p\searrow 3} \partial_p \left(\frac{p+1}{p-1}\right) \\
&= -2\lim_{p\searrow 3} \left(-\frac{4}{(p-1)^3}\right) + \frac{1}{2}\lim_{p\searrow 3} \partial_p \left(\frac{2}{p-1}\right) \\
&= 1 - \frac{1}{2}\frac{2}{4} \\
&= \frac{3}{4}.
\end{aligned} \tag{5.30}$$

By L'Hôpital's rule:

$$\begin{aligned}
\lim_{p\searrow 3} c_2^-(p) &= \frac{7}{8}\lim_{p\searrow 3} \frac{\left(6 - \frac{(p+1)^2}{(p-1)^2}\right) c_1^-(p) - \frac{p+1}{p-1}}{6 - \frac{(p+1)^2}{(p-1)^2} - \frac{p+1}{p-1} c_1^-(p)} \\
&= \frac{7}{8}\lim_{p\searrow 3} \frac{\frac{1}{2}\cdot 2\cdot 2c_1^- + 2\partial_p c_1^- + \frac{1}{2}}{\frac{1}{2}\cdot 2\cdot 2 + \frac{1}{2}c_1^- - 2\partial_p c_1^-} \\
&= \frac{7}{8}\frac{\frac{5}{2}+\frac{3}{2}}{\frac{5}{2}-\frac{3}{2}} \\
&= \frac{7}{8}\frac{8}{2} \\
&= \frac{7}{2}.
\end{aligned} \tag{5.31}$$

We now use induction. For $k = 2$, the lemma holds by (5.31). For simplicity, consider:

$$d_k := \lim_{p\searrow 3} c_k^-(-1, p). \tag{5.32}$$

By (5.27), for $k \geq 2$:

$$\begin{aligned}
d_k &= \left(1 - \frac{1}{2(k+1)^2}\right) \frac{(4k^2+2k-4)d_k - 2}{4k^2+2k-4-2d_k} \\
&= \frac{2k^2+4k+1}{2k^2+4k+2}\frac{(2k^2+k-2)d_k - 1}{2k^2+k-2-d_k}.
\end{aligned} \tag{5.33}$$

By (5.33) and the induction hypothesis:

$$\begin{aligned} d_{k+1} &= \frac{2k^2+4k+1}{2k^2+4k+2}\frac{(2k^2+k-2)d_k-1}{2k^2+k-2-d_k} \\ &= \frac{2k^2+4k+1}{2k^2+4k+2}\frac{(2k^2+k-2)(2k^2-1)-k}{(2k^2+k-2)k-2k^2+1} \\ &= \frac{2k^2+4k+1}{2k^2+4k+2}\frac{4k^4+2k^3-6k^2-2k+2}{2k^3-k^2-2k+1} \\ &= \frac{2(k+1)^2-1}{2(k+1)^2}\frac{(k+1)(4k^3-2k^2-4k+2)}{(k-1)(2k^2+k-1)} \\ &= \frac{2(k+1)^2-1}{2(k+1)^2}\frac{2(k+1)^2(2k^2-3k+1)}{(k-1)(k+1)(2k-1)} \\ &= \frac{2(k+1)^2-1}{2(k+1)^2}\frac{2(k+1)^2(k-1)(2k-1)}{(k-1)(k+1)(2k-1)} \\ &= \frac{2(k+1)^2-1}{k+1}. \end{aligned} \tag{5.34}$$

That concludes the proof. □

Corollary 5.3.6 *The following holds true:*

$$\mathcal{C}(1,3) = 0, \tag{5.35}$$

$$\lim_{p\searrow 3}\frac{1}{\mathcal{C}(-1,p)} = 0. \tag{5.36}$$

Remark *Note that $\mathcal{C}(E,p)$ does not possess a continuous continuation to $(E,p) = (-1,3)$. The limit depends on the direction:*

$$\lim_{p\searrow 3}\frac{1}{\mathcal{C}(-1,p)} \neq \lim_{E\searrow -1}\frac{1}{\mathcal{C}(E,3)}. \tag{5.37}$$

This is examined in more detail in chapter 5.7.

5.3.3. Establishing Bounds

Lemma 5.3.7 *Let $p \in (3,5]$ and $k \geq 1$. Assume:*

$$1 \leq c_k^-(p) \leq \frac{p+1}{p-1}\frac{(k+\frac{\sqrt{2}}{p-1})(k-\frac{\sqrt{2}}{p-1})}{k}. \tag{5.38}$$

Then:

$$c_{k+1}^-(p) \leq \frac{p+1}{p-1}\frac{(k+1+\frac{\sqrt{2}}{p-1})(k+1-\frac{\sqrt{2}}{p-1})}{k+1}. \tag{5.39}$$

Proof. By definition:

$$c_{k+1}^- = \frac{(k+1+\frac{\sqrt{2}}{p-1})(k+1-\frac{\sqrt{2}}{p-1})}{(k+1)^2} \frac{\left(4k^2+2k-\frac{(p+1)^2}{(p-1)^2}\right)c_k^- - \frac{p+1}{p-1}}{4k^2+2k-\frac{(p+1)^2}{(p-1)^2}-\frac{p+1}{p-1}c_k^-}. \tag{5.40}$$

It follows:

$$\begin{aligned} & c_{k+1}^- \frac{k+1}{(k+1+\frac{\sqrt{2}}{p-1})(k+1-\frac{\sqrt{2}}{p-1})} \frac{p-1}{p+1} \qquad (5.41)\\ \le\ & \frac{p-1}{p+1}\frac{1}{k+1}\frac{\frac{p+1}{p-1}A_k(k^2-\frac{2}{(p-1)^2})-\frac{p+1}{p-1}k}{A_k k-\frac{(p+1)^2}{(p-1)^2}(k^2-\frac{2}{(p-1)^2})}\\ =\ & \frac{A_k(k^2-\frac{2}{(p-1)^2})-k}{A_k k(k+1)-\frac{(p+1)^2}{(p-1)^2}(k+1)(k^2-\frac{2}{(p-1)^2})}\\ =:\ & \frac{\alpha_k}{\beta_k}. \qquad (5.42)\end{aligned}$$

It suffices to show $\alpha_k - \beta_k \le 0$. We calculate:

$$\begin{aligned}\alpha_k-\beta_k =\ & A_k\left(k^2-\frac{2}{(p-1)^2}-k(k+1)\right)+\frac{(p+1)^2}{(p-1)^2}(k+1)\left(k^2-\frac{2}{(p-1)^2}\right)-k\\ =\ & -\left(4k^2+2k-\frac{(p+1)^2}{(p-1)^2}\right)\left(k+\frac{2}{(p-1)^2}\right)-k\\ & +\frac{(p+1)^2}{(p-1)^2}\left(k^3+k^2-\frac{2}{(p-1)^2}k-\frac{2}{(p-1)^2}\right)\\ =\ & -\left(4k^2+2k-\frac{(p+1)^2}{(p-1)^2}\right)\left(k+\frac{2}{(p-1)^2}\right)-k\\ & +\frac{(p+1)^2}{(p-1)^2}k^3+\frac{(p+1)^2}{(p-1)^2}k^2-\frac{2(p+1)^2}{(p-1)^4}k-\frac{2(p+1)^2}{(p-1)^4}. \qquad (5.43)\end{aligned}$$

It follows:

$$\begin{aligned}\alpha_k-\beta_k =\ & -4k^3-\frac{8}{(p-1)^2}k^2-2k^2-\frac{4}{(p-1)^2}k+\frac{(p+1)^2}{(p-1)^2}k+\frac{2(p+1)^2}{(p-1)^4}-k\\ & +\frac{(p+1)^2}{(p-1)^2}k^3+\frac{(p+1)^2}{(p-1)^2}k^2-\frac{2(p+1)^2}{(p-1)^4}k-\frac{2(p+1)^2}{(p-1)^4}\\ =\ & \frac{(p+1)^2-4(p-1)^2}{(p-1)^2}k^3+\frac{(p+1)^2-8-2(p-1)^2}{(p-1)^2}k^2\\ & +\frac{(p^2-1)^2-4(p-1)^2-2(p+1)^2-(p-1)^4}{(p-1)^4}k. \qquad (5.44)\end{aligned}$$

Consequently:

$$
\begin{aligned}
\alpha_k - \beta_k = & \frac{-(p-3)(3p-1)}{(p-1)^2}k^3 + \frac{-(p-3)^2}{(p-1)^2}k^2 \\
& + \frac{(p-3)(4p^2-2p+2)}{(p-1)^4}k \\
\leq & \frac{-(p-3)(3p-1)}{(p-1)^2}k + \frac{-(p-3)^2}{(p-1)^2}k + \frac{(p-3)(4p^2-2p+2)}{(p-1)^4}k \\
\leq & -\frac{p-3}{(p-1)^4}\Big((3p-1+p-3)(p-1)^2 - (4p^2-2p+2)\Big)\, k \\
\leq & -\frac{2(p-3)^2}{(p-1)^4}(2p^2-2p+1)k \\
\leq & \; 0.
\end{aligned} \tag{5.45}
$$

That concludes the proof. □

Lemma 5.3.8 *Let $p \in (3, 5]$. Then:*

$$
c_2^-(p) = \frac{1}{4}\frac{p+1}{p-1}\frac{2p^3-4p^2-p+1}{2p^3-6p^2+3p-1}\left(4 - \frac{2}{(p-1)^2}\right). \tag{5.46}
$$

Proof. By definition:

$$
c_{k+1}^- = \frac{(k+1+\frac{\sqrt{2}}{p-1})(k+1-\frac{\sqrt{2}}{p-1})}{(k+1)^2}\frac{\left(4k^2+2k-\frac{(p+1)^2}{(p-1)^2}\right)c_k^- - \frac{p+1}{p-1}}{4k^2+2k-\frac{(p+1)^2}{(p-1)^2}-\frac{p+1}{p-1}c_k^-}. \tag{5.47}
$$

Using $c_0 = \infty$:

$$
c_1^- = \left(1 - \frac{2}{(p-1)^2}\right)\frac{p+1}{p-1}. \tag{5.48}
$$

It follows:

$$
\begin{aligned}
&c_2^- \frac{1}{4-\frac{2}{(p-1)^2}}\\
&= \frac{1}{4}\frac{\left(6-\frac{(p+1)^2}{(p-1)^2}\right)\left(1-\frac{2}{(p-1)^2}\right)\frac{p+1}{p-1}-\frac{p+1}{p-1}}{6-\frac{(p+1)^2}{(p-1)^2}-\frac{(p+1)^2}{(p-1)^2}\left(1-\frac{2}{(p-1)^2}\right)}\\
&= \frac{1}{4}\frac{p+1}{p-1}\frac{\left(6-\frac{(p+1)^2}{(p-1)^2}\right)\left(1-\frac{2}{(p-1)^2}\right)-1}{6-\frac{(p+1)^2}{(p-1)^2}\left(2-\frac{2}{(p-1)^2}\right)}\\
&= \frac{1}{4}\frac{p+1}{p-1}\frac{6-\frac{12}{(p-1)^2}-\frac{(p+1)^2}{(p-1)^2}\left(1-\frac{2}{(p-1)^2}\right)-1}{6-\frac{(p+1)^2}{(p-1)^2}\left(2-\frac{2}{(p-1)^2}\right)}\\
&= \frac{1}{4}\frac{p+1}{p-1}\left(1+\frac{-\frac{12}{(p-1)^2}+\frac{(p+1)^2}{(p-1)^2}-1}{6-\frac{(p+1)^2}{(p-1)^2}\left(2-\frac{2}{(p-1)^2}\right)}\right)\\
&= \frac{1}{4}\frac{p+1}{p-1}\left(1+\frac{(p-1)^2(-12+(p+1)^2-(p-1)^2)}{6(p-1)^4-2(p^2-1)^2+2(p+1)^2}\right)\\
&= \frac{1}{4}\frac{p+1}{p-1}\left(1+\frac{4(p-3)(p-1)^2}{2(p-3)(2p^3-6p^2+3p-1)}\right)\\
&= \frac{1}{4}\frac{p+1}{p-1}\left(1+\frac{2p^2-4p+2}{2p^3-6p^2+3p-1}\right).
\end{aligned}
\tag{5.49}
$$

That concludes the proof. □

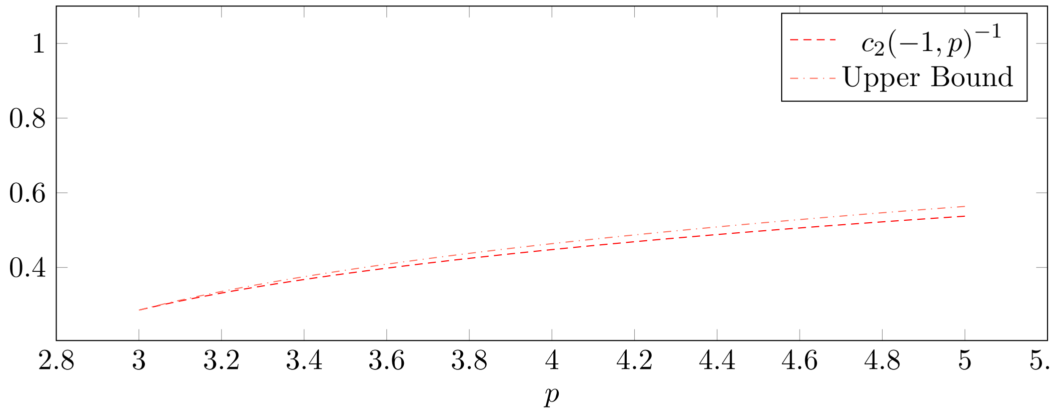
Corollary 5.3.9 *Let $p \in (3,5]$. Let further $\delta = \delta_p = \frac{3}{2}\frac{3p-1}{4(p-2)p}$ and $\varepsilon = \varepsilon_p = \frac{3p-1}{(p-2)p(p+2)^2}$. Then:*

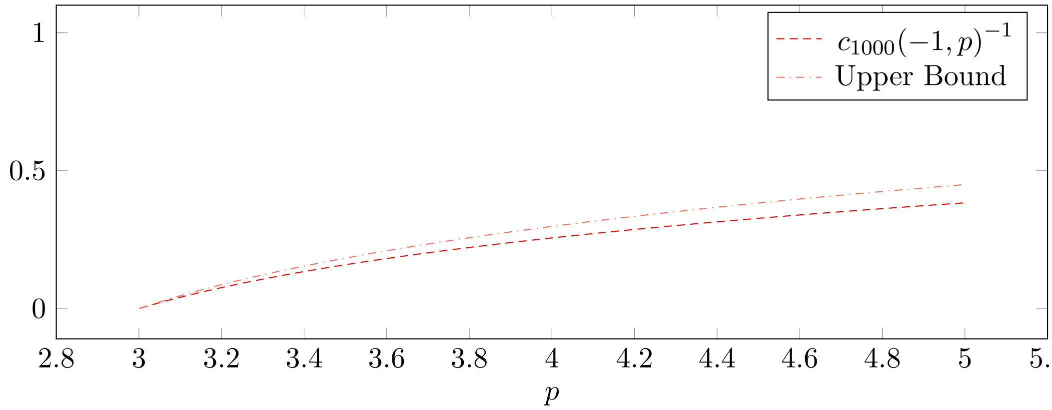
$$
\frac{p+1}{p-1}\frac{(2+\frac{\sqrt{2}}{p-1})(2-\frac{\sqrt{2}}{p-1})}{2} \geq c_2^-(p) \geq \frac{p+1}{p-1}\frac{(1+\varepsilon(p-3))(2+\frac{\sqrt{2}}{p-1})(2-\frac{\sqrt{2}}{p-1})}{2(\delta(p-3)+1)}. \tag{5.50}
$$

Proof. Follows by direct computation from lemma 5.3.8. □

We are now ready to establish an upper bound on $\frac{1}{c_2^-(p)}$. Unfortunately, the calculations needed in order to prove lemma 5.3.10 are extensive.

The following two graphics are meant as a heuristic affirmation of the proof of lemma 5.3.10. The first graphic depicts the base case $k = 2$. The second graphic depicts $k = 1000$, showing that the lemma remains true for large k.





Lemma 5.3.10 *Let $p \in (3,5]$. Let further $\delta = \delta_p = \frac{3}{2}\frac{3p-1}{4(p-2)p}$ and $\varepsilon = \varepsilon_p = \frac{3p-1}{(p-2)p(p+2)^2}$. Then, for every $k \geq 2$:*

$$\frac{p+1}{p-1}\frac{(k+\frac{\sqrt{2}}{p-1})(k-\frac{\sqrt{2}}{p-1})}{k} \geq c_k^-(p) \geq \frac{p+1}{p-1}\frac{(1+\varepsilon(p-3))(k+\frac{\sqrt{2}}{p-1})(k-\frac{\sqrt{2}}{p-1})}{k(\delta(p-3)(k-1)+1)}. \tag{5.51}$$

Proof (Calculation done with a computer). We proceed by induction. The base case is covered by corollary 5.3.9. For the induction step let $k \geq 1$ be given and assume (5.51) holds. By lemma 5.3.7:

$$c_{k+1}^-(p) \leq \frac{p+1}{p-1}\frac{(k+1+\frac{\sqrt{2}}{p-1})(k+1-\frac{\sqrt{2}}{p-1})}{k+1}. \tag{5.52}$$

It remains to show

$$c_{k+1}^-(p) \geq \frac{p+1}{p-1}\frac{(1+\varepsilon(p-3))(k+1+\frac{\sqrt{2}}{p-1})(k+1-\frac{\sqrt{2}}{p-1})}{(k+1)(\delta(p-3)k+1)}. \tag{5.53}$$

Then:

$$c^-_{k+1}\frac{(k+1)(\delta(p-3)k+1)}{(1+\varepsilon(p-3))(k+1+\frac{\sqrt{2}}{p-1})(k+1-\frac{\sqrt{2}}{p-1})}\frac{p-1}{p+1} \tag{5.54}$$

$$\geq \frac{p-1}{p+1}\frac{\delta(p-3)k+1}{(1+\varepsilon(p-3))(k+1)}\frac{\frac{p+1}{p-1}(1+\varepsilon(p-3))A_k(k^2-\frac{2}{(p-1)^2})-\frac{p+1}{p-1}k(\delta(p-3)(k-1)+1)}{A_kk(\delta(p-3)(k-1)+1)-\frac{(p+1)^2}{(p-1)^2}(1+\varepsilon(p-3))(k^2-\frac{2}{(p-1)^2})}$$

$$= \frac{A_k(1+\varepsilon(p-3))(k^2-\frac{2}{(p-1)^2})(\delta(p-3)k+1)-k(\delta(p-3)k+1)(\delta(p-3)(k-1)+1)}{A_k(1+\varepsilon(p-3))k(k+1)(\delta(p-3)(k-1)+1)-\frac{(p+1)^2}{(p-1)^2}(1+\varepsilon(p-3))^2(k+1)(k^2-\frac{2}{(p-1)^2})}$$

$$=: \frac{\alpha_k}{\beta_k}. \tag{5.55}$$

It suffices to show $\alpha_k-\beta_k\leq 0$. We calculate:

$$\begin{aligned}
\alpha_k-\beta_k = {} & A_k(1+\varepsilon(p-3))\left(k^2-\frac{2}{(p-1)^2}-k(k+1)\right)(\delta(p-3)k+1)\\
& + A_k(1+\varepsilon(p-3))k(k+1)\delta(p-3)\\
& + \frac{(p+1)^2}{(p-1)^2}(1+\varepsilon(p-3))^2(k+1)\left(k^2-\frac{2}{(p-1)^2}\right)\\
& - k(\delta(p-3)k+1)(\delta(p-3)(k-1)+1)\\
= {} & -\left(4k^2+2k-\frac{(p+1)^2}{(p-1)^2}\right)(1+\varepsilon(p-3))\left(k+\frac{2}{(p-1)^2}\right)(\delta(p-3)k+1)\\
& + \left(4k^2+2k-\frac{(p+1)^2}{(p-1)^2}\right)(1+\varepsilon(p-3))k(k+1)\delta(p-3)\\
& - \delta^2(p-3)^2k^3+\delta^2(p-3)^2k^2-2\delta(p-3)k^2+\delta(p-3)k-k\\
& + \frac{(p+1)^2}{(p-1)^2}(1+\varepsilon(p-3))^2\left(k^3+k^2-\frac{2}{(p-1)^2}k-\frac{2}{(p-1)^2}\right).
\end{aligned} \tag{5.56}$$

By (5.45):

$$\begin{aligned}
\gamma_k := {} & -\left(4k^2+2k-\frac{(p+1)^2}{(p-1)^2}\right)\left(k+\frac{2}{(p-1)^2}\right)-k\\
& + \frac{(p+1)^2}{(p-1)^2}k^3+\frac{(p+1)^2}{(p-1)^2}k^2-\frac{2(p+1)^2}{(p-1)^4}k-\frac{2(p+1)^2}{(p-1)^4}\\
= {} & \frac{-(p-3)(3p-1)}{(p-1)^2}k^3+\frac{-(p-3)^2}{(p-1)^2}k^2+\frac{(p-3)(4p^2-2p+2)}{(p-1)^4}k.
\end{aligned} \tag{5.57}$$

We conclude:

$$
\begin{aligned}
&\alpha_k - \beta_k - (1+\varepsilon(p-3))\gamma_k \\
= &-\left(4k^2 + 2k - \frac{(p+1)^2}{(p-1)^2}\right)(1+\varepsilon(p-3))\left(k + \frac{2}{(p-1)^2}\right)\delta(p-3)k \\
&+ \left(4k^2 + 2k - \frac{(p+1)^2}{(p-1)^2}\right)(1+\varepsilon(p-3))k(k+1)\delta(p-3) \\
&- \delta^2(p-3)^2k^3 + \delta^2(p-3)^2k^2 - 2\delta(p-3)k^2 + \delta(p-3)k + \varepsilon(p-3)k \\
&+ \frac{(p+1)^2}{(p-1)^2}(1+\varepsilon(p-3))\varepsilon(p-3)\left(k^3 + k^2 - \frac{2}{(p-1)^2}k - \frac{2}{(p-1)^2}\right).
\end{aligned} \tag{5.58}
$$

It follows:

$$
\begin{aligned}
&\frac{\alpha_k - \beta_k - (1+\varepsilon(p-3))\gamma_k}{p-3} \\
= &\left(4k^2 + 2k - \frac{(p+1)^2}{(p-1)^2}\right)(1+\varepsilon(p-3))\left(1 - \frac{2}{(p-1)^2}\right)\delta k \\
&- \delta^2(p-3)k^3 + \delta^2(p-3)k^2 - 2\delta k^2 + \delta k + \varepsilon k \\
&+ \frac{(p+1)^2}{(p-1)^2}\varepsilon(1+\varepsilon(p-3))\left(k^3 + k^2 - \frac{2}{(p-1)^2}k - \frac{2}{(p-1)^2}\right) \\
= &(1+\varepsilon(p-3))\left(4 - \frac{8}{(p-1)^2}\right)\delta k^3 + (1+\varepsilon(p-3))\left(2 - \frac{4}{(p-1)^2}\right)\delta k^2 \\
&- (1+\varepsilon(p-3))\left(\frac{(p+1)^2}{(p-1)^2} - \frac{2(p+1)^2}{(p-1)^4}\right)\delta k \\
&+ \varepsilon(1+\varepsilon(p-3))\left(\frac{(p+1)^2}{(p-1)^2}k^3 + \frac{(p+1)^2}{(p-1)^2}k^2 - \frac{2(p+1)^2}{(p-1)^4}k - \frac{2(p+1)^2}{(p-1)^4}\right) \\
&- \delta^2(p-3)k^3 + \delta^2(p-3)k^2 - 2\delta k^2 + \delta k + \varepsilon k \\
= &\left((1+\varepsilon(p-3))\left(4\delta - \frac{8\delta}{(p-1)^2} + \varepsilon\frac{(p+1)^2}{(p-1)^2}\right) - (p-3)\delta^2\right)k^3 \\
&+ \left((1+\varepsilon(p-3))\left(-\frac{4\delta}{(p-1)^2} + \varepsilon\frac{(p+1)^2}{(p-1)^2}\right) + 2\delta\varepsilon(p-3) + \delta^2(p-3)\right)k^2 \\
&+ \left((1+\varepsilon(p-3))\left(\frac{(p+1)^2(p^2-2p-1)}{(p-1)^4}\delta - \frac{2(p+1)^2}{(p-1)^4}\varepsilon\right) + \delta + \varepsilon\right)k \\
&- \varepsilon(1+\varepsilon(p-3))\frac{2(p+1)^2}{(p-1)^4}.
\end{aligned} \tag{5.59}
$$

In summary:

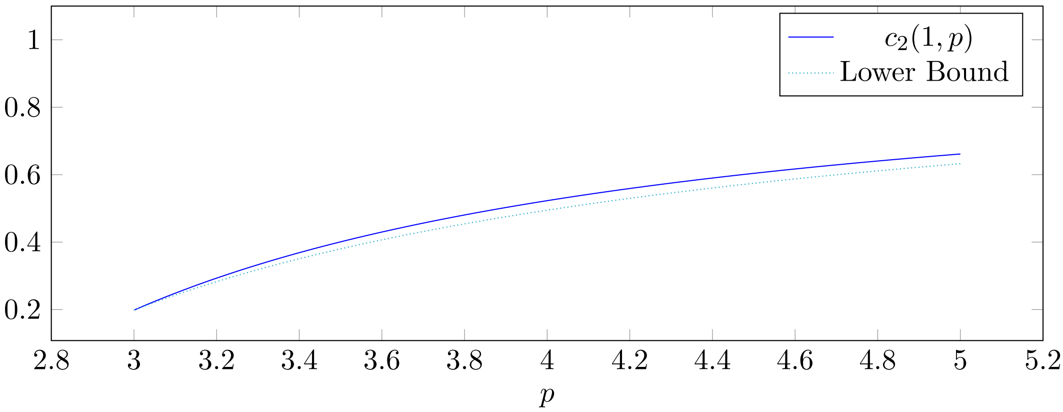
$$
\begin{aligned}
&\frac{\alpha_k-\beta_k}{p-3}\\
=\ &\left((1+\varepsilon(p-3))\left(\frac{1-3p}{(p-1)^2}+4\delta-\frac{8\delta}{(p-1)^2}+\varepsilon\frac{(p+1)^2}{(p-1)^2}\right)-\delta^2(p-3)\right)k^3\\
&+\left((1+\varepsilon(p-3))\left(\frac{3-p}{(p-1)^2}-\frac{4\delta}{(p-1)^2}+\varepsilon\frac{(p+1)^2}{(p-1)^2}\right)+2\delta\varepsilon(p-3)+\delta^2(p-3)\right)k^2\\
&+\left((1+\varepsilon(p-3))\left(\frac{4p^2-2p+2}{(p-1)^4}+\frac{(p+1)^2(p^2-2p-1)}{(p-1)^4}\delta-\frac{2(p+1)^2}{(p-1)^4}\varepsilon\right)+\delta+\varepsilon\right)k\\
&-\varepsilon(1+\varepsilon(p-3))\frac{2(p+1)^2}{(p-1)^4}. \qquad (5.60)
\end{aligned}
$$

We substitute $\delta=\frac{3}{2}\frac{3p-1}{4(p-2)p}$ and $\varepsilon=\frac{3p-1}{(p-2)p(p+2)^2}$. By direct computation:

$$
\begin{aligned}
&\frac{\alpha_k-\beta_k}{p-3}\\
=\ &\frac{15p^9+163p^8+562p^7-2030p^6-7057p^5+5555p^4+26816p^3+7720p^2-9840p+1392}{64(p-2)^2(p-1)^2p^2(p+2)^4}k^3\\
&+\frac{17p^9+125p^8+286p^7-1858p^6-7855p^5-2547p^4+24816p^3+23368p^2-5520p-48}{64(p-2)^2(p-1)^2p^2(p+2)^4}k^2\\
&+\frac{18p^{11}+98p^{10}+85p^9-649p^8-1983p^7-465p^6+5061p^5+6315p^4+2267p^3+897p^2-704p+84}{8(p-2)^2(p-1)^4p^2(p+2)^4}k\\
&-\frac{2(p+1)^2(3p-1)(p^4+2p^3-p^2-18p+3)}{(p-2)^2(p-1)^4p^2(p+2)^4}\\
=:\ &\sum_{l=0}^{3}m_l(p)k^l. \qquad (5.61)
\end{aligned}
$$

By direct computation $2m_3(p)+m_2(p)>0$ and $2m_1(p)+m_0(p)>0$. That concludes the proof. □

For $c_k^+(p)$, we show a lower bound. As in the case of lemma 5.3.10, as a heuristic affirmation of lemma 5.3.11, consider the following graphic, depicting the lower bound for the base case $k=2$.



Lemma 5.3.11 *Let $p \in [3,5]$. Let further $\delta = \delta_p = \frac{3}{2}\frac{3p-1}{4(p-2)p}(1 - \frac{7}{2}\frac{(6p-10)(p-3)}{(p-2)p(p+2)^2})$, $\varepsilon = \varepsilon_p = -\frac{7}{2}\frac{6p-10}{(p-2)p(p+2)^2}$ and $l = l_p = 2 + \frac{\sqrt{2}}{p-1}$. Then:*

$$c_2^+(p) \geq \frac{p-1}{p+1}\frac{l(\delta(p-3)(l-1)+1)}{(1+\varepsilon(p-3))(l+\frac{\sqrt{2}}{p-1})(l-\frac{\sqrt{2}}{p-1})}. \tag{5.62}$$

Proof (Calculation done with a computer). By definition:

$$c_2^+ = \frac{\left(2+\frac{\sqrt{2}}{p-1}\right)^2}{2(2+\frac{2\sqrt{2}}{p-1})}\frac{A_1 c_1^+ - \frac{p+1}{p-1}}{A_1 - \frac{p+1}{p-1}c_1^+} = \frac{l^2}{(l+\frac{\sqrt{2}}{p-1})(l-\frac{\sqrt{2}}{p-1})}\frac{A_1 c_1^+ - \frac{p+1}{p-1}}{A_1 - \frac{p+1}{p-1}c_1^+}. \tag{5.63}$$

We need to show:

$$l\frac{A_1 c_1^+ - \frac{p+1}{p-1}}{A_1 - \frac{p+1}{p-1}c_1^+} \geq \frac{p-1}{p+1}\frac{\delta(p-3)(l-1)+1}{1+\varepsilon(p-3)}. \tag{5.64}$$

We calculate c_1 using $c_0 = \infty$:

$$\begin{aligned} c_1 &= \frac{\left(1+\frac{\sqrt{2}}{p-1}\right)^2}{1+\frac{2\sqrt{2}}{p-1}}\frac{p-1}{p+1}\left(\frac{(p+1)^2}{(p-1)^2} - \frac{8}{(p-1)^2} - \frac{2\sqrt{2}}{p-1}\right) \\ &= \frac{p-1}{p+1}\frac{(l-1)^2}{2l-3}\left(\frac{(p+1)^2}{(p-1)^2} - 4(l-2)^2 - 2(l-2)\right). \end{aligned} \tag{5.65}$$

Further, by definition:

$$A_1 = 4(l-1)^2 + 2(l-1) - \frac{(p+1)^2}{(p-1)^2}. \tag{5.66}$$

We introduce the new variables $q = \frac{p+1}{p-1}$ and $j = l - 2 = \frac{\sqrt{2}}{p-1}$. (5.64) now reads

$$(j+2)\frac{(4(j+1)^2+2(j+1)-q^2)q^{-1}(q^2-4j^2-2j)-q}{(4(j+1)^2+2(j+1)-q^2)-(q^2-4j^2-2j)} \geq q^{-1}\frac{\delta(p-3)(j+1)+1}{1+\varepsilon(p-3)}. \tag{5.67}$$

By definition $p = \frac{q+1}{q-1}$ and $l = \frac{\sqrt{2}}{2}(q-1)$. It follows

$$\delta(p-3) = \frac{3(q-2)(q+2)(37q^4 - 248q^3 + 574q^2 - 600q + 221)}{2(1-3q)^2(q-3)^2(q+1)^2}, \tag{5.68}$$

$$1 + \varepsilon(p-3) = \frac{37q^4 - 248q^3 + 574q^2 - 600q + 221}{(1-3q)^2(q-3)(q+1)}. \tag{5.69}$$

(5.67) is thus equivalent to:

$$\begin{aligned}
\frac{n_1(q)}{n_2(q)} &:= \frac{-(\sqrt{2}q - \sqrt{2} + 4)(q^4 + 6\sqrt{2}q^3 - 8q^3 - 30\sqrt{2}q^2 + 37q^2 + 42\sqrt{2}q - 60q - 18\sqrt{2} + 26)}{4(q^2 + 3\sqrt{2}q - 4q - 3\sqrt{2} + 5)} \\
&= (j+2)\frac{(4(j+1)^2 + 2(j+1) - q^2)(q^2 - 4j^2 - 2j) - q^2}{(4(j+1)^2 + 2(j+1) - q^2) - (q^2 - 4j^2 - 2j)} \\
&\geq \frac{\delta(p-3)(j+1) + 1}{1 + \varepsilon(p-3)} \\
&= \frac{3(q-2)(q+2)(\sqrt{2}q - \sqrt{2} + 2)}{4(q-3)(q+1)} + \frac{(1-3q)^2(q-3)(q+1)}{37q^4 - 248q^3 + 574q^2 - 600q + 221} \\
&= \frac{m_1(q)}{4(q-3)(q+1)(37q^4 - 248q^3 + 574q^2 - 600q + 221)} =: \frac{m_1(q)}{m_2(q)}.
\end{aligned} \tag{5.70}$$

Hereby:

$$\begin{aligned}
m_1(q) = {} & 111\sqrt{2}q^7 - 855\sqrt{2}q^6 + 258q^6 + 2022\sqrt{2}q^5 - 1656q^5 - 102\sqrt{2}q^4 + 2584q^4 \\
& - 7401\sqrt{2}q^3 + 2816q^3 + 13425\sqrt{2}q^2 - 12422q^2 - 9852\sqrt{2}q + 14232q \\
& + 2652\sqrt{2} - 5268.
\end{aligned} \tag{5.71}$$

We calculate:

$$\begin{aligned}
& n_2(q)m_1(q) \\
= {} & 444\sqrt{2}q^9 - 5196\sqrt{2}q^8 + 3696q^8 + 27084\sqrt{2}q^7 - 33936q^7 - 72828\sqrt{2}q^6 + 111040q^6 \\
& + 63348\sqrt{2}q^5 - 114176q^5 + 172860\sqrt{2}q^4 - 218240q^4 - 585084\sqrt{2}q^3 + 811824q^3 \\
& + 756588\sqrt{2}q^2 - 1055872q^2 - 473472\sqrt{2}q + 669024q + 116256\sqrt{2} - 169008.
\end{aligned} \tag{5.72}$$

On the other hand:

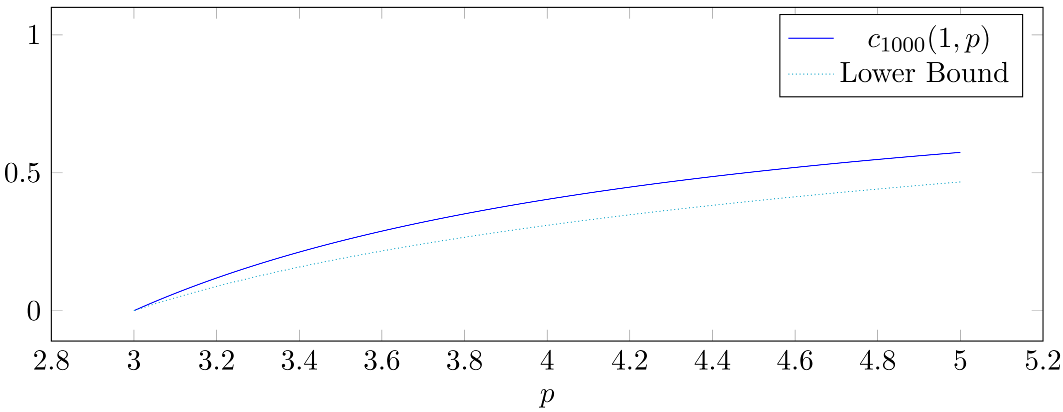
$$\begin{aligned}
& n_1(q)m_2(q) \\
= {} & -148\sqrt{2}q^{11} + 2620\sqrt{2}q^{10} - 2368q^{10} - 25640\sqrt{2}q^9 + 36000q^9 + 159528\sqrt{2}q^8 \\
& - 238544q^8 - 616712\sqrt{2}q^7 + 892576q^7 + 1434904\sqrt{2}q^6 - 2002912q^6 - 1837424\sqrt{2}q^5 \\
& + 2521824q^5 + 736048\sqrt{2}q^4 - 1023872q^4 + 1273980\sqrt{2}q^3 - 1733152q^3 \\
& - 2073204\sqrt{2}q^2 + 2898464q^2 + 1205944\sqrt{2}q - 1715200q - 259896\sqrt{2} + 371280.
\end{aligned} \tag{5.73}$$

It follows:

$$\begin{aligned}
&n_1(q)m_2(q) - n_2(q)m_1(q) \\
=& -148\sqrt{2}q^{11} + 2620\sqrt{2}q^{10} - 2368q^{10} - 26084\sqrt{2}q^9 + 36000q^9 + 164724\sqrt{2}q^8 - 242240q^8 \\
&- 643796\sqrt{2}q^7 + 926512q^7 - 1507732\sqrt{2}q^6 + 2113952q^6 + 1900772\sqrt{2}q^5 - 2636000q^5 \\
&- 563188\sqrt{2}q^4 + 805632q^4 - 1859064\sqrt{2}q^3 + 2544976q^3 + 2829792\sqrt{2}q^2 - 3954336q^2 \\
&- 1679416\sqrt{2}q + 2384224q + 376152\sqrt{2} - 540288. \qquad (5.74)
\end{aligned}$$

To conclude the proof, it suffices to show $n_1(q)m_2(q) - n_2(q)m_1(q) \geq 0$ for $q \in [\frac{3}{2}, 2]$, which can be checked by direct computation. That concludes the proof. ☐

As a heuristic affirmation of lemma 5.3.12, consider the following graphic, depicting the lower bound for the large value $k = 1000$.



Lemma 5.3.12 *Let $p \in [3,5]$. Let further $\delta = \delta_p = \frac{3}{2}\frac{3p-1}{4(p-2)p}(1 - \frac{7}{2}\frac{(6p-10)(p-3)}{(p-2)p(p+2)^2})$, $\varepsilon = -\frac{7}{2}\frac{6p-10}{(p-2)p(p+2)^2}$. Assume $c_k \in (0,\infty)$ for every $k \geq 1$ and let $l = l_{k,p} = k + \frac{\sqrt{2}}{p-1}$. Then:*

$$c_k^+(p) \geq \frac{p-1}{p+1}\frac{l(\delta(p-3)(l-1)+1)}{(1+\varepsilon(p-3))(l+\frac{\sqrt{2}}{p-1})(l-\frac{\sqrt{2}}{p-1})}. \qquad (5.75)$$

Proof (Calculation done with a computer). We proceed by induction. The base case is covered by lemma 5.3.11. For the induction step let $k \geq 1$ be given and assume (5.75) holds. By definition

$$c_{k+1}^{+\ -1} = \frac{(l+1+\frac{\sqrt{2}}{p-1})(l+1-\frac{\sqrt{2}}{p-1})}{(l+1)^2}\frac{\left(4l^2+2l-\frac{(p+1)^2}{(p-1)^2}\right)c_k^{+\ -1} - \frac{p+1}{p-1}}{4l^2+2l-\frac{(p+1)^2}{(p-1)^2} - \frac{p+1}{p-1}c_k^{+\ -1}} \qquad (5.76)$$

which directly mirrors

$$c_{k+1}^- = \frac{(k+1+\frac{\sqrt{2}}{p-1})(k+1-\frac{\sqrt{2}}{p-1})}{(k+1)^2}\frac{\left(4k^2+2k-\frac{(p+1)^2}{(p-1)^2}\right)c_k^- - \frac{p+1}{p-1}}{4k^2+2k-\frac{(p+1)^2}{(p-1)^2} - \frac{p+1}{p-1}c_k^-}. \qquad (5.77)$$

We proceed completely analogously to the proof of lemma 5.3.10. It follows

$$c_{k+1}^{+}{}^{-1}\frac{(l+1)(\delta(p-3)l+1)}{(1+\varepsilon(p-3))(l+1+\frac{\sqrt{2}}{p-1})(l+1-\frac{\sqrt{2}}{p-1})}\frac{p-1}{p+1}\leq\frac{\alpha_k}{\beta_k}, \tag{5.78}$$

whereby

$$\begin{aligned}
&\frac{\alpha_k-\beta_k}{p-3}\\
=&\left((1+\varepsilon(p-3))\left(\frac{1-3p}{(p-1)^2}+4\delta-\frac{8\delta}{(p-1)^2}+\varepsilon\frac{(p+1)^2}{(p-1)^2}\right)-\delta^2(p-3)\right)l^3\\
&+\left((1+\varepsilon(p-3))\left(\frac{3-p}{(p-1)^2}-\frac{4\delta}{(p-1)^2}+\varepsilon\frac{(p+1)^2}{(p-1)^2}\right)+2\delta\varepsilon(p-3)+\delta^2(p-3)\right)l^2\\
&+\left((1+\varepsilon(p-3))\left(\frac{4p^2-2p+2}{(p-1)^4}+\frac{(p+1)^2(p^2-2p-1)}{(p-1)^4}\delta-\frac{2(p+1)^2}{(p-1)^4}\varepsilon\right)+\delta+\varepsilon\right)l\\
&-\varepsilon(1+\varepsilon(p-3))\frac{2(p+1)^2}{(p-1)^4}.
\end{aligned} \tag{5.79}$$

We substitute $\delta=\frac{3}{2}\frac{3p-1}{4(p-2)p}(1-\frac{7}{2}\frac{(6p-10)(p-3)}{(p-2)p(p+2)^2})$ and $\varepsilon=-\frac{7}{2}\frac{6p-10}{(p-2)p(p+2)^2}$. It follows

$$\begin{aligned}
\frac{\alpha_k-\beta_k}{p-3}=&\frac{(p^4+2p^3-25p^2+90p-105)\kappa_3(p)}{64(p-2)^4(p-1)^2p^4(p+2)^4}l^3\\
&+\frac{(p^4+2p^3-25p^2+90p-105)\kappa_2(p)}{64(p-2)^4(p-1)^2p^4(p+2)^4}l^2\\
&+\frac{\kappa_1(p)}{8(p-2)^3(p-1)^4p^3(p+2)^4}l\\
&+\frac{14(p+1)^2(3p-5)(p^4+2p^3-25p^2+90p-105)}{(p-2)^2(p-1)^4p^2(p+2)^4}\\
=:&\sum_{j=0}^{3}m_j(p)l^j.
\end{aligned} \tag{5.80}$$

Hereby

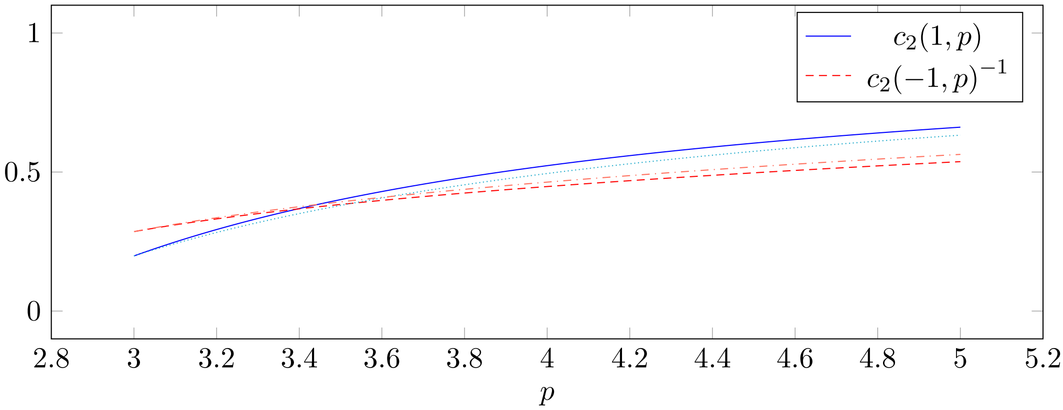
$$\begin{aligned}
\kappa_1(p)=&\,15p^9+73p^8-6287p^7+41457p^6-103800p^5+89440p^4+55665p^3\\
&-129415p^2+46215p-2835, && (5.81)\\
\kappa_2(p)=&\,17p^9+23p^8-6625p^7+49471p^6-157544p^5+257312p^4-211009p^3\\
&+76823p^2-15975p+2835, && (5.82)\\
\kappa_3(p)=&\,18p^{13}+62p^{12}-897p^{11}+1971p^{10}+9428p^9-58194p^8+105267p^7+2097p^6\\
&-286399p^5+288635p^4+229312p^3-290150p^2-143745p+33075. && (5.83)
\end{aligned}$$

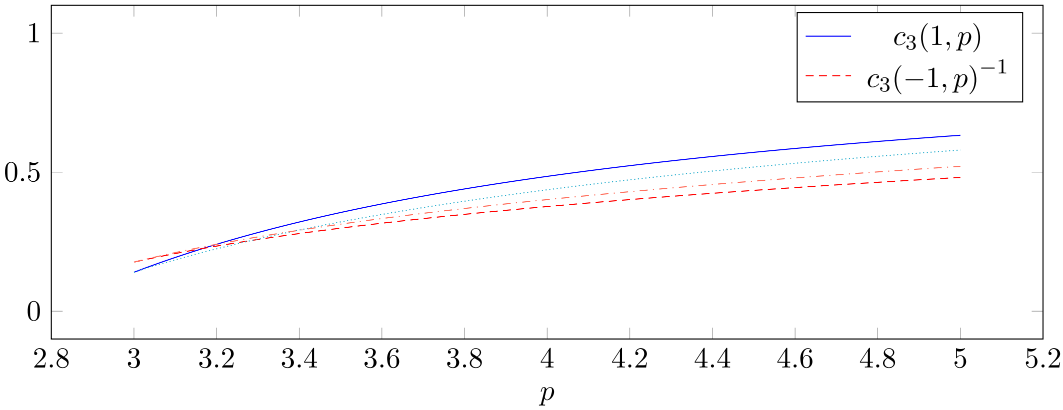
By direct computation $m_3,m_2>0$, $m_1,m_0<0$. Further $8m_3+4m_2+2m_1+m_0>0$. That concludes the proof. $\square$

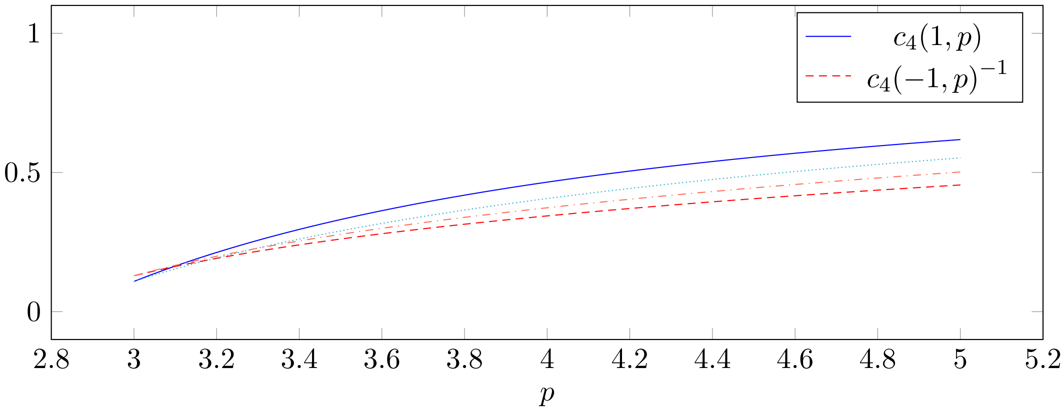
5.3.4. Absence of Resonances

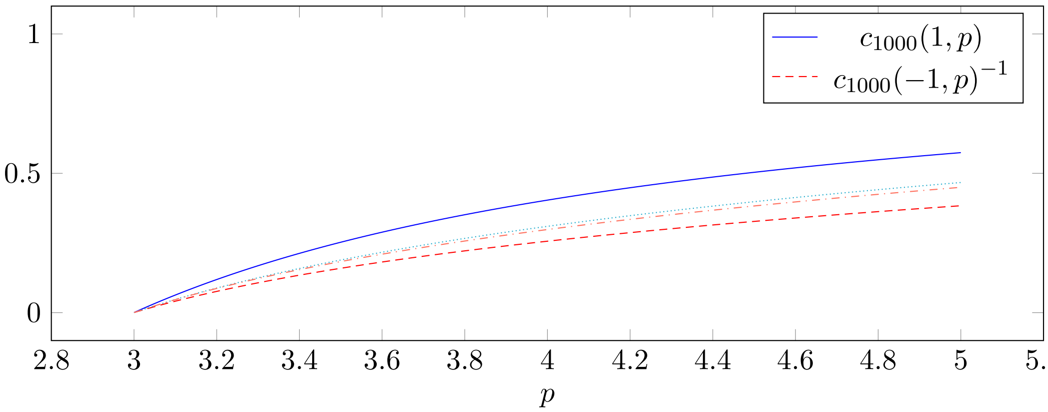
Showing that no resonance exists is now a simple matter of comparing the limits of the lower and upper bounds given by lemma 5.3.10 and lemma 5.3.12.

c_k^+, $\frac{1}{c_k^-}$ and the corresponding bounds are shown in the following graphics for $k = 2$, $k = 3$, $k = 4$ and $k = 1000$.









Lemma 5.3.13 *Let $p \in (3,5]$. Then, -1 and 1 are not even resonances or eigenvalues of iL, meaning no even bounded solutions to $(iL \pm 1)w = 0$ exist.*

Proof. By lemma 5.1.8, it suffices to show

$$\mathcal{C}(1,p) > \frac{1}{\mathcal{C}(-1,p)}. \tag{5.84}$$

Lemma 5.3.10 implies

$$\frac{1}{\mathcal{C}(-1,p)} \in [0,\infty). \tag{5.85}$$

If $c_k^+ \notin (0,\infty)$, for any $k \geq 1$, then due to

$$\begin{aligned} -c_{k+1}^+ &= -\frac{\left(k+1+\frac{\sqrt{2}}{p-1}\right)^2}{(k+1)(k+1+\frac{2\sqrt{2}}{p-1})} \frac{A_k c_k^+ - \frac{p+1}{p-1}}{A_k - \frac{p+1}{p-1}c_k^+} \\ &= \frac{\left(k+1+\frac{\sqrt{2}}{p-1}\right)^2}{(k+1)(k+1+\frac{2\sqrt{2}}{p-1})} \frac{A_k(-c_k^+) + \frac{p+1}{p-1}}{A_k + \frac{p+1}{p-1}(-c_k^+)}, \end{aligned} \tag{5.86}$$

we conclude $\mathcal{C}(1,p) = \lim_{k\to\infty} c_k^+ \in (-\infty, 0)$ and thus $\mathcal{C}(1,p) \neq \frac{1}{\mathcal{C}(-1,p)}$. If $c_k^+ \in (0,\infty)$, for every $k \geq 1$, then, by lemma 5.3.10 and lemma 5.3.12:

$$\begin{aligned} \mathcal{C}(1,p) = \lim_{k\to\infty} c_k^+ &\geq \frac{p-1}{p+1}\frac{3}{2}\frac{3p-1}{4(p-2)p}(p-3) \\ &> \frac{p-1}{p+1}\frac{\frac{3}{2}\frac{3p-1}{4(p-2)p}(p-3)}{1+\frac{3p-1}{(p-2)p(p+2)^2}(p-3)} \\ &\geq \lim_{k\to\infty} c_k^{-\,-1} \\ &= \frac{1}{\mathcal{C}(-1,p)}. \end{aligned} \tag{5.87}$$

That concludes the proof. $\square$

5.4. Odd Resonances

We show for $3 \leq p \leq 5$:

$$\frac{1}{\mathcal{C}(1,p)} < \mathcal{C}(-1,p). \tag{5.88}$$

The calculations are more forgiving than in the previous section and we pursue a slightly more sophisticated strategy.

We begin by showing a straightforward lower bound on $c_k(-1,p)$. Then, we take advantage of a similarity between the recursion characterising $c_k(-1,p)$ and $c_k(1,p)^{-1}$. This allows us to show $\frac{1}{\mathcal{C}(1,p)} < \mathcal{C}(-1,p)$ without separately bounding both terms.

Definition 5.4.1 *Let $E \in [-1,1]$ and $k \geq 0$. We define A_k and B_k by definition 4.2.1, meaning*

$$\begin{aligned} A_k &= 4k^2 + 6k + \frac{8\sqrt{1+E}}{p-1}k + \frac{4(1+E)}{(p-1)^2} + \frac{6\sqrt{1+E}}{p-1} + 2 - \frac{(p+1)^2}{(p-1)^2} \\ &= 4\left(k + \frac{\sqrt{1+E}}{p-1}\right)^2 + 6\left(k + \frac{\sqrt{1+E}}{p-1}\right) + 2 - \frac{(p+1)^2}{(p-1)^2}, \end{aligned} \tag{5.89}$$

$$\begin{aligned} B_k &= 4(k+1)^2 + \frac{8\sqrt{1+E}}{p-1}(k+1) \\ &= 4(k+1)\left(k + \frac{2\sqrt{1+E}}{p-1}\right). \end{aligned} \tag{5.90}$$

Definition 5.4.2 *For $E \in [-1,1]$, we define $(c_k(E))_{k\in\mathbb{N}} = (c_k(E,p))_{k\in\mathbb{N}} \subseteq \mathbb{R} \cup \{\infty\}$ as given by lemma 4.2.6, meaning $c_0 = \infty$ and*

$$c_{k+1} = \left(1 + \frac{8E}{(p-1)^2 B_k}\right)\frac{A_k c_k - \frac{p+1}{p-1}}{A_k - \frac{p+1}{p-1}c_k}. \tag{5.91}$$

Notation 5.4.3 *In chapter 5.4, given $p \in [3,5]$, we often abbreviate $c_k^+ = c_k(1,p)$ and $c_k^- = c_k(-1,p)$.*

Lemma 5.4.4 *Let $p \in [3,5]$. Then, for every $k \geq 1$:*

$$c_k(1,p) \leq -1. \tag{5.92}$$

Proof. By definition:

$$c_{k+1}^+ = \left(1 + \frac{2}{(p-1)^2(k+1)^2 + 2\sqrt{2}(p-1)(k+1)}\right)\frac{A_k c_k^+ - \frac{p+1}{p-1}}{A_k - \frac{p+1}{p-1}c_k^+}. \tag{5.93}$$

It follows:

$$\begin{aligned} c_1^+ = & -\left(1 + \frac{2}{(p-1)^2 + 2\sqrt{2}(p-1)}\right)\frac{p-1}{p+1}\left(\frac{8}{(p-1)^2} + \frac{6\sqrt{2}}{p-1} + 2 - \frac{(p+1)^2}{(p-1)^2}\right) \\ \leq & -1. \end{aligned} \tag{5.94}$$

Note:

$$-c_{k+1}^{+} = -\left(1 + \frac{2}{(p-1)^2(k+1)^2 + 2\sqrt{2}(p-1)(k+1)}\right) \frac{A_k(-c_k^{+}) + \frac{p+1}{p-1}}{A_k + \frac{p+1}{p-1}(-c_k^{+})}. \tag{5.95}$$

The claim consequently follows by induction. □

Note that ${c_k^{+}}^{-1}$ and c_k^{-} fulfil virtually the same recursive equation. On the one hand:

$$\begin{aligned} {c_{k+1}^{+}}^{-1} = & \frac{(k+1+\frac{\sqrt{2}}{p-1}+\frac{\sqrt{2}}{p-1})(k+1+\frac{\sqrt{2}}{p-1}-\frac{\sqrt{2}}{p-1})}{(k+1+\frac{\sqrt{2}}{p-1})^2} \\ & \frac{(4(k+\frac{\sqrt{2}}{p-1})^2+6(k+\frac{\sqrt{2}}{p-1})+2-\frac{(p+1)^2}{(p-1)^2}){c_k^{+}}^{-1}-\frac{p+1}{p-1}}{4(k+\frac{\sqrt{2}}{p-1})^2+6(k+\frac{\sqrt{2}}{p-1})+2-\frac{(p+1)^2}{(p-1)^2}-\frac{p+1}{p-1}{c_k^{+}}^{-1}}. \end{aligned} \tag{5.96}$$

On the other hand:

$$c_{k+1}^{-} = \frac{(k+1+\frac{\sqrt{2}}{p-1})(k+1-\frac{\sqrt{2}}{p-1})}{(k+1)^2} \frac{(4k^2+6k+2-\frac{(p+1)^2}{(p-1)^2})c_k^{-}-\frac{p+1}{p-1}}{4k^2+6k+2-\frac{(p+1)^2}{(p-1)^2}-\frac{p+1}{p-1}c_k^{-}}. \tag{5.97}$$

5.4.1. Lower Bound

We seek to establish a lower bound on c_k^{-}. By definition:

$$c_1^{-} = \left(1 - \frac{2}{(p-1)^2}\right)\left(\frac{p+1}{p-1} - \frac{2(p-1)}{p+1}\right) \in (0,1). \tag{5.98}$$

We substitute $p = q + 3$:

$$c_1^{-} = \frac{(8-q^2)(q^2+4q+2)}{(q+2)^3(q+4)}. \tag{5.99}$$

It follows:

$$\begin{aligned} c_2^{-} &= \frac{2-\frac{1}{(q+2)^2}}{2} \frac{(30-\frac{(q+4)^2}{(q+2)^2})c_1^{-}-\frac{q+4}{q+2}}{30-\frac{(q+4)^2}{(q+2)^2}-\frac{q+4}{q+2}c_1^{-}} \\ &= \frac{(7+8q+2q^2)(128+512q+496q^2+56q^3-115q^4-50q^5-6q^6)}{2(q+2)^3(q+4)(56+128q+115q^2+44q^3+6q^4)}. \end{aligned} \tag{5.100}$$

This function attains its minimum at the boundary $q = 2$, meaning $p = 5$. We conclude:

Lemma 5.4.5 *Let $3 \le p \le 5$. Then:*

$$c_2(-1,p) \ge -\frac{31}{3904}. \tag{5.101}$$

Definition 5.4.6 *Let $(d_k)_{k\geq 2}$ be given by $d_2 = -\frac{1}{100}$ and*

$$d_{k+1} = \frac{(4k^2+6k+2-4)d_k - 2}{4k^2+6k+2-4-2d_k}. \tag{5.102}$$

Lemma 5.4.7 *For every $k \geq 2$:*

$$d_k < 0. \tag{5.103}$$

Proof. Follows easily by induction. □

Lemma 5.4.8 *For every $k \geq 2$ and $3 \leq p \leq 5$:*

$$c_k(-1,p) \geq d_k. \tag{5.104}$$

Proof. We proceed by induction. For $k = 2$ the claim follows from lemma 5.4.5. Assume $c_k(-1,p) \geq d_k$. By lemma 5.4.7, we have $d_{k+1} < 0$. It follows:

$$\begin{aligned} d_{k+1} &\leq \frac{(k+1+\frac{\sqrt{2}}{p-1})(k+1-\frac{\sqrt{2}}{p-1})}{(k+1)^2} \frac{(4k^2+6k+2-\frac{(p+1)^2}{(p-1)^2})d_k - \frac{p+1}{p-1}}{4k^2+6k+2-\frac{(p+1)^2}{(p-1)^2} - \frac{p+1}{p-1}d_k} \\ &\leq \frac{(k+1+\frac{\sqrt{2}}{p-1})(k+1-\frac{\sqrt{2}}{p-1})}{(k+1)^2} \frac{(4k^2+6k+2-\frac{(p+1)^2}{(p-1)^2})c_k^- - \frac{p+1}{p-1}}{4k^2+6k+2-\frac{(p+1)^2}{(p-1)^2} - \frac{p+1}{p-1}c_k^-} \\ &= c_{k+1}^-. \end{aligned} \tag{5.105}$$

That concludes the proof. □

Lemma 5.4.9 *Let $k \geq 2$. Then:*

$$d_k > -\frac{1}{4} + \frac{1}{1000}. \tag{5.106}$$

Proof. By definition $d_2 = -\frac{1}{100}$ and

$$d_{k+1} = \frac{(2k^2+3k-1)d_k - 1}{2k^2+3k-1-d_k}. \tag{5.107}$$

We conclude:

$$d_3 = -\frac{113}{1301}, \tag{5.108}$$

$$d_4 = -\frac{157}{1257}, \tag{5.109}$$

$$d_5 = -\frac{13}{88}, \tag{5.110}$$

$$d_6 = -\frac{184}{1129}, \tag{5.111}$$

$$d_7 = -\frac{389}{2237}, \tag{5.112}$$

$$d_8 = -\frac{529}{2905} \leq -\frac{1}{6}. \tag{5.113}$$

As d_k is decreasing by definition, we conclude $d_k \leq -\frac{1}{6}$ for every $k \geq 8$. It follows, for $k \geq 8$:

$$\begin{aligned} d_{k+1} &= \frac{(2k^2+3k-1)-d_k^{-1}}{2k^2+3k-1-d_k} d_k \\ &\geq \frac{2k^2+3k-1+6}{2k^2+3k-1+\frac{1}{6}} d_k \\ &\geq \frac{2k^2+3k+5}{2k^2+3k-1} d_k \\ &\geq \frac{2k^2+3k+1}{2k^2+3k-5} d_k \\ &= \frac{(k+1)(2k+1)}{(k-1)(2k+5)} d_k. \end{aligned} \tag{5.114}$$

We conclude for every $k \geq 8$:

$$\begin{aligned} d_{k+1} \geq & -\frac{1}{6}\prod_{l=8}^{k}\frac{(l+1)(2l+1)}{(l-1)(2l+5)} \\ = & -\frac{1}{6}\frac{16+1}{8-1}\frac{18+1}{9-1}\frac{l}{2l+3}\frac{l+1}{2l+5} \\ \geq & -\frac{1}{6}\frac{17}{7}\frac{19}{8}\frac{1}{4} \\ > & -\frac{1}{4}+\frac{1}{1000}. \end{aligned} \tag{5.115}$$

That concludes the proof. □

Corollary 5.4.10 *Let $k \geq 2$ and $3 \leq p \leq 5$. Then:*

$$c_k(-1,p) > -\frac{1}{4}+\frac{1}{1000}. \tag{5.116}$$

5.4.2. Recursion

To take advantage of the similarity between (5.96) and (5.97), we formalize the recursions via the following operators.

Definition 5.4.11 *Let $p \in [3,5]$, and $t \geq 0$. We define:*

$$\alpha(t) := 4\left(t+\frac{\sqrt{2}}{p-1}\right)^2 + 6\left(t+\frac{\sqrt{2}}{p-1}\right) + 2 - \frac{(p+1)^2}{(p-1)^2}, \tag{5.117}$$

$$\beta(t) := \frac{(t+1+\frac{\sqrt{2}}{p-1}+\frac{\sqrt{2}}{p-1})(t+1+\frac{\sqrt{2}}{p-1}-\frac{\sqrt{2}}{p-1})}{(t+1+\frac{\sqrt{2}}{p-1})^2}. \tag{5.118}$$

α and β are direct analogues of A and B given by definition 4.1.1. However α and β allow for the real valued argument t, while A and B are dependent on the integer k.

We now redefine the defining recursion of $(c_n)_n$ using α and β instead of A and B.

Definition 5.4.12 *Let $p \in [3,5]$ and $k, l \in \mathbb{Z}_{\geq 1}$ with $l \leq k$. We define*

$$\mu_{l,k}, \nu_{l,k} : \mathbb{R} \cup \{\infty\} \to \mathbb{R} \cup \{\infty\} \tag{5.119}$$

by $\mu_{l,l}(x) = x$ and

$$\mu_{l,k+1}(x) = \beta(k) \frac{\alpha(k)\mu_{l,k}(x) - 1}{\alpha(k) - \mu_{l,k}(x)}, \tag{5.120}$$

$$\nu_{l,k+1}(x) = \beta\left(k - \frac{\sqrt{2}}{p-1}\right) \frac{\alpha\left(k - \frac{\sqrt{2}}{p-1}\right)\nu_{l,k}(x) - 1}{\alpha\left(k - \frac{\sqrt{2}}{p-1}\right) - \nu_{l,k}(x)}. \tag{5.121}$$

We further define:

$$\mu_{l,\infty}(x) = \lim_{k\to\infty} \mu_{l,k}(x), \tag{5.122}$$

$$\nu_{l,\infty}(x) = \lim_{k\to\infty} \nu_{l,k}(x). \tag{5.123}$$

These limits exist due to lemma 5.4.13.

Lemma 5.4.13 *Let $p \in [3,5]$ and $l \in \mathbb{Z}_{\geq 1}$. Then, the limits*

$$\mu_{l,\infty}(x) = \lim_{k\to\infty} \mu_{l,k}(x), \tag{5.124}$$

$$\nu_{l,\infty}(x) = \lim_{k\to\infty} \nu_{l,k}(x). \tag{5.125}$$

are well-defined.

Proof. $\mu_{l,k}$ and $\nu_{l,k}$ are a generalisation of the sequence c_k. Lemma 5.4.13 follows completely analogously to corollary 4.2.7 (or corollary 4.1.7). □

Lemma 5.4.14 *Let $l, k \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$ and $p \in [3,5]$. Assume $k \geq l$. Then*

$$c_k^{+-1} = \mu_{l,k}(c_l^{+-1}), \tag{5.126}$$

$$c_k^{-} = \nu_{l,k}(c_l^{-}). \tag{5.127}$$

Proof. Holds by definition. □

Lemma 5.4.15 *Let $x \in [-1,0]$, $l \in \mathbb{Z}_{\geq 1}$ and $p \in [3,5]$. Then, there exists $\delta = \delta(x) > 0$, such that for every $k > l$:*

$$-1 \leq \mu_{l,k}(x), \nu_{l,k}(x) < -\delta. \tag{5.128}$$

Proof. We only consider $\mu_{l,k}$. The claim follows completely analogously for $\nu_{l,k}$. Note that $\mu_{l,k}(x) \in [-1,0]$ implies $\mu_{l,k+1}(x) \in \left[-1, \frac{(k+1)^2 - \frac{2}{(p-1)^2}}{(k+1)^2}\mu_{l,k}(x)\right]$. That already concludes the proof. □

Lemma 5.4.16 *Let $x, y \in [-1, 1]$ with $x < y$ and $1 \le l \le k$. Then*

$$\mu_{l,k}(x) < \mu_{l,k}(y), \tag{5.129}$$
$$\nu_{l,k}(x) < \nu_{l,k}(y). \tag{5.130}$$

Proof. Follows by induction. □

Lemma 5.4.17 *Let $x, y \in [-1, 0]$ with $x < y$. Let $l \in \mathbb{Z}_{\ge 1}$ and $p \in [3, 5]$. Then*

$$\mu_{l,\infty}(x) < \mu_{l,\infty}(y), \tag{5.131}$$
$$\nu_{l,\infty}(x) < \nu_{l,\infty}(y). \tag{5.132}$$

Proof. We only consider $\mu_{l,k}$. The claim follows completely analogously for $\nu_{l,k}$. Let $z \in (x, y)$. By definition

$$\mu_{l,k+1}(z) = \frac{(k+1+\frac{\sqrt{2}}{p-1})(k+1-\frac{\sqrt{2}}{p-1})}{(k+1)^2}\frac{\alpha(k)\mu_{l,k}(z)-1}{\alpha(k)-\mu_{l,k}(z)}. \tag{5.133}$$

It follows for $k \ge 1$:

$$\begin{aligned}\frac{\partial_z\mu_{l,k+1}(z)}{\mu_{l,k+1}(z)} &= \frac{\alpha(k)\partial_z\mu_{l,k}(z)}{\alpha(k)\mu_{l,k}(z)-1} + \frac{\partial_z\mu_{l,k}(z)}{\alpha(k)-\mu_{l,k}(z)}\\ &= \frac{\partial_z\mu_{l,k}(z)}{\mu_{l,k}(z)}\left(\frac{\alpha(k)\mu_{l,k}(z)}{\alpha(k)\mu_{l,k}(z)-1} + \frac{\mu_{l,k}(z)}{\alpha(k)-\mu_{l,k}(z)}\right)\\ &= \frac{\partial_z\mu_{l,k}(z)}{\mu_{l,k}(z)}\frac{\alpha(k)^2\mu_{l,k}(z)-\alpha(k)\mu_{l,k}(z)^2+\alpha(k)\mu_{l,k}(z)^2-\mu_{l,k}(z)}{(\alpha(k)\mu_{l,k}(z)-1)(\alpha(k)-\mu_{l,k}(z))}\\ &= \frac{\partial_z\mu_{l,k}(z)}{\mu_{l,k}(z)}\frac{(\alpha(k)^2-1)\mu_{l,k}(z)}{(\alpha(k)\mu_{l,k}(z)-1)(\alpha(k)-\mu_{l,k}(z))}\\ &= \frac{\partial_z\mu_{l,k}(z)}{\mu_{l,k}(z)}\frac{(\alpha(k)+1)(\alpha(k)-1)}{(\alpha(k)-\mu_{l,k}(z)^{-1})(\alpha(k)-\mu_{l,k}(z))}.\end{aligned} \tag{5.134}$$

By lemma 5.4.15 and $\mu_{l,k+1}(z) < 0$, we find $M = \delta^{-1} > 0$, such that

$$\frac{\partial_z\mu_{l,k+1}(z)}{\mu_{l,k+1}(z)} \le \frac{\partial_z\mu_{l,k}(z)}{\mu_{l,k}(z)}\frac{(\alpha(k)+1)(\alpha(k)-1)}{(\alpha(k)+M)(\alpha(k)+1)}. \tag{5.135}$$

We conclude

$$\frac{\partial_z\mu_\infty(z)}{\mu_\infty(z)} \le \prod_{k=l}^{\infty}\frac{\alpha(k)-1}{\alpha(k)+M}\frac{\partial_z\mu_{l,l}(z)}{\mu_{l,l}(z)} = \prod_{k=l}^{\infty}\frac{\alpha(k)-1}{\alpha(k)+M}\frac{1}{z} < 0. \tag{5.136}$$

That concludes the proof, as $\mu_\infty(z) < 0$ and $z < 0$. □

Lemma 5.4.18 *Let $x \in [-1, 0]$ and $y \in [-1, 1]$ with $x < y$. Let $l \in \mathbb{Z}_{\ge 1}$ and $p \in [3, 5]$. Then*

$$\mu_{l,\infty}(x) < \mu_{l,\infty}(y), \tag{5.137}$$
$$\nu_{l,\infty}(x) < \nu_{l,\infty}(y). \tag{5.138}$$

Proof. We only consider $\mu_{l,k}$. The claim follows completely analogously for $\nu_{l,k}$. Lemma 5.4.15 implies $\mu_{l,\infty}(x) < 0$. Consequently, we can assume $\mu_{l,\infty}(y) < 0$. We find $k \geq 1$, such that $\mu_{l,k}(y) < 0$. Lemma 5.4.16 implies $\mu_{l,k}(x) < \mu_{l,k}(y)$. By lemma 5.4.17:

$$\mu_{l,\infty}(x) = \mu_{k,\infty}(\mu_{l,k}(x)) < \mu_{k,\infty}(\mu_{l,k}(y)) = \mu_{l,\infty}(y). \tag{5.139}$$

That concludes the proof. □

5.4.3. Absence of Resonances

Lemma 5.4.19 *Let $3 \leq p \leq 5$ and $k \geq 2$. Let $x \in [-\frac{1}{4}, 0)$. Then:*

$$\mu_{k-1,k}(x) \leq \nu_{k,k+1}(x). \tag{5.140}$$

Proof. We have to show:

$$\beta(k-1)\frac{\alpha(k-1)x-1}{\alpha(k-1)-x} \leq \beta\left(k - \frac{\sqrt{2}}{p-1}\right)\frac{\alpha\left(k-\frac{\sqrt{2}}{p-1}\right)x-1}{\alpha\left(k-\frac{\sqrt{2}}{p-1}\right)-x}. \tag{5.141}$$

Due to $x \in [-\frac{1}{4}, 0)$,

$$\begin{aligned} f(t) &:= \beta\left(t - \frac{\sqrt{2}}{p-1}\right)\frac{\alpha\left(t-\frac{\sqrt{2}}{p-1}\right)-x^{-1}}{\alpha\left(t-\frac{\sqrt{2}}{p-1}\right)-x} \\ &= \frac{(t+1)^2 - \frac{2}{(p-1)^2}}{(t+1)^2}\frac{4t^2+6t+2-\frac{(p+1)^2}{(p-1)^2}-\frac{p+1}{p-1}x^{-1}}{4t^2+6t+2-\frac{(p+1)^2}{(p-1)^2}-\frac{p+1}{p-1}x} \end{aligned} \tag{5.142}$$

is decreasing for $t \geq 2$. Consequently, $t \mapsto xf(t)$ is increasing with respect to $t \geq 2$. That concludes the proof. □

Lemma 5.4.20 *Let $3 \leq p \leq 5$. Then:*

$$\frac{1}{\mathcal{C}(1,p)} < \mathcal{C}(-1,p). \tag{5.143}$$

Proof. By corollary 5.4.10, $c_k^- = \nu_{2,k}(c_2^-) > -\frac{1}{4} + \frac{1}{1000}$ for $k \geq 2$. By continuity (of $\nu_{2,\infty}$), we can choose $d_2 < c_2^-$, such that for every $k \geq 2$:

$$d_k := \nu_{2,k}(d_2) \in \left(-\frac{1}{4}, 0\right). \tag{5.144}$$

By lemma 5.4.18, to conclude the proof, it suffices to show ${c_{k-1}^+}^{-1} \leq d_k$ for every $k \geq 2$.

We proceed by induction. The base case follows from corollary 5.4.10 and

$${c_1^+}^{-1} = -\frac{(p-1)^2(p+1)(p+2\sqrt{2}-1)}{(p^2+2\sqrt{2}p-2p-2\sqrt{2}+3)(p^2+6\sqrt{2}p-6p-6\sqrt{2}+9)} < -\frac{1}{4}. \tag{5.145}$$

In order to prove the induction step, assume ${c_{k-1}^{+}}^{-1} \leq d_k$. By lemma 5.4.18 and lemma 5.4.19:

$$ {c_k^{+}}^{-1} = \mu_{k-1,k}({c_{k-1}^{+}}^{-1}) \leq \nu_{k,k+1}({c_{k-1}^{+}}^{-1}) \leq \nu_{k,k+1}(d_k) = d_{k+1}. \tag{5.146} $$

That concludes the proof. $\square$

Lemma 5.4.21 *Let* $p \in [3,5]$*. Then,* -1 *and* 1 *are not odd resonances or eigenvalues of* iL*, meaning no odd bounded solutions to* $(iL \pm 1)w = 0$ *exist.*

Proof. By lemma 5.1.17, it suffices to show

$$ \frac{1}{\mathcal{C}(1,p)} \neq \mathcal{C}(-1,p). \tag{5.147} $$

Lemma 5.4.20 concludes the proof. $\square$

5.5. Cubic Case, Even Eigenvalues

We show that, for $p = 3$, iL admits no even eigenvalues apart from 0 within the spectral gap $(-1, 1)$. As shown in chapter 5.1.1, specifically lemma 5.1.7, $E \in (0, 1)$ is an eigenvalue of iL, if and only if

$$\frac{1}{\mathcal{C}(E,3)} = \mathcal{C}(-E,3). \tag{5.148}$$

Recall the usual definitions.

Definition 5.5.1 (Definition 4.1.1) *Let $E \in [-1, 1]$, $k \geq 0$ (and $p = 3$). Consider*

$$\begin{aligned} A_k(E) &= 4\left(k + \frac{\sqrt{1+E}}{2}\right)^2 + 2\left(k + \frac{\sqrt{1+E}}{2}\right) - 4 \\ &= 4k^2 + 2k + 4\sqrt{1+E}k + E + \sqrt{1+E} - 3, \end{aligned} \tag{5.149}$$

$$B_k(E) = 4(k+1)^2 + 4\sqrt{1+E}(k+1). \tag{5.150}$$

Definition 5.5.2 (Lemma 4.1.5) *Let $E \in [-1, 1]$ (and $p = 3$). We define $(c_k)_{k\in\mathbb{N}} = (c_k(E))_{k\in\mathbb{N}} \subseteq \mathbb{R} \cup \{\infty\}$ by $c_0 = \infty$ and*

$$c_{k+1} = \left(1 + \frac{E}{2B_k}\right)\frac{A_k c_k - 2}{A_k - 2c_k}. \tag{5.151}$$

Notation 5.5.3 *As long as $E \in (0, 1)$ is clear from context, we abbreviate $c_k^+ = c_k(E)$ and $c_k^- = c_k(-E)$, as well as $A_k^+ = A_k(E)$, $A_k^- = A_k(-E)$.*

We use a different Ansatz then before, by calculating a lower bound on $\frac{c_k^{-\,-1} - c_k^+}{E^2}$. This bound is basically established through brute force via 20 pages of computations. A computer algebra system might make it easier to follow along, however the computations were derived using no more than a calculator for external help.

The strategy is relatively simple. Using brute force, we establish a lower bound on $c_6(-E, 3)^{-1} - c_6(E, 3)$. At that point the difference between c_k and c_{k+1} is small enough that we can establish a bound via natural induction.

However, the induction step still requires a variety of technical lemmata. Heuristically, these technical results are necessary to control $c_k^{-\,-1} - c_k^+$ for large E. In the odd case considered in chapter 5.6, such a step is unnecessary as $E \mapsto \frac{c_k^- - c_k^{+\,-1}}{E^2}$ is increasing.

Lemma 5.5.4 *Let $p = 3$ and $E \in [0, 1]$. Then:*

$$c_2^{+\,-1} - c_2^- \geq \frac{47}{72}E^2, \tag{5.152}$$

$$c_2^{-\,-1} - c_2^+ \geq \frac{9}{26}E^2, \tag{5.153}$$

$$\frac{E}{2} \leq 1 - c_2^+, c_2^- - 1 \leq \frac{5}{6}E. \tag{5.154}$$

Proof. For now, consider for $E \in [-1,1]$. We calculate:

$$\begin{aligned}
c_1(E) = &-\left(1+\frac{2E}{B_0(E)}\right)\frac{A_0(E)}{2}\\
=& \left(1+\frac{E}{2+2\sqrt{1+E}}\right)\left(2-\frac{1+E}{2}-\frac{\sqrt{1+E}}{2}\right). \qquad (5.155)
\end{aligned}$$

It follows:

$$\begin{aligned}
c_1(E)-1 =& \left(1+\frac{E}{2\left(\sqrt{1+E}+1\right)}\right)\left(1-\frac{E}{2}-\frac{\sqrt{1+E}-1}{2}\right)-1\\
=& \left(1+\frac{\sqrt{1+E}-1}{2}\right)\left(1-\frac{(\sqrt{1+E}-1)(\sqrt{1+E}+1)}{2}-\frac{\sqrt{1+E}-1}{2}\right)-1\\
=& \frac{\sqrt{1+E}-1}{2}\left(1-\frac{(\sqrt{1+E}-1)(\sqrt{1+E}+1)}{2}-\frac{\sqrt{1+E}-1}{2}\right)\\
&-\frac{(\sqrt{1+E}-1)(\sqrt{1+E}+1)}{2}-\frac{\sqrt{1+E}-1}{2}\\
=& \frac{\sqrt{1+E}-1}{2}\left(-1-\sqrt{1+E}-\frac{(\sqrt{1+E}-1)(\sqrt{1+E}+1)}{2}-\frac{\sqrt{1+E}-1}{2}\right)\\
=& \frac{\sqrt{1+E}-1}{4}\left(-1-3\sqrt{1+E}-E\right)\\
=& -\frac{\sqrt{1+E}-1}{4}\left(3\sqrt{1+E}+1+E\right). \qquad (5.156)
\end{aligned}$$

Calculating further gives:

$$\begin{aligned}
c_2(E) &= \underbrace{\left(1 + \frac{E}{8 + 4\sqrt{1+E}}\right)}_{=:K(E)} \frac{\left(2 + 5\sqrt{1+E} + 1 + E\right) c_1(E) - 2}{2 + 5\sqrt{1+E} + 1 + E - 2c_1(E)} \\
&= K(E) \frac{5\sqrt{1+E} + 1 + E + \left(2 + 5\sqrt{1+E} + 1 + E\right)(c_1(E) - 1)}{5\sqrt{1+E} + 1 + E - 2(c_1(E) - 1)} \\
&= K(E) \frac{5\sqrt{1+E} + 1 + E - \left(2 + 5\sqrt{1+E} + 1 + E\right) \frac{\sqrt{1+E}-1}{4} \left(3\sqrt{1+E} + 1 + E\right)}{5\sqrt{1+E} + 1 + E + 2\frac{\sqrt{1+E}-1}{4}\left(3\sqrt{1+E} + 1 + E\right)} \\
&= K(E) \frac{5 + \sqrt{1+E} - \left(2 + 5\sqrt{1+E} + 1 + E\right) \frac{\sqrt{1+E}-1}{4} \left(3 + \sqrt{1+E}\right)}{5 + \sqrt{1+E} + 2\frac{\sqrt{1+E}-1}{4}\left(3 + \sqrt{1+E}\right)} \qquad (5.157) \\
&= K(E) \left(1 - \frac{\left(4 + 5\sqrt{1+E} + 1 + E\right) \frac{\sqrt{1+E}-1}{4} \left(3 + \sqrt{1+E}\right)}{5 + \sqrt{1+E} + \frac{\sqrt{1+E}-1}{2}\left(3 + \sqrt{1+E}\right)}\right) \\
&= K(E) \left(1 - \frac{\left(4 + \sqrt{1+E}\right)\left(1 + \sqrt{1+E}\right)\left(\sqrt{1+E} - 1\right)\left(3 + \sqrt{1+E}\right)}{20 + 4\sqrt{1+E} + 2\left(\sqrt{1+E} - 1\right)\left(3 + \sqrt{1+E}\right)}\right) \\
&= K(E) \left(1 - E\frac{\left(4 + \sqrt{1+E}\right)\left(3 + \sqrt{1+E}\right)}{14 + 8\sqrt{1+E} + 2\,(1+E)}\right). \qquad (5.158)
\end{aligned}$$

By inverting the sign of E, it follows:

$$c_2(-E) = K(-E) \left(1 + E\frac{\left(4 + \sqrt{1-E}\right)\left(3 + \sqrt{1-E}\right)}{14 + 8\sqrt{1-E} + 2\,(1-E)}\right). \qquad (5.159)$$

We calculate:

$$\begin{aligned}
\frac{\left(4 + \sqrt{1+E}\right)\left(3 + \sqrt{1+E}\right)}{14 + 8\sqrt{1+E} + 2\,(1+E)} &= \frac{1}{2}\frac{12 + 7\sqrt{1+E} + 1 + E}{7 + 4\sqrt{1+E} + 1 + E} \\
&= \frac{1}{2}\left(1 + \frac{5 + 3\sqrt{1+E}}{7 + 4\sqrt{1+E} + 1 + E}\right). \qquad (5.160)
\end{aligned}$$

Clearly, $E \mapsto \frac{1}{2}\left(1 + \frac{5+3\sqrt{1+E}}{7+4\sqrt{1+E}+1+E}\right)$ is a decreasing function on $[-1, 1]$. That allows us to estimate for $E \in [0,1]$:

$$\frac{\left(4+\sqrt{1+E}\right)\left(3+\sqrt{1+E}\right)}{14+8\sqrt{1+E}+2\left(1+E\right)} \leq \frac{1}{2}\left(1+\frac{8}{12}\right) = \frac{5}{6}, \tag{5.161}$$

$$\begin{aligned}\frac{\left(4+\sqrt{1+E}\right)\left(3+\sqrt{1+E}\right)}{14+8\sqrt{1+E}+2\left(1+E\right)} &\geq \frac{1}{2}\left(1+\frac{5+3\sqrt{2}}{7+4\sqrt{2}+2}\right)\\ &\geq \frac{1}{2}\left(1+\frac{5+3\frac{3}{2}}{7+4\frac{3}{2}+2}\right) = \frac{49}{60},\end{aligned} \tag{5.162}$$

$$\frac{\left(4+\sqrt{1-E}\right)\left(3+\sqrt{1-E}\right)}{14+8\sqrt{1-E}+2\left(1-E\right)} \geq \frac{1}{2}\left(1+\frac{8}{12}\right) = \frac{5}{6} \tag{5.163}$$

$$\frac{\left(4+\sqrt{1-E}\right)\left(3+\sqrt{1-E}\right)}{14+8\sqrt{1-E}+2\left(1-E\right)} \leq \frac{1}{2}\left(1+\frac{5}{7}\right) = \frac{6}{7}. \tag{5.164}$$

From this point onwards in the proof, we only consider $E \in [0,1]$. We also switch to the shorthand-notation c_k^+, c_k^-. It follows from (5.161):

$$\begin{aligned}1-c_2^+ &= 1-\left(1+\frac{E}{8+4\sqrt{1+E}}\right)\left(1-E\frac{\left(4+\sqrt{1+E}\right)\left(3+\sqrt{1+E}\right)}{14+8\sqrt{1+E}+2\left(1+E\right)}\right)\\ &\leq 1-\left(1-\frac{5}{6}E\right)\\ &= \frac{5}{6}E.\end{aligned} \tag{5.165}$$

It follows from (5.162):

$$\begin{aligned}1-c_2^+ &= 1-\left(1+\frac{E}{8+4\sqrt{1+E}}\right)\left(1-E\frac{\left(4+\sqrt{1+E}\right)\left(3+\sqrt{1+E}\right)}{14+8\sqrt{1+E}+2\left(1+E\right)}\right)\\ &\geq 1-\left(1+\frac{E}{8}\right)\left(1-\frac{49}{60}E\right)\\ &\geq -\frac{E}{8}+\frac{49}{60}E\\ &\geq \frac{1}{2}E.\end{aligned} \tag{5.166}$$

It follows from (5.163):

$$\begin{aligned} c_2^- - 1 &= \left(1 - \frac{E}{8+4\sqrt{1-E}}\right)\left(1 + E\frac{\left(4+\sqrt{1-E}\right)\left(3+\sqrt{1-E}\right)}{14+8\sqrt{1-E}+2\,(1-E)}\right) - 1 \\ &\geq \left(1-\frac{E}{8}\right)\left(1+\frac{5}{6}E\right) - 1 \\ &= -\frac{1}{8}E + \frac{5}{6}E - \frac{5}{48}E^2 \\ &\geq -\frac{6}{48}E + \frac{40}{48}E - \frac{5}{48}E \\ &\geq \frac{E}{2}. \end{aligned} \tag{5.167}$$

It follows from (5.164):

$$\begin{aligned} c_2^- - 1 &= \left(1 - \frac{E}{8+4\sqrt{1-E}}\right)\left(1 + E\frac{\left(4+\sqrt{1-E}\right)\left(3+\sqrt{1-E}\right)}{14+8\sqrt{1-E}+2\,(1-E)}\right) - 1 \\ &\leq \left(1-\frac{E}{8+4}\right)\left(1+\frac{6}{7}E\right) - 1 \\ &\leq \left(-\frac{1}{12}+\frac{6}{7}\right)E \\ &\leq \frac{5}{6}E. \end{aligned} \tag{5.168}$$

That concludes the proof of (5.154).

Recall (5.158):

$$c_2^+ = K(E)\left(1 - E\frac{\left(4+\sqrt{1+E}\right)\left(3+\sqrt{1+E}\right)}{14+8\sqrt{1+E}+2\,(1+E)}\right), \tag{5.169}$$

$$c_2^- = K(-E)\left(1 + E\frac{\left(4+\sqrt{1-E}\right)\left(3+\sqrt{1-E}\right)}{14+8\sqrt{1-E}+2\,(1-E)}\right). \tag{5.170}$$

Also recall (5.157). We calculate analogously to (5.158):

$$\begin{aligned} \frac{1}{c_2^+} &= \frac{1}{K(E)}\frac{5+\sqrt{1+E}+2\frac{\sqrt{1+E}-1}{4}\left(3+\sqrt{1+E}\right)}{5+\sqrt{1+E}-\left(2+5\sqrt{1+E}+1+E\right)\frac{\sqrt{1+E}-1}{4}\left(3+\sqrt{1+E}\right)} \\ &= \frac{1}{K(E)}\left(1+E\frac{\left(4+\sqrt{1+E}\right)\left(3+\sqrt{1+E}\right)}{20+4\sqrt{1+E}-\left(2+5\sqrt{1+E}+1+E\right)\left(\sqrt{1+E}-1\right)\left(3+\sqrt{1+E}\right)}\right) \\ &= \frac{1}{K(E)}\left(1+E\frac{\left(4+\sqrt{1+E}\right)\left(3+\sqrt{1+E}\right)}{26+15\sqrt{1+E}-9(1+E)-7(1+E)^{\frac{3}{2}}-(1+E)^2}\right). \end{aligned} \tag{5.171}$$

The same calculation holds true, when switching signs. Therefore:

$$\frac{1}{c_2^-} = \frac{1}{K(-E)}\left(1 - E\frac{\left(4+\sqrt{1-E}\right)\left(3+\sqrt{1-E}\right)}{26+15\sqrt{1-E}-9(1-E)-7(1-E)^{\frac{3}{2}}-(1-E)^2}\right). \tag{5.172}$$

We compare c_2^- and c_2^{+-1}. By definition:

$$K(-E) = 1 - \frac{E}{8+4\sqrt{1-E}} \leq 1 - \frac{E}{8+4\sqrt{1+E}}\frac{1}{1+\frac{E}{8+4\sqrt{1+E}}} = \frac{1}{K(E)}. \tag{5.173}$$

We calculate:

$$\begin{aligned}
&\frac{\left(4+\sqrt{1+E}\right)\left(3+\sqrt{1+E}\right)}{26+15\sqrt{1+E}-9(1+E)-7(1+E)^{\frac{3}{2}}-(1+E)^2} - \frac{\left(4+\sqrt{1-E}\right)\left(3+\sqrt{1-E}\right)}{14+8\sqrt{1-E}+2\,(1-E)}\\
&=: \frac{m_1(E)}{n_1(E)} - \frac{m_2(E)}{n_2(E)}\\
&= \frac{m_1(E)-m_2(E)}{n_1(E)} + m_2(E)\frac{n_2(E)-n_1(E)}{n_1(E)n_2(E)}\\
&= \frac{7\left(\sqrt{1+E}-\sqrt{1-E}\right)+2E}{n_1(E)} + m_2(E)\frac{(E+1)^2+7(1+E)^{\frac{3}{2}}+7E-15\sqrt{1+E}+8\sqrt{1-E}-1}{n_1(E)n_2(E)}\\
&= \frac{7\frac{2E}{\sqrt{1+E}+\sqrt{1-E}}+2E}{n_1(E)} + m_2(E)\frac{(E+1)^2-1+7E\sqrt{1+E}+7E-8(\sqrt{1+E}-\sqrt{1-E})}{n_1(E)n_2(E)}\\
&= \frac{\frac{14}{\sqrt{1+E}+\sqrt{1-E}}+2}{n_1(E)}E + m_2(E)\frac{2+E+7\sqrt{1+E}+7-\frac{16}{\sqrt{1+E}+\sqrt{1-E}}}{n_1(E)n_2(E)}E \qquad (5.174)\\
&\geq \frac{9}{24}E + \frac{m_2(E)}{n_2(E)}\frac{8+\left(\sqrt{1+E}+7-\frac{16}{1+E+\sqrt{1-E^2}}\right)\sqrt{1+E}}{n_1(E)}E\\
&\geq \frac{9}{24}E + \frac{5}{6}\frac{8}{24}E = \frac{27}{72}E + \frac{20}{72}E = \frac{47}{72}E. \qquad (5.175)
\end{aligned}$$

In the last inequality we used (5.163), which implies $\frac{m_2(E)}{n_2(E)} \geq \frac{5}{6}$.

(5.173) and (5.175) together imply:

$$\frac{1}{c_2^-} - c_2^+ \geq \frac{47}{72}E^2. \tag{5.176}$$

That concludes the proof of (5.152). By (5.173):

$$K(E) \leq \frac{1}{K(-E)}. \tag{5.177}$$

Analogously to (5.174), we conclude:

$$\frac{\left(4+\sqrt{1+E}\right)\left(3+\sqrt{1+E}\right)}{14+8\sqrt{1+E}+2\,(1+E)} - \frac{\left(4+\sqrt{1-E}\right)\left(3+\sqrt{1-E}\right)}{26+15\sqrt{1-E}-9(1-E)-7(1-E)^{\frac{3}{2}}-(1-E)^2}$$
$$= \frac{\frac{14}{\sqrt{1+E}+\sqrt{1-E}}+2}{n_1(-E)}E + \frac{m_2(-E)}{n_2(-E)}\frac{8+1-E+7\sqrt{1-E}-\frac{16}{\sqrt{1+E}+\sqrt{1-E}}}{n_1(-E)}E. \tag{5.178}$$

Recall $\frac{5}{6} \geq \frac{m_2(-E)}{n_2(-E)} \geq \frac{49}{60} > \frac{4}{5}$ as given by (5.161) and (5.162).

If $8+1-E+7\sqrt{1-E}-\frac{16}{\sqrt{1+E}+\sqrt{1-E}} > 0$, then:

$$\frac{\left(4+\sqrt{1+E}\right)\left(3+\sqrt{1+E}\right)}{14+8\sqrt{1+E}+2\,(1+E)} - \frac{\left(4+\sqrt{1-E}\right)\left(3+\sqrt{1-E}\right)}{26+15\sqrt{1-E}-9(1-E)-7(1-E)^{\frac{3}{2}}-(1-E)^2}$$
$$\geq \frac{\frac{14}{\sqrt{1+E}+\sqrt{1-E}}+2}{n_1(-E)}E + \frac{4}{5}\frac{8+1-E+7\sqrt{1-E}-\frac{16}{\sqrt{1+E}+\sqrt{1-E}}}{n_1(-E)}E$$
$$\geq \frac{2+\frac{32}{5}+\left(14-\frac{64}{5}\right)\frac{1}{\sqrt{1+E}+\sqrt{1-E}}}{n_1(-E)}E \geq \frac{2+\frac{32}{5}+\frac{3}{5}}{26}E = \frac{9}{26}E. \tag{5.179}$$

If $8+1-E+7\sqrt{1-E}-\frac{16}{\sqrt{1+E}+\sqrt{1-E}} \leq 0$, then:

$$\frac{\left(4+\sqrt{1+E}\right)\left(3+\sqrt{1+E}\right)}{14+8\sqrt{1+E}+2\,(1+E)} - \frac{\left(4+\sqrt{1-E}\right)\left(3+\sqrt{1-E}\right)}{26+15\sqrt{1-E}-9(1-E)-7(1-E)^{\frac{3}{2}}-(1-E)^2}$$
$$\geq \frac{\frac{14}{\sqrt{1+E}+\sqrt{1-E}}+2}{n_1(-E)}E + \frac{5}{6}\frac{8+1-E+7\sqrt{1-E}-\frac{16}{\sqrt{1+E}+\sqrt{1-E}}}{n_1(-E)}E$$
$$\geq \frac{2+\frac{20}{3}+\left(14-\frac{40}{3}\right)\frac{1}{\sqrt{1+E}+\sqrt{1-E}}}{n_1(-E)}E \geq \frac{2+\frac{20}{3}+\frac{1}{3}}{26}E = \frac{9}{26}E. \tag{5.180}$$

We conclude from (5.177), (5.179) and (5.180):

$$\frac{1}{c_2^+} - c_2^- \geq \frac{9}{26}E^2. \tag{5.181}$$

That shows (5.153) and concludes the proof. □

Lemma 5.5.5 *Let $p = 3$ and $E \in [0,1]$. Then, for every $k \geq 2$:*

$$0 \leq c_k^- - 1 \leq 2E. \tag{5.182}$$

Proof. As we only consider c_k^-, notate $c_k = c_k^-$, $A_k = A_k^-$ and $B_k = B_k^-$ for short.

We proceed inductively. The base case $k = 2$ follows from lemma 5.5.4. For the induction step, assume (5.182) holds true for $2 \leq k \leq K$. That implies $c_k \in [1,3]$. By definition:

$$c_{k+1} = \frac{B_k - 2E}{B_k}\frac{A_k c_k - 2}{A_k - 2c_k} = \left(1 - \frac{2E}{B_k}\right)\frac{A_k - 2c_k^{-1}}{A_k - 2c_k}c_k \geq \left(1 - \frac{2E}{B_k}\right)c_k. \tag{5.183}$$

It follows from lemma 5.5.4:

$$\begin{aligned}
c_{K+1} &\geq \prod_{k=2}^{K}\left(1-\frac{2E}{B_k}\right)c_2\\
&\geq \prod_{k=2}^{K}\left(1-\frac{E}{2(k+1)^2}\right)\left(1+\frac{E}{2}\right)\\
&\geq \left(1-\sum_{k=2}^{K}\frac{E}{2(k+1)^2}\right)\left(1+\frac{E}{2}\right)\\
&\geq \left(1-\frac{E}{2}\sum_{k=2}^{K}\left(\frac{1}{k}-\frac{1}{k+1}\right)\right)\left(1+\frac{E}{2}\right)\\
&= \left(1-\frac{E}{4}\right)\left(1+\frac{E}{2}\right)\\
&\geq 1.
\end{aligned} \tag{5.184}$$

It remains to show $c_{K+1}-1 \leq 2E$. For $2 \leq k \leq K$, let $\varepsilon_k := \frac{c_k-1}{E} \in [0,2]$. We estimate:

$$\begin{aligned}
c_{k+1}-1 &= \left(1-\frac{2E}{B_k}\right)\frac{A_k c_k - 2}{A_k - 2c_k} - 1\\
&\leq \frac{A_k\left(1+\varepsilon_k E\right)-2}{A_k - 2\left(1+\varepsilon_k E\right)} - 1\\
&= \frac{A_k - 2 + \varepsilon_k E A_k}{A_k - 2 - 2\varepsilon_k E} - 1\\
&= \frac{A_k + 2}{A_k - 2 - \varepsilon_k E}\varepsilon_k E\\
&\leq \left(1+\frac{4+\varepsilon_k}{4k^2+2k-6-\varepsilon_k}\right)\varepsilon_k E.
\end{aligned} \tag{5.185}$$

If $K=2$ or $K=3$, then the claim follows from lemma 5.5.4:

$$c_3 - 1 \leq \frac{5}{6}\left(1+\frac{4+\frac{5}{6}}{16+4-6-\frac{5}{6}}\right)E = \frac{5}{6}\frac{108}{79}E = \frac{90}{79}E, \tag{5.186}$$

$$c_4 - 1 \leq \frac{90}{79}\left(1+\frac{4+\frac{90}{79}}{36+6-6-\frac{90}{79}}\right)E = \frac{90}{79}\frac{3160}{2754}E = \frac{15800}{12087}E. \tag{5.187}$$

Otherwise, if $K \geq 4$, then:

$$\begin{aligned}
c_{K+1} - 1 &\leq \frac{15800}{12087} E \prod_{k=4}^{\infty} \left(1 + \frac{4+2}{4k^2 + 2k - 6 - 2}\right) \\
&= \frac{15800}{12087} E \prod_{k=4}^{\infty} \left(\frac{k^2 + \frac{1}{2}k - \frac{1}{2}}{k^2 + \frac{1}{2}k - 2}\right) \\
&= \frac{15800}{12087} E \prod_{k=4}^{\infty} \frac{(k+1)(k-\frac{1}{2})}{(k+\frac{1}{4})^2} \frac{(k+\frac{1}{4})^2}{k^2 + \frac{1}{2}k - 2} \\
&\leq \frac{15800}{12087} E \prod_{k=4}^{\infty} \frac{(k+1)(k-\frac{1}{2})}{(k+\frac{1}{4})^2} \frac{(k+\frac{1}{4})^2 - 1}{k^2 + \frac{1}{2}k - 2 - 1} \\
&= \frac{15800}{12087} E \prod_{k=4}^{\infty} \frac{(k+1)(k-\frac{1}{2})}{(k+\frac{1}{4})^2} \frac{(k+\frac{5}{4})(k-\frac{3}{4})}{(k+2)(k-\frac{3}{2})} \\
&= \frac{15800}{12087} \frac{4+1}{4-\frac{3}{2}} \frac{4-\frac{3}{4}}{4+\frac{1}{4}} E \\
&= \frac{15800}{12087} \frac{10}{5} \frac{13}{17} E \\
&= \frac{10}{5} \frac{205400}{205479} E \\
&\leq 2E. \qquad (5.188)
\end{aligned}$$

That concludes the proof. □

Lemma 5.5.6 *Let $p = 3$ and $E \in [0, 1]$. Then, for every $k \geq 2$:*

$$c_k^+ \leq 1. \qquad (5.189)$$

Proof. As we only consider c_k^+, notate $c_k = c_k^+$, $A_k = A_k^+$ and $B_k = B_k^+$ for short.

We proceed inductively. The base case $k = 2$ follows from lemma 5.5.4.

Now, assume $c_k \leq 1$ for $2 \leq k \leq K$. By definition:

$$c_{k+1} = \frac{B_k + 2E}{B_k} \frac{A_k c_k - 2}{A_k - 2c_k}. \qquad (5.190)$$

Hence, if $c_k \leq 0$ for any $2 \leq k \leq K$, then $c_{K+1} \leq 0 \leq 1$.

Therefore, for the rest of the proof, we assume $0 < c_k \leq 1$ for every $2 \leq k \leq K$. It follows:

$$c_{k+1} = \left(1 + \frac{2E}{B_k}\right) \frac{A_k - 2c_k^{-1}}{A_k - 2c_k} c_k \leq \left(1 + \frac{2E}{B_k}\right) c_k. \qquad (5.191)$$

By lemma 5.5.4:

$$
\begin{aligned}
c_{K+1} &\leq \prod_{k=2}^{K} \left(1 + \frac{2E}{B_k}\right) c_2 \\
&\leq \prod_{k=2}^{K} \left(1 + \frac{E}{2(k+1)^2}\right) \left(1 - \frac{E}{2}\right) \\
&\leq \prod_{k=2}^{\infty} \left(1 - \frac{E}{2(k+1)^2}\right)^{-1} \left(1 - \frac{E}{2}\right) \\
&\leq \left(1 - \sum_{k=2}^{\infty} \frac{E}{2(k+1)^2}\right)^{-1} \left(1 - \frac{E}{2}\right) \\
&\leq \left(1 - \frac{E}{2} \sum_{k=2}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right)\right)^{-1} \left(1 - \frac{E}{2}\right) \\
&\leq \left(1 - \frac{E}{4}\right)^{-1} \left(1 - \frac{E}{2}\right) \\
&\leq 1.
\end{aligned} \tag{5.192}
$$

That concludes the proof. □

Lemma 5.5.7 *Let $p = 3$, $E \in [0, 1]$ and $k \geq 2$. Then, at least one of the following two estimates is fulfilled:*

$$1 - c_k^+ \leq \frac{4}{3} E, \tag{5.193}$$

$$c_k^+ \leq 0. \tag{5.194}$$

Proof. As we only consider c_k^+, notate $c_k = c_k^+$, $A_k = A_k^+$ and $B_k = B_k^+$ for short.

We proceed inductively. The base case $k = 2$ follows from lemma 5.5.4. For the induction step, assume $1 - c_k \leq \frac{4}{3}E$ or $c_k \leq 0$ for $2 \leq k \leq K$. By definition:

$$c_{k+1} = \frac{B_k + 2E}{B_k} \frac{A_k c_k - 2}{A_k - 2c_k}. \tag{5.195}$$

$c_k \leq 1$ holds by lemma 5.5.6. If $c_k \leq 0$ for any $2 \leq k \leq K$, then $c_{K+1} \leq 0$. Therefore, we can assume $c_k > 0$ for $2 \leq k \leq K + 1$ and consequently $1 - c_k \leq \frac{4}{3}E$ for $2 \leq k \leq K$.

It follows for $2 \leq k \leq K$:

$$
\begin{aligned}
\frac{B_k + 2E}{B_k} \frac{A_k c_k - 2}{A_k - 2c_k} = c_{k+1} > 0 \Rightarrow \frac{A_k c_k - 2}{A_k - 2c_k} &> 0 \\
\Rightarrow c_{k+1} &\geq \frac{A_k c_k - 2}{A_k - 2c_k}.
\end{aligned} \tag{5.196}
$$

For $2 \leq k \leq K$, let $\varepsilon_k := \frac{1-c_k}{E} \in [0, \frac{4}{3}]$. Then:

$$\begin{aligned}
1 - c_{k+1} &\leq 1 - \frac{A_k c_k - 2}{A_k - 2c_k} \\
&= 1 - \frac{A_k - 2 - A_k \varepsilon_k E}{A_k - 2 + 2\varepsilon_k E} \\
&= \frac{A_k + 2}{A_k - 2 + 2\varepsilon_k E} \varepsilon_k E \\
&\leq \left(1 + \frac{4}{4k^2 + 6k - 4}\right) \varepsilon_k E. \qquad (5.197)
\end{aligned}$$

Lemma 5.5.4 implies:

$$\begin{aligned}
1 - c_{K+1} &\leq \frac{5}{6} E \prod_{l=2}^{\infty} \left(1 + \frac{1}{l^2 + \frac{3}{2}l - 1}\right) \\
&= \frac{5}{6} E \prod_{l=2}^{\infty} \frac{l^2 + \frac{3}{2}l}{l^2 + \frac{3}{2}l - 1} \\
&= \frac{5}{6} E \prod_{l=2}^{\infty} \frac{(l + \frac{3}{2})l}{(l - \frac{1}{2})(l + 2)} \\
&= \frac{5}{6} \frac{2}{2 - \frac{1}{2}} \frac{3}{3 - \frac{1}{2}} E \\
&= \frac{5}{6} \frac{4}{3} \frac{6}{5} E \\
&= \frac{4}{3} E. \qquad (5.198)
\end{aligned}$$

That concludes the proof. $\square$

Corollary 5.5.8 *Let $p = 3$ and $E \in [0, 1]$. If $c_l^+ \leq 0$ for any $l \geq 2$, then for every $k \geq l$:*

$$c_k^{-\,-1} - c_k^+ \geq \frac{1}{3} E^2. \qquad (5.199)$$

Proof. By definition:

$$c_{k+1}^+ = \frac{B_k^+ + 2E}{B_k^+} \frac{A_k^+ c_k^+ - 2}{A_k^+ - 2c_k^+}. \qquad (5.200)$$

That implies $c_k^+ \leq 0$ for every $k \geq l$. By lemma 5.5.5:

$$c_k^{-\,-1} - c_k^+ \geq \frac{1}{1 + 2E} - 0 \geq \frac{1}{3} \geq \frac{1}{3} E^2. \qquad (5.201)$$

That concludes the proof. $\square$

Lemma 5.5.9 *Let $p = 3$, $E \in [0, 1]$ and $k \geq 2$. Assume $c_l^+ > 0$ for every $l \geq 2$. Assume further:*

$$\frac{A_k^- - 2c_k^-}{A_k^- - 2{c_k^-}^{-1}}({c_k^-}^{-1} - c_k^+) + \left(\frac{A_k^- - 2c_k^-}{A_k^- - 2{c_k^-}^{-1}} - \frac{A_k^+ - 2{c_k^+}^{-1}}{A_k^+ - 2c_k^+}\right) c_k^+ \geq 0. \tag{5.202}$$

Then:

$${c_{k+1}^-}^{-1} - c_{k+1}^+ \geq \frac{A_k^- - 2c_k^-}{A_k^- - 2{c_k^-}^{-1}}({c_k^-}^{-1} - c_k^+) + \left(\frac{A_k^- - 2c_k^-}{A_k^- - 2{c_k^-}^{-1}} - \frac{A_k^+ - 2{c_k^+}^{-1}}{A_k^+ - 2c_k^+}\right) c_k^+. \tag{5.203}$$

Proof. Lemma 5.5.5 implies $c_l^- \geq 1 > 0$ for every $l \geq 2$. By definition:

$$\begin{aligned} {c_{k+1}^-}^{-1} &= \frac{B_k^-}{B_k^- - 2E}\frac{A_k^- - 2c_k^-}{A_k^- - 2{c_k^-}^{-1}}{c_k^-}^{-1}, &(5.204)\\ c_{k+1}^+ &= \frac{B_k^+ + 2E}{B_k^+}\frac{A_k^+ - 2{c_k^+}^{-1}}{A_k^+ - 2c_k^+}c_k^+. &(5.205)\end{aligned}$$

Due to $B_k^+ > B_k^-$:

$$\frac{B_k^-}{B_k^- - 2E} \geq \frac{B_k^+ + 2E}{B_k^+} \geq 1. \tag{5.206}$$

It follows:

$$\begin{aligned} &{c_{k+1}^-}^{-1} - c_{k+1}^+ \\ \geq\ & \frac{B_k^+ + 2E}{B_k^+}\left(\frac{A_k^- - 2c_k^-}{A_k^- - 2{c_k^-}^{-1}}({c_k^-}^{-1} - c_k^+) + \left(\frac{A_k^- - 2c_k^-}{A_k^- - 2{c_k^-}^{-1}} - \frac{A_k^+ - 2{c_k^+}^{-1}}{A_k^+ - 2c_k^+}\right) c_k^+\right) \\ \geq\ & \frac{A_k^- - 2c_k^-}{A_k^- - 2{c_k^-}^{-1}}({c_k^-}^{-1} - c_k^+) + \left(\frac{A_k^- - 2c_k^-}{A_k^- - 2{c_k^-}^{-1}} - \frac{A_k^+ - 2{c_k^+}^{-1}}{A_k^+ - 2c_k^+}\right) c_k^+. \end{aligned} \tag{5.207}$$

That concludes the proof. □

Lemma 5.5.10 *Let $p = 3$, $E \in [0, 1]$ and $k \geq 2$. Assume $c_l^+ > 0$ for every $l \geq 2$ and ${c_k^-}^{-1} - c_k^+ \geq 0$. Then:*

$$\begin{aligned} &\frac{A_k^- - 2c_k^-}{A_k^- - 2{c_k^-}^{-1}} - \frac{A_k^+ - 2{c_k^+}^{-1}}{A_k^+ - 2c_k^+} \\ \geq\ & \frac{2({c_k^-}^{-1} - c_k^+)(8k^2 + 4(\sqrt{2}+1)k + \sqrt{2} - 10) - \frac{20}{3}E^2(4\sqrt{2}k + 2 + \sqrt{2})}{(A_k^- - 2{c_k^-}^{-1})(A_k^+ - 2c_k^+)}. \end{aligned} \tag{5.208}$$

In particular, if:

$$c_k^{-\,-1} - c_k^+ \geq \frac{5}{3}\sqrt{2}E^2 \frac{8k + 2(\sqrt{2}+1)}{8k^2 + 4(\sqrt{2}+1)k + \sqrt{2} - 10}, \tag{5.209}$$

then:

$$\frac{A_k^- - 2c_k^-}{A_k^- - 2c_k^{-\,-1}} - \frac{A_k^+ - 2c_k^{+\,-1}}{A_k^+ - 2c_k^+} \geq 0. \tag{5.210}$$

Proof. Lemma 5.5.5, lemma 5.5.6 and lemma 5.5.7 imply:

$$0 \leq 1 - c_k^+ \leq \frac{4}{3}E, \tag{5.211}$$

$$0 \leq c_k^- - 1 \leq 2E. \tag{5.212}$$

We calculate:

$$\begin{aligned}
&\left(\frac{A_k^- - 2c_k^-}{A_k^- - 2c_k^{-\,-1}} - \frac{A_k^+ - 2c_k^{+\,-1}}{A_k^+ - 2c_k^+}\right)(A_k^- - 2c_k^{-\,-1})(A_k^+ - 2c_k^+)\\
&= (A_k^- - 2c_k^-)(A_k^+ - 2c_k^+) - (A_k^- - 2c_k^{-\,-1})(A_k^+ - 2c_k^{+\,-1})\\
&= 2(c_k^{+\,-1}A_k^- + c_k^{-\,-1}A_k^+ - c_k^+A_k^- - c_k^-A_k^+) + 4(c_k^+c_k^- - (c_k^+c_k^-)^{-1})\\
&= 2(c_k^{+\,-1} - c_k^-)A_k^- + 2(c_k^{-\,-1} - c_k^+)A_k^+ + 2(c_k^- - c_k^+)A_k^- + 2(c_k^+ - c_k^-)A_k^+\\
&\quad + 4(c_k^+ + c_k^{-\,-1})(c_k^- - c_k^{+\,-1})\\
&= 2(c_k^{+\,-1} - c_k^-)(A_k^- - 2(c_k^+ + c_k^{-\,-1})) + 2(c_k^{-\,-1} - c_k^+)A_k^+ - 2(c_k^- - c_k^+)(A_k^+ - A_k^-).
\end{aligned} \tag{5.213}$$

By (5.211) and (5.212):

$$\begin{aligned}
&\left(\frac{A_k^- - 2c_k^-}{A_k^- - 2c_k^{-\,-1}} - \frac{A_k^+ - 2c_k^{+\,-1}}{A_k^+ - 2c_k^+}\right)(A_k^- - 2c_k^{-\,-1})(A_k^+ - 2c_k^+)\\
&= 2(c_k^{+\,-1} - c_k^-)(A_k^- - 2(c_k^+ + c_k^{-\,-1})) + 2(c_k^{-\,-1} - c_k^+)A_k^+ - 2(c_k^- - c_k^+)(A_k^+ - A_k^-)\\
&\geq 2(c_k^{-\,-1} - c_k^+)\left(\frac{c_k^-}{c_k^+}(A_k^- - 4) + A_k^+\right) - 2\left(\frac{4}{3}E + 2E\right)(A_k^+ - A_k^-)\\
&\geq 2(c_k^{-\,-1} - c_k^+)(A_k^- - 4 + A_k^+) - \frac{20}{3}E(A_k^+ - A_k^-)\\
&\geq 2(c_k^{-\,-1} - c_k^+)(8k^2 + 4k + 4(\sqrt{1+E} + \sqrt{1-E})k + 2 + \sqrt{1+E} + \sqrt{1-E} - 8 - 4)\\
&\quad - \frac{20}{3}E(4(\sqrt{1+E} - \sqrt{1-E})k + 2E + \sqrt{1+E} - \sqrt{1-E})\\
&\geq 2(c_k^{-\,-1} - c_k^+)(8k^2 + 4(\sqrt{2}+1)k + \sqrt{2} - 10) - \frac{20}{3}E^2(4\sqrt{2}k + 2 + \sqrt{2}).
\end{aligned} \tag{5.214}$$

That concludes the proof. □

Lemma 5.5.11 *Let $p = 3$, $E \in [0,1]$ and $l \geq 3$. Assume $c_k^+ > 0$ for every $k \geq 2$. Further, assume:*

$$c_l^{-\,-1} - c_l^+ \geq \frac{5}{3}\sqrt{2}E^2 \frac{8l + 2(\sqrt{2}+1)}{8l^2 + 4(\sqrt{2}+1)l + \sqrt{2} - 10}. \tag{5.215}$$

Then, for every $k \geq l$:

$$c_k^{-\,-1} - c_k^+ \geq \frac{5}{3}\sqrt{2}E^2 \frac{8k + 2(\sqrt{2}+1)}{8k^2 + 4(\sqrt{2}+1)k + \sqrt{2} - 10}. \tag{5.216}$$

Proof. Let:

$$\gamma_k = \gamma_k(E) := \frac{5}{3}\sqrt{2}E^2 \frac{8k + 2(\sqrt{2}+1)}{8k^2 + 4(\sqrt{2}+1)k + \sqrt{2} - 10}. \tag{5.217}$$

We proceed inductively. Assume (5.216) holds for some $k \geq l$. By lemma 5.5.10:

$$\frac{A_k^- - 2c_k^-}{A_k^- - 2c_k^{-\,-1}} - \frac{A_k^+ - 2c_k^{+\,-1}}{A_k^+ - 2c_k^+} \geq 0. \tag{5.218}$$

Lemma 5.5.5 ensures $1 \leq c_k^- \leq 3$. By lemma 5.5.9:

$$\begin{aligned}
&c_{k+1}^{-\,-1} - c_{k+1}^+ \\
\geq\ & (c_k^{-\,-1} - c_k^+)\frac{A_k^- - 2c_k^-}{A_k^- - 2c_k^{-\,-1}} \\
\geq\ & \frac{5}{3}\sqrt{2}E^2 \frac{8k + 2(\sqrt{2}+1)}{8k^2 + 4(\sqrt{2}+1)k + \sqrt{2} - 10} \frac{4k^2 + 2k - 4 - 6}{4k^2 + 2k - 4} \\
\geq\ & \gamma_{k+1} \frac{8k + 2(\sqrt{2}+1)}{8(k+1) + 2(\sqrt{2}+1)} \frac{8(k+1)^2 + 4(\sqrt{2}+1)(k+1) + \sqrt{2} - 10}{8k^2 + 4(\sqrt{2}+1)k + \sqrt{2} - 10} \frac{k^2 + \frac{1}{2}k - \frac{5}{2}}{k^2 + \frac{1}{2}k - 1} \\
\geq\ & \gamma_{k+1} \frac{k + \frac{\sqrt{2}+1}{4}}{k + \frac{\sqrt{2}+5}{4}} \frac{8(k+1)^2 + 4(\sqrt{2}+1)(k+1) - 8}{8k^2 + 4(\sqrt{2}+1)k - 8} \frac{k + \frac{1}{2} - \frac{1}{k} - \frac{3}{2k}}{k + \frac{1}{2} - \frac{1}{k}} \\
\geq\ & \gamma_{k+1} \frac{k + \frac{1}{2}}{k + \frac{3}{2}} \frac{8(k+1)^2 + 12(k+1) - 8}{8k^2 + 12k - 8} \frac{k + \frac{1}{2} - \frac{1}{3} - \frac{3}{2k}}{k + \frac{1}{2} - \frac{1}{3}} \\
\geq\ & \gamma_{k+1} \frac{k + \frac{1}{2}}{k + \frac{3}{2}} \frac{(k+3)(k+\frac{1}{2})}{(k+2)(k-\frac{1}{2})} \frac{k - \frac{1}{3}}{k + \frac{1}{6}} \\
=\ & \gamma_{k+1} \frac{(k+\frac{1}{2})(k+3)}{(k+1)(k+2)} \frac{(k+1)(k+\frac{1}{2})}{(k+\frac{3}{2})k} \frac{(k-\frac{1}{3})k}{(k-\frac{1}{2})(k+\frac{1}{6})} \\
\geq\ & \gamma_{k+1}.
\end{aligned} \tag{5.219}$$

That concludes the proof. □

Corollary 5.5.12 *Let $p = 3$, $E \in [0,1]$ and $l \geq 3$. Assume $c_k^+ > 0$ for every $k \geq 2$. Further, assume:*

$$c_l^{-\,-1} - c_l^+ \geq \frac{5}{3}\sqrt{2}E^2 \frac{8l + 2(\sqrt{2}+1)}{8l^2 + 4(\sqrt{2}+1)l + \sqrt{2} - 10}. \tag{5.220}$$

Then, for every $k \geq l$:

$$c_k^{-\,-1} - c_k^+ \geq \frac{c_l^{-\,-1} - c_l^+}{6} \tag{5.221}$$

Proof. Lemma 5.5.11 implies for $k \geq l$:

$$c_k^{-\,-1} - c_k^+ \geq \frac{5}{3}\sqrt{2}E^2 \frac{8k + 2(\sqrt{2}+1)}{8k^2 + 4(\sqrt{2}+1)k + \sqrt{2} - 10}. \tag{5.222}$$

Consequently, by lemma 5.5.10:

$$\frac{A_k^- - 2c_k^-}{A_k^- - 2c_k^{-\,-1}} - \frac{A_k^+ - 2c_k^{+\,-1}}{A_k^+ - 2c_k^+} \geq 0. \tag{5.223}$$

By lemma 5.5.9:

$$c_{k+1}^{-\,-1} - c_{k+1}^+ \geq (c_k^{-\,-1} - c_k^+)\frac{A_k^- - 2c_k^-}{A_k^- - 2c_k^{-\,-1}}. \tag{5.224}$$

It follows inductively:

$$c_k^{-\,-1} - c_k^+ = (c_l^{-\,-1} - c_l^+)\prod_{n=l}^{k-1} \frac{A_n^- - 2c_n^-}{A_n^- - 2c_n^{-\,-1}}. \tag{5.225}$$

Lemma 5.5.5 implies $1 \leq c_n \leq 3$ for $n \geq 2$. It follows:

$$\begin{aligned}
c_k^{-\,-1} - c_k^+ &\geq (c_l^{-\,-1} - c_l^+)\prod_{n=l}^{k-1} \frac{4n^2 + 2n - 4 - 6}{4n^2 + 2n - 4} \\
&= (c_l^{-\,-1} - c_l^+)\prod_{n=l}^{k-1} \frac{n^2 + \frac{1}{2}n - \frac{5}{2}}{n^2 + \frac{1}{2}n - 1} \\
&\geq (c_l^{-\,-1} - c_l^+)\prod_{n=l}^{\infty} \frac{n^2 + \frac{1}{2}n - 3}{n^2 + \frac{1}{2}n - \frac{1}{2}} \\
&= (c_l^{-\,-1} - c_l^+)\prod_{n=l}^{\infty} \frac{(n+2)(n - \frac{3}{2})}{(n+1)(n - \frac{1}{2})} \\
&= (c_l^{-\,-1} - c_l^+)\frac{l - \frac{3}{2}}{l+1} \\
&\geq (c_l^{-\,-1} - c_l^+)\frac{2 - \frac{3}{2}}{2+1} \\
&= \frac{c_l^{-\,-1} - c_l^+}{6}.
\end{aligned} \tag{5.226}$$

That concludes the proof. ☐

Lemma 5.5.13 *Let $p = 3$ and $E \in [0, 1]$. Assume $c_k^+ > 0$ for every $k \geq 2$. Then:*

$$c_3^{-\,-1} - c_3^+ \geq \frac{3}{13}E^2. \tag{5.227}$$

Proof. Analogously to (5.213):

$$\begin{aligned}
&\left(\frac{A_2^- - 2c_2^-}{A_2^- - 2c_2^{-\,-1}} - \frac{A_2^+ - 2c_2^{+\,-1}}{A_2^+ - 2c_2^+}\right)(A_2^- - 2c_2^{-\,-1})(A_2^+ - 2c_2^+) \\
&= 2(c_2^{+\,-1} - c_2^-)(A_2^- - 2(c_2^+ + c_2^{-\,-1})) + 2(c_2^{-\,-1} - c_2^+)A_2^+ - 2(c_2^- - c_2^+)(A_2^+ - A_2^-).
\end{aligned} \tag{5.228}$$

We estimate using lemma 5.5.4:

$$\begin{aligned}
&\left(\frac{A_2^- - 2c_2^-}{A_2^- - 2c_2^{-\,-1}} - \frac{A_2^+ - 2c_2^{+\,-1}}{A_2^+ - 2c_2^+}\right)(A_2^- - 2c_2^{-\,-1})(A_2^+ - 2c_2^+) \\
&\geq 2\left(\frac{47}{72}E^2(A_2^- - 4) + \frac{9}{26}E^2A_2^+\right) - 2\left(\frac{5}{6}E + \frac{5}{6}E\right)(A_2^+ - A_2^-) \\
&\geq 2E^2\left(\frac{47}{72}(16-4) + \frac{1}{3}\left(16 + 8\sqrt{2} + 2 + \sqrt{2}\right)\right) \\
&\quad - \frac{10}{3}E\left(8(\sqrt{1+E} - \sqrt{1-E}) + 2E + (\sqrt{1+E} - \sqrt{1-E})\right) \\
&= 2E^2\left(\frac{47}{6} + 6 + 3\sqrt{2}\right) - \frac{10}{3}E^2\left(\frac{18}{\sqrt{1+E} + \sqrt{1-E}} + 2\right) \\
&\geq E^2\left(\frac{47}{3} + 12 + 6\sqrt{2}\right) - E^2\left(30\sqrt{2} + \frac{20}{3}\right) \\
&= E^2\left(\frac{27}{3} + 12 - 24\sqrt{2}\right) \\
&= E^2\left(21 - 24\sqrt{2}\right).
\end{aligned} \tag{5.229}$$

It follows:

$$\begin{aligned}
\frac{A_2^- - 2c_2^-}{A_2^- - 2c_2^{-\,-1}} - \frac{A_2^+ - 2c_2^{+\,-1}}{A_2^+ - 2c_2^+} &\geq E^2\frac{21 - 24\sqrt{2}}{(A_2^- - 2c_2^{-\,-1})(A_2^+ - 2c_2^+)} \\
&\geq -E^2\frac{14}{(A_2^- - 2)(A_2^+ - 2)} \\
&= -E^2\frac{14}{(15 + 9\sqrt{1-E} - E)(15 + 9\sqrt{1+E} + E)} \\
&\geq -E^2\frac{14}{14(16 + 9\sqrt{2})} \\
&\geq -\frac{E^2}{28}.
\end{aligned} \tag{5.230}$$

Lemma 5.5.9 implies:

$$\begin{aligned} {c_3^-}^{-1} - c_3^+ &\geq \frac{16 - 2(1+\frac{5}{6})}{16-0}\frac{9}{26}E^2 - \frac{E^2}{28} \\ &\geq \left(\frac{42-5}{48}\frac{9}{26} - \frac{1}{28}\right)E^2 = \left(\frac{333}{1248} - \frac{1}{28}\right)E^2 = \frac{8076}{34944}E^2 \geq \frac{3}{13}E^2. \end{aligned} \tag{5.231}$$

That concludes the proof. □

Lemma 5.5.14 *Let $p = 3$, $E \in [0,1]$ and $k \geq 3$. Assume $c_l^+ > 0$ for $l \geq 2$. Further, assume:*

$$0 \leq {c_k^-}^{-1} - c_k^+ < \frac{5}{3}\sqrt{2}E^2 \frac{8k + 2(\sqrt{2}+1)}{8k^2 + 4(\sqrt{2}+1)k + \sqrt{2} - 10}, \tag{5.232}$$

and:

$$\frac{(k - \frac{\sqrt{2}}{2} + \frac{3}{2})(k - \frac{\sqrt{2}}{2} - 1)}{(k - \frac{\sqrt{2}}{2} + \frac{5}{4})(k - \frac{\sqrt{2}}{2} - \frac{3}{4})}({c_k^-}^{-1} - c_k^+) - \frac{5}{3}\sqrt{2}E^2 \frac{1}{(k-\frac{2}{3})(k+\frac{7}{2})(k-\frac{3}{2})} \geq 0. \tag{5.233}$$

Then:

$$\begin{aligned} {c_{k+1}^-}^{-1} - c_{k+1}^+ \geq\ & \frac{(k - \frac{\sqrt{2}}{2} + \frac{3}{2})(k - \frac{\sqrt{2}}{2} - 1)}{(k - \frac{\sqrt{2}}{2} + \frac{5}{4})(k - \frac{\sqrt{2}}{2} - \frac{3}{4})}({c_k^-}^{-1} - c_k^+) \\ & - \frac{5}{3}\sqrt{2}E^2 \frac{1}{(k-\frac{2}{3})(k+\frac{7}{2})(k-\frac{3}{2})}. \end{aligned} \tag{5.234}$$

Proof. We show:

$$\begin{aligned} \phi_k :=\ & \frac{A_k^- - 2c_k^-}{A_k^- - 2{c_k^-}^{-1}}({c_k^-}^{-1} - c_k^+) \\ & + \frac{2({c_k^-}^{-1} - c_k^+)(8k^2 + 4(\sqrt{2}+1)k + \sqrt{2} - 10) - \frac{20}{3}E^2(4\sqrt{2}k + 2 + \sqrt{2})}{(A_k^- - 2{c_k^-}^{-1})(A_k^+ - 2c_k^+)}c_k^+ \\ \geq\ & \frac{(k - \frac{\sqrt{2}}{2} + \frac{3}{2})(k - \frac{\sqrt{2}}{2} - 1)}{(k - \frac{\sqrt{2}}{2} + \frac{5}{4})(k - \frac{\sqrt{2}}{2} - \frac{3}{4})}({c_k^-}^{-1} - c_k^+) - \frac{5}{3}\sqrt{2}E^2 \frac{1}{(k-\frac{2}{3})(k+\frac{7}{2})(k-\frac{3}{2})} \geq 0. \end{aligned} \tag{5.235}$$

(5.235) suffices to concludes the proof, as lemma 5.5.9 and lemma 5.5.10 then imply:

$${c_{k+1}^-}^{-1} - c_{k+1}^+ \geq \phi_k. \tag{5.236}$$

Lemma 5.5.5 ensures $1 \leq c_k^- \leq 3$. It follows:

$$\frac{A_k^- - 2c_k^-}{A_k^- - 2{c_k^-}^{-1}} \geq \frac{A_k^- - 6}{A_k^-} \geq \frac{4k^2 + 2k - 4 - 6}{4k^2 + 2k - 4} = 1 - \frac{3}{2}\frac{1}{k^2 + \frac{1}{2}k - 1}. \tag{5.237}$$

Further, by (5.232):

$$\frac{2(c_k^{-1} - c_k^+)(8k^2 + 4(\sqrt{2}+1)k + \sqrt{2} - 10) - \frac{20}{3}E^2(4\sqrt{2}k + 2 + \sqrt{2})}{(A_k^- - 2c_k^{-1})(A_k^+ - 2c_k^+)} \leq 0. \tag{5.238}$$

Lemma 5.5.6 implies $c_k^+ \leq 1$. It follows:

$$\begin{aligned}\phi_k \geq &\left(1 - \frac{3}{2}\frac{1}{k^2 + \frac{1}{2}k - 1}\right)(c_k^{-1} - c_k^+) \\ &+ \frac{2(c_k^{-1} - c_k^+)(8k^2 + 4(\sqrt{2}+1)k + \sqrt{2} - 10) - \frac{20}{3}E^2(4\sqrt{2}k + 2 + \sqrt{2})}{(A_k^- - 2c_k^{-1})(A_k^+ - 2c_k^+)}.\end{aligned} \tag{5.239}$$

Lemma 5.5.5 and lemma 5.5.6 ensure $c_k^{-1}, c_k^+ \leq 1$. We estimate:

$$\begin{aligned}0 &\geq \frac{2(c_k^{-1} - c_k^+)(8k^2 + 4(\sqrt{2}+1)k + \sqrt{2} - 10) - \frac{20}{3}E^2(4\sqrt{2}k + 2 + \sqrt{2})}{(A_k^- - 2c_k^{-1})(A_k^+ - 2c_k^+)} \\ &\geq \frac{2(c_k^{-1} - c_k^+)(8k^2 + 4(\sqrt{2}+1)k + \sqrt{2} - 10) - \frac{20}{3}E^2(4\sqrt{2}k + 2 + \sqrt{2})}{(A_k(-1) - 2)(A_k(1) - 2)} \\ &= (c_k^{-1} - c_k^+)\frac{16k^2 + 8(\sqrt{2}+1)k + 2\sqrt{2} - 20}{(A_k(-1) - 2)(A_k(1) - 2)} - \frac{20}{3}E^2\frac{4\sqrt{2}k + 2 + \sqrt{2}}{(A_k(-1) - 2)(A_k(1) - 2)}.\end{aligned} \tag{5.240}$$

We estimate (5.240) further. Firstly:

$$\begin{aligned}&(c_k^{-1} - c_k^+)\frac{16k^2 + 8(\sqrt{2}+1)k + 2\sqrt{2} - 20}{(A_k(-1) - 2)(A_k(1) - 2)} \\ &\geq (c_k^{-1} - c_k^+)\frac{16k^2 + 8(\sqrt{2}+1)k + 2\sqrt{2} - 20}{\frac{1}{4}(A_k(-1) + A_k(1) - 4)^2} \\ &= (c_k^{-1} - c_k^+)\frac{16k^2 + 8(\sqrt{2}+1)k + 2\sqrt{2} - 20}{(4k^2 + 2(\sqrt{2}+1)k + \frac{\sqrt{2}}{2} - 5)^2} \\ &= (c_k^{-1} - c_k^+)\frac{4}{4k^2 + 2(\sqrt{2}+1)k + \frac{\sqrt{2}}{2} - 5} \\ &= (c_k^{-1} - c_k^+)\frac{1}{k^2 + \frac{\sqrt{2}+1}{2}k + \frac{\sqrt{2}-2}{8} - 1}.\end{aligned} \tag{5.241}$$

Secondly:

$$\begin{aligned}
& -\frac{20}{3}E^2\frac{4\sqrt{2}k+2+\sqrt{2}}{(A_k(-1)-2)(A_k(1)-2)} \\
= & -\frac{20}{3}E^2\frac{4\sqrt{2}k+2+\sqrt{2}}{(4k^2+2k-6)(4k^2+2k+4\sqrt{2}k+\sqrt{2}-4)} \\
= & -\frac{5}{3}\sqrt{2}E^2\frac{k+\frac{1+\sqrt{2}}{4}}{(k^2+\frac{k}{2}-\frac{3}{2})(k^2+\frac{1+2\sqrt{2}}{2}k+\frac{\sqrt{2}}{4}-1)} \\
= & -\frac{5}{3}\sqrt{2}E^2\frac{k+\frac{1+\sqrt{2}}{4}}{(k^2+\frac{k}{2}-\frac{1}{16}-\frac{23}{16})(k+\frac{1+2\sqrt{2}}{4}+\frac{5}{2})(k+\frac{1+2\sqrt{2}}{4}-\frac{5}{2})} \\
= & -\frac{5}{3}\sqrt{2}E^2\frac{k+\frac{1+\sqrt{2}}{4}}{((k+\frac{1+\sqrt{2}}{4})(k+\frac{1-\sqrt{2}}{4})-\frac{23}{16})(k+\frac{1+2\sqrt{2}}{4}+\frac{5}{2})(k+\frac{1+2\sqrt{2}}{4}-\frac{5}{2})} \\
= & -\frac{5}{3}\sqrt{2}E^2\frac{1}{(k+\frac{1-\sqrt{2}}{4}-\frac{23}{16k+4+4\sqrt{2}})(k+\frac{1+2\sqrt{2}}{4}+\frac{5}{2})(k+\frac{1+2\sqrt{2}}{4}-\frac{5}{2})} \\
\geq & -\frac{5}{3}\sqrt{2}E^2\frac{1}{(k+\frac{1-\sqrt{2}}{4}-\frac{23}{52+4\sqrt{2}})(k+\frac{1+2\sqrt{2}}{4}+\frac{5}{2})(k+\frac{1+2\sqrt{2}}{4}-\frac{5}{2})} \\
\geq & -\frac{5}{3}\sqrt{2}E^2\frac{1}{(k+\frac{1-\sqrt{2}}{4}-\frac{2}{5})(k+\frac{1+2\sqrt{2}}{4}+\frac{5}{2})(k+\frac{1+2\sqrt{2}}{4}-\frac{5}{2})} \\
= & -\frac{5}{3}\sqrt{2}E^2\frac{1}{(k+\frac{1-\sqrt{2}}{4}-\frac{2}{5})(k-\frac{3-2\sqrt{2}}{4}+\frac{7}{2})(k-\frac{3-2\sqrt{2}}{4}-\frac{3}{2})} \\
\geq & -\frac{5}{3}\sqrt{2}E^2\frac{1}{(k-\frac{3-2\sqrt{2}}{2}+\frac{1-\sqrt{2}}{4}-\frac{2}{5})(k+\frac{7}{2})(k-\frac{3}{2})} \\
\geq & -\frac{5}{3}\sqrt{2}E^2\frac{1}{(k-\frac{2}{3})(k+\frac{7}{2})(k-\frac{3}{2})}.
\end{aligned} \tag{5.242}$$

(5.239), (5.240), (5.241) and (5.242) together yield:

$$\begin{aligned}
\phi_k \geq & \left(1+\frac{1}{k^2+\frac{\sqrt{2}+1}{2}k+\frac{\sqrt{2}-2}{8}-1}-\frac{3}{2}\frac{1}{k^2+\frac{1}{2}k-1}\right)(c_k^{-1}-c_k^{+}) \\
& -\frac{5}{3}\sqrt{2}E^2\frac{1}{(k-\frac{2}{3})(k+\frac{7}{2})(k-\frac{3}{2})}.
\end{aligned} \tag{5.243}$$

We simplify:

$$
\begin{aligned}
&1+\frac{1}{k^2+\frac{1+\sqrt{2}}{2}k+\frac{\sqrt{2}-2}{8}-1}-\frac{3}{2}\frac{1}{k^2+\frac{1}{2}k-1}\\
&=1-\frac{1}{2}\frac{3k^2+3\frac{1+\sqrt{2}}{2}k+3\frac{\sqrt{2}-2}{8}-3-2k^2-k+2}{(k^2+\frac{1+\sqrt{2}}{2}k+\frac{\sqrt{2}-2}{8}-1)(k^2+\frac{1}{2}k-1)}\\
&=1-\frac{1}{2}\frac{k^2+\frac{1+3\sqrt{2}}{2}k+3\frac{\sqrt{2}-2}{8}-1}{(k^2+\frac{1+\sqrt{2}}{2}k+\frac{\sqrt{2}-2}{8}-1)(k^2+\frac{1}{2}k-1)}\\
&\geq 1-\frac{1}{2}\frac{k^2+\frac{1+3\sqrt{2}}{2}k+3\frac{\sqrt{2}-2}{8}-1}{(k^2+\frac{1+\sqrt{2}}{2}k+\frac{\sqrt{2}-2}{8}-1+(\sqrt{2}k+\frac{\sqrt{2}-2}{4}))(k^2+\frac{1}{2}k-1-(\sqrt{2}k+\frac{\sqrt{2}-2}{4}))}\\
&=1-\frac{1}{2}\frac{1}{k^2-\frac{2\sqrt{2}-1}{2}k-1+\frac{2-\sqrt{2}}{4}}\\
&=\frac{k^2-\frac{2\sqrt{2}-1}{2}k-\frac{3}{2}+\frac{2-\sqrt{2}}{4}}{k^2-\frac{2\sqrt{2}-1}{2}k-1+\frac{2-\sqrt{2}}{4}}\\
&=\frac{(k-\frac{2\sqrt{2}-1}{4}+\sqrt{\frac{9}{16}-\frac{1}{2}+\frac{3}{2}})(k-\frac{2\sqrt{2}-1}{4}-\sqrt{\frac{9}{16}-\frac{1}{2}+\frac{3}{2}})}{(k-\frac{2\sqrt{2}-1}{4}+\sqrt{\frac{9}{16}-\frac{1}{2}+1})(k-\frac{2\sqrt{2}-1}{4}-\sqrt{\frac{9}{16}-\frac{1}{2}+1})}\\
&=\frac{(k+\frac{3-\sqrt{2}}{2})(k-\frac{\sqrt{2}+2}{2})}{(k-\frac{\sqrt{2}}{2}+\frac{\sqrt{17}+1}{4})(k-\frac{\sqrt{2}}{2}-\frac{\sqrt{17}-1}{4})}\\
&\geq\frac{(k-\frac{\sqrt{2}}{2}+\frac{3}{2})(k-\frac{\sqrt{2}}{2}-1)}{(k-\frac{\sqrt{2}}{2}+\frac{5}{4})(k-\frac{\sqrt{2}}{2}-\frac{3}{4})}.
\end{aligned}
\tag{5.244}
$$

That concludes the proof. □

Lemma 5.5.15 *Let $k \geq 5$. Then:*

$$\left(k-\frac{2}{3}\right)\left(k+\frac{7}{2}\right)\left(k-\frac{3}{2}\right)\geq k^3. \tag{5.245}$$

Proof. Follows by direct computation. □

Lemma 5.5.16 *Let $p = 3$, $E \in [0,1]$ and $K \geq 6$. Assume $c_k^+ > 0$ for $k \geq 2$. Further, assume for $6 \leq k \leq K$:*

$$0 \leq c_k^{-\,-1}-c_k^+ < \frac{5}{3}\sqrt{2}E^2\frac{8k+2(\sqrt{2}+1)}{8k^2+4(\sqrt{2}+1)k+\sqrt{2}-10}, \tag{5.246}$$

as well as:

$$(c_6^{-\,-1}-c_6^+)\prod_{k=6}^{\infty}\frac{(k-\frac{\sqrt{2}}{2}+\frac{3}{2})(k-\frac{\sqrt{2}}{2}-1)}{(k-\frac{\sqrt{2}}{2}+\frac{5}{4})(k-\frac{\sqrt{2}}{2}-\frac{3}{4})}-\frac{5}{3}\sqrt{2}E^2\sum_{k=6}^{\infty}\frac{1}{k^3}\geq 0. \tag{5.247}$$

Then:

$$c_{K+1}^{-\,-1} - c_{K+1}^{+} \geq (c_6^{-\,-1} - c_6^{+}) \prod_{k=6}^{K} \frac{(k - \frac{\sqrt{2}}{2} + \frac{3}{2})(k - \frac{\sqrt{2}}{2} - 1)}{(k - \frac{\sqrt{2}}{2} + \frac{5}{4})(k - \frac{\sqrt{2}}{2} - \frac{3}{4})} - \frac{5}{3}\sqrt{2}E^2 \sum_{k=6}^{K} \frac{1}{k^3}. \tag{5.248}$$

Proof. We show the lemma by induction. Lemma 5.5.14 provides the base case $K = 6$. For the induction step, assume:

$$\begin{aligned} c_K^{-\,-1} - c_K^{+} \geq (c_6^{-\,-1} - c_6^{+}) &\prod_{k=6}^{K-1} \frac{(k - \frac{\sqrt{2}}{2} + \frac{3}{2})(k - \frac{\sqrt{2}}{2} - 1)}{(k - \frac{\sqrt{2}}{2} + \frac{5}{4})(k - \frac{\sqrt{2}}{2} - \frac{3}{4})} \\ &- \frac{5}{3}\sqrt{2}E^2 \sum_{k=6}^{K-1} \frac{1}{k^3}. \end{aligned} \tag{5.249}$$

By lemma 5.5.14 and lemma 5.5.15:

$$\begin{aligned} &c_{K+1}^{-\,-1} - c_{K+1}^{+} \\ \geq\ &\frac{(K - \frac{\sqrt{2}}{2} + \frac{3}{2})(K - \frac{\sqrt{2}}{2} - 1)}{(K - \frac{\sqrt{2}}{2} + \frac{5}{4})(K - \frac{\sqrt{2}}{2} - \frac{3}{4})}(c_K^{-\,-1} - c_K^{+}) - \frac{5}{3}\sqrt{2}E^2 \frac{1}{(K - \frac{2}{3})(K + \frac{7}{2})(K - \frac{3}{2})} \\ \geq\ &(c_6^{-\,-1} - c_6^{+}) \prod_{k=6}^{K} \frac{(k - \frac{\sqrt{2}}{2} + \frac{3}{2})(k - \frac{\sqrt{2}}{2} - 1)}{(k - \frac{\sqrt{2}}{2} + \frac{5}{4})(k - \frac{\sqrt{2}}{2} - \frac{3}{4})} - \frac{5}{3}\sqrt{2}E^2 \frac{1}{K^3} \\ &- \underbrace{\frac{(K - \frac{\sqrt{2}}{2} + \frac{3}{2})(K - \frac{\sqrt{2}}{2} - 1)}{(K - \frac{\sqrt{2}}{2} + \frac{5}{4})(K - \frac{\sqrt{2}}{2} - \frac{3}{4})}}_{\leq 1} \frac{5}{3}\sqrt{2}E^2 \sum_{k=6}^{K-1} \frac{1}{k^3} \\ \geq\ &(c_6^{-\,-1} - c_6^{+}) \prod_{k=6}^{K} \frac{(k - \frac{\sqrt{2}}{2} + \frac{3}{2})(k - \frac{\sqrt{2}}{2} - 1)}{(k - \frac{\sqrt{2}}{2} + \frac{5}{4})(k - \frac{\sqrt{2}}{2} - \frac{3}{4})} - \frac{5}{3}\sqrt{2}E^2 \sum_{k=6}^{K} \frac{1}{k^3} \geq 0. \end{aligned} \tag{5.250}$$

That concludes the proof. ☐

Lemma 5.5.17 *Let $p = 3$, $E \in [0, 1]$. Assume $c_k^{+} > 0$ for every $k \geq 2$. Then, for $k \geq 3$:*

$$c_k^{-\,-1} - c_k^{+} \geq \frac{1}{6000}E^2. \tag{5.251}$$

Proof. Assume the lemma does not hold. We choose $K \geq 2$ as the smallest such integer fulfilling:

$$c_{K+1}^{-\,-1} - c_{K+1}^{+} < \frac{1}{1000}E^2. \tag{5.252}$$

Lemma 5.5.13 implies $K \geq 3$. As K was chosen minimally, we conclude for every $2 \leq k \leq K$:

$$c_k^{-\,-1} - c_k^{+} \geq \frac{1}{1000}E^2. \tag{5.253}$$

By corollary 5.5.12, for every $2 \leq k \leq K$:

$$c_k^{-1} - c_k^+ < \frac{5}{3}\sqrt{2}E^2 \frac{8k + 2(\sqrt{2}+1)}{8k^2 + 4(\sqrt{2}+1)k + \sqrt{2} - 10}. \tag{5.254}$$

By lemma 5.5.13 and lemma 5.5.14:

$$\begin{aligned}
c_4^{-1} - c_4^+ &\geq \frac{(3 - \frac{\sqrt{2}}{2} + \frac{3}{2})(3 - \frac{\sqrt{2}}{2} - 1)}{(3 - \frac{\sqrt{2}}{2} + \frac{5}{4})(3 - \frac{\sqrt{2}}{2} - \frac{3}{4})} \frac{3E^2}{13} - \frac{5}{3}\sqrt{2}E^2 \frac{1}{(3 - \frac{2}{3})(3 + \frac{7}{2})(3 - \frac{3}{2})} \\
&= \frac{(9 - \sqrt{2})(4 - \sqrt{2})}{(\frac{17}{2} - \sqrt{2})(\frac{9}{2} - \sqrt{2})} \frac{3}{13} E^2 - \frac{20}{21}\sqrt{2}E^2 \frac{1}{13} \\
&= \frac{E^2}{13}\left(3 \frac{36 + 2 - 13\sqrt{2}}{\frac{153}{4} + 2 - 13\sqrt{2}} - \frac{20}{21}\sqrt{2}\right) \\
&= \frac{E^2}{13}\left(3 - \frac{9}{4} \frac{1}{\frac{161}{4} - 13\sqrt{2}} - \frac{20}{21}\sqrt{2}\right) \\
&\geq \frac{E^2}{13}\left(3 - \frac{9}{4} \frac{1}{40 - 19} - \frac{20}{21} \frac{144}{100}\right) \\
&= \frac{E^2}{13}\left(3 - \frac{3}{28} - \frac{48}{35}\right) \\
&\geq \frac{E^2}{13}\left(\frac{420 - 15 - 192}{140}\right) \\
&= \frac{E^2}{13} \frac{213}{140} \\
&\geq \frac{3}{26}E^2.
\end{aligned} \tag{5.255}$$

Further, by lemma 5.5.14:

$$
\begin{aligned}
{c_5^-}^{-1} - c_5^+ &\geq \frac{(4-\frac{\sqrt{2}}{2}+\frac{3}{2})(4-\frac{\sqrt{2}}{2}-1)}{(4-\frac{\sqrt{2}}{2}+\frac{5}{4})(4-\frac{\sqrt{2}}{2}-\frac{3}{4})}\frac{3}{26}E^2 - \frac{5}{3}\sqrt{2}E^2\frac{1}{(4-\frac{2}{3})(4+\frac{7}{2})(4-\frac{3}{2})} \\
&= \frac{(11-\sqrt{2})(6-\sqrt{2})}{(\frac{21}{2}-\sqrt{2})(\frac{13}{2}-\sqrt{2})}\frac{3}{26}E^2 - \frac{5}{3}\frac{3}{10}\frac{2}{15}\frac{2}{5}\sqrt{2}E^2 \\
&= \frac{66+2-17\sqrt{2}}{\frac{273}{4}+2-17\sqrt{2}}\frac{3}{26}E^2 - \frac{2}{75}\sqrt{2}E^2 \\
&= \frac{68-17\sqrt{2}}{\frac{281}{4}-17\sqrt{2}}\frac{3}{26}E^2 - \frac{2}{75}\sqrt{2}E^2 \\
&\geq \frac{68-25}{\frac{281}{4}-25}\frac{3}{26}E^2 - \frac{2}{75}\sqrt{2}E^2 \\
&\geq \frac{42}{45}\frac{3}{26}E^2 - \frac{2}{75}\frac{37}{26}E^2 \\
&\geq \frac{1}{26}\left(\frac{42}{15} - \frac{74}{75}\right)E^2 \\
&\geq \frac{1}{26}\frac{27}{15}E^2 \\
&\geq \frac{1}{15}E^2.
\end{aligned} \tag{5.256}
$$

Again, by lemma 5.5.14 and lemma 5.5.15:

$$
\begin{aligned}
{c_6^-}^{-1} - c_6^+ &\geq \frac{(5-\frac{\sqrt{2}}{2}+\frac{3}{2})(5-\frac{\sqrt{2}}{2}-1)}{(5-\frac{\sqrt{2}}{2}+\frac{5}{4})(5-\frac{\sqrt{2}}{2}-\frac{3}{4})}\frac{1}{15}E^2 - \frac{5}{3}\sqrt{2}E^2\frac{1}{5^3} \\
&\geq \frac{(5-\frac{3}{4}+\frac{3}{2})(5-\frac{3}{4}-1)}{(5-\frac{3}{4}+\frac{5}{4})(5-\frac{3}{4}-\frac{3}{4})}\frac{1}{15}E^2 - \frac{1}{75}\sqrt{2}E^2 \\
&= \frac{23}{22}\frac{13}{14}\frac{1}{15}E^2 - \frac{1}{75}\sqrt{2}E^2 \\
&\geq \frac{1}{22}E^2.
\end{aligned} \tag{5.257}
$$

We conclude $K \geq 6$. By lemma 5.5.16 and (5.254):

$$
\begin{aligned}
{c_{K+1}^-}^{-1} - c_{K+1}^+ &\geq \frac{1}{22}E^2\prod_{k=6}^{\infty}\frac{(k-\frac{\sqrt{2}}{2}+\frac{3}{2})(k-\frac{\sqrt{2}}{2}-1)}{(k-\frac{\sqrt{2}}{2}+\frac{5}{4})(k-\frac{\sqrt{2}}{2}-\frac{3}{4})} - \frac{5}{3}\sqrt{2}E^2\sum_{k=6}^{\infty}\frac{1}{k^3} \\
&\geq \frac{1}{22}E^2\frac{6-\frac{\sqrt{2}}{2}-1}{6-\frac{\sqrt{2}}{2}-\frac{3}{4}}\frac{6-\frac{\sqrt{2}}{2}}{6-\frac{\sqrt{2}}{2}+\frac{1}{4}}\prod_{k=6}^{\infty}\frac{(k-\frac{\sqrt{2}}{2}+\frac{3}{2})(k-\frac{\sqrt{2}}{2}+1)}{(k-\frac{\sqrt{2}}{2}+\frac{5}{4})^2} \\
&\quad - \frac{5}{3}\sqrt{2}E^2\sum_{k=6}^{\infty}\frac{1}{k^3}.
\end{aligned} \tag{5.258}
$$

We calculate:

$$\frac{1}{22}\frac{6-\frac{\sqrt{2}}{2}-1}{6-\frac{\sqrt{2}}{2}-\frac{3}{4}}\frac{6-\frac{\sqrt{2}}{2}}{6-\frac{\sqrt{2}}{2}+\frac{1}{4}} \geq \frac{1}{22}\frac{6-\frac{3}{4}-1}{6-\frac{3}{4}-\frac{3}{4}}\frac{6-\frac{3}{4}}{6-\frac{3}{4}+\frac{1}{4}} = \frac{1}{22}\frac{17}{18}\frac{21}{22}. \tag{5.259}$$

Further:

$$\begin{aligned}\prod_{k=6}^{K}\frac{(k-\frac{\sqrt{2}}{2}+\frac{3}{2})(k-\frac{\sqrt{2}}{2}+1)}{(k-\frac{\sqrt{2}}{2}+\frac{5}{4})^2} &= \prod_{k=6}^{K}\frac{(k-\frac{\sqrt{2}}{2}+\frac{5}{4})^2-\frac{1}{16}}{(k-\frac{\sqrt{2}}{2}+\frac{5}{4})^2}\\ &\geq \prod_{k=6}^{K}\left(1-\frac{1}{16}\frac{1}{(k+\frac{1}{2})^2}\right)\\ &\geq 1-\frac{1}{16}\sum_{k=6}^{\infty}\frac{1}{(k+\frac{1}{2})^2}\\ &\geq 1-\frac{1}{16}\sum_{k=6}^{\infty}\int_k^{k+1}\frac{1}{x^2}dx\\ &= 1-\frac{1}{16}\int_6^{\infty}\frac{1}{x^2}dx\\ &= \frac{95}{96}. \end{aligned}\tag{5.260}$$

Finally:

$$-\sum_{k=6}^{K}\frac{1}{k^3} \geq -\sum_{k=6}^{\infty}\int_{k-\frac{1}{2}}^{k+\frac{1}{2}}\frac{1}{x^3}dx \geq -\int_{\frac{11}{2}}^{\infty}\frac{1}{x^3}dx \geq -\frac{2}{121}. \tag{5.261}$$

We summarize:

$$\begin{aligned}{c_{K+1}^{-}}^{-1}-c_{K+1}^{+} &\geq \frac{1}{22}\frac{17}{18}\frac{21}{22}\frac{95}{96}E^2-\frac{5}{3}\frac{2}{121}\sqrt{2}E^2\\ &= \frac{5}{3}\frac{1}{11^2}\left(\frac{2261}{3}\frac{1}{2^8}-2\sqrt{2}\right)E^2\\ &\geq \frac{5}{363}\left(\frac{2261}{768}-\frac{142}{50}\right)E^2\\ &\geq \frac{5}{363}\left(\frac{2261}{775}-\frac{71}{25}\right)E^2\\ &= \frac{5}{363}\frac{2261-2201}{775}E^2\\ &= \frac{300}{363}\frac{1}{775}E^2\\ &\geq \frac{1}{1000}E^2. \end{aligned}\tag{5.262}$$

That contradicts (5.252) and concludes the proof. □

Lemma 5.5.18 *Let* $p = 3$. *Then* iL *admits no eigenvalues with even eigenfunctions within the spectral gap* $(-1, 1)$, *apart from* 0.

Proof. Follows from lemma 3.4.6, lemma 5.1.7, corollary 5.5.8 and lemma 5.5.17. $\square$

5.6. Cubic Case, Odd Eigenvalues

We show that, for $p = 3$, iL admits no odd eigenvalues apart from 0 within the spectral gap $(-1, 1)$. As before, we make use of lemma 5.1.16. $E \in (0, 1)$ is an eigenvalue of iL, if and only if

$$\mathcal{C}(E, 3) = \frac{1}{\mathcal{C}(-E, 3)}. \tag{5.263}$$

We make use of the same strategy as for even eigenvalues in chapter 5.5. Using brute force, we establish a lower bound on $c_5(-E, 3) - c_5(E, 3)^{-1}$ and then use natural induction to establish an estimate for $k \geq 5$. However, in contrast to the even case, no complicated technical results are needed to complete the induction step.

As before, the computations can be followed without external help, a computer algebra system might nevertheless be helpful.

Recall the usual definitions.

Definition 5.6.1 (Definition 4.2.1) *Let $E \in [-1, 1]$, $k \geq 0$ (and $p = 5$). Consider*

$$\begin{aligned} A_k(E) &= 4\left(k + \frac{\sqrt{1+E}}{2}\right)^2 + 6\left(k + \frac{\sqrt{1+E}}{2}\right) - 2 \\ &= 4k^2 + 6k + 4\sqrt{1+E}k + E + 3\sqrt{1+E} - 1, \end{aligned} \tag{5.264}$$

$$B_k(E) = 4(k+1)^2 + 4\sqrt{1+E}(k+1). \tag{5.265}$$

Definition 5.6.2 (Lemma 4.2.6) *Let $E \in [-1, 1]$ (and $p = 3$). We define $(c_k)_{k\in\mathbb{N}} = (c_k(E))_{k\in\mathbb{N}} \subseteq \mathbb{R} \cup \{\infty\}$ by $c_0 = \infty$ and*

$$c_{k+1} = \left(1 + \frac{E}{2B_k}\right) \frac{A_k c_k - \frac{3}{2}}{A_k - \frac{3}{2}c_k}. \tag{5.266}$$

In the even case, we had $c_k(0) = 1$ for every $k \geq 1$. Now, in the odd, case it holds $c_k(0) = -1$ instead. To improve readability, we introduce:

Definition 5.6.3 *Let $E \in [-1, 1]$. We define $d_k(E) := -c_k(E) \in \mathbb{R} \cup \{\infty\}$.*

Lemma 5.6.4 *Let $E \in [-1, 1]$. $(d_k(E))_{k\in\mathbb{N}} = (d_k)_{k\in\mathbb{N}}$ satisfies the recursion $d_0 = \infty$ and:*

$$d_{k+1} = \left(1 + \frac{8E}{(p-1)^2 B_k}\right) \frac{A_k d_k + \frac{p+1}{p-1}}{A_k + \frac{p+1}{p-1} d_k}. \tag{5.267}$$

Proof. By definition $d_0 = -c_0 = \infty$ and:

$$\begin{aligned} d_{k+1} =& -c_{k+1} \\ =& -\left(1 + \frac{8E}{(p-1)^2 B_k}\right) \frac{A_k c_k - \frac{p+1}{p-1}}{A_k - \frac{p+1}{p-1} c_k} \\ =& \left(1 + \frac{8E}{(p-1)^2 B_k}\right) \frac{A_k d_k + \frac{p+1}{p-1}}{A_k + \frac{p+1}{p-1} d_k}. \end{aligned} \tag{5.268}$$

That concludes the proof. □

Notation 5.6.5 *As long as $E \in (0,1)$ is clear from context, we abbreviate $d_k^+ = d_k(E)$ and $d_k^- = d_k(-E)$, as well as $A_k^+ = A_k(E)$, $A_k^- = A_k(-E)$.*

Lemma 5.6.6 *Let $p = 3$ and $E \in [0,1]$. Then:*

$$d_2^{+-1} - d_2^- \geq \frac{29}{72} E^2, \tag{5.269}$$

$$0 \leq 1 - d_2^- \leq \frac{10}{7} E. \tag{5.270}$$

Proof. For now, consider $E \in [-1,1]$. We calculate:

$$\begin{aligned} d_1(E) =& \left(1 + \frac{2E}{B_0(E)}\right) \frac{A_0(E)}{2} \\ =& \left(1 + \frac{E}{2\left(1+\sqrt{1+E}\right)}\right) \left(1 + \frac{E}{2} + \frac{3\left(\sqrt{1+E}-1\right)}{2}\right). \end{aligned} \tag{5.271}$$

It follows:

$$\begin{aligned} d_1(E) - 1 =& \left(1 + \frac{E}{2\left(1+\sqrt{1+E}\right)}\right) \left(1 + \frac{E}{2} + \frac{3\left(\sqrt{1+E}-1\right)}{2}\right) - 1 \\ =& \left(1 + \frac{\sqrt{1+E}-1}{2}\right) \left(1 + \frac{(\sqrt{1+E}-1)(\sqrt{1+E}+1)}{2} + \frac{3\left(\sqrt{1+E}-1\right)}{2}\right) - 1 \\ =& \frac{\sqrt{1+E}-1}{2} \left(1 + \frac{(\sqrt{1+E}-1)(\sqrt{1+E}+1)}{2} + \frac{3\left(\sqrt{1+E}-1\right)}{2}\right) \\ &+ \frac{(\sqrt{1+E}-1)(\sqrt{1+E}+1)}{2} + \frac{3\left(\sqrt{1+E}-1\right)}{2} \\ =& \frac{\sqrt{1+E}-1}{2} \left(5 + \sqrt{1+E} + \frac{(\sqrt{1+E}-1)(\sqrt{1+E}+1)}{2} + \frac{3\left(\sqrt{1+E}-1\right)}{2}\right) \\ =& \frac{\sqrt{1+E}-1}{4} \left(7 + 5\sqrt{1+E} + E\right). \end{aligned} \tag{5.272}$$

By definition:

$$d_2(E) = \left(1 + \frac{E}{8+4\sqrt{1+E}}\right) \frac{\left(8+7\sqrt{1+E}+1+E\right) d_1(E) + 2}{8+7\sqrt{1+E}+1+E+2d_1(E)}. \tag{5.273}$$

For the sake of readability, let:

$$K(E) := 1 + \frac{E}{8+4\sqrt{1+E}}. \tag{5.274}$$

It follows:

$$\begin{aligned}
&\frac{d_2(-E)}{K(-E)} \\
=\ &\frac{\left(8+7\sqrt{1-E}+1-E\right) d_1(-E) + 2}{8+7\sqrt{1-E}+1-E+2d_1(-E)} \\
=\ &\frac{10+7\sqrt{1-E}+1-E+\left(8+7\sqrt{1-E}+1-E\right)(d_1(-E)-1)}{10+7\sqrt{1-E}+1-E+2(d_1(-E)-1)} \\
=\ &\frac{10+7\sqrt{1-E}+1-E-\left(8+7\sqrt{1-E}+1-E\right)\frac{1-\sqrt{1-E}}{4}\left(6+5\sqrt{1-E}+1-E\right)}{10+7\sqrt{1-E}+1-E-2\frac{1-\sqrt{1-E}}{4}\left(6+5\sqrt{1-E}+1-E\right)}.
\end{aligned} \tag{5.275}$$

Further:

$$\begin{aligned}
\frac{d_2(-E)}{K(-E)} &= 1 - \frac{\left(8+7\sqrt{1-E}+1-E-2\right)\frac{1-\sqrt{1-E}}{4}\left(6+5\sqrt{1-E}+1-E\right)}{10+7\sqrt{1-E}+1-E-2\frac{1-\sqrt{1-E}}{4}\left(6+5\sqrt{1-E}+1-E\right)} \\
&= 1 - \frac{1}{4}\frac{(\sqrt{1-E}+1)(\sqrt{1-E}+6)(1-\sqrt{1-E})(\sqrt{1-E}+2)(\sqrt{1-E}+3)}{(\sqrt{1-E}+2)(\sqrt{1-E}+5)-\frac{1-\sqrt{1-E}}{2}(\sqrt{1-E}+2)(\sqrt{1-E}+3)} \\
&= 1 - \frac{E}{4}\frac{(\sqrt{1-E}+6)(\sqrt{1-E}+3)}{\sqrt{1-E}+5-\frac{1-\sqrt{1-E}}{2}(\sqrt{1-E}+3)} \\
&= 1 - \frac{E}{2}\frac{1-E+9\sqrt{1-E}+18}{2\sqrt{1-E}+10-(3-2\sqrt{1-E}-(1-E))} \\
&= 1 - \frac{E}{2}\frac{1-E+9\sqrt{1-E}+18}{7+4\sqrt{1-E}+1-E} \\
&= 1 - E\frac{18+9\sqrt{1-E}+1-E}{14+8\sqrt{1-E}+2(1-E)} \\
&=: 1 - E\frac{m_2(E)}{n_2(E)}.
\end{aligned} \tag{5.276}$$

We note for $E \in [0,1]$:

$$\frac{m_2(E)}{n_2(E)} = \frac{1}{2}\left(1 + \frac{11+5\sqrt{1-E}}{7+4\sqrt{1-E}+1-E}\right) \le \frac{1}{2}\left(1+\frac{11+5\sqrt{1-E}}{7+4\sqrt{1-E}}\right)$$
$$\le \frac{1}{2}\left(1+\frac{11}{7}\right) = \frac{9}{7}. \tag{5.277}$$

By the same token, for $E \in [0,1]$:

$$\frac{m_2(E)}{n_2(E)} \ge \frac{1}{2}\left(1+\frac{11+5\sqrt{1-E}}{7+5\sqrt{1-E}}\right) \ge \frac{1}{2}\left(1+\frac{16}{12}\right) = \frac{7}{6} \ge 1. \tag{5.278}$$

We conclude:

$$\begin{aligned} d_2^- - 1 &= \left(1 - \frac{E}{8+4\sqrt{1-E}}\right)\left(1 - E\frac{m_2(E)}{n_2(E)}\right) - 1 \\ &\ge 1 - \frac{E}{8+4\sqrt{1-E}} - E\frac{m_2(E)}{n_2(E)} - 1 \\ &\ge -\frac{10}{7}E. \end{aligned} \tag{5.279}$$

Next, we estimate $d_2^{+^{-1}}$. We invert (5.275) and proceed analogously to (5.276):

$$\begin{aligned} \frac{K(E)}{d_2^+} &= \frac{10+7\sqrt{1+E}+1+E+2\frac{\sqrt{1+E}-1}{4}\left(6+5\sqrt{1+E}+1+E\right)}{10+7\sqrt{1+E}+1+E+\left(8+7\sqrt{1+E}+1+E\right)\frac{\sqrt{1+E}-1}{4}\left(6+5\sqrt{1+E}+1+E\right)} \\ &= \frac{\sqrt{1+E}+2}{\sqrt{1+E}+2}\,\frac{\sqrt{1+E}+5+2\frac{\sqrt{1+E}-1}{4}(\sqrt{1+E}+3)}{\sqrt{1+E}+5+\left(8+7\sqrt{1+E}+1+E\right)\frac{\sqrt{1+E}-1}{4}(\sqrt{1+E}+3)} \\ &= 1 - \frac{\left(8+7\sqrt{1+E}+1+E-2\right)\frac{\sqrt{1+E}-1}{4}(\sqrt{1+E}+3)}{\sqrt{1+E}+5+\left(8+7\sqrt{1+E}+1+E\right)\frac{\sqrt{1+E}-1}{4}(\sqrt{1+E}+3)} \\ &= 1 - \frac{1}{4}\,\frac{(\sqrt{1+E}+6)(\sqrt{1+E}+1)(\sqrt{1+E}-1)(\sqrt{1+E}+3)}{\sqrt{1+E}+5+\frac{1}{4}\left(8+7\sqrt{1+E}+1+E\right)(1+E+2\sqrt{1+E}-3)} \\ &= 1 - E\frac{(\sqrt{1+E}+6)(\sqrt{1+E}+3)}{4\sqrt{1+E}+20-24-5\sqrt{1+E}+19(1+E)+9(1+E)^{\frac{3}{2}}+(1+E)^2} \\ &= 1 - E\frac{1+E+9\sqrt{1+E}+18}{(1+E)^2+9(1+E)^{\frac{3}{2}}+19(1+E)-\sqrt{1+E}-4} \\ &=: 1 - E\frac{m_1(E)}{n_1(E)}. \end{aligned} \tag{5.280}$$

Clearly, $1 - \frac{m_1(E)}{n_1(E)}E \ge 0$ for every $E \in [0,1]$. We now compare d_2^- and $d_2^{+^{-1}}$. By definition:

$$K(-E) = 1 - \frac{E}{8+4\sqrt{1-E}} \le 1 - \frac{E}{8+4\sqrt{1+E}}\,\frac{1}{1+\frac{E}{8+4\sqrt{1+E}}} = \frac{1}{K(E)}. \tag{5.281}$$

We calculate:

$$
\begin{aligned}
&d_2^{+-1} - d_2^- \\
&= \frac{1}{K(E)}\left(1 - E\frac{m_1(E)}{n_1(E)}\right) - K(-E)\left(1 - E\frac{m_2(E)}{n_2(E)}\right) \\
&\geq K(-E)E\left(\frac{m_2(E)}{n_2(E)} - \frac{m_1(E)}{n_1(E)}\right) \\
&= K(-E)E\left(\frac{m_2(E) - m_1(E)}{n_1(E)} + \frac{n_1(E) - n_2(E)}{n_1(E)}\frac{m_2(E)}{n_2(E)}\right) \\
&= K(-E)E\left(-\frac{2E + 9\left(\sqrt{1+E} - \sqrt{1-E}\right)}{n_1(E)}\right. \\
&\quad \left. + \underbrace{\frac{-18 - 8\sqrt{1-E} - \sqrt{1+E} + 4E + 17(1+E) + 9(1+E)^{\frac{3}{2}} + (1+E)^2}{n_1(E)}}_{\geq 0}\frac{m_2(E)}{n_2(E)}\right).
\end{aligned} \tag{5.282}
$$

Using (5.278), we further estimate:

$$
\begin{aligned}
&d_2^{+-1} - d_2^- \\
&\geq K(-E)E\left(-\frac{2E + 9\left(\sqrt{1+E} - \sqrt{1-E}\right)}{n_1(E)}\right. \\
&\quad \left. + \frac{-1 - 8\sqrt{1-E} - \sqrt{1+E} + 21E + 9(1+E)^{\frac{3}{2}} + (1+E)^2}{n_1(E)}\right) \\
&= K(-E)E\frac{19E - \sqrt{1+E} + \sqrt{1-E} - 9(\sqrt{1+E} - 1) + 9((1+E)^{\frac{3}{2}} - 1) + (1+E)^2 - 1}{n_1(E)} \\
&\geq K(-E)E\frac{19E - \frac{2E}{\sqrt{1+E}+\sqrt{1-E}} - 9\frac{E}{\sqrt{1+E}+1} + 9\frac{3}{2}E + 2E}{n_1(E)} \\
&= K(-E)E^2\frac{69 - \frac{4}{\sqrt{1+E}+\sqrt{1-E}} - \frac{18}{\sqrt{1+E}+1}}{2n_1(E)} \\
&\geq K(-E)E^2\frac{69 - \frac{4}{\sqrt{1+1}+\sqrt{1-1}} - \frac{18}{\sqrt{1+1}+1}}{2n_1(1)} \\
&= K(-E)E^2\frac{69 - 2\sqrt{2} - 18(\sqrt{2} - 1)}{2(4 + 18\sqrt{2} + 38 - \sqrt{2} - 4)} \\
&= \left(1 - \frac{E}{8 + 4\sqrt{1-E}}\right)E^2\frac{87 - 20\sqrt{2}}{2(38 + 17\sqrt{2})} \\
&\geq \frac{7}{8}E^2\frac{87 - 29}{2(38 + 25)} = \frac{7}{8}\frac{29}{63}E^2 = \frac{29}{72}E^2.
\end{aligned} \tag{5.283}
$$

That concludes the proof. □

Lemma 5.6.7 *Let $p = 3$, $E \in [0, 1]$ and $k \geq 1$. Then:*

$$d_k^+ \geq 1. \tag{5.284}$$

Proof. By definition:

$$d_1^+ = \left(1 + \frac{2E}{B_0^+}\right) \frac{A_0^+}{2} \geq 1. \tag{5.285}$$

Further:

$$d_{k+1}^+ = \left(1 + \frac{2E}{B_k^+}\right) \frac{A_k d_k^+ + 2}{A_k + 2d_k^+}. \tag{5.286}$$

The lemma follows inductively. □

Lemma 5.6.8 *Let $p = 3$, $E \in [0, 1]$ and $k \geq 2$. Then:*

$$0 \leq 1 - d_k^- \leq \frac{67}{42} E. \tag{5.287}$$

Proof. As we only consider d_k^-, notate $d_k = d_k^-$, $A_k = A_k^-$ and $B_k = B_k^-$ for short.

We proceed by induction. The base case $k = 2$ follows from lemma 5.6.6 as $0 \leq 1 - d_2 \leq \frac{10}{7} E$.

For the induction step, assume $0 \leq 1 - d_k \leq \frac{67}{42} E$. It follows $A_k + 2d_k \geq A_k + 2(1 - \frac{67}{42} E) > A_k - 2$. We estimate:

$$\begin{aligned}
1 - d_{k+1} &= 1 - \left(1 - \frac{2E}{B_k}\right) \frac{A_k d_k + 2}{A_k + 2d_k} \\
&= \frac{2E}{B_k} + \left(1 - \frac{2E}{B_k}\right) \left(1 - \frac{A_k d_k + 2}{A_k + 2d_k}\right) \\
&= \frac{2E}{B_k} + \left(1 - \frac{2E}{B_k}\right) \frac{A_k - A_k d_k - 2 + 2d_k}{A_k - 2d_k} \\
&= \frac{2E}{B_k} + \left(1 - \frac{2E}{B_k}\right) \frac{A_k - 2}{A_k + 2d_k} (1 - d_k) \\
&\leq \frac{2E}{B_k} + \left(1 - \frac{2E}{B_k}\right) \frac{A_k + 2d_k}{A_k + 2d_k} (1 - d_k) \\
&\leq \frac{2E}{B_k} + 1 - d_k.
\end{aligned} \tag{5.288}$$

It follows:

$$\begin{aligned}
1-d_{k+1} &\le 1+d_2+\sum_{l=2}^{k}\frac{2}{B_l}\\
&\le \frac{10}{7}E+\sum_{l=2}^{k}\frac{2E}{4(l+1)^2+4(l+1)}\\
&= \frac{10}{7}E+\frac{E}{2}\sum_{l=2}^{k}\left(\frac{1}{l+1}-\frac{1}{l+2}\right)\\
&\le \frac{10}{7}E+\frac{E}{6}\\
&= \frac{67}{42}E. \qquad (5.289)
\end{aligned}$$

Further, $d_k \le 1$ implies $A_k d_k + 2 \le A_k + 2d_k$. Together with $A_k + 2d_k > 0$, it follows:

$$\begin{aligned}
d_{k+1} &= \left(1-\frac{2E}{B_k}\right)\frac{A_k d_k+2}{A_k+2d_k}\\
&\le 1-\frac{2E}{B_k}\\
&\le 1. \qquad (5.290)
\end{aligned}$$

That concludes the proof. □

Lemma 5.6.9 *Let $p=3$, $E\in[0,1]$ and $k\ge 2$. Assume ${d_k^+}^{-1}-d_k^-\ge 0$. Assume further:*

$$\frac{(k+2)(k-\frac{1}{2})}{k(k+\frac{3}{2})}({d_k^+}^{-1}-d_k^-)-\frac{67}{42}\left(\frac{E^2}{k^2+\frac{3}{2}k}-\frac{E^2}{k^2+3k+\frac{3}{2}}\right)\ge 0. \qquad (5.291)$$

Then:

$$\begin{aligned}
&{d_{k+1}^+}^{-1}-d_{k+1}^-\\
\ge\ &\frac{k(k+2)}{(k+1)^2}\left(\frac{(k+2)(k-\frac{1}{2})}{k(k+\frac{3}{2})}({d_k^+}^{-1}-d_k^-)-\frac{67}{42}\left(\frac{E^2}{k^2+\frac{3}{2}k}-\frac{E^2}{k^2+3k+\frac{3}{2}}\right)\right). \qquad (5.292)
\end{aligned}$$

Proof. Note:

$$\left(1-\frac{2E}{B_k^-}\right)\left(1+\frac{2E}{B_k^+}\right)\le 1-\frac{2E}{B_k^-}+\frac{2E}{B_k^+}\le 1. \qquad (5.293)$$

That implies:

$$1-\frac{2E}{B_k^-}\le\left(1+\frac{2E}{B_k^+}\right)^{-1}. \qquad (5.294)$$

By lemma 5.6.7, we have $d_k^+ \geq 1 > 0$. It follows:

$$
\begin{aligned}
{d_{k+1}^+}^{-1} - d_{k+1}^- &= \left(1 + \frac{2E}{B_k^+}\right)^{-1} \frac{A_k^+ + 2d_k^+}{A_k^+ d_k^+ + 2} - \left(1 - \frac{2E}{B_k^-}\right) \frac{A_k^- d_k^- + 2}{A_k^- + 2d_k^-} \\
&\geq \left(1 - \frac{2E}{B_k^-}\right) \left(\frac{A_k^+ + 2d_k^+}{A_k^+ d_k^+ + 2} - \frac{A_k^- d_k^- + 2}{A_k^- + 2d_k^-}\right) \\
&= \left(1 - \frac{2E}{B_k^-}\right) \left(\frac{A_k^+ {d_k^+}^{-1} + 2}{A_k^+ + 2{d_k^+}^{-1}} - \frac{A_k^- d_k^- + 2}{A_k^- + 2d_k^-}\right). \qquad (5.295)
\end{aligned}
$$

We calculate:

$$
\begin{aligned}
&\frac{A_k^+ {d_k^+}^{-1} + 2}{A_k^+ + 2{d_k^+}^{-1}} - \frac{A_k^- d_k^- + 2}{A_k^- + 2d_k^-} \\
&= \frac{(A_k^- + 2d_k^-)(A_k^+ {d_k^+}^{-1} + 2) - (A_k^- d_k^- + 2)(A_k^+ + 2{d_k^+}^{-1})}{(A_k^- + 2d_k^-)(A_k^+ + 2{d_k^+}^{-1})} \\
&= \frac{A_k^- A_k^+ ({d_k^+}^{-1} - d_k^-) + 2(A_k^+ - A_k^-) d_k^- {d_k^+}^{-1} + 2(A_k^- - A_k^+) + 4(d_k^- - {d_k^+}^{-1})}{(A_k^- + 2d_k^-)(A_k^+ + 2{d_k^+}^{-1})} \\
&= \frac{A_k^- A_k^+ - 4}{(A_k^- + 2d_k^-)(A_k^+ + 2{d_k^+}^{-1})} ({d_k^+}^{-1} - d_k^-) + \frac{2(A_k^+ - A_k^-)}{(A_k^- + 2d_k^-)(A_k^+ + 2{d_k^+}^{-1})} (d_k^- {d_k^+}^{-1} - 1).
\end{aligned}
\qquad (5.296)
$$

We estimate:

$$
\begin{aligned}
\frac{A_k^- A_k^+ - 4}{(A_k^- + 2)(A_k^+ + 2)} &= 1 - \frac{2}{A_k^- + 2} - \frac{2}{A_k^+ + 2} \\
&\geq 1 - \frac{4}{A_k(-1) + 2} \\
&\geq 1 - \frac{4}{4k^2 + 6k} \\
&= \frac{k^2 + \frac{3}{2}k - 1}{k^2 + \frac{3}{2}k} \\
&= \frac{(k+2)(k - \frac{1}{2})}{k(k + \frac{3}{2})}. \qquad (5.297)
\end{aligned}
$$

Similarly:

$$\begin{aligned}\frac{2(A_k^+ - A_k^-)}{(A_k^- + 2)(A_k^+ + 2)} &\leq \frac{2((4k+3)\sqrt{2}+2)E}{(A_k(-1)+2)(A_k(1)+2)} \\ &= \frac{2((4k+3)\sqrt{2}+2)E}{(4k^2+6k)(4k^2+6k+(4k+3)\sqrt{2}+2)} \\ &= \frac{2E}{4k^2+6k} - \frac{2E}{4k^2+6k+(4k+3)\sqrt{2}+2} \\ &= \frac{E}{2}\left(\frac{1}{k^2+\frac{3}{2}k} - \frac{1}{k^2+(\frac{3}{2}+\sqrt{2})k+\frac{3}{4}\sqrt{2}+\frac{1}{2}}\right). \end{aligned} \tag{5.298}$$

Using $k \geq 2$, we conclude $k^2 + (\frac{3}{2}+\sqrt{2})k + \frac{3}{4}\sqrt{2} + \frac{1}{2} \leq k^2 + 3k + \frac{3}{2}$. Consequently:

$$\frac{2(A_k^+ - A_k^-)}{(A_k^- + 2)(A_k^+ + 2)} \leq \frac{E}{2}\left(\frac{1}{k^2+\frac{3}{2}k} - \frac{1}{k^2+3k+\frac{3}{2}}\right). \tag{5.299}$$

By lemma 5.6.7, lemma 5.6.8 and $d_k^{+\,-1} - d_k^- \geq 0$, it follows $d_k^{+\,-1} \geq d_k^- \geq 1 - \frac{67}{42}E$. Consequently:

$$d_k^- d_k^{+\,-1} - 1 \geq -\frac{67}{21}E. \tag{5.300}$$

Finally:

$$1 - \frac{2E}{B_k^-} \geq \frac{B_k(-1)-2}{B_k(-1)} \geq \frac{4(k+1)^2-4}{4(k+1)^2} \geq \frac{k(k+2)}{(k+1)^2}. \tag{5.301}$$

The lemma follows from (5.295), (5.296), (5.297), (5.300) and (5.301). ☐

Lemma 5.6.10 *Let $p = 3$ and $E \in [0,1]$. Then:*

$$d_5^{+\,-1} - d_5^- \geq \frac{1}{9}E^2. \tag{5.302}$$

Proof. $d_2^{+\,-1} - d_2^- \geq \frac{29}{72}E^2$ holds by lemma 5.6.6. By lemma 5.6.9:

$$\begin{aligned} d_3^{+\,-1} - d_3^- &\geq \frac{8}{9}\left(\frac{4\frac{3}{2}}{2\frac{7}{2}}\frac{29}{72}E^2 - \frac{67}{42}\left(\frac{1}{7} - \frac{2}{23}\right)E^2\right) \\ &= \frac{8}{9}\left(\frac{1}{7}\frac{29}{12} - \frac{67}{42}\frac{9}{161}\right)E^2 \\ &= \frac{8}{9}\frac{1}{7}\left(\frac{29}{12} - \frac{201}{322}\right)E^2 \\ &\geq \frac{8}{9}\frac{1}{7}\left(\frac{29}{12} - \frac{2}{3}\right)E^2 \\ &= \frac{2}{9}E^2. \end{aligned} \tag{5.303}$$

Further:

$$\begin{aligned}
d_4^{+\,-1} - d_4^- &\geq \frac{15}{16}\left(\frac{5^{\frac{5}{2}}}{3^{\frac{9}{2}}}\frac{2}{9}E^2 - \frac{67}{42}\left(\frac{2}{27}-\frac{2}{45}\right)E^2\right) \\
&\geq \frac{15}{16}\left(\frac{25}{27}\frac{2}{9} - \frac{67}{21}\frac{1}{9}\frac{2}{15}\right)E^2 \\
&\geq \frac{15}{16}\frac{2}{81}\left(\frac{25}{3} - \frac{67}{35}\right)E^2 \\
&\geq \frac{5}{8}\frac{1}{27}\frac{674}{105}E^2 \\
&\geq \frac{1}{8}\frac{1}{27}\frac{672}{21}E^2 \\
&= \frac{4}{27}E^2. \qquad (5.304)
\end{aligned}$$

Finally:

$$\begin{aligned}
d_5^{+\,-1} - d_5^- &\geq \frac{24}{25}\left(\frac{6^{\frac{7}{2}}}{4^{\frac{11}{2}}}\frac{4}{27}E^2 - \frac{67}{42}\left(\frac{1}{22}-\frac{1}{30}\right)E^2\right) \\
&= \frac{24}{25}\left(\frac{21}{22}\frac{4}{27} - \frac{67}{42}\frac{8}{660}\right)E^2 \\
&\geq \frac{8}{25}\left(\frac{14}{33} - \frac{67}{7}\frac{1}{165}\right)E^2 \\
&= \frac{8}{25}\frac{1}{33}\left(14 - \frac{67}{35}\right)E^2 \\
&\geq \frac{1}{3}\frac{12}{33}E^2 \\
&\geq \frac{1}{9}E^2. \qquad (5.305)
\end{aligned}$$

That concludes the proof. □

Lemma 5.6.11 *Let $p = 3$, $E \in [0,1]$ and $K \geq 5$. Assume $d_k^{+\,-1} - d_k^- \geq 0$ for every $5 \leq k \leq K$. Assume further:*

$$\frac{E^2}{9}\prod_{k=5}^{\infty}\frac{k(k+2)}{(k+1)^2}\prod_{k=5}^{\infty}\frac{(k+2)(k-\frac{1}{2})}{k(k+\frac{3}{2})} - \frac{67}{42}E^2\sum_{k=5}^{\infty}\left(\frac{1}{k^2+\frac{3}{2}k} - \frac{1}{k^2+3k+\frac{3}{2}}\right) \geq 0. \qquad (5.306)$$

Then:

$$\begin{aligned}
&d_{K+1}^{+\,-1} - d_{K+1}^- \\
&\geq \frac{E^2}{9}\prod_{k=5}^{K}\frac{k(k+2)}{(k+1)^2}\prod_{k=5}^{K}\frac{(k+2)(k-\frac{1}{2})}{k(k+\frac{3}{2})} - \frac{67}{42}E^2\sum_{k=5}^{K}\left(\frac{1}{k^2+\frac{3}{2}k} - \frac{1}{k^2+3k+\frac{3}{2}}\right). \qquad (5.307)
\end{aligned}$$

Proof. We show the lemma by induction. By lemma 5.6.9 and lemma 5.6.10 the base case $K = 5$ holds. For the induction step, assume:

$$\begin{aligned} &{d_K^+}^{-1} - d_K^- \\ &\geq \frac{E^2}{9} \prod_{k=5}^{K-1} \frac{k(k+2)}{(k+1)^2} \prod_{k=5}^{K-1} \frac{(k+2)(k-\frac{1}{2})}{k(k+\frac{3}{2})} - \frac{67}{42} E^2 \sum_{k=5}^{K-1} \left(\frac{1}{k^2 + \frac{3}{2}k} - \frac{1}{k^2 + 3k + \frac{3}{2}} \right). \end{aligned} \tag{5.308}$$

By lemma 5.6.9:

$$\begin{aligned} &{d_{K+1}^+}^{-1} - d_{K+1}^- \\ &\geq \frac{K(K+2)}{(K+1)^2} \frac{(K+2)(K-\frac{1}{2})}{K(K+\frac{3}{2})} ({d_K^+}^{-1} - d_K^-) - \frac{67}{42} \left(\frac{E^2}{K^2 + \frac{3}{2}K} - \frac{E^2}{K^2 + 3K + \frac{3}{2}} \right) \\ &\geq \frac{E^2}{9} \prod_{k=5}^{K} \frac{k(k+2)}{(k+1)^2} \prod_{k=5}^{K} \frac{(k+2)(k-\frac{1}{2})}{k(k+\frac{3}{2})} - \underbrace{\frac{K(K+2)}{(K+1)^2}}_{\leq 1} \frac{67}{42} \left(\frac{E^2}{K^2 + \frac{3}{2}K} - \frac{E^2}{K^2 + 3K + \frac{3}{2}} \right) \\ &\quad - \underbrace{\frac{K(K+2)}{(K+1)^2} \frac{(K+2)(K-\frac{1}{2})}{K(K+\frac{3}{2})}}_{\leq 1} \frac{67}{42} E^2 \sum_{k=5}^{K-1} \left(\frac{1}{k^2 + \frac{3}{2}k} - \frac{1}{k^2 + 3k + \frac{3}{2}} \right) \\ &\geq \frac{E^2}{9} \prod_{k=5}^{K} \frac{k(k+2)}{(k+1)^2} \prod_{k=5}^{K} \frac{(k+2)(k-\frac{1}{2})}{k(k+\frac{3}{2})} - \frac{67}{42} E^2 \sum_{k=5}^{K} \left(\frac{1}{k^2 + \frac{3}{2}k} - \frac{1}{k^2 + 3k + \frac{3}{2}} \right). \end{aligned} \tag{5.309}$$

That concludes the proof. □

Lemma 5.6.12 *Let $p = 3$, $E \in [0, 1]$ and $k \geq 5$. Then:*

$${d_k^+}^{-1} - d_k^- \geq \frac{E^2}{36}. \tag{5.310}$$

Proof. We proceed inductively. The base case $k = 5$ holds by lemma 5.6.10. For the induction step, assume (5.310) for every $5 \leq k \leq K$. By lemma 5.6.11, it suffices to show:

$$\frac{E^2}{9} \prod_{k=5}^{\infty} \frac{k(k+2)}{(k+1)^2} \prod_{k=5}^{\infty} \frac{(k+2)(k-\frac{1}{2})}{k(k+\frac{3}{2})} - \frac{67}{42} E^2 \sum_{k=5}^{\infty} \left(\frac{1}{k^2 + \frac{3}{2}k} - \frac{1}{k^2 + 3k + \frac{3}{2}} \right) \geq \frac{E^2}{36}. \tag{5.311}$$

We calculate:

$$\begin{aligned} \prod_{k=5}^{\infty} \frac{k(k+2)}{(k+1)^2} \prod_{k=5}^{\infty} \frac{(k+2)(k-\frac{1}{2})}{k(k+\frac{3}{2})} &= \prod_{k=5}^{\infty} \frac{(k+2)^2}{(k+1)^2} \frac{k-\frac{1}{2}}{k+\frac{3}{2}} \\ &= \frac{1}{6^2} \left(5 - \frac{1}{2} \right) \left(6 - \frac{1}{2} \right) \\ &= \frac{11}{16}. \end{aligned} \tag{5.312}$$

Further:

$$\begin{aligned}
\sum_{k=5}^{\infty}\left(\frac{1}{k^2+\frac{3}{2}k}-\frac{1}{k^2+3k+\frac{3}{2}}\right) &\leq \sum_{k=5}^{\infty}\left(\frac{1}{k^2+\frac{3}{2}k}-\frac{1}{k^2+3k+2}\right) \\
&= \sum_{k=5}^{\infty}\left(\frac{1}{k(k+\frac{3}{2})}-\frac{1}{(k+1)(k+2)}\right) \\
&= \frac{2}{3}\sum_{k=5}^{\infty}\left(\frac{1}{k}-\frac{1}{k+\frac{3}{2}}\right)-\sum_{k=5}^{\infty}\left(\frac{1}{k+1}-\frac{1}{k+2}\right) \\
&\leq \frac{1}{3}\sum_{k=5}^{\infty}\left(\frac{1}{k-\frac{1}{2}}+\frac{1}{k+\frac{1}{2}}-\frac{2}{k+\frac{3}{2}}\right)-\frac{1}{6} \\
&= \frac{1}{3}\sum_{k=5}^{\infty}\left(\frac{1}{k-\frac{1}{2}}-\frac{1}{k+\frac{3}{2}}\right)+\frac{1}{3}\sum_{k=5}^{\infty}\left(\frac{1}{k+\frac{1}{2}}-\frac{1}{k+\frac{3}{2}}\right)-\frac{1}{6} \\
&= \frac{1}{3}\left(\frac{1}{5-\frac{1}{2}}+\frac{1}{6-\frac{1}{2}}\right)+\frac{1}{3}\frac{1}{5+\frac{1}{2}}-\frac{1}{6} \\
&= \frac{2}{3}\left(\frac{1}{9}+\frac{1}{11}\right)+\frac{2}{3}\frac{1}{11}-\frac{1}{6} \\
&= \frac{1}{6}\left(\frac{80}{99}+\frac{4}{11}-1\right)=\frac{1}{6}\frac{17}{99}.
\end{aligned} \tag{5.313}$$

It follows:

$${d_{K+1}^{+}}^{-1}-d_{K+1}^{-}\geq\frac{E^2}{9}\frac{11}{16}-\frac{67}{42}E^2\frac{1}{6}\frac{17}{99}\geq\left(\frac{11}{4}-\frac{67}{21}\frac{17}{33}\right)\frac{E^2}{36}\geq\frac{E^2}{36}. \tag{5.314}$$

That concludes the proof. □

Lemma 5.6.13 *Let $p=3$. Then iL admits no eigenvalues or resonances with odd eigenfunctions within the spectral gap $[-1,1]$, apart from 0.*

Proof. Follows from lemma 3.4.6, lemma 5.1.16 and lemma 5.6.12. □

5.7. Controlling the Resonance

Notation 5.7.1 *Given $p \in [3,5]$ and $E \in [0,1]$, we abbreviate $c_k^+ = c_k(E,p)$ and $c_k^- = c_k(-E,p)$.*

We study the resonance that arises for even solutions as $E = 1$ and $p = 3$. We formalise the recursion defining c_k, given by lemma 4.1.5.

Definition 5.7.2 *Let $l \in \mathbb{R}$ and $p \in [3,5]$. We define*

$$\alpha(l,p) = 4l^2 + 2l - \frac{(p+1)^2}{(p-1)^2}. \tag{5.315}$$

Definition 5.7.3 *Let $E \in [0,1]$ and $p \in [3,5]$, such that $(E,p) \neq (1,3)$. Let $l \in \mathbb{R}$. We define the Möbius transform $x \mapsto \eta_{l,p,E}(x) = \eta_l(x)$ by*

$$\eta_{l,p,E}(x) = \frac{(l+1+\frac{\sqrt{1-E}}{p-1})(l+1-\frac{\sqrt{1-E}}{p-1})}{(l+1+\frac{\sqrt{1+E}}{p-1})(l+1-\frac{\sqrt{1+E}}{p-1})} \frac{\alpha(l,p)x - \frac{p+1}{p-1}}{\alpha(l,p) - \frac{p+1}{p-1}x}. \tag{5.316}$$

Definition 5.7.4 *Let $E \in [0,1]$ and $p \in [3,5]$, such that $(E,p) \neq (1,3)$. Let $l \in \mathbb{R}$ and $k \in \mathbb{Z}_{\geq 0}$. We define the Möbius transform $x \mapsto \mu_{l,k}(x)$ by $\mu_{l,0}(x) = x$ and*

$$\mu_{l,k+1}(x) = \eta_{l+k}(\mu_{l,k}(x)). \tag{5.317}$$

We further define

$$\mu_{l,\infty}(x) = \lim_{k\to\infty} \mu_{l,k}(x). \tag{5.318}$$

That this limit exists follows analogously to corollary 4.1.7, corollary 4.2.7 or lemma 5.4.13.

Lemma 5.7.5 *Let $E \in [0,1]$ and $p \in [3,5]$, such that $(E,p) \neq (1,3)$. Let $k \in \mathbb{Z}_{\geq 0}$. Then:*

$$c_{k+1}^+ = \eta_{k+\frac{\sqrt{1+E}}{p-1},p,E}\left(c_k^+\right), \tag{5.319}$$

$${c_{k+1}^-}^{-1} = \eta_{k+\frac{\sqrt{1-E}}{p-1},p,E}\left({c_k^-}^{-1}\right). \tag{5.320}$$

Proof. By definition, on the one hand:

$$c_{k+1}^+ = \frac{(k+1+\frac{\sqrt{1+E}}{p-1}+\frac{\sqrt{1-E}}{p-1})(k+1+\frac{\sqrt{1+E}}{p-1}-\frac{\sqrt{1-E}}{p-1})}{(k+1+\frac{\sqrt{1+E}}{p-1}+\frac{\sqrt{1+E}}{p-1})(k+1+\frac{\sqrt{1+E}}{p-1}-\frac{\sqrt{1+E}}{p-1})} \frac{\left(4\left(k+\frac{\sqrt{1+E}}{p-1}\right)^2 + 2\left(k+\frac{\sqrt{1+E}}{p-1}\right) - \frac{(p+1)^2}{(p-1)^2}\right)c_k^+ - \frac{p+1}{p-1}}{4\left(k+\frac{\sqrt{1+E}}{p-1}\right)^2 + 2\left(k+\frac{\sqrt{1+E}}{p-1}\right) - \frac{(p+1)^2}{(p-1)^2} - \frac{p+1}{p-1}c_k^+}. \tag{5.321}$$

On the other hand:

$$c_{k+1}^{-\ -1} = \frac{(k+1+\frac{\sqrt{1-E}}{p-1}+\frac{\sqrt{1-E}}{p-1})(k+1+\frac{\sqrt{1-E}}{p-1}-\frac{\sqrt{1-E}}{p-1})}{(k+1+\frac{\sqrt{1-E}}{p-1}+\frac{\sqrt{1+E}}{p-1})(k+1+\frac{\sqrt{1-E}}{p-1}-\frac{\sqrt{1+E}}{p-1})}$$
$$\frac{\left(4\left(k+\frac{\sqrt{1-E}}{p-1}\right)^2+2\left(k+\frac{\sqrt{1-E}}{p-1}\right)-\frac{(p+1)^2}{(p-1)^2}\right)c_k^{-\ -1}-\frac{p+1}{p-1}}{4\left(k+\frac{\sqrt{1-E}}{p-1}\right)^2+2\left(k+\frac{\sqrt{1-E}}{p-1}\right)-\frac{(p+1)^2}{(p-1)^2}-\frac{p+1}{p-1}c_k^{-\ -1}}. \tag{5.322}$$

That concludes the proof. □

Lemma 5.7.6 *Let $E \in [0,1]$ and $p \in [3,5]$. Let $l \in \mathbb{R}_{\geq 1}$. Then*

$$x \mapsto \mu_{l,\infty}(x) \tag{5.323}$$

constitutes a Möbius transform.

Proof. As the limit of a sequence of Möbius transforms, $\mu_{l,\infty}$ is either a Möbius transform or a degenerate Möbius transform, i.e. constant. If $\mu_{l,\infty}$ is constant, so is $\mu_{l+k,\infty}$ for every $k \in \mathbb{Z}_{\geq 0}$. That is clearly not the case, as $x < 0$ implies $\mu_{l+k,\infty}(x) < 0$, while $\lim_{k\to\infty} \mu_{l+k,\infty}(1) = 1$. □

What happens, if we extend the definition of $\eta_{l,p,E}$ to $(E,p) = (1,3)$?

For $k \geq 1$, $\eta_{k+\frac{\sqrt{1-E}}{p-1},p,E}$ remains a well-defined Möbius transform. However, $\eta_{\frac{\sqrt{1-E}}{p-1},p,E}$ is degenerate:

$$\eta_{\frac{\sqrt{1-E}}{p-1},p,E}(x) = \frac{2x-2}{2-2x}. \tag{5.324}$$

We conclude:

Lemma 5.7.7 *Let $E \in [0,1]$ and $p \in [3,5]$ with $(E,p) \neq (1,3)$. $c_2(-E,p)$ is a rational function in $\xi = \sqrt{1-E}$ and p by lemma 4.1.5. Let P_1, P_2 be polynomials, such that:*

$$c_2(\xi^2-1,p) = \frac{P_1(\xi,p-3)}{P_2(\xi,p-3)}. \tag{5.325}$$

Then $P_1(0,0) = P_2(0,0) = 0$.

Lemma 5.7.7 allows us to define:

Definition 5.7.8 *Let $\alpha \in [0,\frac{\pi}{2}]$. Let P_1, P_2 as in lemma 5.7.7. We define:*

$$\begin{aligned}\kappa(\alpha) &:= \lim_{\varepsilon\searrow 0}\frac{P_2(\varepsilon\cos(\alpha),\varepsilon\sin(\alpha))}{P_1(\varepsilon\cos(\alpha),\varepsilon\sin(\alpha))}\\ &= \lim_{\varepsilon\searrow 0}\frac{1}{c_2(\varepsilon^2\cos^2(\alpha)-1,\varepsilon\sin(\alpha)+3)}.\end{aligned} \tag{5.326}$$

Lemma 5.7.9 *Let $\alpha \in [0, \frac{\pi}{2}]$. Then:*

$$\kappa(\alpha) = \frac{7\cos(\alpha) + 2\sin(\alpha)}{13\cos(\alpha) + 8\sin(\alpha)}. \tag{5.327}$$

Proof (Calculations are computer assisted). Recall (5.322):

$$\begin{aligned} c_{k+1}^{-\,-1} = &\frac{(k+1+\frac{\sqrt{1-E}}{p-1}+\frac{\sqrt{1-E}}{p-1})(k+1+\frac{\sqrt{1-E}}{p-1}-\frac{\sqrt{1-E}}{p-1})}{(k+1+\frac{\sqrt{1-E}}{p-1}+\frac{\sqrt{1+E}}{p-1})(k+1+\frac{\sqrt{1-E}}{p-1}-\frac{\sqrt{1+E}}{p-1})} \\ &\frac{\left(4\left(k+\frac{\sqrt{1-E}}{p-1}\right)^2+2\left(k+\frac{\sqrt{1-E}}{p-1}\right)-\frac{(p+1)^2}{(p-1)^2}\right)c_k^{-\,-1}-\frac{p+1}{p-1}}{4\left(k+\frac{\sqrt{1-E}}{p-1}\right)^2+2\left(k+\frac{\sqrt{1-E}}{p-1}\right)-\frac{(p+1)^2}{(p-1)^2}-\frac{p+1}{p-1}c_k^{-\,-1}}. \end{aligned} \tag{5.328}$$

Let $\xi = \sqrt{1-E}$ and $q = p-3$. By direct computation, using $c_0^- = \infty$:

$$c_1^{-\,-1} = \frac{(q+2)(q^2+6q+8)(q+2\xi+2)}{(q^2+2q(\xi+2)+2(\xi+1)^2)(q^2-2q(\xi-4)-4(\xi^2+\xi-4))}. \tag{5.329}$$

Further:

$$\frac{(2+\frac{\sqrt{1-E}}{p-1}+\frac{\sqrt{1+E}}{p-1})(2+\frac{\sqrt{1-E}}{p-1}-\frac{\sqrt{1+E}}{p-1})}{(2+\frac{\sqrt{1-E}}{p-1}+\frac{\sqrt{1-E}}{p-1})(2+\frac{\sqrt{1-E}}{p-1}-\frac{\sqrt{1-E}}{p-1})}c_2^{-\,-1} = \frac{X_1(\xi,q)}{X_2(\xi,q)}. \tag{5.330}$$

Hereby, on the one hand:

$$X_1(\xi,q) = \frac{2(q+4)Y_1(\xi,q)}{(q+2)(q^2-2q\xi+8q-4\xi^2-4\xi+16)(q^2+2q\xi+4q+2\xi^2+4\xi+2)}, \tag{5.331}$$

with

$$\begin{aligned} Y_1(\xi,q) = &\, 2q^4+10q^3\xi+12q^3+15q^2\xi^2+52q^2\xi+21q^2+10q\xi^3+56q\xi^2 \\ &+78q\xi+4\xi^4+20\xi^3+44\xi^2+8q+28\xi. \end{aligned} \tag{5.332}$$

On the other hand:

$$X_2(\xi,q) = \frac{Y_2(\xi,q)}{(2+q)^2((q^2+2(1+\xi)^2+2q(2+\xi))(q^2-2q(\xi-4)-4(\xi^2+\xi-4)))} \tag{5.333}$$

with

$$\begin{aligned} Y_2(\xi,q) = &\, 4q^6+8q^5\xi+60q^5-26q^4\xi^2+152q^4\xi+346q^4-120q^3\xi^3-48q^3\xi^2 \\ &+936q^3\xi+944q^3-184q^2\xi^4-560q^2\xi^3+536q^2\xi^2+2448q^2\xi \\ &+1184q^2-128q\xi^5-672q\xi^4-544q\xi^3+1760q\xi^2+2656q\xi \\ &-32\xi^6-256\xi^5-512\xi^4+192\xi^3+1312\xi^2+512q+832\xi. \end{aligned} \tag{5.334}$$

As the zeros are contained within Y_1, Y_2, it follows:

$$\begin{aligned}\kappa(\alpha) &= \frac{\frac{8}{64}}{\frac{1}{128}} \lim_{\varepsilon \searrow 0} \frac{Y_1(\varepsilon\cos(\alpha), \varepsilon\sin(\alpha))}{Y_2(\varepsilon\cos(\alpha), \varepsilon\sin(\alpha))} \\ &= \frac{16}{1} \frac{28\cos(\alpha) + 8\sin(\alpha)}{832\cos(\alpha) + 512\sin(\alpha)}. \end{aligned} \tag{5.335}$$

That concludes the proof. □

Corollary 5.7.10 *Let $\alpha \in [0, \frac{\pi}{2}]$. Then:*

$$\kappa(\alpha) > 0, \tag{5.336}$$

$$\partial_\alpha \kappa(\alpha) < 0. \tag{5.337}$$

Lemma 5.7.11 *There exist $\xi_0 \in (0,1)$ and $p_0 \in (3,5)$, as well as an analytic function $\nu : [0, \xi_0] \times [3, p_0] \to \mathbb{R}\backslash\{0\}$, such that for $(\xi, p) \in [0, \xi_0] \times [3, p_0]\backslash\{(0,3)\}$:*

$$\frac{1}{c_2(\xi^2 - 1, p)} = \kappa\left(\arctan\left(\frac{p-3}{\xi}\right)\right)\nu(\xi, p). \tag{5.338}$$

Proof. Follows from the fact that $\frac{1}{c_2(\xi^2-1,p)}$ is a rational function. □

Lemma 5.7.12 *Let $\xi_0 \in (0,1)$, $p_0 \in (3,5)$ and $\nu : [0, \xi_0] \times [3, p_0] \to \mathbb{R}$ be as in lemma 5.7.11.*

Then, $E = 1 - \xi^2 \in [1 - \xi_0^2, 1]$ is an eigenvalue or resonance of iL with exponent $p \in [3, p_0]$, if and only if

$$\frac{\mu^{-1}_{2+\frac{\xi}{p-1},\infty}\left(\mathcal{C}(1-\xi^2, p)\right)}{\nu(\xi, p)} = \kappa\left(\arctan\left(\frac{p-3}{\xi}\right)\right). \tag{5.339}$$

As always, we need to exclude the case $p = 3$, $E = 1$.

Proof. By lemma 5.7.5:

$$\mu_{2+\frac{\xi}{p-1},\infty}\left(\frac{1}{c_2(-E,p)}\right) = \frac{1}{\mathcal{C}(-E,p)}. \tag{5.340}$$

By lemma 5.7.11, (5.339) is equivalent to

$$\mathcal{C}(E,p) = \frac{1}{\mathcal{C}(-E,p)}. \tag{5.341}$$

That concludes the proof. □

Definition 5.7.13 *For $E \geq 0$, $p \geq 3$, consider the coordinate transform $\tau(E,p) = (r, \alpha)$ into polar coordinates:*

$$\alpha = \arctan\left(\frac{p-3}{\xi}\right), \tag{5.342}$$

$$r = \sqrt{\xi^2 + (p-3)^2}. \tag{5.343}$$

Lemma 5.7.14 *There exist $\xi_1 \in (0,1)$ and $p_1 \in (3,5)$, such that for every $p \in (3, p_1)$, there exists at most one eigenvalue $E \in (1-\xi_1^2, 1]$ of iL.*

Proof. We transform (E,p) into polar coordinates via $(r,\alpha) = \tau(E,p)$. By lemma 5.7.12, we find an analytic function $f : [0,\xi_0] \times [3,p_0] \to \mathbb{R}$, such that $E = 1-\xi^2$ is an eigenvalue or resonance of iL with exponent p, if and only if

$$f(\xi, p) = \kappa\left(\arctan\left(\frac{p-3}{\xi}\right)\right). \tag{5.344}$$

Expressed in polar coordinates, $E = 1 - r^2\cos^2(\alpha)$ is an eigenvalue or resonance of iL with exponent $p = r\sin(\alpha) + 3$, if and only if

$$g(r,\alpha) := f(r\cos(\alpha), r\sin(\alpha)+3) - \kappa(\alpha) = 0. \tag{5.345}$$

By lemma 5.3.4 and lemma 5.3.5:

$$g(0,\pi/2) = 0. \tag{5.346}$$

Consequently, by corollary 5.7.10, for $\alpha < \frac{\pi}{2}$:

$$g(0,\alpha) > 0. \tag{5.347}$$

As f is analytic, we find $r_0 > 0$, such that for $r \in [0, r_0]$ and $\alpha \in [0, \frac{\pi}{2}]$:

$$\begin{aligned} \partial_\alpha g(r,\alpha) = &-\kappa'(\alpha) - \partial_\xi f(r\cos(\alpha), r\sin(\alpha)+3) r\sin(\alpha) \\ &+ \partial_p f(r\cos(\alpha), r\sin(\alpha)+3) r\cos(\alpha) \\ > &\ 0. \end{aligned} \tag{5.348}$$

Suppose we have a solution $(\rho,\beta) \in [0,r_0] \times [0,\pi/2]$ of $g(\rho,\beta) = 0$. By the implicit function theorem, we find $r_1 \in [0, r_0]$ and a unique analytic function $\theta : [0, r_1] \to [0, \pi/2]$, such that $\theta(\rho) = \beta$ and

$$g(r, \theta(r)) = 0. \tag{5.349}$$

From (5.347), we conclude $\theta(0) = \pi/2$, meaning that $r \mapsto (r, \theta(r))$ parametrises every solution of $g(r,\alpha) = 0$ in $[0, r_1] \times [0, \pi/2]$.

We conclude that

$$r \mapsto (\lambda(r), q(r)) := (1 - r^2\cos^2(\theta(r)), r\sin(\theta(r)) + 3) \tag{5.350}$$

parametrises every pair of eigenvalues/resonances E and exponents p close to $(1,3)$.

As $r \mapsto q(r)$ is analytic, it follows that q is either constant or injective on some sufficiently small interval $[0, r_2] \subseteq [0, r_1]$. By lemma 5.5.18, $r \mapsto q(r)$ is not constant. Due to $q(0) = 3$ and $q(r) \geq 3$, we can conclude that $r \mapsto q(r)$ is increasing for $r \in [0, r_2]$.

In conclusion, we find the parametrisation

$$p \mapsto (\lambda(q^{-1}(p)), p) \tag{5.351}$$

for every pair of eigenvalues/resonances E and exponents p close to $(1,3)$. That concludes the proof. $\square$

Corollary 5.7.15 *There exist $E_1 \in [0,1)$ and $p_1 \in (3,5)$, such that for every $p \in (3, p_1]$, there exists exactly one eigenvalue $E \in (E_1, 1)$ of iL.*

Proof. Assume that at least one eigenvalue/exponent pair (E,p) close to $(1,3)$ exists. As seen in the proof of lemma 5.7.14, that allows us to construct a parametrisation $p \mapsto (E,p)$ for every p close to zero.

The fact that there are indeed eigenvalue/exponent pairs (E,p) close to $(1,3)$ is a simple consequence of the intermediate value theorem. As shown in the proof of lemma 5.3.13, for every $p > 3$:

$$\mathcal{C}(1,p) > \frac{1}{\mathcal{C}(-1,p)}. \tag{5.352}$$

By lemma 5.5.17, for every $E < 1$:

$$\mathcal{C}(E,3) < \frac{1}{\mathcal{C}(-E,3)}. \tag{5.353}$$

That concludes the proof. □

Corollary 5.7.16 *Let $p_1 \in (3,5)$ and $E_1 \in [0,1]$ be given by corollary 5.7.15. For $p \in (3, p_1)$, let $E_p \in (E_1, 1)$ be the unique eigenvalue of iL. Let ζ_p denote the corresponding eigenfunction. Then, for $x \in \mathbb{R}$, up to a possibly different sign, the following point-wise convergence holds:*

$$\lim_{p \searrow 3} \frac{\zeta_p(x)}{|\zeta_p(0)|} = \frac{\begin{pmatrix} 1 - Q(x)^2 \\ i \end{pmatrix}}{\sqrt{(1 - Q(0)^2)^2 + 1}}. \tag{5.354}$$

Hereby, $|(x,y)| = \sqrt{|x|^2 + |y|^2}$ denotes the usual norm on $\mathbb{R}^2$.

Proof. In the proof of lemma 5.7.14, we have constructed an analytic parametrisation $p \mapsto (p, E_p)$ for $p \in [0, p_1]$. Consequently $p \mapsto \zeta_p$ is analytic as well. That concludes the proof, as the eigenfunction of the resonance is given by:

$$(iL - 1) \begin{pmatrix} 1 - Q^2 \\ i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5.355}$$

□

5.8. Characterisation of the Spectrum

We summarize the chapter.

Lemma 5.8.1 *There exists $p_0 \in (3,5)$, such that for $p \in (3, p_0]$, iL admits two simple eigenvalues $E_1, E_2 \in (-1,1)\backslash\{0\}$ with $E_1 = -E_2$. iL admits no further eigenvalues or resonances apart from 0. The associated eigenstates of E_1, E_2 are even.*

Proof. Consider the case of odd functions first. 0 is well-known to be an eigenvalue of multiplicity 2 for every $p \in [3,5]$, see [34]. Consequently,

$$\left|\frac{1}{\mathcal{C}(E,p)} - \mathcal{C}(-E,p)\right| \leq CE^2. \tag{5.356}$$

for every $p \in [3,5]$ and $E \in [0,1]$. As the functions are analytic for $p \in [3,5]$ and $E \in [0,1)$, we conclude that

$$f(E,p) := \frac{\frac{1}{\mathcal{C}(E,p)} - \mathcal{C}(-E,p)}{E^2} \tag{5.357}$$

is a well-defined continuous function. By lemma 5.6.12, we conclude

$$f(E,3) \leq -\frac{1}{36}. \tag{5.358}$$

Consequently, we find $p_0 > 3$, such that $f(E,p) < 0$ for every $p \in [3, p_0]$.

Now consider even functions. By corollary 5.7.15, there is $p_1 > 3$ and $0 < \lambda_1 < 1$, such that

$$\mathcal{C}(E,p) = \frac{1}{\mathcal{C}(-E,p)} \tag{5.359}$$

has exactly one solution $E \in [\lambda_1, 1]$ for every $p \in (3, p_1]$. As in the case of even functions, we define

$$g(E,p) := \frac{\mathcal{C}(E,p) - \frac{1}{\mathcal{C}(-E,p)}}{E^2}. \tag{5.360}$$

Then g is a well-defined, continuous function for $E \in [0, \lambda_1]$ and $p \in [3, p_1]$.

By corollary 5.5.8 and lemma 5.5.17, we find $p_2 \in (3, p_1]$, such that $g(E,p) > 0$ for every $E \in (0, \lambda_1]$ and $p \in [3, p_2]$. That conclude the proof. □

Lemma 5.8.2 *There exists $\gamma \in (0,1)$, such that the following holds true. Let $p_0 > 3$ be given by lemma 5.8.1 and let $p \in [p_0, 5]$. Then, iL admits no eigenvalues outside of $(-\gamma, \gamma)$.*

Proof. Assume the lemma does not hold. We find a sequence of exponent-eigenvalue pairs $(p_n, E_n)_{\geq 1} \in [p_0, 5] \times [-1, 1]$ with $\lim_{n\to\infty} |E_n| = 1$. By symmetry, we can assume $E_n \geq 0$ for every $n \geq 1$.

As $[p_0, 5]$ is compact, we can find a convergent subsequence and define $p_\infty := \lim_{n\to\infty} p_n$. By continuity (lemma 5.1.5 and lemma 5.1.14):

$$\mathcal{C}(1, p_\infty) = \frac{1}{\mathcal{C}(-1, p_\infty)} \tag{5.361}$$

We conclude that 1 is an eigenvalue or resonance of iL for $p = p_\infty$. This contradicts lemma 5.3.13 and lemma 5.4.21. □

Notation 5.8.3 *As usual, let $\mathcal{H}$ denote the Hilbert space given by definition 1.9.1, while L denotes the linearised operator given by* (1.35)*.*

We denote $\mathcal{H}_p = \mathcal{H}$ and $L_p = L$ to express the fact that $\mathcal{H}$ and L are dependant on the exponent p.

Definition 5.8.4 *Let $p_0 \in (3, 5)$ be given by lemma 5.8.1. For $p \in [p_0, 5]$, let $\mathcal{P}^p$ denote the spectral projection associated with iL_p on $\mathcal{H}_p$.*

Lemma 5.8.5 *The spectral projection $\mathcal{P}^p_{[-\gamma,\gamma]}$ can be expressed via the resolvent formalism:*

$$\mathcal{P}^p_{[-\gamma,\gamma]} = -\frac{1}{2\pi i} \oint_\Gamma (iL_p - \lambda I)^{-1} d\lambda. \tag{5.362}$$

Hereby, $\Gamma : [0, 2\pi] \to \mathbb{C}, t \mapsto \frac{1+\gamma}{2} e^{it}$.

Proof. The essential spectrum of iL_p is given by $(-\infty, -1] \cup [1, \infty)$. We conclude by lemma 5.8.2, $\sigma(iL_p) \subset (-\infty, -1] \cup [1, \infty) \cup [-\gamma, \gamma]$. That already concludes the proof. □

(5.362) extends the definition of $\mathcal{P}^p_{[-\gamma,\gamma]}$ to $H^1(\mathbb{R})^2$, allowing us to consider $\mathcal{P}^p_{[-\gamma,\gamma]}$ independent of the Hilbert space $\mathcal{H}_p$, which depends on p.

Lemma 5.8.6 *Let $p_0 \in (3, 5)$ be given by lemma 5.8.1. Let $w \in H^1(\mathbb{R})^2$ and $\Gamma : [0, 2\pi] \to \mathbb{C}, t \mapsto \frac{1+\gamma}{2} e^{it}$. Then,*

$$p \mapsto -\frac{1}{2\pi i} \oint_\Gamma (iL_p - \lambda I)^{-1} w d\lambda \tag{5.363}$$

is continuous for $p \in [p_0, 5]$.

Proof. For $r > 0$, let $K_r := \{z \in \mathbb{C} | \, |z| = r\}$ be the circle around the origin with radius r. $|\lambda| = \frac{1+\gamma}{2}$, $(iL_p - \lambda I)^{-1}$ is uniformly bounded with respect to $\lambda \in K_{\frac{1+\gamma}{2}}$. Further, $p \to (iL_p - \lambda I)^{-1}$ is continuous. That already concludes the proof. □

Theorem 5.8.7 (Theorem 1.10.1) *For $p = 3$, iL possesses no eigenvalues apart from 0. 1 and -1 constitute resonances.*

For $p \in (3, 5)$, iL admits two simple eigenvalues $E_1, E_2 \in (-1, 1)\backslash\{0\}$ with $E_1 = -E_2$. iL admits no further eigenvalues or resonances apart from 0. The associated eigenstates of E_1, E_2 are even.

Finally, for $p = 5$, iL admits no eigenvalues or resonances apart from 0.

Proof. Let $p_0 \in (3, 5)$ be given by lemma 5.8.1. By lemma 5.5.18, lemma 5.6.13 and lemma 5.8.1, it remains to consider $p \geq p_0$.

By lemma 5.8.5, the total multiplicity of the eigenvalues of iL_p for $p \in [p_0, 5]$ is given by $\dim \operatorname{ran} \oint_\Gamma (iL_p - \lambda I)^{-1} d\lambda$. This number is continuous and thus constant by lemma 5.8.6. By lemma 5.8.1, $\dim \operatorname{ran} \oint_\Gamma (iL_p - \lambda I)^{-1} d\lambda = 6$. That concludes the proof, as 0 is an eigenvalue of multiplicity 4 for $p \in [3, 5)$ and of multiplicity 6 for $p = 5$. □

Lemma 5.8.8 *For $3 < p < 5$, let $E_1(p)$ be given by theorem 5.8.7. Then, $\lim_{p \searrow 3} E_1(p) = 1$ and $\lim_{p \nearrow 5} E_1(p) = 0$.*

Proof. We only show $\lim_{p \searrow 3} E_1(p) = 1$. $\lim_{p \nearrow 5} E_1(p) = 0$ follows completely analogously.

Because $[0, 1]$ is compact, it suffices to show that 1 is the only cluster point of $E(p)$ for $p \searrow 3$. Assume that another cluster point $\lambda \in [0, 1)$ exists. We find a sequence $(p_k)_{k \geq 0} \subset (3, 5)$ with $\lim_{k \to \infty} p_k = 3$ and $\lim_{k \to \infty} E_1(p_k) = \lambda$.

By continuity (lemma 5.1.5):

$$\mathcal{C}(\lambda, 3) = \frac{1}{\mathcal{C}(-\lambda, 3)}, \tag{5.364}$$

in contradiction to lemma 5.5.18. □

6. Definition: Eigenfunctions and Hilbert Space

We have succeeded in charactering the spectrum of L, which is the main result of this thesis (theorem 1.10.1). The remainder of the thesis considers how theorem 1.10.1 can be used to establish dispersive estimates on the linear equation $\partial_t w = Lw$, in particular by constructing a wave operator for iL.

Defining a wave operator is only sensible for the continuous part of the spectrum of iL. We need to further restrict the Hilbert space $\mathcal{H}$, by projecting away from the eigenfunctions associated with the non-zero eigenvalues of iL, the so-called internal modes.

The point of this short chapter is to define such a Hilbert space H.

6.1. Internal Modes

Definition 6.1.1 *Let $3 < p < 5$. We define $E = E(p) \in (0,1)$ as the unique positive eigenvalues of iL given by theorem 1.10.1. The unique negative eigenvalue is given by $-E$.*

Definition 6.1.2 *Let $3 < p < 5$. We choose*

$$\zeta = \begin{pmatrix} \zeta_u \\ i\zeta_v \end{pmatrix}. \tag{6.1}$$

to be the eigenfunction of iL with eigenvalue E. We further require $||\zeta||_{\mathcal{H}} = 1$ and that ζ_u, ζ_v are real-valued. Note that the eigenfunction of $-E$ is given by:

$$\overline{\zeta} = \begin{pmatrix} \zeta_u \\ -i\zeta_v \end{pmatrix}. \tag{6.2}$$

6.2. Hilbert Space

In order to prove orbital stability, Weinstein [34], [35] constructed the Hilbert space $\mathcal{H}$ given by definition 6.2.1. We recall, for $3 < p < 5$:

Lemma & Definition 6.2.1 (Definition 1.9.1) *Let $p \in (3,5)$. Consider:*

$$\mathcal{H} := \{w = (u,v) \in H^1(\mathbb{R})^2 |\ (1.43) \textit{ holds true}\}. \tag{6.3}$$

Further, let $\langle \cdot, \cdot \rangle_{\mathcal{H}} : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ be given by:

$$\begin{aligned} &\langle (u_1, v_1), (u_2, v_2) \rangle_{\mathcal{H}} \\ = &\int_{\mathbb{R}} \Big(\nabla u_1 \nabla \overline{u}_2 + u_1 \overline{u}_2 + \nabla v_1 \nabla \overline{v}_2 + v_1 \overline{v}_2 - pQ^{p-1} u_1 \overline{u}_2 - Q^{p-1} v_1 \overline{v}_2 \Big)\, dx. \end{aligned} \tag{6.4}$$

Then, $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ constitutes a Hilbert space and L maps $\mathcal{H}$ onto itself. Further, iL is self-adjoint with respect to $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. Finally, $||\cdot||_{\mathcal{H}}$ and $||\cdot||_{H^1(\mathbb{R})^2}$ constitute equivalent norms.

Definition 6.2.2 *Let $3 < p < 5$. Consider the Hilbert space $H \subset \mathcal{H}$ given by*

$$H = \{w \in \mathcal{H} | \langle w, \zeta \rangle_{\mathcal{H}} = \langle w, \overline{\zeta} \rangle_{\mathcal{H}} = 0\}, \tag{6.5}$$

$$\langle \cdot, \cdot \rangle_H = \langle \cdot, \cdot \rangle_{\mathcal{H}}. \tag{6.6}$$

Lemma 6.2.3 *Let $3 < p < 5$. Then $L : H \supset \mathrm{dom}(L) \to H$ admits neither eigenvalues nor resonances.*

Part III.
Wave Operator

For the remainder of the thesis we only consider $3 < p < 5$.

7. Goal

Consider the free Schrödinger equation:

$$i\partial_t w = (-\Delta + 1)w \tag{7.1}$$

$(-\Delta + 1 - E)w = 0$ admits non-trivial bounded solutions for every $E \geq 1$. If we define $E = 1+\xi^2$, $\xi \in \mathbb{R}$, then the solution space is spanned by $e^{i\xi x}$ and $e^{-i\xi x}$. $e^{-i\xi x}$ is precisely the kernel of the Fourier transform $\mathcal{F}$:

$$(\mathcal{F}f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)e^{-i\xi x}dx. \tag{7.2}$$

That allows us to use the Fourier transform to diagonalise $-\Delta + 1$:

$$-\Delta + 1 = \mathcal{F}^{-1}(\xi^2 + 1)\mathcal{F}. \tag{7.3}$$

Now, suppose we have a bounded matrix-valued solution

$$W : \mathbb{R} \times \mathbb{R} \to \mathbb{C}^{2\times 2}, \tag{7.4}$$
$$(\xi, x) \mapsto W(\xi, x) \tag{7.5}$$

of the eigenvalue equation

$$(L - i\,\mathrm{sgn}(\xi)(\xi^2 + 1))w = 0. \tag{7.6}$$

In analogy to the Fourier transform, we can define the distorted Fourier transform F for Schwartz functions w by somewhat incorrectly using the scalar product of H (since $W(\xi, \cdot) \notin H$):

$$\begin{aligned}(Fw)(\xi) &= \frac{1}{\sqrt{2\pi}(1+\xi^2)}\langle w, W(\xi,\cdot)\rangle_H \\ &= \frac{1}{\sqrt{2\pi}(1+\xi^2)}\int_{\mathbb{R}} \left(\nabla\overline{W}\nabla w + \overline{W}w - \begin{pmatrix} pQ^{p-1} & 0 \\ 0 & Q^{p-1}\end{pmatrix}\overline{W}w\right)dx.\end{aligned} \tag{7.7}$$

Heuristically, F satisfies the diagonalisation:

$$FLw = i\,\mathrm{sgn}(\xi)(\xi^2 + 1)Fw. \tag{7.8}$$

In chapter 8, we construct bounded solutions $W_e, W_o : \mathbb{R}^2 \to \mathbb{C}^2$ to (7.6). W_e is even with respect to x, while W_o is odd.

We define W_e, W_o such that (7.7) for $W = (W_e, W_o)$ resembles the Fourier transform (7.2) as closely as possible. Based on theorem 2.4.7, the asymptotic behaviour of W_e, W_o for $x \to -\infty$ is given by

$$W_e(\xi, x) \approx \begin{pmatrix} 1 \\ -i \end{pmatrix} c_e(\xi)\cos(\xi x) + \begin{pmatrix} 1 \\ -i \end{pmatrix} s_e(\xi)\sin(\xi x), \tag{7.9}$$

$$W_o(\xi, x) \approx \begin{pmatrix} 1 \\ -i \end{pmatrix} c_o(\xi)\cos(\xi x) + \begin{pmatrix} 1 \\ -i \end{pmatrix} s_o(\xi)\sin(\xi x). \tag{7.10}$$

By rescaling W_e, W_o, we can ensure

$$|c_e|^2 + |s_e|^2 = |c_o|^2 + |s_o|^2 = 1. \tag{7.11}$$

(7.9), (7.10) now characterise W_e, W_o as some matrix equivalent of Jost solutions.

In chapter 9, we define an operator F based on (7.7) and show that F maps L onto the multiplication operator $i\,\mathrm{sgn}(\xi)(\xi^2+1)$. We prove that F extends to a unitary operator mapping H onto $\{f \in L^2(\mathbb{R})^2 | \xi h \in L^2(\mathbb{R})^2\}$. Further, we generalise H for $s \geq 0$ via

$$H_s = \{w = (u, v) \in H^s(\mathbb{R})^2 |\ (1.43) \text{ holds, and } \langle w, \zeta\rangle_{\mathcal{H}} = \langle w, \overline{\zeta}\rangle_{\mathcal{H}} = 0\}. \tag{7.12}$$

After defining a suitable norm, we show that $F : H_s \to \{f \in L^2(\mathbb{R})^2 | (1+\xi^2)^{\frac{s}{2}} h \in L^2(\mathbb{R})^2\}$ constitutes a unitary map as well.

That allows us to define the wave operator in chapter 10, as finding a unitary transform

$$\mathcal{G} : \{f \in L^2(\mathbb{R})^2 | (1+\xi^2)^{\frac{s}{2}} h \in L^2(\mathbb{R})^2\} \to H^s(\mathbb{R})^2 \tag{7.13}$$

mapping $i\,\mathrm{sgn}(\xi)(\xi^2+1)$ onto $I(-\Delta+1)$ is quite simple. Hereby, I denotes the matrix equivalent of the imaginary unit i.

In chapter 11, we show that the wave operator T and its inverse are bounded $L^q \to L^q$ for every $1 \leq q \leq \infty$, as well as a variety of less significant estimates. The proofs of these bounds are quite straightforward, requiring only some basic results about Fourier multipliers.

Finally, in chapter 12, we establish several dispersive estimates for the linear equation $\partial_t w = Lw$. As a result of the $L^q \to L^q$ bound, both the dispersive estimate given by lemma 1.6.1 and the Strichartz estimates given by 1.6.2 hold for $\partial_t w = Lw$ as well. The actual result of chapter 12 is a non-resonant local smoothing estimate, for which the free equation allows no analogue.

8. Bounded Solutions

8.1. Bounds on Symmetrical Solutions

We examine the bounded solutions to $(L - iE)w = 0$, $E = \operatorname{sgn}(\xi)(\xi^2 + 1)$ and establish some basic bounds. We always assume $3 < p < 5$.

Definition 8.1.1 *For $(c_1, c_2, c_3, c_4) \in \mathbb{C}^4$ let the linear combinations:*

$$w_{c_1,c_2,c_3,c_4} = (u_{c_1,c_2,c_3,c_4}, v_{c_1,c_2,c_3,c_4}) : [0,\infty) \times \mathbb{R} \to \mathbb{R}^2, \tag{8.1}$$

$$\tilde{w}_{c_1,c_2,c_3,c_4} = (\tilde{u}_{c_1,c_2,c_3,c_4}, \tilde{v}_{c_1,c_2,c_3,c_4}) : [\xi_0,\infty) \times \mathbb{R} \to \mathbb{R}^2 \tag{8.2}$$

be given by:

$$w_{c_1,c_2,c_3,c_4} = \sum_{k=1}^{4} c_k w_k, \tag{8.3}$$

$$\tilde{w}_{c_1,c_2,c_3,c_4} = \sum_{k=1}^{4} c_k \tilde{w}_k. \tag{8.4}$$

$(w_k)_{1\le k\le 4}$ and $(\tilde{w}_k)_{1\le k\le 4}$ are given by theorem 2.4.7 and theorem 2.4.12. ξ_0 is given by definition 2.4.6.

Lemma 8.1.2 *Let $\xi \geq 0$, $E = 1 + \xi^2$, $y \in \mathbb{R}$ and $K > 0$. Assume $w = (u, v) \in C^1(\mathbb{R})$ solves* (2.5) *and* (2.6)*. Assume further:*

$$|w(y)| + \frac{|w'(y)|}{1+\xi} \leq K. \tag{8.5}$$

Then, for every $x \in \mathbb{R}$:

$$|w(x)| + \frac{|w'(x)|}{1+\xi} \leq CKe^{\sqrt{2+\xi^2}|x-y|}. \tag{8.6}$$

Proof. Let $\lambda := \sqrt{2+\xi^2} = \sqrt{1+E}$, $U := -\lambda^{-1}u'$ and $V := -\lambda^{-1}v'$. Then (2.5) and (2.6) read:

$$\frac{d}{dx}\begin{pmatrix} u \\ U \\ v \\ V \end{pmatrix} = \begin{pmatrix} 0 & -\lambda & 0 & 0 \\ -\lambda + \frac{p+1}{2\lambda}Q^{p-1} & 0 & \frac{p-1}{2\lambda}Q^{p-1} & 0 \\ 0 & 0 & 0 & -\lambda \\ \frac{p-1}{2\lambda}Q^{p-1} & 0 & \frac{\xi^2}{\lambda} + \frac{p+1}{2\lambda}Q^{p-1} & 0 \end{pmatrix}\begin{pmatrix} u \\ U \\ v \\ V \end{pmatrix}. \tag{8.7}$$

For $f := \sqrt{u^2 + U^2 + v^2 + V^2}$ that implies $f(x)' \leq (\lambda + Ce^{-|x|})f(x)$. The lemma follows from Grönwall's inequality. □

Corollary 8.1.3 *Let $\omega < 0$ be given by definition 2.4.6. Let $\xi \geq 0$, $x \in [\omega, 0]$ and $c_1, c_3, c_4 \in \mathbb{C}$. Then:*

$$|w_{c_1,0,c_3,c_4}(\xi,x)| + \frac{1}{1+\xi}\left|\frac{d}{dx}w_{c_1,0,c_3,c_4}(\xi,x)\right| \leq Ce^{-\sqrt{2+\xi^2}\omega}(|c_1| + |c_3| + |c_4|). \tag{8.8}$$

Proof. Follows from theorem 2.4.7 and lemma 8.1.2. $\square$

Lemma 8.1.4 *Let $\xi \geq 0$ and $(c_1, c_3, c_4) \in \mathbb{C}^3$, such that $w = (u, v) = w_{c_1,0,c_3,c_4}(\xi, \cdot)$ is either even or odd. Then, for every $x \in \mathbb{R}$:*

$$\begin{aligned} u(x) = & -\frac{1}{2\sqrt{2+\xi^2}} \int_x^{\infty} e^{\sqrt{2+\xi^2}(x-y)} Q^{p-1} \left(\frac{p+1}{2} u + \frac{p-1}{2} v \right) dy \\ & - \frac{1}{2\sqrt{2+\xi^2}} \int_{-\infty}^{x} e^{\sqrt{2+\xi^2}(y-x)} Q^{p-1} \left(\frac{p+1}{2} u + \frac{p-1}{2} v \right) dy, \end{aligned} \tag{8.9}$$

$$\begin{aligned} v(x) = & \, c_3 \cos(\xi x) + c_4 \frac{1+\xi}{\xi} \sin(\xi x) \\ & + \int_{-\infty}^{x} \frac{\sin(\xi(x-y))}{\xi} Q^{p-1} \left(\frac{p+1}{2} v + \frac{p-1}{2} u \right) dy. \end{aligned} \tag{8.10}$$

For $\xi \geq \xi_0$, (8.9) *and* (8.10) *can be restated as:*

$$w_{c_1,0,c_3,c_4}(\xi, \cdot) = \tilde{w}_{0,0,c_3,c_4}(\xi, \cdot). \tag{8.11}$$

ξ_0 is given by definition 2.4.6.

Proof. Let $\omega < 0$ be given by definition 2.4.6. We define:

$$\begin{aligned} l_1 := & \frac{c_1}{2\sqrt{2+\xi^2}} \int_{-\infty}^{\infty} e^{-\sqrt{2+\xi^2} y} Q^{p-1} \left(\frac{p+1}{2} u_1 + \frac{p-1}{2} v_1 \right) dy \\ & + \frac{c_3}{2\sqrt{2+\xi^2}} \int_{\omega}^{\infty} e^{-\sqrt{2+\xi^2} y} Q^{p-1} \left(\frac{p+1}{2} u_3 + \frac{p-1}{2} v_3 \right) dy \\ & + \frac{c_4}{2\sqrt{2+\xi^2}} \int_{\omega}^{\infty} e^{-\sqrt{2+\xi^2} y} Q^{p-1} \left(\frac{p+1}{2} u_4 + \frac{p-1}{2} v_4 \right) dy. \end{aligned} \tag{8.12}$$

By definition, for $x \in \mathbb{R}$:

$$\begin{aligned} u(x) = & (c_1 + l_1) e^{\sqrt{2+\xi^2} x} - \frac{1}{2\sqrt{2+\xi^2}} \int_x^{\infty} e^{\sqrt{2+\xi^2}(x-y)} Q^{p-1} \left(\frac{p+1}{2} u + \frac{p-1}{2} v \right) dy \\ & - \frac{1}{2\sqrt{2+\xi^2}} \int_{-\infty}^{x} e^{\sqrt{2+\xi^2}(y-x)} Q^{p-1} \left(\frac{p+1}{2} u + \frac{p-1}{2} v \right) dy, \end{aligned} \tag{8.13}$$

$$\begin{aligned} v(x) = & \, c_3 \cos(\xi x) + c_4 \frac{1+\xi}{\xi} \sin(\xi x) \\ & + \int_{-\infty}^{x} \frac{\sin(\xi(x-y))}{\xi} Q^{p-1} \left(\frac{p+1}{2} v + \frac{p-1}{2} u \right) dy. \end{aligned} \tag{8.14}$$

As u and v are both even or both odd, we conclude $c_1 + l_1 = 0$ and the proof. $\square$

Corollary 8.1.5 *Let $\xi_0 \geq 0$ be given by definition 2.4.6. Let $\xi \geq \xi_0$ and $(c_1, c_3, c_4) \in \mathbb{C}^3$, such that $w = (u, v) = w_{c_1,0,c_3,c_4}(\xi, \cdot)$ is either even or odd. Then, for every $x \in \mathbb{R}$:*

$$|w(x)| \leq C(|c_3| + |c_4|). \tag{8.15}$$

Proof. Lemma 8.1.4 implies $w_{c_1,0,c_3,c_4}(\xi,\cdot) = \tilde{w}_{0,0,c_3,c_4}(\xi,\cdot)$. Theorem 2.4.12 concludes the proof. $\square$

Lemma 8.1.6 *Let $\xi_1 \geq 0$. Let further $\xi \in [0,\xi_1]$ and $(c_1,c_3,c_4) \in \mathbb{C}^3$, such that $w = (u,v) = w_{c_1,0,c_3,c_4}(\xi,\cdot)$ is either even or odd. Then, with a bound only dependant on ξ_1:*

$$|c_1| \leq C_{\xi_1}(|c_3| + |c_4|). \tag{8.16}$$

Proof. Assume w to be even and assume the lemma does not hold.

We find sequences $(\xi^{(n)})_{n\in\mathbb{N}} \subseteq [0,\xi_1]$ and $(c_3^{(n)})_{n\in\mathbb{N}}$, $(c_4^{(n)})_{n\in\mathbb{N}} \subseteq \mathbb{C}$, such that:

1. $w_{1,0,c_3^{(n)},c_4^{(n)}}(\xi^{(n)},\cdot)$ is even for every $n \geq 0$,

2. $\lim_{n\to\infty} c_3^{(n)} = \lim_{n\to\infty} c_4^{(n)} = 0$.

$w_{1,0,c_3^{(n)},c_4^{(n)}}(\xi^{(n)},\cdot)$ being even implies $\frac{d}{dx} w_{1,0,c_3^{(n)},c_4^{(n)}}(\xi^{(n)},0) = (0,0)$. By choosing a suitable subsequence, we find $\xi_2 \in [0,\xi_1]$, with

$$\lim_{n\to\infty} \xi^{(n)} = \xi_2. \tag{8.17}$$

It follows by lemma 2.4.11 and corollary 8.1.3:

$$\begin{aligned}
\left|\frac{d}{dx} w_1(\xi_2,0)\right| &= \lim_{n\to\infty} \left|\frac{d}{dx} w_{1,0,0,0}(\xi^{(n)},0)\right| \\
&\leq \lim_{n\to\infty} \left|\frac{d}{dx} w_{1,0,c_3^{(n)},c_4^{(n)}}(\xi^{(n)},0)\right| + \lim_{n\to\infty} \left|\frac{d}{dx} w_{0,0,c_3^{(n)},c_4^{(n)}}(\xi^{(n)},0)\right| \\
&\leq \lim_{n\to\infty} \left|\frac{d}{dx} w_{1,0,c_3^{(n)},c_4^{(n)}}(\xi^{(n)},0)\right| + C \lim_{n\to\infty} \left(\left|c_3^{(n)}\right| + \left|c_4^{(n)}\right|\right) \\
&= 0.
\end{aligned} \tag{8.18}$$

Hence $w_1(\xi_2,\cdot)$ is even. Theorem 2.4.7 and lemma 2.4.9 imply $w_1(\xi_2,\cdot) \in H^1(\mathbb{R})$ in contradiction to theorem 1.10.1. That concludes the proof if w is even.

The odd case follows analogously. $\square$

Corollary 8.1.7 *Let $\xi \geq 0$ and $(c_1,c_3,c_4) \in \mathbb{C}^3$, such that $w = (u,v) = w_{c_1,0,c_3,c_4}(\xi,\cdot)$ is either even or odd. Then, for every $x \leq 0$:*

$$|u(x)| + \left|v(x) - c_3\cos(\xi x) - c_4\frac{\xi+1}{\xi}\sin(\xi x)\right| \leq C\frac{e^{\frac{1}{2}x}}{1+\xi}(|c_3| + |c_4|), \tag{8.19}$$

$$\left|\frac{d}{dx}u(x)\right| + \left|\frac{d}{dx}\left(v(x) - c_3\cos(\xi x) - c_4\frac{\xi+1}{\xi}\sin(\xi x)\right)\right| \leq Ce^{\frac{1}{2}x}(|c_3| + |c_4|). \tag{8.20}$$

Proof. Let $\xi_0 \geq 0$ and $\omega < 0$ be given by definition 2.4.6. By corollary 8.1.5, for $\xi \geq \xi_0$:

$$|w(x)| \leq C(|c_3| + |c_4|). \tag{8.21}$$

On the other hand, by lemma 8.1.6 for $0 \leq \xi \leq \xi_0$:

$$|c_1| \leq C(|c_3| + |c_4|). \tag{8.22}$$

By theorem 2.4.7, for $\xi \geq 0$ and $x < \omega$:

$$\begin{aligned} |w(x)| &\leq C\left(|c_1| + |c_3| + |c_4| \left(1 + \left|\frac{\sin(\xi x)}{\xi}\right|\right)\right) \\ &\leq C^2(|c_3| + |c_4|\,(1 + |x|)). \end{aligned} \tag{8.23}$$

Lemma 2.4.11 ensures that w_1, w_3 and w_4 are bounded for $(\xi, x) \in [0, \xi_0] \times [\omega, 0]$. It follows:

$$\begin{aligned} |w(x)| \leq\ & |c_1|\,|w_1(\xi, x)| + |c_3|\,|w_3(\xi, x)| + |c_4|\,|w_4(\xi, x)| \\ \leq\ & C(|c_3| + |c_4|). \end{aligned} \tag{8.24}$$

Summarizing, for $\xi \geq 0$ and $x \leq 0$:

$$|w(x)| \leq C(|c_3| + |c_4|\,(1 + |x|)). \tag{8.25}$$

As w is even or odd, (8.25) holds for true for every $x \in \mathbb{R}$. (8.25) applied to lemma 8.1.4 yields the claim for $x \leq 0$ and consequently $x \in \mathbb{R}$. $\square$

8.2. Transmission Matrix

The fundamental systems $(w_k)_k$ and $(\tilde{w}_k)_k$ describe the asymptotic behaviour of solutions as $x \to -\infty$. As (2.5) and (2.6) are symmetric with respect to $x \mapsto -x$, these fundamental systems can also be used to describe the asymptotic behaviour as $x \to \infty$.

8.2.1. Small x

Connecting the asymptotic behaviour at $-\infty$ and $+\infty$, we define the following transmission matrix S, which is quite similar to a scattering matrix.

Definition 8.2.1 *For $\xi \geq 0$, let $S = S(\xi) = (s_{nm})_{1 \leq n,m \leq 4}$:*

$$\begin{aligned} \mathbb{C}^4 &\to \mathbb{C}^4, \\ (k_1, k_2, k_3, k_4) &\mapsto (l_1, l_2, l_3, l_4), \end{aligned}$$

be given by:

$$\forall x \in \mathbb{R} : w_{k_1,k_2,k_3,k_4}(\xi, x) = w_{l_1,l_2,l_3,l_4}(\xi, -x). \tag{8.26}$$

Here, w_{k_1,k_2,k_3,k_4} is given by definition 8.1.1.

Due to symmetry, the following basic properties are immediately apparent.

Lemma 8.2.2 *Let $\xi \geq 0$. Then, $S(\xi) \in \mathbb{R}^{4\times 4}$ and $S(\xi)^2 = \mathrm{Id}$.*

Lemma 8.2.3 *Let and $\xi \geq 0$. Then, the eigenvalues of $S(\xi)$ are given by -1 and 1, both of geometric multiplicity 2.*

Proof. Given $(k_1, k_2, k_3, k_4) \in \mathbb{R}^4$, $w_{k_1,k_2,k_3,k_4}(\xi, \cdot)$ is even, if and only if:

$$S(\xi)(k_1, k_2, k_3, k_4) = (k_1, k_2, k_3, k_4). \tag{8.27}$$

Analogously, $(k_1, k_2, k_3, k_4) \in \mathbb{R}^4$ is an eigenvector of $S(\xi)$ with eigenvalue -1, if and only if $w_{k_1,k_2,k_3,k_4}(\xi, \cdot)$ is odd. That concludes the proof. ☐

Lemma 8.2.4 *$S : [0, \infty) \to \mathbb{R}^{4\times 4}$ is an analytic function.*

Proof. Let $(c_1, c_2, c_3, c_4) \in \mathbb{R}^4$ and $E = \xi^2 + 1$. We define $W_{c_1,c_2,c_3,c_4}(\xi, \cdot)$ as the unique solution (U, V) of (2.5) and (2.6) given initial data $(U, U', V, V')(0) = (c_1, c_2, c_3, c_4)$.

In analogy to S, we define $T(\xi) : (k_1, k_2, k_3, k_4) \mapsto (c_1, c_2, c_3, c_4)$ as given by:

$$\forall x \in \mathbb{R} : w_{k_1,k_2,k_3,k_4}(\xi, x) = W_{c_1,c_2,c_3,c_4}(\xi, x). \tag{8.28}$$

By lemma 2.4.11, $T : [0, \infty) \to \mathbb{R}^{4\times 4}$ is an analytic function. By definition:

$$S(\xi) = T^{-1}(\xi) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} T(\xi). \tag{8.29}$$

That concludes the proof. ☐

8.2.2. Large ξ

Definition 8.2.5 *For $\xi \geq \xi_0$, let $\tilde{S} = \tilde{S}(\xi) = (\tilde{s}_{nm})_{1\leq n,m\leq 4}$:*

$$\mathbb{C}^4 \to \mathbb{C}^4,$$
$$(k_1, k_2, k_3, k_4) \mapsto (l_1, l_2, l_3, l_4),$$

be given by:

$$\forall x \in \mathbb{R} : \tilde{w}_{k_1,k_2,k_3,k_4}(\xi, x) = \tilde{w}_{l_1,l_2,l_3,l_4}(\xi, -x). \tag{8.30}$$

Here, $\tilde{w}_{k_1,k_2,k_3,k_4}$ is given by definition 8.1.1. ξ_0 is given by definition 2.4.6.

Completely analogously to the last section, we infer the following three lemmata.

Lemma 8.2.6 *Let $\xi \geq \xi_0$. Then, $\tilde{S}(\xi) \in \mathbb{R}^{4\times 4}$ and $\tilde{S}(\xi)^2 = \mathrm{Id}$.*

Lemma 8.2.7 *Let $\xi \geq \xi_0$. Then, the eigenvalues of $\tilde{S}(\xi)$ are given by -1 and 1, both of geometric multiplicity 2.*

Lemma 8.2.8 *$\tilde{S} : [\xi_0, \infty) \to \mathbb{R}^{4\times 4}$ is an analytic function.*

We show some asymptotic behaviour of $\tilde{S}$. Consider the following operator $\tilde{T}$, in analogy to the operator T defined in the proof of lemma 8.2.4.

Definition 8.2.9 *For $\xi \geq \xi_0$, let $\tilde{T} = (\tilde{t}_{nm})_{1\leq n,m\leq 4} = \tilde{T}(\xi)$:*

$$\mathbb{C}^4 \to \mathbb{C}^4,$$
$$(k_1, k_2, k_3, k_4) \mapsto (l_1, l_2, l_3, l_4),$$

be given by:

$$\forall x \in \mathbb{R} : \tilde{w}_{k_1,k_2,k_3,k_4}(\xi, x) = W_{l_1,l_2,l_3,l_4}(\xi, -x). \tag{8.31}$$

Hereby, $W_{l_1,l_2,l_3,l_4}(\xi, \cdot) = (U, V)(\xi, \cdot)$ is the unique solution to the pair of equations:

$$\begin{aligned} U(\xi, x) = {} & l_1 e^{\sqrt{2+\xi^2}x} + l_2 e^{-\sqrt{2+\xi^2}x} \\ & + \frac{1}{2\sqrt{2+\xi^2}} \int_0^x e^{\sqrt{2+\xi^2}(x-y)} Q^{p-1} \left(\frac{p+1}{2} U + \frac{p-1}{2} V \right) dy \\ & - \frac{1}{2\sqrt{2+\xi^2}} \int_0^x e^{\sqrt{2+\xi^2}(y-x)} Q^{p-1} \left(\frac{p+1}{2} U + \frac{p-1}{2} V \right) dy, \end{aligned} \tag{8.32}$$

$$\begin{aligned} V(\xi, x) = {} & c_3 \cos(\xi x) + c_4 \frac{1+\xi}{\xi} \sin(\xi x) \\ & + \int_0^x \frac{\sin(\xi(x-y))}{\xi} Q^{p-1} \left(\frac{p+1}{2} V + \frac{p-1}{2} U \right) dy. \end{aligned} \tag{8.33}$$

Lemma 8.2.10 *There is $\tilde{T}_\infty \in \mathbb{R}^{4\times 4}$, such that $\tilde{T}$ fulfils:*

$$\tilde{T}(\xi) = \operatorname{Id} + \xi^{-1}\tilde{T}_\infty + \mathcal{O}(\xi^{-2}), \tag{8.34}$$

as $\xi \to \infty$.

Proof. We show (8.34) for $\tilde{t}_{11}$. The other elements of the matrix are estimated analogously.

Let $\tilde{w}_1 = (\tilde{u}_1, \tilde{v}_1) = \tilde{w}_{1,0,0,0}$ be given by theorem 2.4.12. By definition:

$$\begin{aligned}
\tilde{u}_1(\xi, x) &= e^{\sqrt{2+\xi^2}x} + \frac{1}{2\sqrt{2+\xi^2}} \int_{-\infty}^{x} e^{\sqrt{2+\xi^2}(x-y)} Q^{p-1} \left(\frac{p+1}{2}\tilde{u}_1 + \frac{p-1}{2}\tilde{v}_1 \right) dy \\
&\quad - \frac{1}{2\sqrt{2+\xi^2}} \int_{-\infty}^{x} e^{\sqrt{2+\xi^2}(y-x)} Q^{p-1} \left(\frac{p+1}{2}\tilde{u}_1 + \frac{p-1}{2}\tilde{v}_1 \right) dy \\
&= e^{\sqrt{2+\xi^2}x} \left(1 + \frac{1}{2\sqrt{2+\xi^2}} \int_{-\infty}^{0} e^{-\sqrt{2+\xi^2}y} Q^{p-1} \left(\frac{p+1}{2}\tilde{u}_1 + \frac{p-1}{2}\tilde{v}_1 \right) dy \right) \\
&\quad - e^{-\sqrt{2+\xi^2}x} \frac{1}{2\sqrt{2+\xi^2}} \int_{-\infty}^{0} e^{\sqrt{2+\xi^2}y} Q^{p-1} \left(\frac{p+1}{2}\tilde{u}_1 + \frac{p-1}{2}\tilde{v}_1 \right) dy \\
&\quad + \frac{1}{2\sqrt{2+\xi^2}} \int_{0}^{x} e^{\sqrt{2+\xi^2}(x-y)} Q^{p-1} \left(\frac{p+1}{2}\tilde{u}_1 + \frac{p-1}{2}\tilde{v}_1 \right) dy \\
&\quad - \frac{1}{2\sqrt{2+\xi^2}} \int_{0}^{x} e^{\sqrt{2+\xi^2}(y-x)} Q^{p-1} \left(\frac{p+1}{2}\tilde{u}_1 + \frac{p-1}{2}\tilde{v}_1 \right) dy,
\end{aligned} \tag{8.35}$$

$$\begin{aligned}
\tilde{v}_1(\xi, x) &= \int_{-\infty}^{x} \frac{\sin(\xi(x-y))}{\xi} Q^{p-1} \left(\frac{p+1}{2}\tilde{v}_1 + \frac{p-1}{2}\tilde{u}_1 \right) dy \\
&= -\cos(\xi x) \int_{-\infty}^{0} \frac{\sin(\xi y)}{\xi} Q^{p-1} \left(\frac{p+1}{2}\tilde{v}_1 + \frac{p-1}{2}\tilde{u}_1 \right) dy \\
&\quad + \sin(\xi x) \int_{-\infty}^{0} \frac{\cos(\xi y)}{\xi} Q^{p-1} \left(\frac{p+1}{2}\tilde{v}_1 + \frac{p-1}{2}\tilde{u}_1 \right) dy \\
&\quad + \int_{0}^{x} \frac{\sin(\xi(x-y))}{\xi} Q^{p-1} \left(\frac{p+1}{2}\tilde{v}_1 + \frac{p-1}{2}\tilde{u}_1 \right) dy.
\end{aligned} \tag{8.36}$$

It follows:

$$\tilde{t}_{11}(\xi) = 1 + \frac{1}{2\sqrt{2+\xi^2}} \int_{-\infty}^{0} e^{-\sqrt{2+\xi^2}y} Q^{p-1} \left(\frac{p+1}{2}\tilde{u}_1 + \frac{p-1}{2}\tilde{v}_1 \right) dy. \tag{8.37}$$

By theorem 2.4.12:

$$\begin{aligned}
\tilde{t}_{11}(\xi) &= 1 + \frac{p+1}{4\sqrt{2+\xi^2}} \int_{-\infty}^{0} e^{-\sqrt{2+\xi^2}y} Q^{p-1} e^{\sqrt{2+\xi^2}y} dy \\
&\quad + \frac{1}{2\sqrt{2+\xi^2}} \int_{-\infty}^{0} e^{-\sqrt{2+\xi^2}y} Q^{p-1} \left(\frac{p+1}{2}(\tilde{u}_1 - e^{\sqrt{2+\xi^2}y}) + \frac{p-1}{2}\tilde{v}_1 \right) dy \\
&= 1 + \frac{p+1}{4\sqrt{2+\xi^2}} \int_{-\infty}^{0} Q^{p-1} dy + \mathcal{O}(\xi^{-2}).
\end{aligned} \tag{8.38}$$

That concludes the proof. □

Lemma 8.2.11 *Let $k \geq 1$. Then, as $\xi \to \infty$:*

$$\frac{d^k}{d\xi^k}\tilde{T}(\xi) \in \mathcal{O}(\xi^{-2}). \tag{8.39}$$

Proof. Taking the k-th derivative of (8.37) and applying lemma 2.4.15 yields (8.39) for $\tilde{t}_{11}$. The other elements of the matrix are estimated analogously. □

Corollary 8.2.12 *There is $\tilde{S}_\infty \in \mathbb{R}^{4\times 4}$, such that $\tilde{S}$ fulfils:*

$$\tilde{S}(\xi) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} + \xi^{-1}\tilde{S}_\infty + \mathcal{O}(\xi^{-2}), \tag{8.40}$$

as $\xi \to \infty$.

Proof. Corollary 8.2.12 follows from lemma 8.2.10 and:

$$\tilde{S}(\xi) = \tilde{T}(\xi)^{-1} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tilde{T}(\xi). \tag{8.41}$$

□

Corollary 8.2.13 *Let $k \geq 1$. Then, as $\xi \to \infty$:*

$$\frac{d^k}{d\xi^k}\tilde{S}(\xi) \in \mathcal{O}(\xi^{-2}). \tag{8.42}$$

Proof. Corollary 8.2.13 follows from lemma 8.2.10, lemma 8.2.11 and:

$$\tilde{S}(\xi) = \tilde{T}(\xi)^{-1} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tilde{T}(\xi). \tag{8.43}$$

□

8.3. Constructing Symmetrical Solutions

Consider $\xi > 0$ and $E = 1 + \xi^2$. $w_{l,b,c,s}(\xi, x)$ is bounded and even with respect to x, if and only if $b = 0$ and:

$$S(\xi)(l, b, c, s) = (l, b, c, s). \tag{8.44}$$

By the same token, $w_{l,b,c,s}(\xi, x)$ is bounded and odd with respect to x, if and only if $b = 0$ and:

$$S(\xi)(l, b, c, s) = -(l, b, c, s). \tag{8.45}$$

In case of $\xi = 0$, theorem 1.10.1 tells us that no non-zero bounded solution of (2.5) and (2.6) exists.

Definition 8.3.1 *Let $\xi \geq 0$ and $E = \xi^2 + 1$. We define:*

$$K_{e,\xi} := \{(l, 0, c, s) \in \mathbb{R}^4 | (S(\xi) - \mathrm{Id})(l, 0, c, s) = (0, 0, 0, 0)\}, \tag{8.46}$$
$$K_{o,\xi} := \{(l, 0, c, s) \in \mathbb{R}^4 | (S(\xi) + \mathrm{Id})(l, 0, c, s) = (0, 0, 0, 0)\}. \tag{8.47}$$

Let further $K_\xi := K_{e,\xi} \oplus K_{o,\xi}$.

Lemma 8.3.2 *For every $\xi \geq 0$ with at most finitely many exceptions:*

$$\dim K_{e,\xi} = \dim K_{o,\xi} = 1. \tag{8.48}$$

Proof. By lemma 8.2.3:

$$1 \leq \dim K_{e,\xi}, \dim K_{o,\xi} \leq 2. \tag{8.49}$$

Consequently, it suffices to show $\dim K_\xi \leq 2$ for almost every $\xi \geq 0$. By definition:

$$K_\xi = \{(l, 0, c, s) \in \mathbb{R}^4 | \exists \tilde{l}, \tilde{c}, \tilde{s} \in \mathbb{C} : S(\xi)(l, 0, c, s) = (\tilde{l}, 0, \tilde{c}, \tilde{s})\}, \tag{8.50}$$

Further, $(l, b, c, s) \in K_\xi$, if and only if $w_{l,b,c,s}(\xi, x)$ is bounded with respect to x.

Let ξ_0 be given by definition 2.4.6. Analogously to before, given $\xi \geq \xi_0$, $\tilde{w}_{l,b,c,s}(\xi, x)$ is bounded with respect to x, if and only if:

$$(l, b, c, s) \in \tilde{K}_\xi := \{(l, 0, c, s) \in \mathbb{R}^4 | \exists \tilde{l}, \tilde{c}, \tilde{s} \in \mathbb{C} : \tilde{S}(\xi)(l, 0, c, s) = (\tilde{l}, 0, \tilde{c}, \tilde{s})\}. \tag{8.51}$$

That implies $\dim K_\xi = \dim \tilde{K}_\xi$ for every $\xi \geq \xi_0$. Consequently, for $\xi \geq \xi_0$, $\dim K_\xi = 3$ is equivalent to $\tilde{s}_{21}(\xi) = \tilde{s}_{23}(\xi) = \tilde{s}_{24}(\xi) = 0$.

From corollary 8.2.12, we know $\lim_{\xi\to\infty} \tilde{s}_{21}(\xi) = 1$. That allows us to choose $\xi_1 \geq \xi_0$ with $\tilde{s}_{21}(\xi) \neq 0$ for every $\xi \geq \xi_1$. Consequently, $\dim K_\xi = 3$ is only possible for $\xi \in [0, \xi_1]$.

By definition, $\dim K_\xi = 3$ is equivalent to $s_{21}(\xi) = s_{23}(\xi) = s_{24}(\xi) = 0$. That concludes the proof, as S is analytic by lemma 8.2.4. $\square$

Lemma 8.3.3 *Let $\xi_1 \geq 0$. Then, there exist $\varepsilon > 0$ and real analytic functions:*

$$(l_e, c_e, s_e) : [0, \infty) \cap (\xi_1 - \varepsilon, \xi_1 + \varepsilon) \to \mathbb{R}^3 \backslash \{(0,0,0\}, \tag{8.52}$$
$$(l_o, c_o, s_o) : [0, \infty) \cap (\xi_1 - \varepsilon, \xi_1 + \varepsilon) \to \mathbb{R}^3 \backslash \{(0,0,0\}, \tag{8.53}$$

fulfilling:

$$(l_e, 0, c_e, s_e)(\xi) \in K_{e,\xi}, \tag{8.54}$$
$$(l_o, 0, c_o, s_o)(\xi) \in K_{o,\xi}, \tag{8.55}$$

for every $\xi \geq 0$ with $|\xi - \xi_1| < \varepsilon$.

Proof. We show the lemma for l_e, c_e and s_e. l_o, c_o and s_o are constructed completely analogously.

Lemma 8.2.3 allows to us find linear independent (L_1, B_1, C_1, S_1), $(L_2, B_2, C_2, S_2) \in \mathbb{R}^4$ fulfilling:

$$(S(\xi_1) - \mathrm{Id})(l, b, c, s) = (0, 0, 0, 0). \tag{8.56}$$

That implies:

$$(S(\xi_1) + \mathrm{Id})(L_1, B_1, C_1, S_1) = 2(L_1, B_1, C_1, S_1), \tag{8.57}$$
$$(S(\xi_1) + \mathrm{Id})(L_2, B_2, C_2, S_2) = 2(L_2, B_2, C_2, S_2). \tag{8.58}$$

We define:

$$(l_1, b_1, c_1, s_1)(\xi) := (S(\xi) + \mathrm{Id})(L_1, B_1, C_1, S_1), \tag{8.59}$$
$$(l_2, b_2, c_2, s_2)(\xi) := (S(\xi) + \mathrm{Id})(L_2, B_2, C_2, S_2). \tag{8.60}$$

As S is continuous, we can choose $\delta > 0$, such that $(l_1, b_1, c_1, s_1)(\xi)$ and $(l_2, b_2, c_2, s_2)(\xi)$ are linear independent for every $\xi \geq 0$ with $|\xi - \xi_1| < \delta$.

Lemma 8.2.2 implies $S(\xi)^2 = \mathrm{Id}$ and thus:

$$(S(\xi) + \mathrm{Id})(S(\xi) - \mathrm{Id}) = (0, 0, 0, 0). \tag{8.61}$$

Hence, $(l_1, b_1, c_1, s_1)(\xi)$ and $(l_2, b_2, c_2, s_2)(\xi)$ both fulfil:

$$(S(\xi) - \mathrm{Id})(l, b, c, s) = (0, 0, 0, 0). \tag{8.62}$$

We conclude $(l_1, b_1, c_1, s_1)(\xi) \in K_{e,\xi}$, if and only if $b_1(\xi) = 0$. Analogously, $(l_2, b_2, c_2, s_2)(\xi) \in K_{e,\xi}$, if and only if $b_2(\xi) = 0$.

If $b_1 \equiv 0$ or $b_2 \equiv 0$, then we are already done. Otherwise, since b_1 and b_2 are both real analytic by lemma 8.2.4, there exist real analytic functions $g_1, g_2 : [0, \infty) \cap (\xi_1 - \delta, \xi_1 + \delta) \to \mathbb{R}$, as well as $k_1, k_2 \in \mathbb{N}$, fulfilling:

$$b_1(\xi) = (\xi - \xi_1)^{k_1} g_1(\xi), \qquad g_1(\xi_1) \neq 0, \tag{8.63}$$
$$b_2(\xi) = (\xi - \xi_1)^{k_2} g_2(\xi), \qquad g_2(\xi_1) \neq 0. \tag{8.64}$$

By choosing $0 < \varepsilon < \delta$ small enough, we ensure $g_1(\xi) \neq 0$ and $g_2(\xi) \neq 0$ for $|\xi - \xi_1| < \varepsilon$. Symmetry allows us to assume $k_1 \geq k_2$. We define:

$$(l_e, b_e, c_e, s_e)(\xi) := (\xi - \xi_1)^{k_2 - k_1} g_2(\xi)(l_1, b_1, c_1, s_1)(\xi) - g_1(\xi)(l_2, b_2, c_2, s_2)(\xi). \tag{8.65}$$

It follows:

$$(S(\xi) - \mathrm{Id})(l_e, b_e, c_e, s_e)(\xi) = 0 \tag{8.66}$$

and:

$$b_e(\xi) = (\xi - \xi_1)^{k_2 - k_1} g_2(\xi)(\xi - \xi_1)^{k_1} g_1(\xi) - g_1(\xi)(\xi - \xi_1)^{k_2} g_2(\xi) = 0. \tag{8.67}$$

That implies $(l_e, 0, c_e, s_e)(\xi) \in K_{e,\xi}$ for every $|\xi - \xi_1| < \varepsilon$, $\xi \geq 0$. □

Lemma 8.3.4 *There exist analytic functions:*

$$(l_e, c_e, s_e) : [0, \infty) \to \mathbb{R}^3, \tag{8.68}$$
$$(l_o, c_o, s_o) : [0, \infty) \to \mathbb{R}^3, \tag{8.69}$$

fulfilling for every $\xi \geq 0$:

$$(l_e, 0, c_e, s_e)(\xi) \in K_{e,\xi}, \tag{8.70}$$
$$(l_o, 0, c_o, s_o)(\xi) \in K_{o,\xi}, \tag{8.71}$$

as well as:

$$l_e(\xi)^2 + c_e(\xi)^2 + s_e(\xi)^2 = l_o(\xi)^2 + c_o(\xi)^2 + s_o(\xi)^2 = 1. \tag{8.72}$$

Proof. We show the lemma for l_e, c_e and s_e. l_o, c_o and s_o are constructed completely analogously.

By lemma 8.3.3, there is some $\delta > 0$ and real analytic functions $(\tilde{l}, \tilde{c}, \tilde{s}) : [0, \delta) \to \mathbb{R}^3 \backslash \{(0,0,0\}$ with $(\tilde{l}, 0, \tilde{c}, \tilde{s})(\xi) \in K_{e,\xi}$ for $0 \leq \xi < \delta$. We define:

$$\begin{aligned} (l, c, s) : [0, \delta) &\to \mathbb{R}^3, \\ \xi &\mapsto \frac{(\tilde{l}, \tilde{c}, \tilde{s})(\xi)}{\tilde{l}(\xi)^2 + \tilde{c}(\xi)^2 + \tilde{s}(\xi)^2}. \end{aligned} \tag{8.73}$$

That yields real analytic functions $(l, c, s) : [0, \delta) \to \mathbb{R}^3$ fulfilling:

$$(\mathrm{Id} - S(\xi))(l, 0, c, s)(\xi) = (0, 0, 0, 0), \tag{8.74}$$
$$l(\xi)^2 + c(\xi)^2 + s(\xi)^2 = 1 \tag{8.75}$$

for every $0 \leq \xi < \delta$. Consider:

$$\zeta := \sup\{\xi_1 \geq 0 \mid l, c \text{ and } s \text{ extend analytically to } [0, \xi_1]\} \geq \delta. \tag{8.76}$$

Recall that an analytic extension on an interval is always unique. For given $\xi_1 \in [0, \zeta)$, we denote the unique analytic extension to $[0, \xi_1]$ by $(l_{\xi_1}, c_{\xi_1}, s_{\xi_1})$. Then:

$$(l_e, c_e, s_e) : [0, \zeta) \to \mathbb{R}^3, \tag{8.77}$$

$$\xi \mapsto (l_\xi, c_\xi, s_\xi)(\xi), \tag{8.78}$$

constitutes an analytic extension of (l, c, s) to $[0, \zeta)$.

By the identity theorem for analytic functions, (l_e, c_e, s_e) fulfils (8.74) and (8.75) for every $\xi \in [0, \zeta)$.

Therefore, to conclude the proof, it suffices to show $\zeta = \infty$. We proceed indirectly.

Assume $\zeta < \infty$. Using lemma 8.3.3, we can choose $\varepsilon > 0$ and analytic functions $(\tilde{l}_1, \tilde{c}_1, \tilde{s}_1) : (\zeta - \varepsilon, \zeta + \varepsilon) \to \mathbb{R}^3 \backslash \{(0, 0, 0\}$ fulfilling $(\tilde{l}_1, \tilde{c}_1, \tilde{s}_1)(\xi) \in K_{e,\xi}$ for $|\xi - \zeta| < \varepsilon$.

By choosing ε small enough, lemma 8.3.2 ensures for $\xi \in (\zeta - \varepsilon, \zeta)$:

$$\dim K_{e,\xi} = 1. \tag{8.79}$$

We define:

$$(l_1, c_1, s_1) : (\zeta - \varepsilon, \zeta + \varepsilon) \to \mathbb{R}^3,$$

$$\xi \mapsto \frac{(\tilde{l}_1, \tilde{c}_1, \tilde{s}_1)(\xi)}{\tilde{l}_1(\xi)^2 + \tilde{c}_1(\xi)^2 + \tilde{s}_1(\xi)^2}. \tag{8.80}$$

Both $(l_e, 0, c_e, s_e)$ and $(l_1, 0, c_1, s_1)$ fulfil (8.74) and (8.75) on $(\zeta - \varepsilon, \zeta)$. By (8.79), that implies for every $\xi \in (\zeta - \varepsilon, \xi_1)$:

$$(l_e, 0, c_e, s_e)(\xi) = (l_1, 0, c_1, s_1)(\xi) \vee (l_e, 0, c_e, s_e)(\xi) = -(l_1, 0, c_1, s_1)(\xi). \tag{8.81}$$

By continuity, the sign in (8.81) remains constant. We can assume without loss of generality for $\xi \in (\zeta - \varepsilon, \zeta)$:

$$(l_e, 0, c_e, s_e)(\xi) = (l_1, 0, c_1, s_1)(\xi). \tag{8.82}$$

Hence:

$$(l_2, c_2, s_2) : [0, \zeta + \varepsilon) \to \mathbb{R}^3, \tag{8.83}$$

$$\xi \mapsto \begin{cases} (l_e, c_e, s_e)(\xi), & \text{if } \xi < \zeta, \\ (l_1, c_1, s_1)(\xi), & \text{if } \xi \geq \zeta, \end{cases} \tag{8.84}$$

constitutes an analytic continuation of (l, c, s). That contradicts the definition of ζ. We conclude $\zeta = \infty$ and the proof. □

Theorem 8.3.5 *There exist real analytic functions $l_e, c_e, s_e : [0, \infty) \to \mathbb{R}$ and $l_o, c_o, s_o : [0, \infty) \to \mathbb{R}$, such that $w_e, w_o : [0, \infty) \times \mathbb{R} \to \mathbb{R}^2$ given by:*

$$w_e(\xi, x) := w_{l_e(\xi), 0, c_e(\xi), -s_e(\xi)\frac{\xi}{\xi+1}}(\xi, x), \tag{8.85}$$

$$w_o(\xi, x) := w_{l_o(\xi), 0, c_o(\xi), -s_o(\xi)\frac{\xi}{\xi+1}}(\xi, x), \tag{8.86}$$

fulfil:

1. *For every* $\xi \geq 0$, $w_e(\xi, x)$ *is even with respect to* x.
2. *For every* $\xi \geq 0$, $w_o(\xi, x)$ *is odd with respect to* x.
3. w_e *and* w_o *can be expressed in terms of* $\tilde{w}$:

$$\forall \xi \geq \xi_0 \ \forall x \in \mathbb{R} : w_e(\xi, x) = \tilde{w}_{0,0,c_e(\xi),-s_e(\xi)\frac{\xi}{\xi+1}}(\xi, x), \tag{8.87}$$

$$w_o(\xi, x) = \tilde{w}_{0,0,c_o(\xi),-s_o(\xi)\frac{\xi}{\xi+1}}(\xi, x). \tag{8.88}$$

   ξ_0 *is given by definition 2.4.6.*

4. c_e *and* s_e *resp.* c_o *and* s_o *are normed:*

$$\forall \xi \geq 0 : c_e(\xi)^2 + s_e(\xi)^2 = c_o(\xi)^2 + s_o(\xi)^2 = 1. \tag{8.89}$$

5. *As* $\xi \to \infty$, c_e, s_e *and* c_o, s_o *fulfil:*

$$(c_e, s_e)(\xi) = (1, 0) + \xi^{-1}(c_{e,0}, s_{e,0}) + \mathcal{O}(\xi^{-2}), \tag{8.90}$$

$$(c_o, s_o)(\xi) = (0, 1) + \xi^{-1}(c_{o,0}, s_{o,0}) + \mathcal{O}(\xi^{-2}), \tag{8.91}$$

   with suitable constants $c_{e,0}, s_{e,0}, c_{o,0}, s_{o,0} \in \mathbb{R}$.

6. *As* $\xi \to \infty$, c_e, s_e *and* c_o, s_o *fulfil for every* $k \geq 1$:

$$\frac{d^k}{d\xi^k}(c_e, s_e)(\xi) \in \mathcal{O}(\xi^{-2}), \tag{8.92}$$

$$\frac{d^k}{d\xi^k}(c_o, s_o)(\xi) \in \mathcal{O}(\xi^{-2}). \tag{8.93}$$

Proof. We show the lemma for l_e, c_e and s_e. l_o, c_o and s_o are constructed completely analogously.

Let (l, c, s) be given by lemma 8.3.4. Then $w_{l(\xi),0,c(\xi),s(\xi)}(\xi, \cdot)$ is a non-trivial bounded and even solution to (2.5) and (2.6) for every $\xi > 0$ and $E = \xi^2 + 1$. Corollary 8.1.5 and lemma 8.1.6 imply $(c, s)(\xi) \neq (0, 0)$ for every $\xi > 0$.

On the other hand, $w_{l(0),0,c(0),s(0)}(0, \cdot)$ is bounded, if and only if $s(0) = 0$. Theorem 1.10.1 consequently ensures $s(0) \neq 0$. Hence, $l_e, c_e, s_e : [0, \infty) \to \mathbb{R}$ as given by:

$$l_e(\xi) := \frac{\xi l(\xi)}{\sqrt{\xi^2 c(\xi)^2 + (1+\xi)^2 s(\xi)}}, \tag{8.94}$$

$$c_e(\xi) := \frac{\xi c(\xi)}{\sqrt{\xi^2 c(\xi)^2 + (1+\xi)^2 s(\xi)}}, \tag{8.95}$$

$$s_e(\xi) := \frac{-(1+\xi) s(\xi)}{\sqrt{\xi^2 c(\xi)^2 + (1+\xi)^2 s(\xi)}}, \tag{8.96}$$

are well-defined and real analytic functions on $[0, \infty)$. (8.87) follows from lemma 8.1.4.

It remains to demonstrate (8.90) and (8.92). From (8.87), we can infer for every $\xi \geq \xi_0$:

$$(\tilde{S}(\xi) - \mathrm{Id})\left(0, 0, c_e(\xi), -\frac{\xi s_e(\xi)}{\xi + 1}\right) = (0, 0, 0, 0). \tag{8.97}$$

We define for $\xi \geq \xi_0$:

$$(\tilde{l}, \tilde{b}, \tilde{c}, \tilde{s})(\xi) := (\tilde{S}(\xi) + \mathrm{Id})(0, 0, 1, 0). \tag{8.98}$$

We rewrite (8.98). Thus, for $\xi \geq \xi_0$ and $x \in \mathbb{R}$:

$$\tilde{w}_{\tilde{l},\tilde{b},\tilde{c},\tilde{s}}(\xi, x) = \tilde{w}_{0,0,1,0}(\xi, x) + \tilde{w}_{0,0,1,0}(\xi, -x) = \tilde{w}_3(\xi, x) + \tilde{w}_3(\xi, -x). \tag{8.99}$$

As w_3 is bounded and w_1, w_2 are unbounded by theorem 2.4.12, it follows $\tilde{l}(\xi) = \tilde{b}(\xi) = 0$.

By corollary 8.2.12, there is $\xi_1 \geq \xi_0$ with $(\tilde{c}, \tilde{s})(\xi) \neq (0, 0)$ for every $\xi \geq \xi_1$. If we choose ξ_1 large enough, we ensure $\dim K_{e,\xi} = 1$ by lemma 8.3.2.

Lemma 8.2.6 implies $(\tilde{S}(\xi) - \mathrm{Id})(\tilde{S}(\xi) + \mathrm{Id}) = 0$ and thus:

$$(\tilde{S}(\xi) - \mathrm{Id})(0, 0, \tilde{c}(\xi), \tilde{s}(\xi)) = (0, 0, 0, 0). \tag{8.100}$$

From $\dim K_{e,\xi} = 1$, (8.97) and (8.100), we infer that $(c_e(\xi), -\xi(\xi + 1)^{-1} s_e(\xi))$ and $(\tilde{c}(\xi), \tilde{s}(\xi))$ are linear dependent. Hence, if we define:

$$\tilde{c}_e(\xi) := \frac{\xi \tilde{c}(\xi)}{\xi^2 \tilde{c}(\xi) + (1 + \xi)^2 \tilde{s}(\xi)}, \tag{8.101}$$

$$\tilde{s}_e(\xi) := \frac{-(1 + \xi)\tilde{s}(\xi)}{\xi^2 \tilde{c}(\xi) + (1 + \xi)^2 \tilde{s}(\xi)}, \tag{8.102}$$

for $\xi \geq \xi_0$, we ensure either:

$$\forall \xi \geq \xi_0 : (c_e, s_e)(\xi) = (\tilde{c}_e, \tilde{s}_e)(\xi), \tag{8.103}$$

or:

$$\forall \xi \geq \xi_0 : (c_e, s_e)(\xi) = -(\tilde{c}_e, \tilde{s}_e)(\xi). \tag{8.104}$$

After possibly changing the sign of (l_e, c_e, s_e), we can assume (8.103) to be true. (8.90) and (8.92) now follow from corollary 8.2.12, corollary 8.2.13 and (8.98).

That concludes the proof. $\square$

Definition 8.3.6 *Consider c_e, s_e and c_o, s_o as given by theorem 8.3.5. We extend c_e, s_e and c_o, s_o to functions $\mathbb{R} \to \mathbb{R}$ by:*

$$c_e(\xi) = c_e(|\xi|), \tag{8.105}$$

$$s_e(\xi) = s_e(|\xi|), \tag{8.106}$$

$$c_o(\xi) = c_o(|\xi|), \tag{8.107}$$

$$s_o(\xi) = s_o(|\xi|). \tag{8.108}$$

Corollary 8.3.7 *$c_e, s_e, c_o, s_o \in W^{1,\infty}(\mathbb{R})$. Further, given any $k \geq 0$:*

$$c_e|_{(0,\infty)}, s_e|_{(0,\infty)}, c_o|_{(0,\infty)}, s_o|_{(0,\infty)} \in W^{k,\infty}(0,\infty). \tag{8.109}$$

Lemma 8.3.8 *$l_e(0) = c_e(0) = l_o(0) = c_o(0) = 0$ and $s_e(0)^2 = s_o(0)^2 = 1$.*

Proof. Follows from theorem 1.10.1 and (8.89). □

Corollary 8.3.9 *$s_e'(0) = s_o'(0) = 0$.*

Proof. Follows from (8.89). □

8.4. Remainder Terms

8.4.1. First Order Remainder Terms

Consider the functions $w_e = (u_e, v_e)$, $w_o = (u_o, v_o) : [0,\infty) \times \mathbb{R} \to \mathbb{R}^2$, as given by theorem 8.3.5:

$$w_e(\xi, x) = w_{l_e(\xi), 0, c_e(\xi), -s_e(\xi)\frac{\xi}{\xi+1}}(\xi, x), \tag{8.110}$$

$$w_o(\xi, x) = w_{l_o(\xi), 0, c_o(\xi), -s_o(\xi)\frac{\xi}{\xi+1}}(\xi, x). \tag{8.111}$$

For $\xi \geq 0$ and $x \leq 0$, we establish bounds on the remainder terms $\rho_e = (\rho_{e,u}, \rho_{e,v})$ and $\rho_o = (\rho_{o,u}, \rho_{o,v})$ given by:

$$\rho_{e,u}(\xi, x) := u_e(\xi, x), \tag{8.112}$$

$$\rho_{e,v}(\xi, x) := v_e(\xi, x) - c_e(\xi)\cos(\xi x) + s_e(\xi)\sin(\xi x), \tag{8.113}$$

$$\rho_{o,u}(\xi, x) := u_o(\xi, x), \tag{8.114}$$

$$\rho_{o,v}(\xi, x) := v_o(\xi, x) - c_o(\xi)\cos(\xi x) + s_o(\xi)\sin(\xi x). \tag{8.115}$$

Lemma 8.4.1 *Let w_e, w_o be given by (8.110) and (8.111). Then, $w_e(\xi, x)$ and $w_o(\xi, x)$ are real analytic both in $\xi \geq 0$ and $x \in \mathbb{R}$. Further, $w_e(0, x) = w_o(0, x) = (0, 0)$.*

Proof. Lemma 8.3.8 implies $w_e(0, x) = w_o(0, x) = (0, 0)$. The analyticity follows from lemma 2.4.11 and theorem 8.3.5. □

Lemma 8.4.2 *Let $\rho_e = (\rho_{e,u}, \rho_{e,v})$ and $\rho_o = (\rho_{o,u}, \rho_{o,v})$ be given by (8.112) - (8.115). Then, for $\xi \geq 0$ and $x \leq 0$:*

$$|\rho_{e,u}(\xi, x)| + |\rho_{e,v}(\xi, x)| + |\rho_{o,u}(\xi, x)| + |\rho_{o,v}(\xi, x)| \leq C\frac{e^{\frac{1}{2}x}}{1+\xi}, \tag{8.116}$$

$$|\partial_x \rho_{e,u}(\xi, x)| + |\partial_x \rho_{e,v}(\xi, x)| + |\partial_x \rho_{o,u}(\xi, x)| + |\partial_x \rho_{o,v}(\xi, x)| \leq Ce^{\frac{1}{2}x}. \tag{8.117}$$

Proof. Follows from corollary 8.1.7. □

Lemma 8.4.3 *Let $\rho_e = (\rho_{e,u}, \rho_{e,v})$ and $\rho_o = (\rho_{o,u}, \rho_{o,v})$ be given by* (8.112) - (8.115)*. Then, for $\xi \geq 0$, $x \leq 0$ and $k \geq 0$:*

$$\left|\partial_\xi^k \rho_{e,u}\right| + \left|\partial_\xi^k \rho_{e,v}\right| + \left|\partial_\xi^k \rho_{o,u}\right| + \left|\partial_\xi^k \rho_{o,v}\right| \leq C_k \frac{e^{\frac{1}{2}x}}{1+\xi}. \tag{8.118}$$

Proof. We prove (8.118) for ρ_e. For ρ_o, the proof is identical.

Let ξ_0 be given by definition 2.4.6. Continuity implies for $(\xi, x) \in [0, \xi_0] \times [\omega, 0]$:

$$\left|\frac{d^k}{d\xi^k}\rho_{e,u}(\xi, x)\right| + \left|\frac{d^k}{d\xi^k}\rho_{e,v}(\xi, x)\right| \leq C. \tag{8.119}$$

To conclude the proof, it suffices to show (8.118) for $(\xi, x) \in (\xi_0, \infty) \times (-\infty, 0]$ and $(\xi, x) \in [0, \xi_0] \times (-\infty, \omega)$.

Assume $\xi > \xi_0$ and $x \leq 0$. Recall lemma 2.4.15.

By theorem 8.3.5, $w_e(\xi, x) = c_e(\xi)\tilde{w}_3(\xi, x) - \frac{\xi}{\xi+1}s_e(\xi)\tilde{w}_4(\xi, x)$, which implies:

$$\rho_{e,u}(\xi, x) = c_e(\xi)\tilde{r}_{u,3}(\xi, x) - \frac{\xi}{\xi+1}s_e(\xi)\tilde{r}_{u,4}(\xi, x), \tag{8.120}$$

$$\rho_{e,v}(\xi, x) = c_e(\xi)\tilde{r}_{v,3}(\xi, x) - \frac{\xi}{\xi+1}s_e(\xi)\tilde{r}_{v,4}(\xi, x). \tag{8.121}$$

(8.118) follows from lemma 2.4.15 and corollary 8.3.7.

Assume $0 \leq \xi \leq \xi_0$ and $x < \omega$. Recall lemma 2.4.10.

By theorem 8.3.5, $w_e(\xi, x) = l_e(\xi)w_1(\xi, x) + c_e(\xi)w_3(\xi, x) - \frac{\xi}{\xi+1}s_e(\xi)w_4(\xi, x)$. It follows:

$$\rho_{e,u}(\xi, x) = l_e(\xi)u_1(\xi, x) + c_e(\xi)r_{u,3}(\xi, x) - \frac{\xi}{\xi+1}s_e(\xi)r_{u,4}(\xi, x), \tag{8.122}$$

$$\rho_{e,v}(\xi, x) = l_e(\xi)v_1(\xi, x) + c_e(\xi)r_{v,3}(\xi, x) - \frac{\xi}{\xi+1}s_e(\xi)r_{v,4}(\xi x). \tag{8.123}$$

By continuity, $\frac{d^k}{d\xi^k}l_e$ is bounded on $[0, \xi_0]$. (8.118) follows from lemma 2.4.10 and corollary 8.3.7.

That concludes the proof. $\square$

8.4.2. Second Order Remainder Terms

Lemma 8.4.2 and lemma 8.4.3 estimate the remainder terms with a decay of $\mathcal{O}(\xi^{-1})$. We construct a higher order remainder term exhibiting a decay of $\mathcal{O}(\xi^{-2})$.

We only do the calculations for ρ_e. ρ_o is handled completely analogously.

By lemma 8.1.4:

$$\begin{aligned}
\rho_{e,u} = & -\frac{1}{2\sqrt{2+\xi^2}}\int_x^\infty e^{\sqrt{2+\xi^2}(x-y)}Q^{p-1}\left(\frac{p+1}{2}u_e+\frac{p-1}{2}v_e\right)dy \\
& -\frac{1}{2\sqrt{2+\xi^2}}\int_{-\infty}^x e^{\sqrt{2+\xi^2}(y-x)}Q^{p-1}\left(\frac{p+1}{2}u_e+\frac{p-1}{2}v_e\right)dy, \qquad (8.124)\\
\rho_{e,v} = & \int_{-\infty}^x \frac{\sin(\xi(x-y))}{\xi}Q^{p-1}\left(\frac{p+1}{2}v_e+\frac{p-1}{2}u_e\right)dy. \qquad (8.125)
\end{aligned}$$

Lemma 8.4.2 and lemma 8.4.3 applied to (8.124) already yield stronger bounds for $\rho_{e,u}$:

$$\forall k \geq 0 : \left|\frac{d^k}{d\xi^k}\frac{d}{dx}\rho_{e,u}(\xi,x)\right| \leq C_k\frac{e^{\frac{1}{2}x}}{1+\xi}, \qquad (8.126)$$

$$\left|\frac{d^k}{d\xi^k}\rho_{e,u}(\xi x)\right| \leq C_k\frac{e^{\frac{1}{2}x}}{1+\xi^2}. \qquad (8.127)$$

$\rho_{e,v}$ must be controlled more carefully. We expand (8.125):

$$\begin{aligned}
\rho_{e,v} = & \int_{-\infty}^x \frac{\sin(\xi(x-y))}{\xi}Q^{p-1}\left(\frac{p+1}{2}\rho_{e,v}+\frac{p-1}{2}\rho_{e,u}\right)dy \\
& +\frac{p+1}{2}\int_{-\infty}^x \frac{\sin(\xi(x-y))}{\xi}Q^{p-1}(c_e(\xi)\cos(\xi y)-s_e(\xi)\sin(\xi y))dy \\
= & \int_{-\infty}^x \frac{\sin(\xi(x-y))}{\xi}Q^{p-1}\left(\frac{p+1}{2}\rho_{e,v}+\frac{p-1}{2}\rho_{e,u}\right)dy \\
& +\frac{p+1}{2}\int_{-\infty}^x \frac{\sin(\xi(x-y))}{\xi(1+\xi)}Q^{p-1}(c_e(\xi)\cos(\xi y)-s_e(\xi)\sin(\xi y))dy \\
& +\frac{p+1}{2}\int_{-\infty}^x \frac{\sin(\xi(x-y))}{1+\xi}Q^{p-1}(c_e(\xi)\cos(\xi y)-s_e(\xi)\sin(\xi y))dy. \qquad (8.128)
\end{aligned}$$

Note $\sin(\xi(x-y)) = \sin(\xi x)\cos(\xi y) - \cos(\xi x)\sin(\xi y)$. We conclude:

$$\begin{aligned}
& \int_{-\infty}^x \sin(\xi(x-y))Q^{p-1}(c_e(\xi)\cos(\xi y)-s_e(\xi)\sin(\xi y))dy \\
= & \, c_e(\xi)\sin(\xi x)\int_{-\infty}^x Q^{p-1}\cos(\xi y)^2dy + s_e(\xi)\cos(\xi x)\int_{-\infty}^x Q^{p-1}\sin(\xi y)^2dy \\
& -(c_e(\xi)\cos(\xi x)+s_e(\xi)\sin(\xi x))\int_{-\infty}^x Q^{p-1}\cos(\xi y)\sin(\xi y)dy. \qquad (8.129)
\end{aligned}$$

We apply $\cos(\xi y)^2 = \frac{1}{2}(1+\cos(2\xi y))$, $\sin(\xi y)^2 = \frac{1}{2}(1-\cos(2\xi y))$ and $\cos(\xi y)\sin(\xi y) =$

$\frac{1}{2}\sin(2\xi y)$:

$$\begin{aligned}
&\int_{-\infty}^{x} \sin(\xi(x-y))Q^{p-1}(c_e(\xi)\cos(\xi y) - s_e(\xi)\sin(\xi y))dy \\
&= \frac{1}{2}(c_e(\xi)\sin(\xi x) + s_e(\xi)\cos(\xi x)) \int_{-\infty}^{x} Q^{p-1} dy \\
&\quad + \frac{1}{2}(c_e(\xi)\sin(\xi x) - s_e(\xi)\cos(\xi x)) \int_{-\infty}^{x} Q^{p-1}\cos(2\xi y)dy \\
&\quad - \frac{1}{2}(c_e(\xi)\cos(\xi x) + s_e(\xi)\sin(\xi x)) \int_{-\infty}^{x} Q^{p-1}\sin(2\xi y)dy.
\end{aligned} \tag{8.130}$$

For simplicity, let:

$$\theta_{e,v}(\xi, x) := \frac{p+1}{4}\frac{1}{1+\xi}(c_e(\xi)\sin(\xi x) + s_e(\xi)\cos(\xi x)) \int_{-\infty}^{x} Q^{p-1} dy. \tag{8.131}$$

Using lemma 8.4.2 and lemma 8.4.3 to control the integral terms, we conclude:

$$\forall k \geq 0: \left|\partial_\xi^k \partial_x (\rho_{e,v}(\xi, x) - \theta_{e,v}(\xi, x))\right| \leq C_k \frac{e^{\frac{1}{2}x}}{1+\xi}, \tag{8.132}$$

$$\left|\partial_\xi^k (\rho_{e,v}(\xi, x) - \theta_{e,v}(\xi, x))\right| \leq C_k \frac{e^{\frac{1}{2}x}}{1+\xi^2}. \tag{8.133}$$

We summarize:

Lemma 8.4.4 *Let $\rho_e = (\rho_{e,u}, \rho_{e,v})$ and $\rho_o = (\rho_{o,u}, \rho_{o,v})$ be given by (8.112) - (8.115). Let further $\theta_e, \theta_o : [0, \infty) \times (-\infty, 0] \to \mathbb{R}$ be given by:*

$$\theta_{e,v}(\xi, x) := \frac{p+1}{4}\frac{1}{1+\xi}(c_e(\xi)\sin(\xi x) + s_e(\xi)\cos(\xi x)) \int_{-\infty}^{x} Q^{p-1} dy, \tag{8.134}$$

$$\theta_{o,v}(\xi, x) := \frac{p+1}{4}\frac{1}{1+\xi}(c_o(\xi)\sin(\xi x) + s_o(\xi)\cos(\xi x)) \int_{-\infty}^{x} Q^{p-1} dy. \tag{8.135}$$

Then, for every $\xi \geq 0$, $x \leq 0$ and $k \geq 0$:

$$\left|\partial_\xi^k \partial_x \rho_{e,u}\right|, \left|\partial_\xi^k \partial_x (\rho_{e,v} - \theta_{e,v})\right|, \left|\partial_\xi^k \partial_x \rho_{o,u}\right|, \left|\partial_\xi^k \partial_x (\rho_{o,v} - \theta_{o,v})\right| \leq C_k \frac{e^{\frac{1}{2}x}}{1+\xi}, \tag{8.136}$$

$$\left|\partial_\xi^k \rho_{e,u}\right|, \left|\partial_\xi^k (\rho_{e,v} - \theta_{e,v})\right|, \left|\partial_\xi^k \rho_{o,u}\right|, \left|\partial_\xi^k (\rho_{o,v} - \theta_{o,v})\right| \leq C_k \frac{e^{\frac{1}{2}x}}{1+\xi^2}. \tag{8.137}$$

8.5. Jost Functions

With w_e and w_o as given by theorem 8.3.5, let $(u_e, v_e) := w_e$ and $(u_o, v_o) := w_o$. Consider

$$U_e^+ := u_e + v_e, \tag{8.138}$$

$$V_e^+ := i(u_e - v_e), \tag{8.139}$$

$$U_o^+ := u_o + v_o, \tag{8.140}$$

$$V_o^+ := i(u_o - v_o). \tag{8.141}$$

By definition, $(U_e^+, V_e^+)(\xi, \cdot)$ and $(U_o^+, V_o^+)(\xi, \cdot)$ are solutions to the equation:

$$Lw = iEw \tag{8.142}$$

for every $E = \xi^2 + 1, \xi \geq 0$. By symmetry, $(U_e^+, -V_e^+)(\xi, \cdot)$ and $(U_o^+, -V_o^+)(\xi, \cdot)$ solve:

$$Lw = -iEw. \tag{8.143}$$

Definition 8.5.1 *We define*

$$\begin{aligned} W_e = (U_e, V_e) &: \mathbb{R}^2 \to \mathbb{C}^2, \\ (\xi, x) &\mapsto \begin{cases} (U_e^+, V_e^+)(\xi, x), & \text{if } \xi \geq 0, \\ (U_e^+, -V_e^+)(-\xi, x), & \text{if } \xi < 0, \end{cases} \end{aligned} \tag{8.144}$$

$$\begin{aligned} W_o = (U_o, V_o) &: \mathbb{R}^2 \to \mathbb{C}^2, \\ (\xi, x) &\mapsto \begin{cases} (U_o^+, V_o^+)(\xi, x), & \text{if } \xi \geq 0, \\ (U_o^+, -V_o^+)(-\xi, x), & \text{if } \xi < 0. \end{cases} \end{aligned} \tag{8.145}$$

Lemma 8.5.2 *Let $\xi \in \mathbb{R}$. Then*

$$LW_e(\xi, \cdot) = i \operatorname{sgn}(\xi)(\xi^2 + 1) W_e(\xi, \cdot), \tag{8.146}$$

$$LW_o(\xi, \cdot) = i \operatorname{sgn}(\xi)(\xi^2 + 1) W_o(\xi, \cdot). \tag{8.147}$$

Lemma 8.5.3 *Let $(\xi, x) \in \mathbb{R}^2$. Then, $U_e(\xi, x), U_o(\xi, x) \in \mathbb{R}$, as well as $V_e(\xi, x), V_o(\xi, x) \in i\mathbb{R}$. Further*

$$W_e(\xi, x) = \overline{W_e(-\xi, x)}, \tag{8.148}$$

$$W_o(\xi, x) = \overline{W_o(-\xi, x)}. \tag{8.149}$$

Lemma 8.5.4 *$\xi \mapsto W_e(\xi, x)$ and $\xi \mapsto W_e(\xi, x)$ are real analytic on both $(-\infty, 0]$ and $[0, \infty)$ for every $x \in \mathbb{R}$. $x \mapsto W_e(\xi, x)$ and $x \mapsto W_e(\xi, x)$ are real analytic on $\mathbb{R}$ for every $\xi \in \mathbb{R}$. Further, $W_e(0, x) = W_o(0, x) = (0, 0)$.*

Proof. Follows from lemma 8.4.1. $\square$

Remark *$W_e(0, x) = W_o(0, x) = (0, 0)$ is directly related to the absence of resonances (theorem 1.10.1). In contrast, the integral kernel $e^{i\xi x}$ of the Fourier transform admits no zero. This property of W_e, W_o allows us to show a stronger local smoothing estimate (theorem 12.1.9) for $\partial_t w = -Lw$ than the free equation admits.*

Lemma 8.5.5 *Let $\xi \in \mathbb{R}$. Then*

$$||W_e(\xi, \cdot)||_{L^\infty(\mathbb{R})}, ||W_o(\xi, \cdot)||_{L^\infty(\mathbb{R})} \leq C, \tag{8.150}$$

$$||W_e(\xi, \cdot)||_{W^{1,\infty}(\mathbb{R})}, ||W_o(\xi, \cdot)||_{W^{1,\infty}(\mathbb{R})} \leq C(1 + |\xi|). \tag{8.151}$$

Proof. Follows from corollary 8.1.7 and theorem 8.3.5. $\square$

Mirroring (8.112) - (8.115), we define:

Definition 8.5.6 *We define the remainder terms $R_e = (R_{e,U}, R_{e,V}) : \mathbb{R}^2 \to \mathbb{C}^2$ and $R_o = (R_{o,U}, R_{o,V}) : \mathbb{R}^2 \to \mathbb{C}^2$ by:*

$$R_{e,U}(\xi,x) := U_e(\xi,x) - c_e(\xi)\cos(\xi x) - \operatorname{sgn}(\xi x)s_e(\xi)\sin(\xi x), \tag{8.152}$$
$$R_{e,V}(\xi,x) := V_e(\xi,x) + i\operatorname{sgn}(\xi)c_e(\xi)\cos(\xi x) + i\operatorname{sgn}(x)s_e(\xi)\sin(\xi x), \tag{8.153}$$
$$R_{o,U}(\xi,x) := U_o(\xi,x) + \operatorname{sgn}(x)c_o(\xi)\cos(\xi x) + \operatorname{sgn}(\xi)s_o(\xi)\sin(\xi x), \tag{8.154}$$
$$R_{o,V}(\xi,x) := V_o(\xi,x) - i\operatorname{sgn}(\xi x)c_o(\xi)\cos(\xi x) - is_o(\xi)\sin(\xi x). \tag{8.155}$$

Remark *While the eigenfunctions W_e and W_o are smooth with respect to x, the remainder terms exhibit discontinuity. $R_{o,U}$, $R_{o,V}$ are not continuous in $x = 0$, while the other remainder terms are not continuously differentiable in $x = 0$.*

Lemma 8.5.7 *Let $x \in \mathbb{R}$. Then, $R_e(0,x) = R_o(0,x) = (0,0)$.*

Proof. Follows from lemma 8.3.8 and lemma 8.5.4. □

Lemma 8.5.8 *Let $\xi \in \mathbb{R}$. Then, $x \mapsto R_e(\xi,x)$ and $x \mapsto R_o(\xi,x)$ are analytic on both $(-\infty,0]$ and $[0,\infty)$. Likewise, for $x \in \mathbb{R}$, $\xi \mapsto R_e(\xi,x)$ and $\xi \mapsto R_o(\xi,x)$ are analytic on both $(-\infty,0]$ and $[0,\infty)$.*

Proof. Follows from lemma 8.5.4. □

Lemma 8.5.9 *Let $(\xi,x) \in \mathbb{R}^2$. Then*

$$\left|R_e(\xi,x)\right|, \left|R_o(\xi,x)\right| \le \frac{C}{1+|\xi|}e^{-\frac{1}{2}|x|}, \tag{8.156}$$
$$\left|\partial_x R_e(\xi,x)\right|, \left|\partial_x R_o(\xi,x)\right| \le Ce^{-\frac{1}{2}|x|}. \tag{8.157}$$

For $x = 0$, the derivatives are to be understood as one-sided.

Proof. Follows from lemma 8.4.2. □

Lemma 8.5.10 *Let $(\xi,x) \in \mathbb{R}^2$ and $k \ge 0$. Then*

$$\left|\partial_\xi^k R_e(\xi,x)\right|, \left|\partial_\xi^k R_o(\xi,x)\right| \le \frac{C_k}{1+|\xi|}e^{-\frac{1}{2}|x|}. \tag{8.158}$$

For $\xi = 0$, the derivatives are to be understood as one-sided.

Proof. Follows from lemma 8.4.3. □

Corollary 8.5.11 *Let $\xi, x \in \mathbb{R}$ and $k \ge 0$. Then*

$$\left|\partial_\xi^k W_e(\xi,x)\right|, \left|\partial_\xi^k W_o(\xi,x)\right| \le C_k(1+|x|)^k. \tag{8.159}$$

For $\xi = 0$, the derivatives are to be understood as one-sided.

Proof. Follows from lemma 8.5.10 and corollary 8.3.7. □

We close with a higher order estimate.

Definition 8.5.12 *Consider the function*

$$\tilde{Q}(x) := \frac{p+1}{4} \int_{-\infty}^{-|x|} Q^{p-1} dy. \tag{8.160}$$

We define the second order remainder terms $\tilde{R}_e := (\tilde{R}_{e,U}, \tilde{R}_{e,V})$ *and* $\tilde{R}_o := (\tilde{R}_{o,U}, \tilde{R}_{o,V})$*:*

$$\tilde{R}_{e,U}(\xi, x) := R_{e,U} + \frac{c_e(\xi)}{1+|\xi|} \operatorname{sgn}(\xi x) \sin(\xi x) \tilde{Q}(x) - \frac{s_e(\xi)}{1+|\xi|} \cos(\xi x) \tilde{Q}(x), \tag{8.161}$$

$$\tilde{R}_{e,V}(\xi, x) := R_{e,V} - i \frac{c_e(\xi)}{1+|\xi|} \operatorname{sgn}(x) \sin(\xi x) \tilde{Q}(x) + i \frac{s_e(\xi)}{1+|\xi|} \operatorname{sgn}(\xi) \cos(\xi x) \tilde{Q}(x), \tag{8.162}$$

$$\tilde{R}_{o,U}(\xi, x) := R_{o,U} - \frac{c_o(\xi)}{1+|\xi|} \operatorname{sgn}(\xi) \sin(\xi x) \tilde{Q}(x) + \frac{s_o(\xi)}{1+|\xi|} \operatorname{sgn}(x) \cos(\xi x) \tilde{Q}(x), \tag{8.163}$$

$$\tilde{R}_{o,V}(\xi, x) := R_{o,V} + i \frac{c_o(\xi)}{1+|\xi|} \sin(\xi x) \tilde{Q}(x) - i \frac{s_o(\xi)}{1+|\xi|} \operatorname{sgn}(\xi x) \cos(\xi x) \tilde{Q}(x). \tag{8.164}$$

Lemma 8.5.13 *Let* $\xi, x \in \mathbb{R}$ *and* $k \geq 0$*. Then*

$$\left|\partial_\xi^k \tilde{R}_e(\xi, x)\right|, \left|\partial_\xi^k \tilde{R}_o(\xi, x)\right| \leq C_k \frac{e^{-\frac{1}{2}|x|}}{1+\xi^2}, \tag{8.165}$$

$$\left|\partial_\xi^k \partial_x \tilde{R}_e(\xi, x)\right|, \left|\partial_\xi^k \partial_x \tilde{R}_o(\xi, x)\right| \leq C_k \frac{e^{-\frac{1}{2}|x|}}{1+|\xi|}. \tag{8.166}$$

For $\xi = 0$ *and* $x = 0$ *respectively, the derivatives are to be understood as one-sided.*

Proof. Follows from lemma 8.4.4. □

Corollary 8.5.14 *Let* $\xi, x \in \mathbb{R}$ *and* $k \geq 0$*. Then*

$$\left|\partial_\xi^k \partial_x R_e(\xi, x)\right|, \left|\partial_\xi^k \partial_x R_o(\xi, x)\right| \leq C_k e^{-\frac{1}{2}|x|}. \tag{8.167}$$

For $\xi = 0$ *and* $x = 0$ *respectively, the derivatives are to be understood as one-sided.*

Proof. Follows from lemma 8.5.13. □

9. Distorted Fourier Transform

We have succeeded in defining Jost solutions W_e, W_o.

We now use W_e, W_o to define analogues of the Fourier transform called F and G. Establishing F and G as unitary transforms between reasonable Hilbert spaces is the goal of this chapter.

We generally use the variable names $w = (u, v)$ and $h = (f, g)$ as arguments of F and G respectively. We will often decompose these functions into even and odd parts.

Convention 9.0.1 *Consider functions $\mathbb{R} \to \mathbb{C}^2$, denoted specifically $w = (u, v)$ or $h = (f, g)$ or slight variations thereof (e.g. $\tilde{w}$ instead of w).*

We understand $w_e = (u_e, v_e), w_o = (u_o, v_o)$ and $h_e = (f_e, g_e), h_o = (f_o, g_o)$ to be the unique decomposition into sums of even and odd functions $w = w_e + w_o$, $h = h_e + h_o$.

As always, we assume $3 < p < 5$.

9.1. Overview

The Fourier transform is a well-known operator $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, which for $f \in \mathcal{S}(\mathbb{R})$ is given by:

$$(\mathcal{F}f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ix\xi} dx. \tag{9.1}$$

In this chapter we define a 'distorted' Fourier transform using W_e and W_o as integral kernels. For Schwartz functions $f, g \in \mathcal{S}(\mathbb{R})$ and $w = (u, v) \in \mathcal{S}(\mathbb{R})^2$:

$$(F_e w)(\xi) := \frac{1}{\sqrt{2\pi}(1 + \xi^2)} \langle w, W_e(\xi, \cdot) \rangle_H, \tag{9.2}$$

$$(F_o w)(\xi) := \frac{1}{\sqrt{2\pi}(1 + \xi^2)} \langle w, W_o(\xi, \cdot) \rangle_H, \tag{9.3}$$

$$(G_e f)(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} W_e(\xi, x) f(\xi) d\xi, \tag{9.4}$$

$$(G_o g)(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} W_o(\xi, x) g(\xi) d\xi. \tag{9.5}$$

We show that $F = (F_e, F_o)$ extends to a unitary operator $H \to \{f \in L^2(\mathbb{R})^2 | \xi h \in L^2(\mathbb{R})^2\}$ with inverse $G(f, g) = G_e f + G_o g$.

We also show that F maps the Hamiltonian L onto the multiplication operator $i \operatorname{sgn}(\xi)(\xi^2 + 1)$. That allows us to define the wave operator, as finding a unitary transform $\mathcal{G} : \{f \in L^2(\mathbb{R})^2 | xh \in L^2(\mathbb{R})^2\} \to H^1(\mathbb{R})^2$ mapping $i \operatorname{sgn}(\xi)(\xi^2 + 1)$ onto $I(-\Delta + 1)$ is quite simple. (I denotes the matrix equivalent of the imaginary unit i.)

9.2. G_e and G_o

Heuristically, we can determine $||G_e f||_H^2$, by calculating:

$$\frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\langle W_e(\xi_1,\cdot),W_e(\xi_2,\cdot)\rangle_H f(\xi_1)f(\xi_2)d\xi_1 d\xi_2, \tag{9.6}$$

which can be simplified via:

$$\begin{aligned}&\frac{1}{1+\xi_1^2}\langle W_e(\xi_1,\cdot),W_e(\xi_2,\cdot)\rangle_H\\ =\ &-\operatorname{sgn}(\xi_1)\langle U_e(\xi_1,\cdot),V_e(\xi_2,\cdot)\rangle_{L^2(\mathbb{R})}+\operatorname{sgn}(\xi_1)\langle V_e(\xi_1,\cdot),U_e(\xi_2,\cdot)\rangle_{L^2(\mathbb{R})}.\end{aligned} \tag{9.7}$$

None of the products used in (9.7) are well-defined. We introduce the following scalar-like product in order to remedy this fact.

Definition 9.2.1 *Let $\varepsilon > 0$. Given measurable functions $u_1, u_2 : \mathbb{R} \to \mathbb{C}$ for which the following integral converges, we define:*

$$\langle u_1,u_2\rangle_\varepsilon := \int_{\mathbb{R}} e^{-\varepsilon^2 x^2} u_1 \overline{u}_2 dx. \tag{9.8}$$

Using $\lim_{\varepsilon\to 0}\langle\cdot,\cdot\rangle_\varepsilon$ in place of $\langle\cdot,\cdot\rangle_{L^2}$ makes the Ansatz (9.6), (9.7) viable. We define:

Definition 9.2.2 *For $\xi_1, \xi_2 \in \mathbb{R}$ and $\varepsilon > 0$, we define*

$$\eta_{\varepsilon,e}(\xi_1,\xi_2) := -\operatorname{sgn}(\xi_1)\langle U_e(\xi_1,\cdot),V_e(\xi_2,\cdot)\rangle_\varepsilon+\operatorname{sgn}(\xi_1)\langle V_e(\xi_1,\cdot),U_e(\xi_2,\cdot)\rangle_\varepsilon, \tag{9.9}$$

$$\eta_{\varepsilon,o}(\xi_1,\xi_2) := -\operatorname{sgn}(\xi_1)\langle U_o(\xi_1,\cdot),V_o(\xi_2,\cdot)\rangle_\varepsilon+\operatorname{sgn}(\xi_1)\langle V_o(\xi_1,\cdot),U_o(\xi_2,\cdot)\rangle_\varepsilon. \tag{9.10}$$

9.2.1. Approximate Identity

We show that $\eta_{\varepsilon,e}$ and $\eta_{\varepsilon,o}$ act as approximate identities. We will only be giving the proof for $\eta_{\varepsilon,e}$ as the proof for $\eta_{\varepsilon,o}$ is identical.

Lemma 9.2.3 *For $\xi_1 \neq \xi_2$:*

$$\lim_{\varepsilon\searrow 0}\eta_{\varepsilon,e}(\xi_1,\xi_2) = 0. \tag{9.11}$$

Proof. For the sake of simplicity, let:

$$\eta_\varepsilon(\xi_1,\xi_2) := \operatorname{sgn}(\xi_1)\eta_{\varepsilon,e}(\xi_1,\xi_2) = -\langle U_e(\xi_1,\cdot),V_e(\xi_2,\cdot)\rangle_\varepsilon+\langle V_e(\xi_1,\cdot),U_e(\xi_2,\cdot)\rangle_\varepsilon. \tag{9.12}$$

Lemma 8.5.4 implies $W_e(0,\cdot) = (0,0)$. Hence, it suffices to consider $\xi_1, \xi_2 \neq 0$.

Let $(U_1, V_1) := W_e(\xi_1, \cdot)$, $(U_2, V_2) := W_e(\xi_2, \cdot)$. By lemma 8.5.2:

$$\begin{aligned}
&- i\,\mathrm{sgn}(\xi_1)(\xi_1^2+1)\langle U_1, V_2\rangle_\varepsilon \\
&= \langle L_V V_1, V_2\rangle_\varepsilon \\
&= \int_{\mathbb{R}} e^{-\varepsilon^2x^2}(-\Delta+1-Q^{p-1})V_1\overline{V_2}dx \\
&= \int_{\mathbb{R}} e^{-\varepsilon^2x^2}V_1(-\Delta+1-Q^{p-1})\overline{V_2}dx \\
&\quad + \int_{\mathbb{R}} 4\varepsilon^2xe^{-\varepsilon^2x^2}V_1\partial_x\overline{V_2}dx - \int_{\mathbb{R}}(-2\varepsilon^2+4\varepsilon^4x^2)e^{-\varepsilon^2x^2}V_1\overline{V_2}dx \\
&= \langle V_1, -i\,\mathrm{sgn}(\xi_2)(\xi_2^2+1)U_2\rangle_\varepsilon \\
&\quad + \int_{\mathbb{R}} 4\varepsilon^2xe^{-\varepsilon^2x^2}V_1\partial_x\overline{V_2}dx - \int_{\mathbb{R}}(-2\varepsilon^2+4\varepsilon^4x^2)e^{-\varepsilon^2x^2}V_1\overline{V_2}dx \\
&= i\,\mathrm{sgn}(\xi_2)(\xi_2^2+1)\langle V_1, U_2\rangle_\varepsilon \\
&\quad + \int_{\mathbb{R}} 4\varepsilon^2xe^{-\varepsilon^2x^2}V_1\partial_x\overline{V_2}dx - \int_{\mathbb{R}}(-2\varepsilon^2+4\varepsilon^4x^2)e^{-\varepsilon^2x^2}V_1\overline{V_2}dx. \qquad (9.13)
\end{aligned}$$

We estimate (9.13). By lemma 8.5.5:

$$\left|\int_{\mathbb{R}}(-2\varepsilon^2+4\varepsilon^4x^2)e^{-\varepsilon^2x^2}V_1\overline{V_2}dx\right| \leq C\int_{\mathbb{R}}(2\varepsilon^2+4\varepsilon^4x^2)e^{-\varepsilon^2x^2}dx \leq C^2\varepsilon. \qquad (9.14)$$

By the same token, lemma 8.5.9 allows us to estimate:

$$\begin{aligned}
&\left|\int_{\mathbb{R}} 4\varepsilon^2xe^{-\varepsilon^2x^2}V_1\partial_x\overline{V_2}dx\right| \\
&\leq \left|\int_{\mathbb{R}} 4\varepsilon^2xe^{-\varepsilon^2x^2}e^{-\frac{1}{2}|x|}dx\right| \\
&\quad + \left|\int_{\mathbb{R}} 4\varepsilon^2xe^{-\varepsilon^2x^2}(c_e(\xi_1)\cos(\xi_1x)+s_e(\xi_1)\sin(|\xi_1x|))\partial_x(c_e(\xi_2)\cos(\xi_2x)+s_e(\xi_2)\sin(|\xi_2x|))dx\right| \\
&\leq C\varepsilon^2 + |\xi_2|\left|\int_0^\infty 4\varepsilon^2xe^{-\varepsilon^2x^2}e^{ix(\xi_1+\xi_2)}dx\right| + |\xi_2|\left|\int_0^\infty 4\varepsilon^2xe^{-\varepsilon^2x^2}e^{ix(\xi_1-\xi_2)}dx\right| \\
&\leq C\varepsilon^2 + |\xi_2|\left|\int_0^\infty 4xe^{-x^2}e^{ix\frac{\xi_1+\xi_2}{\varepsilon}}dx\right| + |\xi_2|\left|\int_0^\infty 4xe^{-x^2}e^{ix\frac{\xi_1-\xi_2}{\varepsilon}}dx\right|. \qquad (9.15)
\end{aligned}$$

It follows for $|\xi_1| \neq |\xi_2|$:

$$\left|\int_{\mathbb{R}} 4\varepsilon^2xe^{-\varepsilon^2x^2}V_1\partial_x\overline{V_2}dx\right| \leq \varepsilon\left(\varepsilon + \frac{|\xi_2|}{|\xi_1+\xi_2|} + \frac{|\xi_2|}{|\xi_1-\xi_2|}\right). \qquad (9.16)$$

On the other hand, for $\xi_1 = -\xi_2$, lemma 8.5.3 implies $V_1 = \overline{V_2}$. It follows:

$$\begin{aligned}
\left|\int_{\mathbb{R}} 4\varepsilon^2xe^{-\varepsilon^2x^2}V_1\partial_x\overline{V_2}dx\right| &= \left|\int_{\mathbb{R}} 2\varepsilon^2xe^{-\varepsilon^2x^2}\partial_x(V_1^2)dx\right| \\
&= \left|\int_{\mathbb{R}}(2\varepsilon^2-4\varepsilon^4x^2)e^{-\varepsilon^2x^2}V_1^2dx\right| \\
&\leq C\varepsilon\left|\int_{\mathbb{R}}(2-4x^2)e^{-x^2}dx\right| \leq C^2\varepsilon. \qquad (9.17)
\end{aligned}$$

Completely analogously to (9.13), we conclude:

$$\begin{aligned}
&i\,\mathrm{sgn}(\xi_1)(\xi_1^2+1)\langle V_1, U_2\rangle_\varepsilon \\
=\ &-i\,\mathrm{sgn}(\xi_2)(\xi_2^2+1)\langle U_1, V_2\rangle_\varepsilon \\
&+\int_{\mathbb{R}} 4\varepsilon^2 x e^{-\varepsilon^2x^2} U_1\partial_x\overline{U_2}dx - \int_{\mathbb{R}}(-2\varepsilon^2+4\varepsilon^4x^2)e^{-\varepsilon^2x^2}U_1\overline{U_2}dx. \qquad (9.18)
\end{aligned}$$

Analogously to (9.14), (9.16) and (9.17):

$$\lim_{\varepsilon\searrow 0}\left|\int_{\mathbb{R}} 4\varepsilon^2 x e^{-\varepsilon^2x^2} U_1\partial_x\overline{U_2}dx\right| + \left|\int_{\mathbb{R}}(-2\varepsilon^2+4\varepsilon^4x^2)e^{-\varepsilon^2x^2}U_1\overline{U_2}dx\right| = 0. \qquad (9.19)$$

By (9.13) and (9.18):

$$\begin{aligned}
&i\,\mathrm{sgn}(\xi_1)(\xi_1^2+1)\eta_\varepsilon(\xi_1,\xi_2) - i\,\mathrm{sgn}(\xi_2)(\xi_2^2+1)\eta_\varepsilon(\xi_1,\xi_2) \\
=\ &\int_{\mathbb{R}} 4\varepsilon^2 x e^{-\varepsilon^2x^2} V_1\partial_x\overline{V_2}dx - \int_{\mathbb{R}}(-2\varepsilon^2+4\varepsilon^4x^2)e^{-\varepsilon^2x^2}V_1\overline{V_2}dx \\
&+\int_{\mathbb{R}} 4\varepsilon^2 x e^{-\varepsilon^2x^2} U_1\partial_x\overline{U_2}dx - \int_{\mathbb{R}}(-2\varepsilon^2+4\varepsilon^4x^2)e^{-\varepsilon^2x^2}U_1\overline{U_2}dx. \qquad (9.20)
\end{aligned}$$

It follows:

$$(\mathrm{sgn}(\xi_1)(\xi_1^2+1) - \mathrm{sgn}(\xi_2)(\xi_2^2+1))\lim_{\varepsilon\searrow 0}\eta_\varepsilon(\xi_1,\xi_2) = 0. \qquad (9.21)$$

That concludes the proof. $\square$

To estimate $\eta_{\varepsilon,e}$, we require the following additional functions.

Definition 9.2.4 *We define*

$$\begin{aligned}
k : \mathbb{R} &\to \{z\in\mathbb{C}|\,|z|=1\}, \\
\xi &\mapsto c_e(\xi) - i s_e(\xi). \qquad (9.22)
\end{aligned}$$

Put another way, k is the unique function $\mathbb{R}\to\{|z|=1\}$ fulfilling for every $\xi, x\in\mathbb{R}$:

$$\begin{aligned}
\mathrm{Re}(k(\xi)e^{i|\xi x|}) &= c_e(\xi)\cos(\xi x) + s_e(\xi)\sin(|\xi x|) \\
&= U_e(\xi,x) - R_{e,U}(\xi,x) \\
&= i\,\mathrm{sgn}(\xi)(V_e(\xi,x) - R_{e,V}(\xi,x)). \qquad (9.23)
\end{aligned}$$

Definition 9.2.5 *For $\xi_1,\xi_2\in\mathbb{R}$ and $\varepsilon>0$, we define*

$$\begin{aligned}
&\mu_\varepsilon(\xi_1,\xi_2) \\
:=\ &-\mathrm{sgn}(\xi_1)\int_{\mathbb{R}} e^{-\varepsilon^2x^2}\left(U_e(\xi_1,x) - R_{e,U}(\xi_1,x)\right)\overline{(V_e(\xi_2,x) - R_{e,V}(\xi_2,x))}dx \\
&+\mathrm{sgn}(\xi_1)\int_{\mathbb{R}} e^{-\varepsilon^2x^2}\left(V_e(\xi_1,x) - R_{e,V}(\xi_1,x)\right)\overline{(U_e(\xi_2,x) - R_{e,U}(\xi_2,x))}dx. \qquad (9.24)
\end{aligned}$$

Definition 9.2.6 *For $x \in \mathbb{R}$, consider the Dawson-integral:*

$$D_+(x) := e^{-x^2} \int_0^x e^{y^2} dy. \tag{9.25}$$

Lemma 9.2.7 *Let $\xi_1, \xi_2 \in \mathbb{R}$ and $\varepsilon > 0$. If $\operatorname{sgn}(\xi_1) = \operatorname{sgn}(\xi_2)$, then:*

$$\mu_\varepsilon(\xi_1, \xi_2) = -4i \int_{-\infty}^0 e^{-\varepsilon^2 x^2} \operatorname{Re}(k(\xi_1) e^{-i|\xi_1|x}) \operatorname{Re}(k(\xi_2) e^{-i|\xi_2|x}) dx. \tag{9.26}$$

If $\operatorname{sgn}(\xi_1) \neq \operatorname{sgn}(\xi_2)$, then:

$$\mu_\varepsilon(\xi_1, \xi_2) = 0. \tag{9.27}$$

Proof. We abbreviate $k_1 = k(\xi_1)$ and $k_2 = k(\xi_2)$. By definition:

$$\begin{aligned} \mu_\varepsilon(\xi_1, \xi_2) = & -\operatorname{sgn}(\xi_1) \int_{\mathbb{R}} e^{-\varepsilon^2 x^2} \operatorname{Re}(k_1 e^{i|\xi_1 x|}) \overline{(-i)\operatorname{sgn}(\xi_2) \operatorname{Re}(k_2 e^{i|\xi_2 x|})} dx \\ & + \operatorname{sgn}(\xi_1) \int_{\mathbb{R}} e^{-\varepsilon^2 x^2} (-i) \operatorname{sgn}(\xi_1) \operatorname{Re}(k_1 e^{i|\xi_1 x|}) \overline{\operatorname{Re}(k_2 e^{i|\xi_2 x|})} dx \\ = & -i(\operatorname{sgn}(\xi_1 \xi_2) + 1) \int_{-\infty}^{\infty} e^{-\varepsilon^2 x^2} \operatorname{Re}(k_1 e^{i|\xi_1 x|}) \operatorname{Re}(k_2 e^{i|\xi_2 x|}) dx \\ = & -2i(\operatorname{sgn}(\xi_1 \xi_2) + 1) \int_{-\infty}^{0} e^{-\varepsilon^2 x^2} \operatorname{Re}(k_1 e^{-i|\xi_1|x}) \operatorname{Re}(k_2 e^{-i|\xi_2|x}) dx. \end{aligned} \tag{9.28}$$

That concludes the proof. □

Lemma 9.2.8 *D_+ is an odd function. The following estimates are fulfilled for $x > 0$:*

$$D_+(x) \geq \frac{1 - e^{-x^2}}{2x}, \tag{9.29}$$

$$D_+(x) \leq x, \tag{9.30}$$

$$D_+(x) \leq x e^{-\sqrt{x}} + \frac{1}{2x} + \frac{1}{x^2}. \tag{9.31}$$

Proof. D_+ is odd by definition. (9.30) also holds by definition. We calculate:

$$\frac{1 - e^{-x^2}}{2x} = e^{-x^2} \int_0^x \frac{2y}{2x} e^{y^2} dy \leq D_+(x). \tag{9.32}$$

By (9.30), it suffices to show (9.31) for $x \geq 1$. We estimate:

$$\begin{aligned} D_+(x) &\leq e^{-x^2} \int_0^{x - \frac{1}{\sqrt{x}}} e^{y^2} dy + e^{-x^2} \int_{x - \frac{1}{\sqrt{x}}}^x e^{y^2} dy \\ &\leq e^{-x^2} x e^{\left(x - \frac{1}{\sqrt{x}}\right)^2} + e^{-x^2} \int_{x - \frac{1}{\sqrt{x}}}^x \frac{y}{x} e^{y^2} dy + \int_{x - \frac{1}{\sqrt{x}}}^x \frac{x - y}{x} e^{y^2 - x^2} dy \\ &\leq x e^{-2\sqrt{x} + \frac{1}{x}} + \frac{1}{2x} + \frac{1}{\sqrt{x}} \frac{\frac{1}{\sqrt{x}}}{x} e^0 \\ &\leq x e^{-\sqrt{x}} + \frac{1}{2x} + \frac{1}{x^2}. \end{aligned} \tag{9.33}$$

That concludes the proof. □

Corollary 9.2.9 *D_+ exhibits the following asymptotic behaviour:*

$$\lim_{|x|\to\infty} 2xD_+(x) = 1. \tag{9.34}$$

Corollary 9.2.10 *D_+ exhibits the following asymptotic behaviour. For $x \neq 0$:*

$$\lim_{\varepsilon\searrow 0} \frac{D_+\left(\frac{x}{\varepsilon}\right)}{\varepsilon} = \frac{1}{2x}. \tag{9.35}$$

Proof. By corollary 9.2.9:

$$\lim_{\varepsilon\searrow 0} \frac{D_+\left(\frac{x}{\varepsilon}\right)}{\varepsilon} = \frac{1}{2x} \lim_{\varepsilon\searrow 0} \frac{2x}{\varepsilon} D_+\left(\frac{x}{\varepsilon}\right) = \frac{1}{2x}. \tag{9.36}$$

That concludes the proof. $\square$

Lemma 9.2.11 *Let $\xi \in \mathbb{R}$ and $\varepsilon > 0$. Then:*

$$\int_{-\infty}^{0} e^{-\varepsilon^2 x^2} e^{-i\xi x} dx = \sqrt{\pi}\frac{e^{-\frac{\xi^2}{4\varepsilon^2}}}{2\varepsilon} + i\frac{D_+(\frac{\xi}{4\varepsilon})}{\varepsilon}. \tag{9.37}$$

Proof. We calculate:

$$\begin{aligned}
\int_{-\infty}^{0} e^{-\varepsilon^2 x^2} e^{-i\xi x} dx &= \frac{1}{\varepsilon}\int_{-\infty}^{0} e^{-z^2} e^{-iz\frac{\xi}{\varepsilon}} dz \\
&= \frac{e^{-\frac{\xi^2}{4\varepsilon^2}}}{\varepsilon} \int_{-\infty}^{0} e^{-(z+i\frac{\xi}{2\varepsilon})^2} dz \\
&= \frac{e^{-\frac{\xi^2}{4\varepsilon^2}}}{\varepsilon} \int_{(-\infty,0]+i\frac{\xi}{4\varepsilon}} e^{-z^2} dz \\
&= \frac{e^{-\frac{\xi^2}{4\varepsilon^2}}}{\varepsilon} \int_{(-\infty,0]} e^{-z^2} dz + \frac{e^{-\frac{\xi^2}{4\varepsilon^2}}}{\varepsilon} \int_{\left(0,i\frac{\xi}{4\varepsilon}\right]} e^{-z^2} dz \\
&= \frac{e^{-\frac{\xi^2}{4\varepsilon^2}}}{\varepsilon}\frac{\sqrt{\pi}}{2} + i\frac{e^{-\frac{\xi^2}{4\varepsilon^2}}}{\varepsilon}\int_{0}^{\frac{\xi}{4\varepsilon}} e^{y^2} dy \\
&= \frac{e^{-\frac{\xi^2}{4\varepsilon^2}}}{2\varepsilon}\sqrt{\pi} + i\frac{D_+(\frac{\xi}{4\varepsilon})}{\varepsilon}.
\end{aligned} \tag{9.38}$$

That concludes the proof. $\square$

Lemma 9.2.12 *Let $\varepsilon > 0$ and $\xi_1, \xi_2 \in \mathbb{R}$. Assume $\operatorname{sgn}(\xi_1) = \operatorname{sgn}(\xi_2)$. Then:*

$$\begin{aligned}
\mu_\varepsilon(\xi_1,\xi_2) = 2i\operatorname{Im}(k(\xi_1)k(\xi_2))\frac{D_+(\frac{|\xi_1|+|\xi_2|}{4\varepsilon})}{\varepsilon} + 2i\operatorname{Im}(k(\xi_1)\overline{k(\xi_2)})\frac{D_+(\frac{|\xi_1|-|\xi_2|}{4\varepsilon})}{\varepsilon} \\
- \sqrt{\pi}i\operatorname{Re}(k(\xi_1)k(\xi_2))\frac{e^{-\frac{(\xi_1+\xi_2)^2}{4\varepsilon^2}}}{\varepsilon} - \sqrt{\pi}i\operatorname{Re}(k(\xi_1)\overline{k(\xi_2)})\frac{e^{-\frac{(\xi_1-\xi_2)^2}{4\varepsilon^2}}}{\varepsilon}.
\end{aligned} \tag{9.39}$$

Proof. We abbreviate $k_1 = k(\xi_1)$ and $k_2 = k(\xi_2)$. By lemma 9.2.7 and lemma 9.2.11:

$$
\begin{aligned}
&\mu_\varepsilon(\xi_1,\xi_2)\\
&= -i\int_{-\infty}^{0} e^{-\varepsilon^2x^2}(k_1e^{-i|\xi_1|x}+\overline{k_1}e^{i|\xi_1|x})(k_2e^{-i|\xi_2|x}+\overline{k_2}e^{i|\xi_2|x})dx\\
&= k_1k_2\frac{D_+(\frac{|\xi_1|+|\xi_2|}{4\varepsilon})}{\varepsilon}+k_1\overline{k_2}\frac{D_+(\frac{|\xi_1|-|\xi_2|}{4\varepsilon})}{\varepsilon}+\overline{k_1}k_2\frac{D_+(\frac{-|\xi_1|+|\xi_2|}{4\varepsilon})}{\varepsilon}+\overline{k_1k_2}\frac{D_+(\frac{-|\xi_1|-|\xi_2|}{4\varepsilon})}{\varepsilon}\\
&\quad-\frac{\sqrt{\pi}}{2}i\left(k_1k_2\frac{e^{-\frac{(|\xi_1|+|\xi_2|)^2}{4\varepsilon^2}}}{\varepsilon}+k_1\overline{k_2}\frac{e^{-\frac{(|\xi_1|-|\xi_2|)^2}{4\varepsilon^2}}}{\varepsilon}+\overline{k_1}k_2\frac{e^{-\frac{(-|\xi_1|+|\xi_2|)^2}{4\varepsilon^2}}}{\varepsilon}+\overline{k_1k_2}\frac{e^{-\frac{(-|\xi_1|-|\xi_2|)^2}{4\varepsilon^2}}}{\varepsilon}\right)\\
&= 2i\operatorname{Im}(k_1k_2)\frac{D_+(\frac{|\xi_1|+|\xi_2|}{4\varepsilon})}{\varepsilon}+2i\operatorname{Im}(k_1\overline{k_2})\frac{D_+(\frac{|\xi_1|-|\xi_2|}{4\varepsilon})}{\varepsilon}\\
&\quad-\sqrt{\pi}i\operatorname{Re}(k_1k_2)\frac{e^{-\frac{(\xi_1+\xi_2)^2}{4\varepsilon^2}}}{\varepsilon}-\sqrt{\pi}i\operatorname{Re}(k_1\overline{k_2})\frac{e^{-\frac{(\xi_1-\xi_2)^2}{4\varepsilon^2}}}{\varepsilon}.
\end{aligned}
\tag{9.40}
$$

That concludes the proof. ☐

Corollary 9.2.13 *Let $\varepsilon > 0$ and $\xi_1, \xi_2 \in \mathbb{R}$. Assume $\operatorname{sgn}(\xi_1) = \operatorname{sgn}(\xi_2)$ and $\xi_1 \neq \xi_2$. Then:*

$$\lim_{\varepsilon\searrow 0}\mu_\varepsilon(\xi_1,\xi_2) = 4i\frac{\operatorname{Im}(k(\xi_1)k(\xi_2))}{|\xi_1|+|\xi_2|}+4i\frac{\operatorname{Im}(k(\xi_1)\overline{k(\xi_2)})}{|\xi_1|-|\xi_2|}. \tag{9.41}$$

Proof. Follows from corollary 9.2.10 and lemma 9.2.12. ☐

Definition 9.2.14 *Let $\varepsilon > 0$ and $\xi_1, \xi_2 \in \mathbb{R}$. Let $\mathbb{1}_M$ denote the characteristic function of any given set $M \subseteq \mathbb{R}^2$. We define:*

$$Y_1(\varepsilon)(\xi_1,\xi_2) := \lim_{\delta\searrow 0}(\eta_{\varepsilon,e}(\xi_1,\xi_2)-\eta_{\delta,e}(\xi_1,\xi_2)-\mu_\varepsilon(\xi_1,\xi_2)+\mu_\delta(\xi_1,\xi_2)), \tag{9.42}$$

$$Y_2(\varepsilon)(\xi_1,\xi_2) := -i\operatorname{Re}(k(\xi_1)k(\xi_2))\sqrt{\pi}\frac{e^{-\frac{(\xi_1+\xi_2)^2}{4\varepsilon^2}}}{\varepsilon}\mathbb{1}_{\{\xi_1\xi_2>0\}}, \tag{9.43}$$

$$Y_3(\varepsilon)(\xi_1,\xi_2) := i\operatorname{Re}(k(\xi_1)\overline{k(\xi_2)})\sqrt{\pi}\frac{e^{-\frac{(\xi_1-\xi_2)^2}{4\varepsilon^2}}}{\varepsilon}\mathbb{1}_{\{\xi_1\xi_2<0\}}, \tag{9.44}$$

$$K(\varepsilon)(\xi_1,\xi_2) := -i\operatorname{Re}(k(\xi_1)\overline{k(\xi_2)})\sqrt{\pi}\frac{e^{-\frac{(\xi_1-\xi_2)^2}{4\varepsilon^2}}}{\varepsilon}. \tag{9.45}$$

For $\xi_1 \neq \xi_2$, we define:

$$Y_4(\varepsilon)(\xi_1,\xi_2) := 2i\operatorname{Im}(k(\xi_1)k(\xi_2))\left(\frac{D_+(\frac{|\xi_1|+|\xi_2|}{4\varepsilon})}{\varepsilon}-\frac{2}{|\xi_1|+|\xi_2|}\right)\mathbb{1}_{\{\xi_1\xi_2>0\}}, \tag{9.46}$$

$$Y_5(\varepsilon)(\xi_1,\xi_2) := 2i\operatorname{Im}(k(\xi_1)\overline{k(\xi_2)})\left(\frac{D_+(\frac{|\xi_1|-|\xi_2|}{4\varepsilon})}{\varepsilon}-\frac{2}{|\xi_1|-|\xi_2|}\right)\mathbb{1}_{\{\xi_1\xi_2>0\}}. \tag{9.47}$$

Lemma 9.2.15 *Let $\varepsilon > 0$ and $\xi_1 \in \mathbb{R}$. Let further $f \in \mathcal{S}(\mathbb{R})$ be a Schwartz function. Then:*

$$\left|\int_{\mathbb{R}} Y_1(\varepsilon)(\xi_1,\xi_2)f(\xi_2)d\xi_2\right| \leq C\varepsilon^2 \left|\left|f\right|\right|_{L^1(\mathbb{R})}. \tag{9.48}$$

Proof. Let $\xi_2 \in \mathbb{R}$ and $\delta \in (0,\varepsilon)$. By lemma 8.5.9:

$$\begin{aligned}
&|\eta_{\varepsilon,e}(\xi_1,\xi_2) - \eta_{\delta,e}(\xi_1,\xi_2) - (\mu_\varepsilon(\xi_1,\xi_2) - \mu_\delta(\xi_1,\xi_2))| \\
&\leq C\left|\left|(e^{-\varepsilon^2x^2} - e^{-\delta^2x^2})(|R_{e,U}(\xi_1,x)| + |R_{e,V}(\xi_1,x)| + |R_{e,U}(\xi_2,x)| + |R_{e,V}(\xi_2,x)|)\right|\right|_{L^1_x(\mathbb{R})} \\
&\leq C^2\left|\left|(e^{-\varepsilon^2x^2} - 1)e^{-\frac{1}{2}|x|}\right|\right|_{L^1_x(\mathbb{R})} \\
&\leq 2C^2\left|\left|\varepsilon^2x^2e^{-\frac{1}{2}|x|}\right|\right|_{L^1_x(\mathbb{R})} \\
&\leq C^3\varepsilon^2.
\end{aligned} \tag{9.49}$$

That concludes the proof. □

Lemma 9.2.16 *Let $\varepsilon > 0$, $\xi_1 \in \mathbb{R}$ and $f \in \mathcal{S}(\mathbb{R})$. Then:*

$$\left|\int_{\mathbb{R}} Y_2(\varepsilon)(\xi_1,\xi_2)f(\xi_2)d\xi_2\right|, \left|\int_{\mathbb{R}} Y_3(\varepsilon)(\xi_1,\xi_2)f(\xi_2)d\xi_2\right| \leq Ce^{-\frac{\xi_1^2}{4\varepsilon^2}}\left|\left|f\right|\right|_{L^\infty(\mathbb{R})}. \tag{9.50}$$

Proof. By definition:

$$\begin{aligned}
\left|\left|Y_2(\varepsilon)(\xi_1,\cdot)\right|\right|_{L^1(\mathbb{R})}, \left|\left|Y_3(\varepsilon)(\xi_1,\cdot)\right|\right|_{L^1(\mathbb{R})} &\leq C\int_0^\infty \frac{e^{-\frac{(|\xi_1|+\xi)^2}{4\varepsilon^2}}}{\varepsilon}d\xi \\
&= C\int_0^\infty e^{-\frac{(|\xi_1|+\varepsilon\xi)^2}{4\varepsilon^2}}d\xi \\
&\leq Ce^{-\frac{|\xi_1|^2}{4\varepsilon^2}}\int_0^\infty e^{-\frac{\xi^2}{4}}d\xi \\
&= C\sqrt{\pi}e^{-\frac{\xi_1^2}{4\varepsilon^2}}.
\end{aligned} \tag{9.51}$$

That concludes the proof. □

Lemma 9.2.17 *Let $1 \geq \varepsilon > 0$, $\xi_1 \in \mathbb{R}$ and $f \in \mathcal{S}(\mathbb{R})$. Then:*

$$\left|\int_{\mathbb{R}} Y_4(\varepsilon)(\xi_1,\xi_2)f(\xi_2)d\xi_2\right| \leq C\sqrt{\varepsilon}\left|\left|f\right|\right|_{L^\infty(\mathbb{R})}. \tag{9.52}$$

Proof. By symmetry, we can assume $\xi_1 \geq 0$. Let $\xi_2 \geq 0$ be given. We abbreviate $k_1 = k(\xi_1)$ and $k_2 = k(\xi_2)$.

Denote $x = \frac{\xi_1+\xi_2}{4\varepsilon}$. By definition:

$$|Y_4(\varepsilon)(\xi_1,\xi_2)| = |\mathrm{Im}(k_1k_2)|\frac{4x}{\xi_1+\xi_2}\left|D_+(x) - \frac{1}{2x}\right|. \tag{9.53}$$

Lemma 8.3.8 implies $\mathrm{Im}(k(0)^2) = 0$. Using corollary 8.3.7, we conclude:

$$|\mathrm{Im}(k_1 k_2)| \leq C(\xi_1 + \xi_2). \tag{9.54}$$

Corollary 8.3.7 also implies $|\mathrm{Im}(k_1 k_2)| \leq C$. It follows:

$$|\mathrm{Im}(k_1 k_2)| \leq C\sqrt{\xi_1 + \xi_2}. \tag{9.55}$$

By (9.53) and lemma 9.2.8:

$$\begin{aligned}
|Y_4(\varepsilon)(\xi_1, \xi_2)| &\leq \frac{4Cx}{\sqrt{\xi_1 + \xi_2}} \left| D_+(x) - \frac{1}{2x} \right| \\
&\leq \frac{4Cx}{\sqrt{\xi_1 + \xi_2}} \left(\frac{e^{-x^2}}{2x} + x e^{-\sqrt{x}} + \frac{1}{x^2} \right) \\
&\leq \frac{1}{\sqrt{\xi_1 + \xi_2}} \frac{C^2}{x} \\
&= 4C^2 \varepsilon (\xi_1 + \xi_2)^{-\frac{3}{2}}.
\end{aligned} \tag{9.56}$$

Further, by lemma 9.2.8, (9.53) and (9.54), we conclude for $\xi_1 + \xi_2 \leq 4\varepsilon$ resp. $x \leq 1$:

$$|Y_4(\varepsilon)(\xi_1, \xi_2)| \leq 4Cx \left| D_+(x) - \frac{1}{2x} \right| \leq 4Cx \left(\frac{e^{-x^2}}{2x} + x \right) \leq C^2. \tag{9.57}$$

(9.56) and (9.57) imply the desired estimate. We define $\delta := \max(4\varepsilon - \xi_1, 0)$. Then:

$$\begin{aligned}
&\left| \int_{\mathbb{R}} Y_4(\varepsilon)(\xi_1, \xi_2) f(\xi_2) d\xi_2 \right| \\
&\leq 2 \int_0^\infty |Y_4(\varepsilon)(\xi_1, \xi_2)| \, |f(\xi_2)| \, d\xi_2 \\
&\leq 2 \int_0^\delta |Y_4(\varepsilon)(\xi_1, \xi_2)| \, d\xi_2 \, ||f||_{L^\infty(\mathbb{R})} + 2 \int_\delta^\infty |Y_4(\varepsilon)(\xi_1, \xi_2)| \, d\xi_2 \, ||f||_{L^\infty(\mathbb{R})} \\
&\leq \int_0^\delta C d\xi_2 \, ||f||_{L^\infty(\mathbb{R})} + \int_\delta^\infty C\varepsilon (\xi_1 + \xi_2)^{-\frac{3}{2}} d\xi_2 \, ||f||_{L^\infty(\mathbb{R})} \\
&\leq \int_0^{4\varepsilon} C d\xi_2 \, ||f||_{L^\infty(\mathbb{R})} + \int_{4\varepsilon}^\infty C\varepsilon \xi_2^{-\frac{3}{2}} d\xi_2 \, ||f||_{L^\infty(\mathbb{R})} \\
&\leq 4C\varepsilon \, ||f||_{L^\infty(\mathbb{R})} + 2C\varepsilon (4\varepsilon)^{-\frac{1}{2}} \, ||f||_{L^\infty(\mathbb{R})} \\
&\leq 5C\sqrt{\varepsilon} \, ||f||_{L^\infty(\mathbb{R})}.
\end{aligned} \tag{9.58}$$

That concludes the proof. □

Lemma 9.2.18 *Let* $1 \geq \varepsilon > 0$, $\xi_1 \in \mathbb{R}$ *and* $f \in \mathcal{S}(\mathbb{R})$. *Then:*

$$\left| \int_{\mathbb{R}} Y_5(\varepsilon)(\xi_1, \xi_2) f(\xi_2) d\xi_2 \right| \leq C\sqrt{\varepsilon} \, ||f||_{L^\infty(\mathbb{R})}. \tag{9.59}$$

Proof. By symmetry, we can assume $\xi_1 \geq 0$. Let $\xi_2 \geq 0$ with $\xi_2 \neq \xi_1$ be given. We abbreviate $k_1 = k(\xi_1)$ and $k_2 = k(\xi_2)$.

Denote $x = \frac{|\xi_1 - \xi_2|}{4\varepsilon}$. By definition:

$$|Y_5(\varepsilon)(\xi_1, \xi_2)| \leq \left|\mathrm{Im}(k_1\overline{k_2})\right| \frac{4x}{|\xi_1 - \xi_2|} \left|D_+(x) - \frac{1}{2x}\right|. \tag{9.60}$$

Clearly, $\mathrm{Im}(k_1\overline{k_1}) = \mathrm{Im}(|k_1|^2) = 0$. Corollary 8.3.7 implies:

$$\left|\mathrm{Im}(k_1\overline{k_2})\right| = \frac{1}{2}\left|k_1\overline{k_2} - k_2\overline{k_1}\right| \leq C\left|\xi_1 - \xi_2\right|. \tag{9.61}$$

Corollary 8.3.7 further implies $\left|\mathrm{Im}(k_1\overline{k_2})\right| \leq C$. It follows:

$$\left|\mathrm{Im}(k_1\overline{k_2})\right| \leq C\sqrt{|\xi_1 - \xi_2|}. \tag{9.62}$$

By (9.60) and lemma 9.2.8:

$$\begin{aligned}
|Y_5(\varepsilon)(\xi_1, \xi_2)| &\leq \frac{4Cx}{\sqrt{|\xi_1 - \xi_2|}}\left|D_+(x) - \frac{1}{2x}\right| \\
&\leq \frac{4Cx}{\sqrt{|\xi_1 - \xi_2|}}\left(\frac{e^{-x^2}}{2x} + xe^{-\sqrt{x}} + \frac{1}{x^2}\right) \\
&\leq \frac{1}{\sqrt{|\xi_1 - \xi_2|}}\frac{C^2}{x} \\
&\leq \frac{4C^2\varepsilon}{|\xi_1 - \xi_2|^{\frac{3}{2}}}.
\end{aligned} \tag{9.63}$$

Further, by lemma 9.2.8, (9.60) and (9.61), we conclude for $|\xi_1 - \xi_2| \leq 4\varepsilon$ resp. $x \leq 1$:

$$|Y_5(\varepsilon)(\xi_1, \xi_2)| \leq 4Cx\left|D_+(x) - \frac{1}{2x}\right| \leq 4Cx\left(\frac{e^{-x^2}}{2x} + x\right) \leq C^2. \tag{9.64}$$

(9.63) and (9.64) imply the desired estimate:

$$\begin{aligned}
&\left|\int_{\mathbb{R}} Y_5(\varepsilon)(\xi_1, \xi_2)f(\xi_2)d\xi_2\right| \\
&\leq 2\int_{\xi_1-4\varepsilon}^{\xi_1+4\varepsilon} |Y_5(\varepsilon)(\xi_1, \xi_2)|\, d\xi_2\, ||f||_{L^\infty(\mathbb{R})} + \int_{\mathbb{R}\setminus(\xi_1-4\varepsilon,\xi_1+4\varepsilon)} |Y_5(\varepsilon)(\xi_1, \xi_2)|\, d\xi_2\, ||f||_{L^\infty(\mathbb{R})} \\
&\leq \int_{\xi_1-4\varepsilon}^{\xi_1+4\varepsilon} C d\xi_2\, ||f||_{L^\infty(\mathbb{R})} + \int_{\mathbb{R}\setminus(\xi_1-4\varepsilon,\xi_1+4\varepsilon)} \frac{C\varepsilon}{|\xi_1 - \xi_2|^{\frac{3}{2}}} d\xi_2\, ||f||_{L^\infty(\mathbb{R})} \\
&\leq 8C\varepsilon\, ||f||_{L^\infty(\mathbb{R})} + 4C\varepsilon(4\varepsilon)^{-\frac{1}{2}}\, ||f||_{L^\infty(\mathbb{R})} \\
&\leq 10C\sqrt{\varepsilon}\, ||f||_{L^\infty(\mathbb{R})}.
\end{aligned} \tag{9.65}$$

That concludes the proof. $\square$

Lemma 9.2.19 *Let $\varepsilon > 0$, $\xi_1 \in \mathbb{R}$ and $f \in \mathcal{S}(\mathbb{R})$. Then:*

$$\left|\int_{\mathbb{R}} K(\varepsilon)(\xi_1,\xi_2)f(\xi_2)d\xi_2 + 2\pi i f(\xi_1)\right| \leq C\varepsilon\, ||f||_{W^{1,\infty}(\mathbb{R})}. \tag{9.66}$$

Proof. That is a well-known property of the Gaussian e^{-x^2}. We still give a proof.

By the mean value theorem or Taylor's formula, we can find, for every $\xi_1, \xi_2 \in \mathbb{R}$, some $m(\xi_2) \in [\xi_1, \xi_2]$ fulfilling:

$$\begin{aligned}\mathrm{Re}(k_1\overline{k_2})f(\xi_2) = &\ \mathrm{Re}(k_1\overline{k_1})f(\xi_1) + (\xi_2-\xi_1)\frac{d}{d\xi}\left(\mathrm{Re}(k(\xi_1)\overline{k(\xi)})f(\xi)\right)\Big|_{\xi=m(\xi_2)}\\ =:&\ f(\xi_1) + (\xi_2-\xi_1)g(\xi_1,\xi_2).\end{aligned} \tag{9.67}$$

Corollary 8.3.7 implies $k \in W^{1,\infty}(\mathbb{R})$. We conclude:

$$||g(\xi_1,\cdot)||_{L^\infty(\mathbb{R})} \leq C\, ||f||_{W^{1,\infty}(\mathbb{R})}. \tag{9.68}$$

It follows:

$$\begin{aligned}&\left|\int_{\mathbb{R}} K(\varepsilon)(\xi_1,\xi_2)f(\xi_2)d\xi_2 + 2\pi i f(\xi_1)\right|\\ =&\left|-i\int_{\mathbb{R}} \mathrm{Re}(k_1\overline{k_2})\sqrt{\pi}\frac{e^{-\frac{(\xi_1-\xi_2)^2}{4\varepsilon^2}}}{\varepsilon}f(\xi_2)d\xi_2 + 2\pi i f(\xi_1)\right|\\ =&\sqrt{\pi}\left|\int_{\mathbb{R}} f(\xi_1)\frac{e^{-\frac{(\xi_1-\xi_2)^2}{4\varepsilon^2}}}{\varepsilon}d\xi_2 - 2\sqrt{\pi}f(\xi_1) + \int_{\mathbb{R}}(\xi_2-\xi_1)g(\xi_1,\xi_2)\frac{e^{-\frac{(\xi_1-\xi_2)^2}{4\varepsilon^2}}}{\varepsilon}d\xi_2\right|\\ =&\sqrt{\pi}\left|2\sqrt{\pi}f(\xi_1) - 2\sqrt{\pi}f(\xi_1) + \int_{\mathbb{R}}\xi g(\xi_1,\xi+\xi_1)\frac{e^{-\frac{\xi^2}{4\varepsilon^2}}}{\varepsilon}d\xi\right|\\ \leq&\sqrt{\pi}\int_{\mathbb{R}}|\xi|\,\frac{e^{-\frac{\xi^2}{4\varepsilon^2}}}{\varepsilon}d\xi\, ||g(\xi_1,\cdot)||_{L^\infty(\mathbb{R})}\\ \leq&\, C\varepsilon\int_0^\infty \xi e^{-\frac{\xi^2}{4}}d\xi\, ||f||_{W^{1,\infty}(\mathbb{R})}\\ \leq&\, C\varepsilon\, ||f||_{W^{1,\infty}(\mathbb{R})}.\end{aligned} \tag{9.69}$$

That concludes the proof. □

Lemma 9.2.20 *For every $\xi_1 \in \mathbb{R}$, $f, g \in \mathcal{S}(\mathbb{R})$ and $1 \geq \varepsilon > 0$:*

$$\left|\int_{\mathbb{R}} \eta_{\varepsilon,e}(\xi_1,\xi_2)f(\xi_2)d\xi_2 + 2\pi i f(\xi_1)\right| \leq C(\sqrt{\varepsilon} + e^{-\frac{\xi_1^2}{4\varepsilon^2}})(||f||_{W^{1,\infty}(\mathbb{R})} + ||f||_{L^1(\mathbb{R})}), \tag{9.70}$$

$$\left|\int_{\mathbb{R}} \eta_{\varepsilon,o}(\xi_1,\xi_2)g(\xi_2)d\xi_2 + 2\pi i g(\xi_1)\right| \leq C(\sqrt{\varepsilon} + e^{-\frac{\xi_1^2}{4\varepsilon^2}})(||g||_{W^{1,\infty}(\mathbb{R})} + ||g||_{L^1(\mathbb{R})}). \tag{9.71}$$

Proof. Lemma 9.2.3, lemma 9.2.12 and corollary 9.2.13 imply for $\xi_1 \in \mathbb{R}$ with $\operatorname{sgn}(\xi_1) = \operatorname{sgn}(\xi_2)$ and $\xi_1 \neq \xi_2$:

$$\begin{aligned}
\eta_{\varepsilon,e}(\xi_1,\xi_2) = &\lim_{\delta\searrow 0}(\eta_{\varepsilon,e}(\xi_1,\xi_2) - \eta_{\delta,e}(\xi_1,\xi_2) - \mu_\varepsilon(\xi_1,\xi_2) + \mu_\delta(\xi_1,\xi_2)) \\
&+ \lim_{\delta\searrow 0}\eta_{\delta,e}(\xi_1,\xi_2) + \mu_\varepsilon(\xi_1,\xi_2) - \lim_{\delta\searrow 0}\mu_\delta(\xi_1,\xi_2) \\
= &\, Y_1(\varepsilon)(\xi_1,\xi_2) + 2i\operatorname{Im}(k_1k_2)\frac{D_+(\frac{|\xi_1|+|\xi_2|}{4\varepsilon})}{\varepsilon} + 2i\operatorname{Im}(k_1\overline{k_2})\frac{D_+(\frac{|\xi_1|-|\xi_2|}{4\varepsilon})}{\varepsilon} \\
&- \sqrt{\pi}i\operatorname{Re}(k_1k_2)\frac{e^{-\frac{(\xi_1+\xi_2)^2}{4\varepsilon^2}}}{\varepsilon} - \sqrt{\pi}i\operatorname{Re}(k_1\overline{k_2})\frac{e^{-\frac{(\xi_1-\xi_2)^2}{4\varepsilon^2}}}{\varepsilon} \\
&- 4i\frac{\operatorname{Im}(k_1k_2)}{|\xi_1|+|\xi_2|} - 4i\frac{\operatorname{Im}(k_1\overline{k_2})}{|\xi_1|-|\xi_2|}.
\end{aligned} \tag{9.72}$$

Lemma 9.2.3 and lemma 9.2.7 imply for $\xi_1 \in \mathbb{R}$ with $\operatorname{sgn}(\xi_1) \neq \operatorname{sgn}(\xi_2)$:

$$\begin{aligned}
\eta_{\varepsilon,e}(\xi_1,\xi_2) = &\lim_{\delta\searrow 0}(\eta_{\varepsilon,e}(\xi_1,\xi_2) - \eta_{\delta,e}(\xi_1,\xi_2) - \mu_\varepsilon(\xi_1,\xi_2) + \mu_\delta(\xi_1,\xi_2)) \\
&+ \lim_{\delta\searrow 0}\eta_{\delta,e}(\xi_1,\xi_2) + \mu_\varepsilon(\xi_1,\xi_2) - \lim_{\delta\searrow 0}\mu_\delta(\xi_1,\xi_2) \\
= &\, Y_1(\varepsilon)(\xi_1,\xi_2).
\end{aligned} \tag{9.73}$$

We conclude for every $\xi_1 \neq \xi_2 \in \mathbb{R}$:

$$\eta_{\varepsilon,e}(\xi_1,\xi_2) = (Y_1(\varepsilon) + Y_2(\varepsilon) + Y_3(\varepsilon) + K(\varepsilon) - Y_4(\varepsilon))(\xi_1,\xi_2). \tag{9.74}$$

(9.70) follows form lemma 9.2.15 - 9.2.19. (9.71) follows completely analogously.
That concludes the proof. □

9.2.2. Unitarity

Theorem 9.2.21 *Let $f, g \in \mathcal{S}(\mathbb{R})$. Then, $G_e f, G_o f, G_e g, G_o g \in H$. Further:*

$$\langle G_e f, G_e g\rangle_H = \langle G_o f, G_o g\rangle_H = \left\langle f, (1+\cdot^2)g\right\rangle_{L^2(\mathbb{R})}, \tag{9.75}$$

$$||G_e f||_H = ||G_o f||_H = \left|\left|\sqrt{1+\cdot^2}f\right|\right|_{L^2(\mathbb{R})}. \tag{9.76}$$

Proof. We show the even case. The odd case follows analogously.
Consider $|||f||| := ||f||_{W^{1,\infty}(\mathbb{R})} + ||f||_{L^1(\mathbb{R})}$.

By Lebesgue's dominated convergence theorem:

$$\begin{aligned}
&\langle G_e f, G_e g\rangle_H \\
&= \frac{1}{2\pi}\left\langle \int_{\mathbb{R}} W_e(\xi,\cdot)f(\xi)d\xi, \int_{\mathbb{R}} W_e(\xi,\cdot)g(\xi)d\xi \right\rangle_H \\
&= \frac{1}{2\pi}\left\langle \int_{\mathbb{R}} \begin{pmatrix} L_U U_e(\xi_1,\cdot) \\ L_V V_e(\xi_1,\cdot)\end{pmatrix} f(\xi_1)d\xi_1, \int_{\mathbb{R}} \begin{pmatrix} U_e(\xi_2,\cdot) \\ V_e(\xi_2,\cdot)\end{pmatrix} g(\xi_2)d\xi_2 \right\rangle_{L^2(\mathbb{R})^2} \\
&= \frac{1}{2\pi}\left\langle \int_{\mathbb{R}} \begin{pmatrix} V_e(\xi_1,\cdot) \\ -U_e(\xi_1,\cdot)\end{pmatrix} i\,\mathrm{sgn}(\xi_1)(\xi_1^2+1) f(\xi_1)d\xi_1, \int_{\mathbb{R}} \begin{pmatrix} U_e(\xi_2,\cdot) \\ V_e(\xi_2,\cdot)\end{pmatrix} g(\xi_2)d\xi_2 \right\rangle_{L^2(\mathbb{R})^2} \\
&= \frac{i}{2\pi}\lim_{\varepsilon\searrow 0}\left\langle \int_{\mathbb{R}} \mathrm{sgn}(\xi_1)V_e(\xi_1,\cdot)(\xi_1^2+1)f(\xi_1)d\xi_1, \int_{\mathbb{R}} U_e(\xi_2,\cdot)g(\xi_2)d\xi_2 \right\rangle_\varepsilon \\
&\quad - \frac{i}{2\pi}\lim_{\varepsilon\searrow 0}\left\langle \int_{\mathbb{R}} \mathrm{sgn}(\xi_1)U_e(\xi_1,\cdot)(\xi_1^2+1)f(\xi_1)d\xi_1, \int_{\mathbb{R}} V_e(\xi_2,\cdot)g(\xi_2)d\xi_2 \right\rangle_\varepsilon \\
&= \frac{i}{2\pi}\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}}\int_{\mathbb{R}} \eta_{e,\varepsilon}(\xi_1,\xi_2)(\xi_1^2+1)f(\xi_1)g(\xi_2)d\xi_2 d\xi_1. \qquad (9.77)
\end{aligned}$$

By lemma 9.2.20:

$$\begin{aligned}
&\left|\langle G_e f, G_e g\rangle_H - \langle f, (1+\cdot^2)g\rangle_{L^2(\mathbb{R})}\right| \\
&= \lim_{\varepsilon\searrow 0}\left|\frac{i}{2\pi}\int_{\mathbb{R}}(\xi_1^2+1)f(\xi_1)\int_{\mathbb{R}}\eta_{e,\varepsilon}(\xi_1,\xi_2)g(\xi_2)d\xi_2 d\xi_1 + i^2\int_{\mathbb{R}}(\xi_1^2+1)f(\xi_1)g(\xi_1)d\xi_1\right| \\
&= \frac{1}{2\pi}\lim_{\varepsilon\searrow 0}\left|\int_{\mathbb{R}}(\xi_1^2+1)f(\xi_1)\left(\int_{\mathbb{R}}\eta_{e,\varepsilon}(\xi_1,\xi_2)g(\xi_2)d\xi_2 + 2\pi i g(\xi_1)\right)d\xi_1\right| \\
&\leq C\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}}(\xi_1^2+1)\,|f(\xi_1)|\,(\sqrt{\varepsilon}+e^{-\frac{\xi_1^2}{4\varepsilon^2}})\,|||g|||\,d\xi_1 \\
&\leq C\lim_{\varepsilon\searrow 0}\sqrt{\varepsilon}\left|\left|(\cdot^2+1)f\right|\right|_{L^1(\mathbb{R})}|||g||| + C\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}}e^{-\frac{\xi_1^2}{4\varepsilon^2}}d\xi_1\left|\left|(\cdot^2+1)f\right|\right|_{L^\infty(\mathbb{R})}|||g||| \\
&= 0 + 2C\lim_{\varepsilon\searrow 0}\varepsilon\int_{\mathbb{R}}e^{-\xi^2}d\xi\left|\left|(\cdot^2+1)f\right|\right|_{L^\infty(\mathbb{R})}|||g||| \\
&= 0. \qquad (9.78)
\end{aligned}$$

$G_e f, G_e g \in H$ holds by lemma 8.5.2. That concludes the proof. □

Definition 9.2.22 *Let $s\in\mathbb{R}$. We define the Hilbert space*

$$\hat{H}^s := \{f\in L^2(\mathbb{R}) | (1+\cdot^2)^{\frac{s}{2}}f\in L^2(\mathbb{R})\} \qquad (9.79)$$

with scalar product resp. norm:

$$\begin{aligned}
\langle f,g\rangle_{\hat{H}^s} &:= \langle (1+\cdot^2)^{\frac{s}{2}}f, (1+\cdot^2)^{\frac{s}{2}}g\rangle_{L^2(\mathbb{R})}, \qquad (9.80)\\
||f||_{\hat{H}^s} &:= \left|\left|(1+\cdot^2)^{\frac{s}{2}}f\right|\right|_{L^2(\mathbb{R})}. \qquad (9.81)
\end{aligned}$$

Definition 9.2.23 *Consider the linear subspaces $H_e, H_o \subset H$ given by:*

$$H_e := \{w \in H | w \text{ is even}\}, \tag{9.82}$$

$$H_o := \{w \in H | w \text{ is odd}\}. \tag{9.83}$$

Using these spaces we conclude the definition of G.

Definition 9.2.24 *By theorem 9.2.21, G_e and G_o extend to linear operators*

$$G_e : \hat{H}^1 \to H_e, \tag{9.84}$$

$$G_o : \hat{H}^1 \to H_o, \tag{9.85}$$

which we again denote by G_e, G_o. We define $G : \hat{H}^1 \times \hat{H}^1 \to H$ by:

$$G(f,g) = G_e f + G_o g. \tag{9.86}$$

Lemma 9.2.25 *Let $h = (f,g) \in \hat{H}^1 \times \hat{H}^1$. Then:*

$$||h||^2_{\hat{H}^1 \times \hat{H}^1} = ||f||^2_{\hat{H}^1} + ||g||^2_{\hat{H}^1} = ||Gh||^2_H. \tag{9.87}$$

Proof. By theorem 9.2.21, we only need to show $||Gh||^2_H = ||G_e f||^2_H + ||G_o g||^2_H$. That follows directly from the fact that $G_e f$ is even and $G_o g$ is odd. □

9.2.3. Alternative Definition

Lemma 9.2.26 *Let $f \in \mathcal{S}(\mathbb{R})$. Then, for every $x \in \mathbb{R}$:*

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \cos(\xi x) f(\xi) d\xi = (\mathcal{F} f_e)(x) = (\mathcal{F}^{-1} f_e)(x), \tag{9.88}$$

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \sin(\xi x) f(\xi) d\xi = i(\mathcal{F} f_o)(x) = -i(\mathcal{F}^{-1} f_o)(x). \tag{9.89}$$

Proof. We calculate:

$$\begin{aligned}\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \cos(\xi x) f(\xi) d\xi &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \cos(-\xi x) f_e(\xi) d\xi \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x} f_e(\xi) d\xi \\ &= (\mathcal{F} f_e)(x). \end{aligned} \tag{9.90}$$

Analogously:

$$\begin{aligned}\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \sin(\xi x) f(\xi) d\xi &= \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} i \sin(-\xi x) f_o(\xi) d\xi \\ &= \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x} f_o(\xi) d\xi \\ &= i(\mathcal{F} f_o)(x). \end{aligned} \tag{9.91}$$

That concludes the proof. □

Lemma 9.2.27 *Let $f, g \in \hat{H}^1$. Then, for $x \in \mathbb{R}$:*

$$(G_e f)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \begin{pmatrix} R_{e,U}(\xi, x) \\ R_{e,V}(\xi, x) \end{pmatrix} f(\xi) d\xi + \begin{pmatrix} \mathcal{F}(c_e f_e) \\ -i\mathcal{F}(c_e \operatorname{sgn}(\cdot) f_o) \end{pmatrix} + \operatorname{sgn}(x) \begin{pmatrix} i\mathcal{F}(s_e \operatorname{sgn}(\cdot) f_e) \\ \mathcal{F}(s_e f_o) \end{pmatrix}, \tag{9.92}$$

$$(G_o g)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \begin{pmatrix} R_{o,U}(\xi, x) \\ R_{o,V}(\xi, x) \end{pmatrix} g(\xi) d\xi - \operatorname{sgn}(x) \begin{pmatrix} \mathcal{F}(c_o g_e) \\ -i\mathcal{F}(c_o \operatorname{sgn}(\cdot) g_o) \end{pmatrix} - \begin{pmatrix} i\mathcal{F}(s_o \operatorname{sgn}(\cdot) g_e) \\ \mathcal{F}(s_o g_o) \end{pmatrix}. \tag{9.93}$$

The remainder terms R_e, R_o are given by definition 8.5.6.

Proof. It suffices to consider Schwartz functions. By definition 8.5.6:

$$(G_e f)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \begin{pmatrix} R_{e,U}(\xi, x) \\ R_{e,V}(\xi, x) \end{pmatrix} f(\xi) d\xi + \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} c_e(\xi) \begin{pmatrix} \cos(\xi x) \\ -i \operatorname{sgn}(\xi) \cos(\xi x) \end{pmatrix} f(\xi) d\xi + \frac{\operatorname{sgn}(x)}{\sqrt{2\pi}} \int_{\mathbb{R}} s_e(\xi) \begin{pmatrix} \operatorname{sgn}(\xi) \sin(\xi x) \\ -i \sin(\xi x) \end{pmatrix} f(\xi) d\xi, \tag{9.94}$$

$$(G_o g)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \begin{pmatrix} R_{o,U}(\xi, x) \\ R_{o,V}(\xi, x) \end{pmatrix} g(\xi) d\xi - \frac{\operatorname{sgn}(x)}{\sqrt{2\pi}} \int_{\mathbb{R}} c_o(\xi) \begin{pmatrix} \cos(\xi x) \\ -i \operatorname{sgn}(\xi) \cos(\xi x) \end{pmatrix} g(\xi) d\xi - \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} s_o(\xi) \begin{pmatrix} \operatorname{sgn}(\xi) \sin(\xi x) \\ -i \sin(\xi x) \end{pmatrix} g(\xi) d\xi. \tag{9.95}$$

Lemma 9.2.26 concludes the proof. □

9.3. Definition of F

We define operators $F_e, F_o : H^1(\mathbb{R})^2 \to \hat{H}^1$ by extending the definitions given by (9.2) and (9.3). Let $w = (u, v) \in \mathcal{S}(\mathbb{R})^2$. (9.2) can be simplified via:

$$\begin{aligned}
(F_e w)(\xi) &= \frac{1}{\sqrt{2\pi}(1+\xi^2)} \langle w, W_e(\xi, \cdot)\rangle_H \\
&= \frac{1}{\sqrt{2\pi}(1+\xi^2)} \left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} L_U U_e(\xi, \cdot) \\ L_V V_e(\xi, \cdot) \end{pmatrix} \right\rangle_{L^2(\mathbb{R})^2} \\
&= -\frac{i \operatorname{sgn}(\xi)}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(u(x)\overline{V_e(\xi, x)} - v(x)\overline{U_e(\xi, x)} \right) dx \\
&= \frac{i \operatorname{sgn}(\xi)}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(V_e(\xi, x) u(x) + U_e(\xi, x) v(x) \right) dx. \qquad (9.96)
\end{aligned}$$

In the last step, we used lemma 8.5.3. By definition 8.5.6:

$$U_e(\xi, x) = R_{e,U}(\xi, x) + c_e(\xi)\cos(\xi x) + \operatorname{sgn}(\xi x) s_e(\xi)\sin(\xi x), \qquad (9.97)$$

$$V_e(\xi, x) = R_{e,V}(\xi, x) - i\operatorname{sgn}(\xi) c_e(\xi)\cos(\xi x) - i\operatorname{sgn}(x) s_e(\xi)\sin(\xi x). \qquad (9.98)$$

It follows:

$$\begin{aligned}
(F_e w)(\xi) = {} & \frac{i \operatorname{sgn}(\xi)}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(R_{e,V}(\xi, x) u(x) + R_{e,U}(\xi, x) v(x) \right) dx + \frac{c_e(\xi)}{\sqrt{2\pi}} \int_{\mathbb{R}} \cos(\xi x) u(x) dx \\
& + \frac{i c_e(\xi)\operatorname{sgn}(\xi)}{\sqrt{2\pi}} \int_{\mathbb{R}} \cos(\xi x) v(x) dx + \frac{s_e(\xi)\operatorname{sgn}(\xi)}{\sqrt{2\pi}} \int_{\mathbb{R}} \sin(\xi x)\operatorname{sgn}(x) u(x) dx \\
& + \frac{i s_e(\xi)}{\sqrt{2\pi}} \int_{\mathbb{R}} \sin(\xi x)\operatorname{sgn}(x) v(x) dx. \qquad (9.99)
\end{aligned}$$

Analogously:

$$\begin{aligned}
(F_o w)(\xi) = {} & \frac{i \operatorname{sgn}(\xi)}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(R_{o,V}(\xi, x) u(x) + R_{o,U}(\xi, x) v(x) \right) dx \\
& - \frac{c_e(\xi)}{\sqrt{2\pi}} \int_{\mathbb{R}} \cos(\xi x)\operatorname{sgn}(x) u(x) dx - \frac{i c_e(\xi)\operatorname{sgn}(\xi)}{\sqrt{2\pi}} \int_{\mathbb{R}} \cos(\xi x)\operatorname{sgn}(x) v(x) dx \\
& - \frac{s_e(\xi)\operatorname{sgn}(\xi)}{\sqrt{2\pi}} \int_{\mathbb{R}} \sin(\xi x) u(x) dx - \frac{i s_e(\xi)}{\sqrt{2\pi}} \int_{\mathbb{R}} \sin(\xi x) v(x) dx. \qquad (9.100)
\end{aligned}$$

We can use (9.99), (9.100) and lemma 9.2.26 to define F_e and F_o on $H^1(\mathbb{R})^2$.

Definition 9.3.1 *We define $F_e, F_o : H^1(\mathbb{R})^2 \to \hat{H}^1$ by:*

$$\begin{aligned}
F_e w = {} & \frac{i \operatorname{sgn}(\xi)}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(R_{e,V}(\xi, x) u_e(x) + R_{e,U}(\xi, x) v_e(x) \right) dx \\
& + c_e \mathcal{F} u_e + i c_e \operatorname{sgn}(\xi) \mathcal{F} v_e + i s_e \operatorname{sgn}(\xi) \mathcal{F}(\operatorname{sgn}(\cdot) u_e) - s_e \mathcal{F}(\operatorname{sgn}(\cdot) v_e), \qquad (9.101) \\
F_o w = {} & \frac{i \operatorname{sgn}(\xi)}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(R_{o,V}(\xi, x) u_o(x) + R_{o,U}(\xi, x) v_o(x) \right) dx \\
& - c_o \mathcal{F}(\operatorname{sgn}(\cdot) u_o) - i c_o \operatorname{sgn}(\xi) \mathcal{F}(\operatorname{sgn}(\cdot) v_o) - i s_o \operatorname{sgn}(\xi) \mathcal{F} u_o + s_o \mathcal{F} v_o. \qquad (9.102)
\end{aligned}$$

Definition 9.3.2 *With F_e and F_o as above, we define:*

$$F : H^1(\mathbb{R})^2 \to \hat{H}^1 \times \hat{H}^1, \tag{9.103}$$
$$w \mapsto (F_e w, F_o w). \tag{9.104}$$

Lemma 9.3.3 *Let $h \in \hat{H}^1 \times \hat{H}^1$. Then:*

$$FGh = h. \tag{9.105}$$

Proof. It suffices to show $F_e G_e f = f$ and $F_o G_o g = g$ for $f, g \in \mathcal{S}(\mathbb{R})$.

For $\varepsilon > 0$ and any w_1, w_2 for which the integral converges, we define following the scalar-like product:

$$\langle w_1, w_2 \rangle_{H_\varepsilon} := \langle w_1, e^{-\varepsilon^2 \cdot^2} w_2 \rangle_H. \tag{9.106}$$

$\langle \cdot, \cdot \rangle_H$ is given by (6.4). Lemma 9.2.20 implies for $\xi \in \mathbb{R}$:

$$\begin{aligned}
(F_e G_e f)(\xi) &= \frac{\lim_{\varepsilon \searrow 0} \langle \int_{\mathbb{R}} W_e(\xi_1, \cdot) f(\xi_1) d\xi_1, W_e(\xi, \cdot) \rangle_{H_\varepsilon}}{2\pi(1+\xi^2)} \\
&= \frac{i}{2\pi} \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}} \eta_{e,\varepsilon}(\xi_1, \xi) f(\xi_1) d\xi_1 \\
&= f(\xi).
\end{aligned} \tag{9.107}$$

$F_o G_o g = g$ follows analogously. That concludes the proof. $\square$

9.4. Injectivity of F

The goal of this chapter is to prove that $F|_H$ is injective, which we formulate as the following lemma.

Lemma 9.4.1 $\ker F_e|_{H_e} = \{0\}$ *and* $\ker F_o|_{H_o} = \{0\}$.

Together with lemma 9.2.25 and lemma 9.3.3 that shows $F|_H$ and G to be unitary transformations and inverses of each other.

9.4.1. Spectral Theorem and Projection-Valued Measures

We need some preparation before proving lemma 9.4.1. Let us briefly recall the definition of projection-valued measures and how they can be used to formulate the spectral theorem for self-adjoint operators.

Definition 9.4.2 *Consider a measurable space $(X, \mathcal{A})$ and a Hilbert space $\mathcal{B}$.*
A mapping P from $\mathcal{A}$ to the set of orthogonal projections on $\mathcal{B}$ is called a ***projection-valued measure****, if and only if the following two properties are fulfilled. Firstly,*

$$P(X) = id_{\mathcal{B}}, \tag{9.108}$$

and, secondly, for every $x, y \in \mathcal{B}$, the function:

$$\mathcal{A} \to \mathbb{C}, \tag{9.109}$$
$$M \mapsto \langle P(M)x, y\rangle_{\mathcal{B}} \tag{9.110}$$

constitutes a complex measure.

Lemma 9.4.3 (Spectral Theorem) *Let $A : \mathcal{B} \supseteq \mathrm{dom}(A) \to \mathcal{B}$ be a self-adjoint operator on a Hilbert space $\mathcal{B}$. There exists a unique projection-valued measure E defined on the Borel subsets of $\sigma(A)$, such that:*

$$A = \int_{\sigma(A)} \lambda dE_\lambda. \tag{9.111}$$

That is to say, for every $x, y \in \mathcal{B}$:

$$\langle Ax, y\rangle = \int_{\sigma(A)} \lambda d\langle E_\lambda x, y\rangle. \tag{9.112}$$

Proof. See, e.g. [18]. □

Note the following lemma about the most basic properties of projection-valued measures.

Lemma 9.4.4 *Let P be a projection-valued measure on Hilbert space $\mathcal{B}$ and measurable space $(X, \mathcal{A})$. Then, $P(A)P(B) = P(A \cap B)$ for every $A, B \in \mathcal{A}$. Further, it necessarily holds $P(\emptyset) = 0$.*

Proof. By definition, for every $x \in \mathcal{B}$:

$$||P(\emptyset)x||_{\mathcal{B}}^2 = \langle P(\emptyset)x, P(\emptyset)x\rangle_{\mathcal{B}} = \langle P(\emptyset)x, x\rangle_{\mathcal{B}} = 0. \tag{9.113}$$

That proves $P(\emptyset) = 0$.

Assume for now $A \cap B = \emptyset$ and let $C := X \backslash (A \cup B)$. Given $x \in \mathcal{B}$, we define $y = P(B)x$ and conclude:

$$\begin{aligned} 0 \leq\ & ||P(A)P(B)x||_{\mathcal{B}}^2 + ||P(C)P(B)x||_{\mathcal{B}}^2 \\ &= \langle P(A)y, y\rangle_{\mathcal{B}} + \langle P(C)y, y\rangle_{\mathcal{B}} \\ &= \langle P(A \cup C)y, y\rangle_{\mathcal{B}} \\ &= \langle P(X)y, y\rangle_{\mathcal{B}} - \langle P(B)y, y\rangle_{\mathcal{B}} \\ &= 0. \end{aligned} \tag{9.114}$$

That shows $P(A)P(B) = 0$. We now consider arbitrary sets $A, B \subseteq X$:

$$\begin{aligned} P(A)P(B) &= (P(A\backslash B) + P(A \cap B))(P(B\backslash A) + P(A \cap B)) \\ &= P(A \cap B)^2 \\ &= P(A \cap B). \end{aligned} \tag{9.115}$$

That concludes the proof. □

Lemma 9.4.5 *Let P be a projection-valued measure on Hilbert space $\mathcal{B}$ and measurable space $(X, \mathcal{A})$. Consider a sequence of sets $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ with $A_n \subseteq A_m$ for every $n \geq m$. Then, for every $w \in \mathcal{B}$:*

$$P\left(\bigcap_{n=0}^{\infty} A_n\right) w = \lim_{n\to\infty} P(A_n)w. \tag{9.116}$$

Proof. Let $A := \bigcap_{n=0}^{\infty} A_n$ and $B_n := A_n \backslash A_{n+1}$. By definition:

$$\begin{aligned} ||P(A_n)w||_{\mathcal{B}}^2 &= \langle P(A_n)w, w\rangle_{\mathcal{B}} \\ &= \langle P(A)w, w\rangle_{\mathcal{B}} + \sum_{k=n}^{\infty} \langle P(B_k)w, w\rangle_{\mathcal{B}} \\ &= ||P(A)w||_{\mathcal{B}}^2 + \sum_{k=n}^{\infty} ||P(B_k)w||_{\mathcal{B}}^2 . \end{aligned} \tag{9.117}$$

It follows $\sum_{k=n}^{\infty} ||P(B_k)w||_{\mathcal{B}}^2 < \infty$ and consequently $\lim_{n\to\infty} \sum_{k=n}^{\infty} ||P(B_k)w||_{\mathcal{B}}^2 = 0$. We conclude:

$$\lim_{n\to\infty} ||P(A_n)w - P(A)w||_{\mathcal{B}}^2 = \lim_{n\to\infty} \sum_{k=n}^{\infty} ||P(B_k)w||_{\mathcal{B}}^2 = 0. \tag{9.118}$$

That concludes the proof. □

Lemma 9.4.6 *Let $A : \mathcal{B} \supseteq \mathrm{dom}(A) \to \mathcal{B}$ be a self-adjoint operator on a Hilbert space $\mathcal{B}$ with spectral measure E. Let further $\lambda \in \sigma(A)$ and $x \in \mathcal{B}$. Then, $y := E_{\{\lambda\}}x$ fulfils:*

$$Ay = \lambda y. \tag{9.119}$$

Proof. By definition, for every $z \in \mathcal{B}$:

$$\langle Ay, z\rangle = \int_{\sigma(A)} \mu d\langle E_\mu y, z\rangle. \tag{9.120}$$

Let us examine the measure $M \mapsto \langle E_M y, z\rangle$. If $\lambda \notin M$:

$$\langle E_M y, z\rangle = \langle E_M E_{\{\lambda\}} x, z\rangle = \langle E_\emptyset x, z\rangle = 0. \tag{9.121}$$

If $\lambda \in M$:

$$\langle E_M y, z\rangle = \langle E_M E_{\{\lambda\}} x, z\rangle = \langle E_{\{\lambda\}} x, z\rangle = \langle y, z\rangle. \tag{9.122}$$

We conclude:

$$\langle E_M y, z\rangle = \delta_\lambda(M)\langle y, z\rangle. \tag{9.123}$$

Hereby, δ_λ denotes the Dirac measure. In summary:

$$\langle Ay, z\rangle = \int_{\sigma(A)} \mu\langle y, z\rangle d\delta_\lambda(\mu) = \lambda\langle y, z\rangle. \tag{9.124}$$

That concludes the proof. □

Let us briefly discuss the basic idea of proof for lemma 9.4.1.

It is well-known that the spectral measure E_M corresponding to the Laplace operator $-\Delta : L^2(\mathbb{R}) \supset H^2(\mathbb{R}) \to L^2(\mathbb{R})$ can be expressed via the Fourier transform:

$$E_M f = \mathcal{F}^{-1}\left(\mathbb{1}_M \mathcal{F} f\right). \tag{9.125}$$

Hereby, $\mathbb{1}_M$ denotes the characteristic function of a given set $M \subseteq \sigma(-\Delta) = [0,\infty)$.

As it turns out (but will not be proven in this thesis), for $M \subset \mathbb{R}\backslash(-1,1)$, the spectral measure E_M corresponding to $-iL$ can similarly be expressed via:

$$E_M w = G\left(\mathbb{1}_M F w\right). \tag{9.126}$$

This illustrates the natural connection between F and G on the one hand and the spectral measure on the other hand.

We will show $F|_H$ to be injective by examining the spectral measure E_λ corresponding to $-iL$, In particular, the spectral density $\lim_{\varepsilon\searrow 0} \frac{1}{\varepsilon} E_{[x_0-\varepsilon, x_0+\varepsilon]} w$ for $w \in H$ and $x_0 \in \sigma(-iL)$ will be of importance.

9.4.2. Regularity

We characterise the image of $\mathcal{S}(\mathbb{R})^2$ under the map GF. That allows us to only consider sufficiently smooth functions when proving lemma 9.4.1.

Definition 9.4.7 *Let $\hat{\mathcal{S}} \subset \hat{H}^1$ and $\hat{\hat{\mathcal{S}}} \subset H^1(\mathbb{R})$ be given by:*

$$\hat{\mathcal{S}} := \left\{ f \in C(\mathbb{R}) \;\middle|\; f(0) = 0 \wedge \forall k, l \geq 0 : \sup_{\xi \neq 0} \left|(1+\xi^2)^{\frac{l}{2}} f^{(k)}(\xi)\right| < \infty \right\}, \tag{9.127}$$

$$\hat{\hat{\mathcal{S}}} := \left\{ u \in C(\mathbb{R}) \;\middle|\; \forall k \geq 0 : \sup_{x \in \mathbb{R}} \left|(1+x^2) u^{(k)}(x)\right| < \infty \right\}. \tag{9.128}$$

Lemma 9.4.8 *Let $w \in \mathcal{S}(\mathbb{R})^2$. Then $F_e w, F_o w \in \hat{\mathcal{S}}$.*

Proof. We show the even case. The odd case is handled analogously.

Let $f = F_e w$. $f(0) = 0$ follows directly from lemma 8.5.4. Due to $LW_e(\xi,\cdot) = i\,\mathrm{sgn}(\xi)(1+\xi^2)W_e(\xi,\cdot)$, we only need to show:

$$\forall k \geq 0\ \exists C > 0\ \forall \xi \neq 0 : \left|\frac{d^k}{d\xi^k} f(\xi)\right| < C. \tag{9.129}$$

That follows directly from definition 9.3.1, corollary 8.3.7 and lemma 8.5.10. ☐

Lemma 9.4.9 *Let $w \in \mathcal{S}(\mathbb{R})^2$. Then, $GFw \in \hat{\hat{\mathcal{S}}}^2 \cap H$.*

Proof. Let $\hat{w} := GFw$. By definition, for $k \geq 0$:

$$(-iL)^k \hat{w} = GF(-iL)^k w. \tag{9.130}$$

That already shows $\hat{w} \in C^\infty(\mathbb{R})^2$. It only remains to show, for every $x \in \mathbb{R}$:

$$|\hat{w}(x)| \leq |\hat{w}_e(x)| + |\hat{w}_o(x)| \leq \frac{C}{x^2+1}. \tag{9.131}$$

Let $f = F_e w$, $g = F_o w$ and in consequence $G_e f = \hat{w}_e$, $G_o g = \hat{w}_o$. By definition:

$$\sqrt{2\pi}(G_e f)(x) = \int_{\mathbb{R}} W_e(\xi, x) f(\xi) d\xi = \int_{\mathbb{R}} \begin{pmatrix} U_e(\xi, x) \\ V_e(\xi, x) \end{pmatrix} f(\xi) d\xi, \tag{9.132}$$

$$\sqrt{2\pi}(G_o g)(x) = \int_{\mathbb{R}} W_o(\xi, x) g(\xi) d\xi = \int_{\mathbb{R}} \begin{pmatrix} U_o(\xi, x) \\ V_o(\xi, x) \end{pmatrix} g(\xi) d\xi. \tag{9.133}$$

By definition 8.5.6:

$$\begin{aligned} &\int_{\mathbb{R}} U_e(\xi, x) f(\xi) d\xi - \int_{\mathbb{R}} R_{e,U}(\xi, x) f(\xi) d\xi \\ = &\int_{\mathbb{R}} c_e(\xi) f(\xi) \cos(\xi x) d\xi + \int_{\mathbb{R}} s_e(\xi) f(\xi) \sin(|\xi x|) d\xi \\ = &\, 2\int_0^\infty c_e(\xi) f_e(\xi) \cos(\xi x) d\xi + 2\int_0^\infty s_e(\xi) f_e(\xi) \sin(\xi\,|x|) d\xi. \end{aligned} \tag{9.134}$$

Analogously:

$$\begin{aligned} &\int_{\mathbb{R}} V_e(\xi, x) f(\xi) d\xi - \int_{\mathbb{R}} R_{e,V}(\xi, x) f(\xi) d\xi \\ = &-2i\int_0^\infty c_e(\xi) f_o(\xi) \cos(\xi x) d\xi - 2i\int_0^\infty s_e(\xi) f_o(\xi) \sin(\xi\,|x|) d\xi, \end{aligned} \tag{9.135}$$

$$\begin{aligned} &\int_{\mathbb{R}} U_o(\xi, x) g(\xi) d\xi - \int_{\mathbb{R}} R_{o,U}(\xi, x) g(\xi) d\xi \\ = &-\operatorname{sgn}(x)\int_0^\infty c_o(\xi) g_e(\xi) \cos(\xi x) d\xi - \operatorname{sgn}(x)\int_0^\infty s_o(\xi) g_e(\xi) \sin(\xi\,|x|) d\xi, \end{aligned} \tag{9.136}$$

$$\begin{aligned} &\int_{\mathbb{R}} V_o(\xi, x) g(\xi) d\xi - \int_{\mathbb{R}} R_{o,V}(\xi, x) g(\xi) d\xi \\ = &\, 2i\operatorname{sgn}(x)\int_0^\infty c_o(\xi) g_o(\xi) \cos(\xi x) d\xi + 2i\operatorname{sgn}(x)\int_0^\infty s_o(\xi) g_o(\xi) \sin(\xi\,|x|) d\xi. \end{aligned} \tag{9.137}$$

By lemma 8.5.9 and lemma 9.4.8:

$$\begin{aligned} &\left|\int_{\mathbb{R}} R_{e,U}(\xi, x) f(\xi) d\xi\right|, \left|\int_{\mathbb{R}} R_{e,V}(\xi, x) f(\xi) d\xi\right|, \\ &\left|\int_{\mathbb{R}} R_{o,U}(\xi, x) g(\xi) d\xi\right|, \left|\int_{\mathbb{R}} R_{o,V}(\xi, x) g(\xi) d\xi\right| \leq \frac{C}{x^2+1}. \end{aligned} \tag{9.138}$$

Hence, to prove (9.131) it suffices to show:

$$\begin{aligned}
&\forall f \in \hat{\mathcal{S}}\ \exists C > 0\ \forall x \geq 0 : \\
&\left|\int_0^\infty c_e(\xi) f(\xi)\cos(\xi x) d\xi\right| + \left|\int_0^\infty s_e(\xi) f(\xi)\sin(\xi x) d\xi\right| \leq \frac{C}{x^2+1}, \qquad (9.139)\\
&\left|\int_0^\infty c_o(\xi) f(\xi)\cos(\xi x) d\xi\right| + \left|\int_0^\infty s_o(\xi) f(\xi)\sin(\xi x) d\xi\right| \leq \frac{C}{x^2+1}. \qquad (9.140)
\end{aligned}$$

For $x < 1$, (9.139) and (9.140) follow from corollary 8.3.7. We calculate for $x \geq 1$, using $f(0) = 0$:

$$\begin{aligned}
&(1+x^2)\int_0^\infty c_e(\xi) f(\xi)\cos(\xi x) d\xi \\
=\ & -(1+x^2)\int_0^\infty \frac{d}{d\xi}(c_e(\xi) f(\xi))\frac{\sin(\xi x)}{x} d\xi - (1+x^2)c_e(0)\underbrace{f(0)}_{=0}\frac{\sin(0x)}{x} \\
=\ & \frac{1+x^2}{x^2}\int_0^\infty \frac{d^2}{d\xi^2}(c_e(\xi) f(\xi))\cos(\xi x) d\xi + \frac{1+x^2}{x^2}\lim_{\xi\searrow 0}\frac{d}{d\xi}(c_e(\xi) f(\xi)). \qquad (9.141)
\end{aligned}$$

Corollary 8.3.7 implies:

$$(1+x^2)\left|\int_0^\infty c_e(\xi) f(\xi)\cos(\xi x) d\xi\right| \leq C. \qquad (9.142)$$

The rest of (9.139) and (9.140) follows analogously. That concludes the proof. □

9.4.3. Variation of Constants and Wronskian for $-iLw = w_0$

Lemma 9.4.10 (Variation of constants) *Let $A : \mathbb{R} \to \mathbb{C}^{n\times n}$ and $b : \mathbb{R} \to \mathbb{C}^n$ be continuous functions and $\Phi(x) = (y_1(x), y_2(x), .., y_n(x))$ a solution matrix of the homogeneous equation $y'(x) = A(x)y(x)$.*

Let further $\Phi_k(x)$ be the matrix one obtains when replacing the kth column of $\Phi(x)$ by $b(x)$. Then:

$$y(x) := \sum_{k=1}^n c_k(x) y_k(x), \qquad (9.143)$$

with:

$$c_k(x) := \int_{x_0}^x \frac{\det \Phi_k(s)}{\det \Phi(s)} ds, \qquad (9.144)$$

solves the inhomogeneous equation $y'(x) = A(x)y(x) + b(x)$, $y(x_0) = 0$.

Proof. See e.g. [33]. □

To be able to make use of lemma 9.4.10, we reformulate $-iLw = w_0$ for $w = (u,v) \in C^2(\mathbb{R})^2$ and $w_0 = (u_0, v_0) \in C(\mathbb{R})^2$. By (1.35), $-iLw = w_0$ can be stated as:

$$\begin{pmatrix} \Delta v - v + Q^{p-1}v \\ -\Delta u + u - pQ^{p-1}u \end{pmatrix} = i \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}. \tag{9.145}$$

This can be easily restated in a form compatible with lemma 9.4.10:

$$\frac{d}{dx}\begin{pmatrix} u \\ u' \\ v \\ v' \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 - pQ^{p-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 - pQ^{p-1} & 0 \end{pmatrix}}_{=:A} \begin{pmatrix} u \\ u' \\ v \\ v' \end{pmatrix} + \begin{pmatrix} 0 \\ -iv_0 \\ 0 \\ iu_0 \end{pmatrix}. \tag{9.146}$$

Hence $-iLw = \lambda w + w_0$ is given by:

$$\frac{d}{dx}\begin{pmatrix} u \\ u' \\ v \\ v' \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 - pQ^{p-1} & 0 & -i\lambda & 0 \\ 0 & 0 & 0 & 1 \\ i\lambda & 0 & 1 - pQ^{p-1} & 0 \end{pmatrix}}_{=:A_\lambda} \begin{pmatrix} u \\ u' \\ v \\ v' \end{pmatrix} + \begin{pmatrix} 0 \\ -iv_0 \\ 0 \\ iu_0 \end{pmatrix}. \tag{9.147}$$

Lemma 9.4.11 *Let $\lambda \in \mathbb{R}$ and $A_\lambda : \mathbb{R} \to \mathbb{C}^{4\times 4}$ be as in* (9.147)*. Let $\Phi = (y_1, y_2, y_3, y_4)$ be a solution matrix of the homogeneous equation $y'(x) = A_\lambda(x)y(x)$. Then, $x \mapsto \det \Phi(x)$ is constant.*

Proof. Let $D_0 := \det \Phi$. It follows:

$$\begin{aligned} D_0' &= \det(y_1', y_2, y_3, y_4) + \det(y_1, y_2', y_3, y_4) + \det(y_1, y_2, y_3', y_4) + \det(y_1, y_2, y_3, y_4') \\ &=: D_1. \end{aligned} \tag{9.148}$$

We also define:

$$D_2 := \det(y_1', y_2', y_3, y_4) + \det(y_1', y_2, y_3', y_4) + .. + \det(y_1, y_2, y_3', y_4'), \tag{9.149}$$

$$D_3 := \det(y_1', y_2', y_3', y_4) + \det(y_1', y_2', y_3, y_4') + .. + \det(y_1, y_2', y_3', y_4'), \tag{9.150}$$

$$D_4 := \det(y_1', y_2', y_3', y_4'). \tag{9.151}$$

With $\Phi' = (y_1', y_2', y_3', y_4')$, it follows for $\mu \in \mathbb{C}$:

$$\det(\Phi + \mu\Phi') = \sum_{k=0}^{4} \mu^k D_k. \tag{9.152}$$

We conclude:

$$
\begin{aligned}
4D_1 &= \sum_{k=0}^{4} D_k \sum_{l=0}^{3} (i^{k-1})^l \\
&= \sum_{l=0}^{3} i^{-l} \sum_{k=0}^{4} (i^l)^k D_k \\
&= \sum_{l=0}^{3} i^{-l} \det(\Phi + i^l \Phi') \\
&= D_0 \sum_{l=0}^{3} i^{-l} \det(\operatorname{Id} + i^l A_\lambda) \\
&= D_0 \sum_{l=0}^{3} i^{-l} \det(i^{-l} \operatorname{Id} + A_\lambda). \qquad (9.153)
\end{aligned}
$$

We calculate for every $\mu \in \mathbb{C}$:

$$
\begin{aligned}
\det(A_\lambda - \mu \operatorname{Id}) &= \det \begin{pmatrix} -\mu & 1 & 0 & 0 \\ 1 - pQ^{p-1} & -\mu & -i\lambda & 0 \\ 0 & 0 & -\mu & 1 \\ i\lambda & 0 & 1 - pQ^{p-1} & -\mu \end{pmatrix} \\
&= \det \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 - pQ^{p-1} - \mu^2 & -\mu & -i\lambda & 0 \\ 0 & 0 & 0 & 1 \\ i\lambda & 0 & 1 - pQ^{p-1} - \mu^2 & -\mu \end{pmatrix} \\
&= \det \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 - pQ^{p-1} - \mu^2 & 0 & -i\lambda & 0 \\ 0 & 0 & 0 & 1 \\ i\lambda & 0 & 1 - pQ^{p-1} - \mu^2 & 0 \end{pmatrix} \\
&= \det \begin{pmatrix} 1 - pQ^{p-1} - \mu^2 & -i\lambda \\ i\lambda & 1 - pQ^{p-1} - \mu^2 \end{pmatrix} \\
&= (1 - pQ^{p-1} - \mu^2)^2 - \lambda^2. \qquad (9.154)
\end{aligned}
$$

(9.153) and (9.154) imply $4D_1 = 0$. (9.148) then implies $\frac{d}{dx}(\det \Phi) = 0$.
That concludes the proof. $\square$

9.4.4. Definition of Φ

Let $w_1 = (u_1, v_1) : [0, \infty) \times \mathbb{R} \to \mathbb{C}^2$ be as in theorem 2.4.7 and lemma 2.4.9.

By theorem 2.4.7, w_1 solves (2.5), (2.6) and exhibits exponential decay for $x \to -\infty$. Analogously to definition 8.5.1, if we define

$$U_1 := u_1 + v_1, \qquad (9.155)$$

$$V_1 := i(u_1 - v_1), \qquad (9.156)$$

then $W_1 = (U_1, V_1)$ solves $Lw = i(1+\xi^2)w$. By symmetry, $W_2 = (U_2, V_2)$ given by $W_2(x) := W_1(-x)$ saves $Lw = i(1+\xi^2)w$ while exhibiting exponential decay for $x \to \infty$.

Definition 9.4.12 *Let $W_1 = (U_1, V_1), W_2 = (U_2, V_2) : [0,\infty) \times \mathbb{R} \to \mathbb{C}^2$ be as above. We define $\Phi = (y_1, y_2, y_3, y_4) : (0,\infty) \times \mathbb{R} \to \mathbb{C}^{4\times 4}$ by:*

$$\Phi := \begin{pmatrix} U_1 & U_2 & U_e & U_o \\ \partial_x U_1 & \partial_x U_2 & \partial_x U_e & \partial_x U_o \\ V_1 & V_2 & V_e & V_o \\ \partial_x V_1 & \partial_x V_2 & \partial_x V_e & \partial_x V_o \end{pmatrix}. \tag{9.157}$$

Definition 9.4.13 *For $\xi \geq 0$, let $S = S(\xi) = (s_{nm})_{1\leq n,m\leq 4}$ be as in definition 8.2.1. We define for $\varepsilon > 0$:*

$$N := \{\xi \geq 0 | s_{21}(\xi) = 0\}, \tag{9.158}$$

$$\tilde{N} := \{1+\xi^2 | \xi \in N\}. \tag{9.159}$$

Lemma 9.4.14 *Let $\xi > 0$, $\lambda = 1+\xi^2$ and $A_\lambda : \mathbb{R} \to \mathbb{C}^{4\times 4}$ be as in (9.147). Then, $\Phi(\xi, \cdot)$ is a solution matrix of $y' = A_\lambda y$.*

Proof. Follows from theorem 2.4.7, lemma 2.4.9 and lemma 8.5.2. □

Lemma 9.4.15 *N and $\tilde{N}$ are discrete sets.*

Proof. s_{21} is analytical by lemma 8.2.4. Hence, either N and $\tilde{N}$ are discrete or $s_{21}(\xi) = 0$ for every $\xi \in \mathbb{R}$. Assume the latter holds true.

Let $K_\xi = K_{e,\xi} \oplus K_{o,\xi}$ be as in definition 8.3.1. By definition, for every $\xi \geq 0$:

$$(1,0,0,0) \in K_\xi. \tag{9.160}$$

By lemma 8.2.2:

$$(1+s_{11}(\xi), 0, s_{31}(\xi), s_{41}(\xi)) \in K_{e,\xi}, \tag{9.161}$$

$$(1-s_{11}(\xi), 0, -s_{31}(\xi), -s_{41}(\xi)) \in K_{o,\xi}. \tag{9.162}$$

By lemma 8.3.2, we can choose $\xi_1 \geq 0$, such that $\dim K_{e,\xi} = \dim K_{o,\xi} = 1$ for every $\xi \geq \xi_1$. Consequently, we find continuous coefficient functions $\alpha_e, \alpha_o : [\xi_1, \infty) \to \mathbb{C}$, such that, for $\xi \geq \xi_1$ and $x \in \mathbb{R}$:

$$W_e(\xi, x) = \alpha_e(\xi) w_{1+s_{11}(\xi),0,s_{31}(\xi),s_{41}(\xi)}(\xi, x), \tag{9.163}$$

$$W_o(\xi, x) = \alpha_o(\xi) w_{1-s_{11}(\xi),0,-s_{31}(\xi),-s_{41}(\xi)}(\xi, x). \tag{9.164}$$

By theorem 8.3.5:

$$\alpha_e(\xi) s_{31}(\xi) = c_e(\xi), \tag{9.165}$$

$$\alpha_e(\xi) s_{41}(\xi) = -s_e(\xi)\frac{\xi}{1+\xi}, \tag{9.166}$$

$$-\alpha_o(\xi) s_{31}(\xi) = c_o(\xi), \tag{9.167}$$

$$-\alpha_o(\xi) s_{41}(\xi) = -s_o(\xi)\frac{\xi}{1+\xi}. \tag{9.168}$$

Theorem 8.3.5 further states:

$$\lim_{\xi\to\infty}(c_e, s_e, c_o, s_o)(\xi) = (1, 0, 0, 1). \tag{9.169}$$

Hence, for sufficiently large ξ, we have $\alpha_e(\xi) \neq 0$ and $\alpha_o(\xi) \neq 0$.

Again, for sufficiently large ξ, (9.165), (9.166) and (9.169) imply $|s_{31}(\xi)| > |s_{41}(\xi)|$. Analogously, (9.167), (9.168) and (9.169) imply $|s_{31}(\xi)| < |s_{41}(\xi)|$.

That constitutes a contradiction. □

Lemma 9.4.16 *Let $\xi > 0$, $\xi \notin N$. Then, for every $x \in \mathbb{R}$:*

$$C_\xi^{-1} e^{\sqrt{2+\xi^2}x} \leq \ |y_1(\xi, x)| \leq C_\xi e^{\sqrt{2+\xi^2}x}, \tag{9.170}$$

$$C_\xi^{-1} e^{-\sqrt{2+\xi^2}x} \leq \ |y_2(\xi, x)| \leq C_\xi e^{-\sqrt{2+\xi^2}x}. \tag{9.171}$$

Proof. Let $w_1 = (u_1, v_1) : [0, \infty) \times \mathbb{R} \to \mathbb{R}^2$ be as in theorem 2.4.7. It suffices to show:

$$C_\xi^{-1} e^{\sqrt{2+\xi^2}x} \leq |w_1(\xi, x)| + \frac{1}{1+\xi}\left|\frac{d}{dx} w_1(\xi, x)\right| \leq C_\xi e^{\sqrt{2+\xi^2}x}. \tag{9.172}$$

Theorem 2.4.7 and lemma 2.4.9 allow us to find $x_0 \leq \omega$, such that (9.172) holds for $x \leq x_0$. By definition 8.2.1:

$$w_1(\xi, x) = \sum_{k=1}^{4} s_{k1}(\xi) w_k(\xi, -x). \tag{9.173}$$

$\xi \notin N$ implies $s_{21}(\xi) \neq 0$. By theorem 2.4.7 and lemma 2.4.9, we find $x_1 = x_{1,\xi} \geq -\omega$, such that (9.172) holds for $x \geq x_1$.

For $x \in [x_0, x_1]$, (9.172) follows from lemma 8.1.2. That concludes the proof. □

Lemma 9.4.17 *Let $\xi > 0$ and $x \in \mathbb{R}$. Then:*

$$C_\xi^{-1} \leq |y_3(\xi, x)| \leq C_\xi, \tag{9.174}$$

$$C_\xi^{-1} \leq |y_4(\xi, x)| \leq C_\xi. \tag{9.175}$$

Proof. It suffices to show:

$$C_\xi^{-1} \leq |W_e(\xi, x)| + \frac{1}{1+\xi}\left|\frac{d}{dx} W_e(\xi, x)\right| \leq C_\xi, \tag{9.176}$$

$$C_\xi^{-1} \leq |W_o(\xi, x)| + \frac{1}{1+\xi}\left|\frac{d}{dx} W_o(\xi, x)\right| \leq C_\xi. \tag{9.177}$$

By symmetry, we only need to consider $x \leq 0$. (9.176) and (9.177) follow from lemma 8.5.9 and lemma 8.1.2. □

Lemma 9.4.18 *Let $\xi > 0$, $\xi \notin N$ and $x \in \mathbb{R}$. Then,* $\det \Phi(\xi, x) \neq 0$.

Proof. Let $\vec{c} = (c_1, c_2, c_3, c_4)^T \in \mathbb{C}^4$, $x_0 \in \mathbb{R}$ and assume $\Phi(\xi, x_0)\vec{c} = 0$. As $x \mapsto \Phi(\xi, x)\vec{c}$ solves $y' = A_\lambda y$, it follows for every $x \in \mathbb{R}$:

$$y(\xi, x) := \Phi(\xi, x)\vec{c} = (0,0,0,0)^T. \tag{9.178}$$

By lemma 9.4.16 and lemma 9.4.17:

$$\begin{aligned} 0 = |y(\xi,x)| &\geq |c_1|\,|y_1(\xi,x)| - |c_2|\,|y_2(\xi,x)| - |c_3|\,|y_3(\xi,x)| - |c_4|\,|y_4(\xi,x)| \\ &\geq C_\xi^{-1}\,|c_1|\,e^{\sqrt{2+\xi^2}x} - |c_2|\,|y_1(\xi,-x)| - C_\xi(|c_3| + |c_4|) \\ &\geq C_\xi^{-1}\,|c_1|\,e^{\sqrt{2+\xi^2}x} - C_\xi\,|c_2|\,e^{-\sqrt{2+\xi^2}x} - C_\xi(|c_3| + |c_4|). \end{aligned} \tag{9.179}$$

By considering $x >> 0$ that implies $c_1 = 0$. Analogously:

$$0 = |y(\xi,x)| \geq C_\xi\,|c_2|\,e^{-\sqrt{2+\xi^2}x} - C_\xi^{-1}\,|c_1|\,e^{\sqrt{2+\xi^2}x} - C_\xi(|c_3| + |c_4|). \tag{9.180}$$

That implies $c_2 = 0$. It follows $c_3 y_3 + c_4 y_4 = 0$. By definition $c_3 y_3 + c_4 y_4 = 0$ is equivalent to:

$$\forall x \in \mathbb{R} : c_3 W_e(\xi,x) + c_4 W_o(\xi,x) = 0. \tag{9.181}$$

As $W_e(\xi,\cdot)$ is even and $W_o(\xi,\cdot)$ is odd, it follows:

$$\forall x \in \mathbb{R} : c_3 W_e(\xi,x) = c_4 W_o(\xi,x) = 0. \tag{9.182}$$

We conclude $c_3 = c_4 = 0$ and the proof. ☐

Lemma 9.4.19 *Let $\xi > 0$, $\xi \notin N$, $x \leq 0$ and $c_1, c_2, c_3, c_4 \in \mathbb{C}$. Then:*

$$|c_1|\,e^{\sqrt{2+\xi^2}x} + |c_2|\,e^{-\sqrt{2+\xi^2}x} + |c_3| + |c_4| \leq C_\xi \left|\sum_{l=1}^{4} c_l y_l\right|. \tag{9.183}$$

Proof. We calculate:

$$\begin{aligned} \left|\sum_{l=1}^{4} c_l y_l\right| &= \left|\sum_{l=1}^{4} c_l\,|y_l|\,\frac{y_l}{|y_l|}\right| \\ &= \left|\sum_{l=1}^{4} \frac{c_l\,|y_l|}{\sum_{n=1}^{4}|c_n|\,|y_n|}\,\frac{y_l}{|y_l|}\right| \sum_{n=1}^{4}|c_n|\,|y_n| \\ &\geq \inf_{|k_1|+|k_2|+|k_3|+|k_4|=1} \left|\sum_{l=1}^{4} k_l\,\frac{y_l}{|y_l|}\right| \sum_{n=1}^{4}|c_n|\,|y_n| \\ &\geq \frac{1}{4} \inf_{|k_1|^2+|k_2|^2+|k_3|^2+|k_4|^2=1} \left|\sum_{l=1}^{4} k_l\,\frac{y_l}{|y_l|}\right| \sum_{n=1}^{4}|c_n|\,|y_n| \\ &= \frac{1}{4}\left|\det\left(\frac{y_1}{|y_1|}, \frac{y_2}{|y_2|}, \frac{y_3}{|y_3|}, \frac{y_4}{|y_4|}\right)\right| \sum_{n=1}^{4}|c_n|\,|y_n| \\ &= \frac{1}{4}\,|\det\Phi| \prod_{l=1}^{4}|y_l|^{-1} \sum_{n=1}^{4}|c_n|\,|y_n|. \end{aligned} \tag{9.184}$$

Lemma 9.4.11, lemma 9.4.16, lemma 9.4.17 and lemma 9.4.18 conclude the proof. ☐

Lemma 9.4.20 *Let $w, w_0 \in H$, $1 > \varepsilon > 0$ and $\xi > 0$, $\xi \notin N$, $\lambda = 1 + \xi^2$. Assume:*

1. $(-iL - \lambda)w = w_0$,
2. $||w||_H \leq \frac{1}{\varepsilon}$,
3. $||w_0||_H \leq \varepsilon$.

Then:

$$||w||_{W^{1,\infty}(\mathbb{R})} \leq C_\xi. \tag{9.185}$$

Proof. Let $(u, v) := w$ and $(u_0, v_0) := w_0$.

We show $||w||_{W^{1,\infty}(-\infty,0)} \leq C_\xi$. By considering $\tilde{w}(x) := w(-x)$ and $\tilde{w}_0(x) := w_0(-x)$, that also implies $||w||_{W^{1,\infty}(0,\infty)} \leq C_\xi$.

By lemma 9.4.10 and lemma 9.4.18, we find continuous coefficient functions $c_l : \mathbb{R} \to \mathbb{C}$, $l \in \{1, 2, 3, 4\}$, such that for every $x \in \mathbb{R}$:

$$(u, u', v, v')(x) = \sum_{l=1}^{4} c_l(x) y_l(\xi, x). \tag{9.186}$$

By lemma 9.4.16 and lemma 9.4.17, it suffices to show for $x \leq 0$:

$$|c_1(x)|\, e^{\sqrt{2+\xi^2}x}, |c_2(x)|\, e^{-\sqrt{2+\xi^2}x}, |c_3(x)|\,, |c_4(x)| \leq C_\xi, \tag{9.187}$$

as well as:

$$|\partial_x c_1(x)|\, e^{\sqrt{2+\xi^2}x}, |\partial_x c_2(x)|\, e^{-\sqrt{2+\xi^2}x}, |\partial_x c_3(x)|\,, |\partial_x c_4(x)| \leq C_\xi. \tag{9.188}$$

Inspired by lemma 9.4.10, we define $\Phi_k(x)$ to be the matrix one obtains when replacing the kth column of $\Phi(\xi, x)$ by $b(x) := (0, -iv_0, 0, iu_0)^T$.

By lemma 9.4.16 and lemma 9.4.17:

$$|\det \Phi_1| \leq |w_0|\,|y_2|\,|y_3|\,|y_4| \leq C_\xi\,|w_0|\, e^{-\sqrt{2+\xi^2}x}, \tag{9.189}$$

$$|\det \Phi_2| \leq C_\xi\,|w_0|\, e^{\sqrt{2+\xi^2}x}, \tag{9.190}$$

$$|\det \Phi_3| \leq C_\xi\,|w_0|\,, \tag{9.191}$$

$$|\det \Phi_4| \leq C_\xi\,|w_0|\,. \tag{9.192}$$

By lemma 9.4.18, $K = K_\xi := \det \Phi \neq 0$. Lemma 9.4.10 and (9.147) imply for every $x, y \in \mathbb{R}$:

$$c_k(x) - c_k(y) = \int_y^x \frac{\det \Phi_k(z)}{K_\xi} dz. \tag{9.193}$$

Using the Sobolev embedding $||w_0||_{L^\infty} \leq ||w_0||_{H^1} \leq C\,||w_0||_H$, (9.188) follows from (9.189) - (9.193).

It remains to prove (9.187). Lemma 9.4.19 and $w \in L^2(\mathbb{R})$ imply $c_1 e^{\sqrt{2+\xi^2}x} \in L^2(\mathbb{R})$. It follows:

$$\begin{aligned}
|c_1(x)| &\leq \int_x^\infty \frac{|\det \Phi_1(z)|}{|K|} dz \\
&\leq \frac{C}{|K|} \int_x^\infty |w_0(z)|\, e^{-\sqrt{2+\xi^2}z} dz \\
&\leq \frac{C}{|K|} \left|\left|w_0\right|\right|_{L^2(\mathbb{R})} \left|\left| e^{-\sqrt{2+\xi^2}z} \right|\right|_{L^2_z(x,\infty)} \\
&\leq \frac{C^2}{\sqrt{2+\xi^2}\,|K|} \varepsilon e^{-\sqrt{2+\xi^2}x}.
\end{aligned} \tag{9.194}$$

We conclude $|c_1(x)|\, e^{\sqrt{2+\xi^2}x} \leq C\varepsilon$. Analogously, note $c_2 e^{-\sqrt{2+\xi^2}x} \in L^2(\mathbb{R})$ and:

$$|c_2(x)| \leq \int_{-\infty}^x \frac{|\det \Phi_2(z)|}{|K|} dz \leq \frac{C}{\sqrt{2+\xi^2}\,|K|} \varepsilon e^{\sqrt{2+\xi^2}x}. \tag{9.195}$$

It follows $|c_2(x)|\, e^{-\sqrt{2+\xi^2}x} \leq C\varepsilon$.

To prove $|c_3(x)|, |c_4(x)| \leq C$, fix any $x_0 \in \mathbb{R}$ with $|c_3(x_0)| \geq 1$. By (9.193):

$$|c_3(x) - c_3(x_0)| \leq \frac{C}{|K|} \int_{x_0}^x |w_0(z)|\, dz \leq \frac{C^2}{|K|} \left|\left|1\right|\right|_{L^2(x_0,x)} \left|\left|w_0\right|\right|_H \leq \frac{C^2}{|K|} |x - x_0|^{\frac{1}{2}}\, \varepsilon. \tag{9.196}$$

With $C > 0$ as in (9.196), we define $\delta := \frac{|K|}{2C^2} > 0$. (9.196) implies for $x \in (x_0 - \frac{\delta^2}{\varepsilon^2}, x_0 + \frac{\delta^2}{\varepsilon^2})$:

$$|c_3(x)| + \frac{1}{2} \geq |c_3(x_0)| \geq \frac{1}{2}\, |c_3(x_0)| + \frac{1}{2}. \tag{9.197}$$

We conclude:

$$|c_3(x)| \geq \frac{1}{2}\, |c_3(x_0)|\, . \tag{9.198}$$

By lemma 9.4.19:

$$\begin{aligned}
C \left|\left|w\right|\right|_H &\geq \left|\left|c_3\right|\right|_{L^2(\mathbb{R})} \\
&\geq \left|\left|c_3\right|\right|_{L^2(x_0 - \frac{\delta^2}{\varepsilon^2}, x_0 + \frac{\delta^2}{\varepsilon^2})} \\
&\geq \inf_{|x - x_0| \leq \frac{\delta^2}{\varepsilon^2}} |c_3(x)|\, \left|\left|1\right|\right|_{L^2(x_0 - \frac{\delta^2}{\varepsilon^2}, x_0 + \frac{\delta^2}{\varepsilon^2})} \\
&\geq \frac{1}{2}\, |c_3(x_0)|\, \frac{\sqrt{2}\delta}{\varepsilon}.
\end{aligned} \tag{9.199}$$

Due to $\left|\left|w\right|\right|_H \leq \frac{1}{\varepsilon}$, it follows $\frac{C}{\delta} \geq |c_3(x_0)|$ for every $x_0 \in \mathbb{R}$. $\frac{C}{\delta} \geq |c_4(x_0)|$ follows analogously.

That concludes the proof. □

Lemma 9.4.21 *Let $(w_n)_{n\in\mathbb{N}}, (w_{0,n})_{n\in\mathbb{N}} \subseteq H$, $(\varepsilon_n)_{n\in\mathbb{N}} \subseteq (0,\infty)$, $\xi > 0$, $\xi \notin N$ and $\lambda = 1+\xi^2$. Assume:*

1. $\forall n \in \mathbb{N} : (-iL-\lambda)w_n = w_{0,n}$,
2. $\forall n \in \mathbb{N} : ||w_n||_H \leq \frac{1}{\varepsilon_n}$,
3. $\forall n \in \mathbb{N} : ||w_{0,n}||_H \leq \varepsilon_n$,
4. $\lim_{n\to\infty} \varepsilon_n = 0$,
5. $\lim_{n\to\infty} w_n(0)$ *and* $\lim_{n\to\infty} w_n'(0)$ *converge.*

Then, there are $k_e, k_s \in \mathbb{C}$, fulfilling for $x \in \mathbb{R}$:

$$\lim_{n\to\infty} w_n(x) = k_e W_e(\xi,x) + k_o W_o(\xi,x), \tag{9.200}$$

$$\lim_{n\to\infty} w_n'(x) = k_e \frac{d}{dx} W_e(\xi,x) + k_o \frac{d}{dx} W_o(\xi,x). \tag{9.201}$$

Proof. Let $w_n = (u_n, v_n)$. For every $n \in \mathbb{N}$, we find continuous coefficient functions $c_{l,n} : \mathbb{R} \to \mathbb{C}$, $l \in \{1,2,3,4\}$, fulfilling:

$$(u_n, u_n', v_n, v_n')(x) = \sum_{l=1}^{4} c_{l,n}(x) y_l(x). \tag{9.202}$$

Analogously to (9.194) and (9.195) in the proof of lemma 9.4.20, it follows:

$$|c_{1,n}(x)|\, e^{\sqrt{2+\xi^2}x}, |c_{2,n}(x)|\, e^{-\sqrt{2+\xi^2}x} \leq C\varepsilon_n. \tag{9.203}$$

Hence, for every $x \in \mathbb{R}$:

$$\lim_{n\to\infty} c_{1,n}(x) = \lim_{n\to\infty} c_{2,n}(x) = 0. \tag{9.204}$$

As $(u_n, u_n', v_n, v_n')(0)$ converges against some limit for $n \to \infty$, we also find limits $k_e := \lim_{n\to\infty} c_{3,n}(0)$ and $k_o := \lim_{n\to\infty} c_{4,n}(0)$.

Analogously to (9.196) in the proof of lemma 9.4.20, we conclude:

$$|c_{3,n}(x) - c_{3,n}(x_0)|, |c_{4,n}(x) - c_{4,n}(x_0)| \leq C\, |x-x_0|^{\frac{1}{2}}\, \varepsilon_n. \tag{9.205}$$

Hence, for every $x \in \mathbb{R}$:

$$\lim_{n\to\infty} c_{3,n}(x) = \lim_{n\to\infty} c_{3,n}(0) = k_e, \tag{9.206}$$

$$\lim_{n\to\infty} c_{4,n}(x) = \lim_{n\to\infty} c_{4,n}(0) = k_o. \tag{9.207}$$

Due to $y_3 := (U_e, U_e', V_e, V_e')(\xi,\cdot)$ and $y_4 := (U_o, U_o', V_o, V_o')(\xi,\cdot)$, that concludes the proof. □

9.4.5. Proof of Injectivity

Proof (of lemma 9.4.1). We show the lemma indirectly.

Assume there exists $0 \neq w_1 \in H$ with $Fw_1 = 0$.

By choosing $w_2 \in \mathcal{S}(\mathbb{R})^2 \cap H$, such that $||w_1 - w_2||_H$ is sufficiently small, it follows for $h = (f, g) := Fw_2 \in \hat{H}^1 \times \hat{H}^1$:

$$||h||^2_{\hat{H}^1 \times \hat{H}^1} = ||f||^2_{\hat{H}^1} + ||g||^2_{\hat{H}^1} < ||w_2||^2_H . \tag{9.208}$$

Lemma 9.2.25 implies $w := w_2 - Gh \neq 0$ and lemma 9.3.3 further implies $Fw = 0$. By lemma 9.4.9, $w \in \hat{\hat{\mathcal{S}}}^2 \cap H$.

Let E_λ be the spectral measure corresponding to the self-adjoint operator $-iL : H \supset \mathrm{dom}(L) \to H$ as given by the spectral theorem (lemma 9.4.3). By lemma 6.2.3:

$$0 \neq w = E_{\sigma(-iL)} w = E_{(-\infty,-1]} w + E_{[1,\infty)} w. \tag{9.209}$$

Symmetry allows us to assume $E_{[1,\infty)} w \neq 0$.

Lemma 9.4.6 together with theorem 1.10.1 implies $E_{\tilde{N} \cup \{1\}} w = 0$. By lemma 9.4.15, we find $1 < a_0 < b_0$ with $b_0 - a_0 < 1$, $[a_0, b_0] \subset (1, \infty) \backslash \tilde{N}$ and $E_{[a_0,b_0]} w \neq 0$. We can assume:

$$\left|\left| E_{[a_0,b_0]} w \right|\right|_H = 1. \tag{9.210}$$

By lemma 9.4.4:

$$E_{[a_0,b_0]} w = E_{[a_0, \frac{a_0+b_0}{2}]} w + E_{[\frac{a_0+b_0}{2}, b_0]} w. \tag{9.211}$$

Moreover, as $E_{[a_0, \frac{a_0+b_0}{2}]}$ and $E_{[\frac{a_0+b_0}{2}, b_0]}$ are orthogonal to each other:

$$1 = \left|\left| E_{[a_0,b_0]} w \right|\right|^2_H = \left|\left| E_{[a_0, \frac{a_0+b_0}{2}]} w \right|\right|^2_H + \left|\left| E_{[\frac{a_0+b_0}{2}, b_0]} w \right|\right|^2_H . \tag{9.212}$$

The intermediate value theorem applied to $t \mapsto \left|\left| E_{[a_0+t, \frac{a_0+b_0}{2}+t]} w \right|\right|^2_H$ allows us to find $[a_1, b_1] \subset [a_0, b_0]$ with $b_1 - a_1 = \frac{b_0 - a_0}{2} \leq \frac{1}{2}$ and:

$$\left|\left| E_{[a_1,b_1]} w \right|\right|^2_H = \frac{1}{2}. \tag{9.213}$$

By inductively repeating this argument, we find sequences $(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ with $b_n - a_n \leq 2^{-n}$, $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ and:

$$\left|\left| E_{[a_n,b_n]} w \right|\right|_H = 2^{-\frac{n}{2}} \tag{9.214}$$

for every $n \in \mathbb{N}$. By definition, $(a_n)_n$ and $(b_n)_n$ constitute Cauchy sequences. We define:

$$\lambda_1 = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n \in [a_0, b_0] \subset (1,\infty)\backslash\tilde{N}. \tag{9.215}$$

Let $\xi_1 > 0$, such that $\lambda_1 = 1 + \xi_1^2$. $\lambda_1 \notin \tilde{N}$ implies $\xi_1 \notin N$. To improve readability, we define $w_n := E_{[a_n,b_n]}w$.

By definition, for every $k \geq 0$:

$$(-iL)^k w_n = E_{[a_n,b_n]}(-iL)^k w. \tag{9.216}$$

We conclude $w_n \in C^\infty(\mathbb{R})^2$ for every $n \in \mathbb{N}$. By the spectral theorem:

$$||(-iL - \lambda_1)2^n w_n||_H \leq \sup_{\mu\in[a_n,b_n]} |\mu - \lambda_1|\, ||2^n w_n||_H \leq |b_n - a_n|\, 2^{\frac{n}{2}} \leq 2^{-\frac{n}{2}}. \tag{9.217}$$

By lemma 9.4.20, (9.214) and (9.217), we conclude $||2^n w_n||_{W^{1,\infty}(\mathbb{R})} < C$ independently of $n \in \mathbb{N}$.

That allows us to choose some index sequence $(n_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$, such that $\lim_{k\to\infty} w_{n_k}(0)$ and $\lim_{k\to\infty} w'_{n_k}(0)$ exist. Lemma 9.4.21 yields $k_e, k_o \in \mathbb{C}$, fulfilling for every $x \in \mathbb{R}$:

$$\lim_{k\to\infty} w_{n_k}(x) = k_e W_e(\xi_1, x) + k_o W_e(\xi_1, x), \tag{9.218}$$

$$\lim_{k\to\infty} w'_{n_k}(x) = k_e \frac{d}{dx} W_e(\xi_1, x) + k_o \frac{d}{dx} W_s(\xi_1, x). \tag{9.219}$$

We estimate:

$$\begin{aligned} |w(x)2^{n_k} w_{n_k}(x)| + \left|w'(x)2^{n_k} w'_{n_k}(x)\right| &\leq (|w(x)| + |w'(x)|)\, ||2^{n_k} w_{n_k}||_{W^{1,\infty}(\mathbb{R})} \\ &\leq C(|w(x)| + |w'(x)|). \end{aligned} \tag{9.220}$$

By Lebesgue's dominated convergence theorem and $w \in W^{1,1}(\mathbb{R})$:

$$\begin{aligned} \lim_{k\to\infty} \langle w, 2^{n_k} w_{n_k}\rangle_H &= \langle w, k_e W_e(\xi_1, \cdot) + k_o W_o(\xi_1, \cdot)\rangle_H \\ &= \sqrt{2\pi}(1 + \xi_1^2)\,(k_e(F_e w)(\xi_1) + k_o(F_o w)(\xi_1)) \\ &= 0. \end{aligned} \tag{9.221}$$

(9.221) contradicts:

$$\begin{aligned} \forall n \geq 0 : \langle w, 2^n w_n\rangle_H &= \langle w, 2^n E_{[a_n,b_n]} w\rangle_H \\ &= \langle E_{[a_n,b_n]} w, 2^n E_{[a_n,b_n]} w\rangle_H \\ &= \left|\left|2^{\frac{n}{2}} E_{[a_n,b_n]} w\right|\right|_H^2 \\ &= 1. \end{aligned} \tag{9.222}$$

That concludes the proof. $\square$

9.5. L^2-Isometry

Theorem 9.5.1 *$F|_H : H \to \hat{H}^1 \times \hat{H}^1$ and $G : \hat{H}^1 \times \hat{H}^1 \to H$ are unitary operators fulfilling $F|_H^{-1} = G$.*

Proof. This is a consequence of lemma 9.2.25, lemma 9.3.3 and lemma 9.4.1. □

Lemma 9.5.2 *Let $w \in H$. Then, the following is equivalent:*

1. $w \in \mathrm{dom}(L)$.
2. $(\xi^2 + 1)Fw \in \hat{H}^1 \times \hat{H}^1$.

Every $w \in H$ realizing one of the above properties fulfils:

$$FLw = i\,\mathrm{sgn}(\xi)(\xi^2 + 1)Fw. \tag{9.223}$$

Proof. Follows from lemma 8.5.2. □

In this section we generalize lemma 9.5.2. We show that F unitarily maps H_k (see below) onto $\hat{H}^k \times \hat{H}^k$ for any $k \geq 0$.

Recall the orthogonality conditions (1.43), which characterise $\mathcal{H}$:

$$\begin{aligned} 0 &= \langle u, Q\rangle_{L^2} = \langle u, xQ\rangle_{L^2} \\ &= \langle v, Q_x\rangle_{L^2} = \left\langle v, \left(\frac{2}{p-1}Q + xQ_x\right)\right\rangle_{L^2}. \end{aligned} \tag{9.224}$$

H is given by (6.3). We restate the orthogonality conditions in terms of the L^2 scalar product:

$$\begin{aligned} 0 &= \langle w, \zeta\rangle_{\mathcal{H}} = \langle w, \overline{\zeta}\rangle_{\mathcal{H}}. \\ \Leftrightarrow 0 &= \langle w, IL\zeta\rangle_{L^2} = \langle w, IL\overline{\zeta}\rangle_{L^2} \\ \Leftrightarrow 0 &= \langle w, I\zeta\rangle_{L^2} = \langle w, I\overline{\zeta}\rangle_{L^2}. \end{aligned} \tag{9.225}$$

I denotes the matrix equivalent of the imaginary unit i:

$$I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{9.226}$$

Definition 9.5.3 *Let $k \geq 0$. We define the vector space:*

$$H_k := \{w = (u, v) \in H^k(\mathbb{R})^2 |\ (9.224),\ (9.225)\ \textit{hold}\}. \tag{9.227}$$

For odd $k = 2l + 1, l \in \mathbb{N}$, we define a scalar product and norm on H_k:

$$\langle w_1, w_2\rangle_{H_k} := \langle L^l w_1, L^l w_2\rangle_H, \tag{9.228}$$

$$||w||_{H_k} := \left|\left|L^l w\right|\right|_H. \tag{9.229}$$

We can use H_k and $\hat{H}^k$ to generalize theorem 9.5.1.

Lemma 9.5.4 *Let $k \in 2\mathbb{N}+1$. Then, $F|_{H_k}$ maps H_k unitarily onto $\hat{H}^k \times \hat{H}^k$ with inverse $G|_{\hat{H}^k \times \hat{H}^k} = F|_{H_k}^{-1}$. For every $w \in H_k$:*

$$||w||_{H_k} = ||Fw||_{\hat{H}^k \times \hat{H}^k}. \tag{9.230}$$

Proof. Let $k = 2l+1$. By lemma 9.5.2, $\mathrm{dom}(L^l) = H_k$. Further, $w \in H_k$ is equivalent to $(\xi^2+1)^l Fw \in \hat{H}^1 \times \hat{H}^1$. By definition, $(\xi^2+1)^l Fw \in \hat{H}^1 \times \hat{H}^1$ is equivalent to $Fw \in \hat{H}^k \times \hat{H}^k$. It follows:

$$\begin{aligned}
||w||_{H_k} &= \left|\left|L^l w\right|\right|_H \\
&= \left|\left|FL^l w\right|\right|_{\hat{H}^1 \times \hat{H}^1} \\
&= \left|\left|(i\,\mathrm{sgn}(\xi)(\xi^2+1))^l Fw\right|\right|_{\hat{H}^1 \times \hat{H}^1} \\
&= \left|\left|(\xi^2+1)^l Fw\right|\right|_{\hat{H}^1 \times \hat{H}^1} \\
&= ||Fw||_{\hat{H}^k \times \hat{H}^k}.
\end{aligned} \tag{9.231}$$

That concludes the proof. □

Lemma 9.5.4 can be used to invert L.

Lemma 9.5.5 *Let $k \in 2\mathbb{N}+1$ and $w_0 \in H_k$. Then, there exists $w \in H_{k+2} \subset H_k$ with $Lw = w_0$ and $||w||_{H_k} \leq ||w||_{H_{k+2}} = ||w_0||_{H_k}$.*

Proof. By lemma 9.5.4, $Fw_0 \in \hat{H}^k \times \hat{H}^k$. That implies $(\xi^2+1)^{-1} Fw_0 \in \hat{H}^{k+2} \times \hat{H}^{k+2}$. We define $w \in H_{k+2}$ by:

$$Fw = \frac{Fw_0}{i\,\mathrm{sgn}(\xi)(\xi^2+1)}. \tag{9.232}$$

By lemma 9.5.4, $||w||_{H_{k+2}} = ||w_0||_{H_k}$. Lemma 9.5.2 implies $w_0 = Lw$.

Finally, $||w||_{H_k} = ||Fw||_{\hat{H}^k \times \hat{H}^k} \leq ||Fw||_{\hat{H}^{k+2} \times \hat{H}^{k+2}} = ||w||_{H_{k+2}}$ holds true for every $w \in H_{k+2}$.

That concludes the proof. □

Lemma 9.5.6 *Let $k \in 2\mathbb{N}+1$. Then, $||\cdot||_{H_k}$ and $||\cdot||_{H^k(\mathbb{R})^2}$ constitute equivalent norms on H_k.*

Proof. We show the lemma inductively. The case of $k = 1$ is covered by lemma 6.2.1.

Let $k \in 2\mathbb{N}+1$. Assume:

$$\forall w \in H_k : C^{-1} ||w||_{H^k(\mathbb{R})^2} \leq ||w||_{H_k} \leq C\, ||w||_{H^k(\mathbb{R})^2}. \tag{9.233}$$

To conclude the proof, we need to demonstrate:

$$\forall w \in H_{k+2} : C^{-1} \left|\left|w\right|\right|_{H^{k+2}(\mathbb{R})^2} \leq \left|\left|w\right|\right|_{H_{k+2}} \leq C \left|\left|w\right|\right|_{H^{k+2}(\mathbb{R})^2} . \tag{9.234}$$

By lemma 9.5.5, $\left|\left|w\right|\right|_{H_{k+2}} = \left|\left|Lw\right|\right|_{H_k}$. By (1.35) and (9.233):

$$\left|\left|w\right|\right|_{H_{k+2}} = \left|\left|Lw\right|\right|_{H_k} \leq C \left|\left|Lw\right|\right|_{H^k(\mathbb{R})^2} \leq C^2 \left|\left|w\right|\right|_{H^{k+2}(\mathbb{R})^2} . \tag{9.235}$$

On the other hand, by (1.35) and (9.233):

$$\begin{aligned} \left|\left|w\right|\right|_{H^{k+2}(\mathbb{R})^2} &\leq \left|\left|Lw\right|\right|_{H^k(\mathbb{R})^2} + C \left|\left|w\right|\right|_{H^k(\mathbb{R})^2} \\ &\leq \left|\left|Lw\right|\right|_{H_k} + C^2 \left|\left|w\right|\right|_{H_k} \\ &\leq (1 + C^2) \left|\left|w\right|\right|_{H_{k+2}} . \end{aligned} \tag{9.236}$$

That concludes the proof. □

We use complex interpolation to expand the definition of $\left|\left|\cdot\right|\right|_{H_s}$ to include every $s \geq 1$.

Lemma 9.5.7 *Let $k \in 2\mathbb{N} + 1$, $0 < \theta < 1$ and $s = (1-\theta)k + \theta(k+2)$. Then:*

$$\left(H_s, \left|\left|\cdot\right|\right|_{H^s(\mathbb{R})^2}\right) = \left(\left(H_k, \left|\left|\cdot\right|\right|_{H^k(\mathbb{R})^2}\right), \left(H_{k+2}, \left|\left|\cdot\right|\right|_{H^{k+2}(\mathbb{R})^2}\right)\right)_\theta . \tag{9.237}$$

Proof. It is well-known (see, e.g. [3]) that complex interpolation between Sobolev spaces $H^k(\mathbb{R})$ and $H^{k+2}(\mathbb{R})$ yields another Sobolev space:

$$(H^k(\mathbb{R}), H^{k+2}(\mathbb{R}))_\theta = H^s(\mathbb{R}). \tag{9.238}$$

That already concludes the proof. □

Lemma 9.5.8 *Let $k \in 2\mathbb{N} + 1$, $0 < \theta < 1$ and $s = (1-\theta)k + \theta(k+2)$. Then, there exists a norm $\left|\left|\cdot\right|\right|_{H_s}$ on H_s, such that:*

$$\left(H_s, \left|\left|\cdot\right|\right|_{H_s}\right) = (H_k, H_{k+2})_\theta. \tag{9.239}$$

Proof. Follows from lemma 9.5.6 and lemma 9.5.7. □

Definition 9.5.9 *Let $s \geq 1$ with $s \notin 2\mathbb{N} + 1$. Let $k \in 2\mathbb{N} + 1$ and $0 < \theta < 1$ be chosen, such that $s = (1-\theta)k + \theta(k+2)$. We define $\left|\left|\cdot\right|\right|_{H_s}$ as the norm given by lemma 9.5.8.*

Lemma 9.5.10 *Let $k, l \in \mathbb{R}$, $0 < \theta < 1$ and $s = (1-\theta)k + \theta l$. Then:*

$$(\hat{H}^k, \hat{H}^l)_\theta = \hat{H}^s. \tag{9.240}$$

Proof. The Fourier transform acts as an isometry between $(\hat{H}^s, \left|\left|\cdot\right|\right|_{\hat{H}^s})$ and $(H^s(\mathbb{R}), \left|\left|\cdot\right|\right|_{H^s(\mathbb{R})})$ for every $s \in \mathbb{R}$. The lemma follows from:

$$(H^k(\mathbb{R}), H^l(\mathbb{R}))_\theta = H^s(\mathbb{R}). \tag{9.241}$$

That concludes the proof. □

Lemma 9.5.11 *Lemma 9.5.4 - 9.5.6 hold true for every $s \geq 1$, i.e.:*

1. *F maps H_s unitarily onto $\hat{H}^s \times \hat{H}^s$ with inverse $G|_{\hat{H}^s \times \hat{H}^s} = F|_{H_s}^{-1}$.*
2. *Let $w_0 \in H_s$. There is $w \in H_{s+2}$ with $Lw = w_0$ and $||w||_{H_s} \leq ||w||_{H_{s+2}} = ||w_0||_{H_s}$.*
3. *$||\cdot||_{H_s}$ and $||\cdot||_{H^s(\mathbb{R})^2}$ constitute equivalent norms on H_s.*

Proof. Follows by complex interpolation from lemma 9.5.4. □

By lemma 9.2.27 and definition 9.3.1, F and G can be continuously extended to bounded operators:

$$F, G : L^2(\mathbb{R})^2 \to L^2(\mathbb{R})^2. \tag{9.242}$$

To be able to extend lemma 9.5.11 to include every $s \geq 0$, we have to define a norm $||\cdot||_{H_0}$ on H_0.

We take inspiration from the following identity:

$$||f||_{L^2(\mathbb{R})} = \sup_{||g||_{H^2(\mathbb{R})}=1} \langle f, g \rangle_{H^1(\mathbb{R})}. \tag{9.243}$$

Definition 9.5.12 *We define the norm $||\cdot||_{H_0}$ on H_0 by:*

$$||w||_{H_0} = \sup_{||\tilde{w}||_{H_2}=1} \langle w, \tilde{w} \rangle_{H_1}. \tag{9.244}$$

Lemma 9.5.13 *$||\cdot||_{H_0}$ is equivalent to $||\cdot||_{L^2(\mathbb{R})^2}$ on H_1.*

Proof. Let $w_1 \in H_1$ be given. By lemma 9.5.11, we find $w_3 \in H_3$ with $Lw_3 = w_1$. Lemma 9.5.11 implies:

$$\begin{aligned} ||w_1||_{H_0} &= \sup_{||\tilde{w}_2||_{H_2}=1} \langle Lw_3, \tilde{w}_2 \rangle_{H_1} \\ &= \sup_{||\tilde{w}_4||_{H_4}=1} \langle Lw_3, L\tilde{w}_4 \rangle_{H_1} \\ &= \sup_{||\tilde{w}_4||_{H_4}=1} \langle w_3, \tilde{w}_4 \rangle_{H_3} \\ &= \sup_{||h||_{\hat{H}^4 \times \hat{H}^4}=1} \langle Fw_3, h \rangle_{\hat{H}^3 \times \hat{H}^3} \\ &= ||Fw_3||_{\hat{H}^2 \times \hat{H}^2} \\ &= ||w_3||_{H_2}. \end{aligned} \tag{9.245}$$

It follows $||w_1||_{L^2(\mathbb{R})^2} = ||Lw_3||_{L^2(\mathbb{R})^2} \leq C\, ||w_3||_{H^2(\mathbb{R})^2} \leq C^2\, ||w_3||_{H_2} = C^2\, ||w_1||_{H_0}$.

On the other hand:

$$\begin{aligned} ||w_1||_{H_0} &= \sup_{||\tilde{w}_2||_{H_2}=1} \langle w_1, \tilde{w}_2 \rangle_{H_1} \\ &\leq C \sup_{||\tilde{w}_2||_{H^2(\mathbb{R})^2}=1} \langle w_1, \tilde{w}_2 \rangle_{H^1(\mathbb{R})^2} \\ &= C \, ||w_1||_{L^2(\mathbb{R})^2} . \end{aligned} \tag{9.246}$$

That concludes the proof. □

Corollary 9.5.14 $||\cdot||_{H_0}$ *is equivalent to* $||\cdot||_{L^2(\mathbb{R})^2}$ *on* H_0.

Proof. By definition, H_0 is the topological closure of H_1 with respect to $||\cdot||_{L^2(\mathbb{R})}$. By lemma 9.5.13, $||\cdot||_{L^2(\mathbb{R})}$ and $||\cdot||_{H_0}$ are equivalent on H_1. That concludes the proof. □

Lemma 9.5.15 $F : H_0 \to \hat{H}^0 \times \hat{H}^0$ *constitutes a unitary operator with inverse* $G = F|_{H_0}^{-1}$.

Proof. It suffices to show $||w||_{H_0} = ||Fw||_{\hat{H}^0 \times \hat{H}^0}$ for every $w \in H_1$. We calculate:

$$\begin{aligned} ||w||_{H_0} &= \sup_{||\tilde{w}||_{H_2}=1} \langle w, \tilde{w} \rangle_{H_1} \\ &= \sup_{||h||_{\hat{H}^2 \times \hat{H}^2}=1} \langle Fw, h \rangle_{\hat{H}^1 \times \hat{H}^1} \\ &= \sup_{||h||_{\hat{H}^0 \times \hat{H}^0}=1} \langle Fw, h \rangle_{\hat{H}^0 \times \hat{H}^0} \\ &= ||Fw||_{\hat{H}^0 \times \hat{H}^0} . \end{aligned} \tag{9.247}$$

That concludes the proof. □

Lemma 9.5.16 *L continuously extends to an operator $L : H_0 \supset H_2 \to H_0$. Let further $w_0 \in H_0$. Then, there is $w \in H_2$ with $Lw = w_0$ and $||w||_{H_0} \leq ||w||_{H_2} = ||w_0||_{H_0}$.*

Proof. L extends to an operator $L : H_0 \supset H_2 \to H_0$ simply by the definition (1.35).

As H_0 is the topological closure of H_2 under $||\cdot||_{H_0}$, it suffices to find $w \in H_2$ with $Lw = w_0$ and $||w||_{H_0} \leq ||w||_{H_2} = ||w_0||_{H_0}$ for $w_0 \in H_2$.

Lemma 9.5.11 yields $w \in H_4$ with $Lw = w_0$. It follows:

$$||w||_{H_0} \leq ||w||_{H_2} = ||Lw||_{H_0} = ||w_0||_{H_0} . \tag{9.248}$$

That concludes the proof. □

Definition 9.5.17 *Let $s \in (0, 1)$. We define $||\cdot||_{H_s}$ as the complex interpolation norm between $||\cdot||_{H_0}$ and $||\cdot||_{H_1}$.*

Theorem 9.5.18 *Let $s \geq 0$. Then:*

1. *F maps H_s unitarily onto $\hat{H}^s \times \hat{H}^s$ with inverse $G|_{\hat{H}^s \times \hat{H}^s} = F|_{H_s}^{-1}$.*
2. *Let $w_0 \in H_s$. There is $w \in H_{s+2}$ with $Lw = w_0$ and $||w||_{H_s} \leq ||w||_{H_{s+2}} = ||w_0||_{H_s}$.*
3. *$||\cdot||_{H_s}$ and $||\cdot||_{H^s(\mathbb{R})^2}$ constitute equivalent norms on H_s.*

Proof. Follows via complex interpolation from lemma 9.5.11 and corollary 9.5.14, lemma 9.5.15, lemma 9.5.16. □

We restate lemma 9.5.2 for the extended operator $L : H_0 \supset H_2 \to H_0$.

Theorem 9.5.19 *Let $w \in H_0$. Then, the following is equivalent:*

1. *$w \in \mathrm{dom}(L) = H_2$.*
2. *$Fw \in \hat{H}^2 \times \hat{H}^2$.*

Every $w \in H$ realizing one of the above properties fulfils:

$$FLw = i\,\mathrm{sgn}(\xi)(\xi^2 + 1)Fw. \tag{9.249}$$

Theorem 9.5.18 and 9.5.19 allow us to consider the linearised equation (1.34) entirely in an L^2-setting.

9.6. Zeros of F

We end the chapter by giving a simple lemma related to the zeros of W_e and W_o laid out by lemma 8.5.4. The Fourier transform $\mathcal{F}$ allows no equivalent to this lemma. It holds true due to L not allowing resonances (theorem 1.10.1).

Lemma 9.6.1 will be used to show a local smoothing estimate for $\partial_t w = -Lw$. This smoothing estimate will be stronger than the usual local smoothing estimate for the free equation.

Lemma 9.6.1 *Let $w \in H_0$ and $h = Fw$. If $w \in L^1(\mathbb{R})^2$, then $h(0) = (0, 0)$. If $xw \in L^2(\mathbb{R})^2$, then $\xi^{-1}h \in L^2(\mathbb{R})^2$.*

Proof. The first claim follows from (9.96), lemma 8.5.4 and lemma 8.5.5.

To prove the second claim, recall definition 9.3.1:

$$\begin{aligned} F_e w = {} & \frac{i\,\mathrm{sgn}(\xi)}{\sqrt{2\pi}} \int_{\mathbb{R}} (R_{e,V}(\xi, x)u_e(x) + R_{e,U}(\xi, x)v_e(x))\, dx \\ & + c_e \mathcal{F} u_e + i c_e\,\mathrm{sgn}(\xi)\mathcal{F} v_e + i s_e\,\mathrm{sgn}(\xi)\mathcal{F}(\mathrm{sgn}(\cdot)u_e) - s_e \mathcal{F}(\mathrm{sgn}(\cdot)v_e), \end{aligned} \tag{9.250}$$

$$\begin{aligned} F_o w = {} & \frac{i\,\mathrm{sgn}(\xi)}{\sqrt{2\pi}} \int_{\mathbb{R}} (R_{o,V}(\xi, x)u_o(x) + R_{o,U}(\xi, x)v_o(x))\, dx \\ & - c_o \mathcal{F}(\mathrm{sgn}(\cdot)u_o) - i c_o\,\mathrm{sgn}(\xi)\mathcal{F}(\mathrm{sgn}(\cdot)v_o) - i s_o\,\mathrm{sgn}(\xi)\mathcal{F} u_o + s_o \mathcal{F} v_o. \end{aligned} \tag{9.251}$$

Due to lemma 8.3.8, we only need to show:

$$\xi^{-1}\int_{\mathbb{R}}\left(R_{e,V}(\xi,x)u_e(x)+R_{e,U}(\xi,x)v_e(x)\right)dx\in L^2(\mathbb{R}), \tag{9.252}$$

$$\xi^{-1}\int_{\mathbb{R}}\left(R_{o,V}(\xi,x)u_o(x)+R_{o,U}(\xi,x)v_o(x)\right)dx\in L^2(\mathbb{R}). \tag{9.253}$$

Lemma 8.5.7, lemma 8.5.10 and the mean value theorem conclude the proof. □

10. Wave Operator

As before, given w or h, we use w_e, w_o and h_e, h_o to denote the even or odd functions described in convention 9.0.1.

So far, we have defined unitary operators $H_0 \underset{G}{\overset{F}{\rightleftarrows}} L^2(\mathbb{R})^2 = \hat{H}^0 \times \hat{H}^0$.

F and G are closely linked to the spectral measure of L. As a consequence - noted in theorem 9.5.19 - we have $FLw = i\,\mathrm{sgn}(\xi)(\xi^2+1)Fw$, which directly mirrors the Fourier transform and the Laplace operator.

Recall the definition of the operator L, as given by (1.35):

$$L\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \Delta v - v + Q^{p-1}v \\ -\Delta u + u - pQ^{p-1}u \end{pmatrix}, \tag{10.1}$$

Definition 10.0.1 *Consider the imaginary unit i expressed as a matrix when identifying $\mathbb{C}$ with $\mathbb{R}^2$:*

$$I := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{10.2}$$

Definition 10.0.2 *We define the 'unperturbed' operator $L_0 : L^2(\mathbb{R})^2 \supset H^2(\mathbb{R})^2 \to L^2(\mathbb{R})^2$:*

$$L_0 = (-\Delta + 1)I. \tag{10.3}$$

Linked to L_0 we define 'unperturbed' or 'undistorted' versions of F and G, which we name $\mathcal{G}^{-1}$ and $\mathcal{G}$. We then define a unitary operator $T = \mathcal{G}F$.

T fulfils the commutation property $TL = L_0T$, meaning that T is actually the wave operator $T = \lim_{t\to\infty} e^{-tL_0}e^{tL}$.

We also show that T and T^{-1} are well-defined as bounded operators $L^q \to L^q$ for every $1 \le q \le \infty$, by proving $C^{-1}\,||w||_{L^q} \le ||Tw||_{L^q} \le C\,||w||_{L^q}$.

10.1. Classical Fourier Transformation

Recall definition 9.3.1. For $w = (u,v) \in L^2(\mathbb{R})^2$:

$$\begin{aligned} F_e w = {} & \frac{i\,\mathrm{sgn}(\xi)}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(R_{e,V}(\xi,x)u(x) + R_{e,U}(\xi,x)v(x)\right) dx \\ & + c_e\mathcal{F}u_e + ic_e\,\mathrm{sgn}(\xi)\mathcal{F}v_e + is_e\,\mathrm{sgn}(\xi)\mathcal{F}(\mathrm{sgn}(\cdot)u_e) - s_e\mathcal{F}(\mathrm{sgn}(\cdot)v_e), \end{aligned} \tag{10.4}$$

$$\begin{aligned} F_o w = {} & \frac{i\,\mathrm{sgn}(\xi)}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(R_{o,V}(\xi,x)u(x) + R_{o,U}(\xi,x)v(x)\right) dx \\ & - c_o\mathcal{F}(\mathrm{sgn}(\cdot)u_o) - ic_o\,\mathrm{sgn}(\xi)\mathcal{F}(\mathrm{sgn}(\cdot)v_o) - is_o\,\mathrm{sgn}(\xi)\mathcal{F}u_o + s_o\mathcal{F}v_o. \end{aligned} \tag{10.5}$$

Definition 10.1.1 *Consider the 'undistorted' Fourier transforms* $\mathcal{F}_e, \mathcal{F}_o : L^2(\mathbb{R})^2 \to L^2(\mathbb{R})$*:*

$$\mathcal{F}_e w := \mathcal{F}u_e + i \operatorname{sgn}(\cdot)\mathcal{F}v_e, \tag{10.6}$$

$$\mathcal{F}_o w := -i \operatorname{sgn}(\cdot)\mathcal{F}u_o + \mathcal{F}v_o. \tag{10.7}$$

(10.6) is derived from (10.4) by substituting $R_{e,U} = R_{e,V} = 0$, $s_e = 0$ and $c_e = 1$. (10.7) is derived from (10.5) by substituting $R_{o,U} = R_{o,V} = 0$, $s_o = 1$ and $c_o = 0$. Those are the limits of R_e, c_e and s_e, as well as R_o, c_o and s_o for $|\xi| \to \infty$.

We also define 'undistorted' Fourier transformations mirroring G_e and G_o. By lemma 9.2.27:

$$\begin{aligned} G_e f = & \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \begin{pmatrix} R_{e,U}(\xi,x) \\ R_{e,V}(\xi,x) \end{pmatrix} f(\xi) d\xi \\ & + \begin{pmatrix} \mathcal{F}(c_e f_e) \\ -i\mathcal{F}(c_e \operatorname{sgn}(\cdot) f_o) \end{pmatrix} + \operatorname{sgn}(x) \begin{pmatrix} i\mathcal{F}(s_e \operatorname{sgn}(\cdot) f_e) \\ \mathcal{F}(s_e f_o) \end{pmatrix}, \end{aligned} \tag{10.8}$$

$$\begin{aligned} G_o g = & \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \begin{pmatrix} R_{o,U}(\xi,x) \\ R_{o,V}(\xi,x) \end{pmatrix} g(\xi) d\xi \\ & - \operatorname{sgn}(x) \begin{pmatrix} \mathcal{F}(c_o g_e) \\ -i\mathcal{F}(c_o \operatorname{sgn}(\cdot) g_o) \end{pmatrix} - \begin{pmatrix} i\mathcal{F}(s_o \operatorname{sgn}(\cdot) g_e) \\ \mathcal{F}(s_o g_o) \end{pmatrix}. \end{aligned} \tag{10.9}$$

We define $\mathcal{G}_e f$ and $\mathcal{G}_o g$ by substituting limits as well.

Definition 10.1.2 *Consider the 'undistorted' Fourier transforms* $\mathcal{G}_e, \mathcal{G}_o : L^2(\mathbb{R})^2 \to L^2(\mathbb{R})$*:*

$$\mathcal{G}_e f := \begin{pmatrix} \mathcal{F} f_e \\ -i\mathcal{F}(\operatorname{sgn}(\cdot) f_o) \end{pmatrix}, \tag{10.10}$$

$$\mathcal{G}_o g := -\begin{pmatrix} i\mathcal{F}(\operatorname{sgn}(\cdot) g_e) \\ \mathcal{F} g_o \end{pmatrix}. \tag{10.11}$$

Definition 10.1.3 *We define:*

$$\mathcal{G} : L^2(\mathbb{R})^2 \to L^2(\mathbb{R})^2, \tag{10.12}$$

$$(f, g) \mapsto \mathcal{G}_e f + \mathcal{G}_o g, \tag{10.13}$$

$$\mathcal{G}^{-1} : L^2(\mathbb{R})^2 \to L^2(\mathbb{R})^2, \tag{10.14}$$

$$w \mapsto (\mathcal{F}_e w, \mathcal{F}_o w). \tag{10.15}$$

The notation $\mathcal{G}^{-1}$ is used to prevent conflation with the classical Fourier transformation $\mathcal{F}$ and is justified due to the following lemma.

Lemma 10.1.4 *$\mathcal{G}$ and $\mathcal{G}^{-1}$ are inverses of one another. Further, $\mathcal{G}$ and $\mathcal{G}^{-1}$ constitute unitary operators $L^2(\mathbb{R})^2 \to L^2(\mathbb{R})^2$.*

Proof. That $\mathcal{G}$ and $\mathcal{G}^{-1}$ are inverses of one another follows by direct computation. We calculate for $w = (u, v) \in L^2(\mathbb{R})^2$:

$$\begin{aligned}
&\left|\left|\mathcal{G}^{-1}w\right|\right|_{L^2}^2 \\
=\ & ||\mathcal{F}_e w||_{L^2}^2 + ||\mathcal{F}_o w||_{L^2}^2 \\
=\ & \int_{\mathbb{R}} (\mathcal{F}u_e + i\,\mathrm{sgn}(\xi)\mathcal{F}v_e)\overline{(\mathcal{F}u_e + i\,\mathrm{sgn}(\xi)\mathcal{F}v_e)}d\xi \\
& + \int_{\mathbb{R}} (-i\,\mathrm{sgn}(\xi)\mathcal{F}u_o + \mathcal{F}v_o)\overline{(-i\,\mathrm{sgn}(\xi)\mathcal{F}u_o + \mathcal{F}v_o)}d\xi \\
=\ & \int_{\mathbb{R}} ((\mathcal{F}u_e)^2 + (\mathrm{sgn}(\xi)^2(\mathcal{F}v_e)^2)d\xi + \int_{\mathbb{R}} (\mathrm{sgn}(\xi)^2(\mathcal{F}u_o)^2 + (\mathcal{F}v_o)^2)d\xi \\
=\ & ||\mathcal{F}u_e||_{L^2}^2 + ||\mathcal{F}v_e||_{L^2}^2 + ||\mathcal{F}u_o||_{L^2}^2 + ||\mathcal{F}v_o||_{L^2}^2 \\
=\ & ||u_e||_{L^2}^2 + ||v_e||_{L^2}^2 + ||u_o||_{L^2}^2 + ||v_o||_{L^2}^2 \\
=\ & ||u||_{L^2}^2 + ||v||_{L^2}^2 \\
=\ & ||w||_{L^2}^2\,.
\end{aligned} \tag{10.16}$$

That concludes the proof. □

L_0, $\mathcal{G}$ and $\mathcal{G}^{-1}$ closely mirror L, G and F, as demonstrated by the following equivalent to theorem 9.5.19.

Lemma 10.1.5 *For $w \in L^2(\mathbb{R})^2$, the following is equivalent:*

1. $w \in \mathrm{dom}(L_0) = H^2(\mathbb{R})^2$.
2. $\mathcal{G}^{-1}w \in \hat{H}^2 \times \hat{H}^2$.

Every $w \in L^2(\mathbb{R})^2$ realizing one of the above properties fulfils:

$$L_0 w = \mathcal{G}(i\,\mathrm{sgn}(\xi)(\xi^2+1)\mathcal{G}^{-1}w). \tag{10.17}$$

Proof. The equivalence of the two properties follows from $-\Delta\mathcal{F}f = \mathcal{F}(\xi^2 f)$ for every $f \in L^2(\mathbb{R})$.

Let $w = (u, v) \in H^2(\mathbb{R})^2$. Then:

$$\begin{aligned}
\mathcal{G}^{-1}L_0 w &= \mathcal{G}^{-1}\begin{pmatrix} -(-\Delta+1)v \\ (-\Delta+1)u \end{pmatrix} \\
&= \begin{pmatrix} -\mathcal{F}(-\Delta+1)v_e + i\,\mathrm{sgn}(\xi)\mathcal{F}(-\Delta+1)u_e \\ i\,\mathrm{sgn}(\xi)\mathcal{F}(-\Delta+1)v_o + \mathcal{F}(-\Delta+1)u_o \end{pmatrix} \\
&= (\xi^2+1)\begin{pmatrix} -\mathcal{F}v_e + i\,\mathrm{sgn}(\xi)\mathcal{F}u_e \\ i\,\mathrm{sgn}(\xi)\mathcal{F}v_o + \mathcal{F}u_o \end{pmatrix} \\
&= i\,\mathrm{sgn}(\xi)(\xi^2+1)\begin{pmatrix} i\,\mathrm{sgn}(\xi)\mathcal{F}v_e + \mathcal{F}u_e \\ \mathcal{F}v_o - i\,\mathrm{sgn}(\xi)\mathcal{F}u_o \end{pmatrix} \\
&= i\,\mathrm{sgn}(\xi)(\xi^2+1)\mathcal{G}^{-1}w.
\end{aligned} \tag{10.18}$$

That concludes the proof. □

10.2. Definition of the Wave Operator

Definition 10.2.1 *We define:*

$$T : H_0 \to L^2(\mathbb{R})^2, \tag{10.19}$$
$$w \mapsto \mathcal{G}Fw. \tag{10.20}$$

By theorem 9.5.18 and lemma 10.1.4, T constitutes a unitary operator.

Convention 10.2.2 *By definition 9.3.2, F and thus $T = \mathcal{G}F$ are well-defined on $L^2(\mathbb{R})^2$. Of course, both operators are no longer unitary if we extend them to $L^2(\mathbb{R})^2$. Indeed, by definition, the eigenvalues of L are all mapped onto zero:*

$$F\begin{pmatrix} \frac{2}{p-1}Q + x\partial_x Q \\ 0 \end{pmatrix} = F\begin{pmatrix} 0 \\ Q \end{pmatrix} = F\begin{pmatrix} 0 \\ xQ \end{pmatrix} = F\begin{pmatrix} \partial_x Q \\ 0 \end{pmatrix} = F\zeta = F\overline{\zeta} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{10.21}$$

Consequently, we also understand $T = \mathcal{G}F$ as an operator $L^2(\mathbb{R})^2 \to L^2(\mathbb{R})^2$ by projecting $L^2(\mathbb{R})^2$ onto H_0 via (10.21)*.*

By $T^{-1} = G\mathcal{G}^{-1} : L^2(\mathbb{R})^2 \to H_0 \subset L^2(\mathbb{R})^2$, we always refers to the inverse of the unitary operator $T : H_0 \to L^2(\mathbb{R})^2$.

Combining lemma 9.5.2 and lemma 10.1.5 immediately implies:

Lemma 10.2.3 *For $w \in H_0$, the following is equivalent:*

1. $w \in \mathrm{dom}(L) = H_2$.
2. $Tw \in \mathrm{dom}(L_0) = H^2(\mathbb{R})^2$.

Every $w \in H_0$ realizing one of the above properties fulfils:

$$TLw = L_0 Tw. \tag{10.22}$$

11. Bounds on the Wave Operator

We show that both the wave operator $T : L^2(\mathbb{R})^2 \to L^2(\mathbb{R})^2$ (defined on L^2 by projecting onto H_0) and its inverse $T^{-1} : L^2(\mathbb{R})^2 \to H_0 \hookrightarrow L^2(\mathbb{R})^2$ extend to bounded operators $L^q(\mathbb{R})^2 \to L^q(\mathbb{R})^2$, for every $1 \leq q \leq \infty$. We also show a variety of other bounds that might prove useful in proving asymptotic stability of the NLS.

As always, we assume $3 < p < 5$.

Theorem 11.0.1 *Let $1 \leq q \leq \infty$. Then, for $w \in L^q(\mathbb{R})^2 \cap L^2(\mathbb{R})^2$:*

$$||Tw||_{L^q} \leq C\, ||w||_{L^q}\,, \tag{11.1}$$

$$\left|\left|T^{-1}w\right|\right|_{L^q} \leq C\, ||w||_{L^q}\,. \tag{11.2}$$

Proof. Follows from corollary 11.4.4 and corollary 11.4.6. □

Corollary 11.0.2 *Let $1 \leq q \leq \infty$ and $k \geq 0$. Then, for $w \in W^{k,q}(\mathbb{R})^2$:*

$$||Tw||_{W^{k,q}} \leq C_k\, ||w||_{W^{k,q}}\,, \tag{11.3}$$

$$\left|\left|T^{-1}w\right|\right|_{W^{k,q}} \leq C_k\, ||w||_{W^{k,q}}\,. \tag{11.4}$$

Proof. Follows from theorem 11.0.1 and lemma 10.2.3. □

11.1. Smooth Remainder

In order to prove an L^q-bound, we will decompose T and T^{-1} based on lemma 9.2.27 and definition 9.3.1.

However, such a decomposition introduces discontinuities both in $\xi = 0$ and $x = 0$, despite the fact that the integral kernels $W_e(\xi, x)$, $W_o(\xi, x)$ of T, T^{-1} are continuous in ξ and x.

We remedy this problem by introducing continuous remainder terms, replacing R_e and R_o. In analogy to definition 8.5.6:

Definition 11.1.1 *Let $\chi : \mathbb{R} \to [0, 1]$ be a smooth and even function, fulfilling $\chi(x) = 1$ for $|x| \geq 2$ and $\chi(x) = 0$ for $|x| \leq 1$.*

We define the remainder terms $\rho_e = (\rho_{e,U}, \rho_{e,V}) : \mathbb{R}^2 \to \mathbb{C}^2$ and $\rho_o = (\rho_{o,U}, \rho_{o,V}) : \mathbb{R}^2 \to \mathbb{C}^2$ by:

$$\rho_{e,U}(\xi, x) := U_e(\xi, x) - \chi(\xi)\chi(x)(c_e(\xi)\cos(\xi x) - \operatorname{sgn}(\xi x)s_e(\xi)\sin(\xi x)), \tag{11.5}$$

$$\rho_{e,V}(\xi, x) := V_e(\xi, x) + i\chi(\xi)\chi(x)(\operatorname{sgn}(\xi)c_e(\xi)\cos(\xi x) + \operatorname{sgn}(x)s_e(\xi)\sin(\xi x)), \tag{11.6}$$

$$\rho_{o,U}(\xi, x) := U_o(\xi, x) + \chi(\xi)\chi(x)(\operatorname{sgn}(x)c_o(\xi)\cos(\xi x) + \operatorname{sgn}(\xi)s_o(\xi)\sin(\xi x)), \tag{11.7}$$

$$\rho_{o,V}(\xi, x) := V_o(\xi, x) - i\chi(\xi)\chi(x)(\operatorname{sgn}(\xi x)c_o(\xi)\cos(\xi x) + s_o(\xi)\sin(\xi x)). \tag{11.8}$$

Lemma 11.1.2 *Let $x \in \mathbb{R}$. Then, $\rho_e(0, x) = \rho_o(0, x) = (0, 0)$.*

Proof. Follows from lemma 8.5.7. $\square$

Lemma 11.1.3 *Let $\xi \in \mathbb{R}$. Then, $x \mapsto \rho_e(\xi, x)$ and $x \mapsto \rho_o(\xi, x)$ are smooth on $\mathbb{R}$. $\xi \mapsto \rho_e(\xi, x)$ and $\xi \mapsto \rho_o(\xi, x)$ are continuous on $\mathbb{R}$ and smooth on $\mathbb{R}\backslash\{0\}$ for every $x \in \mathbb{R}$.*

Proof. Follows from lemma 8.5.4 and lemma 8.5.8. $\square$

Lemma 11.1.4 *For $(\xi, x) \in \mathbb{R}^2$:*

$$|\rho_e(\xi, x)|, |\rho_o(\xi, x)| \leq \frac{C}{1+|\xi|} e^{-\frac{1}{2}|x|}, \tag{11.9}$$

$$|\partial_x \rho_e(\xi, x)|, |\partial_x \rho_o(\xi, x)| \leq C e^{-\frac{1}{2}|x|}. \tag{11.10}$$

Proof. Follows from lemma 8.5.4 and lemma 8.5.9. $\square$

Lemma 11.1.5 *For $(\xi, x) \in \mathbb{R}^2$ and $k \geq 0$:*

$$\left|\partial_\xi^k \rho_e(\xi, x)\right|, \left|\partial_\xi^k \rho_o(\xi, x)\right| \leq \frac{C_k}{1+|\xi|} e^{-\frac{1}{2}|x|}. \tag{11.11}$$

For $\xi = 0$, the derivatives are to be understood as one-sided.

Proof. Follows from lemma 8.5.4 and lemma 8.5.10. $\square$

We also define analogues to definition 8.5.12.

Definition 11.1.6 *Consider the function:*

$$\tilde{Q}(x) := \frac{p+1}{4} \int_{-\infty}^{-|x|} Q^{p-1} dy. \tag{11.12}$$

Let $\chi : \mathbb{R} \to [0, 1]$ be as in definition 11.1.1. We define the second order remainder terms $\tilde{\rho}_e := (\tilde{\rho}_{e,U}, \tilde{\rho}_{e,V})$ and $\tilde{\rho}_o := (\tilde{\rho}_{o,U}, \tilde{\rho}_{o,V})$ by

$$\tilde{\rho}_{e,U} := \rho_{e,U} + \chi(\xi)\chi(x)\tilde{Q}(x)\left(\frac{c_e(\xi)}{1+|\xi|}\operatorname{sgn}(\xi x)\sin(\xi x) - \frac{s_e(\xi)}{1+|\xi|}\cos(\xi x)\right), \tag{11.13}$$

$$\begin{aligned}\tilde{\rho}_{e,V} := {} & \rho_{e,V} \\ & - i\chi(\xi)\chi(x)\tilde{Q}(x)\left(\frac{c_e(\xi)}{1+|\xi|}\operatorname{sgn}(x)\sin(\xi x) - \frac{s_e(\xi)}{1+|\xi|}\operatorname{sgn}(\xi)\cos(\xi x)\right),\end{aligned} \tag{11.14}$$

$$\begin{aligned}\tilde{\rho}_{o,U} := {} & \rho_{o,U} \\ & - \chi(\xi)\chi(x)\tilde{Q}(x)\left(\frac{c_o(\xi)}{1+|\xi|}\operatorname{sgn}(\xi)\sin(\xi x) - \frac{s_o(\xi)}{1+|\xi|}\operatorname{sgn}(x)\cos(\xi x)\right),\end{aligned} \tag{11.15}$$

$$\tilde{\rho}_{o,V} := \rho_{o,V} + i\chi(\xi)\chi(x)\tilde{Q}(x)\left(\frac{c_o(\xi)}{1+|\xi|}\sin(\xi x) - \frac{s_o(\xi)}{1+|\xi|}\operatorname{sgn}(\xi x)\cos(\xi x)\right). \tag{11.16}$$

In contrast to the remainder terms given by definition 8.5.12, $\tilde{\rho}_e$ and $\tilde{\rho}_o$ do not exhibit discontinuity in $x = 0$ or $\xi = 0$.

Lemma 11.1.7 *Let $x \in \mathbb{R}$. Then, $\tilde{\rho}_e(0, x) = \tilde{\rho}_o(0, x) = (0, 0)$.*

Proof. Follows from lemma 8.5.4. □

Lemma 11.1.8 *Let $\xi \in \mathbb{R}$. Then, $x \mapsto \tilde{\rho}_e(\xi, x)$ and $x \mapsto \tilde{\rho}_o(\xi, x)$ are smooth on $\mathbb{R}$. $\xi \mapsto \tilde{\rho}_e(\xi, x)$ and $\xi \mapsto \tilde{\rho}_o(\xi, x)$ are continuous on $\mathbb{R}$ and smooth on $\mathbb{R}\backslash\{0\}$ for every $x \in \mathbb{R}$.*

Proof. Follows from lemma 8.5.4. □

Lemma 11.1.9 *For $\xi, x \in \mathbb{R}$ and $k \geq 0$:*

$$\left|\partial_\xi^k \tilde{\rho}_e(\xi, x)\right|, \left|\partial_\xi^k \tilde{\rho}_o(\xi, x)\right| \leq C_k \frac{e^{-\frac{1}{2}|x|}}{1 + \xi^2}, \tag{11.17}$$

$$\left|\partial_\xi^k \partial_x \tilde{\rho}_e(\xi, x)\right|, \left|\partial_\xi^k \partial_x \tilde{\rho}_o(\xi, x)\right| \leq C_k \frac{e^{-\frac{1}{2}|x|}}{1 + |\xi|}. \tag{11.18}$$

For $\xi = 0$, the derivatives are to be understood as one-sided.

Proof. Follows from lemma 8.5.4 and lemma 8.5.13. □

Corollary 11.1.10 *For $\xi, x \in \mathbb{R}$ and $k \geq 0$:*

$$\left|\partial_\xi^k \partial_x \rho_e(\xi, x)\right|, \left|\partial_\xi^k \partial_x \rho_o(\xi, x)\right| \leq C_k e^{-\frac{1}{2}|x|}. \tag{11.19}$$

For $\xi = 0$, the derivatives are to be understood as one-sided.

Proof. Follows from lemma 8.5.4 and lemma 8.5.14. □

Using these new remainder terms, we can restate F and G.

Lemma 11.1.11 *Let $f, g \in \mathcal{S}(\mathbb{R})$. Then, for $x \in \mathbb{R}$:*

$$\begin{aligned}
(G_e f)(x) = {} & \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \begin{pmatrix} \rho_{e,U}(\xi, x) \\ \rho_{e,V}(\xi, x) \end{pmatrix} f(\xi) d\xi \\
& + \chi(x) \begin{pmatrix} \mathcal{F}(\chi c_e f_e) \\ -i\mathcal{F}(\chi c_e \operatorname{sgn}(\cdot) f_o) \end{pmatrix} + \chi(x) \operatorname{sgn}(x) \begin{pmatrix} i\mathcal{F}(\chi s_e \operatorname{sgn}(\cdot) f_e) \\ \mathcal{F}(\chi s_e f_o) \end{pmatrix},
\end{aligned} \tag{11.20}$$

$$\begin{aligned}
(G_o g)(x) = {} & \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \begin{pmatrix} \rho_{o,U}(\xi, x) \\ \rho_{o,V}(\xi, x) \end{pmatrix} g(\xi) d\xi \\
& - \chi(x) \operatorname{sgn}(x) \begin{pmatrix} \mathcal{F}(\chi c_o g_e) \\ -i\mathcal{F}(\chi c_o \operatorname{sgn}(\cdot) g_o) \end{pmatrix} - \chi(x) \begin{pmatrix} i\mathcal{F}(\chi s_o \operatorname{sgn}(\cdot) g_e) \\ \mathcal{F}(\chi s_o g_o) \end{pmatrix}.
\end{aligned} \tag{11.21}$$

Proof. Follows from lemma 9.2.27. □

Lemma 11.1.12 *$F_e, F_o : L^2(\mathbb{R})^2 \to L^2(\mathbb{R})$ are given by:*

$$\begin{aligned} F_e w = {} & \frac{i\,\mathrm{sgn}(\xi)}{\sqrt{2\pi}} \int_{\mathbb{R}} (\rho_{e,V}(\xi,x)u_e(x) + \rho_{e,U}(\xi,x)v_e(x))\,dx + \chi c_e \mathcal{F}(\chi u_e) \\ & + i\chi c_e\,\mathrm{sgn}(\xi)\mathcal{F}(\chi v_e) + i\chi s_e\,\mathrm{sgn}(\xi)\mathcal{F}(\chi\,\mathrm{sgn}(\cdot)u_e) - \chi s_e \mathcal{F}(\chi\,\mathrm{sgn}(\cdot)v_e), \quad (11.22) \\ F_o w = {} & \frac{i\,\mathrm{sgn}(\xi)}{\sqrt{2\pi}} \int_{\mathbb{R}} (\rho_{o,V}(\xi,x)u_o(x) + \rho_{o,U}(\xi,x)v_o(x))\,dx - \chi c_o \mathcal{F}(\chi\,\mathrm{sgn}(\cdot)u_o) \\ & - i\chi c_o\,\mathrm{sgn}(\xi)\mathcal{F}(\chi\,\mathrm{sgn}(\cdot)v_o) - i\chi s_o\,\mathrm{sgn}(\xi)\mathcal{F}(\chi u_o) + \chi s_o \mathcal{F}(\chi v_o). \quad (11.23) \end{aligned}$$

Proof. Follows from definition 9.3.1. □

11.2. Fourier Multiplier

We give some basic facts about Fourier multipliers. For more information on the topic see, e.g. [12].

Definition 11.2.1 *Given a sufficiently regular function $m : \mathbb{R} \to \mathbb{C}$, the associated multiplier operator B_m is given by:*

$$B_m f := \mathcal{F}^{-1}(m\mathcal{F}f). \tag{11.24}$$

m is called a multiplier or a symbol.

Lemma 11.2.2 *Let $H : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the Hilbert transform. Then, for every $f \in L^2(\mathbb{R})$:*

$$2\pi i H f = \mathcal{F}^{-1}(\mathrm{sgn}(\cdot)\mathcal{F}f). \tag{11.25}$$

Proof. See [12, chapter 3]. □

Corollary 11.2.3 *Let $f \in L^2(\mathbb{R}) \cap L^q(\mathbb{R})$, $1 < q < \infty$. Then:*

$$\left|\left|\mathcal{F}^{-1}(\mathrm{sgn}(\cdot)\mathcal{F}f)\right|\right|_{L^q} \leq C_q \,||f||_{L^q}. \tag{11.26}$$

Proof. The Hilbert transform is a well known bounded operator $L^q(\mathbb{R}) \to L^q(\mathbb{R})$ for every $1 < q < \infty$. The bound was originally shown by Marcel Riesz in 1928, see [28]. □

Lemma 11.2.4 *Consider $m_0 \in \mathbb{C}$ and $m \in W^{1,\infty}(\mathbb{R})$ fulfilling:*

$$\exists C > 0\ \forall \xi \in \mathbb{R} :\ |m(\xi) - m_0|\,, |m'(\xi)| \leq \frac{C}{1+|\xi|}. \tag{11.27}$$

Then, for every $1 \leq q \leq \infty$, the multiplier operator B_m is bounded $L^q(\mathbb{R}) \to L^q(\mathbb{R})$.

Proof. For $f \in L^q(\mathbb{R})$:

$$B_m = B_{m-m_0} + m_0 \,\mathrm{Id}\,. \tag{11.28}$$

Hence, we can assume $m_0 = 0$. By definition:

$$B_m f = \mathcal{F}^{-1}(m\mathcal{F}f) = \frac{1}{\sqrt{2\pi}}(\mathcal{F}^{-1}m) * f. \tag{11.29}$$

By (11.27):

$$\begin{aligned} \|\mathcal{F}m\|_{L^1} &\leq \left\|\frac{1}{1+|x|}\right\|_{L^2} \|(1+|x|)\mathcal{F}m\|_{L^2} \\ &\leq 2\,\|\mathcal{F}m\|_{L^2} + 2\,\|x\mathcal{F}m\|_{L^2} \\ &= 2\,\|m\|_{L^2} + 2\,\|m'\|_{L^2} \\ &\leq C. \end{aligned} \tag{11.30}$$

Young's convolution inequality concludes the proof. □

Lemma 11.2.5 *Consider $m_0 \in \mathbb{C}$ and $m \in W^{1,\infty}(\mathbb{R})$ fulfilling:*

$$\exists C > 0\ \forall \xi \in \mathbb{R}:\ |m(\xi) - m_0|\,, |m'(\xi)| \leq \frac{C}{1+|\xi|}. \tag{11.31}$$

Then, for every $1 < q < \infty$, the multiplier operators B_m and $B_{\mathrm{sgn}(\cdot)m}$ are bounded $L^q(\mathbb{R}) \to L^q(\mathbb{R})$.

Proof. Follows from corollary 11.2.3 and lemma 11.2.4. □

Corollary 11.2.6 *Consider $m \in W^{1,\infty}(\mathbb{R})$ fulfilling:*

$$\exists C > 0\ \forall \xi \in \mathbb{R}:\ |m(\xi)|\,, |m'(\xi)| \leq \frac{C}{1+|\xi|}. \tag{11.32}$$

Then, for every $1 \leq q \leq 2$, the multiplier operator B_m is bounded $L^q(\mathbb{R}) \to L^2(\mathbb{R})$.

Proof. Recall the proof of lemma 11.2.4. By the Plancherel theorem:

$$\|\mathcal{F}m\|_{L^2} = \|m\|_{L^2} \leq C. \tag{11.33}$$

Together with (11.29), (11.30) and Young's convolution inequality the claim follows. □

11.3. Decomposition of the Wave Operator

Note the basic identities:

$$T = \mathcal{G}_e F_e + \mathcal{G}_o F_o, \tag{11.34}$$

$$T^{-1} = G_e \mathcal{F}_e + G_o \mathcal{F}_o. \tag{11.35}$$

We further decompose $\mathcal{G}_e F_e$ and $\mathcal{G}_o F_o$, as well as $G_e \mathcal{F}_e$ and $G_o \mathcal{F}_o$ into a variety of operators.

Definition 11.3.1 *For $w=(u,v)\in L^2(\mathbb{R})^2$, we define:*

$$A_{e,1}w := \begin{pmatrix} \mathcal{F}^{-1}((\chi c_e - 1)\mathcal{F}(\chi u)) \\ \mathcal{F}^{-1}((\chi c_e - 1)\mathcal{F}(\chi v)) \end{pmatrix}. \tag{11.36}$$

To improve readability, we introduce the shorthand notation:

$$A_{e,1}w = \mathcal{F}^{-1}((\chi c_e - 1)\mathcal{F}(\chi w)). \tag{11.37}$$

We further define:

$$A_{e,2}w := i\mathcal{F}^{-1}(\chi \operatorname{sgn}(\xi) s_e \mathcal{F}(\chi \operatorname{sgn}(y) w)), \tag{11.38}$$
$$A_{o,1}w := \mathcal{F}^{-1}((\chi s_o - 1)\mathcal{F}(\chi w)), \tag{11.39}$$
$$A_{o,2}w := i\mathcal{F}^{-1}(\chi \operatorname{sgn}(\xi) c_o \mathcal{F}(\chi \operatorname{sgn}(y) w)), \tag{11.40}$$
$$B_{e,1}w := \chi\mathcal{F}^{-1}((\chi c_e - 1)\mathcal{F}w), \tag{11.41}$$
$$B_{e,2}w := -i\chi \operatorname{sgn}(x)\mathcal{F}^{-1}(\chi \operatorname{sgn}(\xi) s_e \mathcal{F}w), \tag{11.42}$$
$$B_{o,1}w := \chi\mathcal{F}^{-1}((\chi s_o - 1)\mathcal{F}w), \tag{11.43}$$
$$B_{o,2}w := i\chi \operatorname{sgn}(x)\mathcal{F}^{-1}(\chi \operatorname{sgn}(\xi) c_o \mathcal{F}w). \tag{11.44}$$

Also consider:

$$A_{e,3}w := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \begin{pmatrix} i\mathcal{F}(\operatorname{sgn}(\xi)\rho_{e,V}(\xi,y))u(y) \\ \mathcal{F}(\rho_{e,U}(\xi,y))v(y) \end{pmatrix} dy, \tag{11.45}$$

$$A_{o,3}w := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \begin{pmatrix} \mathcal{F}(\rho_{o,V}(\xi,y))u(y) \\ -i\mathcal{F}(\operatorname{sgn}(\xi)\rho_{o,U}(\xi,y))v(y) \end{pmatrix} dy, \tag{11.46}$$

$$B_{e,3}w := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \begin{pmatrix} \rho_{e,U}(\xi,x)\mathcal{F}u \\ i\rho_{e,V}(\xi,x)\operatorname{sgn}(\xi)\mathcal{F}v \end{pmatrix} d\xi, \tag{11.47}$$

$$B_{o,3}w := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \begin{pmatrix} -i\rho_{o,U}(\xi,x)\operatorname{sgn}(\xi)\mathcal{F}u \\ \rho_{o,V}(\xi,x)\mathcal{F}v \end{pmatrix} d\xi. \tag{11.48}$$

Lemma 11.3.2 *Let $w=(u,v)\in L^2(\mathbb{R})^2$. Then:*

$$\mathcal{G}_e F_e w = \chi w_e + A_{e,1}w_e + A_{e,2}w_e + A_{e,3}w_e, \tag{11.49}$$
$$\mathcal{G}_o F_o w = \chi w_o + A_{o,1}w_o + A_{o,2}w_o + A_{o,3}w_o, \tag{11.50}$$
$$G_e \mathcal{F}_e w = \chi w_e + B_{e,1}w_e + B_{e,2}w_e + B_{e,3}w_e, \tag{11.51}$$
$$G_o \mathcal{F}_o w = \chi w_o + B_{o,1}w_o + B_{o,2}w_o + B_{o,3}w_o. \tag{11.52}$$

Proof. Direct computation using lemma 11.1.11, lemma 11.1.12, definition 10.1.1 and definition 10.1.2 gives:

$$\begin{aligned}
\mathcal{G}_e F_e w = {} & \mathcal{F}(\chi c_e \mathcal{F}(\chi w_e)) + i\mathcal{F}(\chi \operatorname{sgn}(\xi) s_e \mathcal{F}(\chi \operatorname{sgn}(y) w_e)) \\
& + \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \begin{pmatrix} i\mathcal{F}(\operatorname{sgn}(\xi)\rho_{e,V}(\xi,y))u_e(y) \\ \mathcal{F}(\rho_{e,U}(\xi,y))v_e(y) \end{pmatrix} dy, && (11.53) \\
\mathcal{G}_o F_o w = {} & -\mathcal{F}(\chi s_o \mathcal{F}(\chi w_o)) - i\mathcal{F}(\chi \operatorname{sgn}(\xi) c_o \mathcal{F}(\chi \operatorname{sgn}(y) w_o)) \\
& + \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \begin{pmatrix} \mathcal{F}(\rho_{o,V}(\xi,y))u_o(y) \\ -i\mathcal{F}(\operatorname{sgn}(\xi)\rho_{o,U}(\xi,y))v_o(y) \end{pmatrix} dy, && (11.54) \\
G_e \mathcal{F}_e w = {} & \mathcal{F}(\chi c_e \mathcal{F}(\chi w_e)) + i\operatorname{sgn}(x)\mathcal{F}(\chi \operatorname{sgn}(\xi) s_e \mathcal{F}(\chi w_e)) \\
& + \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \begin{pmatrix} \rho_{e,U}(\xi,x)\mathcal{F}u_e \\ i\rho_{e,V}(\xi,x)\operatorname{sgn}(\xi)\mathcal{F}v_e \end{pmatrix} d\xi, && (11.55) \\
G_o \mathcal{F}_o w = {} & -\mathcal{F}(\chi s_o \mathcal{F}(\chi w_o)) + i\operatorname{sgn}(x)\mathcal{F}(\chi \operatorname{sgn}(\xi) c_o \mathcal{F}(\chi w_o)) \\
& + \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \begin{pmatrix} -i\rho_{o,U}(\xi,x)\operatorname{sgn}(\xi)\mathcal{F}u_o \\ \rho_{o,V}(\xi,x)\mathcal{F}v_o \end{pmatrix} d\xi. && (11.56)
\end{aligned}$$

We now use the fact that even functions fulfil $\mathcal{F}f_e = \mathcal{F}^{-1}f_e$, while odd functions fulfil $\mathcal{F}f_o = -\mathcal{F}^{-1}f_o$. □

Definition 11.3.3 *Consider $\tilde{Q} : \mathbb{R} \to \mathbb{R}$ as given by definition 11.1.6:*

$$\tilde{Q}(x) = \frac{p+1}{4} \int_{-\infty}^{-|x|} Q^{p-1} dy. \tag{11.57}$$

For $w = (u,v) \in L^2(\mathbb{R})^2$, we define:

$$\begin{aligned}
A_{e,4}w &:= \mathcal{F}^{-1}\left(\chi \frac{s_e}{1+|\xi|}\mathcal{F}(\chi\tilde{Q}w)\right), && (11.58) \\
A_{e,5}w &:= -i\mathcal{F}^{-1}\left(\chi \operatorname{sgn}(\xi)\frac{c_e}{1+|\xi|}\mathcal{F}(\chi \operatorname{sgn}(y)\tilde{Q}w)\right), && (11.59) \\
A_{o,4}w &:= -\mathcal{F}^{-1}\left(\chi \frac{c_o}{1+|\xi|}\mathcal{F}(\chi\tilde{Q}w)\right), && (11.60) \\
A_{o,5}w &:= -i\mathcal{F}^{-1}\left(\chi \operatorname{sgn}(\xi)\frac{s_o}{1+|\xi|}\mathcal{F}(\chi \operatorname{sgn}(y)\tilde{Q}w)\right), && (11.61) \\
B_{e,4}w &:= \chi\tilde{Q}\mathcal{F}^{-1}\left(\chi \frac{s_e}{1+|\xi|}\mathcal{F}w\right), && (11.62) \\
B_{e,5}w &:= i\chi \operatorname{sgn}(x)\tilde{Q}\mathcal{F}^{-1}\left(\chi \operatorname{sgn}(\xi)\frac{c_e}{1+|\xi|}\mathcal{F}w\right), && (11.63) \\
B_{o,4}w &:= -\chi\tilde{Q}\mathcal{F}^{-1}\left(\chi \frac{c_o}{1+|\xi|}\mathcal{F}w\right), && (11.64) \\
B_{o,5}w &:= i\chi \operatorname{sgn}(x)\tilde{Q}\mathcal{F}^{-1}\left(\chi \operatorname{sgn}(\xi)\frac{s_o}{1+|\xi|}\mathcal{F}w\right). && (11.65)
\end{aligned}$$

Further, using the remainder terms given by definition 11.1.6:

$$A_{e,6}w := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \begin{pmatrix} i\mathcal{F}(\operatorname{sgn}(\xi)\tilde{\rho}_{e,V}(\xi,y))u(y) \\ \mathcal{F}(\tilde{\rho}_{e,U}(\xi,y))v(y) \end{pmatrix} dy, \tag{11.66}$$

$$A_{o,6}w := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \begin{pmatrix} \mathcal{F}(\tilde{\rho}_{o,V}(\xi,y))u(y) \\ -i\mathcal{F}(\operatorname{sgn}(\xi)\tilde{\rho}_{o,U}(\xi,y))v(y) \end{pmatrix} dy, \tag{11.67}$$

$$B_{e,6}w := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \begin{pmatrix} \tilde{\rho}_{e,U}(\xi,x)\mathcal{F}u \\ i\tilde{\rho}_{e,V}(\xi,x)\operatorname{sgn}(\xi)\mathcal{F}v \end{pmatrix} d\xi, \tag{11.68}$$

$$B_{o,6}w := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \begin{pmatrix} -i\tilde{\rho}_{o,U}(\xi,x)\operatorname{sgn}(\xi)\mathcal{F}u \\ \tilde{\rho}_{o,V}(\xi,x)\mathcal{F}v \end{pmatrix} d\xi. \tag{11.69}$$

Lemma 11.3.4 *Let $w = (u,v) \in L^2(\mathbb{R})^2$. Then:*

$$A_{e,3}w_e = A_{e,4}w_e + A_{e,5}w_e + A_{e,6}w_e, \tag{11.70}$$

$$A_{o,3}w_e = A_{o,4}w_o + A_{o,5}w_o + A_{o,6}w_o, \tag{11.71}$$

$$B_{e,3}w_e = B_{e,4}w_e + B_{e,5}w_e + B_{e,6}w_e, \tag{11.72}$$

$$B_{o,3}w_e = B_{o,4}w_o + B_{o,5}w_o + B_{o,6}w_o. \tag{11.73}$$

Proof. Follows by direct computation and lemma 9.2.26. □

11.4. L^q-Boundedness

Lemma 11.4.1 *Let $1 \leq q < \infty$ and $\rho = \operatorname{sgn}(\xi)\rho_{e,V}$, $\rho = \rho_{e,U}$, $\rho = \rho_{o,V}$ or $\rho = \operatorname{sgn}(\xi)\rho_{e,U}$. Then, for every $y \in \mathbb{R}$:*

$$||(\mathcal{F}\rho)(x,y)||_{L^q_x} \leq C_q e^{-\frac{1}{2}|y|}. \tag{11.74}$$

If $2 \leq q < \infty$, then:

$$||x(\mathcal{F}\rho)(x,y)||_{L^q_x} \leq C_q e^{-\frac{1}{2}|y|}. \tag{11.75}$$

Proof. For every $r \geq 2$ and $k \in \{0,1\}$, by lemma 11.1.5:

$$\left|\left|x^k(\mathcal{F}\rho)(x,y)\right|\right|_{L^r_x} \leq C \left|\left|\partial_\xi^k \rho(\xi,y)\right|\right|_{L^{r'}_\xi} \leq C_r e^{-\frac{1}{2}|y|}. \tag{11.76}$$

(Weak) differentiability in 0 is ensured by lemma 11.1.2 and lemma 11.1.3.

We choose $r \geq 2$ with $r \geq q$. It follows:

$$||(\mathcal{F}\rho)(x,y)||_{L^q_x} \leq C\,||(\mathcal{F}\rho)(x,y)||_{L^r_x} + C\,||x(\mathcal{F}\rho)(x,y)||_{L^r_x} \leq C_q e^{-\frac{1}{2}|y|}. \tag{11.77}$$

That concludes the proof. □

Lemma 11.4.2 *Let $\rho = \operatorname{sgn}(\xi)\tilde{\rho}_{e,V}$, $\rho = \tilde{\rho}_{e,U}$, $\rho = \tilde{\rho}_{o,V}$ or $\rho = \operatorname{sgn}(\xi)\tilde{\rho}_{e,U}$. Then, for every $y \in \mathbb{R}$ and $1 \leq q \leq \infty$:*

$$||(\mathcal{F}\rho)(x,y)||_{L_x^q} \leq Ce^{-\frac{1}{2}|y|}. \tag{11.78}$$

If $1 \leq q < \infty$, then:

$$||\partial_x(\mathcal{F}\rho)(x,y)||_{L_x^q} \leq Ce^{-\frac{1}{2}|y|}. \tag{11.79}$$

If $2 \leq q < \infty$, then:

$$||x(\mathcal{F}\rho)(x,y)||_{L_x^q}, ||x\partial_x(\mathcal{F}\rho)(x,y)||_{L_x^q}, ||\partial_y(\mathcal{F}\rho)(x,y)||_{L_x^q} \leq C_q e^{-\frac{1}{2}|y|}. \tag{11.80}$$

Proof. It suffices to show for every $1 \leq r \leq 2$:

$$||\rho(\xi,y)||_{L_\xi^r}, ||\partial_\xi\rho(\xi,y)||_{L_\xi^r} \leq Ce^{-\frac{1}{2}|y|}, \tag{11.81}$$

as well as for $1 < r \leq 2$:

$$||\xi\rho(\xi,y)||_{L_\xi^r}, ||\partial_\xi(\xi\rho(\xi,y))||_{L_\xi^r}, ||\partial_y\rho(\xi,y)||_{L_\xi^r} \leq C_r e^{-\frac{1}{2}|y|}. \tag{11.82}$$

Lemma 11.1.9 shows just that. Lemma 11.1.7 and lemma 11.1.8 ensure differentiability in $\xi = 0$. That concludes the proof. □

Lemma 11.4.3 *Let $1 \leq q \leq \infty$ and $w = (u,v) \in L^q(\mathbb{R})^2$. Then, for every $1 \leq k \leq 6$:*

$$||A_{e,k}w||_{L^q}, ||A_{o,k}w||_{L^q} \leq C\,||w||_{L^q}. \tag{11.83}$$

Proof. The claim follows, for $k = 1,2,4,5$, from lemma 11.2.4 and theorem 8.3.5, specifically (8.90) - (8.93). For $k = 6$, lemma 11.4.2 yields the claim. Lemma 11.3.4 concludes the proof. □

Corollary 11.4.4 *Let $1 \leq q \leq \infty$ and $w = (u,v) \in L^q(\mathbb{R})^2$. Then:*

$$||Tw||_{L^q} \leq C\,||w||_{L^q}. \tag{11.84}$$

Proof. Follows from lemma 11.4.3. □

Lemma 11.4.5 *Let $1 \leq q \leq \infty$ and $w = (u,v) \in L^q(\mathbb{R})^2$. Then, for every $1 \leq k \leq 6$:*

$$||B_{e,k}w||_{L^q}, ||B_{o,k}w||_{L^q} \leq C\,||w||_{L^q}. \tag{11.85}$$

Proof. The claim follows, for $k = 1,2,4,5$, from lemma 11.2.4 and theorem 8.3.5, specifically (8.90) - (8.93). For $k = 6$, lemma 11.4.2 yields the claim. Lemma 11.3.4 concludes the proof. □

Corollary 11.4.6 *Let $1 \leq q \leq \infty$ and $w = (u,v) \in L^q(\mathbb{R})^2$. Then:*

$$\left|\left|T^{-1}w\right|\right|_{L^q} \leq C\,||w||_{L^q}. \tag{11.86}$$

Proof. Follows from lemma 11.4.5. □

Lemma 11.4.7 *Let $w = (u, v) \in L^2(\mathbb{R})^2$. Then:*

$$||(x\partial_x T - Tx\partial_x)w||_{L^2} \leq C\,||w||_{L^2}\,. \tag{11.87}$$

Proof. Let $A = A_{e,1}$ or $A_{e,2}$, $A_{e,4}$, $A_{e,5}$.

$x\partial_x A - Ax\partial_x$ cancels largely out. The remainder can be bounded using lemma 11.2.5. For $A = A_{e,6}$ the bound follows from lemma 11.4.2.

$A_{o,1}$, $A_{o,2}$, $A_{o,4}$, $A_{o,5}$ and $A_{e,6}$ are bounded analogously. □

Lemma 11.4.8 *Let $w = (u, v) \in L^2(\mathbb{R})^2$ and $1 < q < \infty$. Then:*

$$||xTw||_{L^q} \leq C_q(||w||_{L^q} + ||xw||_{L^q}). \tag{11.88}$$

Proof. By definition:

$$\begin{aligned} xA_{e,1}w &= i\mathcal{F}^{-1}\partial_\xi((\chi c_e - 1)\mathcal{F}(\chi w)) \\ &= i\mathcal{F}^{-1}(\partial_\xi(\chi c_e)\mathcal{F}(\chi w)) + \mathcal{F}^{-1}((\chi c_e - 1)\mathcal{F}(\chi y w)). \end{aligned} \tag{11.89}$$

Lemma 11.2.5 yields the desired estimate for $A_{e,1}$. $A_{e,2}$, $A_{o,1}$ and $A_{o,2}$ are bounded analogously. The bound for $A_{e,3}$ and $A_{o,3}$ follows from lemma 11.4.1.

That concludes the proof. □

11.5. Galilean Operator

Consider the self-adjoint operator $J_t = \frac{x}{2} + It\partial_x$. Through the following identity, J_t is strongly connected to the Galilean invariance of the free Schrödinger equation:

$$J_t e^{It\Delta} w_0 = e^{It\Delta}\left(\frac{x}{2} w_0\right). \tag{11.90}$$

For that reason we refer to J_t as the Galilean operator.

It seems likely that J_t will play a major role in any proof of asymptotic stability of the NLS. For that reason, we establish a variety of bounds involving J_t and T, that might prove useful. In particular, we estimate $||J_t Th||_{L^2}$ and $||J_t T^{-1}h||_{L^2}$.

We also show bounds involving the commutators $T_1 = IT - TI$ and $T_2 = x\partial_x T - Tx\partial_x$.

11.5.1. Bounds on T

Lemma 11.5.1 *Let $2 \leq q \leq \infty$, $k \in \{1, 2, 4, 5\}$ and $w \in L^q(\mathbb{R})^2$. Then:*

$$||xA_{e,k}w - A_{e,k}(xw)||_{L^q}\,, ||xA_{o,k}w - A_{o,k}(xw)||_{L^q} \leq C\,||w||_{L^q}\,. \tag{11.91}$$

Proof. Follows by direct computation from theorem 8.3.5, specifically (8.90) - (8.93), and lemma 11.2.4. □

Lemma 11.5.2 *Let $2 \leq q < \infty$ and $w \in L^q(\mathbb{R})^2$. Then:*

$$||xA_{e,6}w||_{L^q}, ||xA_{o,6}w||_{L^q} \leq C_q ||w||_{L^q}. \tag{11.92}$$

Proof. Follows from lemma 11.4.2. ☐

Lemma 11.5.3 *Let $2 \leq q \leq \infty$, $k \in \{1, 2, 4, 5\}$ and $w \in L^q(\mathbb{R})^2$. Then:*

$$||\partial_x A_{e,k}w - A_{e,k}\partial_x w||_{L^q}, ||\partial_x A_{o,k}w - A_{o,k}\partial_x w||_{L^q} \leq C \left|\left|e^{-\frac{1}{2}|x|}w\right|\right|_{L^q}. \tag{11.93}$$

Proof. By direct computation

$$\partial_x A_{e,1}w - A_{e,1}\partial_x w = \mathcal{F}^{-1}((\chi c_e - 1)\mathcal{F}(\partial_y \chi w)). \tag{11.94}$$

Lemma 11.2.4 shows the claim. The other bounds follow completely analogously. ☐

Lemma 11.5.4 *Let $2 \leq q < \infty$ and $w \in L^q(\mathbb{R})^2$. Then:*

$$||\partial_x A_{e,6}w||_{L^q}, ||\partial_x A_{o,6}w||_{L^q} \leq C_q \left|\left|e^{-\frac{1}{3}|x|}w\right|\right|_{L^q}. \tag{11.95}$$

Proof. Follows from lemma 11.4.2. ☐

Lemma 11.5.5 *Let $2 \leq q \leq \infty$, $t \geq 0$ and $w \in L^q(\mathbb{R})^2$. Let $J_t = \frac{x}{2} + It\partial_x$. Consider the operator:*

$$\tilde{T}w := \sum_{k=1}^{5} A_{e,k}w_o + \sum_{k=1}^{5} A_{o,k}w_e. \tag{11.96}$$

Then:

$$\left|\left|\tilde{T}w\right|\right|_{L^q} \leq C ||w||_{L^q}. \tag{11.97}$$

Further, for $2 \leq q < \infty$:

$$\left|\left|J_t T w - \tilde{T} J_t w\right|\right|_{L^q} \leq C_q ||w||_{L^q} + C_q t \left|\left|e^{-\frac{1}{3}|x|}w\right|\right|_{L^q}. \tag{11.98}$$

Proof. Follows from lemma 11.5.1 - lemma 11.5.4 and lemma 11.3.2, lemma 11.3.4. ☐

Corollary 11.5.6 *Let $2 \leq q \leq \infty$, $t \geq 0$ and $w \in L^q(\mathbb{R})^2$. Let $J_t = \frac{x}{2} + It\partial_x$. Then, for $2 \leq q < \infty$:*

$$||J_t T w||_{L^q} \leq C_q ||w||_{L^q} + C ||J_t w||_{L^q} + C_q t \left|\left|e^{-\frac{1}{3}|x|}w\right|\right|_{L^q}. \tag{11.99}$$

Using $t \left|\left|e^{-\frac{1}{3}|x|}w\right|\right|_{L^q}$ as an upper bound is totally fine. By the smoothing theorem at the end of chapter 12 (theorem 12.1.9), $t \left|\left|e^{-\frac{1}{3}|x|}w\right|\right|_{L^q}$ behaves similarly to $||J_t w||_{L^q}$.

11.5.2. Bounds on T^{-1}

Lemma 11.5.7 *Let $2 \leq q \leq \infty$, $k \in \{1, 2, 4, 5\}$ and $w \in L^q(\mathbb{R})^2$. Then:*

$$||xB_{e,k}w - B_{e,k}(xw)||_{L^q}, ||xB_{o,k}w - B_{o,k}(xw)||_{L^q} \leq C\, ||w||_{L^q}. \tag{11.100}$$

Proof. Follows by direct computation from theorem 8.3.5, specifically (8.90) - (8.93), and lemma 11.2.4. □

Lemma 11.5.8 *Let $2 \leq q \leq \infty$ and $w \in L^q(\mathbb{R})^2$. Then:*

$$||xB_{e,6}w||_{L^q}, ||xB_{o,6}w||_{L^q} \leq C\, ||w||_{L^q}. \tag{11.101}$$

Proof. Follows from lemma 11.4.2. □

Lemma 11.5.9 *Let $2 \leq q \leq \infty$, $t \geq 0$ and $w \in L^q(\mathbb{R})^2$. Let $J_t = \frac{x}{2} + It\partial_x$ and assume $J_t w \in L^2(\mathbb{R})^2$, as well as $(\mathcal{F}w)(0) = (0, 0)$. Then:*

$$t\, ||\partial_x B_{e,3}w||_{L^q}, t\, ||\partial_x B_{o,3}w||_{L^q} \leq C\, ||J_t w||_{L^q} + C\, ||w||_{L^q}. \tag{11.102}$$

Proof. By lemma 11.1.2, lemma 11.1.3 and lemma 11.1.5:

$$\begin{aligned}
t\, ||\partial_x B_{e,3}w||_{L^q} &\leq \frac{t}{\sqrt{2\pi}} \left|\left| \int_{\mathbb{R}} \begin{pmatrix} \partial_x \rho_{e,U}(\xi, x)\mathcal{F}u \\ i\partial_x \rho_{e,V}(\xi, x)\,\mathrm{sgn}(\xi)\mathcal{F}v \end{pmatrix} d\xi \right|\right|_{L^q} \\
&\leq \frac{t}{\sqrt{2\pi}} \left|\left| \int_{\mathbb{R}} \begin{pmatrix} \partial_x \rho_{e,U}(\xi, x)\mathcal{F}u \\ i\partial_x \rho_{e,V}(\xi, x)\,\mathrm{sgn}(\xi)\mathcal{F}v \end{pmatrix} d\xi \right|\right|_{L^q} \\
&= \frac{1}{\sqrt{2\pi}} \left|\left| \int_{\mathbb{R}} \begin{pmatrix} \xi^{-1}\partial_x \rho_{e,U}(\xi, x)\mathcal{F}(t\partial_y u) \\ i\xi^{-1}\partial_x \rho_{e,V}(\xi, x)\,\mathrm{sgn}(\xi)\mathcal{F}(t\partial_y v) \end{pmatrix} d\xi \right|\right|_{L^q} \\
&\leq C\, ||J_t w||_{L^q} + \left|\left| \int_{\mathbb{R}} \begin{pmatrix} \xi^{-1}\partial_x \rho_{e,U}(\xi, x)\mathcal{F}(yv) \\ i\xi^{-1}\partial_x \rho_{e,V}(\xi, x)\,\mathrm{sgn}(\xi)\mathcal{F}(yu) \end{pmatrix} d\xi \right|\right|_{L^q}.
\end{aligned} \tag{11.103}$$

We make use of $(\mathcal{F}u)(0) = (\mathcal{F}v)(0) = 0$. By lemma 11.1.2, corollary 11.1.10 and the mean value theorem:

$$\begin{aligned}
&\left|\left| \int_{\mathbb{R}} \begin{pmatrix} \xi^{-1}\partial_x \rho_{e,U}(\xi, x)\mathcal{F}(yv) \\ i\xi^{-1}\partial_x \rho_{e,V}(\xi, x)\,\mathrm{sgn}(\xi)\mathcal{F}(yu) \end{pmatrix} d\xi \right|\right|_{L^q} \\
&\leq \left|\left| \int_{\mathbb{R}} \begin{pmatrix} \partial_\xi \left(\xi^{-1}\partial_x \rho_{e,U}(\xi, x)\right) \mathcal{F}v \\ i\partial_\xi \left(\xi^{-1}\partial_x \rho_{e,V}(\xi, x)\right) \mathrm{sgn}(\xi)\mathcal{F}u \end{pmatrix} d\xi \right|\right|_{L^q} \\
&\leq C\, ||w||_{L^q}.
\end{aligned} \tag{11.104}$$

$t\, ||\partial_x B_{o,3}w||_{L^q}$ is bounded completely analogously. That concludes the proof. □

Lemma 11.5.10 *Let $2 \leq q \leq \infty$, $t \geq 0$ and $w \in L^q(\mathbb{R})^2$. Let $J_t = \frac{x}{2} + It\partial_x$ and assume $J_t w \in L^2(\mathbb{R})^2$, as well as $(\mathcal{F}w)(0) = (0,0)$. Then:*

$$\left|\left|J_t T^{-1} w\right|\right|_{L^q} \leq C \left|\left|J_t w\right|\right|_{L^q} + C \left|\left|w\right|\right|_{L^q} + Ct \left|\left|Bw\right|\right|_{L^q}. \tag{11.105}$$

Hereby:

$$\begin{aligned} Bw = &\ \partial_x \chi w_e + \partial_x B_{e,1} w_e - B_{e,1} \partial_x w_e + \partial_x B_{e,2} w_e - B_{e,2} \partial_x w_e \\ &+ \partial_x \chi w_o + \partial_x B_{o,1} w_o - B_{o,1} \partial_x w_o + \partial_x B_{o,2} w_o - B_{o,2} \partial_x w_o. \end{aligned} \tag{11.106}$$

Proof. Follows from lemma 11.5.7 - lemma 11.5.9. □

In order to control $||tBw||_{L^q}$, we use the following local smoothing estimate. More generalised local smoothing estimates will be proven later in the form of lemma 12.1.8, theorem 12.1.9 and theorem 12.1.10.

Lemma 11.5.11 *Let $2 \leq q \leq \infty$, $t \geq 0$ and $w_0 \in L^2(\mathbb{R})^2$. Consider $w = e^{It\Delta} w_0$ and*

$$\begin{aligned} Bw = &\ \partial_x \chi w_e + \partial_x B_{e,1} w_e - B_{e,1} \partial_x w_e + \partial_x B_{e,2} w_e - B_{e,2} \partial_x w_e \\ &+ \partial_x \chi w_o + \partial_x B_{o,1} w_o - B_{o,1} \partial_x w_o + \partial_x B_{o,2} w_o - B_{o,2} \partial_x w_o. \end{aligned} \tag{11.107}$$

Then:

$$t \left|\left|Bw\right|\right|_{L^q} \leq C \left|\left|e^{It\Delta} w_0\right|\right|_{L^q} + C \left|\left|e^{It\Delta}(x w_0)\right|\right|_{L^q}. \tag{11.108}$$

Proof. By direct computation and lemma 12.1.2:

$$\begin{aligned} \partial_x \chi w_e + \partial_x B_{e,1} w - B_{e,1} \partial_x w &= \partial_x \chi \mathcal{F}^{-1}(\chi c_e \mathcal{F} w) \\ &= \partial_x \chi \tilde{\mathcal{F}}^{-1}(\chi c_e \tilde{\mathcal{F}} w). \end{aligned} \tag{11.109}$$

By definition, $\partial_x \chi$ is a smooth function with $\partial_x \chi(x) = 0$ for $|x| \geq 2$ and $|x| \leq 1$. It follows:

$$\begin{aligned}
&t\,||\partial_x\chi w_e + \partial_x B_{e,1}w - B_{e,1}\partial_x w||_{L^q} \\
= &\sup_{||h||_{L^{q'}}=1} t\langle \partial_x\chi\tilde{\mathcal{F}}^{-1}(\chi c_e\tilde{\mathcal{F}}w), h\rangle_{L^2} \\
= &\sup_{||h||_{L^{q'}}=1} t\langle e^{-It\xi^2}\tilde{\mathcal{F}}w_0, \chi c_e\tilde{\mathcal{F}}(\partial_x\chi h)\rangle_{L^2} \\
= &\sup_{||h||_{L^{q'}}=1} \int_{\mathbb{R}} (-2It\xi)e^{-It\xi^2}\tilde{\mathcal{F}}w_0\cdot\frac{1}{2}\frac{\chi(\xi)}{\xi}c_e I\tilde{\mathcal{F}}(\partial_x\chi h)d\xi \\
= &\sup_{||h||_{L^{q'}}=1} \int_{\mathbb{R}} e^{-It\xi^2}\tilde{\mathcal{F}}w_0\cdot\partial_\xi\left(\frac{1}{2}\frac{\chi(\xi)}{\xi}c_e I\tilde{\mathcal{F}}(\partial_x\chi h)\right)d\xi \\
&+\sup_{||h||_{L^{q'}}=1} \int_{\mathbb{R}} e^{-It\xi^2}\tilde{\mathcal{F}}(xw_0)\cdot\frac{1}{2}\frac{\chi(\xi)}{\xi}c_e I\tilde{\mathcal{F}}(\partial_x\chi h)d\xi \\
= &\sup_{||h||_{L^{q'}}=1} \int_{\mathbb{R}} e^{It\Delta}w_0\cdot\tilde{\mathcal{F}}^{-1}\partial_\xi\left(\frac{1}{2}\frac{\chi(\xi)}{\xi}c_e I\tilde{\mathcal{F}}(\partial_x\chi h)\right)d\xi \\
&+\sup_{||h||_{L^{q'}}=1} \int_{\mathbb{R}} e^{It\Delta}(xw_0)\cdot\tilde{\mathcal{F}}^{-1}\left(\frac{1}{2}\frac{\chi(\xi)}{\xi}c_e I\tilde{\mathcal{F}}(\partial_x\chi h)\right)d\xi.
\end{aligned} \tag{11.110}$$

By lemma 11.2.4:

$$\begin{aligned}
&t\,||\partial_x\chi w_e + \partial_x B_{e,1}w - B_{e,1}\partial_x w||_{L^q} \\
\le &C\left|\left|e^{It\Delta}w_0\right|\right|_{L^q}\sup_{||h||_{L^{q'}}=1}\left(||\partial_x\chi h||_{L^{q'}} + ||x\partial_x\chi h||_{L^{q'}}\right) \\
&+C\left|\left|e^{It\Delta}(xw_0)\right|\right|_{L^q}\sup_{||h||_{L^{q'}}=1}||\partial_x\chi h||_{L^{q'}} \\
\le &C^2\left|\left|e^{It\Delta}w_0\right|\right|_{L^q} + C^2\left|\left|e^{It\Delta}(xw_0)\right|\right|_{L^q}.
\end{aligned} \tag{11.111}$$

Completely analogously:

$$\begin{aligned}
&||\partial_x B_{e,2}w_e - B_{e,2}\partial_x w_e||_{L^q} + ||\partial_x\chi w_o + \partial_x B_{o,1}w_o - B_{o,1}\partial_x w_o||_{L^q} \\
&+||\partial_x B_{o,2}w_o - B_{o,2}\partial_x w_o||_{L^q} \\
\le &C\left|\left|e^{It\Delta}w_0\right|\right|_{L^q} + C\left|\left|e^{It\Delta}(xw_0)\right|\right|_{L^q}.
\end{aligned} \tag{11.112}$$

That concludes the proof. □

Corollary 11.5.12 *Let $2 \le q \le \infty$, $t \ge 0$ and $w \in L^q(\mathbb{R})^2$. Let $J_t = \frac{x}{2} + It\partial_x$ and assume $(\mathcal{F}w)(0) = (0,0)$. Then:*

$$\left|\left|J_t T^{-1}w\right|\right|_{L^q} \le C\,||J_t w||_{L^q} + C\,||w||_{L^q}. \tag{11.113}$$

Proof. Consider $w_0 = e^{-It\Delta}w$. Using the identity, $J_t e^{It\Delta}w_0 = e^{It\Delta}\left(\frac{x}{2}w_0\right)$, the claim follows from lemma 11.5.10 and lemma 11.5.11. □

11.6. Commutators

When trying to show asymptotic stability of the NLS the commutators $T_1 = IT - TI$ and $T_2 = x\partial_x T - Tx\partial_x$ are very natural quantities arising from the phase and scaling invariance of the NLS.

Similarly, the translation and Galilean invariance give rise to $T_3 = \partial_x T - T\partial_x$ and $T_4 = xT - Tx$.

There are two reasons, we only consider T_1 and T_2. Firstly, by restricting ourself to radially symmetric solutions, the question of asymptotic stability can be examined without considering T_3 and T_4. Secondly, T_4 admits significantly weaker bounds. Even without involving J_t, the strongest L^q-bound on T_4 is given by

$$||T_4 w||_{L^q} \leq C \, ||w||_{L^q} + C \, ||xw||_{L^q} \, . \tag{11.114}$$

For that reason alone, examining asymptotic stability without restricting oneself to radially symmetric solutions, makes any proof significantly more technically challenging. T_3 on the other hand can be bounded in much the same way as T_2. We will still ignore T_3 in this chapter.

11.6.1. Bounds on T_1

Lemma 11.6.1 *Let $1 \leq q \leq \infty$ and $w = (u, v) \in L^q(\mathbb{R})^2$. Then:*

$$||(T + ITI)w||_{L^q} \leq C \left|\left| e^{-\frac{1}{2}|x|} w \right|\right|_{L^q} . \tag{11.115}$$

If $1 \leq q < \infty$, then:

$$||\partial_x (T + ITI)w||_{L^q} \leq C \left|\left| e^{-\frac{1}{2}|x|} w \right|\right|_{L^q} . \tag{11.116}$$

If $2 \leq q < \infty$, then:

$$||x(T + ITI)w||_{L^q} \leq C \left|\left| e^{-\frac{1}{2}|x|} w \right|\right|_{L^q} . \tag{11.117}$$

Proof. I commutes with $A_{e,k}$ and $A_{o,k}$ for $k \in \{1, 2, 4, 5\}$. Hence:

$$T + ITI = A_{e,6} + IA_{e,6}I + A_{o,6} + IA_{o,6}I. \tag{11.118}$$

Lemma 11.4.2 concludes the proof. □

Corollary 11.6.2 *Let $t \geq 0$ and $w \in L^2(\mathbb{R})^2$. Let $J_t = \frac{x}{2} + It\partial_x$ and assume $J_t w \in L^2(\mathbb{R})^2$. Then:*

$$||J_t(T + ITI)w||_{L^2} \leq C(1 + t) \left|\left| e^{-\frac{1}{2}|x|} w \right|\right|_{L^2} . \tag{11.119}$$

11.6.2. Bounds on T_2

Lemma 11.6.3 *Let $t \geq 0$, $w \in L^2(\mathbb{R})^2$ and $J_t = \frac{x}{2} + It\partial_x$. Then:*

$$||J_t(x\partial_x T - Tx\partial_x)w||_{L^2} \leq C \, ||w||_{L^2} + Ct \left|\left| e^{-\frac{1}{2}|x|} w \right|\right|_{L^2} + C \, ||J_t w||_{L^2} . \tag{11.120}$$

Proof. Follows analogously to lemma 11.4.7 and lemma 11.5.5. □

11.7. H^1-Boundedness

As a consequence of lemma 10.2.3, whatever $L^2 \to L^2$ estimates we showed for T, T^{-1} and T_k also hold $H^1 \to H^1$.

Lemma 11.7.1 *Let $1 \leq q \leq \infty$. Then, for every $w \in W^{1,q}(\mathbb{R})^2$:*

$$||Tw||_{W^{1,q}} \leq C\,||w||_{W^{1,q}}\,, \tag{11.121}$$

$$\left|\left|T^{-1}w\right|\right|_{W^{1,q}} \leq C\,||w||_{W^{1,q}}\,, \tag{11.122}$$

$$||(T+ITI)w||_{W^{1,q}} \leq C\left|\left|e^{-\frac{1}{2}|x|}w\right|\right|_{W^{1,q}}. \tag{11.123}$$

Further, for $w \in H^1(\mathbb{R})^2$:

$$||(x\partial_x T - Tx\partial_x)w||_{H^1} \leq C\,||w||_{H^1}\,. \tag{11.124}$$

12. Dispersive Estimates

The wave operator being bounded $L^q \to L^q$ by theorem 11.0.1 ensures that the linearised equation $\partial_t w = -Lw$ fulfils the same dispersive estimates as the free Schrödinger equation. As always, we assume $3 < p < 5$.

Theorem 12.0.1 *Let $3 < p < 5$. Consider the linearised Schrödinger equation*

$$\partial_t w(t,x) = -Lw(t,x), \quad w : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2 \tag{12.1}$$

Let the solution for initial data $w(0,\cdot) = w_0 \in H_0$ be given by $e^{-tL}w_0$. Let further $q \in [2,\infty]$ with dual exponent $q' \in [1,2]$. Then, for every $t > 0$:

$$||w(t,\cdot)||_{L^q} \leq Ct^{-\frac{1}{2}+\frac{1}{q}} ||w_0||_{L^{q'}} . \tag{12.2}$$

Proof. Follows from lemma 1.6.1, lemma 10.2.3 and theorem 11.0.1. ☐

Theorem 12.0.2 (Strichartz estimate) *Let $3 < p < 5$. Consider the linearised Schrödinger equation (12.1). Let the solution for initial data $w(0,\cdot) = w_0 \in H_0$ be given by $w = e^{-tL}w_0$. Assume $q \in [4,\infty]$ and $r \in [2,\infty]$ satisfy $\frac{2}{q} + \frac{1}{r} = \frac{1}{2}$. Further assume $\tilde{q}' \in [1, \frac{4}{3}]$ and $\tilde{r}' \in [1,2]$ satisfy $\frac{2}{\tilde{q}'} + \frac{1}{\tilde{r}'} = \frac{5}{2}$. Then, the following holds true:*

1. *The homogeneous Strichartz estimates:*

$$\left|\left|e^{-tL}w_0\right|\right|_{L^q_t L^r_x} \leq C\,||w_0||_{L^2} . \tag{12.3}$$

2. *The dual homogeneous Strichartz estimates:*

$$\left|\left|\int_{\mathbb{R}} e^{sL}h(s,\cdot)ds\right|\right|_{L^2} \leq C\,||h||_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x} . \tag{12.4}$$

3. *The inhomogeneous Strichartz estimates:*

$$\left|\left|\int_0^t e^{-(t-s)L}h(s,\cdot)ds\right|\right|_{L^q_t L^r_x} \leq C\,||h||_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x} . \tag{12.5}$$

Proof. Follows from lemma 1.6.2, lemma 10.2.3 and theorem 11.0.1. ☐

12.1. Local Smoothing Estimate

The linearised equation $\partial_t w = -Lw$ admits a strong local smoothing estimate. We make use of the fact that the absence of resonances (theorem 1.10.1) introduces a zero $W_e(0,x) = W_o(0,x) = (0,0)$ in the Jost functions W_e, W_o used to define F and G.

For the free Schrödinger equation $(i\partial_t + \Delta)w = 0$, a local smoothing estimate denotes something of the form

$$\left|\left|(1+|x|)^{-\frac{1}{2}+\varepsilon}\,|\nabla|^{\frac{1}{2}}\,w\right|\right|_{L^2_t L^2_x} \leq C\,||w_0||_{L^2} . \tag{12.6}$$

The zero introduced by W_e, W_o means that we do not need to consider a derivative. We instead establish a bound on $||t(1+|x|)^{-1}w||_{L^4_t L^2_x}$.

Definition 12.1.1 *Let I be given by definition 10.0.1. We define the matrix equivalent of the Fourier transform for Schwartz functions $h : \mathbb{R} \to \mathbb{C}^2$ by:*

$$(\tilde{\mathcal{F}}h)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-I\xi x} h(x) dx. \tag{12.7}$$

$\tilde{\mathcal{F}}$ naturally extends to a unitary operator $L^2(\mathbb{R})^2 \to L^2(\mathbb{R})^2$.

Lemma 12.1.2 *Let $h_e, h_o \in L^2(\mathbb{R})^2$ be even and odd functions respectively. Then:*

$$\tilde{\mathcal{F}}h_e = \mathcal{F}h_e, \tag{12.8}$$

$$\tilde{\mathcal{F}}h_o = -iI\mathcal{F}h_o. \tag{12.9}$$

The Fourier transform $\mathcal{F}$ is to be understood as component-wise, $\mathcal{F}(f,g) = (\mathcal{F}f, \mathcal{F}g)$.

Proof. Follows from lemma 9.2.26. □

Lemma 12.1.3 *Consider $\tilde{\mathcal{F}}$ as given by definition 12.1.1 and $\mathcal{G}$ as given by definition 10.1.3. Then, for every $h = (f,g) \in L^2(\mathbb{R})^2$:*

$$\tilde{\mathcal{F}}\mathcal{G}h = \begin{pmatrix} f_e + ig_o \\ -i\operatorname{sgn}(\cdot)f_o + \operatorname{sgn}(\cdot)g_e \end{pmatrix}. \tag{12.10}$$

Proof. By definition:

$$\mathcal{G}h = \begin{pmatrix} \mathcal{F}f_e \\ -i\mathcal{F}(\operatorname{sgn}(\cdot)f_o) \end{pmatrix} - \begin{pmatrix} i\mathcal{F}(\operatorname{sgn}(\cdot)g_e) \\ \mathcal{F}g_o \end{pmatrix}. \tag{12.11}$$

Lemma 12.1.2 concludes the proof. □

Lemma 12.1.4 *Let G^* denote the L^2-adjoint of G. Then, for $w = (u,v) \in L^2(\mathbb{R})^2$:*

$$\begin{aligned} G^*w = &\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \begin{pmatrix} \rho_{e,U}(\xi,x)u_e(x) - \rho_{e,V}(\xi,x)v_e(x) \\ \rho_{o,U}(\xi,x)u_o(x) - \rho_{o,V}(\xi,x)v_o(x) \end{pmatrix} dx \\ &+ \begin{pmatrix} \chi c_e \mathcal{F}(\chi u_e) + i\chi c_e \operatorname{sgn}(\xi)\mathcal{F}(\chi v_e) \\ -\chi c_o \mathcal{F}(\chi \operatorname{sgn}(\cdot) u_o) - i\chi c_o \operatorname{sgn}(\xi)\mathcal{F}(\chi \operatorname{sgn}(\cdot) v_o) \end{pmatrix} \\ &+ \begin{pmatrix} i\chi s_e \operatorname{sgn}(\xi)\mathcal{F}(\chi \operatorname{sgn}(\cdot) u_e) - \chi s_e \mathcal{F}(\chi \operatorname{sgn}(\cdot) v_e) \\ -i\chi s_o \operatorname{sgn}(\xi)\mathcal{F}(\chi u_o) + \chi s_o \mathcal{F}(\chi v_o) \end{pmatrix}. \end{aligned} \tag{12.12}$$

ρ_e, ρ_o and χ are given by definition 11.1.1.

Proof. By lemma 11.1.11:

$$
\begin{aligned}
(G_e f)(x) = {}& \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \begin{pmatrix} \rho_{e,U}(\xi,x) \\ \rho_{e,V}(\xi,x) \end{pmatrix} f(\xi) d\xi \\
&+ \chi(x) \begin{pmatrix} \mathcal{F}(\chi c_e f_e) \\ -i\mathcal{F}(\chi c_e \operatorname{sgn}(\cdot) f_o) \end{pmatrix} + \chi(x)\operatorname{sgn}(x) \begin{pmatrix} i\mathcal{F}(\chi s_e \operatorname{sgn}(\cdot) f_e) \\ \mathcal{F}(\chi s_e f_o) \end{pmatrix}, \qquad (12.13) \\
(G_o g)(x) = {}& \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \begin{pmatrix} \rho_{o,U}(\xi,x) \\ \rho_{o,V}(\xi,x) \end{pmatrix} g(\xi) d\xi \\
&- \chi(x)\operatorname{sgn}(x) \begin{pmatrix} \mathcal{F}(\chi c_o g_e) \\ -i\mathcal{F}(\chi c_o \operatorname{sgn}(\cdot) g_o) \end{pmatrix} - \chi(x) \begin{pmatrix} i\mathcal{F}(\chi s_o \operatorname{sgn}(\cdot) g_e) \\ \mathcal{F}(\chi s_o g_o) \end{pmatrix}. \qquad (12.14)
\end{aligned}
$$

Let any $h = (f,g) \in L^2(\mathbb{R})^2$ be given. By definition, $G_e f$ is an even function, while $G_o g$ is odd. Therefore:

$$
\langle Gh, w \rangle_{L^2} = \langle G_e f, w_e \rangle_{L^2} + \langle G_o g, w_o \rangle_{L^2} \qquad (12.15)
$$

The lemma follows from (12.13) and (12.14). Note that lemma 8.5.3 and definition 11.1.1 imply $\overline{\rho_{e,U}} = \rho_{e,U}$ and $\overline{\rho_{o,U}} = \rho_{o,U}$, as well as $\overline{\rho_{e,V}} = -\rho_{e,V}$ and $\overline{\rho_{o,V}} = -\rho_{o,V}$. □

Corollary 12.1.5 *Let G^* denote the L^2-adjoint of G. Then, for $w = (u,v) \in L^2(\mathbb{R})^2$:*

$$
\begin{aligned}
\tilde{\mathcal{F}}\mathcal{G}G^* w = {}& \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \begin{pmatrix} \rho_{e,U}(\xi,x) u_e(x) - i\rho_{o,V}(\xi,x) v_o(x) \\ i\operatorname{sgn}(\xi)\rho_{e,V}(\xi,x) v_e(x) + \operatorname{sgn}(\xi)\rho_{o,U}(\xi,x) u_o(x) \end{pmatrix} dx \\
&+ \chi c_e \mathcal{F}(\chi w_e) - I\chi c_o \operatorname{sgn}(\xi)\mathcal{F}(\chi \operatorname{sgn}(\cdot) w_o) \\
&+ i\chi s_e \operatorname{sgn}(\xi)\mathcal{F}(\chi\operatorname{sgn}(\cdot) w_e) - iI\chi s_o \mathcal{F}(\chi w_o). \qquad (12.16)
\end{aligned}
$$

Proof. By lemma 12.1.3 and lemma 12.1.4:

$$
\begin{aligned}
\tilde{\mathcal{F}}\mathcal{G}G^* w = {}& \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \begin{pmatrix} \rho_{e,U}(\xi,x) u_e(x) - i\rho_{o,V}(\xi,x) v_o(x) \\ i\operatorname{sgn}(\xi)\rho_{e,V}(\xi,x) v_e(x) + \operatorname{sgn}(\xi)\rho_{o,U}(\xi,x) u_o(x) \end{pmatrix} dx \\
&+ \begin{pmatrix} \chi c_e \mathcal{F}(\chi u_e) + \chi c_o \operatorname{sgn}(\xi)\mathcal{F}(\chi\operatorname{sgn}(\cdot) v_o) \\ \chi c_e \mathcal{F}(\chi v_e) - \chi c_o \operatorname{sgn}(\xi)\mathcal{F}(\chi\operatorname{sgn}(\cdot) u_o) \end{pmatrix} \\
&+ \begin{pmatrix} i\chi s_e \operatorname{sgn}(\xi)\mathcal{F}(\chi\operatorname{sgn}(\cdot) u_e) + i\chi s_o \mathcal{F}(\chi v_o) \\ i\chi s_e \operatorname{sgn}(\xi)\mathcal{F}(\chi\operatorname{sgn}(\cdot) v_e) - i\chi s_o \mathcal{F}(\chi u_o) \end{pmatrix}. \qquad (12.17)
\end{aligned}
$$

That concludes the proof. □

Lemma 12.1.6 (Local Smoothing in L^2) *Let $w_0 \in L^2(\mathbb{R})^2$ and assume $(\mathcal{F}w_0)(0) = (0,0)$. Let further $2 \le q < \infty$ and $1 < q' \le 2$ with $\frac{1}{q} + \frac{1}{q'} = 1$. Then, for every $t \ge 0$:*

$$
t \left\| (1+|x|)^{-1} T^{-1} e^{It\Delta} w_0 \right\|_{L^2_x} \le C \left\| \frac{x}{1+x^2} \right\|_{L^{q'}} \left(\left\| e^{It\Delta} w_0 \right\|_{L^q_x} + \left\| e^{It\Delta}(x w_0) \right\|_{L^q_x} \right). \qquad (12.18)
$$

Proof. Recall $T^{-1} = G\mathcal{G}^{-1}$. Let G^* denote the L^2-adjoint of G. It follows, for $t \geq 0$:

$$t \left|\left|(1+|x|)^{-1}T^{-1}e^{It\Delta}w_0\right|\right|_{L^2_x} = t \sup_{||h||_{L^2_x}=1} \left\langle e^{-It\xi^2}\tilde{\mathcal{F}}w_0, \tilde{\mathcal{F}}\mathcal{G}G^*((1+|x|)^{-1}h)\right\rangle_{L^2}. \quad (12.19)$$

Let any $h \in L^2(\mathbb{R})^2$ with $||h||_{L^2} = 1$ be given. For simplicity, let $\tilde{h} = (1+|x|)^{-1}h$. Then:

$$t\left\langle e^{-It\xi^2}\tilde{\mathcal{F}}w_0, \tilde{\mathcal{F}}\mathcal{G}G^*\tilde{h}\right\rangle_{L^2} \leq C\left|\int_{\mathbb{R}}(-It\xi)e^{-It\xi^2}\tilde{\mathcal{F}}w_0 \cdot \frac{\tilde{\mathcal{F}}\mathcal{G}G^*\tilde{h}}{\xi}d\xi\right|. \quad (12.20)$$

Note that $\xi^{-1}\tilde{\mathcal{F}}\mathcal{G}G^*\tilde{h}$ is bounded by corollary 12.1.5, as well as lemma 11.1.2, lemma 11.1.3, lemma 11.1.5 and the mean value theorem. However, despite being smooth for $\xi \neq 0$, it is not necessarily continuous in $\xi = 0$.

We use partial integration in order to bound (12.20):

$$\begin{aligned}
&t\left\langle e^{-It\xi^2}\tilde{\mathcal{F}}w_0, \tilde{\mathcal{F}}\mathcal{G}G^*\tilde{h}\right\rangle_{L^2} \\
&\leq C\left|\int_{\mathbb{R}} e^{-It\xi^2}\tilde{\mathcal{F}}(xw_0)\cdot\frac{\tilde{\mathcal{F}}\mathcal{G}G^*\tilde{h}}{\xi}d\xi\right| + C\left|\int_{\mathbb{R}} e^{-It\xi^2}\tilde{\mathcal{F}}w_0\cdot\partial_\xi\left(\frac{\tilde{\mathcal{F}}\mathcal{G}G^*\tilde{h}}{\xi}\right)d\xi\right| \\
&\leq C\left|\int_{\mathbb{R}} e^{It\Delta}(xw_0)\cdot\tilde{\mathcal{F}}^{-1}\left[\frac{\tilde{\mathcal{F}}\mathcal{G}G^*\tilde{h}}{\xi}\right]dx\right| + C\left|\int_{\mathbb{R}} e^{It\Delta}w_0\cdot\tilde{\mathcal{F}}^{-1}\left[\partial_\xi\left(\frac{\tilde{\mathcal{F}}\mathcal{G}G^*\tilde{h}}{\xi}\right)\right]dx\right|.
\end{aligned} \quad (12.21)$$

$(\mathcal{F}w_0)(0) = (0,0) \Leftrightarrow (\tilde{\mathcal{F}}w_0)(0) = (0,0)$ allows us to consider $\partial_\xi(\xi^{-1}\tilde{\mathcal{F}}\mathcal{G}G^*\tilde{h})$ as a pointwise derivative, instead of the distributional derivative that arises from the discontinuity of $\xi^{-1}\tilde{\mathcal{F}}\mathcal{G}G^*\tilde{h}$ in $\xi = 0$.

Both integrals in (12.21) are bounded completely analogously. We show the bound for the second integral.

Let $(\tilde{f}, \tilde{g}) = \tilde{h}$ denote the components of $\tilde{h}$. By corollary 12.1.5:

$$\begin{aligned}
\tilde{\mathcal{F}}\mathcal{G}G^*\tilde{h} = &\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\begin{pmatrix}\rho_{e,U}(\xi,y)\tilde{f}_e(y) - i\rho_{o,V}(\xi,y)\tilde{g}_o(y)\\ i\,\mathrm{sgn}(\xi)\rho_{e,V}(\xi,y)\tilde{g}_e(y) + \mathrm{sgn}(\xi)\rho_{o,U}(\xi,y)\tilde{f}_o(y)\end{pmatrix}dy \\
&+ \chi c_e\mathcal{F}(\chi\tilde{h}_e) - I\chi c_o\,\mathrm{sgn}(\xi)\mathcal{F}(\chi\,\mathrm{sgn}(\cdot)\tilde{h}_o) \\
&+ i\chi s_e\,\mathrm{sgn}(\xi)\mathcal{F}(\chi\,\mathrm{sgn}(\cdot)\tilde{h}_e) - iI\chi s_o\mathcal{F}(\chi\tilde{h}_o).
\end{aligned} \quad (12.22)$$

We define:

$$\psi_1(\xi) := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\begin{pmatrix}\rho_{e,U}(\xi,y)\tilde{f}_e(y) - i\rho_{o,V}(\xi,y)\tilde{g}_o(y)\\ i\,\mathrm{sgn}(\xi)\rho_{e,V}(\xi,y)\tilde{g}_e(y) + \mathrm{sgn}(\xi)\rho_{o,U}(\xi,y)\tilde{f}_o(y)\end{pmatrix}dy, \quad (12.23)$$

$$\begin{aligned}
\psi_2(\xi) := &\chi[c_e\mathcal{F}(\chi\tilde{h}_e) - Ic_o\,\mathrm{sgn}(\xi)\mathcal{F}(\chi\,\mathrm{sgn}(\cdot)\tilde{h}_o)] \\
&+ i\chi[s_e\,\mathrm{sgn}(\xi)\mathcal{F}(\chi\,\mathrm{sgn}(\cdot)\tilde{h}_e) - Is_o\mathcal{F}(\chi\tilde{h}_o)].
\end{aligned} \quad (12.24)$$

By definition:

$$\begin{aligned}
&\left|\int_{\mathbb{R}} e^{It\Delta} w_0 \cdot \tilde{\mathcal{F}}^{-1}\left[\partial_\xi\left(\frac{\tilde{\mathcal{F}}\mathcal{G}G^*\tilde{h}}{\xi}\right)\right] dx\right| \\
\leq &\left|\int_{\mathbb{R}} e^{It\Delta} w_0 \cdot \tilde{\mathcal{F}}^{-1}\left[\partial_\xi\left(\xi^{-1}\psi_1\right)\right] dx\right| + \left|\int_{\mathbb{R}} e^{It\Delta} w_0 \cdot \tilde{\mathcal{F}}^{-1}\left[\partial_\xi\left(\xi^{-1}\psi_2\right)\right] dx\right| \qquad (12.25)
\end{aligned}$$

Recall $\chi(\xi) = 0$ for $|\xi| \leq 1$. Lemma 11.2.4 implies

$$\left|\left|\tilde{\mathcal{F}}^{-1}\left[\partial_\xi\left(\xi^{-1}\psi_2\right)\right]\right|\right|_{L^{q'}} \leq C\left|\left|\tilde{h}\right|\right|_{L^{q'}} \leq C^2\,||h||_{L^2}. \qquad (12.26)$$

Consequently:

$$\left|\int_{\mathbb{R}} e^{It\Delta} w_0 \cdot \tilde{\mathcal{F}}^{-1}\left[\partial_\xi\left(\xi^{-1}\psi_2\right)\right] d\xi\right| \leq C\left|\left|e^{It\Delta}w_0\right|\right|_{L^{q'}} ||h||_{L^2}. \qquad (12.27)$$

It remains to bound $\left|\int_{\mathbb{R}} e^{It\Delta} w_0 \cdot \tilde{\mathcal{F}}^{-1}\left[\partial_\xi\left(\xi^{-1}\psi_1\right)\right] d\xi\right|$. Recall that $\rho_{e,U}$ and $\rho_{e,V}$ are even with respect to x, while $\rho_{o,U}$ and $\rho_{o,V}$ are odd. Hence:

$$\begin{aligned}
\psi_1 &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \begin{pmatrix} \rho_{e,U}(\xi,y)\tilde{f}(y) - i\rho_{o,V}(\xi,y)\tilde{g}(y) \\ i\,\mathrm{sgn}(\xi)\rho_{e,V}(\xi,y)\tilde{g}(y) + \mathrm{sgn}(\xi)\rho_{o,U}(\xi,y)\tilde{f}(y)\end{pmatrix} dy \\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \begin{pmatrix} \rho_{e,U}(\xi,y) & -i\rho_{o,V}(\xi,y) \\ \mathrm{sgn}(\xi)\rho_{o,U}(\xi,y) & i\,\mathrm{sgn}(\xi)\rho_{e,V}(\xi,y)\end{pmatrix}\begin{pmatrix}\tilde{f}(y) \\ \tilde{g}(y)\end{pmatrix} dy \\
&=: \int_{\mathbb{R}} P(\xi,y)\tilde{h}(y)dy. \qquad (12.28)
\end{aligned}$$

By lemma 11.1.2, lemma 11.1.3, lemma 11.1.5 and the mean value theorem, for $\xi \in \mathbb{R}\backslash\{0\}$ and $y \in \mathbb{R}$:

$$\left|\xi^{-1}P(\xi,y)\right|, \left|\partial_\xi(\xi^{-1}P(\xi,y))\right|, \left|\partial_\xi^2(\xi^{-1}P(\xi,y))\right| \leq \frac{Ce^{-\frac{1}{2}|x|}}{1+\xi^2}. \qquad (12.29)$$

We define $\Psi : \mathbb{R} \to \mathbb{C}^{2\times 2}$ by

$$\Psi(y) := \lim_{\xi \searrow 0}\left(\partial_\xi(\xi^{-1}P)(\xi,y) - \partial_\xi(\xi^{-1}P)(-\xi,y)\right). \qquad (12.30)$$

By (12.29), for every $y \in \mathbb{R}$:

$$|\Psi(y)| \leq Ce^{-\frac{1}{2}|y|}. \qquad (12.31)$$

Let $\delta_0 f = f(0)$ denote the Dirac distribution. From the discontinuity of $\partial_\xi(\xi^{-1}P)$ in $\xi = 0$, it follows:

$$\begin{aligned}
&x\tilde{\mathcal{F}}^{-1}\left[\partial_\xi\left(\xi^{-1}\psi_1\right)\right] \\
&= i\tilde{\mathcal{F}}^{-1}\left[\partial_\xi^2\left(\xi^{-1}\psi_1\right)\right] + i\tilde{\mathcal{F}}^{-1}\left[\delta_0\int_{\mathbb{R}}\Psi(y)\tilde{h}(y)dy\right] \\
&= i\tilde{\mathcal{F}}^{-1}\left[\int_{\mathbb{R}}\partial_\xi^2\left(\xi^{-1}P(\xi,y)\right)\tilde{h}(y)dy\right] + \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\Psi(y)\tilde{h}(y)dy. \qquad (12.32)
\end{aligned}$$

We conclude:

$$
\begin{aligned}
&\left|\int_{\mathbb{R}} e^{It\Delta} w_0 \cdot \tilde{\mathcal{F}}^{-1}\left[\partial_\xi\left(\xi^{-1}\psi_1\right)\right] dx\right| \\
&\leq \left|\int_{\mathbb{R}} e^{It\Delta} w_0 \cdot \frac{1}{1+x^2}\tilde{\mathcal{F}}^{-1}\left[\partial_\xi\left(\xi^{-1}\psi_1\right)\right] dx\right| \\
&\quad + \left|\int_{\mathbb{R}} e^{It\Delta} w_0 \cdot \frac{x^2}{1+x^2}\tilde{\mathcal{F}}^{-1}\left[\partial_\xi\left(\xi^{-1}\psi_1\right)\right] dx\right| \\
&\leq C\left\|\int_{\mathbb{R}} e^{It\Delta} w_0\right\|_{L^q}\left\|\frac{1}{1+x^2}\tilde{\mathcal{F}}^{-1}\left[\partial_\xi\left(\xi^{-1}\psi_1\right)\right]\right\|_{L^{q'}} \\
&\quad + C\left\|\int_{\mathbb{R}} e^{It\Delta} w_0\right\|_{L^q}\left\|\frac{x}{1+x^2}\tilde{\mathcal{F}}^{-1}\left[\int_{\mathbb{R}}\partial_\xi^2\left(\xi^{-1}P(\xi,y)\right)\tilde{h}(y)dy\right]\right\|_{L^{q'}} \\
&\quad + C\left|\int_{\mathbb{R}} e^{It\Delta} w_0 \cdot \frac{x}{1+x^2}\int_{\mathbb{R}}\Psi(y)\tilde{h}(y)dydx\right| \\
&\leq C^2\left\|\int_{\mathbb{R}} e^{It\Delta} w_0\right\|_{L^q} + C^2\left|\int_{\mathbb{R}}\frac{x}{1+x^2}e^{It\Delta}w_0 dx\right| \qquad (12.33)
\end{aligned}
$$

That concludes the proof. □

Lemma 12.1.7 *Let $2 \leq q < \infty$ and $1 < q' \leq 2$ with $\frac{1}{q} + \frac{1}{q'} = 1$. Then:*

$$
\left\|\frac{x}{1+x^2}\right\|_{L^{q'}} \leq C^2(q-1)^{1-\frac{1}{q}}. \qquad (12.34)
$$

Proof. We estimate:

$$
\begin{aligned}
\left\|\frac{x}{1+x^2}\right\|_{L^{q'}} &\leq C\left(\int_1^\infty \frac{1}{x^{q'}}dx\right)^{\frac{1}{q'}} \\
&\leq C^2\left(\frac{1}{q'-1}\right)^{\frac{1}{q'}} \\
&= C^2\left(\frac{\frac{1}{q'}}{1-\frac{1}{q'}}\right)^{\frac{1}{q'}} \\
&= C^2\left(\frac{1-\frac{1}{q}}{\frac{1}{q}}\right)^{1-\frac{1}{q}} \\
&= C^2(q-1)^{1-\frac{1}{q}}. \qquad (12.35)
\end{aligned}
$$

That conclude the proof. □

Lemma 12.1.6 can be generalised to include the first derivative.

Lemma 12.1.8 (Local Smoothing in $\boldsymbol{H^1}$) *Let $w_0 \in L^2(\mathbb{R})^2$ and assume $(\mathcal{F}w_0)(0) = (0,0)$. Then, for every $t \geq 0$ and $2 \leq q < \infty$:*

$$
t\left\|(1+x^2)^{-1}T^{-1}e^{It\Delta}w_0\right\|_{H_x^1} \leq C(q-1)^{1-\frac{1}{q}}\left(\left\|e^{It\Delta}w_0\right\|_{L_x^q} + \left\|e^{It\Delta}(xw_0)\right\|_{L_x^q}\right). \qquad (12.36)
$$

Proof. Consider the Galilean operator $J_t = \frac{x}{2} + It\partial_x$. By corollary 11.4.6, as well as lemma 11.5.10 and lemma 11.5.11:

$$\begin{aligned}
&t\left|\left|(1+x^2)^{-1}\partial_x T^{-1}e^{It\Delta}w_0\right|\right|_{L^2_x} \\
&\leq C\left|\left|J_t T^{-1}e^{It\Delta}w_0\right|\right|_{L^q_x} + C\left|\left|T^{-1}e^{It\Delta}w_0\right|\right|_{L^q_x} \\
&\leq C^2\left|\left|J_t e^{It\Delta}w_0\right|\right|_{L^q_x} + C^2\left|\left|e^{It\Delta}w_0\right|\right|_{L^q_x} + C^2\left|\left|e^{It\Delta}w_0\right|\right|_{L^q} + C^2\left|\left|e^{It\Delta}(xw_0)\right|\right|_{L^q} \\
&\leq 2C^2\left|\left|e^{It\Delta}w_0\right|\right|_{L^q} + 2C^2\left|\left|e^{It\Delta}(xw_0)\right|\right|_{L^q}. \qquad (12.37)
\end{aligned}$$

Together with lemma 12.1.6 and lemma 12.1.7, that concludes the proof. □

Theorem 12.1.9 (Local Smoothing) *Let $w_0 \in L^2(\mathbb{R})^2$ and assume $(\mathcal{F}w_0)(0) = (0,0)$. Then:*

$$\begin{aligned}
&\left|\left|\frac{t}{\ln(2+t)^3}(1+x^2)^{-1}T^{-1}e^{It\Delta}w_0\right|\right|_{L^4_t(0,\infty)H^1_x} + \left|\left|t(1+x^2)^{-1}T^{-1}e^{It\Delta}w_0\right|\right|_{L^\infty_t(0,\infty)H^1_x} \\
&\leq C\left|\left|(1+|x|)w_0\right|\right|_{L^2}. \qquad (12.38)
\end{aligned}$$

Proof. By lemma 12.1.8, it suffices to show:

$$\left|\left|\frac{t}{\ln(2+t)^3}(1+x^2)^{-1}T^{-1}e^{It\Delta}w_0\right|\right|_{L^4_t(1,\infty)H^1_x} \leq C\left|\left|(1+|x|)w_0\right|\right|_{L^2}. \qquad (12.39)$$

Given $n \geq 0$, we choose $q_n \in (4,\infty]$, such that $\frac{2}{q} + \frac{1}{2n+2} = \frac{1}{2}$. Note that this implies:

$$\frac{1}{4} = \frac{1}{q_n} + \frac{1}{4n+4}. \qquad (12.40)$$

By lemma 12.1.8, for $n \geq 0$:

$$\begin{aligned}
&\left|\left|\frac{t}{\ln(2+t)^3}(1+x^2)^{-1}T^{-1}e^{It\Delta}w_0\right|\right|_{L^4_t(e^n,e^{n+1})H^1_x} \\
&\leq C(n+1)\left|\left|\ln(2+t)^{-3}e^{It\Delta}w_0\right|\right|_{L^4_t(e^n,e^{n+1})L^{2n+2}_x} \\
&\quad + C(n+1)\left|\left|\ln(2+t)^{-3}e^{It\Delta}(xw_0)\right|\right|_{L^4_t(e^n,e^{n+1})L^{2n+2}_x} \\
&\leq C^2(n+1)\left|\left|\ln(2+t)^{-3}\right|\right|_{L^{4n+4}_t(e^n,e^{n+1})}\left|\left|e^{It\Delta}w_0\right|\right|_{L^{q_n}_t(e^n,e^{n+1})L^{2n+2}} \\
&\quad + C^2(n+1)\left|\left|\ln(2+t)^{-3}\right|\right|_{L^{4n+4}_t(e^n,e^{n+1})}\left|\left|e^{It\Delta}(xw_0)\right|\right|_{L^{q_n}_t(e^n,e^{n+1})L^{2n+2}} \\
&\leq C^3(n+1)e^{\frac{n}{4n+4}}\ln(2+e^{n+1})^{-3}\left(||w_0||_{L^2} + ||xw_0||_{L^2}\right) \\
&\leq C^4\frac{1}{(n+1)^2}\left(||w_0||_{L^2} + ||xw_0||_{L^2}\right). \qquad (12.41)
\end{aligned}$$

In conclusion:

$$\begin{aligned}
&\left|\left|\frac{t}{\ln(2+t)^3}(1+x^2)^{-1}T^{-1}e^{It\Delta}w_0\right|\right|_{L_t^4(1,\infty)H_x^1}\\
&= \sum_{n=0}^{\infty}\left|\left|\frac{t}{\ln(2+t)^3}(1+x^2)^{-1}T^{-1}e^{It\Delta}w_0\right|\right|_{L_t^4(e^n,e^{n+1})H_x^1}\\
&\leq C\sum_{n=0}^{\infty}\frac{1}{(n+1)^2}\left(||w_0||_{L^2}+||xw_0||_{L^2}\right)\\
&\leq C^2\left(||w_0||_{L^2}+||xw_0||_{L^2}\right). \qquad (12.42)
\end{aligned}$$

That concludes the proof. □

Theorem 12.1.10 (Local Smoothing for Duhamel's formula) *Let the dual of a Strichartz pair be given, meaning $q' \in [1, \frac{4}{3}]$, $r' \in [1,2]$, such that $\frac{2}{q'} + \frac{1}{r'} = \frac{5}{2}$. Let $J_t = \frac{x}{2} + It\partial_x$. Let $w \in L^{q'}(0,\infty)L^{r'}(\mathbb{R})^2$ and assume $(\mathcal{F}w)(t,0) = (0,0)$ for every $t \geq 0$. Then:*

$$\begin{aligned}
&\left|\left|\frac{t}{\ln(2+t)^3}(1+x^2)^{-1}T^{-1}\int_0^t e^{I(t-s)\Delta}w(s,x)ds\right|\right|_{L_t^4(0,\infty)H_x^1}\\
&+\left|\left|t(1+x^2)^{-1}T^{-1}\int_0^t e^{I(t-s)\Delta}w(s,x)ds\right|\right|_{L_t^\infty(0,\infty)H_x^1}\\
&\leq C\,||w||_{L_t^{q'}(0,\infty)L_x^{r'}} + C\,||J_tw||_{L_t^{q'}(0,\infty)L_x^{r'}}. \qquad (12.43)
\end{aligned}$$

More generally, for $k > 2$:

$$\begin{aligned}
&\left|\left|\frac{t}{\ln(2+t)^k}(1+x^2)^{-1}T^{-1}\int_0^t e^{I(t-s)\Delta}w(s,x)ds\right|\right|_{L_t^4(0,\infty)H_x^1}\\
&+\left|\left|\frac{t}{\ln(2+t)^k}(1+x^2)^{-1}T^{-1}\int_0^t e^{I(t-s)\Delta}w(s,x)ds\right|\right|_{L_t^\infty(0,\infty)H_x^1}\\
&\leq C\,||w||_{L_t^{q'}(0,\infty)L_x^{r'}} + C\,||J_tw||_{L_t^{q'}(0,1)L_x^{r'}} + C\sum_{n=0}^{\infty}\frac{1}{(n+1)^{k-1}}\,||J_tw||_{L_t^{q'}(0,e^{n+1})L_x^{r'}}. \qquad (12.44)
\end{aligned}$$

Proof. Note, $J_te^{It\Delta}w_0 = \frac{1}{2}e^{It\Delta}(xw_0)$. The theorem follows completely analogously to theorem 12.1.9. □

Remark *Theorem 12.1.9 suggest decay of order $\sim t^{-\frac{5}{4}}$. This decay came about by using the absence of a resonance to establish decay of $\sim t^{-1}$, and a Strichartz estimate to ensure the resulting function lives in L^4 with respect to the time t.*

If, instead of a Strichartz estimate, we where to utilise the dispersive estimate $\left|\left|e^{it\Delta}f\right|\right|_{L^\infty} \leq Ct^{-\frac{1}{2}}\,||f||_{L^1}$, decay of order $\sim t^{-\frac{3}{2}}$ could be established as well, under the additional assumption $xw_0 \in L^1(\mathbb{R})^2$. For theorem 12.1.10, decay of order $\sim t^{-\frac{3}{2}}$ does not seem feasible while maintaining reasonable assumptions.

Part IV.
Outlook: Stability of the nonlinear Schrödinger Equation

We can restate (1.1) as a system of equations by identifying the imaginary unit i with its matrix equivalent $I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$:

$$I\partial_t W = -\Delta W - |W|^{p-1} W, \qquad (t,x) \in [0,\infty) \times \mathbb{R}. \tag{12.45}$$

$W_0 \in L^2(\mathbb{R})^2$ denotes the initial data. As before, we consider $3 < p < 5$.

Conjecture 12.1.11 (Asymptotic Stability) *There exist $\varepsilon_0 > 0$ and $p_1 \in (3,5)$, such that the following holds true. Consider equation (12.45) with radially symmetric initial data $W_0 \in H^1(\mathbb{R})^2$ and exponent $p \in (3, p_1)$. Assume:*

$$w_0 := W_0 - \begin{pmatrix} Q \\ 0 \end{pmatrix} \in \mathcal{H} \tag{12.46}$$

and $||w_0||_{H^1}, ||xw_0||_{L^2} < \varepsilon_0$. Then, there are functions $\nu : [0,\infty) \to \mathbb{R}$, $\lambda : [0,\infty) \to (0,\infty)$ and $w : [0,\infty) \times \mathbb{R} \to \mathbb{C}^2$, such that:

$$W = \tau_{\nu,0,0,\lambda,t} \begin{pmatrix} Q \\ 0 \end{pmatrix} + w. \tag{12.47}$$

For every $t \geq 0$, w admits the dispersive estimate:

$$||w(t,x)||_{L^\infty_x} \leq C_p (1+t)^{-\frac{1}{2}} \left(||w_0||_{H^1} + \left|\left| x^2 w_0 \right|\right|_{L^2} \right). \tag{12.48}$$

Further, given a Strichartz pair $q, r \in [2,\infty]$, $\frac{2}{q} + \frac{1}{r} = \frac{1}{2}$:

$$||w(t,x)||_{L^q_t W^{1,r}_x} \leq C_p \left(||w_0||_{H^1} + \left|\left| x^2 w_0 \right|\right|_{L^2} \right). \tag{12.49}$$

ν and λ fulfil for $0 \leq s \leq t$:

$$|\lambda(t) - \lambda(s)|, |\nu(t) - \nu(s)| \leq C_p (1+s)^{-\frac{1}{2}} \left(||w_0||_{H^1} + \left|\left| x^2 w_0 \right|\right|_{L^2} \right). \tag{12.50}$$

While an actual proof is beyond the scope of this thesis, we give some ideas on how its results might be used to show asymptotic stability of the NLS. Everything presented in this outlook is based on an, as of yet, unfinished proof of asymptotic stability, which I originally planned to include in this thesis.

The reason I ultimately decided against including such a proof is threefold. Firstly, I have not yet finished the proof. Secondly, the proof is quite long. It would easily make this thesis longer by a third. Thirdly, the techniques used in the proof are quite different to both of the already very distinct parts of this thesis.

I remain optimistic that the linear results contained within this thesis are sufficient to analytically show asymptotic stability of the one-dimensional focusing non-linear Schrödinger equation. Most likely even without restricting oneself to radially symmetric initial data.

13. Stability for Small Data

We already discussed the general approaches when proving asymptotic stability in chapter 1.3. The approach we take is to show dispersive estimates (theorem 12.0.1, 12.0.2, 12.1.9) for the linearised equation $\partial_t w = Lw$ and then treat the perturbation as a small data Cauchy problem.

Thus, in proving asymptotic stability, the first step is to recall how the NLS acts on small initial data. The information presented in chapter 13 is largely taken from [19].

Definition 13.0.1 *Given $t \in \mathbb{R}$, we introduce the following operator:*

$$J_t w = \frac{x}{2} w + It\partial_x w. \tag{13.1}$$

J_t is linked to the Galilean invariance of the Schrödinger equation. A consequence of this relationship is lemma 13.0.2.

Lemma 13.0.2 *Let $w_0 \in L^2(\mathbb{R})^2$, such that $xw_0 \in L^2(\mathbb{R})^2$. Then, for every $t \in \mathbb{R}$:*

$$J_t e^{It\Delta} w_0 = e^{It\Delta} \left(\frac{x}{2} w_0 \right). \tag{13.2}$$

Proof. Let $w(t,x) = e^{It\Delta} w_0$. w satisfies the equation:

$$I\partial_t w = -\Delta w. \tag{13.3}$$

Applying J_t yields:

$$I\partial_t J_t w + \partial_x w = -\Delta J_t w + \partial_x w. \tag{13.4}$$

$J_0 w(0,x) = \frac{x}{2} w_0$ concludes the proof. $\square$

Lemma 13.0.3 *Consider* (12.45) *and assume $||xW_0||_{L^2(\mathbb{R})^2}, ||W_0||_{L^2(\mathbb{R})^2} < \varepsilon$ for some sufficiently small $\varepsilon > 0$. Then, for every $q, r \in [2, \infty]$ satisfying $\frac{2}{q} + \frac{1}{r} = \frac{1}{2}$:*

$$||W||_{L^q_t L^r_x} \leq C\varepsilon. \tag{13.5}$$

Further, for every $t > 0$:

$$||W(t,\cdot)||_{L^\infty} \leq C\varepsilon t^{-\frac{1}{2}}. \tag{13.6}$$

Proof. Note:

$$
\begin{aligned}
& J_t(|W|^{p-1} W) \\
= \ & J_t(\langle W, W\rangle^{\frac{p-1}{2}} W) \\
= \ & |W|^{p-1} J_t W + \frac{p-1}{2}\langle It\partial_x W, W\rangle |W|^{p-3} W - \frac{p-1}{2}\langle W, It\partial_x W\rangle |W|^{p-3} W \\
= \ & |W|^{p-1} J_t W + \frac{p-1}{2}\langle J_t W, W\rangle |W|^{p-3} W - \frac{p-1}{2}\langle W, J_t W\rangle |W|^{p-3} W. \qquad (13.7)
\end{aligned}
$$

It follows:

$$
\left|J_t(|W|^{p-1} W)\right| \leq C\,|W|^{p-1}\,|J_t W|\,. \qquad (13.8)
$$

We define the following quantities for $t \geq 0$:

$$
\varepsilon_1(t) := \sup_{\frac{2}{q}+\frac{1}{r}=\frac{1}{2}} ||W(s,x)||_{L^q(0,t)L^r}\,, \qquad (13.9)
$$

$$
\varepsilon_2(t) := \sup_{\frac{2}{q}+\frac{1}{r}=\frac{1}{2}} ||J_s W(s,x)||_{L^q(0,t)L^r}\,, \qquad (13.10)
$$

$$
\varepsilon_3(t) := \sup_{0<s\leq t} s^{\frac{1}{2}}\,||W(s,x)||_{L^\infty_x}\,. \qquad (13.11)
$$

Clearly, $\varepsilon_1(0), \varepsilon_2(0), \varepsilon_3(0) < \varepsilon$. By definition:

$$
\int_0^t ||W(s,x)||_{L^\infty_x}^{p-1}\,ds \leq C\varepsilon_1(t)^{p-1} + C\varepsilon_3(t)^{p-1}. \qquad (13.12)
$$

We use Duhamel's principle to estimate W:

$$
W(t,x) = e^{It\Delta}W_0 + I\int_0^t e^{I(t-s)\Delta}\left(|W|^{p-1} W\right) ds \qquad (13.13)
$$

Using a Strichartz estimate (lemma 1.6.2), it follows:

$$
\begin{aligned}
\varepsilon_1(t) &\leq \varepsilon + C\,||W^p||_{L^1_s L^2_x} \\
&\leq \varepsilon + C\int_0^t ||W(s,x)||_{L^\infty_x}^{p-1}\,ds\,||W||_{L^\infty_s L^2_x} \\
&\leq \varepsilon + C^2\varepsilon_1(t)(\varepsilon_1(t)^{p-1} + \varepsilon_3(t)^{p-1}). \qquad (13.14)
\end{aligned}
$$

Analogously:

$$
\begin{aligned}
\varepsilon_2(t) &\leq C\varepsilon + C\left|\left||W|^{p-1}\,|J_s W|\right|\right|_{L^1_s L^2_x} \\
&\leq C\varepsilon + C^2\varepsilon_2(t)(\varepsilon_1(t)^{p-1} + \varepsilon_3(t)^{p-1}). \qquad (13.15)
\end{aligned}
$$

Further, for $t > 0$:

$$
\begin{aligned}
||W||_{L^\infty}^2 &\leq \left|\left|e^{-I\frac{x^2}{4t}}W\right|\right|_{L^2}\left|\left|\partial_x(e^{-I\frac{x^2}{4t}}W)\right|\right|_{L^2} \\
&\leq ||W||_{L^2}\left|\left|t^{-1}J_t W\right|\right|_{L^2}. \qquad (13.16)
\end{aligned}
$$

Consequently:

$$\varepsilon_3(t) \leq \varepsilon_1(t)^{\frac{1}{2}} \varepsilon_2(t)^{\frac{1}{2}}. \tag{13.17}$$

(13.14), (13.15) and (13.17) conclude the proof. $\square$

14. Asymptotic Stability, Possible Ansatz

Consider the NLS in the form given by (12.45):

$$I\partial_t W = -\Delta W - |W|^{p-1} W, \qquad (t,x) \in [0,\infty) \times \mathbb{R}. \tag{14.1}$$

$W_0 \in L^2(\mathbb{R})^2$ denotes the initial data. The exponent fulfils $3 < p < 5$.

We seek to treat (14.1) in the neighbourhood of a soliton in the same manner as lemma 13.0.3. What complications arise compared to lemma 13.0.3?

Consider first the basic idea $W(t,x) = e^{It}(Q(x) + w(t,x))$. This Ansatz leads to an equation

$$I\partial_t w = Lw - |w|^{p-1} w + R \tag{14.2}$$

with the remainder term $R(t,x)$ containing nonlinear terms of order ≥ 2 with exponentially decaying coefficients. If the orthogonal condition $w \in H$ is fulfilled, we can control R using the local smoothing estimates given by theorem 12.1.9 and theorem 12.1.10. $|w|^{p-1} w$ is controlled analogously to lemma 13.0.3, using the dispersive estimates given by theorem 12.0.1 and theorem 12.0.2.

Clearly, the condition $w \in H$ can not be presupposed. While the flow generated by the linear equation $\partial_t w = Lw$ stays contained within the sub-manifold H, such a thing does not hold in the presence of nonlinear interactions.

Recall that $H \subset \mathcal{H}$ is of co-dimension 6 with $\mathcal{H}$ being of co-dimension 4. $\mathcal{H}$ is characterised by (1.43), four orthogonal conditions in L^2 related to the eigenspace of L associated with the eigenvalue 0. H is then defined as the orthogonal complement (with respect to $\langle \cdot, \cdot \rangle_{\mathcal{H}}$) of the internal modes $\zeta, \overline{\zeta}$, which are the eigenfunctions of the non-zero eigenvalues E, $-E$ of L.

By introducing geometric parameters, which we use to variate the soliton along its symmetries (compare chapter 1.4), it is possible to control the eigenspace of 0 and ensure $w \in \mathcal{H}$.

That leaves the internal modes $\zeta, \overline{\zeta}$, which can not be controlled in such an elegant manner. We are forced to make the decomposition $w = w_1 + \mu\zeta + \overline{\mu}\overline{\zeta}$, whereby $w_1 \in H$, $\mu \in \mathbb{C}$. w_1 is to be controlled as a radiative term in the same manner we discussed above. However, controlling the coefficient $\mu(t)$ of the internal modes requires yet another idea, the co-called nonlinear version of Fermi's golden rule.

By examining the quadratic terms of the evolution equation describing w_1 and μ respectively, Fermi's golden rule describes the interaction between the internal modes and w_1, specifically at the frequency $2E - 1$.

At this point I want to mention [5], where Buslaev and Sulem examine a class of non-linear Schrödinger equations with internal modes, and making use of Fermi's golden rule. While [5] makes stronger assumptions than we are afforded, Fermi's golden rule still works in much the same way.

We present three ingredients for a possible proof of conjecture 12.1.11. In chapter 14.1, we introduce the geometric parameters that ensure $w \in \mathcal{H}$.

In chapter 14.2, we give a possible structure for the proof in a way that mirrors lemma 13.0.3.

Finally, in chapter 14.3, we affirm that the nonlinear version of Fermi's golden rule can be used to control the internal modes and explain the form Fermi's golden rule takes when using the terminology developed in this thesis.

14.1. Orthogonality Conditions

Definition 14.1.1 *Given $t \geq 0$, $\lambda > 0$, $\nu, \beta, y \in \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{C}^2$, consider:*

$$\tau_{\nu,\beta,y,\lambda,t} h(x) := e^{I\nu} e^{I\lambda^2 t} e^{I\frac{\beta}{2}\left(x - \frac{\beta}{2}t\right)} \lambda^{\frac{2}{p-1}} h(\lambda(x - \beta t + y)). \tag{14.3}$$

τ is based on the invariances of the NLS laid out in chapter 1.4. Instead of $\mu, \beta, y \in \mathbb{R}$ and $\lambda > 0$, consider differentiable functions $\nu, \beta, y : [0, \infty) \to \mathbb{R}$ and $\lambda : [0, \infty) \to (0, \infty)$. Using the short-hand notation, $\tau_t = \tau_{\nu(t),\beta(t),y(t),\lambda(t),t}$, it follows by direct computation:

$$\begin{aligned} &I\tau_t^{-1}\partial_t \tau_t h + \tau_t^{-1} \Delta \tau_t h \\ &= I\partial_t h + \lambda^2 \Delta h - \lambda^2 h - \alpha_1 h - \alpha_2 x h - I\alpha_3 \partial_x h - I\alpha_4 \left(\frac{2}{p-1} h + x\partial_x h \right). \end{aligned} \tag{14.4}$$

The coefficients α_k are given by

$$\alpha_1(t) := \nu'(t) + 2\lambda(t)\lambda'(t)t - \frac{\beta'(t)}{2} y(t), \tag{14.5}$$

$$\alpha_2(t) := \frac{\beta'(t)}{2\lambda(t)}, \tag{14.6}$$

$$\alpha_3(t) := \lambda(t)\beta'(t)t - \lambda(t)y'(t), \tag{14.7}$$

$$\alpha_4(t) := -\frac{\lambda'(t)}{\lambda(t)}. \tag{14.8}$$

In order to ensure $w \in \mathcal{H}$, the following lemma can be used.

Lemma 14.1.2 *Consider equation (12.45) with exponent $3 < p < 5$ and initial data $W_0 \in H^1(\mathbb{R})^2$. Given some sufficiently small $\varepsilon > 0$, assume*

$$w_0 := W_0 - \begin{pmatrix} Q \\ 0 \end{pmatrix} \in \mathcal{H} \tag{14.9}$$

and $||w_0||_{H^1} < \varepsilon$. Then, one can choose $0 < t_0 \leq \infty$ and $\nu, \beta, y : [0, t_0) \to \mathbb{R}$, $\lambda : [0, t_0) \to (0, \infty)$ and $w_1 : [0, t_0) \times \mathbb{R} \to \mathbb{R}^2$, such that, for every $t \in [0, t_0)$:

1. *$\nu(0) = \beta(0) = y(0) = 0$ and $\lambda(0) = 1$.*

2. $W = \tau_{\nu,\beta,y,\lambda,t}\left(\begin{pmatrix} Q \\ 0 \end{pmatrix} + w_1\right)$.

3. $w_1(t,\cdot) \in \mathcal{H}$.

4. *With* $\alpha_1, .., \alpha_4$ *as given by* (14.5) - (14.8)*:*

$$\sum_{k=1}^{4} |\alpha_k(t)| \le C\lambda(t)^2 \int_{\mathbb{R}} e^{-\frac{|x|}{2}} |w_1(t,x)|^2 \, dx. \tag{14.10}$$

5. $t_0 = \infty$ *or* $\lim_{t\to t_0} ||w_1(t,\cdot)||_{H^1} \ge \sqrt{\varepsilon}$.

Proof. The proof is straightforward, but omitted, to prevent this outlook chapter from becoming bloated. □

14.2. The Continuity Argument

We give a plausible structure for a proof of conjecture 12.1.11, which is essentially a more technical version of the proof of lemma 13.0.3.

Let initial data $w_0 \in \mathcal{H}$ fulfil the assumptions of conjecture 12.1.11. Lemma 14.1.2 gives rise to $w_1 : [0, t_0) \times \mathbb{R} \to \mathbb{R}^2$, which fulfils $w_1(t,\cdot) \in \mathcal{H}$ for every time $t \in [0, t_0)$. As this w_1 is radially symmetric, we conclude $\beta(t) = y(t) = 0$. Consider $w = \tau_{\nu,0,0,\lambda,t} w_1$. w fulfils $W = \tau_{\nu,0,0,\lambda,t}\begin{pmatrix} Q \\ 0 \end{pmatrix} + w$ by lemma 14.1.2.

We introduce the first half of the continuity argument. Consider

$$t_1 = \sup\{0 \le t < t_0 \,|\forall 0 \le r \le s \le t : \text{ (LAMBDA1) holds true}\}. \tag{14.11}$$

(LAMBDA1) is a condition on the mode λ, ensuring that $\lambda(t)$ converges as t increases. Given some large $K_1 > 0$, I would suggest the following:

$$|\lambda(s) - \lambda(r)| \le 2K_1(\varepsilon^{-2} + r)^{-\frac{1}{2}}. \tag{LAMBDA1}$$

It might even be possible to utilize a condition as strong as:

$$|\lambda(s) - \lambda(r)| \le 2K_1(\varepsilon^{-2} + r)^{-1}. \tag{LAMBDA2}$$

We can now define $\lambda_0 := \lim_{t\to t_1} \lambda(t)$. Also consider

$$\tilde{\nu}(t) := \nu(t) + \left(\lambda(t)^2 - \lambda_0^2\right) t. \tag{14.12}$$

We introduce the shorthand $\tau_{t,s} = \tau_{\tilde{\nu}(t),0,0,\lambda_0,s}$ and define

$$w_2 = \tau_{t,t}^{-1} w = \tau_{t,t}^{-1} \tau_{\nu(t),0,0,\lambda(t),t} w_1. \tag{14.13}$$

Put another way, this reads

$$w_2(t,x) = (\lambda\lambda_0^{-1})^{\frac{2}{p-1}} w_1\left(t, \lambda\lambda_0^{-1} x\right). \tag{14.14}$$

Clearly, we can not assume $w_2 \in \mathcal{H}$. However, the error is small. By projecting onto $\mathcal{H}$ and H, we find a decomposition

$$w_2 = w_3 + \mu\zeta + \overline{\mu\zeta} + \mathfrak{w}_1. \tag{14.15}$$

w_3 fulfils $w_3 \in H$, μ is a coefficient term for the non-zero eigenfunctions and $\mathfrak{w}_1$ is small to the point of being negligible. We define

$$\mathfrak{w}_2 = \mu\zeta + \overline{\mu\zeta} = \begin{pmatrix} 2\mu_u\zeta_u \\ -2\mu_v\zeta_v \end{pmatrix}. \tag{14.16}$$

Finally, the wave operator $T = \mathcal{G}F$ comes into play. Consider:

$$w_4 = Fw_3, \tag{14.17}$$
$$w_5 = Tw_3 = \mathcal{G}w_4, \tag{14.18}$$
$$w_6 = \tau_{t,t}Tw_3 = \tau_{t,t}w_5. \tag{14.19}$$

We introduce the second half of the continuity argument:

$$t_2 = \ \sup\left\{0 \leq t < t_1 \mid (14.21) \text{ - } (14.25) \text{ hold true}\right\}. \tag{14.20}$$

Given some large constants $K_2, K_3, K_4, K_5, K_6 > 0$, (14.21) - (14.25) are given by:

1. For every Strichartz pair $q, r \in [2, \infty]$, $\frac{2}{q} + \frac{1}{r} = \frac{1}{2}$:
$$||w_3(s,x)||_{L^q_s(0,t)W^{1,r}_x} \leq 2K_2\varepsilon. \tag{14.21}$$

2. w_3 fulfils the dispersive estimate:
$$\left|\left|(\varepsilon^{-2} + s)^{\frac{1}{2}} w_3(s,x)\right|\right|_{L^\infty_s(0,t)L^\infty_x} \leq 2K_3. \tag{14.22}$$

3. Let $J_t = \frac{x}{2} + It\partial_x$. Let $\gamma > 0$ be small. For every Strichartz pair $q, r \in [2, \infty]$, $\frac{2}{q} + \frac{1}{r} = \frac{1}{2}$:
$$\left|\left|(1+s)^{-\frac{1}{4}-\gamma} J_{\lambda_0^2 s} w_3(s,x)\right|\right|_{L^q_s(0,t)L^r_x} \leq 2K_4. \tag{14.23}$$

4. w_3 fulfils the smoothing estimate for every $q \in [4, \infty]$:
$$\left|\left|\left(\varepsilon^{-1} + \frac{s^{\frac{3}{4}}}{\ln(2+s)^3}\right)\frac{w_3(s,x)}{1+x^2}\right|\right|_{L^q_s(0,t)H^1_x} \leq 2K_5. \tag{14.24}$$

5. μ fulfils:
$$\left|\left|(\varepsilon^{-2} + s)^{\frac{1}{2}} \mu(s)\right|\right|_{L^\infty(0,t)} \leq 2K_6. \tag{14.25}$$

In order to prove asymptotic stability, it suffices to show that (LAMBDA1) and (14.21) - (14.25) hold true with smaller bounds.

14.3. Fermi's Golden Rule

The conditions (LAMBDA1), as well as (14.21) - (14.24) should be provable in a straightforward manner using the dispersive estimates from chapter 12 and the Duhamel evolution equation of w_6:

$$\begin{aligned} w_6(t,x) = e^{It\Delta}w_6(0,x) &- \int_0^t \tilde{\nu}'(s)e^{I(t-s)\Delta}\tau_{\tilde{\nu},0,0,\lambda_0,s}(TI-IT)w_2 ds \\ &+ \lambda_0^2 \int_0^t e^{I(t-s)\Delta}\tau_{\tilde{\nu},0,0,\lambda_0,s}TI(\left|w_2\right|^{p-1} w_2) ds \\ &+ \lambda_0^2 \int_0^t e^{I(t-s)\Delta}\tau_{\tilde{\nu},0,0,\lambda_0,s}TB_2[w_2,w_2] ds \\ &- \int_0^t e^{I(t-s)\Delta}\tau_{\tilde{\nu},0,0,\lambda_0,s}TIR ds. \end{aligned} \tag{14.26}$$

R denotes a suitably small remainder term. B_2 is a quadratic form arising from the power non-linearity. For $h=(f,g):\mathbb{R}\to\mathbb{C}^2$:

$$B_2[h,h] = \begin{pmatrix} -(p-1)Q^{p-2}fg \\ \frac{(p+9)(p-1)}{8}Q^{p-2}f^2 + \frac{p-1}{2}Q^{p-2}g^2 \end{pmatrix}. \tag{14.27}$$

The condition (14.25) is more technically complex. Consider polar coordinates

$$\mu(t) = e^{i\varphi(t)}\rho(t). \tag{14.28}$$

Then, the evolution of ρ is given by

$$\rho(t) = \rho(t_2) + \sum_{k=1}^{5}\int_t^{t_2} \mathrm{Re}\left(e^{-i\varphi(s)}\eta_k(s)\right) ds. \tag{14.29}$$

Hereby

$$\eta_1(t) = \tilde{\nu}'(t)\langle Iw_2,\zeta\rangle_{\mathcal{H}}, \tag{14.30}$$
$$\eta_2(t) = -\lambda_0^2\langle I\left|w_2\right|^{p-1} w_2,\zeta\rangle_{\mathcal{H}}, \tag{14.31}$$
$$\eta_3(t) = -\lambda_0^2\langle B_2[w_2,w_2]-B_2[\mathfrak{w}_2,w_3],\zeta\rangle_{\mathcal{H}}, \tag{14.32}$$
$$\eta_4(t) = \langle IR,\zeta\rangle_{\mathcal{H}}, \tag{14.33}$$
$$\eta_5(t) = -\lambda_0^2\langle B_2[\mathfrak{w}_2,w_3],\zeta\rangle_{\mathcal{H}}. \tag{14.34}$$

The crucial term is η_5, meaning

$$\mathcal{B}(t) := -\lambda_0^2\int_t^{t_2} \mathrm{Re}\left(e^{-i\varphi(s)}\left\langle B_2[\mathfrak{w}_2,w_3],\zeta\right\rangle_{\mathcal{H}}\right) ds. \tag{14.35}$$

$\mathcal{B}$ governs the interaction between the continuous component w_3 and the pure-point component $\mathfrak{w}_2$, specifically how $\mathfrak{w}_2$ interacts with the frequency $2E-1$. Through careful examination of the term, it is possible to show

$$\mathcal{B}(t) = \frac{\pi}{2}\frac{Z(\sqrt{2E-1})}{\sqrt{2E-1}}\int_t^{t_2}\rho(s)^3 ds + o((\varepsilon^{-2}+t)^{-\frac{1}{2}}). \tag{14.36}$$

If $Z(\sqrt{2E-1}) \neq 0$, then (14.36) implies that (14.25) holds with a bound independent of K_6:

$$\left|\left|(\varepsilon^{-2}+s)^{\frac{1}{2}}\mu(s)\right|\right|_{L^\infty(0,t)} \leq C_p. \tag{14.37}$$

Z is an analytic function given by

$$Z(\xi) = \, ||\mathfrak{z}_1(\xi)||_2^2 + ||\mathfrak{z}_2(\xi)||_2^2 + 2\,\langle \mathfrak{z}_1, \mathfrak{z}_2\rangle_2\,, \tag{14.38}$$

whereby

$$\mathfrak{z}_1 = G^* \begin{pmatrix} \frac{p-1}{2}\left(\frac{p+9}{4}\zeta_u^2 - \zeta_v^2\right) E Q^{p-2} \\ 0 \end{pmatrix}, \tag{14.39}$$

$$\mathfrak{z}_2 = \, -iG^* \begin{pmatrix} 0 \\ (p-1)E\zeta_u\zeta_v Q^{p-2} \end{pmatrix}. \tag{14.40}$$

G^* denotes the L^2-adjoint of the unitary operator G.

$Z(\sqrt{2E-1}) \neq 0$ serves the role of a non-degeneracy condition. Clearly, the non-zero eigenvalue E must fulfil $E > \frac{1}{2}$ for the condition to be well-posed. If we can show that $p \mapsto Z(\sqrt{2E-1})$ does not uniformly equal zero, then analyticity ensures $Z(\sqrt{2E-1}) \neq 0$ for almost every p with $E(p) > \frac{1}{2}$.

Analytically showing $Z(\sqrt{2E-1}) \neq 0$ may not be trivial. In $p = 3$, $p \mapsto Z(\sqrt{2E-1})$ features a zero of at least order 3.

References

[1] Stefan Banach. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fundamenta Mathematicae*, 3:133–181, 1922.

[2] Henri Berestycki and Pierre-Louis Lions. Nonlinear scalar field equations, i existence of a ground state. *Archive for Rational Mechanics and Analysis*, 82:313–345, 1983.

[3] Jöran Bergh and Jörgen Löfström. *Interpolation Spaces*. Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg, 1976.

[4] Jean Bourgain. *Global Solutions of Nonlinear Schrödinger Equations*, volume 46. American Mathematical Soc., 1999.

[5] Vladimir S. Buslaev and Catherine Sulem. On asymptotic stability of solitary waves for nonlinear Schrödinger equations. *Annales de l'I.H.P. Analyse non linéaire*, 20(3):419–475, 2003.

[6] Thierry Cazenave and Fred B. Weissler. The cauchy problem for the critical nonlinear schrödinger equation in H^s. *Nonlinear Analysis: Theory, Methods & Applications*, 14(10):807–836, 1990.

[7] J.M. Delort and N. Masmoudi. *Long-time Dispersive Estimates for Perturbations of a Kink Solution of One-dimensional Cubic Wave Equations*. Memoirs of the European Mathematical Society. EMS Press, 2022.

[8] *NIST Digital Library of Mathematical Functions*. http://dlmf.nist.gov/, Release 1.1.7 of 2022-10-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.

[9] Roland Donninger. Asymptotics and analytic modes for the wave equation in similarity coordinates. *Journal of Evolution Equations*, 9:511–523, 2009.

[10] Roland Donninger. The radial wave operator in similarity coordinates. *Journal of Mathematical Physics*, 51(2), 02 2010. 023527.

[11] Roland Donninger. Strichartz estimates in similarity coordinates and stable blowup for the critical wave equation. *Duke Mathematical Journal*, 166(9):1627 – 1683, 2017.

[12] Javier Duoandikoetxea. *Fourier analysis*, volume 29. American Mathematical Soc., 2001.

[13] Gadi Fibich, Frank Merle, and Pierre Raphaël. Proof of a spectral property related to the singularity formation for the L^2 critical nonlinear schrödinger equation. *Physica D: Nonlinear Phenomena*, 220(1):1–13, 2006.

[14] Alberto Fiorenza, Maria Rosaria Formica, Tomáš G. Roskovec, and Filip Soudský. Detailed proof of classical gagliardo–nirenberg interpolation inequality with historical remarks. *Z. Anal. Anwend.*, 40(2):217–236, 2021.

[15] Clifford S. Gardner, John M. Greene, Martin D. Kruskal, and Robert M. Miura. Method for solving the korteweg-devries equation. *Phys. Rev. Lett.*, 19:1095–1097, Nov 1967.

[16] B. Gidas, Wei Ming Ni, and L. Nirenberg. Symmetry and related properties via the maximum principle. *Communications in Mathematical Physics*, 68(3):209 – 243, 1979.

[17] J. Ginibre and G. Velo. On a class of nonlinear schrödinger equations. i. the cauchy problem, general case. *Journal of Functional Analysis*, 32(1):1–32, 1979.

[18] Brian C Hall. *Quantum theory for mathematicians*, volume 267. Springer, 2013.

[19] Mihaela Ifrim and Daniel Tataru. Global bounds for the cubic nonlinear schrödinger equation (nls) in one space dimension. *Nonlinearity*, 28(8):2661, jun 2015.

[20] Herbert Koch, Daniel Tataru, and Monica Vişan. *Dispersive equations and nonlinear waves. Generalized Korteweg-de Vries, nonlinear Schrödinger, wave and Schrödinger maps*, volume 45 of *Oberwolfach Semin.* Basel: Birkhäuser/Springer, 2014.

[21] Michał Kowalczyk, Yvan Martel, and Claudio Muñoz. On asymptotic stability of nonlinear waves. *Séminaire Laurent Schwartz—EDP et applications*, pages 1–27, 2017.

[22] Man Kam Kwong. Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in R^n. *Archive for Rational Mechanics and Analysis*, 105(3):243–266, 1989.

[23] Jeremy Louis Marzuola and Gideon Simpson. Spectral analysis for matrix hamiltonian operators. *Nonlinearity*, 24:389 – 429, 2010.

[24] Frank Merle and Pierre Raphael. The blow-up dynamic and upper bound on the blow-up rate for critical nonlinear schrödinger equation. *Annals of Mathematics*, 161(1):157–222, 2005.

[25] Cathleen S. Morawetz. *Notes on Time Decay and Scattering for Some Hyperbolic Problems*. Society for Industrial and Applied Mathematics, 1975.

[26] L. Nirenberg. On elliptic partial differential equations. *Annali della Scuola Normale Superiore di Pisa - Scienze Fisiche e Matematiche*, Ser. 3, 13(2):115–162, 1959.

[27] Richard Palais. The symmetries of solitons. *Bulletin of the American Mathematical Society*, 34, 09 1997.

[28] Marcel Riesz. Sur les fonctions conjuguées. *Mathematische Zeitschrift*, 27(1):218–244, 1928.

[29] Walter A Strauss. Existence of solitary waves in higher dimensions. *Communications in Mathematical Physics*, 55(2):149–162, 1977.

[30] Robert S. Strichartz. Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations. *Duke Mathematical Journal*, 44(3):705 – 714, 1977.

[31] Terence Tao. *Nonlinear dispersive equations: Local and global analysis*, volume 106 of *CBMS Regional Conference Series in Mathematics*. AMS and CBMS, 2006.

[32] Terence Tao. Why are solitons stable? *Bulletin of the American Mathematical Society*, 46(1):1–33, 2009.

[33] W. Walter. *Gewöhnliche Differentialgleichungen: Eine Einführung*. Springer-Lehrbuch. Springer Berlin Heidelberg, 2013.

[34] Michael I. Weinstein. Modulational stability of ground states of nonlinear schrödinger equations. *SIAM Journal on Mathematical Analysis*, 16(3):472–491, 1985.

[35] Michael I. Weinstein. Lyapunov stability of ground states of nonlinear dispersive evolution equations. *Communications on Pure and Applied Mathematics*, 39(1):51–67, 1986.

[36] N. J. Zabusky and M. D. Kruskal. Interaction of "solitons" in a collisionless plasma and the recurrence of initial states. *Phys. Rev. Lett.*, 15:240–243, Aug 1965.