Two-pion exchange contributions for coupled-channel $B^{(*)}B^{(*)}$ and $B^{(*)}\bar{B}^{(*)}$ scattering

Dissertation
zur
Erlangung des Doktorgrades (Dr. rer. nat.)
der
Mathematisch-Naturwissenschaftlichen Fakultät
der
Rheinischen Friedrich-Wilhelms-Universität Bonn

vorgelegt von
Jabez Tom Chacko
aus
Umm Al Quwain, Vereinigte Arabische Emirate

June 2024

Angefertigt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät
der Rheinischen Friedrich-Wilhelms-Universität Bonn

Gutachter 1:	Professor Dr. Christoph Hanhart
Gutachter 2:	PD Dr. Bastian Kubis

Tag der Promotion:	17th June 2024
Erscheinungsjahr:	2024

Abstract

In the paper [Phys.Rev.D 99 (2019) 9, 094013] of V. Baru et al., the LO and incomplete NLO potentials were utilised to investigate the line shapes of $Z_b(10610)$ and $Z_b(10650)$ and their (yet to be discovered) spin partners. The spin partners of $Z_b^{\pm}$'s are conventionally refereed to as W_{bJ}'s with quantum numbers $J^{PC} = 0^{++}, 1^{++}$ and 2^{++}.
In this thesis, the full next-to-leading order calculations of the effective potentials of $B^{(*)}\bar{B}^{(*)} \to B^{(*)}\bar{B}^{(*)}$ and $B^{(*)}B^{(*)} \to B^{(*)}B^{(*)}$ are presented. The Feynman rules are constructed based on heavy meson chiral perturbation theory and effective potentials up to NLO are obtained. The subsequent effective potentials are partial wave decomposed into quantum numbers $J^{PC} = 0^{++}, 1^{++}, 1^{+-}$ and 2^{++}. For the first time, we also present results for non-diagonal transition potentials between different channels. Our results deviate in some aspects from earlier calculations. Arguments are provided supporting that ours are correct.

Contents

1 Motivation

The study of exotic hadrons within the context of Quantum Chromodynamics (QCD) presents a challenge to our understanding of the strong force [1]. Exotic mesons with heavy quarks, known as XYZ states, diverge from conventional quark model predictions, raising intriguing issues concerning their fundamental structure. An ideal platform for exploring deeper into exotic states are the Z_b states, namely $Z_b(10610)$ and $Z_b(10650)$, which were found by the Belle collaboration [2]. Some of their decays consist of a heavy quarkonium and a pion as final states. Thus, they contain a $\bar{b}b$ pair and have isospin one, calling for four valence quarks. Both $Z_b(10610)$ and $Z_b(106650)$ have $J^{PC} = 1^{+-}$ and appear as two narrow peaks separated by 45 MeV in the invariant mass distributions of the $\pi^{\pm}\Upsilon(nS)$ ($n = 1, 2, 3$) and $\pi^{\pm}h_b(mP)$ ($m = 1, 2$) subsystems in the dipion transitions from the vector bottomonium $\Upsilon(10860)$ [3]. Furthermore, these states have been observed in the $B\bar{B}^*$ [1] and $B^*\bar{B}^*$ invariant mass distributions in the decays $\Upsilon(10860) \to \pi B^{(*)}\bar{B}^*$ with dominant branching fractions [4, 5]. The masses of $Z_b(10610)$ and $Z_b(10650)$ using Breit-Wigner analysis were found close to the $B\bar{B}^*$ and $B^*\bar{B}^*$ threshold [6]

$$\begin{aligned}(10607.8 \pm 2.0)\,\text{MeV} &\approx m_B + m_{B^*} \approx 10603\,\text{MeV}\\ (10652.2 \pm 1.5)\,\text{MeV} &\approx 2m_{B^*} \approx 10648\,\text{MeV}\end{aligned}$$

The proximity of $Z_b(10610)$ and $Z_b(10650)$ to the $B\bar{B}^*$ and $B^*\bar{B}^*$ thresholds, respectively, and the prevalence of open-flavor branching fractions, strongly support their molecular interpretation [7].
In the recent works of Q. Wang et al. [8] and V. Baru et al. [9], a chiral EFT-based method was developed to analyse the experimental data for all measured production and decay channels of the bottomonium-like states $Z_b(10610)$ and $Z_b(10650)$. This thesis will continue the work done in [8] and [9] in investigating the line shapes of $Z_b^{\pm}$'s and their undiscovered spin partners. These studies applied HMχPT to find the effective potentials of $B^{(*)}\bar{B}^{(*)} \to B^{(*)}\bar{B}^{(*)}$ scattering in Leading order (LO) and incomplete Next-to-Leading order (NLO). The NLO contained just the Contact Interactions (CI). The resultant potentials were then partial wave projected to quantum numbers $J^{PC} = 1^{+-}$ and provided as input for the coupled-channel Lippmann-Schwinger Equation (LSE). The parameters were determined by fitting line shapes in elastic $B^{(*)}\bar{B}^{(*)}$-channels and inelastic $h_b(mP)$ channels with $m = 1, 2$ [8]. Some of the challenges they encountered are:

- It turns out that one-pion exchange (1PE) in S-D transitions at momenta

[1] Here, a properly normalized C-odd combination of the $B\bar{B}^*$ and $\bar{B}B^*$ components is understood.

about 100 MeV is expected to be relevant. In contrast, this significance is not demonstrated experimentally.

- In their investigations, to tame severe regulator dependency originating from higher-momentum 1PE contributions, especially when several open-flavor coupled channels are included, a $\mathcal{O}(Q^2)$ S-to-D-wave counter term is promoted to leading order. In addition to this, two momentum-dependent $\mathcal{O}(Q^2)$ S-wave-to-S-wave contact terms also appear.

- Additionally, treating the pion mass m_π and the binding momenta as soft ($p_{\text{typ}} \sim$ 500MeV) and the hadronic scale as hard ($\Lambda_\chi \sim$ 1GeV) provides a slow convergence of the expansion parameter χ of HMχPT

$$\chi = Q/\Lambda_\chi \sim p_{\text{typ}}/\Lambda_\chi \simeq 1/2 \ . \tag{1}$$

Therefore a NLO calculation is needed to address these issues. Anticipating the yet-to-be-found spin partners, this thesis provides such potentials deconstructed into its partial waves beyond $J^{PC} = 1^{+-}$. This work will lay the framework for future inquiries, such as determining the theory's convergence and the impact of NLO contributions when employed in the scattering equations.
This thesis has the following structure: Sec. 2 begins with fundamental QCD principles, its features and concludes with a brief description on exotic states. Sec. 3 starts with an introduction to effective field theories (EFTs) and moves on to chiral perturbation theory (χPT) and heavy-quark effective field theory (HQFT). This section also includes a theoretical framework for heavy meson chiral perturbation theory (HMχPT), the EFT utilized in this thesis. The section concludes with the power counting scheme employed in this work and a brief description of dimensional regularization. Sec. 4 entails the Lagrangian and the vertices derived from it. In Sec. 5, the effective potentials at $\mathcal{O}(\chi)$ and a brief discussion on effective potentials at $\mathcal{O}(\chi^2)$ are presented, and Sec. 6 contains partial wave projected potentials in $J^{PC} = 0^{++}, 1^{++}, 1^{+-}$ and 2^{++} channels. Appendices A and B provide the calculation of loop integrals found in potentials and partial wave projectors, respectively. The explicit form of the potentials at $\mathcal{O}(\chi^2)$, its partial wave decomposed (PWD) form, and calculation of the integrals obtained from partial wave decomposition are mentioned in Appendices C, D and E, respectively. In addition, Appendix F contains the complete computations of loop integrals where q_0 was treated perturbatively. Sec. 7 summarizes the various checks conducted on our PWD potentials and in Appendix G the potentials for triangle diagrams in $\bar{B}B \to \bar{B}B$ and $BB \to BB$ scatterings are expressed in particle basis. In Section 8, we compare our potentials to those of previous works. Sec. 9 concludes this thesis with a summary and outlook.

2 Introduction

The later half of the 20th century saw the development of the Standard Model of particle physics, which led to the unified description of the fundamental forces of nature, with the exception of gravity. The Standard Model is a quantum field theory that describes electroweak and strong interactions [10–12]. It consists of two types of particles, fermions (matter particles) and bosons (force-carrying particles). The fermions are particles with spin 1/2 and they follow Pauli's exclusion principle [13]. The fermions consist of quarks and leptons with the former taking part in electroweak and strong interactions whereas the latter in just the electroweak interactions.

The quarks and leptons as well as their antiparticles come in three generations. Each generation consists of a doublet under the weak interaction. For leptons, they are electron e and electron-neutrino ν_e, muon μ and the muon-neutrino ν_μ and tauon τ and tau-neutrino ν_τ. Similarly, the quark families are up u and down d, charm c and strange s and top t and bottom b quarks.

The electroweak and the strong force can be combined into a $\mathrm{SU}(3)_c\times\mathrm{SU}(2)_L\times\mathrm{U}(1)_Y$ gauge theory in the Standard Model. The $\mathrm{SU}(3)_c$ gauge group is denoted by the color corresponding to the strong force. Accordingly, the $\mathrm{SU}(2)_L\times\mathrm{U}(1)_Y$ refers to the electroweak force [14–16]. The strong force's gauge bosons are represented by the eight gluons, which can be represented by particular combinations of color and anti-color. Likewise, the $W^\pm$ and the Z bosons correspond to the weak interaction and the photon to the electromagnetic force. In contrast to the electromagnetic interactions, the gluons themselves carry color charge leading to gluon self-interactions and resulting in QCD being a non-abelian gauge theory.

Additionally, the Higgs mechanism allows for the inclusion of masses without breaking the symmetry [17, 18]. This was done by introducing a scalar Higgs field which transforms as a doublet in $\mathrm{SU}(2)_L$ and is coupled to quark and lepton field via a Yukawa term. The Higgs potential has a non-zero vacuum expectation value which leads to spontaneous symmetry breaking and through this, the $W^\pm$ and Z bosons as well as quarks and leptons gain masses. Although the mass of neutrinos can be detected experimentally via neutrino oscillations [19–21], the mechanism for accumulating neutrino mass remains unexplained.

Since the thesis's primary focus is on the strong force, the QCD Lagrangian and its characteristics will be discussed in the sections that follow.

2.1 QCD Lagrangian

The QCD Lagrangian is given by [22]:

$$\mathcal{L}_{QCD} = \sum_{f=u,d,s,c,b,t} \bar{q}_f(i\not{D} - m_f)q_f - \frac{1}{4}\mathcal{G}_{\mu\nu,a}\mathcal{G}^{\mu\nu}_a - \theta\frac{g_s^2}{64\pi^2}\varepsilon^{\mu\nu\rho\sigma}\mathcal{G}^a_{\mu\nu}\mathcal{G}^a_{\rho\sigma} \ , \tag{2}$$

where q_f is the quark field with f as the sub-index which denotes the six flavours and the quark fields are color triplets (with red, blue and green for particles and the complementary colors for antiparticles), m_f is the quark mass.
$\not{D}$ is the covariant derivative of QCD and is given by [23]

$$D_\mu = \partial_\mu - ig_s\sum_{a=1}^{8}\frac{\lambda^a}{2}A^a_\mu \ , \tag{3}$$

with g_s as the strong coupling constant and A^a_μ is the gluon field with color index a and λ^a are the Gell-Mann matrices which are given by

$$\lambda^1 = \begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}, \quad \lambda^2 = \begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix}, \quad \lambda^3 = \begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}, \quad \lambda^4 = \begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}$$

$$\lambda^5 = \begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix}, \quad \lambda^6 = \begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}, \quad \lambda^7 = \begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix}, \quad \lambda^8 = \begin{pmatrix}\frac{1}{\sqrt{3}}&0&0\\0&\frac{1}{\sqrt{3}}&1\\0&1&\frac{-2}{\sqrt{3}}\end{pmatrix}.$$

They are the eight generators of SU(3), which is similar to the Pauli matrices in SU(2). They are traceless and Hermitian matrices. The structure of the SU(3) group is given by the commutation relations of the Gell-Mann matrices

$$\left[\frac{\lambda_a}{2}, \frac{\lambda_b}{2}\right] = if_{abc}\frac{\lambda_c}{2} \tag{4}$$

and the gluon field strength tensor is given by

$$\mathcal{G}^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f^{abc}A^b_\mu A^c_\nu \ , \tag{5}$$

where f^{abc} are the structure constants of QCD and given in Eq. (4). Their values are

$$f_{123} = 1$$

$$f_{147} = f_{246} = f_{257} = f_{345} = \frac{1}{2}$$

$$f_{156} = f_{367} = -\frac{1}{2}$$

$$f_{458} = f_{678} = \frac{\sqrt{3}}{2}$$

$$\text{all other } f_{abc} = 0 \ \ .$$

The 3-gluon and 4-gluon vertices are generated from the $\mathcal{G}^a_{\mu\nu}\mathcal{G}^{a\mu\nu}$ term in the Lagrangian density.
The last term of the Lagrangian contains $\varepsilon^{\mu\nu\rho\sigma}$- the antisymmetric Levi-Civita tensor. Thus a non-vanishing value of θ implies an explicit P and CP violation of QCD. Since $\theta < 10^{-10}$ [24], from here on, this term is dropped from the Lagrangian and CP conservation of the strong interaction can be safely assumed. With QCD Lagrangian defined, one can now focus on the different features of QCD.

2.2 Features of QCD

2.2.1 Asymptotic freedom (running of α_s)

Due to gluonic self-interactions, the QCD coupling constant behaves differently from the QED coupling constant. In QED, the nearby vacuum of an electric charge produces virtual particle-antiparticle pairs, which 'screens' the effect of the charge over distance. Thus, the running coupling constant "α" of QED decreases with distance.
In QCD, since the gluons themselves carry color charge, the overall effect of the gluon cloud surrounding a quark is to change its color (anti-screening) which strengthens the coupling constant, as opposed to the screening of color charge by the virtual quark-antiquark pairs. Since, three colors and less than 16 quark flavours have been observed (6 flavours), the anti-screening effect of the virtual gluon cloud dominates over the screening effect of virtual quarks. Thus, the QCD coupling increases with an increase in distance, in contrast to the QED coupling. This is the difference between the running of α (QED) compared to the running of α_s (QCD). Therefore, quarks and gluons interact weakly at very short distances. This is the "asymptotic freedom" of QCD [25, 26].
The running of strong coupling (α_s) which was explained phenomenologically in the above paragraphs, can also be derived formally. The scaling (running) behavior can be explained using the beta (β) function [27]. In quantum field theory, the beta function ($\beta(g)$) is used to explain the dependence of the coupling parameter (g) on an energy scale (Q) [28]. Using this, the running of the strong coupling constant is expressed as

$$Q^2 \frac{\partial \alpha_s}{\partial Q^2} = \frac{\partial \alpha_s}{\partial \ln Q^2} = \beta(\alpha_s) \ . \tag{6}$$

In the above equation, it is seen that the running of the strong coupling is logarithmic. Further, the beta function is defined as

$$\beta(\alpha_s) = -\alpha_s^2 \left(b_0 + b_1\alpha_s + b_2\alpha_s^2 + ...\right) \ , \tag{7}$$

where b_0 is the coefficient for the 1-loop interactions and b_1 is the coefficient for 2-loop interactions and they are defined as [29]

$$b_0 = \frac{11C_A - 4T_R n_f}{12\pi} \ ,$$

$$b_1 = \frac{17C_A^2 - 10T_R C_A n_f - 6T_R C_A n_f}{24\pi^2} \ ,$$

where $C_A = 3$, $T_R = \frac{1}{2}$, and n_f is the number of quark flavours.
Single gluon loops account for the first term in the b_0 coefficient, while single quark loops account for the second. Comparably, the first term of the b_1 coefficient is caused by double gluon loops, whereas the second and third terms are caused by quark-gluon mixed loops.
At higher orders, the coefficient b_i depends on the renormalisation scheme [29]. Solving the equation for the beta function gives

$$\alpha_s(Q^2) = \alpha_s(\mu^2)\left(\frac{1}{1 + b_0\alpha_s(\mu^2)\ln\frac{Q^2}{\mu^2} + \mathcal{O}(\alpha_s^2)}\right) \ , \tag{8}$$

where the strong coupling constant $\big(\alpha_s(Q^2)\big)$ is defined using a renormalisation scale, $Q^2 = \mu^2$.
It is noticed that the coefficients b_0 and b_1 do depend on the number of flavours, via the presence of n_f. This changes the behavior of the running of strong coupling in relation to the number of flavours (n_f). For example, the running of the strong coupling for $n_f = 3$ (charm threshold), will be different when compared to $n_f = 6$ (top threshold). Therefore, if any new colored particles are discovered, then the running of the coupling should be modified at the threshold of the rest mass of the newly discovered colored particle.
With the definition of the beta function in Eq. (7) and the result '$b_0 > 0$' (since $n_f < 16$), leads to the conclusion that the QCD coupling decreases with an increase in energy (asymptotic freedom) [29, 30]. For the discovery of asymptotic freedom in 1973, David Gross, Frank Wilczek and David Politzer were awarded the 2004 Nobel Prize in physics.
Perturbative computations are possible at very high energies because of asymptotic freedom. It is observed experimentally that the running of α_s is what causes the Bjørken scaling violation. [29].

2.2.2 Color confinement

Another phenomenon of QCD is "color confinement" which is seen as the absence of free colored quarks in nature, only color neutral hadrons exist in nature [31]. The three colors of QCD combine together to form a color neutral state such as

baryons and mesons. Due to confinement, the coupling between a quark pair gets stronger as we try to separate them from each other. Eventually, the force between the quark pair will get so strong that energy supplied to the field to separate the pair, will in turn create a new quark-antiquark pair.
With the coupling constant being much greater at lower energies (around 1 GeV), combined with the formation of gluon flux tube, leads to color confinement [12]. Thus, perturbation theory solely cannot explain the full physical behavior of QCD [29]. The scale at which the perturbatively described strong coupling diverges is called the confinement scale or QCD scale and its value depends on the renormalisation scheme. For example, rewriting the previous equation for $\alpha_s(Q^2)$

$$\alpha_s(Q^2) = \frac{1}{b_0 \ln \frac{Q^2}{\Lambda^2}} \ ,$$

where

$$\Lambda_{QCD} = (332 \pm 17)\text{MeV}$$

with $n_f = 3$ (below the charm threshold) and the renormalisation scheme used is "modified minimal subtraction" ($\overline{MS}$) scheme [32]. For $\sqrt{Q^2}$ below this scale, the perturbative theory for α_s fails and just non-perturbative methods prevail. The phenomenon of color confinement is experimentally observed in many particle accelerators, as the final state consists of jets of many color neutral particles (mesons and baryons). Additionally, it requires us to use techniques that are outside the realm of perturbative QCD like: effective theories (such as the Operator product expansion, Chiral perturbation theory), phenomenological models (such as Regge theory) and lattice QCD [29]. We still can't analytically explain the process of hadronization using QCD and to this day it remains an open problem in physics. Fig. 1 describes the running of α_s against energy.

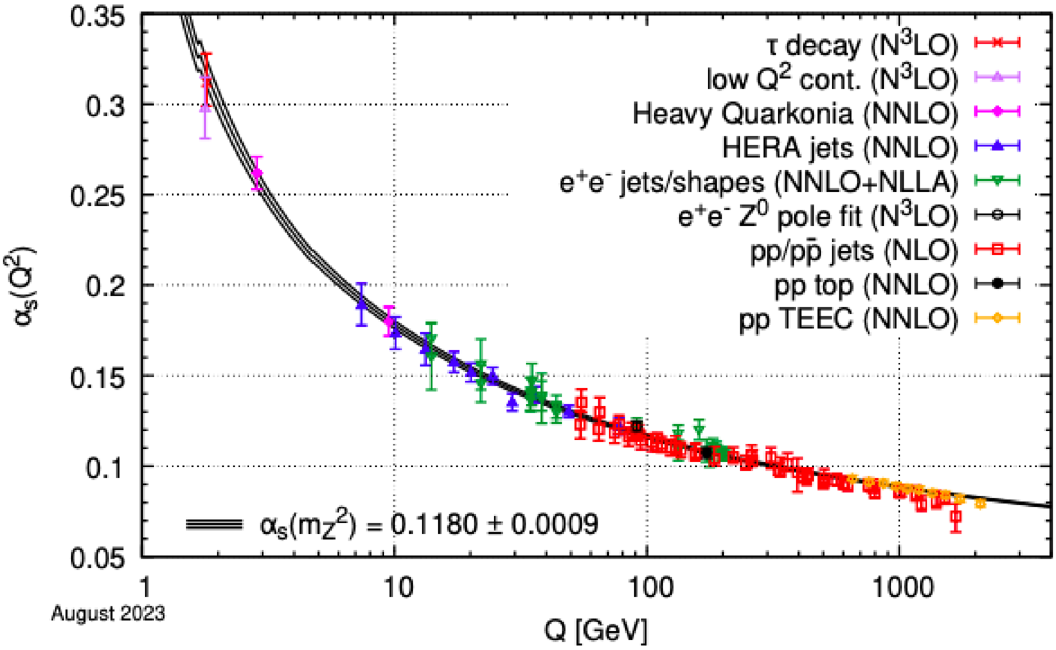


Figure 1: A pictorial representation of the running of α_s in different experimental processes at different scales [33].

2.3 Exotic states

The traditional quark model [34, 35] assumes that mesons are characterized as quark-antiquark systems and baryons as three quarks in a color singlet. As a result, states that do not have the quantum numbers allowed by this model can be termed exotic [36].

With the advent of QCD, our understanding of the structure of hadrons became more complicated as compared to what the classical quark model allows because of possible structures like glueballs (consists of only valence gluons), hybrid (consists of both valence gluons and valence quarks) and multiquarks (such as pentaquarks and tetraquarks) [36–41]. With the discovery of the $X(3872)$ (an exotic state due to unusual decay properties) in 2003, many new exotic states beyond the simple, yet successful quark model were discovered [42].

States like $Z_b^{\pm}(10610)$, $Z_b^{\pm}(10650)$, $Z_c^{\pm}(3900)$ [43] and $Z_c^{\pm}(4020)$ [44] and their decays into bottomium or charmonium states and a pion exclude a simple $Q\bar{Q}$ meson, with $Q(\bar{Q})$ denoting heavy quark (anti-quark). Hence, they consist of at least four quarks. One notices that all these exotic states of the Z family are located above the first heavy quark open flavor threshold, which is $D\bar{D}$, seen in Fig. 2 [45], for charmonium and $B\bar{B}$, seen in Fig. 3 [45], for bottomium. There are many different interpretations of these exotic hadrons with compact tetraquarks (containing four quarks clustered into (anti) diquarks) being the simplest extensions of the quark model. Many of them have masses close to the meson-meson threshold and if assumed to be in S-wave, often share their quantum numbers with the exotic states and in turn, make them ideal candidates for hadronic molecules- composite systems built out of hadrons [46].

There are many candidates for exotic states in the $c\bar{c}$-sector, however, in the $b\bar{b}$-sector there are only two: $Z_b^{\pm}(10610)$ and $Z_b^{\pm}(10650)$, with quantum numbers $J^{PC} = 1^{+-}$ which will be main focus of this thesis. But Heavy Quark Spin Symmetry (HQSS) which we will discuss later, predicts an abundance of states in the $b\bar{b}$-sector, with different quantum numbers $J^{++}(J = 0, 1, 2)$. These states are the spin partners W_{bJ} of the $Z_b^{\pm}(10610)$ and $Z_b^{\pm}(10650)$.

As mentioned in Sec. 1, the $Z_b^{\pm}(10610)$ and $Z_b^{\pm}(10650)$ states have a strong argument for molecular interpretation due to their proximity to the $B\bar{B}^*$ and $B^*\bar{B}^*$ threshold, coupled with the prevalence of open-flavor branching fractions. We also mention that a compact tetraquark interpretation of these states is also compatible with the available data [47]. If the molecular picture is correct, it leads to the prediction of spin partners W_{bJ} of the Z_b states [9, 48–50]. The hadronic molecular picture would be strongly confirmed if the spin partners are detectable in the decays $\Upsilon(5S) \to W_{bJ}\gamma \to \chi_{bJ}(\eta_b)\pi\gamma$ at the predicted masses since the pattern of spin symmetry violation is specific for the assumed structure.

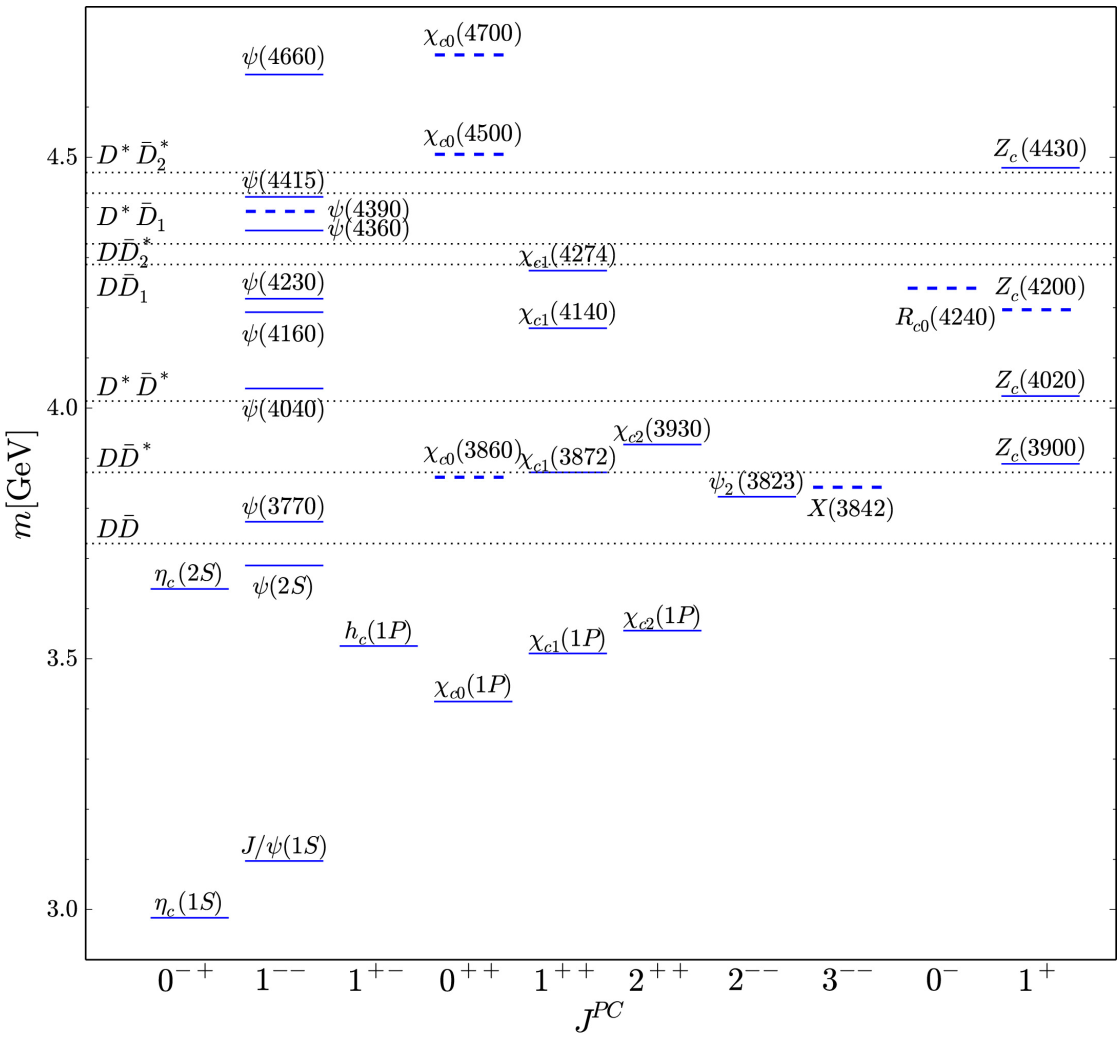


Figure 2: Charmonium-like states with completely determined quantum numbers and without hidden strangeness (as of July 2019). Established states are depicted with solid lines, not (yet) established ones with dashed lines. Dashed lines show some relevant thresholds that open in the considered mass range (details in [45]). $D_1 \equiv D_1(2420)$, $D_2^* \equiv D_2^*(2460)$

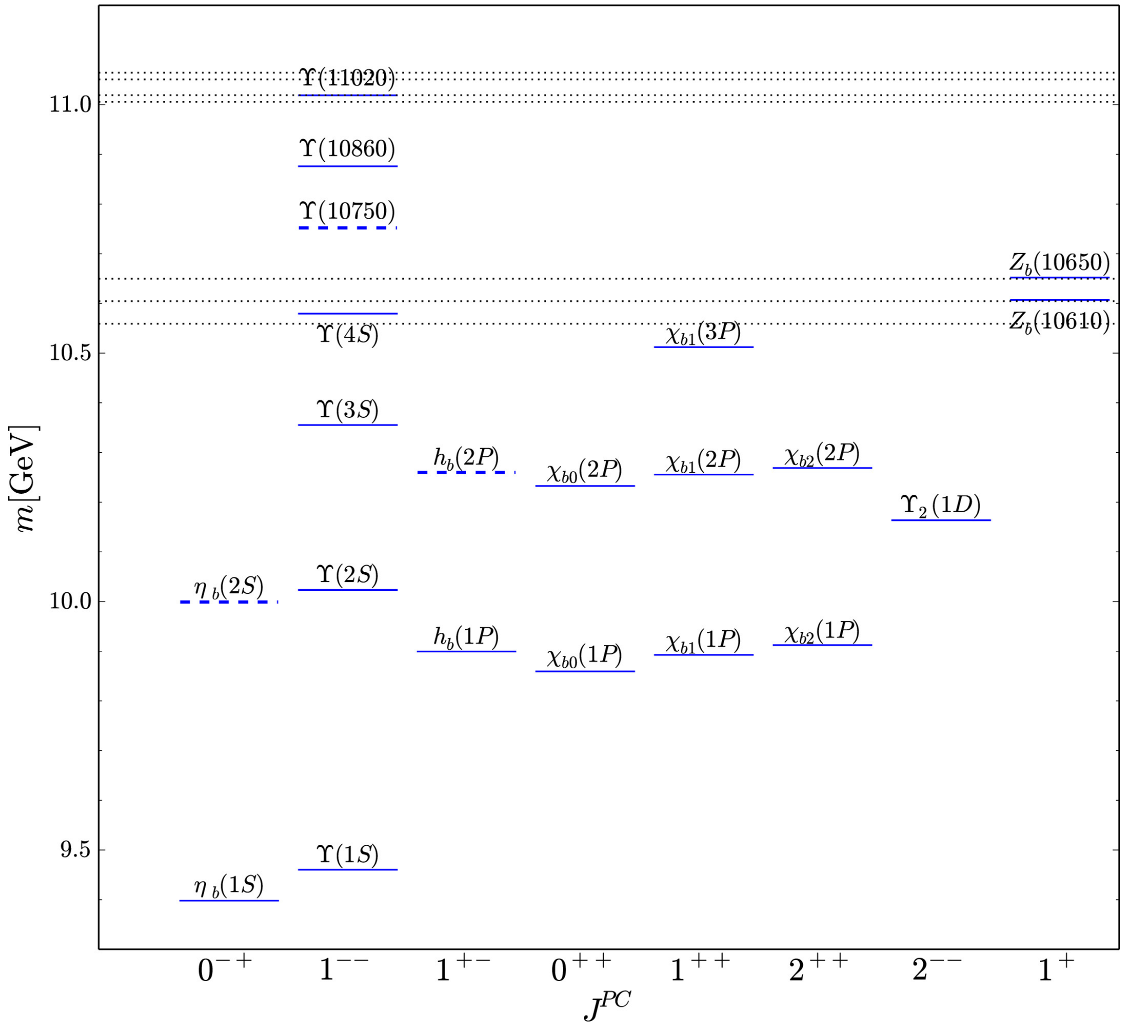


Figure 3: Bottomium-like states with completely determined quantum numbers and without hidden strangeness (as of July 2019). Established states are depicted with solid lines, not (yet) established ones with dashed lines. The fine dots depict open-bottom thresholds from lowest to highest: $B\bar{B}$, $B^*\bar{B}$, $B^*\bar{B}^*$, $\bar{B}B_1(5721)$, $\bar{B}B_2^*(5747)$, $\bar{B}^*B_1(5721)$, $\bar{B}^*B(5747)$.

3 Effective field theory (EFTs)

The essential notion behind an EFT is that one does not need to know everything in order to provide a reasonable explanation of the specific part of physics in question [51, 52]. Effective field theories are usually low-energy approximations of fundamental theories [53].

Applying EFTs, allows one to describe the phenomena of low energy QCD using

perturbative method and make accurate predictions, but in turn, one loses renormalizability at all scales and is no longer working in a fundamental theory [54, 55]. EFTs are not renormalisable in the traditional sense [56]. However, divergences that occur in calculations up to a particular order of p/Λ can be renormalized by redefining fields and parameters of the EFT Lagrangian [57, 58], where p represents the momenta or masses that are smaller than a certain momentum scale Λ. This exercise will be explicitly performed later in Sec. 7.
For this thesis, we will use Chiral Perturbation Theory which is the EFT, that arises due to the global flavor symmetries of QCD at the massless limit of light quarks.

3.1 Chiral perturbation theory (χPT)

To derive χPT, we first separate the masses of the six quark flavors as light and heavy quarks. The first three quark flavors (u,d,s) can be titled as light quarks, since they have masses less than Λ_{QCD} and the other three flavors (c, b, t) as heavy quarks as their masses are above Λ_{QCD}. Using the above inference, we can rewrite the $\mathcal{L}_{QCD}$ for light quarks in the chiral limit (m_u, m_d, m_s=0)

$$\mathcal{L}_{QCD} = \sum_{l=u,d,s} \bar{q}_l (i\not{D}) q_l \ , \tag{9}$$

where we only have the quark terms in the equation above. From Eq. (9), it is possible to decouple the fields by applying the projector operators (P_L, P_R) on the quark fields and in turn, they decompose as left and right fields

$$\mathcal{L}_{QCD} = \sum_{l=u,d,s} \bar{q}_{R,l}(i\not{D})q_{R,l} + \bar{q}_{L,l}(i\not{D})q_{L,l} \ , \tag{10}$$

where $q_L = P_L q$ is the left-handed quark $q_R = P_R\, q$ is the right-handed quark field and this Lagrangian in the chiral limit is invariant under flavor rotations

$$q_L \rightarrow \exp\bigg(-i \sum_{a=1}^{3} \theta_a^L \frac{\lambda_a}{2} \bigg) e^{-i\theta^L} q_L \tag{11}$$

and

$$q_R \rightarrow \exp\bigg(-i \sum_{a=1}^{3} \theta_a^R \frac{\lambda_a}{2} \bigg) e^{-i\theta^R} q_R \, . \tag{12}$$

The Eq. (11) and Eq. (12) represent the left and right chiral rotations, where λ_a is the Gell-Mann matrix. This symmetry generates two conserved currents from the Noether's theorem.

$$L^{\mu,a} = \bar{q}_L \gamma^\mu \frac{\lambda_a}{2} q_L \quad \text{with} \quad \partial_\mu L^{\mu,a} = 0 \ , \tag{13}$$

$$R^{\mu,a} = \bar{q}_R \gamma^\mu \frac{\lambda_a}{2} q_R \quad \text{with} \quad \partial_\mu R^{\mu,a} = 0 \ , \tag{14}$$

$$L^\mu = \bar{q}_L \gamma^\mu q_L \quad \text{with} \quad \partial_\mu L^\mu = 0 \quad \text{and} \tag{15}$$

$$R^\mu = \bar{q}_R \gamma^\mu q_R \quad \text{with} \quad \partial_\mu R^\mu = 0 \ . \tag{16}$$

Instead of using left and right currents, a linear combination of these two currents is used, which transforms as a vector and an axial-vector given by

$$V^\mu = R^\mu + L^\mu \quad \text{and} \tag{17}$$

$$A^\mu = R^\mu - L^\mu \ . \tag{18}$$

Hence, the chiral symmetry can be decomposed as $U(3)_L \times U(3)_R = SU(3)_V \times SU(3)_A \times U(1)_V \times U(1)_A$, where $SU(3)_V$ and $SU(3)_A$ is the corresponding symmetry of vector and axial-vector currents respectively. Furthermore, $U(1)_V$ is related to Baryon number conservation and is always conserved, while $U(1)_A$ is explicitly broken by the axial-anomaly [57]. The currents and their subsequent divergences are

$$V^\mu = \bar{q}\gamma^\mu q \ , \qquad \partial_\mu V^\mu = 0 \ , \tag{19}$$

$$A^\mu = \bar{q}\gamma^\mu \gamma_5 q \ , \qquad \partial_\mu A^\mu = \frac{3g_s^2}{32\pi^2} \varepsilon^{\mu\nu\rho\sigma} \mathcal{G}^a_{\mu\nu} \mathcal{G}^a_{\rho\sigma} \ . \tag{20}$$

Similarly, for the octet currents one finds

$$V^{\mu,a} = \bar{q}\gamma^\mu \frac{\lambda_a}{2} q \ , \qquad \partial_\mu V^{\mu,a} = i\bar{q}\,[M, \frac{\lambda_a}{2}]\, q \ , \tag{21}$$

$$A^{\mu,a} = \bar{q}\gamma^\mu \gamma_5 \frac{\lambda_a}{2} q \ , \qquad \partial_\mu A^{\mu,a} = i\bar{q}\,\{\frac{\lambda_a}{2}, M\}\gamma_5\, q \ , \tag{22}$$

where for finite u-, d- and s-quark masses, one incorporates these quark masses into a quark-mass matrix M [59] given by

$$M = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix} \ . \tag{23}$$

As mentioned above, the singlet vector current V^μ is always conserved, whereas the singlet axial-vector current A^μ is never conserved due to the axial-anomaly. The octet vector current $V^{\mu,a}$ is conserved in the limit of massless quarks ($M = 0$). $V^{\mu,a}$ is also conserved in the case of equal quark masses, $m_u = m_d = m_s$, because $[\lambda_a, 1] = 0$ [60]. Finally, the octet axial-vector current is only conserved in the limit of massless quarks, $M = 0$.

3.2 Spontaneous symmetry breaking (SSB)

Another feature of χPT is spontaneous symmetry breaking (SSB), which arises due to chiral symmetry being spontaneously broken. To demonstrate this, consider the charges of the axial and vector currents as $Q^a_A = R^a_A - L^a_A$ and $Q^a_V = R^a_V + L^a_V$ respectively.
Now, it was shown by Vafa and Witten that the vector symmetries, like Q^a_V cannot be spontaneously broken given the structure of QCD [61]. But in the case of axial charge Q^a_A, we see that QCD ground state is not invariant under axial transformations. Furthermore, due to the absence of parity doubling (which is two particle multiplets of equal masses but with opposite parity) in experimental observations, Q^a_A does not annihilate the QCD vacuum leading to the symmetry being broken spontaneously [57]. This in turn leads us to the Goldstone theorem, which states that for every broken symmetry there exists a massless scalar boson (Goldstone boson) with the related quantum numbers that do not interact for vanishing momenta [62, 63]. In our case, since the symmetry being broken is $U(3)_L \times U(3)_R$, we expect nine Goldstone bosons. However, $U_A(1)$ is broken explicitly hence we are left with eight Goldstone bosons which are three pions (π^+,π^-,π^0), four kaons (K^+,K^-,K^0,$\bar{K}^0$) and η.
The chiral symmetry is preserved much better in the u-quark and d-quark systems or in the SU(2) isospin limit as compared to the s-quark system, as the s-quark is an order magnitude heavier than the u and d. Through out this thesis, we will work in the SU(2) isospin limit for the following reasons: due to the higher mass of the strange-quark, it plays a lesser role as compared the up and down sector of the Goldstone bosons. Additionally, we focus on the Z_b states that are close to the $B^{(*)}\bar{B}^{(*)}$ threshold and not $B_s^{(*)}\bar{B}_s^{(*)}$ threshold . Finally, the expansion in terms of Goldstone bosons converge better in SU(2)χPT as compared to SU(3)χPT.

3.3 Chiral perturbation theory at leading order

In this section, we will define the leading order Lagrangian for χPT from the aspects mentioned in the earlier sections. The leading order Lagrangian can be obtained from Weinberg conjecture [64, 65] which states that any Quantum Field Theory (QFT) only consists of analyticity, unitarity, cluster decomposition and symmetries and in order the compute the S-matrix for any theory below a particular scale , one must use the most general effective Lagrangian that is consistent with the above mentioned principles expressed in the appropriate asymptotic states (mesons and baryons) [65]. To make this useful, one needs to combine it with a power counting scheme.
We need an effective field theory that preserves $SU(3)_V \times SU(3)_A \times U(1)_V$ in the chiral limit, and contains the interactions of Goldstone bosons which comes from

Spontaneous symmetry breaking (SSB). As we are working in the SU(2) isospin limit, there will be three Goldstone bosons generated from SSB. The three bosons or in this case pions, can be collected and represented in the matrix $U(x)$ [66].

$$U(x) = \exp\left(\frac{i\phi(x)}{f_\pi}\right) , \tag{24}$$

with

$$\phi(x) = \sum_{a=1}^{3} \pi_i \tau_i = \begin{pmatrix} \pi_3 & \pi_1 - i\pi_2 \\ \pi_1 + i\pi_2 & -\pi_3 \end{pmatrix} , \tag{25}$$

where f_π is the pion decay constant with $f_\pi = 92.4$ MeV [6, 67]. τ_i are the Pauli matrices which are summed over ϕ_i. The above representation of ϕ is in Cartesian basis which when translated to physical fields is

$$\phi(x) = \vec{\tau} \cdot \vec{\pi} = \begin{pmatrix} \pi_3 & \pi_1 - i\pi_2 \\ \pi_1 + i\pi_2 & -\pi_3 \end{pmatrix} = \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix} . \tag{26}$$

Now, we have introduced Goldstone boson dynamics into our theory and as they originate from axial transformations they have the quantum numbers $J^P = 0^-$. Since EFT is renormalisable order by order therefore we introduce a power counting scheme that allows us to estimate the prominence of different terms. In our kinematic range (500 MeV), the momenta of the Goldstone bosons are very small as compared to hadronic scale Λ_χ, an expansion in terms of p_{typ}/Λ_χ is chosen, with p_{typ} being the typical momentum of Goldstone boson. We can now formulate an effective Lagrangian [68] as an expansion in p_{typ}

$$\mathcal{L}^{\text{eff}} = \mathcal{L}^2 + \mathcal{L}^4 + \mathcal{L}^6 \dots , \tag{27}$$

where the superscripts refer to the chiral dimension which denotes the Lagrangian's number of derivatives or the order in a momentum. In the standard power counting, the quark masses are treated as p_{typ}^2, which is different in our case since we use a different power counting scheme. The even powers of $\mathcal{L}$ is due to Lorentz symmetry. Using the definition of U-matrix and effective Lagrangian, we can write the most simple Lagrangian density with the minimal number of derivatives and obeys chiral and Lorentz symmetry [57, 69]

$$\mathcal{L}^2 = \frac{f_\pi^2}{4} \operatorname{Tr}\left[\partial_\mu U \partial^\mu U^\dagger\right] , \tag{28}$$

where the order of the above term is $\mathcal{O}((p_{typ}/\Lambda_\chi)^2)$ and the trace is over flavor space. The masses of the pions need to be included in the Lagrangian due to the pions being massive, which is caused by approximate chiral symmetry breaking (m_u, $m_d \neq 0$) [69].

$$\mathcal{L}^2 = \frac{f_\pi^2}{4} \mathrm{Tr}\left[\partial_\mu U \partial^\mu U^\dagger + m_\pi^2 (U + U^\dagger)\right] \ , \tag{29}$$

where $m_\pi^2 = B_0(m_u + m_d)$, which is obtained from the Gell-Mann-Okubo relation and B_0 is related to the chiral quark condensate as $3f_\pi^2 B_0^2 = -\langle \bar{q}q \rangle$ [70].
In this thesis, the lowest order of effective Lagrangian $\mathcal{L}^2$ will suffice since $\mathcal{L}^4$ contributes to the matrix elements of interest here only at Next-to-Next-Leading order (N^{2}LO) which is already beyond Next-to-Leading order (NLO). So, we have now traded the fundamental theory of QCD for an Effective Field Theory (χPT), that explains the dynamics of QCD at low energy scales.

3.4 Heavy-quark effective theory (HQFT)

In this section, we will include the heavy quarks in our existing theory.
The Chiral Lagrangian derived in the earlier section can explain the dynamics of mesons made up of light quarks like pions and kaons, but not for heavy mesons which consist of charm and bottom quarks (as top quark does not hadronize). Since, the masses of the charm and bottom quarks are above Λ_{QCD}, we can study the charm and bottom physics in perturbation theory by expanding in Λ_{QCD}/m_c and Λ_{QCD}/m_b respectively.
Let us consider an example of a meson system $Q\bar{q}$, where Q is the heavy quark with $m_Q \gg \Lambda_{QCD}$ and $\bar{q}$ is the light quark with $m_q \ll \Lambda_{QCD}$, this heavy-light system has a size of the order Λ_{QCD}^{-1}. Now in the $m_Q \to \infty$ limit, the change in velocity of the heavy meson due to the interactions of the light quarks is negligible, since $\Delta v = \Delta p/m_Q$ [71]. Hence, the heavy quark acts as a static color source [72] and the meson dynamics can be reduced to the light quarks interacting with this color source. It is evident right away that in the $m_Q \to \infty$ limit [73], the mass of the heavy quark has no bearing at all, meaning that all heavy quarks interact within the heavy mesons in the identical way, which leads us to **heavy quark flavor symmetry (HQFS)** [72]. We can also deduce that, the heavy quark flavor symmetry breaking occurs at the level of $(1/m_{Q_i} - 1/m_{Q_j})$, where Q_i and Q_j are two different heavy quark flavors. Also in the same limit, the static heavy quark can only interact through gluons which is spin-independent, due to spin dependent interactions scaling as Λ_{QCD}/m_Q, leading to **heavy quark spin symmetry (HQSS)**. In the leading order of Λ_{QCD}/m_b, due to heavy quark spin symmetry, we find that the B and B^* meson are degenerate.
To derive an effective theory with the inclusion of heavy quarks, we decompose the 4-momenta of the heavy meson as [74, 75]

$$p^\mu = m_Q v^\mu + k^\mu \ , \tag{30}$$

where v^μ is the 4-velocity of the heavy quark defined as $v^\mu = (1,0,0,0)$, with $v^2 = 1$ and k^μ is the small residual momentum with $k^\mu \ll m_Q$. The light quarks typically have a momenta of $k^\mu \sim \Lambda_{QCD}$ in the meson and hence the change in velocity v^μ of the heavy quark is of the order of Λ_{QCD}/m_Q. From this v^μ can act as a conserved quantum number for heavy quark up to an order of m_Q^{-1} [74]. The momentum of the light quarks is the largest dynamical scale within this framework and we usually denote it as p_{typ}, thus we can expand our heavy meson fields in p_{typ}/m_Q while applying a non-relativistic treatment of the heavy quark and neglecting terms of the first order $\mathcal{O}(p_{typ}/m_Q)$.
A suitable technique is to treat the multiplet of degenerate states (such as B and B^*) as a single object that transforms linearly under heavy quark symmetry. This is done through **Superfields**, which is a linear combination of the physical B and B^* fields, expressed as [76, 77]

$$H_a^{(\bar{Q})} = \frac{(1+\not{v})}{2}(B^*_{a\mu}\gamma^\mu - B_a\gamma_5) \; , \tag{31}$$

where $H_a^{(\bar{Q})}$ annihilates a superfield with velocity v, $B^*_{a\mu}$ is a field operator that annihilates a B^*_a meson with four-velocity v and B_a is a field operator that annihilates a B_a meson with four-velocity v. Here, B_a is a pseudoscalar field that transforms as a spin singlet on the other hand, $B^*_{a\mu}$ is a vector field that transforms as a spin triplet and carries spin λ. $H_a^{(\bar{Q})}$ contains the heavy antiquark $\bar{Q}$, which is the bottom antiquark $\bar{b}$ in our case. Analogously, the conjugate field [78] is

$$\bar{H}_b^{(\bar{Q})} = \gamma^0 H_b^{(\bar{Q})\dagger}\gamma^0 = (B^{*\dagger}_{b\mu}\gamma^\mu + B_b^\dagger\gamma_5)\frac{(1+\not{v})}{2} \; , \tag{32}$$

where $\bar{H}_b^{(\bar{Q})}$ creates a superfield with velocity v, $B^{*\dagger}_{b\mu}$ is a field operator that creates a B^*_b meson with four-velocity v and $B_b^\dagger$ is a field operator that creates a B_b meson with four-velocity v. Similarly, the superfield for the corresponding anti-B-mesons are

$$H_a^{(Q)} = (B^*_{a\mu}\gamma^\mu - B_a\gamma_5)\frac{(1-\not{v})}{2} \; . \tag{33}$$

Here, $H_a^{(Q)}$ contains the bottom quark b and the conjugate field is

$$\bar{H}_b^{(Q)} = \frac{(1-\not{v})}{2}(B^{*\dagger}_{b\mu}\gamma^\mu + B_b^\dagger\gamma_5) \; . \tag{34}$$

We have also used the projection operator $P_\pm$ which retains only the particle components of the heavy quark Q [79] and are defined as

$$P_\pm = \frac{1 \pm \not{v}}{2} \tag{35}$$

and they follow the relations, $P_+ + P_- = 1$, $P_+P_- = P_-P_+ = 0$ and $P_\pm^2 = P_\pm$.
But in this thesis, a two-component notation (2×2 matrix) [49, 80] instead of a four-component notation (4×4 matrix) is used since it is simpler and preserves the existing symmetries [80].

$$H_a = B_a + B_a^{*\,i}\sigma^i \tag{36}$$

and the conjugate field

$$H_b^\dagger = B_b^\dagger + (B_b^{*i})^\dagger\sigma^i \ , \tag{37}$$

where σ^i are Pauli matrices with index $i = 1, 2, 3$. The superfields for the anti-B-mesons are provided in Sec. 4.
One can start by constructing the terms of the Lagrangian in velocity dependent quark fields $Q_v(x)$ which is related to original quark fields $Q(x)$ as [81]

$$Q(x) = e^{im_Q v\cdot x}[Q_v(x) + \mathcal{Q}_v(x)] \ , \tag{38}$$

where

$$Q_v(x) = e^{im_Q v\cdot x}\frac{(1+\not{v})}{2}Q(x), \qquad \mathcal{Q}_v(x) = e^{im_Q v\cdot x}\frac{(1-\not{v})}{2}Q(x) \ , \tag{39}$$

$Q_v(x)$ produces effects in leading order and $\mathcal{Q}_(x)$ effects are suppressed by powers of $1/m_Q$ hence we can ignore them in this discussion. We substitute this relation into the heavy quark field part of the QCD Lagrangian $\bar{Q}(i\not{D} - m_Q)Q$ and multiplying $(1+\not{v})/2$ on either side of $\not{D}$ gives

$$\mathcal{L} = \bar{Q}_v(iv\cdot D)Q_v \tag{40}$$

and the Q_v propagator from the Lagrangian [73, 82] is

$$\left(\frac{1+\not{v}}{2}\right)\frac{i}{v\cdot k + i\epsilon} \ , \tag{41}$$

we can arrive at the same propagator by defining the momentum of the heavy quark as $p^\mu = m_Q v^\mu + k^\mu$, applying this definition of p^μ in Dirac quark propagator

$$i\frac{\not{p} + m_Q}{p^2 - m_Q^2 + i\epsilon} = i\frac{m_Q\not{v} + m_Q + \not{k}}{2m_Q v\cdot k + k^2 + i\epsilon} \to i\frac{1+\not{v}}{2v\cdot k + i\epsilon} \tag{42}$$

and ignoring k^2 terms then it is equivalent to Eq. (41).
The polarisation vectors of the heavy vector meson fields follow the convention $\epsilon_\mu^*(\lambda)\epsilon^\mu(\lambda) = -1$ and $\epsilon\cdot v = 0$ for a given spin λ, where ϵ and ϵ^* describes the polarisation of an incoming meson and an outgoing meson, respectively. The polarization vectors may be broken down to $\vec{\epsilon}^{\,*}\cdot\vec{\epsilon} = 1$ since heavy mesons are treated as static and the contribution of the 0-th component is of the order $\mathcal{O}(\vec{k}^{\,2}/(2m_H))$.

When adding up all possible spins in the case of heavy internal mesons, one discovers

$$\sum_\lambda \vec{\epsilon}^{\,*}_\mu(\lambda)\vec{\epsilon}_\nu(\lambda) = -g_{\mu\nu} + \frac{p_\mu p_\nu}{m_H^2} = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} + \mathcal{O}\left(\frac{\vec{k}^{\,2}}{m_H^2}\right) \to \epsilon_i^*(\lambda)\epsilon_j(\lambda) = \delta_{ij} \ . \tag{43}$$

The polarization vectors in this thesis will be labeled as 1 for incoming particles and $1'$ for outgoing ones. The anti-mesons shall be indicated as 2 when incoming and $2'$ when outgoing.

3.5 Heavy meson chiral perturbation theory (HMχPT)

HMχPT can be described using the spontaneous symmetry breaking of $SU(3)_V \times SU(3)_A \times U(1)_V \times U(1)_A$ symmetry for light quarks and the conservation of spin-flavor symmetry of the heavy quarks. The Lagrangian for HMχPT should describe the strong interactions of the heavy mesons (in our case B and B^*) with low momentum pseudo-Goldstone bosons (pions in our case), it should preserve all the chiral and heavy quark symmetries and contain the least number of derivatives at leading order with the inclusion of light quark matrix. The superfields H_a which contain B and B^* are referred to as matter fields.
For the construction of a chiral Lagrangian, we use H_a fields that preserve the $SU(3)_L \times SU(3)_R$ chiral symmetry as [81]

$$H_a \to H_b K^\dagger_{ba} \ , \tag{44}$$

where $K^\dagger_{ba}$ is a 3×3 special unitary matrix which is a complicated nonlinear function of L, R and pseudo-Goldstone-boson field $U(x)$ and the repeated index b is summed over 1,2,3. Similarly, for the light mesons interactions, it is convenient to introduce [83]

$$u(x) = \sqrt{U(x)} \ , \tag{45}$$

which under chiral $SU(3)_L \times SU(3)_R$ transformation [66, 76]

$$u \to LuK^\dagger = KuR^\dagger \ . \tag{46}$$

Hence, we see that u transforms using L, R and K whereas H_a transforms using only K. In the chiral Lagrangian, it is better to have combinations of $u(x)$ such that the transformations only depend on $K(x)$. Two such combinations are [76]

$$D^\mu = \partial^\mu \delta_{ab} + i\Gamma^\mu_{ba} \ , \text{ with } \Gamma_\mu = \frac{1}{2}(u^\dagger \partial_\mu u + u\partial_\mu u^\dagger), \tag{47}$$

and

$$u_\mu = i(u^\dagger \partial_\mu u - u\partial_\mu u^\dagger), \tag{48}$$

where u_μ has the quantum numbers of a axial-vector field and Γ_μ has the quantum numbers of a vector field. The leading effective Lagrangian containing the interactions of the pseudo-Goldstone bosons with the heavy mesons having just one derivative is [81, 84]

$$\mathcal{L}^{(1)} = -i\mathrm{Tr}[\bar{H}_a\, v_\mu \partial^\mu\, H_a] + \mathrm{Tr}[\bar{H}_a H_b] v_\mu \Gamma^\mu_{ba} + g_\pi \mathrm{Tr}[\bar{H}_a H_b \gamma_\mu \gamma_5] u^\mu_{ba} \ , \tag{49}$$

which can be more simplified using a chiral covariant derivative D^μ

$$\mathcal{L}^{(1)} = -i\mathrm{Tr}[\bar{H}_a\, v_\mu D^\mu\, H_b] + g_\pi \mathrm{Tr}[\bar{H}_a H_b \gamma_\mu \gamma_5] u^\mu_{ba} \ , \tag{50}$$

where $\mathcal{L}^{(1)}$ denotes the leading order term for the interactions of heavy mesons with pseudo-Goldstone bosons, where the trace is taken over spin space and $v_\mu = (1, 0, 0, 0)$. The first term in Eq. (49) denotes the kinetic term of heavy mesons, the second and the third term denotes the vertices of heavy mesons with the pseudo-Goldstone bosons. The last term has a factor of g_π which indicates the coupling strength of the heavy mesons to the axial-vector field u^μ_{ba} [85], where the value of g_π is given in Sec. 4.
Using the expansion of $u = 1 + \frac{i\phi}{2f_\pi}$ and similarly $u^\dagger$, in Γ^μ_{ba} and u^μ_{ba}

$$\Gamma^\mu_{ba} = \frac{i}{4f_\pi^2}\,(\vec{\pi} \times \partial^\mu \vec{\pi}) \cdot \vec{\tau}_{ba} + \mathcal{O}(\vec{\pi}^{\,4}) \ , \tag{51}$$

$$u^\mu_{ba} = \frac{-1}{f_\pi}(\partial^\mu \vec{\pi} \cdot \vec{\tau}_{ba}) + \mathcal{O}(\vec{\pi}^{\,3}) \ . \tag{52}$$

The expansion of Γ^μ_{ba} gives us an even number of pseudo-Goldstone bosons and the expansion of u^μ_{ba} gives us an odd number of pseudo-Goldstone bosons [86, 87]. The Lagrangian we have discussed only contains long-range components or terms where the heavy mesons interact with pseudo-Goldstone boson, but the Lagrangian should also contain short-range interactions which are the **Contact interactions** (**CI**). These short range interactions which do not include pseudo-Goldstone bosons are seen as contact-range operators and they act as counter-terms in our theory.
Thus, the HMχPT Lagrangian of relevance for this work reads [88]

$$\mathcal{L}^{(1)} = \mathcal{L}^{(0)}_{4H} + \mathcal{L}^{(1)}_{\pi HH} \ , \tag{53}$$

where the second term $\mathcal{L}^{(1)}_{\pi HH}$ denotes the long-range part which we derived earlier and the $\mathcal{L}^{(0)}_{4H}$ denotes vertices with four heavy mesons, an explicit form the $\mathcal{L}^{(0)}_{4H}$ is given in [88].
In contrast to the Lagrangian mentioned in Eq. (50) which uses a four-component notation, this thesis will use a two-component Lagrangian with many counter terms, as seen in Eq. (88).

3.6 Power counting

The Weinberg conjecture says an EFT Lagrangian will have an infinite number of terms with an infinite number of free parameters, which is impractical to use in any case [65]. Hence, to make it practical to use, we first need a scheme to arrange the effective Lagrangian and an efficient method to assess the "weight" or importance of the different diagrams which come from the interaction terms in the effective Lagrangian.
To make a successful power counting scheme, we need an expansion parameter $\chi < 1$, such that a sum of all contributions converges sufficiently fast. In our case, we take $\chi = p_{typ}/\Lambda_\chi$, where p_{typ} is the typical momentum and Λ_χ is the hadronic scale (~ 1 GeV) where χPT fails. We calculate the chiral dimension of the diagrams using the rules provided by Hanhart [89]. In order to find the chiral dimension of a loop diagram, we need to replace every piece that appears in the potential by its value when all momenta are of their typical size.
The rules are

- the integral measure scales as
$$\int \frac{d^4l}{(2\pi)^4} \sim \mathcal{O}\left(\frac{p_0 p_{typ}^3}{(4\pi)^2}\right) .$$
- The momenta at the vertices scale as $\mathcal{O}(p_{typ})$.
- The pion propagator scale as $\mathcal{O}(1/p_{typ}^2)$.

and $p_0 \sim p_{typ}$ for irreducible diagrams which will be discussed later.
With these rules for power counting, we can discuss the power counting scheme which will be used in this thesis. The power counting of the pion loops in the B-sector is dependent on the heavy and light scales with the heavy scale consisting of the average mass of the B meson ($M_B = 5314$ MeV) and the typical hadronic scale ($\Lambda_\chi = 1$ GeV) and the light scale consisting of the mass difference of M_B and M_{B^*} ($\delta \approx 45$ MeV), the pion mass (m_π= 139 MeV) and the typical momentum ($p_{typ} \approx \sqrt{M_B\delta}$ = 490MeV). The LS equation's solution generates the binding energies of the Z_b states dynamically, therefore the power counting of the potentials does not require their consideration. As we are dealing with $Z_b(10610)$ and $Z_b(10650)$, we in turn require a power counting scheme that covers the energy range from the $B\bar{B}$ threshold up to the $B^*\bar{B}^*$ (energy range of $\approx$ 90 MeV) and therefore need to treat p_{typ} dynamically. For an accurate prediction, we need to keep track of the momentum scales which drive the loop contributions, then the expansion parameters are

$$\chi_1 = \frac{p_{\rm typ}}{\Lambda_\chi} \ , \ \chi_2 = \frac{m_\pi}{\Lambda_\chi} \ , \ \chi_3 = \frac{p_{\rm typ}}{M_B} \ , \ \chi_4 = \frac{\delta}{\Lambda_\chi} \ , \tag{54}$$

which numerically take values of about 1/2, 1/7 , 1/10 and 1/20 respectively. Since in the charm system the mass splitting between the pseudoscalar and vector ground state mesons are of the order of the pion mass and to keep the scheme simple we treat these expansion parameters as a one-parameter expansion

$$\chi \sim \chi_1 \ , \ \chi^2 \sim \chi_2 \sim \chi_3 \sim \chi_4 \ . \tag{55}$$

The consequences of using this power counting can be seen in two examples, namely two-pion exchange (2PE) triangle contribution to the $B\bar{B} \to B\bar{B}$ potential and the one loop vertex correction, as seen in Fig. 4.

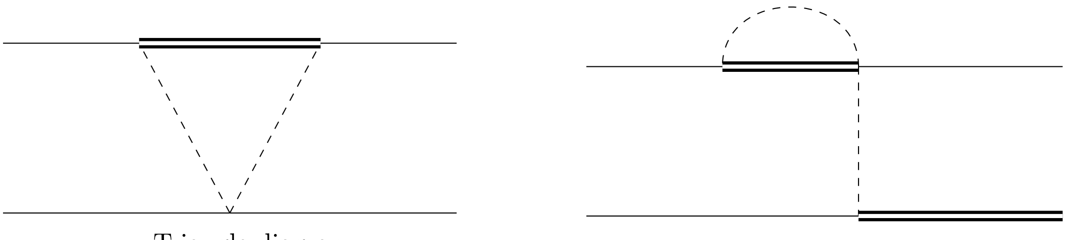


Figure 4: Typical one loop diagrams that appear at NLO (left panel) and NNLO (right panel) in the momentum expansion.

The effective potential for the triangle diagram is written as

$$V_{tr} = \sum_{\lambda} \int \frac{d^4 l}{(2\pi)^4} \frac{1}{4f_\pi^2} \Big((2l_0 + q_0) \epsilon_{cdh} (\tau_1^c)_h \Big) \frac{i}{2(E_4 - l_0 - M_{B^*} - (\vec{p'} + \vec{l})^2/(2M_{B^*}))}$$
$$\left(\frac{g}{2f_\pi} \Big(\epsilon_j(\lambda)(-l-q)_j \Big) (\tau_2)_d \right) \frac{i}{(l+q)^2 - m_\pi^2} \left(\frac{g}{2f_\pi} \Big(\epsilon_i^*(\lambda)(l)_i \Big) (\tau_2)_c \right) \frac{i}{l^2 - m_\pi^2} \ , \tag{56}$$

where

$$q = (E_3, \vec{p'}) - (E_1, \vec{p}) = (0, \vec{q}) \tag{57}$$

and

$$k = (E_4, \vec{p'} - l) \tag{58}$$

with $\vec{p}$ and $\vec{p'}$ as initial and final momenta, E_1 (E_2) and E_3 (E_4) are the energies of the meson (anti-meson) in the initial and final states respectively and $q \sim p_{typ.}$. The external states are assumed to be on their mass shell and the total energy to be at the $B^*\bar{B}^*$ threshold for the purpose of this discussion. Using

$$\epsilon_{cdh} (\tau_1)_h (\tau_2)_d (\tau_2)_c = -2i(\vec{\tau_1} \cdot \vec{\tau_2}) \tag{59}$$

and

$$\sum_{\lambda} \epsilon_i^*(\lambda)\epsilon_j(\lambda) = \delta_{ij} \ , \tag{60}$$

we get

$$V_T = \frac{g^2}{8f_\pi^4}(\vec{\tau_1}\cdot\vec{\tau_2})I_{tr} \ , \tag{61}$$

where the pertinent integral is given by

$$I_{tr} = \frac{i}{2}\int \frac{d^4l}{(2\pi)^4}\left(\frac{2l_0+q_0}{l_0+\delta+(2\vec{p}'\vec{l}+\vec{l}^2)/(2M_{B^*})}\right) \times \frac{(\vec{l}+\vec{q})\cdot\vec{l}}{\left[(l+q)^2-m_\pi^2+i\epsilon\right]\left[l^2-m_\pi^2+i\epsilon\right]} \ . \tag{62}$$

The mentioned scales enter the integral through $|\vec{q}| \sim |\vec{p}'|$, and the energy of the pion propagator l_0 scale as $l_0 \sim p_{typ}$ [89]. Due to this, we can drop all terms in the heavy-meson propagator except l_0, since all other terms appear to be suppressed as δ/p_{typ} or p_{typ}/M_B which are both counted equally. Then the integral to be evaluated is

$$I_{tr} = \int \frac{d^4l}{(2\pi)^4}\frac{(\vec{l}+\vec{q})\cdot\vec{l}}{\left[(l+q)^2-m_\pi^2+i\epsilon\right]\left[l^2-m_\pi^2+i\epsilon\right]} \ . \tag{63}$$

In the two scale expansion used here, a particular loop simultaneously contributes in different orders [89, 90]. However, the power counting simply shows us the lowest order at which the loop begins to contribute. As an example to clarify the above statement, solving the integral in Eq. (63) using dimensional regularisation, one finds

$$\begin{aligned} I_{tr} = -\frac{1}{16\pi^2}\Bigg\{&\left(\frac{5}{12}\vec{q}^{\,2}+\frac{3}{2}m_\pi^2\right)\mathcal{R}-\frac{13}{36}\vec{q}^{\,2}-\frac{m_\pi^2}{3}+\left(\frac{5}{6}\vec{q}^{\,2}+3m_\pi^2\right)\ln\left(\frac{m_\pi}{\mu}\right) \\ &+\left(\frac{5}{6}\vec{q}^{\,2}+\frac{4}{3}m_\pi^2\right)L(q)\Bigg\} \\ = -\frac{5\vec{q}^{\,2}}{96\pi^2}&\left\{\frac{\mathcal{R}}{2}-\frac{13}{30}+\ln\left(\frac{m_\pi}{\mu}\right)+L(q)\right\}+\mathcal{O}(\chi^4) \ , \end{aligned} \tag{64}$$

where R and $L(q)$ are given by

$$\mathcal{R} = -\frac{2}{\xi}+\gamma_E-1-\ln(4\pi) \tag{65}$$

and

$$L(q) = \frac{\sqrt{4m_\pi^2 + q^2}}{q} \ln\left(\frac{\sqrt{4m_\pi^2 + q^2} + q}{2m_\pi}\right). \tag{66}$$

Since the leading order for the scattering of two heavy particles appears at $\mathcal{O}(\chi^0)$ and there are no contributions at $\mathcal{O}(\chi)$, Eq. (64) has contributions at $\mathcal{O}(\chi^2)$. The explicit calculation of Eq. (63) is provided in Appendix A.1.

Therefore in the main part of this thesis, we will just present the lowest order contribution $\mathcal{O}(\chi^2)$ that comes from the loops and the loops containing higher order contributions $\mathcal{O}(\chi^4)$ will be mentioned in Appendices C and F.

In the vertex case, the pertinent loop integral in our scheme can be written as

$$I_{vert} = \int \frac{d^4l}{(2\pi)^4} \frac{\frac{\vec{l}^2}{D-1}}{\left[l_0 + i\epsilon\right]\left[l^2 - m_\pi^2 + i\epsilon\right]} , \tag{67}$$

where the $D = 4 - \epsilon$ is the number of dimensions and l_0 is energy of the pion propagator. For the vertex correction (the right panel in Fig. 4) the integration variable can always be chosen such that the pion propagator in the loop does not contain any external variable. Therefore, contrary to Eq. (63), the momentum $q \sim p_{\rm typ}$ does not enter the pion propagator in the loop. Hence we can set l_0 as m_π. At the same time, the momentum scale is also given by m_π. Including all these points, one can conclude that the vertex correction is suppressed as compared to the 2PE triangle contribution by a factor of $\chi^2 = (m_\pi/p_{typ})^2$ [91].

From this observation, all the vertex contributions are ignored in this thesis as they start to contribute only at order N^{3}LO. In Sec. 5.5.3, we will see if the aforementioned observation is true by comparing the triangle loop integral to the vertex correction integral. The explicit calculations of the vertices and potentials will presented later in this thesis.

3.7 Dimensional regularisation

Dimensional regularisation is a tool that allows one to isolate the divergence of integrals in a way that preserves all symmetries [92]. The degree of divergence of an integral can be estimated by counting the powers of momenta. To provide some examples, if the integral behaves asymptotically as $\int d^4l/l^2$ or $\int d^4l/l^4$, then it diverges quadratically or logarithmically respectively. We demonstrate the technique here, by computing the crossed box loop integral, $I^{(2)}_{box}$, and the planar box integral, $I^{(1)}_{box}$, encountered in 2PE $B\bar{B} \to B\bar{B}$ scattering, seen in Fig. 5.

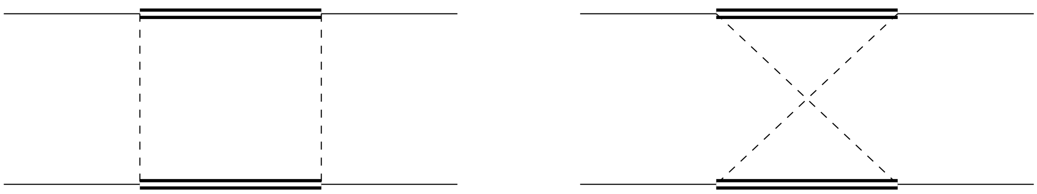


Figure 5: 2PE box diagrams for $B\bar{B} \to B\bar{B}$ scattering

The $I^{(2)}_{box}$ integral is given by

$$I^{(2)}_{box} = i \int \frac{d^4 l}{(2\pi)^4} \frac{(\vec{q_1} \cdot \vec{q_2})^2}{(l_0 - i\epsilon)^2 \left[q_2^2 - m_\pi^2 + i\epsilon\right] \left[q_1^2 - m_\pi^2 + i\epsilon\right]} , \tag{68}$$

where q_1 and q_2 are the momenta of the two pions and can be read from the Fig. 8 as $q_1{=}(l_0, \vec{p}{-}\vec{l})$, $q_2{=}(l_0, \vec{p'}{-}\vec{l})$ and $q{=}(0, \vec{p'}{-}\vec{p}){=}(0, \vec{q})$ with l as the loop momentum and ϵ as a small positive real number to be taken to zero at the end of the calculation. The terms q_1^2 and q_2^2 are expanded as $q_1^2 = (l^0)^2 - \vec{q_1}^2$ and $q_2^2 = (l^0)^2 - \vec{q_2}^2$, respectively. The two pion propagators are joined using Feynman parameters and then shifting $l \to l - qx$. All the odd powers of l are dropped as the integrand is antisymmetric under $l \to -l$ and when integrating over all l, the integral will vanish. Hence, only terms with even powers of k are kept in the numerator

$$I^{(2)}_{box} = \int_0^1 dx \int \frac{d^3 \vec{l}}{(2\pi)^3} (\vec{q_1} \cdot \vec{q_2})^2 \int \frac{dl_0}{2\pi} \frac{1}{(l_0 - i\epsilon)^2 \left[(l_0)^2 - (\vec{q_2}^2 - \vec{q_1}^2)x - \vec{q_1}^2 - m_\pi^2 + i\epsilon\right]^2} . \tag{69}$$

The l_0-integration can be performed using the residue theorem and setting $\epsilon \to 0$ as there is no longer a singularity in the integral

$$I^{(2)}_{box} = \frac{-3}{4} \int_0^1 dx \int \frac{d^3 \vec{l}}{(2\pi)^3} \frac{(\vec{q_1} \cdot \vec{q_2})^2}{\left[(\vec{q_2}^2 - \vec{q_1}^2)x + \vec{q_1}^2 + m_\pi^2\right]^{5/2}} . \tag{70}$$

Now, shift $\vec{l} \to \vec{l} + \vec{p}$ such that $\vec{q_1} = \vec{p} - \vec{l} \to -\vec{l}$ and $\vec{q_2} = \vec{p'} - \vec{l} \to -\vec{l} + \vec{q}$ with $\vec{q} = \vec{p'} - \vec{p}$:

$$I^{(2)}_{box} = \frac{-3}{4} \int_0^1 dx \int \frac{d^3 \vec{l}}{(2\pi)^3} \frac{\left(\vec{l} \cdot (\vec{l} - \vec{q})\right)^2}{\left[\vec{l}^2 + (-2\vec{l} \cdot \vec{q} + \vec{q}^{\,2})x + m_\pi^2\right]^{5/2}} . \tag{71}$$

In the denominator, complete the square by using $\vec{l} \to \vec{l} + \vec{q}x$. The above integral is divergent for $D = 4$ and convergent for $D < 4$. Therefore, one can generalise the four-dimensional integral as $4 \to D$, where $D = 4 - \xi$ with ξ being small positive real number. For consistency, we have introduced a dimensional parameter μ so that the integral has the same dimension for arbitrary D. Furthermore, the angular integration is performed in spherical coordinates using, $d^{D-1}\vec{l} = l^{D-2} dl\, d\Omega_{D-1}$, where $d\Omega_{D-1}$ is

$$\int d\Omega_{D-1} = \frac{2\pi^{(D-1)/2}}{\Gamma\left(\frac{D-1}{2}\right)} \ , \tag{72}$$

where $\Gamma(x)$ is the Gamma function, then

$$I^{(2)}_{box} = \frac{-3\sqrt{\pi}}{(4\pi)^{D/2}} \frac{\mu^{4-D}}{\Gamma(\frac{D-1}{2})} \int_0^1 dx$$
$$\int_0^\infty dl \frac{l^{D+2} + l^D \vec{q}^{\,2}(2\frac{D+1}{D-1}x^2 - 2\frac{D+1}{D-1}x + \frac{1}{D-1}) + l^{D-2}\big(\vec{q}^{\,2}x(1-x)\big)^2}{\big[\vec{l}^{\,2} + \vec{q}^{\,2}x(1-x) + m_\pi^2\big]^{5/2}} \ , \tag{73}$$

carrying out the l-integration, inserting $D = 4 - \xi$ in the limit of $\xi \to 0$ and expanding $a^\xi = 1 + \xi \cdot a + \mathcal{O}(\xi^2)$. Then

$$\begin{aligned} I^{(2)}_{box} = \frac{-1}{16\pi^2} \int_0^1 dx \Bigg\{ & \left(\left[-\frac{35}{2}x^2 + \frac{35}{2}x - 1 \right] \vec{q}^{\,2} + \frac{15}{2} m_\pi^2 \right) \mathcal{R} \\ & + (-22x^2 + 22x - 1)\vec{q}^{\,2} + 8m_\pi^2 + \frac{2\vec{q}^{\,4}x^2(1-x)^2}{m_\pi^2 + \vec{q}^{\,2}x(1-x)} \\ & + \left(\left[-\frac{35}{2}x^2 + \frac{35}{2}x - 1 \right] \vec{q}^{\,2} + \frac{15}{2} m_\pi^2 \right) \ln\left(\frac{\vec{q}^{\,2}x(1-x) + m_\pi^2}{\mu^2} \right) \Bigg\} \ , \end{aligned} \tag{74}$$

where $\mathcal{R}$ is the singularity of the loop and is given by

$$\mathcal{R} = -\frac{2}{\xi} + \gamma_E - 1 - \ln(4\pi) \tag{75}$$

and γ_E is the Euler-Mascheroni constant. The radial integration in Euclidean space was performed using

$$\int_0^\infty dl_E \frac{l_E^a}{[l_E^2 + \Xi]^b} = \Xi^{\frac{a+1}{2} - b} \frac{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(b - \frac{a+1}{2}\right)}{2\Gamma(b)} \ . \tag{76}$$

Finally, performing the x- integration:

$$I_{box}^{(2)} = \frac{-1}{16\pi^2}\Bigg\{\left(\frac{23}{12}\vec{q}^{\,2} + \frac{15}{2}m_\pi^2\right)\mathcal{R} + \frac{5}{36}\vec{q}^{\,2} + \frac{8}{3}m_\pi^2 + \left(\frac{23}{6}\vec{q}^{\,2} + 15m_\pi^2\right)\ln\left(\frac{m_\pi}{\mu}\right)$$
$$+ \left(\frac{23}{6}\vec{q}^{\,2} + \frac{10}{3}m_\pi^2 + \frac{8m_\pi^4}{4m_\pi^2 + \vec{q}^{\,2}}\right)L(q)\Bigg\}$$
$$= \frac{-23\vec{q}^{\,2}}{96\pi^2}\left\{\frac{\mathcal{R}}{2} + \frac{5}{138} + \ln\left(\frac{m_\pi}{\mu}\right) + L(q)\right\} + \mathcal{O}(\chi^4)\ , \quad (77)$$

where $L(q)$ is given by

$$L(q) = \frac{\sqrt{4m_\pi^2 + q^2}}{q}\ln\left(\frac{\sqrt{4m_\pi^2 + q^2} + q}{2m_\pi}\right)\ . \quad (78)$$

Next, we will look into the planar box integral $I_{box}^{(1)}$ in the $B\bar{B} \to B\bar{B}$ scattering. The integral splits into repeated 1PEs and a 2PE contribution. Since the Lippmann-Schwinger equation (LSE) takes care of iterated diagrams, the double 1PE needs to be removed from the potential and one finds that the remaining integral agrees with the crossed box integral that was evaluated above [93].

$$I_{box}^{(1)} = i\int \frac{d^4l}{(2\pi)^4}\frac{1}{\left[l_0 + i\epsilon\right]\left[l_0 - i\epsilon\right]}\frac{(\vec{q_2}\cdot\vec{q_1})^2}{\left[q_2^2 - m_\pi^2 + i\epsilon\right]\left[q_1^2 - m_\pi^2 + i\epsilon\right]}\ . \quad (79)$$

Like in the earlier case, we expand $q_1^2 = (l_0)^2 - \vec{q_1}^{\,2}$ and $q_2^2 = (l_0)^2 - \vec{q_2}^{\,2}$ and use Feynman parameters

$$I_{box}^{(1)} = i\int_0^1 dx \int \frac{d^3\vec{l}}{(2\pi)^3}(\vec{q_1}\cdot\vec{q_2})^2 \int \frac{dl^0}{2\pi}\frac{1}{\left[l^0 + i\epsilon\right]\left[l^0 - i\epsilon\right]}\frac{1}{\left[(l_0)^2 - a^2 + i\epsilon\right]^2}\ , \quad (80)$$

where $a^2 = (\vec{q_2}^{\,2} - \vec{q_1}^{\,2})x + \vec{q_1}^{\,2} + m_\pi^2$ and x is the Feynman parameter.
The l^0-integration diverges as $\epsilon \to 0$. This is due to the contour of integration γ being squeezed between the two poles at $l_0 = \pm i\epsilon$ leading to a pinch singularity [94], which is seen in Fig. 6. In contrast, both poles of the crossed box integral are on the same side of the integration contour. The pinch singularity can be avoided by including a second order term, which shifts the pole locations and evades the singularity

$$i\epsilon \to i\zeta = \frac{\vec{p}^{\,2}}{2M_B} - \frac{\vec{l}^{\,2}}{2M_B} + i\epsilon\ . \quad (81)$$

Now, replace $i\epsilon \to i\zeta$,

$$I_{box}^{(1)} = i \int_0^1 dx \int \frac{d^3\vec{l}}{(2\pi)^3} (\vec{q_1} \cdot \vec{q_2})^2 \int \frac{dl_0}{2\pi} \frac{1}{\left[l_0 + i\zeta\right]\left[l_0 - i\zeta\right]} \frac{1}{\left[(l_0)^2 - a^2 + i\epsilon\right]^2} \; . \tag{82}$$

Like in the previous case, we perform the l_0- integration using the residue theorem and setting $\epsilon \to 0$, one finds

$$I_{box}^{(1)} = i \int_0^1 dx \int \frac{d^3\vec{l}}{(2\pi)^3} (\vec{q_1} \cdot \vec{q_2})^2 \frac{i}{4} \frac{2\,a - i\zeta}{\left[i\zeta a^3 (a - i\zeta)^2\right]} \; . \tag{83}$$

Expand the fraction for $a \gg i\zeta$, and terms of order $\mathcal{O}(\zeta)$ are dropped.

$$I_{box}^{(1)} = i \int_0^1 dx \int \frac{d^3\vec{l}}{(2\pi)^3} (\vec{q_1} \cdot \vec{q_2})^2 \frac{i}{4} \left(\frac{3}{a^5} + \frac{2}{i\zeta\, a^4} + \mathcal{O}(\zeta) \right) \; . \tag{84}$$

The first term of the expansion matches the integral from the crossed box $I_{box}^{(2)}$ as claimed above. This is the irreducible contribution from $I_{box}^{(1)}$. The dropped terms scale as $i\zeta/a^6$. Since $i\zeta \sim p_{typ}^2/m_B$ and $a^2 \sim p_{typ}^2$, the neglected term is suppressed by a factor of $\delta/p_{\rm typ} = \chi^2$ in comparison to $I_{box}^{(2)}$.
To understand the second term, insert $i\zeta = (\vec{p}^{\,2} - \vec{l}^{\,2})/2M_B$ to find

$$I_{box}^{(1)} \;=\; I_{box}^{(2)} \;+\; i^2 \int_0^1 dx \int \frac{d^3\vec{l}}{(2\pi)^3} \frac{(\vec{q_2} \cdot \vec{q_1})^2}{\left[\vec{p}^{\,2} - \vec{l}^{\,2}\right]\left[(\vec{q_2}^{\,2} - \vec{q_1}^{\,2})x + \vec{q_1}^{\,2} + m_\pi^2\right]} \quad , \tag{85}$$

undoing the Feynman parameters, one finds

$$I_{box}^{(1)} - I_{box}^{(2)} = i^2 \int_0^1 dx \int \frac{d^3\vec{l}}{(2\pi)^3} \frac{(\vec{q_2} \cdot \vec{q_1})^2}{\left[\vec{p}^{\,2} - \vec{l}^{\,2}\right]\left[\vec{q_2}^{\,2} + m_\pi^2\right]\left[\vec{q_1}^{\,2} + m_\pi^2\right]} \; , \tag{86}$$

which is the non-relativistic version of double 1-pion exchange [95] and the reducible contribution from $I_{box}^{(1)}$. Therefore, the iterated potential can be written as

$$V_{2PE,it}^{eff}(\vec{p'}, \vec{p}) = - \int_0^1 dx \int \frac{d^3\vec{l}}{(2\pi)^3} \frac{V_{1PE}^{eff}(\vec{p'}, \vec{l})\, V_{1PE}^{eff}(\vec{l}, \vec{p})}{\left[\vec{p}^{\,2} - \vec{l}^{\,2} + i\epsilon\right]} \; . \tag{87}$$

As a result of this exercise, we can replace $I_{box}^{(1)}$ by its irreducible part which is $I_{box}^{(2)}$ in the potential.

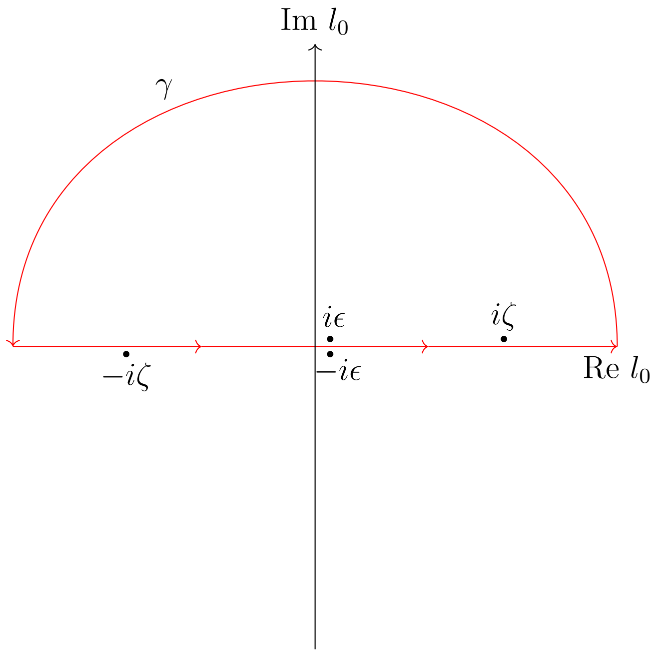


Figure 6: The contour of integration γ (shown in red) for $l_0 = \pm i\epsilon$ and $l_0 = \pm i\zeta$ is shown here. γ is pinched between the two poles at $l_0 = \pm i\epsilon$. Pinch singularity is avoided for $l_0 = \pm i\zeta$.

4 Lagrangian and vertices

The effective Lagrangian describing $B^{(*)}\ \bar{B}^{(*)}$ scattering at low energies reads [8, 9] reads:

$$
\begin{aligned}
\mathcal{L} = {} & \mathrm{Tr}[H_a^\dagger\,(iD_0)_{ba}\,H_b] + \frac{\delta}{4}\mathrm{Tr}[H_a^\dagger\,\sigma^i\,H_a\,\sigma^i] + \mathrm{Tr}[\bar{H}_a^\dagger\,(iD_0)_{ab}\,\bar{H}_b] + \frac{\delta}{4}[\bar{H}_a^\dagger\,\sigma^i\,\bar{H}_b\sigma] \\
& - \frac{g_Q}{2}\mathrm{Tr}[\vec{\sigma}\cdot\vec{u}_{ab}\,H_a^\dagger H_b] + \frac{g_Q}{2}\mathrm{Tr}[\bar{H}_a\bar{H}_b^\dagger\,\vec{\sigma}\cdot\vec{u}_{ab}] - \frac{C_{10}}{8}\mathrm{Tr}[\bar{H}_a^\dagger\,\tau^A_{aa'}H^\dagger_{a'}H_b\tau^A_{bb'}\bar{H}_{b'}] \\
& - \frac{C_{11}}{8}\mathrm{Tr}[\bar{H}_a^\dagger\,\tau^A_{aa'}\sigma^i H^\dagger_{a'}H_b\tau^A_{bb'}\sigma^i\bar{H}_{b'}] \\
& - \frac{D_{10}}{8}\Big\{\mathrm{Tr}[\nabla^i\bar{H}_a^\dagger\,\tau^A_{aa'}\nabla^i H^\dagger_{a'}H_b\tau^A_{bb'}\,\bar{H}_{b'}] + \mathrm{Tr}[\bar{H}_a^\dagger\,\tau^A_{aa'}H^\dagger_{a'}\nabla^i H_b\tau^A_{bb'}\nabla^i\bar{H}_{b'}]\Big\} \\
& - \frac{D_{11}}{8}\Big\{\mathrm{Tr}[\nabla^i\bar{H}_a^\dagger\,\tau^A_{aa'}\sigma^j\nabla^i H^\dagger_{a'}H_b\tau^A_{bb'}\,\sigma^j\bar{H}_{b'}] + \mathrm{Tr}[\bar{H}_a^\dagger\,\tau^A_{aa'}\,\sigma^j H^\dagger_{a'}\nabla^i H_b\tau^A_{bb'}\,\sigma^j\nabla^i\bar{H}_{b'}]\Big\} \\
& - \frac{D_{12}}{8}\Bigg\{\mathrm{Tr}\Bigg[\Big(\nabla^i\bar{H}_a^\dagger\,\tau^A_{aa'}\,\sigma^i\nabla^j H^\dagger_{a'} + \nabla^j\bar{H}_a^\dagger\,\tau^A_{aa'}\,\sigma^i\nabla^i H^\dagger_{a'} - \frac{2}{3}\delta^{ij}\nabla^k\bar{H}_a^\dagger\,\tau^A_{aa'}\sigma^i\nabla^k H^\dagger_{a'}\Big) \\
& \times H_b\tau^A_{bb'}\,\sigma^j\bar{H}_{b'}\Bigg] + \mathrm{Tr}\Bigg[\bar{H}_a^\dagger\,\tau^A_{aa'}\,\sigma^i H^\dagger_{a'}\Big(\nabla^i H_b\tau^A_{bb'}\,\sigma^j\nabla^j\bar{H}_{b'} + \nabla^j H_b\tau^A_{bb'}\,\sigma^j\nabla^i\bar{H}_{b'} \\
& - \frac{2}{3}\delta^{ij}\nabla^k H_b\tau^A_{bb'}\,\sigma^j\nabla^k\bar{H}_{b'}\Big)\Bigg]\Bigg\}\ ,
\end{aligned}
\tag{88}
$$

where a and b are isospin indices and σ's and τ's are the spin and isospin Pauli matrices, respectively. The isospin and spin matrices are normalized as $\tau^A_{ab}\tau^B_{ba} = 2\delta^{AB}$ and the trace is taken over spin space. The terms proportional to $\delta = M_{B^*} - M_B \approx 45$ MeV are the leading terms that violate spin symmetry and by our power counting scheme δ scales as $\delta \to \vec{p}^{\,2}/M_B$, hence they can be neglected here. They only affect the two-body propagators in the Lippmann-Schwinger equation (LS-equation), not the loops in the potentials at the order we are working (NLO in the chiral expansion and in the heavy quark expansion).
The H_a and $\bar{H}_a$ are super-fields that contain the $B^{(*)}$ and $\bar{B}^{(*)}$ fields, respectively, with $H_a = B_a + B_a^{*\,i}\sigma^i$ and $\bar{H}_a = (\bar{B}\tau_2)_a - (\bar{B}^{*\,i}\tau_2)_a\sigma^i$, where $B_a(\bar{B}_a)$ and $B_a^{*\,i}(\bar{B}_a^{*\,i})$ are the pseudoscalar and vector B mesons, respectively. The τ_2, acting as the charge conjugation matrix in isospin space, appears in the expressions for the anti-B-mesons, since they contain light antiquarks. H_1 contains B^0 and $(B^0)^*$ and H_2 contains B^+ and $(B^+)^*$, while $\bar{H}_1$ and $\bar{H}_2$ contain the respective antiparticles [96].
The zeroth component of the chiral covariant derivative is given by $D_0 = \partial_0 + \Gamma_0$ with

$$
\Gamma_0 = \frac{i}{4f_\pi^2}\,(\vec{\pi}\times\partial_0\vec{\pi})\cdot\vec{\tau} + \mathcal{O}(\vec{\pi}^{\,4})\ . \tag{89}
$$

The spatial components of the axial current read

$$
\vec{u} = -\partial_i(\vec{\tau}\cdot\vec{\pi})/f_\pi + \mathcal{O}(\vec{\pi}^{\,3})\ . \tag{90}
$$

In both cases $\vec{\tau}$ and $\vec{\pi}$ are 3-dimensional vectors made off the Pauli matrices and the pions fields, respectively (given in Eq. (26)).
Employing heavy quark spin symmetry, the pion-heavy meson coupling constant is set to

$$g_Q \approx g_b \approx g_c \approx g = 0.57 \ ,$$

extracted from the partial decay width $D^* \to D\pi$ provided in Ref. [33] (this value agrees within 10% with that extracted in lattice QCD for static sources [97]).
The terms proportional to C_{10} and C_{11} corresponds to the $\mathcal{O}(p^0)$ S-wave contact interactions, whereas the terms proportional to D_{10} and D_{11} corresponds to $\mathcal{O}(p^2)$ S-wave contact interactions. The term D_{12} gives rise to S-D transitions — this is the counter term formally appearing at NLO, however, promoted to LO as detailed in [8, 9]. As we are only interested in S-S and S-D transitions, terms proportional to $\nabla^i H^\dagger \nabla^j H$, leading to P-wave interactions, are ignored [8]. Following Ref. [9], we define the following linear combinations

$$\begin{aligned}
\mathcal{C}_d &= \frac{1}{8}(C_{11}+C_{10}), & \mathcal{C}_f &= \frac{1}{8}(C_{11}-C_{10}), \\
\mathcal{D}_d &= \frac{1}{8}(D_{11}+D_{10}), & \mathcal{D}_f &= \frac{1}{8}(D_{11}-D_{10}), \\
\mathcal{D}_{SD} &= \frac{2\sqrt{2}}{3} D_{12} \ , & &
\end{aligned} \tag{91}$$

where the sub-index d and f label diagonal and off-diagonal terms, respectively. From the Lagrangian provided in Eq. (88), we now derive the vertex structures relevant for this thesis.

4.1 Pion-emission vertex

The Lagrangian term for the pion-emission vertex is given by

$$\mathcal{L}_{B^{(*)}\to B^{(*)}\pi} = -\frac{g}{2}\mathrm{Tr}[\vec{\sigma}\cdot\vec{u}_{ab}\, H_a^\dagger H_b] \ . \tag{92}$$

Using the definition of H_a

$$\mathcal{L}_{B^{(*)}\to B^{(*)}\pi} = -\frac{g}{4}\mathrm{Tr}\Big[\big(B^\dagger + (B_j^*)^\dagger\sigma^j\big)_a\big(B + B_k^*\sigma^k\big)_b\,\sigma^i\Big]\left(\frac{\partial_i(\vec{\tau}\cdot\vec{\pi})}{f_\pi}\right) \ , \tag{93}$$

where the indices i, j, k refer to the spatial index of the derivative or the B^* polarisation vectors—summation over those is assumed and the factor of (1/2) comes from the normalisation of the heavy-meson field as shown in Appendix A of Ref. [98]. When expanded and using trace relations,

$$\mathcal{L}_{B^{(*)}B^{(*)}\pi}=\frac{g}{2f_\pi}\partial_i\vec{\pi}\Big[B^\dagger\vec{\tau}(B_i^*)+(B_i^*)^\dagger\vec{\tau}B+i\epsilon_{jki}(B_j^*)^\dagger\vec{\tau}(B_k^*)\Big]\,, \tag{94}$$

using $\partial_i\vec{\pi} \to -ik_i$, where k_i is the momentum of the outgoing pion, then

$$\mathcal{L}_{B^{(*)}\to B^{(*)}\pi_a} = \frac{-ig}{2f_\pi}k_i\Big[B^\dagger\tau_a(B_i^*) + (B_i^*)^\dagger\tau_a B + i\,\epsilon_{jki}\,(B_j^*)^\dagger\tau_a(B_k^*)\Big]\ , \tag{95}$$

where a denotes the isospin index of the pion. We thus get the following vertices [99],

$$\begin{aligned}
v_{B^*\to B\pi_a} &= \frac{g}{2f_\pi}(\vec{\epsilon}\cdot\vec{k})\tau_a\ ,\\
v_{B\to B^*\pi_a} &= \frac{g}{2f_\pi}(\vec{\epsilon}^{\,*}\cdot\vec{k})\tau_a\ ,\\
v_{B^*\to B^*\pi_a} &= -i\frac{g}{2f_\pi}\tau_a(\vec{\epsilon}\times\vec{\epsilon}^{\,*})\vec{k}\ .
\end{aligned}$$

The respective vertices for the antimesons are

$$\begin{aligned}
v_{\bar{B}^*\to\bar{B}\pi_a} &= \frac{g}{2f_\pi}(\vec{\epsilon}\cdot\vec{k})\tau_a^c\ ,\\
v_{\bar{B}\to\bar{B}^*\pi_a} &= \frac{g}{2f_\pi}(\vec{\epsilon}^{\,*}\cdot\vec{k})\tau_a^c\ ,\\
v_{\bar{B}^*\to\bar{B}^*\pi_a} &= -i\frac{g}{2f_\pi}\tau_a^c(\vec{\epsilon}\times\vec{\epsilon}^{\,*})\vec{k}\ .
\end{aligned}$$

For the antimesons, we use the charge-conjugated Pauli matrix which is related to the antifundamental representation of the isospin group, which is expressed as $\vec{\tau}^{\,c} = \tau_2\,\vec{\tau}^{\,T}\,\tau_2 = -\vec{\tau}$ [9].

4.2 Weinberg-Tomozawa vertex

The term in the Lagrangian for the Weinberg-Tomozawa vertex is given by

$$\mathcal{L}_{B^{(*)}\pi\to B^{(*)}\pi} = \mathrm{Tr}[H_a^\dagger\,(i\Gamma_0)_{ba}\,H_b]\ , \tag{96}$$

where Γ_0 is given in Eq. (89). To the second order in $\vec{\pi}$ fields, gives

$$\mathcal{L}_{B^{(*)}B^{(*)}\pi\pi} = -\frac{1}{4f_\pi^2}\epsilon_{abe}\pi_a\partial_0\pi_b\Big[B^\dagger\tau_e B + B^{*j\,\dagger}\tau_e B^{*j}\Big] + \mathcal{O}(\vec{\pi}^{\,4})\ . \tag{97}$$

From this expression, the different vertices are

$$\begin{aligned}
v_{B\pi_a\to B\pi_b} &= \frac{1}{4f_\pi^2}\epsilon_{abe}\tau_e(k_0' + k_0)\ ,\\
v_{B^*\pi_a\to B^*\pi_b} &= \frac{1}{4f_\pi^2}\epsilon_{abe}\tau_e(\vec{\epsilon}\cdot\vec{\epsilon}^{\,*})(k_0' + k_0)\ ,
\end{aligned}$$

where k_0 (k_0') denotes the zeroth component of the incoming (outgoing) pion four-momentum. Again, for the corresponding vertices of the anti-B-mesons, one needs replace the the τ matrices by their charge conjugate counterparts τ^c, and they are given by

$$
\begin{aligned}
v_{\bar{B}\pi_a \to \bar{B}\pi_b} &= \frac{1}{4f_\pi^2}\epsilon_{abe}\tau_e^c(k_0' + k_0) \ , \\
v_{\bar{B}^*\pi_a \to \bar{B}^*\pi_b} &= \frac{1}{4f_\pi^2}\epsilon_{abe}\tau_e^c(\vec{\epsilon}\cdot\vec{\epsilon}^*)(k_0' + k_0) \ .
\end{aligned}
$$

5 Effective potentials

With the vertices derived, in this section we discuss the effective potentials of $B^{(*)}\bar{B}^{(*)} \to B^{(*)}\bar{B}^{(*)}$ and $B^{(*)}B^{(*)} \to B^{(*)}B^{(*)}$. Further details of the calculations are provided in the Appendix A.

5.1 Leading order diagrams

At the leading-order, there are contact interactions (CI) and the 1-pion exchange (1PE) diagrams. The contact interactions consist of momentum independent and the promoted $S-D$ transition terms.

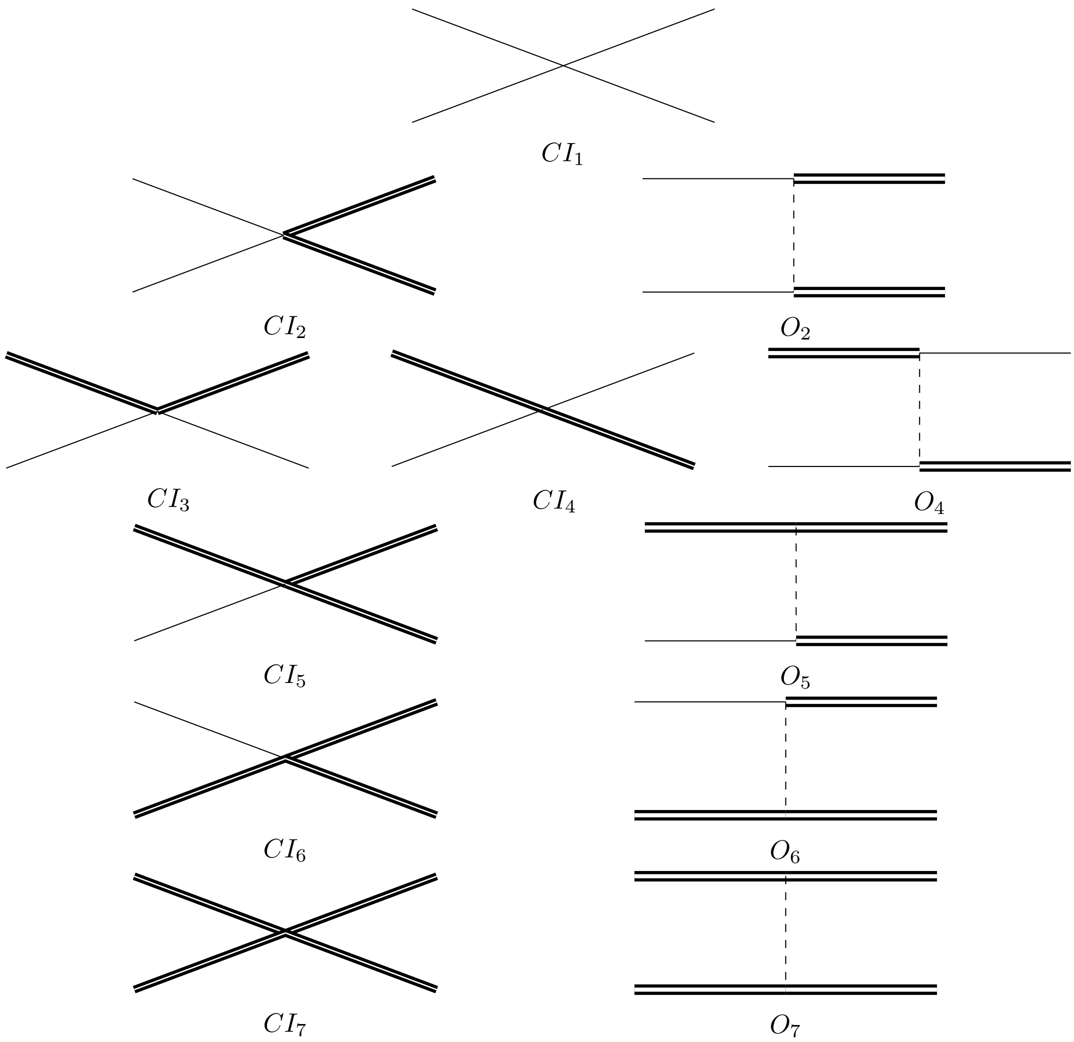


Figure 7: LO contributions to the $B^{(*)}\bar{B}^{(*)} \to B^{(*)}\bar{B}^{(*)}$ scattering potential. Here, single (double) solid lines denote B (B^*) mesons and dashed lines pions. The third line in the diagram shows a properly normalized C-parity of the $B\bar{B}^*$ and $\bar{B}B^*$ components

5.1.1 LO ($\mathcal{O}(\chi^0)$) contact interactions (CI)

The relevant CIs diagrams at leading order are shown in the left side of Fig. 7. The CIs contain the momentum independent term proportional to C_{10} and C_{11} as well as the $S-D$ transition term, D_{12}, promoted to leading order as described in Sec. 1,4. As an example, we present the CI for the $\bar{B}\bar{B} \to \bar{B}\bar{B}$ seen in sub-figure CI_1 of Fig. 7

$$iV_{\mathrm{LO,CI_1}} = i(\mathcal{C}_d + \frac{1}{2}\mathcal{C}_f) \ , \tag{98}$$

where $\mathcal{C}_d$ and $\mathcal{C}_f$ are given in Eq. (91). The potentials for all the $\mathcal{O}(\chi^0)$ CIs diagrams are presented in the partial wave decomposed form in Sec. 6.

5.1.2 LO one-pion exchange (1PE)

The vertices for the one-pion exchange were derived in section 4 with q as the pion momentum. Here, we may safely put the energy of the heavy mesons to zero because they are suppressed relative to the momenta by $p_{\text{typ}}/m_B \sim \mathcal{O}(\chi^2)$.
At the same time, this also implies that the energy transfer, $q^0 = E_p - E_{p'} = 0 + \mathcal{O}(\vec{k}^{\,2}/(2m_H))$, can be dropped to the order we are working. Taking all this together, we get for $B^*\bar{B} \to B\bar{B}^*$ potential in Fig. 7

$$V_{O_4} \;=\; -\frac{g^2}{4f_\pi^2}(\vec{\tau_1}\cdot\vec{\tau_2})(\epsilon^*_{2',n}\epsilon_{1,i})\frac{q_i q_n}{\vec{q}^{\,2}+m_\pi^2}\;, \tag{99}$$

where we used $\vec{\tau}^c = -\vec{\tau}$, $\vec{q} = \vec{p'} - \vec{p}$.
The external polarisation vectors for incoming and outgoing B^* meson ($\bar{B}^*$) is represented as $\epsilon_{1,i}$ and $\epsilon^*_{1',k}$ ($\epsilon_{2,l}$ and $\epsilon^*_{2',n}$) respectively. The isospin factor for the 1PE and 2PE potential is $\vec{\tau}_A \cdot \vec{\tau}_B = 3 - 2I(I+1)$ which evaluates to 3 for isoscalar and -1 for isotriplet states. The effective potentials for all the 1PE diagrams in the $\bar{B}B$ and BB cases are presented in Sec. 5.3 and Sec. 5.4, respectively.

5.2 Next-to-leading order diagrams

At next-to-leading-order, $\mathcal{O}(\chi^2)$, there are momentum-dependent contact interactions (CI) and the 2-pion exchange (2PE) diagrams.
There are three types of 2PE diagrams: triangle-diagrams, football-diagrams and box-diagrams. This section will cover the general properties of a few diagrams in the $\bar{B}\bar{B} \to \bar{B}\bar{B}$ case, while the complete expressions for each NLO diagram are presented in Sec. 5.3.

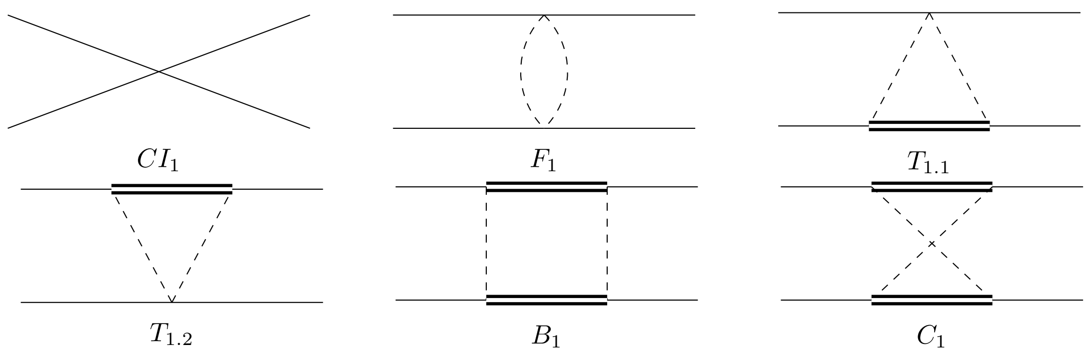


Figure 8: All NLO diagrams for $B\bar{B} \to B\bar{B}$

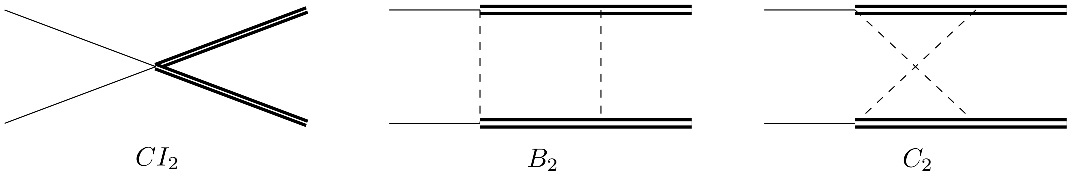


Figure 9: All NLO diagrams for $B\bar{B} \to B^*\bar{B}^*$

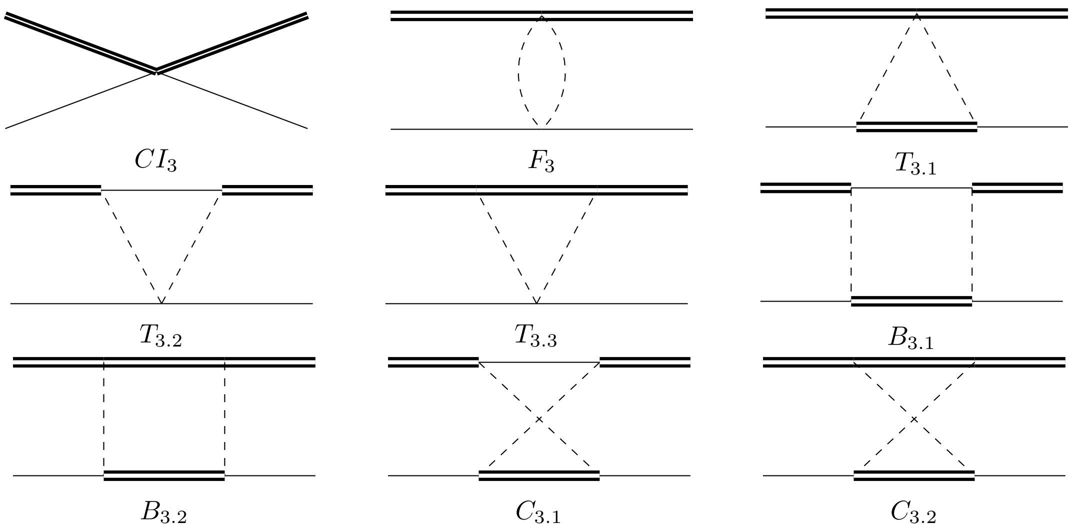


Figure 10: All NLO diagrams for $B^*\bar{B} \to B^*\bar{B}$

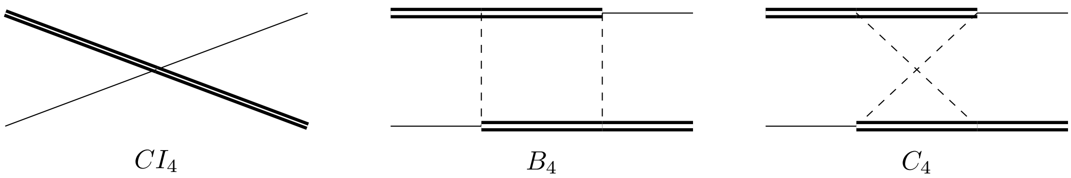


Figure 11: All NLO diagrams for $B^*\bar{B} \to B\bar{B}^*$

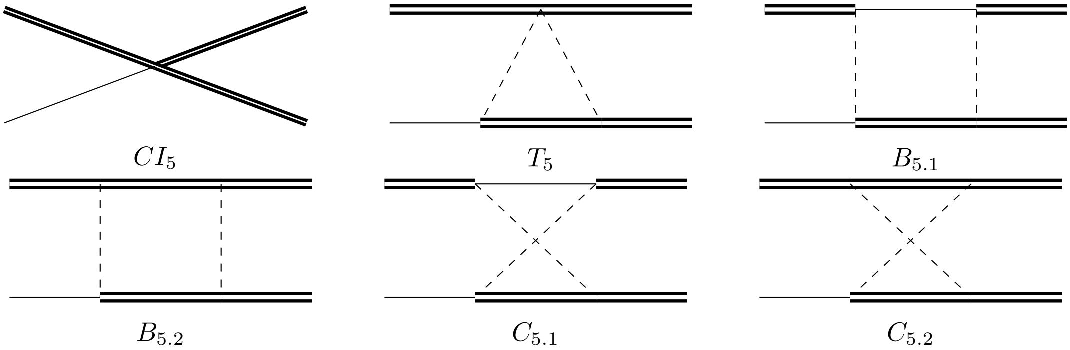


Figure 12: All NLO diagrams for $B^*\bar{B} \to B^*\bar{B}^*$

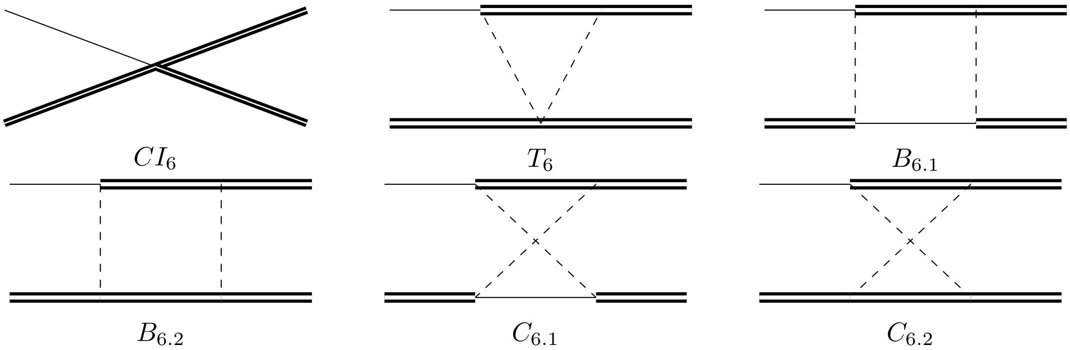


Figure 13: All NLO diagrams for $B\bar{B}^* \to B^*\bar{B}^*$

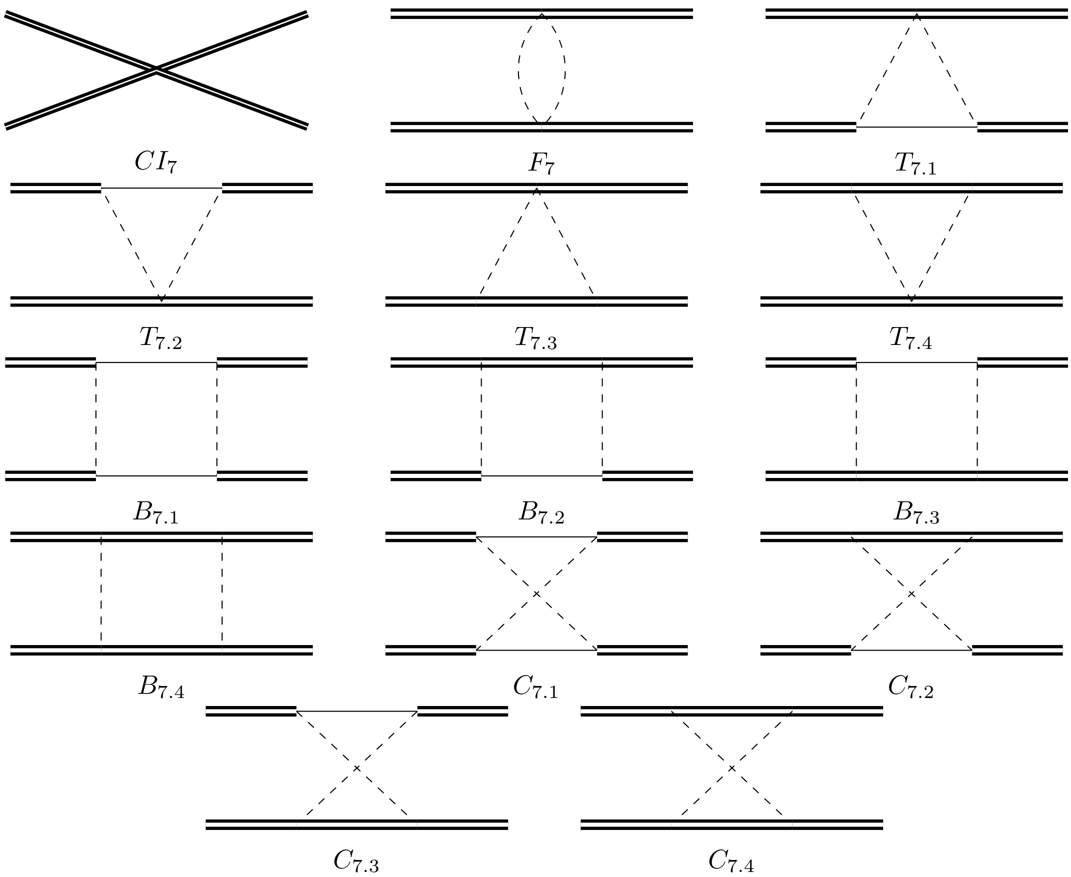


Figure 14: All NLO diagrams for $B^*\bar{B}^* \to B^*\bar{B}^*$

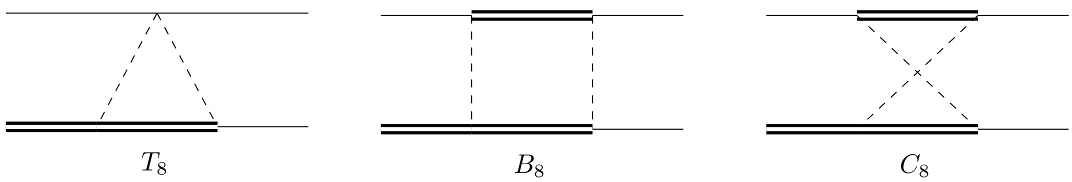


Figure 15: All NLO diagrams for $B\bar{B}^* \to B\bar{B}$

5.2.1 NLO, $\mathcal{O}(\chi^2)$, contact interactions

The relevant CIs for the NLO case are listed above. The CIs contain the momentum-dependent term $\mathcal{O}(\chi^2)$ which is proportional to D_{10} and D_{11} as seen in the Lagrangian. Formally, the chiral expansion also produces momentum independent subleading contact terms proportional to m_π^2 (linear in the quark mass matrix), which would likewise appear at the same order in standard power counting. In the momentum counting scheme employed here, however, those are suppressed by

$(m_\pi/p_{\rm typ})^2{\sim}\chi^2$ and thus do not need to be taken into account. Like in the previous cases, we present the CI for the $\bar{B}\bar{B} \to \bar{B}\bar{B}$ seen in sub-figure CI_1 of Fig. 8

$$iV_{\rm NLO,CI_1} = i(p^2 + p'^{\,2})(\mathcal{D}_d + \frac{1}{2}\mathcal{D}_f) \ , \tag{100}$$

where p and (p') stands for the relative momentum of the initial (final) heavy-meson pair, $\mathcal{D}_d$ and $\mathcal{D}_f$ are given in Eq. (91). The potentials for all the $\mathcal{O}(\chi^2)$ CIs diagrams are presented in the partial wave decomposed form in Sec. 6.

5.2.2 Triangle diagrams

In the $B^{(*)}\bar{B}^{(*)}$ case, the sum of all triangle diagrams vanishes. In this section, we demonstrate this explicitly for the $B\bar{B} \to B\bar{B}$ channel, however, the same pattern applies to all the other potentials analogously. For the $B\bar{B} \to B\bar{B}$ potential, we have two triangle diagrams. The potential from the first diagram is

$$iV_{T_{1.1}} = \sum_\lambda \int \frac{d^4l}{(2\pi)^4} \frac{1}{4f_\pi^2} \Big(2l_0\epsilon_{cdh}(\tau_1)_h\Big) \frac{i}{l^2 - m_\pi^2} \left(\frac{g}{2f_\pi} \Big(\epsilon_j(\lambda)(-l)_j\Big)(\tau_2^c)_d \right) \\ \times \frac{i}{2(-l_0 + i\epsilon)} \left(\frac{g}{2f_\pi} \Big(\epsilon_i^*(\lambda)(l+q)_i\Big)(\tau_2^c)_c \right) \frac{i}{(l+q)^2 - m_\pi^2} \ , \tag{101}$$

where the labels on the potentials refer to those in Fig. 8. The potential can thus be written as

$$V_{T_{1.1}} = \frac{g^2}{8f_\pi^4}(\vec{\tau}_1 \cdot \vec{\tau}_2) I_{tr} \ , \tag{102}$$

where we used $\sum_\lambda \epsilon_i^*(\lambda)\epsilon_j(\lambda) = \delta_{ij}$ and integral I_{tr} is given by

$$I_{tr} = i\int \frac{d^4l}{(2\pi)^4} \frac{(\vec{l} + \vec{q}) \cdot \vec{l}}{\left[(l+q)^2 - m_\pi^2 + i\epsilon\right]\left[l^2 - m_\pi^2 + i\epsilon\right]} \ . \tag{103}$$

The closed form of this integral is given in Appendix A.1. The second diagram simplifies to

$$V_{T_{1.2}} = -\frac{g^2}{8f_\pi^4}(\vec{\tau}_1 \cdot \vec{\tau}_2) I_{tr} \ , \tag{104}$$

where the change in sign resulted from the appearance of the charge conjugate isospin matrix at the lower, $\pi\bar{B} \to \pi\bar{B}$ vertex. The total contribution therefore cancels. If we had instead looked at the BB potential, it is apparent that the two triangle contributions would be summed. This can be seen in Appendix G, where we have calculated the triangle contribution in particle basis for the $B^{(*)}B^{(*)}$ case. The effective potentials for all the triangle diagrams in the BB case are presented in Sec. 5.4.

5.2.3 Football diagrams

The football diagram consists of two contracting Weinberg-Tomozawa vertices with two pion propagators. The football diagrams are denoted by F_i, where sub-index i denotes the type of $\bar{B}B$ scattering. We present the potential for the football diagram in the $B\bar{B} \to B\bar{B}$ case, seen in sub-figure F_1 of Fig. 8

$$iV_{F_1} = \int \frac{d^4l}{(2\pi)^4} \frac{1}{4f_\pi^2}\Big(2l_0\epsilon_{cde}(\tau_1)_e\Big)\frac{i}{(l^2-m_\pi^2)}\frac{1}{4f_\pi^2}\Big((-2l_0)\epsilon_{cdf}(\tau_2^c)_f\Big)\frac{i}{(l+q)^2-m_\pi^2} \ . \tag{105}$$

Using $\epsilon_{cde}\epsilon_{cdf} = 2\delta_{ef}$, then

$$V_{F_1} = \frac{1}{2f_\pi^4}(\vec{\tau_1}\cdot\vec{\tau_2})I_{fb} \ , \tag{106}$$

where, we used $\epsilon_{cde}\epsilon_{cdf} = 2\delta_{ef}$ and I_{fb} is given by

$$I_{fb} = i\int \frac{d^4l}{(2\pi)^4}\frac{(l^0)^2}{\big[(l+q)^2-m_\pi^2+i\epsilon\big]\big[l^2-m_\pi^2+i\epsilon\big]} \ . \tag{107}$$

A closed form expression for this integral is provided in Eq. (272) in Appendix A.2. Once again, the effective potentials for all the football diagrams in the $\bar{B}B$ and BB cases are presented in Sec. 5.3 and Sec. 5.4, respectively.

5.2.4 Box diagrams

The box diagrams consist of either a planar box or a crossed box, formed by contracting four 1PE vertices with two pion propagators. We refer to the planar and crossed box as B_i and C_i in the figures with the sub-index i denoting the type of $\bar{B}B$ scattering. To prevent double counting, we keep just the two-heavy meson irreducible part of the planar box in the potential, see in Sec.3.7, as the Lippmann-Schwinger equation generates the reducible parts. Again, for illustration the $B\bar{B} \to B\bar{B}$ case is discussed explicitly here. For the planar box, seen in Fig. 8, one finds

$$\begin{aligned} iV_{B_1} = \sum_{\lambda_1,\lambda_2}\int \frac{d^4l}{(2\pi)^4}&\left(\frac{g}{2f_\pi}\Big(\epsilon_i(\lambda_1)(-q_2)_i\Big)(\tau_1)_d\right)\frac{i}{2(-l_0+i\epsilon)} \\ &\left(\frac{g}{2f_\pi}\Big(\epsilon_j^*(\lambda_1)(q_1)_j\Big)(\tau_1)_c\right)\frac{i}{q_1^2-m_\pi^2}\left(\frac{g}{2f_\pi}\Big(\epsilon_n(\lambda_2)(q_2)_n\Big)(\tau_2^c)_d\right)\frac{i}{q_2^2-m_\pi^2} \\ &\left(\frac{g}{2f_\pi}\Big(\epsilon_m^*(\lambda_2)(-q_1)_m\Big)(\tau_2^c)_c\right)\frac{i}{2(l_0+i\epsilon)} \ . \end{aligned} \tag{108}$$

Here, q_1 and q_2 are the momenta of the two pions and we define $q_1{=}(l_0, \vec{p}{-}\vec{l})$, $q_2{=}(l_0, \vec{p'}{-}\vec{l})$ and $q{=}(0, \vec{p'}{-}\vec{p}){=}(0, \vec{q})$ where l is the loop momentum. This can be written as

$$V_{B_1} = \frac{g^4}{64 f_\pi^4}\Big(3 - 2(\vec{\tau_1}\cdot\vec{\tau_2})\Big)\, I^{(1)}_{box}\ , \tag{109}$$

where we used $(\tau_1)_d(\tau_1)_c(\tau_2)_d(\tau_2)_c = \Big(3 - 2(\vec{\tau_1}\cdot\vec{\tau_2})\Big)$ and

$$I^{(1)}_{box} = i\int \frac{d^4 l}{(2\pi)^4}\frac{(\vec{q_2}\cdot\vec{q_1})^2}{\left[l^0{+}i\epsilon\right]\left[l^0{-}i\epsilon\right]\left[q_2^2{-}m_\pi^2\right]\left[q_1^2{-}m_\pi^2\right]}\ . \tag{110}$$

For crossed box

$$\begin{aligned} iV_{C_1} = \sum_{\lambda_1,\lambda_2}\int \frac{d^4 l}{(2\pi)^4}&\left(\frac{g}{2f_\pi}\Big(\epsilon_i(\lambda_1)(-q_2)_i\Big)(\tau_1)_d\right)\frac{i}{2(-l_0 + i\epsilon)} \\ &\left(\frac{g}{2f_\pi}\Big(\epsilon^*_j(\lambda_1)(q_1)_j\Big)(\tau_1)_c\right)\frac{i}{q_1^2 - m_\pi^2}\left(\frac{g}{2f_\pi}\Big(\epsilon_n(\lambda_2)(-q_1)_n\Big)(\tau_2^c)_c\right) \\ &\frac{i}{2(-l_0 + i\epsilon)}\left(\frac{g}{2f_\pi}\Big(\epsilon^*_m(\lambda_2)(q_2)_m\Big)(\tau_2^c)_d\right)\frac{i}{q_2^2 - m_\pi^2}\ . \end{aligned} \tag{111}$$

The above expression can be simplified to

$$V_{C_1} = \frac{g^4}{64 f_\pi^4}\Big(-3 - 2(\vec{\tau_1}\cdot\vec{\tau_2})\Big)\, I^{(2)}_{box}\ , \tag{112}$$

where $(\tau_1)_d(\tau_1)_c(\tau_2)_c(\tau_2)_d = 3 + 2(\vec{\tau_1}\cdot\vec{\tau_2})$ is used. The loop integral $I^{(2)}_{box}$ is given by

$$I^{(2)}_{box} = i\int \frac{d^4 l}{(2\pi)^4}\frac{(\vec{q_2}\cdot\vec{q_1})^2}{\left[l_0{-}i\epsilon\right]^2\left[q_2^2{-}m_\pi^2\right]\left[q_1^2{-}m_\pi^2\right]}\ . \tag{113}$$

A closed expression for this integral is provided in Eq. (77). The sign in the $i\epsilon$ term in one of the heavy meson propagators is the only distinction between $I^{(1)}_{box}$ and $I^{(2)}_{box}$. The difference in sign indicates that the former integral has a reducible contribution while the latter does not.

In Ref. [93] and Sec. 3.7, it was shown that the $I^{(1)}_{box}$ splits into reducible double 1PE and irreducible crossed box integral $I^{(2)}_{box}$. Thus, in Eq. (110) we may replace $I^{(1)}_{box}$ by $I^{(2)}_{box}$. Accordingly, the irreducible box contribution to the effective potential for $B\bar{B} \to B\bar{B}$ reads

$$V^{box}_{B\bar{B}\to B\bar{B}} = V_{B_1} + V_{C_1} = -\frac{g^4}{16 f_\pi^4}(\vec{\tau_1}\cdot\vec{\tau_2})\, I^{(2)}_{box}\ . \tag{114}$$

Additionally, we also present the calculation of the box diagram in the $B^*\bar{B}^* \to B^*\bar{B}^*$ case, due to its complicated structure.
The $B^*\bar{B}^* \to B^*\bar{B}^*$ potential, seen in Fig. 14, has eight box diagrams and the total contribution is

$$\begin{aligned} V^{box}_{B^*\bar{B}^*\to B^*\bar{B}^*} &= V_{B7.1} + V_{B7.2} + V_{B7.3} + V_{B7.4} + V_{C7.1} + V_{C7.2} + V_{C7.3} + V_{C7.4} \\ &= \frac{g^4}{64 f_\pi^4} (\epsilon_{1,i}\epsilon^*_{1',k}\epsilon_{2,l}\epsilon^*_{2',n}) i \int \frac{d^4 l}{(2\pi)^4} \frac{3(A) - 2(\vec{\tau_1}\cdot\vec{\tau_2})(B)}{[l_0 - i\epsilon]^2 \big[q_2^2 - m_\pi^2\big]\big[q_1^2 - m_\pi^2\big]} \,, \end{aligned} \tag{115}$$

where

$$\begin{aligned} A = 2\big[&(q_2)_k(q_1)_i(q_2)_n(q_1)_l - (q_2)_i(q_1)_k(q_2)_n(q_1)_l - (q_2)_k(q_1)_i(q_2)_l(q_1)_n \\ &+ (q_2)_i(q_1)_k(q_2)_l(q_1)_n\big] + 2\delta_{ik}(\vec{q_2}\cdot\vec{q_1})\big[(q_2)_n(q_1)_l - (q_2)_l(q_1)_n)\big] \end{aligned} \tag{116}$$

and

$$B = 2\delta_{ik}\delta_{ln}(\vec{q_2}\cdot\vec{q_1})^2 + 2\delta_{ln}(\vec{q_2}\cdot\vec{q_1})^2\big[(q_2)_k(q_1)_i - (q_2)_i(q_1)_k\big] \,. \tag{117}$$

The last term in A (proportional to δ_{ik}) and the second term in B (proportional to $\delta_{ln}(\vec{q_2}\cdot\vec{q_1})^2$) will vanish, due to tensor decomposition being symmetric while the mentioned terms in A and B being antisymmetric under exchange of indices. Solving the remaining terms using tensor decomposition

$$\begin{aligned} V^{box}_{B^*\bar{B}^*\to B^*\bar{B}^*} = \frac{g^4}{16 f_\pi^4}(\epsilon_{1,i}\epsilon^*_{1',k}\epsilon_{2,l}\epsilon^*_{2',n}) \Bigg\{ &-(\vec{\tau_1}\cdot\vec{\tau_2})\delta_{ik}\delta_{ln} I^{(2)}_{box} + \frac{1}{2}\big[\delta_{il}\delta_{kn} - \delta_{in}\delta_{kl}\big]\big[I^{(3)}_{box} \\ - I^{(2)}_{box} + \frac{1}{\vec{q}^2}(I^{(4)}_{box} - I^{(5)}_{box})\big] + \frac{3}{4\vec{q}^4}&\big[q_k q_n \delta_{il} - q_k q_l \delta_{in} + q_i q_l \delta_{kn} - q_i q_n \delta_{kl}\big]\big[I^{(5)}_{box} - I^{(4)}_{box}\big] \Bigg\} \,, \end{aligned} \tag{118}$$

where

$$\begin{aligned} I^{(3)}_{box} &= i\int \frac{d^4 l}{(2\pi)^4} \frac{\vec{q_1}^2 \vec{q_2}^2}{(l^0 - i\epsilon)^2\big[q_2^2 - m_\pi^2 + i\epsilon\big]\big[q_1^2 - m_\pi^2 + i\epsilon\big]} \,, \\ I^{(4)}_{box} &= i\int \frac{d^4 l}{(2\pi)^4} \frac{\vec{q_1}^2 (\vec{q_2}\cdot\vec{q})^2}{(l^0 - i\epsilon)^2\big[q_2^2 - m_\pi^2 + i\epsilon\big]\big[q_1^2 - m_\pi^2 + i\epsilon\big]} \,, \\ I^{(5)}_{box} &= i\int \frac{d^4 l}{(2\pi)^4} \frac{\vec{q_2}^2 (\vec{q_1}\cdot\vec{q})^2}{(l^0 - i\epsilon)^2\big[q_2^2 - m_\pi^2 + i\epsilon\big]\big[q_1^2 - m_\pi^2 + i\epsilon\big]} \,. \end{aligned} \tag{119}$$

The detailed calculation for $I^{(3)}_{box}$, $I^{(4)}_{box}$, $I^{(5)}_{box}$ are given in the Appendix A.3. The effective potentials for all the box diagrams in the $\bar{B}B$ and BB cases are given in Sec. 5.3 and Sec. 5.4, respectively.

5.3 Effective potentials of $\mathbf{B}^{(*)}\bar{\mathbf{B}}^{(*)} \to \mathbf{B}^{(*)}\bar{\mathbf{B}}^{(*)}$ at the $\mathcal{O}(\chi^2)$

Following all the steps outlined in the calculation of $B\bar{B} \to B\bar{B}$ potential. Here, we present the 1PE and 2PE potentials of all $B^{(*)}\bar{B}^{(*)} \to B^{(*)}\bar{B}^{(*)}$ scatterings.

5.3.1 $B\bar{B} \to B\bar{B}$

$$V_{1PE}(B\bar{B} \to B\bar{B}) = 0 \ , \tag{120}$$

$$\begin{aligned} V_{2PE}(B\bar{B} \to B\bar{B}) &= \frac{1}{2f_\pi^4}(\vec{\tau}_1 \cdot \vec{\tau}_2)\left(I_{fb} - \frac{g^4}{8}I_{box}^{(2)}\right) \\ &= \frac{\vec{q}^2}{16\pi^2 f_\pi^4}(\vec{\tau}_1 \cdot \vec{\tau}_2)\left\{\mathcal{R}\left[\frac{23}{192}g^4 + \frac{1}{24}\right] + \left(\frac{5}{576}g^4 - \frac{5}{72}\right)\right. \\ &\left. + \left(\frac{23}{96}g^4 + \frac{1}{12}\right)\ln\left(\frac{m_\pi}{\mu}\right) + L(q)\left(\frac{23}{96}g^4 + \frac{1}{12}\right)\right\} . \end{aligned} \tag{121}$$

5.3.2 $B^*\bar{B} \to B^*\bar{B}$

$$V_{1PE}(B^*\bar{B} \to B^*\bar{B}) = 0 \ , \tag{122}$$

$$V_{2PE}(B^*\bar{B} \to B^*\bar{B}) = (\epsilon_{1'}^* \cdot \epsilon_1)V_{2PE}(B\bar{B} \to B\bar{B}) \ . \tag{123}$$

5.3.3 $B^*\bar{B} \to B\bar{B}^*$

$$V_{1PE}(B^*\bar{B} \to B\bar{B}^*) = -\frac{g^2}{4f_\pi^2}(\vec{\tau}_1 \cdot \vec{\tau}_2)(\epsilon_{2',n}^*\epsilon_{1,i})\frac{q_i q_n}{\vec{q}^2 + m_\pi^2} \ , \tag{124}$$

$$\begin{aligned} V_{2PE}(B^*\bar{B} \to B\bar{B}^*) = \frac{3}{512\pi^2 f_\pi^4}g^4(\epsilon_{2',n}^*\epsilon_{1,i})(\delta_{in}\vec{q}^2 - q_i q_n)\Bigg\{ &-\mathcal{R} + 1 - 2L(q) \\ &- 2\ln\left(\frac{m_\pi}{\mu}\right)\Bigg\} . \end{aligned} \tag{125}$$

5.3.4 $B^*\bar{B}^* \to B^*\bar{B}^*$

$$V_{1PE}(B^*\bar{B}^* \to B^*\bar{B}^*) = \frac{g^2}{4f_\pi^2}(\vec{\tau}_1 \cdot \vec{\tau}_2)\epsilon_{ikr}\epsilon_{lns}(\epsilon_{1,i}\epsilon_{1',k}^*\,\epsilon_{2,l}\epsilon_{2',n}^*)\frac{q_r q_s}{\vec{q}^2 + m_\pi^2} \ , \tag{126}$$

$$
\begin{aligned}
V_{2PE}(B^*\bar{B}^* \to B^*\bar{B}^*) = \frac{\vec{q}^{\,2}}{16\pi^2 f_\pi^4}\big(\vec{\tau}_1\cdot\vec{\tau}_2\big)(\epsilon_1\cdot\epsilon_{1'}^*)(\epsilon_2\cdot\epsilon_{2'}^*)\Bigg\{\mathcal{R}\left[\frac{23}{192}g^4+\frac{1}{24}\right] \\
+\left(\frac{5}{576}g^4-\frac{5}{72}\right)+\ln\left(\frac{m_\pi}{\mu}\right)\left(\frac{23}{96}g^4+\frac{1}{12}\right) \\
+L(q)\left(\frac{23}{96}g^4+\frac{1}{12}\right)\Bigg\}+\frac{\vec{q}^{\,2}g^4}{512\pi^2 f_\pi^4}(\epsilon_{1,i}\epsilon_{1',k}^*\epsilon_{2,l}\epsilon_{2',n}^*) \\
\big(\delta_{in}\delta_{kl}-\delta_{il}\delta_{kn}\big)\left\{\frac{-7}{3}\mathcal{R}-5\ln\left(\frac{m_\pi}{\mu}\right)-\frac{15}{2}L(q)\right\} \\
+\frac{g^4}{16\pi^2 f_\pi^4}(\epsilon_{1,i}\epsilon_{1',k}^*\epsilon_{2,l}\epsilon_{2',n}^*)\big[q_k q_n\delta_{il}-q_k q_l\delta_{in}+q_i q_l\delta_{kn} \\
-q_i q_n\delta_{kl}\big]\times\frac{3}{64}\left\{\frac{-1}{3}\mathcal{R}-\frac{1}{9}-\ln\left(\frac{m_\pi}{\mu}\right)-\frac{7}{2}L(q)\right\}.
\end{aligned}
\tag{127}
$$

5.3.5 $B\bar{B} \to B^*\bar{B}^*$

$$
V_{1PE}(B\bar{B} \to B^*\bar{B}^*) = -\frac{g^2}{4f_\pi^2}(\vec{\tau}_1\cdot\vec{\tau}_2)(\epsilon_{1',k}^*\epsilon_{2',n}^*)\frac{q_k q_n}{\vec{q}^{\,2}+m_\pi^2}\ , \tag{128}
$$

$$
\begin{aligned}
V_{2PE}(B\bar{B} \to B^*\bar{B}^*) = \frac{3}{512\pi^2 f_\pi^4}g^4(\epsilon_{1',k}^*\epsilon_{2',n}^*)\big(\delta_{kn}\vec{q}^{\,2}-q_k\,q_n\big)\Bigg\{-\mathcal{R}+1-2L(q) \\
-2\ln\left(\frac{m_\pi}{\mu}\right)\Bigg\}.
\end{aligned}
\tag{129}
$$

5.3.6 $B\bar{B}^* \to B\bar{B}$

$$
V_{1PE}(B\bar{B}^* \to B\bar{B}) = V_{2PE}(B\bar{B}^* \to B\bar{B}) = 0\ . \tag{130}
$$

5.3.7 $B^*\bar{B} \to B^*\bar{B}^*$

$$
V_{1PE}(B^*\bar{B} \to B^*\bar{B}^*) = i\frac{g^2}{4f_\pi^2}(\vec{\tau}_1\cdot\vec{\tau}_2)\epsilon_{ikr}(\epsilon_{1,i}\epsilon_{1',k}^*\epsilon_{2',n}^*)\frac{q_r q_n}{\vec{q}^{\,2}+m_\pi^2}\ , \tag{131}
$$

$$
\begin{aligned}
V_{2PE}(B^*\bar{B} \to B^*\bar{B}^*) = i\,\frac{3}{512\pi^2 f_\pi^4}g^4(\epsilon_{1,i}\epsilon_{1',k}^*\epsilon_{2',n}^*)\left(\epsilon_{nku}q_u q_i-\epsilon_{niu}q_u q_k\right) \\
\left\{-\mathcal{R}+1-2L(q)-2\ln\left(\frac{m_\pi}{\mu}\right)\right\}.
\end{aligned}
\tag{132}
$$

5.3.8 $B\bar{B}^* \to B^*\bar{B}^*$

$$V_{1PE}(B\bar{B}^* \to B^*\bar{B}^*) = i\frac{g^2}{4f_\pi^2}(\vec{\tau_1}\cdot\vec{\tau_2})\epsilon_{lns}(\epsilon_{2,l}\epsilon^*_{1',k}\epsilon^*_{2',n})\frac{q_k q_s}{\vec{q}^2+m_\pi^2} \ , \tag{133}$$

$$\begin{aligned} V_{2PE}(B\bar{B}^* \to B^*\bar{B}^*) = i\,&\frac{3}{512\pi^2 f_\pi^4}g^4(\epsilon_{2,l}\epsilon^*_{1',k}\epsilon^*_{2',n})\left(\epsilon_{knu}q_u q_l - \epsilon_{klu}q_u q_n\right) \\ &\left\{-\mathcal{R}+1-2L(q)-2\ln\left(\frac{m_\pi}{\mu}\right)\right\} . \end{aligned} \tag{134}$$

The potentials presented above, only have terms at the $\mathcal{O}(\chi^2)$, and terms of higher order have been neglected. The potentials containing all terms from the loop contribution are mentioned in Appendix C.

5.4 Effective potentials of $\mathbf{B}^{(*)}\mathbf{B}^{(*)} \to \mathbf{B}^{(*)}\mathbf{B}^{(*)}$ at the $\mathcal{O}(\chi^2)$

The same topologies discussed above also contribute to $B^{(*)}B^{(*)}$ scattering, except as already mentioned in Section 5.2.2, in this case the triangle diagrams do not cancel. For the same reason the one-pion exchange and the football diagrams change their sign. The reason for these differences is the absence of charge-conjugated Pauli matrix ($\vec{\tau^c} = -\vec{\tau}$). It is evident that this change does not affect the box diagrams as there is an even number of pion-emission vertices on each heavy meson line. The evaluated forms of the $B^{(*)}B^{(*)} \to B^{(*)}B^{(*)}$ potentials are mentioned below.

5.4.1 $BB \to BB$

$$V_{1PE}(BB \to BB) = 0 \ , \tag{135}$$

$$\begin{aligned} V_{2PE}(BB \to BB) &= \frac{1}{2f_\pi^4}(\vec{\tau_1}\cdot\vec{\tau_2})\left(-I_{fb}+\frac{g^2}{2}I_{tr}-\frac{g^4}{8}I^{(2)}_{box}\right) \\ &= \frac{\vec{q}^2}{16\pi^2 f_\pi^4}(\vec{\tau_1}\cdot\vec{\tau_2})\left\{\mathcal{R}\left[\frac{23}{192}g^4-\frac{5}{48}g^2-\frac{1}{24}\right]+\left(\frac{5}{576}g^4\right.\right. \\ &\left.+\frac{13}{144}g^2+\frac{5}{72}\right)+\ln\left(\frac{m_\pi}{\mu}\right)\left(\frac{23}{96}g^4-\frac{5}{24}g^2-\frac{1}{12}\right) \\ &\left.+L(q)\left(\frac{23}{96}g^4-\frac{5}{24}g^2-\frac{1}{12}\right)\right\} . \end{aligned} \tag{136}$$

5.4.2 $B^*B \to B^*B$

$$V_{1PE}(B^*B \to B^*B) = 0 \ , \tag{137}$$

$$V_{2PE}(B^*B \to B^*B) = (\epsilon^*_{1'} \cdot \epsilon_1) V_{2PE}(BB \to BB) \ . \tag{138}$$

5.4.3 $B^*B \to BB^*$

$$V_{1PE}(B^*B \to BB^*) = \frac{g^2}{4f_\pi^2}(\vec{\tau}_1 \cdot \vec{\tau}_2)(\epsilon^*_{2',n}\epsilon_{1,i})\frac{q_i q_n}{\vec{q}^2 + m_\pi^2} \ , \tag{139}$$

$$V_{2PE}(B^*B \to BB^*) = \frac{3}{512\pi^2 f_\pi^4} g^4 (\epsilon^*_{2',n}\epsilon_{1,i})(\delta_{in}\vec{q}^2 - q_i q_n)\Bigg\{ -\mathcal{R} + 1 - 2L(q) - 2\ln\left(\frac{m_\pi}{\mu}\right)\Bigg\} \ . \tag{140}$$

5.4.4 $B^*B^* \to B^*B^*$

$$V_{1PE}(B^*B^* \to B^*B^*) = -\frac{g^2}{4f_\pi^2}(\vec{\tau}_1 \cdot \vec{\tau}_2)\epsilon_{ikr}\epsilon_{lns}(\epsilon_{1,i}\epsilon^*_{1',k}\, \epsilon_{2,l}\epsilon^*_{2',n})\frac{q_r q_s}{\vec{q}^2 + m_\pi^2} \ , \tag{141}$$

$$\begin{aligned} V_{2PE}(B^*B^* \to B^*B^*) = {} & \frac{\vec{q}^2}{16\pi^2 f_\pi^4}\big(\vec{\tau}_1 \cdot \vec{\tau}_2\big)(\epsilon_1 \cdot \epsilon^*_{1'})(\epsilon_2 \cdot \epsilon^*_{2'})\Bigg\{\mathcal{R}\left[\frac{23}{192}g^4 - \frac{5}{48}g^2 - \frac{1}{24}\right] \\ & + \left(\frac{5}{576}g^4 + \frac{13}{144}g^2 + \frac{5}{72}\right) + \ln\left(\frac{m_\pi}{\mu}\right)\left(\frac{23}{96}g^4 - \frac{5}{24}g^2 \right. \\ & \left. - \frac{1}{12}\right) + L(q)\left(\frac{23}{96}g^4 - \frac{5}{24}g^2 - \frac{1}{12}\right)\Bigg\} \\ & + \frac{\vec{q}^2 g^4}{512\pi^2 f_\pi^4}(\epsilon_{1,i}\epsilon^*_{1',k}\epsilon_{2,l}\epsilon^*_{2',n})\big(\delta_{in}\delta_{kl} - \delta_{il}\delta_{kn}\big)\Bigg\{\frac{-7}{3}\mathcal{R} \\ & - 5\ln\left(\frac{m_\pi}{\mu}\right) - \frac{15}{2}L(q)\Bigg\} + \frac{g^4}{16\pi^2 f_\pi^4}(\epsilon_{1,i}\epsilon^*_{1',k}\epsilon_{2,l}\epsilon^*_{2',n}) \\ & \big[q_k q_n \delta_{il} - q_k q_l \delta_{in} + q_i q_l \delta_{kn} - q_i q_n \delta_{kl}\big] \\ & \times \frac{3}{64}\Bigg\{\frac{-1}{3}\mathcal{R} - \frac{1}{9} - \ln\left(\frac{m_\pi}{\mu}\right) - \frac{7}{2}\vec{q}^2 L(q)\Bigg\} \ . \end{aligned} \tag{142}$$

5.4.5 $BB \to B^*B^*$

$$V_{1PE}(BB \to B^*B^*) = \frac{g^2}{4f_\pi^2}(\vec{\tau}_1 \cdot \vec{\tau}_2)(\epsilon^*_{1',k}\epsilon^*_{2',n})\frac{q_k q_n}{\vec{q}^2 + m_\pi^2} \ , \tag{143}$$

$$V_{2PE}(BB \to B^*B^*) = \frac{3}{512\pi^2 f_\pi^4} g^4(\epsilon^*_{1',k}\epsilon^*_{2',n})\big(\delta_{kn}\vec{q}^2 - q_k\, q_n\big)\Bigg\{ -\mathcal{R} + 1 - 2L(q) - 2\ln\left(\frac{m_\pi}{\mu}\right)\Bigg\} \ . \tag{144}$$

5.4.6 $BB^* \to BB$

$$V_{1PE}(BB^* \to BB) = V_{2PE}(BB^* \to BB) = 0 \ . \tag{145}$$

5.4.7 $B^*B \to B^*B^*$

$$V_{1PE}(B^*B \to B^*B^*) = -i\frac{g^2}{4f_\pi^2}(\vec{\tau}_1 \cdot \vec{\tau}_2)\epsilon_{ikr}(\epsilon_{1,i}\epsilon^*_{1',k}\epsilon^*_{2',n})\frac{q_r q_n}{\vec{q}^2 + m_\pi^2} \ , \tag{146}$$

$$V_{2PE}(B^*B \to B^*B^*) = i\,\frac{3}{512\pi^2 f_\pi^4} g^4(\epsilon_{1,i}\epsilon^*_{1',k}\epsilon^*_{2',n})\left(\epsilon_{nku}q_u q_i - \epsilon_{niu}q_u q_k\right) \Bigg\{ -\mathcal{R} + 1 - 2L(q) - 2\ln\left(\frac{m_\pi}{\mu}\right)\Bigg\} \ . \tag{147}$$

5.4.8 $B\bar{B}^* \to B^*\bar{B}^*$

$$V_{1PE}(B\bar{B}^* \to B^*\bar{B}^*) = -i\frac{g^2}{4f_\pi^2}(\vec{\tau}_1 \cdot \vec{\tau}_2)\epsilon_{lns}(\epsilon_{2,l}\epsilon^*_{1',k}\epsilon^*_{2',n})\frac{q_k q_s}{\vec{q}^2 + m_\pi^2} \ , \tag{148}$$

$$V_{2PE}(B\bar{B}^* \to B^*\bar{B}^*) = i\,\frac{3}{512\pi^2 f_\pi^4} g^4(\epsilon_{2,l}\epsilon^*_{1',k}\epsilon^*_{2',n})\left(\epsilon_{knu}q_u q_l - \epsilon_{klu}q_u q_n\right) \Bigg\{ -\mathcal{R} + 1 - 2L(q) - 2\ln\left(\frac{m_\pi}{\mu}\right)\Bigg\} \ . \tag{149}$$

As mentioned in Section 5.3, the potentials containing all terms from the loop contribution are mentioned in Appendix C.

5.5 Potentials with q_0 treated perturbatively

The calculations done so far, follow the power counting scheme mentioned Sec 3.6, which means we have dropped terms like q_0, δ and m_π^2 from our loop integrals that appear in our potentials. But in this part, we perform the calculations in which all terms that appear in our potentials are kept to confirm that there are no large enhancements of the corrections generated by the integrals. We also confirm that the vertex correction is suppressed by a factor of $\chi^2 = (m_\pi/p_{typ})^2$ when compared to the 2PE triangle contribution.

When including the mass difference $\delta = M_{B^*} - M_B$ in the loop integrals, we find that the heavy meson kinetic terms or the recoil terms ($\vec{p}^{\,2}/2M_B$) scale as the mass difference term, since $\vec{p} \sim \sqrt{M_B\,\delta}$, such that $\vec{p}^{\,2}/2M_B \to \delta/2$. Due to this, we cannot drop the recoil terms when including the mass difference term in the heavy meson propagator. As δ and $\vec{p}^{\,2}/2M_B$ scale equally, therefore if we keep δ in the expression then we need to keep the recoil terms also in the expression. To solve the loop integrals with δ and recoil terms, we treat δ and $\vec{p}^{\,2}/2M_B$ perturbatively (i.e. they appear only in the numerator and not in the denominator). With the perturbative approach of q_0, we expand the heavy meson propagator in inverse power of M_B. One then finds that the results for the loop integral results closely match the power counting expectations.

To confirm these statements, we conduct this exercise for I_{tr}, $I_{box}^{(2)}$, and the pertinent integral for the vertex correction. Furthermore, we employ the most general form of the vertices, the ones containing 4-momenta instead of 3-momenta. We start with the triangle contribution for $B\bar{B} \to B\bar{B}$ scattering. The explicit calculations of the integrals appearing in this section are provided in Appendix F.

5.5.1 The triangle loop integral

Consider once again the triangle diagram in the $B\bar{B} \to B\bar{B}$ case,

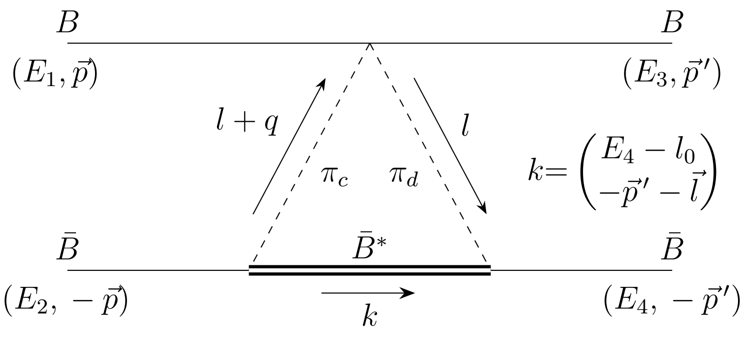


Figure 16: Triangle diagram for $B\bar{B} \to B\bar{B}$

The potential is given by

$$
iV_\Delta = \sum_\lambda \int \frac{d^4l}{(2\pi)^4}\frac{1}{4f_\pi^2}\Big((2l_0+q_0)\epsilon_{cdh}(\tau_1)_h\Big)\frac{i}{l^2-m_\pi^2}
$$

$$
\left(\frac{g}{2f_\pi}\Big(\epsilon_\mu^*(\lambda)(-l)^\mu\Big)(\tau_2)_d\right)\frac{i}{k^2-M_{B*}^2}\left(\frac{g}{2f_\pi}\Big(\epsilon_\nu(\lambda)(l+q)_\nu\Big)(\tau_2)_c\right)\frac{i}{(l+q)^2-m_\pi^2}\ , \tag{150}
$$

which simplifies to

$$
V_\Delta = \frac{g^2}{16f_\pi^4}(\vec{\tau}_1\cdot\vec{\tau}_2)I_\Delta\ , \tag{151}
$$

with the loop integral

$$
I_\Delta \ = \ i\int\frac{d^4l}{(2\pi)^4}\left(\frac{2l_0+q_0}{k^2-M_{B^*}^2}\right)\times\left(\frac{[l^\mu(l^\nu+q^\nu)](-g_{\mu\nu}+v_\mu v_\nu)}{\big[(l+q)^2-m_\pi^2\big]\big[l^2-m_\pi^2\big]}\right)\ , \tag{152}
$$

where we used $\sum_\lambda \epsilon_\mu^*\epsilon_\nu = -g_{\mu\nu}+v_\mu v_\nu$. The loop integral is simplified to

$$
I_\Delta = i\int\frac{d^4l}{(2\pi)^4}\frac{(2l_0+q_0)(-l\cdot(l+q)+l_0^2+l_0q_0)}{\big[(l+q)^2-m_\pi^2+i\epsilon\big]\big[l^2-m_\pi^2+i\epsilon\big]\big[(\bar{P}-l)^2-M_{B^*}^2+i\epsilon\big]}\ , \tag{153}
$$

where $\bar{P}=(E_4,\vec{p'})$ and $\bar{P}^2=M_B^2$. We split the momentum $\bar{P}^\mu = M_Bv^\mu + r^\mu$ and $v^\mu=(1,\vec{0})$, then

$$
\begin{aligned}
(\bar{P}-l)^2 &= M_B^2 - 2\bar{P}\cdot l + l^2 \\
&= M_B^2 - 2M_Bv\cdot l - 2\,r\cdot l + l^2\ .
\end{aligned} \tag{154}
$$

Applying these expressions

$$
\begin{aligned}
I_\Delta = i\int\frac{d^4l}{(2\pi)^4}&\frac{(2l_0+q_0)(-l\cdot(l+q)+l_0^2+l_0q_0)}{\big[(l+q)^2-m_\pi^2+i\epsilon\big]\big[l^2-m_\pi^2+i\epsilon\big]} \\
&\times\left(\frac{(2l_0+q_0)(-l\cdot(l+q)+l_0^2+l_0q_0)}{\big[-2M_Bv\cdot l-2\,r\cdot l+l^2+M_B^2-M_{B^*}^2+i\epsilon\big]}\right)\ .
\end{aligned}
$$

Now, we expand the heavy meson propagator in $(M_B)^{-1}$

$$
\begin{aligned}
I_\Delta &= i\int\frac{d^4l}{(2\pi)^4}\frac{(2l_0+q_0)(-l\cdot(l+q)+l_0^2+l_0q_0)}{\big[(l+q)^2-m_\pi^2+i\epsilon\big]\big[l^2-m_\pi^2+i\epsilon\big]\big[-2v\cdot l+i\epsilon\big]M_B} \\
&- i\int\frac{d^4l}{(2\pi)^4}\frac{(2l_0+q_0)(-l\cdot(l+q)+l_0^2+l_0q_0)\big[l^2-2\,r\cdot l+M_B^2-M_{B^*}^2\big]}{\big[(l+q)^2-m_\pi^2+i\epsilon\big]\big[l^2-m_\pi^2+i\epsilon\big]\big[-2v\cdot l+i\epsilon\big]^2M_B^2}\ ,
\end{aligned} \tag{155}
$$

which when simplified gives

$$
\begin{aligned}
I_\Delta &= \underbrace{i\int\frac{d^4l}{(2\pi)^4}\frac{(2l_0+q_0)(-l\cdot(l+q)+l_0^2+l_0q_0)}{\big[(l+q)^2-m_\pi^2+i\epsilon\big]\big[l^2-m_\pi^2+i\epsilon\big]\big[-2v\cdot l+i\epsilon\big]M_B}}_{I_{1,\Delta}} \\
&\quad \underbrace{-i\int\frac{d^4l}{(2\pi)^4}\frac{(2l_0+q_0)(-l\cdot(l+q)+l_0^2+l_0q_0)}{\big[(l+q)^2-m_\pi^2+i\epsilon\big]\big[-2v\cdot l+i\epsilon\big]^2M_B^2}}_{I_{2,\Delta}} \\
&\quad \underbrace{-i\int\frac{d^4l}{(2\pi)^4}\frac{(2l_0+q_0)(-l\cdot(l+q)+l_0^2+l_0q_0)\big[m_\pi^2+M_B^2-M_{B^*}^2\big]}{\big[(l+q)^2-m_\pi^2+i\epsilon\big]\big[l^2-m_\pi^2+i\epsilon\big]\big[-2v\cdot l+i\epsilon\big]^2M_B^2}}_{I_{3,\Delta}} \qquad (156) \\
&\quad \underbrace{+\,i\int\frac{d^4l}{(2\pi)^4}\frac{(2l_0+q_0)(-l\cdot(l+q)+l_0^2+l_0q_0)(2\,r\cdot l)}{\big[(l+q)^2-m_\pi^2+i\epsilon\big]\big[l^2-m_\pi^2+i\epsilon\big]\big[-2v\cdot l+i\epsilon\big]^2M_B^2}}_{I_{4,\Delta}} \\
&= I_{1,\Delta}+I_{2,\Delta}+I_{3,\Delta}+I_{4,\Delta}\ .
\end{aligned}
$$

The explicit calculations of these four integrals are presented in Appendix F.1. As an additional note, most of the integrals appearing in this calculation are also seen in Appendix B of [100]. The integrals that are not presented there will be solved in Appendix F.1. The explicit forms of $I_{1,\Delta}$, $I_{2,\Delta}$, $I_{3,\Delta}$, $I_{4,\Delta}$ are provided in Eq. (464), Eq. (466), Eq. (470), and Eq. (485), respectively.
We can verify that the results of the calculation match our expectations from the power counting. This is accomplished by examining the influence of the correction terms on the formally leading term. One observes from the expansion of I_Δ, that $I_{1,\Delta}$ is the leading order term (since $I_{1,\Delta}\sim\mathcal{O}(M_B^{-1})$) and $I_{2,\Delta}$, $I_{3,\Delta}$ and $I_{4,\Delta}$ are the NLO terms (as they $\sim\mathcal{O}(M_B^{-2})$) hence, we take the ratio of the sum of $I_{2,\Delta}$, $I_{3,\Delta}$ and $I_{4,\Delta}$ over $I_{1,\Delta}$ to see the effect of corrections.
The total effect of corrections of I_Δ integral

$$
\frac{I_{2,\Delta}+I_{3,\Delta}+I_{4,\Delta}}{I_{1,\Delta}}\approx 14\%\ , \qquad (157)
$$

where we used $\vec{r}=\vec{q}=500$ MeV, $r_0=q_0=50$ MeV and $\delta=46$ MeV. This result is consistent with the power counting, which predicts for this ratio $\approx 11\%$, which is in the order of $(m_\pi/p_{\rm typ})^2\!\sim\!\chi^2$.

5.5.2 The crossed-box loop integral

Consider the crossed box diagram for $B\bar{B}\to B\bar{B}$ scattering,

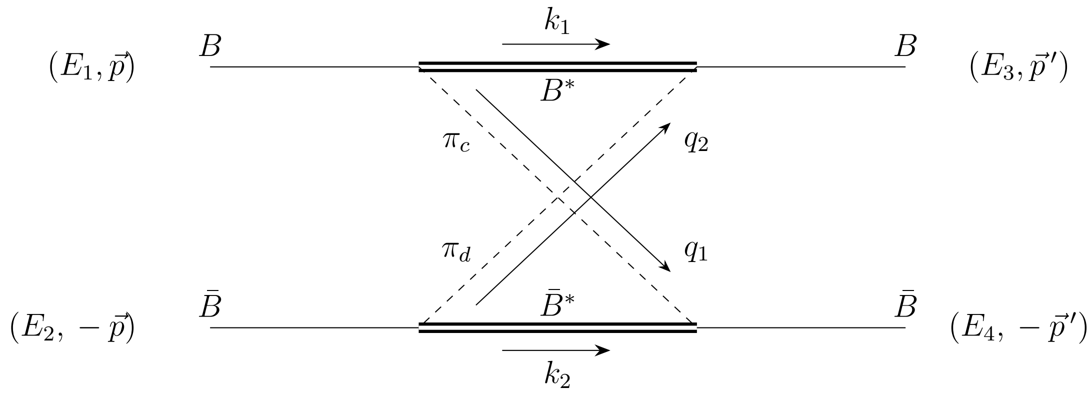


Figure 17: Crossed box diagram for $B\bar{B} \to B\bar{B}$

$$iV_{box} = \sum_{\lambda_1,\lambda_2} \int \frac{d^4l}{(2\pi)^4} \left(\frac{g}{2f_\pi} \Big(\epsilon_\mu(\lambda_1)(-q_2)^\mu \Big) (\tau_1)_d \right) \frac{i}{k_1^2 - m_B^2} \left(\frac{g}{2f_\pi} \Big(\epsilon^*_\nu(\lambda_1)(q_1)^\nu \Big) (\tau_1)_c \right)$$
$$\frac{i}{q_1^2 - m_\pi^2} \left(\frac{g}{2f_\pi} \Big(\epsilon_\rho(\lambda_2)(-q_1)^\rho \Big) (\tau_2^c)_c \right) \frac{i}{k_2^2 - m_B^2} \left(\frac{g}{2f_\pi} \Big(\epsilon^*_\sigma(\lambda_2)(q_2)^\sigma \Big) (\tau_2^c)_d \right) \frac{i}{q_2^2 - m_\pi^2} \,, \tag{158}$$

where $q_1 = l = (l_0, \vec{l})$, $q_2 = l + q$ and $q = (E_3 - E_1, \vec{p'} - \vec{p})$. Simplifying the above relation gives

$$V_{box} = \frac{g^4}{16 f_\pi^4} \Big(-3 - 2(\vec{\tau_1} \cdot \vec{\tau_2}) \Big) \, I_{cb} \ . \tag{159}$$

The loop integral is given by

$$I_{cb} = i \int \frac{d^4l}{(2\pi)^4} \left(\frac{(-g_{\mu\nu} + v_\mu v_\nu)(-g_{\rho\sigma} + v_\rho v_\sigma)}{\big[k_1^2 - M_{B^*}^2\big]\big[k_1^2 - M_{B^*}^2\big]} \right) \times \left(\frac{[l^\mu(l^\nu + q^\nu)][l^\rho(l^\sigma + q^\sigma)]}{\big[(l+q)^2 - m_\pi^2\big]\big[l^2 - m_\pi^2\big]} \right) , \tag{160}$$

where $k_1 = P - l$, $k_2 = \bar{P}' - l$ with $P = (E_1, \vec{p})$ and $\bar{P}' = (E_4, -\vec{p'})$. Renaming the numerator as $\Pi = (-g_{\mu\nu} + v_\mu v_\nu)(-g_{\rho\sigma} + v_\rho v_\sigma)[l^\mu(l^\nu + q^\nu)][l^\rho(l^\sigma + q^\sigma)]$ for simplicity

$$I_{cb} = i \int \frac{d^4l}{(2\pi)^4} \frac{\Pi}{\big[(l+q)^2 - m_\pi^2 + i\epsilon\big]\big[l^2 - m_\pi^2 + i\epsilon\big]\big[(\bar{P}' - l)^2 - M_{B^*}^2 + i\epsilon\big]} \times \left(\frac{1}{\big[(P-l)^2 - M_{B^*}^2 + i\epsilon\big]} \right) , \tag{161}$$

where $\bar{P'}^2 = P^2 = M_B^2$. We split the momentum as $P^\mu = M_B v^\mu + r^\mu$ and $(\bar{P}')^\mu = M_B v^\mu + s^\mu$

$$
\begin{aligned}
(\bar{P}' - l)^2 &= M_B^2 - 2\bar{P}' \cdot l + l^2 \\
&= M_B^2 - 2M_B v \cdot l - 2\, s \cdot l + l^2
\end{aligned}
\tag{162}
$$

and

$$
\begin{aligned}
(P^2 - l)^2 &= M_B^2 - 2P \cdot l + l^2 \\
&= M_B^2 - 2M_B v \cdot l - 2\, r \cdot l + l^2 \ .
\end{aligned}
\tag{163}
$$

Expanding the heavy meson propagator in $(M_B)^{-1}$ as in the previous case

$$
\begin{aligned}
I_{cb} = i \int \frac{d^4 l}{(2\pi)^4} & \frac{\Pi}{\big[(l+q)^2 - m_\pi^2 + i\epsilon\big]\big[l^2 - m_\pi^2 + i\epsilon\big]\big[-2v \cdot l + i\epsilon\big]^2 M_B^2} \\
- 2i \int \frac{d^4 l}{(2\pi)^4} & \frac{\Pi(l^2 - r \cdot l - s \cdot l + M_B^2 - M_{B^*}^2)}{\big[(l+q)^2 - m_\pi^2 + i\epsilon\big]\big[l^2 - m_\pi^2 + i\epsilon\big]\big[2v \cdot l + i\epsilon\big]^3 M_B^3} \ .
\end{aligned}
\tag{164}
$$

By adding and subtracting m_π^2 in the numerator and using $v \cdot l = l_0$, then

$$
\begin{aligned}
I_{cb} = & \underbrace{i \int \frac{d^4 l}{(2\pi)^4} \frac{\Pi}{4M_B^2\big[(l+q)^2 - m_\pi^2 + i\epsilon\big]\big[l^2 - m_\pi^2 + i\epsilon\big]\big[l_0 - i\epsilon\big]^2}}_{I_{1,cb}} \\
& \underbrace{+ i \int \frac{d^4 l}{(2\pi)^4} \frac{\Pi}{4M_B^3\big[(l+q)^2 - m_\pi^2 + i\epsilon\big]\big[l_0 - i\epsilon\big]^3}}_{I_{2,cb}} \\
& \underbrace{+ i \int \frac{d^4 l}{(2\pi)^4} \frac{\Pi(m_\pi^2 + M_B^2 - M_{B^*}^2)}{4M_B^3\big[(l+q)^2 - m_\pi^2 + i\epsilon\big]\big[l^2 - m_\pi^2 + i\epsilon\big]\big[l_0 - i\epsilon\big]^3}}_{I_{3,cb}} \\
& \underbrace{-i \int \frac{d^4 l}{(2\pi)^4} \frac{\Pi(r \cdot l + s \cdot l)}{4M_B^3\big[(l+q)^2 - m_\pi^2 + i\epsilon\big]\big[l^2 - m_\pi^2 + i\epsilon\big]\big[l_0 - i\epsilon\big]^3}}_{I_{4,cb}} \\
& = I_{1,cb} + I_{2,cb} + I_{3,cb} + I_{4,cb} \ .
\end{aligned}
\tag{165}
$$

As in the previous case, we have four integrals to solve and the calculations of these four integrals are presented in Appendix F.2.
The integrated forms of $I_{1,cb}$, $I_{2,cb}$, $I_{3,cb}$, $I_{4,cb}$ are mentioned in Eq. (488), Eq. (489), Eq. (494) and Eq. (499), respectively.

Once again, we verify the results of the calculation by checking the power counting expectations. From the expansion of I_{cb}, we see that $I_{1,cb}$ is the leading order term and $I_{2,cb}$, $I_{3,cb}$ and $I_{4,cb}$ are the NLO terms. So, we take the ratio of the sum of $I_{2,cb}$, $I_{3,cb}$ and $I_{4,cb}$ over I_1 to see the effect of corrections.
and the total effect of the corrections with respect to the leading order term

$$\frac{I_{2,cb}+I_{3,cb}+I_{4,cb}}{I_{1,cb}} \approx 12.3\% \ , \tag{166}$$

where, $\vec{q}$ = 500 MeV, q_0 = 50 MeV, $\vec{r} = \vec{s}$ = 250 MeV and $r_0 = s_0$ = 25 MeV with δ = 46 MeV. Again, this result is consistent with the power counting, which predicts for this ratio $\approx$ 9%. Therefore, the effect of corrections are in the order of $(m_\pi/p_{\text{typ}})^2 \sim \chi^2$.

5.5.3 The vertex loop integral

Consider the vertex diagram

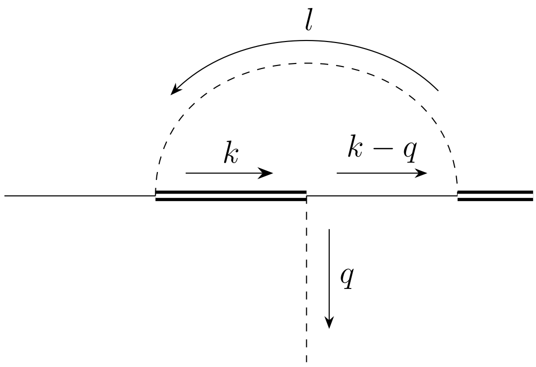


Figure 18: Vertex diagram

$$iV_{vert} = \sum_{\lambda_1} \int \frac{d^4 l}{(2\pi)^4} \left(\frac{g}{2f_\pi} \left(\epsilon^*_\mu(\lambda_1)(-l)^\mu \right)(\tau_1)_d \right) \frac{i}{k^2 - M^2_{B*}}$$
$$\left(\frac{g}{2f_\pi} \left(\epsilon^*_\rho(\lambda_1)(q)^\rho \right)(\tau_1)_c \right) \frac{i}{(k-q)^2 - M^2_B} \left(\frac{g}{2f_\pi} \left(\epsilon_\nu(\lambda_2)(l)^\nu \right)(\tau_1)_d \right) \frac{i}{l^2 - m^2_\pi} \ , \tag{167}$$

where $k = (E_p + l_0, \vec{p} + \vec{l})$, simplifying the potential gives

$$
\begin{aligned}
V_{vert} &= \frac{g^3}{8f_\pi^3}(\tau_1)_c(-g_{\mu\rho}+v_\rho v_\mu)q^\rho\epsilon_\nu\frac{g^{\mu\nu}}{D}I_{vert} \\
&= -\frac{g^3}{8f_\pi^3}(\tau_1)_c\frac{(\epsilon\cdot q)}{D}I_{vert}\ .
\end{aligned}
\tag{168}
$$

The loop integral is

$$
I_{vert} = i\int\frac{d^4l}{(2\pi)^4}\left(\frac{1}{\big[k^2-M_{B^*}^2\big]\big[(k-q)^2-M_B^2\big]}\right)\times\left(\frac{l^2}{\big[l^2-m_\pi^2\big]}\right)\ , \tag{169}
$$

where we used $l_\mu l_\nu \to \frac{1}{D}l^2 g_{\mu\nu}$ due to symmetry inside loops, D is the number of dimensions $D=4-\epsilon$ and $\sum_\lambda \epsilon^*_\mu\epsilon_\rho = -g_{\mu\rho}+v_\mu v_\rho$.
We split the momentum $\bar{P}^\mu = M_B v^\mu + r^\mu$, such that

$$
\begin{aligned}
k^2-M_{B^*}^2 &= (l+P)^2-M_{B^*}^2 \\
&= M_B^2-M_{B^*}^2+2M_Bv\cdot l+2\,r\cdot l+l^2
\end{aligned}
\tag{170}
$$

and

$$
\begin{aligned}
(k-q)^2-M_B^2 &= (l+P-q)^2-M_{B^*}^2 \\
&= l^2+q^2+2M_Bv\cdot l+2\,r\cdot l-2M_Bv\cdot q-2\,r\cdot q-2\,l\cdot q\ .
\end{aligned}
\tag{171}
$$

Next, we expand the heavy meson propagator in the inverse powers of M_B like in the previous cases

$$
\begin{aligned}
I_{vert} &= i\int\frac{d^4l}{(2\pi)^4}\frac{l^2}{M_B^2\big[l^2-m_\pi^2+i\epsilon\big]\big[2v\cdot l+i\epsilon\big]^2} \\
&\quad - i\int\frac{d^4l}{(2\pi)^4}\frac{\big[2l^2+q^2-2l\cdot q+4\,r\cdot l-2r\cdot q+M_B^2-M_{B^*}^2\big]l^2}{M_B^3\big[l^2-m_\pi^2+i\epsilon\big]\big[2v\cdot l+i\epsilon\big]^3}\ ,
\end{aligned}
\tag{172}
$$

by adding and subtracting m_π^2 in the numerator and $v\cdot l = l_0$

$$
\begin{aligned}
I_{vert} &= \underbrace{i\int\frac{d^4l}{(2\pi)^4}\frac{l^2}{4M_B^2\big[l^2-m_\pi^2+i\epsilon\big]\big[l_0+i\epsilon\big]^2}}_{I_{1,vert}} \\
&\quad \underbrace{-i\int\frac{d^4l}{(2\pi)^4}\frac{2l^2}{8M_B^3\big[l_0+i\epsilon\big]^3}}_{I_{2,vert}}\underbrace{-i\int\frac{d^4l}{(2\pi)^4}\frac{\big[2m_\pi^2+q^2-2\,r\cdot q+M_B^2-M_{B^*}^2\big]l^2}{8M_B^3\big[l^2-m_\pi^2+i\epsilon\big]\big[l_0+i\epsilon\big]^3}}_{I_{3,vert}} \\
&\quad \underbrace{-i\int\frac{d^4l}{(2\pi)^4}\frac{\big[4\,r\cdot l-2\,l\cdot q\big]l^2}{8M_B^3\big[l^2-m_\pi^2+i\epsilon\big]\big[l_0+i\epsilon\big]^3}}_{I_{4,vert}} \\
&= I_{1,vert}+I_{2,vert}+I_{3,vert}+I_{4,vert}
\end{aligned}
\tag{173}
$$

We solve for each of the four integrals like in the previous case, the calculations of the four integrals are given in Appendix F.3. The integrated forms of $I_{1,vert}$, $I_{2,vert}$, $I_{3,vert}$ and $I_{4,vert}$ are provided in Eq. (511), Eq. (512), Eq. (514) and Eq. (516), respectively.

Like in the earlier cases, we check if the results of the calculation are in line with the power counting. From the expansion of I_{vert}, we see that $I_{1,vert}$ is the LO term and $I_{2,vert}$, $I_{3,vert}$ and $I_{4,vert}$ are the NLO terms.

The total effect of corrections with respect to the LO term is

$$\frac{I_{2,vert} + I_{3,vert} + I_{4,vert}}{I_{1,vert}} \approx 39\% \ , \tag{174}$$

for $\vec{q} = \vec{r} = 500$ MeV, $q_0 = r_0 = 50$ MeV and $\delta m = 46$ MeV.
Once again, this result is consistent with the power counting, which predicts for this ratio $\approx 37\%$.

From our power counting scheme in Sec. 3.6, we observed that the vertex correction potential is suppressed by a factor of $(m_\pi/p_{\text{typ}})^2 = \chi^2$ in comparison to 2PE triangle diagram potential. We can check this statement, by comparing the effect of the leading term in I_Δ to the leading term in I_{vert}, employing the leading term $I_{1,\Delta}$ as the loop integral in V_Δ (Eq. (151)) and similarly $I_{1,vert}$ in V_{vert} (Eq. (168)). One finds for the ratio of potentials when considering the integrals

$$\frac{V_\Delta}{V_{vert}} \approx 0.06 \sim \chi^2 \ , \tag{175}$$

for $\vec{q} = 500$ MeV and $q_0 = 50$ MeV. This result agrees with the power counting which predicts for this ratio $\approx 0.05 = (m_\pi/p_{\text{typ}})^2 = \chi^2$. Therefore, the vertex corrections are indeed suppressed, in line with the power counting.

6 Partial wave decomposition

In this section, the effective potentials from the Sec. 5.3 are decomposed into four channels $J^{PC} = 0^{++}, 1^{++}, 1^{+-}, 2^{++}$. For those we picked the following bases [9]:

$$
\begin{aligned}
0^{++} &: \alpha,\beta = \Big\{B\bar{B}(^1S_0), B^*\bar{B}^*(^1S_0), B^*\bar{B}^*(^5D_0)\Big\} \\
1^{++} &: \alpha,\beta = \Big\{B\bar{B}^*(^3S_1,+), B\bar{B}^*(^3D_1,+), B^*\bar{B}^*(^5D_1)\Big\} \\
1^{+-} &: \alpha,\beta = \Big\{B\bar{B}^*(^3S_1,-), B\bar{B}^*(^3D_1,-), \\
&\qquad\qquad B^*\bar{B}^*(^3S_1), B^*\bar{B}^*(^3D_1)\Big\} \\
2^{++} &: \alpha,\beta = \Big\{B\bar{B}(^1D_2), B\bar{B}^*(^3D_2), B^*\bar{B}^*(^5S_2), \\
&\qquad B^*\bar{B}^*(^1D_2), B^*\bar{B}^*(^5D_2), B^*\bar{B}^*(^5G_2)\Big\} \ . \qquad (176)
\end{aligned}
$$

The individual partial waves are labeled as $^{2S+1}L_J$ with S, L and J denoting the total spin, the angular momentum and the total momentum, respectively. For $B\bar{B}^*$ states, the sign included in parenthesis corresponds to their C-parity, which is provided by

$$
|B\bar{B}^*, \pm\rangle = \frac{1}{\sqrt{2}}\big(|B\bar{B}^*\rangle \pm |B^*\bar{B}\rangle\big) \ . \qquad (177)
$$

The partial wave projection of the potentials is done using the formalism of Ref. [9], which gives

$$
V_{\alpha\beta}(J^{PC}) = \frac{1}{2J+1}\int \frac{d\Omega_n}{4\pi}\frac{d\Omega_{n'}}{4\pi}\,\mathrm{Tr}\big[P^\dagger(\alpha,\vec{n})VP(\beta,\vec{n'})\big] \ , \qquad (178)
$$

where $\vec{n} = \vec{p}/|\vec{p}|$, $\vec{n'} = \vec{p'}/|\vec{p'}|$, $P^\dagger(\alpha,\vec{n})$ and $P(\beta,\vec{n'})$ are outgoing and incoming normalised projectors respectively with α and β being the bases states mentioned in Eq. (176). The projectors are normalised as:

$$
\int \frac{d\Omega_n}{4\pi} P^\dagger(\alpha,\vec{n})P(\alpha,\vec{n}) = 2J+1 \ . \qquad (179)
$$

Finally, V are the potentials calculated in Sec. 5.3. Due to the spatial symmetry of this $2 \to 2$ reaction, we are only required to consider the angle θ between the incoming and outgoing momentum, denoted by $x = \vec{n}\cdot\vec{n'} = \cos(\theta)$. Then the above expression simplifies to

$$
V_{\alpha\beta}\big(J^{PC}\big) = \frac{1}{2J+1}\int_{-1}^{+1}\frac{dx}{2}\,\mathrm{Tr}\big[P^\dagger(\alpha,\vec{n'})\,V\,P(\beta,\vec{n})\big] \ , \qquad (180)
$$

with the trace taken over the indices of angular momentum, since J is conserved. The partial wave projectors used to calculate the potentials are presented in Appendix B.

The partial wave projected potentials for $B^{(*)}\bar{B}^{(*)} \to B^{(*)}\bar{B}^{(*)}$ in the $J^{PC} = 0^{++}, 1^{++}, 1^{+-}, 2^{++}$ channel are presented here. The pertinent integrals of the decomposition are abbreviated as $Q(p', p)$, $R(p', p)$, $S(p', p)$. Their explicit expression are given in Appendix E. We denote $|\vec{p'}| = p'$ and $|\vec{p}| = p$.

6.1 $B^{(*)}\bar{B}^{(*)} \to B^{(*)}\bar{B}^{(*)}$

6.1.1 $\mathbf{J^{PC} = 0^{++}}$

$$V_{CI}^{NLO}(0^{++}) = \begin{pmatrix} \mathcal{C}_d + \frac{1}{2}\mathcal{C}_f + (\mathcal{D}_d + \frac{1}{2}\mathcal{D}_f)(p^2 + p'^2) & \frac{1}{2}\sqrt{3}(\mathcal{C}_f + \mathcal{D}_f(p^2 + p'^2)) & -\sqrt{3}\mathcal{D}_{SD}p'^2 \\ \frac{1}{2}\sqrt{3}(\mathcal{C}_f + \mathcal{D}_f(p^2 + p'^2)) & \mathcal{C}_d - \frac{1}{2}\mathcal{C}_f + (\mathcal{D}_d - \frac{1}{2}\mathcal{D}_f)(p^2 + p'^2) & -\mathcal{D}_{SD}p'^2 \\ -\sqrt{3}\mathcal{D}_{SD}p^2 & -\mathcal{D}_{SD}p^2 & 0 \end{pmatrix} . \tag{181}$$

$$V_{1PE}(0^{++}) = -\frac{g^2}{4f_\pi^2}(\vec{\tau}_1 \cdot \vec{\tau}_2) \begin{pmatrix} 0 & \frac{1}{\sqrt{3}}Q_2 & \frac{1}{\sqrt{6}}(Q_2 - 3Q_{n'}) \\ \frac{1}{\sqrt{3}}Q_2 & -\frac{2}{3}Q_2 & \frac{1}{\sqrt{2}}(\frac{Q_2}{3} - Q_{n'}) \\ \frac{1}{\sqrt{6}}(Q_2 - 3Q_n) & \frac{1}{\sqrt{2}}(\frac{Q_2}{3} - Q_n) & \frac{3}{2}(5Q_2 - 6Q_n - 6Q_{n'} + 18Q_x - 9Q_{x^2}) \end{pmatrix} \tag{182}$$

and

$$V_{2PE}(0^{++}) = \frac{1}{16\pi^2 f_\pi^4} \begin{pmatrix} S_0(p', p) & \frac{\sqrt{3}}{16}g^4(V_{2PE}^{0^{++}})_{12} & \frac{3\sqrt{3}}{32\sqrt{2}}g^4(V_{2PE}^{0^{++}})_{13} \\ \frac{\sqrt{3}}{16}g^4(V_{2PE}^{0^{++}})_{21} & (V_{2PE}^{0^{++}})_{22} & \frac{-1}{64\sqrt{2}}g^4(V_{2PE}^{0^{++}})_{23} \\ \frac{3\sqrt{3}}{32\sqrt{2}}g^4(V_{2PE}^{0^{++}})_{31} & \frac{-1}{64\sqrt{2}}g^4(V_{2PE}^{0^{++}})_{32} & (V_{2PE}^{0^{++}})_{33} \ , \end{pmatrix} \tag{183}$$

where

$$\begin{aligned} S_0 = & \int_{-1}^{1} \frac{dx}{2}(\vec{\tau}_1 \cdot \vec{\tau}_2)\vec{q}^2 \Bigg\{ \mathcal{R}\left[\frac{23}{192}g^4 + \frac{1}{24}\right] + \left(\frac{5}{576}g^4 - \frac{5}{72}\right) + \ln\left(\frac{m_\pi}{\mu}\right)\left(\frac{23}{96}g^4 + \frac{1}{12}\right) \\ & + L(q)\left(\frac{23}{96}g^4 + \frac{1}{12}\right) + \mathcal{O}(\chi^4) \Bigg\} \\ = & (\vec{\tau}_1 \cdot \vec{\tau}_2) \Bigg\{ \mathcal{R}(p'^2 + p^2)\left(\frac{23}{192}g^4 + \frac{1}{24}\right) + (p'^2 + p^2)\left(\frac{5}{576}g^4 - \frac{5}{18}\right) + (p'^2 + p^2) \\ & \left(\frac{23}{96}g^4 + \frac{1}{12}\right)\ln\left(\frac{m_\pi}{\mu}\right) + R_2(p', p)\left(\frac{23}{96}g^4 + \frac{1}{12}\right) \Bigg\} + \mathcal{O}(\chi^4) \ , \end{aligned} \tag{184}$$

$$(V_{2PE}^{0^{++}})_{12} = (V_{2PE}^{0^{++}})_{21} = (p'^{\,2} + p^2)\Bigg[-\mathcal{R} + 1 - 2\ln\left(\frac{m_\pi}{\mu}\right)\Bigg] + 2R_2(p',p) + \mathcal{O}(\chi^4) \quad (185)$$

$$(V_{2PE}^{0^{++}})_{13} = \sqrt{\frac{2}{3}}(p'^{\,2})\Bigg[-\mathcal{R} + 1 - 2\ln\left(\frac{m_\pi}{\mu}\right)\Bigg] - 2R_{n'}(p',p) + \frac{2}{3}R_2(p',p) + \mathcal{O}(\chi^4)\ , \quad (186)$$

$$(V_{2PE}^{0^{++}})_{22} = S_0 + \frac{g^4}{16}\Bigg\{ 2\mathcal{R}(p'^{\,2} + p^2) - \frac{1}{9}(p'^{\,2} + p^2) + \big[4(p'^{\,2} + p^2)\big]\ln\left(\frac{m_\pi}{\mu}\right) + 4R_2(p',p)\Bigg\} + \mathcal{O}(\chi^4)\ , \quad (187)$$

$$(V_{2PE}^{0^{++}})_{23} = 2p'^{\,2}\Bigg[2\mathcal{R} - \frac{1}{9} - \ln\left(\frac{m_\pi}{\mu}\right)\Bigg] - \frac{7}{2}\Bigg\{3R_{n'}(p',p) - R_2(p',p)\Bigg\} + \mathcal{O}(\chi^4)\ , \quad (188)$$

$$(V_{2PE}^{0^{++}})_{31} = \sqrt{\frac{2}{3}}(p^2)\Bigg[-\mathcal{R} + 1 - 2\ln\left(\frac{m_\pi}{\mu}\right)\Bigg] - 2R_n(p',p) + \frac{2}{3}R_2(p',p) + \mathcal{O}(\chi^4)\ , \quad (189)$$

$$(V_{2PE}^{0^{++}})_{32} = 2p^2\Bigg[2\mathcal{R} - \frac{1}{9} - \ln\left(\frac{m_\pi}{\mu}\right)\Bigg] - \frac{7}{2}\Bigg\{3R_n(p',p) - R_2(p',p)\Bigg\} + \mathcal{O}(\chi^4)\ , \quad (190)$$

$$(V_{2PE}^{0^{++}})_{33} = S_2 + \frac{g^4}{16}\Bigg\{\frac{-15}{64p'p}\left[\frac{\rho^4}{4} - \rho^4 L(\rho)\right]_{p'-p}^{p'+p} - \frac{45}{8}R_{x^2}(p',p) - \frac{21}{8}\Bigg(R_n(p',p) + R_{n'}(p',p) - \frac{1}{2}R_2(p',p) - 3R_x(p',p)\Bigg)\Bigg\} + \mathcal{O}(\chi^4)\ , \quad (191)$$

and

$$S_2 = \int_{-1}^{1} \frac{dx}{2} \frac{(3x^2-1)}{2} (\vec{\tau}_1 \cdot \vec{\tau}_2) \vec{q}^{\,2} \Bigg\{ \mathcal{R} \left[\frac{23}{192} g^4 + \frac{1}{24} \right] + \left(\frac{5}{576} g^4 - \frac{5}{72} \right) + \ln\left(\frac{m_\pi}{\mu}\right) \left(\frac{23}{96} g^4 + \frac{1}{12} \right) + L(q) \left(\frac{23}{96} g^4 + \frac{1}{12} \right) + \mathcal{O}(\chi^4) \Bigg\}$$

$$= \frac{(\vec{\tau}_1 \cdot \vec{\tau}_2)}{8 p'^2 p^2} \Bigg\{ \Big[3R_6 - 6(p'^2 + p^2) R_4 + \big(3(p'^2+p^2)^2 - 4p'^2 p^2\big) R_2(p',p) \Big] \left(\frac{23}{96} g^4 + \frac{1}{12} \right) \Bigg\} + \mathcal{O}(\chi^4) \ . \quad (192)$$

6.1.2 $\mathbf{J^{PC} = 1^{++}}$

$$V_{CI}^{NLO}(1^{++}) = \begin{pmatrix} \mathcal{C}_d + \mathcal{C}_f + (\mathcal{D}_d + \mathcal{D}_f)(p^2 + p'^2) & -\mathcal{D}_{SD} p'^2 & -\sqrt{3}\mathcal{D}_{SD} p'^2 \\ -\mathcal{D}_{SD} p'^2 & 0 & 0 \\ -\sqrt{3}\mathcal{D}_{SD} p^2 & 0 & 0 \end{pmatrix} , \quad (193)$$

$$V_{1PE}(1^{++}) = -\frac{g^2}{4 f_\pi^2} (\vec{\tau}_1 \cdot \vec{\tau}_2) \begin{pmatrix} \frac{1}{3} Q_2 & -\frac{1}{\sqrt{18}}(3Q_{n'} - Q_2) & \frac{7}{18\sqrt{6}}(3Q_{n'} - Q_2) \\ \frac{1}{\sqrt{18}}(3Q_n - Q_2) & -\frac{1}{6}(V_{1PE}^{1^{++}})_{22} & \frac{1}{18\sqrt{3}}(V_{1PE}^{1^{++}})_{23} \\ \frac{7}{18\sqrt{6}}(3Q_n - Q_2) & \frac{1}{18\sqrt{3}}(V_{1PE}^{1^{++}})_{32} & \frac{5}{162}(V_{1PE}^{1^{++}})_{33} \end{pmatrix} , \quad (194)$$

$$V_{2PE}(1^{++}) = \frac{1}{16\pi^2 f_\pi^4} \begin{pmatrix} (V_{2PE}^{1^{++}})_{11} & \frac{3}{32\sqrt{2}} g^4 (V_{2PE}^{1^{++}})_{12} & -\frac{9}{16\sqrt{24}} g^4 (V_{2PE}^{1^{++}})_{13} \\ \frac{3}{32\sqrt{2}} g^4 (V_{2PE}^{1^{++}})_{21} & (V_{2PE}^{1^{++}})_{22} & \frac{3\sqrt{3}}{32} g^4 (V_{2PE}^{1^{++}})_{23} \\ -\frac{9}{16\sqrt{24}} g^4 (V_{2PE}^{1^{++}})_{31} & \frac{3\sqrt{3}}{32} g^4 (V_{2PE}^{1^{++}})_{32} & (V_{2PE}^{1^{++}})_{33} \end{pmatrix} , \quad (195)$$

where

$$(V_{2PE}^{1^{++}})_{11} = S_0 + \frac{g^4}{16} \Bigg\{ (p'^{\,2} + p^2) \left[-\mathcal{R} + 1 - 2\ln\left(\frac{m_\pi}{\mu}\right) \right] - 2R_2(p',p) \Bigg\} + \mathcal{O}(\chi^4) \ , \quad (196)$$

$$(V_{2PE}^{1^{++}})_{12} = (V_{2PE}^{1^{++}})_{13} = \frac{2}{3}(p'^{\,2}) \left[-\mathcal{R} + 1 - 2\ln\left(\frac{m_\pi}{\mu}\right) \right] + \frac{2}{3} R_2(p',p) - 2R_{n'}(p',p) + \mathcal{O}(\chi^4) \ , \quad (197)$$

$$(V_{2PE}^{1^{++}})_{21} = (V_{2PE}^{1^{++}})_{31} = \frac{2}{3}(p^2)\Big[-\mathcal{R} + 1 - 2\ln\left(\frac{m_\pi}{\mu}\right)\Big] + \frac{2}{3}R_2(p',p) - 2R_n(p',p) + \mathcal{O}(\chi^4) \ , \quad (198)$$

$$(V_{2PE}^{1^{++}})_{22} = S_2 + g^4\frac{3}{32}\Big\{ R_0(p',p) - 3R_{2x}(p',p) + 3R_x(p',p) - R_{n'}(p',p) - R_n(p',p) + \frac{1}{3}R_2(p',p)\Big\} + \mathcal{O}(\chi^4) \ , \quad (199)$$

$$(V_{2PE}^{1^{++}})_{23} = (V_{2PE}^{1^{++}})_{32} = \frac{2}{3}R_2(p',p) + 3R_x(p',p) - R_{n'}(p',p) - R_n(p',p) - R_{x^2}(p',p) + \mathcal{O}(\chi^4) \ , \quad (200)$$

$$(V_{2PE}^{1^{++}})_{33} = 7S_2 + \frac{g^4}{16}\Big\{ \frac{45}{2}R_2(p',p) - \frac{135}{2}R_{x^2}(p',p) - \frac{7}{12}\Big(15R_n(p',p) + 15R_{n'}(p',p) + 8R_2(p',p) - 45R_x(p',p) - 39R_{x^2}(p',p)\Big)\Big\} + \mathcal{O}(\chi^4) \ . \quad (201)$$

6.1.3 $\mathbf{J^{PC} = 1^{+-}}$

$$V_{CI}^{NLO}(1^{+-}) = \begin{pmatrix} \mathcal{C}_d + \mathcal{D}_d(p^2+p'^2) & \mathcal{D}_{SD}p'^{\,2} & \mathcal{C}_f + \mathcal{D}_f(p^2+p'^2) & \mathcal{D}_{SD}p'^{\,2} \\ \mathcal{D}_{SD}p^2 & 0 & \mathcal{D}_{SD}p^2 & 0 \\ \mathcal{C}_f + \mathcal{D}_f(p^2+p'^2) & \mathcal{D}_{SD}p'^{\,2} & \mathcal{C}_d + \mathcal{D}_d(p^2+p'^2) & \mathcal{D}_{SD}p'^{\,2} \\ \mathcal{D}_{SD}p^2 & 0 & \mathcal{D}_{SD}p^2 & 0 \end{pmatrix} , \quad (202)$$

$$V_{1PE}(1^{+-}) = -\frac{g^2}{4f_\pi^2}(\vec{\tau}_1\cdot\vec{\tau}_2) \times \begin{pmatrix} -\frac{1}{3}Q_2 & \frac{1}{\sqrt{18}}(3Q_{n'}-Q_2) & \frac{2}{3}Q_2 & \frac{1}{\sqrt{18}}(3Q_{n'}-Q_2) \\ \frac{1}{\sqrt{18}}(3Q_n-Q_2) & -\frac{1}{6}(V_{1PE}^{1^{+-}})_{22} & \frac{1}{\sqrt{18}}(3Q_n-Q_2) & (V_{1PE}^{1^{+-}})_{24} \\ \frac{2}{3}Q_2 & \frac{1}{\sqrt{18}}(3Q_{n'}-Q_2) & -\frac{1}{3}Q_2 & \frac{1}{\sqrt{18}}(3Q_{n'}-Q_2) \\ \frac{1}{\sqrt{18}}(3Q_n-Q_2) & (V_{1PE}^{1^{+-}})_{24} & \frac{1}{\sqrt{18}}(3Q_n-Q_2) & -\frac{1}{6}(V_{1PE}^{1^{+-}})_{22} \end{pmatrix} , \quad (203)$$

where

$$(V_{1PE}^{1^{+-}})_{22} = 3Q_n(p',p) + 3Q_{n'}(p',p) - 9Q_x(p',p) - Q_{x^2}(p',p) \ , \quad (204)$$

$$(V_{1PE}^{1^{+-}})_{24} = \frac{1}{2}Q_n(p',p) + \frac{1}{2}Q_{n'}(p',p) - \frac{3}{2}Q_x(p',p) + \frac{3}{2}Q_{x^2}(p',p) - \frac{2}{3}Q_2(p',p)\ . \quad (205)$$

$$V_{2PE}(1^{+-}) = \frac{1}{16\pi^2 f_\pi^4} \begin{pmatrix} (V_{2PE}^{1^{+-}})_{11} & \frac{-3}{32\sqrt{2}}\,g^4(V_{2PE}^{1^{+-}})_{12} & (-2g^4)(V_{2PE}^{1^{+-}})_{13} & \frac{-3}{32\sqrt{2}}\,g^4(V_{2PE}^{1^{+-}})_{14} \\ \frac{-3}{32\sqrt{2}}\,g^4(V_{2PE}^{1^{+-}})_{21} & (V_{2PE}^{1^{+-}})_{22} & \frac{-3}{\sqrt{2}}\,g^4(V_{2PE}^{1^{+-}})_{23} & g^4(V_{2PE}^{1^{+-}})_{24} \\ (-2g^4)(V_{2PE}^{1^{+-}})_{31} & \frac{-3}{\sqrt{2}}\,g^4(V_{2PE}^{1^{+-}})_{32} & (V_{2PE}^{1^{+-}})_{33} & \frac{1}{64\sqrt{2}}\,g^4(V_{2PE}^{1^{+-}})_{34} \\ \frac{-3}{32\sqrt{2}}\,g^4(V_{2PE}^{1^{+-}})_{41} & g^4(V_{2PE}^{1^{+-}})_{42} & \frac{1}{64\sqrt{2}}\,g^4(V_{2PE}^{1^{+-}})_{43} & (V_{2PE}^{1^{+-}})_{44} \end{pmatrix}\ , \quad (206)$$

where

$$(V_{2PE}^{1^{+-}})_{11} = S_0 - \frac{g^4}{16}\left\{(p'^{\,2}+p^2)\left[-\mathcal{R}+1-2\ln\left(\frac{m_\pi}{\mu}\right)\right] - 2R_2(p',p)\right\} + \mathcal{O}(\chi^4)\ , \quad (207)$$

$$(V_{2PE}^{1^{+-}})_{12} = (V_{2PE}^{1^{+-}})_{14} = (V_{2PE}^{1^{+-}})_{32} = \frac{2}{3}(p'^{\,2})\left[-\mathcal{R}+1-2\ln\left(\frac{m_\pi}{\mu}\right)\right] + \frac{2}{3}R_2(p',p) - 2R_{n'}(p',p) + \mathcal{O}(\chi^4)\ , \quad (208)$$

$$(V_{2PE}^{1^{+-}})_{13} = (V_{2PE}^{1^{+-}})_{31} = \frac{1}{16}(p'^{\,2}+p^2)\left[-\mathcal{R}+1-2\ln\left(\frac{m_\pi}{\mu}\right)\right] - 2R_2(p',p) + \mathcal{O}(\chi^4)\ , \quad (209)$$

$$(V_{2PE}^{1^{+-}})_{21} = (V_{2PE}^{1^{+-}})_{23} = (V_{2PE}^{1^{+-}})_{41} = \frac{2}{3}(p^2)\left[-\mathcal{R}+1-2\ln\left(\frac{m_\pi}{\mu}\right)\right] + \frac{2}{3}R_2(p',p) - 2R_n(p',p) + \mathcal{O}(\chi^4)\ , \quad (210)$$

$$(V_{2PE}^{1^{+-}})_{22} = S_2 - g^4\frac{3}{2}\left\{R_0(p',p) - 3R_{2x}(p',p) + 3R_x(p',p) - R_{n'}(p',p) - R_n(p',p) + \frac{1}{3}R_2(p',p)\right\} + \mathcal{O}(\chi^4)\ , \quad (211)$$

$$(V_{2PE}^{1^{+-}})_{24} = (V_{2PE}^{1^{+-}})_{42} = \frac{9}{4}R_{x^2}(p',p) - \frac{27}{4}R_x(p',p) + \frac{9}{4}R_{n'}(p',p) + \frac{9}{4}R_n(p',p) - \frac{3}{2}R_2(p',p) + \mathcal{O}(\chi^4)\ , \quad (212)$$

$$(V_{2PE}^{1+-})_{33} = S_0 - \frac{g^4}{32}\left\{-2\mathcal{R} + \frac{1}{9}(p^2+p'^{\,2}) - 4(p^2+p'^{\,2})\ln\left(\frac{m_\pi}{\mu}\right) - 4R_2(p',p)\right\} + \mathcal{O}(\chi^4)\ , \tag{213}$$

$$(V_{2PE}^{1+-})_{34} = 2p'^{\,2}\left[2\mathcal{R} - \frac{1}{9} - \ln\left(\frac{m_\pi}{\mu}\right)\right] - \frac{7}{2}\left[3R_{n'}(p',p) - R_2(p',p)\right] + \mathcal{O}(\chi^4)\ , \tag{214}$$

$$(V_{2PE}^{1+-})_{43} = 2p^2\left[2\mathcal{R} - \frac{1}{9} - \ln\left(\frac{m_\pi}{\mu}\right)\right] - \frac{7}{2}\left[3R_n(p',p) - R_2(p',p)\right] + \mathcal{O}(\chi^4)\ , \tag{215}$$

$$(V_{2PE}^{1+-})_{44} = -S_2 + \frac{g^4}{64}\left\{\frac{15}{2}R_2(p',p) - \frac{45}{2}R_{x^2}(p',p) + \frac{21}{2}\left(2R_n(p',p) + 2R_{n'}(p',p) - \frac{8}{3}R_2(p',p) - 6R_x(p',p) + 6R_{x^2}(p',p)\right)\right\} + \mathcal{O}(\chi^4)\ . \tag{216}$$

6.1.4 $\mathbf{J^{PC} = 2^{++}}$

$$
V_{CI}^{NLO}(2^{++}) =
\begin{pmatrix}
0 & 0 & -\sqrt{\frac{3}{5}}\mathcal{D}_{SD}p'^{\,2} & 0 & 0 & 0 \\
0 & 0 & -\frac{3}{\sqrt{5}}\mathcal{D}_{SD}p^2 & 0 & 0 & 0 \\
-\sqrt{\frac{3}{5}}\mathcal{D}_{SD}p'^{\,2} & -\frac{3}{\sqrt{5}}\mathcal{D}_{SD}p'^{\,2} & \mathcal{C}_d + \mathcal{C}_f + (\mathcal{D}_d + \mathcal{D}_f)(p^2 + p'^2) & -\frac{1}{\sqrt{5}}\mathcal{D}_{SD}p'^{\,2} & \sqrt{\frac{7}{5}}\mathcal{D}_{SD}p'^{\,2} & \\
0 & 0 & -\frac{1}{\sqrt{5}}\mathcal{D}_{SD}p^2 & 0 & 0 & 0 \\
0 & 0 & \sqrt{\frac{7}{5}}\mathcal{D}_{SD}p^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} . \tag{217}
$$

$$
V_{1PE}(2^{++}) = -\frac{g^2}{4f_\pi^2}(\vec{\tau_1}\cdot\vec{\tau_2})
$$

$$
\begin{pmatrix}
0 & 0 & -\frac{1}{\sqrt{30}}(3Q_{n'} - Q_2) & \frac{1}{\sqrt{12}}(3Q_{x^2} - Q_2) & -\frac{1}{\sqrt{21}}K_{15} & -\frac{3}{\sqrt{560}}K_{16} \\
0 & 0 & \frac{1}{\sqrt{20}}(3Q_{n'} - Q_2) & 0 & \frac{1}{\sqrt{504}}K_{25} & -\frac{1}{\sqrt{1400}}K_{26} \\
-\frac{1}{\sqrt{30}}(3Q_n - Q_2) & \frac{1}{\sqrt{20}}(3Q_n - Q_2) & -\frac{1}{12}Q_2 & \frac{1}{\sqrt{90}}(3Q_n - Q_2) & -\sqrt{\frac{7}{90}}(3Q_n - Q_2) & 0 \\
\frac{1}{\sqrt{12}}(3Q_{x^2} - Q_2) & 0 & \frac{1}{\sqrt{90}}(3Q_{n'} - Q_2) & \frac{1}{6}(3Q_{x^2} - Q_2) & \frac{1}{\sqrt{63}}K_{45} & \frac{1}{8\sqrt{35}}K_{46} \\
-\frac{1}{\sqrt{21}}K_{51} & \frac{1}{\sqrt{504}}K_{52} & -\sqrt{\frac{7}{90}}(3Q_{n'} - Q_2) & \frac{1}{\sqrt{63}}K_{54} & \frac{3}{14}K_{55} & -\frac{1}{56\sqrt{5}}K_{56} \\
-\frac{3}{\sqrt{560}}K_{61} & -\frac{1}{\sqrt{1400}}K_{62} & 0 & \frac{1}{8\sqrt{35}}K_{64} & -\frac{1}{56\sqrt{5}}K_{65} & \frac{1}{28}K_{66}
\end{pmatrix} , \tag{218}
$$

where

$$K_{51} = K_{15} = 3Q_{x^2} - 9Q_x + 3Q_x + 3Q_{n'} - 2Q_2 \ , \tag{219}$$

$$K_{61} = 35Q_{n'x^2} - Q_{x^2} + 20Q_x - 5Q_{n'} + 2Q_n + Q_2 \ , \tag{220}$$

$$K_{16} = 35Q_{nx^2} - Q_{x^2} + 20Q_x - 5Q_n + 2Q_{n'} + Q_2 \ , \tag{221}$$

$$K_{52} = K_{25} = 18Q_{x^2} - 18Q_x + 6Q_{n'} + 6Q_n + 8Q_2 \ , \tag{222}$$

$$K_{62} = 35Q_{n'x^2} - 5Q_{x^2} - 20Q_x - 5Q_{n'} + 2Q_n + Q_2 \ , \tag{223}$$

$$K_{26} = 35Q_{nx^2} - 5Q_{x^2} - 20Q_x - 5Q_n + 2Q_{n'} + Q_2 \ , \tag{224}$$

$$K_{54} = K_{45} = 3Q_{x^2} - Q_x + 3Q_{n'} + 3Q_n - 2Q_2 \ , \tag{225}$$

$$K_{64} = 35Q_{n'x^2} - 35Q_{x^2} - 140Q_x - 5Q_{n'} + 2Q_n + Q_2 \ , \tag{226}$$

$$K_{46} = 35Q_{nx^2} - 35Q_{x^2} - 140Q_x - 5Q_n + 2Q_{n'} + Q_2 \ , \tag{227}$$

$$K_{55} = -8Q_{x^2} + 11Q_x - 6(Q_{n'} + Q_n) + 7Q_2 \ , \tag{228}$$

$$K_{65} = 35Q_{n'x^2} - 5Q_{x^2} + 20Q_x - 5Q_{n'} + 2Q_n + Q_2 \ , \tag{229}$$

$$K_{56} = 35Q_{nx^2} - 5Q_{x^2} + 20Q_x - 5Q_n + 2Q_{n'} + Q_2 \ , \tag{230}$$

$$K_{66} = 245Q_{x^3} - 105(Q_{n'x^2} + Q_{nx^2}) + 15Q_{x^2} + 15(Q_{n'} + Q_n) + 5Q_x - 3Q_2 \ . \tag{231}$$

$$V_{2PE}(2^{++}) = \frac{1}{512\pi^2 f_\pi^4}$$

$$\begin{pmatrix}
32S_2(p',p) & 0 & \tfrac{3\sqrt{3}}{\sqrt{10}}\, g^4(V^{2^{++}}_{2PE})_{13} & \sqrt{3}\, g^4(V^{2^{++}}_{2PE})_{14} & \tfrac{6\sqrt{3}}{\sqrt{7}}\, g^4(V^{2^{++}}_{2PE})_{15} & \tfrac{1}{4\sqrt{105}}\, g^4(V^{2^{++}}_{2PE})_{16} \\
0 & 0 & \tfrac{9}{\sqrt{5}}\, g^4(V^{2^{++}}_{2PE})_{23} & 0 & \tfrac{-9}{\sqrt{14}}\, g^4(V^{2^{++}}_{2PE})_{25} & \tfrac{-3}{\sqrt{70}}\, g^4(V^{2^{++}}_{2PE})_{26} \\
\tfrac{3\sqrt{3}}{\sqrt{10}}\, g^4(V^{2^{++}}_{2PE})_{31} & \tfrac{9}{\sqrt{5}}\, g^4(V^{2^{++}}_{2PE})_{32} & 32(V^{2^{++}}_{2PE})_{33} & \tfrac{-1}{2\sqrt{10}}\, g^4(V^{2^{++}}_{2PE})_{34} & \tfrac{\sqrt{7}}{2\sqrt{10}}\, g^4(V^{2^{++}}_{2PE})_{35} & 0 \\
\sqrt{3} g^4(V^{2^{++}}_{2PE})_{41} & 0 & \tfrac{-1}{2\sqrt{10}}\, g^4(V^{2^{++}}_{2PE})_{43} & 32(V^{2^{++}}_{2PE})_{44} & \tfrac{1}{2\sqrt{7}}\, g^4(V^{2^{++}}_{2PE})_{45} & \tfrac{-3}{2\sqrt{35}}\, g^4(V^{2^{++}}_{2PE})_{46} \\
\tfrac{6\sqrt{3}}{\sqrt{7}}\, g^4(V^{2^{++}}_{2PE})_{51} & \tfrac{-9}{\sqrt{14}}\, g^4(V^{2^{++}}_{2PE})_{52} & \tfrac{\sqrt{7}}{2\sqrt{10}}\, g^4(V^{2^{++}}_{2PE})_{53} & \tfrac{1}{2\sqrt{7}}\, g^4(V^{2^{++}}_{2PE})_{54} & 32(V^{2^{++}}_{2PE})_{55} & \tfrac{3}{28\sqrt{5}}\, g^4(V^{2^{++}}_{2PE})_{56} \\
\tfrac{1}{4\sqrt{105}}\, g^4(V^{2^{++}}_{2PE})_{61} & \tfrac{-3}{\sqrt{70}}\, g^4(V^{2^{++}}_{2PE})_{62} & 0 & \tfrac{-3}{2\sqrt{35}}\, g^4(V^{2^{++}}_{2PE})_{64} & \tfrac{3}{28\sqrt{5}}\, g^4(V^{2^{++}}_{2PE})_{65} & 32(V^{2^{++}}_{2PE})_{66}
\end{pmatrix}, \tag{232}$$

where

$$(V^{2^{++}}_{2PE})_{13} = (V^{2^{++}}_{2PE})_{23} = \frac{2}{3}p^2\left[-\mathcal{R}+1-2\ln\left(\frac{m_\pi}{\mu}\right)\right]+\frac{2}{3}R_2(p',p)-2R_n(p',p)+\mathcal{O}(\chi^4)\ , \tag{233}$$

$$(V^{2^{++}}_{2PE})_{14} = (V^{2^{++}}_{2PE})_{41} = 3R_{x^2}(p',p) - 9R_{2x}(p',p) + 3R_0(p',p) - R_2(p',p) + \mathcal{O}(\chi^4)\ , \tag{234}$$

$$(V^{2^{++}}_{2PE})_{15} = (V^{2^{++}}_{2PE})_{25} = (V^{2^{++}}_{2PE})_{51} = (V^{2^{++}}_{2PE})_{52} = \frac{2}{3}R_2(p',p)+3R_x(p',p)-R_n(p',p)-R_{n'}(p',p) - R_{x^2}(p',p) + \mathcal{O}(\chi^4)\ , \tag{235}$$

$$(V^{2^{++}}_{2PE})_{16} = 63R_{n'x^2}(p',p) - 90R_{x^2}(p',p) - 360R_x(p',p) - 90R_{n'}(p',p) + 36R_n(p',p) + 18R_2(p',p) + \mathcal{O}(\chi^4)\ , \tag{236}$$

$$(V^{2^{++}}_{2PE})_{26} = 35R_{n'x^2}(p',p) - 5R_{x^2}(p',p) - 2R_x(p',p) - 5R_{n'}(p',p) + 2R_n(p',p) + R_2(p',p) + \mathcal{O}(\chi^4)\ , \tag{237}$$

$$(V^{2^{++}}_{2PE})_{31} = (V^{2^{++}}_{2PE})_{32} = \frac{2}{3}p'^{\,2}\left[-\mathcal{R}+1-2\ln\left(\frac{m_\pi}{\mu}\right)\right]+\frac{2}{3}R_2(p',p)-2R_{n'}(p',p)+\mathcal{O}(\chi^4)\ , \tag{238}$$

$$(V^{2^{++}}_{2PE})_{33} = S_0+\frac{g^4}{32}\left\{-2\mathcal{R}+\frac{1}{9}(p^2+p'^{\,2})-4(p^2+p'^{\,2})\ln\left(\frac{m_\pi}{\mu}\right)-4R_2(p',p)\right\}+\mathcal{O}(\chi^4)\ , \tag{239}$$

$$(V^{2^{++}}_{2PE})_{34} = (V^{2^{++}}_{2PE})_{35} = 2p'^{\,2}\left[2\mathcal{R}-\frac{1}{9}-\ln\left(\frac{m_\pi}{\mu}\right)\right]-\frac{7}{2}\left[3R_{n'}(p',p)-R_2(p',p)\right]+\mathcal{O}(\chi^4)\ , \tag{240}$$

$$(V^{2^{++}}_{2PE})_{43} = (V^{2^{++}}_{2PE})_{53} = 2p^2\left[2\mathcal{R}-\frac{1}{9}-\ln\left(\frac{m_\pi}{\mu}\right)\right]-\frac{7}{2}\left[3R_n(p',p)-R_2(p',p)\right]+\mathcal{O}(\chi^4)\ , \tag{241}$$

$$(V_{2PE}^{2++})_{44} = S_2 + \frac{g^4}{32}\Bigg\{\frac{45}{2}R_{x^2}(p',p) - \frac{15}{2}R_2(p',p) + \frac{7}{2}R_0(p',p) - \frac{21}{2}R_{2x}(p',p)\Bigg\} + \mathcal{O}(\chi^4)\ , \tag{242}$$

$$(V_{2PE}^{2++})_{45} = (V_{2PE}^{2++})_{54} = -\frac{7}{2}\Bigg[3R_n(p',p) + 3R_{n'}(p',p) - 9R_x(p',p) + 3R_{x^2}(p',p) - 2R_2(p',p)\Bigg] + \mathcal{O}(\chi^4)\ , \tag{243}$$

$$(V_{2PE}^{2++})_{46} = (V_{2PE}^{2++})_{56} = -\frac{7}{2}\Bigg[35R_{n'x^2}(p',p) - 5R_{n'}(p',p) - 20R_x(p',p) + R_2(p',p) + 2R_n(p',p) - 5R_{x^2}(p',p)\Bigg] + \mathcal{O}(\chi^4)\ , \tag{244}$$

$$(V_{2PE}^{2++})_{55} = S_2 + \frac{g^4}{64}\Bigg\{\frac{15}{2}R_2(p',p) - \frac{45}{2}R_{x^2}(p',p) - \frac{1}{4}\Bigg(- 9R_n(p',p) - 9R_{n'}(p',p) - 51R_{x^2}(p',p) + 27R_x(p',p) + 20R_2(p',p)\Bigg)\Bigg\} + \mathcal{O}(\chi^4)\ , \tag{245}$$

$$(V_{2PE}^{2++})_{61} = 63R_{nx^2}(p',p) - 90R_{x^2}(p',p) - 360R_x(p',p) - 90R_n(p',p) + 36R_{n'}(p',p) + 18R_2(p',p) + \mathcal{O}(\chi^4)\ , \tag{246}$$

$$(V_{2PE}^{2++})_{62} = 35R_{nx^2}(p',p) - 5R_{x^2}(p',p) - 2R_x(p',p) - 5R_n(p',p) + 2R_{n'}(p',p) + R_2(p',p)\ , \tag{247}$$

$$(V_{2PE}^{2++})_{64} = (V_{2PE}^{2++})_{65} = -\frac{7}{2}\Bigg[35R_{nx^2}(p',p) - 5R_n(p',p) - 20R_x(p',p) + R_2(p',p) + 2R_{n'}(p',p) - 5R_{x^2}(p',p)\Bigg] + \mathcal{O}(\chi^4)\ , \tag{248}$$

$$(V_{2PE}^{2++})_{66} = S_4 - \frac{15g^4}{512}\left[35R_{x^4}(p',p) - 30R_{x^2}(p',p) + 3R_2(p',p)\right] + \frac{3g^4}{448}\Bigg\{ -\frac{7}{2}$$

$$\Bigg(105R_{n'x^2}(p',p) + 105R_{nx^2}(p',p) - 15R_{n'}(p',p) - 15R_n(p',p) + 45R_x(p',p)$$

$$+ 3R_2(p',p) - 15R_{x^2}(p',p) - 245R_{nn'x^3}(p',p)\Bigg)\Bigg\} + \mathcal{O}(\chi^4) \quad (249)$$

and

$$S_4 = \int_{-1}^{1} \frac{dx}{2} \frac{(35x^4 - 30x^2 + 3)}{8} (\vec{\tau_1} \cdot \vec{\tau_2})\vec{q}^{\,2} \Bigg\{ \mathcal{R}\left[\frac{23}{192}g^4 + \frac{1}{24}\right] + \left(\frac{5}{576}g^4 - \frac{5}{72}\right)$$

$$+ \ln\left(\frac{m_\pi}{\mu}\right)\left(\frac{23}{96}g^4 + \frac{1}{12}\right) + L(q)\left(\frac{23}{96}g^4 + \frac{1}{12}\right) + \mathcal{O}(\chi^4)\Bigg\}$$

$$= \frac{(\vec{\tau_1} \cdot \vec{\tau_2})}{128p'^4p^4}\Bigg\{\Bigg[\left(35p^8 + 35p'^8 + 20p^6p'^2 + 18p^4p'^4 + 20p^2p'^6\right)R_2 + (210p^4 + 300p^2p'^2$$

$$+ 210p'^4)R_6 - \left(140p^2 + 140p'^2\right) R_8 - \left(140p^6 + 180p^4p'^2 + 180p^2p'^4 + 140p'^6\right) R_4$$

$$+ 35R_{10}\Bigg]\left(\frac{23}{96}g^4 + \frac{1}{24}\right)\Bigg\} + \mathcal{O}(\chi^4)\ . \quad (250)$$

All the Q and R functions mentioned above are functions of p and p' (as in $Q(p',p)$). Here, only the terms contributing at NLO in the momentum counting scheme are shown. The PWD potentials containing all terms that arise from the loop integrals (namely m_π^2) are mentioned in Appendix D.

6.2 $B^{(*)}B^{(*)} \to B^{(*)}B^{(*)}$

The PWD potentials are exactly the same as $B^{(*)}\bar{B}^{(*)} \to B^{(*)}\bar{B}^{(*)}$ except with additional triangle diagram contribution and with an opposite sign for 1PE and football contribution as mentioned in the section 5.4. The additional triangle contributions are included in the PWD potentials of $B^{(*)}B^{(*)}$ by changing the $S_k(p',p)$ functions (which includes $S_0(p',p)$, $S_2(p',p)$ and $S_4(p',p)$) to $T_k(p',p)$ functions ($T_0(p',p)$, $T_2(p',p)$ and $T_4(p',p)$), where $T_k(p',p)$ is given by

$$T_k(p',p) = \int_{-1}^{1} \frac{dx}{2} P_k(x) \big(\vec{\tau}_1 \cdot \vec{\tau}_2\big) \vec{q}^{\,2} \Bigg\{ \mathcal{R}\left[\frac{23}{192} g^4 - \frac{5}{48} g^2 - \frac{1}{24}\right] + \left(\frac{5}{576} g^4 + \frac{13}{144} g^2 + \frac{5}{72}\right)$$
$$+ \ln\left(\frac{m_\pi}{\mu}\right)\left(\frac{23}{96} g^4 - \frac{5}{24} g^2 - \frac{1}{12}\right) + L(q)\left(\frac{23}{96} g^4 - \frac{5}{24} g^2 - \frac{1}{12}\right) + \mathcal{O}(\chi^4) \Bigg\} , \tag{251}$$

where $P_k(x)$ denotes the k-th Legendre polynomial ($k = 0, 2, 4$; in our case).

7 Checks of the results

A non-trivial verification of our results is provided by the renormalisation of the formally divergent loops, which must work at each order in the power counting separately. For that we need to relate the divergent terms ($\mathcal{R}$-terms—see Eq. (65)), contained in the loop contributions of $B^{(*)}\bar{B}^{(*)}$ and $B^{(*)}B^{(*)}$ 2PE potentials, to the low-energy constants (LECs) of the CI. We take the case of $(V_{2PE}^{0^{++}})_{11}$ and $(V_{CI}^{0^{++}})_{11}$ in the 0^{++} channel as an example. By equating the divergence of 2PE loop integral with the LECs ($\mathcal{C}_d, \mathcal{C}_f$, etc..) for g^4 contribution, one finds for the divergent parts of the counter terms

$$\mathcal{C}_d = 0 , \tag{252}$$

$$\mathcal{C}_f = 0 , \tag{253}$$

$$\mathcal{D}_d = \frac{\mathcal{R} g^4}{256\pi^2 f_\pi^4}\left(\frac{23}{12}(\vec{\tau}_1 \cdot \vec{\tau}_2) + 1\right) \text{ and} \tag{254}$$

$$\mathcal{D}_f = \frac{-\mathcal{R} g^4}{128\pi^2 f_\pi^4} . \tag{255}$$

Using the same approach for $(V_{CI}^{0^{++}})_{13}$ and $(V_{2PE}^{0^{++}})_{13}$ with its prefactors, one finds the divergent part of $\mathcal{D}_{SD}$ as

$$\mathcal{D}_{SD} = \frac{\sqrt{2}\mathcal{R} g^4}{512\pi^2 f_\pi^4} . \tag{256}$$

Similarly, when performing this exercise for g^0 contribution, one finds

$$\mathcal{C}_d = \mathcal{C}_f = \mathcal{D}_f = \mathcal{D}_{SD} = 0 , \tag{257}$$

$$\mathcal{D}_d = \frac{\mathcal{R}}{768\pi^2 f_\pi^4}(\vec{\tau}_1 \cdot \vec{\tau}_2) . \tag{258}$$

In the g^2 case

$$\mathcal{C}_d = \mathcal{C}_f = \mathcal{D}_f = \mathcal{D}_{SD} = 0 \ , \tag{259}$$

$$\mathcal{D}_d = \frac{-5\mathcal{R}g^2}{768\pi^2 f_\pi^4}(\vec{\tau_1} \cdot \vec{\tau_2}) \ . \tag{260}$$

The renormalisation program provides a non-trivial cross check of the calculations. For example, the divergent terms of the PWD potentials of $B\bar{B}(^1S_0)\rightarrow B^*\bar{B}^*(^5D_0)$ and $B^*\bar{B}^*(^1S_0)\rightarrow B^*\bar{B}^*(^5D_0)$ must be related such that their infinities can be absorbed into one single $\mathcal{D}_{SD}$ term. Repeating the calculation for remaining transitions, we confirm that $\mathcal{C}_d$, $\mathcal{C}_f$, $\mathcal{D}_d$ and $\mathcal{D}_f$ absorb all the divergences of $B\bar{B}(^1S_0)\rightarrow B\bar{B}(^1S_0)$, $B\bar{B}(^1S_0)\rightarrow B^*\bar{B}^*(^1S_0)$ and $B^*\bar{B}^*(^1S_0)\rightarrow B^*\bar{B}^*(^1S_0)$ PWD potentials.

As a further non-trivial cross-check, we find that the triangle contributions add up in the $B^{(*)}B^{(*)}$ case and cancel for $B^{(*)}\bar{B}^{(*)}$, by performing the triangle diagram calculations in particle basis for both the cases. This is shown in Appendix G.

8 Comparison to earlier works

We compared our 1PE results with the ones calculated by Q. Wang et al. [8] (given in Eqs. (22)-(28)) and found that the results agreed with ours (seen in Eq. (203) for $I = 0$ (since, $\vec{\tau_1} \cdot \vec{\tau_2} = -1$). Furthermore, the PWD contact interactions of V. Baru et al. [9] (seen in Eqs. (12)-(15)) agree with ours (seen in Eqs. (181, (193, 202, 217).

For the 2PE contribution in $B^{(*)}\bar{B}^{(*)}$, however, our results disagree with those of B. Wang et al. [101] (given in Eq. (10)). In the mentioned paper, there is a net triangle contribution (terms proportional to g^2) in the total 2PE potentials, which is not present in our case. At the same time, their total 2PE potential does not have any box contributions (terms proportional to g^4), which are present in our case. We agree with Ref. [101] results only in the case of the football diagrams (terms proportional to g^0).

Furthermore, we compared our results to the 2PE potentials of B. Wang et al. [102] for $\bar{B}^{(*)}\bar{B}^{(*)}$ scattering. In contrast to the meson-anti-meson scattering amplitudes of Ref. [101] mentioned above, the 2PE potentials for anti-meson–anti-meson scattering in [102] contain all types, namely football, triangle and box contributions. However, a direct comparison with our potentials is difficult, since the results are given in a quite different form compared to ours.

9 Summary and outline

In this thesis, we calculated the full potential of $B^{(*)}\bar{B}^{(*)}$ and $B^{(*)}B^{(*)}$ systems up to to NLO and decomposed them into partial waves with quantum numbers $J^{PC} = 0^{++}, 1^{++}, 1^{+-}$ and 2^{++}. The motivation to work beyond the LO stemmed from the relatively slow convergence of the chiral expansion $\chi \sim 0.5$, provided in Q. Wang et al. [8] and V. Baru et al. [9]. Furthermore, observations of exotic multiquark states in the $c\bar{c}$-sector coupled with HQFS suggests the existence of several, still to be discovered, spin partners W_{bJ} of the $Z_b^{\pm}(10610)$ and $Z_b^{\pm}(10650)$. Thus, we performed the partial wave analysis beyond the quantum number $J^{PC} = 1^{+-}$.
Following an introduction to QCD and its properties, we utilised HMχPT to calculate the effective potentials in the expansion of $\chi = p_{typ}/\Lambda_\chi$. Using our power counting scheme detailed in 3.6, we could separate the terms in the potentials as $\mathcal{O}(\chi^0)$, $\mathcal{O}(\chi^2)$ and $\mathcal{O}(\chi^4)$, with $\mathcal{O}(\chi^4)$ terms ignored in the main part of the thesis. The effective potentials were computed at LO, $\mathcal{O}(\chi^0)$, comprising the CI and 1PE, and at NLO, $\mathcal{O}(\chi^2)$, comprising the $\mathcal{O}(\chi^2)$ CI and 2PE. Moreover, it was observed that 2PE loops were made up of terms contributing at $\vec{q}^{\,2} \sim \mathcal{O}(\chi^2)$ and $m_\pi^2 \sim \mathcal{O}(\chi^4)$, with the $\mathcal{O}(\chi^4)$ potentials presented in the Appendix C. We also calculated the triangle (I_{tr}), crossed-box ($I_{box}^{(2)}$) and vertex correction loop integrals by treating q_0, δ and the recoil terms perturbatively and expanding the heavy meson propagator in $(M_B)^{-1}$. The results of these calculations were validated to comply with the power counting expectations. The renormalisation program further confirmed our results by providing us with a non-trivial check. Finally, we compared our potentials with those mentioned in previous works.
This thesis lays the groundwork for future investigations into the yet-to-be-discovered spin partners W_{bJ} of the $Z_b^{\pm}(10610)$ and $Z_b^{\pm}(10650)$. The potentials reported in this thesis can be used as input for the full scattering equation (coupled channel LSE). Subsequently, the scattering equation output can be utilized to investigate the theory's convergence and the impact of NLO contributions.

Acknowledgement

I wish to express my deepest gratitude to my supervisor Prof. Dr. Christoph Hanhart for allowing me to be a part of his group and for his invaluable scientific advises, constant assistance and supportive feedback throughout the course of the project. I am grateful for our fruitful discussions and his patience with my projects, even when they presented unanticipated obstacles. I am also appreciative of the many non-physics discussions I had with him, which aided my integration into society.

I would also like to sincerely thank PD Dr. Bastian Kubis for being my second supervisor. I would also like to thank him and his group for their assistance with tutoring in master's courses.

Next, I would like to thank Dr. Vadim Baru and Dr. Jambul Gegelia for their scientific guidance and supportive comments.

Furthermore, I would like to thank Leon von Detten, Anuvind Ashokan, Shivam Rawat, and Paul Pütz for the amazing office talks that increased my knowledge of physics while also providing some laughs.

Next, I would like to thank my parents Thomas Chacko and Rachel Chacko, and brother Johan Tom Chacko for their constant emotional support. This project would not have been possible without their help.

Also, I'd like to thank my girlfriend, Harini Balaji, for her presence throughout the duration of this endeavor, supporting and strengthening me.

Also, I want to thank my good friends, Aishwarya Sweety and Savani for the many discussions. Finally, I want to thank my football friends at the International FC-Köln for allowing me to express my frustrations through sports and weekend activities.

Appendices

A Calculation of pertinent Integrals

Here, we present the calculation of all the loop integrals encountered in this thesis. The calculation follow the same steps as mentioned in Sec. 3.7.

A.1 Calculation of triangle Integral

We start with the triangle loop integral I_{tr}

$$I_{tr} = i\int \frac{d^4l}{(2\pi)^4}\,\frac{(\vec{l}+\vec{q})\cdot\vec{l}}{\left[(l+q)^2 - m_\pi^2 + i\epsilon\right]\left[l^2 - m_\pi^2 + i\epsilon\right]}\ . \tag{261}$$

Now, introduce the Feynman parameters, shift $l \to l - qx$ and dropping all odd powers of l due to symmetry

$$= i\int_0^1 dx \int \frac{d^3\vec{l}}{(2\pi)^3}\int \frac{dl_0}{2\pi}\,\frac{\vec{l}^2 - \vec{q}^2x(1-x)}{\left[(l_0)^2 - \vec{l}^2 - \vec{q}^{\,2}x(1-x) - m_\pi^2 + i\epsilon\right]^2}\ . \tag{262}$$

Executing the l_0- integration with the residue theorem and setting $\epsilon \to 0$, since there is no singularity in the integral

$$= \frac{i^2}{4}\int_0^1 dx \int \frac{d^3\vec{l}}{(2\pi)^3}\frac{\vec{l}^2 - \vec{q}^{\,2}x(1-x)}{\left[\vec{l}^2 + \vec{q}^{\,2}x(1-x) + m_\pi^2\right]^2}\ . \tag{263}$$

Going to $(D-1)$ - dimensional spherical coordinates as the integral is divergent at four-dimensions and inserting dimensional scale parameter μ to to keep the correct units for the integral

$$= \frac{-\sqrt{\pi}}{(4\pi)^{D/2}}\frac{\mu^{4-D}}{\Gamma(\frac{D-1}{2})}\int_0^1 dx \int_0^\infty dl\,\frac{l^D - l^{D-2}\vec{q}^{\,2}x(1-x)}{\left[\vec{l}^2 + \vec{q}^{\,2}x(1-x) + m_\pi^2\right]^2}\ . \tag{264}$$

Performing the l-integration and inserting $D = 4 - \xi$ and doing the x-integration

$$\begin{aligned} I_{tr} &= \frac{-1}{16\pi^2}\Bigg\{\left(\frac{5}{12}\vec{q}^{\,2} + \frac{3}{2}m_\pi^2\right)\mathcal{R} - \frac{13}{36}\vec{q}^{\,2} - \frac{m_\pi^2}{3} + \left(\frac{5}{6}\vec{q}^{\,2} + 3m_\pi^2\right)\ln\left(\frac{m_\pi}{\mu}\right) \\ &\qquad\qquad + \left(\frac{5}{6}\vec{q}^{\,2} + \frac{4}{3}m_\pi^2\right)L(q)\Bigg\} \\ &= \frac{-\vec{q}^{\,2}}{16\pi^2}\left\{\frac{5}{12}\mathcal{R} - \frac{13}{36} + \frac{5}{6}\ln\left(\frac{m_\pi}{\mu}\right) + \frac{5}{6}L(q)\right\} + \mathcal{O}(\chi^4)\ , \end{aligned} \tag{265}$$

where $\mathcal{R}$ and $L(q)$ are given by

$$\mathcal{R} = -\frac{2}{\xi} + \gamma_E - 1 - \ln(4\pi) \ , \tag{266}$$

$$L(q) = \frac{\sqrt{4m_\pi^2 + q^2}}{q} \ln\left(\frac{\sqrt{4m_\pi^2 + q^2} + q}{2m_\pi}\right) \tag{267}$$

and γ_E is the Euler-Mascheroni constant.

A.2 Calculation of football integral

$$I_{fb} = i \int \frac{d^4 l}{(2\pi)^4} \frac{(l_0)^2}{\left[(l+q)^2 - m_\pi^2 + i\epsilon\right]\left[l^2 - m_\pi^2 + i\epsilon\right]} \ . \tag{268}$$

Like in the previous calculation, Introduce Feynman parameters, shift $l \to l - qx$, drop all odd powers of l due to symmetry and perform the l_0-integration.

$$= \frac{1}{4} \int_0^1 dx \int \frac{d^3\vec{l}}{(2\pi)^3} \frac{1}{\left[\vec{l}^2 + \vec{q}^{\,2} x(1-x) + m_\pi^2\right]^{1/2}} \ . \tag{269}$$

Going to $(D-1)$ - dimensional spherical coordinates and inserting μ for consistent units for the integral

$$= \frac{\sqrt{\pi}}{(4\pi)^{D/2}} \frac{\mu^{4-D}}{\Gamma(\frac{D-1}{2})} \int_0^1 dx \int_0^\infty dl \frac{l^{D-2}}{\left[\vec{l}^2 + \vec{q}^{\,2} x(1-x) + m_\pi^2 + i\epsilon\right]^{1/2}} \ . \tag{270}$$

Executing the l-integration and inserting $D = 4 - \xi$

$$= \frac{1}{32\pi^2} \int_0^1 dx \left(\vec{q}^{\,2} x(1-x) + m_\pi^2\right) \left[-\frac{2}{\xi} + \gamma_E - 1 - \ln(4\pi) + \ln\left(\frac{\vec{q}^{\,2} x(1-x) + m_\pi^2}{\mu^2}\right)\right] \tag{271}$$

and doing the x-integration

$$I_{fb} = \frac{-1}{16\pi^2}\Bigg\{ -\left(\frac{\vec{q}^{\,2}}{12} + \frac{m_\pi^2}{2}\right)\mathcal{R} + \frac{5}{36}\vec{q}^{\,2} + \frac{2m_\pi^2}{3} - \left(\frac{\vec{q}^{\,2}}{6} + m_\pi^2\right)\ln\left(\frac{m_\pi}{\mu}\right) - \left(\frac{\vec{q}^{\,2}}{6} + \frac{4}{6}m_\pi^2\right)L(q)\Bigg\}$$

$$= \frac{\vec{q}^{\,2}}{96\pi^2}\left\{\frac{\mathcal{R}}{2} - \frac{5}{6} + \ln\left(\frac{m_\pi}{\mu}\right) + L(q)\right\} + \mathcal{O}(\chi^4) \ . \tag{272}$$

A.3 Calculation of box integrals

Now, we present the calculation of $I^{(3)}_{box}$, $I^{(4)}_{box}$, $I^{(5)}_{box}$, $I^{(6)}_{box}$ and $I^{(7)}_{box}$, as the calculation of I^{1}_{box} and I^{2}_{box} were shown in Sec. 3.7.

A.3.1 $I^{(3)}_{box}$

We see this loop integral in $B^*\bar{B}^* \to B^*\bar{B}^*$ and $B^*B^* \to B^*B^*$ potential, given by

$$I^{(3)}_{box} = i\int \frac{d^4l}{(2\pi)^4}\frac{\vec{q_1}^{\,2}\vec{q_2}^{\,2}}{(l_0-i\epsilon)^2\left[q_2^2-m_\pi^2+i\epsilon\right]\left[q_1^2-m_\pi^2+i\epsilon\right]} \; . \tag{273}$$

Expanding $q_1^2 = (l_0)^2 - \vec{q_1}^{\,2}$ and $q_2^2 = (l_0)^2 - \vec{q_2}^{\,2}$, using Feynman parameters and executing the l^0-integration and setting $\epsilon \to 0$

$$I^{(3)}_{box} = \frac{-3}{4}\int_0^1 dx \int \frac{d^3\vec{l}}{(2\pi)^3}\frac{\vec{q_1}^{\,2}\vec{q_2}^{\,2}}{\left[(\vec{q_2}^{\,2}-\vec{q_1}^{\,2})x+\vec{q_1}^{\,2}+m_\pi^2\right]^{5/2}} \; . \tag{274}$$

Now, shift $\vec{l} \to \vec{l}+\vec{p}$ such that $\vec{q_1} = \vec{p}-\vec{l} \to -\vec{l}$ and $\vec{q_2} = \vec{p'}-\vec{l} \to -\vec{l}+\vec{q}$ with $\vec{q} = \vec{p'}-\vec{p}$:

$$I^{(3)}_{box} = \frac{-3}{4}\int_0^1 dx \int \frac{d^3\vec{l}}{(2\pi)^3}\frac{\vec{l}^{\,2}(\vec{l}-\vec{q})^2}{\left[\vec{l}^2+(-2\vec{l}\cdot\vec{q}+\vec{q}^{\,2})x+m_\pi^2\right]^{5/2}} \; . \tag{275}$$

As in earlier calculation, complete the square through $\vec{l} \to \vec{l}+\vec{q}x$:

$$I^{(3)}_{box} = \frac{-3}{4}\int_0^1 dx \int \frac{d^3\vec{l}}{(2\pi)^3}\frac{\vec{l}^4+\vec{l}^2\vec{q}^{\,2}(2x^2-x+1)+4(\vec{l}\cdot\vec{q})^2x(x-1)+\left(\vec{q}^{\,2}x(1-x)\right)^2}{\left[\vec{l}^2+\vec{q}^{\,2}x(1-x)+m_\pi^2\right]^{5/2}} \; . \tag{276}$$

Transforming to $(D-1)$-dimensional spherical coordinates to make the integral convergent and inserting μ

$$= \frac{-3\sqrt{\pi}}{(4\pi)^{D/2}}\frac{\mu^{4-D}}{\Gamma(\frac{D-1}{2})}\int_0^1 dx \int_0^\infty dl \frac{l^{D+2}+l^D\vec{q}^{\,2}(2x^2\frac{D+1}{D-1}-2x\frac{D+1}{D-1}+1)+l^{D-2}\left(\vec{q}^{\,2}x(1-x)\right)^2}{\left[\vec{l}^2+\vec{q}^{\,2}x(1-x)+m_\pi^2\right]^{5/2}} \; , \tag{277}$$

solving the l-integration, inserting $D = 4 - \xi$ and then performing x-integration

$$I^{(3)}_{box} = \frac{-1}{16\pi^2}\Bigg\{\left(\frac{15}{2}m_\pi^2 - \frac{1}{12}\vec{q}^{\,2}\right)\mathcal{R} + \frac{1}{4}\vec{q}^{\,2} + \frac{8}{3}m_\pi^2 + \Bigg(15m_\pi^2$$
$$- \frac{1}{6}\vec{q}^{\,2}\Bigg)\ln\left(\frac{m_\pi}{\mu}\right) + \left(\frac{10}{3}m_\pi^2 - \frac{1}{6}\vec{q}^{\,2} + \frac{8m_\pi^4}{4m_\pi^2 + \vec{q}^{\,2}}\right)L(q)\Bigg\}$$
$$= \frac{-\vec{q}^{\,2}}{16\pi^2}\left\{\frac{-1}{12}\mathcal{R} + \frac{1}{4} - \frac{1}{6}\ln\left(\frac{m_\pi}{\mu}\right) - \frac{1}{6}L(q)\right\} + \mathcal{O}(\chi^4) \quad (278)$$

A.3.2 $I^{(4)}_{box}$

We also see this loop integral in $B^*\bar{B}^* \to B^*\bar{B}^*$ and $B^*B^* \to B^*B^*$ potential, given by

$$I^{(4)}_{box} = i\int \frac{d^4l}{(2\pi)^4}\frac{\vec{q_1}^{\,2}(\vec{q_2}\cdot\vec{q})^2}{(l^0 - i\epsilon)^2\left[q_2^2 - m_\pi^2 + i\epsilon\right]\left[q_1^2 - m_\pi^2 + i\epsilon\right]} \quad (279)$$

where $\vec{q} = \vec{p'} - \vec{p}$. Using Feynman parameters

$$I^4_{box} = i\int \frac{d^4l}{(2\pi)^4}\frac{\vec{q_1}^{\,2}(\vec{q_2}\cdot\vec{q})^2}{(l_0 - i\epsilon)^2\left[(l^0)^2 - (\vec{q_2}^{\,2} - \vec{q_1}^{\,2})x - \vec{q_1}^{\,2} - m_\pi^2 + i\epsilon\right]^2} \,. \quad (280)$$

Executing the l_0-integration, setting $\epsilon \to 0$ and shifting $\vec{l} \to \vec{l} + \vec{p}$ such that $\vec{q_1} = \vec{p} - \vec{l} \to -\vec{l}$ and $\vec{q_2} = \vec{p'} - \vec{l} \to -\vec{l} + \vec{q}$

$$I^4_{box} = i^2\frac{3}{4}\int_0^1 dx \int \frac{d^3\vec{l}}{(2\pi)^3}\frac{\vec{l}^2[(\vec{l} - \vec{q})\cdot\vec{q}]^2}{\left[\vec{l}^2 + (-2\vec{l}\cdot\vec{q} + \vec{q}^{\,2})x + m_\pi^2\right]^{5/2}} \,. \quad (281)$$

Completing the square in the denominator through $\vec{l} \to \vec{l} + \vec{q}x$ and dropping terms odd in $\vec{l}$:

$$I^4_{box} = \frac{-3}{4}\int_0^1 dx$$
$$\int \frac{d^3\vec{l}}{(2\pi)^3}\frac{\frac{\vec{l}^4\vec{q}^2}{D-1} + \frac{\vec{l}^2\vec{q}^4}{D-1}[x^2(D-4) - 2x(D+2) + D - 1] + \vec{q}^{\,6}x^2(1-x)^2}{\left[\vec{l}^2 + \vec{q}^{\,2}x(1-x) + m_\pi^2\right]^{5/2}} \,. \quad (282)$$

Like in the previous case, going to $(D-1)$-dimensional spherical coordinates and inserting μ

$$= \frac{-3\sqrt{\pi}}{(4\pi)^{D/2}} \frac{\mu^{4-D}}{\Gamma(\frac{D-1}{2})} \int_0^1 dx$$

$$\int_0^\infty dl \frac{\frac{l^{D+2}\vec{q}^2}{D-1} + \frac{l^D \vec{q}^4}{D-1}[x^2(D-4) - 2x(D+2) + D - 1] + l^{D-2}\vec{q}^6 x^2(1-x)^2}{\left[\vec{l}^2 + \vec{q}^2 x(1-x) + m_\pi^2\right]^{5/2}} . \tag{283}$$

Solving the l-integration, inserting $D = 4 - \xi$ and doing the x- integration:

$$I_{box}^{(4)} = \frac{-1}{16\pi^2}\vec{q}^2 \Bigg\{ \left(\frac{5}{2}m_\pi^2 - \frac{7}{12}\vec{q}^2\right)\mathcal{R} - \frac{7}{36}\vec{q}^2 - \frac{7}{3}m_\pi^2 + \Bigg(5m_\pi^2$$

$$- \frac{7}{6}\vec{q}^2\Bigg) \ln\left(\frac{m_\pi}{\mu}\right) + \left(\frac{10}{3}m_\pi^2 - \frac{7}{6}\vec{q}^2 + \frac{8m_\pi^4}{4m_\pi^2 + \vec{q}^2}\right) L(q) \Bigg\}$$

$$= \frac{-\vec{q}^4}{16\pi^2}\left\{ \frac{-7}{12}\mathcal{R} - \frac{7}{36} - \frac{7}{6}\ln\left(\frac{m_\pi}{\mu}\right) + -\frac{7}{6}L(q) \right\} + \mathcal{O}(\chi^4) . \tag{284}$$

A.3.3 $I_{box}^{(5)}$

The $I_{box}^{(5)}$ integral seen in $B^*\bar{B}^* \to B^*\bar{B}^*$ and $B^*B^* \to B^*B^*$ potential is given by

$$I_{box}^{(5)} = i \int \frac{d^4 l}{(2\pi)^4} \frac{\vec{q_2}^2 (\vec{q_1}\cdot\vec{q})^2}{(l_0 - i\epsilon)^2 \left[q_2^2 - m_\pi^2 + i\epsilon\right]\left[q_1^2 - m_\pi^2 + i\epsilon\right]} , \tag{285}$$

following the same steps as in the earlier integrals,

$$I_{box}^{(5)} = i \int \frac{d^4 l}{(2\pi)^4} \frac{\vec{q_2}^2 (\vec{q_1}\cdot\vec{q})^2}{(l_0 - i\epsilon)^2 \left[(l^0)^2 - (\vec{q_2}^2 - \vec{q_1}^2)x - \vec{q_1}^2 - m_\pi^2 + i\epsilon\right]^2} . \tag{286}$$

Executing the l_0-integration, setting $\epsilon \to 0$ and shifting $\vec{l} \to \vec{l} + \vec{p}$ such that $\vec{q_1} = \vec{p} - \vec{l} \to -\vec{l}$ and $\vec{q_2} = \vec{p'} - \vec{l} \to -\vec{l} + \vec{q}$.

$$I_{box}^{I^{(5)}} = \frac{-3}{4}\int_0^1 dx \int \frac{d^3\vec{l}}{(2\pi)^3} \frac{(\vec{l}\cdot\vec{q})^2(\vec{l} - \vec{q})^2}{\left[\vec{l}^2 + (-2\vec{l}\cdot\vec{q} + \vec{q}^2)x + m_\pi^2\right]^{5/2}} . \tag{287}$$

Completing the square in the denominator through $\vec{l} \to \vec{l} + \vec{q}x$ and dropping terms odd in $\vec{l}$:

$$I_{box}^{(5)} = \frac{-3}{4}\int_0^1 dx \int \frac{d^3\vec{l}}{(2\pi)^3} \frac{\frac{\vec{l}^4\vec{q}^2}{D-1} + \frac{\vec{l}^2\vec{q}^4}{D-1}[x^2(D+4) - 6x + 1] - \vec{q}^6 x^2(1-x)^2}{\left[\vec{l}^2 + \vec{q}^2 x(1-x) + m_\pi^2\right]^{5/2}} . \tag{288}$$

Now, going to $(D-1)$-dimensional spherical coordinates and inserting μ

$$= \frac{-3\sqrt{\pi}}{(4\pi)^{D/2}} \frac{\mu^{4-D}}{\Gamma(\frac{D-1}{2})} \int_0^1 dx$$
$$\int_0^\infty dl \frac{\frac{l^{D+2}\vec{q}^2}{D-1} + \frac{l^D\vec{q}^4}{D-1}[x^2(D+4) - 6x + 1] + l^{D-2}\vec{q}^6x^2(1-x)^2}{\left[\vec{l}^2 + \vec{q}^2x(1-x) + m_\pi^2\right]^{5/2}} \ . \quad (289)$$

Solving the l-integration, inserting $D = 4 - \xi$ and doing the x- integration:

$$I^{(5)}_{box} = \frac{-1}{16\pi^2}\vec{q}^2\left\{\left(\frac{5}{2}m_\pi^2 - \frac{1}{4}\vec{q}^2\right)\mathcal{R} - \frac{1}{12}\vec{q}^2 + 3m_\pi^2 + \left(2m_\pi^2 - \frac{1}{6}\vec{q}^2\right)\ln\left(\frac{m_\pi}{\mu}\right)\right.$$
$$\left. + \left(\frac{7}{3}\vec{q}^2 - \frac{8}{3}m_\pi^2 + \frac{8m_\pi^4}{4m_\pi^2 + \vec{q}^2}\right)L(q)\right\}$$
$$= \frac{-\vec{q}^4}{16\pi^2}\left\{\frac{-1}{4}\mathcal{R} - \frac{1}{12} - \frac{1}{6}\ln\left(\frac{m_\pi}{\mu}\right) + \frac{7}{3}L(q)\right\} + \mathcal{O}(\chi^4) \ . \quad (290)$$

A.3.4 $(I^{(6)}_{box})_{in}$

We encounter this integral in $B^*\bar{B} \to B\bar{B}^*$ and $B\bar{B} \to B^*\bar{B}^*$ and in the subsequent $B^{(*)}B^{(*)}$ counterparts. The $(I^{(6)}_{box})_{in}$ integral is given by,

$$(I^{(6)}_{box})_{in} = i\int \frac{d^4l}{(2\pi)^4} \frac{\epsilon_{ijs}\epsilon_{nrm}(q_2)_j(q_1)_s(q_2)_r(q_1)_m}{[l_0 - i\epsilon]^2\left[q_2^2 - m_\pi^2 + i\epsilon\right]\left[q_1^2 - m_\pi^2 + i\epsilon\right]} \ , \quad (291)$$

repeating the same steps as in the earlier cases

$$(I^{(6)}_{box})_{in} = \frac{-3}{4}\mu^{4-D}\int_0^1 dx \int \frac{d^{D-1}\vec{l}}{(2\pi)^{D-1}}\epsilon_{ijs}\epsilon_{nrm}$$
$$\frac{[l_j + q_j(x-1)](l_s + q_s s)[l_r + q_r(x-1)](l_m + q_m x)}{\left[\vec{l}^2 + \vec{q}^2x(1-x) + m_\pi^2\right]^{5/2}} \ , \quad (292)$$

due to anti-symmetry of the Levi-Civita tensor, most of the numerator vanishes,

$$= \frac{-3}{4}\mu^{4-D}\int_0^1 dx \int \frac{d^{D-1}\vec{l}}{(2\pi)^{D-1}} \frac{\epsilon_{ijs}\epsilon_{nrm}(-l_sq_j)(-l_mq_r)}{\left[\vec{l}^2 + \vec{q}^2x(1-x) + m_\pi^2\right]^{5/2}} \ . \quad (293)$$

Again, Going to $(D-1)$-dimensional spherical coordinates and using $l_i l_j \to \frac{\vec{l}^2\,\delta_{ij}}{D-1}$, we get

$$= \frac{-3\sqrt{\pi}}{(4\pi)^{D/2}} \frac{\mu^{4-D}}{\Gamma(\frac{D-1}{2})} \int_0^1 dx(\delta_{in}\vec{q}^{\,2} - q_i q_n) \int_0^\infty dl \frac{\frac{l^D}{D-1}}{\left[\vec{l}^2 + \vec{q}^{\,2}x(1-x) + m_\pi^2\right]^{5/2}} \,. \quad (294)$$

Performing the l- and x- integration and inserting $D = 4 - \xi$

$$(I^{(6)}_{box})_{in} = \frac{-1}{16\pi^2}(\delta_{in}\vec{q}^{\,2} - q_i q_n)\left\{ -\mathcal{R} + 1 - 2L(q) - 2\ln\left(\frac{m_\pi}{\mu}\right)\right\} . \quad (295)$$

A.3.5 $(I^{(7)}_{box})_{ikn}$

We encounter this integral in $B^*\bar{B} \to B^*\bar{B}^*$, $B\bar{B}^* \to B^*\bar{B}^*$ and in the subsequent $B^{(*)}B^{(*)}$ counterparts . The $(I^{(7)}_{box})_{ikn}$ integral is given by,

$$(I^{(7)}_{box})_{ikn} = i\int \frac{d^4l}{(2\pi)^4} \frac{\big((q_2)_k\,(q_1)_i - (q_2)_i\,(q_1)_k\big)\epsilon_{num}(q_2)_u(q_1)_m}{[l_0 - i\epsilon]^2\left[q_2^2 - m_\pi^2\right]\left[q_1^2 - m_\pi^2\right]} \,. \quad (296)$$

Introducing Feynman parameters, executing the l_0-integration, and shifting $\vec{l} \to \vec{l} + \vec{p}$ such that $\vec{q_1} = \vec{p} - \vec{l} \to -\vec{l}$ and $\vec{q_2} = \vec{p'} - \vec{l} \to -\vec{l} + \vec{q}$ with $\vec{q} = \vec{p'} - \vec{p}$:

$$= \frac{-3}{4}\int_0^1 dx \int \frac{d^3\vec{l}}{(2\pi)^3} \frac{\big((l-q)_k\,(l)_i - (l-q)_i\,(l)_k\big)\epsilon_{num}(l-q)_u(l)_m}{\left[\vec{l}^2 + (-2\vec{l}\cdot\vec{q} + \vec{q}^{\,2})x + m_\pi^2\right]^{5/2}} \,. \quad (297)$$

All terms proportional to $\epsilon_{num} l_u l_m$ or $\epsilon_{num} q_u q_m$ and every term of odd power in l vanishes due to symmetry:

$$= \frac{-3}{4}\mu^{4-D}\int_0^1 dx \int \frac{d^{D-1}\vec{l}}{(2\pi)^{D-1}} \frac{\big(l_k\,q_i - l_i\,q_k\big)\epsilon_{num}\big(-l_m q_u\big)}{\left[\vec{l}^2 + (-2\vec{l}\cdot\vec{q} + \vec{q}^{\,2})x + m_\pi^2\right]^{5/2}} \,. \quad (298)$$

Solving the remaining integral like in the earlier integral,

$$(I^{(7)}_{box})_{ikn} = \frac{-1}{16\pi^2}(\epsilon_{nku} q_u q_i - \epsilon_{niu} q_u q_k)\left\{ -\mathcal{R} + 1 - 2L(q) - 2\ln\left(\frac{m_\pi}{\mu}\right)\right\} . \quad (299)$$

B Partial wave projectors

In this section, we have presented the complete set of partial wave projectors used for calculating the potentials [9].

$$P(B\bar{B}(^1S_0)) = 1 \ , \tag{300}$$

$$P(B^*\bar{B}^*(^1S_0)) = \frac{1}{\sqrt{3}}\big(\vec{\epsilon_1}^* \cdot \vec{\epsilon_2}\big) \ , \tag{301}$$

$$P(B^*\bar{B}^*(^5D_0)) = -\sqrt{\frac{3}{8}}S_{xy}v_{xy} \ , \tag{302}$$

$$P(B\bar{B}^*(^3S_1))_x = \epsilon_{2,x} \ , \tag{303}$$

$$P(B^*\bar{B}(^3S_1))_h = \epsilon^*_{1,h} \ , \tag{304}$$

$$P(B\bar{B}^*(^3D_1))_x = -\frac{3}{\sqrt{2}}\epsilon_{2,y}v_{xy} \ , \tag{305}$$

$$P(B^*\bar{B}(^3D_1))_x = -\frac{3}{\sqrt{2}}\epsilon^*_{1,y}v_{xy} \ , \tag{306}$$

$$P(B^*\bar{B}^*(^5D_1))_h = -\frac{\sqrt{3}}{2}i\epsilon_{hxj}S_{xy}v_{jy} \ , \tag{307}$$

$$P(B^*\bar{B}^*(^3S_1))_x = A_x \ , \tag{308}$$

$$P(B^*\bar{B}^*(^3D_1))_h = -\frac{3}{\sqrt{2}}A_x v_{hx} \ , \tag{309}$$

$$P(B\bar{B}(^1D_2))_{xy} = -\sqrt{\frac{15}{2}}v_{xy} \ , \tag{310}$$

$$P(B\bar{B}^*(^3D_2))_{zy} = -\frac{\sqrt{5}}{2}i\epsilon_{2,h}\big(\epsilon_{zhx}v_{xy} + \epsilon_{yhx}v_{xz}\big) \ , \tag{311}$$

$$P(B^*\bar{B}^*(^5S_2))_{xy} = \frac{1}{2}S_{xy} \ , \tag{312}$$

$$P(B^*\bar{B}^*(^1D_2))_{xy} = -\sqrt{\frac{5}{2}}\big(\vec{\epsilon_1}^* \cdot \vec{\epsilon_2}\big)v_{xy} \ , \tag{313}$$

$$P(B^*\bar{B}^*(^5D_2))_{zy} = -\sqrt{\frac{45}{56}}\Big(S_{zx}v_{xy} + S_{yx}v_{xz} - \frac{2}{3}\delta_{xy}S_{lx}v_{lx}\Big) \ , \tag{314}$$

$$P(B^*\bar{B}^*(^5G_2))_{zw} = \sqrt{\frac{175}{32}}S_{xy}v_{xyzw} \ , \tag{315}$$

where

$$v_{xy} = n_x n_y - \frac{1}{3}\delta_{xy} \ , \tag{316}$$

$$S_{xy} = \epsilon^*_{1,x}\epsilon_{2,y} + \epsilon^*_{1,y}\epsilon_{2,x} - \frac{2}{3}\delta_{xy}\big(\vec{\epsilon_1}^* \cdot \vec{\epsilon_2}\big) \ , \tag{317}$$

$$A_x = \frac{i}{\sqrt{2}}\epsilon_{xyz}\epsilon^*_{1,y}\epsilon_{2,z} \ , \tag{318}$$

and

$$v_{xyzw} = n_x n_y n_z n_w - \frac{1}{7}\big(n_x n_y \delta_{zw} + n_x n_z \delta_{yw} + n_x n_w \delta_{yz} + n_y n_z \delta_{xw} + n_y n_w \delta_{xz} + n_z n_w \delta_{xy}\big) + \frac{1}{35}\big(\delta_{xy}\delta_{zw} + \delta_{xz}\delta_{yw} + \delta_{xw}\delta_{yz}\big) \tag{319}$$

The projectors are normalised as:

$$\int \frac{d\Omega_n}{4\pi} P^\dagger(\alpha, \vec{n}) P(\alpha, \vec{n}) = 2J + 1$$

C The effective potentials with whole loop contribution

In this section, all the 2PE potentials of $B^{(*)}\bar{B}^{(*)} \to B^{(*)}\bar{B}^{(*)}$ and $B^{(*)}B^{(*)} \to B^{(*)}B^{(*)}$ scatterings containing all the terms arising from the loop integrals are presented.

C.1 $B\bar{B} \to B\bar{B}$

$$\begin{aligned} V_{2PE}(B\bar{B} \to B\bar{B}) &= \frac{1}{2f_\pi^4}(\vec{\tau}_1 \cdot \vec{\tau}_2)\left(I_{fb} - \frac{g^4}{8} I^{(2)}_{box}\right) \\ &= \frac{1}{16\pi^2 f_\pi^4}(\vec{\tau}_1 \cdot \vec{\tau}_2)\Bigg\{\mathcal{R}\left[\left(\frac{23}{192}\vec{q}^{\,2} + \frac{15}{32}m_\pi^2\right)g^4 + \frac{\vec{q}^{\,2}}{24} + \frac{m_\pi^2}{4}\right] \\ &+ \vec{q}^{\,2}\left(\frac{5}{576}g^4 - \frac{5}{72}\right) + m_\pi^2\left(\frac{1}{6}g^4 - \frac{1}{3}\right) + \ln\left(\frac{m_\pi}{\mu}\right)\Bigg[\vec{q}^{\,2} \\ &\left(\frac{23}{96}g^4 + \frac{1}{12}\right) + m_\pi^2\left(\frac{15}{16}g^4 + \frac{1}{2}\right)\Bigg] + L(q)\Bigg[\vec{q}^{\,2}\left(\frac{23}{96}g^4 + \frac{1}{12}\right) \\ &+ m_\pi^2\left(\frac{5}{24}g^4 + \frac{1}{3}\right) + \frac{g^4 m_\pi^4}{2(4m_\pi^2 + \vec{q}^{\,2})}\Bigg]\Bigg\} \ . \end{aligned} \tag{320}$$

C.2 $B^*\bar{B} \to B^*\bar{B}$

$$V_{2PE}(B^*\bar{B} \to B^*\bar{B}) = (\epsilon^*_{1'} \cdot \epsilon_1) V_{2PE}(B\bar{B} \to B\bar{B}) \ . \tag{321}$$

C.3 $B^*\bar{B} \to B\bar{B}^*$

$$V_{2PE}(B^*\bar{B} \to B\bar{B}^*) = \frac{3}{512\pi^2 f_\pi^4} g^4 (\epsilon^*_{2',n}\epsilon_{1,i})(\delta_{in}\vec{q}^{\,2} - q_i q_n)\Bigg\{ -\mathcal{R} + 1 - 2L(q) - 2\ln\left(\frac{m_\pi}{\mu}\right)\Bigg\} . \quad (322)$$

C.4 $B^*\bar{B}^* \to B^*\bar{B}^*$

$$\begin{aligned} V_{2PE}(B^*\bar{B}^* \to B^*\bar{B}^*) = {} & \frac{1}{16\pi^2 f_\pi^4}\big(\vec{\tau}_1\cdot\vec{\tau}_2\big)(\epsilon_1\cdot\epsilon^*_{1'})(\epsilon_2\cdot\epsilon^*_{2'})\Bigg\{\mathcal{R}\Bigg[\left(\frac{23}{192}\vec{q}^{\,2} + \frac{15}{32}m_\pi^2\right)g^4 \\ & + \frac{\vec{q}^{\,2}}{24} + \frac{m_\pi^2}{4}\Bigg] + \vec{q}^{\,2}\left(\frac{5}{576}g^4 - \frac{5}{72}\right) + m_\pi^2\left(\frac{1}{6}g^4 - \frac{1}{3}\right) \\ & + \ln\left(\frac{m_\pi}{\mu}\right)\left[\vec{q}^{\,2}\left(\frac{23}{96}g^4 + \frac{1}{12}\right) + m_\pi^2\left(\frac{15}{16}g^4 + \frac{1}{2}\right)\right] \\ & + L(q)\left[\vec{q}^{\,2}\left(\frac{23}{96}g^4 + \frac{1}{12}\right) + m_\pi^2\left(\frac{5}{24}g^4 + \frac{1}{3}\right) + \frac{g^4 m_\pi^4}{2(4m_\pi^2 + \vec{q}^{\,2})}\right]\Bigg\} \\ & + \frac{g^4}{512\pi^2 f_\pi^4}(\epsilon_{1,i}\epsilon^*_{1',k}\epsilon_{2,l}\epsilon^*_{2',n})\big(\delta_{in}\delta_{kl} - \delta_{il}\delta_{kn}\big)\Bigg\{\frac{-7}{3}\vec{q}^{\,2}\mathcal{R} - \frac{16}{3}m_\pi^2 \\ & - 5\vec{q}^{\,2}\ln\left(\frac{m_\pi}{\mu}\right) - \frac{15}{2}\vec{q}^{\,2}L(q)\Bigg\} + \frac{g^4}{16\pi^2 f_\pi^4}(\epsilon_{1,i}\epsilon^*_{1',k}\epsilon_{2,l}\epsilon^*_{2',n}) \\ & \big[q_k q_n\delta_{il} - q_k q_l\delta_{in} + q_i q_l\delta_{kn} - q_i q_n\delta_{kl}\big] \times \frac{3}{64\vec{q}^{\,4}}\Bigg\{\frac{-1}{3}\vec{q}^{\,2}\mathcal{R} \\ & - \frac{1}{9}\vec{q}^{\,2} - \frac{16}{3}m_\pi^2 + \ln\left(\frac{m_\pi}{\mu}\right)\big[3m_\pi^2 - \vec{q}^{\,2}\big] \\ & + L(q)\big[6m_\pi^2 - \frac{7}{2}\vec{q}^{\,2}\big]\Bigg\} . \end{aligned} \quad (323)$$

C.5 $B\bar{B} \to B^*\bar{B}^*$

$$V_{2PE}(B\bar{B} \to B^*\bar{B}^*) = \frac{3}{512\pi^2 f_\pi^4} g^4 (\epsilon^*_{1',k}\epsilon^*_{2',n}) \big(\delta_{kn}\vec{q}^{\,2} - q_k\, q_n\big) \Bigg\{ -\mathcal{R} + 1 - 2L(q) - 2\ln\left(\frac{m_\pi}{\mu}\right) \Bigg\} . \quad (324)$$

C.6 $B\bar{B}^* \to B\bar{B}$

$$V_{2PE}(B\bar{B}^* \to B\bar{B}) = 0 \; . \quad (325)$$

C.7 $B^*\bar{B} \to B^*\bar{B}^*$

$$V_{2PE}(B^*\bar{B} \to B^*\bar{B}^*) = i\, \frac{3}{512\pi^2 f_\pi^4} g^4 (\epsilon_{1,i}\epsilon^*_{1',k}\epsilon^*_{2',n}) \left(\epsilon_{nku} q_u q_i - \epsilon_{niu} q_u q_k\right) \Bigg\{ -\mathcal{R} + 1 - 2L(q) - 2\ln\left(\frac{m_\pi}{\mu}\right) \Bigg\} . \quad (326)$$

C.8 $B\bar{B}^* \to B^*\bar{B}^*$

$$V_{2PE}(B\bar{B}^* \to B^*\bar{B}^*) = i\, \frac{3}{512\pi^2 f_\pi^4} g^4 (\epsilon_{2,l}\epsilon^*_{1',k}\epsilon^*_{2',n}) \left(\epsilon_{knu} q_u q_l - \epsilon_{klu} q_u q_n\right) \Bigg\{ -\mathcal{R} + 1 - 2L(q) - 2\ln\left(\frac{m_\pi}{\mu}\right) \Bigg\} . \quad . \quad (327)$$

C.9 $BB \to BB$

$$
\begin{aligned}
V_{2PE}(BB \to BB) &= \frac{1}{2f_\pi^4}(\vec{\tau}_1 \cdot \vec{\tau}_2)\bigg(- I_{fb} + \frac{g^2}{2} I_{tr} - \frac{g^4}{8} I_{box}^{(2)} \bigg) \\
&= \frac{1}{16\pi^2 f_\pi^4}(\vec{\tau}_1 \cdot \vec{\tau}_2)\Bigg\{ \mathcal{R}\bigg[\bigg(\frac{23}{192}\vec{q}^{\,2} + \frac{15}{32}m_\pi^2\bigg)g^4 - \bigg(\frac{5}{48}\vec{q}^{\,2} \\
&+ \frac{3}{8}m_\pi^2\bigg)g^2 - \frac{\vec{q}^{\,2}}{24} - \frac{m_\pi^2}{4}\bigg] + \vec{q}^{\,2}\bigg(\frac{5}{576}g^4 + \frac{13}{144}g^2 + \frac{5}{72}\bigg) \\
&+ m_\pi^2\bigg(\frac{1}{6}g^4 + \frac{1}{12}g^2 + \frac{1}{3}\bigg) + \ln\left(\frac{m_\pi}{\mu}\right)\bigg[\vec{q}^{\,2}\bigg(\frac{23}{96}g^4 - \frac{5}{24}g^2 - \frac{1}{12}\bigg) \\
&+ m_\pi^2\bigg(\frac{15}{16}g^4 - \frac{3}{4}g^2 - \frac{1}{2}\bigg)\bigg] + L(q)\bigg[\vec{q}^{\,2}\bigg(\frac{23}{96}g^4 - \frac{5}{24}g^2 - \frac{1}{12}\bigg) \\
&+ m_\pi^2\bigg(\frac{5}{24}g^4 - \frac{1}{3}g^2 - \frac{1}{3}\bigg) + \frac{g^4 m_\pi^4}{2(4m_\pi^2 + \vec{q}^{\,2})}\bigg]\Bigg\}\ .
\end{aligned} \tag{328}
$$

C.10 $B^*B \to B^*B$

$$
V_{2PE}(B^*B \to B^*B) = (\epsilon_{1'}^* \cdot \epsilon_1) V_{2PE}(BB \to BB)\ . \tag{329}
$$

C.11 $B^*B \to BB^*$

$$
\begin{aligned}
V_{2PE}(B^*B \to BB^*) = \frac{3}{512\pi^2 f_\pi^4} g^4 (\epsilon_{2',n}^* \epsilon_{1,i})(\delta_{in}\vec{q}^{\,2} - q_i q_n)\Bigg\{ &- \mathcal{R} + 1 - 2L(q) \\
&- 2\ln\left(\frac{m_\pi}{\mu}\right)\Bigg\}\ .
\end{aligned} \tag{330}
$$

C.12 $B^*B^* \to B^*B^*$

$$
\begin{aligned}
V_{2PE}(B^*B^* \to B^*B^*) = \frac{1}{16\pi^2 f_\pi^4}\big(\vec{\tau}_1\cdot\vec{\tau}_2\big)(\epsilon_1\cdot\epsilon_{1'}^*)(\epsilon_2\cdot\epsilon_{2'}^*)\Bigg\{\mathcal{R}\Bigg[\bigg(\frac{23}{192}\vec{q}^{\,2}+\frac{15}{32}m_\pi^2\bigg)g^4 \\
-\bigg(\frac{5}{48}\vec{q}^{\,2}+\frac{3}{8}m_\pi^2\bigg)g^2-\frac{\vec{q}^{\,2}}{24}-\frac{m_\pi^2}{4}\Bigg]+\vec{q}^{\,2}\bigg(\frac{5}{576}g^4+\frac{13}{144}g^2 \\
+\frac{5}{72}\bigg)+m_\pi^2\bigg(\frac{1}{6}g^4+\frac{1}{12}g^2+\frac{1}{3}\bigg)+\ln\bigg(\frac{m_\pi}{\mu}\bigg)\bigg[\vec{q}^{\,2}\bigg(\frac{23}{96}g^4 \\
-\frac{5}{24}g^2-\frac{1}{12}\bigg)+m_\pi^2\bigg(\frac{15}{16}g^4-\frac{3}{4}g^2-\frac{1}{2}\bigg)\bigg]+L(q)\bigg[\vec{q}^{\,2} \\
\bigg(\frac{23}{96}g^4-\frac{5}{24}g^2-\frac{1}{12}\bigg)+m_\pi^2\bigg(\frac{5}{24}g^4-\frac{1}{3}g^2-\frac{1}{3}\bigg) \\
+\frac{g^4 m_\pi^4}{2(4m_\pi^2+\vec{q}^{\,2})}\bigg]\Bigg\}+\frac{g^4}{512\pi^2 f_\pi^4}(\epsilon_{1,i}\epsilon_{1',k}^*\epsilon_{2,l}\epsilon_{2',n}^*) \\
\big(\delta_{in}\delta_{kl}-\delta_{il}\delta_{kn}\big)\bigg\{\frac{-7}{3}\vec{q}^{\,2}\mathcal{R}-\frac{16}{3}m_\pi^2-5\vec{q}^{\,2}\ln\bigg(\frac{m_\pi}{\mu}\bigg) \\
-\frac{15}{2}\vec{q}^{\,2}L(q)\bigg\}+\frac{g^4}{16\pi^2 f_\pi^4}(\epsilon_{1,i}\epsilon_{1',k}^*\epsilon_{2,l}\epsilon_{2',n}^*)\big[q_k q_n\delta_{il} \\
-q_k q_l\delta_{in}+q_i q_l\delta_{kn}-q_i q_n\delta_{kl}\big]\times\frac{3}{64\vec{q}^{\,4}}\bigg\{\frac{-1}{3}\vec{q}^{\,2}\mathcal{R}-\frac{1}{9}\vec{q}^{\,2} \\
-\frac{16}{3}m_\pi^2+\ln\bigg(\frac{m_\pi}{\mu}\bigg)\big[3m_\pi^2-\vec{q}^{\,2}\big]+L(q)\big[6m_\pi^2-\frac{7}{2}\vec{q}^{\,2}\big]\bigg\}\ .
\end{aligned} \tag{331}
$$

C.13 $BB \to B^*B^*$

$$
\begin{aligned}
V_{2PE}(BB \to B^*B^*) = \frac{3}{512\pi^2 f_\pi^4}g^4(\epsilon_{1',k}^*\epsilon_{2',n}^*)\big(\delta_{kn}\vec{q}^{\,2}-q_k\,q_n\big)\bigg\{-\mathcal{R}+1-2L(q) \\
-2\ln\bigg(\frac{m_\pi}{\mu}\bigg)\bigg\}
\end{aligned} \tag{332}
$$

C.14 $BB^* \to BB$

$$
V_{2PE}(BB^* \to BB) = 0\ . \tag{333}
$$

C.15 $B^*B \to B^*B^*$

$$V_{2PE}(B^*B \to B^*B^*) = i\,\frac{3}{512\pi^2 f_\pi^4} g^4(\epsilon_{1,i}\epsilon^*_{1',k}\epsilon^*_{2',n})\left(\epsilon_{nku}q_u q_i - \epsilon_{niu}q_u q_k\right)$$
$$\left\{-\mathcal{R} + 1 - 2L(q) - 2\ln\left(\frac{m_\pi}{\mu}\right)\right\}. \tag{334}$$

C.16 $B\bar{B}^* \to B^*\bar{B}^*$

$$V_{2PE}(B\bar{B}^* \to B^*\bar{B}^*) = i\,\frac{3}{512\pi^2 f_\pi^4} g^4(\epsilon_{2,l}\epsilon^*_{1',k}\epsilon^*_{2',n})\left(\epsilon_{knu}q_u q_l - \epsilon_{klu}q_u q_n\right)$$
$$\left\{-\mathcal{R} + 1 - 2L(q) - 2\ln\left(\frac{m_\pi}{\mu}\right)\right\}. \tag{335}$$

D PWD with whole loop contribution

D.1 $\mathbf{J^{PC} = 0^{++}}$

Here, we present the PWD potentials which contains all terms (namely m_π^2 terms) which arise from loop integrals

$$V_{2PE}(0^{++}) = \frac{1}{16\pi^2 f_\pi^4}\begin{pmatrix} S_0(p',p) & \frac{\sqrt{3}}{16}\,g^4(V_{2PE}^{0^{++}})_{12} & \frac{3\sqrt{3}}{32\sqrt{2}}\,g^4(V_{2PE}^{0^{++}})_{13} \\ \frac{\sqrt{3}}{16}\,g^4(V_{2PE}^{0^{++}})_{21} & (V_{2PE}^{0^{++}})_{22} & \frac{-1}{64\sqrt{2}}\,g^4(V_{2PE}^{0^{++}})_{23} \\ \frac{3\sqrt{3}}{32\sqrt{2}}\,g^4(V_{2PE}^{0^{++}})_{31} & \frac{-1}{64\sqrt{2}}\,g^4(V_{2PE}^{0^{++}})_{32} & (V_{2PE}^{0^{++}})_{33} \end{pmatrix} \tag{336}$$

where,

$$S_0 = \int_{-1}^{1}\frac{dx}{2}(\vec{\tau}_1\cdot\vec{\tau}_2)\Bigg\{\mathcal{R}\left[\left(\frac{23}{192}\vec{q}^{\,2} + \frac{15}{32}m_\pi^2\right)g^4 + \frac{\vec{q}^{\,2}}{24} + \frac{m_\pi^2}{4}\right] + \vec{q}^{\,2}\left(\frac{5}{576}g^4 - \frac{5}{72}\right)$$
$$+ m_\pi^2\left(\frac{1}{6}g^4 - \frac{1}{3}\right) + \ln\left(\frac{m_\pi}{\mu}\right)\left[\vec{q}^{\,2}\left(\frac{23}{96}g^4 + \frac{1}{12}\right) + m_\pi^2\left(\frac{15}{16}g^4 + \frac{1}{2}\right)\right] + L(q)\Bigg[\vec{q}^{\,2}$$
$$\left(\frac{23}{96}g^4 + \frac{1}{12}\right) + m_\pi^2\left(\frac{5}{24}g^4 + \frac{1}{3}\right) + \frac{g^4 m_\pi^4}{2(4m_\pi^2 + \vec{q}^{\,2})}\Bigg]\Bigg\} \tag{337}$$

$$(V_{2PE}^{0^{++}})_{12} = (V_{2PE}^{0^{++}})_{21} = (p'^{\,2} + p^2)\left[-\mathcal{R} + 1 - 2\ln\left(\frac{m_\pi}{\mu}\right)\right] + 2R_2(p',p) \tag{338}$$

$$(V_{2PE}^{0^{++}})_{13} = \sqrt{\frac{2}{3}}(p'^{\,2})\left[-\mathcal{R} + 1 - 2\ln\left(\frac{m_\pi}{\mu}\right)\right] - 2R_{n'}(p',p) + \frac{2}{3}R_2(p',p) \tag{339}$$

$$(V_{2PE}^{0++})_{22} = S_0 + \frac{g^4}{16}\Bigg\{2\mathcal{R}(p'^{\,2}+p^2) - \frac{1}{9}(p'^{\,2}+p^2) + \big[4(p'^{\,2}+p^2)+3m_\pi^2\big]$$
$$\ln\left(\frac{m_\pi}{\mu}\right) + 4R_2(p',p) + 6m_\pi^2 R_0(p',p)\Bigg\} \quad (340)$$

$$(V_{2PE}^{0++})_{23} = 2p'^{\,2}\left[2\mathcal{R} - \frac{1}{9} - \ln\left(\frac{m_\pi}{\mu}\right)\right] - \frac{7}{2}\Big[3R_{n'}(p',p) - R_2(p',p)\Big] + 6m_\pi^2$$
$$\Big[3R_{n'/q^2}(p',p) - R_0(p',p)\Big] + \left[\frac{-16}{3}m_\pi^2 + 3m_\pi^2\ln\left(\frac{m_\pi}{\mu}\right)\right]\left(\frac{1}{16pp'^{\,3}}\right)\Big[20pp'^{\,3}$$
$$-12p^3p' + 3(p^2-p'^{\,2})^2\ln\left(\frac{(p+p')^2}{(p-p')^2}\right)\Bigg] \quad (341)$$

$$(V_{2PE}^{0++})_{31} = \sqrt{\frac{2}{3}}(p^2)\left[-\mathcal{R} + 1 - 2\ln\left(\frac{m_\pi}{\mu}\right)\right] - 2R_n(p',p) + \frac{2}{3}R_2(p',p) \quad (342)$$

$$(V_{2PE}^{0++})_{32} = 2p^2\left[2\mathcal{R} - \frac{1}{9} - \ln\left(\frac{m_\pi}{\mu}\right)\right] - \frac{7}{2}\Big[3R_n(p',p) - R_2(p',p)\Big] + 6m_\pi^2$$
$$\Big[3R_{n/q^2}(p',p) - R_0(p',p)\Big] + \left[\frac{-16}{3}m_\pi^2 + 3m_\pi^2\ln\left(\frac{m_\pi}{\mu}\right)\right]\left(\frac{1}{16p'p^3}\right)\Big[20p^3p'$$
$$-12p'^{\,3}p + 3(p'^{\,2}-p^2)^2\ln\left(\frac{(p+p')^2}{(p-p')^2}\right)\Bigg] \quad (343)$$

$$(V_{2PE}^{0++})_{33} = S_2 + \frac{g^4}{16}\Bigg\{\frac{15}{8}R_2(p',p) - \frac{45}{8}R_{x^2}(p',p) + \frac{9}{2}m_\pi^2\Big(R_{n/q^2}(p',p) + R_{n'/q^2}(p',p)$$
$$-\frac{1}{2}R_0(p',p) - 3R_{x/q^2}(p',p)\Big) - \frac{21}{8}\Big(R_n(p',p) + R_{n'}(p',p) - \frac{1}{2}R_2(p',p) - 3R_x(p',p)\Big)$$
$$+\left[\frac{-16}{3}m_\pi^2 + 3m_\pi^2\ln\left(\frac{m_\pi}{\mu}\right)\right]\left(\frac{1}{32p'^{\,3}p^3}\right)\Big[4pp'(3p^4 - 2p^2p'^{\,2} + 3p'^{\,4})$$
$$+3(p^2-p'^{\,2})^2\,(p^2+p'^{\,2})\ln\left(\frac{(p-p')^2}{(p+p')^2}\right)\Big]\Bigg\} \quad (344)$$

and,

$$S_2 = \int_{-1}^{1} \frac{dx}{2} \frac{(3x^2-1)}{2} (\vec{\tau}_1 \cdot \vec{\tau}_2) \Bigg\{ \mathcal{R} \Bigg[\Bigg(\frac{23}{192}\vec{q}^{\,2} + \frac{15}{32} m_\pi^2 \Bigg) g^4 + \frac{\vec{q}^{\,2}}{24} + \frac{m_\pi^2}{4} \Bigg] + \vec{q}^{\,2} \Bigg(\frac{5}{576} g^4 - \frac{5}{72} \Bigg) + m_\pi^2 \Bigg(\frac{1}{6} g^4 - \frac{1}{3} \Bigg) + \ln\left(\frac{m_\pi}{\mu}\right) \Bigg[\vec{q}^{\,2} \Bigg(\frac{23}{96} g^4 + \frac{1}{12} \Bigg) + m_\pi^2 \Bigg(\frac{15}{16} g^4 + \frac{1}{2} \Bigg) \Bigg] + L(q) \Bigg[\vec{q}^{\,2} \Bigg(\frac{23}{96} g^4 + \frac{1}{12} \Bigg) + m_\pi^2 \Bigg(\frac{5}{24} g^4 + \frac{1}{3} \Bigg) + \frac{g^4 m_\pi^4}{2(4m_\pi^2 + \vec{q}^{\,2})} \Bigg] \Bigg\} \tag{345}$$

D.2 $\mathbf{J^{PC} = 1^{++}}$

$$V_{2PE}(1^{++}) = \frac{1}{16\pi^2 f_\pi^4} \begin{pmatrix} (V_{2PE}^{1^{++}})_{11} & \frac{3}{32\sqrt{2}} g^4 (V_{2PE}^{1^{++}})_{12} & -\frac{9}{16\sqrt{24}} g^4 (V_{2PE}^{1^{++}})_{13} \\ \frac{3}{32\sqrt{2}} g^4 (V_{2PE}^{1^{++}})_{21} & (V_{2PE}^{1^{++}})_{22} & \frac{3\sqrt{3}}{32} g^4 (V_{2PE}^{1^{++}})_{23} \\ -\frac{9}{16\sqrt{24}} g^4 (V_{2PE}^{1^{++}})_{31} & \frac{3\sqrt{3}}{32} g^4 (V_{2PE}^{1^{++}})_{32} & (V_{2PE}^{1^{++}})_{33} \end{pmatrix} \tag{346}$$

where,

$$(V_{2PE}^{1^{++}})_{11} = S_0 + \frac{g^4}{16} \Bigg\{ (p'^{\,2} + p^2) \Bigg[[-\mathcal{R} + 1 - 2\ln\left(\frac{m_\pi}{\mu}\right) \Bigg] - 2R_2(p', p) \Bigg\} \tag{347}$$

$$(V_{2PE}^{1^{++}})_{12} = (V_{2PE}^{1^{++}})_{13} = \frac{2}{3}(p'^{\,2}) \Bigg[[-\mathcal{R} + 1 - 2\ln\left(\frac{m_\pi}{\mu}\right) \Bigg] + \frac{2}{3} R_2(p', p) - 2R_{n'}(p', p) \tag{348}$$

$$(V_{2PE}^{1^{++}})_{21} = (V_{2PE}^{1^{++}})_{31} = \frac{2}{3}(p^2) \Bigg[[-\mathcal{R} + 1 - 2\ln\left(\frac{m_\pi}{\mu}\right) \Bigg] + \frac{2}{3} R_2(p', p) - 2R_n(p', p) \tag{349}$$

$$(V_{2PE}^{1^{++}})_{22} = S_2 + g^4 \frac{3}{32} \Bigg\{ R_0(p', p) - 3R_{2x}(p', p) + 3R_x(p', p) - R_{n'}(p', p) - R_n(p', p) + \frac{1}{3} R_2(p', p) \Bigg\} \tag{350}$$

$$(V_{2PE}^{1^{++}})_{23} = (V_{2PE}^{1^{++}})_{32} = \frac{2}{3} R_2(p', p) + 3R_x(p', p) - R_{n'}(p', p) - R_n(p', p) - R_{x^2}(p', p) \tag{351}$$

$$(V_{2PE}^{1++})_{33} = 7S_2 + \frac{g^4}{16}\Bigg\{\frac{45}{2}R_2(p',p) - \frac{135}{2}R_{x^2}(p',p) + 6m_\pi^2\Bigg(15R_{n/q^2}(p',p)$$

$$+ 15R_{n'/q^2}(p',p) + 8R_0(p',p) - 45R_{x/q^2}(p',p) - 39R_x(p',p)\Bigg) - \frac{7}{12}\Bigg(15R_n(p',p)$$

$$+ 15R_{n'}(p',p) + 8R_2(p',p) - 45R_x(p',p) - 39R_{x^2}(p',p)\Bigg) + \Bigg[\frac{-16}{3}m_\pi^2$$

$$+ 3m_\pi^2 \ln\left(\frac{m_\pi}{\mu}\right)\Bigg]\left(\frac{5}{192p'^{\,3}p^3}\right)\Bigg[4pp'(3p^4 - 2p^2p'^{\,2} + 3p'^{\,4}) + 3(p^2 - p'^{\,2})^2\,(p^2 + p'^{\,2})$$

$$\ln\left(\frac{(p-p')^2}{(p+p')^2}\right)\Bigg]\Bigg\} \quad (352)$$

D.3 $\mathbf{J^{PC} = 1^{+-}}$

$$V_{2PE}(1^{+-}) = \frac{1}{16\pi^2 f_\pi^4}$$

$$\begin{pmatrix} (V_{2PE}^{1+-})_{11} & \frac{-3}{32\sqrt{2}}\,g^4(V_{2PE}^{1+-})_{12} & (-2g^4)(V_{2PE}^{1+-})_{13} & \frac{-3}{32\sqrt{2}}\,g^4(V_{2PE}^{1+-})_{14} \\ \frac{-3}{32\sqrt{2}}\,g^4(V_{2PE}^{1+-})_{21} & (V_{2PE}^{1+-})_{22} & \frac{-3}{\sqrt{2}}\,g^4(V_{2PE}^{1+-})_{23} & g^4(V_{2PE}^{1+-})_{24} \\ (-2g^4)(V_{2PE}^{1+-})_{31} & \frac{-3}{\sqrt{2}}\,g^4(V_{2PE}^{1+-})_{32} & (V_{2PE}^{1+-})_{33} & \frac{-1}{64\sqrt{2}}\,g^4(V_{2PE}^{1+-})_{34} \\ \frac{-3}{32\sqrt{2}}\,g^4(V_{2PE}^{1+-})_{41} & g^4(V_{2PE}^{1+-})_{24} & \frac{-1}{64\sqrt{2}}\,g^4(V_{2PE}^{1+-})_{43} & (V_{2PE}^{1+-})_{44} \end{pmatrix} \quad (353)$$

where,

$$(V_{2PE}^{1+-})_{11} = S_0 - \frac{g^4}{16}\Bigg\{(p'^{\,2} + p^2)\Bigg[[-\mathcal{R} + 1 - 2\ln\left(\frac{m_\pi}{\mu}\right)\Bigg] - 2R_2(p',p)\Bigg\} \quad (354)$$

$$(V_{2PE}^{1+-})_{12} = (V_{2PE}^{1+-})_{14} = (V_{2PE}^{1+-})_{32} = \frac{2}{3}(p'^{\,2})\Bigg[[-\mathcal{R} + 1 - 2\ln\left(\frac{m_\pi}{\mu}\right)\Bigg] + \frac{2}{3}R_2(p',p)$$

$$- 2R_{n'}(p',p) \quad (355)$$

$$(V_{2PE}^{1+-})_{13} = (V_{2PE}^{1+-})_{31} = \frac{1}{16}(p'^{\,2} + p^2)\Bigg[[-\mathcal{R} + 1 - 2\ln\left(\frac{m_\pi}{\mu}\right)\Bigg] - 2R_2(p',p) \quad (356)$$

$$(V_{2PE}^{1+-})_{21} = (V_{2PE}^{1+-})_{23} = \frac{2}{3}(p^2)\Bigg[[-\mathcal{R} + 1 - 2\ln\left(\frac{m_\pi}{\mu}\right)\Bigg] + \frac{2}{3}R_2(p',p) - 2R_n(p',p)$$

$$(357)$$

$$(V_{2PE}^{1+-})_{22} = S_2 - g^4\frac{3}{2}\Bigg\{R_0(p',p) - 3R_{2x}(p',p) + 3R_x(p',p) - R_{n'}(p',p) - R_n(p',p)$$

$$+ \frac{1}{3}R_2(p',p)\Bigg\} \quad (358)$$

$$(V_{2PE}^{1+-})_{24} = (V_{2PE}^{1+-})_{42} = \frac{9}{4}R_{x^2}(p',p) - \frac{27}{4}R_x(p',p) + \frac{9}{4}R_{n'}(p',p) + \frac{9}{4}R_n(p',p) - \frac{3}{2}R_2(p',p) \quad (359)$$

$$(V_{2PE}^{1+-})_{33} = S_0 - \frac{g^4}{32}\Bigg\{ -2\mathcal{R} + \frac{1}{9}(p^2 + p'^{\,2}) - [3m_\pi^2 + 4(p^2 + p'^{\,2})]\ln\left(\frac{m_\pi}{\mu}\right) - 6m_\pi^2 R_0(p',p) - 4R_2(p',p)\Bigg\} \quad (360)$$

$$(V_{2PE}^{1+-})_{34} = 2p'^{\,2}\left[2\mathcal{R} - \frac{1}{9} - \ln\left(\frac{m_\pi}{\mu}\right)\right] - \frac{7}{2}\left[3R_{n'}(p',p) - R_2(p',p)\right] + 6m_\pi^2\left[3R_{n'/q^2}(p',p) - R_0(p',p)\right] + \left[\frac{-16}{3}m_\pi^2 + 3m_\pi^2\ln\left(\frac{m_\pi}{\mu}\right)\right]\left(\frac{1}{16pp'^{\,3}}\right) \left[20pp'^{\,3} - 12p^3p' + 3(p^2 - p'^{\,2})^2\ln\left(\frac{(p+p')^2}{(p-p')^2}\right)\right] \quad (361)$$

$$(V_{2PE}^{1+-})_{43} = 2p^2\left[2\mathcal{R} - \frac{1}{9} - \ln\left(\frac{m_\pi}{\mu}\right)\right] - \frac{7}{2}\left[3R_n(p',p) - R_2(p',p)\right] + 6m_\pi^2\left[3R_{n/q^2}(p',p) - R_0(p',p)\right] + \left[\frac{-16}{3}m_\pi^2 + 3m_\pi^2\ln\left(\frac{m_\pi}{\mu}\right)\right]\left(\frac{1}{16p'p^3}\right) \left[20p^3p' - 12p'^{\,3}p + 3(p'^{\,2} - p^2)^2\ln\left(\frac{(p+p')^2}{(p-p')^2}\right)\right] \quad (362)$$

$$(V_{2PE}^{1+-})_{44} = -S_2 + \frac{g^4}{64}\Bigg\{\frac{15}{2}R_2(p',p) - \frac{45}{2}R_{x^2}(p',p) - 18m_\pi^2\Big(2R_{n/q^2}(p',p) + 2R_{n'/q^2}(p',p) + 6R_{2x}(p',p) - 6R_{x/q^2}(p',p) - \frac{8}{3}R_0(p',p)\Big) + \frac{21}{2}\Big(2R_n(p',p) + 2R_{n'}(p',p) - \frac{8}{3}R_2(p',p) - 6R_x(p',p) + 6R_{x^2}(p',p)\Big) - \left[\frac{-16}{3}m_\pi^2 + 3m_\pi^2\ln\left(\frac{m_\pi}{\mu}\right)\right] \left(\frac{1}{16p'^{\,3}p^3}\right)\left[4pp'(3p^4 - 2p^2p'^{\,2} + 3p'^{\,4}) + 3(p^2 - p'^{\,2})^2\,(p^2 + p'^{\,2})\ln\left(\frac{(p-p')^2}{(p+p')^2}\right)\right]\Bigg\} \quad (363)$$

D.4 $\mathbf{J^{PC} = 2^{++}}$

$$V_{2PE}(2^{++}) = \frac{1}{512\pi^2 f_\pi^4}$$
$$\begin{pmatrix}
32S_2(p',p) & 0 & \frac{3\sqrt{3}}{\sqrt{10}}\, g^4(V_{2PE}^{2^{++}})_{13} & \sqrt{3}\, g^4(V_{2PE}^{2^{++}})_{14} & \frac{6\sqrt{3}}{\sqrt{7}}\, g^4(V_{2PE}^{2^{++}})_{15} & \frac{1}{4\sqrt{105}}\, g^4(V_{2PE}^{2^{++}})_{16} \\
0 & 0 & \frac{9}{\sqrt{5}}\, g^4(V_{2PE}^{2^{++}})_{23} & 0 & \frac{-9}{\sqrt{14}}\, g^4(V_{2PE}^{2^{++}})_{25} & \frac{-3}{\sqrt{70}}\, g^4(V_{2PE}^{2^{++}})_{26} \\
\frac{3\sqrt{3}}{\sqrt{10}}\, g^4(V_{2PE}^{2^{++}})_{31} & \frac{9}{\sqrt{5}}\, g^4(V_{2PE}^{2^{++}})_{32} & 32(V_{2PE}^{2^{++}})_{33} & \frac{-1}{2\sqrt{10}}\, g^4(V_{2PE}^{2^{++}})_{34} & \frac{\sqrt{7}}{2\sqrt{10}}\, g^4(V_{2PE}^{2^{++}})_{35} & 0 \\
\sqrt{3}g^4(V_{2PE}^{2^{++}})_{41} & 0 & \frac{-1}{2\sqrt{10}}\, g^4(V_{2PE}^{2^{++}})_{43} & 32(V_{2PE}^{2^{++}})_{44} & \frac{1}{2\sqrt{7}}\, g^4(V_{2PE}^{2^{++}})_{45} & \frac{-3}{2\sqrt{35}}\, g^4(V_{2PE}^{2^{++}})_{46} \\
\frac{6\sqrt{3}}{\sqrt{7}}\, g^4(V_{2PE}^{2^{++}})_{51} & \frac{-9}{\sqrt{14}}\, g^4(V_{2PE}^{2^{++}})_{52} & \frac{\sqrt{7}}{2\sqrt{10}}\, g^4(V_{2PE}^{2^{++}})_{53} & \frac{1}{2\sqrt{7}}\, g^4(V_{2PE}^{2^{++}})_{54} & 32(V_{2PE}^{2^{++}})_{55} & \frac{3}{28\sqrt{5}}\, g^4(V_{2PE}^{2^{++}})_{56} \\
\frac{1}{4\sqrt{105}}\, g^4(V_{2PE}^{2^{++}})_{61} & \frac{-3}{\sqrt{70}}\, g^4(V_{2PE}^{2^{++}})_{62} & 0 & \frac{-3}{2\sqrt{35}}\, g^4(V_{2PE}^{2^{++}})_{64} & \frac{3}{28\sqrt{5}}\, g^4(V_{2PE}^{2^{++}})_{65} & 32(V_{2PE}^{2^{++}})_{66}
\end{pmatrix} \tag{364}$$

where,

$$(V_{2PE}^{2^{++}})_{13} = (V_{2PE}^{2^{++}})_{23} = \frac{2}{3}p^2\left[-\mathcal{R} + 1 - 2\ln\left(\frac{m_\pi}{\mu}\right)\right] + \frac{2}{3}R_2(p',p) - 2R_n(p',p) \tag{365}$$

$$(V_{2PE}^{2^{++}})_{14} = (V_{2PE}^{2^{++}})_{41} = 3R_{x^2}(p',p) - 9R_{2x}(p',p) + 3R_0(p',p) - R_2(p',p) \tag{366}$$

$$\begin{aligned}(V_{2PE}^{2^{++}})_{15} = (V_{2PE}^{2^{++}})_{25} = (V_{2PE}^{2^{++}})_{51} = (V_{2PE}^{2^{++}})_{52} = \frac{2}{3}R_2(p',p) + 3R_x(p',p) \\ - R_n(p',p) - R_{n'}(p',p) - R_{x^2}(p',p)\end{aligned} \tag{367}$$

$$\begin{aligned}(V_{2PE}^{2^{++}})_{16} = 63R_{n'x^2}(p',p) - 90R_{x^2}(p',p) - 360R_x(p',p) - 90R_{n'}(p',p) + 36R_n(p',p) \\ + 18R_2(p',p)\end{aligned} \tag{368}$$

$$(V_{2PE}^{2^{++}})_{26} = 35R_{n'x^2}(p',p) - 5R_{x^2}(p',p) - 2R_x(p',p) - 5R_{n'}(p',p) + 2R_n(p',p) + R_2(p',p) \tag{369}$$

$$(V_{2PE}^{2^{++}})_{31} = (V_{2PE}^{2^{++}})_{32} = \frac{2}{3}p'^{\,2}\left[-\mathcal{R} + 1 - 2\ln\left(\frac{m_\pi}{\mu}\right)\right] + \frac{2}{3}R_2(p',p) - 2R_{n'}(p',p) \tag{370}$$

$$\begin{aligned}(V_{2PE}^{2^{++}})_{33} = S_0 + \frac{g^4}{32}\Bigg\{ -2\mathcal{R} + \frac{1}{9}(p^2 + p'^{\,2}) - [3m_\pi^2 + 4(p^2 + p'^{\,2})]\ln\left(\frac{m_\pi}{\mu}\right) - 6m_\pi^2 R_0(p',p) \\ - 4R_2(p',p)\Bigg\}\end{aligned} \tag{371}$$

$$(V_{2PE}^{2++})_{34} = (V_{2PE}^{2++})_{35} = 2p'^{\,2}\left[2\mathcal{R} - \frac{1}{9} - \ln\left(\frac{m_\pi}{\mu}\right)\right] - \frac{7}{2}\left[3R_{n'}(p',p) - R_2(p',p)\right]$$
$$+ 6m_\pi^2\left[3R_{n'/q^2}(p',p) - R_0(p',p)\right] + \left[\frac{-16}{3}m_\pi^2 + 3m_\pi^2\ln\left(\frac{m_\pi}{\mu}\right)\right]\left(\frac{1}{16pp'^{\,3}}\right)$$
$$\left[20pp'^{\,3} - 12p^3p' + 3(p^2 - p'^{\,2})^2\ln\left(\frac{(p+p')^2}{(p-p')^2}\right)\right] \quad (372)$$

$$(V_{2PE}^{2++})_{43} = (V_{2PE}^{2++})_{53} = 2p^2\left[2\mathcal{R} - \frac{1}{9} - \ln\left(\frac{m_\pi}{\mu}\right)\right] - \frac{7}{2}\left[3R_n(p',p) - R_2(p',p)\right]$$
$$+ 6m_\pi^2\left[3R_{n/q^2}(p',p) - R_0(p',p)\right] + \left[\frac{-16}{3}m_\pi^2 + 3m_\pi^2\ln\left(\frac{m_\pi}{\mu}\right)\right]\left(\frac{1}{16p'p^3}\right)$$
$$\left[20p^3p' - 12p'^{\,3}p + 3(p'^{\,2} - p^2)^2\ln\left(\frac{(p+p')^2}{(p-p')^2}\right)\right] \quad (373)$$

$$(V_{2PE}^{2++})_{44} = S_2 + \frac{g^4}{32}\left\{\frac{45}{2}R_{x^2}(p',p) - \frac{15}{2}R_2(p',p) + \frac{7}{2}R_0(p',p) - 6m_\pi^2R_{1/q^2}(p',p)\right.$$
$$+ 18m_\pi^2R_{x^2/q^2}(p',p) - \frac{21}{2}R_{2x}(p',p) + \left[\frac{16}{3}m_\pi^2 - 3m_\pi^2\ln\left(\frac{m_\pi}{\mu}\right)\right]\left(\frac{1}{16p'^{\,3}p^3}\right)$$
$$\left.\left[12pp'(p^2 + p'^{\,2}) + (3p^4 + 2p^2p'^{\,2} + 3p'^{\,4})\ln\left(\frac{(p-p')^2}{(p+p')^2}\right)\right]\right\} \quad (374)$$

$$(V_{2PE}^{2++})_{45} = (V_{2PE}^{2++})_{54} = 6m_\pi^2\left[3R_{n'/q^2}(p',p) + 3R_{n/q^2}(p',p) - 9R_{x/q^2}(p',p)\right.$$
$$\left.+ 3R_{2x}(p',p) - 2R_0(p',p)\right] - \frac{7}{2}\left[3R_n(p',p) + 3R_{n'}(p',p) - 9R_x(p',p) + 3R_{x^2}(p',p)\right.$$
$$\left.- 2R_2(p',p)\right] + \left(\frac{1}{3p'^{\,3}p^3}\right)\left[4pp'(3p^4 - 2p^2p'^{\,2} + 3p'^{\,4}) + 3(p^2 - p'^{\,2})^2\,(p^2 + p'^{\,2})\right.$$
$$\left.\ln\left(\frac{(p-p')^2}{(p+p')^2}\right)\right] \quad (375)$$

$$(V_{2PE}^{2++})_{46} = (V_{2PE}^{2++})_{56} = 6m_\pi^2\Big[35R_{n'/q^2}(p',p) - 5R_{n'/q^2}(p',p) - 20R_{x/q^2}(p',p)$$

$$-5R_{2x}(p',p)+2R_{n/q^2}(p',p)+R_0(p',p)\Big]-\frac{7}{2}\Big[35R_{n'x^2}(p',p)-5R_{n'}(p',p)-20R_x(p',p)$$

$$+ R_2(p',p) + 2R_n(p',p) - 5R_{x^2}(p',p)\Big] + \left[\frac{16}{3}m_\pi^2 - 3m_\pi^2\left(\frac{1}{192p'^{\,5}p^3}\right)\right]\Big[-4pp'(105p^6$$

$$-145p^4p'^{\,2}+15p^2p'^{\,4}+9p'^{\,6})+3(p^2-p'^{\,2})^2\,(35p^4+10p^2p'^{\,2}+3p'^{\,4})\ln\left(\frac{(p+p')^2}{(p-p')^2}\right)\Big] \quad (376)$$

$$(V_{2PE}^{2++})_{55} = S_2 + \frac{g^4}{64}\Bigg\{\frac{15}{2}R_2(p',p) - \frac{45}{2}R_{x^2}(p',p) + \frac{3}{7}m_\pi^2\Big(-9R_{n/q^2}(p',p)$$

$$-9R_{n'/q^2}(p',p) - 51R_{2x}(p',p) + 27R_{x/q^2}(p',p) + 20R_0(p',p)\Big) - \frac{1}{4}$$

$$\Big(-9R_n(p',p) - 9R_{n'}(p',p) - 51R_{x^2}(p',p) + 27R_x(p',p) + 20R_2(p',p)\Big)$$

$$+ \left[\frac{-16}{3}m_\pi^2 + 3m_\pi^2\ln\left(\frac{m_\pi}{\mu}\right)\right]\left(\frac{1}{448p'^{\,3}p^3}\right)\Big[-12pp'(3p^4 - 2p^2p'^{\,2} + 3p'^{\,4})$$

$$+ 9(p^2 - p'^{\,2})^2\,(p^2 + p'^{\,2})\ln\left(\frac{(p+p')^2}{(p-p')^2}\right)\Big]\Bigg\} \quad (377)$$

$$(V_{2PE}^{2++})_{61} = 63R_{nx^2}(p',p) - 90R_{x^2}(p',p) - 360R_x(p',p) - 90R_n(p',p) + 36R_{n'}(p',p) + 18R_2(p',p) \quad (378)$$

$$(V_{2PE}^{2++})_{62} = 35R_{nx^2}(p',p)-5R_{x^2}(p',p)-2R_x(p',p)-5R_n(p',p)+2R_{n'}(p',p)+R_2(p',p) \quad (379)$$

$$(V_{2PE}^{2++})_{64} = (V_{2PE}^{2++})_{65} = 6m_\pi^2\Big[35R_{n/q^2}(p',p) - 5R_{n/q^2}(p',p) - 20R_{x/q^2}(p',p)$$

$$-5R_{2x}(p',p)+2R_{n'/q^2}(p',p)+R_0(p',p)\Big]-\frac{7}{2}\Big[35R_{nx^2}(p',p)-5R_n(p',p)-20R_x(p',p)$$

$$+ R_2(p',p) + 2R_{n'}(p',p) - 5R_{x^2}(p',p)\Big] + \left(\frac{1}{192p'^{\,3}p^5}\right)\Big[-4pp'(105p'^{\,6} - 145p'^{\,4}p^2$$

$$+ 15p'^{\,2}p^4 + 9p^6) + 3(p^2 - p'^{\,2})^2\,(35p'^{\,4} + 10p^2p'^{\,2} + 3p^4)\ln\left(\frac{(p+p')^2}{(p-p')^2}\right)\Big] \quad (380)$$

$$
(V_{2PE}^{2++})_{66} = S_4 - \frac{15g^4}{512}\left[35R_{x^4}(p',p) - 30R_{x^2}(p',p) + 3R_2(p',p)\right] + \frac{3g^4}{448}\Bigg\{6m_\pi^2
$$

$$
\Bigg(105R_{n'x^2/q^2}(p',p) + 105R_{nx^2/q^2}(p',p) - 15R_{n'/q^2}(p',p) - 15R_{n/q^2}(p',p)
$$

$$
+45R_{x/q^2}(p',p)+3R_0(p',p)-15R_{2x}(p',p)-245R_{nn'x^3/q^2}(p',p)\Bigg)-\frac{7}{2}\Bigg(105R_{n'x^2}(p',p)
$$

$$
+105R_{nx^2}(p',p)-15R_{n'}(p',p)-15R_n(p',p)+45R_x(p',p)+3R_2(p',p)-15R_{x^2}(p',p)
$$

$$
-245R_{nn'x^3}(p',p)\Bigg)+\left[\frac{-16}{3}m_\pi^2+3m_\pi^2\ln\left(\frac{m_\pi}{\mu}\right)\right]\left(\frac{1}{384p'^{\,5}p^5}\right)\Big[4(105p^9p'-40p^7p'^{\,3}
$$

$$
-34p^5p'^{\,5}-40p^3p'^{\,7}+105p'^{\,9}p)+15(p-p')^2\,(p+p')^2(p^2+p'^{\,2})\ln\left(\frac{(p-p')^2}{(p+p')^2}\right)\Big]\Bigg\} \tag{381}
$$

and

$$
S_4 = \int_{-1}^{1}\frac{dx}{2}\frac{(35x^4-30x^2+3)}{8}(\vec{\tau}_1\cdot\vec{\tau}_2)\Bigg\{\mathcal{R}\Bigg[\left(\frac{23}{192}\vec{q}^{\,2}+\frac{15}{32}m_\pi^2\right)g^4+\frac{\vec{q}^{\,2}}{24}+\frac{m_\pi^2}{4}\Bigg]
$$

$$
+\vec{q}^{\,2}\left(\frac{5}{576}g^4-\frac{5}{72}\right)+m_\pi^2\left(\frac{1}{6}g^4-\frac{1}{3}\right)+\ln\left(\frac{m_\pi}{\mu}\right)\left[\vec{q}^{\,2}\left(\frac{23}{96}g^4+\frac{1}{12}\right)+m_\pi^2\left(\frac{15}{16}g^4+\frac{1}{2}\right)\right]
$$

$$
+L(q)\left[\vec{q}^{\,2}\left(\frac{23}{96}g^4+\frac{1}{12}\right)+m_\pi^2\left(\frac{5}{24}g^4+\frac{1}{3}\right)+\frac{g^4m_\pi^4}{2(4m_\pi^2+\vec{q}^{\,2})}\right]\Bigg\} \tag{382}
$$

E Calculation of integrals of partial wave decomposition

The calculations of integrals from partial wave decomposition are shown here. Specifically, we present the integration of Q, R, S and T terms encountered in the Partial wave decomposition here.
With $|\vec{p'}| = p'$, $|\vec{p}| = p$,$|\vec{q}| = q$, $\vec{n} = \vec{p}/p$, $\vec{n'} = \vec{p'}/p'$. In addition, $\vec{q}^{\,2} = {p'}^2 + p^2 - 2p'px$ and $\vec{n'}\cdot\vec{n}$ and

$$
\vec{n}\cdot\vec{q} = p'x - p = \frac{{p'}^2 - p^2 - q^2}{2p} \tag{383}
$$

$$
\vec{n'}\cdot\vec{q} = p' - px = \frac{{p'}^2 - p^2 + q^2}{2p'} \tag{384}
$$

E.1 Q-integrals

$$Q_2(p',p) = \int_{-1}^{1} \frac{dx}{2} \frac{\vec{q}^{\,2}}{\vec{q}^{\,2}+m_\pi^2} = 1 - \frac{m_\pi^2}{2p'p} \operatorname{arctanh}\left(\frac{2p'p}{p'^2+p^2+m_\pi^2}\right) = 1 + \mathcal{O}(\chi^4) \tag{385}$$

$$\begin{aligned} Q_n(p',p) = \int_{-1}^{1} \frac{dx}{2} \frac{(\vec{n}\cdot\vec{q})^2}{\vec{q}^{\,2}+m_\pi^2} &= 1 - \frac{p'^2+p^2+m_\pi^2}{4p^2} + \frac{(p'^2-p^2+m_\pi^2)^2}{8p'p^3} \\ &\operatorname{arctanh}\left(\frac{2p'p}{p'^2+p^2+m_\pi^2}\right) \\ &= 1 - \frac{p'^2+p^2}{4p^2} + \frac{(p'^2-p^2)^2}{8p'p^3} \operatorname{arctanh}\left(\frac{2p'p}{p'^2+p^2+m_\pi^2}\right) + \mathcal{O}(\chi^4) \end{aligned} \tag{386}$$

$$\begin{aligned} Q_{n'}(p',p) = \int_{-1}^{1} \frac{dx}{2} \frac{(\vec{n'}\cdot\vec{q})^2}{\vec{q}^{\,2}+m_\pi^2} &= 1 - \frac{p'^2+p^2+m_\pi^2}{4p'^2} + \frac{(-p'^2+p^2+m_\pi^2)^2}{8p'^3p} \\ &\operatorname{arctanh}\left(\frac{2p'p}{p'^2+p^2+m_\pi^2}\right) \\ &= 1 - \frac{p'^2+p^2}{4p'^2} + \frac{(-p'^2+p^2)^2}{8p'^3p} \operatorname{arctanh}\left(\frac{2p'p}{p'^2+p^2+m_\pi^2}\right) + \mathcal{O}(\chi^4) \end{aligned} \tag{387}$$

$$\begin{aligned} Q_x(p',p) &= \int_{-1}^{1} \frac{dx}{2} \frac{(\vec{n'}\cdot\vec{q})(\vec{n}\cdot\vec{q})x}{\vec{q}^{\,2}+m_\pi^2} = \frac{5}{12} - \frac{p'^4+p^4-m_\pi^4}{8p'^2p^2} \\ &+ \frac{\big((p'^2-p^2)^2-m_\pi^4\big)(p'^2+p^2+m_\pi^2)}{16p'^3p^3} \times \operatorname{arctanh}\left(\frac{2p'p}{p'^2+p^2+m_\pi^2}\right) \\ &= \frac{5}{12} - \frac{p'^4+p^4}{8p'^2p^2} + \frac{\big((p'^2-p^2)^2\big)(p'^2+p^2)}{16p'^3p^3} \operatorname{arctanh}\left(\frac{2p'p}{p'^2+p^2+m_\pi^2}\right) + \mathcal{O}(\chi^4) \end{aligned} \tag{388}$$

$$\begin{aligned} &Q_{x^2}(p',p) \\ &= \int_{-1}^{1} \frac{dx}{2} \frac{(\vec{q}^{\,2}x^2)}{\vec{q}^{\,2}+m_\pi^2} = \frac{1}{3} + \frac{m_\pi^2(p'^2+p^2+m_\pi^2)}{4p'^2p^2} - \frac{m_\pi^2(p'^2+p^2+m_\pi^2)^2}{8p'^3p^3} \\ &\operatorname{arctanh}\left(\frac{2p'p}{p'^2+p^2+m_\pi^2}\right) = \frac{1}{3} + \mathcal{O}(\chi^4) \end{aligned} \tag{389}$$

$$
\begin{aligned}
Q_{nx^2}(p',p) &= \int_{-1}^{1}\frac{dx}{2}\frac{(\vec{n}\cdot\vec{q})^2x^2}{\vec{q}^{\,2}+m_\pi^2} \\
&= \frac{p^4(5p'^2+m_\pi^2)-p^6-m_\pi^6-3m_\pi^4p'^2-3m_\pi^2p'^4-p'^6}{16p'^2p^4} \\
&+\frac{3m_\pi^4+2m_\pi^2p'^2-p'^4}{48p^2p'^2}+\frac{(p^4-(p'^2+m_\pi^2)^2)^2}{32p'^3p^5}\operatorname{arctanh}\left(\frac{2p'p}{p'^2+p^2+m_\pi^2}\right) \\
&= \frac{p^4(5p'^2)-p^6-p'^6}{16p'^2p^4}+\frac{-p'^4}{48p^2p'^2}+\frac{(p^4-p'^4)^2}{32p'^3p^5}\operatorname{arctanh}\left(\frac{2p'p}{p'^2+p^2+m_\pi^2}\right)+\mathcal{O}(\chi^4)
\end{aligned} \tag{390}
$$

$$
\begin{aligned}
Q_{n'x^2}(p',p) &= \int_{-1}^{1}\frac{dx}{2}\frac{(\vec{n'}\cdot\vec{q})^2x^2}{\vec{q}^{\,2}+m_\pi^2} \\
&= \frac{p'^4(5p^2+m_\pi^2)-p'^6-m_\pi^6-3m_\pi^4p^2-3m_\pi^2p^4-p^6}{16p^2p'^4} \\
&+\frac{3m_\pi^4+2m_\pi^2p^2-p^4}{48p'^2p^2}+\frac{(p'^4-(p^2+m_\pi^2)^2)^2}{32p'^5p^3}\operatorname{arctanh}\left(\frac{2p'p}{p'^2+p^2+m_\pi^2}\right) \\
&= \frac{p'^4(5p^2)-p'^6-p^6}{16p^2p'^4}+\frac{-p^4}{48p'^2p^2}+\frac{(p'^4-p^4)^2}{32p'^5p^3}\operatorname{arctanh}\left(\frac{2p'p}{p'^2+p^2+m_\pi^2}\right)+\mathcal{O}(\chi^4)
\end{aligned} \tag{391}
$$

$$
\begin{aligned}
Q_{x^3}(p',p) &= \int_{-1}^{1}\frac{dx}{2}\frac{(\vec{n'}\cdot\vec{q})(\vec{n}\cdot\vec{q})x^3}{\vec{q}^{\,2}+m_\pi^2} \\
&= \frac{3m_\pi^6+8m_\pi^4p'^2+3m_\pi^2p'^4-2p'^6-p^4(3m_\pi^2+2p'^2)}{48p'^4p^2} \\
&+\frac{15m_\pi^2+59p'^2}{240p'^2}+\frac{m_\pi^8-p^8+2m_\pi^6p'^2-2m_\pi^2p'^6-p'^8}{32p'^4p^4} \\
&+\frac{((p'^2-p^2)-m_\pi^4)(p'^2+p^2+m_\pi^2)^3}{64p'^5p^5}\operatorname{arctanh}\left(\frac{2p'p}{p'^2+p^2+m_\pi^2}\right) \\
&= \frac{-2p'^6-p^4(2p'^2)}{48p'^4p^2}+\frac{59p'^2}{240p'^2}+\frac{-p^8-p'^8}{32p'^4p^4}+\frac{((p'^2-p^2))(p'^2+p^2)^3}{64p'^5p^5} \\
&\qquad \operatorname{arctanh}\left(\frac{2p'p}{p'^2+p^2+m_\pi^2}\right)+\mathcal{O}(\chi^4)
\end{aligned} \tag{392}
$$

E.2 R-integrals

Since $\vec{q}^{\,2} = p'^2 + p^2 - 2p'px$, we can substitute x inside the R-integrals:

$$\frac{dx}{2} = -\frac{q}{2p'p}dq \tag{393}$$

In the following, q is relabeled as ρ to avoid ambiguity between the transferred momentum and an integration variable. The limits of integration are:

$$x_b = 1 \rightarrow \rho_b = p' - p \tag{394}$$

$$x_a = -1 \rightarrow \rho_a = p' + p \tag{395}$$

The R-integrals are now written as,

$$\begin{aligned} R_{(-2)}(p', p) = \int_{-1}^{1} \frac{dx}{2} \frac{L(q)}{\vec{q}^{\,2}} = \frac{1}{2p'p} \int_{p'-p}^{p'+p} d\rho \, \frac{L(\rho)}{\rho} = \frac{1}{2p'p} \ln\left(\frac{p'+p}{p'-p}\right) \\ + \frac{1}{2p'p}\left[\frac{1}{2}\frac{\rho^2}{4m_\pi^2 + \rho^2} L^2(\rho) - L(\rho)\right]_{p'-p}^{p'+p} \end{aligned} \tag{396}$$

$$\begin{aligned} R_0(p', p) = \int_{-1}^{1} \frac{dx}{2} L(q) = \frac{1}{2p'p} \int_{p'-p}^{p'+p} d\rho \, \rho L(\rho) = -\frac{1}{2} + \frac{1}{2p'p}\Bigg[\frac{m_\pi^2 \rho^2}{4m_\pi^2 + \rho^2} L^2(\rho) \\ + \frac{\rho^2}{2} L(\rho)\Bigg]_{p'-p}^{p'+p} \\ = -\frac{1}{2} + \frac{1}{4p'p}\left[\rho^2 L(\rho)\right]_{p'-p}^{p'+p} + \mathcal{O}(\chi^4) \end{aligned} \tag{397}$$

$$\begin{aligned} R_2(p', p) = \int_{-1}^{1} \frac{dx}{2} \vec{q}^{\,2} L(q) = \frac{1}{2p'p} \int_{p'-p}^{p'+p} d\rho \, \rho^3 L(\rho) = \frac{-1}{2p'p}\Bigg[\frac{\rho^2}{16}(4m_\pi^2 + \rho^2) \\ + \frac{m_\pi^4 \rho^2}{4m_\pi^2 + \rho^2} L^2(\rho) - \frac{\rho^2}{4}(2m_\pi^2 + \rho^2) L(\rho)\Bigg]_{p'-p}^{p'+p} \\ = \frac{-1}{8p'p}\left[\frac{\rho^4}{4} - \rho^4 L(\rho)\right]_{p'-p}^{p'+p} + \mathcal{O}(\chi^4) \end{aligned} \tag{398}$$

$$R_4(p',p) = \int_{-1}^{1} \frac{dx}{2} \vec{q}^4 L(q) = \frac{1}{4p'p} \bigg[- \frac{\rho^6}{18} - \frac{m_\pi^2 \rho^4}{12} + m_\pi^4 \rho^2 + \frac{4 m_\pi^6 \rho^2}{(4m_\pi^2 + \rho^2)} L^2(\rho)$$
$$+ \bigg(- 2m_\pi^4 + \frac{m_\pi^2 \rho^2}{3} + \frac{\rho^4}{3} \bigg) \rho^2 L(\rho) \bigg]_{p'-p}^{p'+p}$$
$$= \frac{1}{12p'p} \bigg[- \frac{\rho^6}{6} + \rho^6 L(\rho) \bigg]_{p'-p}^{p'+p} + \mathcal{O}(\chi^4) \quad (399)$$

$$R_6(p',p) = \int_{-1}^{1} \frac{dx}{2} \vec{q}^6 L(q) = -\frac{1}{16p'p} \bigg[\frac{\rho^8}{8} + \frac{m_\pi^2 \rho^6}{9} - \frac{5 m_\pi^4 \rho^4}{6} + 10 m_\pi^6 \rho^2$$
$$+ 40 m_\pi^8 \frac{\rho^2}{4m_\pi^2 + \rho^2} L^2(\rho) - \rho^2 L(\rho) \bigg(\rho^6 + \frac{2 m_\pi^2 \rho^4}{3} - \frac{10 m_\pi^4 \rho^2}{3} + 20 m_\pi^6 \bigg) \bigg]_{p'-p}^{p'+p}$$
$$= -\frac{1}{16p'p} \bigg[\frac{\rho^8}{8} - \rho^8 L(\rho) \bigg]_{p'-p}^{p'+p} + \mathcal{O}(\chi^4) \quad (400)$$

$$R_8(p',p) = \int_{-1}^{1} \frac{dx}{2} \vec{q}^8 L(q) = \frac{1}{8p'p} \bigg[- \frac{\rho^{10}}{25} - \frac{m_\pi^2 \rho^8}{40} + \frac{7 m_\pi^4 \rho^6}{45} - \frac{7 m_\pi^6 \rho^4}{6} + 14 m_\pi^8 \rho^2$$
$$+ 56 m_\pi^{10} \frac{\rho^2}{4m_\pi^2 + \rho^2} L^2(\rho) + \rho^2 L(\rho) \bigg(\frac{2\rho^8}{5} + \frac{m_\pi^2 \rho^6}{5} - \frac{14 m_\pi^4 \rho^4}{15} + \frac{14 m_\pi^6 \rho^2}{3} - 28 m_\pi^8 \bigg) \bigg]_{p'-p}^{p'+p}$$
$$= \frac{1}{8p'p} \bigg[\frac{-\rho^{10}}{25} + \frac{2\rho^{10}}{5} L(\rho) \bigg]_{p'-p}^{p'+p} + \mathcal{O}(\chi^4) \quad (401)$$

$$R_{10}(p',p) = \int_{-1}^{1} \frac{dx}{2} \vec{q}^{10} L(q) = \frac{1}{2p'p}$$
$$\bigg[\frac{\rho^2(-75600 m_\pi^{10} + 6300 m_\pi^8 \rho^2 - 840 m_\pi^6 \rho^4 + 135 m_\pi^4 \rho^6 - 24 m_\pi^2 \rho^8 - 50 \rho^{10})}{7200}$$
$$- 42 m_\pi^{12} \frac{\rho^2}{4m_\pi^2 + \rho^2} L^2(\rho) + \frac{\rho^2}{60} L(\rho) \big(1260 m_\pi^{10} - 210 m_\pi^8 \rho^2 + 42 m_\pi^6 \rho^4 - 9 m_\pi^4 \rho^6$$
$$+ 2 m_\pi^2 \rho^8 + 5 \rho^{10} \big) \bigg]_{p'-p}^{p'+p}$$
$$= \frac{1}{2p'p} \bigg[\frac{-\rho^{12}}{144} + \frac{\rho^{12}}{12} L(\rho) \bigg]_{p'-p}^{p'+p} + \mathcal{O}(\chi^4) \quad (402)$$

$$R_n(p',p) = \int_{-1}^{1}\frac{dx}{2}(\vec{n}\cdot\vec{q})^2 L(q)$$
$$= \frac{1}{2p'p^3}\int_{p'-p}^{p'+p} d\rho\big(\rho^4 - 2\rho^2({\rho'}^2-\rho^2) + ({\rho'}^2-\rho^2)^2\big)\rho\cdot L(\rho)$$
$$= \frac{1}{4p^2}\Big(R_4(p',p) - 2({p'}^2-p^2)R_2(p',p) + ({p'}^2-p^2)^2 R_0(p',p)\Big) \quad (403)$$

$$R_{n'}(p',p) = \int_{-1}^{1}\frac{dx}{2}(\vec{n'}\cdot\vec{q})^2 L(q)$$
$$= \frac{1}{4{p'}^2}\Big(R_4(p',p) + 2({p'}^2-p^2)R_2(p',p) + ({p'}^2-p^2)^2 R_0(p',p)\Big) \quad (404)$$

$$R_{n/q^2}(p',p) = \int_{-1}^{1}\frac{dx}{2}\frac{(\vec{n}\cdot\vec{q})^2}{\vec{q})^2}L(q)$$
$$= \frac{1}{4p^2}\Big(R_2(p',p) - 2({p'}^2-p^2)R_0(p',p) + ({p'}^2-p^2)^2 R_{(-2)}(p',p)\Big) \quad (405)$$

$$R_{n'/q^2}(p',p) = \int_{-1}^{1}\frac{dx}{2}(\vec{n'}\cdot\vec{q})^2 L(q)$$
$$= \frac{1}{4{p'}^2}\Big(R_2(p',p) + 2({p'}^2-p^2)R_0(p',p) + ({p'}^2-p^2)^2 R_{(-2)}(p',p)\Big) \quad (406)$$

$$R_x(p',p) = \int_{-1}^{1}\frac{dx}{2}(\vec{n'}\cdot\vec{q})(\vec{n}\cdot\vec{q})xL(q) = \frac{1}{8{p'}^2p^2}\Big(R_6(p',p) - ({p'}^2+p^2)R_4(p',p)$$
$$- ({p'}^2-p^2)^2R_2(p',p) + ({p'}^2-p^2)^2({p'}^2+p^2)R_0(p',p)\Big) \quad (407)$$

$$R_{x/q^2}(p',p) = \int_{-1}^{1}\frac{dx}{2}\frac{(\vec{n'}\cdot\vec{q})(\vec{n}\cdot\vec{q})x}{\vec{q}^2}L(q) = \frac{1}{8{p'}^2p^2}\Big(R_4(p',p) - ({p'}^2+p^2)R_2(p',p)$$
$$- ({p'}^2-p^2)^2R_0(p',p) + ({p'}^2-p^2)^2({p'}^2+p^2)R_{(-2)}(p',p)\Big) \quad (408)$$

$$R_{x^2}(p',p) = \int_{-1}^{1} \frac{dx}{2} \vec{q}^{\,2} x^2 L(q)$$
$$= \frac{1}{4p'^2 p^2}\Big(R_6(p',p) - 2(p'^2+p^2)R_4(p',p) + (p'^2+p^2)^2 R_2(p',p)\Big) \quad (409)$$

$$R_{2x}(p',p) = \int_{-1}^{1} \frac{dx}{2} x^2 L(q)$$
$$= \frac{1}{4p'^2 p^2}\Big(R_4(p',p) - 2(p'^2+p^2)R_2(p',p) + (p'^2+p^2)^2 R_0(p',p)\Big) \quad (410)$$

$$R_{x^2/q^2}(p',p) = \int_{-1}^{1} \frac{dx}{2} \frac{x^2}{\vec{q}^{\,2}} L(q)$$
$$= \frac{1}{4p'^2 p^2}\Big(R_2(p',p) - 2(p'^2+p^2)R_0(p',p) + (p'^2+p^2)^2 R_{(-2)}(p',p)\Big) \quad (411)$$

$$R_{4x}(p',p) = \int_{-1}^{1} \frac{dx}{2} x^4 L(q) = \frac{1}{16p'^4 p^4}\Big(R_8(p',p) - 4(p'^2+p^2)R_6(p',p)$$
$$+ 6(p'^2+p^2)R_4(p',p) - 4(p'^2+p^2)^3 R_2(p',p) + (p'^2+p^2)^4 R_0(p',p)\Big) \quad (412)$$

$$R_{x^4}(p',p) = \int_{-1}^{1} \frac{dx}{2} x^4 \vec{q}^{\,2} L(q) = \frac{1}{16p'^4 p^4}\Big(R_{10}(p',p) - 4(p'^2+p^2)R_8(p',p)$$
$$+ 6(p'^2+p^2)R_6(p',p) - 4(p'^2+p^2)^3 R_4(p',p) + (p'^2+p^2)^4 R_2(p',p)\Big) \quad (413)$$

$$R_{nx^2}(p',p) = \int_{-1}^{1} \frac{dx}{2} (\vec{n}\cdot\vec{q}) x^2 L(q)$$
$$= \frac{1}{16p'^2 p^4}\Big(R_8(p',p) - 4p'^2 R_6(p',p) + (6p'^4 - 2p^4)R_4(p',p)$$
$$+ 4p'^2(p'^4 - p^4)R_2(p',p) + (p'^4 - p^4)^2 R_0(p',p)\Big) \quad (414)$$

$$R_{nx^2/q^2}(p',p) = \int_{-1}^{1} \frac{dx}{2} \frac{(\vec{n}\cdot\vec{q})x^2}{\vec{q}^2} L(q) = \frac{1}{16p'^2p^4}\Big(R_6(p',p) - 4p'^2 R_4(p',p)$$
$$+ (6p'^4 - 2p^4)R_2(p',p) + 4p'^2(p'^4 - p^4)R_0(p',p) + (p'^4 - p^4)^2 R_{(-2)}(p',p)\Big) \quad (415)$$

$$R_{n'x^2}(p',p) = \int_{-1}^{1} \frac{dx}{2} (\vec{n'}\cdot\vec{q})x^2 L(q) = \frac{1}{16p'^2p^4}\Big(R_8(p',p) - 4p^2 R_6(p',p)$$
$$+ (6p^4 - 2p'^4)R_4(p',p) + 4p^2(p^4 - p'^4)R_2(p',p) + (p^4 - p'^4)^2 R_0(p',p)\Big) \quad (416)$$

$$R_{n'x^2/q^2}(p',p) = \int_{-1}^{1} \frac{dx}{2} \frac{(\vec{n'}\cdot\vec{q})x^2}{\vec{q}^2} L(q) = \frac{1}{16p'^2p^4}\Big(R_6(p',p) - 4p^2 R_4(p',p)$$
$$+ (6p^4 - 2p'^4)R_2(p',p) + 4p^2(p^4 - p'^4)R_0(p',p) + (p^4 - p'^4)^2 R_{(-2)}(p',p)\Big) \quad (417)$$

$$R_{nn'x^3}(p',p) = \int_{-1}^{1} \frac{dx}{2} (\vec{n'}\cdot\vec{q})(\vec{n}\cdot\vec{q})x^3 L(q) = \frac{1}{16p'^2p^4}\Big(R_{10}(p',p) - 3(p'^2+p^2)R_8(p',p)$$
$$+ 2(p^4 + 4p'^2p^2 + p'^4)R_6(p',p) + 2(p^6 - 3p'^2p^4 - 3p'^4p^2 + p'^6)R_4(p',p)$$
$$- 3(p^4 - p'^4)^2 R_2(p',p) + (p^2 - p'^2)^2(p^2 + p'^2)^4 R_0(p',p)\Big) \quad (418)$$

$$R_{nn'x^3/q^2}(p',p) = \int_{-1}^{1} \frac{dx}{2} \frac{(\vec{n'}\cdot\vec{q})(\vec{n}\cdot\vec{q})x^3}{\vec{q}^2} L(q) = \frac{1}{16p'^2p^4}\Big(R_8(p',p)$$
$$-3(p'^2+p^2)R_6(p',p) + 2(p^4+4p'^2p^2+p'^4)R_4(p',p) + 2(p^6-3p'^2p^4-3p'^4p^2+p'^6)R_2(p',p)$$
$$- 3(p^4 - p'^4)^2 R_0(p',p) + (p^2 - p'^2)^2(p^2 + p'^2)^4 R_{(-2)}(p',p)\Big) \quad (419)$$

$L(\rho)$ is given by,

$$L(\rho) = \frac{\sqrt{4m_\pi^2 + \rho^2}}{\rho} \ln\left(\frac{\sqrt{4m_\pi^2 + \rho^2} + \rho}{2m_\pi}\right) \quad (420)$$

E.3 S-integrals and T-integrals

The S-integrals can be written in general as,

$$
\begin{aligned}
S_k(p',p) &= \int_{-1}^{1}\frac{dx}{2}P_k(x)\big(\vec{\tau}_1\cdot\vec{\tau}_2\big)\Bigg\{\mathcal{R}\left[\left(\frac{23}{192}\vec{q}^{\,2}+\frac{15}{32}m_\pi^2\right)g^4+\frac{\vec{q}^{\,2}}{24}+\frac{m_\pi^2}{4}\right] \\
&+\vec{q}^{\,2}\left(\frac{5}{576}g^4-\frac{5}{72}\right)+m_\pi^2\left(\frac{1}{6}g^4-\frac{1}{3}\right)+\ln\left(\frac{m_\pi}{\mu}\right)\left[\vec{q}^{\,2}\left(\frac{23}{96}g^4+\frac{1}{12}\right)+m_\pi^2\left(\frac{15}{16}g^4\right.\right. \\
&\left.\left.+\frac{1}{2}\right)\right]+L(q)\left[\vec{q}^{\,2}\left(\frac{23}{96}g^4+\frac{1}{12}\right)+m_\pi^2\left(\frac{5}{24}g^4+\frac{1}{3}\right)+\frac{g^4m_\pi^4}{2(4m_\pi^2+\vec{q}^{\,2})}\right]\Bigg\} \\
&= \int_{-1}^{1}\frac{dx}{2}P_k(x)\big(\vec{\tau}_1\cdot\vec{\tau}_2\big)\vec{q}^{\,2}\Bigg\{\mathcal{R}\left[\frac{23}{192}g^4+\frac{1}{24}\right]+\left(\frac{5}{576}g^4-\frac{5}{72}\right)+\ln\left(\frac{m_\pi}{\mu}\right)\left(\frac{23}{96}g^4\right. \\
&\left.+\frac{1}{12}\right)+L(q)\left(\frac{23}{96}g^4+\frac{1}{12}\right)+\mathcal{O}(\chi^4)\Bigg\} \qquad (421)
\end{aligned}
$$

where $P_k(x)$ denotes the k-th Legendre polynomial.

$$
\begin{aligned}
S_0(p',p) &= \mathcal{R}\left[(p'^2+p^2)\left(\frac{23}{192}g^4+\frac{1}{24}\right)+m_\pi^2\left(\frac{15}{32}g^4+\frac{1}{4}\right)\right]+(p'^2+p^2)\left(\frac{5}{576}g^4-\frac{5}{18}\right) \\
&+m_\pi^2\left(\frac{1}{6}g^4-\frac{1}{3}\right)+\left[(p'^2+p^2)\left(\frac{23}{96}g^4+\frac{1}{12}\right)+m_\pi^2\left(\frac{15}{16}g^4+\frac{1}{2}\right)\right]\ln\left(\frac{m_\pi}{\mu}\right)+R_2\left(\frac{23}{96}g^4\right. \\
&\left.+\frac{1}{12}\right)+\left(\frac{5}{24}g^4+\frac{1}{3}\right)m_\pi^2\,R_0+\frac{g^4m_\pi^4}{8p'p}\left[\frac{\rho^2}{4m_\pi^2+\rho^2}L^2(\rho)\right]_{p'-p}^{p'+p} \\
&= \big(\vec{\tau}_1\cdot\vec{\tau}_2\big)\Bigg\{\mathcal{R}(p'^2+p^2)\left(\frac{23}{192}g^4+\frac{1}{24}\right)+(p'^2+p^2)\left(\frac{5}{576}g^4-\frac{5}{18}\right)+(p'^2+p^2)\left(\frac{23}{96}g^4\right. \\
&\left.+\frac{1}{12}\right)\ln\left(\frac{m_\pi}{\mu}\right)+\frac{-1}{8p'p}\left[\frac{\rho^4}{4}-\rho^4L(\rho)\right]_{p'-p}^{p'+p}\left(\frac{23}{96}g^4+\frac{1}{12}\right)\Bigg\}+\mathcal{O}(\chi^4) \qquad (422)
\end{aligned}
$$

In the case of $S_2(p',p)$ and $S_4(p',p)$, with the exception of $L(q)$ terms, every other term vanishes after the x-integration.

$$
\begin{aligned}
S_2(p',p) = &\int_{-1}^{1}\frac{dx}{2}\left(\frac{3x^2-1}{2}\right)L(q)\left\{\vec{q}^{\,2}\left(\frac{23}{96}g^4+\frac{1}{12}\right)+m_\pi^2\left(\frac{5}{24}g^4+\frac{1}{3}\right)\right.\\
&\left.+\frac{g^4m_\pi^4}{2(4m_\pi^2+\vec{q}^{\,2})}\right\}\\
= &\frac{1}{8p'^2p^2}\left\{\Big(3R_6-6(p'^2+p^2)R_4+\big(3(p'^2+p^2)^2-4p'^2p^2\big)R_2\Big)\left(\frac{23}{96}g^4+\frac{1}{12}\right)+\Big(3R_4\right.\\
&-6(p'^2+p^2)R_2+\big(3(p'^2+p^2)^2-4p'^2p^2\big)R_0\Big)m_\pi^2\left(\frac{5}{24}g^4+\frac{1}{3}\right)+\frac{1}{16}\left\{\frac{3g^4m_\pi^4}{p'p}\Big[3m_\pi^2\rho^2\right.\\
&-\frac{\rho^4}{4}+\rho^2(-6m_\pi^2+\rho^2)L(\rho)+\frac{12\rho^2m_\pi^4}{4m_\pi^2+\rho^2}L^2(\rho)\Big]_{p'-p}^{p'+p}+\frac{6g^4m_\pi^4}{p'p}(p'^2+p^2)\Big[\rho^2-2\rho^2L(\rho)\\
&\left.\left.+\frac{4\rho^2m_\pi^2}{4m_\pi^2+\rho^2}L^2(\rho)\Big]_{p'-p}^{p'+p}+\frac{2g^4m_\pi^4\big[3(p'^2+p^2)^2-4p'^2p^2\big]}{p'p}\Big[\frac{\rho^2}{4m_\pi^2+\rho^2}L^2(\rho)\Big]_{p'-p}^{p'+p}\right\}\right\}\\
= &\frac{(\vec{\tau}_1\cdot\vec{\tau}_2)}{8p'^2p^2}\left\{\Big(3R_6-6(p'^2+p^2)R_4+\big(3(p'^2+p^2)^2-4p'^2p^2\big)R_2\Big)\left(\frac{23}{96}g^4+\frac{1}{12}\right)\right\}+\mathcal{O}(\chi^4)
\end{aligned}
\tag{423}
$$

$$
\begin{aligned}
S_4(p',p) &= \int_{-1}^{1} \frac{dx}{2}\left(\frac{35x^4-30x^2+3}{8}\right)L(q)\left\{\vec{q}^{\,2}\left(\frac{23}{96}g^4+\frac{1}{24}\right)+m_\pi^2\left(\frac{5}{24}g^4+\frac{1}{3}\right)\right.\\
&\qquad\left.+\frac{g^4m_\pi^4}{2(4m_\pi^2+\vec{q}^{\,2})}\right\}\\
&= \frac{1}{128p'^4p^4}\left\{\left[\left(35p^8+35p'^8+20p^6p'^2+18p^4p'^4+20p^2p'^6\right)R_2+(210p^4+300p^2p'^2\right.\right.\\
&\quad +210p'^4)R_6-\left(140p^2+140p'^2\right)R_8-\left(140p^6+180p^4p'^2+180p^2p'^4+140p'^6\right)R_4\\
&\quad \left.+35R_{10}\right]\left(\frac{23}{96}g^4+\frac{1}{12}\right)+\left[\left(35p^8+35p'^8+20p^6p'^2+18p^4p'^4+20p^2p'^6\right)R_0+(210p^4\right.\\
&\quad +300p^2p'^2+210p'^4)R_4-\left(140p^2+140p'^2\right)R_6-(140p^6+180p^4p'^2+180p^2p'^4\\
&\quad \left.+140p'^6)R_2+35R_8\right]m_\pi^2\left(\frac{5}{24}g^4+\frac{1}{3}\right)+\frac{1}{16}\left\{\frac{140g^4m_\pi^4}{p'p}\right.\\
&\quad \left[\frac{\rho^2(70m_\pi^4\rho^2-420m_\pi^6-14m_\pi^2\rho^4+3\rho^6)}{24}L(\rho)+\frac{35m_\pi^8\rho^2}{4m_\pi^2+\rho^2}L^2(\rho)\right.\\
&\quad \left.+\frac{5040m^6\rho^2-420m^4\rho^4+56m^2\rho^6-9\rho^8}{576}\right]_{p'-p}^{p'+p}+\frac{4g^4m_\pi^4}{p'p}(140p^2+140p'^2)\\
&\quad \times\left[\frac{\rho^2(30m_\pi^4-5m_\pi^2\rho^2+\rho^4)}{6}L(\rho)-\frac{10m_\pi^6\rho^2}{4m_\pi^2+\rho^2}L^2(\rho)-\frac{\rho^6}{36}+\frac{5m_\pi^2\rho^4-60m_\pi^4\rho^2}{24}\right]_{p'-p}^{p'+p}\\
&\quad +\frac{4g^4m_\pi^4}{p'p}(210p^4+300p^2p'^2+210p'^4)\left[\frac{\rho^2(\rho^2-6m_\pi^2)}{4}L(\rho)+\frac{3m_\pi^4\rho^2}{4m_\pi^2+\rho^2}L^2(\rho)-\frac{\rho^4}{16}\right.\\
&\quad \left.+\frac{3m_\pi^2\rho^2}{4}\right]_{p'-p}^{p'+p}-\frac{4g^4m_\pi^4}{p'p}(140p^6+180p^4p'^2+180p^2p'^4+140p'^6)\left[\frac{\rho^2}{2}L(\rho)\right.\\
&\quad \left.-\frac{m_\pi^2\rho^2}{4m_\pi^2+\rho^2}L^2(\rho)-\frac{\rho^2}{4}\right]_{p'-p}^{p'+p}+\frac{4g^4m_\pi^4\left[35p^8+35p'^8+20p^6p'^2+18p^4p'^4+20p^2p'^6\right]}{p'p}\\
&\quad \left.\left.\left[\frac{\rho^2}{4m_\pi^2+\rho^2}L^2(\rho)\right]_{p'-p}^{p'+p}\right\}\right\}\\
&= \frac{(\vec{\tau}_1\cdot\vec{\tau}_2)}{128p'^4p^4}\left\{\left[\left(35p^8+35p'^8+20p^6p'^2+18p^4p'^4+20p^2p'^6\right)R_2+(210p^4+300p^2p'^2\right.\right.\\
&\quad +210p'^4)R_6-\left(140p^2+140p'^2\right)R_8-\left(140p^6+180p^4p'^2+180p^2p'^4+140p'^6\right)R_4\\
&\quad \left.\left.+35R_{10}\right]\left(\frac{23}{96}g^4+\frac{1}{24}\right)\right\}+\mathcal{O}(\chi^4) \qquad (424)
\end{aligned}
$$

The T-integrals which differs from S-integrals by an additional triangle diagram contribution,

$$
\begin{aligned}
T_k(p',p) = \int_{-1}^{1} \frac{dx}{2} P_k(x)\big(\vec{\tau}_1\cdot\vec{\tau}_2\big)\Bigg\{\mathcal{R}\Bigg[\bigg(\frac{23}{192}\vec{q}^{\,2}+\frac{15}{32}m_\pi^2\bigg)g^4-\bigg(\frac{5}{48}\vec{q}^{\,2}+\frac{3}{8}m_\pi^2\bigg)g^2-\frac{\vec{q}^{\,2}}{24} \\
-\frac{m_\pi^2}{4}\Bigg] + \vec{q}^{\,2}\bigg(\frac{5}{576}g^4+\frac{13}{144}g^2+\frac{5}{72}\bigg) + m_\pi^2\bigg(\frac{1}{6}g^4+\frac{1}{12}g^2+\frac{1}{3}\bigg) + \ln\left(\frac{m_\pi}{\mu}\right)\Bigg[\vec{q}^{\,2} \\
\bigg(\frac{23}{96}g^4-\frac{5}{24}g^2-\frac{1}{12}\bigg) + m_\pi^2\bigg(\frac{15}{16}g^4-\frac{3}{4}g^2-\frac{1}{2}\bigg)\Bigg] + L(q)\Bigg[\vec{q}^{\,2}\bigg(\frac{23}{96}g^4-\frac{5}{24}g^2-\frac{1}{12}\bigg) \\
+ m_\pi^2\bigg(\frac{5}{24}g^4-\frac{1}{3}g^2-\frac{1}{3}\bigg) + \frac{g^4 m_\pi^4}{2(4m_\pi^2+\vec{q}^{\,2})}\Bigg]\Bigg\} \\
= \int_{-1}^{1} \frac{dx}{2} P_k(x)\big(\vec{\tau}_1\cdot\vec{\tau}_2\big)\vec{q}^{\,2}\Bigg\{\mathcal{R}\bigg[\frac{23}{192}g^4-\frac{5}{24}g^2-\frac{1}{48}\bigg] + \bigg(\frac{5}{576}g^4+\frac{13}{144}g^2+\frac{5}{72}\bigg) \\
+ \ln\left(\frac{m_\pi}{\mu}\right)\bigg(\frac{23}{96}g^4-\frac{5}{24}g^2-\frac{1}{12}\bigg) + L(q)\bigg(\frac{23}{96}g^4-\frac{5}{24}g^2-\frac{1}{12}\bigg)\Bigg\} + \mathcal{O}(\chi^4) \quad (425)
\end{aligned}
$$

Starting with $T_0(p',p)$

$$
\begin{aligned}
T_0(p',p) = \mathcal{R}\Bigg[(p'^2+p^2)\bigg(\frac{23}{192}g^4-\frac{5}{48}g^2-\frac{1}{24}\bigg) + m_\pi^2\bigg(\frac{15}{32}g^4-\frac{3}{8}g^2-\frac{1}{4}\bigg)\Bigg] \\
+(p'^2+p^2)\bigg(\frac{5}{576}g^4+\frac{13}{144}g^2+\frac{5}{72}\bigg)+m_\pi^2\bigg(\frac{1}{6}g^4+\frac{1}{12}g^2+\frac{1}{3}\bigg)+\Bigg[(p'^2+p^2)\bigg(\frac{23}{96}g^4-\frac{5}{24}g^2 \\
-\frac{1}{12}\bigg) + m_\pi^2\bigg(\frac{15}{16}g^4-\frac{3}{4}g^2-\frac{1}{2}\bigg)\Bigg]\ln\left(\frac{m_\pi}{\mu}\right) + R_2\bigg(\frac{23}{96}g^4-\frac{5}{24}g^2-\frac{1}{12}\bigg) \\
+\bigg(\frac{5}{24}g^4-\frac{1}{3}g^2-\frac{1}{3}\bigg)m_\pi^2 R_0 + \frac{g^4 m_\pi^4}{8p'p}\left[\frac{\rho^2}{4m_\pi^2+\rho^2}L^2(\rho)\right]_{p'-p}^{p'+p} \\
= \big(\vec{\tau}_1\cdot\vec{\tau}_2\big)\Bigg\{\mathcal{R}(p'^2+p^2)\bigg(\frac{23}{192}g^4-\frac{5}{48}g^2-\frac{1}{24}\bigg) + (p'^2+p^2)\bigg(\frac{5}{576}g^4+\frac{13}{144}g^2+\frac{5}{72}\bigg) \\
+(p'^2+p^2)\bigg(\frac{23}{96}g^4-\frac{5}{24}g^2-\frac{1}{12}\bigg)\ln\left(\frac{m_\pi}{\mu}\right) + R_2\bigg(\frac{23}{96}g^4-\frac{5}{24}g^2-\frac{1}{12}\bigg)\Bigg\} + \mathcal{O}(\chi^4) \\
(426)
\end{aligned}
$$

Similar to the earlier case ($S_2(p',p)$, $S_4(p',p)$), only the $L(q)$ terms contribute in

$T_2(p',p)$ and $T_4(p',p)$.

$$\begin{aligned}
T_2(p',p) &= \int_{-1}^{1}\frac{dx}{2}\left(\frac{3x^2-1}{2}\right)L(q)\Bigg\{\vec{q}^{\,2}\left(\frac{23}{96}g^4-\frac{5}{24}g^2-\frac{1}{12}\right)+m_\pi^2\left(\frac{5}{24}g^4-\frac{1}{3}g^2-\frac{1}{3}\right) \\
&\qquad + \frac{g^4 m_\pi^4}{2(4m_\pi^2+\vec{q}^{\,2})}\Bigg\} \\
&= \frac{1}{8p'^2p^2}\Bigg\{\Big(3R_6-6(p'^2+p^2)R_4+\big(3(p'^2+p^2)^2-4p'^2p^2\big)R_2\Big)\left(\frac{23}{96}g^4-\frac{5}{24}g^2-\frac{1}{12}\right) \\
&+\Big(3R_4-6(p'^2+p^2)R_2+\big(3(p'^2+p^2)^2-4p'^2p^2\big)R_0\Big)m_\pi^2\left(\frac{5}{24}g^4-\frac{1}{3}g^2-\frac{1}{3}\right)+\frac{1}{16}\Bigg\{\frac{3g^4m_\pi^4}{p'p} \\
&\left[3m_\pi^2\rho^2-\frac{\rho^4}{4}+\rho^2(-6m_\pi^2+\rho^2)L(\rho)+\frac{12\rho^2m_\pi^4}{4m_\pi^2+\rho^2}L^2(\rho)\right]_{p'-p}^{p'+p}+\frac{6g^4m_\pi^4}{p'p}(p'^2+p^2)\Bigg[\rho^2 \\
&\qquad -2\rho^2L(\rho)+\frac{4\rho^2m_\pi^2}{4m_\pi^2+\rho^2}L^2(\rho)\Bigg]_{p'-p}^{p'+p}+\frac{2g^4m_\pi^4\big[3(p'^2+p^2)^2-4p'^2p^2\big]}{p'p} \\
&\qquad \left[\frac{\rho^2}{4m_\pi^2+\rho^2}L^2(\rho)\right]_{p'-p}^{p'+p}\Bigg\}\Bigg\} \\
&= \frac{(\vec{\tau}_1\cdot\vec{\tau}_2)}{8p'^2p^2}\Bigg\{\Big(3R_6-6(p'^2+p^2)R_4+\big(3(p'^2+p^2)^2-4p'^2p^2\big)R_2\Big)\left(\frac{23}{96}g^4-\frac{5}{24}g^2-\frac{1}{12}\right)\Bigg\} \\
&\qquad + \mathcal{O}(\chi^4) \qquad (427)
\end{aligned}$$

$$
\begin{aligned}
T_4(p',p) &= \int_{-1}^{1} \frac{dx}{2}\left(\frac{35x^4-30x^2+3}{8}\right)L(q)\Bigg\{\vec{q}^{\,2}\left(\frac{23}{96}g^4-\frac{5}{24}g^2-\frac{1}{12}\right) \\
&\qquad + m_\pi^2\left(\frac{5}{24}g^4-\frac{1}{3}g^2-\frac{1}{3}\right)+\frac{g^4m_\pi^4}{2(4m_\pi^2+\vec{q}^{\,2})}\Bigg\} \\
&= \frac{1}{128p'^4p^4}\Bigg\{\Bigg[\Big(35p^8+35p'^8+20p^6p'^2+18p^4p'^4+20p^2p'^6\Big)R_2+(210p^4+300p^2p'^2 \\
&\quad +210p'^4)R_6-\left(140p^2+140p'^2\right)R_8-\left(140p^6+180p^4p'^2+180p^2p'^4+140p'^6\right)R_4 \\
&\quad +35R_{10}\Bigg]\left(\frac{23}{96}g^4-\frac{5}{24}g^2-\frac{1}{12}\right)+\Bigg[\Big(35p^8+35p'^8+20p^6p'^2+18p^4p'^4+20p^2p'^6\Big)R_0 \\
&\quad +(210p^4+300p^2p'^2+210p'^4)R_4-\left(140p^2+140p'^2\right)R_6-\big(140p^6+180p^4p'^2+180p^2p'^4 \\
&\quad +140p'^6R_2+35R_8\big)\Bigg]m_\pi^2\left(\frac{5}{24}g^4-\frac{1}{3}g^2-\frac{1}{3}\right)+\frac{1}{16}\Bigg\{\frac{140g^4m_\pi^4}{p'p} \\
&\quad \Bigg[\frac{\rho^2(70m_\pi^4\rho^2-420m_\pi^6-14m_\pi^2\rho^4+3\rho^6)}{24}L(\rho)+\frac{35m_\pi^8\rho^2}{4m_\pi^2+\rho^2}L^2(\rho) \\
&\quad +\frac{5040m^6\rho^2-420m^4\rho^4+56m^2\rho^6-9\rho^8}{576}\Bigg]_{p'-p}^{p'+p}+\frac{4g^4m_\pi^4}{p'p}(140p^2+140p'^2) \\
&\quad \Bigg[\frac{\rho^2(30m_\pi^4-5m_\pi^2\rho^2+\rho^4)}{6}L(\rho)-\frac{10m_\pi^6\rho^2}{4m_\pi^2+\rho^2}L^2(\rho)-\frac{\rho^6}{36}+\frac{5m_\pi^2\rho^4-60m_\pi^4\rho^2}{24}\Bigg]_{p'-p}^{p'+p} \\
&\quad +\frac{4g^4m_\pi^4}{p'p}(210p^4+300p^2p'^2+210p'^4)\Bigg[\frac{\rho^2(\rho^2-6m_\pi^2)}{4}L(\rho)+\frac{3m_\pi^4\rho^2}{4m_\pi^2+\rho^2}L^2(\rho)-\frac{\rho^4}{16} \\
&\quad +\frac{3m_\pi^2\rho^2}{4}\Bigg]_{p'-p}^{p'+p}-\frac{4g^4m_\pi^4}{p'p}\left(140p^6+180p^4p'^2+180p^2p'^4+140p'^6\right)\Bigg[\frac{\rho^2}{2}L(\rho) \\
&\quad -\frac{m_\pi^2\rho^2}{4m_\pi^2+\rho^2}L^2(\rho)-\frac{\rho^2}{4}\Bigg]_{p'-p}^{p'+p} \\
&\quad +\frac{4g^4m_\pi^4\big[35p^8+35p'^8+20p^6p'^2+18p^4p'^4+20p^2p'^6\big]}{p'p}\Bigg[\frac{\rho^2}{4m_\pi^2+\rho^2}L^2(\rho)\Bigg]_{p'-p}^{p'+p}\Bigg\}\Bigg\} \\
&= \frac{(\vec{\tau}_1\cdot\vec{\tau}_2)}{128p'^4p^4}\Bigg\{\Bigg[\Big(35p^8+35p'^8+20p^6p'^2+18p^4p'^4+20p^2p'^6\Big)R_2+(210p^4+300p^2p'^2 \\
&\quad +210p'^4)R_6-\left(140p^2+140p'^2\right)R_8-\left(140p^6+180p^4p'^2+180p^2p'^4+140p'^6\right)R_4 \\
&\quad +35R_{10}\Bigg]\left(\frac{23}{96}g^4-\frac{5}{24}g^2-\frac{1}{12}\right)\Bigg\}+\mathcal{O}(\chi^4) \qquad (428)
\end{aligned}
$$

F Potentials with q_0 terms

This section discusses the explicit computations of loop integrals containing the perturbative treatment of q_0, as well as the recoil terms of the heavy meson propagator. We begin with the triangle diagram for $B\bar{B} \to B\bar{B}$ potential as indicated in Sec. (5.5.1)

F.1 Triangle diagram

We start with the first integral mentioned in Eq. (156).

F.1.1 The first integral

The first integral is given as

$$\begin{aligned} I_{1,\Delta} &= i\int \frac{d^4l}{(2\pi)^4}\frac{(2l_0+q_0)[-l\cdot(l+q)+l_0^2+l_0q_0]}{\big[(l+q)^2-m_\pi^2+i\epsilon\big]\big[l^2-m_\pi^2+i\epsilon\big]\big[-2v\cdot l+i\epsilon\big]M_B} \\ &= \left(\frac{-i}{2M_B}\right)\int \frac{d^4l}{(2\pi)^4}\frac{(2l_0+q_0)[-l\cdot(l+q)+l_0^2+l_0q_0]}{\big[(l+q)^2-m_\pi^2+i\epsilon\big]\big[l^2-m_\pi^2+i\epsilon\big]\big[l_0-i\epsilon\big]} \end{aligned} \tag{429}$$

where, we used $v\cdot l = l_0$.
We will start with the $[-l\cdot(l+q)]$ term in the Eq. (429), which can be decomposed as,

$$l\cdot(l+q) = \underbrace{\frac{1}{2}\big((l+q)^2-m_\pi^2\big)}_{\text{a}}+\underbrace{\frac{1}{2}\big(l^2-m_\pi^2\big)}_{\text{b}}+\underbrace{m_\pi^2-\frac{q^2}{2}}_{\text{c}} \tag{430}$$

For simplicity, the numerator will be solved term by term. $I_{1a,\Delta}$ is the integral from the first term of the $l\cdot(l+q)$ decomposition, and it is given by

$$I_{1a,\Delta} = \left(\frac{i}{2M_B}\right)\frac{1}{2}\int\frac{d^4l}{(2\pi)^4}\left(\frac{2l_0+q_0}{l_0-i\epsilon}\right)\times\left(\frac{1}{\big[l^2-m_\pi^2\big]}\right) \tag{431}$$

For the l_0 term in the numerator of Eq. (431,

$$\left(\frac{i}{2M_B}\right)\int\frac{d^4l}{(2\pi)^4}\frac{1}{\big[l^2-m_\pi^2\big]} = \left(\frac{i}{2M_B}\right)(-i\,\Delta_\pi) \tag{432}$$

where Δ_π is mentioned in [100] and evaluated as,

$$\Delta_\pi = 2m_\pi^2\left(\frac{1}{32\pi^2}\mathcal{R}+\frac{1}{16\pi^2}\ln\frac{m_\pi}{\mu}\right)+\mathcal{O}(n-4) \tag{433}$$

and for the q_0 term in the numerator of Eq. (431,

$$\left(\frac{i}{2M_B}\right)\left(\frac{q_0}{2}\right)\int\frac{d^4l}{(2\pi)^4}\frac{1}{[l_0-i\epsilon]}\frac{1}{\left[l^2-m_\pi^2\right]}=\left(\frac{i}{2M_B}\right)\left(\frac{-iq_0}{2}\right)J_0(0) \qquad (434)$$

where $J_0(\omega)$ is,

$$\frac{1}{i}\int\frac{d^4l}{(2\pi)^4}\frac{1}{(v\cdot l-\omega)(m_\pi^2-l^2)}=J_0(\omega) \qquad (435)$$

and the evaluated expression is [100],

$$J_0(\omega)=\frac{-1}{8\pi^2}\mathcal{R}\,\omega+\frac{\omega}{8\pi^2}\left(1-2\ln\frac{m_\pi}{\mu}\right)-\frac{1}{4\pi^2}\sqrt{m_\pi^2-\omega^2}\arccos\left(\frac{-\omega}{m_\pi}\right) \qquad (436)$$

here $\omega=0$ hence $J_0(0)=\frac{-m_\pi}{8\pi}$.
The evaluated expression for integral $I_{1a,\Delta}$ is

$$I_{1a,\Delta}=\left(\frac{i}{2M_B}\right)\frac{1}{2}\int\frac{d^4l}{(2\pi)^4}\left(\frac{2l_0+q_0}{l_0-i\epsilon}\right)\times\left(\frac{1}{\left[l^2-m_\pi^2\right]}\right)=\left(\Delta_\pi+\frac{q_0}{2}J_0(0)\right)\frac{1}{2M_B}\ . \qquad (437)$$

$I_{1b,\Delta}$ is the integral from the second term of the $l\cdot(l+q)$ decomposition ,

$$I_{1b,\Delta}=\left(\frac{i}{2M_B}\right)\frac{1}{2}\int\frac{d^4l}{(2\pi)^4}\left(\frac{2l_0+q_0}{l_0-i\epsilon}\right)\times\left(\frac{1}{\left[(l+q)^2-m_\pi^2\right]}\right) \qquad (438)$$

we perform a shift $l\to l-q$, but $q_0=0$ in the denominator since we treat q_0 perturbatively. After following the same steps as in the earlier case,

$$I_{1b,\Delta}=\left(\frac{i}{2M_B}\right)\frac{1}{2}\int\frac{d^4l}{(2\pi)^4}\left(\frac{2l_0+q_0}{l_0-i\epsilon}\right)\times\left(\frac{1}{\left[(l+q)^2-m_\pi^2\right]}\right)$$
$$=\left(\Delta_\pi-\frac{q_0}{2}J_0(0)\right)\frac{1}{2M_B} \qquad (439)$$

Finally, the last term of the $l\cdot(l+q)$ decomposition,

$$I_{1c,\Delta}=\left(\frac{i}{2M_B}\right)\left(m_\pi^2-\frac{q^2}{2}\right)\int\frac{d^4l}{(2\pi)^4}\left(\frac{2l_0+q_0}{l_0-i\epsilon}\right)\times\left(\frac{1}{\left[(l+q)^2-m_\pi^2\right]\left[l^2-m_\pi^2\right]}\right) \qquad (440)$$

For the l_0 term in the numerator of Eq. (440),

$$\left(\frac{i}{M_B}\right)\left(m_\pi^2-\frac{q^2}{2}\right)\int\frac{d^4l}{(2\pi)^4}\frac{1}{\left[(l+q)^2-m_\pi^2\right]\left[l^2-m_\pi^2\right]} \qquad (441)$$

Since this integral is not given [100], we perform the calculation of this integral here.
Introducing Feynman parameters, shifting $l \to l - qx$, dropping all odd powers of l due to symmetry and using $q^0 = 0$ in the denominator

$$= \left(\frac{i}{M_B}\right)\left(m_\pi^2 - \frac{q^2}{2}\right)\int_0^1 dx \int \frac{d^3\vec{l}}{(2\pi)^3}\int \frac{dl_0}{2\pi}\frac{1}{\left[(l_0)^2 - \vec{l}^2 - \vec{q}^2 x(1-x) - m_\pi^2 + i\epsilon\right]^2} \tag{442}$$

executing the l^0- integration with the residue theorem and setting $\epsilon \to 0$.

$$= \left(\frac{i}{M_B}\right)\left(m_\pi^2 - \frac{q^2}{2}\right)\frac{i}{4}\int_0^1 dx \int \frac{d^3\vec{l}}{(2\pi)^3}\frac{1}{\left[\vec{l}^2 + \vec{q}^2 x(1-x) + m_\pi^2\right]^{3/2}} , \tag{443}$$

going to $(D-1)$ - dimensional spherical coordinates, inserting μ and using $d^{D-1}\vec{l} = l^{D-2} dl\, d\Omega_{D-1}$, where $d\Omega_{D-1}$ is,

$$\int d\Omega_{D-1} = \frac{2\pi^{(D-1)/2}}{\Gamma\left(\frac{D-1}{2}\right)} , \tag{444}$$

then

$$= \left(\frac{i}{M_B}\right)\left(m_\pi^2 - \frac{q^2}{2}\right)\frac{i\sqrt{\pi}}{(4\pi)^{D/2}}\frac{\mu^{4-D}}{\Gamma(\frac{D-1}{2})}\int_0^1 dx \int_0^\infty dl \frac{l^{D-2}}{\left[\vec{l}^2 + \vec{q}^2 x(1-x) + m_\pi^2\right]^2} . \tag{445}$$

Using,

$$\int_0^\infty dl_E \frac{l_E^a}{[l_E^2 + \Xi]^b} = \Xi^{\frac{a+1}{2} - b}\,\frac{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(b - \frac{a+1}{2}\right)}{2\Gamma(b)} , \tag{446}$$

we perform the l-integration

$$= \left(\frac{i}{M_B}\right)\left(m_\pi^2 - \frac{q^2}{2}\right)\frac{i}{(4\pi)^{D/2}}\mu^{4-D}\int_0^1 dx \left[\left(\vec{q}^2 x(1-x) + m_\pi^2\right)^{\frac{D-4}{2}}\Gamma\left(\frac{4-D}{2}\right)\right] . \tag{447}$$

Using $\Gamma(x+1) = x\,\Gamma(x)$ and inserting $D = 4 - \epsilon$, then

$$= \left(\frac{i}{M_B}\right)\left(m_\pi^2 - \frac{q^2}{2}\right)\frac{-i}{16\pi^2}\int_0^1 dx \left[\left(\vec{q}^2 x(1-x) + m_\pi^2\right)^{-\epsilon/2}\Gamma\left(\frac{2-D}{2}\right)\right] , \tag{448}$$

expanding the Γ-function,

$$\Gamma\left(\frac{2-D}{2}\right) = -\frac{2}{\epsilon} + \gamma_E - 1 + \mathcal{O}(\epsilon) . \tag{449}$$

We get,

$$\left(\frac{i}{M_B}\right)\left(m_\pi^2 - \frac{q^2}{2}\right)\frac{-i}{16\pi^2}\int_0^1 dx\left[\frac{-2}{\epsilon} + \gamma_E - 1 - \ln(4\pi) + \ln\left(\frac{\vec{q}^{\,2}x(1-x) + m_\pi^2}{\mu^2}\right)\right], \tag{450}$$

performing the x-integration

$$= \left(\frac{i}{M_B}\right)\left(m_\pi^2 - \frac{q^2}{2}\right)\frac{i}{16\pi^2}\left[-\mathcal{R} + 1 - 2\,\frac{\sqrt{4m_\pi^2 + \vec{q}^{\,2}}}{\vec{q}}\,\ln\left(\frac{\sqrt{4m_\pi^2 + \vec{q}^{\,2}} + \vec{q}}{2m_\pi}\right) - 2\ln\left(\frac{m_\pi}{\mu}\right)\right] \tag{451}$$

where,

$$\mathcal{R} = \frac{-2}{\xi} + \gamma_E - 1 - \ln(4\pi) \tag{452}$$

Therefore,

$$\left(\frac{i}{M_B}\right)\left(m_\pi^2 - \frac{q^2}{2}\right)\int\frac{d^4l}{(2\pi)^4}\frac{1}{\left[(l+q)^2 - m_\pi^2\right]\left[l^2 - m_\pi^2\right]} = \left(\frac{-1}{M_B}\right)\left(m_\pi^2 - \frac{q^2}{2}\right)I_0(\vec{q}) \tag{453}$$

where,

$$I_0(\vec{q}) = \frac{1}{16\pi^2}\left[-\mathcal{R} + 1 - 2\,\frac{\sqrt{4m_\pi^2 + \vec{q}^{\,2}}}{\vec{q}}\,\ln\left(\frac{\sqrt{4m_\pi^2 + \vec{q}^{\,2}} + \vec{q}}{2m_\pi}\right) - 2\ln\left(\frac{m_\pi}{\mu}\right)\right] \tag{454}$$

Now for the q_0 term in the numerator of Eq. (440) ,

$$\left(\frac{i}{2M_B}\right)q_0\left(m_\pi^2 - \frac{q^2}{2}\right)\int\frac{d^4l}{(2\pi)^4}\frac{1}{(l_0 - i\epsilon)}\frac{1}{\left[(l+q)^2 - m_\pi^2\right]\left[l^2 - m_\pi^2\right]} \tag{455}$$

This is also not a standard integral, so we do the same procedures we did in the earlier case.
Introducing Feynman parameters, shifting $l \to l - qx$, executing the l^0- integration and setting $\epsilon \to 0$

$$= \left(\frac{i}{2M_B}\right)\left(m_\pi^2 - \frac{q^2}{2}\right)\left(\frac{i\,q_0}{2}\right)\int_0^1 dx\int\frac{d^3\vec{l}}{(2\pi)^3}\frac{1}{\left[\vec{l}^{\,2} + \vec{q}^{\,2}x(1-x) + m_\pi^2\right]^2} \tag{456}$$

Following the same steps as in the earlier calculation,

$$\begin{aligned}
&\left(\frac{i}{2M_B}\right) q_0 \left(m_\pi^2 - \frac{q^2}{2}\right) \int \frac{d^4l}{(2\pi)^4} \frac{1}{(l_0 - i\epsilon)} \frac{1}{\left[(l+q)^2 - m_\pi^2\right]\left[l^2 - m_\pi^2\right]} \\
&\qquad = \left(\frac{i}{2M_B}\right)\left(m_\pi^2 - \frac{q^2}{2}\right) \frac{i\, q_0}{16\pi} \left\{ \frac{2}{\vec{q}} \arctan\left(\frac{\vec{q}}{2m_\pi}\right)\right\} \\
&\qquad = \left(\frac{-1}{2M_B}\right)\left(m_\pi^2 - \frac{q^2}{2}\right) \frac{q_0}{16\pi} S_0(\vec{q})
\end{aligned} \tag{457}$$

where

$$S_0(\vec{q}) = \frac{2}{\vec{q}} \arctan\left(\frac{\vec{q}}{2m_\pi}\right) \tag{458}$$

Now, including $I_{1a,\Delta}$, $I_{1b,\Delta}$, $I_{1c,\Delta}$,

$$\begin{aligned}
&\left(\frac{i}{2M_B}\right) \int \frac{d^4l}{(2\pi)^4} \frac{(2l_0 + q_0)\big(l\cdot(l+q)\big)}{\left[(l+q)^2 - m_\pi^2 + i\epsilon\right]\left[l^2 - m_\pi^2 + i\epsilon\right]\left[l_0 - i\epsilon\right]} \\
&\qquad = \left\{ 2\Delta_\pi - \left[2I_0(\vec{q}) + \frac{q_0}{16\pi} S_0(\vec{q})\right]\left(m_\pi^2 - \frac{q^2}{2}\right)\right\} \frac{1}{2M_B}
\end{aligned} \tag{459}$$

Now, for the second term in the Eq. (429)

$$\left(\frac{-i}{2M_B}\right) \int \frac{d^4l}{(2\pi)^4} \frac{(2l_0 + q_0)l_0^2}{\left[(l+q)^2 - m_\pi^2 + i\epsilon\right]\left[l^2 - m_\pi^2 + i\epsilon\right]\left[l_0 - i\epsilon\right]} \tag{460}$$

The first term in Eq. (460) is the football integral I_{fb} which we solved earlier and the second term (q_0) in Eq. (460) is of the $\mathcal{O}(\epsilon)$. Therefore,

$$\begin{aligned}
&\left(\frac{-i}{2M_B}\right) \int \frac{d^4l}{(2\pi)^4} \frac{(2l_0 + q_0)l_0^2}{\left[(l+q)^2 - m_\pi^2 + i\epsilon\right]\left[l^2 - m_\pi^2 + i\epsilon\right]\left[l_0 - i\epsilon\right]} \\
&= \left(\frac{1}{M_B}\right)\frac{1}{16\pi^2}\left\{ -\left(\frac{\vec{q}^2}{12} + \frac{m_\pi^2}{2}\right)\mathcal{R} + \frac{5}{36}\vec{q}^2 + \frac{2m_\pi^2}{3} - \left(\frac{\vec{q}^2}{6} + m_\pi^2\right)\ln\left(\frac{m_\pi}{\mu}\right) \right. \\
&\qquad\qquad \left. - \left(\frac{\vec{q}^2}{6} + \frac{4}{6}m_\pi^2\right) L(q)\right\}
\end{aligned} \tag{461}$$

Finally for the last term, for the second term in the Eq. (429)

$$\left(\frac{-i}{2M_B}\right) \int \frac{d^4l}{(2\pi)^4} \frac{(2l_0 + q_0)l_0 q_0}{\left[(l+q)^2 - m_\pi^2 + i\epsilon\right]\left[l^2 - m_\pi^2 + i\epsilon\right]\left[l_0 - i\epsilon\right]} \tag{462}$$

The first term of Eq. (462) is of the $\mathcal{O}(\epsilon)$ and the second term of Eq. (462) is $I_0(\vec{q}$ which was solved earlier in Eq. (454).

$$\left(\frac{-i}{2M_B}\right)\int\frac{d^4l}{(2\pi)^4}\frac{(2l_0+q_0)l_0q_0}{\left[(l+q)^2-m_\pi^2+i\epsilon\right]\left[l^2-m_\pi^2+i\epsilon\right]\left[l_0-i\epsilon\right]}$$
$$=\left(\frac{q_0^2}{2M_B}\right)\frac{1}{16\pi^2}\left[-\mathcal{R}+1-2\,\frac{\sqrt{4m_\pi^2+\vec{q}^{\,2}}}{\vec{q}}\,\ln\left(\frac{\sqrt{4m_\pi^2+\vec{q}^{\,2}}+\vec{q}}{2m_\pi}\right)-2\ln\left(\frac{m_\pi}{\mu}\right)\right] \tag{463}$$

Finally, the evaluated expression for Eq. (429) is

$$\left(\frac{-i}{2M_B}\right)\int\frac{d^4l}{(2\pi)^4}\frac{(2l_0+q_0)[-l\cdot(l+q)+l_0^2+l_0q_0]}{\left[(l+q)^2-m_\pi^2+i\epsilon\right]\left[l^2-m_\pi^2+i\epsilon\right]\left[l_0-i\epsilon\right]}=$$
$$\frac{1}{2M_B}\left\{2\Delta_\pi+\left[2I_0(\vec{q})-\frac{q_0}{16\pi}S_0(\vec{q})\right]\left(m_\pi^2-\frac{q^2}{2}\right)+q_0I_0(\vec{q})+\frac{1}{8\pi^2}\left[-\left(\frac{\vec{q}^{\,2}}{12}+\frac{m_\pi^2}{2}\right)\right.\right.$$
$$\left.\left.\mathcal{R}+\frac{5}{36}\vec{q}^{\,2}+\frac{2m_\pi^2}{3}-\left(\frac{\vec{q}^{\,2}}{6}+m_\pi^2\right)\ln\left(\frac{m_\pi}{\mu}\right)-\left(\frac{\vec{q}^{\,2}}{6}+\frac{4}{6}m_\pi^2\right)L(q)\right]\right\}. \tag{464}$$

F.1.2 The second integral

The second integral is defined as,

$$I_{2,\Delta}=\left(\frac{-i}{4M_B^2}\right)\int\frac{d^4l}{(2\pi)^4}\frac{(2l_0+q_0)[-l\cdot(l+q)+l_0^2+l_0q_0]}{\left[(l+q)^2-m_\pi^2+i\epsilon\right]\left[l_0-i\epsilon\right]^2} \tag{465}$$

Following all the same steps as in the earlier case,

$$I_{2,\Delta}=\left(\frac{-i}{4M_B^2}\right)\int\frac{d^4l}{(2\pi)^4}\frac{(2l_0+q_0)\big(l\cdot(l+q)\big)}{\left[(l+q)^2-m_\pi^2+i\epsilon\right]\left[l_0-i\epsilon\right]^2}$$
$$=\left\{\left[2J_0(0)-q_0G_0(0)\right]\left(m_\pi^2-q^2\right)+q_0\Delta_\pi-4q_0^2J_0(0)+2\,q^2J_0(0)\right.$$
$$\left.-\,q^2q_0G_0(0)+4q_0J_0(0)\right\}\frac{1}{4M_B^2} \tag{466}$$

where, we have used the ref ([100]) for $G_0(\omega)$ and it is defined as

$$G_0(\omega)=\frac{\partial}{\partial\omega}J_0(\omega) \tag{467}$$

with $J_0(\omega)$ is given in Eq. (436). The evaluated expression for $G_0(\omega)$,

$$G_0(\omega)=\frac{-1}{32\pi^2}\mathcal{R}-\frac{1}{8\pi^2}-\frac{1}{4\pi^2}\ln\left(\frac{m_\pi}{\mu}\right) \tag{468}$$

F.1.3 The third integral

The third integral is defined as,

$$I_{3,\Delta} = \left(\frac{-i}{4M_B^2}\right)(m_\pi^2 + M_B^2 - M_{B^*}^2) \int \frac{d^4l}{(2\pi)^4} \frac{(2l_0+q_0)[-l\cdot(l+q)+l_0^2+l_0q_0]}{\left[(l+q)^2-m_\pi^2+i\epsilon\right]\left[l^2-m_\pi^2+i\epsilon\right]\left[l_0-i\epsilon\right]^2} \tag{469}$$

After following the same as for the above integrals, one obtains

$$\left(\frac{-i}{4M_B^2}\right)(m_\pi^2 + M_B^2 - M_{B^*}^2) \int \frac{d^4l}{(2\pi)^4} \frac{(2l_0+q_0)[-l\cdot(l+q)+l_0^2+l_0q_0]}{\left[(l+q)^2-m_\pi^2+i\epsilon\right]\left[l^2-m_\pi^2+i\epsilon\right]\left[l_0-i\epsilon\right]^2}$$
$$= \left\{2J_0(0) - \left[\frac{q_0}{8\pi^2}P_0(\vec{q}) + \frac{1}{8\pi}S_0(\vec{q})\right]\left(m_\pi^2 - \frac{q^2}{2}\right) + 3q_0I_0(\vec{q}) + \frac{q_0}{8\pi^2}S_0(\vec{q})\right\} \frac{(m_\pi^2 + M_B^2 - M_{B^*}^2)}{4M_B^2} \tag{470}$$

where $P_0(\vec{q})$ is gained from the solving the integral,

$$\int \frac{d^4l}{(2\pi)^4} \frac{1}{(l_0-i\epsilon)^2} \frac{1}{\left[(l+q)^2-m_\pi^2\right]\left[l^2-m_\pi^2\right]} \tag{471}$$

Since this is not a standard integral, we solve this like the earlier integrals by introducing Feynman parameters and executing the l_0 integration,

$$= \frac{3i}{4}\int_0^1 dx \int \frac{d^3\vec{l}}{(2\pi)^3} \frac{1}{\left[\vec{l}^2 + \vec{q}^{\,2}x(1-x) + m_\pi^2\right]^{5/2}} \tag{472}$$

and following the same steps as in the $S_0(\vec{q})$ integral,

$$\int \frac{d^4l}{(2\pi)^4} \frac{1}{(l_0-i\epsilon)^2} \frac{1}{\left[(l+q)^2-m_\pi^2\right]\left[l^2-m_\pi^2\right]} = \frac{i}{8\pi^2}P_0(\vec{q}) \tag{473}$$

where

$$P_0(\vec{q}) = \frac{-4}{\vec{q}\sqrt{-4m_\pi^2-\vec{q}^{\,2}}} \arctan\left(\frac{\vec{q}}{\sqrt{-4m_\pi^2-\vec{q}^{\,2}}}\right) \tag{474}$$

F.1.4 The fourth integral

Here, we have

$$I_{4,\Delta} = \left(\frac{i}{4M_B^2}\right)(2\,r^\mu) \int \frac{d^4l}{(2\pi)^4} \frac{l_\mu(2l_0+q_0))[-l\cdot(l+q)+l_0^2+l_0q_0]}{\left[(l+q)^2-m_\pi^2+i\epsilon\right]\left[l^2-m_\pi^2+i\epsilon\right]\left[l_0-i\epsilon\right]^2} \tag{475}$$

which can be decomposed as

$$\left(\frac{i}{4M_B^2}\right)(2\,r^\mu)\int\frac{d^4l}{(2\pi)^4}\frac{l_\mu(2l_0+q_0)[-l\cdot(l+q)+l_0^2+l_0q_0]}{\left[(l+q)^2-m_\pi^2+i\epsilon\right]\left[l^2-m_\pi^2+i\epsilon\right]\left[l_0-i\epsilon\right]^2}$$
$$=\left(\frac{i}{4M_B^2}\right)2\,r^\mu[q_\mu B+v_\mu C] \quad (476)$$

where

$$B=\left(\frac{q_0}{q_0^2-q^2}\right)\underbrace{\int\frac{d^4l}{(2\pi)^4}\frac{l_0(2l_0+q_0)[-l\cdot(l+q)+l_0^2+l_0q_0]}{\left[(l+q)^2-m_\pi^2+i\epsilon\right]\left[l^2-m_\pi^2+i\epsilon\right]\left[l_0-i\epsilon\right]^2}}_{\alpha}$$
$$-\left(\frac{1}{q_0^2-q^2}\right)\underbrace{\int\frac{d^4l}{(2\pi)^4}\frac{l\cdot q\,(2l_0+q_0)[-l\cdot(l+q)+l_0^2+l_0q_0]}{\left[(l+q)^2-m_\pi^2+i\epsilon\right]\left[l^2-m_\pi^2+i\epsilon\right]\left[l_0-i\epsilon\right]^2}}_{\beta} \quad (477)$$

similarly for C

$$C=\left(\frac{q_0}{q_0^2-q^2}\right)\underbrace{\int\frac{d^4l}{(2\pi)^4}\frac{l\cdot q\,(2l_0+q_0)[-l\cdot(l+q)+l_0^2+l_0q_0]}{\left[(l+q)^2-m_\pi^2+i\epsilon\right]\left[l^2-m_\pi^2+i\epsilon\right]\left[l_0-i\epsilon\right]^2}}_{\beta}$$
$$-\left(\frac{q^2}{q_0^2-q^2}\right)\underbrace{\int\frac{d^4l}{(2\pi)^4}\frac{l_0(2l_0+q_0)[-l\cdot(l+q)+l_0^2+l_0q_0]}{\left[(l+q)^2-m_\pi^2+i\epsilon\right]\left[l^2-m_\pi^2+i\epsilon\right]\left[l_0-i\epsilon\right]^2}}_{\alpha} \quad (478)$$

where we have two integers labelled as α and β. We start with α term, solving the integral term by term gives,

$$\alpha=\int\frac{d^4l}{(2\pi)^4}\frac{(2l_0+q_0)[-l\cdot(l+q)+l_0^2+l_0q_0]}{\left[(l+q)^2-m_\pi^2+i\epsilon\right]\left[l^2-m_\pi^2+i\epsilon\right]\left[l_0-i\epsilon\right]}$$
$$=i\,2\Delta_\pi-iq_0J_0(0)+i\frac{q_0^2}{2}G_0(0)+\left[2i\,I_0(\vec{q})-\frac{i\,q_0}{16\pi}S_0(\vec{q})\right]\left(m_\pi^2-\frac{q^2}{2}\right)$$
$$-\,iq_0^2I_0(\vec{q})-i2I_{fb} \quad (479)$$

where

$$I_{fb}=\frac{-1}{16\pi^2}\Bigg\{-\left(\frac{\vec{q}^{\,2}}{12}+\frac{m_\pi^2}{2}\right)\mathcal{R}+\frac{5}{36}\vec{q}^{\,2}+\frac{2m_\pi^2}{3}-\left(\frac{\vec{q}^{\,2}}{6}+m_\pi^2\right)\ln\left(\frac{m_\pi}{\mu}\right)$$
$$-\left(\frac{\vec{q}^{\,2}}{6}+\frac{4}{6}m_\pi^2\right)L(q)\Bigg\}\,. \quad (480)$$

For β term,

$$\beta = \int \frac{d^4l}{(2\pi)^4} \frac{l\cdot q\,(2l_0+q_0)[-l\cdot(l+q)+l_0^2+l_0q_0]}{\big[(l+q)^2-m_\pi^2+i\epsilon\big]\big[l^2-m_\pi^2+i\epsilon\big]\big[l_0-i\epsilon\big]^2} \tag{481}$$

we can decompose $l\cdot q$ as,

$$l\cdot q = \frac{1}{2}\big((l+q)^2-m_\pi^2\big) - \frac{1}{2}\big(l^2-m_\pi^2\big) - \frac{q^2}{2} \tag{482}$$

and repeating the same steps as in the earlier cases,

$$\begin{aligned}
\beta = \int \frac{d^4l}{(2\pi)^4} &\frac{l\cdot q\,(2l_0+q_0)[-l\cdot(l+q)+l_0^2+l_0q_0]}{\big[(l+q)^2-m_\pi^2+i\epsilon\big]\big[l^2-m_\pi^2+i\epsilon\big]\big[l_0-i\epsilon\big]^2} = \\
& i\,m_\pi^2 J_0(0) + \frac{iq_0}{2}\,m_\pi^2 G_0(0) + i\,q_0\Delta_\pi + i\,\frac{q_0^2}{2}J_0(0) + \frac{1}{2}\Big\{\big[-2\,iJ_0(0) \\
& + i\,q_0G_0(0)\big]\big(m_\pi^2-q^2\big) + 2\,i\,q_0\Delta_\pi - i\,q_0^2J_0(0) - 2\,i\,q^2J_0(0) + i\,q^2q_0G_0(0)\Big\} \\
& + \frac{q^2}{2}\Big\{-2\,i\,J_0(0) + \left[\frac{i\,q_0}{8\pi^2}P_0(\vec{q}) + \frac{i}{8\pi}S_0(\vec{q})\right]\left(m_\pi^2-\frac{q^2}{2}\right)\Big\} - \frac{3i}{2}I_0(\vec{q}) + \frac{iq_0^2}{32\pi}S_0(\vec{q}) \\
& - \frac{i}{2}\Big[3q_0\Delta_\pi - 5q_0^2J_0(0) + 4q_0J_0(0)\Big] - \frac{q^2}{2}\left[-3iq_0I_0(\vec{q}) + \frac{iq_0^2}{16\pi}S_0(\vec{q})\right]
\end{aligned} \tag{483}$$

Now we can write the fourth integral,

$$\begin{aligned}
I_{4,\Delta} &= \left(\frac{i}{4M_B^2}\right)(2\,r^\mu)\int \frac{d^4l}{(2\pi)^4}\frac{l_\mu(2l_0+q_0)[-l\cdot(l+q)+l_0^2+l_0q_0]}{\big[(l+q)^2-m_\pi^2+i\epsilon\big]\big[l^2-m_\pi^2+i\epsilon\big]\big[l_0-i\epsilon\big]^2} \\
&= \left(\frac{i}{4M_B^2}\right)2\,r^\mu[q_\mu B + v_\mu C] = \left(\frac{i}{2M_B^2}\right)[r\cdot q\,B + r_0\,C]
\end{aligned} \tag{484}$$

Putting all the terms of B and C together we can write I_4 as

$$
\begin{aligned}
I_{4,\Delta} = &\left(\frac{r\cdot q}{2M_B^2}\right)\left(\frac{q_0}{q_0^2-q^2}\right)\Bigg\{-2\Delta_\pi + q_0 J_0(0) - \frac{q_0^2}{2}G_0(0) + \Big[-2I_0(\vec{q}) + \frac{q_0}{16\pi}S_0(\vec{q})\Big] \\
&\left(m_\pi^2 - \frac{q^2}{2}\right) + q_0^2 I_0(\vec{q}) + 2I_{fb}\Bigg\} \\
&-\left(\frac{r\cdot q}{2M_B^2}\right)\left(\frac{1}{q_0^2-q^2}\right)\Bigg\{-m_\pi^2 J_0(0) - \frac{q_0}{2}\,m_\pi^2 G_0(0) - q_0\Delta_\pi - \frac{q_0^2}{2}J_0(0) + \frac{1}{2}\Big\{\big[2J_0(0) \\
&-q_0 G_0(0)\big]\big(m_\pi^2 - q^2\big) - 2\,q_0\Delta_\pi + q_0^2 J_0(0) + 2\,q^2 J_0(0) - q^2 q_0 G_0(0)\Big\} \\
&+\frac{q^2}{2}\Big\{+2J_0(0) + \Big[\frac{-q_0}{8\pi^2}P_0(\vec{q}) - \frac{1}{8\pi}S_0(\vec{q})\Big]\left(m_\pi^2 - \frac{q^2}{2}\right)\Big\} + \frac{3}{2}I_0(\vec{q}) - \frac{q_0^2}{32\pi}S_0(\vec{q}) \\
&+\frac{1}{2}\Big[3q_0\Delta_\pi - 5q_0^2 J_0(0) + 4q_0 J_0(0)\Big] - \frac{q^2}{2}\Big[3q_0 I_0(\vec{q}) - \frac{q_0^2}{16\pi}S_0(\vec{q})\Big]\Bigg\} \\
&+\left(\frac{r_0}{2M_B^2}\right)\left(\frac{q_0}{q_0^2-q^2}\right)\Bigg\{-m_\pi^2 J_0(0) - \frac{q_0}{2}\,m_\pi^2 G_0(0) - q_0\Delta_\pi - \frac{q_0^2}{2}J_0(0) + \frac{1}{2}\Big\{\big[2J_0(0) \\
&-q_0 G_0(0)\big]\big(m_\pi^2 - q^2\big) - 2\,q_0\Delta_\pi + q_0^2 J_0(0) + 2\,q^2 J_0(0) - q^2 q_0 G_0(0)\Big\} \\
&+\frac{q^2}{2}\Big\{+2J_0(0) + \Big[\frac{-q_0}{8\pi^2}P_0(\vec{q}) - \frac{1}{8\pi}S_0(\vec{q})\Big]\left(m_\pi^2 - \frac{q^2}{2}\right)\Big\} + \frac{3}{2}I_0(\vec{q}) - \frac{q_0^2}{32\pi}S_0(\vec{q}) \\
&+\frac{1}{2}\Big[3q_0\Delta_\pi - 5q_0^2 J_0(0) + 4q_0 J_0(0)\Big] - \frac{q^2}{2}\Big[3q_0 I_0(\vec{q}) - \frac{q_0^2}{16\pi}S_0(\vec{q})\Big]\Bigg\} \\
&-\left(\frac{r_0}{2M_B^2}\right)\left(\frac{q^2}{q_0^2-q^2}\right)\Bigg\{-2\Delta_\pi + q_0 J_0(0) - \frac{q_0^2}{2}G_0(0) + \Big[-2I_0(\vec{q}) + \frac{q_0}{16\pi}S_0(\vec{q})\Big]\left(m_\pi^2 - \frac{q^2}{2}\right) \\
&+ q_0^2 I_0(\vec{q}) + 2I_{fb}\Bigg\} \qquad (485)
\end{aligned}
$$

Hence we have solved for all the four integrals that make up I_Δ, once again I_Δ is given by

$$
\begin{aligned}
I_\Delta = i\int \frac{d^4l}{(2\pi)^4}\frac{(2l_0+q_0)[-l\cdot(l+q)+l_0^2+l_0q_0]}{\big[(l+q)^2-m_\pi^2+i\epsilon\big]\big[l^2-m_\pi^2+i\epsilon\big]\big[-2v\cdot l+i\epsilon\big]M_B} \\
- i\int \frac{d^4l}{(2\pi)^4}\frac{(2l_0+q_0)(l^2-2\,r\cdot l+M_B^2-M_{B^*}^2)[-l\cdot(l+q)+l_0^2+l_0q_0]}{\big[(l+q)^2-m_\pi^2+i\epsilon\big]\big[l^2-m_\pi^2+i\epsilon\big]\big[-2v\cdot l+i\epsilon\big]^2M_B^2} \\
= I_{1,\Delta}+I_{2,\Delta}+I_{3,\Delta}+I_{4,\Delta} \qquad (486)
\end{aligned}
$$

where $I_{1,\Delta}$, $I_{2,\Delta}$, $I_{3,\Delta}$ and $I_{4,\Delta}$ are given in Eq. (464), Eq. (466), Eq. (470) and Eq. (485) respectively.

F.2 Crossed-box diagram

Here, we will perform the same exercise for crossed-box diagram mentioned in Sec. 5.5.2. We start with the first integral mentioned in Eq. (165).

F.2.1 The first integral

The first integral is

$$
I_{1,cb} = i\int \frac{d^4l}{(2\pi)^4}\frac{(-g_{\mu\nu}+v_\mu v_\nu)(-g_{\rho\sigma}+v_\rho v_\sigma)[l^\mu(l^\nu+q^\nu)][l^\rho(l^\sigma+q^\sigma)]}{4M_B^2\big[(l+q)^2-m_\pi^2+i\epsilon\big]\big[l^2-m_\pi^2+i\epsilon\big]\big[l_0-i\epsilon\big]^2}\ , \qquad (487)
$$

which when solved gives

$$
\begin{aligned}
I_{1,cb} = \left(\frac{i}{4M_B^2}\right)\int \frac{d^4l}{(2\pi)^4}\frac{(-g_{\mu\nu}+v_\mu v_\nu)(-g_{\rho\sigma}+v_\rho v_\sigma)[l^\mu(l^\nu+q^\nu)][l^\rho(l^\sigma+q^\sigma)]}{\big[(l+q)^2-m_\pi^2+i\epsilon\big]\big[l^2-m_\pi^2+i\epsilon\big]\big[l_0-i\epsilon\big]^2} \\
= \left(\frac{1}{4M_B^2}\right)\Bigg\{\frac{3}{2}m_\pi^2G_0(0)-\frac{q^2}{2}G_0(0)+\left(m_\pi^2-\frac{q^2}{2}\right)\left[G_0(0)-\frac{1}{8\pi^2}\left(m_\pi^2-\frac{q^2}{2}\right)P_0(\vec{q})\right] \\
-2\Delta_\pi+2\left(m_\pi^2-\frac{q^2}{2}\right)\left[-\frac{q_0}{16\pi}S_0(\vec{q})-I_0(\vec{q})\right]+I_{fb}+I_0(\vec{q})\Bigg\} \qquad (488)
\end{aligned}
$$

F.2.2 The second integral

The second integral can be written as,

$$I_{2,cb} = \left(\frac{i}{4M_B^3}\right) \int \frac{d^4l}{(2\pi)^4} \frac{(-g_{\mu\nu} + v_\mu v_\nu)(-g_{\rho\sigma} + v_\rho v_\sigma)[l^\mu(l^\nu + q^\nu)][l^\rho(l^\sigma + q^\sigma)]}{\big[(l+q)^2 - m_\pi^2 + i\epsilon\big]\big[l_0 - i\epsilon\big]^3}$$

$$= \left(\frac{1}{4M_B^3}\right)\Bigg\{ - m_\pi^4 H_0(0) - 2q_0 m_\pi^2 G_0(0) - \frac{q^2}{3}\Big[m_\pi^2 H_0(0) + J_0(0)\Big] + \frac{q_0^2}{3}[4J_0(0)$$
$$+ m_\pi^2 H_0(0)] + 2(m_\pi^2 - q^2)\big[q_0^2 H_0(0) - J_0(0) + 2q_0 G_0(0)\big] - 2q_0 G_0(0) + 2q^2 H_0(0)$$
$$- 2q_0\Delta_\pi + q_0^2 J_0(0)\Bigg\} \quad (489)$$

where $H_0(0)$ obtained from the solving the integral

$$\int \frac{d^4l}{(2\pi)^4} \frac{1}{\big[l^2 - m_\pi^2 + i\epsilon\big]\big[l_0 - i\epsilon\big]^3}\ , \quad (490)$$

after performing the l_0 integration, one finds

$$\int \frac{d^3\vec{l}}{(2\pi)^3} \frac{1}{\big[\vec{l}^2 + m_\pi^2\big]^2}\ , \quad (491)$$

which when integrated over gives

$$\int \frac{d^4l}{(2\pi)^4} \frac{1}{\big[l^2 - m_\pi^2 + i\epsilon\big]\big[l_0 - i\epsilon\big]^3} = iH_0(0) \quad (492)$$

where

$$H_0(0) = \frac{-1}{16\pi m_\pi^2} \quad (493)$$

F.2.3 The third integral

The third integral can be written as,

$$
\begin{aligned}
I_{3,cb} = & \left(\frac{i(m_\pi^2 + M_B^2 - M_{B^*}^2)}{4M_B^3} \right) \\
& \int \frac{d^4l}{(2\pi)^4} \frac{(-g_{\mu\nu} + v_\mu v_\nu)(-g_{\rho\sigma} + v_\rho v_\sigma)[l^\mu(l^\nu + q^\nu)][l^\rho(l^\sigma + q^\sigma)]}{\left[(l+q)^2 - m_\pi^2 + i\epsilon\right]\left[l^2 - m_\pi^2 + i\epsilon\right]\left[l_0 - i\epsilon\right]^3} \\
= & \left(\frac{m_\pi^2 + M_B^2 - M_{B^*}^2}{(4M_B^3)} \right) \Bigg\{ \frac{3}{2} m_\pi^2 H_0(0) - \frac{1}{2}\left(m_\pi^2 - \frac{q^2}{2}\right)[G_0(0) + H_0(0)] \\
& - \left(m_\pi^2 - \frac{q^2}{2}\right)^2 K_0(0) - 2J_0(0) + q_0^2 H_0(0) + 2q_0 G_0(0) + \frac{1}{8\pi}\left(m_\pi^2 - \frac{q^2}{2}\right) S_0(\vec{q}) \\
& + 2q_0 G_0(0) + q_0^2 H_0(0) - \frac{q_0}{4\pi^2}\left(m_\pi^2 - \frac{q^2}{2}\right) P_0(\vec{q}) - \frac{q_0^2}{16\pi} S_0(\vec{q}) - 2q_0 I_0(\vec{q}) \Bigg\} \quad (494)
\end{aligned}
$$

where $K_0(0)$ is obtained from computing the integral,

$$
\int \frac{d^4l}{(2\pi)^4} \frac{1}{\left[(l+q)^2 - m_\pi^2 + i\epsilon\right]\left[l^2 - m_\pi^2 + i\epsilon\right]\left[l_0 - i\epsilon\right]^3} \quad (495)
$$

with the l_0 integration one obtains,

$$
\int_0^1 dx \int \frac{d^3\vec{l}}{(2\pi)^3} \frac{1}{2\pi} \frac{-2i\pi}{\left[\vec{l}^2 + \vec{q}^2 x(1-x) + m_\pi^2\right]^3} \quad (496)
$$

where x is the Feynman parameter and solving the integral

$$
\int \frac{d^4l}{(2\pi)^4} \frac{1}{\left[(l+q)^2 - m_\pi^2 + i\epsilon\right]\left[l^2 - m_\pi^2 + i\epsilon\right]\left[l_0 - i\epsilon\right]^3} = iK_0(0) \quad (497)
$$

where,

$$
K_0(0) = \frac{1}{8\pi m_\pi (4m_\pi^2 + \vec{q}^2)} \quad (498)
$$

F.2.4 The fourth integral

The fourth integral

$$
\begin{aligned}
I_{4,cb} = &\left(\frac{-i}{4M_B^3}\right) \\
&\int \frac{d^4l}{(2\pi)^4} \frac{(r\cdot l + s\cdot l)(-g_{\mu\nu}+v_\mu v_\nu)(-g_{\rho\sigma}+v_\rho v_\sigma)[l^\mu(l^\nu+q^\nu)][l^\rho(l^\sigma+q^\sigma)]}{\big[(l+q)^2-m_\pi^2+i\epsilon\big]\big[l^2-m_\pi^2+i\epsilon\big]\big[l_0-i\epsilon\big]^3} \\
= &\frac{1}{4M_B^3}\Bigg\{\frac{m_\pi^2}{2}r_0 G_0(0) + \frac{(r\cdot q)}{6}\big[m_\pi^2 H_0(0)+J_0(0)\big] - \frac{r_0 q_0}{2}\big[4J_0(0)+m_\pi^2 H_0(0)\big] \\
&+(m_\pi^2-q^2)\left[\frac{r_0}{2}G_0(0)-\frac{q_0}{2}H_0(0)\right] - \frac{1}{2}\left[\frac{(r\cdot q)}{3}(m_\pi^2 H_0(0)+J_0(0)) - \frac{r_0 q_0}{3}(m_\pi^2 H_0(0)\right. \\
&\left.+4J_0(0))\right] + \frac{q^2 r_0}{2}G_0(0) + \frac{(r\cdot q)q_0}{2}H_0(0) - \frac{(r\cdot q)q^2}{2}H_0(0) + \left(m_\pi^2-\frac{q^2}{2}\right)\left[r_0 G_0(0)\right. \\
&\left.+\frac{(r\cdot q)}{2}H_0(0)\right] + \left(m_\pi^2-\frac{q^2}{2}\right)^2\left[\frac{(r\cdot q)}{\vec{q}^2}\left(\frac{q_0}{8\pi^2}P_0(\vec{q})-\frac{q^2}{2}K_0(0)\right)+\frac{r_0}{\vec{q}^2}\left(\frac{q_0 q^2}{2}K_0(0)\right.\right. \\
&\left.\left.-\frac{q^2}{8\pi^2}P_0(\vec{q})\right)\right] + 2r_0\Delta_\pi - r_0 q_0^2 G_0(0) - (r\cdot q)J_0(0) + q_0^2(r\cdot q)H_0(0) - q_0(r\cdot q)G_0(0) \\
&+2\left(m_\pi^2-\frac{q^2}{2}\right)\left[\frac{-(r\cdot q)}{\vec{q}^2}\left(q_0 I_0(\vec{q})-\frac{q^2}{32\pi}S_0(\vec{q})\right)+\frac{r_0}{\vec{q}^2}\left(\frac{q_0 q^2}{2}S_0(\vec{q})-q^2 I_0(\vec{q})\right)\right] \\
&-2q_0\left(m_\pi^2-\frac{q^2}{2}\right)\left[\frac{(r\cdot q)}{\vec{q}^2}\left(\frac{q_0}{16\pi}S_0(\vec{q})-\frac{q^2}{16\pi^2}P_0(\vec{q})\right)+\frac{r_0}{\vec{q}^2}\left(\frac{q_0 q^2}{16\pi^2}P_0(\vec{q})-\frac{q^2}{16\pi}S_0(\vec{q})\right)\right] \\
&-r_0 I_{fb} + 2(r\cdot q)q_0 I_0(\vec{q}) + q_0^2\left[\frac{(r\cdot q)}{\vec{q}^2}\left(q_0 I_0(\vec{q})-\frac{q^2}{32\pi}S_0(\vec{q})\right)+\frac{r_0}{\vec{q}^2}\left(\frac{q_0 q^2}{32\pi}S_0(\vec{q})\right.\right. \\
&\left.\left.-q^2 I_0(\vec{q})\right)\right]\frac{m_\pi^2}{2}s_0 G_0(0) + \frac{(s\cdot q)}{6}\big[m_\pi^2 H_0(0)+J_0(0)\big] - \frac{s_0 q_0}{2}\big[4J_0(0)+m_\pi^2 H_0(0)\big] \\
&+(m_\pi^2-q^2)\left[\frac{s_0}{2}G_0(0)-\frac{q_0}{2}H_0(0)\right] - \frac{1}{2}\left[\frac{(s\cdot q)}{3}(m_\pi^2 H_0(0)+J_0(0)) - \frac{s_0 q_0}{3}(m_\pi^2 H_0(0)\right. \\
&\left.+4J_0(0))\right] + \frac{q^2 s_0}{2}G_0(0) + \frac{(s\cdot q)q_0}{2}H_0(0) - \frac{(s\cdot q)q^2}{2}H_0(0) + \left(m_\pi^2-\frac{q^2}{2}\right)\left[s_0 G_0(0)\right. \\
&\left.+\frac{(s\cdot q)}{2}H_0(0)\right] + \left(m_\pi^2-\frac{q^2}{2}\right)^2\left[\frac{(s\cdot q)}{\vec{q}^2}\left(\frac{q_0}{8\pi^2}P_0(\vec{q})-\frac{q^2}{2}K_0(0)\right)+\frac{s_0}{\vec{q}^2}\left(\frac{q_0 q^2}{2}K_0(0)\right.\right. \\
&\left.\left.-\frac{q^2}{8\pi^2}P_0(\vec{q})\right)\right] + 2s_0\Delta_\pi - s_0 q_0^2 G_0(0) - (s\cdot q)J_0(0) + q_0^2(s\cdot q)H_0(0) - q_0(s\cdot q)G_0(0) \\
&+2\left(m_\pi^2-\frac{q^2}{2}\right)\left[\frac{-(s\cdot q)}{\vec{q}^2}\left(q_0 I_0(\vec{q})-\frac{q^2}{32\pi}S_0(\vec{q})\right)+\frac{s_0}{\vec{q}^2}\left(\frac{q_0 q^2}{2}S_0(\vec{q})-q^2 I_0(\vec{q})\right)\right] - 2q_0 \\
&\left(m_\pi^2-\frac{q^2}{2}\right)\left[\frac{(s\cdot q)}{\vec{q}^2}\left(\frac{q_0}{16\pi}S_0(\vec{q})-\frac{q^2}{16\pi^2}P_0(\vec{q})\right)+\frac{s_0}{\vec{q}^2}\left(\frac{q_0 q^2}{16\pi^2}P_0(\vec{q})-\frac{q^2}{16\pi}S_0(\vec{q})\right)\right] \\
&-s_0 I_{fb} + 2(s\cdot q)q_0 I_0(\vec{q}) + q_0^2\left[\frac{(s\cdot q)}{\vec{q}^2}\left(q_0 I_0(\vec{q})-\frac{q^2}{32\pi}S_0(\vec{q})\right)+\frac{s_0}{\vec{q}^2}\left(\frac{q_0 q^2}{32\pi}S_0(\vec{q})-q^2 I_0(\vec{q})\right)\right]\Bigg\}
\end{aligned}
\tag{499}
$$

Since all the four integrals have been worked out we can write I_{cb}

$$I_{cb} = i\int \frac{d^4l}{(2\pi)^4}\frac{\Pi}{\big[(l+q)^2-m_\pi^2+i\epsilon\big]\big[l^2-m_\pi^2+i\epsilon\big]\big[-2v\cdot l+i\epsilon\big]^2 M_B^2}$$
$$-2i\int \frac{d^4l}{(2\pi)^4}\frac{\Pi\big[l^2-r\cdot l-s\cdot l+M_B^2-M_{B^*}^2\big]}{\big[(l+q)^2-m_\pi^2+i\epsilon\big]\big[l^2-m_\pi^2+i\epsilon\big]\big[2v\cdot l+i\epsilon\big]^3 M_B^3}$$
$$= I_{1,cb}+I_{2,cb}+I_{3,cb}+I_{4,cb} \quad (500)$$

where $I_{1,cb}$, $I_{2,cb}$, $I_{3,cb}$ and $I_{4,cb}$ are given by Eq. (488), Eq. (489), Eq. (494) and Eq. (499) respectively and $\Pi = (-g_{\mu\nu}+v_\mu v_\nu)(-g_{\rho\sigma}+v_\rho v_\sigma)[l^\mu(l^\nu+q^\nu)][l^\rho(l^\sigma+q^\sigma)]$.

F.3 Vertex diagram

Here, we will solve the four integrals encountered in Sec. 5.5.3.

F.3.1 The first integral

$$I_{1,vert} = i\int \frac{d^4l}{(2\pi)^4}\frac{l^2}{4M_B^2\big[l^2-m_\pi^2+i\epsilon\big]\big[l_0+i\epsilon\big]^2} \ . \quad (501)$$

Using $l^2 = l_0^2 - \vec{l}^2$

$$I_{1,vert} = i\int \frac{d^4l}{(2\pi)^4}\frac{l_0^2-\vec{l}^2}{4M_B^2\big[l^2-m_\pi^2+i\epsilon\big]\big[l_0+i\epsilon\big]^2} \ . \quad (502)$$

for l_0^2 term,

$$= i\int \frac{d^4l}{(2\pi)^4}\frac{l_0^2}{4M_B^2\big[l^2-m_\pi^2+i\epsilon\big]\big[l_0+i\epsilon\big]^2} \ , \quad (503)$$

doing the l_0 integration,

$$= \frac{i}{4M_B^2}\frac{1}{D}\int \frac{d^3\vec{l}}{(2\pi)^3}\frac{1}{2\pi}\frac{i\pi}{\big[\vec{l}^2-m_\pi^2\big]^{1/2}} \ , \quad (504)$$

going to $(D-1)$ - dimensional spherical coordinates, inserting μ and using $d^{D-1}\vec{l} = l^{D-2}dl\, d\Omega_{D-1}$,

$$= \frac{-1}{4M_B^2}\frac{1}{2}\frac{4\sqrt{\pi}}{(4\pi)^{D/2}}\frac{\mu^{4-D}}{\Gamma(\frac{D-1}{2})}\int_0^\infty dl \frac{l^{D-2}}{\big[l^2+m_\pi^2\big]^{1/2}} \quad (505)$$

Performing the l integration and expanding in $D = 4 - \epsilon$,

$$= \frac{-1}{4M_B^2} \frac{m_\pi^2}{16\pi^2} \left[R + 2 \ln \left(\frac{m_\pi}{\mu} \right) \right] \tag{506}$$

Now for $\vec{l}^2$ term,

$$= i \int \frac{d^4 l}{(2\pi)^4} \frac{-\vec{l}^2}{4M_B^2 \left[l^2 - m_\pi^2 + i\epsilon \right] \left[l_0 + i\epsilon \right]^2} , \tag{507}$$

doing the l_0 integration,

$$= \frac{i}{4M_B^2} \int \frac{d^3 \vec{l}}{(2\pi)^3} \frac{\vec{l}^2}{2\pi} \frac{-i\pi}{\left[\vec{l}^2 + m_\pi^2 \right]^{3/2}} , \tag{508}$$

which can be simplified as,

$$= \frac{1}{4M_B^2} \frac{1}{2} \int \frac{d^3 \vec{l}}{(2\pi)^3} \frac{\vec{l}^2}{\left[\vec{l}^2 + m_\pi^2 \right]^{3/2}} . \tag{509}$$

Following the same steps as in the previous case we get,

$$= \frac{1}{4M_B^2} \frac{3m_\pi^2}{16\pi^2} \left[R + 2 \ln \left(\frac{m_\pi}{\mu} \right) \right] . \tag{510}$$

Now, we write I_1 as

$$\begin{aligned} I_{1,vert} &= i \int \frac{d^n l}{(2\pi)^4} \frac{l_0^2 - \vec{l}^2}{4M_B^2 \left[l^2 - m_\pi^2 + i\epsilon \right] \left[l_0 + i\epsilon \right]^2} \\ &= \frac{-1}{4M_B^2} \frac{m_\pi^2}{16\pi^2} \left[R + 2 \ln \left(\frac{m_\pi}{\mu} \right) \right] + \frac{1}{4M_B^2} \frac{3m_\pi^2}{16\pi^2} \left[R + 2 \ln \left(\frac{m_\pi}{\mu} \right) \right] \\ &= \frac{2}{4M_B^2} \frac{m_\pi^2}{16\pi^2} \left[R + 2 \ln \left(\frac{m_\pi}{\mu} \right) \right] \end{aligned} \tag{511}$$

F.3.2 The second integral

$$I_{2,vert} = -i \int \frac{d^4 l}{(2\pi)^4} \frac{2(l_0^2 - \vec{l}^2)}{8M_B^3 \left[l_0 + i\epsilon \right]^3} \tag{512}$$

The $d\vec{l}$ integration tends to infinity. Hence with upper and lower included, this term vanishes.

F.3.3 The third integral

The third integral is

$$I_{3,vert} = -i \int \frac{d^4 l}{(2\pi)^4} \frac{\big[2m_\pi^2 + q^2 - 2\, r \cdot q + M_B^2 - M_{B^*}^2\big] l^2}{8 M_B^3 \big[l^2 - m_\pi^2 + i\epsilon\big]\big[l_0 + i\epsilon\big]^3} \ . \tag{513}$$

Following the same procedures as before

$$\begin{aligned} I_{3,vert} &= -i \int \frac{d^4 l}{(2\pi)^4} \frac{\big[2m_\pi^2 + q^2 - 2\, r \cdot q + M_B^2 - M_{B^*}^2\big] l^2}{8 M_B^3 \big[l^2 - m_\pi^2 + i\epsilon\big]\big[l_0 + i\epsilon\big]^3} \\ &= \frac{-m_\pi \big[2m_\pi^2 + q^2 - 2\, r \cdot q + M_B^2 - M_{B^*}^2\big]}{128 \pi M_B^3} \end{aligned} \tag{514}$$

F.3.4 The fourth integral

The fourth integral is

$$I_{4,vert} = -i \int \frac{d^4 l}{(2\pi)^4} \frac{\big[4\, r \cdot l - 2\, l \cdot q\big] l^2}{8 M_B^3 \big[l^2 - m_\pi^2 + i\epsilon\big]\big[l_0 + i\epsilon\big]^3} \ . \tag{515}$$

Solving this integral as in the earlier cases,

$$\begin{aligned} I_{4,vert} &= -i \int \frac{d^4 l}{(2\pi)^4} \frac{\big[4\, r \cdot l - 2\, l \cdot q\big] l^2}{8 M_B^3 \big[l^2 - m_\pi^2 + i\epsilon\big]\big[l_0 + i\epsilon\big]^3} \\ &= \frac{1}{M_B^3} \frac{m_\pi^2}{16\pi^2} \left[R + 2 \ln\left(\frac{m_\pi}{\mu}\right)\right] \left(\frac{q_0}{2} - r_0\right) \end{aligned} \tag{516}$$

Having solved all the four integrals that make up I_{vert}, we write

$$\begin{aligned} I_{vert} = i \int & \frac{d^4 l}{(2\pi)^4} \frac{1}{M_B^2 \big[l^2 - m_\pi^2 + i\epsilon\big]\big[2v \cdot l + i\epsilon\big]^2} \\ & - i \int \frac{d^4 l}{(2\pi)^4} \frac{2l^2 + q^2 - 2l \cdot q + 4\, r \cdot l - 2r \cdot q + M_B^2 - M_{B^*}^2}{M_B^3 \big[l^2 - m_\pi^2 + i\epsilon\big]\big[2v \cdot l + i\epsilon\big]^3} \\ & = I_{1,vert} + I_{2,vert} + I_{3,vert} + I_{4,vert} \end{aligned} \tag{517}$$

where $I_{1,vert}$, $I_{2,vert}$, $I_{3,vert}$ and $I_{4,vert}$ are given in Eq. (511), Eq. (512), Eq. (514) and Eq. (516) respectively.

G Particle basis

In this section, we derive the vertices from the Lagrangian (given in Eq. (88)) in particle basis. Then using these vertices, we will express the triangle diagrams in $\bar{B}B \to \bar{B}B$ and $BB \to BB$ case.
We will start with the pion-emission vertex.

G.1 Pion-emission vertex

The Lagrangian is given by Eq. (92), but here we assign $a = 1$ and $b = 1$

$$\mathcal{L}_{B^{0(*)} \to B^{0(*)} \pi^0} = -\frac{g}{2} \mathrm{Tr}[H_1^\dagger H_1 \boldsymbol{\sigma}] \cdot \mathbf{A_{11}} \tag{518}$$

Using the definition of H_1,

$$\mathcal{L}_{B^{0(*)} \to B^{0(*)} \pi^0} = -\frac{g}{4} \mathrm{Tr}\Big[\big(B^{0\dagger} + (B_j^{0*})^\dagger \sigma^j\big)\big(B^0 + B_k^{0*}\sigma^k\big)\,\sigma^i\Big] \left(\frac{-\partial_i \pi^0}{f_\pi}\right) \tag{519}$$

where used Eq. (26).
Using the trace relations, one finds

$$\mathcal{L}_{B^{0(*)} \to B^{0(*)} \pi^0} = \frac{g}{2f_\pi} \partial_i \pi^0 \Big[B^{0\dagger} B_i^{0*} + (B_i^{0*})^\dagger B^{0\dagger} + i\,\epsilon_{jki}\,(B_j^{0*})^\dagger B_k^{0*}\Big] , \tag{520}$$

replacing $\partial_i \boldsymbol{\pi} = -i\mathbf{k}$, where $\mathbf{k}$ is the momentum of the outgoing pion, then

$$\mathcal{L}_{B^{0(*)} \to B^{0(*)} \pi^0} = \frac{-ig}{2f_\pi} k_i \Big[B^{0\dagger} B_i^{0*} + (B_i^{0*})^\dagger B^{0\dagger} + i\,\epsilon_{jki}\,(B_j^{0*})^\dagger B_k^{0*}\Big] \tag{521}$$

From here, we get the various vertices

$$\begin{aligned}
v_{B^{0*} \to B^0 \pi^0} &= \frac{g}{2f_\pi} (\vec{\epsilon} \cdot \vec{k}) , \\
v_{B^0 \to B^{0*} \pi^0} &= \frac{g}{2f_\pi} (\vec{\epsilon}^{\,*} \cdot \vec{k}) , \\
v_{B^{0*} \to B^{0*} \pi^0} &= i \frac{g}{2f_\pi} \epsilon_{jki}\, \vec{\epsilon_j}^{\,*} \vec{\epsilon_k} k_i .
\end{aligned}$$

Similarly, the vertices for all the combinations of a and b isospin index.

$$
\begin{aligned}
v_{B^{+*}\to B^0\pi^+} &= \frac{g\sqrt{2}}{2f_\pi}(\vec{\epsilon}\cdot\vec{k})\ , & v_{B^{+}\to B^{0*}\pi^+} &= \frac{g\sqrt{2}}{2f_\pi}(\vec{\epsilon}^{\,*}\cdot\vec{k})\ , \\
v_{B^{+*}\to B^{0*}\pi^+} &= i\frac{g\sqrt{2}}{2f_\pi}\epsilon_{jki}\,\vec{\epsilon_j}^{\,*}\vec{\epsilon_k}k_i\ , & v_{B^{0*}\to B^{+}\pi^-} &= \frac{g\sqrt{2}}{2f_\pi}(\vec{\epsilon}\cdot\vec{k})\ , \\
v_{B^{0}\to B^{+*}\pi^-} &= \frac{g\sqrt{2}}{2f_\pi}(\vec{\epsilon}^{\,*}\cdot\vec{k})\ , & v_{B^{0*}\to B^{+*}\pi^-} &= i\frac{g\sqrt{2}}{2f_\pi}\epsilon_{jki}\,\vec{\epsilon_j}^{\,*}\vec{\epsilon_k}k_i\ , \\
v_{B^{+*}\to B^{+}\pi^0} &= \frac{-g}{2f_\pi}(\vec{\epsilon}\cdot\vec{k})\ , & v_{B^{+}\to B^{+*}\pi^0} &= \frac{-g}{2f_\pi}(\vec{\epsilon}^{\,*}\cdot\vec{k})\ , \\
v_{B^{+*}\to B^{+*}\pi^0} &= -\,i\frac{g}{2f_\pi}\epsilon_{jki}\,\vec{\epsilon_j}^{\,*}\vec{\epsilon_k}k_i\ .
\end{aligned}
$$

In the same way, we can also derive the corresponding vertices for the anti-mesons with the only difference being to use the charge-conjugated Pauli matrix, which reads as $\boldsymbol{\tau^c} = \tau_2\,\boldsymbol{\tau}^T\,\tau_2 = -\boldsymbol{\tau}$.

G.2 Weinberg-Tomozawa vertex

The Lagrangian for Weinberg-Tomozawa vertex is

$$
\mathcal{L}_{B^{(*)}\pi\to B^{(*)}\pi} = \mathrm{Tr}[H_a^\dagger\,(iD_0)_{ba}\,H_b] \tag{522}
$$

where $(D_0)_{ba}$ the chiral covariant derivative is given by $D_0 = \partial_0 + \Gamma_0$ and

$$
\begin{aligned}
\Gamma_0 &= \frac{1}{2}\left[u^\dagger\partial_0 u + u\partial_0 u^\dagger\right] \\
&= \frac{\phi\cdot\partial_0\phi - \partial_0\phi\cdot\phi}{8f_\pi^2} \\
&= i\frac{\epsilon_{\alpha\beta c}}{4f_\pi^2}\phi_\alpha\partial_0\phi_\beta\tau_c\ .
\end{aligned} \tag{523}
$$

Expanding the H fields

$$
\mathcal{L}_{B^{(*)}\pi\to B^{(*)}\pi} = -\frac{\epsilon_{\alpha\beta c}}{4f_\pi^2}\phi_\alpha\partial_0\phi_\beta\left[B_a^\dagger(\tau_c)_{ba}B_b + (B_j^\dagger)_a(\tau_c)_{ba}(B_j)_b\right] \tag{524}
$$

We disregard the vector mesons term because this exercise is limited to pseudoscalar scatterings.

$$
\mathcal{L}_{B\pi\to B\pi} = -\frac{\epsilon_{\alpha\beta c}}{4f_\pi^2}\phi_\alpha\partial_0\phi_\beta\left[B_a^\dagger(\tau_c)_{ba}B_b\right] \tag{525}
$$

Deriving the term in the square brackets for different cases of τ_c.
For $c = 1$,

$$\begin{aligned} B_a^\dagger(\tau_1)_{ba}B_b &= B_1^\dagger(\tau_1)_{21}B_2 + B_2^\dagger(\tau_1)_{12}B_1 \\ &= B^{0\dagger}B^+ + (B^+)^\dagger B^0 \end{aligned} \tag{526}$$

where $(\tau_1)_{21} = (\tau_1)_{12} = 1$.
Similarly for $c = 2$,

$$B_a^\dagger(\tau_1)_{ba}B_b = i(B^{0\dagger}B^+ - (B^+)^\dagger B^0 \tag{527}$$

and finally for $c = 3$,

$$B_a^\dagger(\tau_1)_{ba}B_b = B^{0\dagger}B^0 - (B^+)^\dagger B^+ \ . \tag{528}$$

Solving for π^+ as the initial and π^- final pions, one obtains for the non vanishing vertices,

$$\begin{aligned} v_{B^0\pi^+\to B^0\pi^+} &= \frac{i}{4f_\pi^2}(k_0' + k_0) \ , \\ v_{B^+\pi^+\to B^+\pi^+} &= -\frac{i}{4f_\pi^2}(k_0' + k_0) \ . \end{aligned} \tag{529}$$

where, k' and k are the momentum of the initial and final pions respectively .
Similarly, the vertices for all combinations of pions and B mesons

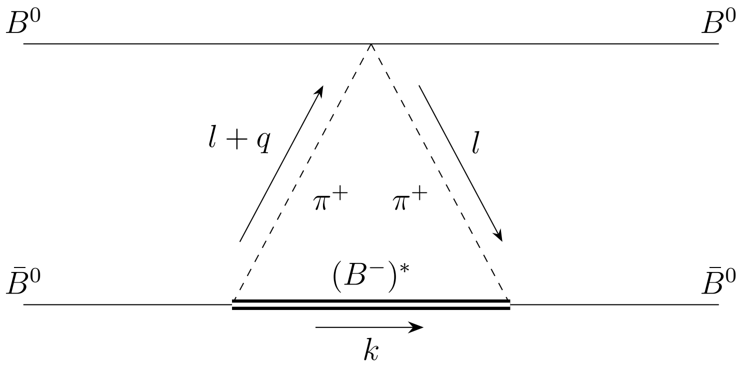
$$\begin{aligned} v_{B^0\pi^-\to B^0\pi^-} &= \frac{-i}{4f_\pi^2}(k_0' + k_0) \ , & v_{B^+\pi^-\to B^+\pi^-} &= \frac{i}{4f_\pi^2}(k_0' + k_0) \ , \\ v_{B^+\pi^0\to B^0\pi^+} &= \frac{i\sqrt{2}}{4f_\pi^2}(k_0' + k_0) \ , & v_{B^0\pi^0\to B^+\pi^-} &= \frac{-i\sqrt{2}}{4f_\pi^2}(k_0' + k_0) \ . \end{aligned}$$

Following the same steps, one can derive the corresponding vertices for the anti-mesons by replacing the τ matrices with the charge-conjugated counterparts.
Using these vertices, we express the triangle diagram for $B\bar{B} \to B\bar{B}$ and $BB \to BB$

G.3 Triangle Diagrams for $B\bar{B} \to B\bar{B}$

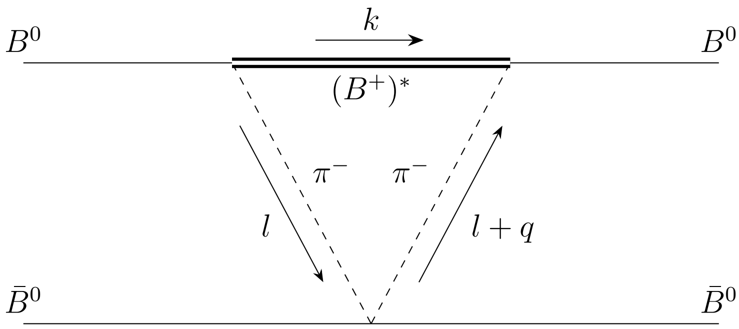
In this section, $B = (B^0, B^+)$ and $\bar{B} = (\bar{B}^0, B^-)$ and starting with $B^0\bar{B}^0 \to B^0\bar{B}^0$ scattering, which consists of two diagrams,

G.3.1 $B^0\bar{B}^0 \to B^0\bar{B}^0$



$$iV = \int \frac{d^4l}{(2\pi)^4}\frac{i}{4f_\pi^2}(l_0+q_0+l_0)\left(\frac{-g\sqrt{2}}{2f_\pi}(\vec{\epsilon}_j^{\,*}\,(l+q)_j)\right)\frac{i}{k^2-m_{B^*}^2}$$
$$\frac{i}{l^2-m_\pi^2}\left(\frac{-g\sqrt{2}}{2f_\pi}(\vec{\epsilon}_i\,(-l)_i)\right)\frac{i}{(l+q)^2-m_\pi^2} \quad (530)$$

The second diagram,

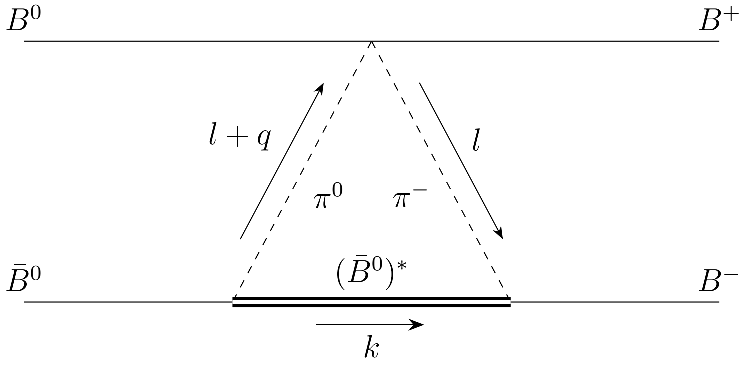


$$iV = \int \frac{d^4l}{(2\pi)^4}\frac{-i}{4f_\pi^2}(l_0+q_0+l_0)\left(\frac{g\sqrt{2}}{2f_\pi}(\vec{\epsilon}_j^{\,*}\,(l)_j)\right)\frac{i}{k^2-m_{B^*}^2}$$
$$\frac{i}{l^2-m_\pi^2}\left(\frac{g\sqrt{2}}{2f_\pi}(\vec{\epsilon}_i\,(-l-q)_i)\right)\frac{i}{(l+q)^2-m_\pi^2} \quad (531)$$

Adding the two diagrams,

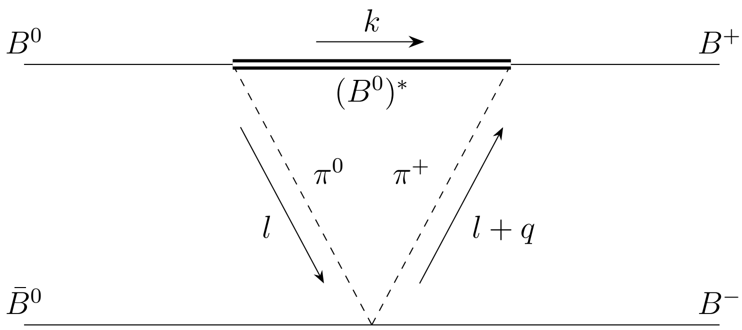
$$V_{tr}(B^0\bar{B}^0 \to B^0\bar{B}^0) = 0 \quad (532)$$

G.3.2 $B^0\bar{B}^0 \to B^+B^-$



$$iV = \int \frac{d^4l}{(2\pi)^4}\frac{-i\sqrt{2}}{4f_\pi^2}(2l_0+q_0)\left(\frac{g}{2f_\pi}(\vec{\epsilon_j}^*\,(l+q)_j)\right)\frac{i}{k^2-m_{B^*}^2}$$
$$\frac{i}{l^2-m_\pi^2}\left(\frac{g\sqrt{2}}{2f_\pi}(\vec{\epsilon_i}\,(-l)_i)\right)\frac{i}{(l+q)^2-m_\pi^2} \quad (533)$$

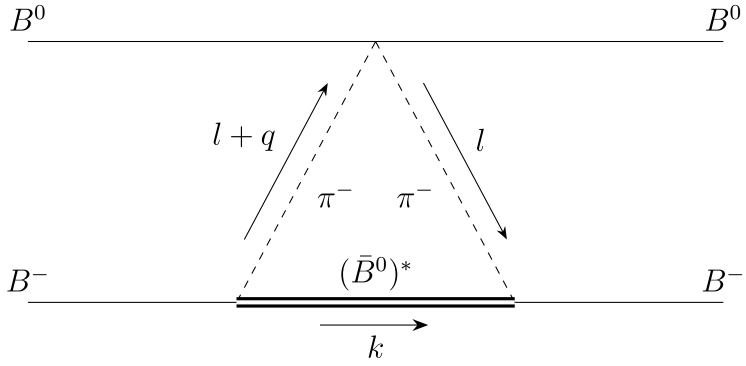
The second diagram,



$$iV = \int \frac{d^4l}{(2\pi)^4}\frac{i\sqrt{2}}{4f_\pi^2}(2l_0+q_0)\left(\frac{g}{2f_\pi}(\vec{\epsilon_j}^*\,(l)_j)\right)\frac{i}{k^2-m_{B^*}^2}$$
$$\frac{i}{l^2-m_\pi^2}\left(\frac{g\sqrt{2}}{2f_\pi}(\vec{\epsilon_i}\,(-l-q)_i)\right)\frac{i}{(l+q)^2-m_\pi^2} \quad (534)$$

Adding the two diagrams, $V_{tr}(B^0\bar{B}^0 \to B^+B^-) = 0$

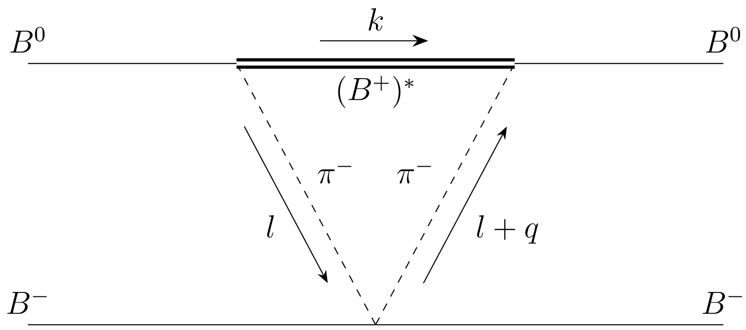
G.3.3 $B^0B^- \to B^0B^-$



$$iV = \int \frac{d^4l}{(2\pi)^4}\frac{-i}{4f_\pi^2}(2l_0+q_0)\left(\frac{g\sqrt{2}}{2f_\pi}(\vec{\epsilon}_j^{\,*}(l+q)_j)\right)\frac{i}{k^2-m_{B^*}^2}$$

$$\frac{i}{l^2-m_\pi^2}\left(\frac{g\sqrt{2}}{2f_\pi}(\vec{\epsilon}_i\,(-l)_i)\right)\frac{i}{(l+q)^2-m_\pi^2} \quad (535)$$

The second diagram,

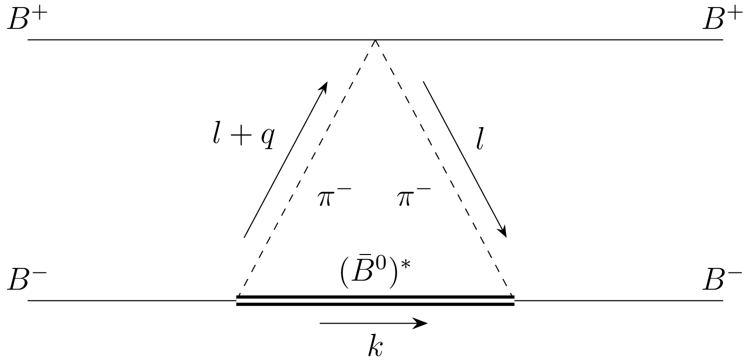


$$iV = \int \frac{d^4l}{(2\pi)^4}\frac{i}{4f_\pi^2}(2l_0+q_0)\left(\frac{g\sqrt{2}}{2f_\pi}(\vec{\epsilon}_j^{\,*}(l)_j)\right)\frac{i}{k^2-m_{B^*}^2}$$

$$\frac{i}{l^2-m_\pi^2}\left(\frac{g\sqrt{2}}{2f_\pi}(\vec{\epsilon}_i\,(-l-q)_i)\right)\frac{i}{(l+q)^2-m_\pi^2} \quad (536)$$

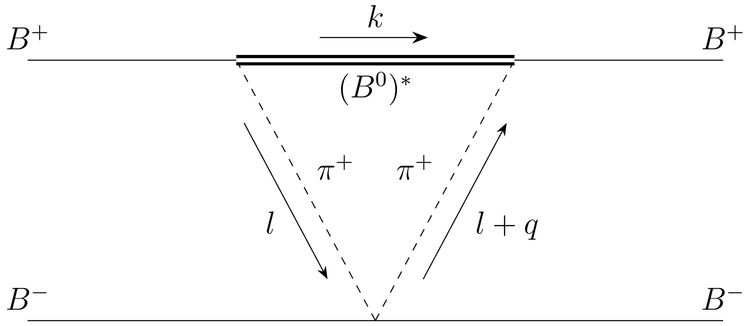
and, $V_{tr}(B^0B^- \to B^0B^-) = 0$

G.3.4 $B^+B^- \to B^+B^-$



$$iV = \int \frac{d^4l}{(2\pi)^4} \frac{i}{4f_\pi^2}(2l_0+q_0)\left(\frac{g\sqrt{2}}{2f_\pi}(\vec{\epsilon}_j^{\,*}\,(l+q)_j)\right)\frac{i}{k^2-m_{B^*}^2} \frac{i}{l^2-m_\pi^2}\left(\frac{g\sqrt{2}}{2f_\pi}(\vec{\epsilon}_i\,(-l)_i)\right)\frac{i}{(l+q)^2-m_\pi^2} \tag{537}$$

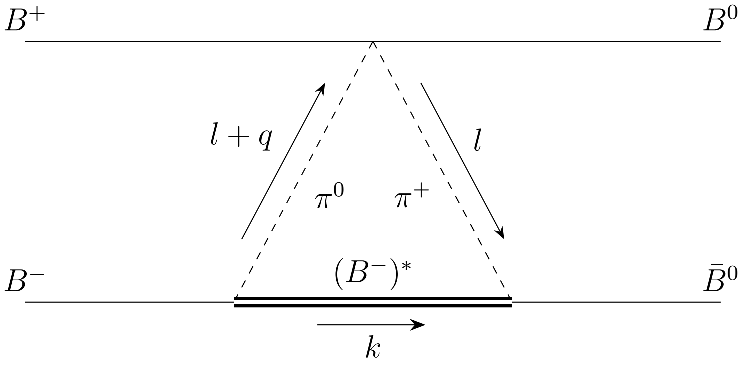
The second diagram,



$$iV = \int \frac{d^4l}{(2\pi)^4} \frac{-i}{4f_\pi^2}(2l_0+q_0)\left(\frac{g\sqrt{2}}{2f_\pi}(\vec{\epsilon}_j^{\,*}\,(l)_j)\right)\frac{i}{k^2-m_{B^*}^2} \frac{i}{l^2-m_\pi^2}\left(\frac{g\sqrt{2}}{2f_\pi}(\vec{\epsilon}_i\,(-l-q)_i)\right)\frac{i}{(l+q)^2-m_\pi^2} \tag{538}$$

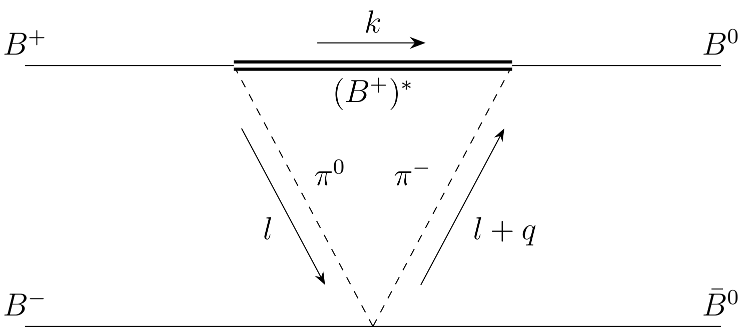
and the total contribution vanishes.

G.3.5 $B^+B^- \to B^0\bar{B}^0$



$$iV = \int \frac{d^4l}{(2\pi)^4}\frac{i\sqrt{2}}{4f_\pi^2}(2l_0+q_0)\left(\frac{-g}{2f_\pi}(\vec{\epsilon}_j^{\,*}\,(l+q)_j)\right)\frac{i}{k^2-m_{B^*}^2}\frac{i}{l^2-m_\pi^2}\left(\frac{g\sqrt{2}}{2f_\pi}(\vec{\epsilon}_i\,(-l)_i)\right)\frac{i}{(l+q)^2-m_\pi^2} \quad (539)$$

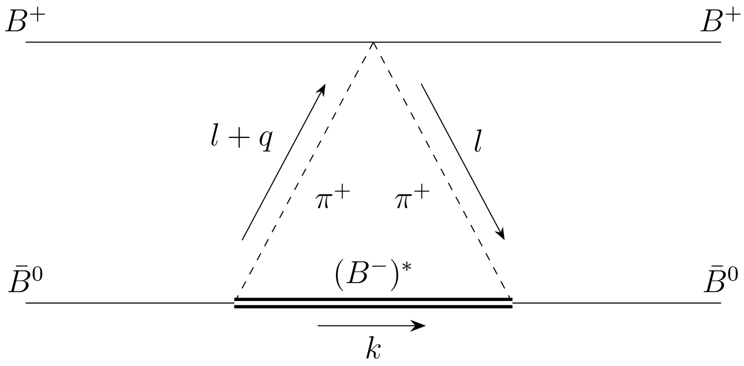
The second diagram,



$$iV = \int \frac{d^4l}{(2\pi)^4}\frac{-i\sqrt{2}}{4f_\pi^2}(2l_0+q_0)\left(\frac{-g}{2f_\pi}(\vec{\epsilon}_j^{\,*}\,(l)_j)\right)\frac{i}{k^2-m_{B^*}^2}\frac{i}{l^2-m_\pi^2}\left(\frac{g\sqrt{2}}{2f_\pi}(\vec{\epsilon}_i\,(-l-q)_i)\right)\frac{i}{(l+q)^2-m_\pi^2} \quad (540)$$

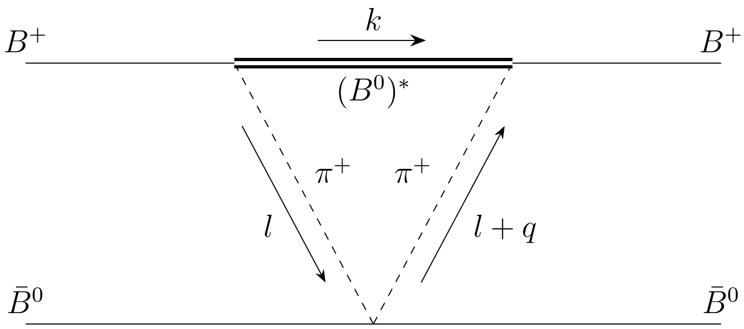
the total contribution is 0.

G.3.6 $B^+\bar{B}^0 \to B^+\bar{B}^0$



$$iV = \int \frac{d^4l}{(2\pi)^4}\frac{-i}{4f_\pi^2}(2l_0+q_0)\left(\frac{g\sqrt{2}}{2f_\pi}(\vec{\epsilon}_j^{\,*}\,(l+q)_j)\right)\frac{i}{k^2-m_{B^*}^2} \frac{i}{l^2-m_\pi^2}\left(\frac{g\sqrt{2}}{2f_\pi}(\vec{\epsilon}_i\,(-l)_i)\right)\frac{i}{(l+q)^2-m_\pi^2} \quad (541)$$

The second diagram,



$$iV = \int \frac{d^4l}{(2\pi)^4}\frac{i}{4f_\pi^2}(2l_0+q_0)\left(\frac{g\sqrt{2}}{2f_\pi}(\vec{\epsilon}_j^{\,*}\,(l)_j)\right)\frac{i}{k^2-m_{B^*}^2} \frac{i}{l^2-m_\pi^2}\left(\frac{g\sqrt{2}}{2f_\pi}(\vec{\epsilon}_i\,(-l-q)_i)\right)\frac{i}{(l+q)^2-m_\pi^2} \quad (542)$$

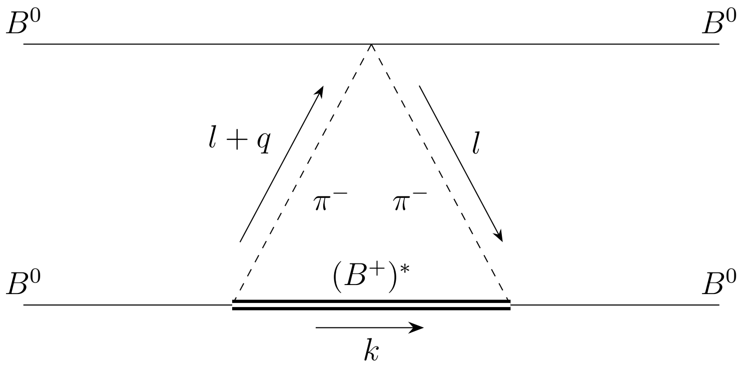
Adding the two diagrams gives 0

Therefore all the triangle diagrams cancel with each other for $B\bar{B} \to B\bar{B}$.

G.4 Triangle Diagrams for $BB \to BB$

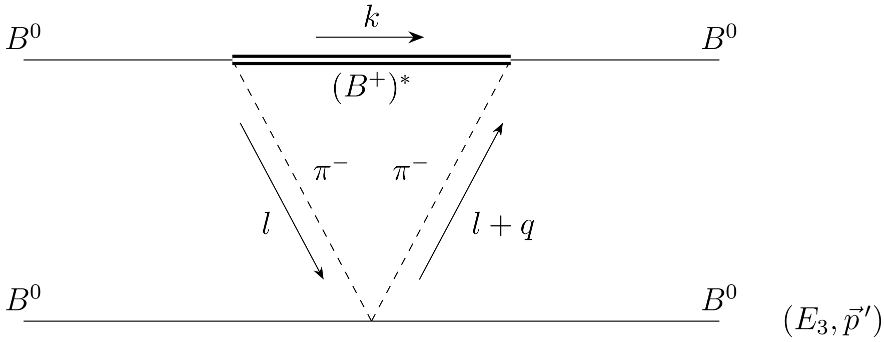
In this section, we express the triangle contribution for $BB \to BB$ scattering

G.4.1 $B^0 B^0 \to B^0 B^0$



$$iV = \int \frac{d^4 l}{(2\pi)^4} \frac{-i}{4f_\pi^2}(2l_0 + q_0)\left(\frac{g\sqrt{2}}{2f_\pi}(\vec{\epsilon}_j^{\,*}\,(l+q)_j)\right)\frac{i}{k^2 - m_{B^*}^2}$$
$$\frac{i}{l^2 - m_\pi^2}\left(\frac{g\sqrt{2}}{2f_\pi}(\vec{\epsilon}_i\,(-l)_i)\right)\frac{i}{(l+q)^2 - m_\pi^2} \quad (543)$$

The second diagram,

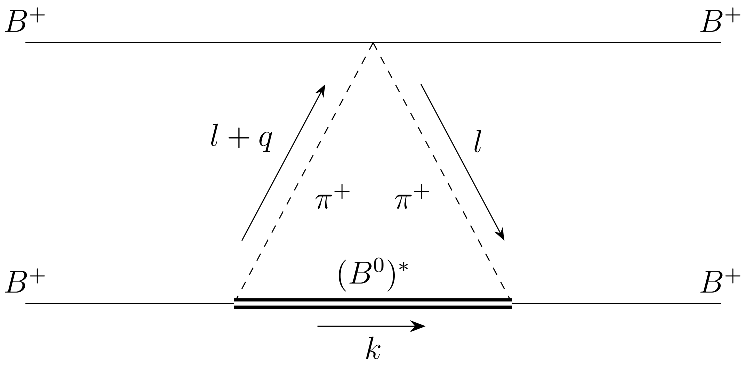


$$iV = \int \frac{d^4 l}{(2\pi)^4} \frac{-i}{4f_\pi^2}(2l_0 + q_0)\left(\frac{g\sqrt{2}}{2f_\pi}(\vec{\epsilon}_j^{\,*}\,(l)_j)\right)\frac{i}{k^2 - m_{B^*}^2}$$
$$\frac{i}{l^2 - m_\pi^2}\left(\frac{g\sqrt{2}}{2f_\pi}(\vec{\epsilon}_i\,(-l-q)_i)\right)\frac{i}{(l+q)^2 - m_\pi^2} \quad (544)$$

The total contribution is

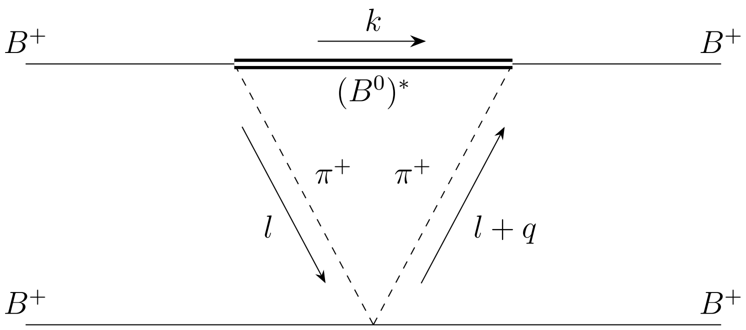
$$V_{tr}(B^0\bar{B}^0 \to B^0\bar{B}^0) = \int \frac{d^4l}{(2\pi)^4}\frac{-1}{2f_\pi^2}(2l_0+q_0)\left(\frac{g\sqrt{2}}{2f_\pi}(\vec{\epsilon_j}^*(l)_j)\right)\frac{i}{k^2-m_{B^*}^2}$$
$$\frac{i}{l^2-m_\pi^2}\left(\frac{g\sqrt{2}}{2f_\pi}(\vec{\epsilon_i}(-l-q)_i)\right)\frac{i}{(l+q)^2-m_\pi^2} \quad (545)$$

G.4.2 $B^+B^+ \to B^+B^+$



$$iV = \int \frac{d^4l}{(2\pi)^4}\frac{-i}{4f_\pi^2}(2l_0+q_0)\left(\frac{g\sqrt{2}}{2f_\pi}(\vec{\epsilon_j}^*(l+q)_j)\right)\frac{i}{k^2-m_{B^*}^2}$$
$$\frac{i}{l^2-m_\pi^2}\left(\frac{g\sqrt{2}}{2f_\pi}(\vec{\epsilon_i}(-l)_i)\right)\frac{i}{(l+q)^2-m_\pi^2} \quad (546)$$

The second diagram,

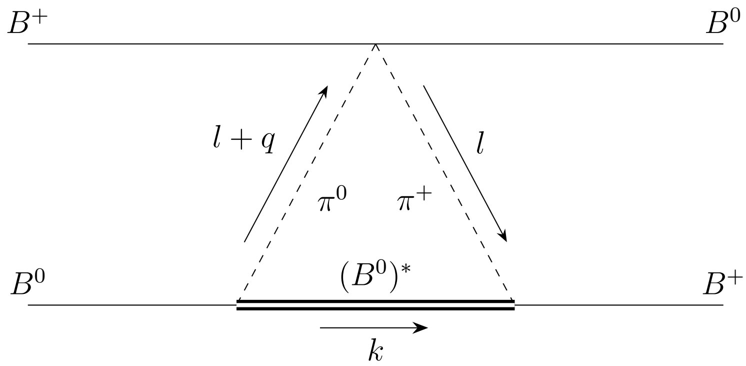


$$iV = \int \frac{d^4l}{(2\pi)^4} \frac{-i}{4f_\pi^2}(2l_0+q_0)\left(\frac{g\sqrt{2}}{2f_\pi}(\vec{\epsilon}_j^{\,*}(l)_j)\right)\frac{i}{k^2-m_{B^*}^2}$$
$$\frac{i}{l^2-m_\pi^2}\left(\frac{g\sqrt{2}}{2f_\pi}(\vec{\epsilon}_i(-l-q)_i)\right)\frac{i}{(l+q)^2-m_\pi^2} \quad (547)$$

Adding both the diagrams,

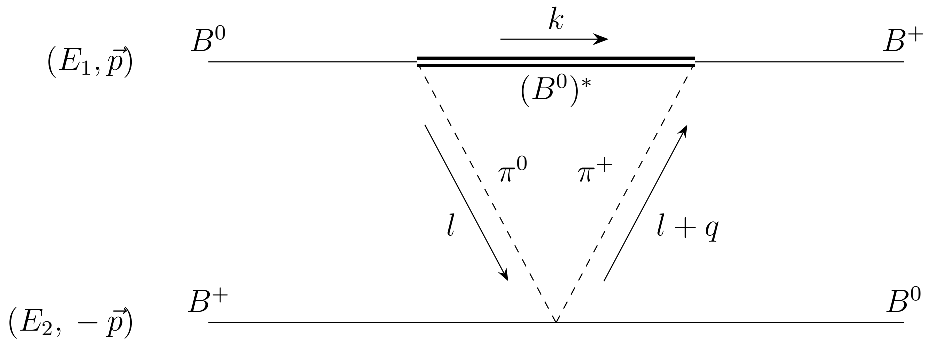
$$V_{tr}(B^+\bar{B}^0 \to B^+\bar{B}^0) = \int \frac{d^4l}{(2\pi)^4} \frac{-1}{2f_\pi^2}(2l_0+q_0)\left(\frac{g\sqrt{2}}{2f_\pi}(\vec{\epsilon}_j^{\,*}(l)_j)\right)\frac{i}{k^2-m_{B^*}^2}$$
$$\frac{i}{l^2-m_\pi^2}\left(\frac{g\sqrt{2}}{2f_\pi}(\vec{\epsilon}_i(-l-q)_i)\right)\frac{i}{(l+q)^2-m_\pi^2} \quad (548)$$

G.4.3 $B^+B^0 \to B^+B^0$



$$iV = \int \frac{d^4l}{(2\pi)^4} \frac{i\sqrt{2}}{4f_\pi^2}(2l_0+q_0)\left(\frac{g}{2f_\pi}(\vec{\epsilon}_j^{\,*}(l+q)_j)\right)\frac{i}{k^2-m_{B^*}^2}$$
$$\frac{i}{l^2-m_\pi^2}\left(\frac{g\sqrt{2}}{2f_\pi}(\vec{\epsilon}_i(-l)_i)\right)\frac{i}{(l+q)^2-m_\pi^2} \quad (549)$$

The second diagram,



$$iV = \int \frac{d^4l}{(2\pi)^4} \frac{i\sqrt{2}}{4f_\pi^2}(2l_0+q_0)\left(\frac{g}{2f_\pi}(\vec{\epsilon_j}^*(l)_j)\right)\frac{i}{k^2-m_{B^*}^2}$$
$$\frac{i}{l^2-m_\pi^2}\left(\frac{g\sqrt{2}}{2f_\pi}(\vec{\epsilon_i}(-l-q)_i)\right)\frac{i}{(l+q)^2-m_\pi^2} \quad (550)$$

with the total contribution

$$V_{tr}(B^+\bar{B}^0 \to B^+\bar{B}^0) = \int \frac{d^4l}{(2\pi)^4} \frac{\sqrt{2}}{2f_\pi^2}(2l_0+q_0)\left(\frac{g\sqrt{2}}{2f_\pi}(\vec{\epsilon_j}^*(l)_j)\right)\frac{i}{k^2-m_{B^*}^2}$$
$$\frac{i}{l^2-m_\pi^2}\left(\frac{g\sqrt{2}}{2f_\pi}(\vec{\epsilon_i}(-l-q)_i)\right)\frac{i}{(l+q)^2-m_\pi^2} \quad (551)$$

Therefore, we can conclude from this exercise that the triangle is contribution vanishes for $B\bar{B} \to B\bar{B}$ scattering and adds up in $BB \to BB$ scattering.

References

[1] P. Langacker.
"Introduction to the Standard Model and Electroweak Physics".
In: *Theoretical Advanced Study Institute in Elementary Particle Physics: The Dawn of the LHC Era*. 2010, pp. 3–48.
DOI: 10.1142/9789812838360_0001. arXiv: 0901.0241 [hep-ph].

[2] I. Adachi. "Observation of two charged bottomonium-like resonances".
In: *9th Conference on Flavor Physics and CP Violation*. May 2011.
arXiv: 1105.4583 [hep-ex].

[3] A. Bondar et al. "Observation of two charged bottomonium-like resonances in Y(5S) decays". In: *Phys. Rev. Lett.* 108 (2012), p. 122001.
DOI: 10.1103/PhysRevLett.108.122001. arXiv: 1110.2251 [hep-ex].

[4] I. Adachi et al. "Study of Three-Body Y(10860) Decays". In: Sept. 2012.
arXiv: 1209.6450 [hep-ex].

[5] A. Garmash et al.
"Observation of Zb(10610) and Zb(10650) Decaying to B Mesons".
In: *Phys. Rev. Lett.* 116.21 (2016), p. 212001.
DOI: 10.1103/PhysRevLett.116.212001. arXiv: 1512.07419 [hep-ex].

[6] M. Tanabashi et al. "Review of Particle Physics".
In: *Phys. Rev. D* 98.3 (2018), p. 030001.
DOI: 10.1103/PhysRevD.98.030001.

[7] A. E. Bondar et al. "Heavy quark spin structure in Z_b resonances".
In: *Phys. Rev. D* 84 (2011), p. 054010.
DOI: 10.1103/PhysRevD.84.054010. arXiv: 1105.4473 [hep-ph].

[8] Q. Wang et al. "Line shapes of the $Z_b(10610)$ and $Z_b(10650)$ in the elastic and inelastic channels revisited".
In: *Phys. Rev. D* 98 (7 Oct. 2018), p. 074023.
DOI: 10.1103/PhysRevD.98.074023.
URL: https://link.aps.org/doi/10.1103/PhysRevD.98.074023.

[9] V. Baru et al.
"Spin partners W_{bJ} from the line shapes of the $Z_b(10610)$ and $Z_b(10650)$".
In: *Phys. Rev. D* 99 (9 May 2019), p. 094013.
DOI: 10.1103/PhysRevD.99.094013.
URL: https://link.aps.org/doi/10.1103/PhysRevD.99.094013.

[10] H. Weyl. "Electron and Gravitation. 1. (In German)".
In: *Z. Phys.* 56 (1929), pp. 330–352. DOI: 10.1007/BF01339504.

[11] C.-N. Yang and R. L. Mills.
"Conservation of Isotopic Spin and Isotopic Gauge Invariance".
In: *Phys. Rev.* 96 (1954). Ed. by J.-P. Hsu and D. Fine, pp. 191–195.
DOI: 10.1103/PhysRev.96.191.

[12] H. Fritzsch, M. Gell-Mann, and H. Leutwyler.
"Advantages of the Color Octet Gluon Picture".
In: *Phys. Lett. B* 47 (1973), pp. 365–368.
DOI: 10.1016/0370-2693(73)90625-4.

[13] I. G. Kaplan. "The Pauli Exclusion Principle and the Problems of Its Experimental Verification". In: *Symmetry* 12.2 (2020), p. 320.
DOI: 10.3390/sym12020320. arXiv: 1902.00499 [quant-ph].

[14] S. L. Glashow. "Partial Symmetries of Weak Interactions".
In: *Nucl. Phys.* 22 (1961), pp. 579–588.
DOI: 10.1016/0029-5582(61)90469-2.

[15] S. Weinberg. "A Model of Leptons".
In: *Phys. Rev. Lett.* 19 (1967), pp. 1264–1266.
DOI: 10.1103/PhysRevLett.19.1264.

[16] A. Salam. "Weak and Electromagnetic Interactions".
In: *Conf. Proc. C* 680519 (1968), pp. 367–377.
DOI: 10.1142/9789812795915_0034.

[17] P. W. Higgs. "Broken symmetries, massless particles and gauge fields".
In: *Phys. Lett.* 12 (1964), pp. 132–133.
DOI: 10.1016/0031-9163(64)91136-9.

[18] P. W. Higgs.
"Spontaneous Symmetry Breakdown without Massless Bosons".
In: *Phys. Rev.* 145 (1966), pp. 1156–1163.
DOI: 10.1103/PhysRev.145.1156.

[19] Z. Maki, M. Nakagawa, and S. Sakata.
"Remarks on the unified model of elementary particles".
In: *Prog. Theor. Phys.* 28 (1962), pp. 870–880. DOI: 10.1143/PTP.28.870.

[20] B. Pontecorvo. "Neutrino Experiments and the Problem of Conservation of Leptonic Charge". In: *Zh. Eksp. Teor. Fiz.* 53 (1967), pp. 1717–1725.

[21] K. S. Hirata et al.
"Experimental Study of the Atmospheric Neutrino Flux".
In: *Phys. Lett. B* 205 (1988). Ed. by J. Tran Thanh Van, p. 416.
DOI: 10.1016/0370-2693(88)91690-5.

[22] M. E. Peskin and D. V. Schroeder.
An Introduction to quantum field theory.
Reading, USA: Addison-Wesley, 1995. ISBN: 978-0-201-50397-5.

[23] M. D. Schwartz. *Quantum Field Theory and the Standard Model.*
Cambridge University Press, Mar. 2014.
ISBN: 978-1-107-03473-0, 978-1-107-03473-0.

[24] K. Ottnad et al. "New insights into the neutron electric dipole moment".
In: *Phys. Lett. B* 687 (2010), pp. 42–47.
DOI: 10.1016/j.physletb.2010.03.005. arXiv: 0911.3981 [hep-ph].

[25] A. Ali and G. Kramer. "Jets and QCD: A Historical Review of the Discovery of the Quark and Gluon Jets and its Impact on QCD".
In: *The European Physical Journal H* 36.2 (2011), pp. 245–326.

[26] H. D. Politzer. "Reliable Perturbative Results for Strong Interactions?"
In: *Phys. Rev. Lett.* 30 (1973). Ed. by J. C. Taylor, pp. 1346–1349.
DOI: 10.1103/PhysRevLett.30.1346.

[27] D. J. Gross and F. Wilczek.
"Ultraviolet Behavior of Nonabelian Gauge Theories".
In: *Phys. Rev. Lett.* 30 (1973). Ed. by J. C. Taylor, pp. 1343–1346.
DOI: 10.1103/PhysRevLett.30.1343.

[28] D. J. Gross and F. Wilczek. "Asymptotically Free Gauge Theories. I".
In: *Phys. Rev. D* 8 (10 Nov. 1973), pp. 3633–3652.
DOI: 10.1103/PhysRevD.8.3633.
URL: https://link.aps.org/doi/10.1103/PhysRevD.8.3633.

[29] P. Skands. "Introduction to QCD".
In: *Searching for New Physics at Small and Large Scales: TASI 2012.*
World Scientific, 2013, pp. 341–420.

[30] M. L. Mangano. *Introduction to QCD.* 1999.

[31] T. Hebbeker. "Tests of quantum chromodynamics in hadronic decays of Z0 bosons produced in e+ e- annihilation".
In: *Physics reports* 217.2-3 (1992), pp. 69–157.

[32] C. Patrignani et al. "9.1. Basics". In: ().

[33] R. L. Workman et al. "Review of Particle Physics".
In: *PTEP* 2022 (2022), p. 083C01. DOI: 10.1093/ptep/ptac097.

[34] G. Zweig. "An SU(3) model for strong interaction symmetry and its breaking. Version 2". In: *DEVELOPMENTS IN THE QUARK THEORY OF HADRONS. VOL. 1. 1964 - 1978.*
Ed. by D. B. Lichtenberg and S. P. Rosen. Feb. 1964, pp. 22–101.

[35] M. Amaryan. "History and geography of light pentaquark searches: challenges and pitfalls". In: *Eur. Phys. J. Plus* 137.6 (2022), p. 684. DOI: 10.1140/epjp/s13360-022-02888-0. arXiv: 2201.04885 [hep-ex].

[36] F.-K. Guo et al. "Hadronic molecules". In: *Rev. Mod. Phys.* 90.1 (2018). [Erratum: Rev.Mod.Phys. 94, 029901 (2022)], p. 015004. DOI: 10.1103/RevModPhys.90.015004. arXiv: 1705.00141 [hep-ph].

[37] A. Esposito, A. Pilloni, and A. D. Polosa. "Multiquark Resonances". In: *Phys. Rept.* 668 (2017), pp. 1–97. DOI: 10.1016/j.physrep.2016.11.002. arXiv: 1611.07920 [hep-ph].

[38] S. L. Olsen, T. Skwarnicki, and D. Zieminska. "Nonstandard heavy mesons and baryons: Experimental evidence". In: *Rev. Mod. Phys.* 90.1 (2018), p. 015003. DOI: 10.1103/RevModPhys.90.015003. arXiv: 1708.04012 [hep-ph].

[39] L. Meng et al. "Chiral perturbation theory for heavy hadrons and chiral effective field theory for heavy hadronic molecules". In: *Phys. Rept.* 1019 (2023), pp. 1–149. DOI: 10.1016/j.physrep.2023.04.003. arXiv: 2204.08716 [hep-ph].

[40] F.-K. Guo, X.-H. Liu, and S. Sakai. "Threshold cusps and triangle singularities in hadronic reactions". In: *Prog. Part. Nucl. Phys.* 112 (2020), p. 103757. DOI: 10.1016/j.ppnp.2020.103757. arXiv: 1912.07030 [hep-ph].

[41] H.-X. Chen et al. "An updated review of the new hadron states". In: *Rept. Prog. Phys.* 86.2 (2023), p. 026201. DOI: 10.1088/1361-6633/aca3b6. arXiv: 2204.02649 [hep-ph].

[42] S. K. Choi et al. "Observation of a narrow charmonium-like state in exclusive $B^{\pm} \to K^{\pm}\pi^{+}\pi^{-}J/\psi$ decays". In: *Phys. Rev. Lett.* 91 (2003), p. 262001. DOI: 10.1103/PhysRevLett.91.262001. arXiv: hep-ex/0309032.

[43] M. Ablikim et al. "Observation of a Charged Charmoniumlike Structure in $e^{+}e^{-} \to \pi^{+}\pi^{-}J/\psi$ at $\sqrt{s}$ =4.26 GeV". In: *Phys. Rev. Lett.* 110 (2013), p. 252001. DOI: 10.1103/PhysRevLett.110.252001. arXiv: 1303.5949 [hep-ex].

[44] M. Ablikim et al. "Observation of a Charged Charmoniumlike Structure $Z_c(4020)$ and Search for the $Z_c(3900)$ in $e^{+}e^{-} \to \pi^{+}\pi^{-}h_c$". In: *Phys. Rev. Lett.* 111.24 (2013), p. 242001. DOI: 10.1103/PhysRevLett.111.242001. arXiv: 1309.1896 [hep-ex].

[45] N. Brambilla et al.
"The XYZ states: experimental and theoretical status and perspectives".
In: *Phys. Rept.* 873 (2020), pp. 1–154.
DOI: 10.1016/j.physrep.2020.05.001. arXiv: 1907.07583 [hep-ex].

[46] C. Hanhart. "Theory of hadronic molecules applied to the XYZ states".
In: *EPJ Web Conf.* 218 (2019). Ed. by A. Denig and C. .-. Redmer,
p. 08002. DOI: 10.1051/epjconf/201921808002.
arXiv: 1709.09920 [hep-ph].

[47] A. Ali, C. Hambrock, and W. Wang.
"Tetraquark interpretation of the charged bottomonium-like states
$Z_b^{\pm}(10610)$ and $Z_b^{\pm}(10650)$ and implications".
In: *Phys. Rev. D* 85 (5 Mar. 2012), p. 054011.
DOI: 10.1103/PhysRevD.85.054011.
URL: https://link.aps.org/doi/10.1103/PhysRevD.85.054011.

[48] M. B. Voloshin.
"Radiative transitions from Upsilon(5S) to molecular bottomonium".
In: *Phys. Rev. D* 84 (2011), p. 031502.
DOI: 10.1103/PhysRevD.84.031502. arXiv: 1105.5829 [hep-ph].

[49] T. Mehen and J. W. Powell. "Heavy Quark Symmetry Predictions for
Weakly Bound B-Meson Molecules".
In: *Phys. Rev. D* 84 (2011), p. 114013.
DOI: 10.1103/PhysRevD.84.114013. arXiv: 1109.3479 [hep-ph].

[50] V. Baru et al. "Spin partners of the Z_b (10610) and Z_b (10650) revisited".
In: *JHEP* 06 (2017), p. 158. DOI: 10.1007/JHEP06(2017)158.
arXiv: 1704.07332 [hep-ph].

[51] A. Manohar and H. Georgi.
"Chiral Quarks and the Nonrelativistic Quark Model".
In: *Nucl. Phys. B* 234 (1984), pp. 189–212.
DOI: 10.1016/0550-3213(84)90231-1.

[52] H. Georgi. "Effective field theory".
In: *Ann. Rev. Nucl. Part. Sci.* 43 (1993), pp. 209–252.
DOI: 10.1146/annurev.ns.43.120193.001233.

[53] B. Kubis. "An Introduction to chiral perturbation theory". In: *Workshop on Physics and Astrophysics of Hadrons and Hadronic Matter*. Mar. 2007.
arXiv: hep-ph/0703274.

[54] S. Scherer. "Introduction to chiral perturbation theory". In: *Adv. Nucl. Phys.* 27 (2003). Ed. by J. W. Negele and E. W. Vogt, p. 277.
arXiv: hep-ph/0210398.

[55] A. Pich. "Effective field theory: Course". In: *Les Houches Summer School in Theoretical Physics, Session 68: Probing the Standard Model of Particle Interactions*. June 1998, pp. 949–1049. arXiv: hep-ph/9806303.

[56] A. V. Manohar. "Effective field theories". In: *Lect. Notes Phys.* 479 (1997). Ed. by H. Latal and W. Schweiger, pp. 311–362. DOI: 10.1007/BFb0104294. arXiv: hep-ph/9606222.

[57] S. Scherer and M. R. Schindler. *A Chiral Perturbation Theory Primer*. 2005. arXiv: hep-ph/0505265 [hep-ph].

[58] D. B. Kaplan. "Effective field theories". In: *7th Summer School in Nuclear Physics Symmetries*. June 1995. arXiv: nucl-th/9506035.

[59] J. Gasser and H. Leutwyler. "Chiral Perturbation Theory: Expansions in the Mass of the Strange Quark". In: *Nucl. Phys. B* 250 (1985), pp. 465–516. DOI: 10.1016/0550-3213(85)90492-4.

[60] M. Gell-Mann. "The Eightfold Way: A Theory of strong interaction symmetry". In: (Mar. 1961). DOI: 10.2172/4008239.

[61] C. Vafa and E. Witten. "Restrictions on Symmetry Breaking in Vector-Like Gauge Theories". In: *Nucl. Phys. B* 234 (1984), pp. 173–188. DOI: 10.1016/0550-3213(84)90230-X.

[62] J. Goldstone, A. Salam, and S. Weinberg. "Broken Symmetries". In: *Phys. Rev.* 127 (1962), pp. 965–970. DOI: 10.1103/PhysRev.127.965.

[63] J. Goldstone. "Field Theories with Superconductor Solutions". In: *Nuovo Cim.* 19 (1961), pp. 154–164. DOI: 10.1007/BF02812722.

[64] S. Weinberg. "Effective Field Theory, Past and Future". In: *PoS* CD09 (2009), p. 001. DOI: 10.22323/1.086.0001. arXiv: 0908.1964 [hep-th].

[65] S. Weinberg. "Phenomenological Lagrangians". In: *Physica A* 96.1-2 (1979). Ed. by S. Deser, pp. 327–340. DOI: 10.1016/0378-4371(79)90223-1.

[66] J. F. Amundson et al. "Radiative D* decay using heavy quark and chiral symmetry". In: *Phys. Lett. B* 296 (1992), pp. 415–419. DOI: 10.1016/0370-2693(92)91341-6. arXiv: hep-ph/9209241.

[67] M. Knecht et al. “The Low-energy pi pi amplitude to one and two loops”. In: *Nucl. Phys. B* 457 (1995), pp. 513–576. DOI: 10.1016/0550-3213(95)00515-3. arXiv: hep-ph/9507319.

[68] J. Gasser and H. Leutwyler. “Chiral Perturbation Theory to One Loop”. In: *Annals Phys.* 158 (1984), p. 142. DOI: 10.1016/0003-4916(84)90242-2.

[69] T.-M. Yan et al. “Heavy quark symmetry and chiral dynamics”. In: *Phys. Rev. D* 46 (1992). [Erratum: Phys.Rev.D 55, 5851 (1997)], pp. 1148–1164. DOI: 10.1103/PhysRevD.46.1148.

[70] M. Gell-Mann, R. J. Oakes, and B. Renner. “Behavior of current divergences under SU(3) x SU(3)”. In: *Phys. Rev.* 175 (1968), pp. 2195–2199. DOI: 10.1103/PhysRev.175.2195.

[71] N. Isgur and M. B. Wise. “WEAK TRANSITION FORM-FACTORS BETWEEN HEAVY MESONS”. In: *Phys. Lett. B* 237 (1990), pp. 527–530. DOI: 10.1016/0370-2693(90)91219-2.

[72] N. Isgur and M. B. Wise. “Weak Decays of Heavy Mesons in the Static Quark Approximation”. In: *Phys. Lett. B* 232 (1989), pp. 113–117. DOI: 10.1016/0370-2693(89)90566-2.

[73] B. Grinstein. “The Static Quark Effective Theory”. In: *Nucl. Phys. B* 339 (1990), pp. 253–268. DOI: 10.1016/0550-3213(90)90349-I.

[74] H. Georgi. “An Effective Field Theory for Heavy Quarks at Low-energies”. In: *Phys. Lett. B* 240 (1990), pp. 447–450. DOI: 10.1016/0370-2693(90)91128-X.

[75] A. F. Falk, B. Grinstein, and M. E. Luke. “Leading mass corrections to the heavy quark effective theory”. In: *Nucl. Phys. B* 357 (1991), pp. 185–207. DOI: 10.1016/0550-3213(91)90464-9.

[76] M. B. Wise. “Chiral perturbation theory for hadrons containing a heavy quark”. In: *Phys. Rev. D* 45.7 (1992), R2188. DOI: 10.1103/PhysRevD.45.R2188.

[77] H. Georgi. “Heavy quark effective field theory”. In: *Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 91): Perspectives in the Standard Model*. Aug. 1991.

[78] B. Grinstein et al.
"Chiral perturbation theory for f D(s) / f D and B B(s) / B B".
In: *Nucl. Phys. B* 380 (1992), pp. 369–376.
DOI: 10.1016/0550-3213(92)90248-A. arXiv: hep-ph/9204207.

[79] T. Mannel, W. Roberts, and Z. Ryzak.
"A Derivation of the heavy quark effective Lagrangian from QCD".
In: *Nucl. Phys. B* 368 (1992), pp. 204–217.
DOI: 10.1016/0550-3213(92)90204-O.

[80] J. Hu and T. Mehen.
"Chiral Lagrangian with heavy quark-diquark symmetry".
In: *Phys. Rev. D* 73 (2006), p. 054003.
DOI: 10.1103/PhysRevD.73.054003. arXiv: hep-ph/0511321.

[81] A. V. Manohar and M. B. Wise. *Heavy quark physics*. Vol. 10. 2000.
ISBN: 978-0-521-03757-0.

[82] E. Eichten and B. R. Hill. "An Effective Field Theory for the Calculation of Matrix Elements Involving Heavy Quarks".
In: *Phys. Lett. B* 234 (1990), pp. 511–516.
DOI: 10.1016/0370-2693(90)92049-O.

[83] G. Burdman and J. F. Donoghue.
"Union of chiral and heavy quark symmetries".
In: *Phys. Lett. B* 280 (1992), pp. 287–291.
DOI: 10.1016/0370-2693(92)90068-F.

[84] C. G. Boyd and B. Grinstein.
"Chiral and heavy quark symmetry violation in B decays".
In: *Nucl. Phys. B* 442 (1995), pp. 205–227.
DOI: 10.1016/S0550-3213(95)00005-4. arXiv: hep-ph/9402340.

[85] E. E. Jenkins. "Heavy meson masses in chiral perturbation theory with heavy quark symmetry". In: *Nucl. Phys. B* 412 (1994), pp. 181–200.
DOI: 10.1016/0550-3213(94)90499-5. arXiv: hep-ph/9212295.

[86] C. G. Callan Jr. et al. "Structure of phenomenological Lagrangians. 2."
In: *Phys. Rev.* 177 (1969), pp. 2247–2250.
DOI: 10.1103/PhysRev.177.2247.

[87] H.-Y. Cheng et al.
"Chiral Lagrangians for radiative decays of heavy hadrons".
In: *Phys. Rev. D* 47 (1993), pp. 1030–1042.
DOI: 10.1103/PhysRevD.47.1030. arXiv: hep-ph/9209262.

[88] M. Pavón Valderrama. “Power counting and perturbative one pion exchange in heavy meson molecules”. In: *Phys. Rev. D* 85 (11 June 2012), p. 114037. DOI: 10.1103/PhysRevD.85.114037. URL: https://link.aps.org/doi/10.1103/PhysRevD.85.114037.

[89] C. HANHART. “Meson production in nucleon?nucleon collisions close to the threshold”. In: *Physics Reports* 397.3–4 (July 2004), pp. 155–256. ISSN: 0370-1573. DOI: 10.1016/j.physrep.2004.03.007. URL: http://dx.doi.org/10.1016/j.physrep.2004.03.007.

[90] V. Baru, C. Hanhart, and F. Myhrer. “Effective field theory calculations of $NN \to NN\pi$”. In: *Int. J. Mod. Phys. E* 23.4 (2014), p. 1430004. DOI: 10.1142/S0218301314300045. arXiv: 1310.3505 [nucl-th].

[91] C. Hanhart and N. Kaiser. “Complete next-to-leading order calculation for pion production in nucleon-nucleon collisions at threshold”. In: *Phys. Rev. C* 66 (2002), p. 054005. DOI: 10.1103/PhysRevC.66.054005. arXiv: nucl-th/0208050.

[92] G. Leibbrandt. “Introduction to the Technique of Dimensional Regularization”. In: *Rev. Mod. Phys.* 47 (1975), p. 849. DOI: 10.1103/RevModPhys.47.849.

[93] R. Machleidt and D. R. Entem. “Chiral effective field theory and nuclear forces”. In: *Phys. Rept.* 503 (2011), pp. 1–75. DOI: 10.1016/j.physrep.2011.02.001. arXiv: 1105.2919 [nucl-th].

[94] S. Weinberg. “Effective chiral Lagrangians for nucleon - pion interactions and nuclear forces”. In: *Nucl. Phys. B* 363 (1991), pp. 3–18. DOI: 10.1016/0550-3213(91)90231-L.

[95] N. Kaiser, R. Brockmann, and W. Weise. “Peripheral nucleon-nucleon phase shifts and chiral symmetry”. In: *Nucl. Phys. A* 625 (1997), pp. 758–788. DOI: 10.1016/S0375-9474(97)00586-1. arXiv: nucl-th/9706045.

[96] T. Mehen and J. W. Powell. “Line shapes in $\Upsilon(5S) \to B^{(*)}\overline{B}^{(*)}\pi$ with $Z(10610)$ and $Z(10650)$ using effective field theory”. In: *Phys. Rev. D* 88 (3 Aug. 2013), p. 034017. DOI: 10.1103/PhysRevD.88.034017. URL: https://link.aps.org/doi/10.1103/PhysRevD.88.034017.

[97] F. Bernardoni et al.
"Precision lattice QCD computation of the $B^*B\pi$ coupling".
In: *Phys. Lett. B* 740 (2015), pp. 278–284.
DOI: 10.1016/j.physletb.2014.11.051. arXiv: 1404.6951 [hep-lat].

[98] S. Fleming et al. "Pion interactions in the $X(3872)$".
In: *Phys. Rev. D* 76 (3 Aug. 2007), p. 034006.
DOI: 10.1103/PhysRevD.76.034006.
URL: https://link.aps.org/doi/10.1103/PhysRevD.76.034006.

[99] R. Casalbuoni et al. "Phenomenology of heavy meson chiral Lagrangians".
In: *Phys. Rept.* 281 (1997), pp. 145–238.
DOI: 10.1016/S0370-1573(96)00027-0. arXiv: hep-ph/9605342.

[100] V. BERNARD, N. KAISER, and U.-G. MEIßNER.
"CHIRAL DYNAMICS IN NUCLEONS AND NUCLEI".
In: *International Journal of Modern Physics E* 04.02 (June 1995),
pp. 193–344. ISSN: 1793-6608. DOI: 10.1142/s0218301395000092.
URL: http://dx.doi.org/10.1142/S0218301395000092.

[101] B. Wang, L. Meng, and S.-L. Zhu. "Deciphering the charged heavy quarkoniumlike states in chiral effective field theory".
In: *Phys. Rev. D* 102 (2020), p. 114019.
DOI: 10.1103/PhysRevD.102.114019. arXiv: 2009.01980 [hep-ph].

[102] B. Wang, Z.-W. Liu, and X. Liu.
"$\bar{B}^{(*)}\bar{B}^{(*)}$ interactions in chiral effective field theory".
In: *Phys. Rev. D* 99.3 (2019), p. 036007.
DOI: 10.1103/PhysRevD.99.036007. arXiv: 1812.04457 [hep-ph].