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A note on source term representation for control-and-state-constrained parabolic control problems with purely time-dependent control

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A note on source term representation for control-and-state-constrained parabolic control problems with purely time-dependent control

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We revisit the concept of source term representation, originally developed for boundary control problems subject to pointwise state constraints in the domain, and apply it to linear-quadratic parabolic control problems with purely time-dependent control subject to pointwise state constraints and control bounds. Since control and state are defined on different domains, classical Lavrentiev regularization cannot be applied. Instead, in the source term representation ansatz, the adjoint operator is applied to an auxiliary control that is defined in the same domain as the state. By this ansatz, additional pure control bounds would be transformed into artificial state constraint and were therefore previously excluded from the discussion. For purely time dependent control we observe that control constraints will become averaged in space and pointwise in time state constraints, and can apply available theory for the existence of Lagrange multipliers. We briefly visit elliptic problems with finitely many control parameters, where this regularization of pointwise state constraints transforms additional simple bounds on the control parameters into easy-to-handle integral-state constraints.

1 Introduction

In this paper, we revisit a source term regularization method developed for boundary control problems with pointwise state constraints, cf. [38, 37] for elliptic problems, or the work [33] by Tröltzsch and the author for parabolic optimal problems. We will apply

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this to a parabolic problem with purely time dependent control and pointwise state constraints as well as an additional bound on the control, and also visit briefly an elliptic counterpart with finitely many control parameters. Additional control constraints could not be handled in the original works on boundary control. In this way, we will be able to narrow an interesting theoretical gap in the theory of this regularization concept.

More precisely, we will be concerned with the following simple parabolic model problem, whose complete functional analytic setting will be defined in the next section.

$$\min J(y,u) = \frac{1}{2}\|y - y_d\|^2_{L^2(Q)} + \frac{\nu}{2}\|u\|^2_{L^2(0,T)} \tag{1.1}$$

subject to the heat equation

$$\begin{aligned} y_t - \Delta y &= u(t)e(x) && \text{in } Q := \Omega \times (0,T) \\ y &= 0 && \text{in } \Sigma := \Gamma \times (0,T) \\ y(\cdot,0) &= 0 && \text{in } \Omega, \end{aligned} \tag{1.2}$$

and constraints on the state as well as the control

$$y_a \le y \le y_b \quad \text{in } Q, \quad u \le u_b \quad \text{in } (0,T). \tag{1.3}$$

Here, the control u is a purely time-dependent function $u\colon (0,T) \mapsto \mathbb{R}$ and for a given fixed shape function $e\colon \Omega \mapsto \mathbb{R}$ we set

$$Bu = u(t)e(x).$$

We note in passing that the results in this manuscript can be extended to more than one time dependent control, i.e. a time dependent control vector

$$u\colon (0,T) \mapsto \mathbb{R}^m, \quad Bu = \sum_{i=1}^m u_i(t)e_i(x),$$

for $m > 1$ given shape functions $e_i\colon \Omega \to \mathbb{R}$.

Problems with state constraints are meanwhile well-investigated. The main challenges of pointwise state constraints are due to the fact that a constraint qualification is usually needed to obtain Lagrange multipliers for the state constraints in a Karush-Kuhn-Tucker system. Often, one relies on the Slater condition, which requires the existence of a strictly feasible point with respect to the state constraints, and usually continuity of the state for all admissible controls as in [5, 6] or [34]. Then, the Lagrange multipliers are usually obtained in the space of regular Borel measures. For elliptic problems, improved regularity of the multiplier under certain conditions has been obtained in [9]. For parabolic problems, the question of existence of Lagrange multipliers is more delicate than in the elliptic setting and might lead to restrictions on the spatial dimension or regularity requirements on the control, not only for the existence of Lagrange multipliers but also for second order sufficient conditions, see e.g. [12]. However, even the discussion of semilinear problems, [12] or certain quasilinear problems, see [18, 8] is meanwhile possible.

Very recently, results on improved regularity without Slater point assumptions but rather compatibility conditions between the bound and the differential operator were obtained in [10]. Let us also refer to the introduction of these last more recent papers for further references, also including e.g. finite element error analysis, which is not within the scope of the present paper.

Different regularization concepts have been studied in a variety of contexts. It is probably safe to say that the Moreau-Yosida regularization concept, see e.g. [19], is often the method of choice, yet it removes the constraints by a penalization method. We also like to point out [1, 2, 3, 21, 16]. Another regularization concept, that explicitly keeps certain constraints, is the classical Lavrentiev regularization originally introduced for elliptic problems with distributed controls by Meyer, Rösch, and Tröltzsch in [31], and Meyer and Tröltzsch in [32], leading to mixed pointwise control-state constraints. Since this method is not applicable for e.g. boundary control problems with state constraints in the domain, further methods have been developed for this problem class.

The source term extension of Lavrentiev regularization to the case of elliptic or parabolic boundary control and pointwise constraints in the domain, including a limit analysis, was suggested in [37, 38] and [33], respectively. Let us also mention the virtual control concept from [20]. Here, we apply the idea of source term representation. For purely time dependent control, as for boundary control, it is not possible to consider the expression $w = \lambda u + y$ since u is defined on the time interval $I := (0, T)$, while y is defined on the whole space time domain Q. The main idea from [37, 38, 33] was to introduce an auxiliary control $v \in L^2(Q)$ and using for u the ansatz $u = S^*v$, where S denotes the control to state mapping for the state equation with range in $L^2(Q)$. The motivation behind this choice comes from the fact that in case a Lagrange multiplier for the state constraints exists, it will often be a measure entering the adjoint equation, and one will obtain a coupling of the optimal control $\bar{u}$ with this adjoint state. By the ansatz $u = S^*v$, smoothing is performed. This idea can readily be applied to different control domains as in our model problem (1.1)-(1.3). Then the state $y = y(v)$ is given by $y = SS^*v$ and the state constraint $y_a \leq SS^*v \leq y_b$ can be relaxed by

$$y_a \leq \lambda v + SS^*v \leq y_b,$$

similar to classical Lavrentiev regularization, cf. for instance [31]. Note that then the control bound $u \leq u_b$ is transformed into

$$S^*v \leq u_b,$$

i.e. an artificial state constraint is obtained, and all difficulties mentioned previously will in principle apply to this problem. However, in our model problem formulation due to having purely time-dependent controls, these constraints will only hold in time. We will see that in fact

$$\int_\Omega z(x,t)e(x)dx \leq u_b(t)$$

is obtained, where z is a solution of an adjoint equation with right-hand-side $v \in L^2(Q)$. Due to regularity properties of parabolic PDEs, these constraints can be handled pointwise in time. In [27], similar problems of this type have been analyzed with respect to finite element discretization error estimates, and first order optimality conditions of Karush-Kuhn-Tucker type have been used.

The analysis of this problem closes a gap in the analysis of source term representation relying on a specific, yet practically typical structure of the controls. Pointwise state constraints, that cause difficulties even with respect to existence of Lagrange multipliers are transformed into more regular mixed constraints with regular multipliers. On the downside, simple control bounds are transformed into more difficult state constraints, but of integral type in space pointwise in time. Rating the different types of constraints by their difficulty, both the easiest and the hardest type of constraint are transformed into constraints with medium difficulty, so to speak. In a similar way, this is the case for elliptic problems, that we will briefly discuss at the end of the paper. While the analysis of the resulting problems does not pose particularly new challenges, they have not been analysed as a result of a regularization strategy.

To the best of the author's knowledge, this direct application of source term representation regularization to time-dependent controls has not been considered in the literature, even though problems with purely-time-dependent controls and pointwise state constraints have been analyzed in a variety of settings. We only mention results on second order sufficient conditions for semilinear problems in [12], or quasilinear problems in [4, 18]. For finite element error analysis we refer to [17]. They are also often considered in the context of model order reduction such as proper orthogonal decomposition. In this context, pointwise state constraints and boundary control have been analyzed by a virtual control regularization, see [26]. Mixed constraints similar to our setting, but transformed into the control space $U = L^2(0, T)$, have been considered in [15].

Before we analyse our model problem in detail, let us note that in practice of course also pointwise in time state constraints need some attention in the numerical treatment, such as regularization.

2 Assumptions and challenges of the unregularized problem

2.1 Preliminaries

In this paper, we agree on the following notation and general assumptions that we choose close to the ones considered in [33] on parabolic boundary control problems, since the application of the regularization technique is the main point of our paper. We nevertheless point out that parabolic PDEs can also be discussed on rough domains for instance, see e.g. [28]. We collect all assumptions needed at some point in the paper, and point out that some results require less regularity of the given data.

Assumption 2.1. *The set $\Omega \subset \mathbb{R}^n$, $n \in \{1, 2, 3\}$, is a nonempty, convex polygonal* *or polyhedral domain in $\mathbb{R}^2$ or $\mathbb{R}^3$, respectively, or an intervall in $\mathbb{R}$. The real number $T > 0$ is a fixed final time, and we denote the time interval by $I := (0, T)$. Further, we have positive real numbers ν, λ, and ε, functions $y_d \in L^2(Q), y_0 \in H_0^1(\Omega) \cap C(\bar{\Omega})$, $e \in L^2(\Omega)$ and bounds $y_a, y_b \in C(\bar{Q}) \cap L^2(0, T, V)$, $y_a < y_b$ in Q, $y_a < 0 < y_b$ on Γ, $y_a(\cdot, 0) < y_0 < y(\cdot, 0)$, and $u_b \in C([0, T])$.*

We will denote $V = H_0^1(\Omega)$, and by $H = L^2(\Omega)$, so that we obtain the Gelfand triple property $V \hookrightarrow H \hookrightarrow V^*$ with dense, compact, and continuous embeddings. The state space Y will be given by

$$Y := W(0, T) = \{v \colon I \times \Omega \to \mathbb{R} | v \in L^2(I, V) \text{ and } v_t \in L^2(0, T, V^*)\}.$$

The embedding $W(0, T) \hookrightarrow L^2(Q)$ is compact, cf. for instance [23], and we also have the well-known continuous embedding $W(0, T) \hookrightarrow C([0, T], H)$, [39]. The latter is important for our further considerations of (transformed) control constraints for the regularized problem. For the control space, we set $U = L^2(0, T)$. Moreover, let us define the set of admissible controls $U_{\text{ad}} = \{u \in L^2(0, T)\} \colon u \le u_b$ a. e. in $(0, T)\}$ and the set of feasible controls, $U_{\text{feas}} = \{u \in U_{\text{ad}} \colon y_a \le Su \le y_b$ a. e. in $Q\}$. Last, inner products and norms will be abbreviated as follows

$$\begin{array}{lllllll} (v, w) & := & (v, w)_{L^2(\Omega)}, & \|v\| & := & \|v\|_{L^2(\Omega)}, \\ (v, w)_I & := & (v, w)_{L^2(0,T)}, & \|v\|_I & := & \|v\|_{L^2(0,T)}, \\ (v, w)_Q & := & (v, w)_{L^2(Q)}, & \|v\|_I & := & \|v\|_{L^2(Q)}. \end{array} \tag{2.1}$$

All other norms will be indicated by a corresponding subscript. The duality pairings will be denoted by $\langle \cdot, \cdot, \rangle$ with corresponding subscripts. For the duality pairing between $C([0, T])$ and $C([0, T])^*$ we use the short notation $\langle \cdot, \cdot \rangle$ without further subscript. Before analyzing our specific control problem, let us discuss solution theory for the parabolic equations.

Proposition 2.2. *Let Assumption 2.1 hold. For every $f \in L^2(0, T, V^*)$ and initial state $y_0 \in H$, the initial boundary value problem*

$$\begin{aligned} \int_0^T \langle\, y_t, \varphi \rangle_{V^*,V}\, dt + (\nabla y, \nabla \varphi)_Q = (f, \varphi)_Q \quad \forall\, \varphi \in L^2(0, T, V), \\ y(0, \cdot) = y_0 \quad \text{in } \Omega, \end{aligned} \tag{2.2}$$

admits a unique weak solution $y \in W(0, T)$. For $f \in L^2(0, T, H)$ and $y_0 \in V$, the improved regularity $y \in L^2(0, T, H^2(\Omega) \cap V) \cap L^\infty(0, T, V) \cap H^1(0, T, H)$ holds. If $f \in L^r(Q)$ with $r > n/2 + 1$ and $y_0 \in C(\bar{\Omega})$, we obtain $y \in C(\bar{Q})$.

In these cases, the norm estimates

$$\begin{array}{rcl} \|y\|_Q + \|y\|_{L^\infty(0,T,H)} + \|y\|_{W(0,T)} & \le & c\,(\|f\|_{L^2(0,T,V^*)} + \|y_0\|), \\ \|\nabla y\|_{L^\infty(0,T,H} + \|\nabla^2 y\|_Q + \|y_t\|_Q & \le & c\,(\|f\|_I + \|y_0\|_V), \\ \|y\|_{C(\bar{Q})} & \le & c(r)(\|f\|_{L^p(Q)} + \|y_0\|_{C(\bar{\Omega})}). \end{array} \tag{2.3}$$

hold with constants $c, c(r) > 0$.

Proof. For the existence of a unique solution $y \in W(0,T)$ we refer to [22] or [11]. Continuity of solutions was discussed in [7],[34], or [36], the regularity $y \in L^2(0,T,H^2(\Omega)\cap V)\cap L^\infty(0,T,V)\cap H^1(0,T,H)$ is proven in [13]. We also point out the existence theorems in [15] and [27] for problems close to our model problem. $\square$

We note in passing that the existence and regularity results from Proposition 2.2 also hold for backward parabolic PDEs such as

$$\begin{aligned} -p_t - \Delta p &= f && \text{in } Q, \\ p(T,\cdot) &= p_T && \text{in } \Omega, \\ p &= 0 && \text{on } \Sigma \end{aligned} \tag{2.4}$$

by a simple time transformation argument. Equations of this type will appear in the optimality conditions and the regularized problem formulation. If measures are present in the right-hand side, i.e.

$$\begin{aligned} -p_t + \Delta p &= \mu_Q && \text{in } Q, \\ p(T,\cdot) &= \mu_T && \text{in } \Omega, \\ p &= 0 && \text{on } \Sigma. \end{aligned} \tag{2.5}$$

existence results are obtained in [7]. From there, it is known that for $\mu \in C_0(\bar{Q})^*$, (2.5) admits a unique solution $p \in L^\tau(I, W_0^{1,s}(\Omega))$ for all $\tau, s \in [1,2)$ with $(2/\tau)+(n/s) > n+1$.

2.2 Existence of solutions and optimality conditions

After the preliminary discussion we now observe for our parabolic model problem (1.1)-(1.3) that for every control $u \in L^2(0,T)$, we obtain a unique state $y \in W(0,T)$, which is not necessarily continuous. Let us denote by

$$\mathcal{G}\colon L^2(Q) \to W(0,T) \hookrightarrow C([0,T],H)$$

the solution operator of the state equation with right-hand-side from $L^2(Q)$, such that the control-to-state mapping G is obtained by concatenation of $\mathcal{G}$ and B, the latter being defined by

$$B\colon L^2(0,T) \to L^2(Q), \quad B\varphi = \int_\Omega \varphi e(x)dx,$$

i.e.

$$G\colon L^2(0,T) \to W(0,T) \hookrightarrow C([0,T],H), \quad G = \mathcal{G}B.$$

In what follows, we will denote by $\mathcal{S}$ the operator $\mathcal{G}$ with range in $L^2(Q)$ and by S the control-to-state mapping G with range in $L^2(Q)$. We can rewrite the problem in its usual reduced form:

$$\min\, f(u) := J(Su,u) \quad \text{s.t. } y_a \le Su \le y_b \quad \text{a.e. in } Q, \quad u \le u_b \quad \text{a.e. in } (0,T). \tag{P}$$

Note that here both types of constraints are prescribed in an L^2 almost everywhere sense and L^2 regularity of y_0, u_b, y_a, y_b would be sufficient. By standard arguments we can then prove existence of a unique optimal control $\bar{u} \in L^2(0,T)$ with associated optimal state $\bar{y} \in W(0,T)$, provided that the feasible set U_{feas} is not empty. To obtain optimality conditions of Karush-Kuhn-Tucker type, we use the higher regularity of y_0, y_a, y_b from Assumption 2.1, as well as of the optimal control $\bar{u}$, along with $e \in L^r(\Omega)$, $r > n/2+1$. Assuming for instance $\bar{u} \in L^r(0,T)$, $r > n/2+1$, we obtain continuity of the optimal state, and the state inequality constraints can be considered in the space $C(\bar{Q})$, i.e.

$$y_a(t,x) \leq y(t,x) \leq y_b(t,x) \quad \forall (t,x) \in Q.$$

A posteriori, the space of controls U can be taken as $L^r(0,T)$. Note that this regularity of the optimal control can easily be guaranteed by bilateral control bounds in $L^\infty(0,T)$ and a more regular shape function e, say $e \in L^\infty(\Omega)$ for simplicity. In the optimality system we would then obtain Lagrange multipliers in the dual space of the state space, i.e. the space of regular Borel measures. The following theorem on first order optimality conditions follows from the Pontryagin principle in [6], cf. also the presentation in [33]. For time-dependent controls specifically, we also refer to Theorem 4 in [12]. In this paper, regularity conditions for e and u are explored in more detail.

Assumption 2.3. *Let the shape function fulfill the regularity $e \in L^\infty(\Omega)$ and let there exist a Slater point $\tilde{u} \in U_{ad} \cap L^r(0,T)$, $r > n/2+1$, such that $\tilde{y} := y(\tilde{u})$ satisfies*

$$y_a(t,x) + \delta \leq \tilde{y}(t,x) \leq y_b(t,x) - \delta \quad \forall (t,x) \in \bar{Q}$$

for some $\delta > 0$ (Slater condition).

Theorem 2.4. *Let Assumptions 2.1 and 2.3 hold and let the optimal solution pair $(\bar{u}, \bar{y})$ of (P) fulfill the regularity*

$$(\bar{u}, \bar{y}) \in L^r(0,T) \times W(0,T) \cap C(\bar{Q}), r > n/2+1.$$

Then there exist $\mu_a, \mu_b \in M(\bar{Q})$ and an adjoint state $p \in L^\rho(0,T; W^{1,\sigma}(\Omega))$, for all $\rho, \sigma \in [1,2)$ with $\frac{2}{\rho} + \frac{n}{\sigma} > n+1$, such that the adjoint equation (2.6), the variational inequality (2.7), and the complementary slackness condition (2.8) hold true:

$$(\varphi_t, p)_Q + (\nabla\varphi, \nabla p)_Q = (\bar{y} - y_d, \varphi)_Q + \langle \varphi, \mu_b - \mu_a \rangle_{C(\bar{Q}), M(\bar{Q})} \tag{2.6}$$

for all $\varphi \in C^1(\bar{Q}) \cap C_0(Q \cup \Omega_T)$,

$$\langle \nu\bar{u} + \int_\Omega p(t,x)e(x)dx, u - \bar{u} \rangle_{L^{r'}(I), L^r(0,T)} \geq 0 \quad \forall u \leq u_b, \text{ a.e. in } (0,T), u \in L^r(O,T), \tag{2.7}$$

$$\int_{\bar{Q}} (y_a - \bar{y}) d\mu_a = 0, \quad \mu_a \geq 0, \quad \int_{\bar{Q}} (\bar{y} - y_b) d\mu_b = 0, \quad \mu_b \geq 0, \tag{2.8}$$

where $\mu_{iQ} = \mu_{i|Q}$, $\mu_{iT} = \mu_{i_{|\bar{\Omega}\times T}}$ denote the restrictions of μ_i to the indicated sets.

Note that in the last theorem $\mu_b a_{|\Sigma} = \mu_{b|\Sigma} = 0$ since due to Assumption 2.1 the state constraints cannot be active on the boundary with homogeneous Dirichlet boundary conditions.

3 The regularized problem

3.1 Problem formulation

In order to be able to treat the problem for L^2 controls without any additional regularity assumptions or restrictions of the dimension to obtain a Lagrange multiplier, we regularize the pointwise constraints on the state as explained in the introduction. Along the lines of [33], we introduce an auxiliary control $v \in L^2(Q)$ and use the ansatz $u = S^*v$ in order to be able to replace the pointwise state constraints

$$y_a(t,x) \le y(t,x) \le y_b(t,x) \quad \forall (t,x) \in Q$$

by mixed control-state constraints

$$y_a \le \lambda v + SS^*v \le y_b \quad \text{a. e. in } Q, \quad S^*v = \int_\Omega z(t,x)e(x)dx,$$

where $z \in W(0,T) \cap H^1(0,T,V) \cap L^2(0,T,H^2(\Omega) \cap V)$ solves

$$\begin{aligned} -(z_t, \varphi)_Q + (\nabla z, \nabla \varphi)_Q = (v, \varphi)_Q \quad \forall\, \varphi \in L^2(0,T,V) \\ z(T,\cdot) = 0 \quad \text{in } \Omega, \end{aligned} \tag{3.1}$$

cf., for instance [24] or [36] for the adjoint of $\mathcal{G}$. The concatenation with B and its adjoint B^*, respectively, is obvious. Note that the use of the L^2-inner product for the term containing the time derivative is justified by the improved regularity results. Moreover, recall that S is the control-to-state mapping G with range in $L^2(Q)$, hence $S^*\colon L^2(Q) \to L^2(0,T)$. With meanwhile well-known arguments we expect existence of Lagrange multipliers associated with the mixed constraints in the more regular space $L^2(Q)$. This will follow from a transformation with a new control $w := \frac{1}{\lambda}(y - v)$, cf. eg. [31], where Lagrange multipliers can be constructed due to the simple structure of the transformed constraints. We would also like to mention the paper [35], that deals with regularity of Lagrange multipliers, including those for mixed constraints. Having said this, we will not be able to use these results directly.

The source term representation concept has so far only been used in the case of boundary control without control constraints, and these in fact require closer attention. By the ansatz

$$u := S^*v,$$

the control now takes the role of an additional state in the regularized problem, and the control bounds become state constraints. Hence, all remarks on the difficulty of pointwise state constraints now in principle hold for the control. Especially, they cannot be treated with Slater-type arguments when prescribed in $L^2(0,T)$. However, as pointed out in the introduction, the specific control structure provides a remedy. We note that z takes values in $W(0,T) \hookrightarrow C([0,T],H)$, which allows to consider the bound $u \le u_b$ pointwise in time, without any additional regularity assumption on the optimal controls.

For this constraint, we can rely on a Slater condition to ensure existence of an associated Lagrange multiplier. We obtain the regularized problem formulation

$$\min\ J_\varepsilon(y,z,v) := \frac{1}{2}\|y-y_d\|_Q^2 + \frac{\nu}{2}\|B^*z\|_I^2 + \frac{\varepsilon}{2}\|v\|_Q^2 \tag{P$_\lambda$}$$

subject to

$$(y_t,\varphi)_Q + (\nabla y,\nabla\varphi)_Q = (BB^*z,\varphi)_Q \quad \forall\,\varphi\in L^2(I,V),\quad y(0,\cdot)=y_0 \quad \text{in } \Omega, \tag{3.2}$$

$$-(z_t,\varphi)_Q + (\nabla z,\nabla\varphi)_Q = (v,\varphi)_Q \quad \forall\,\varphi\in L^2(I,V),\quad z(T,\cdot)=0 \quad \text{in } \Omega, \tag{3.3}$$

and the constraints

$$y_a \le \lambda v + y \le y_b \quad \text{a. e. in } Q, \qquad \int_\Omega z(t,x)e(x)dx \le u_b(t) \quad \forall t\in[0,T].$$

Here, the dual of the operator $B\colon L^2(0,T)\to L^2(Q)$, $(B\omega)(t,x)=\omega(t)e(x)$, takes the form $B^*\colon L^2(Q)\to L^2(0,T)$, $B^*\varphi = \int_\Omega \varphi(t,x)e(x)dx$.

3.2 Existence of regularized solutions and regular Lagrange multipliers

In what follows, we now combine the arguments from [33] and [27] to obtain an optimality system. We consider the solution operator for the auxiliary state equation for z in two different function space settings, as is typical for problems with pointwise state constraint. We have already explained that for $v\in L^2(Q)$ we obtain a unique $z=\mathcal{S}^*v\in L^2(Q)$, and $B^*\mathcal{S}^*v = B^*z\in L^2(Q)$. For y we then observe $y=SB^*\mathcal{S}^*v = SS^*v$. These expressions can be inserted in the objective functional and in the mixed control-state-constraints. To consider the former control bound $u\le u_b$, let $\mathcal{A}\colon L^2(Q)\to W(0,T)\hookrightarrow C([0,T],H)$ denote the operator $\mathcal{S}^*$ with range in $W(0,T)\hookrightarrow C([0,T],H)$. If we denote by $\mathcal{B}^*$ the restriction of B^* to $W(0,T)$, it is clear that $\mathcal{B}^*\mathcal{A}$ maps $L^2(Q)$ into $C([0,T])$ by construction. We define

$$g\colon L^2(Q)\to\mathbb{R},\quad g(v)=\mathcal{B}^*\mathcal{A}v - u_b,$$

and obtain the reduced problem formulation

$$\min\ f_\varepsilon(v) \quad \text{subject to } g(v)(t)\le 0 \quad \forall t\in I,\quad \lambda v + SS^*v \le y_b \quad \text{a.e. in } Q. \tag{P$_\lambda$}$$

The following assumption is needed to prove the existence of a Lagrange multiplier for the constraints on z. The mixed constraints on v and y do not require a Slater condition for Lagrange multipliers to exist.

Assumption 3.1. *There is Slater point $\tilde v\in L^2(Q)$, such that $y_a\le\lambda\tilde v+\tilde y\le y_b$ a.e. and*

$$\int_\Omega \tilde z(t,x)e(x)dx \le u_b(t)-\delta \quad \forall t\in[0,T]$$

for some $\delta>0$, where $\tilde y = G\tilde v$ and $\tilde z=\mathcal{A}\tilde v$.

Existence of a unique optimal control $\bar{v}_\lambda \in L^2(Q)$ with associated $\bar{u}_\lambda \in C([0,T],H)$ and $\bar{y}_\lambda \in W(0,T)$ follows by standard arguments due to the feasibility of the Slater point. Along the lines of [33], we now consider the linear equation

$$(\lambda\mathcal{I} + SS^*)v = w,$$

where $\mathcal{I}$ denotes the identity operator in $L^2(Q)$. We refer to [31] for the original transformation argument in the case of elliptic distributed control. Since SS^* is positive definite and compact in $L^2(Q)$, $(\lambda\mathcal{I} + SS^*)$ has, for every $\lambda > 0$, a bounded and linear inverse operator $K\colon L^2(Q) \to L^2(Q)$,

$$K = (\lambda\mathcal{I} + SS^*)^{-1} \tag{3.4}$$

due to the Fredholm alternative. We also mention [15, Lemma 2.1] where the existence of a bounded inverse has been shown for a slightly different operator $\mathcal{F}$ without relation to S^*. Thus, with

$$v = Kw,\ F(w) := f_\varepsilon(Kw),\ g_w := g(Kw),\ W_{ad} := \{w \in L^2(Q)\ |y_a \leq\ w \leq y_b \ \text{ a.e. in } Q\}$$

we obtain a further transformation of our problem into

$$\min\ F(w) = \frac{1}{2}\|SS^*Kw - y_d\|_Q^2 + \frac{\nu}{2}\|S^*Kw\|_I^2 + \frac{\varepsilon}{2}\|Kw\|_Q^2 \tag{P$_w$}$$

subject to

$$w \in W_{ad}, \quad g_w(w)(t) \leq 0 \quad \forall t \in [0,T].$$

We will obtain a multiplier $\eta \in C([0,T])^*$ for the pointwise in time constraint on z as in [27], and will construct regular Lagrange multipliers for the mixed control-state constraints, following [33, 31]. Since J_ε is differentiable we also have differentiability of f_ε. Since K is a linear and continuous operator F is also differentiable by the chain rule. Likewise, one sees differentiability of g and hence g_w.

Lemma 3.2. *Under the Slater condition from Assumption 3.1 there exists a nonnegative Lagrange multiplier $\eta \in C([0,T])^*$ such that*

$$\langle g_w(\bar{w}), \eta\rangle = 0,\ g(\bar{w}) \leq 0,\ \textit{and}\ F'(\bar{w})(w - \bar{w}) + \langle g_w'(\bar{w})(w-\bar{w}), \eta\rangle \geq 0 \quad \forall w \in W_{ad}.$$

Proof. We observe that W_{ad} is a nonempty, closed and convex subset of $K^2(Q)$ and that $F\colon L^2(Q) \to \mathbb{R}$ is convex and differentiable. We define the cone

$$C([0,T]) \supset K := \{\varphi \in C([0,T])\colon v \leq 0 \text{ in } [0,T]\}$$

and observe that $g_w\colon L^2(Q) \to C([0,T])$. With this, we can rewrite (P$_w$) into

$$\min F(w) \quad \text{s.t. } w \in W_{\mathrm{ad}}, \quad g_w(w) \in K.$$

The claim follows by known generalized KKT theory, see [25] or also the presentation in [36, Theorem 6.2], noting that Assumption 3.1 provides a Slater point $\tilde{w} = \lambda\tilde{v} + \tilde{y}$. $\square$

For the simple bound constraint on w we can now derive existence of Lagrange multipliers by a constructive argument, see [31]. The arguments in [33] only need to be adapted to the presence of η. Note that $(g'_w(\bar{w}))^*$ is a mapping from $C([0,T])^*$ to $L^2(0,T,V)$. Hence, the variational inequality form the last lemma can be rewritten as

$$(F'(\bar{w}) + (g'_w(\bar{w}))^*\eta, w - \bar{w})_Q \geq 0 \quad \forall w \in W_{\text{ad}}.$$

We set $\mu_{\lambda,a} := (F'(\bar{w}) + (g'_w(\bar{w}))^*(\eta))_+$ and $\mu_{\lambda,b} := (F'(\bar{w}) + g'_w(\bar{w})^*(\bar{\eta}))_-$, which directly implies $\mu_{\lambda,a}, \mu_{\lambda,b} \geq 0$ and

$$F'(\bar{w}) + (g'_w(\bar{w}))^*\eta + \mu_{\lambda,b} - \mu_{\lambda,a} = 0. \tag{3.5}$$

We further observe, cf. for instance [31], $\mu_{\lambda,a}(t,x) = 0$ almost everywhere where $y_a(t,x) < \bar{w}(t,x)$ and $\mu_{\lambda,b}(t,x) = 0$ almost everywhere where $\bar{w}(t,x) < y_b(t,x)$, and $\bar{w}(t,x) = y_a(t,x)$ where $\mu_{\lambda,a}(t,x) > 0$ as well as $\bar{w}(t,x) = y_b(t,x)$ where $\mu_{\lambda,b}(t,x) > 0$. This implies

$$(\mu_{\lambda,a}, y_a - \bar{w})_Q = (\mu_{\lambda,b}, \bar{w} - y_b)_Q = 0. \tag{3.6}$$

It remains to reformulate the obtained results in terms of our control problem in v.

Theorem 3.3. *A control $\bar{v}_\lambda \in L^2(Q))$ is the optimal control for (P_λ) with associated states $\bar{y}_\lambda, \bar{z}_\lambda \in W(0,T)$ if and only if $g(\bar{v}_\lambda) \leq 0$, $y_a \leq \lambda\bar{v} + \bar{y} \leq y_b$ and there exist nonnegative Lagrange multipliers $\mu_{\lambda,a}, \mu_{\lambda,b} \in L^2(Q)$, $\eta \in C([0,T])^*$ and adjoint states $p_\lambda \in W(0,T)$ and $q_\lambda \in L^2(0,T,V)$ such that:*

$$\begin{aligned} -(\varphi, p_{\lambda},t)_Q + (\nabla\varphi, \nabla p_\lambda)_Q &= (\bar{y}_\lambda - y_d + \mu_{\lambda,b} - \mu_{\lambda,a}, \varphi)_Q \; \forall \varphi \in L^2(0,T,V) \\ p_\lambda(T,\cdot) &= 0, \end{aligned} \tag{3.7}$$

$$\begin{aligned} -(q_\lambda, \varphi_t)_Q + (\nabla q_\lambda, \nabla\varphi)_Q &= (BB^*(\nu\bar{z}_\lambda + p_\lambda), \varphi)_Q + \langle B^*\varphi, \eta\rangle \; \forall \varphi \in W(0,T) \\ q_\lambda(0,\cdot) &= 0, \end{aligned} \tag{3.8}$$

$$(\mu_{\lambda,a}, y_a - \lambda\bar{v}_\lambda + \bar{y}_\lambda)_Q = 0, \qquad \mu_{\lambda,a} \geq 0 \tag{3.9}$$

$$(\mu_{\lambda,b}, \lambda\bar{v}_\lambda + \bar{y}_\lambda - y_b)_Q = 0, \qquad \mu_{\lambda,b} \geq 0 \tag{3.10}$$

$$\langle (B^*\bar{z}_\lambda)(t) - u_b(t), \eta\rangle = 0, \qquad \eta \geq 0 \tag{3.11}$$

$$\varepsilon\bar{v}_\lambda + q_\lambda + \lambda\mu_{\lambda,b} - \lambda\mu_{\lambda,a} = 0. \tag{3.12}$$

Proof. The proof follows along the lines of the regularized boundary control setting in [33], taking into account the presence of the transformed control constraints. First, we express F' in terms of f_ε and $\bar{v}_\lambda$. With $F(\bar{w}) = f_\varepsilon(K\bar{w})$, $g_w(\bar{w}) = g(v)$, K defined by (3.4), using the chain rule yields

$$\begin{aligned} F'(\bar{w})w &= f'_\varepsilon(K\bar{w})K'(\bar{w})w = f'_\varepsilon(K\bar{w})Kw, \\ g'_w(\bar{w})w &= g'(K\bar{w})K'(\bar{w})w = g'(K\bar{w})Kw. \end{aligned}$$

This makes equation (3.5) equivalent to

$$f'_\varepsilon(K\bar{w})Kw + ((g'(\bar{w}))^*\eta, Kw)_Q + (\mu_{\lambda,b} - \mu_{\lambda,a}, w)_Q = 0 \qquad \forall\, w \in L^2(Q).$$

If we now substitute $v = Kw$ and $\bar{v}_\lambda = K\bar{w}$, we obtain

$$f'_\varepsilon(\bar{v}_\lambda)v + \big((g'(\bar{v}_\lambda))^*\eta, v\big)_Q + (\mu_{\lambda,b} - \mu_{\lambda,a}, (\lambda\mathcal{I} + SS^*)v)_Q = 0. \tag{3.13}$$

The first and third term can be treated exactly along the lines of [33]. We know that

$$\begin{aligned} f'_\varepsilon(\bar{v}_\lambda)v &= (SS^*\bar{v}_\lambda - y_d, SS^*v)_Q + \nu(S^*\bar{v}_\lambda, S^*v)_I + \varepsilon(\bar{v}_\lambda, v)_Q \\ &= (SS^*(\bar{y}_\lambda - y_d) + \nu SS^*\bar{v}_\lambda + \varepsilon\bar{v}_\lambda, v)_Q = (SB^*p_1 + \nu SB^*\bar{z}_\lambda + \varepsilon\bar{v}, v)_Q \\ &= (S(B^*p_1 + \nu B^*\bar{z}_\lambda) + \varepsilon\bar{v}_\lambda, v)_Q = (q_1 + \varepsilon\bar{v}_\lambda, v)_Q, \end{aligned}$$

where $\bar{z}_\lambda = S^*\bar{v}_\lambda$, and p_1, q_1 are the solutions of

$$-(\varphi, p_{1,t})_Q + (\nabla\varphi, \nabla p_1)_Q = (\bar{y}_\lambda - y_d, \varphi)_Q \;\; \forall\varphi \in L^2(0,T,V), \;\; p_1(T,\cdot) = 0, \tag{3.14}$$

$$(q_{1,t}, \varphi)_Q + (\nabla q_1, \nabla\varphi)_Q = (BB^*(p_1 + \nu\bar{z}_\lambda), \varphi)_Q \; \forall\varphi \in L^2(0,T,V), \;\; q_1(0,\cdot) = 0, \tag{3.15}$$

respectively, according to the definitions of S and S^*. We further see that

$$\begin{aligned} (\mu_{\lambda,b} - \mu_{\lambda,a}, (\lambda\mathcal{I} + SS^*)\,v)_Q &= \lambda(\mu_{\lambda,b} - \mu_{\lambda,a}, v)_Q + (\mu_{\lambda,b} - \mu_{\lambda,a}, SS^*v)_Q \\ &= \lambda(\mu_{\lambda,b} - \mu_{\lambda,a}, v)_Q + (SS^*(\mu_{\lambda,b} - \mu_{\lambda,a}), v)_Q \\ &= \lambda(\mu_{\lambda,b} - \mu_{\lambda,a}, v)_Q + (SB^*p_2, v)_Q \\ &= \lambda(\mu_{\lambda,b} - \mu_{\lambda,a}, v)_Q + (q_2, v)_Q, \end{aligned}$$

where $p_2 = S^*(\mu_{\lambda,b} - \mu_{\lambda,a})$ and $q_2 = SB^*p_2$ solve

$$-(\varphi, p_{2,t})_Q + (\nabla\varphi, \nabla p_2)_Q = (\mu_{\lambda,b} - \mu_{\lambda,a}, \varphi)_Q \; \forall\varphi \in L^2(0,T,V) \;\; p_2(T,\cdot) = 0, \tag{3.16}$$

$$(q_{2,t}, \varphi)_Q + (\nabla q_2, \nabla\varphi)_Q = (BB^*p_2, \varphi)_Q \; \forall\varphi \in L^2(0,T,V), \;\; q_2(0,\cdot) = 0. \tag{3.17}$$

It remains to discuss the second term in (3.13). Observe again that $(g'(\bar{v}_\lambda))^*$ maps continuously from $C([0,T])^*$ into $L^2(I,V)$, hence there is $q_3 \in L^2(Q)$ such that

$$(q_3, v)_Q \;=\; ((g'(\bar{v}_\lambda))^*\eta, v)_Q = \langle \eta, g'(\bar{v}_\lambda)v\rangle = \langle \eta, \mathcal{B}^*\mathcal{A}v\rangle.$$

Similarly to the setting with unregularized state constraints in Q, we find that q_3 fulfills

$$-(\varphi_t, q_3)_Q + (\nabla\varphi, \nabla q_3)_Q = \langle \mathcal{B}(\varphi), \eta\rangle \quad \forall\varphi \in W(0,T).$$

The adjoint variables p_1, p_2 and q_1, q_2, q_3 are functions in $W(0,T)$. It is clear that $p := p_1 + p_2$ and $q := q_1 + q_2 + q_3$ solve the adjoint systems (3.7) and (3.8), respectively, and that the gradient equation (3.12) is fulfilled. The conditions (3.9) and (3.10) follow immediately from (3.6) and the definition of $\mu_{\lambda,b}$. Likewise, the conditions (3.11) follow immediately from the complementary slackness conditions in Lemma 3.2 and the definition of g_w. By convexity of the problem, these necessary conditions are also sufficient for optimality. □

3.3 Further aspects

From the optimality conditions obtained in the last section, we can now obtain additional regularity for the optimal control $\bar{v}_\lambda$ by typical bootstrapping arguments. It is known that the conditions (3.9) and (3.10) are equivalent to

$$\begin{aligned}\mu_{\lambda,a} &= \max\left(0, \mu_{\lambda,b} - \mu_{\lambda,a} + c(\lambda\bar{v}_\lambda + \bar{y}_\lambda - y_b)\right)\\ \mu_{\lambda,b} &= \max\left(0, \mu_{\lambda,b} - \mu_{\lambda,a} + c(y_a - \lambda\bar{v}_\lambda + \bar{y}_\lambda)\right)\end{aligned}$$

for any $c > 0$. Choosing $c = \frac{\varepsilon}{\lambda^2}$ and employing the gradient equation (3.12) we arrive at

$$\mu_{\lambda,a} = \max\left(0, -\frac{1}{\lambda}q_\lambda + \frac{\varepsilon}{\lambda^2}(y_a - \bar{y}_\lambda)\right), \quad \mu_{\lambda,b} = \max\left(0, -\frac{1}{\lambda}q_\lambda + \frac{\varepsilon}{\lambda^2}(\bar{y}_\lambda - y_b)\right).$$

We know that $\bar{y}_\lambda, q_\lambda \in L^2(0,T,V)$. If $y_a, y_b \in L^2(0,T,V)$, the same regularity holds for $\mu_{\lambda,a}, \mu_{\lambda,b}$ due to the H^1 stability of the max-operator. In turn, the gradient equation (3.12) directly yields the same regularity for the optimal control $\bar{v}_\lambda$.

When comparing this to the regularity discussion in [27], we observe that there additional $L^\infty(I,H)$-regularity of the control is required for the numerical analysis. In this work, this is immediately obtained due to the presence of control bounds in $L^\infty(Q)$. In our setting this regularity is not immediately obvious since the adjoint $q_\lambda \in L^2(0,T,V)$ appears in the gradient equation, whose regularity is limited by the measure η in the right-hand-side. However, the bilateral (mixed) state constraints

$$y_a \le \lambda v + y \le y_b$$

lead to $\bar{w} := \lambda\bar{v}_\lambda + \bar{y}_\lambda \in L^\infty(Q)$. Moreover, we have $L^\infty(0,T,H)$-regularity of $\bar{y}_\lambda$, therefore $\bar{v}_\lambda = \frac{1}{\lambda}(\bar{w} - \bar{y})$ is an element of $L^\infty(0,T,H)$. We summarize this in the following

Corollary 3.4. *Under the assumptions of Theorem 3.3, the optimal control $\bar{v}_\lambda \in L^2(Q)$ admits the additional regularity*

$$\bar{v}_\lambda \in L^2(0,T,H^1(\Omega)) \cap L^\infty(0,T,H),$$

provided that the bounds fulfill $y_a, y_b \in L^2(0,T,V)$.

With this result at hand, the problem is now feasible for numerical analysis along the lines of [27].

Let us end this discussion with a short comment on convergence results for $\lambda \to 0$. The proof of Theorem 3.3 and the preceding discussions in [33] need to be amended by a feasibility discussion with respect to the control constraints. First, once a minimizing sequence $\{v_n\}$ of (P_λ) is given, it is rather straight forward to show that the weak limit of $\{u_n\} := \{S^* v_n\}$ fulfills the control bounds L^2-a.e., since all u_n fulfill the bounds even in a pointwise manner. It is more difficult to prove that the sequence $\tilde{v}_k^0$ constructed with the help of a Slater point $\tilde{v} \in C(\bar{Q})$ by $\tilde{v}_k = v_k + \frac{2c}{k\delta}\tilde{v}$ is feasible with respect to the pointwise transformed constraints on z of the regularized problem. This step is necessary to show that the weak limit of S^*v_n is actually optimal. Looking into the proof of Theorem 3.3 in [33], $v_k \in C(\bar{Q})$ is chosen such that $\|S^*v_k - \bar{u}\|_Q \leq \frac{1}{k}$, hence this results in convergence and thus feasibility in $L^2(0,T)$. The way out seems to be an approximability assumption as in [37]. If there is $v_k \in C(\bar{Q})$ such that $\|\mathcal{B}^*\mathcal{A}^*v_k - \bar{u}|_{C(\bar{[0,T]})} \leq \frac{1}{k}$, then convergence can be expected if ε is chosen appropriately depending on λ, cf. [33, Theorem 3.3]. We will not discuss convergence in detail here.

4 An elliptic model problem with finitely many control parameters

To end this manuscript, we comment very briefly on the elliptic problem

$$\min\ J(y,u) = \frac{1}{2}\|y - y_d\|^2 + \frac{\nu}{2}u^2 \tag{4.1}$$

subject to

$$\begin{aligned} -\Delta y &= ue(x) \quad \text{in } \Omega \\ y &= 0 \qquad\quad \text{on } \Gamma \end{aligned} \tag{4.2}$$

and for simplicity unilateral constraints on the state as well as the control

$$y \leq y_b \quad \text{in } \Omega, \quad u \leq u_b. \tag{4.3}$$

Here, the control u is a control parameter and $e \in H$ is a given shape function as in the previous sections. Also here, it is possible to consider more than one control, i.e. a control vector $u \in \mathbb{R}^m$ for $m >$ given shape functions $e_i \colon \Omega \to \mathbb{R}$. It is well known by the Lax-Milgram lemma that for every $f \in V^*$ there is a unique solution of Poisson's equation $y \in V$. Under our assumptions, y admits improved regularity $y \in H^2(\Omega) \hookrightarrow C(\bar{\Omega})$, see e.g. [14], where the embedding into the space of continuous functions holds up to spatial dimension $n = 3$. Hence, the unregularized optimal control problem can be analyzed with respect to existence of Lagrange multipliers without further assumptions other than a Slater condition, see for instance [12] or e.g. [29, 30]. In the latter works, slightly higher regularity of the shape function is assumed, but only needed for the numerical analysis.

This problem belongs to the class of semi-infinite optimization problems and its analysis is specifically interesting due the often observed structure of having finitely many points where the state constraints become active. The Lagrange multipliers will then in fact be Dirac measures, and a regularization for numerical treatment seems appropriate. On the other side, despite this low regularity comparibly high rates of convergence of finite element error estimates can be obtained under certain conditions, cf. [29, 30].

For the analysis of the regularized problem, we will not need continuity results for the state, in contrast to the parabolic problem, where we needed at least continuity with respect to time. Redefining the previous solution and control-to-state operators to their elliptic counterparts, observe

$$\begin{array}{ll} \mathcal{G}\colon H \to V \cap C(\bar{\Omega}), & f \mapsto y \\ B\colon \mathbb{R} \to H, & Bu = ue(x) \\ G\colon H \to V \cap C(\bar{\Omega}), & G = \mathcal{G}B, \end{array}$$

and similar to before denote by $\mathcal{S}$ and S the operators $\mathcal{G}$ and G with range in H, respectively. Note that $\mathcal{S}$ is self adjoint. Clearly, $B^*\colon H \to \mathbb{R}$ is given by $B^*\varphi = \int_\Omega \varphi(x)e(x)dx$. Repeating the principal steps from Sections 3.1 to 3.2, we see that it suffices to work with the operators S and S^* instead of e.g. G, see also [37, 38] for source term representation. Observing that $S\colon \mathbb{R} \to H$, hence $S^*\colon H \to \mathbb{R}$, we can directly insert the ansatz $u = S^*v$, $y = SS^*v$ in the objective function, amend it by a cost term for v, and replace the state constraints by mixed constraints $\lambda v + SS^*v \leq y_b$ in H as before. Now, the control bound $u \leq u_b \in \mathbb{R}$ becomes $S^*v \leq u_b$, hence the constraint function g is a function mapping H into the real numbers, $g\colon H \to \mathbb{R}$, $g(v) := S^*v - u_b$. Since $\mathbb{R}$ has nonempty interior, Slater type assumptions are reasonable. We can directly analyse the transformed problem

$$\min\ f_\varepsilon(v) := \frac{1}{2}\|SS^*v - y_d\|^2 + \frac{\nu}{2}|S^*v| + \frac{\varepsilon}{2}\|v\|^2 \tag{P^e_λ}$$

subject to the constraints

$$\lambda v + SS^*v \leq y_b \quad \text{a. e. in } \Omega, \quad g(v) \leq 0. \tag{4.4}$$

Now the discussion of existence of solutions and Lagrange multipliers is straightforward by adjusting the arguments shown in the previous sections. Eventually, assuming existence of a Slater point $\tilde{v} \in H$ such that $\lambda\tilde{v} + \tilde{y} \leq y_b$, $\tilde{z} \leq u_b - \delta$ for some $\delta > 0$, the following optimality system is obtained:

Corollary 4.1. *A control $\bar{v}_\lambda \in H$ is the optimal control for (P^e_λ) with associated states $\bar{y}_\lambda, \bar{z}_\lambda \in V$ fulfilling*

$$(\nabla\bar{y}_\lambda, \nabla\varphi) = (BB * \bar{z}_\lambda, \varphi)\ \forall\varphi \in V \tag{4.5}$$

$$(\nabla\bar{z}_\lambda, \nabla\varphi) = (v, \varphi) + (B\eta, \varphi)\ \forall\varphi \in V, \tag{4.6}$$

if and only if $g(\bar{v}_\lambda) \leq 0$, $\lambda\bar{v} + \bar{y} \leq y_b$ and there exist nonnegative Lagrange multipliers $\mu_{\lambda,b} \in H$, $\eta \in \mathbb{R}$ and adjoint states $p_\lambda, q_\lambda \in V$ such that:

$$(\nabla\varphi, \nabla p_\lambda) = (\bar{y}_\lambda - y_d + \mu_{\lambda,b}, \varphi)\ \forall\varphi \in V \tag{4.7}$$

$$(\nabla q_\lambda, \nabla\varphi) = (BB^*(\nu\bar{z}_\lambda + p_\lambda), \varphi) + (B\eta, \varphi)\ \forall\varphi \in V \tag{4.8}$$

$$(\mu_{\lambda,b}, \lambda\bar{v}_\lambda + \bar{y}_\lambda - y_b)_Q = 0, \qquad \mu_{\lambda,b} \geq 0 \tag{4.9}$$

$$\eta(B^*\bar{z}_\lambda - u_b) = 0, \qquad \eta \geq 0 \tag{4.10}$$

$$\varepsilon\bar{v}_\lambda + q_\lambda + \lambda\mu_{\lambda,b} = 0. \tag{4.11}$$

Note that $\bar{v}_\lambda \in H$ implies $\bar{z}_\lambda \in V \cap H^2(\Omega)$, hence $\bar{y}_\lambda$ and $\mu_{b,\lambda} \in H$ as well as $B\eta \in H$ imply $p_\lambda, q_\lambda \in V \cap H^2$. With the same arguments as in the parabolic setting we further observe

$$\mu_{b,\lambda} = \max(0, -\frac{1}{\lambda} q_\lambda + \frac{\varepsilon}{\lambda^2}(\bar{y}_\lambda - y_b)),$$

and due to H^1-stability of the max-operator we obtain $\mu_{b,\lambda} \in H^1(\Omega)$, provided $y_b \in H^1(\Omega)$. In turn, the gradient equation implies $\bar{v}_\lambda \in H^1(\Omega)$. If boundedness of $\bar{v}_\lambda$ is required, this can be obtained for bilateral mixed constraints in $L^\infty(\Omega)$, since then $\bar{\lambda}\bar{v}_\lambda + \bar{y}_\lambda$ is an L^∞ function, and so is $\bar{y}_\lambda$ due to the embedding $H^2(\Omega \in C(\bar{\Omega}))$ for $n \leq 3$.

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