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Sparse optimal control of a quasilinear elliptic PDE in measure spaces

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Abstract. We prove existence of optimal controls for sparse optimal control of a quasilinear elliptic equation in measure spaces and derive first-order necessary optimality conditions. Under additional assumptions also second-order necessary and sufficient optimality conditions are obtained.

1. Introduction

This paper is concerned with sparse optimal control of a quasilinear elliptic partial differential equation (PDE) in measure spaces. We prove well-posedness of the state equation, and existence of optimal controls, and derive first- and (in space dimension 2) second-order optimality conditions for the following prototypical model problem:

$$\text{(P)} \qquad \min_{u\in\mathcal{M}_D(\overline{\Omega})} J(y,u) := \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \gamma\|u\|_{\mathcal{M}_D(\overline{\Omega})},$$

$$\text{(Eq)} \qquad \text{s.t.} \quad \begin{cases} -\nabla\cdot\xi(y)\rho\nabla y = u, & \text{in } \Omega\cup\Gamma_N, \\ y = 0, & \text{on } \Gamma_D. \end{cases}$$

For the notation and the detailed assumptions we refer the reader to Section 1.1 below. Our problem is of interest because it combines two challenging aspects in the field of PDE-constrained optimization: sparse optimal control in measure spaces, and optimal control of quasilinear PDEs. In particular, the consideration of nonlinear PDEs with measure data is known to be delicate; see, e.g., [**3**, **13**, **38**]. Nevertheless, utilizing the so-called Kirchhoff transform as the main tool of our investigation, we are able to obtain results for (P) as they may be expected from a similar analysis of problems with semilinear elliptic equations in [**13**].

In the last years there have been many contributions to both fields, sparse optimal control and optimal control of quasilinear PDEs. In case of control-constrained optimal control of quasilinear elliptic PDEs we refer, e.g., to [**11**, **16**] for first- and second-order optimality conditions, to [**17**, **18**] for finite element discretization error estimates, or to [**20**, **21**] for a nonsmooth nonlinearity. Optimality conditions in the quasilinear parabolic setting have been derived, e.g., in [**4**, **8**] for control- and in [**33**] for state-constraints, respectively. As typical for problems governed by any nonlinear PDE, first-order necessary optimality conditions for optimal control problems with quasilinear PDEs are in general not sufficient for optimality, due to nonconvexity of the problem. Consequently, second-order optimality conditions need to be addressed. Here, a careful analysis of existence and regularity of solutions to the underlying PDE and its linearizations typically poses the main difficulty.

Sparse optimal control is a highly active area of research, see, e.g., [**7**] for a concise overview, but to the best of our knowledge the case of sparse optimal control of quasilinear PDEs has not been addressed so far. The general idea of sparsity in PDE-constrained optimization is to enforce small support of the optimal control in an optimal control problem. This may be favorable in practical applications, e.g., if actuators that implement such an

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optimal control cannot act on the whole spatial or spatio-temporal-domain. Beside other approaches, as, e.g., L^1-penalization [**41**], directional sparsity [**12, 32**], mixed approaches [**10, 35**], or L^p-penalization with $p \in [0,1)$ [**19, 34**], choosing controls from a measure space is a prominent approach in sparse optimal control. We refer, e.g., to [**9**] for problems with linear elliptic, to [**14**] for problems with linear parabolic, and to [**13, 38**] for problems with semilinear elliptic PDEs. Compared to distributed control with L^1-penalization this has the advantage that point- or certain surface-sources are included in the control space. Since the space L^1 embedds isometrically into the space of Borel measures, this approach can be seen as generalization of L^1-penalization. However, there are three typical difficulties: First, one has to prove well-posendness of the optimal control problem by ensuring existence, uniqueness, and sufficient regularity of solutions to the state equation for controls of very low regulartiy. Second, in the case of a nonlinear state equation the investigation of differentiability properties of the control-to-state becomes particulary challenging. This is due to the fact that differentiability of the nonlinear terms has to be addressed in appropriate, sufficiently regular function spaces, while solutions to PDEs with measure right-hand sides tend to have low regularity. The third difficulty arises from the presence of the $\|\cdot\|_{\mathcal{M}_D(\overline{\Omega})}$-norm in the functional, which makes (P) a nonsmooth problem. In the present paper, the first two problems form the main challenge, and we deal with them by transforming (Eq) into a linear elliptic equation utilizing the so-called Kirchhoff transform; see Section 2.2. The two aforementioned problems basically reduce to checking invertibility and differentiability, respectively, of the Kirchhoff transform between the respective function spaces that are determined by the regularity of solutions of the linear elliptic equation with measure right-hand side. Since second-order differentiability of the Kirchhoff transform is needed this limits our second-order analysis to space dimension 2. Regarding the analysis of the state equation, let us also point out that the nonlinearity in (Eq) is of non-monotone type in general, see, e.g., the counterexample in [**26**], and hence not covered by, e.g., [**3**]. Having obtained the required properties of the control-to-state map, optimality conditions for the nonsmooth and nonconvex problem (P) can be discussed following in principal the techniques applied to the semilinear setting in [**13**].

The structure of the the paper is the following: First, we state and discuss our minimal assumptions in Section 1.1 below. After that, we analyze well-posedness of the state equation and the optimal control problem in Section 2. Moreover, we derive first-order necessary optimality conditions in Section 3. Under appropriate additional assumptions and restriction to space dimension 2 we also prove second-order necessary and sufficient optimality conditions in Section 4. In Section 4.4 we indicate how the restriction to dimension 2 can be avoided.

1.1. Notation and Assumptions. We introduce some notation and conventions, and state our minimal assumptions that hold troughout the rest of the paper. First, we clarify some basic notation: Given real Banach spaces X and Y, we denote by $X \hookrightarrow Y$ that X is continuosly embedded into Y, by $\mathcal{L}(X,Y)$ the space of bounded linear maps $X \to Y$, equipped with the operator norm, and by $X^* := \mathcal{L}(X,\mathbb{R})$ the topological dual of X. Moreover, $\mathbb{B}^X_r(x) \subset X$ denotes the open ball of radius $r > 0$ around some $x \in X$. By "left-hand side $\lesssim$ right-hand side" we indicate that "left-hand side $\leq c$ right-hand side" with some constant $c > 0$ whose exact value is not relevant in the respective context.

With respect to the domain, its boundary, and the boundary conditions we will rely on the following conditions:

Assumption 1.1. $\Omega \subset \mathbb{R}^d$, $d \in \{1,2,3\}$, is a bounded, open, and connected set. Its boundary $\partial\Omega$ is divided into two disjoint subsets Γ_N and $\Gamma_D := \partial\Omega \setminus \Gamma_N$ of which Γ_N is relatively open. We assume that $\Omega \cup \Gamma_N$ is regular in the sense of Gröger [**28**] and that Γ_D has nonzero surface measure within $\partial\Omega$.

For an explanation why Γ_D needs to have nonzero surface measure, which excludes pure Neumann boundary conditions, we refer the reader to the end of Section 2.2. Note that regularity of $\Omega \cup \Gamma_N$ in the sense of Gröger implies that Ω is a Lipschitz-domain in the sense of [**27**, Definition 1.2.1.2]. Moreover, Assumption 1.1 is in particular fulfilled for any domain with a Lipschitz boundary ("strong Lipschitz domain", in the sense of [**27**, Definition 1.2.1.1]) in the case $\Gamma_D = \partial\Omega$; cf. [**31**, Remark 3.3]. Nevertheless, there are also domains without Lipschitz boundary fulfilling Assumption 1.1, e.g., a pair of crossing beams in 3D with pure homogeneous Dirichlet boundary conditions; cf. [**31**, Section 7.3]. For a geometric characterization of regularity in the sense of Gröger in space dimension 2 and 3 we refer the reader to [**29**] for instance.

By a subscript D we indicate from now on that spaces of functions defined on Ω carry homogeneous Dirichlet boundary conditions on Γ_D. By $W^{1,q}_D(\Omega)$, $q \in [1,\infty)$, for instance, we denote the Sobolev space with vanishing trace on Γ_D, and define $W^{-1,q}_D(\Omega) := (W^{1,q'}_D(\Omega))^*$ with $q^{-1} + (q')^{-1} = 1$. Let us point out that the regularity assumptions on the domain, its boundary, and the boundary conditions ensure in particular that $(W^{-1,q}_D(\Omega))_{q\in(1,\infty)}$ forms an interpolation scale, cf. [**2**], which will be used and explained in more detail in Section 2.1 below. Moreover, the classical Sobolev spaces $W^{1,q}_D(\Omega)$, consisting of L^q functions on Ω with weak derivatives in $L^q(\Omega)$ and vanishing trace on Γ_D, coincide with Sobolev spaces obtained by restriction of $W^{1,q}_D(\mathbb{R}^d)$; cf. [**2**, Proposition B.3]. In particular, the classical Sobolev embeddings into Lebesgue and Hölder spaces stay valid. Finally, we will use the notation $C^\infty_D(\Omega) := \{\varphi|_\Omega\colon\ \varphi \in C^\infty_c(\mathbb{R}^d),\ \operatorname{supp}(\varphi) \cap \Gamma_D = \emptyset\}$.

By $C_D(\overline{\Omega})$ we denote the space of continuous functions $\overline{\Omega} \to \mathbb{R}$ vanishing on Γ_D. The space of regular Borel measures $\mathcal{M}_D(\overline{\Omega}) := \mathcal{M}(\Omega \cup \Gamma_N)$ on $\Omega \cup \Gamma_N$ is identified with the dual of $C_D(\overline{\Omega})$, cf. [**40**, Theorem 6.19], and equipped with the norm

$$\|\mu\|_{\mathcal{M}_D(\overline{\Omega})} = \sup_{\|\varphi\|_{C_D(\overline{\Omega})}\le 1} \langle \mu, \varphi\rangle_{\mathcal{M}_D, C_D} := \sup_{\|\varphi\|_{C_D(\overline{\Omega})}\le 1} \int_\Omega \varphi \,\mathrm{d}\mu.$$

The space $\mathcal{M}_D(\overline{\Omega})$ can be viewed as space of Borel measures on $\overline{\Omega}$ with homogeneous boundary condition on Γ_D; cf., e.g., [**5**, Appendix C].

Next, we state our assumptions on the coefficients of the state equation, the desired state and the regularization parameter:

Assumption 1.2. 1. The function $\xi\colon \mathbb{R} \to (0,\infty)$ is continuous, and bounded from below and above, i.e.

$$0 < \xi_\bullet \le \xi(s) \le \xi^\bullet, \qquad \forall s \in \mathbb{R},$$

and $\rho\colon \Omega \to \mathbb{R}^{d\times d}$ is measurable, uniformly bounded, and elliptic, i.e.

$$0 < \rho_\bullet := \operatorname{essinf}_{x\in\Omega} \inf_{z\in\mathbb{R}^d\setminus\{0\}} \frac{z^T\rho(x)z}{z^Tz}, \quad \rho^\bullet := \operatorname{esssup}_{x\in\Omega} \sup_{1\le i,j\le d} |\rho_{i,j}(x)| < \infty.$$

2. We assume that there is some $\bar{q} > d$ such that

$$-\nabla\cdot\rho^T\nabla\colon\ W^{1,\bar{q}}_D \to W^{-1,\bar{q}}_D \tag{1.1}$$

is a topological isomorphism, and fix this choice of $\bar{q}$.

Assumption 1.3. Let $\gamma > 0$ and $y_d \in L^2(\Omega)$.

The assumptions on ξ ensure that the Kirchhoff transform and its inverse, introduced in Section 2 below, are well defined maps between the respective function spaces. The isomorphism property for the elliptic operator $-\nabla\cdot\rho^T\nabla$ for some $\bar{q} > d$, that is crucial for the well-posedness of (Eq), cf. Section 2, is certainly nontrivial, in particular in space dimension 3. However, there are several interesting constellations in which our assumptions are satisfied:

Example 1.4. 1. If Ω is a bounded domain with Lipschitz boundary, $\Gamma_D = \partial\Omega$, and ρ is symmetric-valued and uniformly continuous, then (1.1) is an isomophism for some $\bar{q} > 3$; see [**25**, Theorem 3.12]. This means that Assumptions 1.1 and 1.2 cover in particular the classical "regular" setting of domains with Lipschitz boundary in dimensions $d = 2, 3$ with pure Dirichlet boundary conditions and symmetric, uniformly continuous ρ.

2. It is a well known result of Gröger [**28**] that (1.1) is a topological isomorphism for some $\bar{q} > 2$, possibly arbitrarily close to 2, if Assumptions 1.1 and 1.2.1 are satisfied. In fact, this is even true under more general assumptions; cf. [**30**]. Therefore, Assumption 1.2.2 does not pose an additional restriction in space dimension $d = 2$.
3. In [**23**], for instance, several real-world constellations in dimension $d = 3$ have been described that fulfill Assumptions 1.1 and 1.2. Two crossing beams, e.g., with constant ρ and $\Gamma_D = \partial\Omega$, fulfill our requirements.

2. Well-posedness of the state equation and the optimal control problem

First, we define the notion of a solution to (Eq), and verify that (Eq) is well-posed in this sense. This is done in Section 2.2 by application of the so-called Kirchhoff transform, i.e. a nonlinear superposition operator that allows to transfer (Eq) into a linear elliptic equation. The well known existence, uniqueness, continuity, and regularity theory of the latter, summarized in Section 2.1, can be transferred back to (Eq). The crucial point is to discuss the mapping properties of the nonlinear superposition operator of the Kirchhoff transform between the respective spaces. Having shown well-posedness of the state equation and appropriate continuity properties of the control-to-state map, we prove existence of a solution to (P) in Section 2.3.

First, let us state precisely what a solution of (Eq) means in the following. Unlike in the case of the linear elliptic equation (2.3) we cannot introduce a very weak formulation of (Eq) because shifting all derivatives to the test-function via integration by parts would also require us to take derivatives of $\xi(y)$.

Definition 2.1 (Solutions of (Eq)). We call y a solution to (Eq) if

$$y \in W_D^{1,\bar{q}'}(\Omega), \qquad \text{s.t.} \quad \int_\Omega \xi(y)\rho\nabla y\nabla\varphi \,\mathrm{d}x = \int_{\overline{\Omega}} \varphi \,\mathrm{d}u, \quad \forall \varphi \in C_D^\infty(\Omega). \tag{2.1}$$

The reason for defining a solution of (Eq) a priori to be an element of $W_D^{1,\bar{q}'}(\Omega)$, instead of, e.g., $W_D^{1,1}(\Omega)$, is to ensure uniqueness of solutions. We will explain this in more detail below Proposition 2.4. Note that if y is a solution to (Eq) with additional regularity $y \in W_D^{1,q}(\Omega)$ with some $q \in [\bar{q}', \frac{d}{d-1})$, it will even hold that

$$\int_\Omega \xi(y)\rho\nabla y\nabla\varphi \,\mathrm{d}x = \int_{\overline{\Omega}} \varphi \,\mathrm{d}u, \qquad \forall \varphi \in W_D^{1,q'}(\Omega), \tag{2.2}$$

because $C_D^\infty(\Omega)$ is dense in $W_D^{1,q'}(\Omega)$. Here, the right-hand side of (2.2) is well defined, because $W_D^{1,q'}(\Omega) \hookrightarrow C_D(\overline{\Omega})$ due to $q' > d$. Moreover, let us point out that the left-hand side in (2.2) is well defined because boundedness and measurability of ξ ensure $\xi(y) \in L^\infty(\Omega)$.

2.1. Linear elliptic equations in measure spaces. Before addressing the analysis of (Eq), we recall some well known theory concerning the existence, uniqueness, and regularity of solutions to linear elliptic PDEs for convenience of the reader. More precisely, for some $q \in [\bar{q}', 2]$ we consider the equation

$$\begin{aligned} -\nabla \cdot \rho\nabla w &= u \quad \text{on } \Omega \cup \Gamma_N, \\ w &= 0 \quad \text{on } \Gamma_D, \end{aligned} \tag{2.3}$$

with $u \in W_D^{-1,q}(\Omega)$. We follow a duality-based concept going back to Stampacchia [**42**], cf. also [**36**, Section 2], and call w a solution of (2.3) if

$$w \in W_D^{1,q}(\Omega) \quad \text{s.t.} \quad \int_\Omega \rho\nabla w\nabla\varphi \,\mathrm{d}x = \langle u, \varphi\rangle_{W_D^{-1,q}, W_D^{1,q'}}, \quad \forall \varphi \in C_D^\infty(\Omega). \tag{2.4}$$

As in Definition 2.1, a priori $W^{1,q}$-regularity of the solution is part of this solution concept. This is necessary to ensure uniqueness of solutions, because it is well known that there may be several $\tilde{w} \in W_D^{1,1}(\Omega)$ that solve the varitional formulation in (2.4); see, e.g., [**36**] and the references therein. Note that once (2.4) is fulfilled for all $\varphi \in C_D^\infty(\Omega)$, it is also fulfilled for all $\varphi \in W_D^{1,q'}(\Omega)$ by a density argument.

Proposition 2.2. *Let Assumptions 1.1 and 1.2 hold.*

1. *Let $q \in [\bar{q}', 2]$. For any $u \in W_D^{-1,q}(\Omega)$ there exists a unique solution $w \in W_D^{1,q}(\Omega)$ of (2.3) in the sense (2.4). Moreover, the solution map $(-\nabla\cdot\rho\nabla)^{-1}\colon u \mapsto w$ is a bounded linear map $W_D^{-1,q}(\Omega) \to W_D^{1,q}(\Omega)$.*
2. *Let $q \in [\bar{q}', \frac{d}{d-1})$. For any $u \in \mathcal{M}_D(\overline{\Omega})$ there exists a unique solution $w \in W_D^{1,q}(\Omega)$ of (2.3) in the sense*
$$\int_\Omega \rho\nabla w\nabla\varphi \;\mathrm{d}x = \int_{\overline{\Omega}} \varphi \;\mathrm{d}u, \qquad \forall \varphi \in C_D^\infty(\Omega),$$
and the solution map $(-\nabla\cdot\rho\nabla)^{-1}\colon u \mapsto w$ is a bounded linear and weak-$\star$-to-strong continuous map $\mathcal{M}_D(\overline{\Omega}) \to W_D^{1,q}(\Omega)$.

Proof. 1. The elliptic operator $-\nabla\cdot\rho^T\nabla$ provides an isomorphism $W_D^{1,q'}(\Omega) \to W_D^{-1,q'}(\Omega)$ for $q' = \bar{q}$ by Assumption 1.2.2, and for $q' = 2$ by Lax-Milgram and the Poincaré-Friedrich inequality, because Γ_D has non-zero surface measure. Since Assumption 1.1 implies the requirements of [**2**], see, e.g., [**4**, Appendix A] for the verification, $(W_D^{1,q'}(\Omega))_{q'\in(1,\infty)}$ forms an interpolation scale according to [**2**, Theorem 1.2]. Consequently, we obtain by interpolation that $-\nabla\cdot\rho^T\nabla$ provides a topolocial isomorphism $W_D^{1,q'}(\Omega) \to W_D^{-1,q'}(\Omega)$ for all $q' \in [\bar{q}', 2]$. Hence, the claim follows by a standard duality argument as, e.g., in [**36**, Section 2].

2. For $q \in [1, \frac{d}{d-1})$ it holds $\mathcal{M}_D(\overline{\Omega}) \hookrightarrow W_D^{-1,q}(\Omega)$, which follows from $W_D^{1,q'}(\Omega) \hookrightarrow C_D(\overline{\Omega})$ for $q' > d$. Consequently, measure right-hand sides $u \in \mathcal{M}_D(\overline{\Omega})$ in (2.3) are included in the formulation (2.4) for these q via $\langle u, \varphi\rangle_{W_D^{-1,q}, W_D^{1,q'}} := \int_{\overline{\Omega}} \varphi \;\mathrm{d}u$. Weak-$\star$-to-strong continuity follows from compactness of $W_D^{1,q'}(\Omega) \hookrightarrow C_D(\overline{\Omega})$. ☐

In the above duality argument we have crucially made use of the isomophism property in Assumption 1.2.2 when obtaining unique solutions to (2.3), to be understood in the sense of (2.4). For an overview of different solution concepts of linear elliptic PDEs with measures on the right-hand side without such an isomorphism-property we refer the reader to, e.g., [**36**].

2.2. Well-posedness of the state equation. We now address existence, uniqueness, and additional regularity of solutions to (2.1). These results will be obtained from those of Section 2.1 by application of the so-callend Kirchhoff (or enthalpy) transform; see, e.g., [**39**, Example 2.74], [**44**, Chapter V]. Note that this has already been used in the context of optimal control of quasilinear elliptic equations in [**20, 21**] for instance. We define

$$\Xi\colon \mathbb{R} \to \mathbb{R}, \qquad s \mapsto \int_0^s \xi(t)\;\mathrm{d}t,$$

and observe that, at least on a formal level, $y \in W_D^{1,q}(\Omega)$, satisfies (2.2) if and only if $w = \Xi(y) \in W_D^{1,q}(\Omega)$ satisfies (2.4), i.e. w is a solution to (2.3). To make this idea mathematically precise in the following, we have to investigate the properties of the superposition operators associated with Ξ and Ξ^{-1} first:

Lemma 2.3.

1. *$\Xi\colon \mathbb{R} \to \mathbb{R}$ is stricly increasing, Lipschitz continuous, surjective, and hence bijective with continuous inverse $\Xi^{-1}\colon \mathbb{R} \to \mathbb{R}$, and the following linear growth-estimates hold:*
$$\xi_\bullet|s| \le |\Xi(s)| \le \xi^\bullet|s|, \qquad (\xi^\bullet)^{-1}|s| \le |\Xi^{-1}(s)| \le \xi_\bullet^{-1}|s|, \qquad \forall s \in \mathbb{R}.$$
Moreover, Ξ^{-1} is continuously differentiable with bounded derivative $(\Xi^{-1})' = 1/(\xi\circ\Xi^{-1})$.
2. *The superposition operators associated with Ξ and Ξ^{-1} are well defined as maps $W_D^{1,q}(\Omega) \to W_D^{1,r}(\Omega)$ for $1 \le r \le q \le \infty$. For $r = q$ they are inverse to each other; for $r < q$ they are continuous.*

3. *The superposition operators associated with* Ξ *and* Ξ^{-1} *are well defined and continuous as maps* $L^s(\Omega) \to L^r(\Omega)$ *for* $1 \leq r \leq s \leq \infty$. *Moreover, if* $r < s$ *they are continuously Fréchet differentiable.*

Proof. 1. is a direct consequence of the definition of Ξ and Assumption 1.2.1. Well-definedness in 2. follows from the chain rule, see, e.g., [**39**, Proposition 1.28], and continuity for $r < q$ can be shown by a short calculation. 3. follows from [**43**, Chapter 4.3.3]. □

Note that a positive lower bound for ξ is necessary to ensure surjectivity of Ξ and the linear growth bound for Ξ^{-1}. Boundedness of ξ ensures the linear growth bound for Ξ and therefore well definedness of the superposition operator associated with Ξ. Moreover, we already have used boundedness of ξ when considering the weak formulation (2.2). Therefore, the conditions on ξ from Assumption 1.2.1 are necessary for our further analysis.

We are now able to analyze the state equation (Eq):

Proposition 2.4. *Let Assumptions 1.1 and 1.2 hold. For any* $u \in \mathcal{M}_D(\overline{\Omega})$ *there exists a unique solution* y *to* (Eq) *in the sense of Definition 2.1. The solution exhibits additional regularity* $y \in W_D^{1,q}(\Omega)$, $q \in [\bar{q}', \frac{d}{d-1})$. *Moreover, the solution map* $S\colon u \mapsto y$ *of* (Eq) *is continuous and weak-$\star$-to-strong continuous as map*

$$S = \Xi^{-1} \circ (-\nabla \cdot \rho \nabla)^{-1}\colon \mathcal{M}_D(\overline{\Omega}) \to W_D^{1,q}(\Omega), \qquad q \in [\bar{q}', \frac{d}{d-1}).$$

In the following we will also refer to S as the control-to-state map.

Proof. According to Lemma 2.3.2 and the chain rule some $y \in W_D^{1,\bar{q}'}(\Omega)$ satisfies (2.1) if and only if $w = \Xi(w) \in W_D^{1,\bar{q}'}(\Omega)$ satisfies (2.4). Therefore, existence, uniqueness and regularity of solutions $y = \Xi^{-1}(w)$ to (2.1) follows directly from the respective theory for solutions w of (2.3) obtained in Proposition 2.2.2. □

Since Ξ^{-1} acts as a bijective map on $W_D^{1,q}(\Omega)$ for any $q \in [1, \infty]$, it is clear that solutions of (Eq) are unique in $W_D^{1,\bar{q}'}(\Omega)$, but not in $W_D^{1,1}(\Omega)$ in general. This is due to the fact that, as pointed out above in Section 2.1, solutions to (2.3) are in general not unique in $W_D^{1,1}(\Omega)$. Moreover, we note that in terms of Sobolev-regularity solutions of (Eq) are neither more nor less regular than the solutions of the $-\nabla \cdot \rho \nabla$-problem with measures on the right-hand side.

Finally, let us briefly explain why Γ_D is required to have nonzero surface measure in Assumption 1.1. In the case of pure Neumann boundary conditions it is well known that (Eq) would not even be well-posed with right-hand sides from $H_D^{-1}(\Omega)$ due to the lack of $H_D^1(\Omega)$-coercivity of the elliptic operator in the case $\Gamma_D = \emptyset$. Therefore, it would be necessary to add a zero-order linear term to the elliptic operator, e.g., by considering the modified equation

$$\text{(Eq')} \qquad \begin{cases} -\nabla \cdot \xi(y) \rho \nabla y + y = u, & \text{in } \Omega \cup \Gamma_N, \\ y = 0, & \text{on } \Gamma_D. \end{cases}$$

Now, (Eq') is well-posed also in the case $\Gamma_D = \emptyset$, at least if $u \in H_D^{-1}(\Omega)$. When considering $u \in \mathcal{M}_D(\overline{\Omega})$, however, the Kirchhoff transform of (Eq') leads to

$$\text{(2.5)} \qquad \begin{cases} -\nabla \cdot \rho \nabla w + (d \circ \Xi^{-1})(w) = u, & \text{in } \Omega \cup \Gamma_N, \\ w = 0, & \text{on } \Gamma_D, \end{cases}$$

which is a *semilinear* elliptic equation with a measure right-hand side although the original equation (Eq') was linear. Similarly as before, the solution map S of (Eq') would be given by $S = \Xi^{-1} \circ G$, where G denotes the solution map of (2.5) (associated with an appropriate solution concept and spaces); see, e.g., [**13**] for the analysis of an equation of type (2.5) under certain additional assumptions. Since the consideration of semilinear terms in (2.5) infers specific new difficulties, we exclude pure Neumann boundary conditions in Assumption 1.1.

2.3. Existence of solutions to (P)**.** Having at hand the previously shown properties of the control-to-state map, we can apply standard techniques to prove wellposedness of the optimal control problem. We introduce the abbreviations

$$F(u) := \frac{1}{2}\|S(u) - y_d\|^2_{L^2(\Omega)}, \qquad j(u) := \gamma\|u\|_{\mathcal{M}_D(\overline{\Omega})}, \qquad \hat{J}(u) := F(u) + j(u).$$

Proposition 2.5. *Under Assumptions 1.1-1.3 there exists a solution to* (P)*.*

Proof. F is weak-$\star$ continuous due to Proposition 2.4, and j is weak-$\star$ sequentially lower semicontinuous [**6**, Theorem 3.13iii] and coercive due to $\gamma > 0$. Consequently, $\hat{J}$ is bounded from below, weak-$\star$-sequentially lower semicontinuous, and coercive. Therefore, standard arguments that ensure existence of a minimizer apply. ☐

For the necessity of $\gamma > 0$, which ensures coercivity of $\hat{J}$, to guarantee existence of solutions to (P) we refer the reader to Remark 3.4 below.

Remark 2.6. Under the same assumptions as in Proposition 2.5 also the problem with the additional control-constraint "$u \in U_{\text{ad}}$" with a weak-$\star$ sequentially closed set $U_{\text{ad}} \subset \mathcal{M}_D(\overline{\Omega})$, is well-posed. For instance, consider U_{ad} to be the subset of nonnegative measures in $\mathcal{M}_D(\overline{\Omega})$. The condition $\gamma > 0$ can be dropped, if U_{ad} is weak-$\star$-compact, which is the case, e.g., if U_{ad} is weak-$\star$-sequentially closed and bounded [**6**, Theorem 3.16]. For instance, given $\mu_a, \mu_b \in \mathcal{M}_D(\overline{\Omega})$ such that $\mu_a \leq \mu_b$, the choice $U_{\text{ad}} := \{\mu \in \mathcal{M}_D(\overline{\Omega})\colon \mu_a \leq \mu \leq \mu_b\}$, is possible. Here, "$\leq$" denotes the ordering on $\mathcal{M}_D(\overline{\Omega})$ induced by the natural ordering on $C_D(\overline{\Omega})$. One may think of such U_{ad} as analogon of classical box-constraints in the space of Borel-measures.

3. First-order optimality conditions

In this section we derive first-order necessary optimality conditions for local solutions of (P). Since the analysis of the nonsmooth part of J can be adapted from [**13**] without any changes, we concentrate on the analysis of the smooth part which differs from this setting. As a first step, we analyze differentiability properties of the control-to-state map in Section 3.1. As in the previous section, our results are based on the Kirchhoff transform. Now, the crucial point is to analyze differentiability of the nonlinear superposition operator of the Kirchhoff transform. After these preliminaries we derive first-order conditions for (P) using in principal the same ideas as in [**13**]. Surprisingly, the Kirchhoff transform allows to address first-order differentiability of the control-to-state map as well as first-order necessary optimality conditions without requiring a derivative of ξ.

3.1. Differentiability of the control-to-state map. A direct computation of the derivative of S by application of the implicit function theorem seems to be difficult. It would require to discuss in particular invertibility of $-\nabla \cdot \xi(y)\rho\nabla - \nabla \cdot \xi'(y) \cdot \rho\nabla y$ on the low-regularity space $\mathcal{M}_D(\overline{\Omega})$, which is particularly difficult due to the low regularity of $\xi(y)$ and $\xi'(y)$. Therefore we follow the same approach as in the last section and utilize the Kirchhoff transform:

Proposition 3.1. *The control-to-state map S is continuously Fréchet differentiable as map $\mathcal{M}_D(\overline{\Omega}) \to L^r(\Omega)$ for each $1 \leq r < \frac{d}{d-2}$. It holds*

$$S'(u)v = \xi(y)^{-1}(-\nabla \cdot \rho\nabla)^{-1}v, \qquad u, v \in \mathcal{M}_D(\overline{\Omega}),\ y = S(u). \tag{3.1}$$

Proof. Recall from Proposition 2.2 that $(-\nabla \cdot \rho\nabla)^{-1} \in \mathcal{L}(\mathcal{M}_D(\overline{\Omega}), W^{1,q}_D(\Omega))$ for each $q \in [1, \frac{d}{d-1})$. By Sobolev embedding it follows that $(-\nabla \cdot \rho\nabla)^{-1} \in \mathcal{L}(\mathcal{M}_D(\overline{\Omega}), L^s(\Omega))$ for all $s \in [1, \frac{d}{d-2})$. Hence the claim follows from the chain rule, applied to $S = \Xi^{-1} \circ (-\nabla \cdot \rho\nabla)^{-1}$, and the differentiability properties of $\Xi^{-1}\colon L^s(\Omega) \to L^r(\Omega)$; cf. Lemma 2.3.4. ☐

A purely formal computation shows that the formula for the derivative is the same we could expect from the implicit function theorem.

3.2. First-order necessary optimality conditions. From Section 2.3 recall the definition of $\hat{J}$, F and j. For a discussion of directional derivatives and properties of the subdifferential of j we refer the reader to [**13**, Propositions 3.2, 3.3] for instance. Due to Proposition 3.1 and $d \le 3$ the functional F is continuously Fréchet differentiable. The derivative of F for $u, v \in \mathcal{M}_D(\overline{\Omega})$ is computed as follows:

$$F'(u)v = \langle S(u) - y_d, S'(u)v\rangle_{L^2(\Omega)} = \langle y - y_d, \xi(y)^{-1}(-\nabla\cdot\rho\nabla)^{-1}v\rangle_{L^2(\Omega)} \\ = \langle p, v\rangle_{C_D(\overline{\Omega}),\mathcal{M}_D(\overline{\Omega})},$$

where we have introduced the so-called adjoint state $p = (-\nabla\cdot\rho^T\nabla)^{-1}(\xi(y)^{-1}(y - y_d)) \in W_D^{1,\bar{q}}(\Omega) \hookrightarrow C_D(\overline{\Omega})$. Here, recall Assumption 1.2.2 and that $\xi(y)^{-1}(y - y_d) \in L^2(\Omega) \hookrightarrow W_D^{-1,\bar{q}}(\Omega)$ due to $d \le 3$ and $\bar{q} \le 6$. Consequently, the linear form $F'(u)$ is not only an element of the highly abstract dual space $\mathcal{M}_D(\overline{\Omega})^*$ but can be represented by some p in the predual $C_D(\overline{\Omega})$. Therefore the same arguments as in [**13**] can be utilized to derive first-order necessary optimality conditions. In fact, these first-order conditions even hold at a local solution of (P) for a notion of local optimality that is weaker than local optimality in the $\mathcal{M}_D(\overline{\Omega})$-topology.

Theorem 3.2. *Let Assumptions 1.1-1.3 hold and $\bar{u} \in \mathcal{M}_D(\overline{\Omega})$ be a local solution of* (P) *with respect to the $W_D^{-1,q}(\Omega)$-topology for some $q \in (1, \frac{d}{d-1})$. Then there exists a so-called adjoint state $\bar{p} \in W_D^{1,\bar{q}}(\Omega)$ such that*

$$\begin{cases} -\nabla\cdot\xi(\bar{y})\rho\nabla\bar{y} = u, & \text{on } \Omega\cup\Gamma_N, \\ \bar{y} = 0, & \text{on } \Gamma_D, \end{cases} \tag{3.2}$$

$$\begin{cases} -\nabla\cdot\rho^T\nabla\bar{p} = \xi(\bar{y})^{-1}(\bar{y} - y_d) & \text{on } \Omega, \\ \bar{p} = 0, & \text{on } \Gamma_D, \end{cases} \tag{3.3}$$

$$\gamma\|\bar{u}\|_{\mathcal{M}_D(\overline{\Omega})} + \int_{\overline{\Omega}} \bar{p}\ \mathrm{d}\bar{u} = 0, \quad \text{and} \quad \|\bar{p}\|_{C_D(\overline{\Omega})} \begin{cases} = \gamma & \text{if } \bar{u} \neq 0, \\ \le \gamma & \text{if } \bar{u} = 0. \end{cases} \tag{3.4}$$

Moreover, if $\bar{u} \neq 0$ it holds

$$\operatorname{supp}(\bar{u}^+) \subset \{x \in \overline{\Omega} \colon \bar{p}(x) = -\gamma\}, \quad \text{and} \quad \operatorname{supp}(\bar{u}^-) \subset \{x \in \overline{\Omega} \colon \bar{p}(x) = +\gamma\}. \tag{3.5}$$

Condition (3.5) means that the optimal control $\bar{u}$ only acts on a certain, possibly small, compact subset of $\Omega\cup\Gamma_N$, which we call "sparse" optimal control.

Proof. The proof works completely analogous to the proof of [**13**, Theorem 3.4] in the semilinear elliptic case. We utilize $\|\cdot\|_{W_D^{-1,q}(\Omega)} \lesssim \|\cdot\|_{\mathcal{M}_D(\overline{\Omega})}$, differentiability of F, and convexity of j to conclude from $\hat{J}(u) \ge \hat{J}(\bar{u})$ for all $u \in \mathbb{B}_\varepsilon^{W_D^{-1,q}(\Omega)}(\bar{u})$ that

$$0 \le \lim_{t\to 0} \frac{\hat{J}(\bar{u} + t(u - \bar{u})) - \hat{J}(\bar{u})}{t} \le F'(\bar{u})(u - \bar{u}) + j(u) - j(\bar{u}), \qquad \forall u \in \mathcal{M}_D(\overline{\Omega}),$$

and hence

$$-\int_{\overline{\Omega}} \bar{p}\ \mathrm{d}(u - \bar{u}) \le j(u) - j(\bar{u}), \qquad \forall u \in \mathcal{M}_D(\overline{\Omega}).$$

By definition, this means $-\bar{p} \in \partial j(\bar{u})$. Due to $\bar{p} \in C_D(\overline{\Omega})$, the characterization of $\partial j(u) \cap C_D(\overline{\Omega})$ from [**13**, Proposition 3.2] implies (3.4) and (3.5). □

Utilizing a purely formal computation, the adjoint equation (3.3) could be rewritten as

$$-\nabla\cdot\xi(\bar{y})\rho^T\nabla\bar{p} + \xi'(\bar{y})\rho\nabla\bar{y}\nabla\bar{p} = \bar{y} - y_d \quad \text{on } \Omega,$$

which coincides with the result that would be obtained by –also purely formal– application of the implicit function theorem.

Remark 3.3. The approach used in the proof of Theorem 3.2 is no longer applicable if minimization does not take place over the whole space $\mathcal{M}_D(\overline{\Omega})$, e.g., in the setting described in Remark 2.6. In case of certain linear-quadratic problems with the nonnegative measures as admissible set, first-order necessary optimality conditions have been derived with the help of Fenchel duality [**22**]. Whether a similar result may be obtained also for nonconvex problems is to our knowledge an open problem.

Remark 3.4. Actually, we do not need the assumption $\gamma > 0$ in Theorem 3.2. For $\gamma = 0$, however, it follows $\bar{p} = 0$, and hence $\bar{y} = y_d$ a.e. on Ω. Consequently, if $\bar{u}$ is a local solution to (P) for $\gamma = 0$, then the associated state $\bar{y}$ has to satisfy $\bar{y} = y_d$. We conclude that there are functions $y_d \in L^2(\Omega)$ such that (P) does not have a local solution for $\gamma = 0$.

4. Second-order optimality conditions in 2D

In general, (P) is nonconvex, and hence the first-order conditions derived in the last section are not necessarily sufficient for optimality. Therefore, we address second-order optimality conditions in the present section. First, we discuss second-order derivatives of the control-to-state map and the objective functional in Section 4.1. This works analogous to the first-order analysis in Section 3.1. However, as it will become clear in Section 4.1 below we have to restrict our second-order analysis of (P) to space dimension $d \leq 2$ at the moment. We follow roughly the techniques of [**13**], and derive necessary and sufficient second-order conditions in Sections 4.2 and 4.3, respectively. For a modified version of (P) that allows to avoid the restriction to $d \leq 2$ we refer the reader to Section 4.4.

4.1. Second derivative of the control-to-state map.

In order to discuss second-order derivatives of the objective functional, we require an additional assumption on the coefficient function ξ that ensures second-order differentiability of the control-to-state map.

Assumption 4.1. Let ξ be continuously differentiable, and suppose that there are $a, b \in \mathbb{R}$ such that $|\xi'(s)| \leq a + b|s|$, for all $s \in \mathbb{R}$.

Under this additional assumption, the map Ξ^{-1} is twice continuously differentiable with second derivative $(\Xi^{-1})'' = -\frac{\xi'}{\xi^3} \circ \Xi^{-1}$ satisfying a linear growth bound.

Proposition 4.2. *Under Assumptions 1.1-1.3 and 4.1 the control-to-state map S is twice continuously Fréchet differentiable as map $\mathcal{M}_D(\overline{\Omega}) \to L^r(\Omega)$ for $r \in [1, \frac{d}{2(d-2)})$ with second derivative*

$$S''(u)[v_1, v_2] = -\frac{\xi'(y)}{\xi(y)} z_1 z_2 = -\frac{\xi'(y)}{\xi(y)^3} (-\nabla \cdot \rho \nabla)^{-1} v_1 (-\nabla \cdot \rho \nabla)^{-1} v_2,$$

with $y = S(u)$, $z_i = S'(u) v_i$, for each $u, v_1, v_2 \in \mathcal{M}_D(\overline{\Omega})$.

Proof. As in the proof of Proposition 3.1 it holds $(-\nabla \cdot \rho \nabla)^{-1} \in \mathcal{L}(\mathcal{M}_D(\overline{\Omega}), L^r(\Omega))$ for $r \in [1, \frac{d}{d-2})$. Moreover, due to the linear growth bound on $(\Xi^{-1})''$ the superposition operator Ξ^{-1} is now C^2-differentiable as map $L^r(\Omega) \to L^s(\Omega)$ for $r > 2s$; cf. [**43**, Chapter 4.3.3]. Therefore, second-order differentiability and the formula for the derivative of S follows from the chain rule applied to $S = \Xi^{-1} \circ (-\nabla \cdot \rho \nabla)^{-1}$. □

Since the objective functional is only well defined for at least L^2-integrable states, the above result excludes discussion of second-order optimality conditions in space dimension $d = 3$. For an approach to avoid this restriction by introducing a modified variant of (P) we refer the reader to Section 4.4. Nevertheless, at the moment we have obtained the following property for F: For $d \leq 2$, F is twice continuously differentiable with second derivative given by

$$F''(u)[v_1, v_2] = \int_\Omega \left[1 - \frac{\xi'(y)}{\xi(y)}(y - y_d)\right] z_1 z_2 \ \mathrm{d}x, \qquad y = S(u), z_i = S'(u) v_i, \tag{4.1}$$

for each $u, v_1, v_2 \in \mathcal{M}_D(\overline{\Omega})$. Interestingly we do not require second derivatives of ξ for the second-order analysis of (P). This is again due to the special structure of (Eq) and the Kirchhoff transform.

4.2. Second-order necessary optimality conditions. To prove second-order necessary optimality conditions, we follow the technique of the proof of [**13**, Theorem 3.7]. Given a local solution $\bar u \in \mathcal{M}_D(\overline{\Omega})$ to (P) we introduce the critical cone

$$C_{\bar u} := \{v \in \mathcal{M}_D(\overline{\Omega})\colon F'(\bar u)v + \gamma j'(\bar u, v) = 0\}.$$

From [**13**, Proposition 3.5] we recall that $C_{\bar u}$ is a closed convex cone that can equivalently be expressed as $C_{\bar u} = \{v \in \mathcal{M}_D(\overline{\Omega})\colon \int_{\overline{\Omega}} \bar u \;\mathrm{d}v_s + \gamma \|v_s\|_{\mathcal{M}_D(\overline{\Omega})} = 0\}$. Hereby, for some $v \in \mathcal{M}_D(\overline{\Omega})$ we denote by $v_a, v_s \in \mathcal{M}_D(\overline{\Omega})$ the uniquely determined absolutely continuous and singular parts, respectively, with respect to $|\bar u|$, i.e. $v = v_a + v_s$ with $v_a = g_v \;\mathrm{d}|\bar u|$ and $g_v := \frac{\mathrm{d}v}{\mathrm{d}|\bar u|} \in L^1(\overline{\Omega}, \;\mathrm{d}|\bar u|)$; cf. [**40**, Theorem 6.10]. Using this, the critical cone also may be written as $C_{\bar u} = \{v \in \mathcal{M}_D(\overline{\Omega})\colon \operatorname{supp}(v_s^+) \subset \Omega_{-\gamma},\ \operatorname{supp}(v_s^-) \subset \Omega_{+\gamma}\}$, with $\Omega_{\pm\gamma} := \{x \in \Omega \cup \Gamma_N\colon \bar p(x) = \pm\gamma\}$; cf. [**13**, Remark 3.6]. As in [**13**, Remark 3.8] the following second-order necessary conditions even hold for local solutions with respect to a weaker notion of local optimality.

Theorem 4.3. *Let Assumptions 1.1-1.3 and 4.1, and $d \le 2$ hold. Assume that $\bar u \in \mathcal{M}_D(\overline{\Omega})$ is a local solution to* (P) *w.r.t. the $W_D^{-1,q}(\Omega)$-topology for some $q \in (1,2)$. Then it holds $F''(\bar u)v^2 \ge 0$ for all $v \in C_{\bar u}$.*

Proof. This works in completely the same way as the proof of [**13**, Theorem 3.7], since $\|\cdot\|_{W_D^{-1,q}(\overline{\Omega})} \lesssim \|\cdot\|_{\mathcal{M}_D(\overline{\Omega})}$ and we can show that $u_k \to \bar u$, $v_k \to v$ in $\mathcal{M}_D(\overline{\Omega})$ implies $F''(u_k)v_k^2 \to F''(\bar u)v^2$: Employing formulas (4.1) and (3.1) we have to show

$$\int_\Omega \left[1 - \frac{\xi'(y_k)}{\xi(y_k)}(y_k - y_d)\right] (\xi(y_k)^{-1}(-\nabla\cdot\rho\nabla)^{-1}v_k)^2 \;\mathrm{d}x \\ \to \int_\Omega \left[1 - \frac{\xi'(\bar y)}{\xi(\bar y)}(\bar y - y_d)\right] (\xi(\bar y)^{-1}(-\nabla\cdot\rho\nabla)^{-1}v)^2 \;\mathrm{d}x, \tag{4.2}$$

with $y_k = S(u_k)$. First, note that $u_k \to \bar u$ in $\mathcal{M}_D(\overline{\Omega})$ implies $u_k \to \bar u$ in $W_D^{-1,q}(\Omega)$, $q \in (1,2)$, and therefore $y_k \to \bar y$ in $W_D^{1,q}(\Omega)$, $q \in (\bar q, 2)$ by Proposition 2.4. In particular, we obtain convergence $y_k \to \bar y$ in $L^r(\Omega)$ for any $r \in (1,\infty)$, and it also follows $\frac{\xi'(y_k)}{\xi(y_k)} \to \frac{\xi'(\bar y)}{\xi(\bar y)}$ and $\xi(y_k)^{-1} \to \xi(\bar y)^{-1}$ in $L^r(\Omega)$. Similarly, $v_k \to v$ in $\mathcal{M}_D(\overline{\Omega})$ implies $(-\nabla\cdot\rho\nabla)^{-1}v_k \to (-\nabla\cdot\rho\nabla)^{-1}v$ in $L^r(\Omega)$ for any $r \in (1,\infty)$. Therefore, (4.2) follows from an application of Hölder's inequality. ☐

4.3. Second-order sufficient optimality conditions. Having established second-order necessary optimality conditions, we now turn towards sufficient conditions. As in many other situations, we cannot prove a sufficient second-order condition with minimal gap, i.e. a sufficient condition that is imposed on the same cone of directions as the corresponding second-order necessary condition. Instead, given some $\tau > 0$ we have to introduce the exended cone of critical directions as follows:

$$C_{\bar u}^\tau := \{v \in \mathcal{M}_D(\overline{\Omega})\colon F'(\bar u)v + \gamma j'(\bar u, v) \le \tau \|z_v\|_{L^2(\Omega)}^2\},$$

where $z_v = S'(\bar u)v$. With respect to this larger cone we prove two different second-order sufficient conditions in the following. The first one is of the same type and can also be proven in exactly the same way as [**13**, Theorem 4.2]. The lack of regularity of the second derivative of F arising from dealing with low-regularity controls is compensated by demanding coercivity of F'' not only at $\bar u$, but also in an appropriate neighbourhood of $\bar u$. Exploiting the particular structure of our optimal control problem we are able to obtain also a result that only requires coercivity of $F''(\bar u)$. The price to pay is that this condition is no longer sufficient for a local solution in the classical sense, but only for a certain weaker implication.

Before stating and proving our results, let us briefly comment on a further issue that is specific for sufficient second-order conditions in the context of optimal control by measures: Local optimality in the following results is always meant with respect to the $W_D^{-1,q}(\Omega)$-topology instead of the $\mathcal{M}_D(\overline{\Omega})$-topology. For the reason why this is the clearly more appropriate choice we refer the reader to [**13**, Remark 4.1] that applies vice-versa to the present case.

Theorem 4.4. *Let Assumptions 1.1-1.3 and 4.1, and $d \leq 2$ hold. Let $\bar u \in \mathcal{M}_D(\overline{\Omega})$ satisfy the first-order necessary optimality conditions* (3.2)-(3.4) *and*

$$F''(u)v^2 \geq \kappa \|z_v\|^2_{L^2(\Omega)}, \qquad \forall v \in C^\tau_{\bar u},\ u \in \mathbb{B}_\rho^{W_D^{-1,q}(\Omega)}(\bar u), \tag{4.3}$$

with some $\tau, \rho, \kappa > 0$ and $q \in [\max(\bar q', \frac{3}{2}), 2)$. Moreover, let $y_d \in L^s(\Omega)$ with $s \geq (q^{-1} - \frac{1}{2})^{-1}$. Then there are $\varepsilon, \delta > 0$ such that

$$\hat J(u) \geq \hat J(\bar u) + \frac{\delta}{2}\|z_{u-\bar u}\|^2_{L^2(\Omega)}, \qquad \forall u \in \mathbb{B}_\varepsilon^{W_D^{-1,q}(\Omega)}(\bar u). \tag{4.4}$$

In particular, $\bar u$ is a strict $W_D^{-1,q}(\Omega)$-local solution to (P).

The proof works in completely the same way as the proof of [**13**, Theorem 4.2] in the semilinear elliptic case. For convenience, we nevertheless provide the details.

Proof. Assume that the contrary of (4.4) holds: There is a sequence $(u_k)_k \subset \mathcal{M}_D(\overline{\Omega})$ such that $u_k \to \bar u$ in $W_D^{-1,q}(\Omega)$ and

$$\hat J(u_k) < \hat J(\bar u) + \frac{1}{2k}\|z_{u_k - \bar u}\|^2_{L^2(\Omega)}, \qquad \forall k \in \mathbb{N}. \tag{4.5}$$

We want to prove that $u_k - \bar u \in C^\tau_{\bar u}$ for k sufficiently large: Taylor expansion yields

$$\begin{aligned} \frac{1}{2k}\|z_{u_k-\bar u}\|^2_{L^2(\Omega)} &> \hat J(u_k) - \hat J(\bar u) \\ &= F'(\bar u)(u_k - \bar u) + \frac{1}{2}F''(u_k^\theta)(u_k - \bar u)^2 + \gamma j'(\bar u, u_k - \bar u), \end{aligned} \tag{4.6}$$

with $u_k^\theta = (1-\theta_k)\bar u + \theta_k u_k$ and $\theta_k \in [0,1]$. In particular, it holds

$$F'(\bar u)(u_k - \bar u) + \gamma j'(\bar u, u_k - \bar u) \leq \frac{1}{2k}\|z_{u_k-\bar u}\|^2_{L^2(\Omega)} + \frac{1}{2}\left|F''(u_k^\theta)(u_k - \bar u)^2\right|. \tag{4.7}$$

The first summand is estimated as follows:

$$\|z_{u_k-\bar u}\|^2_{L^2(\Omega)} \lesssim \|u_k - \bar u\|_{W^{-1,q}}\|z_{u_k-\bar u}\|_{L^2}, \tag{4.8}$$

where we have used that $\|(-\nabla\cdot\rho\nabla)^{-1}\|_{\mathcal{L}(W_D^{-1,q},L^2)} < \infty$ due to $\bar q' \leq q < 2$. Regarding the second summand, recall formulas (4.1) and (3.1), and apply Hölder's inequality to obtain:

$$\left|F''(u_k^\theta)(u_k - \bar u)^2\right| \leq \left\| \left(1 - \frac{\xi'(y_k^\theta)}{\xi(y_k^\theta)}(y_k^\theta - y_d)\right) \frac{\xi(\bar y)}{\xi(y_k^\theta)^2}[(-\nabla\cdot\rho\nabla)^{-1}(u_k - \bar u)]\right\|_{L^2} \cdot \|z_{u_k-\bar u}\|_{L^2}.$$

Fix $r^{-1} := q^{-1} - \frac{1}{2}$. From $u_k^\theta \to \bar u$ in $W_D^{-1,q}(\Omega)$ it follows that $y_k^\theta := S(u_k^\theta) \to \bar y$ and $\frac{\xi'(y_k^\theta)}{\xi(y_k^\theta)} \to \frac{\xi'(\bar y)}{\xi(\bar y)}$ in $L^r(\Omega)$. Together with $y_d \in L^s(\Omega) \hookrightarrow L^r(\Omega)$ we conclude that there is a constant $C > 0$ such that $\|1 - \frac{\xi'(y_k^\theta)}{\xi(y_k^\theta)}(y_k^\theta - y_d)\|_{L^{r/2}} \leq C$ for all k. Moreover, it holds $\|(-\nabla\cdot\rho\nabla)^{-1}\|_{\mathcal{L}(W_D^{-1,q},L^r)} < \infty$. Due to $\frac{3}{r} \leq \frac{1}{2}$ we can apply Hölder's inequality to obtain:

$$\begin{aligned} \left|F''(u_k^\theta)(u_k - \bar u)^2\right| &\lesssim \left\|1 - \frac{\xi'(y_k^\theta)}{\xi(y_k^\theta)}(y_k^\theta - y_d)\right\|_{L^{r/2}} \|(-\nabla\cdot\rho\nabla)^{-1}(u_k - \bar u)\|_{L^r}\|z_{u_k-\bar u}\|_{L^2} \\ &\lesssim \|u_k - \bar u\|_{W_D^{-1,q}}\|z_{u_k-\bar u}\|_{L^2}. \end{aligned} \tag{4.9}$$

Combining our estimates for (4.8) and (4.9) with (4.7) we find that

$$F'(\bar u)(u_k - \bar u) + \gamma j'(\bar u, u_k - \bar u) \lesssim \rho(k)\|z_{u_k-\bar u}\|_{L^2(\Omega)}, \qquad \forall k,$$

with $\rho(k) \to 0$ as $k \to \infty$. Hence, for any fixed $\tau > 0$ there is $k_0 \in \mathbb{N}$ such that $u_k - \bar u \in C^\tau_{\bar u}$ for all $k \geq k_0$. W.l.o.g. we can assume that $u_k^\theta \in \mathbb{B}_\varepsilon^{W_D^{-1,q}}(\bar u)$ for these k. Therefore, combining (4.3) and (4.6) we obtain

$$\frac{\kappa}{2}\|z_{u_k-\bar u}\|^2_{L^2} \le \frac{1}{2}F''(u_k^\theta)(u_k-\bar u)^2 \le \frac{1}{2}F''(u_k^\theta)(u_k-\bar u)^2 + F'(\bar u)(u_k-\bar u) + \gamma j'(\bar u, u_k-\bar u)$$
$$= \hat J(u_k) - \hat J(\bar u) < \frac{1}{2k}\|z_{u_k-\bar u}\|^2_{L^2}, \qquad \forall k \ge k_0,$$

which yields the desired contradiction. □

Coercivity of the second derivative in (4.3) has not only to hold at $\bar u$, but also in a neighbourhood of $\bar u$, which is a rather untypical condition. Therefore, we analyze which kind of conclusion can be drawn from imposing coercivity of the second derivative at $\bar u$, only:

Theorem 4.5. *Let Assumptions 1.1-1.3, 4.1, $d \le 2$, and $y_d \in L^\infty(\Omega)$ hold. Let $\bar u \in \mathcal{M}_D(\overline\Omega)$ satisfy (3.2)-(3.4) with $\bar y \in L^\infty(\Omega)$. If*

$$F''(\bar u)v^2 \ge \kappa\|z_v\|^2_{L^2(\Omega)}, \qquad \forall v \in C^\tau_{\bar u}, \tag{4.10}$$

holds with some $\tau, \kappa > 0$, then there are $\varepsilon, \delta > 0$ such that

$$\hat J(u) \ge \hat J(\bar u) + \frac{\delta}{2}\|z_{u-\bar u}\|^2_{L^2(\Omega)}, \quad \forall u \in \mathcal{M}_D(\overline\Omega) \text{ s.t. } \|S(u)-\bar y\|_{L^\infty(\Omega)} < \varepsilon. \tag{4.11}$$

The quadratic growth condition (4.11) implies that

$$\hat J(u) > \hat J(\bar u), \qquad \forall u \in \mathcal{M}_D(\Omega)\setminus\{\bar u\} \quad \text{s.t.} \quad \|S(u)-\bar y\|_{L^\infty(\Omega)} < \varepsilon.$$

Consequently, $\bar u$ is a strong local minimum of (P) in the sense of [**1**, Definition 2.6]. However, in our case this property is weaker than being a local minimum in the "classical" sense (weak local minimum in [**1**, Definition 2.6]), since (4.11) only refers to those $u \in \mathcal{M}_D(\overline\Omega)$ such that $S(u) \in L^\infty(\Omega)$. We give a more detailed explanation in Section 4.4. Before proving Theorem 4.5 we state an auxiliary result that has some similarity to [**15**, Lemma 2.4]:

Lemma 4.6. *Let $(u_k)_k \subset \mathcal{M}_D(\overline\Omega)$, $\bar u \in \mathcal{M}_D(\overline\Omega)$ such that $y_k = S(u_k) \in L^\infty(\Omega)$, $\bar y = S(\bar u) \in L^\infty(\Omega)$ and $y_k \to \bar y$ in $L^\infty(\Omega)$. Then it holds $z_{u_k-\bar u} = S'(\bar u)(u_k-\bar u) \in L^\infty(\Omega)$ for all k and there is a constant $C>0$ such that*

$$\|z_{u_k-\bar u}\|_{L^\infty(\Omega)} \le C\|y_k-\bar y\|_{L^\infty(\Omega)}.$$

Again, the Kirchhoff transform plays a central role in the argument.

Proof of Lemma 4.6. By Propositions 3.1 and 2.4 it holds

$$z_{u_k-\bar u} = \xi(\bar y)^{-1}(-\nabla\cdot\rho\nabla)^{-1}(u_k-\bar u) = \xi(\bar y)^{-1}\left(\Xi(y_k)-\Xi(\bar y)\right) \in L^\infty(\Omega),$$

because Ξ acts as continuous superposition operator on $L^\infty(\Omega)$. Taylor expansion and Proposition 4.2 yield

$$\|y_k-\bar y-z_{u_k-\bar u}\|_{L^\infty} = \left\|\frac{\xi'(y_k^\theta)}{\xi(y_k^\theta)^3}[(-\nabla\cdot\rho\nabla)^{-1}(u_k-\bar u)]^2\right\|_{L^\infty} \tag{4.12}$$
$$\le \left\|\frac{\xi'(y_k^\theta)}{\xi(y_k^\theta)^3}\right\|_{L^\infty}\left\|(-\nabla\cdot\rho\nabla)^{-1}(u_k-\bar u)\right\|^2_{L^\infty} = \left\|\frac{\xi'(y_k^\theta)}{\xi(y_k^\theta)^3}\right\|_{L^\infty}\left\|\Xi(y_k)-\Xi(\bar y)\right\|^2_{L^\infty},$$

with $u_k^\theta = (1-\theta_k)\bar u + \theta_k u_k$, $y_k^\theta = S(u_k^\theta)$, and some $\theta_k \in [0,1]$. Finally, note that

$$\left\|\Xi(y_k)-\Xi(\bar y)\right\|_{L^\infty} \le \xi^\bullet\|y_k-\bar y\|_{L^\infty}, \quad \text{and} \tag{4.13}$$

$$y_k^\theta = \Xi^{-1}((1-\theta_k)\Xi(\bar y)+\theta_k\Xi(y_k)) \to \bar y, \quad \text{in } L^\infty(\Omega), \tag{4.14}$$

due to the continuity of Ξ and Ξ^{-1} on $L^\infty(\Omega)$. Combinig (4.14) and (4.13) with (4.12) yields the claim. □

Proof of Theorem 4.5. Assuming the contrary of (4.11), we find a sequence $(u_k)_k \subset \mathcal{M}_D(\overline\Omega)$ such that $y_k \in L^\infty(\Omega)$, $y_k \to \bar y$ in $L^\infty(\Omega)$ and

$$\hat J(u_k) < \hat J(\bar u) + \frac{1}{2k}\|z_{u_k-\bar u}\|^2_{L^2(\Omega)}, \qquad \forall k \in \mathbb{N}. \tag{4.15}$$

Note that, unlike in the proof of Theorem 4.4, we do not have information on the behaviour of $(u_k)_k$. Nevertheless, we can proof $u_k - \bar{u} \in C_{\bar{u}}^{\tau}$ for sufficiently large k by a similar technique as before: Using Taylor expansion as in (4.6) we obtain (4.7). The delicate point is to avoid the usage of $u_k - \bar{u}$ when estimating the two summands of (4.7). This can be done with the help of Lemma 4.6. Instead of (4.8) we obtain by Lemma 4.6:

$$\|z_{u_k-\bar{u}}\|_{L^2(\Omega)}^2 \lesssim \|z_{u_k-\bar{u}}\|_{L^\infty(\Omega)}\|z_{u_k-\bar{u}}\|_{L^2(\Omega)} \lesssim \|y_k - \bar{y}\|_{L^\infty}\|z_{u_k-\bar{u}}\|_{L^2(\Omega)}, \tag{4.8'}$$

and, similarly, we find instead of (4.9):

$$\begin{aligned} |F''(u_k^\theta)(u_k-\bar{u})^2| &\leq \left\| \left(1 - \frac{\xi'(y_k^\theta)}{\xi(y_k^\theta)}(y_k^\theta - y_d)\right) \frac{\xi(\bar{y})^2}{\xi(y_k^\theta)^2} \right\|_{L^\infty} \|z_{u_k-\bar{u}}\|_{L^2}^2 \\ &\lesssim \|y_k - \bar{y}\|_{L^\infty}\|z_{u_k-\bar{u}}\|_{L^2(\Omega)}. \end{aligned} \tag{4.9'}$$

As in the proof of Theorem 4.4 combination of (4.7), (4.8'), and (4.9') yields $u_k - \bar{u} \in C_{\bar{u}}^{\tau}$ for all sufficiently large k. It remains to modify the last step of the proof of Theorem 4.4 in which the contradiction was obtained: Now, we obtain

$$\begin{aligned} \frac{\kappa}{2}\|z_{u_k-\bar{u}}\|_{L^2}^2 &\leq \frac{1}{2}F''(\bar{u})(u_k-\bar{u})^2 \leq \hat{J}(u_k) - \hat{J}(\bar{u}) + \frac{1}{2}[F''(\bar{u}) - F''(u_k^\theta)](u_k-\bar{u})^2 \\ &< \frac{1}{2k}\|z_{u_k-\bar{u}}\|_{L^2}^2 + \frac{1}{2}[F''(\bar{u}) - F''(u_k^\theta)](u_k-\bar{u})^2, \qquad \forall k \geq k_0. \end{aligned}$$

To obtain a contradiction, we will show

$$|[F''(\bar{u}) - F''(u_k^\theta)](u_k-\bar{u})^2| = o\left(\|z_{u_k-\bar{u}}\|_{L^2}^2\right) \qquad \text{as } k \to \infty, \tag{4.16}$$

in the remaining part of the proof: Employing formulas (4.1) and (3.1) yields

$$\begin{aligned} &[F''(\bar{u}) - F''(u_k^\theta)](u_k-\bar{u})^2 \\ &\quad = \int_\Omega \left[\left(1 - \frac{\xi'(\bar{y})}{\xi(\bar{y})}(\bar{y} - y_d)\right) - \frac{\xi(\bar{y})^2}{\xi(y_k^\theta)^2}\left(1 - \frac{\xi'(y_k^\theta)}{\xi(y_k^\theta)}(y_k^\theta - y_d)\right)\right] z_{u_k-\bar{u}}^2 \,\mathrm{d}x, \end{aligned}$$

and, consequently, in order to prove (4.16) it suffices to show

$$\left\|\left(1 - \frac{\xi'(\bar{y})}{\xi(\bar{y})}(\bar{y} - y_d)\right) - \frac{\xi(\bar{y})^2}{\xi(y_k^\theta)^2}\left(1 - \frac{\xi'(y_k^\theta)}{\xi(y_k^\theta)}(y_k^\theta - y_d)\right)\right\|_{L^\infty} \to 0, \text{ as } k \to \infty. \tag{4.17}$$

The latter, however, is clear due to $y_k^\theta \to \bar{y}$ in $L^\infty(\Omega)$, which is obtained as in (4.14), and continuity of the superposition operators associated to ξ and ξ' on $L^\infty(\Omega)$. □

The proof of Theorem 4.5 crucially relies on the structural properties of the solution map of (Eq) when concluding $y_k^\theta \to \bar{y}$ in $L^\infty(\Omega)$ from convergence of y_k toward $\bar{y}$. This behaviour of the control-to-state map is again due to the Kirchhoff transform. Moreover, to prove the continuity condition (4.16) by Hölder's inequality, it is necessary to show (4.17). Hence, we need convergence of y_k in $L^\infty(\Omega)$ and the respective assumptions in Theorem 4.5 cannot be weakened.

4.4. The case $d = 3$. To conclude the paper, we sketch how the restriction to dimension $d = 2$ in the second-order analysis can be avoided by restricting (P) to a certain subspace of $\mathcal{M}_D(\overline{\Omega})$ that we may imagine to consist of more regular measures. Similarly to [**13**, Section 6], we introduce the Banach space

$$\mathcal{M}_D^\infty(\overline{\Omega}) := \left\{\mu \in \mathcal{M}_D(\overline{\Omega})\colon (-\nabla \cdot \rho\nabla)^{-1}\mu \in L^\infty(\Omega)\right\},$$

equipped with the norm $\|\mu\|_{\mathcal{M}_D^\infty} := \|\mu\|_{\mathcal{M}_D} + \|(-\nabla \cdot \rho\nabla)^{-1}\mu\|_{L^\infty}$, and consider:

$$\min_{u \in \mathcal{M}_D^\infty(\overline{\Omega})} J(y,u) := \frac{1}{2}\|y - y_d\|_{L^2(\Omega)}^2 + \gamma\|u\|_{\mathcal{M}_D(\overline{\Omega})}, \qquad \text{s.t.} \quad \text{(Eq)}. \tag{P$^\infty$}$$

Let us briefly explain why (P$^\infty$) is of interest: First, we will be able to prove that the control-to-state map is twice continuously Fréchet differentiable as map $\mathcal{M}_D^\infty(\overline{\Omega}) \to L^\infty(\Omega)$ without restriction on the space dimension. This allows to overcome the restriction to space dimension $d \leq 2$ in Sections 4.2 and 4.3. Second, the statement of Theorem 4.5

becomes more natural for (P^∞): Recall, that in Theorem 4.5 we have obtained a second-order sufficient condition of a quite unusual type: We required the additional assumption $\bar{y} = S(\bar{u}) \in L^\infty(\Omega)$, and showed optimality of $\bar{u}$ for (P) only among those controls $u \in \mathcal{M}_D(\overline{\Omega})$ that satisfy $y = S(u) \in L^\infty(\Omega)$ and $\|y - \bar{y}\|_{L^\infty} < \varepsilon$. We will see below that this means local optimality (in the classical sense) for (P^∞). However, there are specific new difficulties associated with (P^∞), in particular concerning the well-posedness of the problem.

In the following we give a short summary on results and open problems related to (P^∞). First, note that $\mathcal{M}_D^\infty(\overline{\Omega}) \hookrightarrow \mathcal{M}_D(\overline{\Omega})$, and that (P^∞) differs from (P) only by the fact that minimization in (P^∞) now takes place over $\mathcal{M}_D^\infty(\overline{\Omega})$ instead of the whole space $\mathcal{M}_D(\overline{\Omega})$. Moreover, it follows by same argument as for [**13**, Lemma 5.5] that $\mathcal{M}_D^\infty(\overline{\Omega}) \hookrightarrow H_D^{-1}(\Omega)$; in this sense $\mathcal{M}_D^\infty(\overline{\Omega})$ contains more regular measures.

Existence of optimal controls in $\mathcal{M}_D^\infty(\overline{\Omega})$. Since $\mathcal{M}_D^\infty(\overline{\Omega})$ is not weak-$\star$ sequentially closed in $\mathcal{M}_D(\overline{\Omega})$, the proof of Proposition 2.5 does not apply to (P^∞). For semilinear elliptic counterparts of (P) and (P^∞), it has been proven in [**13**, Theorem 5.1] that given $y_d \in L^\infty(\Omega)$ the (global) solution $\bar{u}$ of (P) is also a solution to (P^∞). This result is based on an observation, that is also well known for linear elliptic equations; cf. [**37**]: The state $\bar{y}$ associated with $\bar{u}$ satisfies $\bar{y} \in L^\infty(\Omega)$ and $\|\bar{y}\|_{L^\infty} \le \|y_d\|_{L^\infty}$. For the setting of [**13**], i.e. Ω being a Lipschitz domain, pure homogeneous Dirichlet boundary conditions, and $\rho = \mathrm{id}$, the argument can be easily adapted to our problem utilizing the Kirchhoff transform. Consequently, existence of optimal controls for (P^∞) can be guaranteed at least for the setting just described. We omit the respective details.

First- and second-order analysis for (P^∞). Let us now briefly describe the main steps of the first- and second-order analysis of (P^∞). Due to $\mathcal{M}_D^\infty(\overline{\Omega}) \hookrightarrow \mathcal{M}_D(\overline{\Omega})$, existence and uniqueness of solutions to (Eq) obtained in Proposition 2.4 and the formula $S = \Xi^{-1} \circ (-\nabla \cdot \rho \nabla)^{-1}$ stay valid. Now, it holds $(-\nabla \cdot \rho\nabla)^{-1} \in \mathcal{L}(\mathcal{M}_D^\infty(\overline{\Omega}), L^\infty(\Omega))$ and the superposition operator $\Xi^{-1}\colon L^\infty(\Omega) \to L^\infty(\Omega)$ is twice continuously Fréchet differentiable if ξ is continuously differentiable. Consequently, $S\colon \mathcal{M}_D^\infty(\overline{\Omega}) \to L^\infty(\Omega)$ and $F\colon \mathcal{M}_D^\infty(\overline{\Omega}) \to \mathbb{R}$ are twice continuously Fréchet differentiable; the formulas for the respective derivatives from Sections 3.1, 3.2 and 4.1 remain valid. Moreover, by the chain rule, see, e.g., [**24**, Proposition 5.7], the subdifferential of j on $\mathcal{M}_D^\infty(\overline{\Omega})$ is given be the restriction of the elements of the subdifferential of j on $\mathcal{M}_D(\overline{\Omega})$. Therefore, we obtain first-order necessary conditions for (P^∞) by the same proof as for Theorem 3.2:

Theorem 4.7. *Under Assumptions 1.1-1.3 let $\bar{u} \in \mathcal{M}_D^\infty(\overline{\Omega})$ be a local solution to (P^∞) with respect to the $W_D^{-1,q}(\Omega)$-topology with some $q \in (1, \frac{d}{d-1})$. Then, there exists an adjoint state $\bar{p} \in W_D^{1,\bar{q}}(\Omega)$ such that (3.2)-(3.5) hold true. Moreover, we have $\bar{y} \in H_D^1(\Omega) \cap L^\infty(\Omega)$.*

Here, the additional regularity of the state is obtained along the lines of the proof of [**13**, Lemma 5.5]. Following the proof of Theorem 4.5 we obtain second-order sufficient conditions for (P^∞):

Theorem 4.8. *Let Assumptions 1.1,1.2, and $y_d \in L^\infty(\Omega)$ hold, and suppose that ξ' is continuously differentiable. If $\bar{u} \in \mathcal{M}_D^\infty(\overline{\Omega})$ satisfies (3.2)-(3.4) and*

$$F''(\bar{u})v^2 \ge \kappa \|z_v\|^2_{L^2(\Omega)}, \qquad \forall v \in C_{\bar{u}}^\tau \cap \mathcal{M}_D^\infty(\overline{\Omega}),$$

with some $\tau, \kappa > 0$, there are $\varepsilon, \delta > 0$ such that

$$\hat{J}(u) \ge \hat{J}(\bar{u}) + \frac{\delta}{2}\|z_{u-\bar{u}}\|^2_{L^2(\Omega)}, \qquad \forall u \in \mathcal{M}_D^\infty(\overline{\Omega}) \quad s.t.\ \|S(u) - \bar{y}\|_{\mathcal{M}_D^\infty(\overline{\Omega})} < \varepsilon.$$

In particular, $\bar{u}$ is a strict local solution to (P^∞) w.r.t. the $\mathcal{M}_D^\infty(\overline{\Omega})$-topology.

Proving second-order necessary optimality conditions for (P^∞) seems to be a bit more delicate and is possible at least in the setting of [**13**]: For Ω being a Lipschitz domain, $\Gamma_D = \partial\Omega$, and $\rho = \mathrm{id}$, the same argument as in the proof of [**13**, Theorem 6.3] allows to generalize the proof of Theorem 4.3 to (P^∞).

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