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Abstract On the example of the simplest C^0 spline quarklet construction of [1], we demonstrate the possible reduction of the complexity estimates for the approximation of singularity functions on the unit interval given in [3], to come closer to the complexity estimates known for hp-methods. To come up with a simplified argument, we explore the fact that the CDF biorthogonal spline wavelets [2] used in the construction of quarklets can be obtained by the lifting scheme [4]. Similar results are also possible for spline quarklet systems built from smooth B-splines with higher order of vanishing moments.

Keywords Quarklets · subexponential rates · hp-approximation

Mathematics Subject Classification (2000) 65F10

1 Introduction

It is known that the singularity function

$$g_\alpha(x) = x^\alpha, \qquad x \in I = [0,1], \qquad \alpha > 1/2,$$

admits subexponential convergence rates in $H^1(I)$ with free-knot, variable-degree splines in terms of the number of degrees of freedom of the associated spline space. More precisely, there is a C^0 spline function $s_\alpha(x)$ over the geometric partition

$$T_p : 0 < 2^{-p} < 2^{-p+1} < 2^{-p+2} < \ldots < 2^{-2} < 2^{-1} < 1 \tag{1}$$

with degree distribution $\mathbf{d}_p := (1,2,3,\ldots,p,p+1)$ such that

$$\|g_\alpha - s_\alpha\|_{H^1(I)} \le C_\alpha e^{-c_\alpha p}, \qquad p = 1,2,\ldots. \tag{2}$$

The dimension of the spline space s_α belongs to is obviously proportional to p^2. In other words, with N parameters to represent the approximant s_α in the standard Lagrange finite element basis, subexponential approximation rates of the form $\mathrm{O}(q_\alpha^{-\sqrt{N}})$ with $q_\alpha < 1$ can be expected. Details can be found in a series of papers by Babuska and Gui [5]. Starting from this result, a convergence theory for the hp-FEM applied to elliptic and parabolic problems in polygonal ($d = 2$) and polyhedral domains ($d = 3$) has been developed by many contributors.

As was evident from talks delivered at the recent MFO workshop 1936, new approximation schemes (anisotropic hp-approximation, tensor-product quarklets, nonlinear TT formats, deep

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ReLU networks) are under investigation that target the same class of problems, where smooth functions with structured singularities appear. Interestingly enough, for the particular family of singularity functions $\{g_\alpha,\, \alpha > 1/2\}$ and its multivariate generalizations on cubes, the current complexity estimates for these new schemes (in terms of the numbers of parameters involved) look weaker than in the hp-case. E.g., for the quarklet systems from [1], in [3] an N-term approximation rate of

$$\|g_\alpha - q_{\alpha,N}\|_{H^1(I)} \le C_\alpha e^{-c_\alpha N^{1/5}}, \qquad N = 1,2,\ldots. \tag{3}$$

was shown. Here, $q_{\alpha,N}$ is a linear combination of at most N quarklets obtained by decomposing a suitable free-knot variable-degree spline function (such as s_α) with respect to the quarklet system.

Our goal is to show that the exponent $1/5$ in (3) can be reduced to $1/3$ by using the structure of the quarklet system. We do this for the lowest order quarklet system with $m = \tilde{m} = 2$ (for short: $(2,2)$-quarklet system) associated with the CDF spline wavelet

$$\psi(x) := \phi(2x-1) - \frac{1}{4}(\phi(x) + \phi(x-1))$$

and the hat function $\phi(x) = (1-|x|)_+$ as scaling function. It is clear from this example how to extend the argument to the arbitrary quarklet systems considered in [1,3]. As in [3], we have used the integer shift-invariant quarklet system defined on $\mathbb{R}$ to approximate $g_\alpha(x)$ on the bounded interval $I = [0,1]$, and not the boundary adapted quarklet system introduced in [6], it remains to check if our argument is valid also in the boundary adapted case. What we do not know is if the exponent $1/3$ can be further reduced, say, by avoiding the intermediate approximation step via hp-splines and approximating g_α by suitable quarklet quasi-interpolants directly. Finally, extensions to higher spatial dimensions and to compositions of singularity functions typical for the solutions of PDE problems in polygonal and polyhedral domains are widely open.

2 Details

The strategy is the same as in [3]: We take a free-knot variable-degree C^0 spline function over T_p with degree distribution $\mathbf{d}_p$ and write it in an economic way as linear combination of quarks. In a second step, we determine the index sets of the quarklets needed to express the quarks appearing in this decomposition, and carefully estimate the size of their union.

We introduce the necessary notation. The basic $(2,2)$-quarks and $(2,2)$-quarklets on level $l = 0,1,\ldots$ are defined by

$$\phi^l(x) = x^l\phi(x), \qquad \psi^l(x) = \frac{3}{4}\phi^l(2x-1) - \frac{1}{4}(\phi^l(2x)+\phi^l(2x-2)) - \frac{1}{8}(\phi^l(2x+1)+\phi^l(2x-3), \tag{4}$$

this follows from the definition in [1] and the fact that $\phi^0 = \phi$ and $\psi^0 = \psi$. The $(2,2)$-quarklets on level $j = 0$ are given by the integer shifts of the quarks ϕ^l, $l \ge 0$, while for level $j > 0$ the 2^{-j+1}-shifts of the dilated quarklets $\psi^l(2^{j-1}x)$, $l \ge 0$, are taken. In other words, the $(2,2)$-quarklet system $\mathscr{Q}_{(2,2)}$ is given by

$$\mathscr{Q}_{(2,2)} = \{\phi^l(x-k), \psi^l(2^{j-1}x-k) :\ k,j,l \in \mathbb{Z},\ j \ge 1,\ l \ge 0\} \tag{5}$$

This is, up to weighting and enumeration, the system introduced in [1], formula (22), for which the frame property in $H^s(\mathbb{R})$, $0 \le s < 3/2$, was proved.

We use the following notation:

$$\phi_k^{l,j}(x) := \phi^l(2^jx-k), \qquad \tilde{\phi}_k^{l,j}(x) := x^l\phi(2^jx-k), \quad k,j,l \in \mathbb{Z},\ j,l \ge 0\}, \tag{6}$$

$$\psi_k^{l,j}(x) := \psi^l(2^{j-1}x-k), \qquad \tilde{\psi}_k^{l,j}(x) := x^l\psi(2^{j-1}x-k), \quad k,j,l \in \mathbb{Z},\ j \ge 1,\ l \ge 0\}. \tag{7}$$

Obviously,

$$\operatorname{span}\{\phi_k^{s,j}(x):\ 0\le s\le l\} = \operatorname{span}\{\tilde{\phi}_k^{s,j}(x):\ 0\le s\le l\}, \tag{8}$$

coincides with the set of all products of polynomials of degree $\le l$ times $\phi(2^j x-k)$ for all choices of indices.

For later use, we write down a more explicit expression for $\psi_k^{l,j}(x)$. From the definition (4), we see that for $l=0$

$$\begin{aligned}
\psi_k^{0,j}(x) &= \tilde{\psi}_k^{0,j}(x) = \psi(2^{j-1}x-k)\\
&= \frac{3}{4}\phi(2^j x-2k-1)-\frac{1}{4}(\phi(2^j x-2k)+\phi(2^j x-2k-2))\\
&\qquad -\frac{1}{8}(\phi(2^j x-2k+1)+\phi(2^j x-2k-3))\\
&= \frac{3}{4}\phi_{2k+1}^{0,j}(x)-\frac{1}{4}(\phi_{2k}^{0,j}(x)+\phi_{2k+2}^{0,j}(x))-\frac{1}{8}(\phi_{2k-1}^{0,j}(x)+\phi_{2k+3}^{0,j}(x)).
\end{aligned}$$

However, for $l\ge 1$ the two functions $\psi_k^{l,j}$ and $\tilde{\psi}_k^{l,j}$ do not coincide, and their expression is more involved. We have

$$\begin{aligned}
\psi_k^{l,j}(x) &= \frac{3}{4}(2^j x-2k-1)^l\phi(2^j x-2k-1)\\
&\qquad -\frac{1}{4}((2^j x-2k)^l\phi(2^j x-2k)+(2^j x-2k-2)^l\phi(2^j x-2k-2))\\
&\qquad -\frac{1}{8}((2^j x-2k+1)^l\phi(2^j x-2k+1)+(2^j x-2k-3)^l\phi(2^j x-2k-3))\\
&= (2^j x)^l\left(\frac{3}{4}\phi(2^j x-2k-1)-\frac{1}{4}(\phi(2^j x-2k)+\phi(2^j x-2k-2))\right.\\
&\qquad \left.-\frac{1}{8}(\phi(2^j x-2k+1)+\phi(2^j x-2k-3)\right)+\sum_{i=2k-1}^{2k+2}\phi_i^{0,j}(x)q_{k,i}^{l,j}(x)\\
&= \frac{3}{4}2^{jl}\tilde{\psi}_k^{l,j}(x)+\sum_{i=2k-1}^{2k+2}\sum_{s=0}^{l-1}c_{k,i,s}^{l,j}\tilde{\phi}_i^{s,j}(x).
\end{aligned}$$

Here, by $q_{k,i}^{l,j}$ we denoted some polynomials of degree $l-1$, and by $c_{k,i,s}^{l,j}$ constants whose concrete values do not matter. The fact of later use is that

$$\tilde{\psi}_k^{l,j}-\frac{4}{3}2^{-jl}\psi_k^{l,j}\in\operatorname{span}\{\tilde{\phi}_i^{s,j}:\ i=2k-1,\ldots,2k+3,\ 0\le s<l\} \tag{9}$$

holds for all indices $l,j\ge 1$ and $k\in\mathbb{Z}$.

Lemma 1 *Let S_p be the set of all splines with degree distribution $\mathbf{d}_p$ on the partition T_p. The restriction of the set*

$$\{\tilde{\phi}_0^{0,p}\}\cup\left(\cup_{j=1}^p\{\tilde{\phi}_1^{0,p+1-j},\ldots,\tilde{\phi}_1^{j,p+1-j}\}\right)\cup\{\tilde{\phi}_1^{0,0},\ldots,\tilde{\phi}_1^{p,0}\}$$

to the unit interval I is a spanning set for S_p.

Proof. Let $s\in S_p$ be arbitrary. Subtracting a suitable linear combination of $\tilde{\phi}_{p,k}^0=\phi_{p,k}^0$, $k=0,1$, from s, we arrive at a spline $s_1\in S_p$ which vanishes on $[0,2^{-p}]$. On the interval $I_1:=[2^{-p},2^{-p+1}]$, s_1 coincides with quadratic polynomial that vanishes at the left endpoint. Thus, for $x\in I_1$ we have

$$s_1(x)=(x-2^{-p})(ax+b)=\phi_2^{0,p}(x)(ax+b)=(\phi_1^{0,p-1}(x)-\frac{1}{2}\phi_1^{0,p}(x))(ax+b).$$

This shows that by subtracting a suitable linear combination of $\tilde{\phi}_1^{l,p-1}(x)$, $\tilde{\phi}_1^{l,p}(x)$, $l=0,1$, we arrive at a spline $s_2 \in S_p$ that vanishes on $[0,2^{-p+1}]$. By similar reasoning, for $x \in I_2 := [2^{-p+1}, 2^{-p+2}]$ we have

$$s_2(x) = (x-2^{-p+1})p_2(x) = \phi_2^{0,p-1}(x)p_2(x) = (\phi_1^{0,p-2}(x) - \phi_1^{0,p-1}(x))p_2(x)$$

for some quadratic polynomial $p_2(x)$. Consequently, by subtracting a linear combination of $\tilde{\phi}_1^{l,p-2}(x)$, $\tilde{\phi}_1^{l,p-1}(x)$, $l=0,1,2$, from $s_2(x)$, one arrives at $s_3 \in S_p$ such that $s_3(x)=0$ for $x \in [0,2^{-p+2}]$, and so on. After altogether p such subtraction steps, we arrive at $s_{p+1}(x)=0$ on $I=[0,1]$. This shows the statement, and finishes the proof of Lemma 1.

As a by-product of the proof of this lemma and (8), we have shown that any $s \in S_p$ can be represented by $\le (1+2+\ldots+2(p+1)) = (p+2)(p+3)/2 = \mathrm{O}(p^2)$ quark functions $\phi_k^{l,j}$.

Now we proceed with trying to find a minimal spanning set of quarklets $\phi_k^{l,0}$, $\psi_k^{l,j}$ for S_p by expressing each $\tilde{\phi}_k^{l,j}$ appearing in the statement of Lemma 1. This will be done in a recursive way using the following notation. Denote by

$$S_m^{l,j} = \mathrm{span}\{\tilde{\phi}_k^{s,j} : (k,s) \in J_m^{l,j}\}|_I, \qquad m=1,3,\ldots,$$

where $J_m^{l,j} := \{k=0,1,\ldots,\min(m,2^j),\, s=0,\ldots,l\}$ is the corresponding set of indices (k,s), and m is odd. By Lemma 1, we have

$$S_p \subset \sum_{j=0}^{p} S_1^{p-j,j}. \tag{10}$$

The recursion argument uses the following

Lemma 2 *If $j,l \ge 1$, $j+l \le p$, and $m \ge 0$ is odd then*

$$S_m^{l,j} \subset \mathrm{span}\{\psi_k^{l,j}|_I\}_{k=0,\ldots,(m-1)/2} + S_{m+2}^{l-1,j} + S_{m'}^{l,j-1},$$

where $m' = 1+2\lfloor (m+1)/4 \rfloor$.

Proof. Functions with support outside of I are silently dropped in the computations below. For even $0 \le k = 2r < m$, using the refinement equation for the scaling function ϕ, we can write

$$\tilde{\phi}_{2r}^{l,j}(x) = x^l \phi_{2r}^{l,j}(x) = \tilde{\phi}_r^{l,j-1}(x) - \frac{1}{2}(\tilde{\phi}_{2r-1}^{l,j}(x) + \tilde{\phi}_{2r+1}^{l,j}(x)).$$

Thus any linear combination of $\tilde{\phi}_k^{l,j}(x)$ can be expressed by functions $\tilde{\phi}_k^{l,j}(x)$ with $k=1,3,\ldots,m$ and $\tilde{\phi}_r^{l,j-1}(x)$ with $0 \le r \le (m-1)/2 \le m'$. The latter belong to $S_{m'}^{l,j-1}$.

Let us now consider the $\tilde{\phi}_k^{l,j}(x)$ with odd index $0 < k = 2r+1 \le m$. By the definition of ψ, $\psi_k^{l,j}$, and $\tilde{\psi}_k^{l,j}$, we have

$$\begin{aligned} \tilde{\phi}_{2r+1}^{l,j}(x) &= x^l \phi_{2r+1}^{l,j}(x) = x^l(\psi_r^{l,j}(x) + \frac{1}{4}(\phi_r^{l,j-1}(x) + \phi_{r+1}^{l,j-1}(x))) \\ &= \tilde{\psi}_r^{l,j}(x) + \frac{1}{4}(\tilde{\phi}_r^{l,j-1}(x) + \tilde{\phi}_{r+1}^{l,j-1}(x))). \end{aligned}$$

Consequently, up to functions $\tilde{\phi}_r^{l,j-1}$ and $\tilde{\phi}_{r+1}^{l,j-1}$ belonging to $S_{m'}^{l,j-1}$ (since the largest appearing index is $(m+1)/2 \le m'$), these $\tilde{\phi}_{2r+1}^{l,j}(x)$ can be expressed by linear combinations of $\tilde{\psi}_r^{l,j}(x)$, $r=0,\ldots,(m-1)/2$. But by (9), $\tilde{\psi}_r^{l,j}(x)$ can be replaced by $\psi_r^{l,j}(x)$, at the expense of some corrections leading to the additional space $S_{m+2}^{l-1,j}$. This proves Lemma 2.

We now use an induction argument based on the previous lemma, starting from the representation (10). Spaces $S_1^{p,0}$ and $S_1^{0,p}$ are not touched, to all other spaces $S_1^{p-j,j}$ we apply Lemma

2 with $m = 1$. Since $m' = 1$ and $m + 2 = 3$, this shows that any $s \in S_p$ can be represented by a linear combination of $n_1 = p - 1$ quarklets $\psi_0^{p-j,j}$, $j = 1, \ldots, p-1$, and a spline

$$\begin{aligned} s_1 &\in S_1^{p,0} + \sum_{j=1}^{p-1} (S_1^{p-j,j-1} + S_3^{p-j-1,j}) + S_1^{0,p} \\ &= S_1^{p,0} + \sum_{j=1}^{p-2} S_3^{p-j-1,j} + (S_3^{0,p-1} + S_1^{0,p}) =: S_{p,1}. \end{aligned}$$

Here and in the following, we use that $S_m^{l',0} \subset S_1^{l,j}$, $l' < l$, and $S_n^{l,j} \subset S_m^{l,j}$, $n < m$.

The second step is the application of Lemma 2 with $m = 3$. We compute $m' = 3 < m + 2 = 5$ and see that $s_1 \in S_{p,1}$ can be expressed as linear combination of $n_2 = 2(p-2)$ quarklets $\psi_k^{p-1-j,j}$, $k = 0, 1$, $j = 1, \ldots, p-2$, and a spline

$$\begin{aligned} s_2 &\in S_1^{p,0} + S_3^{p-2,0} + \sum_{j=1}^{p-1} (S_3^{p-j,j-1} + S_5^{p-j-1,j}) + S_3^{0,p-1} + S_1^{0,p} \\ &= S_1^{p,0} + \sum_{j=1}^{p-3} S_5^{p-j-2,j} + (S_5^{0,p-2} + S_3^{0,p-1} + S_1^{0,p}) =: S_{p,2}. \end{aligned}$$

It is clear that we can continue this procedure for altogether $p - 1$ steps. In the t-th step, we apply Lemma 2 to spaces $S_m^{p+1-t-j,j}$, $j = 1, \ldots, p-t$, where $m = 2t - 1$. This produces $n_t = t(p-t)$ new quarklets $\psi_k^{p+1-t-j,j}$, and a new remainder spline s_t. After $p - 1$ steps, we arrive for each $s \in S_p$ at a representation

$$s = \sum_{t=1}^{p} \sum_{j=1}^{p-t} \sum_{k=0}^{t-1} d_k^{p+1-t-j,j} \psi_k^{p+1-t-j,j} + s_{p-1},$$

where $d_k^{p+1-t-j,j}$ are suitable constants, and s_{p-1} is a spline in

$$S_{p,p-1} := S_1^{p,0} + \sum_{t=1}^{p} S_{2t-1}^{0,p+1-t}.$$

Obviously, $S_1^{p,0}$ is the space of polynomials of degree $\leq p+1$, spanned by $p+2$ suiably chosen quarklet functions $\phi_k^{l,0}$, $k = 0, 1$, of level 0. The remaining spaces $S_{2t-1}^{0,p+1-t}$ are spanned by the CDF wavelets of level $j \geq 1$. By a similar argument, using the formulas

$$\phi_{2r+1}^{0,l} = \psi_r^{0,l} + \frac{1}{4}(\phi_r^{0,l-1} + \phi_{r+1}^{0,l-1})$$

and

$$\phi_{2r}^{0,l} = -\frac{1}{2}(\psi_{r-1}^{0,l} + \psi_r^{0,l}) + \frac{1}{8}(\phi_{r-1}^{0,l-1} + 6\phi_r^{0,l-1} + \phi_{r+1}^{0,l-1}),$$

one shows that for $t = 1, \ldots, p$ any $s \in S_{2t-1}^{0,p+1-t}$ can be represented by a linear combination of t CDF wavelets $\psi_r^{0,p+1-t}$, $r = 0, \ldots, t-1$, and an element of $S_t^{0,p-t} \subset S_{2t+1}^{0,p-t}$. Thus, by recursion no more than $1 + 2 + \ldots + p = p(p+1)/2$ CDF wavelets $\psi_r^{0,j}$ with $j = p, \ldots, 1$ (plus an element from $S_1^{0,0} \subset S_1^{p,0}$) are needed to represent s_{p-1}.

Altogether, we have

Theorem 1 *Any $s \in S_p$ (and in particular the approximant s_α in (2)) can be represented by*

$$N_p := \dim(S_p) \leq \sum_{t=1}^{p} t(p-t) + (p+2) + \frac{p(p+1)}{2} \asymp p^3$$

quarklets from the quarklet system $\mathscr{Q}_{(2,2)}$.

Remarks. 1) The estimate for N_p is not sharp. E.g., we have not used the fact that also $\tilde{\phi}_k^{l,j}$ with $k > 2^j$ can be neglected. Such quarks appear in the above recursions for small j. However, we claim that the estimate is sharp in order, i.e., a bound $\mathrm{O}(p^{3-\varepsilon})$ is not expected to hold for any $\varepsilon > 0$. A formal proof is probably very technical, and imho not worth pursuing.

2) The $\mathrm{O}(p^3)$ result is expected to hold for all quarklet systems $\mathscr{Q}_{(m,\tilde{m})}$ constructed in [1], with some obvious changes. E.g., S_p must be replaced by a space of C^{m-2}-smooth splines over T_p with degree distribution $(m-2)+\mathbf{d}_p$, and s_α by the corresponding Hermite interpolant used in [3]. The basic scaling funcion ϕ will be a (centered or not) B-spline of order m. The counterparts of Lemma 1 and 2 will hold, with constants depending on m and $\tilde{m}$ (obviously, the m in Lemma 2 is not related to the spline order m in the definition of $\mathscr{Q}_{(m,\tilde{m})}$). This is because essentially all we use is the boundedness of supports of the masks of the primal and dual MRA associated with the underlying spline wavelet system. If one keeps $m = 2$ (which is enough for conforming FEM-type applications to second-order PDE problems) then only Lemma 2 needs minor changes: Instead of $S_{m+2}^{l-1,j}$, the space $S_{m+\tilde{m}}^{l-1,j}$ will appear which is due to the larger dual mask size. Moreover, the formula for m' also depends on $\tilde{m}$, however $m' \leq m+\tilde{m}$ will always remain true.

3) The boundary modification introduced in [6] has not been taken into account but should not present serious difficulties.

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