Institut für Numerische Simulation
Rheinische Friedrich-Wilhelms-Universität Bonn

Wegelerstraße 6 • 53115 Bonn • Germany
phone +49 228 73-3427 • fax +49 228 73-7527
www.ins.uni-bonn.de

P. Oswald

Estimates for Generalized Sparse Grid Hierarchical Basis Preconditioners

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Peter Oswald[a]

[a] *Institute for Numerical Simulation (INS), University of Bonn, Wegelerstr. 6-8, D-53211 Bonn*

Abstract

We reconsider some estimates from the 1994 paper [6] concerning the hierarchical basis preconditioner for sparse grid discretizations. The improvement is in three directions: We consider arbitrary space dimensions $d > 1$, give bounds for generalized sparse grid spaces with arbitrary monotone index set Λ, and show that the bounds are sharp up to constants depending only on d, at least for a subclass of Λ containing full grid, standard sparse grid spaces, and energy-norm optimized sparse grid spaces.

Keywords: Generalized sparse grid spaces, hierarchical basis, tensor-product Faber-Schauder systems, preconditioners for elliptic problems.
2000 MSC: 65F10

1. Notation

We consider the hierarchical basis (HB) preconditioner for generalized sparse grid discretizations for generic $H_0^1(I^d)$-elliptic problems (I^d is the d-dimensional unit cube) which has been analyzed for $d = 2, 3$ in [6]. The underlying hierarchical basis is a finite collection of dyadic blocks of the tensor-product Faber-Schauder system on I^d (detailed definitions will be given in the next section). The HB preconditioner considered in this note is different from the HB preconditioners of Bank, Dupont, and Yserentant (see [2, 10]) which correspond to isotropic multivariate versions of the Faber-Schauder system. The HB preconditioner for sparse grid discretizations is not optimal, nor suboptimal (better preconditioners have already been discussed in [6, 8] and more recently in [5]). Nevertheless, due to the popularity of sparse grid methods we find it worthwhile to have a complete understanding of its properties.

In this note, we consider generalized sparse grid spaces S_Λ for arbitrary dimension $d > 1$ generated by a monotone index set $\Lambda \subset \mathbb{N}^d$ which comes with a direct splitting

$$S_\Lambda = \sum_{\beta \in \Lambda} S_\beta$$

where each S_β is the span of nodal basis functions associated with the dyadic block of the tensor-product Faber-Schauder system with index $\beta \in \mathbb{N}^d$. Correspondingly, each

Email address: agp.oswald@gmail.com (Peter Oswald)

$v_\Lambda \in S_\Lambda$ has a unique decomposition

$$v_\Lambda = \sum_{\beta\in\Lambda} s_\beta, \qquad s_\beta \in S_\beta.$$

The associated Galerkin discretization for a generic $H_0^1(I^d)$-elliptic problem using the finite section of the tensor-product Faber-Schauder system associated with S_Λ leads, after appropriate diagonal scaling, to a symmetric positive-definite algebraic system

$$A_\Lambda x_\Lambda = b_\Lambda,$$

with stiffness matrix A_Λ, right-hand side b_Λ, and solution vector x_Λ representing the HB coefficients of the Galerkin projection of the H_0^1-elliptic problem onto S_Λ. We seek as good as possible estimates of the spectral bounds $\lambda_{\max}(A_\Lambda)$ and $\lambda_{\min}(A_\Lambda)$, and, consequently, of the spectral condition number

$$\kappa_{S_\Lambda,HB} := \kappa(A_\Lambda) = \frac{\lambda_{\max}(A_\Lambda)}{\lambda_{\min}(A_\Lambda)}.$$

This is equivalent to estimating the stability bounds of the direct space splitting

$$\{S_\Lambda; (\cdot,\cdot)_{H_0^1}\} = \sum_{\beta\in\Lambda}\{S_\beta; 2^{2|\beta|_\infty}(\cdot,\cdot)_{L_2}\}. \tag{1}$$

This standard fact from additive Schwarz theory is explained in [6]. In particular, estimating $\lambda_{\max}(A_\Lambda)$ (up to generic constants depending on the ellipticity constants and d) is equivalent to finding the best constant C_Λ in the inequality

$$\|\sum_{\beta\in\Lambda} s_\beta\|_{H_0^1}^2 \le C_\Lambda \sum_{\beta\in\Lambda} 2^{2|\beta|_\infty}\|s_\beta\|_{L_2}^2, \tag{2}$$

valid for all $s_\beta \in S_\beta$ with $\beta \in \Lambda$. Upper and lower estimates for C_Λ are obtained in Section 3, they are matching up to constants depending on d but not on Λ.

Estimates for $\lambda_{\min}(A_\Lambda)$ require bounds for the best constant c_Λ in the inequality

$$c_\Lambda \sum_{\beta\in\Lambda} 2^{2|\beta|_\infty}\|s_\beta\|_{L_2}^2 \le \|\sum_{\beta\in\Lambda} s_\beta\|_{H_0^1}^2, \qquad s_\beta \in S_\beta, \tag{3}$$

opposite to (2), and will be given in Section 4. In the final section we summarize the results and show that the resulting condition number estimates for HB preconditioners are asymptotically sharp for certain families of S_Λ with $d > 1$ arbitrarily fixed, including the full grid spaces V_k and standard sparse grid spaces S_k if $k \to \infty$.

2. Notation and auxiliary facts

2.1. Faber-Schauder functions

Denote by T_k the uniform dyadic partition of $I = [0,1]$ of step-size 2^{-k}, $k \in \mathbb{N}$. The univariate Faber-Schauder system on the unit interval (obeying zero boundary conditions) consists of dyadic shifts and dilates of the unit hat function $\phi(t) = (1 - |t|)_+$,

$t \in \mathbb{R}$, defined blockwise as follows: For $k = 1, 2, ...$, define the k-th dyadic block of the Faber-Schauder system as the collection of 2^{k-1} hat functions

$$\phi_{2^{k-1}+i}(t) := \phi(2^k t - (2i+1)), \quad t \in [0,1], \qquad i = 0, 1, \ldots, 2^{k-1} - 1,$$

with non-overlapping support. These are the standard nodal basis functions of the linear finite element space over T_k associated with the interior nodal points of T_k of the form $(2i+1)2^{-k}$. The index set for this block is denoted J_k. The union of all blocks is the univariate Faber-Schauder system (with zero boundary conditions).

The multivariate tensor-product Faber-Schauder system is defined as the collection of the functions

$$\phi_\alpha(x) := \prod_{i=1}^{d} \phi_{\alpha_i}(x_i), \qquad \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d.$$

We organize it into blocks associated with the multi-index sets $J_\beta := J_{\beta_1} \times \ldots \times J_{\beta_d}$, where $\beta \in \mathbb{N}^d$. The Faber-Schauder functions ϕ_α in the same block have non-overlapping support, and are shifts of each other. Together with their tensor-product structure this allows us to obtain explicit formulas for their L_2 and H_0^1 norms (for convenience, the latter is defined as

$$\|u\|_{H_0^1}^2 := \int_{I^d} |\nabla u|^2 \, dx, \qquad u \in H_0^1).$$

More precisely,

$$\|\phi_\alpha\|_{L_2}^2 = \frac{2^d}{3^d} 2^{-|\beta|_1}, \qquad \|\phi_\alpha\|_{H_0^1}^2 = \frac{2^d}{3^{d-1}} 2^{-|\beta|_1} \sum_{i=1}^{d} 2^{2\beta_i}, \qquad \alpha \in J_\beta. \tag{4}$$

We use the notation $|\beta|_1 := \sum_{i=1}^d \beta_i$ and $|\beta|_\infty = \max_{i=1,\ldots,d} \beta_i$.

2.2. Discretization spaces and decomposition norms

Let S_β denote the finite-dimensional space spanned by all ϕ_α with $\alpha \in J_\beta$. Then

$$V_\beta = \sum_{\beta' \le \beta} S_{\beta'}, \qquad \beta \in \mathbb{N}^d,$$

is a direct sum splitting of the anisotropic full-grid space V_β of all d-linear finite element functions over the anisotropic tensor-product partition $T_\beta := T_{\beta_1} \times \ldots \times T_{\beta_d}$ (by $\beta' \le \beta$ we mean that $\beta_i' \le \beta_i$ for all $i = 1, \ldots, d$). The full grid space V_k refers to the isotropic case $\beta = (k, \ldots, k)$ needed in approximation schemes related to uniform grid refinement.

In addition to the subspace families $\{S_\beta\}$ and $\{V_\beta\}$, we also need the family

$$W_\beta := V_\beta \ominus (\sum_{\beta' < \beta} V_{\beta'}) = \otimes_{i=1}^d (V_{\beta_i} \ominus V_{\beta_i - 1})$$

of L_2 orthogonal subspaces which provides splittings of the various spaces of interest to us (the convention is $V_0 = \{0\}$). In particular,

$$L_2(I^d) = \oplus_{\beta \in \mathbb{N}^d} W_\beta, \qquad V_\beta = \oplus_{\beta' \leq \beta} W_{\beta'}. \tag{5}$$

The main focus is the investigation of the generalized sparse grid space

$$S_\Lambda := \mathrm{span}(\{V_\beta :\ \beta \in \Lambda\}) = \mathrm{span}(\{W_\beta :\ \beta \in \Lambda\}) = \mathrm{span}(\{S_\beta :\ \beta \in \Lambda\}), \tag{6}$$

where $\Lambda \subset \mathbb{N}^d$ is a monotone set, i.e., $\beta \in \Lambda$ implies $\beta' \in \Lambda$ for all $\beta' \leq \beta$. The standard sparse grid spaces S_k correspond to the choice $\Lambda = \{\beta \in \mathbb{N}^d :\ |\beta|_1 \leq k + d - 1\}$. Note that the anisotropic full grid spaces V_β are also special instances of the family of S_Λ spaces (take $\Lambda = \{\beta' \in \mathbb{N}^d :\ \beta' \leq \beta\}$).

Each $v_\Lambda \in S_\Lambda$ can be non-uniquely decomposed as

$$v_\Lambda = \sum_{\beta \in \Lambda} v_\beta, \qquad v_\beta \in V_\beta,$$

but also uniquely represented with respect to either the S_β or W_β subspaces of V_Λ:

$$v_\Lambda = \sum_{\beta \in \Lambda} s_\beta = \sum_{\beta \in \Lambda} w_\beta, \qquad s_\beta \in S_\beta, \quad w_\beta \in W_\beta. \tag{7}$$

Each of these representations of $v_\Lambda \in S_\Lambda$ has its own merits. In particular, L_2 orthogonal representations (also called prewavelet (PW) representations) with respect to the W_β subspaces allow us to effectively express L_2 and H_0^1 norms. For any $v_\Lambda \in S_\Lambda$ we have the identity

$$\|v_\Lambda\|_{L_2}^2 = \sum_{\beta \in \Lambda} \|w_\beta\|_{L_2}^2,$$

and the two-sided norm equivalence

$$c_{PW} \|v_\Lambda\|_{PW}^2 \leq \|v_\Lambda\|_{H_0^1}^2 \leq C_{PW} \|v_\Lambda\|_{PW}^2, \qquad \|v_\Lambda\|_{PW}^2 := \sum_{\beta \in \Lambda} 2^{2|\beta|_\infty} \|w_\beta\|_{L_2}^2, \tag{8}$$

which holds with constants $0 < c_{PW} \leq C_{PW} < \infty$ depending on d, only. We also need an inequality for arbitrary decompositions with respect to the isotropic full grid spaces V_k, namely

$$\|\sum_{k=1}^{K} v_k\|_{H_0^1}^2 \leq C_{BPX} \sum_{k=1}^{K} 2^{2k} \|v_k\|_{L_2}^2, \qquad v_k \in V_k, \tag{9}$$

which holds with a constant depending on d. It is related to the BPX preconditioner, and can be obtained from (8). We refer to [8, 10] for more details on the inequalities (8) and (9).

In the remainder of this paper, we will be interested in identifying the constants in a similar two-sided estimate associated with the decomposition with respect to the S_β spaces, namely for comparing the H_0^1 norm with the HB norm

$$\|v_\Lambda\|_{HB}^2 := \sum_{\beta \in \Lambda} 2^{2|\beta|_\infty} \|s_\beta\|_{L_2}^2, \tag{10}$$

instead of the PW norm in (8). As will be demonstrated in the next two sections, these constants also depend on characteristics of Λ, and not only on d.

2.3. Norms of some FE functions

We start with stating an immediate consequence of (4) and the non-overlapping support property of the nodal basis functions spanning S_β:

$$\|s_\beta\|_{L_2}^2 = \frac{2^d}{3^d} 2^{-|\beta|_1} \sum_{\alpha\in J_\beta} c_\alpha^2, \qquad s_\beta = \sum_{\alpha\in J_\beta} c_\alpha\phi_\alpha \in S_\beta. \tag{11}$$

A similar equality holds for the H_0^1 norm but we do not need it.

Next, we estimate the HB norm of the tensor-product hat function

$$\psi_\beta(x) = \prod_{i=1}^{d} \phi(2^{\beta_i}x - 2^{\beta_i-1}), \qquad \beta\in\mathbb{N}^d.$$

This is the nodal basis function in V_β associated with the center of the cube I^d (which is not in the tensor-product Faber-Schauder system unless $\beta=(1,\ldots,1)$). Since it is the tensor product of d univariate hat functions $\phi(2^{\beta_i}x - 2^{\beta_i-1})$ associated with the nodal point 1/2, its HB decomposition is the tensor product of the univariate HB decompositions of the latter. It is easy to see that the univariate HB decompositions are implied by the formula

$$\phi(2^m t - 2^{m-1}) = \phi_{2^0}(t) - \sum_{l=2}^{m} \frac{1}{2}(\phi_{2^{l-1}+2^{l-2}-1}(t) + \phi_{2^{l-1}+2^{l-2}}(t)), \quad t\in[0,1], \quad m\in\mathbb{N},$$

by setting $m=\beta_i$ and $t=x_i$, $i=1,\ldots,d$. Thus, if the multi-index $\beta'\le\beta$ has r components $\beta_i'>1$ and $d-r$ components $\beta_i'=1$ then the HB block $s_{\beta'}$ of ψ_β is the sum of 2^r nodal basis functions with coefficient $(-1/2)^r$. Thus, by (11)

$$\|s_{\beta'}\|_{L_2}^2 = \frac{2^d}{3^d} 2^{-|\beta'|_1} 2^r (1/2)^{2r} \ge 3^{-d} 2^{-|\beta'|_1}, \qquad \beta'\le\beta.$$

Here, and throughout the paper, we denote by $c, C > 0$ generic positive constants depending only on d (which may be different in different places). Moreover, we use the notation $A \approx B$ if $c\,A \le B \le C\,A$. Substitution into the expression for the HB norm gives

$$\|\psi_\beta\|_{HB}^2 = \sum_{\beta'\le\beta} 2^{2|\beta'|_\infty} \|s_{\beta'}\|_{L_2}^2 \ge c \sum_{\beta'\le\beta} 2^{2|\beta'|_\infty - |\beta'|_1} \ge c 2^{|\beta|_\infty}. \tag{12}$$

The last inequality follows since among the indices $\beta'\le\beta$ there is at least one with the property $|\beta'|_1 - (d-1) = |\beta'|_\infty = |\beta|_\infty$. There is also a matching upper bound $\|\psi_\beta\|_{HB}^2 \le C 2^{|\beta|_\infty}$ but we do not need it in the sequel.

Finally, we consider a different construction (a kind of lacunary HB series representation) which we need in the next section. For the following estimates to hold the set Λ can be arbitrary, i.e., not necessarily monotone. Let

$$\bar{s}_\beta = \sum_{\alpha\in J_\beta} \phi_\alpha \in S_\beta, \qquad \beta\in\mathbb{N}^d,$$

and set

$$\bar{s}_\Lambda = \sum_{\beta\in\Lambda} \bar{s}_\beta \in S_\Lambda.$$

The functions $\bar{s}_\beta$ are tensor products of their univariate counterparts

$$\bar{s}_k(t) = 2^k \int_0^t r_k(\xi)\, d\xi, \qquad t \in [0,1], \qquad k = 1, 2, \dots,$$

where $r_k(t) = \mathrm{sign}(\sin(2^k \pi t))$ denotes the univariate Rademacher functions. Below we need that the shifted functions

$$\tilde{s}_k(t) = \bar{s}_k(t) - 1/2, \qquad k = 1, 2, \dots,$$

form an orthogonal system in $L_2([0,1])$ (and are orthogonal to constants as well), with L_2 norm given by

$$\|\tilde{s}_k\|_{L_2}^2 = \|\bar{s}_k\|_{L_2}^2 - \|\bar{s}_k\|_{L_1}^2 + \frac{1}{4} = \frac{1}{3} - \frac{1}{2} + \frac{1}{4} = \frac{1}{12}.$$

Here we have used the case $d = 1$ of the identity

$$\|\bar{s}_\beta\|_{L_2}^2 = \frac{2^d}{3^d} 2^{-|\beta|_1} \sum_{\alpha\in J_\beta} 1 = \frac{2^d}{3^d} 2^{-|\beta|_1} 2^{|\beta|_1 - d} = 3^{-d},$$

which is a consequence of (11). From the last equality, we compute the HB norm of s_Λ as

$$\|\bar{s}_\Lambda\|_{HB}^2 = \sum_{\beta\in\Lambda} 2^{2|\beta|_\infty} \|\bar{s}_\beta\|_{L_2}^2 = 3^{-d} \sum_{\beta\in\Lambda} 2^{2|\beta|_\infty}. \tag{13}$$

For the L_2 norm of $\bar{s}_\Lambda$ we can obtain the lower bound

$$\|\bar{s}_\Lambda\|_{L_2}^2 \geq 4^{-d} |\Lambda|^2. \tag{14}$$

Indeed, since

$$\bar{s}_\beta(x) = \prod_{i=1}^d (\tilde{s}_{\beta_i}(x_i) + \frac{1}{2}),$$

using the orthogonality properties of the system $\{\tilde{s}_k(t)\}$ mentioned before, namely that

$$\int_0^1 (\bar{s}_k(t) + \frac{1}{2})(\bar{s}_{k'}(t) + \frac{1}{2})\, dt = \frac{\delta_{kk'}}{12} + \frac{1}{4} \geq \frac{1}{4},$$

we have

$$\begin{aligned} \|\bar{s}_\Lambda\|_{L_2}^2 &\geq \sum_{\beta\in\Lambda} \sum_{\beta'\in\Lambda} \prod_{i=1}^d \int_0^1 (\bar{s}_{\beta_i}(x_i) + \frac{1}{2})(\bar{s}_{\beta'_i}(x_i) + \frac{1}{2})\, dx_i \\ &\geq \sum_{\beta\in\Lambda} \sum_{\beta'\in\Lambda} 4^{-d} = 4^{-d} |\Lambda|^2. \end{aligned}$$

3. Upper estimates: C_Λ

We follow the approach adopted for $d = 2$ in [6]. Take an arbitrary $v_\Lambda \in S_\Lambda$, and consider its unique decomposition (7) into HB blocks $s_\beta \in S_\beta$. Define a partition of Λ into non-overlapping index subsets

$$\Lambda_k := \{\beta \in \Lambda : \ |\beta|_\infty = k\}, \qquad k = 1, \dots, k_\Lambda, \quad k_\Lambda := \max_{\beta\in\Lambda} |\beta|_\infty, \tag{15}$$

and gather the s_β into blocks associated with these Λ_k. Obviously,

$$v_k := \sum_{\beta\in\Lambda_k} s_\beta \in V_k, \qquad k = 1, \dots, k_\Lambda,$$

and according to (9)

$$\|v_\Lambda\|^2_{H^1_0} = \|\sum_{k=1}^{k_\Lambda} v_k\|^2_{H^1_0} \le C_{BPX} \sum_{k=1}^{\infty} 2^{2k} \|v_k\|^2_{L_2}.$$

If $|\Lambda_k|$ denotes the number of indices in Λ_k then by the Cauchy-Schwarz inequality

$$\|v_k\|^2_{L_2} = \|\sum_{\beta\in\Lambda_k} s_\beta\|^2_{L_2} \le |\Lambda_k| \sum_{\beta\in\Lambda_k} \|s_\beta\|^2_{L_2},$$

and, since $k = |\beta|_\infty$ for all $\beta \in \Lambda_k$, after substitution we see that the best constant in (2) satisfies

$$C_\Lambda \le C_{BPX} \cdot n_\Lambda, \qquad n_\Lambda := \max_{1\le k\le k_\Lambda} |\Lambda_k|. \tag{16}$$

A matching lower bound for the best possible C_Λ in (2) is given by the following example which shows that the appearance of n_Λ is natural. Let k denote the index for which $|\Lambda_k| = n_\Lambda$, and consider the index subsets

$$\Lambda_{k,i} = \{\beta \in \Lambda_k : \ \beta_i = k\}, \qquad i = 1, \dots, d.$$

Obviously, since $\cup_i \Lambda_{k,i} = \Lambda_k$, the largest of these subsets has size at least n_Λ/d. Without loss of generality, we can assume that $\Lambda_{k,1}$ is this largest subset, therefore the monotone index set

$$\Lambda' := \{\beta' \in \mathbb{N}^{d-1} : \ (k, \beta') \in \Lambda_{k,1}\} \subset \mathbb{N}^{d-1}$$

satisfies $|\Lambda'| = |\Lambda_{k,1}| \ge n_\Lambda/d$.

Consider now the function $\bar{s}_{\Lambda_{k,1}} \in S_\Lambda$ defined at the end of Section 2. According to (13), we have

$$\|\bar{s}_{\Lambda_{k,1}}\|^2_{HB} = 3^{-d} \sum_{\beta\in\Lambda_{k,1}} 2^{2|\beta|_\infty} = 3^{-d} 2^{2k} |\Lambda'|.$$

A lower bound for the H^1_0 norm of $\bar{s}_{\Lambda_{k,1}}$ is obtained as follows. Since

$$\bar{s}_{\Lambda_{k,1}}(x) = \bar{s}_k(x_1)\bar{s}_{\Lambda'}(x'),$$

by the construction of $\Lambda_{k,1}$, we get

$$\left|\frac{\partial}{\partial x_1}\bar{s}_{\Lambda_{k,1}}(x)\right| = |\bar{s}_k'(x_1)||\bar{s}_{\Lambda'}(x')| = 2^k|\bar{s}_{\Lambda'}(x')|$$

where we adopted the notation $x = (x_1, x')$ with $x' \in I^{d-1}$. Thus,

$$\|\bar{s}_{\Lambda_{k,1}}\|_{H_0^1}^2 \geq \|\frac{\partial}{\partial x_1}\bar{s}_{\Lambda_{k,1}}\|_{L_2}^2 = 2^{2k}\|\bar{s}_{\Lambda'}\|_{L_2}^2.$$

Now we use (14) for $\bar{s}_{\Lambda'}$. This gives

$$\|\bar{s}_{\Lambda_{k,1}}\|_{H_0^1}^2 \geq 4^{-d}2^{2k}|\Lambda'|^2.$$

Altogether we found that

$$C_\Lambda \geq \frac{\|\bar{s}_{\Lambda_{k,1}}\|_{H_0^1}^2}{\|\bar{s}_{\Lambda_{k,1}}\|_{HB}^2} \geq \frac{3^d}{4^d}|\Lambda'| \geq \frac{3^d}{d4^d}n_\Lambda. \tag{17}$$

To summarize, according to (16) and (17) the best possible constant in (2) is proportional to n_Λ up to constants only depending on d.

4. Lower estimates: c_Λ

To estimate the best constant c_Λ in (3), we proceed again as in [6] but start with a different decomposition of an arbitrary $v_\Lambda \in S_\Lambda$. Namely, for $k = 1, \ldots, k_\Lambda$, we define the index subset $\Lambda_{k,0} \subset \Lambda_k$ by collecting into it all maximal multi-indices in Λ_k, i.e., all $\beta \in \Lambda_k$ such that $\beta' \geq \beta$ and $\beta' \in \Lambda_k$ implies $\beta' = \beta$ (recall that Λ_k and k_Λ are given by (15)). Consider the L_2 orthogonal prewavelet representation

$$v_\Lambda = \sum_{\beta\in\Lambda} w_\beta = \sum_{k=1}^{k_\Lambda}\sum_{\beta'\in\Lambda_k} w'_\beta,$$

and partition, for each $k = 1, \ldots, k_\Lambda$, the index set Λ_k into subsets $\Lambda_{k,\beta}$ corresponding to the maximal indices $\beta \in \Lambda_{k,0}$ such that each β' belongs to exactly one $\Lambda_{k,\beta}$, and $\beta' \in \Lambda_{k,\beta}$ implies $\beta' \leq \beta$ (this partitioning is always possible but not unique). This gives a new decomposition

$$v_\Lambda = \sum_{k=1}^{k_\Lambda}\sum_{\beta\in\Lambda_{k,0}} v_\beta, \qquad v_\beta := \sum_{\beta'\in\Lambda_{k,\beta}} w_{\beta'}, \qquad \beta \in \Lambda_{k,0},$$

into $v_\beta \in V_\beta$ for which

$$\|v_\beta\|_{L_2}^2 = \sum_{\beta'\in\Lambda_{k,\beta}} \|w_{\beta'}\|_{L_2}^2$$

due to the L_2 orthogonality of the $w_{\beta'}$. Thus, according to (8) we obtain the following lower bound for the H_0^1 norm of v_Λ:

$$\|v_\Lambda\|_{H_0^1}^2 \geq c_{PW} \sum_{k=1}^{k_\Lambda} \sum_{\beta \in \Lambda_{k,0}} \sum_{\beta' \in \Lambda_{k,\beta}} \|w_{\beta'}\|_{L_2}^2 = c_{PW} \sum_{k=1}^{k_\Lambda} \sum_{\beta \in \Lambda_{k,0}} \|v_\beta\|_{L_2}^2. \tag{18}$$

We will next estimate the HB norms of the individual v_β with $\beta \in \Lambda_{k,0}$ by their L_2 norms. For fixed but arbitrary $\beta \in \Lambda_{k,0}$, consider the HB decomposition

$$v_\beta = \sum_{\beta' \leq \beta} s_{\beta'}.$$

Each $s_{\beta'}$ is a telescoping sum of at most 2^d multi-linear spline interpolants $I_{\beta''} v_\beta \in V_{\beta''}$ of v_β with respect to the nodal point set $\Sigma_{\beta''}$ associated with tensor-product partition $T_{\beta''}$, where $\beta'' \leq \beta'$ and $|\beta' - \beta''|_\infty \leq 1$. Thus, the HB norm of v_β is bounded by

$$\begin{aligned} \|v_\beta\|_{HB}^2 &= \sum_{\beta' \leq \beta} 2^{2|\beta'|_\infty} \|s_{\beta'}\|_{L_2}^2 \leq 2^d \sum_{\beta' \leq \beta} 2^{2|\beta'|_\infty} \sum_{\beta'' \leq \beta' :\, |\beta' - \beta''|_\infty \leq 1} \|I_{\beta''} v_\beta\|_{L_2}^2 \\ &\leq C \sum_{\beta' \leq \beta} 2^{2|\beta'|_\infty} \|I_{\beta'} v_\beta\|_{L_2}^2 \leq C \sum_{\beta' \leq \beta} 2^{2|\beta'|_\infty - |\beta'|_1} \sum_{P \in \Sigma_{\beta'}} |v_\beta(P)|^2. \end{aligned}$$

Here and in the following we use the L_2-stability

$$c\|v_\beta\|_{L_2}^2 \leq 2^{-|\beta|_1} \sum_{P \in \Sigma_\beta} |v_\beta(P)|^2 \leq C\|v_\beta\|_{L_2}^2, \qquad v_\beta \in V_\beta, \tag{19}$$

of the nodal basis in the full grid spaces V_β (in the above, the lower L_2 stability bound in (19) was applied with v_β replaced by $I_{\beta'} v_\beta \in V_{\beta'}$). Since the nodal point sets Σ_β form a monotone family with respect to the multi-index order, i.e., $\beta' \leq \beta$ implies $\Sigma_{\beta'} \subset \Sigma_\beta$, we have

$$\|v_\beta\|_{HB}^2 \leq C \sum_{P \in \Sigma_\beta} |v_\beta(P)|^2 \sum_{\beta' \leq \beta} 2^{2|\beta'|_\infty - |\beta'|_1}.$$

A straightforward calculation shows that

$$\sum_{\beta' \leq \beta} 2^{2|\beta'|_\infty - |\beta'|_1} \leq \sum_{i=1}^{d} \sum_{\beta' \leq \beta :\, |\beta'|_\infty = \beta_i} 2^{2|\beta'|_\infty - |\beta'|_1} \leq d \sum_{k_1=1}^{k} 2^{k_1} \sum_{1 \leq k_2, \ldots, k_d \leq k_1} 2^{-k_2 - \ldots - k_d} < d 2^{k+1}.$$

Thus,

$$\|v_\beta\|_{HB}^2 \leq C 2^k \sum_{P \in \Sigma_\beta} |v_\beta(P)|^2 \leq C 2^{k + |\beta|_1} \|v_\beta\|_{L_2}^2, \qquad \beta \in \Lambda_{k,0}, \tag{20}$$

for $k = 1, \ldots, k_\Lambda$, where we have used the upper L_2 stability bound in (19).

The combination of (18) and (20) leads to the bound

$$
\begin{aligned}
\|v_\Lambda\|_{HB}^2 &\le \|\sum_{k=1}^{k_\Lambda} \sum_{\beta\in\Lambda_{k,0}} 2^{(|\beta|_1-k)/2} 2^{(k-|\beta|_1)/2} v_\beta\|_{HB}^2 \\
&\le \left(\sum_{k=1}^{k_\Lambda} 2^{-k} \sum_{\beta\in\Lambda_{k,0}} 2^{|\beta|_1}\right)\left(\sum_{k=1}^{k_\Lambda} 2^{k} \sum_{\beta\in\Lambda_{k,0}} 2^{-|\beta|_1}\|v_\beta\|_{HB}^2\right) \\
&\le C\tilde{n}_\Lambda \sum_{k=1}^{k_\Lambda} 2^{2k} \sum_{\beta\in\Lambda_{k,0}} \|v_\beta\|_{L_2}^2 \le C\tilde{n}_\Lambda \|v_\Lambda\|_{H_0^1}^2,
\end{aligned}
$$

where

$$
\tilde{n}_\Lambda := \sum_{k=1}^{k_\Lambda} \sum_{\beta\in\Lambda_{k,0}} 2^{|\beta|_1-|\beta|_\infty}. \tag{21}
$$

Thus, the best constant c_Λ in (3) satisfies

$$
c_\Lambda^{-1} \le C\tilde{n}_\Lambda, \tag{22}
$$

with $\tilde{n}_\Lambda$ defined in (21).

The lower bound

$$
c_\Lambda^{-1} \ge c\tilde{n}'_\Lambda, \qquad \tilde{n}'_\Lambda := \max_{\beta\in\Lambda} 2^{|\beta|_1-|\beta|_\infty} \tag{23}
$$

is implied by considering the hat functions $\psi_\beta \in S_\Lambda$ with $\beta \in \Lambda$. Indeed, (12) implies for each of them

$$
\|\psi_\beta\|_{HB}^2 \ge c2^{|\beta|_\infty},
$$

while the same calculation that led to (4) gives

$$
\|\psi_\beta\|_{H_0^1}^2 \le C2^{2|\beta|_\infty-|\beta|_1}.
$$

Thus,

$$
c_\Lambda^{-1} \ge \max_{\beta\in\Lambda} \frac{\|\psi_\beta\|_{HB}^2}{\|\psi_\beta\|_{H_0^1}^2} \ge c\max_{\beta\in\Lambda} 2^{|\beta|_1-|\beta|_\infty}
$$

yields (23).

As we will see in the next section, for some important classes of generalized sparse grid spaces including V_k and S_k, the two estimates (22) and (23) are of the same order but in general they do not match. Since $\Lambda_{k,0} \subset \Lambda_k \subset \Lambda$, we have

$$
1 \le \frac{\tilde{n}_\Lambda}{\tilde{n}'_\Lambda} \le \sum_{k=1}^{k_\Lambda} \sum_{\beta\in\Lambda_{k,0}} 1 = \sum_{k=1}^{k_\Lambda} |\Lambda_{k,0}| \le C\sum_{k=1}^{k_\Lambda} k^{d-2} \le Ck_\Lambda^{d-1}.
$$

Here we have used that Λ_k is the union of d sets $\Lambda_{k,i}$ each of which is essentially equivalent to a certain monotone set $\Lambda' \subset \mathbb{N}^{d-1}$ with $k_{\Lambda'} \le k$. Therefore, the cardinality of the set

$\Lambda_{k,0}$ of maximal indices in Λ_k cannot exceed d times the maximal cardinality of the set of maximal indices of such Λ' which is bounded by Ck^{d-2}.

Thus, in the worst case upper and lower bounds for c_Λ^{-1} may be off by a factor Ck_Λ^{d-1}. To see that such a gap is indeed possible, consider the index sets

$$\Lambda = \{(2k, \beta') : \ \beta' \in \mathbb{N}^{d-1}, |\beta'|_1 < k + d - 1\}, \qquad k = 1, 2, \ldots,$$

for which $k_\Lambda = 2k$, and

$$\sum_{\beta \in \Lambda_{m,0}} 2^{|\beta|_1 - |\beta|_\infty} = \sum_{(m,\beta') \in \Lambda_{m,0}} 2^{|\beta'|_1} = 2^k |\Lambda_{m,0}| \geq ck^{d-2} 2^k, \qquad m = k+1, \ldots, 2k.$$

This gives $\tilde{n}_\Lambda \geq ck^{d-1} 2^k$ and $\tilde{n}'_\Lambda = 2^k$. We currently do not know if the gap can be reduced by constructing more sophisticated examples in order to improve the bound (23).

5. Summary

To summarize, we have established bounds for HB preconditioning of H_0^1-elliptic variational problems in generalized sparse grid spaces S_Λ that are close to optimal for large classes of monotone Λ, in particular, for V_k, S_k, and the energy-optimized sparse grid spaces defined in [4], see also [3, 1].

Theorem 1 *Let $d > 1$, and $\Lambda \subset \mathbb{N}^d$ be a monotone index set. The condition number $\kappa_{S_\Lambda, HB}$ of the tensor-product HB preconditioner of a discretization of a symmetric H_0^1-elliptic variational problem with the generalized sparse grid space S_Λ satisfies the two-sided estimate*

$$c\, n_\Lambda \tilde{n}'_\Lambda \leq \kappa_{S_\Lambda, HB} \leq C\, n_\Lambda \tilde{n}_\Lambda,$$

where $n_\Lambda, \tilde{n}_\Lambda, \tilde{n}'_\Lambda$ are defined in (16), (21), (23), respectively, and the constants c, C depend solely on d.
For the standard sparse grid spaces S_k, the estimate turns into

$$\kappa_{S_k, HB} \approx k^{d-1} 2^{k(d-1)/d}, \qquad k \to \infty.$$

For the isotropic full grid spaces V_k, we have

$$\kappa_{V_k, HB} \approx k^{d-1} 2^{k(d-1)}, \qquad k \to \infty.$$

More generally, for the energy-optimized sparse-grid spaces $S_k^a := S_{\Lambda_{k,a}}$, $-\infty < a < 1$, given by the index set

$$\Lambda_{k,a} := \{\beta \in \mathbb{N}^d : \ |\beta|_1 - a|\beta|_\infty \leq (1-a)k + d - 1\}, \tag{24}$$

we have

$$\kappa_{S_k^a, HB} \approx k^{d-1} 2^{k(d-1)(1-a)/(d-a)}, \qquad k \to \infty, \tag{25}$$

with constants that depend on d (and may depend on a).

Before proving the asymptotic condition number behavior for the families V_k, S_k, and S_k^a, let us mention that the result for $\kappa_{S_k,HB}$ improves our previous estimates for $d=2$ and $d=3$ stated in [6] by a factor k and k^3, respectively. It is interesting to note that the condition number growth of the tensor-product HB preconditioner for V_k is always worse compared to the isotropic HB preconditioner (papers by Yserentant [9, 10] for $d=2$, Ong [7] for $d=3$, see [8] for arbitrary $d>3$) by roughly an exponential factor of 2^k.

As to the energy-optimized sparse grid spaces S_k^a which are defined in [4, Section 4.1.2] as V_J^T using different notation, it is obvious that $S_k = S_k^0$, S_k^a becomes V_k if $a \to -\infty$, and S_k^a deteriorates for $a=1$ into a sum of essentially univariate spline spaces

$$S_k^1 = V_{k,1,\dots,1} + V_{1,k,\dots,1} + \dots + V_{1,1,\dots,k}.$$

Proof of Theorem 1. The condition number estimate for general Λ is an immediate consequence of the results of Section 3 and 4. The gap between upper and lower bounds is related to the quotient $\tilde{n}_\Lambda/\tilde{n}'_\Lambda$ which may grow as k_Λ^{d-1} in the worst case.

The result for V_k is obvious since for the associated Λ we have $k_\Lambda = k$, and the sets $\Lambda_{r,0}$ consist of a single index $(r,\dots,r)$, $r=1,\dots,k$. Therefore,

$$\tilde{n}'_\Lambda = 2^{k(d-1)}, \qquad \tilde{n}_\Lambda = \sum_{r=1}^{k} 2^{r(d-1)} \le C2^{k(d-1)},$$

while $n_\Lambda = |\Lambda_k| \approx k^{d-1}$.

To prove (25) for the generalized sparse grid spaces S_k^a (which includes $S_k = S_k^0$ as a special case) we observe the following: As long as the integer r satisfies the inequality $dr - ar \le (1-a)k + d - 1$ or, equivalently, $r \le r_0 := \lfloor ((1-a)k+d-1)/(d-a) \rfloor$, we have $V_r \subset S_k^a$ by the definition (24) of the index set $\Lambda = \Lambda_k^a$. Thus, the sets $\Lambda_{r,0}$ of extremal points in Λ_r consist of a single index $(r,r,\dots,r)$ for $r=1,\dots,r_0$ and

$$\sum_{r=1}^{r_0} \sum_{\beta \in \Lambda_{r,0}} 2^{|\beta|_1 - |\beta|_\infty} = \sum_{r=1}^{r_0} 2^{(d-1)r} \le C2^{(d-1)r_0}.$$

For the remaining $r = r_0+1,\dots,k$ (note that $k_\Lambda = k$ since

$$(1-a)|\beta|_\infty + (d-1) \le |\beta|_1 - a|\beta|_\infty \le k(1-a) + d - 1, \qquad \beta \in \Lambda_k^a,$$

and $(k,1,\dots,1) \in \Lambda_k^a$ for any $a<1$), we split Λ_r and thus $\Lambda_{r,0}$ into d (not necessarily disjoint) sets $\Lambda_r^i = \{\beta \in \Lambda_r :\ \beta_i = r\}$. Obviously, the maximal elements of Λ_r^i the set of which is denoted $\Lambda_{r,0}^i$ belong to $\Lambda_{r,0}$, and, vice versa, each index in $\Lambda_{r,0}$ belongs to at least one $\Lambda_{r,0}^i$. Since

$$\sum_{\beta \in \Lambda_{r,0}} 2^{|\beta|_1 - |\beta|_\infty} \le \sum_{i=1}^{d} \sum_{\beta \in \Lambda_{r,0}^i} 2^{|\beta|_1 - |\beta|_\infty} = d \sum_{\beta \in \Lambda_{r,0}^1} 2^{|\beta|_1 - |\beta|_\infty}$$

due to the invariance of Λ_k^a with respect to index permutations, we estimate the sum for

$$\Lambda_{r,0}^1 = \{(r,\beta') : \beta' \in \mathbb{N}^{d-1},\ |\beta'|_\infty \le r,\ |\beta'|_1 = \lfloor (1-a)(k-r)+d-1 \rfloor\}.$$

This description of $\Lambda_{r,0}^1$ holds because any $\beta \in \Lambda_r^1$ is of the form $\beta = (r,\beta')$ with $|\beta'|_\infty \le r$ and satisfies

$$|\beta|_1 - a|\beta|_\infty = r(1-a) + |\beta'|_1 \le (1-a)k + d - 1,$$

and therefore any maximal index β for Λ_r^1 must satisfy equality $|\beta'|_1 = \lfloor (1-a)(k-r) + d - 1\rfloor$. This implies that

$$\sum_{\beta \in \Lambda_{r,0}^1} 2^{|\beta|_1 - |\beta|_\infty} = \sum_{\beta \in \Lambda_{r,0}^1} 2^{|\beta'|_1} = 2^{\lfloor (1-a)(k-r)+d-1\rfloor} |\Lambda_{r,0}^1|.$$

Note that $2^{\lfloor (1-a)(k-r)+d-1\rfloor} \le C 2^{(1-a)(k-r_0)+d-1} 2^{-(1-a)(r-r_0)}$, where the upper bound decays geometrically for any $a<1$. Thus, if we prove that

$$|\Lambda_{r,0}^1| \le C(r-r_0)^{d-2}, \qquad r = r_0+1,\dots,k, \tag{26}$$

then

$$\sum_{r=r_0+1}^{k} \sum_{\beta \in \Lambda_{r,0}} 2^{|\beta|_1 - |\beta|_\infty} \le C 2^{(1-a)(k-r_0)+d-1} \sum_{r=r_0+1}^{k} (r-r_0)^{d-2} 2^{-(1-a)(r-r_0)} \le C 2^{(d-1)r_0},$$

where we have used that $(d-1)r_0 = (d-a)r_0 - (1-a)r_0 \approx (1-a)k + d - 1 - (1-a)r_0 = (1-a)(k-r_0) + d - 1$ as $k \to \infty$. Thus, upper and lower bound for $c_{\Lambda_k^a}^{-1}$ are up to constant factors matching since

$$\tilde{n}_{\Lambda_k^a} \le C 2^{(d-1)r_0} \le C \tilde{n}'_{\Lambda_k^a} \le C \tilde{n}_{\Lambda_k^a}, \qquad k = 1,2,\dots.$$

(recall that $(r_0, r_0, \dots, r_0) \in \Lambda_k^a$ and consequently $\tilde{n}'_{\Lambda_k^a} \ge 2^{(d-1)r_0}$). The result is

$$c_{\Lambda_k^a}^{-1} \approx 2^{(d-1)r_0} \le C 2^{(d-1)(1-a)k/(d-a)}, \qquad k = 1,2,\dots. \tag{27}$$

On the other hand, $C_{\Lambda_k^a} \approx n_{\Lambda_k^a} \approx k^{d-1}$ (we leave this to the reader). Together this implies (25).

It remains to show (26). For $d=2$ this is obvious, since $\beta' \in \mathbb{N}$ and fixing $|\beta'|_1$ means to fix β', i.e., in this case $|\Lambda_{r,0}^1| = 1$ for all $r = r_0+1,\dots,k$. For $d>2$, we set $\gamma_i = r - \beta'_i \ge 0$, $i = 1,\dots,d-1$, and observe that

$$\begin{aligned} |\Lambda_{r,0}^1| &\le |\{\gamma \in \mathbb{Z}_+^{d-1} : |\gamma|_1 = (d-1)r - \lfloor (1-a)(k-r)+d-1\rfloor\}| \\ &\le C((d-1)r - \lfloor (1-a)(k-r)+d-1\rfloor)^{d-2}. \end{aligned}$$

This follows from $|\beta'|_\infty \le r$ and $|\gamma|_1 = (d-1)r - |\beta'|_1$ for all $\beta = (r,\beta') \in \Lambda_{r,0}^1$. Since

$$\begin{aligned} (d-1)r - \lfloor (1-a)(k-r)+d-1\rfloor &= (d-a)r - ((1-a)k + d - 1) + \epsilon \\ &= (d-a)(r-r_0) + \epsilon - \epsilon'(d-a), \end{aligned}$$

where $0 \le \epsilon, \epsilon' < 1$, we have

$$|(d-1)r - \lfloor (1-a)(k-r) + d - 1 \rfloor| \le C(r - r_0), \qquad r = r_0 + 1, \dots, k,$$

with a constant C depending on d and a. This gives (26) and concludes the proof of Theorem 1. $\square$

We mention that in [1, Section 3.2] a modification of $\Lambda_k^{1/5}$ is used to define a so-called energy-based sparse grid space, in order to optimize error bounds in the H_0^1 norm, and claim that the method used for (25) covers this example as well.

Finding the correct order of the HB condition numbers $\kappa_{V_\beta,HB}$ for arbitrary anisotropic full grid spaces V_β as $\beta \to \infty$ is currently an open problem. By inspecting the subsets $\Lambda_{k,0}$ for the index set Λ associated with V_β reveals that here the gap between $\tilde{n}'_\Lambda$ and $\tilde{n}_\Lambda$ may be as large as $k_\Lambda = |\beta|_\infty$ (but not larger), independently of d. Indeed, without loss of generality, set $\beta = (k_\Lambda, \beta')$ with $\beta' \in \mathbb{N}^{d-1}$ satisfying $|\beta'|_\infty \le k_\Lambda$. Then $\Lambda_{k,0} = \{(k, \min(\beta', (k, \dots, k))\}$, $k = 1, \dots, k_\Lambda$, where the minimum of the two index vectors is taken componentwise. Thus,

$$\tilde{n}'_\Lambda = \max_{k=1,\dots,k_\Lambda} 2^{|\min(\beta',(k,\dots,k))|_1} = 2^{|\beta'|_1},$$

while

$$\tilde{n}_\Lambda = \sum_{k=1}^{k_\Lambda} 2^{|\min(\beta',(k,\dots,k))|_1} \le k_\Lambda 2^{|\beta'|_1}.$$

In the last estimate, equality is attained for $\beta' = (1, \dots, 1)$. A direct inspection of this extreme case of V_β with $\beta = (k_\Lambda, 1, \dots, 1)$ shows that $c_\Lambda^{-1} \approx 1$ independently of k_Λ and d. In other words, in this case the lower bound (23) gives the correct behavior which indicates that improvements in the proof of the upper bound (22) for c_Λ^{-1} should be possible.

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