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On the expected uniform error of geometric Brownian motion approximated by the Lévy-Ciesielski construction

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Abstract

It is known that the Brownian bridge or Lévy-Ciesielski construction of Brownian paths almost surely converges uniformly to the true Brownian path. In the present article the focus is on the error. In particular, we show for geometric Brownian motion that at level N, at which there are $d = 2^N$ points evaluated on the Brownian path, the expected uniform error has an upper bound of order $\mathcal{O}(\sqrt{N/2^N})$, or equivalently, $\mathcal{O}(\sqrt{\ln d/d})$. This upper bound matches the known order for the expected uniform error of the standard Brownian motion. We apply the result to an option pricing example.

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1. Introduction

Geometric Brownian motion is the solution $S(t) = S(\omega)(t)$ at time t of the stochastic differential equation

$$\mathrm{d}S(t) \,=\, S(t)\,(r\,\mathrm{d}t + \sigma\,\mathrm{d}B(t)) \quad t \in [0,1] \tag{1.1}$$

for given initial data $S(0)$, where $B(t) = B(\omega)(t)$ denotes standard Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$. That is, for each $t \in [0,1]$, $B(t)$ is a zero-mean Gaussian random variable, and for each pair $t, s \in [0,1]$ the covariance is

$$\mathbb{E}[B(t)B(s)] \,=\, \min(t,s).$$

The solution to (1.1) is given explicitly by

$$S(t) \,=\, S(0)\exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma B(t)\right), \quad 0 \le t \le 1. \tag{1.2}$$

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In this paper we are concerned with the Lévy-Ciesielski (or Brownian bridge) construction of the Brownian paths. The Lévy-Ciesielski construction expresses the Brownian path $B(t)$ in terms of a Faber-Schauder basis $\{\eta_0, \eta_{n,i} : n \in \mathbb{N}, i = 1, \ldots, 2^{n-1}\}$ of continuous functions on $[0,1]$, where $\eta_0(t) := t$ and

$$\eta_{n,i}(t) \;:=\; \begin{cases} 2^{(n-1)/2}\left(t - \dfrac{2i-2}{2^n}\right), & t \in \left[\dfrac{2i-2}{2^n}, \dfrac{2i-1}{2^n}\right], \\ 2^{(n-1)/2}\left(\dfrac{2i}{2^n} - t\right), & t \in \left[\dfrac{2i-1}{2^n}, \dfrac{2i}{2^n}\right], \\ 0 & \text{otherwise.} \end{cases}$$

For a proof that this is a basis in $\mathcal{C}[0,1]$, see [10, Theorem 2.1(iii)] or [11]. The Brownian path corresponding to the sample point $\omega \in \Omega$ is in this construction given by

$$B(t) \;=\; X_0(\omega)\,\eta_0(t) + \sum_{n=1}^{\infty}\sum_{i=1}^{2^{n-1}} X_{n,i}(\omega)\,\eta_{n,i}(t), \tag{1.3}$$

where X_0 and all the $X_{n,i}, i = 1, \ldots, 2^{n-1}, n \in \mathbb{N}$ are independent standard normal random variables. For $N \in \mathbb{N}$ we define the truncated Lévy-Ciesielski expansion by

$$B_N(t) \;:=\; X_0(\omega)\,\eta_0(t) + \sum_{n=1}^{N}\sum_{i=1}^{2^{n-1}} X_{n,i}(\omega)\,\eta_{n,i}(t). \tag{1.4}$$

Then $B_N(t)$ is for each $\omega \in \Omega$ a piecewise-linear function of t coinciding with $B(t)$ at special values of t: we easily see that

$$B(0) = B_N(0) = 0, \quad B(1) = B_N(1) = X_0,$$

and with $t = (2\ell - 1)/2^N$ we have

$$B\left(\frac{2\ell-1}{2^N}\right) \;=\; B_N\left(\frac{2\ell-1}{2^N}\right), \quad \ell = 1, \ldots, 2^{N-1},$$

because the terms in (1.3) with $n > N$ vanish at these points. The successive values returned by the usual (discrete) Brownian bridge construction are the values of $B(t)$ at the corresponding special t values $0, 1, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \ldots$.

The Lévy-Ciesielski construction has the important property that it converges almost surely to a continuous Brownian path, see the original works by [2, 5], or for example [9]: on defining the uniform norm with respect to t by

$$\|u\|_\infty := \sup_{t\in[0,1]} |u(t)|, \quad u \in \mathcal{C}[0,1],$$

the statement is that, almost surely,

$$\|B - B_N\|_\infty \to 0 \quad \text{as } N \to \infty.$$

The precise convergence rate for the expected uniform error of the Lévy-Ciesielski expansion was obtained in [7, Theorem 2]: written in the language of this paper, we have

$$\mathbb{E}\left[\|B - B_N\|_\infty\right] \sim \sqrt{\frac{\ln d}{2d}}, \tag{1.5}$$

where d is the dimension of the Faber-Schauder basis,

$$d = 1 + \sum_{n=1}^{N} 2^{n-1} = 2^N.$$

The meaning of the expected value $\mathbb{E}$ will be made precise in the next section. The asymptotic notation $\alpha(x) \sim \beta(x)$ means that $\lim_{x\to\infty} |\alpha(x)/\beta(x)| \to 1$. Thus (1.5) gives the precise leading term for the expected uniform error of the Lévy-Ciesielski expansion. Actually, the article [7] included much more general results and it showed also that the Lévy-Ciesielski approximation is in a certain sense optimal.

The main result of this paper is the following theorem which states that the expected uniform error of the geometric Brownian motion has an upper bound of the same order as (1.5).

Theorem 1. *Let S be the geometric Brownian motion given by* (1.2)*, and let S_N be the approximation defined by*

$$S_N(t) := S(0)\exp\left(\left(r - \tfrac{\sigma^2}{2}\right)t + \sigma B_N(t)\right), \quad 0 \le t \le 1. \tag{1.6}$$

where B_N is the truncated Lévy-Ciesielski approximation of B given by (1.4)*. Then*

$$\mathbb{E}\left[\|S - S_N\|_\infty\right] = \mathcal{O}\left(\frac{\sqrt{N}}{2^{N/2}}\right) = \mathcal{O}\left(\sqrt{\frac{\ln d}{d}}\right),$$

where $d = 2^N$ and the implied constants depend only on r and σ.

The remainder of the paper is organized as follows. In Section 2 we express the expected value of a geometric Brownian motion, i.e. its path integral, as an infinite dimensional integral over a sequence space. To this end, we focus on the Lévy-Ciesielski expansion and discuss its properties. In Section 3 we show that $\mathbb{E}\left[\|B - B_N\|_{L_2}^2\right] = 2^{-N}/6$, which is of the same order as that for the Karhunen-Loève expansion known to be optimal in L_2. In Section 4 we first recall the result from [7] which gives the asymptotic bound (1.5) on $\mathbb{E}\left[\|B - B_N\|_\infty\right]$. Then we proceed to prove an upper bound of the same order using a different line of argument to [7], namely extreme value statistics. In Section 5 we generalize this new line of argument to geometric Brownian motion, and give the proof of Theorem 1. In Section 6, we give an application to the problem of pricing an arithmetic Asian option. Finally in Section 7 we give some brief concluding remarks.

2. The expected value as an integral over a sequence space

In this section we show that the expected value in Theorem 1 can be expressed as an integral over a sequence space. We remark that we will sometimes find it convenient to use interchangeably the language of measure and integration and that of probability and expectation.

Recall that the Lévy-Ciesielski expansion (1.3) expresses the Brownian path $B(t)$ in terms of an infinite sequence $\boldsymbol{X}(\omega) = (X_0, (X_{n,i})_{n\in\mathbb{N}, i=1,\ldots,2^{n-1}})$ of independent standard normal random variables. In the following we will denote a particular realization of this sequence $\boldsymbol{X}$ by

$$\boldsymbol{x} = \big(x_0, (x_{n,i})_{n\in\mathbb{N}, i=1,\ldots,2^{n-1}}\big) = (x_1, x_2, \ldots) \in \mathbb{R}^\infty,$$

where for convenience we will switch freely between the double-index labeling $(x_0, x_{1,1}, x_{2,1}, x_{2,2}, \ldots)$ and a single-index labeling $(x_1, x_2, \ldots)$ as appropriate, with the indexing convention that $x_1 \equiv x_0$, and $x_{2^{n-1}+i} \equiv x_{n,i}$ for $n \geq 1$ and $1 \leq i \leq 2^{n-1}$.

It is clear from (1.3) that, for $t \in [0,1]$ and a fixed $\omega \in \Omega$,

$$\begin{aligned} |B_N(t)| \;&\leq\; |X_0| \,+\, \sum_{n=1}^{N} \left(\max_{1\leq i\leq 2^{n-1}} |X_{n,i}| \right) \left(\sum_{i=1}^{2^{n-1}} \eta_{n,i}(t) \right) \\ &\leq\; |X_0| \,+\, \sum_{n=1}^{N} \max_{1\leq i\leq 2^{n-1}} |X_{n,i}|\, 2^{-(n+1)/2}, \end{aligned} \tag{2.1}$$

where in the last step we used the fact that for a given $n \geq 1$ the disjoint nature of the Faber-Schauder functions ensures that at most one value of i contributes to the sum over i, and also that the $\eta_{n,i}$ for $i = 1, \ldots, 2^{n-1}$ have the same maximum value $2^{-(n+1)/2}$.

Motivated by the bound (2.1), and following [4], we define a norm of the sequence $\boldsymbol{x} = \big(x_0, (x_{n,i})_{n\in\mathbb{N}, i=1,\ldots,2^{n-1}}\big)$ by

$$\|\boldsymbol{x}\|_{\mathcal{X}} \;:=\; |x_0| + \sum_{n=1}^{\infty} \max_{1\leq i\leq 2^{n-1}} |x_{n,i}|\, 2^{-(n+1)/2},$$

and we define a corresponding normed space $\mathcal{X}$ by

$$\mathcal{X} \;:=\; \{\boldsymbol{x} \in \mathbb{R}^\infty \;:\; \|\boldsymbol{x}\|_{\mathcal{X}} < \infty\}.$$

It is easily seen that $\mathcal{X}$ is a Banach space.

Each choice of $\boldsymbol{x} \in \mathcal{X}$ corresponds to a particular $\omega \in \Omega$ (but not *vice versa*, since there are sample points $\omega \in \Omega$ corresponding to sequences $\boldsymbol{x}$ for which the norm $\|\boldsymbol{x}\|_{\mathcal{X}}$ is not finite). Hence to each $\boldsymbol{x} \in \mathcal{X}$ there corresponds a particular Brownian path via (1.3), or expressed in terms of $\boldsymbol{x}$,

$$B(\boldsymbol{x})(t) \;=\; x_0\, \eta_0(t) + \sum_{n=1}^{\infty} \sum_{i=1}^{2^{n-1}} x_{n,i}\, \eta_{n,i}(t), \quad t \in [0,1]. \tag{2.2}$$

That the resulting path is continuous on $[0,1]$ follows from the observation that the path is the pointwise limit of the truncated series

$$B_N(\boldsymbol{x})(t) \;=\; x_0\,\eta_0(t) + \sum_{n=1}^{N}\sum_{i=1}^{2^{n-1}} x_{n,i}\,\eta_{n,i}(t), \quad t\in[0,1], \tag{2.3}$$

which is uniformly convergent since

$$\|B_N\|_\infty \;\le\; |x_0| + \sum_{n=1}^{\infty} \max_{1\le i\le 2^{n-1}} |x_{n,i}|\,2^{-(n+1)/2} \;=\; \|\boldsymbol{x}\|_{\mathcal{X}} \;<\; \infty \quad \text{for} \quad \boldsymbol{x}\in\mathcal{X},$$

so that (2.2) does indeed define a continuous function for $\boldsymbol{x}\in\mathcal{X}$.

We define $\mathcal{A}_{\mathbb{R}^\infty}$ to be the σ-algebra generated by products of Borel sets of $\mathbb{R}$, see [1, p. 372]. On the Banach space $\mathcal{X}$, we now define a product Gaussian measure (see [1, p. 392 and Example 2.35])

$$\rho(\mathrm{d}\boldsymbol{x}) \;:=\; \bigotimes_{j=1}^{\infty} \phi(x_j)\,\mathrm{d}x_j,$$

where ϕ is the standard normal probability density

$$\phi(x) \;:=\; \frac{1}{\sqrt{2\pi}}\exp(-\tfrac{1}{2}x^2). \tag{2.4}$$

We next show that the space $\mathcal{X}$ has full Gaussian measure, i.e. that

$$\mathbb{P}\left(|X_0| + \sum_{n=1}^{\infty} \max_{1\le i\le 2^{n-1}} |X_{n,i}|\,2^{-(n+1)/2} < \infty\right) \;=\; 1.$$

This fact is at the heart of the classical proof that the Lévy-Ciesielski construction almost surely converges uniformly to the Brownian path. For a brief explanation, we define

$$H_n(\omega) \;:=\; \begin{cases} |X_0(\omega)| & \text{for } n=0,\\ \max_{1\le i\le 2^{n-1}} |X_{n,i}(\omega)|\,2^{-(n+1)/2} & \text{for } n\ge 1.\end{cases}$$

It is known that (see [3, Proof of Theorem 3]), as a consequence of the Borel-Cantelli lemma, one can construct a sequence $(\beta_n)_{n\ge 1}$ of positive numbers such that

$$\sum_{n=1}^{\infty}\beta_n < \infty, \quad \text{and} \quad \mathbb{P}\left(H_n(\cdot) > \beta_n \text{ infinitely often}\right) = 0.$$

We now define $\tilde{\Omega}$ to be the subset of Ω consisting of the sample points ω for which $H_n(\omega) > \beta_n$ for only finitely many values of n. Then $\tilde{\Omega}$ is of full Gaussian measure, and for each $\omega\in\tilde{\Omega}$ there exists $N(\omega)\in\mathbb{N}$ such that

$$H_n(\omega) \le \beta_n \quad \text{for} \quad n > N(\omega),$$

leading to

$$\sum_{n=1}^{\infty} H_n(\omega) \;\le\; \sum_{n=1}^{N(\omega)} H_n(\omega) + \sum_{n=N(\omega)+1}^{\infty} \beta_n \;<\; \infty \quad \text{for } \omega \in \tilde{\Omega}.$$

Thus $\mathbb{P}(\sum_{n=0}^{\infty} H_n < \infty) = 1$, as claimed, and the proof that $\mathcal{X}$ is of full Gaussian measure is complete.

We now study integration on the measure space $(\mathcal{X}, \mathcal{A}_{\mathbb{R}^\infty}, \rho)$, and we denote the integral, or the expected value, of a measurable function f by

$$\mathbb{E}[f] \;:=\; \int_{\mathcal{X}} f(\boldsymbol{x})\, \rho(\mathrm{d}\boldsymbol{x}).$$

In particular, for the proof of Theorem 1 we need the expected value of the uniform error $\|S - S_N\|_\infty$. Before we proceed with that proof, it is instructive to first obtain bounds on the expected value of the L_2 error $\|B - B_N\|_{L_2}$ and the uniform error $\|B - B_N\|_\infty$; we do this in the next two sections.

3. Expected L_2 error of standard Brownian motion

It follows from (1.3) and (1.4) that the random variable $B(t) - B_N(t)$ is the sum

$$B(t) - B_N(t) \;=\; \sum_{n=N+1}^{\infty} \sum_{i=1}^{2^{n-1}} X_{n,i}\, \eta_{n,i}(t), \quad t \in [0,1]. \tag{3.1}$$

It turns out that there is an explicit formula for the expected value of the squared L_2 norm of $B - B_N$.

Lemma 2. *Let B be the Lévy-Ciesielski expansion of the standard Brownian motion as in* (1.3)*, and let B_N be the corresponding truncated expansion as in* (1.4)*. Then, with $d = 2^N$,*

$$\mathbb{E}\,[\|B - B_N\|_{L_2}^2] \;=\; \frac{1}{6 \cdot 2^N} \;=\; \frac{1}{6\,d}\,.$$

Proof. A proof can be found in e.g., [12] (note that the indexing there differs from ours by 1). The lemma is also a direct consequence of the results in [7] for the L_2 norm (take $q = 2$ and $p = 1$ in that paper). For completeness and for expository reasons (later we will consider the L_∞ norm) we give a short proof here.

The key is

$$\mathbb{E}\,[X_{n,i}\, X_{n',i'}] \;=\; \delta_{n,n'}\, \delta_{i,i'}\, \mathbb{E}\,[X_{n,i}^2] \;=\; \delta_{n,n'}\, \delta_{i,i'},$$

which holds because the $X_{n,i}$ are independent mean-zero random variables with variance one. As a result we have

$$\mathbb{E}\left[\|B - B_N\|_{L_2}^2\right] = \mathbb{E}\left[\left\| \sum_{n=N+1}^{\infty} \sum_{i=1}^{2^{n-1}} X_{n,i}\, \eta_{n,i}(\cdot) \right\|_{L_2}^2\right]$$

$$= \sum_{n=N+1}^{\infty} \sum_{i=1}^{2^{n-1}} \sum_{n'=N+1}^{\infty} \sum_{i'=1}^{2^{n'-1}} \underbrace{\mathbb{E}\left[X_{n,i}\, X_{n',i'}\right]}_{\delta_{n,n'}\,\delta_{i,i'}} \int_0^1 \eta_{n,i}(t)\, \eta_{n',i'}(t)\, \mathrm{d}t$$

$$= \sum_{n=N+1}^{\infty} \sum_{i=1}^{2^{n-1}} \int_0^1 (\eta_{n,i}(t))^2\, \mathrm{d}t = \sum_{n=N+1}^{\infty} 2^{n-1} \frac{1}{3 \cdot 2^{2n}} = \frac{1}{6} \sum_{n=N+1}^{\infty} 2^{-n} = \frac{1}{6}\, 2^{-N},$$

which completes the proof. □

It is interesting to compare this result with the corresponding result based on the *Karhunen-Loève expansion* of the standard Brownian motion which is known to be optimal in L_2, see e.g., [6],

$$\mathbb{E}\left[\|B^{\mathrm{KL}} - B_d^{\mathrm{KL}}\|_{L_2}^2\right] = \frac{1}{\pi^2} \sum_{j=d+1}^{\infty} \frac{1}{j^2} \le \frac{1}{\pi^2\, d}.$$

We see that the Lévy-Ciesielski expansion yields the same order, with only a slightly worse constant.

4. Expected uniform error of standard Brownian motion

The following theorem is a special case of [7, Theorem 2] where $p = 1$.

Theorem 3. *Let B be the Lévy-Ciesielski expansion of the standard Brownian motion as in* (1.3)*, and let B_N be the corresponding truncated expansion as in* (1.4)*. Then, with $d = 2^N$,*

$$\mathbb{E}\left[\|B - B_N\|_{\infty}\right] \sim \left(\frac{\ln d}{2d}\right)^{1/2}.$$

This article [7] shows also that the Brownian bridge approximation is optimal among all constructions that use information at d points and Wiener measure. Note that, in contrast to here, zero boundary conditions are employed in [7] at both ends of the time interval, but this is immaterial, since the only difference is that in the present work we have an additional basis function $\eta_0(t) = t$, which disappears in the difference $B(t) - B_N(t)$.

We will now derive the upper bound part of this result (with a slightly worse constant, larger by a factor of $2 + \sqrt{2} \approx 3.41421$), using a different proof which relies on extreme value theory. It will lay the foundations for our proof for the case of geometric Brownian motion.

Proposition 4. *Let B be the Lévy-Ciesielski expansion of the standard Brownian motion as in* (1.3)*, and let B_N be the corresponding truncated expansion as in* (1.4)*. Then, with $d = 2^N$,*

$$\mathbb{E}[\|B - B_N\|_\infty \leq (2+\sqrt{2}) \left(\frac{\ln d}{2d}\right)^{1/2} \left(1 + \mathcal{O}\left(\frac{1}{\sqrt{\ln d}}\right)\right).$$

We devote the remainder of this section to proving Proposition 4. We have from (1.3) and (1.4)

$$\|B - B_N\|_\infty \;=\; \sup_{t\in[0,1]} |B(t) - B_N(t)| \;\leq\; \sum_{n=N+1}^{\infty} \max_{1\leq i\leq 2^{n-1}} |X_{n,i}|\, 2^{-(n+1)/2}.$$

Thus

$$\begin{aligned}
\mathbb{E}\left[\|B - B_N\|_\infty\right] \;&\leq\; \sum_{n=N+1}^{\infty} \mathbb{E}\left[\max_{1\leq i\leq 2^{n-1}} |X_{n,i}|\right] 2^{-(n+1)/2} \\
&=\; \sum_{\ell=2^N,2^{N+1},2^{N+2},\ldots} \frac{\mathbb{E}\left[M_\ell\right]}{2\sqrt{\ell}},
\end{aligned} \tag{4.1}$$

where we substituted $\ell := 2^{n-1}$ and introduced a new random variable

$$M_\ell \;:=\; \max_{1\leq i\leq \ell} |X_i| \quad \text{for } \ell \text{ a power of } 2\,, \tag{4.2}$$

for independent $\mathcal{N}(0,1)$ random variables $X_1, X_2, \ldots, X_\ell$.

Now we are in the territory of extreme value statistics. It is known that the distribution function of the maximum of the absolute value of ℓ independent Gaussian random variables converges (after appropriate centering and scaling, as below) to the Gumbel distribution. A first step is to obtain an explicit expression for the distribution function of M_ℓ. Because $X_1, X_2, \ldots, X_\ell$ are $\mathcal{N}(0,1)$ random variables, for $x \in \mathbb{R}^+$ and $i = 1, \ldots, \ell$, we have

$$\mathbb{P}(X_i \leq x) \;=\; \int_{-\infty}^{x} \phi(t)\,\mathrm{d}t \;=:\; \Phi(x),$$

where ϕ is the standard normal density defined by (2.4). Similarly,

$$\mathbb{P}(|X_i| \leq x) \;=\; \int_{-x}^{x} \phi(t)\,\mathrm{d}t \;=\; \Phi(x) - \Phi(-x) \;=\; 2\Phi(x) - 1.$$

Therefore (since $|X_1|, |X_2|, \ldots, |X_\ell|$ are independent random variables) we have

$$\mathbb{P}(M_\ell \leq x) \;=\; \mathbb{P}(|X_1| \leq x \text{ and } |X_2| \leq x \text{ and } \cdots \text{and } |X_\ell| \leq x) \;=\; (2\Phi(x) - 1)^\ell.$$

Thus the distribution function of M_ℓ is

$$\Psi_\ell(x) \;:=\; (2\Phi(x) - 1)^\ell, \quad x \in \mathbb{R}^+. \tag{4.3}$$

We now define a new random variable Y_ℓ, which is a recentered and rescaled version of M_ℓ:

$$Y_\ell \;:=\; \frac{M_\ell - a_\ell}{b_\ell}, \quad \ell \geq 0,\, a_\ell > 0,\, b_\ell > 0, \tag{4.4}$$

or equivalently

$$M_\ell \;=\; a_\ell + b_\ell Y_\ell. \tag{4.5}$$

It is known (see below) to be appropriate to take a_ℓ and b_ℓ to satisfy

$$a_\ell = \sqrt{2\ln\ell} + o(1), \quad \text{and} \quad b_\ell = \frac{1}{a_\ell}. \tag{4.6}$$

More precisely, for later convenience we will define a_ℓ to be the unique solution of

$$\frac{1}{\ell} = \sqrt{\frac{2}{\pi}}\frac{e^{-a_\ell^2/2}}{a_\ell} = 2\frac{\phi(a_\ell)}{a_\ell}. \tag{4.7}$$

We now show that (4.7) implies (4.6).

Lemma 5. *Equation* (4.7) *for* $\ell \geq 1$ *has a unique positive solution of the form* $a_\ell = \sqrt{2\ln\ell} + o(1)$. *Moreover, for* $\ell \geq 3$ *we have* $a_\ell \in (1, \sqrt{2\ln\ell})$.

Proof. The fact that any solution of (4.7) is positive is immediate. Now observe that

$$g(y) \;:=\; \sqrt{\frac{2}{\pi}}\frac{e^{-y^2/2}}{y}$$

is monotonically decreasing from $+\infty$ to 0 for $y \in (0,\infty)$. It follows immediately that there is a unique solution $a_\ell \in (0,\infty)$ for (4.7). Moreover, we have

$$a_\ell > 1 \quad \Leftrightarrow \quad \frac{1}{\ell} < \sqrt{\frac{2}{\pi}}\frac{e^{-1^2/2}}{1} = \sqrt{\frac{2}{\pi e}} = 0.484\ldots,$$

which holds if and only if $\ell \geq 3$. Now observe that (4.7) is equivalent to

$$a_\ell = \sqrt{2\left(\ln\ell - \ln\left(\sqrt{\frac{\pi}{2}}a_\ell\right)\right)}. \tag{4.8}$$

For $\ell \geq 3$ we have $a_\ell > 1$ and hence $\ln(\sqrt{\pi/2}\,a_\ell) > \ln(\sqrt{\pi/2}) > 0$, so from (4.8) we have $a_\ell < \sqrt{2\ln\ell}$. In turn it follows that

$$a_\ell > \sqrt{2\ln\ell - \ln\left(\sqrt{\frac{\pi}{2}}\sqrt{2\ln\ell}\right)}.$$

Thus for $\ell \geq 3$ we have $1 \leq a_\ell = \sqrt{2\ln\ell} + o(1) \leq \sqrt{2\ln\ell}$. This completes the proof. □

It is well known that the distribution function of Y_ℓ converges in distribution to a random variable with the Gumbel distribution $\exp(-e^{-y})$. For later convenience we state this as a lemma and give a short proof.

Lemma 6. *The random variable Y_ℓ defined in* (4.4)*, with a_ℓ defined by* (4.7) *and $b_\ell = 1/a_\ell$, converges in distribution to a random variable Y with Gumbel distribution function $\mathbb{P}(Y \le y) = \exp(-e^{-y})$.*

Proof. The proof is based on the asymptotic version of Mill's ratio [8],

$$1 - \Phi(x) \sim \frac{\phi(x)}{x}, \quad x \to +\infty.$$

From this it follows that for $y \in \mathbb{R}$

$$\begin{aligned}
\mathbb{P}(Y_\ell \le y) &= \mathbb{P}(M_\ell \le a_\ell + b_\ell y) = (2\Phi(a_\ell + b_\ell y) - 1)^\ell \\
&= (1 - 2[1 - \Phi(a_\ell + b_\ell y)])^\ell \\
&\sim \left(1 - \sqrt{\frac{2}{\pi}} \frac{\exp(-\frac{1}{2}(a_\ell + b_\ell y)^2)}{a_\ell + b_\ell y}\right)^\ell \\
&\sim \left(1 - \sqrt{\frac{2}{\pi}} \frac{\exp(-\frac{1}{2}a_\ell^2 - a_\ell b_\ell y)}{a_\ell}\right)^\ell \\
&= \left(1 - \frac{\exp(-y)}{\ell}\right)^\ell \sim \exp(-e^{-y}) \quad \text{as } \ell \to \infty,
\end{aligned}$$

where in the second step we dropped a higher order term, and in the second last step we used (4.7) and $a_\ell b_\ell = 1$, thus proving the lemma. □

A deeper result, which we need, is that Y_ℓ converges in expectation to the limit Y. This is proved in the following lemma.

Lemma 7. *The random variable Y_ℓ defined in* (4.4)*, with a_ℓ defined by* (4.7) *and $b_\ell = 1/a_\ell$, converges in expectation to a random variable Y with Gumbel distribution* $\exp(-e^{-y})$*, thus*

$$\lim_{\ell \to \infty} \mathbb{E}[Y_\ell] = \mathbb{E}[Y] = \int_{-\infty}^{\infty} y \exp(-y - e^{-y}) \mathrm{d}y = \gamma,$$

where γ is Euler's constant.

Proof. For a sequence of real-valued random variables $Y_1, Y_2, \ldots$ converging in distribution to a random variable Y, it is well known that a sufficient condition for convergence in expectation is uniform integrability of the Y_ℓ. In turn a sufficient condition for uniform integrability is that for sufficiently large ℓ

$$\begin{aligned}
&\mathbb{P}(Y_\ell \ge y) \le Q(y) \text{ for } y > 0, \text{ and} \\
&\mathbb{P}(Y_\ell \le y) \le R(y) \text{ for } y < 0,
\end{aligned}$$

where $Q(y)$ is integrable on $\mathbb{R}^+$ and $R(y)$ is integrable on $\mathbb{R}^-$.

First assume $y > 0$. We have from (4.3) that

$$\begin{aligned}\mathbb{P}(Y_\ell \geq y) &= \mathbb{P}(M_\ell \geq a_\ell + b_\ell y) = 1 - \mathbb{P}(M_\ell \leq a_\ell + b_\ell y) \\ &= 1 - (2\Phi(a_\ell + b_\ell y) - 1)^\ell = 1 - (1 - 2[1 - \Phi(a_\ell + b_\ell y)])^\ell \\ &\leq 1 - \left(1 - 2\frac{\phi(a_\ell + b_\ell y)}{a_\ell + b_\ell y}\right)^\ell = 1 - \left(1 - \sqrt{\frac{2}{\pi}}\frac{\exp(-\frac{1}{2}(a_\ell + b_\ell y)^2)}{a_\ell + b_\ell y}\right)^\ell \\ &\leq 1 - \left(1 - \sqrt{\frac{2}{\pi}}\frac{\exp\left(-\frac{1}{2}a_\ell^2 - a_\ell b_\ell y\right)}{a_\ell}\right)^\ell,\end{aligned}$$

where we used the upper bound form of Mills' ratio [8],

$$1 - \Phi(x) < \frac{\phi(x)}{x}, \quad x \in \mathbb{R}^+, \tag{4.9}$$

and dropped harmless terms in both the denominator and the exponent in the numerator. Using now (4.7) and also $a_\ell b_\ell = 1$ we have

$$\mathbb{P}(Y_\ell \geq y) \leq 1 - \left(1 - \frac{\exp(-y)}{\ell}\right)^\ell \leq \exp(-y) =: Q(y),$$

where we used the fact that the function $(1 - 1/x)^x$ is increasing on $[1, \infty)$, and hence takes its minimum at $x = 1$. It follows that

$$\int_0^\infty \mathbb{P}(Y_\ell \geq y)\,\mathrm{d}y \leq \int_0^\infty \exp(-y)\,\mathrm{d}y = 1.$$

Now we consider $y < 0$. Note first that $M_\ell = a_\ell + b_\ell Y_\ell$ takes only non-negative values, thus we may restrict y to $y \geq -a_\ell/b_\ell$. We have

$$\mathbb{P}(Y_\ell \leq y) = \mathbb{P}(M_\ell \leq a_\ell + b_\ell y) = (2\Phi(a_\ell + b_\ell y) - 1)^\ell.$$

Now for $t > 0$ the standard normal distribution Φ has negative second derivative,

$$\Phi''(t) = \phi'(t) < 0 \quad \text{for } t > 0,$$

and first derivative $\Phi'(t) = \phi(t)$, from which it follows that

$$\Phi(a_\ell + b_\ell y) \leq \Phi(a_\ell) + b_\ell y\,\phi(a_\ell) \quad \text{for } y \geq -a_\ell/b_\ell.$$

Thus on using $b_\ell = 1/a_\ell$, we obtain

$$\begin{aligned}\mathbb{P}(Y_\ell \leq y) &\leq \left(2\Phi(a_\ell) + 2a_\ell^{-1} y\phi(a_\ell) - 1\right)^\ell \\ &\leq \left(1 - 2a_\ell^{-1}\phi(a_\ell)(1 - y - a_\ell^{-2})\right)^\ell = \left(1 - \frac{1}{\ell}(1 - y - a_\ell^{-2})\right)^\ell,\end{aligned}$$

where in the second last step we used the lower bound form of Mills' ratio, see [8, p. 44]

$$1-\Phi(t) \geq \frac{\phi(t)}{t}(1-t^{-2}) \quad \text{for } t>0, \tag{4.10}$$

and in the last step we used (4.7). If we now take $\ell \geq L$ then we have

$$\begin{aligned}\mathbb{P}(Y_\ell \leq y) \;&\leq\; \left(1-\frac{1}{\ell}(1-y-a_L^{-2})\right)^{\ell} \\ &\leq\; \exp\left(-(1-y-a_L^{-2})\right) \;=\; \exp\left(-(1-a_L^{-2})\right)\exp(y) \;=:\; R(y),\end{aligned}$$

since the convergence in the last limit is monotone increasing. The function $R(y)$ so defined is integrable on $\mathbb{R}^-$, completing the proof that Y_ℓ converges in expectation.

It then follows from the previous lemma that this limit of $\mathbb{E}[Y_\ell]$ is precisely $\mathbb{E}[Y]=\gamma$. □

Since the above lemma establishes the convergence of $\mathbb{E}[Y_\ell]$ as $\ell \to \infty$, it can be inferred that there exists a positive constant c such that

$$\mathbb{E}[Y_\ell] \leq c \quad \text{for } \ell \geq 0.$$

From this and (4.5) it follows that

$$\mathbb{E}[M_\ell] \;\leq\; a_\ell + b_\ell\, c \;\leq a_\ell + c,$$

where we used $b_\ell = a_\ell^{-1} \leq 1$ for $\ell \geq 3$. We then conclude from (4.1) that

$$\mathbb{E}\left[\|B-B_N\|_\infty\right] \;\leq\; \frac{1}{2}\sum_{\ell=2^N,2^{N+1},2^{N+2},\ldots} \frac{a_\ell + c}{\sqrt{\ell}}. \tag{4.11}$$

It only remains to estimate the sum in (4.11). Using Lemma 5 with $N \geq 2$ (and hence $\ell \geq 3$), we have $a_\ell < \sqrt{2\ln \ell}$, and on setting $\ell = 2^{N+j}$,

$$\begin{aligned}\sum_{\ell=2^N,2^{N+1},2^{N+2},\ldots} \frac{a_\ell}{\sqrt{\ell}} \;&\leq\; \sum_{j=0}^{\infty} \frac{\sqrt{2(\ln 2)(N+j)}}{\sqrt{2^{N+j}}} \\ &=\; 2^{-(N-1)/2}\sqrt{\ln 2}\sum_{j=0}^{\infty}\frac{\sqrt{N+j}}{2^{j/2}}. \\ &\leq\; 2^{-(N-1)/2}\sqrt{\ln 2}\,\sqrt{N}\,(2+\sqrt{2})\,(1+\mathcal{O}(N^{-1/2})),\end{aligned} \tag{4.12}$$

where in the final step we used $\sqrt{N+j} \leq \sqrt{N}+\sqrt{j}$ and $\sum_{j=0}^{\infty} 1/2^{j/2} = 2+\sqrt{2}$, while noting that $\sum_{j=0}^{\infty}\sqrt{j}/2^{j/2}$ is finite and independent of N. Moreover, by a similar argument we conclude that $\sum_{\ell=2^N,2^{N+1},2^{N+2},\ldots} c/\sqrt{\ell} = \mathcal{O}(2^{-N/2})$, thus altogether we obtain from (4.11)

$$
\begin{aligned}
\mathbb{E}\left[\|B-B_N\|_\infty\right] &\le \frac{1}{2}\cdot 2^{-(N-1)/2}\sqrt{\ln 2}\,\sqrt{N}\,(2+\sqrt{2})\,(1+\mathcal{O}(N^{-1/2})) \\
&= (2+\sqrt{2})\frac{\sqrt{N\ln 2}}{\sqrt{2\cdot 2^N}}(1+\mathcal{O}(N^{-1/2})) \\
&= (2+\sqrt{2})\frac{\sqrt{\ln d}}{\sqrt{2d}}\left(1+\mathcal{O}\left(\frac{1}{\sqrt{\ln d}}\right)\right),
\end{aligned}
$$

which concludes the proof of Proposition 4.

5. Expected uniform error of geometric Brownian motion

We are now in the position to give a proof of Theorem 1. From (1.2) and (1.6) it follows that

$$
S(t)-S_N(t) \;=\; S(0)\,e^{(r-\sigma^2/2)t}\big(\exp(\sigma B(t))-\exp(\sigma B_N(t))\big),
$$

and thus

$$
\|S-S_N\|_\infty \;\le\; S(0)\,e^{|r-\sigma^2/2|}\,\big\|\exp(\sigma B)-\exp(\sigma B_N)\big\|_\infty. \tag{5.1}
$$

In turn it follows that

$$
\begin{aligned}
\|S-S_N\|_\infty &\le S(0)\,e^{|r-\sigma^2/2|}\,\big\|\exp(\sigma B_N)\big(\exp(\sigma(B-B_N))-1\big)\big\|_\infty \\
&\le S(0)\,e^{|r-\sigma^2/2|}\,\exp\big(\sigma\|B_N\|_\infty\big)\big(\exp(\sigma\|B-B_N\|_\infty)-1\big). \qquad (5.2)
\end{aligned}
$$

We now consider the expected value of $\|S-S_N\|_\infty$. We recall that the random variable $B-B_N$ is the sum (3.1), in which the set of i.i.d. Gaussian normal random variables $X_{n,i}$ (with $n \ge N+1$) is disjoint from the $X_{n,i}$ appearing in B_N (for which $n \le N$). Thus we can express the expected value of the right-hand side as the product

$$
\mathbb{E}\left[\|S-S_N\|_\infty\right] \;\le\; S(0)\,e^{|r-\sigma^2/2|}\,\mathcal{J}_N\,\mathcal{K}_N,
$$

where

$$
\mathcal{J}_N \;:=\; \mathbb{E}\left[\exp(\sigma\|B_N\|_\infty)\right] \quad\text{and}\quad \mathcal{K}_N \;:=\; \mathbb{E}\left[\exp(\sigma\|B-B_N\|_\infty)\right]-1\,.
$$

To show that the $\mathcal{J}_N$ are bounded uniformly in N it is convenient to briefly switch to the language of measure and integration. We have

$$
\mathcal{J}_N \;=\; \int_{\mathcal{X}} \exp\big(\sigma\|B_N(\boldsymbol{x})\|_\infty\big)\rho(\mathrm{d}\boldsymbol{x}) \;\le\; \int_{\mathcal{X}} \exp\big(\sigma\|B(\boldsymbol{x})\|_\infty\big)\rho(\mathrm{d}\boldsymbol{x}) \;=:\; \mathcal{J} \;<\; \infty.
$$

The integral $\mathcal{J}$ is finite because Fernique's theorem [1, Theorem 2.8.5] (applicable because the measure is a centered Gaussian measure) asserts that there exists $\alpha > 0$ such that

$$
\int_{\mathcal{X}} \exp\big(\alpha\|\boldsymbol{x}\|_\infty^2\big)\rho(\mathrm{d}\boldsymbol{x}) < \infty,
$$

and because $\|B(\boldsymbol{x})\|_\infty \le \|\boldsymbol{x}\|_{\mathcal{X}} \le (2+1/\sqrt{2})\|\boldsymbol{x}\|_\infty$, which in turn is dominated by $(\alpha/\sigma)\|\boldsymbol{x}\|_\infty^2$ for large $\|\boldsymbol{x}\|_\infty$.

Switching back to the language of probability and expectation, we now have

$$\mathbb{E}\left[\|S - S_N\|_\infty\right] \;\le\; S(0)\, e^{|r-\sigma^2/2|}\, \mathcal{J}\,\mathcal{K}_N. \tag{5.3}$$

It follows, using (3.1), that

$$\begin{aligned}
\mathcal{K}_N \;&\le\; \mathbb{E}\left[\exp\left(\sigma \sum_{n=N+1}^{\infty} \max_{1\le i\le 2^{n-1}} |X_{n,i}|\, 2^{-(n+1)/2}\right)\right] - 1 \\
&=\; \mathbb{E}\left[\prod_{n=N+1}^{\infty} \exp\left(\sigma \max_{1\le i\le 2^{n-1}} |X_{n,i}|\, 2^{-(n+1)/2}\right)\right] - 1 \\
&=\; \prod_{n=N+1}^{\infty} \left(\mathbb{E}\left[\exp\left(\sigma \max_{1\le i\le 2^{n-1}} |X_{n,i}|\, 2^{-(n+1)/2}\right)\right]\right) - 1,
\end{aligned}$$

On substituting $\ell := 2^{n-1}$, we have

$$\mathcal{K}_N \;\le\; \prod_{\ell=2^N,2^{N+1},2^{N+2},\ldots} \mathcal{I}_\ell \;-\; 1, \tag{5.4}$$

where

$$\mathcal{I}_\ell \;:=\; \mathbb{E}\left[\exp\left(\frac{\sigma}{2\sqrt{\ell}} M_\ell\right)\right] \quad \text{for } \ell \text{ a power of 2,} \tag{5.5}$$

with M_ℓ as defined in (4.2). We thus need explicit bounds on the factors $\mathcal{I}_\ell$.

We follow the argument in the previous section. Recall that the random variable Y_ℓ defined in (4.4), which is a recentred and rescaled version of M_ℓ, converges in expectation to a random variable Y with Gumbel distribution $\exp(-e^{-y})$. We now introduce another random variable W_ℓ which also converges in expectation to the limit Y. For $\sigma > 0$ we define

$$W_\ell \;:=\; \frac{2\sqrt{\ell}}{\sigma b_\ell}\left(\exp\left(\frac{\sigma b_\ell}{2\sqrt{\ell}} Y_\ell\right) - 1\right). \tag{5.6}$$

It follows from the definitions of Y_ℓ and W_ℓ that

$$\begin{aligned}
\exp\left(\frac{\sigma}{2\sqrt{\ell}} M_\ell\right) \;&=\; \exp\left(\frac{\sigma}{2\sqrt{\ell}}(a_\ell + b_\ell Y_\ell)\right) \;=\; \exp\left(\frac{\sigma a_\ell}{2\sqrt{\ell}}\right)\exp\left(\frac{\sigma b_\ell}{2\sqrt{\ell}} Y_\ell\right) \\
&=\; \exp\left(\frac{\sigma a_\ell}{2\sqrt{\ell}}\right)\left(1 + \frac{\sigma b_\ell}{2\sqrt{\ell}} W_\ell\right),
\end{aligned}$$

and hence from (5.5)

$$\mathcal{I}_\ell \;=\; \exp\left(\frac{\sigma a_\ell}{2\sqrt{\ell}}\right)\left(1 + \frac{\sigma b_\ell}{2\sqrt{\ell}}\,\mathbb{E}\left[W_\ell\right]\right). \tag{5.7}$$

Lemma 8. *The random variable* W_ℓ *defined by* (5.6) *converges in expectation to a random variable* Y *with distribution* $\exp(-e^{-y})$, *thus*

$$\lim_{\ell\to\infty} \mathbb{E}[W_\ell] \;=\; \mathbb{E}[Y] \;=\; \gamma,$$

where γ *is Euler's constant.*

Proof. We follow closely the proof of Lemma 7. We note, from (5.6) and (4.4), that $W_\ell \ge w$ if and only if $M_\ell \ge x(w)$, where

$$x(w) \;:=\; a_\ell + \frac{2}{\sigma}\sqrt{\ell}\ln\left(1 + \frac{\sigma b_\ell}{2\sqrt{\ell}}w\right). \tag{5.8}$$

First assume $w > 0$. We have

$$\begin{aligned}
\mathbb{P}(W_\ell \ge w) \;&=\; \mathbb{P}(M_\ell \ge x(w)) \\
&=\; 1 - (2\Phi(x(w)) - 1)^\ell = 1 - (1 - 2[1 - \Phi(x(w))])^\ell \\
&\le\; 1 - \left(1 - 2\frac{\phi(x(w))}{x(w)}\right)^\ell \;=\; 1 - \left(1 - \sqrt{\frac{2}{\pi}}\frac{\exp(-\frac{1}{2}x(w)^2)}{x(w)}\right)^\ell \\
&\le\; 1 - \left(1 - \sqrt{\frac{2}{\pi}}\frac{\exp\left(-\frac{1}{2}a_\ell^2 - \frac{2a_\ell\sqrt{\ell}}{\sigma}\ln\left(1 + \frac{\sigma b_\ell}{2\sqrt{\ell}}w\right)\right)}{a_\ell}\right)^\ell \\
&=\; 1 - \left(1 - \sqrt{\frac{2}{\pi}}\frac{\exp\left(-\frac{1}{2}a_\ell^2\right)}{a_\ell}\left(1 + \frac{\sigma b_\ell}{2\sqrt{\ell}}w\right)^{-2a_\ell\sqrt{\ell}/\sigma}\right)^\ell \\
&=\; 1 - \left(1 - \ell^{-1}\left(1 + \frac{\sigma b_\ell}{2\sqrt{\ell}}w\right)^{-2\sqrt{\ell}/(\sigma b_\ell)}\right)^\ell \;=:\; Q_\ell(w)\,,
\end{aligned}$$

where we used the upper bound form of Mills' ratio (4.9), dropped some terms in both the denominator and the exponent in the numerator, and used (4.7) and also $a_\ell = 1/b_\ell$.

It is well known that $(1 + 1/x)^{-x}$ is a decreasing function of $x \in \mathbb{R}^+$. Similarly, $(1 - 1/x)^x$ is an increasing function for $x \in [1, \infty)$. Using the first of these monotonicity properties, for arbitrary $L \in \mathbb{N}$ and $\ell \ge L$ we have

$$\mathbb{P}(W_\ell \ge w) \;\le\; 1 - \left(1 - \ell^{-1}\left(1 + \frac{\sigma b_L}{2\sqrt{L}}w\right)^{-\frac{2\sqrt{L}}{\sigma b_L}}\right)^\ell,$$

and the second monotonicity property then gives

$$\mathbb{P}(W_\ell \ge w) \;\le\; 1 - \left(1 - L^{-1}\left(1 + \frac{\sigma b_L}{2\sqrt{L}}w\right)^{-\frac{2\sqrt{L}}{\sigma b_L}}\right)^L \;=\; Q_L(w),$$

and hence

$$\int_0^\infty \mathbb{P}(W_\ell \geq w)\,\mathrm{d}w \;\leq\; \int_0^\infty Q_L(w)\,\mathrm{d}w.$$

To show that the integral $\int_0^\infty Q_L(w)\,\mathrm{d}w$ is finite, it suffices to use the binomial theorem,

$$Q_L(w) = \sum_{j=1}^{L} \binom{L}{j} (-1)^{j-1} L^{-j} \left(1 + \frac{\sigma b_L}{2\sqrt{L}} w\right)^{-2\sqrt{L}\, j/(\sigma b_L)},$$

in which it is important that there is no $j = 0$ term, and as a result this finite sum is integrable over $\mathbb{R}^+$ term by term. Thus for the case $w > 0$ the result is proved.

Now we consider $w < 0$. We have

$$\mathbb{P}(W_\ell \leq w) = \mathbb{P}(M_\ell \leq x(w))$$

where as above $x(w)$ is given by (5.8). Note that $x(w)$ is an increasing function of w, and, for $w \leq 0$, it has the maximum value a_ℓ at $w = 0$. On the other hand $\mathbb{P}(M_\ell \leq x(w))$ vanishes if $x(w) < 0$, thus the left-hand side vanishes for $w < \tilde{w}$, where

$$\tilde{w} := -\frac{2\sqrt{\ell}}{\sigma b_\ell}\left(1 - \exp\left(-\frac{\sigma a_\ell}{2\sqrt{\ell}}\right)\right) = -\frac{a_\ell}{b_\ell}\left[\frac{2\sqrt{\ell}}{\sigma a_\ell}\left(1 - \exp\left(-\frac{\sigma a_\ell}{2\sqrt{\ell}}\right)\right)\right].$$

We note that $\tilde{w} = -a_\ell/b_\ell + o(1) = -2\ln(\ell) + o(1)$. For $w \in [\tilde{w}, 0]$ we have, from (4.3),

$$\mathbb{P}(W_\ell \leq w) = \mathbb{P}(M_\ell \leq x(w)) = (2\Phi(x(w)) - 1)^\ell.$$

Now again for $t > 0$ the standard normal distribution Φ has negative second derivative, and so

$$\begin{aligned}
\Phi(x(w)) \;&\leq\; \Phi(a_\ell) + (x(w) - a_\ell)\phi(a_\ell) \\
&=\; \Phi(a_\ell) + \frac{2\sqrt{\ell}}{\sigma} \ln\left(1 + \frac{\sigma b_\ell}{2\sqrt{\ell}} w\right)\phi(a_\ell) \;\leq\; \Phi(a_\ell) + b_\ell w \phi(a_\ell),
\end{aligned}$$

where in the last step we used $\ln(1+t) \leq t$ for $t > -1$. Thus, on using $b_\ell = 1/a_\ell$, we have

$$\begin{aligned}
\mathbb{P}(W_\ell \leq w) \;&\leq\; \left(2\Phi(a_\ell) + 2a_\ell^{-1} w\phi(a_\ell) - 1\right)^\ell \leq \left(1 - 2a_\ell^{-1}\phi(a_\ell)(1 - w - a_\ell^{-2})\right)^\ell \\
&= \left(1 - \frac{1}{\ell}(1 - w - a_\ell^{-2})\right)^\ell
\end{aligned}$$

where in the second last step we used the lower bound form of Mills' ratio (4.10) and in the last step used (4.7). If we now take $\ell \geq L$ then we have

$$\mathbb{P}(W_\ell \leq w) \;\leq\; \left(1 - \frac{1}{\ell}(1 - w - a_L^{-2})\right)^\ell \leq \exp\left(-(1 - w - a_L^{-2})\right) \;=:\; R(w),$$

since the convergence in the last limit is monotone increasing, The function $R(w)$ so defined is integrable on $\mathbb{R}^-$, completing the proof of uniform integrability of the random variable W_ℓ, from which it follows that W_ℓ converges in expectation.

The random variable W_ℓ has the same limit in distribution as the Gumbel distribution. The argument follows the line of the proof above, with the upper bounds of $\mathbb{P}(W_\ell \leq w)$ replaced by asymptotics. Hence we conclude that W_ℓ converges in expectation to the expectation of the Gumbel distribution. $\square$

Since the above lemma establishes the convergence of $\mathbb{E}[W_\ell]$ as $\ell \to \infty$, it follows that there exists a positive constant c_σ such that

$$\mathbb{E}[W_\ell] \leq c_\sigma \quad \text{for } \ell \geq 0.$$

From this and (5.7) it follows that

$$\mathcal{I}_\ell \leq \exp\left(\frac{\sigma a_\ell}{2\sqrt{\ell}}\right)\left(1 + \frac{\sigma b_\ell}{2\sqrt{\ell}} c_\sigma\right).$$

It is convenient now to use the (non-sharp) inequality

$$\exp(t) \leq 1 + 2t \quad \text{for } t \in [0,1]. \tag{5.9}$$

Noting that $\sigma a_\ell/(2\sqrt{\ell}) \to 0$ as $\ell \to \infty$, we define

$$\ell_0 := \left\lceil \max\left\{\ell \in \mathbb{N} : \frac{\sigma a_\ell}{2\sqrt{\ell}} > 1\right\}\right\rceil,$$

so that for $\ell \geq \ell_0$ we have $\sigma a_\ell/(2\sqrt{\ell}) \leq 1$. Then for $\ell \geq \ell_0$ we have

$$\mathcal{I}_\ell \leq \left(1 + \frac{\sigma a_\ell}{\sqrt{\ell}}\right)\left(1 + \frac{\sigma b_\ell}{2\sqrt{\ell}} c_\sigma\right) \leq 1 + (2 + 3c_\sigma)\frac{\sigma a_\ell}{2\sqrt{\ell}},$$

where in the second step we used the elementary inequality

$$(1 + at)(1 + bt) \leq 1 + (a + b + ab)t \quad \text{for } t \in [0,1],$$

and also, for the sake of simplicity, $b_\ell = a_\ell^{-1} \leq a_\ell$.

It now follows from (5.4) that

$$\begin{aligned}
\mathcal{K}_N &\leq \prod_{\ell=2^N,2^{N+1},2^{N+2},\ldots} \left(1 + (2 + 3c_\sigma)\frac{\sigma a_\ell}{2\sqrt{\ell}}\right) - 1 \\
&= \exp\left(\sum_{\ell=2^N,2^{N+1},2^{N+2},\ldots} \ln\left(1 + (2 + 3c_\sigma)\frac{\sigma a_\ell}{2\sqrt{\ell}}\right)\right) - 1 \\
&\leq \exp\left(\frac{\sigma(2 + 3c_\sigma)}{2} \sum_{\ell=2^N,2^{N+1},2^{N+2},\ldots} \frac{a_\ell}{\sqrt{\ell}}\right) - 1.
\end{aligned}$$

$$N_0 := \left\lceil \max\left\{N \in \mathbb{N} : \frac{\sigma(2 + 3c_\sigma)}{2} \sum_{\ell=2^N,2^{N+1},2^{N+2},\ldots} \frac{a_\ell}{\sqrt{\ell}} \geq 1\right\}\right\rceil,$$

then for $N \geq N_0$ we have, using again (5.9),

$$\mathcal{K}_N \;\leq\; \sigma(2+3c_\sigma) \sum_{\ell=2^N,2^{N+1},2^{N+2},\ldots} \frac{a_\ell}{\sqrt{\ell}}.$$

Our final upper bound on $\mathbb{E}\left[\|S - S_N\|_\infty\right]$ is obtained by substituting the above bound on $\mathcal{K}_N$ into (5.3) and using the bound (4.12). This finally completes the proof of Theorem 1.

6. Application to option pricing

Now we consider a continuous version of a path-dependent call option with strike price K in a Black-Scholes model with risk-free interest rate $r > 0$ and constant volatility $\sigma > 0$. Recall that the asset price $S(t)$ at time t is given explicitly by (1.2). The discounted payoff for the case of a continuous arithmetic Asian option with terminal time $T = 1$ is therefore

$$\begin{aligned} P \;&:=\; e^{-rT} \max\left(\frac{1}{T}\int_0^T S(t)\,\mathrm{d}t - K, 0\right) \\ &=\; e^{-r} \max\left(S(0)\int_0^1 \exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma B(t)\right)\mathrm{d}t - K, 0\right). \end{aligned} \tag{6.1}$$

The pricing problem is then to compute the expected value $\mathbb{E}(P)$.

We use the Lévy-Ciesielski expansion for the Brownian motion $B(t)$, see (2.2), and we define

$$P_N \;:=\; e^{-r} \max\left(S(0)\int_0^1 \exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma B_N(t)\right)\mathrm{d}t - K, 0\right), \tag{6.2}$$

where $B_N(t)$ is as in (2.3). We are interested in estimating how fast $\mathbb{E}\left[|P - P_N|\right]$ converges to 0 as $N \to \infty$.

Corollary 9. *For P and P_N defined by* (6.1) *and* (6.2), *we have*

$$\mathbb{E}\left[|P - P_N|\right] \;=\; \mathcal{O}\left(\frac{\sqrt{N}}{2^{N/2}}\right) \;=\; \mathcal{O}\left(\sqrt{\frac{\ln d}{d}}\right),$$

where $d = 2^N$ and the implied constants are independent of N.

Proof. Note that we have

$$|\max(\alpha - K, 0) - \max(\beta - K, 0)| \;\leq\; |\alpha - \beta|. \tag{6.3}$$

Indeed, if both $\alpha \leq K$ and $\beta \leq K$ then the left-hand side is 0 and the result holds trivially. If $\alpha \geq K$ and $\beta \geq K$ then the equality holds. If $\alpha \leq K \leq \beta$ then the left-hand side is $\beta - K \leq \beta - \alpha = |\alpha - \beta|$. The case $\beta \leq K \leq \alpha$ holds analogously.

Using (6.3) we obtain

$$
\begin{aligned}
|P - P_N| &\le \left| e^{-r} S(0) \int_0^1 e^{(r-\sigma^2/2)t} \Big(\exp(\sigma B(t)) - \exp(\sigma B_N(t)) \Big) \, \mathrm{d}t \right| \\
&\le e^{-r} S(0) \, e^{|r-\sigma^2/2|} \int_0^1 \Big| \exp(\sigma B(t)) - \exp(\sigma B_N(t)) \Big| \, \mathrm{d}t \\
&\le e^{-r} S(0) \, e^{|r-\sigma^2/2|} \| \exp(\sigma B) - \exp(\sigma B_N) \|_\infty,
\end{aligned}
$$

where the last upper bound differs from the upper bound (5.1) on $\|S - S_N\|_\infty$ only by a factor of e^{-r}. Hence the result follows from Theorem 1. □

7. Concluding remarks

We obtained an upper bound on the expected uniform error of geometric Brownian motion under the Lévy-Ciesielski expansion, with a convergence rate matching that of the standard Brownian motion. We used a new proof technique based on extreme value statistics, and we extended our argument to a path dependent option pricing application – a continuous version of an arithmetic Asian option.

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