Institut für Numerische Simulation
Rheinische Friedrich-Wilhelms-Universität Bonn
Wegelerstraße 6 • 53115 Bonn • Germany
phone +49 228 73-3427 • fax +49 228 73-7527
www.ins.uni-bonn.de

D. Gallistl, D. Peterseim

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INS Preprint No. 1619

August 2016

Computation of local and quasi-local effective diffusion tensors in elliptic homogenization

D. Gallistl* D. Peterseim* †

August 7, 2016

Abstract

This paper gives a re-interpretation of the multiscale method of Målqvist and Peterseim [Math. Comp. 2014] by means of a discrete integral operator acting on standard finite element spaces. The exponential decay of the involved integral kernel motivates the use of a diagonal approximation and, hence, a localized piecewise constant coefficient. This local model turns out to be appropriate when the localized coefficient satisfies a certain homogenization criterion, which can be verified a posteriori. An a priori error analysis of the local model is presented and illustrated in numerical experiments.

Keywords numerical homogenization, multiscale method, upscaling, a priori error estimates, a posteriori error estimates
AMS subject classification 65N12, 65N15, 65N30, 73B27, 74Q05

1 Introduction

Homogenization is a tool of mathematical modeling to identify reduced descriptions of the macroscopic response of multiscale models. In the context of the prototypical elliptic model problem

$$-\operatorname{div} A_\varepsilon \nabla u = f$$

microscopic features on some characteristic length scale ε are encoded in the diffusion coefficient A_ε and homogenization studies the limit as ε tends to zero. It turns out that suitable limits represented by the so-called effective or homogenized coefficient exist in fairly general settings in the framework of G-, H-, or two-scale convergence [Spa68, DG75, MT78, Ngu89, All92]. However, the effective coefficient is rarely given explicitly and even its implicit representation based on cell problems usually requires structural assumptions on the sequence of coefficients A_ε such as local periodicity and scale separation [BLP78]. Under such assumptions, efficient numerical methods for the approximate evaluation of

*Institut für Numerische Simulation, Universität Bonn, Wegelerstraße 6, D-53115 Bonn, Germany, {gallistl,peterseim}(at)ins.uni-bonn.de

†D. Peterseim acknowledges support by the Hausdorff Center for Mathematics Bonn, and by Deutsche Forschungsgemeinschaft in the Priority Program 1748 "Reliable simulation techniques in solid mechanics" (PE2143/2-1).

the homogenized model are available, e.g., the Heterogeneous Multiscale Method (HMM) [EE03, AEEV12] or the two-scale finite element method [MS02].

In contrast to this idealized setting of analytical homogenization, in practice one is often concerned with one coefficient A with heterogeneities on multiple nonseperable scales and a corresponding sequence of scalable models can hardly be identified or may not be available at all. That is why we are interested in the computation of effective representations of very rough coefficients beyond structural assumptions such as scale separation and local periodicity. In recent years, many numerical attempts have been developed that conceptually do not rely on analytical homogenization results for rough cases. Amongst them are the multiscale finite element method [HW97, EH09], metric-based upscaling [OZ07], hierarchical matrix compression [GH08, Hac15], the flux-norm approach [BO10], generalized finite elements based on spectral cell problems [BL11, EGH13], the AL basis [GGS12, WS15], rough polyharmonic splines [OZB14], iterative numerical homogenization [KY15], and gamblets [Owh15].

Another construction based on concepts of orthogonal subspace decomposition and the solution of localized microscopic cell problems was given in [MP14] and later optimized in [HP13, HMP15, GP15, Pet15]. The method is referred to as the Localized Orthogonal Decomposition (LOD) method. The approach is inspired by ideas of the variational multiscale method [HFMQ98, HS07, Mål11]. As most of the methods above, the LOD constructs a basis representation of some finite-dimensional operator-dependent subspace with superior approximation properties rather than computing an upscaled coefficient. The effective model is then a discrete one represented by the corresponding stiffness matrix and possibly right-hand side. The computation of an effective coefficient is, however, often favorable and this paper re-interprets and modifies the LOD method in this regard.

For this purpose, we introduce a new scale H, the observation scale or scale of interest and the class $\mathcal{M}(\Omega, \alpha, \beta)$ of matrix-valued coefficients with measurable entries and uniform lower and upper spectral bounds $0 < \alpha \leq \beta$. Our notion of numerical homogenization is as follows. Given some symmetric coefficient $A \in \mathcal{M}(\Omega, \alpha, \beta)$ and the observation scale H associated with some quasi-uniform mesh $\mathcal{T}_H$ of width H, the goal is to find $A_H \in \mathcal{M}(\Omega, \alpha_H, \beta_H)$ such that

(a) The bounds satisfy

$$0 < \alpha_H \approx \alpha \quad \text{and} \quad \beta_H \approx \beta.$$

(b) For some constant C and all $f \in L^2(\Omega)$ there holds

$$\|u - u_H\|_{L^2(\Omega)} \leq CH\|f\|_{L^2(\Omega)},$$

where $u, u_H \in H^1_0(\Omega)$ are solutions to the model problem $-\operatorname{div}(A\nabla u) = f$ and $-\operatorname{div}(A_H \nabla u_H) = f$ with homogeneous Dirichlet boundary conditions.

Here, the L^2 norm is chosen as a measure for macroscopic approximation of the highly oscillating function u. We note that for particular cases of non-oscillating data (e.g., constant A), the desired estimate above may be suboptimal. Hence, we focus on the regime where the observation scale H is coarse in the sense that a standard finite element method (i.e., the piecewise arithmetic mean of A) leads to an error of order $\mathcal{O}(1)$. Discrete models allowing for (a) and (b) seem only possible under additional assumptions, and our estimates involve a posteriori model error terms.

In this paper, we revisit and re-interpret the LOD method of [MP14]. The original method employs finite element basis functions that are modified by a fine-scale correction with a slightly larger support. We show that it is possible to rewrite the method by means of a discrete integral operator acting on standard finite element spaces. The discrete operator is of non-local nature and is not necessarily associated with a differential operator on V. We are able to show that there the discrete effective non-local model represented by an integral kernel $\mathcal{A}_H \in L^\infty(\Omega\times\Omega,\mathbb{R}^{d\times d})$ such that the problem is well-posed on a finite-element space V_H with similar constants and satisfies

$$\sup_{f\in L^2(\Omega)\setminus\{0\}} \frac{\|u(f)-u_H(f)\|_{L^2(\Omega)}}{\|f\|_{L^2(\Omega)}} \lesssim \sup_{f\in L^2(\Omega)\setminus\{0\}} \inf_{v_H\in V_H} \frac{\|u(f)-v_H\|_{L^2(\Omega)}}{\|f\|_{L^2(\Omega)}} + H^2.$$

Based on the exponential decay of that kernel $\mathcal{A}_H$ away from the diagonal, we suggest a quasi-local and sparse formulation as an approximation. The storage requirement for the quasi-local kernel is $\mathcal{O}(H^{-d}|\log H|)$.

For an even stronger compression to $\mathcal{O}(H^{-d})$ information, one can replace $\mathcal{A}_H$ by a local and piecewise constant tensor field $A_H \in \mathcal{M}(\Omega,\alpha_H,\beta_H)$. It turns out that this localized effective coefficient coincides with the homogenized coefficient of classical homogenization results in the periodic case. In the the general non-periodic case, this procedure is still applicable and yields reasonable results whenever a certain homogenization criterion is satisfied, which can be checked a posteriori through a computable model error estimator. For the two-dimensional case, almost optimal convergence rates can be proved under reasonable assumptions on the data. In three dimensions, similar results are conjectured but cannot be proved with the arguments employed in this work (Sobolev embeddings) and, therefore, remain suboptimal. We emphasize that this possible sub-optimality is not an artifact of our numerical method but due to the possible lack of regularity of the homogenized solution on polyhedral domains.

The structure of this article is as follows. After the preliminaries on the model problem and notation from Section 2, we review the LOD method of [MP14] and introduce the quasi-local effective discrete coefficients in Section 3. In Section 4, we present the error analysis for the localized effective coefficient. Section 5 studies the particular case of a periodic coefficient. We present numerical results in Section 6. Supplementary material for some idealized version of the proposed methods is provided as Appendix A.

Standard notation on Lebesgue and Sobolev spaces applies throughout this paper. The notation $a \lesssim b$ abbreviates $a \leq Cb$ for some constant C that is independent of the mesh-size, but may depend on the contrast of the coefficient A; $a \approx b$ abbreviates $a \lesssim b \lesssim a$. The symmetric part of a quadratic matrix M is denoted by $\operatorname{sym}(M)$.

2 Model problem and notation

This section describes the model problem and some notation on finite element spaces.

2.1 Model problem

Let $\Omega \subseteq \mathbb{R}^d$ for $d \in \{1,2,3\}$ be a convex polytope. We consider the prototypical model problem

$$-\operatorname{div}(A\nabla u) = f \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = 0. \tag{2.1}$$

The coefficient $A \in L^\infty(\Omega;\mathbb{R}^{d\times d})$ is assumed to be symmetric and to satisfy the following uniform spectral bounds

$$\alpha \leq \operatorname*{ess\,inf}_{x\in\Omega} \inf_{\xi\in\mathbb{R}^d\setminus\{0\}} \frac{\xi\cdot(A(x)\xi)}{\xi\cdot\xi} \leq \operatorname*{ess\,sup}_{x\in\Omega} \sup_{\xi\in\mathbb{R}^d\setminus\{0\}} \frac{\xi\cdot(A(x)\xi)}{\xi\cdot\xi} \leq \beta. \tag{2.2}$$

The symmetry of A is not essential for our analysis and is assumed for simpler notation. The weak form employs the Sobolev space $V := H_0^1(\Omega)$ and the bilinear form a defined, for any $v, w \in V$, by

$$a(v,w) := (A\nabla v, \nabla w)_{L^2(\Omega)}.$$

Given $f \in L^2(\Omega)$ and the linear functional

$$F : V \to \mathbb{R}, \quad \text{with } F(v) := \int_\Omega f v\,dx \quad \text{for any } v \in V,$$

the weak form seeks $u \in V$ such that

$$a(u,v) = F(v) \quad \text{for all } v \in V. \tag{2.3}$$

2.2 Finite element spaces

Let $\mathcal{T}_H$ be a quasi-uniform regular triangulation of Ω and let V_H denote the standard P_1 finite element space, that is, the subspace of V consisting of piecewise first-order polynomials.

Given any subdomain $S \subseteq \overline{\Omega}$, define its neighbourhood via

$$\mathsf{N}(S) := \operatorname{int}\Big(\cup\{T \in \mathcal{T}_H \,:\, T \cap \overline{S} \neq \emptyset\}\Big).$$

Furthermore, we introduce for any $m \geq 2$ the patch extensions

$$\mathsf{N}^1(S) := \mathsf{N}(S) \qquad \text{and} \qquad \mathsf{N}^m(S) := \mathsf{N}(\mathsf{N}^{m-1}(S)).$$

Throughout this paper, we assume that the coarse-scale mesh $\mathcal{T}_H$ is quasi-uniform. The global mesh-size reads $H := \max\{\operatorname{diam}(T) : T \in \mathcal{T}_H\}$. Note that the shape-regularity implies that there is a uniform bound $C(m)$ on the number of elements in the mth-order patch, $\operatorname{card}\{K \in \mathcal{T}_H \,:\, K \subseteq \overline{\mathsf{N}^m(T)}\} \leq C(m)$ for all $T \in \mathcal{T}_H$. The constant $C(m)$ depends polynomially on m. The set of interior $(d-1)$-dimensional hyper-faces of $\mathcal{T}_H$ is denoted by $\mathcal{F}_H$. For a piecewise continuous function φ, we denote the jump across an interior edge by $[\varphi]_F$, where the index F will be sometimes omitted for brevity. The space of piecewise constant $d \times d$ matrix fields is denoted by $P_0(\mathcal{T}_H;\mathbb{R}^{d\times d})$.

Let $I_H : V \to V_H$ be a surjective quasi-interpolation operator that acts as a H^1-stable and L^2-stable quasi-local projection in the sense that $I_H \circ I_H = I_H$ and that for any $T \in \mathcal{T}_H$ and all $v \in V$ there holds

$$H^{-1}\|v - I_H v\|_{L^2(T)} + \|\nabla I_H v\|_{L^2(T)} \leq C_{I_H}\|\nabla v\|_{L^2(\mathsf{N}(T))} \tag{2.4}$$

$$\|I_H v\|_{L^2(T)} \leq C_{I_H}\|v\|_{L^2(\mathsf{N}(T))}. \tag{2.5}$$

Since I_H is a stable projection from V to V_H, any $v \in V$ is quasi-optimally approximated by $I_H v$ in the $L^2(\Omega)$ norm as well as in the $H^1(\Omega)$ norm. One possible choice is to define $I_H := E_H \circ \Pi_H$, where Π_H is the piecewise L^2 projection onto the space $P_1(\mathcal{T}_H)$ of piecewise affine (possibly discontinuous) functions and E_H is the averaging operator that maps $P_1(\mathcal{T}_H)$ to V_H by assigning to each free vertex the arithmetic mean of the corresponding function values of the neighbouring cells, that is, for any $v \in P_1(\mathcal{T}_H)$ and any free vertex z of $\mathcal{T}_H$,

$$(E_H(v))(z) = \sum_{\substack{T\in\mathcal{T}_H\\ \text{with } z\in T}} v|_T(z) \Big/ \operatorname{card}\{K \in \mathcal{T}_H \,:\, z \in K\}. \tag{2.6}$$

This choice of I_H is employed in our numerical experiments.

3 Non-local effective coefficient

We briefly review the idealized version of the LOD method of [MP14] (following the presentation in [HP13]) and its localization and give a new interpretation by means of a non-local effective coefficient.

3.1 Review of the LOD method

Let $W := \ker I_H \subseteq V$ denote the kernel of I_H. Given any $T \in \mathcal{T}_H$ and $j \in \{1,\dots,d\}$, the element corrector $q_{T,j} \in W$ is the solution of the variational problem

$$a(w, q_{T,j}) = \int_T \nabla w \cdot (Ae_j)\,dx \quad \text{for all } w \in W. \tag{3.1}$$

Here e_j is the j-th standard Cartesian unit vector in $\mathbb{R}^d$. The gradient of any $v \in V$ has the representation

$$\nabla v = \sum_{T\in\mathcal{T}_H}\sum_{j=1}^{d}(\partial_j v|_T)e_j. \tag{3.2}$$

Given any $v_H \in V_H$, define the corrector $\mathcal{C}v_H$ by

$$\mathcal{C}v_H = \sum_{T\in\mathcal{T}_H}\sum_{j=1}^{d}(\partial_j v_H|_T)q_{T,j}. \tag{3.3}$$

We remark that for any $v_H \in V_H$ the gradient ∇v_H is piecewise constant and, thus, $\mathcal{C}v_H$ is a finite linear combination of the element correctors $q_{T,j}$. It is readily verified that, for any $v_H \in V_H$, $\mathcal{C}v_H$ is the a-orthogonal projection on W, i.e.,

$$a(w, v_H - \mathcal{C}v_H) = 0 \quad \text{for all } w \in W. \tag{3.4}$$

Clearly, by (3.4), the projection $\mathcal{C}v \in W$ is well-defined for any $v \in V$. The representation (3.3) for discrete functions will, however, be useful in this work.

The LOD method in its version from [MP14] seeks $\bar{u}_H \in V_H$ such that

$$a((1-\mathcal{C})\bar{u}_H, (1-\mathcal{C})v_H) = F((1-\mathcal{C})v_H) \quad \text{for all } v_H \in V_H.$$

By (3.4), it is clear that this is equivalent to

$$a(\bar{u}_H, (1-\mathcal{C})v_H) = F((1-\mathcal{C})v_H) \quad \text{for all } v_H \in V_H. \tag{3.5}$$

A variant with a problem-independent right-hand side seeks $u_H \in V_H$ such that

$$a((1-\mathcal{C})u_H, (1-\mathcal{C})v_H) = F(v_H) \quad \text{for all } v_H \in V_H.$$

or, equivalently,

$$a(u_H, (1-\mathcal{C})v_H) = F(v_H) \quad \text{for all } v_H \in V_H. \tag{3.6}$$

We define the following worst-case best-approximation error

$$\mathbf{wcba}(A, \mathcal{T}_H) := \sup_{g \in L^2(\Omega)\setminus\{0\}} \inf_{v_H \in V_H} \frac{\|u(g) - v_H\|_{L^2(\Omega)}}{\|g\|_{L^2(\Omega)}} \tag{3.7}$$

where for $g \in L^2(\Omega)$, $u(g) \in V$ solves (2.3) with right-hand side g. Standard interpolation and stability estimates show that always $\mathbf{wcba}(A, \mathcal{T}_H) \lesssim H$, but it may behave better in certain regimes. E.g., in a periodic homogenization problem with some small parameter ε and some smooth homogenized solution $u_0 \in H^2(\Omega)$, the best approximation error is dominated by the best approximation error of u_0 in the regime $H \lesssim \sqrt{\varepsilon}$ where it scales like H^2. By contrast, the error is typically not improved in the regime $\sqrt{\varepsilon} \gtrsim H \gtrsim \varepsilon$. This non-linear behavior of the best-approximation error in the pre-assymptotic regime is prototypical for many homogenization problems and explains why the rough bound H is suboptimal.

The following result states an L^2 error estimate for the method (3.6).

Proposition 1. *The solutions $u \in V$ to* (2.3) *and $u_H \in V_H$ to* (3.6) *for right-hand side $f \in L^2(\Omega)$ satisfy the following error estimate*

$$\frac{\|u - u_H\|_{L^2(\Omega)}}{\|f\|_{L^2(\Omega)}} \lesssim H^2 + \mathbf{wcba}(A, \mathcal{T}_H).$$

Proof. Let $f \in L^2(\Omega) \setminus \{0\}$ and let $\bar{u}_H \in V_H$ solve (3.5). We begin by analyzing the error $e_H := u_H - \bar{u}_H$. Let $z \in V$ denote the solution to

$$a(v, z) = (e_H, I_H v)_{L^2(\Omega)} \quad \text{for all } v \in V.$$

To see that the right-hand side is indeed represented by an L^2 function, note that I_H is continuous on $L^2(\Omega)$ and, hence, the right-hand side has a Riesz representative $\tilde{e}_H \in L^2(\Omega)$ such that $(e_H, I_H v)_{L^2(\Omega)} = (\tilde{e}_H, v)_{L^2(\Omega)}$. In particular,

z solves (2.3) with right-hand side $\tilde{e}_H$. Its L^2 norm is bounded with (2.5) as follows

$$\|\tilde{e}_H\|^2_{L^2(\Omega)} = (e_H, I_H\tilde{e}_H)_{L^2(\Omega)} \lesssim \|e_H\|_{L^2(\Omega)}\|\tilde{e}_H\|_{L^2(\Omega)},$$

hence

$$\|\tilde{e}_H\|_{L^2(\Omega)} \lesssim \|e_H\|_{L^2(\Omega)}. \tag{3.8}$$

We note that, for any $w \in W$, we have $a(w, z) = (e_H, I_H w)_{L^2(\Omega)} = 0$. Thus, we have $a(e_H, \mathcal{C}z) = a(\mathcal{C}e_H, z) = 0$. With $(1 - \mathcal{C})z = (1 - \mathcal{C})I_H z$ we conclude

$$\|e_H\|^2_{L^2(\Omega)} = a(e_H, z) = a(e_H, (1 - \mathcal{C})I_H z). \tag{3.9}$$

Elementary algebraic manipulations with the projection I_H show that

$$-\mathcal{C}I_H z = (1 - I_H)\big((1 - \mathcal{C})I_H z - z\big) + (1 - I_H)z.$$

The relation (3.9) and the solution properties (3.5) and (3.6), thus, lead to

$$\|e_H\|^2_{L^2(\Omega)} = F(\mathcal{C}I_H z) = |F((1 - I_H)((1 - \mathcal{C})I_H z - z)) + F((1 - I_H)z)|. \tag{3.10}$$

We proceed by estimating the two terms on the right-hand side of (3.10) separately. For the second term in (3.10), the L^2-best approximation property of I_H and (3.8) reveal

$$\begin{aligned} |F((1 - I_H)z)| &\lesssim \|f\|_{L^2(\Omega)}\|\tilde{e}_H\|_{L^2(\Omega)} \inf_{v_H \in V_H} \frac{\|z - v_H\|_{L^2(\Omega)}}{\|\tilde{e}_H\|_{L^2(\Omega)}} \\ &\lesssim \|f\|_{L^2(\Omega)}\|e_H\|_{L^2(\Omega)}\mathbf{wcba}(A, \mathcal{T}_H). \end{aligned} \tag{3.11}$$

For the first term in (3.10), we obtain with the stability of I_H and the Cauchy inequality that

$$|F((1 - I_H)((1 - \mathcal{C})I_H z - z))| \lesssim \|f\|_{L^2(\Omega)}\|z - (1 - \mathcal{C})I_H z\|_{L^2(\Omega)}.$$

Let $\zeta \in V$ denote the solution to

$$a(\zeta, v) = (z - (1 - \mathcal{C})I_H z, v)_{L^2(\Omega)} \quad \text{for all } v \in V.$$

As shown in [MP14], the function $I_H z \in V_H$ is the Galerkin approximation to z with method (3.5). We, thus, have by symmetry of a and Galerkin orthogonality that

$$\begin{aligned} \|z - (1 - \mathcal{C})I_H z\|^2_{L^2(\Omega)} &= a(\zeta, z - (1 - \mathcal{C})I_H z) \\ &= a(\zeta - (1 - \mathcal{C})I_H\zeta, z - (1 - \mathcal{C})I_H z). \end{aligned}$$

The error analysis of [MP14] reveals that this is bounded by $H^2\|z - (1 - \mathcal{C})I_H z\|_{L^2(\Omega)}\|\tilde{e}_H\|_{L^2(\Omega)}$. Altogether, with (3.10),

$$\frac{\|e_H\|_{L^2(\Omega)}}{\|f\|_{L^2(\Omega)}} \lesssim H^2 + \mathbf{wcba}(A, \mathcal{T}_H).$$

Since

$$\frac{\|u - \bar{u}_H\|_{L^2(\Omega)}}{\|f\|_{L^2(\Omega)}} \lesssim \mathbf{wcba}(A, \mathcal{T}_H),$$

(which follows from the fact that $\bar{u}_H = I_H u$), the triangle inequality concludes the proof. □

3.2 Localization of the corrector problems

Here, we briefly describe the localization technique of [MP14]. It was shown in [MP14] and [HP13, Lemma 4.9] that the method is localizable in the sense that for any $v_H \in V_H$

$$\|\nabla q_{T,j}\|_{L^2(\Omega\setminus \mathsf{N}^m(T))} \lesssim \exp(-cm)\|e_j\|_{L^2(T)}. \tag{3.12}$$

The exponential decay from (3.12) suggests to localize the computation (3.1) of the corrector belonging to an element $T \in \mathcal{T}_H$ to to a smaller domain, namely the extended element patch $\Omega_T := \mathsf{N}^\ell(T)$ of order ℓ. The nonnegative integer ℓ is referred to as the *oversampling parameter*. Let $W_{\Omega_T} \subseteq W$ denote the space of functions from W that vanish outside Ω_T. On the patch, in analogy to (3.1), for any $v_H \in V_H$, any $T \in \mathcal{T}_H$ and any $j \in \{1, \ldots, d\}$, the function $q^{(\ell)}_{T,j} \in W_{\Omega_T}$ solves

$$\int_{\Omega_T} \nabla w \cdot (A\nabla q^{(\ell)}_{T,j})\,dx = \int_T \nabla w \cdot (Ae_j)\,dx \quad \text{for all } w \in W_{\Omega_T}. \tag{3.13}$$

Given $v_H \in V_H$, we define the corrector $\mathcal{C}^{(\ell)} v_H \in W$ by

$$\mathcal{C}^{(\ell)}\nabla v_H = \sum_{T\in\mathcal{T}_H}\sum_{j=1}^{d}(\partial_j v_H|_T)\nabla q^{(\ell)}_{T,j}. \tag{3.14}$$

A practical variant of (3.6) is to seek $u^{(\ell)}_H \in V_H$ such that

$$a(u^{(\ell)}_H, (1-\mathcal{C}^{(\ell)})v_H) = F(v_H) \quad \text{for all } v_H \in V_H. \tag{3.15}$$

This procedure is indispensable for actual computations and the effect of the truncation of the domain on the error of the multiscale method was analyzed in [MP14] and [HP13]. With similar arguments it is possible to prove that the coupling $\ell \approx |\log H|$ is sufficient to derive the error bound

$$\|u - u^{(\ell)}_H\|_{L^2(\Omega)} \lesssim (H^2 + \mathbf{wcba}(A, \mathcal{T}_H))\,\|f\|_{L^2(\Omega)}. \tag{3.16}$$

The proof is based on a similar argument as in Proposition 1: Since the L^2 distance of $u - \bar{u}^{(\ell)}_H$ is controlled by the right-hand side of (3.16) [HP13] where $\bar{u}^{(\ell)}_H$ solves a modified version of (3.15) with right-hand side $F((1-\mathcal{C}^{(\ell)})v_H)$, it is sufficient to control $u^{(\ell)}_H - \bar{u}^{(\ell)}_H$ in the L^2 norm. This can be done with a duality argument similar to that from the proof of Proposition 1. The additional tool needed therein is the fact that

$$\|\nabla(\mathcal{C} - \mathcal{C}^{(\ell)})I_H z\|_{L^2(\Omega)} \lesssim \exp(-c\ell)C(\ell)\|\nabla z\|_{L^2(\Omega)}$$

for the dual solution z (see [HP13, Proof of Thm. 4.13] for an outline of a proof) where $C(\ell)$ is an overlap constant depending polynomially on ℓ. The choice of $\ell \approx |\log H|$ therefore leads to (3.16). The details are omitted here.

3.3 Definition of the quasi-local effective coefficient

In this subsection, we do not make any specific choice for the oversampling parameter ℓ. In particular, the analysis covers the case that all element patches Ω_T equal the whole domain Ω. We denote the latter case formally by $\ell = \infty$.

We re-interpret the left-hand side of (3.15) as a non-local operator acting on standard finite element functions. To this end, consider any $u_H, v_H \in V_H$. We have

$$a(u_H, (1 - \mathcal{C}^{(\ell)})v_H) = \int_\Omega \nabla u_H \cdot (A \nabla v_H)\, dx - \int_\Omega \nabla u_H \cdot (A \mathcal{C}^{(\ell)} \nabla v_H)\, dx.$$

The second term can be expanded with (3.2) and (3.14) as

$$\begin{aligned}\int_\Omega \nabla u_H \cdot (A \nabla \mathcal{C}^{(\ell)} v_H)\, dx &= \sum_{T \in \mathcal{T}_H} \sum_{j=1}^{d} (\partial_j v_H|_T) \int_\Omega \nabla u_H \cdot (A \nabla q_{T,j}^{(\ell)})\, dx \\ &= \sum_{K,T \in \mathcal{T}_H} \int_K \nabla u_H \cdot \left(\sum_{j=1}^{d} \fint_K (A \nabla q_{T,j}^{(\ell)})\, dx\, (\partial_j v_H|_T) \right) dx \\ &= \sum_{K,T \in \mathcal{T}_H} \int_K \int_T \nabla u_H|_K \cdot (\mathcal{K}_{T,K}^{(\ell)} \nabla v_H|_T)\, dx\end{aligned}$$

for the matrix $\mathcal{K}_{T,K}^{(\ell)}$ defined for any $K, T \in \mathcal{T}_H$ by

$$(\mathcal{K}_{T,K}^{(\ell)})_{j,k} := \frac{1}{|T|\,|K|} e_k \cdot \int_K A \nabla q_{T,j}^{(\ell)}\, dx.$$

Define the piecewise constant matrix field over $\mathcal{T}_H \times \mathcal{T}_H$, for $T, K \in \mathcal{T}_H$ by

$$\mathcal{A}_H^{(\ell)}|_{T,K} := \delta_{T,K} \fint_T A\, dx - \mathcal{K}_{T,K}^{(\ell)}$$

(where δ is the Kronecker symbol) and the bilinear form $\mathfrak{a}^{(\ell)}$ on $V_H \times V_H$ by

$$\mathfrak{a}^{(\ell)}(v_H, z_H) := \int_\Omega \int_\Omega \nabla v_H(x) \cdot (\mathcal{A}_H^{(\ell)}(x,y) \nabla z_H(y))\, dy\, dx \quad \text{for any } v_H, z_H \in V_H.$$

We obtain for all $v_H, z_H \in V_H$ that

$$a(v_H, (1 - \mathcal{C}^{(\ell)})z_H) = \mathfrak{a}^{(\ell)}(v_H, z_H). \tag{3.17}$$

Next, we state the equivalence of two multiscale formulations.

Proposition 2. *A function $u_H^{(\ell)} \in V_H$ solves (3.15) if and only if it solves*

$$\mathfrak{a}^{(\ell)}(u_H^{(\ell)}, v_H) = F(v_H). \tag{3.18}$$

Proof. This follows directly from the representation (3.17). □

Remark 3. For $d = 1$ and I_H the standard nodal interpolation operator, the corrector problems localize to one element and the presented multiscale approach coincides with various known methods (homogenization, MSFEM). The resulting effective coefficient $\mathcal{A}_H^{(\ell)}$ is diagonal and, thus, local. This is no longer the case for $d \geq 2$.

4 Local effective coefficient

Throughout this section we consider oversampling parameters chosen as $\ell \approx |\log H|$. Similar results are also true for the idealized version $\ell = \infty$. For better readability we focus on the practically relevant case and present further technical results for the idealized case in Appendix A.

4.1 Definition of the local effective coefficient

The exponential decay motivates to approximate the non-local bilinear form $\mathfrak{a}^{(\ell)}(\cdot,\cdot)$ by a quadrature-like procedure: Define the piecewise constant coefficient $A_H^{(\ell)} \in P_0(\mathcal{T}_H;\mathbb{R}^{d\times d})$ by

$$A_H^{(\ell)}|_T := \fint_T A\,dx - \sum_{K\in\mathcal{T}_H} |K|\,\mathcal{K}_{T,K}^{(\ell)}.$$

and the bilinear form $\tilde{a}^{(\ell)}$ on $V \times V$ by

$$\tilde{a}^{(\ell)}(u,v) := \int_\Omega \nabla u \cdot (A_H^{(\ell)} \nabla v)\,dx.$$

Remark 4. In analogy to classical periodic homogenization, the local effective coefficient $A_H^{(\ell)}$ can be written as

$$(A_H^{(\ell)})_{j,k}|_T = |T|^{-1} \int_{\Omega_T} (e_k - \nabla q_{T,k}^{(\ell)}) \cdot \big(A(\chi_T e_j - \nabla q_{T,j}^{(\ell)})\big)$$

for the characteristic function χ_T of T and the slightly enlarged averaging domain Ω_T. See Section 5 for further analogies to homogenization theory in the periodic case.

The localized multiscale method is to seek $\tilde{u}_H^{(\ell)} \in V_H$ such that

$$\tilde{a}^{(\ell)}(\tilde{u}_H^{(\ell)}, v_H) = F(v_H) \quad \text{for all } v_H \in V_H. \tag{4.1}$$

The unique solvability of (4.1) is not guaranteed a priori. It must be checked a posteriori whether positive spectral bounds α_H, β_H on $A_H^{(\ell)}$ exist in the sense of (2.2). Throughout this paper we assume that such bounds exist, that is, we assume that there exist positive numbers α_H, β_H such that

$$\alpha_H|\xi|^2 \le \xi \cdot (A_H^{(\ell)}(x)\xi) \le \beta_H|\xi|^2 \tag{4.2}$$

for all $\xi \in \mathbb{R}^d$ and almost all $x \in \Omega$.

4.2 Error analysis

The goal of this section is to establish an error estimate for the error

$$\|u - \tilde{u}_H^{(\ell)}\|_{L^2(\Omega)}.$$

Let $u_H^{(\ell)} \in V_H$ solve (3.15). Then the error estimate (3.16) leads to the a priori error estimate

$$\|u - u_H^{(\ell)}\|_{L^2(\Omega)} \lesssim (H^2 + \mathbf{wcba}(A,\mathcal{T}_H))\,\|f\|_{L^2(\Omega)}. \tag{4.3}$$

We employ the triangle inequality and merely estimate the difference $\|u_H^{(\ell)} - \tilde{u}_H^{(\ell)}\|_{L^2(\Omega)}$.

With the finite localization parameter ℓ, the quasi-local coefficient $\mathcal{A}^{(\ell)}$ is sparse in the sense that $\mathcal{A}^{(\ell)}(x,y) = 0$ whenever $|x-y| > C\ell H$. We note the following lemma which will be employed in the error analysis. An analogous result for $\ell = \infty$ is provided in Lemma 23 in Appendix A.

Lemma 5. *Let $\ell \approx |\log H|$. Given some $x \in \Omega$ with $x \in T$ for some $T \in \mathcal{T}_H$, any p with $1 \leq p < \infty$ satisfies*

$$\|\mathcal{K}^{(\ell)}(x,y)\|_{L^p(\Omega,dy)} \lesssim C H^{-d(p-1)/p}\,|\log H|^d.$$

Proof. From the definition of $\mathcal{K}^{(\ell)}$, the boundedness of A and the Hölder inequality together with the stability of problem (3.13) and $\|e_j\|_{L^2(T)} = |T|^{1/2}$ we obtain

$$|(\mathcal{K}_{T,K}^{(\ell)})_{j,k}| \leq \frac{1}{|T|\,|K|}\|\nabla q_{T,j}\|_{L^1(K)} \leq \frac{1}{|T|\,|K|^{1/2}}\|\nabla q_{T,j}\|_{L^2(K)} \lesssim H^{-d}.$$

Hence, we obtain that

$$\begin{aligned}\|\mathcal{K}^{(\ell)}(x,y)\|^p_{L^p(\Omega,dy)} &= \sum_{K\in\mathcal{T}_H} |K||\mathcal{K}_{T,K}^{(\ell)}|^p \\ &\lesssim H^{-d(p-1)}\operatorname{card}\{K \in \mathcal{T}_H : \operatorname{dist}(T,K) \leq C\ell H\} \\ &\lesssim H^{-d(p-1)}\ell^d.\end{aligned}$$

The result follows with $\ell \approx |\log H|$. □

In what follows, we abbreviate

$$\rho := CH|\log H| \tag{4.4}$$

for some appropriately chosen constant C.

Proposition 6 (error estimate I)**.** *Let $u_H^{(\ell)} \in V_H$ solve (3.18) and let $\tilde{u}_H^{(\ell)}$ solve (4.1). We have for any $1 \leq p < \infty$ and $q \in (0,\infty]$ such that $1/p + 1/q = 1$ (with the convention $1/\infty = 0$) that*

$$\begin{aligned}&\|\nabla(u_H^{(\ell)} - \tilde{u}_H^{(\ell)})\|_{L^2(\Omega)} \\ &\qquad \lesssim H^{-d(p-1)/p}|\log H|^d \Big\| \|\nabla\tilde{u}_H^{(\ell)}(y) - \nabla\tilde{u}_H^{(\ell)}(x)\|_{L^q(B_\rho(x),dy)}\Big\|_{L^2(\Omega,dx)}.\end{aligned} \tag{4.5}$$

Proof. Denote $e_H := \tilde{u}_H^{(\ell)} - u_H^{(\ell)}$. In the idealized case, $\ell = \infty$, the orthogonality (3.4) and relation (3.17) show that

$$\|\nabla(1 - \mathcal{C}^{(\ell)})e_H\|^2_{L^2(\Omega)} \lesssim \mathfrak{a}^{(\ell)}(e_H, e_H).$$

If $\ell \approx |\log H|$ this estimate is true up to a higher-order term $H^r\|\nabla e_H\|^2_{L^2(\Omega)}$ with any algebraic rate $r \geq 2$ on the right-hand side. In this case the hidden constant is proportional to $\log r$.

The proof again follows ideas from [MP14] with the exponential-in-ℓ closeness of $\mathcal{C}$ and $\mathcal{C}_\ell$ and is not discussed here. From the stability of I_H and the properties of the fine-scale projection $\mathcal{C}^{(\ell)}$ we observe (with contrast-dependent constants)

$$\begin{aligned}\|\nabla e_H\|^2_{L^2(\Omega)} = \|\nabla I_H e_H\|^2_{L^2(\Omega)} &= \|\nabla I_H(1-\mathcal{C}^{(\ell)})e_H\|^2_{L^2(\Omega)} \\ &\lesssim \|\nabla(1-\mathcal{C}^{(\ell)})e_H\|^2_{L^2(\Omega)} \\ &\lesssim \mathfrak{a}^{(\ell)}(e_H,e_H) + H^r\|\nabla e_H\|^2_{L^2(\Omega)}.\end{aligned}$$

The term higher-order term $H^r\|\nabla e_H\|^2_{L^2(\Omega)}$ can be absorbed for $H<1$ and a proper choice of ℓ and, thus, we proceed with (3.15) and (3.18) as

$$\|\nabla e_H\|^2_{L^2(\Omega)} \lesssim \mathfrak{a}^{(\ell)}(\tilde u_H^{(\ell)} - u_H^{(\ell)}, e_H) = \mathfrak{a}^{(\ell)}(\tilde u_H^{(\ell)}, e_H) - \tilde a^{(\ell)}(\tilde u_H^{(\ell)}, e_H).$$

The right-hand side can be rewritten as

$$\begin{aligned}&\mathfrak{a}^{(\ell)}(\tilde u_H^{(\ell)}, e_H) - \tilde a^{(\ell)}(\tilde u_H^{(\ell)}, e_H) \\ &= \int_\Omega \left[\int_\Omega \mathcal{A}_H^{(\ell)}(x,y)(\nabla \tilde u_H^{(\ell)}(y) - \nabla \tilde u_H^{(\ell)}(x))\,dy\right]\cdot \nabla e_H(x)\,dx \\ &\quad + \int_\Omega \left[\left(\int_\Omega \mathcal{A}_H^{(\ell)}(x,y)\,dy - A_H^{(\ell)}\right)\nabla \tilde u_H^{(\ell)}(x)\right]\cdot \nabla e_H(x)\,dx.\end{aligned}$$

The second term vanishes by definition of $A_H^{(\ell)}$. Hence, the combination of the preceding arguments with the Cauchy inequality leads to

$$\|\nabla e_H\|^2_{L^2(\Omega)} \lesssim \|\nabla e_H\|_{L^2(\Omega)} \left\| \int_{B_\rho(x)} \mathcal{A}_H^{(\ell)}(x,y)(\nabla \tilde u_H^{(\ell)}(y) - \nabla \tilde u_H^{(\ell)}(x))\,dy \right\|_{L^2(\Omega,dx)},$$

where it was used that $\mathcal{A}_H^{(\ell)}(x,y) = 0$ whenever $|x-y|>\rho$. Division by $\|\nabla e_H\|_{L^2(\Omega)}$ leads to

$$\|\nabla e_H\|_{L^2(\Omega)} \lesssim \sqrt{\left(\int_\Omega \left|\int_{B_\rho(x)} \mathcal{A}_H^{(\ell)}(x,y)(\nabla \tilde u_H^{(\ell)}(y) - \nabla \tilde u_H^{(\ell)}(x))\,dy\right|^2 dx\right)}. \quad (4.6)$$

This term can be bounded with the Hölder inequality and Lemma 5 by

$$\begin{aligned}&\sqrt{\left(\int_\Omega \left| \|\mathcal{A}_H^{(\ell)}(x,y)\|_{L^p(B_\rho(x),dy)} \|\nabla \tilde u_H^{(\ell)}(y) - \nabla \tilde u_H^{(\ell)}(x)\|_{L^q(B_\rho(x),dy)}\right|^2 dx\right)} \\ &\lesssim H^{-d(p-1)/p}|\log H|^d \left\| \|\nabla \tilde u_H^{(\ell)}(y) - \nabla \tilde u_H^{(\ell)}(x)\|_{L^q(B_\rho(x),dy)}\right\|_{L^2(\Omega,dx)}.\end{aligned}$$

This finishes the proof. □

It is worth noting that the error bound in Proposition 6 can be evaluated without knowledge of the exact solution. Hence, Proposition 6 can be regarded as an a posteriori error estimate. Formula (4.6) could also be an option if it is available. In order to prove the main a priori error estimate, Proposition 8 below, the following technical lemma is required.

Lemma 7 (existence of a regularized coefficient)**.** *Let $A_H \in P_0(\mathcal{T}_H;\mathbb{R}^{d\times d})$ be a piecewise constant field of $d\times d$ matrices that satisfies the spectral bounds (4.2). Then there exists a Lipschitz continuous coefficient $A_H^{reg} \in W^{1,\infty}(\Omega;\mathbb{R}^{d\times d})$ satisfying the following three properties. 1) The piecewise integral mean is conserved, i.e.,*

$$\int_T A_H^{reg}\,dx = \int_T A_H\,dx \quad \text{for all } T\in\mathcal{T}_H.$$

2) The eigenvalues of $\operatorname{sym}(A_H^{reg})$ *lie in the interval $[\alpha_H/2, 2\beta_H]$. 3) The derivative satisfies the bound*

$$\|\nabla A_H^{reg}\|_{L^\infty(\Omega)} \le C\eta(A_H)$$

for some constant C that depends on the shape-regularity of $\mathcal{T}_H$ and for the expression

$$\eta(A_H) := H^{-1}\|[A_H]\|_{L^\infty(\mathcal{F}_H)}\big(1+\alpha_H^{-1}\|[A_H]\|_{L^\infty(\mathcal{F}_H)}\big). \tag{4.7}$$

Here $[\cdot]$ defines the inter-element jump and $\mathcal{F}_H$ denotes the set of interior hyper-faces of $\mathcal{T}_H$.

Proof. Consider a refined triangulation $\mathcal{T}_L$ resulting from L uniform refinements of $\mathcal{T}_H$. In particular, the mesh-size in $\mathcal{T}_L$ is of the order $2^{-L}H$. Let $E_L A_H$ denote the $\mathcal{T}_L$-piecewise affine and continuous function that takes at every interior vertex the arithmetic mean of the nodal values of A_H on the adjacent elements of $\mathcal{T}_L$ (similar to (2.6)). Clearly, for this convex combination the eigenvalues of $\operatorname{sym}(E_L A_H)$ range within the interval $[\alpha_H,\beta_H]$. It is not difficult to prove that, for any $T\in\mathcal{T}_H$,

$$⨍_T |A_H - E_L A_H|\,dx \lesssim 2^{-L}\|[A_H]\|_{L^\infty(\mathcal{F}_H(\omega_T))}. \tag{4.8}$$

Here, $\mathcal{F}_H(\omega_T)$ denotes the set of interior hyper-faces of $\mathcal{T}_H$ that share a point with T. Let, for any $T\in\mathcal{T}_H$, $b_T\in H^1_0(T)$ denote a positive polynomial bubble function with $⨍_T b_T\,dx = 1$ and $\|b_T\|_{L^\infty(T)}\approx 1$. The regularized coefficient $A_H^{reg} = E_L(A_H) + b_T ⨍_T (A_H - E_L(A_H))\,dx$ has, for any $T\in\mathcal{T}_H$, the integral mean $⨍_T A_H^{reg}\,dx = ⨍_T A_H\,dx$. If L is chosen to be of the order $|\log(\alpha_H^{-1}\|[A_H]\|_{L^\infty(\mathcal{F}_H)})|$, then, for any $\xi\in\mathbb{R}^d$ with $|\xi|=1$ and any $T\in\mathcal{T}_H$, we have

$$\left|\xi\cdot ⨍_T (A_H - E_L A_H)\,dx\; b_T\xi\right| \le \left|⨍_T (A_H - E_L A_H)\,dx\; b_T\right| \le \alpha_H/2.$$

This and the triangle inequality prove the claimed spectral bound on $\operatorname{sym}(A_H^{reg})$. For the bound on the derivative of A_H^{reg}, let $t\in T_L$ and $T\in\mathcal{T}_H$ such that $t\subseteq T$. The diameter of t is of order $2^{-L}H$. Since $\|\nabla b_T\|_{L^\infty(T)}\lesssim H^{-1}$, the triangle and inverse inequalities therefore yield with the above choice of L (note that $\nabla(A_H|_T)=0$)

$$\begin{aligned}\|\nabla A_H^{reg}\|_{L^\infty(t)} &\lesssim \|\nabla(A_H - E_L(A_H))\|_{L^\infty(t)} + H^{-1}\|A_H - E_L(A_H)\|_{L^\infty(t)}\\ &\lesssim H^{-1}\|[A_H]\|_{L^\infty(\mathcal{F}_H(\omega_T))}\big(1+\alpha_H^{-1}\|[A_H]\|_{L^\infty(\mathcal{F}_H(\omega_T))}\big).\end{aligned}$$

This proves the assertion. □

By Lemma 7, there exists a coefficient $A_H^{reg} \in W^{1,\infty}(\Omega)$ such that $A_H^{(\ell)}$ is the piecewise L^2 projection of A_H^{reg} onto the piecewise constants. Let $u^{reg} \in V$ solve

$$\int_\Omega \nabla u^{reg} \cdot (A_H^{reg} \nabla v)\, dx = F(v) \quad \text{for all } v \in V. \tag{4.9}$$

In particular, $\tilde{u}_H$ is the finite element approximation to u^{reg}. In the following, s refers to the elliptic $W^{1+s,q}$ regularity index of the model problem with a smooth coefficient of class $W^{1,\infty}$ in a convex polygon. We have the following error estimate for $d = 2$.

Proposition 8 (error estimate II)**.** *Let $d = 2$ and assume that $1 \le p \le 2$ and such that for all interior angles ω of the domain Ω the number $2\omega/(p\pi)$ is not an integer, and let $q \in [2, \infty)$ such that $1/p + 1/q = 1$. Assume that the solution u^{reg} to (4.9) belongs to $W^{1+s,q}(\Omega)$ for some $0 < s \le 1$. Let $u_H^{(\ell)}$ solve (3.15) and let $\tilde{u}_H^{(\ell)}$ solve (4.1). Then, for $f \in L^q(\Omega)$,*

$$\begin{aligned}&\|\nabla(u_H^{(\ell)} - \tilde{u}_H^{(\ell)})\|_{L^2(\Omega)}\\ &\lesssim H^{-d(p-1)/p} |\log H|^d \big(H^s + (H|\log H|)^{(d+sq)/q}\big) \left(1 + \eta(A_H^{(\ell)})\right)^{2s} \|f\|_{L^q(\Omega)}.\end{aligned}$$

Proof. Since Ω is convex, it is known [Gri85] that (4.9) is H^2 regular with the bound

$$\|D^2 u^{reg}\|_{L^2(\Omega)} \lesssim \|A_H^{reg}\|_{W^{1,\infty}(\Omega)} \|f\|_{L^2(\Omega)}.$$

Thus, the Sobolev embedding [Ada75, Thm. 5.4] assures that there holds

$$\|\nabla u^{reg}\|_{L^q(\Omega)} \lesssim \|D^2 u^{reg}\|_{L^2(\Omega)} \lesssim \|A_H^{reg}\|_{W^{1,\infty}(\Omega)} \|f\|_{L^q(\Omega)}.$$

Hence, problem (4.9) is stable in $L^q(\Omega)$. Recall that $\tilde{u}_H^{(\ell)}$ is in particular the finite element approximation to u^{reg}. The result thus follows from Propositon 6: The Hölder and triangle inequalites and a priori finite element error estimates [RS82, BS08] bound the right-hand side of (4.5) for any $0 < s < 1$ by

$$\begin{aligned}&H^{-d(p-1)/p} |\log H| \Bigg[\|\nabla(\tilde{u}_H^{(\ell)} - u^{reg})\|_{L^q(\Omega)}\\ &\qquad + \rho^{(d+sq)/q} \left(\int_\Omega \int_{B_\rho(x)} \frac{|\nabla u^{reg}(x) - \nabla u^{reg}(y)|^q}{\rho^{d+sq}}\, dy\, dx \right)^{1/q} \Bigg]\\ &\lesssim H^{-d(p-1)/p} |\log H| (H^s + \rho^{(d+sq)/q}) \|u^{reg}\|_{W^{1+s,q}(\Omega)}.\end{aligned}$$

If u belongs to $W^{2,q}(\Omega)$, then by the above assumptions on p and q, the results of [Gri85, §5.2] lead to

$$\|u^{reg}\|_{W^{2,q}(\Omega)} \lesssim \|A_H^{reg}\|_{W^{1,\infty}(\Omega)} (\|f\|_{L^q(\Omega)} + \|u\|_{W^{1,q}(\Omega)}) \lesssim \|A_H^{reg}\|^2_{W^{1,\infty}(\Omega)} \|f\|_{L^q(\Omega)}$$

(in particular it is required that $2\omega/(p\pi)$ is not an integer for the interior angles ω). The combination with Lemma 7 proves

$$\|u^{reg}\|_{W^{2,q}(\Omega)} \lesssim \left(1 + \eta(A_H^{(\ell)})\right)^2 \|f\|_{L^q(\Omega)}.$$

The assertion in $W^{1+s,q}(\Omega)$ can be proved with an operator interpolation argument. □

Remark 9 (homogenization indicator)**.** If the relations

$$H^{-1}\|[A_H^{(\ell)}]\|_{L^\infty(\mathcal{F}_H)} \lesssim 1 \quad \text{and} \quad \alpha_H^{-1} H \lesssim 1$$

are satisfied, then the multiplicative constant in Proposition 8 is of moderate size. Hence, we interpret $\eta(A_H^{(\ell)})$ as a homogenization indicator and the above relations as a *homogenization criterion.*

Remark 10 (local mesh-refinement)**.** We furthermore remark that local versions of $\eta(A_H^{(\ell)})$ involving the jump information $H^{-1}\|[A_H^{(\ell)}]\|_{L^\infty(F)}$ for interior interfaces F may be used as refinement indicators for local mesh-adaptation. This possibility shall, however, not be further discussed here.

Remark 11 (global homogenized coefficient)**.** If the global variations of $A_H^{(\ell)}$ are small in the sense that there are positive constants c_1, c_2 such that, almost everywhere,

$$c_1|\xi|^2 \leq \xi \cdot (A_H^{(\ell)}\xi) \leq c_2|\xi|^2 \quad \text{for any } \xi \in \mathbb{R}^d$$

holds with $|c_2 - c_1| \lesssim H$, then $A_H^{(\ell)}$ can be replaced by $\fint_\Omega A_H^{(\ell)}\,dx$ without effecting the accuracy.

Remark 12 ($d = 3$, Sobolev embedding)**.** Clearly, Proposition 8 is also valid for $d = 1$. We expect a similar result as Proposition 8 to hold also for $d = 3$. The use of the Sobolev embedding theorem, however, restricts the admissible indices q to the range $2 \leq q \leq 6$, which accordingly gives worse convergence rates.

The combination of Proposition 8 with (4.3) leads to the following a priori error estimate.

Corollary 13. *Under the assumptions of Proposition 8 we have the error estimate in the 2D case*

$$\|u - u_H^{(\ell)}\|_{L^2(\Omega)} \lesssim \Big(H + H^{s-2(p-1)/p}|\log H|^2 \left(1 + \eta(A_H^{(\ell)})\right)^{2s}\Big)\|f\|_{L^q(\Omega)}.$$

In particular, under the homogenization criterion from Remark 9, a (positive) convergence rate is achieved for any $1 \leq p < 2/(2-s)$ such that $\|f\|_{L^q(\Omega)}$ is finite.

Proof. This follows from combining Proposition 8 with (4.3), the triangle inequality and the Friedrichs inequality. □

Remark 14. We mention that the techniques used in Proposition 8 would lead to the almost linear convergence rate $H|\log H|$ under the a priori assumption that $u^{reg} \in W^{2,\infty}(\Omega)$. However, to obtain error estimates under realistic assumptions in polyhedral domains, the balancing with the parameters p and s is necessary.

Remark 15. We emphasize that $\eta(A_H^{(\ell)})$ is not an error estimator for the discretization error. It rather indicates whether the local discrete model is appropriate. If $\eta(A_H^{(\ell)})$ is close to zero, then the multiplicative constant on the right-hand side of the formula in Corollary 13 is of reasonable magnitude.

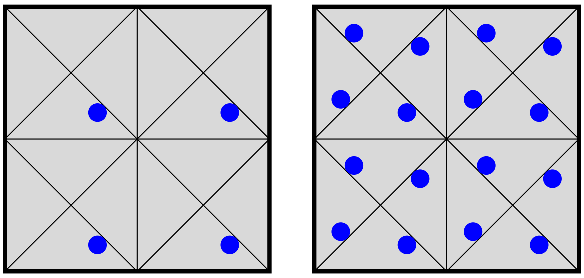

Figure 1: Periodic coefficients with respect to a square grid and triangulations: non-matching (left) and matching (right).

5 The periodic setting

In this section we justify the use of the local effective coefficient A_H in the periodic setting. We illustrate that the procedure in its idealized form with $\ell = \infty$ recovers the classical periodic homogenization limit. Throughout this section, we set $\ell = \infty$ and omit the index ℓ when there is no risk of confusion. We denote by $V := H^1_{\#}(\Omega)/\mathbb{R}$ the space of periodic H^1 functions with vanishing integral mean over Ω. We assume Ω to be a polytope allowing for periodic boundary conditions. We adopt the notation of Section 3, in particular $W \subseteq V$ is the kernel of the quasi-interpolation I_H, V_H is the space of piecewise affine globally continuous functions of V, and $\mathcal{C}$, a, $\tilde{a}$, $\mathfrak{a}$, $\mathcal{A}_H$, A_H, $\mathcal{K}$ are defined as in Section 3 with the underlying space V being $H^1_{\#}(\Omega)/\mathbb{R}$. We assume that the domain Ω matches with integer multiples of the period. We assume the triangulation $\mathcal{T}_H$ to match with the periodicity pattern. For simplicial partitions this implies further symmetry assumptions. In particular, periodicity with respect to a uniform rectangular grid is not sufficient. Instead we require further symmetry within the triangulated macro-cells, see Example 16 for an illustration.

Example 16. Figure 1 displays a periodic coefficient and a matching triangulation. □

We remark that the error estimate of Proposition 8 holds in this case as well. Due to the periodic boundary conditions, the auxiliary solution u^{reg} utilized in the proof of Proposition 8 has the smoothness $u^{reg} \in W^{2,q}(\Omega)$ for any $2 \leq q < \infty$ in two space dimensions, so that those estimates are valid with $s = 1$. In the periodic setting, further properties of A_H can be derived. First, it is not difficult to prove that the coefficient A_H is globally constant. The following result states that A_H is even independent of the mesh-size H and coincides with the classical homogenization limit, where for any $k = 1, \dots, d$, the corrector $\hat{q}_k \in H^1_{\#}(\Omega)/\mathbb{R}$ is the solution to

$$\operatorname{div} A(\nabla \hat{q}_k - e_k) = 0 \text{ in } \Omega \text{ with periodic boundary conditions.} \tag{5.1}$$

Proposition 17. *Let A be periodic and let $\mathcal{T}_H$ be uniform and aligned with the periodicity pattern of A and let V, W be spaces with periodic boundary conditions. Then, for any $T \in \mathcal{T}_H$, the localized coefficient $A_H|_T$ coincides with the homogenized coefficient from the classical homogenization theory. In particular, A_H is globally constant and independent of H.*

Proof. Let $T \in \mathcal{T}_H$ and $j, k \in \{1, \ldots, d\}$. The definitions of $A_H|_T$ and $\mathcal{K}$ lead to

$$\begin{aligned}\fint_T A_{jk}\,dx - (A_H|_T)_{jk} &= |T|^{-1} \sum_{K\in\mathcal{T}_H} \int_K e_k \cdot (A\nabla q_{T,j})\,dx \\ &= |T|^{-1} \int_\Omega e_k \cdot (A\nabla q_{T,j})\,dx.\end{aligned} \tag{5.2}$$

The sum over all element correctors defined by $q_j := \sum_{T\in\mathcal{T}_H} q_{T,j}$ solves

$$a(w, q_j) = (\nabla w, Ae_j)_{L^2(\Omega)} \quad \text{for all } w \in W. \tag{5.3}$$

The definitions of $q_{T,j}$ and q_j and the symmetry of A lead to

$$\begin{aligned}|T|^{-1} \int_\Omega e_k \cdot (A\nabla q_{T,j})\,dx &= |T|^{-1} \int_\Omega \nabla q_k \cdot (A\nabla q_{T,j})\,dx \\ &= \fint_T e_j \cdot (A\nabla q_k)\,dx.\end{aligned} \tag{5.4}$$

Let $v \in V$. We have $(v - I_H v) \in W$ and therefore by (5.3) that

$$\begin{aligned}\int_\Omega \nabla v \cdot (A(\nabla q_k - e_k))\,dx &= \int_\Omega (\nabla I_H v) \cdot ((\nabla q_k - e_k))\,dx \\ &= \sum_{K\in\mathcal{T}_H} \int_K (\nabla I_H v)\,dx \cdot \fint_K A(\nabla q_k - e_k)\,dx\end{aligned}$$

where for the last identity it was used that $\nabla I_H v$ is constant on each element. By periodicity we have that $\fint_K A(\nabla q_k - e_k)\,dx = \fint_\Omega A(\nabla q_k - e_k)\,dx$ for any $K \in \mathcal{T}_H$. Therefore, for all $v \in V$,

$$\int_\Omega \nabla v \cdot (A(\nabla q_k - e_k))\,dx = \int_\Omega (\nabla I_H v)\,dx \cdot \fint_\Omega A(\nabla q_k - e_k)\,dx = 0$$

due to the periodic boundary conditions of $I_H v$. Hence, the difference $\nabla q_k - e_k$ satisfies (5.1). This is the corrector problem from classical homogenization theory and, thus, the proof is concluded by the above formulae (5.2)–(5.4). Indeed,

$$(A_H|_T)_{jk} = \fint_T A_{jk}\,dx - \fint_T e_j \cdot (A\nabla q_k)\,dx.$$

□

Remark 18. For Dirichlet boundary conditions, the method is different from the classical periodic homogenization as it takes the boundary conditions into account.

Next, we prove a direct a priori error estimate for the multiscale method in the periodic setting. Let the coefficient $A = A_\varepsilon$ be periodic, oscillating on the scale ε. We couple $\varepsilon \approx H$, where H is the observation scale represented by the mesh-size of the finite element mesh. Denote denote by $u_H \in V_H$ the solution to (3.6). We use the abbreviation

$$a_\varepsilon(v_H, z_H) := \int_\Omega \nabla v_H(x) \cdot \int_\Omega \mathcal{A}_H(x, y)\nabla z_H(y)\,dy\,dx \quad \text{for } v_H, z_H \in V_H.$$

In this notation, u_H solves

$$a_\varepsilon(u_H, v_H) = F(v_H) \quad \text{for all } v_H \in V_H.$$

Recall from Proposition 17 that the localized coefficient $A_H = A_0$ for a constant coefficient A_0 that is independent of H. It is known (see, e.g., [All97]) that, in the present case of a symmetric coefficient, A_0 satisfies the bounds (4.2). The homogenized bilinear form on V reads

$$a_0(v, z) = \int_\Omega \nabla v \cdot (A_0 \nabla z)\, dx \quad \text{for any } v, z \in V$$

with energy norm $\|\cdot\|_{A_0} := a(\cdot,\cdot)^{1/2}$. Denote by $u_0 \in V$ the solution to

$$a_0(u_0, v) = F(v) \quad \text{for all } v \in V.$$

The aim is to estimate $\|u_0 - u_H\|_{A_0}$.

Proposition 19. *Let $p > 1$ and q with $1/p + 1/q = 1$. In the periodic case*

$$\|u_0 - u_H\|_{A_0} \lesssim H^{1-d(p-1)/p} |\log H|^{(d+q)/q+d} \|f\|_{L^q(\Omega)}.$$

Proof. As in prior sections, abbreviate $\rho := C\varepsilon \log|\varepsilon|$ for some appropriately chosen $C > 0$. The triangle inequality reads

$$\|u_0 - u_H\|_{A_0} \le \|u_0 - I_H u_0\|_{A_0} + \|I_H u_0 - u_H\|_{A_0}.$$

The first term can be estimated with standard estimates. For the analysis of the second term, abbreviate $e_\varepsilon := I_H u_0 - u_H$. From the stability of I_H and the properties of the fine-scale projection $\mathcal{C}$, we observe (with contrast-dependent constants)

$$\|e_\varepsilon\|_{A_0}^2 = \|I_H e_\varepsilon\|_{A_0}^2 = \|I_H(1-\mathcal{C})e_\varepsilon\|_{A_0}^2 \lesssim \|(1-\mathcal{C})e_\varepsilon\|_{A_0}^2 \lesssim a_\varepsilon(e_\varepsilon, e_\varepsilon).$$

Hence, we proceed as

$$\|e_\varepsilon\|_{A_0}^2 \lesssim a_\varepsilon(e_\varepsilon, e_\varepsilon) = a_\varepsilon(I_H u_0, e_\varepsilon) - a(u_0, e_\varepsilon)$$

where we have used the solution properties of u_H and u. With the symmetry of a we conclude

$$\|e_\varepsilon\|_{A_0}^2 \lesssim a_\varepsilon(e_\varepsilon, I_H u) - a(e_\varepsilon, u_0).$$

To analyze this term, consider its split

$$\begin{aligned} &\int_\Omega \nabla e_\varepsilon(x) \Big[\int_\Omega \mathcal{A}_\varepsilon(x,y)\, dy - A_0(x) \Big] \nabla u_0(x)\, dx \\ &\quad + \int_\Omega \nabla e_\varepsilon(x) \int_\Omega \mathcal{A}_\varepsilon(x,y)(\nabla I_H u_0(y) - \nabla u_0(y))\, dy\, dx \\ &\quad + \int_\Omega \nabla e_\varepsilon(x) \int_\Omega \mathcal{A}_\varepsilon(x,y)(\nabla u_0(y) - \nabla u_0(x))\, dy\, dx \\ &=: T_1 + T_2 + T_3. \end{aligned} \tag{5.5}$$

By Proposition 17, term T_1 vanishes in the periodic case. The Hölder inequality (once with $(2,2)$ and once with (p,q)) leads to

$$T_2 \le \|\nabla e_\varepsilon\|_{L^2(\Omega)} \Big\| \|\mathcal{A}_\varepsilon(x,y)\|_{L^p(\Omega,dy)} \Big\|_{L^2(\Omega,dx)} \|\nabla(1 - I_H)u_0\|_{L^q(\Omega)}.$$

Standard estimates for the quasi-interpolation I_H [EG16] together with Lemma 23 from Appendix A lead to

$$T_2 \lesssim \|\nabla e_\varepsilon\|_{L^2(\Omega)} H^{1-d(p-1)/p} \left|\log H\right|^d \|u_0\|_{W^{2,q}(\Omega)}.$$

T_3 can be bounded by $\|\nabla e_\varepsilon\|_{L^2(\Omega)}$ times

$$\begin{aligned}&\sqrt{\left(\int_\Omega \left| \int_{\Omega\setminus B_\rho(x)} \mathcal{A}_\varepsilon(x,y)(\nabla u_0(x) - \nabla u_0(y))\, dy\right|^2 dx\right)}\\ &+ \sqrt{\left(\int_\Omega \left| \int_{B_\rho(x)} \mathcal{A}_\varepsilon(x,y)(\nabla u_0(x) - \nabla u_0(y))\, dy\right|^2 dx\right)}\end{aligned} \tag{5.6}$$

With the L^p bounds from Lemma 23 from Appendix A below and with the techniques from the proof of Proposition 6 and Proposition 8, this can be bounded by

$$H^{(1-d(p-1))/p}|\log H|^{(d-1)/p}\|\nabla u_0\|_{L^q(\Omega)} + H^{-d(p-1)/p}|\log H|^d \rho^{(d+sq)/q}\|u_0\|_{W^{2,q}(\Omega)}.$$

Interior regularity estimates [Gri85] finally show that $\|u_0\|_{W^{2,q}(\Omega)} \leq C(q)\|f\|_{L^q(\Omega)}$. □

Denote for any ε, by $u_\varepsilon \in V$ the solution to (4.9) with coefficient A_ε. The following result recovers the classical homogenization limit $u_\varepsilon \to u_0$ strongly in L^2 as $\varepsilon \to 0$. In particular, it quantifies the convergence speed and states that for $f \in L^\infty(\Omega)$ any sublinear rate can be achieved. The obtained rate is only sublinear, but the result is valid for L^∞ coefficients while known linear-in-ε rates [KLS12] require Hölder continuity of A_ε.

Corollary 20 (quantified homogenization limit)**.** *For any $1 < p \leq 2$ and $2 \leq q < \infty$ with $1/p + 1/q = 1$, we have*

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \lesssim H^{1-d(p-1)/p} \log|H|^{(d+q)/q+d}\|f\|_{L^q(\Omega)}.$$

Proof. Proposition 1 in particular implies the a priori error estimate

$$\|u_\varepsilon - u_H\|_{L^2(\Omega)} \lesssim H\|f\|_{L^2(\Omega)}.$$

On the other hand, Proposition 19 implies, for any $2 \leq q < \infty$, that

$$\|u_0 - u_H\|_{L^2(\Omega)} \lesssim H^{1-d(p-1)/p} \log|H|^{(d+q)/q+d}\|f\|_{L^q(\Omega)},$$

The use of the triangle inequality concludes the proof. □

Remark 21. The regularity assumptions $f \in L^q(\Omega)$ etc. are due to the possible singular behaviour of u_ε and u_0. Under the stronger assumption $u_0 \in W^{2,\infty}(\Omega)$, which we cannot guarantee in general, optimal error bounds can easily be proved.

6 Numerical illustration

In section, we present numerical experiments on the unit square domain $\Omega = (0,1)^2$. We consider the following worst-case error (referred to as the L^2 error) as error measure

$$\sup_{f \in L^2(\Omega)\setminus\{0\}} \frac{\|u(f) - u_{\text{discrete}}(f)\|_{L^2(\Omega)}}{\|f\|_{L^2(\Omega)}}$$

where $u(f)$ is the exact solution to (2.3) with right-hand side f and $u_{\text{discrete}}(f)$ a discrete approximation (standard FEM or local effective coefficient or quasi-local effective coefficient or L^2-best approximation). The error quantity is approximated by solving an eigenvalue problem on the reference mesh.

6.1 First experiment

Consider the scalar coefficient A

$$A(x_1, x_2) = \left(\frac{11}{2} + \sin\left(\frac{2\pi}{\varepsilon_1} x_1\right) \sin\left(\frac{2\pi}{\varepsilon_1} x_2\right) + 4\,\sin\left(\frac{2\pi}{\varepsilon_2} x_1\right) \sin\left(\frac{2\pi}{\varepsilon_2} x_2\right)\right)^{-1}$$

with $\varepsilon_1 = 2^{-3}$ and $\varepsilon_2 = 2^{-5}$. We consider a sequence of uniformly refined meshes of mesh size $H = \sqrt{2} \times 2^{-1}, \ldots, \sqrt{2} \times 2^{-6}$. The corrector problems are solved on a reference mesh of width $h = \sqrt{2} \times 2^{-9}$. The localization (or oversampling) parameter is chosen as $\ell = 2$. Figure 3 displays the coefficient A. The four components of the reconstructed coefficient $A_H^{(\ell)}$ for $H = \sqrt{2} \times 2^{-6}$ are displayed in Figure 4. Figure 2 compares the L^2 errors of the standard FEM, the FEM with the local effective coefficient, the method with the quasi-local effective coefficient, and the L^2-best approximation in dependence of H. For comparison, also the error of the Multiscale Finite Element Method (MSFEM) from [EH09] is displayed. As expected, the error of the FEM is of order $\mathcal{O}(1)$ because the coefficient is not resolved by the mesh-size H. The error for the quasi-local effective coefficient is close the the best-approximation. The local effective coefficient leads to comparable errors on coarse meshes. On the finest mesh, where the coefficient is almost resolved, the error deteriorates. This effect, referred to as "resonance effect", will be studied in the second numerical experiment. Table 1 lists the values of the estimator $\eta(A_H^{(\ell)})$ as well as the bounds α_H and β_H on $(A_H^{(\ell)})$. The estimator $\eta(A_H^{(\ell)})$ is small on the first meshes, which corresponds to an effective coefficient close to a constant. The estimator increases for the meshes approaching the resonance regime. The values of the coefficient A range in the interval $[\alpha, \beta] = [0.096, 1.55]$. In this example, the discrete bounds α_H, β_H stay in this interval.

6.2 Second experiment: Resonance effects

In this experiment we investigate so-called resonance effects of our homogenization procedure. We consider a fixed mesh of width $H = \sqrt{2} \times 2^{-4}$ and the scalar coefficient

$$A(x_1, x_2) = \left(5 + 4\,\sin\left(\frac{2\pi}{\varepsilon} x_1\right) \sin\left(\frac{2\pi}{\varepsilon} x_2\right)\right)^{-1}$$

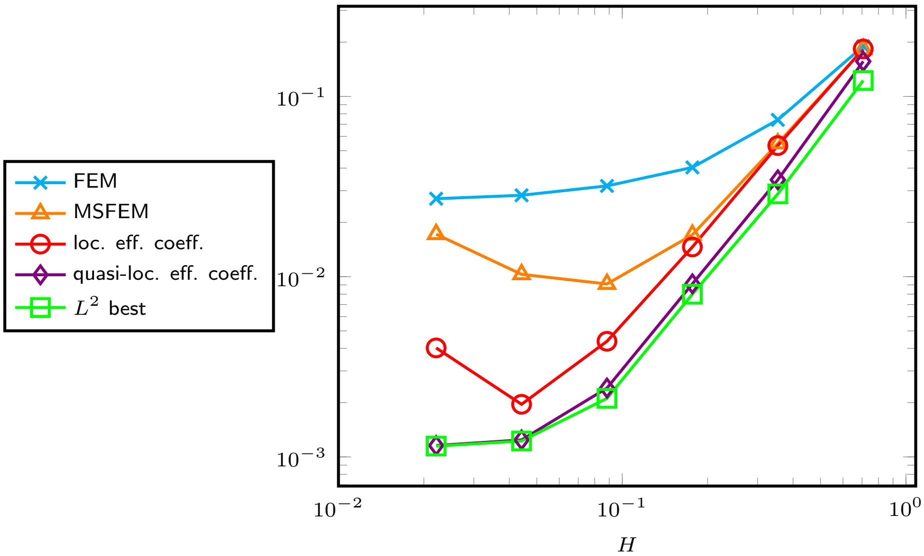

Figure 2: Convergence history under uniform mesh refinement.

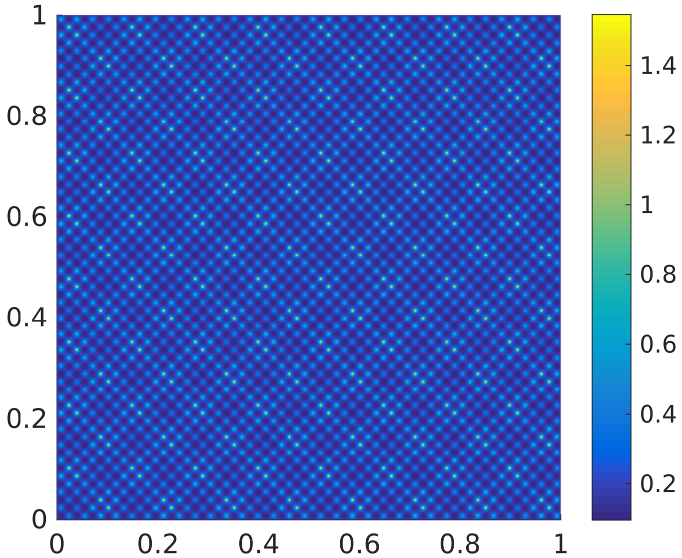

Figure 3: The scalar coefficient A for the first experiment.

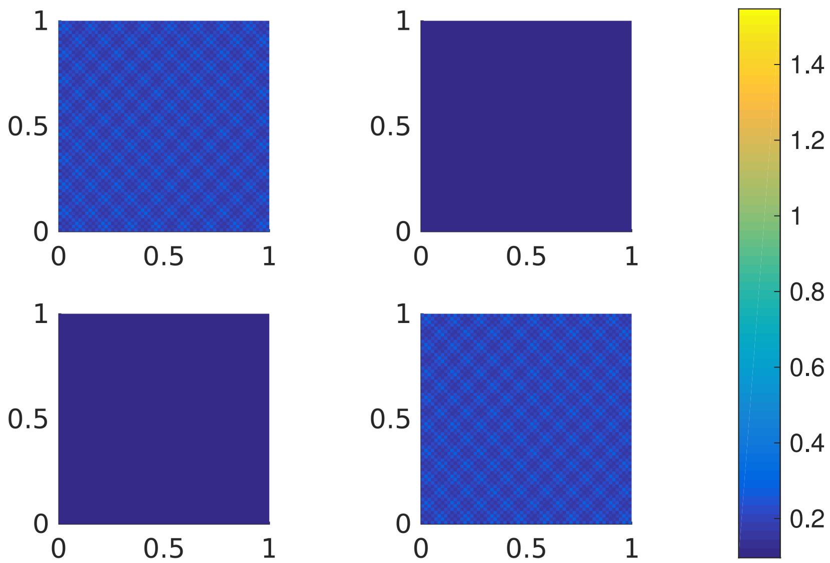

Figure 4: Matrix entries of the reconstructed localized coefficient $(A_H^{(\ell)})$ in the first experiment for $H = \sqrt{2} \times 2^{-6}$.

H	$\eta(A_H^{(\ell)})$	α_H	β_H
$\sqrt{2} \times 2^{-1}$	3.2108e-02	1.9223e-01	2.0786e-01
$\sqrt{2} \times 2^{-2}$	1.1267e-02	1.9568e-01	1.9954e-01
$\sqrt{2} \times 2^{-3}$	1.4765e-02	1.9579e-01	1.9986e-01
$\sqrt{2} \times 2^{-4}$	5.3952e-01	1.8323e-01	2.1992e-01
$\sqrt{2} \times 2^{-5}$	1.7199e+00	1.6909e-01	2.3257e-01
$\sqrt{2} \times 2^{-6}$	1.5538e+01	1.4070e-01	3.0277e-01

Table 1: Values of the estimator $\eta(A_H^{(\ell)})$ and the bounds α_H and β_H on A_H for the first experiment. The values of the coefficient A range in the interval $[\alpha, \beta] = [0.096, 1.55]$.

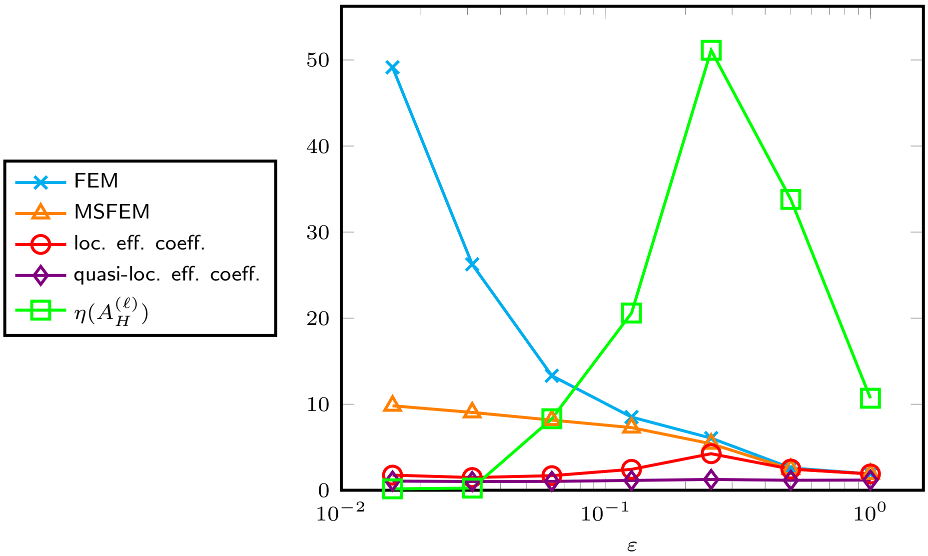

Figure 5: Resonance effect: Normalized (by L^2-best error) errors of FEM, local effective model and quasi-local effecitve model; and values of the estimator $\eta(A_H^{(\ell)})$.

for a sequence of parameters $\varepsilon = 2^0, 2^{-1}, \dots, 2^{-6}$. The coefficient $(A_H^{(\ell)})$ was computed with the same reference mesh and the same oversampling parameter as in the first experiment. Figure 5 displays the L^2 errors normalized by the L^2 error of the L^2-best approximation. On the third mesh, where H and ε have the same order of magnitude, the local effective coefficient leads to a larger error compared to the coarser meshes (where the coefficient is resolved by H) and finer meshes, where H is much coarser than ε and the effective coefficient is close to a constant. We observe that the values of the estimator $\eta(A_H^{(\ell)})$ are large in the resonance regime where also the error of the method the local effective coefficient is large. For smaller values of ε, the values of $\eta(A_H^{(\ell)})$ are close to zero, which indicates that the homogenization criterion from Remark 9 is satisfied, cf. also Remark 15.

A Exponential decay of the non-local effective coefficient

In this section, we illustrate in two lemmas the exponential decay of the entries of $\mathcal{K} := \mathcal{K}^{(\infty)}$, i.e., in the case where the corrector problems are not localized.

Lemma 22. *The coefficient $\mathcal{K}$ satisfies*

$$\left|\left[\mathcal{K}|_{T\times K}\right]_{jk}\right| \lesssim \frac{1}{|K|^{1/2}|T|^{1/2}} \exp\left(-c\,\mathrm{dist}(T,K)/H\right) \tag{A.1}$$

for any $T, K \in \mathcal{T}_H$ and $j, k = 1, \dots, d$.

Proof. From the definition of $\mathcal{K}$, the boundedness of A and the Hölder inequality we obtain

$$|(\mathcal{K}_{T,K})_{j,k}| \leq \frac{1}{|T|\,|K|}\|\nabla q_{T,j}\|_{L^1(K)} \leq \frac{1}{|T|\,|K|^{1/2}}\|\nabla q_{T,j}\|_{L^2(K)}.$$

The combination with the exponential decay from (3.12) and $\|e_j\|_{L^2(T)} = |T|^{1/2}$ proves the result. □

Lemma 23. *Given some $x \in \Omega$ with $x \in T$ for some $T \in \mathcal{T}_H$, any p with $1 \leq p < \infty$ satisfies*

$$\|\mathcal{K}(x,y)\|_{L^p(\Omega,dy)} \lesssim CH^{-d(p-1)/p}|\log H|^d. \tag{A.2}$$

Furthermore, if $m \geq C_1|\log H|$ for a sufficiently large constant C_1 that only depends on c from (A.1) *and the shape-regularity of $\mathcal{T}_H$, then*

$$\|\mathcal{K}(x,y)\|_{L^p(\Omega\setminus\mathsf{N}^m(T),dy)} \lesssim H^{(1-d(p-1))/p}\,|\log H|^{(d-1)/p}. \tag{A.3}$$

Proof. We begin with the proof of the second stated inequality. From the quasi-uniformity of $\mathcal{T}_H$ we obtain with Lemma 22 that

$$\begin{aligned}\|\mathcal{K}(x,y)\|^p_{L^p(\Omega\setminus\mathsf{N}^m(T),dy)} &= \sum_{\substack{K\in\mathcal{T}_H\\ K\not\subseteq\mathsf{N}^m(T)}} |K||\mathcal{K}_{T,K}|^p\\ &\lesssim H^{-d(p-1)} \sum_{\substack{K\in\mathcal{T}_H\\ K\not\subseteq\mathsf{N}^m(T)}} \exp(-c\,\operatorname{dist}(T,K)/H).\end{aligned}$$

Since the mesh $\mathcal{T}_H$ is quasi-uniform, for any positive integer k, the number of elements of $\mathsf{N}^k(T)\setminus\mathsf{N}^{k-1}(T)$ can be bounded by $\mathfrak{p}(k)$ with some polynomial $\mathfrak{p}$ of degree $d-1$. Thus, we can estimate

$$\|\mathcal{K}(x,y)\|^p_{L^p(\Omega\setminus\mathsf{N}^m(T),dy)} \lesssim H^{-d(p-1)} \sum_{k\geq m} \mathfrak{p}(k)\exp(-c\,k).$$

With the above choice of $m \geq C_1|\log H|$, the sum can be rewritten

$$\begin{aligned}\sum_{k\geq m}\mathfrak{p}(k)\exp(-c\,k) &= \sum_{k\geq 0}\mathfrak{p}(m+k)\exp(-c\,(m+k))\\ &\lesssim H\mathfrak{p}(|\log H|)\sum_{k\geq 0}\mathfrak{p}(k)\exp(-c\,k).\end{aligned}$$

The combination with the foregoing displayed estimate proves the second stated estimate. The proof of the first one follows from the combination of this and the arguments from Lemma 5. □

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