zef
Center for
Development Research
University of Bonn

ZEF-Discussion Papers on
Development Policy No. 338

Oded Stark

A note on Sen's representation of the Gini coefficient: Revision and repercussions

Bonn, February 2024.

Oded Stark. A note on Sen's representation of the Gini coefficient: Revision and repercussions. ZEF Discussion Papers on Development Policy No. 338, Center for Development Research, Bonn, February 2024, pp. 8.

ISSN: 1436-9931

Published by:
Zentrum für Entwicklungsforschung (ZEF)
Center for Development Research
Genscherallee 3
D – 53113 Bonn
Germany
Phone: +49-228-73-1861
Fax: +49-228-73-1869
E-Mail: zef@uni-bonn.de
www.zef.de

The author:

Oded Stark, University of Bonn and University of Warsaw. Contact: ostark@uni-bonn.de

Acknowledgements

I am indebted to the reviewers for their sound advice and thoughtful commentary.

This note is dedicated with affection and admiration to Amartya Sen on his 90th birthday.

Abstract

Sen (1973 and 1997) presents the Gini coefficient of income inequality in a population as follows. “In any pair-wise comparison the man with the lower income can be thought to be suffering from some depression on finding his income to be lower. Let this depression be proportional to the difference in income. The sum total of all such depressions in all possible pair-wise comparisons takes us to the Gini coefficient.” (This citation is from Sen 1973, p. 8.) Sen’s verbal account is accompanied by a formula (Sen 1997, p. 31, eq. 2.8.1), which is replicated in the text of this note as equation (1). The formula yields a coefficient bounded from above by a number smaller than 1. This creates a difficulty, because the “mission” of a measure of inequality defined on the unit interval is to accord 0 to perfect equality (maximal equality) and 1 to perfect inequality (maximal inequality). In this note we show that when the Gini coefficient is elicited from a neat measure of the aggregate income-related depression of the population that consists of the people who experience income-related depression, then the obtained Gini coefficient is “well behaved” in the sense that it is bounded from above by 1. We conjecture a reason for a drawback of Sen’s definition, and we present repercussions of the usage of the “well-behaved” Gini coefficient.

Keywords: Sen’s definition of the Gini coefficient of income inequality; Aggregate income-related depression of a population; A “well-behaved” Gini coefficient

JEL classification: D31; D63; I31

1. Introduction

When the distribution of income in a population is such that one person receives all the income, then the Gini coefficient, as defined by Sen (1973 and 1997), registers a value that is smaller than 1. This creates a difficulty, because the "mission" of a measure of inequality defined on the unit interval is to accord 0 to perfect equality (maximal equality) and 1 to perfect inequality (maximal inequality). This is how Sen (1973, p. 8) presents the Gini coefficient. "In any pair-wise comparison the man with the lower income can be thought to be suffering from some depression on finding his income to be lower. Let this depression be proportional to the difference in income. The sum total of all such depressions in all possible pair-wise comparisons takes us to the Gini coefficient." In addition, Sen (1973, p. 6) comments as follows. "Undoubtedly one appeal of the Gini coefficient, or of the relative mean difference, lies in the fact that it is a very direct measure of income difference, taking note of differences between *every* pair of incomes."

In this note we suggest an explanation for a drawback in Sen's presentation of the Gini coefficient, and we propose an amendment. We show that when the Gini coefficient of income inequality in a population is elicited from a "clean" measure of the aggregate depression of the population that experiences income-related depression, then the obtained Gini coefficient is "well behaved:" when the income distribution is such that one person receives all the income, then the Gini coefficient registers the value of 1.

The income-related depression of an individual is the depression that the individual experiences when he observes that his income is lower than the incomes of other individuals in his reference (comparison) group. Drawing on a neat measure of the sum of the levels of depression of the depressed individuals turns out to be a way of ensuring that the Gini coefficient obtains the value of 1 when one person receives all the income.

To begin with, we present the Gini coefficient in the form specified by Sen (1997, p. 31, eq. 2.8.1). In population $N = \{1, 2, \ldots, n\}$, $n \geq 2$, let $y = (y_1, ..., y_n)$ be the vector of the incomes of the individuals. Then, *G*, the Gini coefficient of population *N*, is

$$G \equiv \frac{\sum_{j=1}^{n}\sum_{i=1}^{n}\left|y_i - y_j\right|}{2n^2\overline{y}}, \tag{1}$$

where $\bar{y} = (1/n)\sum_{i=1}^{n} y_i$ is the population's average income.

We can replace the representation in (1), which is based on unordered incomes, with a representation based on ordered incomes, that is, we can let the incomes be arranged in ascending order: $0 \leq y_1 \leq y_2 \leq \ldots \leq y_n$. As in Stark and Budzinski (2021), on noting that $\sum_{j=1}^{n}\sum_{i=1}^{n}\left|y_i - y_j\right| = 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(y_j - y_i)$, an equivalent representation of G in (1), thereby eliminating the need to operate with absolute values, is

$$G = \frac{\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(y_j - y_i)}{n\sum_{i=1}^{n} y_i}. \tag{1'}$$

On the basis of Sen's representation, the sum of the income-related levels of depression of the members of population *N* is obtained in the following way. Let IRD_i denote the income-related depression of individual *i*, $i = 1, 2, ..., n-1$, whose income is y_i. IRD_i is defined as

$$IRD_i \equiv \frac{1}{n}\sum_{j=i+1}^{n}(y_j - y_i).$$

This cumulative measure collects the income excesses to which individual *i* is subjected (the "drivers" of his income-related depression) and then (wrongly, as will be argued momentarily) divides the sum by the size of the population.[1]

Let *TIRD* denote the sum of the levels of IRD_i:

$$TIRD = \frac{1}{n}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(y_j - y_i). \tag{2}$$

On substituting (2) into (1'), we obtain

[1] An intuitive exposition of the formation of this measure and an account of its "embrace" by economists are provided in two appendices in Stark (2023).

$$G = \frac{\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(y_j - y_i)}{n\sum_{i=1}^{n} y_i} = \frac{TIRD}{\sum_{i=1}^{n} y_i}. \qquad (3)$$

With this equivalence, when the income distribution is $y = (0,0,...,0, y_n)$, then $TIRD = \frac{n-1}{n} y_n$, and the Gini coefficient as per (3) is

$$G\big|_{y=(0,0,...,0,y_n)} = \frac{\frac{n-1}{n} y_n}{y_n} = \frac{n-1}{n}.$$

To yield $G = 1$, a correction (multiplication) by $\frac{n}{n-1}$ is required.

2. A revision

This "predicament" is not unavoidable, however. The source of the shortcoming is definition (1): the denominator there should be $2(n-1)n\bar{y}$, not $2n^2\bar{y}$. The reason for stating this relates to the logic that underlies the construction of the *TIRD* measure. Specifically, the "mission" of *TIRD* is to collect the levels of IRD_i contributed by the members of the population. Suppose that the top position in the income hierarchy is occupied by a single individual. In the group of those who contribute to the "pot" of aggregate depression we need not include this top-income individual because his contribution is zero: whereas every individual $1,2,...,n-1$ "collects" and "delivers" depression by observing a higher-income individual higher up in the income hierarchy, the top-income individual has no one higher up in the income hierarchy. In the ascending income distribution there is no one to his right. Thus, because he does not compare himself to anyone, he is not, so to speak, "in the game." It is the case that $n-1$ individuals contribute to the aggregate "pot." Therefore, when we assemble the contributing comparisons (income differences), we collect them from $n-1$ individuals, which leads to a "cleansed" *TIRD* that we denote by $TIRD^*$:

$$TIRD^* = \frac{1}{n-1}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(y_j - y_i).$$

Upon using $TIRD^*$ and rewriting (3) in reverse, we get a "well-behaved" Gini coefficient G^*:

$$\frac{TIRD^*}{\sum_{i=1}^{n} y_i} = \frac{\frac{1}{n-1}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(y_j - y_i)}{\sum_{i=1}^{n} y_i} \equiv G^*. \qquad (4)$$

When the income distribution is $y = (0,0,...,0,y_n),$ then each of the individuals whose income is zero has a level of income-related depression y_n, so that the sum of the "deliveries" of the levels of the income-related depression is $(n-1)y_n$. Dividing this sum by the number of contributors, which is $n-1$ (this yields the numerator of the middle term in (4)), and then by the population's aggregate income, y_n, yields $G^* = 1$; G^* is a "well-behaved" Gini coefficient.

This construction protocol unravels an asymmetry between two distinct and disparate methods of normalization: income per individual (income per capita), which invites drawing on *n* as a denominator, and income-related depression per depressed individual, which invites drawing on $n-1$ as a denominator.

3. Repercussions

Remark 1. Why was the $\frac{n}{n-1}$ deficiency overlooked? It seems that the reason relates to the difference between the standard measure of the average of a phenomenon that is based on the entire membership of a population, and a measure of the average of a phenomenon that is based on members of a population who are responsible for "producing" (that is, for giving rise to) the phenomenon. If in calculating the aggregate income-related depression we count those who are subjected to income-related depression and then formulate the aggregate income-related depression per contributor, then the division needs to be by $n-1$. We cannot exclude the high-income individual from the calculation of income per capita because this individual's contribution is "pivotal," but we can (and should) exclude this same individual from the aggregation of the levels of depression of the individuals who are subjected to income-related depression, as he is not one of these individuals.

Remark 2. Let $y = (y_1, y_2)$ be the vector of the incomes of the individuals, and let these incomes be ordered, $0 \le y_1 < y_2$. Then

$$G^* = \frac{y_2 - y_1}{y_1 + y_2}.$$

If $y_1 = 0$ and $y_2 > 0$, then $G^* = \frac{y_2}{y_2} = 1$. This is what we expect the Gini coefficient to be. For sure, this is better - for the same magnitudes of $y_1 = 0$ and $y_2 > 0$ - than having

$$G = \frac{\frac{1}{2}(y_2 - y_1)}{y_1 + y_2} = \frac{\frac{1}{2}y_2}{y_2} = \frac{1}{2}.$$

Remark 3. Sen (1973 and 1997; 1976; and 1982) sought to measure social welfare by means of the function *SWF*, formulated as $\mu(1-G)$, namely as the product of income per capita, $\mu = \frac{\sum_{i=1}^{n} y_i}{n}$, and $1-G$, where G is the Gini coefficient as defined in (3). Expanding *SWF* while substituting from (3), and then incorporating the case of $n = 2$, we get

$$SWF = \frac{\sum_{i=1}^{n} y_i}{n}\left(1 - \frac{TIRD}{\sum_{i=1}^{n} y_i}\right) = \frac{y_1 + y_2}{2}\left(1 - \frac{\frac{1}{2}(y_2 - y_1)}{y_1 + y_2}\right) = \frac{3y_1 + y_2}{4}.$$

If, however, we rewrite Sen's *SWF* for the case of $n = 2$ with G^* instead of with *G*, then we obtain

$$SWF^* = \frac{y_1 + y_2}{2}\left(1 - \frac{y_2 - y_1}{y_1 + y_2}\right) = \frac{2y_1}{2} = y_1.$$

This is an intriguing result: in the modeled case, when embedded with the "well-behaved" Gini coefficient, Sen's social welfare function coincides with the Rawlsian social welfare function.[2]

Remark 4. Let there be a population of five farmers named R, S, T, U, V who are arranged in a layout such that the farm of farmer T occupies a high ground, whereas each of the other four farms occupies a separate valley in valleys that surround the high ground. Whereas farmer T is observed by each of the other four farmers, these other farmers do not observe each other.

[2] Consult Rawls (1999), and Stark (2020).

In a lean crop year, farmer T has crop *y*, while each of the remainder farmers has crop 0. In this case, farmers R, S, U, and V experience crop-related depression at the level of *y* each and, thus, they contribute $4y$ to the population's "pot" of aggregate depression. Farmer T contributes nothing. Dividing $4y$ by the number of farmers who contribute to the population's aggregate depression, which is 4, gives *y*. Dividing by the farmers' total crop, *y*, yields 1. In this constellation, the Gini coefficient, G^*, is "well behaved:" the most unequal crop distribution occurs when the entire crop of the population of farmers is received by a single farmer, in which case we will, indeed, want the Gini coefficient to be equal to 1. We hasten to add that this magnitude of $G^* = 1$ is distinct from $G = \frac{\frac{4y}{5}}{y} = \frac{4}{5}$.

4. In conclusion

The drawback identified in this note is not likely to be too troubling when the Gini coefficient is applied to large populations because then $\frac{n}{n-1} \approx 1$. Large populations do not typically have a Gini coefficient that is close to 1. This is not the case, however, when the populations concerned are small. And, after all, a good measure of inequality should ideally accommodate populations of different sizes.

Although the need for and application of "an $\frac{n}{n-1}$ correction" were already featured a while ago in several papers, for example, in papers by Weiner and Solbrig (1984) and Deltas (2003), there is an obvious difference between correction and avoidance. The *source* or *origin* of the drawback and the corresponding remedial action identified in this note were not singled out previously.

When in 1912 Corrado Gini developed a mathematical formula for measuring dispersion, he did so independently of social-psychological principles and preferences, his being a sociologist (not just a statistician) notwithstanding. Surely, Gini had no idea that his index, the Gini coefficient, would become "the most commonly used measure of inequality in empirical work" (Sen, 1973, p. 149); Ceriani and Verme (2012) present an illuminating account of the thinking that led Gini to formulate his index. As implied by this note, whereas in the construction of a measure of dispersion, symmetry is a natural attribute, this does not carry over to the construction of a measure of inequality, where asymmetry prevails: in the latter case - using Sen's vocabulary - while the individual with the top income influences the depression of (inflicts depression on) lower-income individuals, that individual is not subject to depression inflicted by lower-income individuals. It is the exploitation of this asymmetry that enabled us to define a "well-behaved" Gini coefficient: for *any* population, when the income distribution is such that one person receives all the income, this "well-behaved" coefficient takes a value of 1.

References

Ceriani, Lidia and Verme, Paolo (2012). "The origins of the Gini index: Extracts from Variabilità e Mutabilità (1912) by Corrado Gini." *Journal of Economic Inequality* 10: 421-443.

Deltas, George (2003). "The small-sample bias of the Gini coefficient: Results and implications for empirical research." *Review of Economics and Statistics* 85(1): 226-234.

Rawls, John (1999). *A Theory of Justice*. Cambridge, MA: Harvard University Press.

Sen, Amartya K. (1973 and 1997). *On Economic Inequality*. Oxford: Clarendon Press.

Sen, Amartya K. (1976). "Real national income." *Review of Economic Studies* 43(1): 19-39.

Sen, Amartya K. (1982). *Choice, Welfare and Measurement.* Oxford: Blackwell.

Stark, Oded (2020). "An economics-based rationale for the Rawlsian social welfare function." In Bishop, John A. and Rodriguez, Gabriel (Eds.), *Research on Economic Inequality*, Vol. 28. Bingley: Emerald, pp. 179-186.

Stark, Oded (2023). "On a tendency in health economics to dwell on income inequality and underestimate social stress." *Economics and Human Biology* 49: 101232.

Stark, Oded and Budzinski, Wiktor (2021). "A social-psychological reconstruction of Amartya Sen's measures of inequality and social welfare." *Kyklos* 74: 552-566.

Weiner, Jacob and Solbrig, Otto T. (1984). "The meaning and measurement of size hierarchies in plant populations." *Oecologia* 61(3): 334-336.