GLOBAL HOMOTOPY THEORY VIA PARTIALLY LAX LIMITS

Dissertation
zur
Erlangung des Doktorgrades (Dr. rer. nat.)
der
Mathematisch-Naturwissenschaftlichen Fakultät
der
Rheinischen Friedrich-Wilhelms-Universität Bonn

vorgelegt von
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aus
Rotterdam, Niederlande

Bonn 2024

Angefertigt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät
der Rheinischen Friedrich-Wilhelms-Universität Bonn

Gutachter/Betreuer: Prof. Dr. Stefan Schwede
Gutachter: Prof. Dr. Markus Hausmann

Tag der Promotion: September 5, 2024
Erscheinungsjahr: 2024

To Zoë, with love

Abstract

Global equivariant homotopy theory is often motivated as the study of compatible collections of equivariant objects for some family of compact Lie groups. In this thesis we make this heuristic precise, by exhibiting the ∞-categories of global spaces and global spectra as a partially lax limit of a diagram of equivariant spaces and spectra respectively. An object of such a partially lax limit is precisely a compatible collection of equivariant objects. We in fact present two approaches to this result. The first is of a direct and calculational nature, and works for arbitrary families of compact Lie groups. This method has the advantage of working in related situations, for example we also obtain a description of proper equivariant homotopy theory as a limit. It is the content of joint work [LNP22] of the author with Denis Nardin and Luca Pol, reproduced in this thesis. The second is of a more categorical nature, but only works for families of finite groups. In this generality it provides an interpretation of the partially lax limits above as a cocompletion procedure for ∞-categories parametrized over the global indexing ∞-category. We then identify a parametrized enhancement of global spaces and spectra with cocompletions of parametrized categories of equivariant spaces and spectra, using results of Bastiaan Cnossen, Tobias Lenz and the author [CLL23a; CLL23b]. Additionally, we deduce a new universal property for Fin-global spectra, as the "representation stabilization" of global spaces at the representation spheres.

Acknowledgements

There are so many people who deserve thanks. First, to my supervisor Stefan Schwede. Thank you for your support and guidance throughout this PhD, and for introducing me to such a stimulating area of research.

To Denis Nardin and Luca Pol, collaborating on [LNP22] together with the both of you has been one of the highlights of my PhD.

To Bastiaan Cnossen, for being a constant friend and collaborator. Our weekly meetings while I was isolated in a cottage in the Netherlands helped keep me sane and our constant discussions at the whiteboard kept me honest. Thank you as well for proof reading the introduction of this thesis.

To Fabian Hebestreit, for all your support as my masters thesis supervisor, coauthor and friend. You have contributed immensely to my growth as a mathematician.

To Markus Hausmann, thank you very much for all of your advice and for agreeing to be on my thesis committee.

To David Gepner, thank you for hosting me in Baltimore and for discussions on the representation stability of global categories.

To my other collaborators, William Balderamma, Max Blans, Adrian Clough, Jack Davies, David Gepner, Rune Haugseng, Kaif Hilman, Branko Juran, Niklas Kipp, Tobias Lenz, Lennart Meier and Joost Nuiten. It has been an absolute pleasure to learn so much about mathematics from all of you.

To all my peers and friends who have enriched my six years in Bonn, whether mathematically or not. My time in Bonn has been incredible, largely thanks to all of you. I would like to especially thank, in rough order of appearance, Aleksander, Jona, Marc, Sid, Lucy, Branko, Louis, Silke, Jan, Radu, Kaif, Marco, Lucas, Emma, Jack, Jonas and Felix.

To Gerald and Susan, for the many great times in Geneva and Athens. Thank you so much for welcoming me into your family.

To my family, Pap, Mam, Abe, Ella, Habib, Sijmen, Naeem, and Yusuf, I give all my love and thanks. I rely on your support more than you know.

To Zoë, you have been with me throughout everything and I can't imagine it any other way. I am honored to dedicate this thesis to you.

Finally, to Asher, enough said.

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Part I

Introduction

Introduction

Global homotopy theory studies objects which have a "compatible" action of all (compact Lie) groups within a designated family.[1] For instance, the reader may consider (G-)equivariant K-theory, equivariant (stable) bordism, stable cohomotopy, and Borel cohomology. Each of these G-equivariant cohomology theories admits a definition which is in some sense uniform in the group G. As such each cohomology theory should, and in fact does, define a global stable homotopy type, that is an object of the ∞-category $\mathrm{Sp}_{\mathrm{gl}}$ of *global spectra* in the sense of [Sch18]. In the unstable setting one studies $\mathcal{S}_{\mathrm{gl}}$, the category of *global spaces* as originally defined by [GH07] (where they are called Orb-spaces). Once again the objects of $\mathcal{S}_{\mathrm{gl}}$ should be spaces which are equipped with a collection of compatible actions. Examples of such spaces include the classifying space of Π-principal bundles, where Π is a compact Lie group. This admits an enhancement to a G-space $B_G\Pi$, which classifies G-equivariant Π-bundles, for every compact Lie group G. Once again these enhancements are all compatible in some, not yet well-defined, sense.

In fact the precise definition of neither global spaces nor global spectra is obviously an implementation of this initial motivation, that a global object should be a compatible family of equivariant objects.

As mentioned, the original definition of the ∞-category of global spaces is due to [GH07]. To define it let us first define an ∞-category Glo. Its objects are indexed by compact Lie groups, which we denote by $\mathbf{B}_{\mathrm{gl}}G$. The space of morphisms from $\mathbf{B}_{\mathrm{gl}}H$ to $\mathbf{B}_{\mathrm{gl}}G$ is equivalent to $\hom(H,G)_{hG}$; the homotopy orbits of the conjugation G-action on the space of continuous group homomorphisms from H to G. The composition is induced by the composition of group homomorphisms. Formally we may define Glo as the ∞-category associated to a category enriched in topological groupids. We then define $\mathcal{S}_{\mathrm{gl}}$ to be the presheaf ∞-category PSh(Glo).

In particular, we see that a global space X consists of the data of a "fixed point" space X^G for every compact Lie group G which is functorial in all continuous group homomorphisms. Furthermore, the conjugation actions on these fixed point spaces have been trivialized, reflecting the fact that spaces of isotropy are insensitive to inner automorphisms. An alternative definition in terms of orthogonal spaces is given in [Sch18]. These two definitions agree by a combination of results in [Sch20] and [KÏ8].

[1] A family is a subset of all compact Lie groups closed under isomorphisms, subgroups and quotients.

Models for the ∞-category $\mathrm{Sp}_{\mathrm{gl}}$ of global spectra have been considered by various authors, for example [Boh14; GM97; Len20; Len22] and [Lew+86, Chapter II]. More recently, in [Sch18] Schwede has constructed a model structure on the category of orthogonal spectra, called the global model structure. Passing to the ∞-category underlying this model category we obtain our definition of $\mathrm{Sp}_{\mathrm{gl}}$. Contained in this work is abundant evidence that $\mathrm{Sp}_{\mathrm{gl}}$ is a good setting in which to study global spectra.

For example, in both cases one can construct a forgetful functor $\mathrm{res}_G\colon \mathrm{Sp}_{\mathrm{gl}} \to \mathrm{Sp}_G$ and $\mathrm{res}_G\colon \mathcal{S}_{\mathrm{gl}} \to \mathcal{S}_G$, which extracts the underlying equivariant homotopy type from a global homotopy type. However in neither case is it immediately clear what the additional compatibility contained in a global object amounts to. As the main contribution of this thesis we provide new models for the ∞-categories of global spaces and spectra which do make this explicit. The question is in what way we can capture and make precise the notion of a "compatible action". Schwede in fact conjectured that this is precisely provided by the notion of partially lax limits, which we introduce now.

PARTIALLY LAX LIMITS

Let $\mathcal{I}$ be an ∞-category and consider a functor $F\colon \mathcal{I} \to \mathrm{Cat}_\infty$. Intuitively, the *lax limit of* F is the ∞-category $\operatorname{laxlim} F$ whose objects consist of the data of

- objects $X_i \in F(i)$ for each $i \in \mathcal{I}$;
- and compatible morphisms $f_\alpha\colon F(\alpha)(X_i) \to X_j$ for every arrow $\alpha\colon i \to j$ in $\mathcal{I}$.

A morphism $\{X_i, f_\alpha\} \to \{X'_i, f'_\alpha\}$ is a suitably natural collection of maps $\{g_i\colon X_i \to X'_i\}$. More formally, one may define $\operatorname{laxlim} F$ to be the ∞-category of sections of the cocartesian fibration associated to F. Next we fix a collection of morphisms $\mathcal{W} \subset \mathcal{I}$, which contains all the equivalences in $\mathcal{I}$ and which is stable under homotopy and composition. The *partially lax limit* of F, denoted by $\operatorname{laxlim}^\dagger F$, is the subcategory of $\operatorname{laxlim} F$ spanned by those objects $(\{X_i\}, \{f_\alpha\})$ for which the canonical map f_α is an equivalence for all edges $\alpha \in \mathcal{W}$. Note that if $\mathcal{W}$ contains only equivalences, then we recover the lax limit of F. On the other hand, if $\mathcal{W}$ contains all edges, we recover the usual notion of the limit of F. In particular we obtain canonical functors

$$\lim F \to \operatorname{laxlim}^\dagger F \to \operatorname{laxlim} F,$$

which indicates that a partially lax limit interpolates between the limit and the lax limit of a diagram. Let us also note that partially lax limits admit a universal property, analogous to the universal property of a limit. More specifically, a partially lax limit of the diagram $F\colon \mathcal{I} \to \mathrm{Cat}_\infty$ represents partially lax cones under the diagram F.

For simplicity, we have only defined the partially lax limit of a functor with values in Cat_∞, but there are similar definitions if we replace Cat_∞ with $\mathrm{Cat}_\infty^\otimes$, the ∞-category of symmetric monoidal ∞-categories.

THE MAIN THEOREMS

With the notion of partially lax limits in hand, we can state the main theorems of this thesis. We will write $(\mathrm{Glo}^{\mathrm{op}})^\dagger$ for the marked category $(\mathrm{Glo}^{\mathrm{op}}, \mathrm{Orb}^{\mathrm{op}})$, where $\mathrm{Orb}^{\mathrm{op}}$ is the wide subcategory of $\mathrm{Glo}^{\mathrm{op}}$ spanned by those maps which admit a representation by an injective group homomorphism.

Theorem A. *There exists a functor $\mathcal{S}_\bullet \colon \mathrm{Glo}^{\mathrm{op}} \to \mathrm{Cat}_\infty^\otimes$ which sends a compact Lie group G to the ∞-category of G-spaces $\mathcal{S}_G$ endowed with the cartesian symmetric monoidal structure, and a continuous group homomorphism $\alpha \colon H \to G$ to the restriction-inflation functors. Furthermore, there is a symmetric monoidal equivalence*

$$\mathcal{S}_{\mathrm{gl}} \simeq \underset{G \in (\mathrm{Glo}^{\mathrm{op}})^\dagger}{\mathrm{laxlim}^\dagger} \mathcal{S}_G$$

between the ∞-category of global spaces with the cartesian monoidal structure and the partially lax limit over $\mathrm{Glo}^{\mathrm{op}}$ of the diagram $\mathcal{S}_\bullet$.

By the previous theorem, a global space X consists of the following data:

- a G-space $\mathrm{res}_G X$ for each compact Lie group G,
- an H-equivariant map $f_\alpha \colon \alpha^* \mathrm{res}_G X \to \mathrm{res}_H X$ for each continuous group homomorphism $\alpha \colon H \to G$.
- the maps f_α are functorial, so that $f_{\beta\circ\alpha} \simeq f_\beta \circ \beta^*(f_\alpha)$ for all composable maps α and β, and $f_{\mathrm{id}} = \mathrm{id}$;
- f_α is an equivalence for every continuous *injective* homomorphism α.
- a homotopy between the map f_{c_g} induced by the conjugation isomorphism and the map $l_g \colon c_g^* \mathrm{res}_G X \to \mathrm{res}_G X$ given by left multiplication by g.
- higher coherences for the homotopies.

This shows that a global space X is precisely a compatible collection of G-spaces $\mathrm{res}_G X$. However we must interpret compatibility as structure, given by the existence of the maps f_α with the required properties.

Similarly there is a stable statement.

Theorem B. *There exists a functor $\mathrm{Sp}_\bullet \colon \mathrm{Glo}^{\mathrm{op}} \to \mathrm{Cat}_\infty^\otimes$ which sends a compact Lie group G to the symmetric monoidal ∞-category of G-spectra $\mathrm{Sp}_G^\otimes$, and a continuous* *group homomorphism $\alpha\colon H \to G$ to the restriction-inflation functor. Furthermore, there is a symmetric monoidal equivalence*

$$\mathrm{Sp}_{\mathrm{gl}} \simeq \operatorname*{laxlim^{\dagger}}_{G\in(\mathrm{Glo}^{\mathrm{op}})^{\dagger}} \mathrm{Sp}_G$$

between Schwede's ∞-category of global spectra and the partially lax limit over $(\mathrm{Glo}^{\mathrm{op}})^{\dagger}$ *of the diagram* $\mathrm{Sp}_{\bullet}$.

Once again we may unravel the theorem above and obtain a concrete description of the data contained in a global spectrum, just as we did before for global spaces. Let us note that the definition of global spectra is not intrinsically ∞-categorical, and instead it is given by passing to the ∞-category underlying a specific model category. One upshot of the previous result is an alternate definition of global spectra using the modern language of homotopy theory: higher category theory.

In this thesis we provide two approaches to these theorems. The first is contained in Part II, and proceeds via universal properties in parametrized homotopy theory. We may term this the *parametrized* approach. The second is the content of [LNP22], joint work of Denis Nardin, Luca Pol and the author. This work is reproduced in Appendix A. We term this the *calculational* approach, because the approach proceeds by first explicitely computing the partially lax limit and then constructing an equivalence to the ∞-categories of global spaces and global spectra.

Before we discuss either approach in more detail, let us note some advantages and disadvantages of either approach. Recall that global homotopy theory studies compatible equivariant phenomena across compact Lie groups. In fact there often exists phenomena which only exist for some specified subcollection of compact Lie groups. Therefore the precise definitions of global spaces and global spectra are made with reference to some family of compact Lie groups. The calculational approach has the benefit that it works for arbitrary families of compact Lie groups. Moreover, the methods are flexible and more generally applicable. For example they give calculations in related but different settings: as an example let us note that we obtain a description of proper equivariant spectra as a limit of the diagram $G/H \mapsto \mathrm{Sp}_H$ defined on the proper orbit ∞-category of G, see Theorem 7.7.11.

The parametrized approach proceeds by enhancing the partially lax limits of the theorem to parametrized categories, and then endows them with a universal property in this context. Because it is fundamentally parametrized, it has the pleasant consequence that it immediately provides a generalization of both theorems to G-global spaces and G-global spectra. It also allows us to conclude a new universal property for the parametrized category of global spectra in terms of a property we dub *representation stability*. Unfortunately all of these benefits do require us to restrict to the family of finite groups.

GLOBAL HOMOTOPY THEORY AS A HOMOTOPY THEORY FOR STACKS

We have motivated global homotopy theory via its connection to equivariant homotopy theory. However, there is a more intrinsic motivation for global homotopy theory, which proceeds by connecting it to the geometry of objects which are locally modelled on quotients of group objects. This is the original motivation of [GH07] for example. There are very many ways of encoding such geometric objects, such as orbifolds, seperated stacks, topological groupoids and topological stacks. For the culture of the reader we will explain one manner in which to make precise the connection of global homotopy theory to the geometry of such objects. The material in this section is not strictly connected to the contents of this thesis, and so can safely be ignored.

We begin with an informal, ahistorical and idiosyncratic discussion of classical algebraic topology. We will then use this perspective to motivate equivariant, and then global, homotopy theory. We take for granted a well-behaved formalism with which to do homotopy theory. In other words, we take for granted the theory of ∞-categories. Included in this theory is the ∞-category $\mathcal{S}$ of *spaces*, which is characterized as the free ∞-category generated from the point under (higher categorical) colimits.

The origins of algebraic topology is found in the study of manifolds[2] undertaken by Henri Poincaré in [Poi95]. Already in this work one finds the fundamental insights of algebraic topology: one can extract out of a manifold combinatorial and algebraic invariants, such as its Betti numbers or fundamental group, which still remember a surprising amount about the original manifold in question. From a modern perspective, we recognize these invariants as shadows of a more fundamental object, its *homotopy type*. This is built by observing that a smooth manifold M is glued together out of simple pieces, which are all homeomorphic to Euclidean space. The homotopy type of M remember the combinatorics of this gluing, while forgetting everything about the local structure of Euclidean space.

To make this intuition precise we appeal to a second fundamental observation of algebraic topology: the process of passing from a manifold to its homotopy type, is obtained by identifying smooth maps of manifolds $f, g\colon M \to N$ which are equivalent up to deformation. Such a deformation is the data of a map $H\colon M \times [0,1] \to N$ which restricts to f and g at the two end points of the interval, also known as a *homotopy* from f to g.

To implement this identification we begin with the category Mfld of manifolds. We then pass from this to the universal ∞-category Mfld_∞ in which homotopic maps are equivalent. The image of a manifold M in Mfld_∞ is our definition of the homotopy type of M, which we denote $\Pi_\infty M$.

[2] Which we will always assume are smooth.

Let us explain in what way this encodes the combinatorics of how M was glued out of euclidean space. Moreover, the fact that one can glue maps (and even homotopies) of manifolds along an open cover implies that, for any cover of a manifold M by two open subsets U, V the square

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & V \arrow[d] \\ U \arrow[r] & M \end{tikzcd}$$

is sent to a pushout square in Mfld_∞. This is one version of the Siefert–van Kampen theorem. One can make similar statements for a general cover of a manifold. Because one can cover a manifold by open balls in such a way that any finite intersection is either empty or again an open ball, we obtain a computation for the homotopy type of a manifold as an iterated pushouts of the homotopy type of open balls in $\mathbb{R}^n$. However each open ball has the same homotopy type as the point, and so we in fact obtain a computation of the homotopy type of M as an interated pushout of the point. The only remaining data is the shape over which one takes the colimit, which encodes the manner in which M is glued out of open balls.

Let us note that Mfld_∞ unfortunately does not have all colimits. For example it fails to have the pushout of the span $S^1 \leftarrow * \to S^1$. Since it is often beneficial to be in a categorical context in which we have all colimits, we shall do this. We first add all colimits to the category of manifolds, while imposing the Mayer–Vietoris property. What we obtain is the ∞-category Shv(Mlfd) of space valued sheaves on the 1-category Man of manifolds equipped with the open cover topology. We now pass to homotopy classes of maps, by contracting the interval. More formally, we consider the localization of sheaves on manifolds at the projection morphisms $[0,1] \times M \to M$. This is identified with a subcategory of Shv(Mfld), spanned by those sheaves $\mathcal{F}$ of spaces such that $\mathcal{F}(\mathrm{pr}_1)\colon \mathcal{F}(M) \to \mathcal{F}(M \times \mathbb{R})$ is an equivalence for every manifold M. We call such sheaves *homotopy invariant*, and denote the subcategory by $\mathrm{Shv}^{\mathrm{htp}}(\mathrm{Mfld})$. There exists a functor $L^{\mathrm{htp}}\colon \mathrm{Shv}(\mathrm{Mfld}) \to \mathrm{Shv}^{\mathrm{htp}}(\mathrm{Mfld})$, and one can show that the essential image of $\mathrm{Mfld} \subset \mathrm{Shv}(\mathrm{Mfld})$ under L^{htp} is equivalent to Mfld_∞. The key result is that there is an equivalence

$$\mathcal{S} \simeq \mathrm{Shv}^{\mathrm{htp}}(\mathrm{Mfld})$$

between the ∞-category of spaces and the ∞-category of homotopy invariant sheaves on manifolds. We refer the reader to [ADH21, Theorem I.1] for a proof of this equivalence.

As a result of this theorem, the homotopy type of a manifold is canonically a space. This expresses a remarkable fact because, as mentioned, the ∞-category of spaces is freely generated under colimits by the point and therefore is a fundamental object of higher category theory. Therefore by passing from manifolds to homotopy types we crystalizes a (relatively) computable part of the theory of manifolds which we can attack with the diverse set of tools of homotopy theory.

As mentioned, some of the fundamental invariants encoded by the homotopy type of a manifold M are its Betti numbers. These are nowadays understood to be decategorifications of more structured invariants, the homology and cohomology $H^*(M)$ and $H_*(M)$ of M. Beyond homotopy invariance and Mayer-Vietoris properties, these groups satisfy an extremely useful property, the *suspension isomorphism*, which gives an equivalence $H_{*+1}(\Sigma M) \simeq H_*(M)$. Recall that ΣM is the *suspension* of M, the pushout of the diagram $* \leftarrow M \rightarrow *$ in $\mathcal{S}$. This relation definitely does not hold for all of the invariants of a homotopy type, such as its homotopy groups. In fact the suspension isomorphism drastically simplifies many calculations. For example one might compare the homology of spheres, for which one sees a complete calculation in a first topology course, to the homotopy groups of spheres, which are essentially impossible to completely calculate.

We may construct an ∞-category of objects which represent invariants which satisfy the suspension isomorphism. Formally, this is done by passing from the ∞-category of pointed spaces to the initial ∞-category in which the operation of suspension is an equivalence. The resulting ∞-category is denoted Sp, and is called the ∞-category of *spectra*. The following consequence of the suspension isomorphism is of crucial importance for geometric applications. Given a manifold M, one can extend the integer grading of $H^*(M)$ and $H_*(M)$ to a grading on formal differences $E - F$ of vector bundles E, F over M. Using this one can prove another fundamental property of homology and cohomology: there is an equivalence between the groups $H^V(M)$ and $H_{V-TM}(M)$. In fact this is nothing but a nonstandard expression of *Poincaré duality*[3].

From the example of classical algebraic topology spawned the area of homotopy theory. This is the study of mathematical objects which come equipped with a natural notion of homotopy between maps. At this point there is an inexhaustable list of examples, of various different flavours. One which is important for the story of this thesis is the theory of *G-spaces* for a compact Lie group G. We motivate this in precisely the same way as before. We begin once again from a purely geometric category, smooth manifolds with an action of the group G. We say two maps $f, g \colon M \to N$ between G-manifolds are *equivariantly homotopic* if there exists a map $H \colon M \times [0,1] \to N$ of G-manifolds which restricts to f and g at the boundary of $[0,1]$. In this definition $[0,1]$ is equipped with the trivial G-action. Once again we may cocomplete and identify equivariantly homotopic maps to obtain the ∞-category $\mathcal{S}_G := \mathrm{Shv}^{\mathrm{htp}}(\mathrm{Mfld}_G)$ of homotopy invariant sheaves on G-manifolds. In the case of smooth manifold

[3] Or more properly *Atiyah Duality*.

we observed that all of the local structure of a manifold was forgotten by passing to its associated homotopy type. The crucial reason for this was the fact that every manifold is locally Euclidean. Intuitively, it is clear that contracting the interval will therefore forget all local structure.

We may wonder if the same is the case for manifolds with a G-action. However here one has to be careful what one means when one says that manifolds with a G-action are locally euclidean. For example, a G-manifold is clearly not covered by open subsets of the form $\mathbb{R}^n$, where $\mathbb{R}^n$ is endowed with the trivial G-action. However, by a non-trivial theorem in equivariant differentiable topology, the existence of linear slices [Die87, Theorem 5.6], it is the case that given a point x with isotropy $H \subset G$, x admits a G-equivariant neighborhood of the form $G \times_H D(V)$, where H is a closed subgroup and $D(V)$ is the open disk in a H-representation V. Because the disk $D(V)$ is contractible, we see that this neighborhood is equivariantly homotopic to the G-manifold $G \times_H * = G/H$. This G-space is characterized as the "free" G-space generated by a single point with isotropy H. This suggests that isotropy is the only local property of a G-manifold which is not forgotten by working up to homotopy.

This is precisely the case, and the following theorem makes this precise. To state it we first fix some notation. First we note that there exists a left adjoint to the inclusion of homotopy invariant sheaves into all sheaves, which we denote L^{htp}. We then write Orb_G for the full subcategory of $\mathcal{S}_G$ spanned by the objects $L^{\mathrm{htp}}G/H$ for H a closed subgroup.

Theorem. There exists an equivalence of ∞-categories

$$\mathrm{Shv}^{\mathrm{htp}}(\mathrm{Mfld}_G) \simeq \mathrm{PSh}(\mathrm{Orb}_G).$$

Under this equivalence, a G-manifold is sent to the diagram $L^{\mathrm{htp}}G/H \mapsto \Pi_\infty M^H$ which sends $L^{\mathrm{htp}}G/H$ to the homotopy type of M^H. The previous theorem seems to also have been folklore, we direct the reader to [Cno23b, Theorem 4.4.16] for a proof. By this theorem, $\mathrm{Shv}^{\mathrm{htp}}(\mathrm{Mfld}_G)$ is equivalent to the ∞-category of G-spaces, which is traditionally defines as the localization of the category of G-CW complexes at the equivariant homotopy equivalences, or equivalently the localization of all G-spaces at the weak homotopy equivalences. It is a consequence of Elmendorff's theorem [Elm83] that $\mathcal{S}_G$ is equivalent to the presheaf ∞-category on Orb_G, and therefore equivalent to $\mathrm{Shv}^{\mathrm{htp}}(\mathrm{Mfld}_G)$ by the theorem above.

Just as in the nonequivariant case, it is of crucial importance to study cohomology theories on G-spaces. One attempt is to naively extend the notion of cohomology theory from non-equivariant to equivariant spaces. In modern terminology, this would amount to passing from the ∞-category of G-spaces to its stabilization $\mathrm{Sp}(\mathcal{S}_G)$. Objects of this ∞-category are often called *naive G-spectra*. This is to distinguish them from more refined notions of cohomology theories one may consider. The need for these more refined theories is the fact that, unlike in the non-equivariant situation, cohomology theories indexed on naive G-spectra do not extend to cohomology theories indexed on formal differences of equivariant vector bundles. In particular such theories fail to satisfy equivariant Poincaré duality. Already at the point we can see the obstruction to extending to a theory graded on vector bundles. Note that a vector bundles over the point is precisely a G-representation. The extended grading on a G-representation V should be obtained by taking the zeroth cohomology on the one point compactification $S^V := V \cup \{\infty\}$ of a G-representation V. This is not generally an invertible object in $\mathrm{Sp}(\mathcal{S}_G)$, and so it is not possible to extend the grading to $0 - V$, the formal inverse of V.

However this example also inspires the construction of an improved theory, analogous to our construction of Sp. Namely, we may formally pass to the initial ∞-category under pointed G-spaces for which the suspension functor $\Sigma^V(-) := S^V \wedge -$ is an equivalence for all representations V of G. One can show that an object of this ∞-category is a representing object for a cohomology theories which does admit an extended grading in equivariant vector bunldes. The resulting ∞-category is denoted Sp_G, and referred to as the ∞-category of *(genuine) G-spectra*. Time has shown this to be a very good context within which to study equivariant cohomology theories: on the one hand, most cohomology theories of interest do naturally admit this extended grading. Examples include equivariant topological K-theory, equivariant bordism theories, equivariant cohomotopy, equivariant elliptic cohomology, Swan K-theory, as well as many others. On the other hand, working in genuine G-spectra one has all of the properties one hopes for, such as an equivariant generalization of Poincaré/Atiyah duality.

The use of G-spaces and G-spectra to study the geometry of manifolds with a G-action has a long and profitable history, see for example [Was69], [Die87], [Bre72], and [CF64] for some classic references. The study of G-spectra has also been an integral part of many developments in homotopy theory. For example, one may cite the Atiyah–Segal Completion theorem and its generalizations [AS69], [GM97], Carlsson's proof of the Segal conjecture [Car84] and work on the Sullivan conjecture [Car91]. More recently, we may cite the resolution of the Kervaire invariant one problem [HHR16], the resolution of the triangulation conjecture by Manolescu [Man16], as well as applications to descent in chromatically localized algebraic K-theory [Cla+20].

Finally, we introduce the ∞-category of global spaces. Once again we will motivate this as the homotopical shadow for a interesting class of geometric objects. These geometric objects are seperated stacks: geometric objects which in some sense locally look like the quotient of a manifold by a compact Lie group.

The study of this class of geometric objects is less classical than that of equivariant manifolds. There are many different motivations for there study, here we provide one. We begin with the following observation: when studying some geometric structure, it is often to consider the *moduli space* $\mathcal{M}$ of such structures. This is defined so that the points of $\mathcal{M}$ correspond to a choice of the relevant geometric structure. Historically, one of the first examples is the moduli space of Riemann structures on a surface Σ of genus g. When $g = 1$ we obtain the moduli space of elliptic curves.

However in the study of such moduli spaces one encounters a problem, which is that these spaces are themselves very rarely smooth. This is slightly surprising, since typically the points of the moduli space $\mathcal{M}$ are locally parametrized by open subsets of euclidean space. For example elliptic curves are given by lattices in $\mathbb{C}$, and so naturally admits a parametrization by points in the upper half plane. However this is precisely where the problem lies: the objects represented by the moduli space are often invariant under reparametrizations. This reflects the fact that certain choices of the geometric structure of interest may have non-trivial automorphisms. In the case of elliptic curves these reparametrizations consist precisely of the Mobius transformations. More concretely, what this implies is that the actual moduli space looks locally like D/G, the quotient of a local chart D by the group G of reparametrizations. Any point of D with non-trivial isotropy gives a singular point of the moduli space, and so obstructs the existence of a smooth structure.

The solution to this problem is to improve our moduli spaces to *moduli stacks*. One way to do this is precisely to remember all of the charts which we used to parametrize our moduli space, together with the action on each chart by the relevant group of reparametrizations. In this context the moduli space is smooth in a suitable sense, and amenable to geometric analysis. An additional benefit is that as a moduli stack, $\mathcal{M}$ in fact solves the moduli problem for families of objects in $\mathcal{M}$ fibered over manifolds. This allows one to apply powerful descent-theoretic methods to the study of such moduli stacks. As an example of this, we note that there is a moduli stack $\mathbf{B}G$ for any Lie group G such that maps $M \to \mathbf{B}G$ from a manifold M into $\mathbf{B}G$ precisely encode the *groupoid* of principal G-bundles on M. To continue to emphasize the distinction between moduli stacks and spaces we note that the underlying moduli space of $\mathbf{B}G$ is always simply a point, and so not very interesting. However as a stack $\mathbf{B}G$ represents prinicpal G-bundles, and so is very interesting. Restricting to those moduli stacks whose isotropy at every point is a compact Lie group, we obtain the $(2, 1)$-category SepStk of *seperated stacks*. Objects of this category appear naturally in many other areas, such as foliation theory, symplectic topology and mathematical physics.

Beyond this, one important and basic source of seperated stacks is from equivariant differential geometry. Namely, given a manifold M with an action of a compact Lie group G, we naturally obtain a seperated stack $M \,/\!\!/\, G$, the *global quotient stack* of M. The moduli space underlying this example is given by the quotient M/G. The G-equivariant local charts of M naturally induce charts for the quotient M/G, and so enhance the quotient to a seperated stack. In fact, one can show that essentially any seperated stack coming from geometry is equivalent to a global quotient stack, see for example [Par22, Corollary 1.3]. In particular, there is a strong connection between the theories of seperated stacks and equivariant manifolds. Nevertheless, the passage from equivariant manifolds to seperated stacks is a drastic one. Morally it involves replacing global symmetries, encoded by actions of compact Lie groups on manifolds, by local symmetries, encoded by the data of isotropy groups for each point. This is reflected by the fact that for any injective group homomorphism $G \to K$, $(X \times_G K) /\!\!/ K$ is equivalent as a seperated stack to $X /\!\!/ G$: a simple calculation shows that $(X \times_G K)/K \simeq X/G$ and that under this equivalence each point has isomorphic isotropy. As a second point we note that the groupoid of morphisms between seperated stacks $M /\!\!/ G$ and $N /\!\!/ G$ is very different from the set of maps $M \to N$ of G-manifolds. For example, when M and N are G/H and G/K respectively, the components of the former are given by group homomorphisms from H to K up to conjugation by elements of K, while the latter is just those maps obtained by conjugating by an element of G.

To study seperated stacks from a homotopical perspective, we may consider seperated stacks up to homotopy equivalence. The previous examples suggest a very natural way in which to do this. Namely, we may consider sheaves of spaces on SepStk, where the covers are given by a suitable generalization of open covers, and then contract the interval. Doing this we obtain the ∞-category $\mathrm{Shv}^{\mathrm{htp}}(\mathrm{SepStk})$ of homotopy invariant sheaves on seperated stacks. Just as in the equivariant case, one can show that the only local structure in a seperable stack which survives the process of contracting the interval is the isotropy data at each point. The diagram of such “isotropy spaces" is naturally encoded by a global space, as shown by the following theorem.

Theorem ([CCL24])**.** There exists an equivalence of ∞-categories

$$\mathcal{S}_{\mathrm{gl}} \simeq \mathrm{Shv}^{\mathrm{htp}}(\mathrm{SepStk}).$$

As this theorem shows, the study of global spaces is naturally motivated by the geometry of seperated stacks. The study of global spaces is in its relative infancy, and its uses for studying the geometry of stacks is less well understood then in the equivariant and non-equivariant cases. Nevertheless, for one application of global homotopy theory to the geometry of stacks we refer the reader to [Par23].

Of course we are also interested in studying cohomology theories on global spaces. Just as in the equivariant case, there is a distinction between naive and genuine variants of such cohomology theories. Once again the crucial additional structure encoded by genuine cohomology theories is an extension of the grading to vector bundles of global spaces. By work of [Sch18], the objects of $\mathrm{Sp}_{\mathrm{gl}}$ canonically represent such genuine cohomology theories on seperated stacks. This makes $\mathrm{Sp}_{\mathrm{gl}}$ an extremely useful setting in which to analyse such global cohomology theories. We may highlight [Sch22], [Hau22] and [LV24] for the use of global methods in the study of stable orbifold bordism.

1. The parametrized approach

We now provide more details on the parametrized approach to Theorem A and Theorem B. Because this method is currently only viable for the family of finite groups[1], in this chapter we let Glo denote the global indexing family for the family of finite groups. We note that this is equivalent to the $(2,1)$-category of finite connected groupoids.

The parametrized approach to these theorems begins with the observation that the partially lax limit in Theorem B universally solves a defect of the diagram $\mathrm{Sp}_\bullet$ of equivariant spectra. Explaining this defect is most convenient from the perspective of *global categories*, which we recall now.

In mathematics one often wants to study objects which come equipped with an action of a group. Typically these collections of G-equivariant objects assemble into an ∞-category, which becomes the main object of study. In the process of understanding such categories one crucially uses their functoriality in the group G; i.e. the fact that one can restrict actions along group homomorphisms. Thinking systematically about this functoriality leads to the definition of a *global category*, which is roughly the data of

1. an ∞-category $\mathcal{C}_G$ for every finite group G;
2. a restriction functor $\alpha^*\colon \mathcal{C}(G) \to \mathcal{C}(H)$ for every group homomorphism $\alpha\colon H \to G$;
3. higher structure, which in particular witnesses that conjugate morphisms induce the same restriction functor.

More precisely we define a global category to be a functor

$$\mathcal{C}\colon \mathrm{Glo}^{\mathrm{op}} \to \mathrm{Cat}_\infty, \quad \mathbf{B}G \mapsto \mathcal{C}_G.$$

The study of global categories is the study of the abstract representation theory of finite groups, understood in a very broad sense. This study of course has a long history, but in this exact formalism was begun in [CLL23a]. The perspective taken there was that of parametrized/internal higher category theory, in the sense of [Bar+16] and [MW21] respectively. This is a robust generalization of higher category theory, which comes with its own notions of adjunctions, colimits and so on. Applying these notions to global categories one recovers

[1] With minor additional work one can also extend to an arbitrary family of finite groups.

properties familiar from representation theory. For example a global category $\mathcal{C}$ admitting certain parametrized colimits, which we call *equivariant colimits*, if each category $\mathcal{C}_G$ admits colimits preserved by restriction, and there exist induction functors $\mathrm{ind}_H^G\colon \mathcal{C}_H \mapsto \mathcal{C}_G$ left adjoint to the restriction $i^*\colon \mathcal{C}_G \to \mathcal{C}_H$ along an inclusion $i\colon H \to G$, which furthermore satisfy an analogue of the classical double coset formula. Having all parametrized colimits implies the further existence of a quotient functor $(-)/N\colon \mathcal{C}_G \to \mathcal{C}_{G/N}$ which is left adjoint to the inflation functor $p^*\colon \mathcal{C}_{G/N} \to \mathcal{C}_G$ for every surjective homomorphism $p\colon G \to G/N$ of finite groups.

With this background we can consider the diagram

$$\mathrm{Sp}_\bullet\colon \mathrm{Glo}^{\mathrm{op}} \to \mathrm{Cat}_\infty, \quad G \mapsto \mathrm{Sp}_G$$

sending the group G to Sp_G, the ∞-category of genuine G-spectra. The crucial observation is that as a global category it suffers from one defect: it does not admit all parametrized colimits. This is a consequence of the fact that the restriction functor $\alpha^*\colon \mathrm{Sp}_G \to \mathrm{Sp}_H$ does not admit a left adjoint when α is a *non-injective* group homomorphism, see Example 3.1.18. In other words, it is not possible to construct a quotient functor $(-)/N\colon \mathrm{Sp}_G \to \mathrm{Sp}_{G/N}$ left adjoint to inflation. Nevertheless it does admit equivariant colimits, in the sense of the previous paragraph.

In this part we will show that one can freely add the missing parametrized colimits to a nice global category which admits equivariant colimits, and moreover that the value of the resulting global category at a group G admits an explicit formula in terms of partially lax limits. Motivated by examples, we call this process *globalization*. To connect this to Theorem B, we note that the category $\mathrm{Sp}_{\mathrm{gl}}$ admits a canonical parametrized enhancement

$$\mathrm{Sp}_{\bullet\text{-gl}}\colon \mathrm{Glo}^{\mathrm{op}} \to \mathrm{Cat}_\infty, \quad \mathbf{B}G \mapsto \mathrm{Sp}_{G\text{-gl}}$$

given by sending the groupoid $\mathbf{B}G$ to the ∞-category of G-global spectra, as defined by [Len20]. We call this the global category of *globally equivariant spectra*. As the main result of this part we will prove that this is equivalent to the globalization of $\mathrm{Sp}_\bullet$, which is an improved version of Theorem B. To state the theorems we require some notation; careful definitions will be given later.

Definition 1.0.1. A global category $\mathcal{C}$ is called *equivariantly presentable* if

- $\mathcal{C}\colon \mathrm{Glo}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ factors through the subcategory $\mathsf{Pr}^{\mathrm{L}} \subset \mathrm{Cat}_\infty$ of presentable ∞-categories;
- the functor $\mathrm{res}_H^G\colon \mathcal{C}_G \to \mathcal{C}_H$ admits a left adjoint $\mathrm{ind}_H^G\colon \mathcal{C}_H \to \mathcal{C}_G$ for every subgroup inclusion $H \subset G$;
- and these left adjoints satisfy a categorified double coset formula, also known as the Beck–Chevalley condition.

A global category $\mathcal{C}$ is called *globally presentable* if, moreover, the restriction functors along surjective group homomorphisms also admits left adjoints, which again satisfy the Beck–Chevalley condition.

We define $\mathsf{Pr}^{\mathrm{Orb}}_{\mathrm{Glo}}$ and $\mathsf{Pr}^{\mathrm{L}}_{\mathrm{Glo}}$ to be the categories of equivariantly presentable and globally presentable global categories respectively.

Definition 1.0.2. Let $\mathcal{C}$ be a global category, then we define a functor

$$\mathrm{Glob}(\mathcal{C})\colon \mathrm{Glo}^{\mathrm{op}} \to \mathrm{Cat}_\infty,$$

called the *globalization* of $\mathcal{C}$, via the assignment

$$\mathrm{Glob}(\mathcal{C})_G = \mathrm{laxlim}^\dagger((\mathrm{Glo}_{/G})^{\mathrm{op}} \to \mathrm{Glo}^{\mathrm{op}} \xrightarrow{\mathcal{C}} \mathrm{Cat}_\infty),$$

where an edge is marked in $\mathrm{Glo}^{\mathrm{op}}_{/G}$ is marked if its projection to $\mathrm{Glo}^{\mathrm{op}}$ is a faithful functor of groupoids, i.e. lands in $\mathrm{Orb}^{\mathrm{op}}$. The functoriality of this assignment in $\mathrm{Glo}^{\mathrm{op}}$ is induced by the contravariant functoriality of partially lax limits applied to the pushforward functoriality of the slices $\mathrm{Glo}_{/G}$.

Theorem C. *Suppose $\mathcal{C}$ is an equivariantly presentable global category. Then* $\mathrm{Glob}(\mathcal{C})$ *is a globally presentable global category. Furthermore the restriction*

$$\mathrm{Glob}\colon \mathsf{Pr}^{\mathrm{Orb}}_{\mathrm{Glo}} \to \mathsf{Pr}^{\mathrm{L}}_{\mathrm{Glo}}$$

is left adjoint to the (non-full) inclusion $\mathsf{Pr}^{\mathrm{L}}_{\mathrm{Glo}} \to \mathsf{Pr}^{\mathrm{Orb}}_{\mathrm{Glo}}$.

See Theorem 3.3.10 for a more precise statement, which is stated in the generality of a suitable pair (T, S) of an ∞-category T and a subcategory $S \subset T$. More precisely we require T to be an orbital ∞-category and S to be an orbital subcategory of T. To prove this result we require a long list of results about partially lax limits, which we collect in Section 3.2. For example we give sufficient conditions for the existence of (co)limits in a partially lax limit of categories, and give two criteria via which one obtains adjunctions between two partially lax limits.

We can now apply this to the global category $\mathrm{Sp}_\bullet$. Using the main results of previous joint work [CLL23a] and [CLL23b] of Bastiaan Cnossen, Tobias Lenz and the author we prove:

Theorem D. *There exists an equivalence*

$$\mathrm{Sp}_{\bullet\text{-gl}} \simeq \mathrm{Glob}(\mathrm{Sp}_\bullet)$$

of global categories.

This theorem is however much more then just a consistency check. For example it has the following significant non-parametrized consequence.

Corollary E. *Let G be a finite group. There exists an equivalence*

$$\mathrm{Sp}_{G\text{-gl}} \simeq \underset{(\mathrm{Glo}_{/G})^{\mathrm{op}}}{\mathrm{laxlim}^{\dagger}} \mathrm{Sp}_{\bullet}.$$

In particular applied to $G = e$ we obtain a second proof of Theorem B for the family of finite groups.

By previous work [CLL23a], the global category $\mathrm{Sp}_{\bullet\text{-gl}}$ has the advantage that it admits a universal property: it is the free globally presentable equivariantly stable global category on a point. Informally, an equivariantly presentable global category is equivariantly stable if each category $\mathcal{C}_G$ is stable, and the functors ind_H^G are also right adjoint to restriction. This universal property is in fact the crucial ingredient for the previous result. By [CLL23b], $\mathrm{Sp}_{\bullet}$ is itself the free equivariantly presentable equivariantly stable global category on a point. Therefore the theorem above follows from the fact that globalization preserves equivariant stability, as we show in Proposition 4.1.13.

REPRESENTATION STABILITY

Having recognized the global category $\mathrm{Sp}_{\bullet\text{-gl}}$ of globally equivariant spectra as the globalization of the global category $\mathrm{Sp}_{\bullet}$ of equivariant spectra, we can immediately deduce universal properties for the former from universal properties of the latter. One such universal property is very close to the definition of genuine equivariant spectra: Sp_G is given by inverting the representations spheres in $\mathcal{S}_{G,*}$, the category of pointed G-spaces. To discuss this systematically, we recall that an equivariantly presentable global category $\mathcal{C}$ is pointed if $\mathcal{C}_G$ is pointed for all $\mathbf{B}G \in \mathrm{Glo}$. In this case one can construct a tensoring of $\mathcal{C}$ by $\mathcal{S}_{\bullet,*}$, the global category of pointed equivariant spaces. Given this we can formulate the following definition.

Definition 1.0.3. We say a pointed equivariantly presentable global category $\mathcal{C}$ is Rep-*stable* if for every $G \in \mathrm{Glo}$ and every G-representation V the functor

$$S^V \otimes -\colon \mathcal{C}_G \to \mathcal{C}_G$$

is an equivalence. We write $\mathsf{Pr}^{\mathrm{Orb}}_{\mathrm{Glo,rep\text{-}st}}$ for the full subcategory of $\mathsf{Pr}^{\mathrm{Orb}}_{\mathrm{Glo}}$ spanned by the Rep-stable equivariantly presentable global categories.

Now consider an arbitrary pointed equivariantly presentable global category $\mathcal{C}$. As we make precise in Definition 4.2.16, inverting the action of the representation spheres pointwise defines a new global category, which we denote by $\mathrm{Stab}^{\mathrm{Orb}}(\mathcal{C})$.

One can show that $\mathrm{Stab}^{\mathrm{Orb}}(\mathcal{C})$ is a Rep-stable equivariantly presentable global category, and furthermore that the functor

$$\mathrm{Stab}^{\mathrm{Orb}}\colon \mathsf{Pr}^{\mathrm{Orb}}_{\mathrm{Glo}} \to \mathsf{Pr}^{\mathrm{Orb}}_{\mathrm{Glo,rep\text{-}st}}$$

defines a left adjoint to the inclusion of Rep-stable equivariantly presentable global categories into $\mathsf{Pr}^{\mathrm{Orb}}_{\mathrm{Glo}}$. In particular we conclude that $\mathrm{Sp}_\bullet$ is the free Rep-stable equivariantly presentable global category generated by a point.

We note that the fact that $\mathrm{Stab}^{\mathrm{Orb}}(\mathcal{C})$ is again equivariantly presentable is not a formality. To emphasize this we observe that while $\mathcal{S}_{\bullet,*}$ is globally presentable, $\mathrm{Sp}_\bullet \simeq \mathrm{Stab}^{\mathrm{Orb}}(\mathcal{S}_{\bullet,*})$ is not. So the process of stabilizing pointwise can in general destroy the existence of certain parametrized colimits. This makes the Rep-stabilizations of *globally* presentable global categories much more complicated in general. We can nevertheless prove the following theorem.

Theorem F. *The globalization of a* Rep-*stable global* ∞*-category is again* Rep-*stable. In particular* $\mathrm{Sp}_{\bullet\text{-gl}}$ *is the free* Rep-*stable globally presentable global category on a single generator.*

Such a universal property for global spectra was first suggested by David Gepner and Thomas Nikolaus [Nik15]. In the setting of *global model categories*, a similar universal property was proved in [LS23].

Finally, note that a partially lax limit of symmetric monoidal categories is canonically symmetric monoidal. Therefore $\mathrm{Sp}_{\bullet\text{-gl}}$ is canonically a symmetric monoidal global category. We also prove a symmetric monoidal analogue of the previous theorem.

Corollary G. *The global* ∞*-category* $\mathrm{Sp}_{\bullet\text{-gl}}$ *of globally equivariant spectra is the initial* Rep-*stable globally presentable symmetric monoidal global category.*

2. The calculational approach: an overview of Appendix A

In this chapter we provide a detailed overview of the contents of [LNP22]. The reader can find the original article in its entirety as Part A. As discussed before, in this article we construct equivalences between the ∞-categories $\mathrm{Sp}_{\mathrm{gl}}$ and $\mathcal{S}_{\mathrm{gl}}$ of global spaces/spectra and certain partially lax limits of equivariant spaces/spectra for arbitrary families of compact Lie groups.

The method is fundamentally calculational. It proceeds by first explicitly calculating the partially lax limits in question, and then constructs an equivalence to the ∞-categories of global spaces and global spectra. To accomplish both of these steps requires translating the definitions of both equivariant and global spaces/spectra into terms more suitable for higher categorical manipulation. Much of the article is spent carefully making these translations.

We now begin with our detailed review of the contents of [LNP22]. The reader may benefit by first reading Chapter 5, the introduction of Appendix C. Specifically we suggest the detailed overview of the proof strategy for Theorem 7.1.17 and Theorem 7.6.10, which will give context for the discussion.

Chapter 6: Partially lax limits, promonoidal ∞-categories and Day convolution

In this chapter we introduce the necessary technical machinery and background with which to state and prove our main theorems.

Recall that $\mathrm{Sp}_{\mathrm{gl}}$ is defined as the associated ∞-category of a model category. For this reason we recall in Section 6.1 the basic properties of this passage. We recall that the ∞-category associated to a symmetric monoidal model category inherits a symmetric monoidal structure. As a general theme, we will often show that contructions on the level of model categories are equivalent to certain constructions on the level of ∞-categories. In Section 6.1.3 we already see the first instance of this philosophy; there we explain in what sense passing to categories of pointed/module objects commutes with the passage to ∞-categories.

In Section 6.2 we recall the notion of $\mathcal{O}^{\otimes}$-promonoidal $\otimes$-categories for an arbitrary ∞-operad $\mathcal{O}^{\otimes}$. These form a particular subcategory of ∞-operads over $\mathcal{O}^{\otimes}$ spanned by those ∞-operads $\mathcal{C}^{\otimes}$ over $\mathcal{O}^{\otimes}$ for which, as we explain, Day convolution $\mathrm{Fun}(\mathcal{C}^{\otimes}, -)^{Day}$ is well-defined as an endofunctor of operads over $\mathcal{O}^{\otimes}$.

We then collect various results about Day convolution which will be important. For example we compute the multi-mapping spaces for Day convolution, as well as its fibers. Using this we show that if $\mathcal{D}^{\otimes} \in (\mathrm{Op}_{\infty})_{/\mathcal{O}^{\otimes}}$ is compatible with colimits, in the sense of Definition 6.2.26, then $\mathrm{Fun}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})^{Day}$ is again an $\mathcal{O}^{\otimes}$-monoidal ∞-category compatible with colimits. In the case that $\mathcal{O}^{\otimes}$ is the commutative ∞-operad, we identify the resulting tensor product with the typical coend formula for Day convolution. Next we discuss the functoriality of Day convolution in the source, focusing on when the canonical restriction functoriality admits $\mathcal{O}^{\otimes}$-monoidal adjoints.

In Section 6.2.1 we prove that any closed symmetric monoidal structures on the ∞-category $\mathrm{Fun}(\mathcal{I}, \mathcal{S})$ of copresheafs is equivalent to Day convolution with respect to some promonoidal structure on $\mathcal{I}$. We use this in the following subsection to give a symmetric monoidal ∞-categorical Elmendorf's theorem, which identifies any cocomplete closed symmetric monoidal ∞-category with enough completely compact objects as a presheaf ∞-category equipped with Day convolution.

Next we turn to discuss partially lax limits and colimits in Section 6.3. We begin by providing a definition of partially lax (co)limits in an arbitrary ∞-category (co)tensored by Cat_{∞}. We then focus in on the case of Cat_{∞}, which is tensored over itself by the cartesian product. We recall the description due to [Ber20] of the partially lax colimit of F as a localization of $\mathrm{Un}^{\mathrm{co}}(F)$, the cocartesian unstraightening of F, at the cocartesian edges over marked edges. This also implies that the partially lax limit of F is equivalent to the ∞-category of sections of $\mathrm{Un}^{\mathrm{co}}(F)$ which send marked edges in the base to cocartesian edges. Using this we obtain some basic properties of partially lax limits. For example, we exhibit the commutativity of lax limits with taking functor categories, and explain when a partially lax limit of Bousfield localizations is again a Bousfield localization.

In Section 6.4 we specialize the definition of partially lax limits to the ∞-category of symmetric monoidal categories. The main result of this section is an equivalence between the lax limit of a diagram $F\colon \mathcal{I} \to \mathrm{Cat}_{\infty}^{\otimes}$ and a particular construction $N_{\mathcal{I}^{\amalg}}\mathcal{C}^{\otimes}$. The right hand side is a symmetric monoidal analog of the ∞-category of sections from before, and so this result is analogous to the description of partially lax limits in terms of sections of the Grothendieck construction. We then use this description to prove that partially lax limits commute with the taking commutative algebra and module objects. To make this identification symmetric monoidal we require an additional technical result about the tensor product on ∞-categories of modules, which we prove in Section 6.5. This final section is the content of appendix A of [LNP22].

Chapter 7: ∞-categories of global objects as partially lax limits

In this chapter we prove that various ∞-categories of global objects admit descriptions as partially lax limits of equivariant objects.

The first section of the chapter contains a complete proof of Theorem A. We begin by recalling the definition of global spaces as a presheaf ∞-category on the global indexing ∞-category. We then explain that any orthogonal factorization system on an ∞-category $\mathcal{C}$ exhibits $\mathcal{C}^{\mathrm{op}}$ as the partially lax colimit of its partial slices, see Proposition 7.1.10. We then show that the injective and surjective group homomorphisms give an orthogonal factorization system on the global indexing category, and that the partial slice over $\mathbf{B}_{\mathrm{gl}}G$ is equivalent to the orbit category of G. Combined with the previous proposition, we conclude Theorem A.

The remainder of the chapter is devoted to proving the stable result. In section Section 7.2 we recall the model categories of equivariant and global spectra which we use. Here the analysis in the equivariant and global situation is very similar, and so for simplicity we will focus on the equivariant case. Let G be a Lie group, then the ∞-category of G-spectra is presented by the category of orthogonal G-spectra equipped with the stable model structure. Recall we write Sp_G for the associated ∞-category. Before we can address the stable model structure, begin with a review of the level model structure on orthogonal G-spectra. We write PSp_G for the ∞-category associated to the level model structure, and call its objects G-prespectra. While the details of this construction are well-recorded in the literature, we repeat it to emphasize how the level model structure on orthogonal (G-)spectra is induced from the level model structure on the category $\mathcal{I}\text{–}G\mathcal{T}$, enriched functors from the category of linear isometries to the category of G-spaces, by passing to pointed and then module objects. In particular, because these operations commute with the passage to ∞-categories we obtain a symmetric monoidal equivalence

$$\mathrm{PSp}_G \simeq \mathrm{Mod}_{\mathbb{S}_G}(\mathcal{I}\text{–}G\mathcal{T}[W_{lvl}^{-1}]).$$

of ∞-categories, see Proposition 7.2.22. This is the main result of this section. Next we discuss how the stable model structure is a Bousfield localization of the level model structure. As we show, this implies that Sp_G is an ∞-categorical Bousfield localization of PSp_G. We give an explicit collection of morphisms in the ∞-category of equivariant and global prespectra which test if a prespectrum is a spectrum, see Proposition 7.2.30. As mentioned, a global analog of these results hold. We obtain a description of $\mathrm{Sp}_{\mathrm{fgl}}$ as an explicit Bousfield localization of the ∞-category $\mathrm{PSp}_{\mathrm{fgl}}$ associated to the faithful level model structure on orthogonal spectra. Once again we obtain a symmetric monoidal equivalence

$$\mathrm{PSp}_{\mathrm{fgl}} \simeq \mathrm{Mod}_{\mathbb{S}_{\mathrm{fgl}}}(\mathcal{I}\text{–}\mathcal{T}[W_{f-lvl}^{-1}]).$$

In Section 7.3 we use the generalized symmetric monoidal Elmendorf's theorem from Section 6.2.1 to identify the ∞-categories

$$\mathcal{I}\text{–}G\mathcal{T}[W_{lvl}^{-1}] \quad \text{and} \quad \mathcal{I}\text{–}\mathcal{T}[W_{f-lvl}^{-1}]$$

with ∞-categories of functors into spaces, equipped with a Day convolution symmetric monoidal structure. More precisely we obtain symmetric monoidal equivalences

$$\mathcal{I}\text{–}G\mathcal{T}[W_{lvl}^{-1}] \simeq \mathrm{Fun}(\mathbf{OR}_G, \mathcal{S}) \quad \text{and} \quad \mathcal{I}\text{–}\mathcal{T}[W_{f-lvl}^{-1}] \simeq \mathrm{Fun}(\mathbf{OR}_{\mathrm{fgl}}, \mathcal{S}).$$

The ∞-category $\mathbf{OR}_{\mathrm{fgl}}$ is defined to be the associated ∞-category of a topologically enriched category. It has as objects pairs (G, V) where G is a compact Lie group and V is a faithful G-representation. Its morphism spaces are defined to be

$$\mathbf{OR}_{\mathrm{gl}}((G, V), (H, W)) = |(\mathrm{Hom}(H, G) \times \mathcal{I}(V, W))^H /\!\!/ G|$$

where $| - /\!\!/ G|$ is the geometric realization of the action groupoid of G on $\mathcal{I}(V, W)$, the space of linear isometries from V to W.

Combining this with the identifications before, we obtain a purely ∞-categorical and concrete description of the ∞-categories of faithful global and equivariant prespectra:

$$\mathrm{PSp}_{\mathrm{fgl}} \simeq \mathrm{Mod}_{\mathbb{S}_{\mathrm{fgl}}}(\mathrm{Fun}(\mathbf{OR}_{\mathrm{fgl}}, \mathcal{S})) \quad \text{and} \quad \mathrm{PSp}_G \simeq \mathrm{Mod}_{\mathbb{S}_G}(\mathrm{Fun}(\mathbf{OR}_G, \mathcal{S})).$$

To foreshadow the remainder of this chapter, let us state that using these descriptions we will be able to explicitly calculate the partially lax limit of the diagram $\mathrm{PSp}_\bullet \colon \mathrm{Glo}^{\mathrm{op}} \to \mathrm{Cat}_\infty^\otimes$, and then exhibit the partially lax limit of $\mathrm{Sp}_\bullet$ as a Bousfield localization thereof. This we will then compare to $\mathrm{Sp}_{\mathrm{gl}}$, which we know is also an explicit localization of $\mathrm{PSp}_{\mathrm{fgl}}$.

However before we can do this we have to actually construct the diagram $\mathrm{PSp}_\bullet$ in a manner which will allow us to calculate the partially lax limit explicitly. This is the content of Section 7.4. By the previous results it (essentially) suffices to exhibit the ∞-categories $\mathbf{OR}_G$ as functorial in the global indexing ∞-category. To do this we construct another category $\mathbf{OR}_{\mathrm{gl}}$ which lives over Glo. As we show this is a cocartesian fibration which classifies the required functor. From this one can construct the functoriality of equivariant prespectra and immediately give an explicit presentation of the partially lax limit of $\mathrm{PSp}_\bullet$, which we denote by $\mathrm{PSp}_{\mathrm{gl}}^\dagger$. Finally we construct a natural transformation $\Sigma_\bullet^\infty \colon \mathcal{S}_\bullet \to \mathrm{PSp}_\bullet$ which agrees pointwise with the standard equivariant suspension prespectrum functor.

In the following section we first show that the functor $\mathrm{PSp}_\bullet \colon \mathrm{Glo}^{\mathrm{op}} \to \mathrm{Cat}_\infty^\otimes$ induces a functor $\mathrm{Sp}_\bullet$ by passing to Bousfield localizations pointwise. In particular we obtain a natural transformation $L_\bullet \colon \mathrm{PSp}_\bullet \Rightarrow \mathrm{Sp}_\bullet$. In this section we also compare the functor $\mathrm{Sp}_\bullet$ to the usual functoriality of G-spectra by appealing to the universal property of equivariant spectra given in [GM23, Appendix C]. Finally we compute the partially lax limit of $\mathrm{Sp}_\bullet$ as a Bousfield localization of $\mathrm{PSp}_{\mathrm{gl}}$.

We have so far shown that $\operatorname{laxlim}^\dagger \mathrm{Sp}_\bullet$ is a Bousfield localization of a ∞-category $\mathrm{PSp}^\dagger_{\mathrm{gl}}$, while $\mathrm{Sp}_{\mathrm{gl}}$ is a Bousfield localization of $\mathrm{PSp}_{\mathrm{fgl}}$. These two ∞-categories of prespectra are very closely related; the only difference is that the first is indexed on all representation while the second is indexed on faithful representations. In Section 7.6 we finish the proof of Theorem B by constructing an adjunction between $\mathrm{PSp}^\dagger_{\mathrm{gl}}$ and $\mathrm{PSp}_{\mathrm{fgl}}$, which we then show induces an equivalence on spectrum objects on each side.

Finally, in Section 7.7 we apply the techniques from previous sections to prove another reconstruction result, which identifies the ∞-category of proper G-spectra with the limit of $\mathrm{Sp}_\bullet$ over the proper orbit category of G.

Let us end this technical overview of Part A by clarifying the contribution of the author to [LNP22]. The project was truly collaborative, and each coauthor contributed to each chapter, each section and each result. However, on balance, the author may take credit for Section 6.1, Section 7.1, Section 7.5 and Section 7.6.

Part II

Globalization and stabilization of global categories

3. Globalization

In this part we present the first approach to theorems A and B. Crucial to this approach is the perspective of parametrized higher category theory; specifically the notion of *partial preentability* from [CLL23b]. We therefore begin with some recollections.

3.1 PARTIAL PRESENTABILITY

In this section we recall some basic definitions from parametrized category theory, and introduce the notion of partial presentability. This section is rather terse, and the reader may benefit from consulting [CLL23b, Section 2].

Notation 3.1.1. Given a category[1] T we define $\mathbb{F}_T$, the *finite coproduct completion* of T, as the smallest full subcategory of $\mathrm{PSh}(T)$ which is closed under coproducts and contains the image of the Yoneda embedding.

Definition 3.1.2. We define $\mathrm{Cat}_T := \mathrm{Fun}^{\times}(\mathbb{F}_T^{\mathrm{op}}, \mathrm{Cat})$, the category of *$T$-categories*, as the ∞-category of finite product preserving functors from $\mathbb{F}_T^{\mathrm{op}}$ to Cat.

Remark 3.1.3. The objects of Cat_T are referred to as T-categories, and morphisms in Cat_T are called T-functors. Note that restriction along the inclusion $T^{\mathrm{op}} \subset \mathbb{F}_T^{\mathrm{op}}$ induces an equivalence

$$\mathrm{Cat}_T \xrightarrow{\sim} \mathrm{Fun}(T^{\mathrm{op}}, \mathrm{Cat}).$$

Notation 3.1.4. Given a T-category $\mathcal{C}$ we will typically denote $\mathcal{C}(X)$ by $\mathcal{C}_X$ and $\mathcal{C}(f)$ by f^*. If f^* has a left or right adjoint, we will denote it by $f_!$ and f_* respectively. We will write the component of a T-functor $F\colon \mathcal{C} \to \mathcal{D}$ at X by $F_X\colon \mathcal{C}_X \to \mathcal{D}_X$.

Remark 3.1.5. Cat_T admits an enhancement to a T-parametrized category $\underline{\mathrm{Cat}}_T$ via the assignment $\underline{\mathrm{Cat}}_T(X) := \mathrm{Cat}_{T_{/X}}$.

We fix a wide subcategory inclusion $S \subset T$. Note that the inclusion $S \subset T$ induces a functor $\mathbb{F}_S \to \mathbb{F}_T$, which exhibits $\mathbb{F}_S$ as a wide subcategory of $\mathbb{F}_T$.

Definition 3.1.6. We say $S \subset T$ is an *orbital subcategory* if the pullback of a morphism in $\mathbb{F}_S$ along any morphism in $\mathbb{F}_T$ exists in $\mathbb{F}_T$ and is again in $\mathbb{F}_S$. We say T is *orbital* if it is an orbital subcategory of itself. We say (T, S) is an *orbital pair* if T is orbital and S is an orbital subcategory of T.

[1] In this part we will use category synonymously with ∞-category

Example 3.1.7. We define Glo to be the $(2,1)$-category of finite connected groupoids $\mathbf{B}G$ and Orb the subcategory spanned by the faithful functors. We claim that $(\mathrm{Glo}, \mathrm{Orb})$ is an orbital pair. Observe that $\mathbb{F}_{\mathrm{Glo}}$ is equivalent to the $(2,1)$-category of finite groupoids, which admits all homotopy pullbacks. The subcategory $\mathbb{F}_{\mathrm{Orb}}$ is the wide subcategory on the faithful maps of groupoids, and thus the orbitality of Orb is equivalent to the observation that pullbacks of faithful maps of groupoids are again faithful.

Example 3.1.8. The orbit category Orb_G of a finite group G is orbital.

Example 3.1.9. Suppose (T, S) is an orbital pair. Then $(T_{/X}, \pi_X^{-1}(S))$ is again an orbital pair, where $\pi_X^{-1}(S)$ is the preimage of S under the functor $\pi_X \colon T_{/X} \to T$.

Before we state the definition of S-presentability we recall the following categorical notion:

Definition 3.1.10. Consider a commutative square

$$\begin{tikzcd} \mathcal{C} \arrow[r, "G"] \arrow[d, "F'"'] & \mathcal{C}' \arrow[d, "F"] \\ \mathcal{D} \arrow[r, "G'"] & \mathcal{D}' \end{tikzcd}$$

in Cat such that both F and F' are right adjoints, with left adjoints L and L' respectively. We say such a square is *left adjointable* if the Beck-Chevalley transformation

$$L'G' \xRightarrow{\eta} L'G'FL \xRightarrow{\sim} L'F'GL \xRightarrow{\epsilon} GL$$

is an equivalence. If F and F' are instead left adjoints then we can dually define the notion of right adjointability.

We now introduce the notion of S-presentability for T-categories.

Definition 3.1.11. Let (T, S) be an orbital pair. We say a T-category $\mathcal{C}$ is *S-presentable* if

1. $\mathcal{C}$ is fiberwise presentable, i.e. lifts to a functor $\mathcal{C} \colon \mathbb{F}_T^{\mathrm{op}} \to \mathsf{Pr}^{\mathrm{L}}$;

2. p^* has a left adjoint for all $p \in \mathbb{F}_S$;

3. For every pullback square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

in $\mathbb{F}_T$ such that p (and therefore p') is in $\mathbb{F}_S$ the square

$$\begin{tikzcd} \mathcal{C}_Y \arrow[r, "g^*"] \arrow[d, "(p)^*"'] & \mathcal{C}_{Y'} \arrow[d, "(p')^*"] \\ \mathcal{C}_X \arrow[r, "(g')^*"] & \mathcal{C}_{X'} \end{tikzcd}$$

is left adjointable. We may refer to this condition by saying that $\mathcal{C}$ satisfies base-change for morphisms in $\mathbb{F}_S$.

We say a functor $F\colon \mathcal{C} \to \mathcal{D}$ between S-presentable categories is *S-cocontinuous* if for all $X \in \mathbb{F}_T$ the functor F_X admits a right adjoint and the square

$$\begin{tikzcd} \mathcal{C}_Y \arrow[r, "F_Y"] \arrow[d, "p^*"'] & \mathcal{D}_Y \arrow[d, "p^*"] \\ \mathcal{C}_X \arrow[r, "F_X"] & \mathcal{D}_X \end{tikzcd}$$

is left adjointable for all $p\colon X \to Y$ in $\mathbb{F}_S$.

We define the category of S-presentable T-categories Pr_T^S as the subcategory of Cat_T spanned by the S-presentable T-categories and S-cocontinuous functors. One can show that the assignment $\underline{\mathsf{Pr}}_T^S(X) := \mathsf{Pr}_{T_{/X}}^{\pi_X^{-1}(S)}$ is a parametrized subcategory of $\underline{\mathrm{Cat}}_T$.

Remark 3.1.12. The notion of S-presentability has been previously introduced by [CLL23b] in the generality of a cleft category $S \subset T$ [CLL23b, Definition 3.2]. In certain ways cleft categories are less general than an orbital pair, but in others ways they are much more general. For example in a cleft category we only require that pullbacks of maps in S along maps in T land in the image of $\mathrm{PSh}(S)$ in $\mathrm{PSh}(T)$, instead of in $\mathbb{F}_S$. We expect that the results of this section are true in a generality which encompasses cleft categories, but have been unable to show this so far.

We note that for a $S \subset T$ which is both a cleft category and an orbital pair, a T-category $\mathcal{C}$ is S-presentable in our sense if and only if it is S-presentable in the sense of [CLL23b, Definition 4.3].

Remark 3.1.13. Pr_T^T is equivalent to the category $\mathsf{Pr}_T^{\mathrm{L}}$ of presentable categories internal to the presheaf topos $\mathrm{PSh}(T)$ in the sense of [MW22], see Theorem A of *loc. cit.* Therefore we will denote Pr_T^T by $\mathsf{Pr}_T^{\mathrm{L}}$. By the parametrized adjoint functor theorem [MW22, Proposition 6.3.1] a T-cocontinuous functor between T-presentable categories is equivalent to a parametrized left adjoint, which we may define as an adjunction in the 2-category $\mathrm{Fun}(T^{\mathrm{op}}, \mathrm{Cat})$[2].

[2]Here $\mathrm{Fun}(T^{\mathrm{op}}, \mathrm{Cat})$ inherits a 2-categorical structure from that of Cat

Remark 3.1.14. Let $f\colon X \to Y$ be a map in $\mathbb{F}_T$, and consider the adjunction

$$f_!\colon \mathbb{F}_{T/X} \rightleftarrows \mathbb{F}_{T/Y} : f^*.$$

Note that both functors are coproduct preserving, and so induce an adjunction

$$f^*\colon \mathrm{Cat}_{T_{/Y}} \rightleftarrows \mathrm{Cat}_{T_{/X}} : f_*,$$

where f^* and f_* are given by precomposing by $f_!$ and f^* respectively. By [CLL23a, Lemma 2.3.14] Conditions (2) and (3) of the previous definition together are equivalent to the claim that for all $p\colon X \to Y$ in $\mathbb{F}_S$, the unit functor

$$\pi_Y^*\mathcal{C} \xrightarrow{\underline{p}^*} p_*p^*\pi_Y^*\mathcal{C}$$

of $T_{/Y}$-categories admits a parametrized left adjoint, which we will denote by $\underline{p}_!$.

Example 3.1.15. Applying the previous definitions to the pair $(\mathrm{Glo}, \mathrm{Orb})$ we recover the notion of equivariant presentability from the introduction, first defined in [CLL23b]. Applied to $(\mathrm{Glo}, \mathrm{Glo})$ we obtain the notion of global presentability.

Example 3.1.16. We define the T-category $\mathcal{S}_\bullet^T := \mathrm{PSh}(T)_{/\bullet}$ of *T-spaces*, where $\mathcal{S}_\bullet^T$ is functorial in pullback. We define $\mathcal{S}_\bullet^S$ as the full T-subcategory of $\mathcal{S}_\bullet^T$ which at $X \in T$ is given by the smallest full category closed under colimits and containing those maps $Z \to X$ which are in S. Because S is orbital this forms a parametrized subcategory. By [MW21, Remark 7.3.4] this is the free S-presentable T-category on a point: for every S-presentable T-∞-category there exists an equivalence

$$\mathrm{Hom}_{\mathrm{Pr}_T^S}(\mathcal{S}_\bullet^S, \mathcal{C}) \simeq \mathrm{core}(\Gamma\mathcal{C}),$$

where $\Gamma\mathcal{C} := \lim_{T^{\mathrm{op}}} \mathcal{C}$ and $\mathrm{core}(\Gamma\mathcal{C})$ is the subcategory of $\Gamma\mathcal{C}$ spanned by the equivalences. Applied to $S = T$ we find that $\mathcal{S}_\bullet^T$ is the free T-presentable T-category on a point.

Example 3.1.17. In the special case of $(T, S) = (\mathrm{Glo}, \mathrm{Orb})$, $\mathcal{S}_\bullet^{\mathrm{Orb}}$ is equivalent to the global category $\mathcal{S}_\bullet$, which sends $\mathbf{B}G$ to the category of G-spaces. See [CLL23b, Theorem 5.3] for a proof of this fact. Similarly $\mathcal{S}_\bullet^{\mathrm{Glo}}$ is equivalent to the category of globally equivariant spaces, which sends $\mathbf{B}G$ to the category of G-global spaces in the sense of [Len20], see [CLL23a, Theorem 3.2.2].

Our key motivation for introducing the notion of S-presentability is to have a convenient formalism to engage with the following example.

Example 3.1.18. We define the global category of equivariant spectra $\mathrm{Sp}_\bullet$ by sending $\mathbf{B}G, \mathbf{B}\mathcal{G} \mapsto Sp_G$. Formally one may define it as the initial functor $\mathcal{C}\colon \mathrm{Glo}^{\mathrm{op}} \to \mathrm{CAlg}(\mathsf{Pr}^{\mathrm{L}})$ under the functor

$$\mathcal{S}_{\bullet,*}\colon \mathrm{Glo}^{\mathrm{op}} \to \mathrm{CAlg}(\mathsf{Pr}^{\mathrm{L}}), \quad \mathbf{B}G \mapsto \mathcal{S}_{G,*},$$

such that the representation spheres are pointwise invertible. Such a functor exists by results of [Rob15]; we refer the reader to Section 4.2 for more details.

We can compare this to more classical definitions, including those used in Part A. Namely, because the representation spheres are invertible in the category of genuine G-spectra, we obtain, by the universal property of $\mathcal{C}$, a comparison natural transformation from $\mathcal{C}$ to the global category of equivariant spectra $\mathrm{Sp}_\bullet$, given by applying Dwyer–Kan localization pointwise to the diagram of relative categories sending $\mathbf{B}G$ to orthogonal G-spectra together with the stable equivalences. The latter is the definition of the global category of equivariant spectra given in [CLL23b, Section 9.1]. The resulting natural transformation is pointwise an equivalence by the results of [GM23, Appendix C], and so we conclude that our definition agrees with the usual definition of genuine equivariant spectra.

To connect to the discussion of partial presentability, we observe that $\mathrm{Sp}_\bullet$ is equivariantly presentable. While this is nothing more then a collection of classical statements about equivariant spectra which are surely well-known to experts, it is also a special case of Theorem 4.2.17. However $\mathrm{Sp}_\bullet$ is *not* globally presentable: the restriction functor $q^*\colon \mathrm{Sp}_G \to \mathrm{Sp}_H$ does not have a left adjoint whenever $q\colon H \to G$ is a non-injective group homomorphism. The existence of such a left adjoint is obstructed by the tom Dieck splitting, which implies that q_* does not preserve compact objects when q is not injective. By [BDS16, Theorem 3.3] this implies that q^* cannot preserve all limits.

3.2 PARTIALLY LAX LIMITS

One of the main goals of this part is to give a construction of the relative cocompletion of S-presentable T-categories using partially lax limits. In this section we will recall the definition of partially lax limits of categories, originally due to [Ber20] in the higher categorical context. Then we will collect a variety of facts about them which we require to provide a formula for relative cocompletion. For example in Section 3.2.1 we consider the question of when partially lax limits admit limits and colimits, and how they are computed. Then in Section 3.2.2 we give two methods for constructing adjunctions between partially lax limits.

Definition 3.2.1. A marked category $(\mathcal{I}, \mathcal{W})$ consists of a category $\mathcal{I}$ equipped with a replete subcategory $\mathcal{W}$. We write $\mathrm{Cat}^\dagger$ for the category of marked categories and marking preserving functors.

Definition 3.2.2. Let $(\mathcal{I}, \mathcal{W})$ be a marked category and let $F\colon \mathcal{I} \to \mathrm{Cat}$ be a functor. Then we view the cocartesian unstraightening $\mathrm{Un}^{\mathrm{co}}(F)$ canonically as a marked category by marking all of the cocartesian morphisms which live over morphisms in $\mathcal{W}$.

Definition 3.2.3. Given two marked categories $\mathcal{C}$ and $\mathcal{D}$ we write $\mathrm{Fun}^{\dagger}(\mathcal{C}, \mathcal{D})$ for the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by those functors which preserve marked morphisms. Suppose $\mathcal{C}$ and $\mathcal{D}$ both admit a functor F and G respectively to a category $\mathcal{I}$. Then we define $\mathrm{Fun}_{\mathcal{I}}(\mathcal{C}, \mathcal{D})$ to be the pullback

$$\begin{tikzcd}
\mathrm{Fun}^{\dagger}_{\mathcal{I}}(\mathcal{C}, \mathcal{D}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}^{\dagger}(\mathcal{C}, \mathcal{D}) \arrow[d, "G_*"] \\
\{F\} \arrow[r] & \mathrm{Fun}(\mathcal{C}, \mathcal{I})
\end{tikzcd}$$

Definition 3.2.4. Given a marked category $(\mathcal{I}, \mathcal{W})$ and a functor $F\colon \mathcal{I} \to \mathrm{Cat}$, we define the *partially lax limit of F with respect to $\mathcal{W}$*

$$\underset{(\mathcal{I},\mathcal{W})}{\mathrm{laxlim}^{\dagger}}\, F := \mathrm{Fun}^{\dagger}_{\mathcal{I}}(\mathcal{I}, \mathrm{Un}^{\mathrm{co}}(F))$$

as the category of sections of the cocartesian unstraightening $\mathrm{Un}^{\mathrm{co}}(F) \to \mathcal{I}$ of F which preserve marked edges, i.e. send morphisms in $\mathcal{W}$ to cocartesian edges of $\mathrm{Un}^{\mathrm{co}}(F)$.

We will sometimes drop the reference to the marking on $\mathcal{I}$ when it is either implicit or clear from context.

Remark 3.2.5. Consider a section $s\colon \mathcal{I} \to \mathrm{Un}^{\mathrm{co}}(F)$. Note that for every $i \in \mathcal{I}$, $s(i)$ lives in the fiber of $\mathrm{Un}^{\mathrm{co}}(F)$ over i and so may view $X_i := s(i)$ as an object of $F(i)$. Next we may consider the map $s(\alpha)\colon X_i \to X_{i'}$ associated to a morphism $\alpha\colon i \to i'$ in $\mathcal{I}$. Once again because s is a section, $s(\alpha)$ lives over α. Factoring $s(\alpha)$ into a cocartesian edge followed by a map living in the fiber over i' gives a morphism $s_\alpha\colon F(\alpha)X_i \to X_{i'}$. Note that s is an object of the partially lax limit with respect to $\mathcal{W}$ if and only if s_α is an equivalence for all edges $\alpha \in \mathcal{W}$. The remaining data contained in the section s encodes compatibility and coherence data for the collection of morphisms s_α.

First we state two simple results, which together will imply that the relative cocompletion preserves T-categories.

Proposition 3.2.6. *Consider a diagram $F\colon \mathcal{I} \to \mathrm{Cat}$ and write $\mathcal{I}^{\Pi}$ for the finite product completion of $\mathcal{I}$ and $\tilde{F}\colon \mathcal{I}^{\Pi} \to \mathrm{Cat}$ for the extension of F to $\mathcal{I}^{\Pi}$. Then the canonical map*

$$\underset{\mathcal{W}^{\Pi} \subset \mathcal{I}^{\Pi}}{\mathrm{laxlim}^{\dagger}}\, \tilde{F} \to \underset{\mathcal{W} \subset \mathcal{I}}{\mathrm{laxlim}^{\dagger}}\, F$$

is an equivalence.

Proof. A simple computation shows that relative right Kan extension, in the sense of [Lur09, Definition 4.3.2.2], provides an inverse. □

Suppose $\mathcal{J}_\bullet\colon \mathcal{I} \to \mathrm{Cat}^\dagger$ is a diagram in marked categories, and that $\mathcal{I}$ is itself marked. Then we will canonically consider the unstraightening $\mathrm{Un}^{\mathrm{co}}(\mathcal{J}_\bullet)$ as a marked category by marking both the cocartesian edges in $\mathrm{Un}^{\mathrm{co}}(\mathcal{J}_\bullet)$ which lie over marked edges of $\mathcal{I}$, as well as the marked edges in each fiber. Recall that there exists a functor $\mathrm{Un}^{\mathrm{co}}(\mathcal{J}_\bullet) \to \operatorname{colim} \mathcal{J}_\bullet$ which exhibits the target as a localization of the source at the cocartesian edges.

Proposition 3.2.7. *Consider a diagram $\mathcal{J}_-\colon \mathcal{I} \to \mathrm{Cat}^\dagger$ in marked categories and a cocone $\{F_i\colon \mathcal{J}_i \to \mathrm{Cat}\}_{i\in\mathcal{I}}$, which induces a functor $F\colon \mathrm{Un}^{\mathrm{co}}(\mathcal{J}_\bullet) \to \operatorname{colim} \mathcal{J}_i \to$ Cat. Then there exists an equivalence*

$$\operatorname*{laxlim}^{\dagger}_{\mathrm{Un}^{\mathrm{co}}(\mathcal{J}_\bullet)} F \simeq \operatorname*{laxlim}^{\dagger}_{\mathcal{I}} \operatorname*{laxlim}^{\dagger}_{\mathcal{J}_i} F_i$$

is an equivalence.

Proof. This follows from the following chain of equivalences:

$$\begin{aligned}
\operatorname*{laxlim}^{\dagger}_{\mathrm{Un}^{\mathrm{co}}(\mathcal{J}_\bullet)} F &:= \mathrm{Fun}^{\mathcal{W}-\mathrm{cocart}}_{\mathrm{Un}^{\mathrm{co}}(\mathcal{J}_\bullet)}(\mathrm{Un}^{\mathrm{co}}(\mathcal{J}_\bullet), \mathrm{Un}^{\mathrm{co}}(F)) \\
&\simeq \operatorname*{laxlim}^{\dagger}_{\mathcal{I}} \mathrm{Fun}^{\mathcal{W}_i-\mathrm{cocart}}_{\mathrm{Un}^{\mathrm{co}}(\mathcal{J}_\bullet)}(\mathcal{J}_i, \mathrm{Un}^{\mathrm{co}}(F)) \\
&\simeq \operatorname*{laxlim}^{\dagger}_{\mathcal{I}} \mathrm{Fun}^{\mathcal{W}_i-\mathrm{cocart}}_{\mathcal{J}_i}(\mathcal{J}_i, \mathrm{Un}^{\mathrm{co}}(F_i)) =: \operatorname*{laxlim}^{\dagger}_{\mathcal{I}} \operatorname*{laxlim}^{\dagger}_{\mathcal{J}_i} F_i,
\end{aligned}$$

where the first equivalence is justified by [LNP22, Proposition 4.15]. □

Remark 3.2.8. Suppose that a functor $G\colon \mathcal{I} \to \mathrm{Cat}$ sends a collection of edges $\mathcal{V}$ to equivalences, and that $\mathcal{I}$ is marked by another collection of edges $\mathcal{W}$. Writing $\tilde{G}\colon \mathcal{I}[\mathcal{V}^{-1}] \to \mathrm{Cat}$ for the functor induced by G, we obtain an equivalence

$$\operatorname*{laxlim}^{\dagger}_{\mathcal{I}} G := \mathrm{Fun}^{\mathcal{W}-\mathrm{cocart}}_{\mathcal{I}}(\mathcal{I}, \mathrm{Un}^{\mathrm{co}}(G)) \simeq \mathrm{Fun}^{\mathcal{W}'-\mathrm{cocart}}_{\mathcal{I}[\mathcal{V}^{-1}]}(\mathcal{I}[\mathcal{V}^{-1}], \mathrm{Un}^{\mathrm{co}}(\tilde{G})) =: \operatorname*{laxlim}^{\dagger}_{\mathcal{I}[\mathcal{V}^{-1}]} \tilde{G},$$

where $\mathcal{W}'$ is the image of $\mathcal{W}$ in $\mathcal{I}[\mathcal{V}^{-1}]$. The functor F from the previous proposition is by definition of this form, and so in the left-hand side of the equivalence stated by the previous proposition we may pass to the localization of $\mathrm{Un}^{\mathrm{co}}(F)$ at the cocartesian edges which lie over marked edges in $\mathcal{I}$. In the extreme case that every edge of $\mathcal{I}$ is marked, we obtain an equivalence

$$\operatorname*{laxlim}^{\dagger}_{\operatorname{colim} \mathcal{J}_i} F \to \lim_{\mathcal{I}} \operatorname*{laxlim}^{\dagger}_{\mathcal{J}_i} F_i.$$

3.2.1 Limits and colimits in partially lax limits

In this subsection we give two propositions which respectively provide sufficient conditions for the existence of limits and colimits in partially lax limits. We begin by considering the case of fully lax limits.

Proposition 3.2.9. *Let $F\colon \mathcal{I} \to \mathrm{Cat}$ be a functor such that each category $F(i)$ admits limits of shape $\mathcal{J}$ for all $i \in \mathcal{I}$. Then* laxlim *F admits limits of shape $\mathcal{J}$, and a section $s\colon \mathcal{I} \to \mathrm{Un}^{\mathrm{co}}(F)$ living over a $\mathcal{J}$-shaped diagram $\{s_j\}_{j\in\mathcal{J}}$ is a limit if and only if $s(i) \simeq \lim_{\mathcal{J}} s_j(i)$ for all $i \in \mathcal{I}$.*

Proof. Recall that $\operatorname{laxlim}_{\mathcal{I}} F := \mathrm{Fun}_{\mathcal{I}}(\mathcal{I}, \mathrm{Un}^{\mathrm{co}}(F))$. Therefore this is the dual of [Lur09, Proposition 5.1.2.2]. □

Recall that we write Cat^L for the wide subcategory of Cat spanned by the left adjoint functors.

Proposition 3.2.10. *Let $F\colon \mathcal{I} \to \mathrm{Cat}^L$ be a functor such that each category $F(i)$ admits colimits of shape $\mathcal{J}$ for all $i \in \mathcal{I}$. Then* laxlim *F admits colimits of shape $\mathcal{J}$ and a section $s\colon \mathcal{I} \to \mathrm{Un}^{\mathrm{co}}(F)$ living under a $\mathcal{J}$-shaped diagram $\{s_j\}_{j\in\mathcal{J}}$ is a colimit if and only if $s(i) \simeq \operatorname{colim} s_j(i)$ for all $i \in \mathcal{I}$.*

Proof. Because $F(f)$ is a left adjoint for all $f\colon i \to i'$ in $\mathcal{I}$, $\mathrm{Un}^{\mathrm{co}}(F)$ is also a cartesian fibration by [Lur09, Corollary 5.2.2.5]. Therefore the result follows from [Lur09, Proposition 5.1.2.2]. □

Suppose $\mathcal{I}$ is a marked category. We will now give a criteria for the inclusion $\operatorname{laxlim}^{\dagger} F \subset \operatorname{laxlim} F$ to preserve limits and colimits. We begin with some preparation.

Notation 3.2.11. Consider a cocartesian fibration $\mathrm{Un}^{\mathrm{co}}(F) \to \mathcal{I}$. Given a morphism $f\colon x_i \to x_j$ in X which lives over the morphism $\alpha\colon i \to j$ in $\mathcal{I}$, we write $f_\alpha\colon F(\alpha)x_i \to x_j$ for the morphism obtained by factoring f into a cocartesian followed by a fiberwise edge.

Lemma 3.2.12. *Let $\mathrm{Un}^{\mathrm{co}}(F) \to \mathcal{I}$ be a cocartesian fibration. Consider a pair of composable morphisms*

$$x_i \xrightarrow{f} x_j \xrightarrow{g} x_k$$

in X which lives over the morphisms

$$i \xrightarrow{\alpha} j \xrightarrow{\beta} k$$

in $\mathcal{I}$. Then the induced map $(gf)_{\beta_\alpha}\colon F(\beta\alpha)x_i \to x_k$ is equal to the composite

$$F(\beta)F(\alpha)x_i \xrightarrow{F(\beta)(f_\alpha)} F(\beta)x_j \xrightarrow{g_\beta} x_k$$

Proof. This follows immediately from the commutative diagram

$$\begin{tikzcd}
x_i \arrow[r, tail] \arrow[dr, "f"'] & F(\alpha)x_i \arrow[r, tail] \arrow[d, "f_\alpha"] & F(\beta)F(\alpha)x_i \arrow[d, "F(\beta)(f_\alpha)"] \\
& x_j \arrow[r, tail] \arrow[dr, "g"'] & F(\beta)x_j \arrow[d, "g_\beta"] \\
& & x_k.
\end{tikzcd}$$

in $\mathrm{Un}^{\mathrm{co}}(F)$ which lives over the triangle $i \to j \to k$ in $\mathcal{I}$ as suggested by the notation, and whose tailed morphisms are cocartesian. □

Remark 3.2.13. We would like to understand the structure maps in a limit of sections. To this end we suppose X is a cocartesian fibration over $[1]$ classifying a functor $F\colon \mathcal{C} \to \mathcal{D}$. Then we observe that the inclusion $\mathcal{D} \hookrightarrow X$ given by including the fiber over $\{1\}$ into X preserves limits: given an object $C \in \mathcal{C}$,

$$\begin{aligned}
\mathrm{Hom}_X(C, \lim D_j) &\simeq \mathrm{Hom}_{\mathcal{D}}(F(C), \lim D_j) \\
&\simeq \lim \mathrm{Hom}_{\mathcal{D}}(F(C), D_j) \\
&\simeq \lim \mathrm{Hom}_X(C, D_j).
\end{aligned}$$

Now suppose that $\mathcal{C}$ and $\mathcal{D}$ both admit $\mathcal{J}$-shaped limits, and consider a $\mathcal{J}$-shaped diagram $\{s_j\colon [1] \to X\}_{j\in\mathcal{J}}$ of sections. By Proposition 3.2.9, the limit of this diagram exists in $\mathrm{Fun}_{[1]}([1], X)$, and is given by $\lim s_j(0) \to \lim s_j(1)$. Since $\lim s_j(1)$ is a limit in X, the map $\lim s_j(0) \to \lim s_j(1)$ is induced by the cone

$$\lim s_j(0) \to s_j(0) \to s_j(1).$$

Factoring this through a cocartesian edge $\lim s_j(0) \to F(\lim s_j(0))$ over $0 \to 1$, we obtain a cone $F(\lim s_j(0)) \to s_j(1)$, which induces a map $F(\lim s_j(0)) \to \lim s_j(1)$. Furthermore the equivalence constructed above shows that the composite of these two maps is equivalent to the map $\lim s_j(0) \to \lim s_j(1)$. Now applying Lemma 3.2.12 we find that the cone $F(\lim s_j(0)) \to s_j(1)$ is equivalent to the composite

$$F(\lim s_j(0)) \to F(s_j(0)) \to s_j(1).$$

In particular the map $F(\lim s_j(0)) \to \lim s_j(1)$ factors as a composite

$$F(\lim s_j(0)) \to \lim F(s_j(0)) \to \lim s_j(1),$$

where the first map is the canonical limit comparison map, and the second map is the limit of the maps $F(s_j(0)) \to s_j(1)$ induced by the maps $s_j(0) \to s_j(1)$. In particular suppose each of the maps $s_j(0) \to s_j(1)$ was cocartesian.

Then this second map is an equivalence, and so we conclude that the map $\lim s_j(0) \to \lim s_j(1)$ is cocartesian if and only if F preserves $\mathcal{J}$-limits.

We note that the dual analysis applies to colimits in cartesian fibrations over $[1]$.

We can now give a sufficient condition for the inclusion of the partially lax limit into the lax limit to preserve limits.

Proposition 3.2.14. *Consider a marked category $(\mathcal{I}, \mathcal{W})$, and a diagram $F \colon \mathcal{I} \to \mathrm{Cat}$. Suppose that the value of F on every i admits limits of shape $\mathcal{J}$ and that for every $\alpha \in \mathcal{W}$ the functor $F(\alpha)$ preserves limits of shape $\mathcal{J}$. Then* $\operatorname{laxlim}^{\dagger} F$ *admits limits of shape $\mathcal{J}$, and they are preserved by the inclusion* $\operatorname{laxlim}^{\dagger} F \subset \operatorname{laxlim} F$.

Proof. Consider a $\mathcal{J}$-shaped diagram $\{s_j \colon \mathcal{I} \to \mathrm{Un}^{\mathrm{co}}(F)\}$ in $\operatorname{laxlim}^{\dagger} F$. We have to show that the limit of this diagram in $\operatorname{laxlim} F$ is again in $\operatorname{laxlim}^{\dagger} F$. I.e. given an edge $\alpha \colon i \to i'$, we have to show that the map $\lim s_j(i) \to s_j(i')$ is cocartesian. However this can be checked by first pulling back along $\alpha \colon [1] \to \mathcal{I}$, where the analysis of Remark 3.2.13 gives the conclusion. □

Remark 3.2.15. Under the assumptions of the previous proposition we have shown, using the informal description of objects in a lax limit from Remark 3.2.5, that

$$\lim\{X_i, s_\alpha\} = \{\lim X_i, \lim s_\alpha \circ \phi\},$$

in $\operatorname{laxlim} F$, where $\phi \colon F(\alpha) \lim X_i \to \lim F(\alpha)(X_i)$ is the canonical limit comparison map.

Similarly we can provide sufficient conditions for the inclusion of the partially lax limit to preserve colimits.

Proposition 3.2.16. *Consider a marked category $(\mathcal{I}, \mathcal{W})$, and a diagram $F \colon \mathcal{I} \to \mathrm{Cat}^L$. Suppose that the value of F on every $i \in \mathcal{I}$ admits colimits of shape $\mathcal{J}$. Then* $\operatorname{laxlim}^{\dagger} F$ *admits colimits of shape $\mathcal{J}$, and they are preserved by the inclusion* $\operatorname{laxlim}^{\dagger} F \subset \operatorname{laxlim} F$.

Proof. Write $G \colon \mathcal{I}^{\mathrm{op}} \to \mathrm{Cat}^R$ for the diagram of right adjoint associated to F. Applying the dual analysis of Remark 3.2.13, we find that given a $\mathcal{J}$-shaped diagram $\{s_j \colon \mathcal{I} \to \mathrm{Un}^{\mathrm{co}}(F)\}$ in $\operatorname{laxlim}^{\dagger} F$ and a map $\alpha \colon i \to i'$ in $\mathcal{W}$, the induced map $\operatorname{colim} s_j(i) \to \operatorname{colim} s_j(i')$ factors as the map

$$\phi \colon \operatorname{colim} s_j(i) \to \operatorname{colim} G(\alpha)(s_j(i')) \to G(\alpha)(s_j(i'))$$

followed by a cartesian edge in $\mathrm{Un}^{\mathrm{co}}(F)$ over α. The map $F(\alpha) \operatorname{colim} s_j(i) \to \operatorname{colim} s_j(i')$ given by instead factoring $\operatorname{colim} s_j(i) \to \operatorname{colim} s_j(i')$ through a cocartesian edge is adjoint to ϕ. In particular we compute that it is given by the composite

$$F(\alpha)(\operatorname{colim} s_j(i)) \xrightarrow{\sim} \operatorname{colim} F(\alpha)(s_j(i)) \to \operatorname{colim} s_j(i').$$

Because the original maps $s_j(i) \to s_j(i')$ were cocartesian, this is an equivalence. We conclude that $\operatorname{colim} s_j(i) \to \operatorname{colim} s_j(i')$ is again cocartesian. □

Remark 3.2.17. Under the assumptions of the previous proposition we have shown, using the informal description of objects in a lax limit, that

$$\operatorname{colim}\{X_i, s_\alpha\} = \{\operatorname{colim} X_i, \operatorname{colim} s_\alpha \circ \phi^{-1}\},$$

where $\phi\colon \operatorname{colim} F(\alpha)X_i \to F(\alpha) \operatorname{colim} X_i$ is the canonical colimit comparison map.

3.2.2 Adjunctions of partially lax limits

Let $(\mathcal{I}, \mathcal{W})$ be a relative category, and consider two functors $F, G\colon \mathcal{I} \to \mathrm{Cat}$. Suppose that one has a commutative diagram

$$\begin{tikzcd} \mathrm{Un}^{\mathrm{co}}(F) \arrow[rr, "H"] \arrow[dr, "p"'] & & \mathrm{Un}^{\mathrm{co}}(G) \arrow[dl, "q"] \\ & \mathcal{I} & \end{tikzcd}$$

such that H preserves cocartesian edges which lie over an edge in $\mathcal{W}$. We call such a commutative diagram a *partially lax transformation* from F to G.

Remark 3.2.18. Note that if H in fact preserves all cocartesian edges then it corresponds via straightening to an honest natural transformation. By weakening this condition we obtain laxly commuting naturality squares, explaining the terminology. For further justification see [Hau+23].

Observe that H induces a functor

$$\underset{(\mathcal{I},\mathcal{W})}{\operatorname{laxlim}^\dagger} F := \mathrm{Fun}_{\mathcal{I}}^{\mathcal{W}\text{-co}}(\mathcal{I}, \mathrm{Un}^{\mathrm{co}}(F)) \xrightarrow{H_*} \mathrm{Fun}_{\mathcal{J}}^{\mathcal{W}\text{-co}}(\mathcal{I}, \mathrm{Un}^{\mathrm{co}}(G)) =: \underset{(\mathcal{I},\mathcal{W})}{\operatorname{laxlim}^\dagger} G.$$

To summarize, partially lax limits are functorial in partially lax natural transformations. Furthermore we note that partially lax limits are also clearly functorial in natural transformations $H \Rightarrow H'$ of partially lax natural transformations $\mathrm{Un}^{\mathrm{co}}(F) \to \mathrm{Un}^{\mathrm{co}}(G)$ which lie over the identity of $\mathcal{I}$. We will now give a sufficient condition for the functor H^* constructed above to admit a right adjoint, for which we first recall the following result.

Lemma 3.2.19. *Suppose $F, G\colon \mathcal{I} \to \mathrm{Cat}$ are two diagram, and consider a natural transformation $\eta\colon F \Rightarrow G$ which is pointwise a left adjoint. Then the functor $H\colon \mathrm{Un}^{\mathrm{co}}(F) \to \mathrm{Un}^{\mathrm{co}}(G)$ encoding η is a left adjoint. The associated right adjoint $J\colon \mathrm{Un}^{\mathrm{co}}(G) \to \mathrm{Un}^{\mathrm{co}}(F)$ is again a functor over $\mathcal{I}$ and is given on the fiber over $i \in \mathcal{I}$ by the right adjoint of η_i. Furthermore the unit and counit of the adjunction $H \dashv J$ live over the identity natural transformation on $\mathrm{id}_{\mathcal{I}}$. Finally J preserves cocartesian edges over $f\colon i \to j$ if and only if the commutative square*

$$\begin{tikzcd}
F(i) \arrow[r, "F(f)"] \arrow[d, "\eta_i"'] & F(j) \arrow[d, "\eta_j"] \\
G(i) \arrow[r, "G(f)"] & G(j)
\end{tikzcd}$$

is left adjointable.

Proof. This is the dual of [Lur16, Proposition 7.3.2.6]. □

Remark 3.2.20. The condition that the right adjoint J of H again lies over $\mathcal{I}$ and that the unit and counit natural transformations of the adjunction $H \dashv J$ live over the identity natural transformation of $\mathcal{I}$ can be summarized by saying that H is a left adjoint in the 2-category $\mathrm{Cat}_{/\mathcal{I}}$ with right adjoint J. This is called a *relative left adjoint* in [Lur16].

Remark 3.2.21. the unstraightening of a parametrized left adjoint $L\colon \mathcal{C} \to \mathcal{D}$ of T-categories is a relative left adjoint over T. This follows from the fact that the unstraightening equivalence $\mathrm{Cocart}(\mathbb{F}_T) \simeq \mathrm{Fun}(\mathbb{F}_T, \mathrm{Cat})$ is an equivalence of 2-categories, see [Hau+23]*Theorem 5.3.1.

From the previous result we immediately obtain the following proposition.

Proposition 3.2.22. *Consider a marked category $(\mathcal{I}, \mathcal{W})$, two diagrams $F, G\colon \mathcal{I} \to \mathrm{Cat}$, and a natural transformation $L\colon F \Rightarrow G$ such that each $L_i\colon F(i) \to G(i)$ is a left adjoint and the square*

$$\begin{tikzcd}
F(i) \arrow[r, "L_i"] \arrow[d, "F(f)"'] & G(i) \arrow[d, "G(f)"] \\
F(j) \arrow[r, "L_j"'] & G(j)
\end{tikzcd}$$

is left adjointable for $f \in \mathcal{W}$. Then the functor $L_*\colon \mathrm{laxlim}^{\dagger} F \to \mathrm{laxlim}^{\dagger} G$ is a left adjoint, with right adjoint given by postcomposing by the partially lax natural transformation given by passing to right adjoints pointwise, as in Lemma 3.2.19.

Proof. By Lemma 3.2.19 the unstraightening of L is a left adjoint in $\mathrm{Cat}_{/\mathcal{I}}$, whose right adjoint preserves cocartesian edges over $\mathcal{W}$. By applying the functoriality of partially lax limits in partially lax natural transformations we obtain the required adjunction. □

Next we introduce the contravariant functoriality of partially lax limits. Consider a functor of marked categories $h\colon (\mathcal{I}, \mathcal{W}) \to (\mathcal{J}, \mathcal{W}')$ and a functor $F\colon \mathcal{J} \to \mathrm{Cat}$. Given a section $s\colon \mathcal{J} \to \mathrm{Un}^{\mathrm{co}}(F)$ we may precompose with the functor h to obtain a functor $\mathcal{I} \to \mathrm{Un}^{\mathrm{co}}(F)$, which we may interpret as a section $t\colon \mathcal{I} \to \mathrm{Un}^{\mathrm{co}}(F) \times_{\mathcal{J}} \mathcal{I}$ of the pullback $\mathrm{Un}^{\mathrm{co}}(F) \times_{\mathcal{J}} \mathcal{I} \to \mathcal{I}$. Recall that $\mathrm{Un}^{\mathrm{co}}(F) \times_{\mathcal{J}} \mathcal{I} \to \mathcal{I}$ is a cocartesian fibration which classifies the functor $F \circ h$. Furthermore an edge in $\mathrm{Un}^{\mathrm{co}}(F) \times_{\mathcal{J}} \mathcal{I}$ is cocartesian if it is in $\mathrm{Un}^{\mathrm{co}}(F)$, and so we conclude from the fact that h is a functor of marked categories that if s sent edges in $\mathcal{W}'$ to cocartesian edges of $\mathrm{Un}^{\mathrm{co}}(F)$ then t sends edges of $\mathcal{W}$ to cocartesian edges of $\mathrm{Un}^{\mathrm{co}}(F \circ h)$. In total we obtain a functor

$$\mathrm{Fun}^{\dagger}_{\mathcal{J}}(\mathcal{J}, \mathrm{Un}^{\mathrm{co}}(F)) \xrightarrow{h^*} \mathrm{Fun}^{\dagger}_{\mathcal{J}}(\mathcal{I}, \mathrm{Un}^{\mathrm{co}}(F)) \simeq \mathrm{Fun}^{\dagger}_{\mathcal{I}}(\mathcal{I}, \mathrm{Un}^{\mathrm{co}}(Fh)).$$

Summarizing, partially lax limits are contravariantly functorial in functors of marked categories. We will again give a sufficient condition for the functor $h^*\colon\ \mathrm{laxlim}^{\dagger} F \to \mathrm{laxlim}^{\dagger} Fh$ constructed above to have a left and right adjoint. To do this we begin with a general categorical result.

Proposition 3.2.23. *Consider a diagram*

$$\begin{tikzcd} X \arrow[r, shift left, "\mathbb{L}"] \arrow[d, "p"'] & Y \arrow[l, shift left, "\mathbb{R}"] \arrow[d, "q"] \\ \mathcal{I} \arrow[r, shift left, "L"] & \mathcal{J} \arrow[l, shift left, "R"] \end{tikzcd}$$

of categories such that p and q are cartesian fibrations, both possible squares commute, and p and q map the unit and counit of $\mathbb{L} \dashv \mathbb{R}$ to that of $L \dashv R$. Suppose X classifies the functor $G\colon \mathcal{I}^{\mathrm{op}} \to \mathrm{Cat}$. Then given an object $i \in \mathcal{I}$, the functor $\mathbb{L}\colon X_i \to Y_{L(i)}$ admits a right adjoint given by the composite

$$Y_{L(i)} \xrightarrow{\mathbb{R}} X_{RL(i)} \xrightarrow{G(\eta)} X_i.$$

Proof. Suppose x, y are objects of X_i and $Y_{L(i)}$ respectively. By assumption the bottom square of the following diagram commutes:

$$\begin{tikzcd} \mathrm{Hom}_{Y_{L(i)}}(\mathbb{L}(x), y) \arrow[r, dashed] \arrow[d] & \mathrm{Hom}_{X_i}(x, G(\eta)\mathbb{R}(y)) \arrow[d] \\ \mathrm{Hom}_{Y}(\mathbb{L}(x), y) \arrow[r, "\sim"] \arrow[d, "p"] & \mathrm{Hom}_{X}(x, \mathbb{R}(y)) \arrow[d, "q"] \\ \mathrm{Hom}_{\mathcal{J}}(L(i), L(i)) \arrow[r, "\sim"] & \mathrm{Hom}_{\mathcal{I}}(i, RL(i)) \end{tikzcd}$$

By [Lur09, Proposition 2.4.4.2], the fiber over $\mathrm{id}_{L(i)}$ and η of p and q respectively are given by the top two spaces of the diagram, and therefore we obtain the dashed equivalence. □

Example 3.2.24. Consider an adjunction $L\colon \mathcal{C} \rightleftarrows \mathcal{D} : R$ between two categories admitting pullbacks. Applying the previous proposition to the square

$$\begin{tikzcd}
\mathrm{Ar}(\mathcal{C}) \arrow[r, shift left, "\mathrm{Ar}(L)"] \arrow[d, "\mathrm{ev}_1"] \arrow[r, phantom, "\perp"] & \mathrm{Ar}(\mathcal{D}) \arrow[l, shift left, "\mathrm{Ar}(R)"] \arrow[d, "\mathrm{ev}_1"] \\
\mathcal{C} \arrow[r, shift left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left, "R"]
\end{tikzcd}$$

we conclude that $L\colon \mathcal{C}_{/x} \longrightarrow \mathcal{D}_{/L(x)}$ admits a right adjoint, given by the composite

$$\mathcal{D}_{/L(x)} \xrightarrow{R} \mathcal{C}_{/RL(x)} \xrightarrow{\eta^*} \mathcal{C}_{/x}.$$

This is a well-known fact, proven as [Lur09, Proposition 5.2.5.1] for example.

We also record the dual proposition:

Proposition 3.2.25. *Consider a diagram*

$$\begin{tikzcd}
X \arrow[r, shift left, "\mathbb{L}"] \arrow[d, "p"'] \arrow[r, phantom, "\perp"] & Y \arrow[l, shift left, "\mathbb{R}"] \arrow[d, "q"] \\
\mathcal{I} \arrow[r, shift left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{J} \arrow[l, shift left, "R"]
\end{tikzcd}$$

of categories such that p and q are cocartesian fibrations and both possible squares commute. Suppose q classifies the functor $F\colon \mathcal{J} \longrightarrow \mathrm{Cat}$. Then given an object $j \in \mathcal{J}$, the functor $\mathbb{R}\colon Y_j \longrightarrow X_{R(j)}$ admits a left adjoint given by the composite

$$X_{R(j)} \xrightarrow{\mathbb{L}} Y_{LR(j)} \xrightarrow{F(\epsilon)} Y_j. \qquad \square$$

Example 3.2.26. Applying the previous proposition to the situation of Example 3.2.24 we conclude that $R\colon \mathcal{D}_{/x} \longrightarrow \mathcal{C}_{/R(x)}$ admits a left adjoint, given by the composite

$$\mathcal{C}_{/R(x)} \xrightarrow{L} \mathcal{C}_{/LR(x)} \xrightarrow{\epsilon_!} \mathcal{C}_{/x}.$$

We now apply these results to construct adjoints to the contravariant functoriality of partially lax limits.

Proposition 3.2.27. *Consider a functor $F\colon \mathcal{I} \longrightarrow \mathrm{Cat}^L$ together with an adjunction $L\colon \mathcal{I} \rightleftarrows \mathcal{J} : R$. Then $R^*\colon \operatorname{laxlim} F \longrightarrow \operatorname{laxlim} FR$ is a left adjoint.*

Proof. We can build the square

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{I}, \mathrm{Un}^{\mathrm{co}}(F)) \arrow[r, shift left, "R^*"] \arrow[d, "p_*"'] \arrow[r, phantom, "\perp"] & \mathrm{Fun}(\mathcal{J}, \mathrm{Un}^{\mathrm{co}}(F)) \arrow[l, shift left, "L^*"] \arrow[d, "p_*"] \\
\mathrm{Fun}(\mathcal{I}, \mathcal{I}) \arrow[r, shift left, "R^*"] \arrow[r, phantom, "\perp"] & \mathrm{Fun}(\mathcal{J}, \mathcal{I}). \arrow[l, shift left, "L^*"]
\end{tikzcd}$$

By [Lur09, Corollary 5.2.2.5] the cocartesian fibration $p\colon \mathrm{Un}^{\mathrm{co}}(F) \to \mathcal{I}$ classifying F is also a cartesian fibration. By [Lur09, Proposition 3.1.2.1(1)] the functors p_* are both also cartesian fibrations, and therefore this square is of the form required to apply Proposition 3.2.23. In particular considering the object $\mathrm{id}_{\mathcal{I}}$ in $\mathrm{Fun}(\mathcal{I}, \mathcal{I})$ we obtain an adjunction

$$R^*\colon \mathrm{Fun}_{\mathcal{I}}(\mathcal{I}, \mathrm{Un}^{\mathrm{co}}(F)) \rightleftarrows \mathrm{Fun}_{\mathcal{I}}(\mathcal{J}, \mathrm{Un}^{\mathrm{co}}(F)) :\Theta(\eta^*)L^*,$$

where Θ refers to the functor classified by p_*. Note that the left hand category is equal to sections of $\mathrm{Un}^{\mathrm{co}}(F)$ while the right-hand side is equivalent to sections of $\mathrm{Un}^{\mathrm{co}}(FR)$. These are equivalent to $\mathrm{laxlim}\, F$ and $\mathrm{laxlim}\, FR$ respectively, and so we conclude. □

Remark 3.2.28. We continue to use the same notation as in the proposition above, and further write $G\colon \mathcal{I}^{\mathrm{op}} \to \mathrm{Cat}^R$ for the diagram of right adjoints associated to F. It is potentially illuminating to informally summarize the adjunction constructed above using the notation of Remark 3.2.5. In this notation, the functor R^* sends the object

$$\{i \mapsto X_i \in F(i) \quad \alpha \mapsto f_\alpha\colon F(\alpha)X_i \to X'_i\}$$

in $\mathrm{laxlim}\, F$ to the object

$$\{j \mapsto X_{R(j)} \in F(R(i)) \quad \beta \mapsto f_{R(\beta)}\colon F(R(\beta))X_{R(i)} \to X_{R(j)}\}$$

in $\mathrm{laxlim}\, FR$. The right adjoint sends the object

$$\{j \mapsto Y_j \in FR(j), \quad \beta \mapsto f_\beta\colon FR(\beta)Y_j \to Y_{j'}\}$$

to the object

$$\{i \mapsto G(\eta_i)Y_{L(i)} \in F(i), \quad \alpha \mapsto \left[F(\alpha)G(\eta_i)Y_{L(i)} \xrightarrow{BC} G(\eta_{i'})F(RL(\alpha))Y_{L(i)} \xrightarrow{G(\eta_{i'})f_{L(\alpha)}} G(\eta_{i'})Y_{L(i')}\right]\},$$

where BC denotes the Beck–Chevalley transformation. This description will become clear from the proof of the next proposition.

This informal description suggests when the adjunction constructed above restricts to one between *partially* lax limits. We make this precise in the following proposition.

Proposition 3.2.29. *In the situation of Proposition 3.2.27, suppose further that $\mathcal{I}$ and $\mathcal{J}$ are marked by $\mathcal{W}$ and $\mathcal{W}'$ respectively, that L and R both preserve marked edges, and that the square*

$$\begin{tikzcd} \mathcal{C}_i \arrow[r, "F(\eta_i)"] \arrow[d, "F(\alpha)"'] & \mathcal{C}_{RL(i)} \arrow[d, "F(RL(\alpha))"] \\ \mathcal{C}_{i'} \arrow[r, "F(\eta_{i'})"'] & \mathcal{C}_{RL(i')} \end{tikzcd}$$

induced by the naturality square of η is left adjointable whenever α is marked, Then R^* and its adjoint restrict to an adjunction between partially lax limits.

Proof. Recall that we have constructed an adjunction

$$R^*\colon \operatorname{laxlim} F \rightleftarrows \operatorname{laxlim} FR :\Theta(\eta^*)L^*.$$

We want to show that both functors restrict to partially lax limits. In the case of R^* this is clear, because R is a functor of marked categories. However showing that $\Theta(\eta^*)L^*$ restricts appropriately is more subtle. We fix an object $s \in \operatorname{laxlim}^\dagger FR$, i.e. a functor $s\colon \mathcal{J} \to \mathrm{Un}^{\mathrm{co}}(F)$ living over R which sends edges in $\mathcal{W}'$ to cocartesian edges. Recall that by [Lur09, Proposition 3.1.2.1(2)] a morphism in $\mathrm{Fun}(\mathcal{I}, \mathrm{Un}^{\mathrm{co}}(F))$ is p_*-cartesian if and only if each component is p-cartesian, and therefore the value of $\Theta(\eta^*)s(L(i))$ at i is equal to the source of the essentially unique cartesian arrow $G(\eta_i)s(L(i)) \twoheadrightarrow s(L(i))$ which lives over $\eta\colon i \to LR(i)$, i.e. the image of $Y(L(i))$ under the functor $G(\eta_i)$. Similarly given a map $\alpha\colon i \to i'$ in $\mathcal{I}$, the map $\Theta(\eta^*)L^*s(\alpha)$ is homotopic to the unique dotted map

$$\begin{tikzcd} G(\eta_i)s(L(i)) \arrow[r, two heads] \arrow[d, dashed, "\phi"] & s(L(i)) \arrow[d] \\ G(\eta_j)s(L(i')) \arrow[r, two heads] & s(L(i')) \end{tikzcd}$$

for which the resulting square commutes and lives over the square

$$\begin{tikzcd} i \arrow[r, "\eta_i"] \arrow[d, "f"'] & RL(i) \arrow[d, "RL(f)"] \\ i' \arrow[r, "\eta_{i'}"] & RL(i'). \end{tikzcd}$$

in $\mathcal{I}$. However we can build the following commutative diagram

$$\begin{tikzcd}
G(\eta_i)s(L(i)) \arrow[r, two heads] \arrow[d, tail] & s(L(i)) \arrow[dd, tail] \\
F(\alpha)G(\eta_i)s(L(i)) \arrow[d, "BC"'] & \\
G(\eta_{i'})F(RL(\alpha))s(L(i)) \arrow[r, two heads] \arrow[d, "G(\eta_{i'})f_{L(\alpha)}"'] & F(\alpha)s(L(i)) \arrow[d, "f_{L(\alpha)}"] \\
G(\eta_{i'})s(L(i')) \arrow[r, two heads] & s(L(i'))
\end{tikzcd}$$

in X which lives over

$$\begin{tikzcd}
i \arrow[r] \arrow[d, "\alpha"'] & RL(i) \arrow[dd, "RL(\alpha)"] \\
i' \arrow[d, equal] & \\
i' \arrow[r] \arrow[d, equal] & RL(i') \arrow[d, equal] \\
i' \arrow[r] & RL(i').
\end{tikzcd}$$

In this diagram the two-headed arrows are cartesian and the tailed arrows are cocartesian. By conclude that the composite along the left hand side is homotopic to ϕ. The identification of the map

$$F(\alpha)G(\eta_i)s(L(i)) \to G(\eta_{i'})F(RL(\alpha))s(L(i))$$

with the Beck–Chevalley transformation of the square

$$\begin{tikzcd}
\mathcal{C}_{RL(i)} \arrow[r, "F(\eta_i)"] \arrow[d, "F(\alpha)"'] & \mathcal{C}_i \arrow[d, "F(RL(\alpha))"] \\
\mathcal{C}_{RL(i')} \arrow[r, "F(\eta_{i'})"'] & \mathcal{C}_{i'}
\end{tikzcd}$$

is given in [Hau+23, Proposition 3.2.7]. Our assumptions on L and R imply that this map, as well as the map $G(\eta_{i'})f_{L(\alpha)}$, are equivalences. Therefore the map ϕ is cocartesian, and we conclude that $\Theta(\eta^*)L^*$ preserves partially cocartesian sections. $\square$

We can also give a sufficient condition for the contravariant restriction along a functor of marked categories to admit a left adjoint.

Proposition 3.2.30. *Consider a functor $F\colon \mathcal{J} \to \mathrm{Cat}$ and an adjunction $L\colon \mathcal{I} \rightleftarrows \mathcal{J} : R$. Then $L^*\colon \operatorname{laxlim} F \to \operatorname{laxlim} FL$ is a right adjoint. If we furthermore suppose that $\mathcal{I}$ and $\mathcal{J}$ are marked and the functors L and R preserve the markings, then L^* and its right adjoint preserve objects of the partially lax limit. In particular they restrict to an adjunction between partially lax limits.*

Proof. We can build the square

$$\begin{tikzcd} \mathrm{Fun}(\mathcal{I}, \mathrm{Un}^{\mathrm{co}}(F)) \arrow[r, shift left, "R^*"] \arrow[d, "p_*"'] \arrow[r, phantom, "\perp"] & \mathrm{Fun}(\mathcal{J}, \mathrm{Un}^{\mathrm{co}}(F)) \arrow[l, shift left, "L^*"] \arrow[d, "p_*"] \\ \mathrm{Fun}(\mathcal{I}, \mathcal{J}) \arrow[r, shift left, "R^*"] \arrow[r, phantom, "\perp"] & \mathrm{Fun}(\mathcal{J}, \mathcal{J}). \arrow[l, shift left, "L^*"] \end{tikzcd}$$

Note this square is of the form required by Proposition 3.2.25. In particular considering the object $\mathrm{id}_{\mathcal{J}}$ in $\mathrm{Fun}(\mathcal{J}, \mathcal{J})$ we obtain an adjunction

$$\Gamma(\epsilon^*)R^*\colon \mathrm{Fun}_L(\mathcal{I}, \mathrm{Un}^{\mathrm{co}}(F)) \rightleftarrows \mathrm{Fun}_{\mathcal{J}}(\mathcal{J}, \mathrm{Un}^{\mathrm{co}}(F)) : L^*,$$

where Γ is the functor associated to the cocartesian fibration p_*. Note that the right-hand side is equal to sections of $\mathrm{Un}^{\mathrm{co}}(F)$ while the left-hand side is equivalent to sections of $\mathrm{Un}^{\mathrm{co}}(FL)$. These are equivalent to $\operatorname{laxlim} F$ and $\operatorname{laxlim} FL$ respectively, and so we conclude the first statement. The second statement follows from an analysis of the functor $\Gamma(\epsilon^*)R^*$, as in Proposition 3.2.29. □

Remark 3.2.31. Similarly to before one can show that the left adjoint to L^* sends an object of the form

$$\{i \mapsto Y_i \in FL(i), \quad \alpha \mapsto f_\alpha\colon FL(\alpha)Y_i \to Y_{i'}\}$$

in $\operatorname{laxlim} FL$ to an object of the form

$$\{j \mapsto F(\epsilon_j)Y_{R(j)} \in F(j), \quad \beta \mapsto \left[F(\beta)F(\epsilon_i)Y_{R(j)} \simeq F(\epsilon_{j'})F(LR(\beta))Y_{R(j)} \xrightarrow{F(\epsilon_{j'})f_{L(\beta)}} F(\epsilon_{j'})Y_{L(i')}\right]\}.$$

in $\operatorname{laxlim} F$.

3.3 RELATIVE COCOMPLETION AND GLOBALIZATION

We fix an orbital pair (T, S) for the remainder of the section. Note that there is an obvious forgetful functor

$$\mathrm{fgt}\colon \mathsf{Pr}^{\mathrm{L}}_T \to \mathsf{Pr}^S_T$$

which exhibits $\mathsf{Pr}^{\mathrm{L}}_T$ as a non-full subcategory of Pr^S_T. In this section we will construct the *S-relative cocompletion* $\mathcal{P}^T_S(\mathcal{C})$ of an S-presentable T-category $\mathcal{C}$, and exhibit $\mathcal{P}^T_S$ as a left adjoint to $\mathrm{fgt}\colon \mathsf{Pr}^{\mathrm{L}}_T \to \mathsf{Pr}^S_T$.

Definition 3.3.1. We define a functor

$$\mathcal{P}_S^T : \mathrm{Cat}_T \to \mathrm{Fun}(\mathbb{F}_T^{\mathrm{op}}, \mathrm{Cat})$$

via the assigment $\mathcal{P}_S^T(\mathcal{C})_X = \mathrm{laxlim}^{\dagger}(\pi_X^*\mathcal{C})$, where $\pi_X^*\mathcal{C}$ denotes the functor

$$\pi_X^*\mathcal{C} \colon (\mathbb{F}_{T/X})^{\mathrm{op}} \xrightarrow{\pi_X} \mathbb{F}_T^{\mathrm{op}} \xrightarrow{\mathcal{C}} \mathrm{Cat}$$

and an edge in $(\mathbb{F}_{T/X})^{\mathrm{op}}$ is marked if and only if its projection to $\mathbb{F}_T^{\mathrm{op}}$ lands in $\mathbb{F}_S^{\mathrm{op}}$. The functoriality of this assignment in $\mathbb{F}_T^{\mathrm{op}}$ is induced by the contravariant functoriality of partially lax limits applied to the postcomposition functoriality of the slices $\mathbb{F}_{T/X}$.

Remark 3.3.2. Note that there is a functor

$$(\mathbb{F}_{T/-})^{\mathrm{op}} \colon \mathbb{F}_T \to \mathrm{Cat}^{\dagger}_{/\mathbb{F}_T^{\mathrm{op}}}, \quad X \mapsto (\mathbb{F}_{T/X})^{\mathrm{op}},$$

where $(\mathbb{F}_{T/X})^{\mathrm{op}}$ is marked by the subcategory of edges whose projection to $\mathbb{F}_T^{\mathrm{op}}$ lies in $\mathbb{F}_S^{\mathrm{op}}$. Then one can equivalently define $\mathcal{P}_S^T(\mathcal{C})$ as the composite of $(\mathbb{F}_{T/-})^{\mathrm{op}}$ and the contravariant functor

$$\mathrm{Fun}^{\dagger}_{\mathbb{F}_T^{\mathrm{op}}}(-, \mathrm{Un}^{\mathrm{co}}(\mathcal{C})) \colon \mathrm{Cat}^{\dagger}_{/\mathbb{F}_T^{\mathrm{op}}} \to \mathrm{Cat},$$

where $\mathrm{Un}^{\mathrm{co}}(\mathcal{C})$ is marked as usual by the cocartesian edges living over edges of $\mathbb{F}_S^{\mathrm{op}}$.

Remark 3.3.3. Consider $X \in \mathbb{F}_T$ and note that $\mathbb{F}_{T/X}$ is the finite coproduct completion of $T_{/X}$, the category of elements of $X \in \mathrm{PSh}(T)$. Therefore by Proposition 3.2.6 we obtain that

$$\mathcal{P}_S^T(\mathcal{C})_X \simeq \mathrm{laxlim}^{\dagger}((T_{/X})^{\mathrm{op}} \to T^{\mathrm{op}} \xrightarrow{\mathcal{C}} \mathrm{Cat}).$$

We will make use of all three descriptions of $\mathcal{P}_S^T(\mathcal{C})_X$ throughout this section. Let us begin by showing that $\mathcal{P}_S^T$ restricts to a functor from S-presentable T-categories to T-presentable T-categories.

Lemma 3.3.4. *The functor $\mathcal{P}_S^T$ factors through* $\mathrm{Cat}_T \subset \mathrm{Fun}(\mathbb{F}_T^{\mathrm{op}}, \mathrm{Cat})$.

Proof. Note that $T_{/\coprod X_i} = \coprod T_{/X_i}$. Therefore Remark 3.2.8, together with the previous remark, implies the desired result. $\square$

Remark 3.3.5. Recall that Cat_T is canonically a 2-category. We observe that $\mathcal{P}_S^T$ is a functor of 2-categories. This is easily seen from the description of Remark 3.3.2.

Theorem 3.3.6. *$\mathcal{P}_S^T$ restricts to a functor*

$$\mathcal{P}_S^T \colon \mathsf{Pr}_T^S \to \mathsf{Pr}_T^{\mathrm{L}}.$$

Proof. Let $\mathcal{C}$ be an S-presentable T-category. We have to show that $\mathcal{P}_S^T(\mathcal{C})$ is a T-presentable T-category. As a first step we show that $\mathcal{P}_S^T(\mathcal{C}) \colon \mathbb{F}_T^{\mathrm{op}} \to \mathrm{Cat}$ factors through Pr^{L}. We first note that by the proof of [Lur09, Proposition 5.5.3.17] each category $\mathcal{P}_S^T(\mathcal{C})_X$ is presentable. By Proposition 3.2.16 we conclude that colimits in $\mathrm{laxlim}^\dagger\, \pi_X^*\mathcal{C}$ are computed fiberwise for all $X \in \mathbb{F}_T$. In particular the restriction functors

$$\mathrm{laxlim}^\dagger\, \pi_X^*\mathcal{C} \to \mathrm{laxlim}^\dagger\, \pi_Y^*\mathcal{C}$$

clearly preserve colimits. By the adjoint functor theorem, proven as [Lur09, Corollary 5.5.2.9], $f^* \colon \mathcal{P}_S^T(\mathcal{C})_Y \to \mathcal{P}_S^T(\mathcal{C})_X$ admits a right adjoint, and so $\mathcal{P}_S^T(\mathcal{C})$ factors through Pr^{L}.

Next we show that the functors $f^* \colon \mathcal{P}_S^T(\mathcal{C})_Y \to \mathcal{P}_S^T(\mathcal{C})_X$ admit left adjoints for all morphisms $f \colon X \to Y$ in $\mathbb{F}_T$, and that the squares in Definition 3.1.11(3) are left adjointable. However by Proposition 3.2.14 the functors f^* also preserves limits for every $f \colon X \to Y$ in $\mathbb{F}_T$, and so another application of the adjoint functor theorem implies that it admits a left adjoint. Therefore all that remains is to prove that the required squares are left adjointable. To do this we will explicitly describe the right adjoint of f^*. First we observe that because T is orbital, the functor

$$(f_!)^{\mathrm{op}} \colon (\mathbb{F}_{T/X})^{\mathrm{op}} \to (\mathbb{F}_{T/Y})^{\mathrm{op}}$$

has a left adjoint $(f^*)^{\mathrm{op}}$ given by pulling back, and that both $(f_!)^{\mathrm{op}}$ and $(f^*)^{\mathrm{op}}$ preserve marked edges. Furthermore by the pasting law for pullbacks the square

$$\begin{tikzcd} Z \times_Y X \arrow[r, "\pi_1"] \arrow[d, "g \times_Y X"'] & Z \arrow[d, "g"] \\ Z' \times_Y X \arrow[r, "\pi_1"] & Z' \end{tikzcd}$$

is a pullback square in $\mathbb{F}_T$ for all $g \colon Z \to Z'$ in $\mathbb{F}_T$. This implies that the square

$$\begin{tikzcd} \mathcal{C}_Z \arrow[r, "(\pi_1)^*"] \arrow[d, "g^*"'] & \mathcal{C}_{Z \times_Y X} \arrow[d, "(g \times_Y X)^*"] \\ \mathcal{C}_{Z'} \arrow[r, "(\pi_1)^*"] & \mathcal{C}_{Z' \times_Y X} \end{tikzcd}$$

is left adjointable whenever g is in $\mathbb{F}_S$ because $\mathcal{C}$ is S-presentable. Therefore Proposition 3.2.29 gives an explicit description of the right adjoint f_* of the restriction functor

$$f^* \colon \mathrm{laxlim}^\dagger(\pi_Y^*\mathcal{C}) \to \mathrm{laxlim}^\dagger(\pi_X^*\mathcal{C}).$$

Informally, f_* sends the object

$$\{h\colon Z \to X \mapsto C_h \in \mathcal{C}_Z, \quad [g\colon h \to h'] \mapsto \lambda_g\colon g^*C_{h'} \to C_h\}$$

to the object

$$\begin{aligned}\{h\colon Z \to Y \mapsto (\pi_1)_* C_{h\times_Y X} \in \mathcal{C}_Z,\\ [g\colon h \to h'] \mapsto \big[g^*(\pi_1)_* C_{h'\times_Y X} \xrightarrow{BC} (\pi_1)_*(g \times_X Y)^* C_{h'\times_Y X} \xrightarrow{(\pi_1)_*\lambda_{g\times_X Y}} (\pi_1)_* Y_{h\times_X Y}\big]\}.\end{aligned}$$

We can now show that the required squares are left adjointable: Given a pullback square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"'] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

in $\mathbb{F}_T$ it suffices by passing to total mates to prove that the Beck-Chevalley transformation filling the square

$$\begin{tikzcd} \mathcal{P}^T_S(\mathcal{C})_{X'} \arrow[r, "(g')_*"] & \mathcal{P}^T_S(\mathcal{C})_X \\ \mathcal{P}^T_S(\mathcal{C})_{Y'} \arrow[r, "g_*"] \arrow[u, "(f')^*"] \arrow[ur, phantom, "\Nwarrow"] & \mathcal{P}^T_S(\mathcal{C})_Y \arrow[u, "f^*"'] \end{tikzcd}$$

is an equivalence. However unwinding the definition of the Beck–Chevalley transformation we find that on a section $s\colon (\mathbb{F}_{T/Y})^{\mathrm{op}} \to \mathrm{Un}^{\mathrm{co}}(\pi_Y^*\mathcal{C})$ it is given at $h\colon Z \to X$ by applying $(\pi_1)_*$ to the map $\beta^*(s(fh \times_Y Y')) \xrightarrow{\sim} s(f' \circ (h \times_X X'))$ induced by the base-change equivalence $\beta\colon fh \times_Y Y' \xrightarrow{\sim} f' \circ (h \times_X X')$, i.e. the morphism witnessing the equivalence of $fh \times_Y Y'$ and $f' \circ (h \times_X X')$ in the slice $\mathbb{F}_{T/Y}$. In particular the Beck-Chevalley transformation is an equivalence. Altogether we have shown that $\mathcal{P}^T_S(\mathcal{C})$ is an object of $\mathsf{Pr}^{\mathrm{L}}_T$.

Next we will show that $\mathcal{P}^T_S$ sends S-cocontinuous functors to T-cocontinuous functors. To this end fix a functor $L\colon \mathcal{C} \to \mathcal{D}$ in Pr^S_T and write $\mathbb{L}\colon \mathrm{Un}^{\mathrm{co}}(\mathcal{C}) \to \mathrm{Un}^{\mathrm{co}}(\mathcal{D})$ for its unstraightening. Because the naturality squares in L are left adjointable for maps in $\mathbb{F}_S$, Proposition 3.2.22 implies that $\mathbb{L}$ admits a right adjoint $\mathbb{R}\colon \mathrm{Un}^{\mathrm{co}}(\mathcal{D}) \to \mathrm{Un}^{\mathrm{co}}(\mathcal{C})$ in $\mathrm{Cat}_{/\mathcal{I}}$ which preserves cocartesian edges over $\mathbb{F}_S$. Now consider the description of $\mathcal{P}^T_S(-)$ from Remark 3.3.2. From this it is clear that postcomposition $\mathbb{R}$ gives a T-functor $\mathcal{P}^T_S(\mathcal{D}) \to \mathcal{P}^T_S(\mathcal{C})$ which is right adjoint to $\mathcal{P}^T_S(L)$. Therefore we conclude that $\mathcal{P}^T_S(L)$ is a T-cocontinuous functor, see Remark 3.1.13. In total we have shown that $\mathcal{P}^T_S\colon \mathsf{Pr}^S_T \to \mathrm{Cat}_T$ restricts to the subcategory $\mathsf{Pr}^{\mathrm{L}}_T$. $\square$

Having shown that $\mathcal{P}^T_S$ restricts appropriately, we now turn to showing that it is left adjoint to the forgetful functor $\mathrm{fgt}\colon \mathsf{Pr}^S_T \to \mathsf{Pr}^{\mathrm{L}}_T$. To do this we define the unit and counit of the putative adjunction.

Construction 3.3.7. Let $\mathcal{C} \in \mathsf{Pr}^S_T$ be an S-presentable T-category. We define a T-functor

$$I\colon \mathcal{C} \to \mathcal{P}^T_S(\mathcal{C})$$

as follows. First observe that because each category $(\mathbb{F}_{T/X})^{\mathrm{op}}$ admits a final object we obtain a natural equivalence $\lim \pi^*_X\mathcal{C} \simeq \mathcal{C}_X$, given by evaluating at the final object. After identifying these two categories, we claim that including the limit into the partially lax limit $\lim \pi^*_X\mathcal{C} \to \mathrm{laxlim}^\dagger\, \pi^*_X\mathcal{C}$ gives a natural S-cocontinuous T-functor $I\colon \mathcal{C} \to \mathcal{P}^T_S(\mathcal{C})$.

To see that I is in fact S-cocontinuous we first note that by Proposition 3.2.30, each functor I_X admits a right adjoint given by evaluating an object $s\colon \mathbb{F}_{T/X} \to \mathrm{Un}^{\mathrm{co}}(\pi^*_X\mathcal{C})$ of $\mathrm{laxlim}^\dagger\, \pi^*_X\mathcal{C}$ at the object $\mathrm{id}_X\colon X \to X$ in $\mathbb{F}_{T/X}$. Next we consider the left adjointability of naturality squares for maps in $\mathbb{F}_S$. By passing to total mates, it suffices to show that given any map $f\colon X \to Y$ in $\mathbb{F}_T$, the Beck–Chevalley transformation filling the square

$$\begin{tikzcd}
\mathcal{P}^T_S(\mathcal{C})_X \arrow[r, "\mathrm{ev}_{\mathrm{id}_X}"] & \mathcal{C}_X \\
\mathcal{P}^T_S(\mathcal{C})_Y \arrow[u, "\alpha^*"] \arrow[r, "\mathrm{ev}_{\mathrm{id}_Y}"'] \arrow[ur, phantom, "\Nwarrow"] & \mathcal{C}_Y \arrow[u, "\alpha^*"']
\end{tikzcd}$$

is an equivalence. One can compute that it is given at $s\colon \mathbb{F}_{T/Y} \to \mathrm{Un}^{\mathrm{co}}(F)$ by the lax structure map $s_\alpha\colon \alpha^*s(\mathrm{id}) \to s(\alpha)$. Because the objects of $\mathcal{P}^T_S(\mathcal{C})_X$ are strict on $\mathbb{F}_S \subset \mathbb{F}_T$, we conclude that this is an equivalence whenever α is a map in $\mathbb{F}_S$. We conclude that I is a morphism in Pr^S_T.

Construction 3.3.8. Next we construct the counit of the desired adjunction. Given an object $\mathcal{C}$ in $\mathsf{Pr}^{\mathrm{L}}_T$ we have to construct a functor

$$L\colon \mathcal{P}^T_S(\mathcal{C}) \to \mathcal{C}$$

in $\mathsf{Pr}^{\mathrm{L}}_T$. We will do this by showing that the functor $I\colon \mathcal{C} \to \mathcal{P}^T_S(\mathcal{C})$ has a parametrized left adjoint when $\mathcal{C}$ is T-presentable. To do this we first observe that by Proposition 3.2.14, I_X preserves all limits. Therefore each functor I_X has a left adjoint L_X by another application of the adjoint functor theorem.

To show that I in fact has a left adjoint as a T-functor, it now suffices by [MW21, Lemma 3.2.7] to show that the Beck–Chevalley transformation

$$\begin{tikzcd}
\mathcal{C}_X \arrow[r, "I_X"] \arrow[d, "f_*"'] & \mathcal{P}^T_S(\mathcal{C})_X \arrow[d, "f_*"] \\
\mathcal{C}_Y \arrow[r, "I_Y"'] \arrow[ur, phantom, "\Nearrow"] & \mathcal{P}^T_S(\mathcal{C})_Y
\end{tikzcd}$$

filling the square above is an equivalence for all $f\colon X \to Y$ in $\mathbb{F}_T$. Unwinding the definition of the Beck–Chevalley transformation, we find that it is given at $h\colon Z \to Y$ by the Beck–Chevalley transformation filling the square

$$\begin{tikzcd} \mathcal{C}_X \arrow[r, "(h\times_Y X)^*"] \arrow[d, "f_*"'] \arrow[dr, phantom, "\nearrow"] & \mathcal{C}_{X\times_Y Z} \arrow[d, "(\pi_1)_*"] \\ \mathcal{C}_Y \arrow[r, "h^*"'] & \mathcal{C}_Z \end{tikzcd}$$

This is an equivalence because $\mathcal{C}$ is an object of $\mathsf{Pr}^{\mathrm{L}}_T$ and so satisfies base-change with respect to all pullback squares in $\mathbb{F}_T$. In total we conclude that L is a functor in $\mathsf{Pr}^{\mathrm{L}}_T$.

We will prove that $\mathcal{P}^T_S$ is a left adjoint by showing that both triangles identities hold for the putative unit and counit. To do this it will be useful to have a different description of the composite $\mathcal{P}^T_S\mathcal{P}^T_S(\mathcal{C})$.

Lemma 3.3.9. *There exists an equivalence*

$$\mathcal{P}^T_S(\mathcal{P}^T_S(\mathcal{C}))_X \simeq \underset{\mathrm{Ar}(\mathbb{F}_{T/X})^{\mathrm{op}}}{\mathrm{laxlim}^\dagger}\, \pi_X^*\mathcal{C} \circ s,$$

where $s\colon \mathrm{Ar}(\mathbb{F}_{T/X})^{\mathrm{op}} \to (\mathbb{F}_{T/X})^{\mathrm{op}}$ *is the source projection and* $\mathrm{Ar}(\mathbb{F}_{T/X})^{\mathrm{op}}$ *is marked by those natural transformations for which both maps are in* $\mathbb{F}_S$*. Moreover this equivalence is natural in both* $\mathcal{C}$ *and* X*.*

Proof. This result follows immediately from Proposition 3.2.7, combined with the fact that by definition $\mathrm{Ar}(\mathbb{F}_{T/X})^{\mathrm{op}} \to (\mathbb{F}_{T/X})^{\mathrm{op}}$ is the cocartesian unstraightening of the slice functor $(\mathbb{F}_{T/X})^{\mathrm{op}}_{/\bullet}$. □

Theorem 3.3.10. *The functor* $\mathcal{P}^T_S\colon \mathsf{Pr}^S_T \to \mathsf{Pr}^{\mathrm{L}}_T$ *is left adjoint to* $\mathrm{fgt}\colon \mathsf{Pr}^{\mathrm{L}}_T \to \mathsf{Pr}^S_T$*.*

Proof. We show that I and L satisfy the triangle identities, and so are a unit and counit exhibiting $\mathcal{P}^T_S \dashv \mathrm{fgt}$ as an adjunction. First we consider the composite

$$\mathcal{C} \xrightarrow{I} \mathcal{P}^T_S(\mathcal{C}) \xrightarrow{L} \mathcal{C}.$$

Recall that I_X is a fully faithful right adjoint to L_X. Therefore the counit gives a natural equivalence from the composite to the identity. For the other triangle identity we consider the composite

$$\mathcal{P}^T_S(\mathcal{C}) \xrightarrow{\mathcal{P}^T_S(I)} \mathcal{P}^T_S(\mathcal{P}^T_S(\mathcal{C})) \xrightarrow{L} \mathcal{P}^T_S(\mathcal{C}).$$

We may equivalently show that the composite

$$\mathcal{P}^T_S(\mathcal{C})_X \xleftarrow{\mathcal{P}^T_S(\mathrm{ev}_{\mathrm{id}})} \mathcal{P}^T_S(\mathcal{P}^T_S(\mathcal{C}))_X \xleftarrow{I} \mathcal{P}^T_S(\mathcal{C})_X$$

given by passing to right adjoints is naturally homotopic to the identity. At this point we apply Lemma 3.3.9 to rewrite $\mathcal{P}^T_S(\mathcal{P}^T_S(\mathcal{C}))_X$ as the partially lax limit of $\pi^*_X\mathcal{C} \circ s$ over $\mathrm{Ar}(\mathbb{F}_{T/X})^{\mathrm{op}}$. One can easily show that under this identification I and $\mathcal{P}^T_S(\mathrm{ev}_{\mathrm{id}})$ are given by restricting along the source projection $s\colon \mathrm{Ar}(\mathbb{F}_{T/X})^{\mathrm{op}} \to (\mathbb{F}_{T/X})^{\mathrm{op}}$ and the identity section $c\colon (\mathbb{F}_{T/X})^{\mathrm{op}} \to \mathrm{Ar}(\mathbb{F}_{T/X})^{\mathrm{op}}$ respectively. Therefore there is a natural equivalence between the composite $I \circ \mathcal{P}^T_S(\mathrm{ev}_{\mathrm{id}})$ and restriction along $s \circ c = \mathrm{id}$. □

As an example of the process of freely adding colimits we obtain the following result:

Corollary 3.3.11. $\mathcal{P}^T_S(\mathcal{S}^S_\bullet) \simeq \mathcal{S}^T_\bullet$.

Proof. This follows immediately by comparing universal properties: by Example 3.1.16 both $\mathcal{P}^T_S(\mathcal{S}^S_\bullet)$ and $\mathcal{S}^T_\bullet$ represent the functor $\mathcal{C} \mapsto \mathrm{core}(\Gamma\mathcal{C})$. □

We note that the proof of the previous theorem did not require any knowledge about the left adjoint of $f^*\colon \mathcal{P}^T_S(\mathcal{C})_Y \to \mathcal{P}^T_S(\mathcal{C})_X$, beyond their existence. In fact the author does not know a general explicit description of the left adjoint of $f^*\colon \mathcal{P}^T_S(\mathcal{C})_Y \to \mathcal{P}^T_S(\mathcal{C})_X$ for an arbitrary map $f\colon X \to Y$. Nevertheless, we now show that when f is in $\mathbb{F}_S$ such a description is in fact possible. Therefore, rather fittingly, it is only the left adjoints which we have freely adjoined which will remain mysterious. For this we require the concept of marked finality.

Definition 3.3.12. A functor $F\colon \mathcal{I} \to \mathcal{J}$ of marked categories is marked final if for every functor $G\colon \mathcal{J} \to \mathrm{Cat}$, restriction along F induces an equivalence

$$\underset{\mathcal{J}}{\mathrm{laxlim}}^\dagger\, G \to \underset{\mathcal{I}}{\mathrm{laxlim}}^\dagger\, GF.$$

The following criteria allows us to recognize marked final functors. Before stating it we recall some notation. Given a functor $F\colon \mathcal{I} \to \mathcal{J}$ and an object $j \in \mathcal{J}$, we write $\mathcal{I}_{/j}$ for the comma category $F \downarrow \{j\}$. If $\mathcal{J}$ is marked, then we enhance this to a marked category by marking all the edges whose projection to $\mathcal{J}$ is marked. Furthermore given a marked category $\mathcal{J}$ we write $\mathcal{L}(\mathcal{J})$ for the (Dwyer–Kan) localization of $\mathcal{J}$ at the marked edges.

Proposition 3.3.13 ([AG22, Proposition 5.6$^{\mathrm{op}}$])**.** *$F\colon \mathcal{I} \to \mathcal{J}$ is marked final if and only if for all $j \in \mathcal{J}$ the canonical functor $F\colon \mathcal{I}_{/j} \to \mathcal{J}_{/j}$ induces an equivalence*

$$\mathcal{L}(\mathcal{I}_{/j}) \xrightarrow{\sim} \mathcal{L}(\mathcal{J}_{/j})$$

after localization. □

Proposition 3.3.14. *Let $\mathcal{I}$ and $\mathcal{J}$ be marked categories. Suppose $L\colon \mathcal{I} \rightleftarrows \mathcal{J} :R$ is an adjoint pair such that both L and R preserve the marking. Then the following are equivalent:*

1. *The unit $\eta\colon i \to RL(i)$ is marked for all $i \in \mathcal{I}$;*
2. *The counit $\epsilon\colon LR(j) \to j$ is marked for all $j \in \mathcal{J}$;*
3. *The adjunction equivalence*

$$\mathrm{Hom}_{\mathcal{I}}(L(i), j) \simeq \mathrm{Hom}_{\mathcal{J}}(i, R(j))$$

preserves marked morphisms.

Proof. Since identities are always marked, (3) clearly implies (1) and (2). Let us now show that (3) implies (1). Recall that the adjunction equivalence is given by the following composite

$$\mathrm{Hom}_{\mathcal{I}}(L(i), j) \to \mathrm{Hom}_{\mathcal{J}}(RL(i), R(j)) \xrightarrow{\eta^*} \mathrm{Hom}_{\mathcal{J}}(i, R(j))$$

The first map preserves marked morphisms because R was assumed to be a marked functor, and the second because the unit is marked and marked morphisms form a subcategory. That (3) implies (2) is similar. □

Definition 3.3.15. We say $L\colon \mathcal{I} \rightleftarrows \mathcal{J} :R$ is a marked adjunction if the equivalent conditions of the previous proposition holds.

Proposition 3.3.16. *Let $L\colon \mathcal{I} \rightleftarrows \mathcal{J} :R$ be a marked adjunction, then L is marked final.*

Proof. Let $j \in \mathcal{J}$ and consider the functor

$$L\colon \mathcal{I}_{/j} \to \mathcal{J}_{/j}.$$

This admits a right adjoint given by sending $f\colon j' \to j$ to the pair

$$(R(j'), LR(j') \xrightarrow{f} LR(j) \xrightarrow{\epsilon} j).$$

One computes that the unit and counit are given by the maps

$$\eta\colon i \to RL(i) \quad \text{and} \quad \epsilon\colon LR(i) \to i.$$

respectively. In particular both are marked by Proposition 3.3.14. We conclude that after localizing at the marked morphisms this adjunction is an adjoint equivalence, and so we conclude by Proposition 3.3.13. □

Proposition 3.3.17. *Let (T, S) be an orbital pair and let $f\colon X \to Y$ be a map in $\mathbb{F}_S$. Then*

$$f_!\colon \mathbb{F}_{T/X} \rightleftarrows \mathbb{F}_{T/Y} :f^*$$

is a marked adjunction, where as always both categories are marked by those morphisms which lie in $\mathbb{F}_S$.

Proof. We have previously observed already that both $f_!$ and f^* preserve marked edges. The counit of the adjunction is given on an object $Z \to Y$ by the map π_1 in the pullback square

$$\begin{tikzcd} X \times_Y Z \arrow[r, "\pi_1"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

In particular as a pullback of f it is again in $\mathbb{F}_S$. □

Construction 3.3.18. We will now give a description of the left adjoint of the restriction functor $f^*\colon \mathcal{P}^T_S(\mathcal{C})_Y \to \mathcal{P}^T_S(\mathcal{C})_X$ when f is in $\mathbb{F}_S$. First we note that to simplify notation we may pass to slices and assume that Y is the final object. Now recall that the restriction functor $f^*\colon \mathcal{P}^T_S(\mathcal{C})_Y \to \mathcal{P}^T_S(\mathcal{C})_X$ is given by the functor

$$(f_!)^*\colon \underset{\mathbb{F}_T^{\mathrm{op}}}{\mathrm{laxlim}^\dagger}\, \mathcal{C} \to \underset{(\mathbb{F}_{T/X})^{\mathrm{op}}}{\mathrm{laxlim}^\dagger}\, f^*\mathcal{C}.$$

To understand the left adjoint of this functor we may postcompose by the functor

$$(f^*)^*\colon \underset{(\mathbb{F}_{T/X})^{\mathrm{op}}}{\mathrm{laxlim}^\dagger}\, f^*\mathcal{C} \to \underset{\mathbb{F}_T^{\mathrm{op}}}{\mathrm{laxlim}^\dagger}\, f_*f^*\mathcal{C}, \tag{3.3.18.1}$$

which is an equivalence by combining Propositions 3.3.16 and 3.3.17, and instead construct a left adjoint of the composite $(f_!f^*)^*$. This functor can again be reinterpreted. Note that the counit transformation $\epsilon\colon f_!f^* \Rightarrow \mathrm{id}$ induces a natural transformation from the identity on $\mathrm{Fun}^\dagger_{\mathbb{F}_T^{\mathrm{op}}}(\mathrm{id}, p)$ to $(f_!f^*)^*$. Evaluating this natural transformation on a section s in $\mathrm{laxlim}^\dagger\, \mathcal{C}$ we find that γ is pointwise cocartesian: at an object $Z \in \mathbb{F}_T$, the natural transformation $\gamma\colon s \to s \circ f_!f^*$ is given by applying s to the map $\pi_1\colon X \times_Y Z \to Z$, which as a pullback of f is in $\mathbb{F}_S$. This implies that when restricted to $\mathrm{laxlim}^\dagger\, \mathcal{C}$, the functor $(f_!f^*)^*$ is naturally equivalent to cocartesian pushforward along the counit $\epsilon\colon f_!f^* \Rightarrow \mathrm{id}$. However this is in turn equivalent to postcomposition by the functor $\underline{f}^*\colon \mathrm{Un}^{\mathrm{co}}(\mathcal{C}) \to \mathrm{Un}^{\mathrm{co}}(f_*f^*\mathcal{C})$. We conclude that $f^*\colon \mathcal{P}^T_S(\mathcal{C})_Y \to \mathcal{P}^T_S(\mathcal{C})_X$ is homotopic to the functor

$$\underset{\mathbb{F}_T^{\mathrm{op}}}{\mathrm{laxlim}^\dagger}\, \mathcal{C} := \mathrm{Fun}_{\mathbb{F}_T^{\mathrm{op}}}(\mathbb{F}_T^{\mathrm{op}}, \mathrm{Un}^{\mathrm{co}}(\mathcal{C})) \xrightarrow{(\underline{f}^*)_*} \mathrm{Fun}_{\mathbb{F}_T^{\mathrm{op}}}(\mathbb{F}_T^{\mathrm{op}}, \mathrm{Un}^{\mathrm{co}}(f_*f^*\mathcal{C})) \simeq \underset{\mathbb{F}_T^{\mathrm{op}}}{\mathrm{laxlim}^\dagger}\, f_*f^*\mathcal{C}.$$

In this form, we can make the left adjoint of the functor explicit. Namely, the functor $\underline{f}^*\colon \mathrm{Un}^{\mathrm{co}}(\mathcal{C}) \to \mathrm{Un}^{\mathrm{co}}(f_*f^*\mathcal{C})$ iteslf admits a relative left adjoint $\underline{f}_!$ given by the unstraightening of the parametrized left adjoint from Remark 3.1.14. By Proposition 3.2.22, postcomposition by $\underline{f}_!$ defines a left adjoint to $\underline{f}^*$.

3.3.1 Globalization

We are most interested in the previous results when T is Glo and S is Orb. For example we immediately obtain a proof of Theorem A.

Theorem 3.3.19. *There is an equivalence $\mathcal{P}^{\mathrm{Glo}}_{\mathrm{Orb}}(\mathcal{S}_\bullet) \simeq \mathcal{S}_{\bullet\text{-gl}}$ of global categories. In particular evaluating this at $\mathbf{B}G$ we obtain an equivalence*

$$\operatorname*{laxlim}^{\dagger}_{(\mathrm{Glo}_{/G})^{\mathrm{op}}} \mathcal{S}_\bullet \simeq \mathcal{S}_{G\text{-gl}}.$$

Proof. This follows immediately from Corollary 3.3.11, after making the identifications of Example 3.1.17. □

We have shown that the Orb-relative cocompletion of the global category of equivariant spaces is given by the global category of globally equivariant spaces. In other words, in this case $\mathcal{P}^{\mathrm{Glo}}_{\mathrm{Orb}}$ sends a global category of "equivariant objects" to a global category of "globally equivariant objects". Another example of this phenomena is given by equivariant spectra, whose relative cocompletion is given by globally equivariant spectra as we will show. For this reason we introduce the following notation.

Notation 3.3.20. We will refer to $\mathcal{P}^{\mathrm{Glo}}_{\mathrm{Orb}}(\mathcal{C})$ as the *globalization* of $\mathcal{C}$ and denote it by $\mathrm{Glob}(\mathcal{C})$.

4. Stabilization

4.1 EQUIVARIANT STABILITY

We will now lead up to a proof of Theorem D of the introduction. To do this we begin by recalling the notion of P-semiadditivity and P-stability for T-categories introduced in [CLL23a]. When $(T, P) = (\mathrm{Glo}, \mathrm{Orb})$ we obtain the notions of equivariant semiadditivity and equivariant stability for global categories. We then recall the main results of [CLL23a; CLL23b], which identify the universal globally presentable and equivariantly presentable equivariantly stable global categories with globally equivariant spectra and equivariant spectra respectively.

Finally, as the new results of this section, we show that for a orbital pair (T, S), $\mathcal{P}_S^T(-)$ preserves P-semiadditivity and P-stability whenever P is a subcategory of S. Combining this with the main results of [CLL23a; CLL23b] we conclude Theorem D, which identifies the global category of globally equivariant spectra as the globalization of the global category of equivariant spectra.

4.1.1 Recollection

We begin with a recollection of the relevant material from [CLL23a].

Definition 4.1.1. An orbital subcategory $P \subset T$ is called an *atomic orbital subcategory* if every map in P that admits a section in T is an equivalence.

Throughout this section, we fix an orbital pair (T, S) and an atomic orbital subcategory P of T such that $P \subset S$.

Definition 4.1.2. We say an S-presentable T-category $\mathcal{C}$ is *pointed* if for all $X \in \mathbb{F}_T$, $\mathcal{C}_X$ is pointed. We define $\mathrm{Pr}^S_{T,*}$ to be the full subcategory of Pr^S_T spanned by the pointed global categories.

Construction 4.1.3. Let $\mathcal{C}$ be a pointed S-presentable T-category. For any map $p \colon A \to B$ in $\mathbb{F}_P \subset \mathbb{F}_T$, [CLL23a, Construction 4.6.1] defines an *adjoint norm map*

$$\overline{\mathrm{Nm}}_p \colon p^* p_! \Rightarrow \mathrm{id}.$$

Definition 4.1.4. A S-presentable T-category $\mathcal{C}$ is called *P-semiadditive* if it is pointed and the adjoint norm map $\overline{\mathrm{Nm}}_p \colon p^* p_! \Rightarrow \mathrm{id}$ is a counit transformation exhibiting p^* as a left adjoint of $p_!$ for every $p \in \mathbb{F}_P$.

Remark 4.1.5. This definition is equivalent to [CLL23a, Definition 4.5.1] by Lemma 4.5.4 of *op. cit.* By these results, one can furthermore show that an S-presentable T-category $\mathcal{C}$ is P-semiadditive if and only if it is pointed, and for all $p\colon X \to Y$ in $\mathbb{F}_P$ a natural transformation $\overline{\mathrm{Nm}}_p\colon \underline{p}^*\underline{p}_! \Rightarrow \mathrm{id}$, defined analogously to Construction 4.1.3, is a counit transformation exhibiting $\underline{p}_!\colon \mathrm{Un}^{\mathrm{co}}(f_*f^*\pi_Y^*\mathcal{C}) \to \mathrm{Un}^{\mathrm{co}}(\mathcal{C})$ as a right adjoint of $\underline{p}^*\colon \mathrm{Un}^{\mathrm{co}}(\mathcal{C}) \to \mathrm{Un}^{\mathrm{co}}(f_*f^*\pi_Y^*\mathcal{C})$.

Example 4.1.6. When $P \subset T$ equals $\mathrm{Orb}_G \subset \mathrm{Orb}_G$, the notion of semiadditivity obtained agrees with G-semiadditivity as defined in [Nar17], see [CLL23a, Proposition 4.6.4].

Definition 4.1.7. We write $\mathrm{Pr}^S_{T,P\text{-}\oplus}$ for the full subcategories of Pr^S_T spanned by the P-semiadditive T-categories.

We may additionally impose a fiberwise stability condition.

Definition 4.1.8. We say a S-presentable T-category $\mathcal{C}$ is *fiberwise stable* if $\mathcal{C}_X$ is stable for all $X \in \mathbb{F}_T$. We say a S-presentable T-category $\mathcal{C}$ is *P-stable* if it is P-semiadditive and fiberwise stable. We write $\mathrm{Pr}^S_{T,P\text{-st}}$ for the full subcategory of Pr^S_T spanned by the P-stable T-categories.

We also specialize the notions above to the setting of global categories.

Definition 4.1.9 ([CLL23a])**.** We say an equivariantly presentable global category $\mathcal{C}$ is equivariantly semiadditive or equivariantly stable if it is is Orb-semiadditive or Orb-stable respectively.

The main results of [CLL23a] and [CLL23b] allow us to identify the free equivariantly presentable and globally presentable equivariantly stable global categories on a point.

Definition 4.1.10. We define $\mathrm{Sp}_{\bullet\text{-gl}}$, the global category of *globally equivariant spectra*, to be diagram which sends $\mathbf{B}G$ to the category of *G-global spectra*, in the sense of [Len20]. This in turn is defined to be the localization of the category of symmetric G-spectra at the G-global weak equivalences. See [CLL23a, Section 7.1] for precise definitions.

Theorem 4.1.11 ([CLL23a, Theorem 7.3.2])**.** $\mathrm{Sp}_{\bullet\text{-gl}}$ *is the free globally presentable equivariantly stable global category on a point. That is, given any globally presentable equivariantly stable global category $\mathcal{C}$, evaluating at the global sphere spectrum $\mathbb{S}_{\mathrm{gl}} \in \mathrm{Sp}_{\mathrm{gl}}$ gives an equivalence*

$$\underline{\mathrm{Fun}}^{\mathrm{L}}(\mathrm{Sp}_{\bullet\text{-gl}}, \mathcal{C}) \simeq \mathcal{C},$$

where the left hand side denotes the global category of cocontinuous functors. Evaluating at $\mathbf{B}e$ and taking groupoid cores we obtain an equivalence

$$\mathrm{Hom}_{\mathsf{Pr}^{\mathrm{L}}_{\mathrm{Glo}}}(\mathrm{Sp}_{\bullet\text{-gl}}, \mathcal{C}) \simeq \mathrm{core}(\mathcal{C}(\mathbf{B}e)).$$

Similarly we have an equivariantly presentable version of the previous theorem.

Theorem 4.1.12 ([CLL23b, Theorem 9.4]). $\mathrm{Sp}_\bullet$ *is the free equivariantly presentable equivariantly stable global category on a point. That is, given any equivariantly presentable equivariantly stable global category $\mathcal{C}$, evaluation at the sphere spectrum $\mathbb{S} \in \mathrm{Sp}$ gives an equivalence*

$$\underline{\mathrm{Fun}}^{\text{eq-cc}}(\mathrm{Sp}_\bullet, \mathcal{C}) \simeq \mathcal{C},$$

where the left hand side denotes the global category of equivariantly cocontinuous functors. Evaluating at $\mathbf{B}e$ and taking groupoid cores we obtain an equivalence

$$\mathrm{Hom}_{\mathrm{Pr}^{\mathrm{Orb}}_{\mathrm{Glo}}}(\mathrm{Sp}_\bullet, \mathcal{C}) \simeq \mathrm{core}(\mathcal{C}(\mathbf{B}e)).$$

4.1.2 Globalizing equivariantly semiadditive and stable categories

We now show that relative cocompletion preserves P-semiadditivity and P-stability. We then conclude Theorem D.

Proposition 4.1.13. *Let (T,S) be an orbital pair and suppose P is an atomic orbital subcategory of T such that $P \subset S$. The functor $\mathcal{P}^T_S \colon \mathsf{Pr}^S_T \to \mathsf{Pr}^{\mathrm{L}}_T$ restricts to functors*

$$\mathrm{Pr}^S_{T,P\text{-}\oplus} \to \mathrm{Pr}^{\mathrm{L}}_{T,P\text{-}\oplus} \quad \text{and} \quad \mathrm{Pr}^S_{T,P\text{-st}} \to \mathrm{Pr}^{\mathrm{L}}_{T,P\text{-st}}.$$

Proof. Let $\mathcal{C}$ be a P-semiadditive S-presentable T-category. First we note that since limits and colimits are computed pointwise in $\mathcal{P}^T_S(\mathcal{C})_X$ for all $X \in \mathbb{F}_T$, it is again pointed. Now consider $p\colon X \to Y$ in $\mathbb{F}_P$. By passing to slices we may assume Y is the final object of $\mathbb{F}_T$. We have to show that the adjoint norm map $\overline{\mathrm{Nm}}_p \colon p^*p_! \Rightarrow \mathrm{id}$ is the counit of an adjunction. However recall that by Construction 3.3.18 the adjunction $p_! \dashv p^*$ can be identified up to equivalence with the adjunction

$$(\underline{p}_!)_* \colon \mathrm{laxlim}^\dagger\, \mathcal{C} \rightleftarrows \mathrm{laxlim}^\dagger\, \mathcal{C} \circ p_! p^* : (\underline{p}^*)_*.$$

Because $\mathcal{C}$ is P-semiadditive, by Remark 4.1.5 there exists a natural transformation $\overline{\mathrm{Nm}} \colon \underline{p}^*\underline{p}_! \Rightarrow \mathrm{id}$ which is the counit of an adjunction. By the 2-functoriality of $\mathrm{Fun}^\dagger_{\mathbb{F}_T}(\mathbb{F}_T, -)$, this induces a counit witnessing $(\underline{p}_!)_*$ as a right adjoint to $(\underline{p}^*)_*$. A tedious diagram chase shows that this transformation agrees, after applying suitable equivalences, with the adjoint norm map $\overline{\mathrm{Nm}}_p$ of $\mathcal{P}^T_S(\mathcal{C})$. We conclude that $\mathcal{P}^T_S(\mathcal{C})$ is P-semiadditive.

Finally we note that because colimits and limits in $\mathcal{P}^T_S(\mathcal{C})$ are computed pointwise, $\mathcal{P}^T_S$ clearly preserves fiberwise stable global categories. □

Applying this in the global context we obtain the parametrized analog of Theorem B.

Theorem 4.1.14. *There is an equivalence*

$$\mathrm{Sp}_{\bullet\text{-gl}} \simeq \mathrm{Glob}(\mathrm{Sp}_{\bullet}).$$

Proof. Note that by Proposition 4.1.13, $\mathrm{Glob}(\mathrm{Sp}_{\bullet})$ is again equivariantly stable. Therefore the result follows immediately from Theorem 4.1.11 and Theorem 4.1.12 by comparing universal properties. □

As an immediate corollary we obtain a description of G-global spectra in the sense of [Len20] as a partially lax limit for every finite group G.

Corollary 4.1.15. *Let G be a finite group. Then there is an equivalence*

$$\mathrm{Sp}_{G\text{-gl}} \simeq \operatorname*{laxlim^{\dagger}}_{(\mathrm{Glo}_{/G})^{\mathrm{op}}} \mathrm{Sp}_{\bullet}. \qquad \square$$

Remark 4.1.16. There are equivariantly semiadditive analogues of the statements above. To avoid testing the readers patience we summarize them in this remark. One can define the global categories $\Gamma\mathcal{S}^{\mathrm{spc}}_{\bullet\text{-gl}}$ and $\Gamma\mathcal{S}^{\mathrm{spc}}_{\bullet}$ of special global Γ-spaces and special equivariant Γ-spaces respectively. Evaluating these global categories at the groupoid $\mathbf{B}G$ one obtains the category of special G-global Γ-spaces and the category of special Γ-G-space, as defined in [Len20] and [Shi89] respectively.

By [CLL23a, Theorem 5.3.1], $\Gamma\mathcal{S}^{\mathrm{spc}}_{\bullet\text{-gl}}$ is the free globally presentable equivariantly semiadditive global category on a point, while $\Gamma\mathcal{S}^{\mathrm{spc}}_{\bullet}$ is the free equivariantly presentable equivariantly semiadditive on a point by [CLL23b, Theorem 7.17]. Therefore by comparing universal properties we obtain an equivalence

$$\Gamma\mathcal{S}^{\mathrm{spc}}_{\bullet\text{-gl}} \simeq \mathrm{Glob}(\Gamma\mathcal{S}^{\mathrm{spc}}_{\bullet})$$

of global categories. Evaluating at the groupoid $\mathbf{B}G$, we find that

$$\Gamma\mathcal{S}^{\mathrm{spc}}_{G\text{-gl}} \simeq \operatorname*{laxlim^{\dagger}}_{(\mathrm{Glo}_{/G})^{\mathrm{op}}} \Gamma\mathcal{S}^{\mathrm{spc}}_{\bullet}.$$

4.2 REPRESENTATION STABILITY

We now switch gears and prove a consequence of the fact that $\mathrm{Sp}_{\bullet\text{-gl}}$ is the globalization of $\mathrm{Sp}_{\bullet}$. Namely, in this section we show that $\mathrm{Sp}_{\bullet\text{-gl}}$ is the initial globally presentable global category on which the representation spheres act invertibly. This universal property of global spectra was first suggested by David Gepner and Thomas Nikolaus, see [Nik15]. An analogous universal property has since been proven in the context of global model categories by [LS23]. Our strategy for proving this universal property is first to observe that the condition that representation spheres act invertibly in fact already makes sense for pointed equivariantly presentable global categories, and therefore first consider the analogous question in this context.

Lemma 4.2.1. *Let $\mathcal{C}$ be an object of $\mathrm{Pr}^{\mathrm{Orb}}_{\mathrm{Glo}}$. Then $\mathcal{C}$ admits a unique colimit preserving tensoring by $\mathcal{S}_\bullet$. In particular there exists a canonical T-functor*

$$\mathcal{S}_\bullet \times \mathcal{C} \to \mathcal{C}$$

which preserves Orb*-colimits in each variable, in the sense of [MW22, Definition 8.1.1]. Similarly if $\mathcal{C}$ is in $\mathrm{Pr}^{\mathrm{Orb}}_{\mathrm{Glo},*}$, then it is uniquely tensored over the global category of pointed equivariant spaces $\mathcal{S}_{\bullet,*}$, defined by the assignment $\mathbf{B}G \mapsto \mathcal{S}_{G,*} := (\mathcal{S}_G)_*$.*

Proof. By [MW22, Corollary 8.2.5] the category of $\mathcal{S}_\bullet$-cocomplete global categories $\mathrm{Cat}^{\mathrm{Orb\text{-}cc}}_{\mathrm{Glo}}$ admits a symmetric monoidal structure $- \otimes -$ such that Orb-cocontinuous global functors $\mathcal{C} \otimes \mathcal{D} \to \mathcal{E}$ are equivalent to global functors $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ which are Orb-cocontinuous in each variable. Moreover by [MW22, Remark 8.2.6] $\mathcal{S}_\bullet$ is the unit of this category, and therefore every object in $\mathrm{Pr}^{\mathrm{Orb}}_{\mathrm{Glo}}$ inherits a tensoring by $\mathcal{S}_\bullet$. This shows the first statement. The second statement follows analogously to [Lur16, Proposition 4.8.2.11]. □

Remark 4.2.2. The tensoring of $\mathcal{C}$ by $\mathcal{S}_\bullet$ implies in particular that each category $\mathcal{C}_X$ is tensored by $\mathcal{S}_X$. Furthermore given a map $f\colon X \to Y$ in $\mathbb{F}_{\mathrm{gl}} := \mathbb{F}_{\mathrm{Glo}}$, the functor $f^*\colon \mathcal{C}_Y \to \mathcal{C}_X$ is canonically $\mathcal{S}_Y$-linear, where $\mathcal{C}_X$ is tensored over $\mathcal{S}_Y$ by restricting along the functor $f^*\colon \mathcal{S}_Y \to \mathcal{S}_X$.

An analogous statement holds for the tensoring of a pointed equivariantly presentable global categories $\mathcal{C}$ by $\mathcal{S}_{\bullet,*}$.

Remark 4.2.3. Now suppose that $f\colon X \to Y$ in $\mathbb{F}_{\mathrm{Orb}}$ is a faithful functor. Given a equivariantly presentable global category $\mathcal{C}$, the left adjoint $f_!\colon \mathcal{C}_X \to \mathcal{C}_Y$ of f^* canonically inherits the structure of an oplax $\mathcal{S}_Y$-linear functor. Contained in the statement that the tensoring preserves Orb-colimits in each variable is the fact that $f_!$ with this oplax $\mathcal{S}_Y$-linear structure is in fact a strong $\mathcal{S}_Y$-linear functor. Informally this means that the projection formula holds. Given this one can compute that the tensoring of $\mathcal{S}_G$ on $\mathcal{C}_G$ is given by colimit extending the assignment

$$\mathrm{Orb}_G \times \mathcal{C}_G \to \mathcal{C}_G, \quad (\iota\colon H \hookrightarrow G,\, C) \mapsto \iota_!\iota^*(C).$$

Once again an analogous statement holds for pointed equivariantly presentable global categories.

Definition 4.2.4. We define a parametrized subcategory $\mathbb{S}^{\mathrm{rep}}$ of $\mathcal{S}_\bullet$ by letting $\mathbb{S}^{\mathrm{rep}}_G$ be the full subcategory spanned by the representation spheres $S^V = V \cup \{\infty\}$, where V is any finite dimensional orthogonal representation of G.

Definition 4.2.5. We define $\mathrm{Pr}^{\mathrm{Orb}}_{\mathrm{Glo,rep\text{-}st}}$ to be the full subcategory of $\mathrm{Pr}^{\mathrm{Orb}}_{\mathrm{Glo},*}$ spanned by the objects $\mathcal{C}$ such that $S^V \otimes (-)\colon \mathcal{C}_G \to \mathcal{C}_G$ is an equivalence for all $\mathbf{B}G \in \mathrm{Glo}$ and all $S^V \in \mathbb{S}^{\mathrm{rep}}_G$. We call such global categories Rep-*stable*.

4.2.1 Formal Inversions

Given a equivariantly presentable global category $\mathcal{C}$ we would like to construct the initial global category under $\mathcal{C}$ which is Rep-stable. In other words, we would like to understand the process of (Rep-)stabilizing equivariantly presentable global categories. Just as stabilizing ordinary categories is given by inverting the action of S^1, the topological sphere, the stabilization of an equivariantly presentable global category will be obtained by inverting the action of the representation spheres pointwise. We first recall the relevant definitions.

Definition 4.2.6. Let $\mathcal{D} \in \mathrm{Mod}_{\mathcal{C}}(\mathsf{Pr}^{\mathrm{L}})$ be a presentable category tensored over another presentable symmetric monoidal category $\mathcal{C}$. Furthermore fix a collection of objects $S \in \mathcal{C}$. We say a $\mathcal{C}$-module $\mathcal{D}$ is *S-local* if for every $X \in S$ the functor $X \otimes -\colon \mathcal{D} \to \mathcal{D}$ is an equivalence. We write $\mathrm{Mod}_{\mathcal{C}}(\mathsf{Pr}^{\mathrm{L}})^{S\text{-loc}}$ for the full subcategory of $\mathrm{Mod}_{\mathcal{C}}(\mathsf{Pr}^{\mathrm{L}})$ spanned by the S-local objects.

Proposition 4.2.7 ([Rob15, Proposition 4.10])**.** *The inclusion* $\mathrm{Mod}_{\mathcal{C}}(\mathsf{Pr}^{\mathrm{L}})^{S\text{-loc}} \subset \mathrm{Mod}_{\mathcal{C}}(\mathsf{Pr}^{\mathrm{L}})$ *admits a symmetric monoidal left adjoint, which we denote by* $\mathcal{D} \mapsto \mathcal{D}[S^{-1}]$.

To invert the action of representation spheres pointwise in a global category, requires that inverting the action of a collection of objects is suitably functorial as the category we are tensored ober changes. To capture this functoriality we make the following definitions.

Definition 4.2.8. We define $\mathrm{Cat}_{\infty,\mathrm{aug}}$ to be the full subcategory of $\mathrm{Fun}([1], \mathrm{Cat})$ spanned by the fully faithful functors $S \hookrightarrow \mathcal{C}$ such that S is a small category. The pair $(\mathcal{C}, S)$ is called an augmented category.

Definition 4.2.9. We define the category $\mathrm{CAlg}(\mathsf{Pr}^{\mathrm{L}})_{\mathrm{aug}}$ of augmented presentable symmetric monoidal categories as the following pullback:

$$\begin{tikzcd}
\mathrm{CAlg}(\mathsf{Pr}^{\mathrm{L}})_{\mathrm{aug}} \arrow[r] \arrow[d] & \mathrm{CAlg}(\mathsf{Pr}^{\mathrm{L}}) \arrow[d] \\
\mathrm{Cat}_{\infty,\mathrm{aug}} \arrow[r] & \mathrm{Cat}_{\infty}
\end{tikzcd}$$

Definition 4.2.10. We write $\mathrm{Mod}(\mathsf{Pr}^{\mathrm{L}})$ for the cartesian unstraightening of the functor

$$\mathrm{Mod}_{\bullet}(\mathsf{Pr}^{\mathrm{L}})\colon \mathrm{CAlg}(\mathsf{Pr}^{\mathrm{L}})^{\mathrm{op}} \to \mathrm{Cat}, \quad \mathcal{C} \mapsto \mathrm{Mod}_{\mathcal{C}}(\mathsf{Pr}^{\mathrm{L}}).$$

Objects of $\mathrm{Mod}(\mathsf{Pr}^{\mathrm{L}})$ consist of a pair $(\mathcal{C}, \mathcal{D})$ of a presentable symmetric monoidal category $\mathcal{C}$ and a presentable category $\mathcal{D}$ tensored over $\mathcal{C}$.

Definition 4.2.11. We define $\mathrm{Mod}(\mathsf{Pr}^{\mathrm{L}})_{\mathrm{aug}}$ to be the pullback

$$\begin{tikzcd} \mathrm{Mod}(\mathsf{Pr}^{\mathrm{L}})_{\mathrm{aug}} \arrow[r] \arrow[d] & \mathrm{Mod}(\mathsf{Pr}^{\mathrm{L}}) \arrow[d] \\ \mathrm{CAlg}(\mathsf{Pr}^{\mathrm{L}})_{\mathrm{aug}} \arrow[r] & \mathrm{CAlg}(\mathsf{Pr}^{\mathrm{L}}). \end{tikzcd}$$

We define $\mathrm{Mod}(\mathsf{Pr}^{\mathrm{L}})_{\mathrm{aug}^{-1}}$ to be the full subcategory of $\mathrm{Mod}(\mathsf{Pr}^{\mathrm{L}})_{\mathrm{aug}}$ spanned by those triples $(\mathcal{C}, S, \mathcal{D})$ such that for every $X \in S$, $X \otimes -\colon \mathcal{D} \to \mathcal{D}$ is an equivalence.

Theorem 4.2.12. *The inclusion* $\mathrm{Mod}(\mathsf{Pr}^{\mathrm{L}})_{\mathrm{aug}^{-1}} \hookrightarrow \mathrm{Mod}(\mathsf{Pr}^{\mathrm{L}})_{\mathrm{aug}}$ *admits a left adjoint*

$$\mathbb{I}\colon \mathrm{Mod}(\mathsf{Pr}^{\mathrm{L}})_{\mathrm{aug}} \to \mathrm{Mod}(\mathsf{Pr}^{\mathrm{L}})_{\mathrm{aug}^{-1}}.$$

Furthermore this left adjoint sends a triple $(\mathcal{C}, S, \mathcal{D})$ *to the triple* $(\mathcal{C}, S, \mathcal{D}[S^{-1}])$.

Proof. Note that both $\mathrm{Mod}(\mathsf{Pr}^{\mathrm{L}})_{\mathrm{aug}}$ and $\mathrm{Mod}(\mathsf{Pr}^{\mathrm{L}})_{\mathrm{aug}^{-1}}$ are the total category of a cartesian fibration over $\mathrm{CAlg}(\mathsf{Pr}^{\mathrm{L}})_{\mathrm{aug}}$: The first because it is a pullback of $\mathrm{Mod}(\mathsf{Pr}^{\mathrm{L}})$ and the second because $\mathrm{Mod}(\mathsf{Pr}^{\mathrm{L}})_{\mathrm{aug}^{-1}}$ is clearly a full subcategory of $\mathrm{Mod}(\mathsf{Pr}^{\mathrm{L}})_{\mathrm{aug}}$ closed under cartesian pushforward, and so a cartesian fibration again.

This also shows that the inclusion $\mathrm{Mod}(\mathsf{Pr}^{\mathrm{L}})_{\mathrm{aug}^{-1}} \to \mathrm{Mod}(\mathsf{Pr}^{\mathrm{L}})_{\mathrm{aug}}$ is a map of cartesian fibrations. Given this, the statement is an application of [Lur16, Proposition 7.3.2.6], where the fiberwise left adjoints are given by Proposition 4.2.7. The final statement is clear. □

The following proposition shows that $\mathbb{I}$ preserves products, in a suitable sense.

Proposition 4.2.13. *Consider a set of objects* $(\mathcal{C}_i, S_i, \mathcal{D}_i)$ *in* $\mathrm{Mod}(\mathsf{Pr}^{\mathrm{L}})_{\mathrm{aug}}$. *Then there is an equivalence*

$$\mathbb{I}(\prod \mathcal{C}_i, \prod S_i, \prod \mathcal{D}_i) \simeq (\prod \mathcal{C}_i, \prod S_i, \prod \mathcal{D}_i[S_i^{-1}]).$$

Proof. This follows immediately from the pair of equivalences

$$\mathrm{Mod}_{\prod \mathcal{C}_i}(\mathsf{Pr}^{\mathrm{L}}) \simeq \prod \mathrm{Mod}_{\mathcal{C}_i}(\mathsf{Pr}^{\mathrm{L}})$$

and

$$\mathrm{Mod}_{\prod \mathcal{C}_i}(\mathsf{Pr}^{\mathrm{L}})^{\prod S_i - \mathrm{loc}} \simeq \prod \mathrm{Mod}_{\mathcal{C}_i}(\mathsf{Pr}^{\mathrm{L}})^{S_i - \mathrm{loc}}.$$ □

4.2.2 Equivariantly presentable Rep-stabilization

We are now ready to construct the Rep-stabilization of an equivariantly presentable global category. First we note that the observations of Remark 4.2.2 extend to a coherent statement:

Proposition 4.2.14. *Let $\mathcal{C}$ be a pointed equivariantly presentable global category. The functor $\mathcal{C}\colon \mathbb{F}_{\mathrm{gl}}^{\mathrm{op}} \to \mathsf{Pr}^{\mathrm{L}}$ extends to a functor*

$$\mathcal{C}\colon \mathbb{F}_{\mathrm{gl}}^{\mathrm{op}} \to \mathrm{Mod}(\mathsf{Pr}^{\mathrm{L}}), \quad X \mapsto (\mathcal{S}_{X,*}, \mathcal{C}_X).$$

Proof. By Lemma 4.2.1, $\mathcal{C}$ is canonically an object in $\mathrm{Mod}_{\mathcal{S}_{\bullet,*}}(\mathrm{Cat}_{\mathrm{Glo}})$. We note that $\mathrm{Cat}_{\mathrm{Glo}}$, as a functor category on $\mathrm{Glo}^{\mathrm{op}}$, is in particular an oplax limit of the constant Glo-shaped diagram on Cat. Therefore [LNP22, Theorem 5.10$^{\mathrm{op}}$] implies that $\mathcal{C}$ gives an object in $\mathrm{oplaxlim}\,\mathrm{Mod}_{\mathcal{S}_{\bullet,*}}\mathrm{Cat}$. This is in turn equivalent to a functor $\mathrm{Glo}^{\mathrm{op}} \to \mathrm{Mod}(\mathrm{Cat})$ by [LNP22, Theorem 4.13$^{\mathrm{op}}$], where Mod(Cat) is defined analogously to $\mathrm{Mod}(\mathsf{Pr}^{\mathrm{L}})$. For this functor to factor through $\mathrm{Mod}(\mathsf{Pr}^{\mathrm{L}})$ is a property, guaranteed by Lemma 4.2.1. We then limit extend this to a functor from $\mathbb{F}_{\mathrm{gl}}^{\mathrm{op}}$. ☐

Remark 4.2.15. Note that Definition 4.2.4 specifies a lift of the functor

$$\mathcal{S}_{\bullet,*}\colon \mathbb{F}_{\mathrm{gl}}^{\mathrm{op}} \to \mathrm{CAlg}(\mathsf{Pr}^{\mathrm{L}})$$

to a functor into $\mathrm{CAlg}(\mathsf{Pr}^{\mathrm{L}})_{\mathrm{aug}}$. This also lifts the functor $\mathcal{C}\colon \mathbb{F}_{\mathrm{gl}}^{\mathrm{op}} \to \mathrm{Mod}(\mathsf{Pr}^{\mathrm{L}})$ of Proposition 4.2.14 to a functor into $\mathrm{Mod}(\mathsf{Pr}^{\mathrm{L}})_{\mathrm{aug}}$.

Definition 4.2.16. Let $\mathcal{C}$ be a pointed equivariantly presentable global category. Postcomposing the lift of $\mathcal{C}$ to a functor into $\mathrm{Mod}(\mathsf{Pr}^{\mathrm{L}})_{\mathrm{aug}}$ with the functor $\mathbb{I}\colon \mathrm{Mod}(\mathsf{Pr}^{\mathrm{L}})_{\mathrm{aug}} \to \mathrm{Mod}(\mathsf{Pr}^{\mathrm{L}})_{\mathrm{aug}^{-1}}$ and then forgetting down to Pr^{L} we obtain a new functor

$$\mathsf{Stab}^{\mathrm{Orb}}(\mathcal{C})\colon \mathbb{F}_{\mathrm{gl}}^{\mathrm{op}} \to \mathsf{Pr}^{\mathrm{L}}, \quad X \mapsto \mathcal{C}_X[(\mathbb{S}_X^{\mathrm{rep}})^{-1}].$$

By Proposition 4.2.13 this is again a global category. Therefore $\mathsf{Stab}^{\mathrm{Orb}}$ defines a functor $\mathsf{Pr}^{\mathrm{Orb}}_{\mathrm{Glo},*} \to \mathrm{Cat}_{\mathrm{Glo}}$.

Theorem 4.2.17. *The functor $\mathsf{Stab}^{\mathrm{Orb}}$ lands in the subcategory $\mathsf{Pr}^{\mathrm{Orb}}_{\mathrm{Glo,rep\text{-}st}}$, and is a left adjoint to the inclusion $\mathsf{Pr}^{\mathrm{Orb}}_{\mathrm{Glo,rep\text{-}st}} \subset \mathsf{Pr}^{\mathrm{Orb}}_{\mathrm{Glo},*}$.*

Proof. The functor $\mathcal{C} \to \mathsf{Stab}^{\mathrm{Orb}}(\mathcal{C})$ induced by the unit of $\mathbb{I}$ is by definition the initial natural transformation of left adjoints such that $\mathsf{Stab}^{\mathrm{Orb}}(\mathcal{C})$ is fiberwise presentable and the action of the representation spheres on $\mathsf{Stab}^{\mathrm{Orb}}(\mathcal{C})$ is invertible. Therefore it suffices to show that $\mathsf{Stab}^{\mathrm{Orb}}(\mathcal{C})$ is in $\mathsf{Pr}^{\mathrm{Orb}}_{\mathrm{Glo}}$ and that the extension of $F\colon \mathcal{C} \to \mathcal{D}$ to $F'\colon \mathrm{Stab}^{\mathrm{Orb}}(\mathcal{C}) \to \mathcal{D}$ preserves all equivariant colimits whenever F does.

First we show that for every map $\iota\colon X \to Y$ in $\mathbb{F}_{\mathrm{Orb}}$ the functor

$$\iota^*\colon \mathrm{Stab}^{\mathrm{Orb}}(\mathcal{C})_Y \to \mathrm{Stab}^{\mathrm{Orb}}(\mathcal{C})_X$$

admits a left adjoint. For this the crucial input is the following property of $\mathbb{S}^{\mathrm{rep}}$: the restriction of the regular representation of G to a subgroup H is a multiple of the regular representation of H, and so every H-representation is a summand of the restriction of enough copies of the regular G-representation. This is obviously also true for more general maps $\iota\colon X \to Y$ in $\mathbb{F}_{\mathrm{Orb}}$. By [Cno23a, Lemma 2.22] we conclude that there exists an equivalence $\mathcal{C}_X[(\mathbb{S}^{\mathrm{rep}}_Y)^{-1}] \simeq \mathcal{C}_X[(\mathbb{S}^{\mathrm{rep}}_X)^{-1}]$ of $\mathcal{S}_{Y,*}$-modules. By Remark 4.2.3 $\iota_!$ is a $\mathcal{S}_{Y,*}$-module map and so induces a functor

$$\iota_!\colon \mathrm{Stab}^{\mathrm{Orb}}(\mathcal{C})_X \to \mathrm{Stab}^{\mathrm{Orb}}(\mathcal{C})_Y.$$

Furthermore because $\iota_! \dashv \iota^*$ is an adjunction in $\mathcal{C}_X$-modules, both the unit and counit are canonically $\mathcal{S}_{Y,*}$-linear natural transformations. These therefore also lift to natural transformations witnessing

$$\iota_!\colon \mathrm{Stab}^{\mathrm{Orb}}(\mathcal{C})_X \to \mathrm{Stab}^{\mathrm{Orb}}(\mathcal{C})_Y$$

as a left adjoint to $\iota^*\colon \mathrm{Stab}^{\mathrm{Orb}}(\mathcal{C})_Y \to \mathrm{Stab}^{\mathrm{Orb}}(\mathcal{C})_X$ in $\mathcal{S}_{Y,*}[(\mathbb{S}^{\mathrm{rep}}_Y)^{-1}]$-modules.

Next we show the required left adjointability conditions. Consider a square

$$\begin{tikzcd}
\mathcal{C}_{Y'}[(\mathbb{S}^{\mathrm{rep}}_{Y'})^{-1}] \arrow[r, "j^*"] \arrow[d, "g^*"] & \mathcal{C}_{X'}[(\mathbb{S}^{\mathrm{rep}}_{X'})^{-1}] \arrow[d, "f^*"] \\
\mathcal{C}_{Y}[(\mathbb{S}^{\mathrm{rep}}_{Y})^{-1}] \arrow[r, "i^*"] & \mathcal{C}_{X}[(\mathbb{S}^{\mathrm{rep}}_{X})^{-1}]
\end{tikzcd}$$

induced by a pullback square in $\mathbb{F}_{\mathrm{gl}}$ such that i and j are faithful maps of groupoids. We have to show this square is left adjointable. We first note that because all of the functors in this diagram preserve colimits, it suffices by [AI22, Lemma 1.5.1] to prove that the Beck-Chevalley transformation is an equivalence on objects of the form $S^{-V} \otimes Z$ for $Z \in \mathcal{C}_{X'}$ and $S^V \in \mathbb{S}^{\mathrm{rep}}_{X'}$. Because j^* induces a cofinal map between the augmentations we immediately see that it in fact suffices to prove this for objects of the form $S^{-j^*V} \otimes Z$, where V is now a Y'-representation. One can show that the Beck-Chevalley transformation on $S^{-j^*V} \otimes Z$ is given by tensoring the Beck-Chevalley transformation of X by

S^{-j^*V}. More precisely we claim that the following diagram commutes:

$$\begin{tikzcd}
g^*j_!(S^{-j^*V}\otimes Z) \arrow[r, "\eta"] \arrow[dd, "\sim"'] & i_!i^*g^*j_!(j^*S^{-V}\otimes Z) \arrow[r, "\sim"] \arrow[ddd, "\sim"] & i_!f^*j^*j_!(j^*S^{-V}\otimes Z) \arrow[r, "\epsilon"] \arrow[ddd, "\sim"] & i_!f^*(S^{-j^*V}\otimes Z) \arrow[d, "\sim"] \\
 & & & i_!(S^{-f^*j^*V}\otimes f^*Z) \arrow[d, "\sim"] \\
g^*(S^{-V}\otimes j_!Z) \arrow[d, "\sim"'] & & & i_!(S^{-i^*g^*V}\otimes f^*Z) \arrow[d, "\sim"] \\
S^{-g^*V}\otimes g^*j_!Z \arrow[r, "S^{-g^*V}\otimes\eta"] & S^{-g^*V}\otimes i_!i^*g^*j_!Z \arrow[r, "S^{-g^*V}\otimes\sim"] & S^{-g^*V}\otimes i_!f^*j^*j_!Z \arrow[r, "S^{-g^*V}\otimes\epsilon"] & S^{-g^*V}\otimes i_!f^*Z.
\end{tikzcd}$$

The proof of this claim is a tedious diagram chase which we omit. From this we conclude that it suffices to check that the Beck-Chevalley transformation is an equivalence on objects in the image of the functor $\mathcal{C}_{X'} \to \mathcal{C}_{X'}[(\mathbb{S}^{\mathrm{rep}}_{X'})^{-1}]$. However on such objects the Beck-Chevalley transformation is simply given by the image of the Beck-Chevalley transformation for $\mathcal{C}_\bullet$ and so an equivalence. In exactly the same way one shows that if $F\colon \mathcal{C} \to \mathcal{D}$ is a functor between pointed equivariantly presentable global categories which preserves Orb-colimits then $\mathrm{Stab}^{\mathrm{Orb}}(F)$ again preserves Orb-colimits. □

Proposition 4.2.18. $\mathrm{Stab}^{\mathrm{Orb}}(\mathcal{S}_\bullet) \simeq \mathrm{Sp}_\bullet$.

Proof. By definition the diagram $\mathrm{Sp}_\bullet$ is given by pointwise stabilizing the diagram $\mathcal{S}_\bullet$ at the representation spheres, see Example 3.1.18. □

4.2.3 Globally presentable Rep-stabilization

We now repeat the previous definitions for $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{Glo}}$.

Definition 4.2.19. We say a globally presentable global category $\mathcal{C}$ is pointed if each $\mathcal{C}_G$ is pointed. Given a pointed globally presentable global category $\mathcal{C}$ we say it is Rep-stable if it is Rep-stable as an equivariantly presentable global category. We define $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{Glo},*}$ and $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{Glo,rep\text{-}st}}$ to be the full subcategory of $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{Glo},*}$ spanned by the pointed and Rep-stable global categories respectively.

Definition 4.2.20. Consider a morphism $F\colon \mathcal{C} \to \mathcal{C}'$ in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{Glo},*}$. We say *F exhibits $\mathcal{C}'$ as the globally presentable* Rep-*stabilization of* $\mathcal{C}$ if the map

$$\mathrm{Hom}_{\mathrm{Pr}^{\mathrm{L}}_{\mathrm{Glo},*}}(\mathcal{C}',\mathcal{D}) \to \mathrm{Hom}_{\mathrm{Pr}^{\mathrm{L}}_{\mathrm{Glo},*}}(\mathcal{C},\mathcal{D})$$

is an equivalence for every $\mathcal{D} \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{Glo,rep\text{-}st}}$.

The construction of globally presentable Rep-stabilizations is significantly more subtle then the analogous story of equivariantly presentable Rep-stabilizations. In particular the construction of globally presentable Rep-stabilizations cannot be as simple as pointwise inverting the action of representation spheres. For example we remind the reader that while $\mathcal{S}_\bullet$ is globally presentable, $\mathrm{Sp}_\bullet$ is not.

Nevertheless, imitating arguments of [Rob15, Section 2] one can show that globally presentable Rep-stabilizations always exist. We will not consider the finer aspects of this construction, but instead content ourselves with the observation that when the globally presentable global category $\mathcal{C}$ is presented as the globalization of some other global category, we can obtain an explicit description of $\mathrm{Stab}^{\mathrm{Glo}}(\mathcal{C})$.

Theorem 4.2.21. *Let $\mathcal{C}$ be an object of $\mathsf{Pr}^{\mathrm{Orb}}_{\mathrm{Glo},*}$. The globally presentable global category* $\mathrm{Glob}(\mathrm{Stab}^{\mathrm{Orb}}(\mathcal{C}))$ *is the globally presentable* Rep*-stabilization of* $\mathrm{Glob}(\mathcal{C})$*.*

Proof. We claim that $\mathrm{Glob}(-)$ preserves Rep-stable global categories. To this end fix a $\mathcal{D} \in \mathsf{Pr}^{\mathrm{Orb}}_{\mathrm{Glob,rep\text{-}st}}$. First note that because colimits and limits are computed pointwise in $\mathrm{Glob}(\mathcal{D})$, it is again a pointed global category. Now consider $S^V \in \mathbb{S}^{\mathrm{rep}}_G$. We compute that the tensoring of S^V on $\mathrm{Glob}(\mathcal{D})_G$ is given by

$$\{X_f\}_{f\colon \mathbf{B}H\to\mathbf{B}G} \mapsto \{f^*(S^V)\otimes X_f\}_{f\colon \mathbf{B}H\to\mathbf{B}G} \simeq \{S^{f^*(V)}\otimes X_f\}_{f\colon \mathbf{B}H\to\mathbf{B}G}.$$

Because each $S^{f^*(V)}$ acts invertibly on $\mathcal{D}_H$, we conclude that tensoring by S^V on $\mathrm{Glob}(\mathcal{D})_G$ is an equivalence. Therefore Glob restricts to a functor $\mathsf{Pr}^{\mathrm{Orb}}_{\mathrm{Glo,rep\text{-}st}} \to \mathsf{Pr}^{\mathrm{L}}_{\mathrm{Glo,rep\text{-}st}}$. The result now follows from the following series of natural equivalences:

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{Pr}^{\mathrm{L}}_{\mathrm{Glo,rep\text{-}st}}}(\mathrm{Glob}(\mathrm{Stab}^{\mathrm{Orb}}(\mathcal{C})),\mathcal{D}) &\simeq \mathrm{Hom}_{\mathsf{Pr}^{\mathrm{Orb}}_{\mathrm{Glo,rep\text{-}st}}}(\mathrm{Stab}^{\mathrm{Orb}}(\mathcal{C}),\mathcal{D})\\
&\simeq \mathrm{Hom}_{\mathsf{Pr}^{\mathrm{Orb}}_{\mathrm{Glo},*}}(\mathcal{C},\mathcal{D})\\
&\simeq \mathrm{Hom}_{\mathsf{Pr}^{\mathrm{L}}_{\mathrm{Glo},*}}(\mathrm{Glob}(\mathcal{C}),\mathcal{D}).
\end{aligned}$$

□

Corollary 4.2.22. *The global category of globally equivariant spectra* $\mathrm{Sp}_{\bullet\text{-gl}}$ *is the free globally presentable* Rep*-stable global category on a point.*

Proof. Apply the previous result to $\mathcal{C} = \mathcal{S}_\bullet$, using Proposition 4.2.18. □

Remark 4.2.23. We expect that both localizations $\mathsf{Pr}^{\mathrm{L}}_{\mathrm{Glo}} \to \mathsf{Pr}^{\mathrm{L}}_{\mathrm{Glo,rep\text{-}st}}$ and $\mathsf{Pr}^{\mathrm{L}}_{\mathrm{Glo}} \to \mathsf{Pr}^{\mathrm{L}}_{\mathrm{Glo,Orb\text{-}st}}$ are smashing. This would imply, among other things, that because they agree on the unit $\mathcal{S}_{\bullet\text{-gl}}$ they in fact agree as functors. In particular we would conclude that a globally presentable global category is equivariantly stable if and only if it is representation stable.

4.2.4 Symmetric monoidal structures

Observe that the partially lax limit of symmetric monoidal categories is canonically symmetric monoidal by taking the tensor product pointwise, see [LNP22, Section 3] for a more detailed discussion. This gives a lift of $\mathrm{Sp}_{\bullet\text{-gl}}\colon \mathbb{F}_{\mathrm{gl}}^{\mathrm{op}} \to \mathsf{Pr}^{\mathrm{L}}$ to a functor into $\mathrm{CAlg}(\mathsf{Pr}^{\mathrm{L}})$. We will write $\mathrm{Sp}^{\otimes}_{\bullet\text{-gl}}$ for this functor.

The goal of this subsection is to show that $\mathrm{Sp}^{\otimes}_{\bullet\text{-gl}}$ is the initial globally presentably symmetric monoidal Rep-stable global category. To elaborate on this we recall that by [MW22, Proposition 8.2.9], $\mathsf{Pr}^{\mathrm{L}}_{\mathrm{Glo}}$ admits a symmetric monoidal structure given by a parametrized version of the Lurie tensor product. As usual functors out of the tensor product corepresent functors out of the product which preserve global colimits in both variables. We call an object $\mathcal{C}$ of the category $\mathrm{CAlg}(\mathsf{Pr}^{\mathrm{L}}_{\mathrm{Glo}})$ a globally presentably symmetric monoidal global category. To understand this it is useful to be more explicit about the structure and properties implicit in presentable symmetric monoidality. We will do this in the generality of an arbitrary orbital category T.

Definition 4.2.24. A symmetric monoidal T-category is a finite product preserving functor $\mathbb{F}_T^{\mathrm{op}} \to \mathrm{Cat}^{\otimes}$. We define $\mathrm{Cat}_T^{\otimes}$ to be the functor category $\mathrm{Fun}^{\times}(\mathbb{F}_T^{\mathrm{op}}, \mathrm{Cat}^{\otimes})$.

Definition 4.2.25. We define $\mathsf{PM}^{\mathrm{L}}_T$ to be the subcategory of $\mathrm{Cat}_T^{\otimes}$ spanned on objects by those $\mathcal{C}$ such that

1. the functor $\mathcal{C}$ lifts to a functor $\mathcal{C}\colon \mathbb{F}_T^{\mathrm{op}} \to \mathrm{CAlg}(\mathsf{Pr}^{\mathrm{L}})$;
2. for all $f\colon X \to Y$ in $\mathbb{F}_T$, the functor f^* has a further left adjoint $f_!$;
3. the square obtained by applying $\mathcal{C}$ to a pullback square in $\mathbb{F}_T$ is left adjointable.

On morphisms $\mathsf{PM}^{\mathrm{L}}_T$ is spanned by those functors $F\colon \mathcal{C} \to \mathcal{D}$ in $\mathrm{Cat}_T^{\otimes}$ such that each functor F_G admits a right adjoint and the square

$$\begin{tikzcd} \mathcal{C}_X \arrow[r, "F_Y"] \arrow[d, "f^*"'] & \mathcal{D}_X \arrow[d, "f^*"'] \\ \mathcal{C}_Y \arrow[r, "F_X"] & \mathcal{D}_Y \end{tikzcd}$$

is left adjointable for all $f \in T$.

Proposition 4.2.26. *There exists a fully faithful forgetful functor* $\mathrm{CAlg}(\mathsf{Pr}^{\mathrm{L}}_T) \subset \mathsf{PM}^{\mathrm{L}}_T$*, with essential image those* $\mathcal{C}\colon \mathbb{F}_T^{\mathrm{op}} \to \mathrm{CAlg}(\mathsf{Pr}^{\mathrm{L}})$ *which satisfy the left projection* *formula, i.e. those $\mathcal{C} \in \mathsf{PM}^{\mathrm{L}}_T$ such that for all morphisms $f\colon X \to Y$ in $\mathbb{F}_T$, the canonical natural transformation*

$$f_!(f^*X \otimes Y) \to f_!f^*X \otimes f_!Y \xrightarrow{\epsilon \otimes Y} X \otimes f_!Y$$

is an equivalence.

Proof. Both $\mathrm{CAlg}(\mathsf{Pr}^{\mathrm{L}}_T)$ and $\mathsf{PM}^{\mathrm{L}}_T$ are subcategories of $\mathrm{Cat}^{\otimes}_T$, and therefore it suffices to compare the images. For this we note that an object $\mathcal{C} \in \mathrm{Cat}^{\otimes}_T$ is in $\mathrm{CAlg}(\mathsf{Pr}^{\mathrm{L}}_T)$ if and only if $\mathcal{C}$ is presentable and the tensor product commutes with fiberwise and T-groupoid indexed colimits in each variable. The first two statements are equivalent to the claim that $\mathcal{C}$ factors through $\mathrm{CAlg}(\mathsf{Pr}^{\mathrm{L}})$, while the final statement is equivalent to the claim that the left projection formula holds. □

Write $\mathrm{CAlg}(\mathsf{Pr}^{\mathrm{L}}_{\mathrm{Glo}})_{\text{rep-st}}$ for the full subcategory of $\mathrm{CAlg}(\mathsf{Pr}^{\mathrm{L}}_{\mathrm{Glo}})$ spanned by those $\mathcal{C}$ which are representation stable. Recall that our goal is to show that $\mathrm{Sp}^{\otimes}_{\bullet\text{-gl}}$ is the initial object of this ∞-category. To do this we will take a slightly circuitous route. We define $\mathsf{PM}^{\mathrm{L}}_{\mathrm{Glo},\text{rep-st}}$ via the pullback

$$\begin{tikzcd}
\mathsf{PM}^{\mathrm{L}}_{\mathrm{Glo},\text{rep-st}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{PM}^{\mathrm{L}}_{\mathrm{Glo}} \arrow[d] \\
\mathsf{Pr}^{\mathrm{L}}_{\mathrm{Glo},\text{rep-st}} \arrow[r] & \mathsf{Pr}^{\mathrm{L}}_{\mathrm{Glo}}.
\end{tikzcd}$$

Lemma 4.2.27. *The symmetric monoidal global category $\mathrm{Sp}^{\otimes}_{\bullet\text{-gl}}$ admits a unique map to any object $\mathcal{D} \in \mathrm{CAlg}(\mathsf{Pr}^{\mathrm{L}}_{\mathrm{Glo}})_{\text{rep-st}}$.*

Proof. This is proven by simply repeating all of the constructions and arguments in Sections 3.3 and 4.2, but now for objects $\mathcal{C}\colon \mathbb{F}_{\mathrm{gl}} \to \mathsf{Pr}^{\mathrm{L}}$ which additionally lift to $\mathrm{CAlg}(\mathsf{Pr}^{\mathrm{L}})$. For example one shows that the functor $\mathcal{P}^T_S\colon \mathsf{Pr}^S_T \to \mathsf{Pr}^{\mathrm{L}}_T$ refines to a left adjoint $\mathsf{PM}^S_T \to \mathsf{PM}^{\mathrm{L}}_T$, where the definition of PM^S_T is analogous to that of $\mathsf{PM}^{\mathrm{L}}_T$. Because partially lax limits of symmetric monoidal categories are computed underlying, the proofs of Theorem 3.3.6 and Theorem 3.3.10 go through unchanged. Similarly for all other steps leading to the theorem. Given this, the results follows immediately from the fact that $\mathcal{S}_\bullet$ is an initial object of $\mathrm{CAlg}(\mathsf{Pr}^{\mathrm{L}}_{\mathrm{Glo}})$), see [MW22, Remark 8.2.6]. □

Theorem 4.2.28. $\mathrm{Sp}^{\otimes}_{\bullet\text{-gl}}$ *is the initial globally presentable symmetric monoidal* Rep-stable *global category.*

Proof. By the previous lemma, it suffices to prove that $\mathrm{Sp}^{\otimes}_{\bullet\text{-gl}}$ is actually an object of $\mathrm{CAlg}(\mathsf{Pr}^{\mathrm{L}}_{\mathrm{Glo}})$. By Proposition 4.2.26 this amounts to proving that

$\mathrm{Sp}^{\otimes}_{\bullet\text{-gl}}$ satisfies the left projection formula. Pick a morphism $f\colon X \to Y$ in $\mathbb{F}_{\mathrm{gl}}$. Consider the functor $\Sigma^\infty_\bullet \colon \mathcal{S}_{\bullet\text{-gl},*} \to \mathrm{Sp}_{\bullet\text{-gl}}$ exhibiting $\mathrm{Sp}_{\bullet\text{-gl}}$ as the globally presentable Rep-stabilization of $\mathcal{S}_{\bullet\text{-gl},*}$. The source is the unit of $\mathsf{Pr}^{\mathrm{L}}_{\mathrm{Glo},*}$ and so satisfies the left projection formula. Because Σ^∞_+ is strong monoidal and commutes with global colimits, we conclude that suspension spectra in $\mathrm{Sp}_{X\text{-gl}}$ and $\mathrm{Sp}_{Y\text{-gl}}$ satisfy the left projection formula for f. That is, the map

$$f_!(f^*E \otimes F) \to E \otimes f_!F$$

is an equivalence when E and F are in the image of Σ^∞_+. Now we note that the collection of objects which satisfy the left projection formula for f is closed under desuspensions and colimits in both $\mathcal{C}_X$ and $\mathcal{C}_Y$. Because $\mathrm{Sp}_{\bullet\text{-gl}}$ is generated as a fiberwise stable global category under fiberwise colimits by suspension spectra, we conclude that $\mathrm{Sp}_{\bullet\text{-gl}}$ satisfies the left projection formula. $\square$

Appendix A

Global spectra via partially lax limits

This appendix is a reproduction of [LNP22], joint work of Denis Nardin, Luca Pol and the author. It will appear in Geometry and Topology.

Abstract

We provide new ∞-categorical models for unstable and stable global homotopy theory. We use the notion of partially lax limits to formalize the idea that a global object is a collection of G-objects, one for each compact Lie group G, which are compatible with the restriction-inflation functors. More precisely, we show that the ∞-category of global spaces is equivalent to a partially lax limit of the functor sending a compact Lie group G to the ∞-category of G-spaces. We also prove the stable version of this result, showing that the ∞-category of global spectra is equivalent to the partially lax limit of a diagram of G-spectra. Finally, the techniques employed in the previous cases allow us to describe the ∞-category of proper G-spectra for a Lie group G, as a limit of a diagram of H-spectra for H running over all compact subgroups of G.

5. Introduction

It has been noted since the beginning of equivariant homotopy theory that there are equivariant objects which exist uniformly and compatibly for all compact Lie groups in a certain family, and which exhibit extra functoriality. For example given compact Lie groups Π and G, there exists a construction for the classifying space of G-equivariant Π-principal bundles which is uniform on the group G and which is functorial on all continuous group homomorphism, [Sch18, Remark 1.1.29]. Similarly, there are uniform constructions for many equivariant cohomology theories, such as K-theory, cobordism and stable cohomotopy, just to mention a few. The objects exhibiting such a "global" behaviour are the subject of study of *global homotopy theory*.

In this paper we provide a new ∞-categorical model for global homotopy theory by formalizing the idea that a global stable/unstable object is a collection of G-objects, one for each compact Lie group G, which are compatible with the restriction-inflation functors. The key categorical construction that we will use to make this slogan precise is that of a partially lax limit, which we recall below. The main result of our paper is that this construction agrees with the models of global homotopy theory considered in the literature. Specifically we will compare to the models of [GH07] and [Sch18] in the unstable and stable case respectively. We first present our result in the simpler context of unstable global homotopy theory, and then consider the stable analogue of our main result. Finally we discuss an application of the techniques developed in this paper to proper equivariant homotopy theory.

Unstable global homotopy theory

Global spaces were first proposed in [GH07] as a powerful framework for studying the homotopy theory of topological stacks and topological groupoids, which in turn generalize orbifolds and complexes of groups. This homotopy theory records the isotropy data of such objects as a particular diagram of fixed points spaces. To make this precise, [GH07] defined the ∞-category of *global spaces* as the presheaf ∞-category

$$\mathcal{S}_{gl} = \mathrm{Fun}(\mathrm{Glo}^{\mathrm{op}}, \mathcal{S}).$$

Here Glo is the ∞-category whose objects are all compact Lie groups G, and whose morphism spaces are given by $\hom(H, G)_{hG}$; the homotopy orbits of the conjugation G-action on the space of continuous group homomorphisms. In particular, a global space X consists of the data of a fixed point space X^G

for every compact Lie group G which are functorial in all continuous group homomorphisms. Furthermore, the conjugation actions have been trivialized, reflecting the fact that spaces of isotropy are insensitive to inner automorphisms.

This definition is motivated by Elmendorf's theorem in equivariant homotopy theory which states that the ∞-category of *G-spaces* $\mathcal{S}_G$ is equivalent to the presheaf ∞-category on the G-orbit category $\mathbf{O}_G$. Here $\mathcal{S}_G$ is defined as the ∞-categorical localization of G-CW-complexes at the homotopy equivalences, and $\mathbf{O}_G$ is the full subcategory of G-spaces spanned by the transitive G-spaces G/H for a closed subgroup $H \subseteq G$.

There is in fact a strong connection between equivariant and global homotopy theory. Let Orb denote the wide subcategory of Glo spanned by the injective group homomorphisms. Gepner-Henriques [GH07] observed that the slice ∞-category $\mathrm{Orb}_{/G}$ is equivalent to the G-orbit category $\mathbf{O}_G$. In particular, this allows us to define a restriction functor

$$\mathrm{res}_G \colon \mathcal{S}_{\mathrm{gl}} \to \mathrm{Fun}(\mathbf{O}_G^{\mathrm{op}}, \mathcal{S}) \simeq \mathcal{S}_G$$

by precomposing with forgetful functor $\mathbf{O}_G \simeq \mathrm{Orb}_{/G} \to \mathrm{Glo}$. Thus a global space has an associated underlying G-space for all compact Lie groups G. Furthermore, that all these G-spaces come from the same global object imposes strong compatibility conditions among them.

We would like to understand how to recover a global space X from its restrictions $\mathrm{res}_G X$ to all compact Lie groups G, together with the previously mentioned compatibility conditions. The precise sense in which this is possible requires the notion of a (partially) lax limit, which we now recall following [GHN17] and [Ber20].

Partially lax limits

Let $\mathcal{I}$ be an ∞-category and consider a functor $F \colon \mathcal{I} \to \mathrm{Cat}_\infty$. Intuitively, the *lax limit of F* is the ∞-category $\mathrm{laxlim}\, F$ whose objects consist of the following data

- an object $X_i \in F(i)$ for each $i \in \mathcal{I}$;
- and compatible morphisms $f_\alpha \colon F(\alpha)(X_i) \to X_j$ for every arrow $\alpha \colon i \to j$ in $\mathcal{I}$.

A morphism $\{X_i, f_\alpha\} \to \{X'_i, f'_\alpha\}$ is a suitably natural collection of maps $\{g_i \colon X_i \to X'_i\}$. More precisely, $\mathrm{laxlim}\, F$ is the ∞-category of sections of the cocartesian fibration associated to F. For our description we will require that for certain arrows α in $\mathcal{I}$, the map f_α is an equivalence. We therefore fix a collection of edges $\mathcal{W} \subset \mathcal{I}$, which contains all equivalences and which is stable under homotopy and composition, and denote by $\mathcal{I}^\dagger$ the resulting marked ∞-category. The *partially lax limit* of F is then the subcategory of $\mathrm{laxlim}\, F$ spanned by those objects $(\{X_i\}, \{f_\alpha\})$ for which the canonical map f_α is an equivalence for all edges $\alpha \in \mathcal{W}$. Note that if $\mathcal{W}$ contains only equivalences, then we recover the lax limit of F. On the other hand, if $\mathcal{W}$ contains all edges, we recover the usual notion of the limit of F. In particular we obtain canonical functors

$$\lim F \longrightarrow \mathrm{laxlim}^\dagger F \longrightarrow \mathrm{laxlim}\, F,$$

which indicates that a partially lax limit interpolates between the limit and the lax limit of a diagram. For exposition's sake, we have only defined the partially lax limit of a functor with values in Cat_∞, but there are similar definitions if we replace Cat_∞ with $\mathrm{Cat}_\infty^\otimes$, the ∞-category of symmetric monoidal ∞-categories. We refer the reader to Section 6.3 for more details on this construction.

As mentioned, in this paper we show that a global space can be thought of as a compatible collection of G-spaces. We can formalize what "compatible" means using the language of partially lax limits. To this end, let $(\mathrm{Glo}^{\mathrm{op}})^\dagger$ denote the ∞-category $\mathrm{Glo}^{\mathrm{op}}$ where we marked all the edges in $\mathrm{Orb}^{\mathrm{op}} \subseteq \mathrm{Glo}^{\mathrm{op}}$, i.e. all the injective edges. We prove the following theorem, which summarizes the main result of Section 7.1.

Theorem (7.1.17)**.** There exists a functor $\mathcal{S}_\bullet \colon \mathrm{Glo}^{\mathrm{op}} \longrightarrow \mathrm{Cat}_\infty^\otimes$ which sends a compact Lie group G to the ∞-category of G-spaces $\mathcal{S}_G$ endowed with the cartesian symmetric monoidal structure, and a continuous group homomorphism $\alpha \colon H \longrightarrow G$ to the restriction-inflation functors. Furthermore, there is a symmetric monoidal equivalence

$$\mathcal{S}_{\mathrm{gl}} \simeq \operatorname*{laxlim^\dagger}_{G \in (\mathrm{Glo}^{\mathrm{op}})^\dagger} \mathcal{S}_G$$

between the ∞-category of global spaces with the cartesian monoidal structure and the partially lax limit over $(\mathrm{Glo}^{\mathrm{op}})^\dagger$ of the diagram $\mathcal{S}_\bullet$.

By the above theorem a global space X consists of the following data:

- a G-space $\mathrm{res}_G X$ for each compact Lie group G,
- an H-equivariant map $f_\alpha \colon \alpha^* \mathrm{res}_G X \longrightarrow \mathrm{res}_H X$ for each continuous group homomorphism $\alpha \colon H \longrightarrow G$.
- the maps f_α are functorial, so that $f_{\beta \circ \alpha} \simeq f_\beta \circ \beta^*(f_\alpha)$ for all composable maps α and β, and $f_{\mathrm{id}} = \mathrm{id}$;
- f_α is an equivalence for every continuous *injective* homomorphism α.
- a homotopy between the map f_{c_g} induced by the conjugation isomorphism and the map $l_g \colon c_g^* \mathrm{res}_G X \longrightarrow \mathrm{res}_G X$ given by left multiplication by g.

- higher coherences for the homotopies.

This is a precise formulation of the compatibility conditions encoded in a global space.

Global stable homotopy theory

Our discussion so far has been limited to the homotopy theory of global spaces, but there are also numerous examples of equivariant cohomology theories exhibiting a global behaviour. These cohomology theories are represented by global spectra, and their study is called *global stable homotopy theory*.

The consideration of “global spectra" grew out of the literature on equivariant stable homotopy theory, and was considered in works such as [GM97]. Morally, a global spectrum models a compatible family of equivariant spectra for all compact Lie groups at once. Our main result makes this moral precise, and provides the same description as in the unstable case.

There are multiple models for the homotopy theory of global spectra. In this paper we will use the framework developed by Schwede in [Sch18]. His approach has the advantage of being very concrete; the category of global spectra is modelled by the usual category of orthogonal spectra but with a finer notion of equivalence, the global equivalences. The category of orthogonal spectra with the global stable model structure of [Sch18, Theorem 4.3.17] underlies a symmetric monoidal ∞-category $\mathrm{Sp}_{\mathrm{gl}}$. As any orthogonal spectrum is a global spectrum, this approach comes with a good range of examples. For instance, there are global analogues of the sphere spectrum, cobordism, topological and algebraic K-theory spectra, Borel cohomology, symmetric product spectra and many others. Global spectra have also been shown to give cohomology theories on orbifolds and topological stacks in [Jur20], thereby establishing them as a natural home for (genuine) cohomology theories on topological stacks. As part of the framework developed by Schwede, the ∞-category of global spectra comes with symmetric monoidal restriction functors

$$\mathrm{res}_G \colon \mathrm{Sp}_{\mathrm{gl}} \to \mathrm{Sp}_G$$

into the ∞-category of G-spectra, for all compact Lie groups G. As a first indication that a global spectrum should consist of just this data, together with various comparison maps, note that the functors res_G are jointly conservative by the very definition of global equivalences.

However, not all equivariant spectra admit global refinements. In fact being a “global” object forces strong compatibility conditions between the underlying G-spectra for different G. For example, $\mathrm{res}_G X$ is always a split G-spectrum by [Sch18, Remark 4.1.2] and its G-homotopy groups for all G together admit the structure of a global functor, see [Sch18, Example 4.2.3]. We can again formalize how a global spectrum is determined by its restrictions for all compact Lie groups using the language of partially lax limits. Recall that $(\mathrm{Glo}^{\mathrm{op}})^{\dagger}$ denotes the ∞-category $\mathrm{Glo}^{\mathrm{op}}$, marked by all the edges in $\mathrm{Orb}^{\mathrm{op}}$, i.e. the injective group homomorphisms.

Theorem (7.6.10)**.** There exists a functor $\mathrm{Sp}_{\bullet}\colon \mathrm{Glo}^{\mathrm{op}} \to \mathrm{Cat}_{\infty}^{\otimes}$ which sends a compact Lie group G to the symmetric monoidal ∞-category of G-spectra $\mathrm{Sp}_G^{\otimes}$, and a continuous group homomorphism $\alpha\colon H \to G$ to the restriction-inflation functor. Furthermore, there is a symmetric monoidal equivalence

$$\mathrm{Sp}_{\mathrm{gl}} \simeq \underset{G\in(\mathrm{Glo}^{\mathrm{op}})^{\dagger}}{\mathrm{laxlim}^{\dagger}}\, \mathrm{Sp}_G$$

between Schwede's ∞-category of global spectra and the partially lax limit over $(\mathrm{Glo}^{\mathrm{op}})^{\dagger}$ of the diagram $\mathrm{Sp}_{\bullet}$.

Proper equivariant stable homotopy theory

The techniques employed in the proof of Theorem 7.6.10 can also be used in other settings. Given a (not necessarily compact) Lie group G, we can consider the ∞-category of proper G-spectra $\mathrm{Sp}_{G,\mathrm{pr}}$. This is the ∞-category underlying the category of orthogonal G-spectra with the proper stable model structure of [Deg+23], in which a map $f\colon X \to Y$ is a weak equivalence if and only if for all compact subgroups $H \leq G$, the map induced on homotopy groups $\pi_*^H(f)\colon \pi_*^H(X) \to \pi_*^H(Y)$ is an isomorphism. Write $\mathbf{O}_{G,\mathrm{pr}}$ for the proper G-orbit category, which is defined to be the subcategory of $\mathbf{O}_G$ spanned by the cosets G/H, where H a compact subgroup of G. Our techniques allow us to prove:

Theorem (7.7.11)**.** Let G be a Lie group. There is a symmetric monoidal equivalence

$$\mathrm{Sp}_{G,\mathrm{pr}} \simeq \lim_{H\in\mathbf{O}_{G,\mathrm{pr}}^{\mathrm{op}}} \mathrm{Sp}_H$$

between the ∞-category of proper G-spectra and the limit of the functor $\mathrm{Sp}_{\bullet}$ restricted along the canonical functor $\iota_G\colon \mathbf{O}_{G,\mathrm{pr}}^{\mathrm{op}} \to \mathrm{Glo}^{\mathrm{op}}$ sending G/H to H.

Having introduced the main theorems of this article. We continue the introduction by discussing the proof strategy for each in some detail.

The proof strategy for Theorem 7.1.17

We begin with a discussion of the proof of the unstable result. Implicit in [Rez14] is the following crucial observation (see also Proposition 7.1.13): the space of factorizations of any map $\alpha\colon H \to G$ in Glo into a surjective followed by an injective group homomorphism is contractible. In fewer words, the surjective and injective maps form an orthogonal factorization system on Glo. This is the main ingredient in the proof of Theorem 7.1.17, and moreover, we would like to argue it is at the core of the relationship between global and G-equivariant homotopy theory.

This claim is justified by the following two facts. The first is that the functoriality under the restriction-inflation functors of the different ∞-categories of equivariant spaces is equivalent to the previous observation. The second is that the observation formally implies that one can recover a global space X from the $\mathrm{Glo}^{\mathrm{op}}$-indexed diagram of G-spaces $\mathrm{res}_G X$.

Let us first explain how the ∞-categories of equivariant spaces are functorial in the category $\mathrm{Glo}^{\mathrm{op}}$. Due to the existence of a non-trivial topology on the morphism spaces, this is not immediate. For example, note that exhibiting this functoriality also entails giving a homotopy coherent trivialization of the conjugation action on $\mathcal{S}_G$. The key is that the existence of the orthogonal factorization system allows one to define functors

$$\alpha_!\colon \mathrm{Orb}_{/H} \to \mathrm{Orb}_{/G}, \quad (K \hookrightarrow H) \mapsto (\alpha(K) \hookrightarrow G).$$

On objects $\alpha_!$ factorizes the composite $K \hookrightarrow H \to G$ into a surjection followed by an injection, and then only remembers the injective part. The fact that such factorizations are unique is equivalent to the fact that this functor is well-defined. Precomposing with $\alpha_!^{\mathrm{op}}$ gives the standard restriction functor $\alpha^*\colon \mathcal{S}_G \to \mathcal{S}_H$. Furthermore given this description of the individual restriction functors, it is clear that they are functorial in $\mathrm{Glo}^{\mathrm{op}}$.

Next we explain how the observation implies that one can recover a global space from its restrictions. When one takes an object $(\{\mathrm{res}_G X\}, \{f_\alpha\})$ of the partially lax limit over $\mathrm{Glo}^\dagger$ of the diagram $\mathcal{S}_\bullet$, the functoriality of the associated global space in injections is recorded by restricting to each $\mathrm{res}_G X$, and the functoriality in surjections is given by the morphisms f_α. One recovers the functoriality in all morphisms in Glo by factorizing an arbitrary morphism into an injection followed by a surjection. The ability to split the functoriality in this way again reduces to the observation that the surjective and injective maps form an orthogonal factorization system. We make precise all of the ideas sketched here in Section 7.1.

The proof strategy for Theorem 7.6.10

The proof of Theorem 7.6.10 is considerably more involved than its unstable analogue, and takes up the majority of the second half of the paper. Therefore we now give an overview of the proof as a roadmap for the reader.

Firstly, we discuss the existence of the functor $\mathrm{Sp}_\bullet$. Recall that a G-spectrum can be thought as a pointed G-space together with a compatible collection of deloopings for all representation spheres. With modern tools we can give this construction a universal property: as a symmetric monoidal ∞-category Sp_G is obtained from the ∞-category of pointed G-spaces by freely inverting the representation spheres S^V for every G-representation V, see [GM23, Appendix C]. This universal property, combined with the unstable functor $\mathcal{S}_\bullet$ of Theorem 7.1.17, immediately gives the functoriality of G-spectra in $\mathrm{Glo}^{\mathrm{op}}$ as in our theorem.

Unfortunately, constructing the functor $\mathrm{Sp}_\bullet$ via the universal property of equivariant spectra is unhelpful for our purposes, as it is too inexplicit for calculating the partially lax limit. For example, note that for a surjective group homomorphism $\alpha\colon H \to G$ and G-spectrum E, to obtain the H-spectrum α^*E one has to freely add deloopings with respect to representation spheres not in the image of $\alpha^*\colon \mathrm{Rep}(G) \to \mathrm{Rep}(H)$. This is a process which one cannot easily control.

Therefore, pivotal to our proof is an explicit construction of the functor $\mathrm{Sp}_\bullet$. The calculation of the partially lax limit of $\mathrm{Sp}_\bullet$ will then follow from this by a long series of nontrivial formal arguments. The crucial idea is construct and calculate with a functoriality on prespectrum objects rather than at the level of spectrum objects. In this setting, we are able to built the functoriality of equivariant prespectra explicitly using the functoriality of the ∞-categories $\mathbf{O}_G$ and $\mathrm{Rep}(G)$, the category of representations and linear isometries.

To make this precise, let us first specify our model of G-prespectra. We define an ∞-category $\mathbf{OR}_G$, naturally fibred over $\mathbf{O}_G^{\mathrm{op}}$, whose objects are pairs (H, V), where H is a closed subgroup of G and V is a H-representation, see Definition 7.3.5. This is canonically symmetric promonoidal and so the ∞-category of functors $\mathrm{Fun}(\mathbf{OR}_G, \mathcal{S}_*)$ is symmetric monoidal via Day convolution. There is a functor $S_G\colon \mathbf{OR}_G \to \mathcal{S}_*$ which sends the object (H, V) to the pointed space $(S^V)^H$. This is a commutative algebra object in $\mathrm{Fun}(\mathbf{OR}_G, \mathcal{S}_*)$ via the universal property of Day convolution. The first ingredient of the proof is the following:

Step 1. The ∞-category Sp_G is equivalent to an explicit Bousfield localization of the ∞-category

$$\mathrm{PSp}_G := \mathrm{Mod}_{S_G} \mathrm{Fun}(\mathbf{OR}_G, \mathcal{S}_*).$$

We obtain this description by reinterpreting the construction of G-spectra as a Bousfield localization of the level model structure on orthogonal G-spectra internally to ∞-categories. This identification is the culmination of Sections 7.2 and 7.3, and the reader can find a precise statement as Proposition 7.2.30 and Corollary 7.3.14.

Having obtained this identification, we can build the functoriality of equivariant prespectra by exhibiting the pairs $(\mathbf{OR}_G, S_G)$ as functorial in $\mathrm{Glo}^{\mathrm{op}}$. In fact the categories $\mathbf{OR}_G$ will only be (pro)functorial in $\mathrm{Glo}^{\mathrm{op}}$, but this is a subtlety which we choose to gloss over in this introduction. To exhibit this functoriality, we build a global version of the category $\mathbf{OR}_G$ and the algebra object S_G, which we denote by $\mathbf{OR}_{\mathrm{gl}}$ and S_{gl}, see Definition 7.4.2. The ∞-category $\mathbf{OR}_{\mathrm{gl}}$ is naturally fibred over $\mathrm{Glo}^{\mathrm{op}}$ and has objects (G, V), where G is a compact

Lie group and V is a G-representation, and $S_{gl}\colon \mathbf{OR}_{\mathrm{gl}} \to \mathcal{S}_*$ sends (G,V) to the pointed space $(S^V)^G$.

There is a precise sense in which the pair $(\mathbf{OR}_{\mathrm{gl}}, S_{gl})$ contain all of the functoriality of the pairs $(\mathbf{OR}_G, S_G)$ in Glo. For the group direction this stems from the fact that the surjections and injections form an orthogonal factorization system on Glo, while for the representation direction this follows from the observation that $\mathbf{OR}_{\mathrm{gl}}$ is a cocartesian fibration over $\mathrm{Glo}^{\mathrm{op}}$ classifying the functor $\mathrm{Rep}(-)\colon \mathrm{Glo}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ which sends a compact Lie group G to its category of G-representations, with functoriality given by restriction. These observations allow us to prove the following result, see Proposition 7.4.16.

Step 2. There exists a functor

$$\mathrm{PSp}_\bullet\colon \mathrm{Glo}^{\mathrm{op}} \to \mathrm{Cat}_\infty^\otimes, \quad G \mapsto \mathrm{PSp}_G.$$

Furthermore the partially lax limit of $\mathrm{PSp}_\bullet$ over $(\mathrm{Glo}^{\mathrm{op}})^\dagger$ is given by the ∞-category $\mathrm{Mod}_{S_{gl}}\,\mathrm{Fun}(\mathbf{OR}_{\mathrm{gl}}, \mathcal{S}_*)$.

We have shown in Step 1 that Sp_G is a Bousfield localization of PSp_G. We call a map in PSp_G a stable equivalence if it is inverted by the functor $\mathrm{PSp}_G \to \mathrm{Sp}_G$.

Step 3. The diagram $\mathrm{PSp}_\bullet$ preserves stable equivalences, and therefore induces a diagram $\mathrm{Sp}_\bullet$. Furthermore, as indicated by the notation, this diagram is equivalent to the functoriality of equivariant spectra built at the beginning of this section using the universal property of Sp_G.

In particular, on morphisms this diagram gives the standard restriction-inflation functors on equivariant spectra, see Corollary 7.5.6. The following result follows formally from this.

Step 4. The partially lax limit of $\mathrm{Sp}_\bullet$ is given by an explicit Bousfield localization of the ∞-category

$$\mathrm{Mod}_{S_{gl}}\,\mathrm{Fun}(\mathbf{OR}_{\mathrm{gl}}, \mathcal{S}_*).$$

Finally, we compare this ∞-category to Schwede's model of global spectra, $\mathrm{Sp}_{\mathrm{gl}}$. Once again we do this by first translating his construction into one internal to ∞-categories. We define an ∞-category $\mathbf{OR}_{\mathrm{fgl}}$ as the subcategory of $\mathbf{OR}_{\mathrm{gl}}$ spanned by the objects (G,V), where V is a *faithful* G-representations. Restricting S_{gl} we obtain a commutative algebra object S_{fgl} in $\mathrm{Fun}(\mathbf{OR}_{\mathrm{fgl}}, \mathcal{S}_*)$. We then show:

Step 5. $\mathrm{Sp}_{\mathrm{gl}} \subset \mathrm{Mod}_{S_{\mathrm{fgl}}}(\mathrm{Fun}(\mathbf{OR}_{\mathrm{fgl}}, \mathcal{S}_*))$ is an explicit Bousfield localization.

The precise statement is obtained by combining Proposition 7.2.27 and Corollary 7.3.23. Finally we show in Section 7.6 that the canonical inclusion $j\colon \mathbf{OR}_{\mathrm{fgl}} \to \mathbf{OR}_{\mathrm{gl}}$ induces an adjunction

$$j_!\colon \mathrm{Mod}_{S_{\mathrm{fgl}}}(\mathrm{Fun}(\mathbf{OR}_{\mathrm{fgl}}, \mathcal{S}_*)) \leftrightarrows \mathrm{Mod}_{S_{gl}}(\mathrm{Fun}(\mathbf{OR}_{\mathrm{gl}}, \mathcal{S}_*)) : j^*$$

on prespectrum objects. Then we show that this adjunction descends to an adjunction on the corresponding Bousfield localizations of Steps 4 and 5. Finally we prove that the fibrancy conditions imposed by these localizations cancel out the difference between all and faithful representations, so that we obtain an equivalence

$$\mathrm{Sp}_{\mathrm{gl}} \simeq \mathrm{laxlim}^{\dagger}\, \mathrm{Sp}_{\bullet},$$

concluding the proof of Theorem 7.6.10.

Finally let us note that to fill in all of the details of this argument requires a long list of technical results about the relationship between various constructions applied to model categories and ∞-categories, Day convolution monoidal structures induced by promonoidal categories, and partially lax limits of symmetric monoidal categories. We have included these in Part I to make the paper self-contained, and because we failed to find a convenient reference for many of these facts.

Related work

There are many models of global unstable homotopy theory. The first was given in [GH07], and since then others have been obtained in [Sch18] and [Sch20]. The second of these papers, together with [KÏ8], proves that all these models induce the same ∞-category. Finally, we would like to mention the unpublished manuscript [Rez14], which contains many of the ideas we exploit in Section 7.1.

There has been a lot of work towards finding a good framework for the study of global stable homotopy theory, see [Boh14; GM97] and [Lew+86, Chapter II]. Schwede's model [Sch18] has so far being the most successful one, in part because of its numerous applications to equivariant stable homotopy theory, see for example [Sch17] and [Hau22]. Hausmann [Hau19] gave a model for global homotopy theory for the family of finite groups by endowing the category of symmetric spectra with a global model structure. There is also a model for G-global homotopy theory [Len20] which is a synthesis between classical equivariant homotopy theory and Schwede's global homotopy theory. This specializes to global homotopy theory by setting G to be the trivial group. Recently, Lenz [Len22] gave an ∞-categorical model for global stable homotopy theory for the family of finite groups using spectral Mackey functors. However to the best of our knowledge, our model is the first ∞-categorical model for global stable homotopy theory for the family of all compact Lie groups and not just the finite ones.

Future directions

In this paper we focused only on global and proper equivariant homotopy theory, but it is quite natural to wonder if we can recover our two results as a special case of a more general one. For any Lie group G, we can in fact consider G-global homotopy theory which is a generalization of global and G-equivariant homotopy theory. We conjecture that G-global stable homotopy theory is equivalent to the partially lax limit of the functor $\mathrm{Sp}_\bullet$ restricted along the canonical functor $\mathrm{Glo}_{/G}^{\mathrm{op}} \to \mathrm{Glo}^{\mathrm{op}}$.

Organization of the paper

The paper is divided into three main parts.

In the first part we first discuss the relationship between model and ∞-categories. Then we recall the concept of a promonoidal ∞-category and use this to define the Day convolution product on functor categories. We then introduce the notions of partially lax (co)limits and collect various useful results that we will need throughout the paper. We finish Part I by describing the lax limits of symmetric monoidal ∞-categories in terms of the operadic norm functor.

The second part of the paper contains the proofs of our main results. In Section 7.1 we introduce the ∞-category of global spaces and prove Theorem 7.1.17. This is an unstable version of Theorem 7.6.10, and serves as a warm up for the considerably more involved proof of the stable case. We therefore recommend the reader to read this section before moving forward. In Section 7.2 we recall various model structures on the categories of orthogonal G-spectra for a Lie group G, and hence define the underlying ∞-categories of proper G-spectra and of global spectra. In Section 7.3 we apply a variant of Elmendorf's theorem and use this to provide specific models for the ∞-categories of proper G-prespectra and global prespectra. In Section 7.4 we construct the functor $\mathrm{Sp}_\bullet$ from the introduction, and in Section 7.6 we identify the partially lax limits with the ∞-category of global spectra. Finally in Section 7.7, we apply the same techniques to describe the ∞-category of proper G-spectra as a limit, proving Theorem 7.7.11.

The third part of the paper contains an appendix on the tensor product of modules in an ∞-category.

Acknowledgements

The authors thank Stefan Schwede for first suggesting that a description of global spectra as a partially lax limit should be possible. We would also like to thank Stefan Schwede, Lennart Meier, Markus Hausmann, Branko Juran and Bastiaan Cnossen for numerous helpful conversations on this topic. We also thank the referees for helpful suggestions which improved the paper.

During the preparation of this text the first author was a member of the Hausdorff Center for Mathematics at the University of Bonn funded by the German Research Foundation (DFG). This material is based upon work supported by the Swedish Research Council under grant no. 2016-06596 while the third author was in residence at Institut Mittag-Leffler in Djursholm, Sweden during the semester *Higher algebraic structures in algebra, topology, and geometry*. The first author was supported by the DFG Schwerpunktprogramm 1786 "Homotopy Theory and Algebraic Geometry" (project ID SCHW 860/1-1). The third author was supported by the SFB 1085 Higher Invariants in Regensburg.

6. Partially lax limits, promonoidal ∞-categories and Day convolution

In this part of the paper we introduce the necessary machinery to state and prove our main results. In the first section we give references for the passage from topological/model categories to ∞-categories. We then discuss the Day convolution product for functor ∞-categories, where the source is only assumed to be a promonoidal ∞-category. Finally we recall the notion of partially lax limits of ∞-categories and symmetric monoidal ∞-categories, and proof some useful properties about them.

6.1 FROM TOPOLOGICAL/MODEL CATEGORIES TO ∞-CATEGORIES

In this paper we will often need to pass from topological categories (or operads) and (symmetric monoidal) model categories to ∞-categories. In this section we recall how this is done, and provide relevant references. After this section we will largely leave these identifications implicit for the rest of the paper.

6.1.1 *Topological categories and operads*

We can promote a topological category $\mathcal{C}$ to an ∞-category by first applying the singular functor to the mapping spaces (see [Lur09, Section 1.1.4]) and then applying the coherent (also called simplicial) nerve functor [Lur09, Corollary 1.1.5.12]. This defines a functor

$$\mathrm{TopCat} \to \mathrm{Cat}_\infty$$

from topological categories to ∞-categories. Importantly, applying this functor to a topologically enriched category $\mathcal{C}$ preserves the set of objects and the weak homotopy type of the mapping space between any two objects, see [Lur09, Theorem 1.1.5.13]. Throughout this paper we will not distinguish between the topological category and its ∞-categorical counterpart.

There is a similar functorial construction between topological operads and ∞-operads, which we now recall. Given a topological coloured operad $\mathcal{O}$, we let $\mathcal{O}^{\otimes}$ denote the topological category whose objects are pairs $(I_+, (C_i)_{i\in I})$ where

$I_+ \in \mathrm{Fin}_*$ and C_i are colours in $\mathcal{O}$. Given a pair of objects $C = (I_+, \{C_i\}_{i\in I})$ and $D = (J_+, \{D_j\}_{j\in J})$ in $\mathcal{O}^\otimes$, the morphism space $\mathcal{O}^\otimes(C, D)$ is given by

$$\coprod_{\alpha\colon I_+\to J_+} \prod_{j\in J} \mathcal{O}(\{C_i\}_{\alpha(i)=j}, D_j).$$

Composition is defined in the obvious way. This is the topological analogue of [Lur16, Notation 2.1.1.22]. Note that $\mathcal{O}^\otimes$ admits a functor to Fin_*. By the process before, this induces a functor of ∞-categories $\mathcal{O}^\otimes \to \mathrm{Fin}_*$.

Lemma 6.1.1. *Let $\mathcal{O}$ be a topological coloured operad. Then the forgetful functor $p\colon \mathcal{O}^\otimes \to \mathrm{Fin}_*$ defines an ∞-operad. Moreover this construction is functorial in the sense that it sends maps of topological coloured operads to maps of ∞-operads.*

Proof. Recall that a topological category is seen as an ∞-category by applying the singular functor on mapping spaces and then by applying the coherent nerve functor to the resulting simplicial category. Since the singular functor preserves products and sends every object to a fibrant one, it sends the topological coloured operad $\mathcal{O}$ to a fibrant[1] simplicial operad $\mathcal{O}_s$. Moreover by direct inspection, the singular functor sends the topological category $\mathcal{O}^\otimes$ defined above to $\mathcal{O}_s^\otimes$ as defined in [Lur16, Notation 2.1.1.22]. Applying the coherent nerve to $\mathcal{O}_s^\otimes \to \mathrm{Fin}_*$ we obtain an ∞-operad by [Lur16, Proposition 2.1.1.27], proving the first claim. A simple check shows that the formation of the topological category $\mathcal{O}^\otimes$ is functorial in maps of topological operads. Applying the singular functor and the coherent nerve then gives a functor of ∞-categories over Fin_*. Furthermore the cocartesian edges over inert edges are explicitly constructed in the proof of [Lur16, Proposition 2.1.1.27], and the functor constructed clearly preserves these edges. □

6.1.2 Model categories and ∞-categories

We will very often pass from model categories to ∞-categories. Therefore we explain and give references for this passage.

Let $\mathcal{M}$ be a model category with class of weak equivalences denoted by W. We always assume that $\mathcal{M}$ has functorial factorizations. The model category $\mathcal{M}$ presents an ∞-category which we denote by $\mathcal{M}[W^{-1}]$. We may define $\mathcal{M}[W^{-1}]$ as the Dwyer-Kan localization of $N(\mathcal{M})$ at the weak equivalences of $\mathcal{M}$, i.e. as the initial ∞-category with a functor from $\mathcal{M}$ which inverts the morphisms in W. Write $\mathcal{M}^f$, $\mathcal{M}^c$, and $\mathcal{M}^\circ$ for the full subcategories of $\mathcal{M}$ spanned by the fibrant, cofibrant and bifibrant objects respectively. The composite

$$N(\mathcal{M}^f) \to N(\mathcal{M}) \to \mathcal{M}[W^{-1}]$$

[1] Recall that a simplicial operad is fibrant if each multispace is a fibrant simplicial set, see [Lur16, Definition 2.1.1.26].

is a Dwyer-Kan localization at the restriction of W to $\mathcal{M}^f$, and similarly for the case of cofibrant and bifibrant objects. See for example the discussion in [Lur16, Remark 1.3.4.16].

If $\mathcal{M}$ is a topological model category, then the enriched structure gives another construction of $\mathcal{M}[W^{-1}]$. In this case, $\mathcal{M}[W^{-1}]$ is equivalent to the ∞-category associated to the topologically enriched category $\mathcal{M}^\circ$ as in the previous section, see [Lur16, Theorem 1.3.4.20]. Throughout our paper it will be necessary to use all these different constructions of $\mathcal{M}[W^{-1}]$.

We note that if the model category $\mathcal{M}$ is cofibrantly generated and the underlying category is locally presentable, then $\mathcal{M}[W^{-1}]$ is a presentable ∞-category, see [Lur16, Proposition 1.3.4.22]. Also we note that any Quillen adjunction of model categories $F\colon \mathcal{M}_0 \rightleftarrows \mathcal{M}_1 : G$ induces an adjunction of underlying ∞-categories $F\colon \mathcal{M}_0[W_0^{-1}] \rightleftarrows \mathcal{M}_1[W_1^{-1}] : G$ by [Hin16, Proposition 1.5.1].

Next we may consider symmetric monoidal model categories. By [Lur16, Proposition 4.1.7.6], if $\mathcal{M}$ is a symmetric monoidal model category then the ∞-category $\mathcal{M}[W^{-1}]$ admits a symmetric monoidal structure such that the localization functor $\mathcal{M}^c \to \mathcal{M}[W^{-1}]$ is strong monoidal, and if F is a symmetric monoidal left Quillen functor then F is again symmetric monoidal.

Once again we obtain a different construction of the symmetric monoidal ∞-category $\mathcal{M}[W^{-1}]$ when $\mathcal{M}$ is topological. Namely one can first restrict to bifibrant objects and then form the topological coloured operad $N^{\otimes}(\mathcal{M})$ with colors $X \in \mathcal{M}^\circ$ and multi-morphism spaces

$$\mathrm{Mul}_{N^{\otimes}(\mathcal{M}^\circ)}(\{X_1, \ldots, X_n\}, Y) = \mathrm{Map}_{\mathcal{M}^\circ}(X_1 \otimes \cdots \otimes X_n, Y).$$

This then gives an ∞-operad by Lemma 6.1.1. By [Lur16, Proposition 4.1.7.10] this is in fact a symmetric monoidal ∞-category whose underlying ∞-category is equivalent to $\mathcal{M}[W^{-1}]$. Furthermore, by [Lur16, Corollary 4.1.7.16], these two methods of obtaining a symmetric monoidal structure on $\mathcal{M}[W^{-1}]$ are equivalent.

6.1.3 Pointed categories

Many of the typical constructions one applies to model categories admit an analogue internally to ∞-categories. Furthermore, in many cases these constructions are not only analogous but in fact equivalent.

For example we may consider the formation of pointed objects. Given a model category $\mathcal{M}$ with final object $*$, we can equip the slice category $\mathcal{M}_* = \mathcal{M}_{*/}$ with a model structure in which fibrations, cofibrations and weak equivalences are detected by the forgetful functor $\mathcal{M}_* \to \mathcal{M}$, see [Hov99, Proposition 1.1.8]. If $\mathcal{M}$ is cofibrantly generated with set of generating cofibrations I and set of generating acyclic cofibrations J, then $\mathcal{M}_*$ is also cofibrantly generated by the sets I_+ and J_+, see [Hov99, Lemma 2.1.21]. If $\mathcal{M}$ is symmetric monoidal with cofibrant unit given by $*$, then the slice category $\mathcal{M}_*$ with the smash product is again a symmetric monoidal model category with cofibrant unit, see [Hov99, Proposition 4.2.9].

Let us now discuss the same construction for ∞-categories. Given a presentable symmetric monoidal ∞-category $(\mathcal{C}, \otimes)$, we can endow the slice $\mathcal{C}_* = \mathcal{C}_{*/}$ with a symmetric monoidal structure $\wedge_\otimes$ given as follows: for all $(* \to C), (* \to D) \in \mathcal{C}_*$, we define $C \wedge_\otimes D$ by the following pushout in $\mathcal{C}$:

$$\begin{tikzcd} C \otimes * \sqcup * \otimes D \arrow[r] \arrow[d] & C \otimes D \arrow[d] \\ * \otimes * \arrow[r] & C \wedge_\otimes D. \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

The existence of such symmetric monoidal structure on $\mathcal{C}_*$ is a formal consequence of [Lur16, Proposition 4.8.2.11] as we now explain. Indeed the cited reference shows that the functor $(-)_*\colon \mathrm{Pr}^{\mathrm{L}} \to \mathrm{Pr}^{\mathrm{L}}_*$ from presentable ∞-categories to pointed presentable ∞-categories is a smashing localization, so it induces a functor on commutative algebras $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}) \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_*)$ showing that a symmetric monoidal structure on $\mathcal{C}_*$ exists. Furthermore [Lur16, Proposition 4.8.2.11] implies that this symmetric monoidal structure is uniquely determined by the condition that the tensor product on $\mathcal{C}_*$ commutes with colimits on each variable and makes the functor $(-)_+\colon \mathcal{C} \to \mathcal{C}_*$ into a symmetric monoidal functor. From this one obtains the concrete description of $\wedge_\otimes$ as given above.

Example 6.1.2. Applying this construction to $\mathcal{S}$ with the cartesian product returns $\mathcal{S}_*$, the category of pointed spaces with the smash product. We write $\mathcal{S}^\times$ for the ∞-operad giving the former, and $\mathcal{S}_*^\wedge$ for the latter.

We now give a result that connects these two constructions.

Proposition 6.1.3. *Let $\mathcal{M}$ be a symmetric monoidal model category with cofibrant final object, which is also the monoidal unit. Suppose that the underlying ∞-category $\mathcal{M}[W^{-1}]$ is presentable. Then the functor $(-)_+\colon \mathcal{M} \to \mathcal{M}_*$ induces a symmetric monoidal equivalence*

$$(\mathcal{M}[W^{-1}])_* \simeq \mathcal{M}_*[W^{-1}].$$

Proof. First we note that the underlying ∞-category $\mathcal{M}_*[W^{-1}]$ models the ∞-categorical slice $(\mathcal{M}[W^{-1}])_*$, see for example [Cis19, Corollary 7.6.13]. Note also that $(-)_+\colon \mathcal{M} \to \mathcal{M}_*$ is left Quillen and strong monoidal, and therefore we obtain a strong monoidal colimit preserving functor

$$(-)_+\colon \mathcal{M}[W^{-1}] \to \mathcal{M}_*[W^{-1}]$$

which is equivalent to the standard left adjoint $(-)_+$ under the equivalence $\mathcal{M}_*[W^{-1}] \simeq \mathcal{M}[W^{-1}]_*$ by inspection. Also, $\mathcal{M}_*[W^{-1}]$ is automatically presentable and closed monoidal. Now we can conclude the result, because there is a unique closed symmetric monoidal structure on $\mathcal{M}[W^{-1}]_*$ such that $(-)_+$ is strong monoidal. □

Next we consider the formation of module categories. Recall that given a symmetric monoidal ∞-category $\mathcal{C}$ and a commutative algebra object $S \in \mathrm{CAlg}(\mathcal{C})$, the category of S-modules in $\mathcal{C}$, $\mathrm{Mod}_S(\mathcal{C})$ is a symmetric monoidal category via the relative tensor product, constructed in [Lur16, Section 4.5.2]. We will always consider $\mathrm{Mod}_S(\mathcal{C})$ as symmetric monoidal in this way.

Proposition 6.1.4. *Let $\mathcal{M}$ be a symmetric monoidal and cofibrantly generated model category with weak equivalences W, generating cofibrations I and generating acyclic cofibrations J, and let A be a commutative algebra object in $\mathcal{M}$ whose underlying object is cofibrant. Suppose that $\mathrm{Mod}_A(\mathcal{M})$ admits a symmetric monoidal and cofibrantly generated model structure where fibrations and weak equivalences are tested on underlying objects, and the sets $A \otimes I$ and $A \otimes J$ form a set of generating cofibrations and generating acyclic cofibrations respectively. Write W_m for the class of weak equivalences in $\mathrm{Mod}_A(\mathcal{M})$. Then applying Mod_A to the functor $\mathcal{M}^c \to \mathcal{M}[W^{-1}]$ induces a symmetric monoidal equivalence*

$$\mathrm{Mod}_A(\mathcal{M})[W_m^{-1}] \simeq \mathrm{Mod}_A(\mathcal{M}[W^{-1}]).$$

Proof. This is essentially [Lur16, Theorem 4.3.3.17]. However since the statement there does not literally apply, let us spell out the argument. We need to show that there exists a symmetric monoidal equivalence

$$\theta\colon N(\mathrm{Mod}_A(\mathcal{M})^c)[W_m^{-1}] \xrightarrow{\simeq} \mathrm{Mod}_A(N(\mathcal{M}^c)[W^{-1}]).$$

We start by noting that the forgetful functor $U\colon \mathrm{Mod}_A(\mathcal{M}) \to \mathcal{M}$ is left Quillen. One can verify this by observing that U sends the generating (acyclic) cofibrations to (acyclic) cofibrations, using that A is cofibrant and that $\mathcal{M}$ satisfies the pushout-product axiom. Since a cofibrant A-module is then also cofibrant in $\mathcal{M}$, there exists a symmetric monoidal functor

$$N(\mathrm{Mod}_A(\mathcal{M})^c) \to N(\mathrm{Mod}_A(\mathcal{M}^c)) \simeq \mathrm{Mod}_A(N(\mathcal{M}^c)).$$

Postcomposing with the symmetric monoidal functor $N(\mathcal{M}^c) \to N(\mathcal{M}^c)[W^{-1}]$ and using the universal property of symmetric monoidal localization we obtain a symmetric monoidal functor θ as claimed. To show that θ is an equivalence, we apply [Lur16, Corollary 4.7.3.16] to the diagram

$$\begin{tikzcd}
N(\mathrm{Mod}_A(\mathcal{M})^c)[W_m^{-1}] \arrow[rr, "\theta"] \arrow[dr, "U"'] & & \mathrm{Mod}_A(N(\mathcal{M}^c)[W^{-1}]) \arrow[dl, "U'"] \\
& N(\mathcal{M}^c)[W^{-1}]. &
\end{tikzcd}$$

We need to check:

(a) The ∞-categories $N(\mathrm{Mod}_A(\mathcal{M})^c)[W_m^{-1}]$ and $\mathrm{Mod}_A(N(\mathcal{M}^c)[W^{-1}])$ admit geometric realization of simplicial objects. In fact, both categories admit all colimits. For $N(\mathrm{Mod}_A(\mathcal{M})^c)[W_m^{-1}]$ this is [BHH17, Theorem 2.5.9]. For $\mathrm{Mod}_A(N(\mathcal{M}^c)[W^{-1}])$, we note that $N(\mathcal{M}^c)[W^{-1}]$ admits all colimits by the previous reference and that these can be calculated as homotopy colimits in the model category by [BHH17, Remark 2.5.7]. Since A is cofibrant, the functor $A \otimes -: \mathcal{M} \to \mathcal{M}$ is left Quillen and so it induces a colimit preserving functor $N(\mathcal{M}^c)[W^{-1}] \to N(\mathcal{M}^c)[W^{-1}]$ by [Hin16, Proposition 1.5.1]. Finally, we can invoke [Lur16, Proposition 4.3.3.9] to deduce the existence of all colimits in $\mathrm{Mod}_A(N(\mathcal{M}^c)[W^{-1}])$.

(b) The functors U and U' admits left adjoints F and F'. The existence of a left adjoint to U follows from the fact that U is determined by a right Quillen functor. The existence of a left adjoint to U' follows from [Lur16, Corollary 4.3.3.14].

(c) The functor U' is conservative and preserves geometric realizations of simplicial objects. This follows from [Lur16, Corollary 4.3.3.2, Proposition 4.3.3.9].

(d) The functor U is conservative and preserves geometric realizations of simplicial objects. The first assertion is immediate from the definition of the weak equivalences in $\mathrm{Mod}_A(\mathcal{M})$, and the second follows from the fact that U is also a left Quillen functor.

(e) The natural map $U' \circ F' \to U \circ F$ is an equivalence. Unwinding the definitions, we are reduced to proving that if N is a cofibrant object of $\mathcal{M}$, then the natural map $N \to A \otimes N$ induces an equivalence $F'(N) \simeq A \otimes N$. This follows from the explicit description of F' given in [Lur16, Corollary 4.3.3.13].

□

Remark 6.1.5. Suppose $\mathcal{M}$ is a symmetric monoidal cofibrantly generated model category. If $\mathcal{M}$ is locally presentable, then the existence of the model structure on $\mathrm{Mod}_A(\mathcal{M})$ as in Proposition 6.1.4 holds by [SS00, Remark 4.2].

6.2 PROMONOIDAL ∞-CATEGORIES AND DAY CONVOLUTION

We start this section by recalling the notion of a promonoidal ∞-category. We recall the definition of the operadic norm functor and use this to define the Day convolution product on a functor category. We then collecting various important results about the Day convolution product which will be important later. We finish the section by giving a symmetric monoidal recognition criteria for presheaf categories, inspired by Elmendorf's theorem.

We start off by recalling the following useful notion from [AF20, Definition 0.7].

Definition 6.2.1. A functor $p\colon \mathcal{C} \to \mathcal{B}$ between ∞-categories is an *exponentiable fibration* if the pullback functor $p^*\colon \mathrm{Cat}_{\infty/\mathcal{B}} \to \mathrm{Cat}_{\infty/\mathcal{C}}$ admits a right adjoint p_*, which we call the pushforward.

Example 6.2.2. Both cocartesian and cartesian fibrations are exponentiable, see [AF20, Lemma 2.15].

Example 6.2.3. Exponentiable fibrations are stable under pullbacks, see [AF20, Corollary 1.17]

For any ∞-operad $\mathcal{O}^\otimes$, we let $\mathcal{O}^\otimes_{act} := \mathcal{O}^\otimes \times_{\mathrm{Fin}_*} \mathrm{Fin} \subseteq \mathcal{O}^\otimes$ denote the subcategory of active arrows. We recall the following definition from [Sha21, Definition 10.2].

Definition 6.2.4. Let $\mathcal{O}^\otimes$ be an ∞-operad. A map of ∞-operads $p\colon \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ defines a $\mathcal{O}^\otimes$-*promonoidal* ∞-category if the restricted functor $p_{act}\colon \mathcal{C}^\otimes_{act} \to \mathcal{O}^\otimes_{act}$ is exponentiable. A functor of $\mathcal{O}^\otimes$-promonoidal ∞-categories is simply a map of $\mathcal{O}^\otimes$-operads.

Example 6.2.5. Any $\mathcal{O}^\otimes$-symmetric monoidal ∞-category is $\mathcal{O}^\otimes$-promonoidal by Example 6.2.2.

Example 6.2.6. Let $\mathcal{C}$ be an ∞-category. Then the ∞-operad $\mathcal{C}^\amalg \to \mathrm{Fin}_*$ of [Lur16, Construction 2.4.3.1] is a symmetric promonoidal ∞-category. In fact

$$\mathcal{C}^\amalg \times_{\mathrm{Fin}_*} \mathrm{Fin} \to \mathrm{Fin}$$

is the cartesian fibration which classifies the functor sending I to $\mathrm{Fun}(I, \mathcal{C})$.

Example 6.2.7. Consider a cartesian fibration $p\colon \mathcal{C} \to \mathcal{I}$. Similarly to Example 6.2.6, one can show that the induced map $p^\amalg\colon \mathcal{C}^\amalg \to \mathcal{I}^\amalg$ exhibits $\mathcal{C}^\amalg$ as a $\mathcal{I}^\amalg$-promonoidal ∞-category.

The key property of promonoidal ∞-categories is that they induce operadic norm functors.

Definition 6.2.8. Let $p\colon \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ be a $\mathcal{O}^\otimes$-promonoidal ∞-category. Then the functor

$$p^*\colon (\mathrm{Op}_\infty)_{/\mathcal{O}^\otimes} \to (\mathrm{Op}_\infty)_{/\mathcal{C}^\otimes}$$

has a right adjoint by [Sha21, Theorem/Construction 10.6], which we denote by N_p and call the *norm* along p. Note that p^* also has a left adjoint $p_!$ which is given by postcomposition with p.

The norm interacts well with pullbacks along maps of ∞-operads.

Lemma 6.2.9. *Let $p\colon \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ be a $\mathcal{O}^\otimes$-promonoidal ∞-category and let $f\colon \mathcal{P}^\otimes \to \mathcal{O}^\otimes$ be a map of ∞- operads. Write $p'\colon \mathcal{C}^\otimes \times_{\mathcal{O}^\otimes} \mathcal{P}^\otimes \to \mathcal{P}^\otimes$ and $f'\colon \mathcal{C}^\otimes \times_{\mathcal{O}^\otimes} \mathcal{P}^\otimes \to \mathcal{O}^\otimes$ for the functors obtained via basechange. Then there is a natural equivalence of functors*

$$f^* N_p \simeq N_{p'}(f')^*\colon (\mathrm{Op}_\infty)_{/\mathcal{C}^\otimes} \to (\mathrm{Op}_\infty)_{/\mathcal{P}^\otimes}\,.$$

In other words, for every $\mathcal{D}^\otimes \in (\mathrm{Op}_\infty)_{/\mathcal{C}^\otimes}$ there is an equivalence of ∞-operads over $\mathcal{P}^\otimes$

$$N_p(\mathcal{D}^\otimes) \times_{\mathcal{O}^\otimes} \mathcal{P}^\otimes \simeq N_{p'}(\mathcal{D}^\otimes \times_{\mathcal{O}^\otimes} \mathcal{P}^\otimes)\,.$$

Proof. To check that two right adjoint functors are equivalent it is enough to check that the left adjoints are equivalent. But the left adjoint of f^* is just postcomposition with f, so the thesis is equivalent to the fact that for every $\mathcal{E}^\otimes \in (\mathrm{Op}_\infty)_{/\mathcal{P}^\otimes}$, there is a natural equivalence

$$\mathcal{E}^\otimes \times_{\mathcal{O}^\otimes} \mathcal{C}^\otimes \simeq \mathcal{E}^\otimes \times_{\mathcal{P}^\otimes} (\mathcal{P}^\otimes \times_{\mathcal{O}^\otimes} \mathcal{C}^\otimes)$$

and this is clear. □

Remark 6.2.10. In a similar vein we observe that because $q^*p^* \simeq (pq)^*$, also $N_{pq} \simeq N_p N_q$.

Remark 6.2.11. Recall that passing to underlying ∞-categories gives a functor $U\colon \mathrm{Op}_\infty \to \mathrm{Cat}_\infty$ which admits a left adjoint F with essential image precisely those ∞-operads $q\colon \mathcal{P}^\otimes \to \mathrm{Fin}_*$ such that the functor q factors through $\mathrm{Triv} \subseteq \mathrm{Fin}_*$, see [Lur16, Proposition 2.1.4.11]. In particular for any ∞-operad $\mathcal{O}^\otimes$, we obtain an adjunction on overcategories:

$$F\colon (\mathrm{Cat}_\infty)_{/\mathcal{O}} \rightarrow (\mathrm{Op}_\infty)_{/\mathcal{O}^\otimes} :U,$$

see [Lur09, Proposition 5.2.5.1] Let $p\colon \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ be a $\mathcal{O}^\otimes$-promonoidal ∞-category; we will now describe the effect of N_p on underlying ∞-categories. Observe that the underlying map $U(p)$ on ∞-categories is exponentiable, as it can be described as the pullback of p along $\mathcal{O} \subseteq \mathcal{O}^\otimes$, compare with Example 6.2.3. One can compute that the following diagram of left adjoints

$$\begin{tikzcd}
(\mathrm{Op}_\infty)_{/\mathcal{C}^\otimes} & (\mathrm{Op}_\infty)_{/\mathcal{O}^\otimes} \arrow[l, "p^*"'] \\
(\mathrm{Cat}_\infty)_{/\mathcal{C}} \arrow[u, "F"] & (\mathrm{Cat}_\infty)_{/\mathcal{O}} \arrow[l, "U(p)^*"] \arrow[u, "F"']
\end{tikzcd}$$

commutes. Therefore the associated diagram of right adjoints

$$\begin{tikzcd}
(\mathrm{Op}_\infty)_{/\mathcal{C}^\otimes} \arrow[r, "N_p"] \arrow[d, "U"'] & (\mathrm{Op}_\infty)_{/\mathcal{O}^\otimes} \arrow[d, "U"] \\
(\mathrm{Cat}_\infty)_{/\mathcal{C}} \arrow[r, "U(p)_*"'] & (\mathrm{Cat}_\infty)_{/\mathcal{O}}
\end{tikzcd}$$

also commutes, and we conclude that on underlying categories N_p is given by the pushforward $U(p)_*$.

We can now define the Day convolution functor.

Definition 6.2.12. Let $p\colon \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ be a $\mathcal{O}^\otimes$-promonoidal ∞-category. The *Day convolution functor*

$$\mathrm{Fun}_{\mathcal{O}}(\mathcal{C},-)^{Day}\colon (\mathrm{Op}_\infty)_{/\mathcal{O}^\otimes} \to (\mathrm{Op}_\infty)_{/\mathcal{O}^\otimes}$$

is the right adjoint of the functor

$$p_!p^* = - \times_{\mathcal{O}^\otimes} \mathcal{C}^\otimes\colon (\mathrm{Op}_\infty)_{/\mathcal{O}^\otimes} \to (\mathrm{Op}_\infty)_{/\mathcal{O}^\otimes}.$$

This is a composite of right adjoints, and so we conclude that $\mathrm{Fun}_{\mathcal{O}}(\mathcal{C},-)^{Day} \simeq N_p\, p^*(-)$. This also shows the existence of $\mathrm{Fun}_{\mathcal{O}}(\mathcal{C},-)^{Day}$. When $\mathcal{O} = \mathrm{Fin}_*$, we will omit it from the notation.

Remark 6.2.13. Recall that $\mathrm{Alg}_{\mathcal{C}^\otimes}(\mathcal{D}^\otimes)$ is defined to be the full subcategory of $\mathrm{Fun}_{/\mathrm{Fin}_*}(\mathcal{C}^\otimes, \mathcal{D}^\otimes)$ spanned by the maps of operads, and that taking the maximal sub ∞-groupoid of this category gives the mapping spaces $\mathrm{Op}_\infty(\mathcal{C}^\otimes, \mathcal{D}^\otimes)$. Therefore we may view $\mathrm{Alg}_{(-)}(-)$ as constituting an enrichment of Op_∞ in Cat_∞. A standard argument shows that the adjunction equivalence

$$\mathrm{Op}_\infty(\mathcal{P}^\otimes, \mathrm{Fun}(\mathcal{J}^\otimes, \mathcal{C}^\otimes)^{Day}) \simeq \mathrm{Op}_\infty(\mathcal{P}^\otimes \times_{\mathrm{Fin}_*} \mathcal{J}^\otimes, \mathcal{C}^\otimes)$$

improves to an equivalence

$$\mathrm{Alg}_{\mathcal{P}^\otimes}(\mathrm{Fun}(\mathcal{J}^\otimes, \mathcal{C}^\otimes)^{Day}) \simeq \mathrm{Alg}_{\mathcal{P}^\otimes \times_{\mathrm{Fin}_*} \mathcal{J}^\otimes}(\mathcal{C}^\otimes).$$

Example 6.2.14. Recall from Example 6.2.6 that for any ∞-category $\mathcal{C}$, the ∞-operad $\mathcal{C}^\amalg \to \mathrm{Fin}_*$ is promonoidal. For every ∞-operad $\mathcal{D}^\otimes$, the Day convolution ∞-operad $\mathrm{Fun}(\mathcal{C}^\amalg, \mathcal{D}^\otimes)^{Day}$ is equivalent to the pointwise operad structure on $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. Indeed they corepresent the same functor by [Lur16, Theorem 2.4.3.18].

The description of Day convolution combined with Remark 6.2.11 implies that on underlying categories $\mathrm{Fun}_{\mathcal{O}}(\mathcal{C}^\otimes, -)^{Day}$ is given by $U(p)_*U(p)^*$. We can describe the fibres of this category explicitly.

Construction 6.2.15. Let $p\colon \mathcal{C} \to \mathcal{B}$ be an exponentiable fibration of ∞-categories and $q\colon \mathcal{D} \to \mathcal{B}$ any functor. Fix an arrow $f\colon b_0 \to b_1$ in $\mathcal{B}$ and let us write $\mathcal{C}_{b_i}$ and $\mathcal{D}_{b_i}$ for the fibres of p and q over b_i. The unit of the adjunction (p^*, p_*) gives a canonical functor $p_*p^*\mathcal{D} \to \mathcal{B}$ whose fibre over b_i can be identified with

$$(p_*p^*\mathcal{D})_{b_i} \simeq \mathrm{Fun}_{\mathcal{B}}(\{b_i\}, p_*p^*\mathcal{D}) \simeq \mathrm{Fun}_{\mathcal{C}}(\mathcal{C} \times_{\mathcal{B}} \{b_i\}, \mathcal{C} \times_{\mathcal{B}} \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{C}_i, \mathcal{D}_i)\,. \tag{6.2.15.1}$$

Remark 6.2.16. One should be careful to note that, if the underlying ∞-category $\mathcal{O}$ of $\mathcal{O}^{\otimes}$ is not contractible, then the underlying ∞-category of $\mathrm{Fun}_{\mathcal{O}}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})^{Day}$ is not the same as the ∞-category of functors over $\mathcal{O}$. Rather, it is a fibration over $\mathcal{O}$ whose global sections are $\mathrm{Fun}_{/\mathcal{O}}(\mathcal{C}, \mathcal{D})$. Compare also with the previous construction.

We would like to have a formula for the multimapping spaces for the Day convolution. We will achieve this in Lemma 6.2.25 below. In preparation for this result, we compute the mapping spaces in a pushforward. To state the result we recall the definition of twisted arrow ∞-categories, and the notion of coends.

Definition 6.2.17. Let $\epsilon\colon \Delta \to \Delta$ be the functor $[n] \mapsto [n] \star [n]^{\mathrm{op}} \simeq [2n+1]$. Let $\mathcal{I}$ be an ∞-category. The twisted arrow ∞-category $\mathrm{Tw}(\mathcal{I})$ is the associated ∞-category of the simplicial set $\epsilon^* N\mathcal{I}$. By definition, we have

$$\mathrm{Tw}(\mathcal{I})_n = \mathrm{Map}(\Delta^n \star (\Delta^n)^{\mathrm{op}}, \mathcal{I}).$$

The natural transformations $\Delta^\bullet$ and $(\Delta^\bullet)^{\mathrm{op}} \to \Delta^\bullet \star (\Delta^\bullet)^{\mathrm{op}}$ induce a functor $(s,t)\colon \mathrm{Tw}(\mathcal{I}) \to \mathcal{I} \times \mathcal{I}^{\mathrm{op}}$.

Remark 6.2.18. There are two possible conventions for defining $\mathrm{Tw}(-)$. In this paper we follow that of Lurie [Lur16, Section 5.2.1]. This is the opposite of the convention used in [Bar17].

Example 6.2.19. The objects of $\mathrm{Tw}(\mathcal{I})$ are given by edges of $\mathcal{I}$. An edge from $f\colon x \to y$ to $f'\colon x' \to y'$ in $\mathrm{Tw}(\mathcal{I})$ is represented by a diagram

$$\begin{tikzcd} x \arrow[r] \arrow[d, "f"'] & x' \arrow[d, "f'"] \\ y & y' \arrow[l] \end{tikzcd}$$

Remark 6.2.20. The twisted arrow category is insensitive to taking opposites, meaning that $\mathrm{Tw}(\mathcal{I}^{\mathrm{op}}) \simeq \mathrm{Tw}(\mathcal{I})$. However under this equivalence (s,t) is sent to (t,s).

Definition 6.2.21. Given a functor $F\colon \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$, we define the coend $\int^{x \in \mathcal{C}} F(x,x)$ to equal the colimit of the functor

$$\mathrm{Tw}(\mathcal{C}) \xrightarrow{(s,t)} \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \xrightarrow{F} \mathcal{S}.$$

Dually for a functor $F\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$, we define the end $\int_{x \in \mathcal{C}} F(x,x)$ to be the limit of the functor

$$\mathrm{Tw}(\mathcal{C})^{\mathrm{op}} \xrightarrow{(s,t)^{\mathrm{op}}} \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{F} \mathcal{S}.$$

We are now ready to state the formula for multimapping spaces in the Day convolution.

Lemma 6.2.22. *Suppose we are in the setting of Construction 6.2.15. Let $F_i\colon \mathcal{C}_i \to \mathcal{D}_i$ be two objects of $(p_*p^*\mathcal{D})_{b_i}$, viewed as such via the equivalence (6.2.15.1). Then there is an equivalence*

$$\mathrm{Map}^f_{p_*p^*\mathcal{D}}(F_0, F_1) \simeq \int \mathrm{Map}(\mathrm{Map}^f_{\mathcal{C}}(x_0, x_1), \mathrm{Map}^f_{\mathcal{D}}(F_0x_0, F_1x_1)) \qquad (6.2.22.1)$$

where the left hand side denotes the fibre over $f\colon b_0 \to b_1$ of the canonical map $\mathrm{Map}_{p_*p^*\mathcal{D}}(F_0, F_1) \to \mathrm{Map}_{\mathcal{B}}(b_0, b_1)$.

Proof. Let us write f as a map $\Delta^1 \to \mathcal{B}$. Then, by the definition of p_* there is an equivalence

$$\mathrm{Map}_{\mathcal{B}}(\Delta^1, p_*p^*\mathcal{D}) \simeq \mathrm{Map}_{\mathcal{C}}(\Delta^1 \times_{\mathcal{B}} \mathcal{C}, \mathcal{C} \times_{\mathcal{B}} \mathcal{D}) \simeq \mathrm{Map}_{\Delta^1}(\Delta^1 \times_{\mathcal{B}} \mathcal{C}, \Delta^1 \times_{\mathcal{B}} \mathcal{D})\,.$$

Therefore we have an equivalence

$$\begin{aligned}\mathrm{Map}^f_{p_*p^*\mathcal{D}}(F_0, F_1) &\simeq \{(F_0, F_1)\} \times_{\mathrm{Map}_{\mathcal{B}}(\partial\Delta^1, p_*p^*\mathcal{D})} \mathrm{Map}_{\mathcal{B}}(\Delta^1, p_*p^*\mathcal{D}) \\ &\simeq \{(F_0, F_1)\} \times_{\mathrm{Map}(\mathcal{C}_0,\mathcal{D}_0)\times\mathrm{Map}(\mathcal{C}_1,\mathcal{D}_1)} \mathrm{Map}_{\Delta^1}(\Delta^1 \times_{\mathcal{B}} \mathcal{C}, \Delta^1 \times_{\mathcal{B}} \mathcal{D})\,.\end{aligned}$$

Now from the proof of [AF20, Lemma 4.2] it follows that the map

$$\mathrm{Cat}_{\infty/\Delta^1} \to \mathrm{Cat}_\infty \times \mathrm{Cat}_\infty \qquad [\mathcal{C} \to \Delta^1] \mapsto (\mathcal{C} \times_{\Delta^1} \{0\}, \mathcal{C} \times_{\Delta^1} \{1\})\,,$$

is a right fibration classified by the functor $\mathrm{Cat}_\infty \times \mathrm{Cat}_\infty \to \mathcal{S}$ sending $(\mathcal{C}_0, \mathcal{C}_1)$ to $\mathrm{Map}(\mathcal{C}_0^{op} \times \mathcal{C}_1, \mathcal{S})$. Therefore

$$\begin{aligned}\{(F_0, F_1)\} \times_{\mathrm{Map}(\mathcal{C}_0,\mathcal{D}_0)\times\mathrm{Map}(\mathcal{C}_1,\mathcal{D}_1)} &\mathrm{Map}_{\Delta^1}(\Delta^1 \times_{\mathcal{B}} \mathcal{C}, \Delta^1 \times_{\mathcal{B}} \mathcal{D}) \\ &\simeq \mathrm{Map}^{(F_0,F_1)}_{\mathrm{Cat}_{\infty/\Delta^1}}(\Delta^1 \times_{\mathcal{B}} \mathcal{C}, \Delta^1 \times_{\mathcal{B}} \mathcal{D}) \\ &\simeq \mathrm{Map}_{(\mathrm{Cat}_{\infty/\Delta^1})_{(\mathcal{C}_0,\mathcal{C}_1)}}(\Delta^1 \times_{\mathcal{B}} \mathcal{C}, (F_0, F_1)^*(\Delta^1 \times_{\mathcal{B}} \mathcal{D})) \\ &\simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{C}_0^{\mathrm{op}}\times\mathcal{C}_1,\mathcal{S})}(\mathrm{Map}^f_{\mathcal{C}}(-,-), \mathrm{Map}^f_{\mathcal{D}}(F_0-, F_1-))\,.\end{aligned}$$

But this is exactly the thesis, thanks to [GHN17, Proposition 5.1]. □

Remark 6.2.23. In the setting of Lemma 6.2.22, suppose that q is equal to the projection $\mathcal{D} \times \mathcal{B} \to \mathcal{B}$ and that $\mathcal{D}$ is cocomplete. Then we can interpret formula 6.2.22.1 as saying that $p_*p^*\mathcal{D}$ is a cocartesian fibration and that given $f\colon i \to j$, the induced functor

$$f_!\colon \mathrm{Fun}(\mathcal{C}_i, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}_j, \mathcal{D})$$

evaluated on a functor $F\colon \mathcal{C}_i \to \mathcal{D}$ gives the functor

$$\mathcal{C}_j \to \mathcal{D},\ x_j \mapsto \int^{x_i \in \mathcal{C}_i} \mathrm{Map}_{\mathcal{C}_{ij}}(x_i, x_j) \times F(x_i),$$

where $\mathcal{C}_{ij} := \mathcal{C} \times_{\mathcal{B},f} [1]$. That is, $f_! F$ is computed by left Kan extending F along the inclusion $\mathcal{C}_i \subseteq \mathcal{C}_{ij}$ and then restricting to $\mathcal{C}_j \subseteq \mathcal{C}_{ij}$. In particular, if $\mathcal{C}_{ij} \to [1]$ is a cartesian fibration we have $f_! F \simeq F \circ f^*$ where $f^*\colon \mathcal{C}_1 \to \mathcal{C}_0$ is the pullback.

Recall the following notion of multimapping spaces.

Definition 6.2.24. Let $\mathcal{C}^\otimes \to \mathcal{O}^\otimes$ be a map of ∞-operads and let $\phi\colon \{x_i\} \to y$ be an active morphism of $\mathcal{O}^\otimes$ with target in $\mathcal{O} := (\mathcal{O}^\otimes)_{1_+}$. For every $\{c_i\} \in (\mathcal{C}^\otimes)_{\{x_i\}} \simeq \prod_i \mathcal{C}_{x_i}$ and $d \in \mathcal{C}_y$, objects of $\mathcal{C}^\otimes$ over the source and target of ϕ, we define the ϕ-multimapping space in $\mathcal{C}^\otimes$ as the space of morphisms $\{c_i\} \to d$ above ϕ:

$$\mathrm{Mul}^{\phi}_{\mathcal{C}^\otimes}(\{c_i\}, d) := \mathrm{Map}_{\mathcal{C}^\otimes}(\{c_i\}, d) \times_{\mathrm{Map}_{\mathcal{O}^\otimes}(\{x_i\}, y)} \{\phi\}\,.$$

We say that $\mathcal{C}^\otimes$ is *representable* if for every active morphism ϕ and objects $\{c_i\}$, the functor

$$\mathrm{Mul}^{\phi}_{\mathcal{C}^\otimes}(\{c_i\}, -)\colon \mathcal{C} \to \mathcal{S}$$

is corepresentable. In this case we write $\bigotimes_\phi \{c_i\}$ for the corepresenting object and we call it the ϕ-tensor product of $\{c_i\}$. This is equivalent to the functor $\mathcal{C}^\otimes \to \mathcal{O}^\otimes$ being a locally cocartesian fibration.

We are ready to prove the formula for the multimapping spaces in the Day convolution.

Lemma 6.2.25. *Let $\mathcal{O}^\otimes$ be an ∞-operad, $\mathcal{C}^\otimes$ be an $\mathcal{O}^\otimes$-promonoidal ∞-category and $\mathcal{D}^\otimes$ be an ∞-operad over $\mathcal{O}^\otimes$. Then the multimapping spaces in $\mathrm{Fun}_{\mathcal{O}}(\mathcal{C}^\otimes, \mathcal{D}^\otimes)^{Day}$ are given by the following natural equivalence*

$$\mathrm{Mul}^{\phi}_{\mathrm{Fun}_{\mathcal{O}}(\mathcal{C}^\otimes,\mathcal{D}^\otimes)^{Day}}(\{F_i\}, G) \simeq \int_{c' \in \mathcal{C}_y} \int_{\{c_i\} \in (\prod_i \mathcal{C}_{x_i})^{\mathrm{op}}} \mathrm{Map}\left(\mathrm{Mul}^{\phi}_{\mathcal{C}}(\{c_i\}, c'), \mathrm{Mul}^{\phi}_{\mathcal{D}}(\{F_i c_i\}, Gc')\right)$$

for all active morphisms $\phi\colon \{x_i\} \to y$, and objects $\{F_i\} \in \prod_i \mathrm{Fun}(\mathcal{C}_{x_i}, \mathcal{D}_{x_i})$, $G \in \mathrm{Fun}(\mathcal{C}_y, \mathcal{D}_y)$.

Proof. We will use [Lur16, Proposition 2.2.6.6]. However the cited result has the hypothesis that $\mathcal{C}^\otimes$ is a $\mathcal{O}^\otimes$-monoidal ∞-category. We note that this is only used to ensure the existence of the norm (after replacing the appeal to [Lur09, Proposition 3.3.1.3] with [Lur16, Proposition B.3.14]). Therefore, in view of [Sha21, Theorem-Construction 10.6] we can safely apply this result when $\mathcal{C}^\otimes$ is only $\mathcal{O}^\otimes$-promonoidal.

Then, arguing as in the proof of [Lur16, Proposition 2.2.6.11], we obtain an equivalence

$$\mathrm{Mul}^{\phi}_{\mathrm{Fun}_{\mathcal{O}}(\mathcal{C}^{\otimes},\mathcal{D}^{\otimes})^{Day}}(\{F_i\},G) \simeq \{(F,G)\}\times_{\mathrm{Fun}_{/\mathcal{O}^{\otimes}}(\partial\Delta^1\times_{\mathcal{O}^{\otimes}}\mathcal{C}^{\otimes},\mathcal{D}^{\otimes})}\mathrm{Fun}_{/\mathcal{O}^{\otimes}}(\Delta^1\times_{\mathcal{O}^{\otimes}}\mathcal{C}^{\otimes},\mathcal{D}^{\otimes})$$

where $\Delta^1 \to \mathcal{O}^{\otimes}$ picks out the active arrow ϕ and $F\colon \mathcal{C}^{\otimes}_{\{x_i\}} \to \mathcal{D}^{\otimes}_{\{x_i\}}$ is the functor sending $\{c_i\}$ to $\{F_i c_i\}$. Let $\mathcal{C}^{act} := \mathcal{C}^{\otimes}\times_{\mathcal{O}^{\otimes}}\mathcal{O}^{act}$ and $\mathcal{D}^{act} := \mathcal{D}^{\otimes}\times_{\mathcal{O}^{\otimes}}\mathcal{O}^{act}$ be the subcategories of active arrows. Since $\Delta^1 \to \mathcal{O}^{\otimes}$ factors through $\mathcal{O}^{act}$, we have an equivalence

$$\begin{aligned}\mathrm{Mul}_{\mathrm{Fun}(\mathcal{C}^{\otimes},\mathcal{D}^{\otimes})^{Day}}(\{F_i\}_{i\in I},G) &\simeq \{(F,G)\}\times_{\mathrm{Fun}_{/\mathcal{O}^{act}}(\partial\Delta^1\times_{\mathcal{O}^{act}}\mathcal{C}^{act},\mathcal{D}^{act})}\mathrm{Fun}_{/\mathcal{O}^{act}}(\Delta^1\times_{\mathrm{Fin}}\mathcal{C}^{act},\mathcal{D}^{act})\\ &\simeq \mathrm{Map}_{(p^{act})_*(p^{act})^*\mathcal{D}^{act}}(F,G)\end{aligned}$$

where the last equality makes sense since p^{act} is an exponentiable fibration. Therefore the thesis follows from Lemma 6.2.22. □

Definition 6.2.26. We say that an $\mathcal{O}^{\otimes}$-monoidal ∞-category $\mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes}$ is *compatible with colimits* if for every object $x \in \mathcal{O}$ the fibre $\mathcal{D}_x$ has all small colimits, and if for every active arrow ϕ, the ϕ-tensor product commutes with all small colimits separately in each variable (see [Lur16, Definition 3.1.1.18] for a more precise formulation). If each fibre is moreover presentable, then we say $\mathcal{D}^{\otimes}$ is a presentably $\mathcal{O}^{\otimes}$-monoidal ∞-category.

Example 6.2.27. The underlying ∞-category of a symmetric monoidal model category is compatible with colimits as the tensor product is a left Quillen bifunctor by the pushout-product axiom.

Remark 6.2.28. Recall that every cocomplete ∞-category $\mathcal{C}$ is canonically tensored over $\mathcal{S}$. Namely for every $X \in \mathcal{S}$ and $C \in \mathcal{C}$, we define $X \times C$ to equal $\mathrm{colim}(\mathrm{const}_C\colon X \to \mathcal{C})$, the colimit over X of the constant functor at C.

Corollary 6.2.29. *Fix an ∞-operad $\mathcal{O}^{\otimes}$. Let $\mathcal{C}^{\otimes}$ be a small $\mathcal{O}^{\otimes}$-promonoidal ∞-category and let $\mathcal{D}^{\otimes}$ be a $\mathcal{O}^{\otimes}$-monoidal ∞-category which is compatible with colimits.*

(a) Then $\mathrm{Fun}_{\mathcal{O}}(\mathcal{C}^{\otimes},\mathcal{D}^{\otimes})^{Day}$ is an $\mathcal{O}^{\otimes}$-monoidal ∞-category which is again compatible with colimits.

Suppose furthermore that $\mathcal{O}^{\otimes} \simeq \mathrm{Fin}_*$ is the commutative ∞-operad.

(b) The unit of $\mathrm{Fun}(\mathcal{C}^{\otimes},\mathcal{D}^{\otimes})^{Day}$ is given by $1_{Day} := \mathrm{Mul}_{\mathcal{D}}(\varnothing,-)\times 1_{\mathcal{D}}$, and the tensor product is given by

$$(F\otimes^{Day} G)(-) \simeq \int^{(c_1,c_2)\in\mathcal{C}^2}\mathrm{Mul}_{\mathcal{C}}(\{c_1,c_2\},-)\times(F(c_1)\otimes G(c_2))\,.$$

In particular, when $\mathcal{D}$ is the ∞-category of spaces with the cartesian symmetric monoidal structure, we have

$$\mathrm{Map}_{\mathcal{C}}(x,-) \otimes^{Day} \mathrm{Map}_{\mathcal{C}}(y,-) \simeq \mathrm{Mul}_{\mathcal{C}}(\{x,y\},-)$$

for every $x, y \in \mathcal{C}$.

Proof. If $\mathcal{D}^{\otimes}$ is $\mathcal{O}^{\otimes}$-monoidal, it follows from the formula of Lemma 6.2.25 that $\mathrm{Fun}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})^{Day}$ is representable and that the ϕ-tensor product is given by

$$\bigotimes_{\phi} \{F_i\}_{i \in I} \simeq \int^{\{c_i\} \in \prod_{i \in I} \mathcal{C}_{o_i}} \mathrm{Mul}^{\phi}_{\mathcal{C}^{\otimes}}(\{c_i\}_{i \in I}, -) \times \bigotimes_{\phi} \{F_i(c_i)\}_{i \in I} \,.$$

This shows the existence of locally cartesian edges in $\mathrm{Fun}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})^{Day}$. Because the tensor product functors in $\mathcal{D}^{\otimes}$ commutes with colimits in each variable, one can calculate that the composite of locally cartesian edges is locally cartesian, and therefore $\mathrm{Fun}_{\mathcal{O}}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})^{Day}$ is a $\mathcal{O}^{\otimes}$-monoidal ∞-category. The fibres are clearly cocomplete, and from the formula for the tensor product it follows that the tensor in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\otimes}$ commutes with colimits in each variable.

Finally the statement for the tensor product of corepresentable functors follows from the formula above and the Yoneda lemma. □

Notation 6.2.30. Suppose we are in the situation of the previous corollary, and suppose that $\mathcal{O}^{\otimes} \simeq \mathrm{Fin}_*$. In the case that both $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ are canonically (pro)monoidal, then we write $\mathcal{C}\text{–}\mathcal{D}$ for the symmetric monoidal category given by the ∞-operad $\mathrm{Fun}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})^{Day}$. The two examples which will arise constantly are $\mathcal{C}\text{–}\mathcal{S}$ and $\mathcal{C}\text{–}\mathcal{S}_*$, where $\mathcal{S}$ is symmetric monoidal via the cartesian product, and $\mathcal{S}_*$ via the smash product. Nevertheless when we refer to the ∞-operad inducing the symmetric monoidal structure on $\mathcal{C}\text{–}\mathcal{D}$, we will continue to write $\mathrm{Fun}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})^{Day}$. While this distinction is mathematically meaningless, we find it notationally convenient.

We next turn to the functoriality of Day convolution.

Construction 6.2.31. Let $\mathcal{O}^{\otimes}$ be an ∞-operad and suppose $f\colon \mathcal{I}^{\otimes} \to \mathcal{J}^{\otimes}$ is a map of $\mathcal{O}^{\otimes}$-promonoidal ∞-categories. Then for every two ∞-operads $\mathcal{C}^{\otimes}$ and $\mathcal{P}^{\otimes}$ over $\mathcal{O}^{\otimes}$ we have a natural transformation

$$\begin{aligned}\mathrm{Alg}_{\mathcal{P}^{\otimes}/\mathcal{O}^{\otimes}}(\mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{J}^{\otimes}, \mathcal{C}^{\otimes})^{Day}) \simeq \mathrm{Alg}_{\mathcal{P}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{J}^{\otimes}}(\mathcal{C}^{\otimes}) \to \mathrm{Alg}_{\mathcal{P}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{I}^{\otimes}}(\mathcal{C}^{\otimes}) \simeq \dots \\ \dots \simeq \mathrm{Alg}_{\mathcal{P}^{\otimes}/\mathcal{O}^{\otimes}}(\mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{I}^{\otimes}, \mathcal{C}^{\otimes})^{Day}), \quad (6.2.31.1)\end{aligned}$$

given by precomposition along $\mathcal{P}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{I}^{\otimes} \to \mathcal{P}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{J}^{\otimes}$. Since this is natural in $\mathcal{P}^{\otimes}$, it induces a map in $(\mathrm{Op}_{\infty})_{/\mathcal{O}^{\otimes}}$

$$f^*\colon \mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{J}^{\otimes}, \mathcal{C}^{\otimes})^{Day} \to \mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{I}^{\otimes}, \mathcal{C}^{\otimes})^{Day} \,.$$

Definition 6.2.32. Consider $\mathcal{C}^\otimes, \mathcal{D}^\otimes \in (\mathrm{Op}_\infty)_{/\mathcal{O}^\otimes}$. An *operadic adjunction* between $\mathcal{C}^\otimes$ and $\mathcal{D}^\otimes$ is a relative adjunction over $\mathcal{O}^\otimes$ in the sense of [Lur16, Definition 7.3.2.2] such that both functors are maps of ∞-operads. This notion is equivalent to an adjunction in the $(\infty, 2)$-category of ∞-operads, see [RV16, Observation 4.3.2].

Remark 6.2.33. Note that if $\mathcal{C}^\otimes$ and $\mathcal{D}^\otimes$ are both $\mathcal{O}^\otimes$-monoidal then an operadic left adjoint $f : \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ is automatically $\mathcal{O}^\otimes$-monoidal by [Lur16, Proposition 7.3.2.6].

Proposition 6.2.34. *Let $\mathcal{O}^\otimes$ be an ∞-operad and let $f : \mathcal{I}^\otimes \to \mathcal{J}^\otimes$ a map of $\mathcal{O}^\otimes$-promonoidal ∞-categories. Suppose $\mathcal{C}^\otimes$ is a presentably $\mathcal{O}^\otimes$-monoidal ∞-category. Let us consider the lax $\mathcal{O}^\otimes$-monoidal functor*

$$f^* \colon \mathrm{Fun}_{\mathcal{O}^\otimes}(\mathcal{J}^\otimes, \mathcal{C}^\otimes)^{Day} \to \mathrm{Fun}_{\mathcal{O}^\otimes}(\mathcal{I}^\otimes, \mathcal{C}^\otimes)^{Day} .$$

(a) Suppose that for every active arrow $\phi : \{t_i\}_i \to t$ in $\mathcal{O}^\otimes$ the natural map

$$(f_t)_! \,\mathrm{Mul}^\phi_{\mathcal{I}}(\{x_i\}_i, -) \to \mathrm{Mul}^\phi_{\mathcal{J}}(\{f_{t_i} x_i\}_i, -)$$

adjoint to

$$\mathrm{Mul}^\phi_{\mathcal{I}}(\{x_i\}_i, -) \to \mathrm{Mul}^\phi_{\mathcal{J}}(\{f_{t_i} x_i\}_i, f_t(-))$$

is an equivalence for every family of objects $\{x_i\}_i$. Then f^* has a left operadic adjoint $f_!$ that is $\mathcal{O}^\otimes$-monoidal;

(b) Suppose f has an operadic right adjoint $g \colon \mathcal{J}^\otimes \to \mathcal{I}^\otimes$. Then there is a natural equivalence of maps of ∞-operads $f_! \simeq g^*$, and moreover this functor is $\mathcal{O}^\otimes$-monoidal.

Proof. We will use [Lur16, Proposition 7.3.2.11] applied to the functor f^* over $\mathcal{O}^\otimes$. Since on the fibre over $t_i \in \mathcal{O}$ this is just given by precomposition by f_{t_i}, the functor on the fibre over $\{t_i\}_i$

$$\prod_i \mathrm{Fun}(\mathcal{J}_{t_i}, \mathcal{C}_{t_i}) \to \prod_i \mathrm{Fun}(\mathcal{I}_{t_i}, \mathcal{C}_{t_i})$$

has a left adjoint, given by the left Kan extension $(f_{t_i})_!$ on every component. In particular, this collection of left adjoints commutes with the pushforwards along inert maps. So it suffices to show that this collection of left adjoints commute with the pushforwards along active maps. Let $\phi : (t_i)_i \to t$ be an active map. Then we need to show that the map

$$(f_t)_! \left(\bigotimes_i^{\phi} F_i \right) \to \bigotimes_i^{\phi} (f_{t_i})_! F_i$$

is an equivalence. But then this follows from our hypothesis together with the description of Corollary 6.2.29.

Suppose now that f has an operadic right adjoint g. Since g^* is an operadic left adjoint to f^*, it follows immediately that $f_! = g^*$. So it remains only to check the two final conditions. But we have

$$(f_t)_! \operatorname{Mul}^{\phi}_{\mathcal{I}}(\{x_i\}_i, -) \simeq \operatorname{Mul}^{\phi}_{\mathcal{I}}(\{x_i\}_i, g_t -) \simeq \operatorname{Mul}^{\phi}_{\mathcal{J}}(\{f_{t_i} x_i\}_i, -)$$

since g is an operadic right adjoint of f. □

Remark 6.2.35. Note that if $\mathcal{O}^{\otimes} = \mathrm{Fin}_*$ and $\mathcal{I}^{\otimes}$ and $\mathcal{J}^{\otimes}$ are both symmetric monoidal, then the conditions ensuring the symmetric monoidality of $f_!$ are equivalent to f being a symmetric monoidal functor (since $f_!$ restricts to f on representables). Thus the above proposition gives an alternative proof of [BMS23, Proposition 3.6].

6.2.1 Symmetric monoidal structures on copresheaf categories

We finish this section by classifying all possible closed symmetric monoidal structures on the copresheaf ∞-category $\operatorname{Fun}(\mathcal{I}, \mathcal{S})$ in terms of symmetric promonoidal structures on $\mathcal{I}$, see Theorem 6.2.37.

Lemma 6.2.36. *Let $\mathcal{I}$ be a small ∞-category and let us suppose that the presheaf category $\operatorname{Fun}(\mathcal{I}, \mathcal{S})$ is equipped with a symmetric monoidal structure $\operatorname{Fun}(\mathcal{I}, \mathcal{S})^{\otimes}$ which is compatible with colimits. Equip $\mathcal{I}$ with the full suboperad structure $\mathcal{I}^{\otimes}$ induced by the Yoneda embedding $\mathcal{I} \subseteq \operatorname{Fun}(\mathcal{I}, \mathcal{S})^{\mathrm{op}}$. Then $\mathcal{I}^{\otimes}$ is symmetric promonoidal.*

Proof. For brevity let us write $\mathcal{D}^{\otimes} = \operatorname{Fun}(\mathcal{I}, \mathcal{S})^{\otimes}$. Recall from Definition 6.2.4 that $\mathcal{I}^{\otimes}$ is promonoidal if the functor $\mathcal{I}^{\otimes} \to \mathrm{Fin}_*$ is exponentiable over $\mathrm{Fin} \simeq (\mathrm{Fin}_*)^{act}$. By the characterization of exponentiablility in [AF20, Lemma 1.10.(c)], we need to show that for every map $f : I \to J$ in Fin, every $x \in \mathcal{I}^I$ and every $z \in \mathcal{I}$ the map

$$\int^{y \in \mathcal{I}^J} \operatorname{Mul}_{\mathcal{I}}(\{y_j\}_{j \in J}, z) \times \prod_{j \in J} \operatorname{Mul}_{\mathcal{I}}(\{x_i\}_{i \in f^{-1}j}, y_j) \to \operatorname{Mul}_{\mathcal{I}}(\{x_i\}_{i \in I}, z)$$

is an equivalence. Using that $\mathcal{I} \subseteq \mathcal{D}^{\mathrm{op}}$ is a full suboperad, this is equivalent to asking that the map

$$\int^{y \in \mathcal{I}^J} \prod_{j \in J} \operatorname{Map}_{\mathcal{D}}(y_j, \bigotimes_{i \in f^{-1}j} x_i) \times \operatorname{Map}_{\mathcal{D}}(z, \bigotimes_{j \in J} y_j) \to \operatorname{Map}_{\mathcal{D}}(z, \bigotimes_{i \in I} x_i)$$

is an equivalence of spaces. But since $\operatorname{Map}_{\mathcal{D}}(z, -)$ commutes with all colimits (as $z \in \mathcal{I}$ is tiny) it is enough to show that the map

$$\int^{y \in \mathcal{I}^J} \left(\prod_{j \in J} \operatorname{Map}_{\mathcal{D}}(y_j, \bigotimes_{i \in f^{-1}j} x_i) \right) \otimes \bigotimes_{j \in J} y_j \to \bigotimes_{i \in I} x_i \,,$$

is an equivalence. Since the tensor product in $\mathcal{D}$ commutes with colimits in each variable, we can bring all the colimits inside (using that $\mathrm{Tw}(\mathcal{I}^J) \simeq \mathrm{Tw}(\mathcal{I})^J$). We are reduced to proving that the map

$$\bigotimes_{j\in J} \int^{y_j \in \mathcal{C}} \mathrm{Map}_{\mathcal{D}}(y_j, \bigotimes_{i\in f^{-1}j} x_i) \otimes y_j \to \bigotimes_{i\in I} x_i$$

is an equivalence. But this follows from the fact that for any $j \in J$ and $w \in \mathcal{D}$, the map

$$\int^{y_j \in \mathcal{C}} \mathrm{Map}(y_j, w) \times y_j \simeq \operatorname*{colim}_{y_j \in \mathcal{C}_{/w}} y_j \to w$$

is an equivalence, which is just another form of the Yoneda lemma. □

We are ready to prove our classification result.

Theorem 6.2.37. *Let $\mathcal{I}$ be a small ∞-category and suppose $\mathrm{Fun}(\mathcal{I}, \mathcal{S})$ is equipped with a symmetric monoidal structure $\mathrm{Fun}(\mathcal{I}, \mathcal{S})^{\otimes}$ which is compatible with colimits. Equip $\mathcal{I}^{\otimes}$ with the ∞-operad structure induced by the Yoneda embedding $\mathcal{I} \subseteq \mathrm{Fun}(\mathcal{I}, \mathcal{S})^{\mathrm{op}}$. Then $\mathcal{I}^{\otimes}$ is symmetric promonoidal and the symmetric monoidal structure on $\mathrm{Fun}(\mathcal{I}, \mathcal{S})$ is equivalent to the one induced by Day convolution with the symmetric promonoidal structure on $\mathcal{I}^{\otimes}$.*

Proof. It follows from Lemma 6.2.36 that $\mathcal{I}^{\otimes}$ is symmetric promonoidal. Consider the composite

$$\mathcal{I}^{\otimes} \times_{\mathrm{Fin}_*} \mathrm{Fun}(\mathcal{I}, \mathcal{S})^{\otimes} \to (\mathrm{Fun}(\mathcal{I}, \mathcal{S})^{\mathrm{op}})^{\otimes} \times_{\mathrm{Fin}_*} \mathrm{Fun}(\mathcal{I}, \mathcal{S})^{\otimes} \to \mathcal{S}^{\times}$$

of lax symmetric monoidal functors, where the first functor is induced by the Yoneda embedding and the second is the lax symmetric monoidal enhancement of the mapping space functor constructed in [Gla16, Section 3]. By the universal property of the Day convolution, we obtain a map of ∞-operads

$$\mathrm{Fun}(\mathcal{I}, \mathcal{S})^{\otimes} \to \mathrm{Fun}(\mathcal{I}^{\otimes}, \mathcal{S}^{\times})^{\mathrm{Day}}$$

which is the identity on underlying ∞-categories. Therefore to prove our thesis it will suffice to show that this functor is symmetric monoidal. Since $\mathrm{Fun}(\mathcal{I}, \mathcal{S})$ is generated under colimits by the corepresentable functors and both tensor products commute with colimits in each variable, it is enough to check that the maps $\mathrm{Mul}_{\mathcal{I}}(\varnothing, -) \simeq 1 \to 1^{\mathrm{Day}}$ and

$$\mathrm{Mul}_{\mathcal{I}}(\{x, y\}, -) \simeq \mathrm{Map}_{\mathcal{I}}(x, -) \otimes \mathrm{Map}_{\mathcal{I}}(y, -) \to \mathrm{Map}_{\mathcal{I}}(x, -) \otimes^{\mathrm{Day}} \mathrm{Map}_{\mathcal{I}}(y, -)$$

are equivalences for all $x, y \in \mathcal{I}$. But this follows from Corollary 6.2.29. □

Recall that the ∞-category of pointed objects in a presentably symmetric monoidal ∞-category is canonically symmetric monoidal. For later use we also record how taking pointed objects in a category of diagram spaces interacts with the Day convolution symmetric monoidal structure.

Proposition 6.2.38. *Consider a small promonoidal ∞-category $\mathcal{I}$, and a presentably symmetric monoidal ∞-category $\mathcal{C}$. There exists a symmetric monoidal equivalence*

$$(\mathcal{I}\text{–}\mathcal{C})_* \simeq \mathcal{I}\text{–}\mathcal{C}_*$$

Proof. Consider the lax monoidal functor $\mathcal{I}\text{–}\mathcal{C} \to \mathcal{I}\text{–}\mathcal{C}_*$ induced by the universal property of Day convolution by the composite

$$\mathrm{Fun}(\mathcal{I}^\otimes, \mathcal{C}^\otimes) \times_{\mathrm{Fin}_*} \mathcal{I}^\otimes \to \mathcal{C}^\otimes \xrightarrow{(-)_+} (\mathcal{C}_*)^{\wedge\otimes}.$$

Because $(-)_+$ is strong monoidal and colimit preserving, one calculates that this functor is in fact strong monoidal. By [Lur16, Proposition 4.8.2.11] we obtain an induced strong monoidal functor $(\mathcal{I}\text{–}\mathcal{C})_* \to \mathcal{I}\text{–}\mathcal{C}_*$, which is easily seen to be the identity on underlying categories. □

6.2.2 A symmetric monoidal Elmendorf's theorem

In this subsection we give a general ∞-categorical version of Elmendorf's theorem. We then enhance this to a symmetric monoidal equivalence.

Theorem 6.2.39 (Elmendorf)**.** *Let $\mathcal{C}$ be a cocomplete ∞-category and let $i\colon \mathcal{C}_0 \to \mathcal{C}$ be the inclusion of a small full subcategory satisfying the following conditions:*

(a) The objects of $\mathcal{C}_0$ are tiny: for all $c \in \mathcal{C}_0$, the functor $\mathrm{Map}_{\mathcal{C}}(c, -)$ preserves small colimits;

(b) The collection of objects $\{c_0 \in \mathcal{C}_0\}$ is jointly conservative: an arrow f in $\mathcal{C}$ is an equivalence if and only if $\mathrm{Map}_{\mathcal{C}}(c_0, f)$ is so for all $c_0 \in \mathcal{C}_0$.

Then the restricted Yoneda functor induces an equivalence $j\colon \mathcal{C} \simeq \mathcal{P}(\mathcal{C}_0)$ of ∞-categories.

Proof. By the universal property of the category of presheaves [Lur09, Theorem 5.1.5.6], there exists a colimit preserving functor $L\colon \mathcal{P}(\mathcal{C}_0) \to \mathcal{C}$ such that $Lj_0 \simeq i$ where $j_0\colon \mathcal{C}_0 \to \mathcal{P}(\mathcal{C}_0)$ denotes the Yoneda embedding. By the adjoint functor theorem [NRS20, Corollary 4.1.4], the functor L admits a right adjoint $R\colon \mathcal{C} \to \mathcal{P}(\mathcal{C}_0)$ which is defined via the formula

$$R_c(c_0) = \mathrm{Map}_{\mathcal{C}}(Lj_0(c_0), c) \simeq \mathrm{Map}_{\mathcal{C}}(i(c_0), c)$$

for all $c \in \mathcal{C}$ and $c_0 \in \mathcal{C}_0$. Therefore R can be identified with the restricted Yoneda functor $j\colon \mathcal{C} \to \mathcal{P}(\mathcal{C}_0)$. We note that the functor j preserves all small colimits since for all $c_0 \in \mathcal{C}_0$, the functor

$$\mathrm{Map}_{\mathcal{C}}(c_0, -)\colon \mathcal{C} \xrightarrow{j} \mathcal{P}(\mathcal{C}_0) \xrightarrow{\mathrm{ev}_{c_0}} \mathcal{C}_0$$

does so by condition (a). As equivalences in $\mathcal{P}(\mathcal{C}_0)$ are detected pointwise, the same argument as above using condition (b) then shows that j is conservative. Note that the unit map $\eta\colon 1 \to jL$ is an equivalence on all objects in the image of j_0 as by construction $jLj_0 \simeq ji = j_0$. It follows that the unit map is an equivalence on all objects as $\mathcal{P}(\mathcal{C}_0)$ is generated under colimits by the representable functors and all the functors involved preserve colimits. Using the triangle identities of the adjunction we then find that $j(\epsilon)$ is an equivalence and so the counit map $\epsilon\colon Lj \to 1$ is an equivalence by conservativity of j. Thus j and L are inverse equivalences. □

Example 6.2.40. Let G be a topological group and let $G\mathcal{T}$ be a convenient category of G-spaces. There is a model structure on $G\mathcal{T}$ where a map $f\colon X \to Y$ of G-spaces is a weak equivalence (resp., fibration) if $f^H\colon X^H \to Y^H$ is a weak homotopy equivalence (resp., Serre fibration) for all closed subgroups $H \le G$, see [Sch18, Proposition B.7]. Let $\mathcal{S}_G$ denote the underlying ∞-category of this model category, which is cocomplete by [BHH17, Theorem 2.5.9]. Moreover colimits in $\mathcal{S}_G$ of projective cofibrant diagrams can be calculated as homotopy colimits in $G\mathcal{T}$ by [BHH17, Remark 2.5.7]. Let $\mathbf{O}_G \le \mathcal{S}_G$ be the full subcategory of G-spaces spanned by the cosets G/H where H runs over all closed subgroups of G. Note that $G/H \in \mathcal{S}_G$ corepresents the H-fixed points functors so the collection of cosets $\{G/H \mid H \le G\}$ is jointly conservative by definition of weak equivalences in $G\mathcal{T}$. The fact that $G/H \in \mathcal{S}_G$ is tiny then follows from the fact that the H-fixed points functor commutes with all small homotopy colimits [Sch18, Proposition B.1, (i) and (ii)]. Then the theorem above gives an equivalence $\mathbf{O}_G^{\mathrm{op}}\text{–}\mathcal{S} \simeq \mathcal{S}_G$. Therefore the previous theorem is a generalization of the classical Elmendorf's theorem [Elm83].

Under suitable assumptions we now enhance this to a symmetric monoidal equivalence, where we endow the presheaf category with Day convolution for a promonoidal structure on subcategory of tiny objects.

Corollary 6.2.41. *Suppose we are in the setting of Theorem 6.2.39 and that furthermore the following holds:*

(a) $\mathcal{C}$ admits a symmetric monoidal structure $\mathcal{C}^\otimes$ which is compatible with colimits;

(b) $\mathcal{C}_0$ admits an ∞-operad structure $\mathcal{C}_0^\otimes$;

(c) $i\colon \mathcal{C}_0 \to \mathcal{C}$ lifts to a fully faithful functor of ∞-operads $i^\otimes\colon \mathcal{C}_0^\otimes \to \mathcal{C}^\otimes$.

Then $\mathcal{C}_0^{\otimes}$ is a symmetric promonoidal ∞-category and the restricted Yoneda embedding induces a symmetric monoidal equivalence $\mathcal{P}(\mathcal{C}_0)^{Day} \simeq \mathcal{C}^{\otimes}$.

Proof. By Theorem 6.2.39 there is a commutative diagram

$$\begin{tikzcd} \mathcal{C}_0 \arrow[r, "i"] \arrow[d, "j_0"'] & \mathcal{C} \arrow[dl, "j", "\sim"'] \\ \mathcal{P}(\mathcal{C}_0) & \end{tikzcd} .$$

We can equip $\mathcal{P}(\mathcal{C}_0)$ with a symmetric monoidal structure $\mathcal{P}(\mathcal{C}_0)^{\otimes}$ induced by $\mathcal{C}^{\otimes}$ via j, and hence obtain a symmetric monoidal equivalence $j^{\otimes}\colon \mathcal{C}^{\otimes} \to \mathcal{P}(\mathcal{C}_0)^{\otimes}$. Combining this with condition (c) we obtain another commutative diagram

$$\begin{tikzcd} \mathcal{C}_0^{\otimes} \arrow[r, "i^{\otimes}"] \arrow[d, "j_0^{\otimes}"'] & \mathcal{C}^{\otimes} \arrow[dl, "j^{\otimes}", "\sim"'] \\ \mathcal{P}(\mathcal{C}_0)^{\otimes} & \end{tikzcd}$$

of ∞-operads. It is only left to note that by Theorem 6.2.37, the ∞-category $\mathcal{C}_0^{\otimes}$ is symmetric promonoidal and that the symmetric monoidal structure on $\mathcal{P}(\mathcal{C}_0)^{\otimes}$ coincides with the Day convolution product. □

6.3 PARTIALLY LAX LIMITS

In this section we recall the necessary background on (partially) lax (co)limits and collect some important properties that we will use throughout the paper. The main references for this material are [GHN17; Ber20].

The notion of a partially lax limit over an ∞-category $\mathcal{I}$ is defined with reference to a collection of edges of $\mathcal{I}$. To make this precise we make the following definition.

Definition 6.3.1. A marked ∞-category is an ∞-category $\mathcal{C}$ along with a collection of edges $\mathcal{W} \subseteq \mathrm{Map}(\Delta^1, \mathcal{C})$ which contains all equivalences and which is stable under homotopy and composition. Given two marked ∞-categories $\mathcal{C}$ and $\mathcal{D}$, we write $\mathrm{Fun}^{\dagger}(\mathcal{C}, \mathcal{D})$ for the subcategory spanned by marked functors; those functors that preserve marked edges. We write $\mathrm{Cat}_{\infty}^{\dagger}$ for the ∞-category of marked ∞-categories. For the existence see [Lur16, Construction 4.1.7.1].

Example 6.3.2. Let $\mathcal{C}$ be an ∞-category.

(a) There is a maximal marking $\mathcal{C}^{\sharp}$ where all morphisms are marked;

(b) There is a minimal marking $\mathcal{C}^{\flat}$ where only the equivalences are marked;

(c) Given a (co)cartesian fibration $p\colon \mathcal{C} \to \mathcal{I}^\dagger$ over a marked ∞-category, there is a marking $\mathcal{C}^p$ in which the (co)cartesian morphisms living over marked edges are marked.

Partially lax limits in an ∞-category $\mathcal{C}$ are also defined with reference to a cotensoring of $\mathcal{C}$ by Cat_∞. For the purposes of this paper, this is nothing but a functor $[-,-]\colon \mathrm{Cat}_\infty^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$. The following examples are all naturally cotensored over Cat_∞.

Example 6.3.3. In the following $\mathcal{I}$ is an ∞-category.

(a) Clearly Cat_∞ is cotensored over itself with cotensor given by $[\mathcal{I},\mathcal{C}] = \mathrm{Fun}(\mathcal{I},\mathcal{C})$.

(b) The ∞-category $\mathrm{Cat}_\infty^\dagger$ is cotensored over Cat_∞ by considering $[\mathcal{I},\mathcal{C}^\dagger] = \mathrm{Fun}(\mathcal{I},\mathcal{C}^\dagger)$, where we mark all those natural transformations whose components are all marked in $\mathcal{C}^\dagger$.

(c) The ∞-category of symmetric monoidal categories $\mathrm{Cat}_\infty^\otimes$ is cotensored over Cat_∞ by endowing the ∞-category $\mathrm{Fun}(\mathcal{I},\mathcal{C})$ with the pointwise symmetric monoidal structure $q\colon \mathrm{Fun}(\mathcal{I},\mathcal{C})^\otimes \to \mathrm{Fin}_*$ which is defined as follows. If $p\colon \mathcal{C}^\otimes \to \mathrm{Fin}_*$ is the cocartesian fibration witnessing the symmetric monoidal structure of $\mathcal{C}$, then we construct the following pullback

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{I},\mathcal{C})^\otimes \arrow[r, "q"] \arrow[d] & \mathrm{Fin}_* \arrow[d, "\mathrm{const}"] \\
\mathrm{Fun}(\mathcal{I},\mathcal{C}^\otimes) \arrow[r, "p_*"] & \mathrm{Fun}(\mathcal{I},\mathrm{Fin}_*).
\end{tikzcd}$$

Note that by construction we have $\mathrm{Fun}(\mathcal{I},\mathcal{C})^\otimes_{\langle n\rangle} \simeq \mathrm{Fun}(\mathcal{I},\mathcal{C}^\otimes_{\langle n\rangle})$ for all $\langle n\rangle \in \mathrm{Fin}_*$. From this we immediately see that q satisfies the Segal conditions. The map p_* is a cocartesian fibration by the dual of [Lur09, Proposition 3.1.2.1] and so by base-change [Lur09, Proposition 2.4.2.3] q is too. Therefore q gives a symmetric monoidal structure on $\mathrm{Fun}(\mathcal{I},\mathcal{C})$.

(d) We can generalize the previous example as follows. Let $\mathcal{O}^\otimes \to \mathrm{Fin}_*$ be an ∞-operad. The ∞-category of ∞-operads Op_∞ is cotensored over Cat_∞ by endowing the ∞-category $\mathrm{Fun}(\mathcal{I},\mathcal{O})$ with the pointwise operadic structure induced by the map $\mathrm{Fun}(\mathcal{I},\mathcal{O}^\otimes) \times_{\mathrm{Fun}(\mathcal{I},\mathrm{Fin}_*)} \mathrm{Fin}_* \to \mathrm{Fin}_*$.

Similarly partially lax colimits in $\mathcal{C}$ are defined with reference to a tensoring of $\mathcal{C}$ by Cat_∞. Once again, while more structured tensorings are typically useful, for our purposes it suffices for this to be a functor $(-)\otimes(-)\colon \mathrm{Cat}_\infty \times \mathcal{C} \to \mathcal{C}$. The most important example will be Cat_∞, for which the cartesian product gives a tensoring.

We now move on to the definition of partially lax (co)limits. For this we need to recall some categorical constructions. Recall the following result.

Lemma 6.3.4 ([Lur16, Proposition 4.1.7.2])**.** *The minimal functor* $(-)^\flat\colon \mathrm{Cat}_\infty \to \mathrm{Cat}_\infty^\dagger$ *admits a left adjoint denoted by* $|-|$.

The ∞-category $|\mathcal{C}^\dagger|$ is obtained from $\mathcal{C}$ by adjoining formal inverses to all the marked morphisms, and so we call $|-|$ the localization functor.

Example 6.3.5. Given a model category $\mathcal{M}$, we may view it as a marked ∞-category by marking the weak equivalences in $\mathcal{M}$. Then $|\mathcal{M}| \simeq \mathcal{M}[W^{-1}]$.

Next we define marked slice categories.

Construction 6.3.6. Let $\mathcal{C}$ be an ∞-category. There is a functor $\mathcal{C}_{/-}\colon \mathcal{C} \to \mathrm{Cat}_\infty$ sending $x \in \mathcal{C}$ to the slice category $\mathcal{C}_{/x}$. This is obtained by straightening the cocartesian fibration given by the target map $t\colon \mathrm{Ar}(\mathcal{C}) := \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$. One checks that a diagram

$$\begin{tikzcd} f_0 \arrow[r] \arrow[d, "f"'] & g_0 \arrow[d, "g"] \\ f_1 \arrow[r] & g_1 \end{tikzcd}$$

is a t-cocartesian edge if the top horizontal arrow is an equivalence. If $\mathcal{C}^\dagger$ is marked, then $\mathcal{C}^\dagger_{/x}$ has an induced marking where a morphism is marked if its image under the forgetful functor $\mathcal{C}^\dagger_{/x} \to \mathcal{C}^\dagger$ is a marked morphism. It is easy to see that this construction is functorial on x, and so we obtain a functor $\mathcal{C}^\dagger_{/-}\colon \mathcal{C} \to \mathrm{Cat}_\infty^\dagger$.

We are finally ready to introduce the notion of partially lax (co)limit. Recall the definition of the twisted arrow ∞-category from Definition 6.2.17.

Definition 6.3.7. Consider a functor $F\colon \mathcal{I} \to \mathcal{C}$ and choose a marking $\mathcal{I}^\dagger$.

(a) If $\mathcal{C}$ is cotensored over Cat_∞, then the partially lax limit of F is the limit of the composite

$$\mathrm{Tw}(\mathcal{I})^{\mathrm{op}} \xrightarrow{(s,t)^{\mathrm{op}}} \mathcal{I}^{\mathrm{op}} \times \mathcal{I} \xrightarrow{|\mathcal{I}^\dagger_{/-}| \times F} \mathrm{Cat}_\infty^{\mathrm{op}} \times \mathcal{C} \xrightarrow{[-,-]} \mathcal{C}.$$

We abbreviate this by $\mathrm{laxlim}^\dagger F$.

(b) If $\mathcal{C}$ is tensored over Cat_∞, then the partially lax colimit of F is the colimit of the composite

$$\mathrm{Tw}(\mathcal{I}) \xrightarrow{(s,t)} \mathcal{I} \times \mathcal{I}^{\mathrm{op}} \xrightarrow{F \times |(\mathcal{I}^{\mathrm{op}})^\dagger_{/-}|} \mathcal{C} \times \mathrm{Cat}_\infty \xrightarrow{-\otimes-} \mathcal{C}.$$

We abbreviate this by $\mathrm{laxcolim}^\dagger F$.

Remark 6.3.8. If we choose the minimal marking $\mathcal{I}^\flat$, then we recover the notion of lax (co)limit of [GHN17]. If we choose the maximal marking $\mathcal{I}^\sharp$, then we recover the usual notion of (co)limit, see [Ber20, Proposition 3.6].

In some cases we have a concrete description of the partial lax (co)limit.

Theorem 6.3.9 ([Ber20, Theorem 4.4])**.** *Let $\mathcal{I}^\dagger$ be a small marked ∞-category and let $F\colon \mathcal{I} \to \mathrm{Cat}_\infty$ be a functor. Consider the source of the (co)cartesian fibrations $\mathrm{Un}^{\mathrm{ct}}(F) \to \mathcal{I}^{\mathrm{op}}$ and $\mathrm{Un}^{\mathrm{co}}(F) \to \mathcal{I}$ as marked via Example 6.3.2(c).*

a) The partially lax limit of F is equivalent to the ∞-category of marked sections of $p\colon \mathrm{Un}^{\mathrm{co}}(F) \to \mathcal{I}^\dagger$. In other words we have

$$\mathrm{laxlim}^\dagger F \simeq \mathrm{Fun}^\dagger_{/I^\dagger}(\mathcal{I}^\dagger, \mathrm{Un}^{\mathrm{co}}(F))$$

(b) The partially lax colimit of F is equivalent to the localization of $\mathrm{Un}^{\mathrm{ct}}(F)$ at the marked edges. In other words we have

$$\mathrm{laxcolim}^\dagger F \simeq |\mathrm{Un}^{\mathrm{ct}}(F)|.$$

Remark 6.3.10. The previous result gives a more explicit description of the partially lax limit of F. Recall that informally the Grothendieck construction $\mathrm{Un}^{\mathrm{co}}(F)$ is the ∞-category whose objects are pairs (X, i) where $i \in \mathcal{I}$ and $X \in F(i)$. A morphism from (X, i) to (Y, j) is a pair (φ, f) where $f\colon i \to j$ is a morphism in $\mathcal{I}$ and $\varphi\colon F(f)(X) \to Y$ is a morphism in $F(j)$. Then the previous result informally implies that $\mathrm{laxlim}^\dagger F$ is equivalent to the ∞-category whose objects are coherent collections of objects $(X_i \in F(i))_{i\in\mathcal{I}}$ together with maps $\varphi_f\colon F(f)(X_i) \to X_j$ for every arrow $f\colon i \to j$ in $\mathcal{I}$, such that the map φ_f is an equivalence whenever f is marked.

We record some useful properties of partially lax (co)limits.

Proposition 6.3.11. *Let $\mathcal{I}^\dagger$ be a marked ∞-category and let $F\colon \mathcal{I} \to \mathrm{Cat}_\infty$ be a functor. Given any other ∞-category $\mathcal{C}$, we have an equivalence*

$$\mathrm{Fun}(\underset{\mathcal{I}}{\mathrm{laxcolim}}^\dagger F, \mathcal{C}) \simeq \underset{\mathcal{I}^{\mathrm{op}}}{\mathrm{laxlim}}^\dagger \mathrm{Fun}(F(-), \mathcal{C}).$$

Proof. The partially lax colimit of $F\colon \mathcal{I} \to \mathrm{Cat}_\infty$ is by definition calculated via the formula

$$\mathrm{laxcolim}^\dagger F = \underset{\mathrm{Tw}(\mathcal{I})}{\mathrm{colim}}\ F \times |(\mathcal{I}^{\mathrm{op}})^\dagger_{/-}|$$

Postcomposing by the limit preserving functor $\mathrm{Fun}(-, \mathcal{C})\colon \mathrm{Cat}_\infty^{\mathrm{op}} \to \mathrm{Cat}_\infty$, we deduce that the ∞-category $\mathrm{Fun}(\mathrm{laxcolim}^\dagger F, \mathcal{C})$ is the limit of the diagram

$$\mathrm{Tw}(\mathcal{I})^{\mathrm{op}} \xrightarrow{(s,t)^{\mathrm{op}}} \mathcal{I}^{\mathrm{op}} \times \mathcal{I} \xrightarrow{(F, |(\mathcal{I}^{\mathrm{op}})^\dagger_{/-}|)^{\mathrm{op}}} \mathrm{Cat}_\infty^{\mathrm{op}} \times \mathrm{Cat}_\infty^{\mathrm{op}} \xrightarrow{-\times-} \mathrm{Cat}_\infty^{\mathrm{op}} \xrightarrow{\mathrm{Fun}(-,\mathcal{C})} \mathrm{Cat}_\infty \tag{6.3.11.1}$$

By adjunction, we find that the composite of the final three functors is equivalent to

$$\mathrm{Fun}(-,-) \circ (|(\mathcal{I}^{\mathrm{op}})^{\dagger}_{/-}|, \mathrm{Fun}(F(-), \mathcal{C})) \circ \sigma \colon \mathcal{I}^{\mathrm{op}} \times \mathcal{I} \to \mathrm{Cat}_\infty,$$

where σ is the symmetry isomorphism of the product. As indicated in Remark 6.2.20, the following triangle commutes:

$$\begin{tikzcd} \mathrm{Tw}(\mathcal{I})^{\mathrm{op}} \arrow[rr, "\sim"] \arrow[dr, "(s,t)^{\mathrm{op}}"'] & & \mathrm{Tw}(\mathcal{I}^{\mathrm{op}})^{\mathrm{op}} \arrow[dl, "(t,s)^{\mathrm{op}}"] \\ & \mathcal{I}^{\mathrm{op}} \times \mathcal{I} & \end{tikzcd}$$

These two observations allow us to rewrite Equation (6.3.11.1) and conclude that $\mathrm{Fun}(\mathrm{laxcolim}^{\dagger} F, \mathcal{C})$ is the limit of the functor

$$\mathrm{Tw}(\mathcal{I}^{\mathrm{op}})^{\mathrm{op}} \xrightarrow{(s,t)^{\mathrm{op}}} \mathcal{I} \times \mathcal{I}^{\mathrm{op}} \xrightarrow{(|(\mathcal{I}^{\mathrm{op}})^{\dagger}_{/-}|, \mathrm{Fun}(F(-),\mathcal{C}))} \mathrm{Cat}_\infty^{\mathrm{op}} \times \mathrm{Cat}_\infty \xrightarrow{\mathrm{Fun}(-,-)} \mathrm{Cat}_\infty,$$

which is exactly the definition of the partially lax limit of $\mathrm{Fun}(F(-), \mathcal{C}) \colon \mathcal{I}^{\mathrm{op}} \to \mathrm{Cat}_\infty$. □

We finish this section by discussing how (partially) lax limits interact with localizations. Later on we will use these results to pass from (partially) lax limits of prespectra to that of spectra.

Lemma 6.3.12. *Let $\mathcal{I}$ be an ∞-category and let $F \colon \mathcal{I} \to \mathrm{Cat}_\infty$ be a functor. Suppose that for every $i \in \mathcal{I}$ we are given a reflexive subcategory $Gi \subseteq Fi$ with left adjoint $L_i \colon Fi \to Gi$. If for every arrow $f \colon i \to j$ of $\mathcal{I}$, the pushforward functor $f_* \colon Fi \to Fj$ sends L_i-equivalences to L_j-equivalences, then there is a functor $G \colon \mathcal{I} \to \mathrm{Cat}_\infty$ and a natural transformation $L \colon F \Rightarrow G$ whose i-th component is given by $L_i \colon Fi \to Gi$. Furthermore, the functor*

$$\operatorname*{laxlim}_{\mathcal{I}} L \colon \operatorname*{laxlim}_{\mathcal{I}} F \to \operatorname*{laxlim}_{\mathcal{I}} G$$

has a fully faithful right adjoint.

Proof. Let us consider the Grothendieck construction $\mathrm{Un}^{\mathrm{co}}(F) \to \mathcal{I}$ of F. This is the cocartesian fibration classified by F under the unstraightening equivalence, so in particular the fibre over $i \in \mathcal{I}$ can be canonically identified with Fi. Let $\mathcal{E} \subseteq \mathrm{Un}^{\mathrm{co}}(F)$ be the full subcategory spanned by the objects of $Gi \subseteq \mathrm{Un}^{\mathrm{co}}(F)$ for all $i \in \mathcal{I}$.

We claim that $\mathcal{E} \to \mathcal{I}$ is a cocartesian fibration whose cocartesian edges are those that can be factored in $\mathrm{Un}^{\mathrm{co}}(F)$ as a cocartesian edge of $\mathrm{Un}^{\mathrm{co}}(F)$ followed by a L_i-equivalence in the fibre over i. More precisely if $f \colon i \to j$ is an arrow of

$\mathcal{I}$ and $x \in Gi$, then the cocartesian lift of f starting from x is the composition $x \to f_*x \to L_j(f_*x)$ where the first arrow is the cocartesian lift of f in $\mathrm{Un}^{\mathrm{co}}(F)$. Indeed for every $z \in Gj$, we have

$$\mathrm{Map}^f_{\mathcal{E}}(x, z) \simeq \mathrm{Map}_{Fj}(f_*x, z) \simeq \mathrm{Map}_{Gj}(L_j f_*x, z)$$

and so those edges are locally cocartesian. Furthermore it is easy to see they are stable under composition (using the fact that L-equivalences are stable under pushforward), therefore they are cocartesian arrows by [Lur09, Lemma 2.4.2.7].

The inclusion $\iota\colon \mathcal{E} \subseteq \mathrm{Un}^{\mathrm{co}}(F)$ has a relative left adjoint which is a map of cocartesian fibrations by [Lur16, Proposition 7.3.2.11]. Therefore there is a functor $G\colon \mathcal{I} \to \mathrm{Cat}_\infty$ and a natural transformation $L\colon F \Rightarrow G$ such that $\mathcal{E}$ can be identified with $\mathrm{Un}^{\mathrm{co}}(G)$ in such a way that the induced map $L\colon \mathrm{Un}^{\mathrm{co}}(F) \to \mathrm{Un}^{\mathrm{co}}(G)$ agrees with $L_i\colon Fi \to Gi$ on each fibre.

Finally by Theorem 6.3.9 the lax limit of F and G are computed by the ∞-categories of sections of the respective cocartesian fibrations and $\mathrm{laxlim}_{\mathcal{I}} L$ is given by postcomposition with L. Therefore postcomposition with ι gives a fully faithful right adjoint to $\mathrm{laxlim}_{\mathcal{I}} L$. □

Lemma 6.3.13. *Suppose we are in the situation of Lemma 6.3.12, and suppose $\mathcal{I}$ is equipped with a marking $\mathcal{I}^\dagger$ such that for every marked edge $f: i \to j$ the pushforward functor $f_*\colon Fi \to Fj$ sends Gi into Gj. Then the functor*

$$\operatorname*{laxlim}_{\mathcal{I}^\dagger} L\colon \operatorname*{laxlim}_{\mathcal{I}^\dagger} F \to \operatorname*{laxlim}_{\mathcal{I}^\dagger} G$$

has a fully faithful right adjoint. In particular $\mathrm{laxlim}_{\mathcal{I}^\dagger} L$ *is a localization functor.*

Proof. It suffices to show that the right adjoint of Lemma 6.3.12 restricts to a functor from $\mathrm{laxlim}_{\mathcal{I}^\dagger} G$ to $\mathrm{laxlim}_{\mathcal{I}^\dagger} F$. Recall that the partially lax limit can be calculated as the subcategory of sections spanned by those sending marked edges to cocartesian arrows. Thus, we ought to show that the right adjoint preserves cocartesian arrows lying over marked edges. But the right adjoint is given by postcomposing a section with the inclusion $\mathrm{Un}^{\mathrm{co}}(G) \to \mathrm{Un}^{\mathrm{co}}(F)$, and so by the description of cocartesian edges given in Lemma 6.3.12 and by our hypothesis, it sends cocartesian arrows over marked edges to cocartesian arrows (here we are implicitly using that an L_i-equivalence between objects of Gi is automatically an equivalence in Fi and so in particular a cocartesian arrow). □

For later reference we record the following corollary of Lemma 6.3.12.

Corollary 6.3.14. *Let $\mathcal{I}$ be an ∞-category and let $F\colon \mathcal{I} \to$ Cat_∞ be a functor. Suppose that for every $i \in \mathcal{I}$, we are given a reflexive subcategory $Gi \subseteq Fi$ with left* *adjoint $L_i\colon Fi \to Gi$ which is compatible with the symmetric monoidal structure in the sense of [Lur16, Definition 2.2.1.6]. Suppose furthermore that for every arrow $f\colon i \to j$ in I, the pushforward functor $f_*\colon Fi \to Fj$ sends L_i-equivalences to L_j-equivalences. Then there exists a functor $G\colon \mathcal{I} \to \mathrm{Cat}_\infty^\otimes$ and a symmetric monoidal natural transformation $L\colon F \Rightarrow G$ whose ith component is given by $L_i\colon Fi \to Gi$.*

Proof. Since $\mathrm{Cat}_\infty^\otimes$ embeds as a subcategory of $\mathrm{Fun}(\mathrm{Fin}_*, \mathrm{Cat}_\infty)$ we can consider the functor $\tilde{F}\colon \mathrm{Fin}_* \times \mathcal{I} \to \mathrm{Cat}_\infty$ induced by F, so that $\tilde{F}(A_+, i) \simeq (Fi)^A$ (the fibre over A of $Fi \to \mathrm{Fin}_*$). If we let $\tilde{G}(A_+, i) = (Gi)^A \subseteq \tilde{F}(A_+, i)$, we can apply Lemma 6.3.12 to $\tilde{F}$. To see that the pushforwards respect local equivalences, it suffices to prove this separately for maps of the form (σ, id) and (id, f) in $\mathrm{Fin}_* \times \mathcal{I}$. However both of these cases are ensured by our assumptions. Therefore there exists a functor

$$\tilde{G}\colon \mathrm{Fin}_* \times \mathcal{I} \to \mathrm{Cat}_\infty$$

and a natural transformation $\tilde{L}\colon \tilde{F} \Rightarrow \tilde{G}$ as desired. By construction $\tilde{G}$ satisfies the Segal conditions, and so it induces a functor $G\colon \mathcal{I} \to \mathrm{Cat}_\infty^\otimes$ with a symmetric monoidal natural transformation $L\colon F \Rightarrow G$ as desired. □

6.4 PARTIALLY LAX LIMITS OF SYMMETRIC MONOIDAL ∞-CATEGORIES

Recall that Op_∞ is canonically cotensored over Cat_∞ by Example 6.3.3. Therefore we immediately obtain a definition of partially lax limits of diagrams in Op_∞. In this section we will collect some important properties of partially lax limits of symmetric monoidal ∞-categories and ∞-operads. In particular the calculations of Proposition 6.4.8, and Theorem 6.4.10 are used repeatedly in part two. The first is analogous to the calculation of the (partially) lax limit of a diagram of ∞-categories, and as such it is stated in terms of an unstraightening equivalence for symmetric monoidal categories, which we recall in Proposition 6.4.5.

Remark 6.4.1. If $\mathcal{P}^\otimes$ is another ∞-operad, it follows from the definition and [Lur16, Remark 2.1.3.4] that there is a natural equivalence

$$\mathrm{Alg}_{\mathcal{P}^\otimes}(\operatorname*{laxlim}_{i\in I} \mathcal{O}_i^\otimes) \simeq \operatorname*{laxlim}_{i\in I} \mathrm{Alg}_{\mathcal{P}^\otimes}(\mathcal{O}^\otimes)\,.$$

Such a natural equivalence then also uniquely determines the lax limit. Since $\mathrm{Cat}_\infty^\otimes \subseteq \mathrm{Op}_\infty$ is a subcategory closed under limits and cotensoring, it is also closed under partially lax limits. In particular we conclude that for every family of symmetric monoidal ∞-categories $\mathcal{C}_\bullet$ and every symmetric monoidal ∞-category $\mathcal{D}$, there is a natural equivalence

$$\mathrm{Fun}^\otimes(\mathcal{D}, \operatorname*{laxlim}_{i\in I} \mathcal{C}_i) \simeq \operatorname*{laxlim}_{i\in I} \mathrm{Fun}^\otimes(\mathcal{D}, \mathcal{C}_i)\,.$$

We note that the underlying ∞-category functor $U\colon \mathrm{Op}_\infty \to \mathrm{Cat}_\infty$ preserves limits and commutes with cotensoring, and therefore preserves partially lax limits. Therefore the previous construction equips the partially lax limit of a family of symmetric monoidal ∞-categories with a canonical symmetric monoidal structure, which satisfies the expected universal property.

Remark 6.4.2. Note that there is always a canonical map $\mathrm{laxlim}^\dagger \mathcal{O}_i^\otimes \to \mathrm{laxlim}\, \mathcal{O}_i^\otimes$. This functor is induced on limits by a natural transformation which is pointwise given by the inclusion of a fully faithful suboperad. Therefore we conclude that the partially lax limit is always a fully faithful suboperad of the lax limit. In practice this means that we can determine which suboperad by considering the induced map on underlying categories.

In the second part of the paper we will build diagrams of symmetric monoidal ∞-categories indexed on $\mathrm{Glo}^{\mathrm{op}}$. Central to our constructions of these diagrams is an operadic variant of straightening/unstraightening, which we will recall now.

Notation 6.4.3. Recall from [Lur16, p. 2.4.3.5] that for every ∞-category $\mathcal{I}$ there is a functor of ∞-operads $c\colon \mathcal{I} \times \mathrm{Fin}_* \to \mathcal{I}^{\amalg}$ sending (x, A_+) to the constant family $\{x\}_{a\in A} \in \mathcal{I}^{\amalg}_{A_+}$.

Construction 6.4.4. Let $\mathcal{I}$ be an ∞-category and let $\mathcal{C}^\otimes$ be an $\mathcal{I}^{\amalg}$-monoidal ∞-category. Then the commutative diagram of cocartesian fibrations

$$\begin{tikzcd}
\mathcal{C}^\otimes \times_{\mathcal{I}^{\amalg}} (\mathcal{I} \times \mathrm{Fin}_*) \arrow[r, "\mathrm{pr}_2"] \arrow[dr, "\mathrm{pr}_{\mathcal{I}}"'] & \mathcal{I} \times \mathrm{Fin}_* \arrow[d, "\mathrm{pr}_1"] \\
& \mathcal{I}
\end{tikzcd}$$

is classified by a functor $\mathcal{C}_\bullet : \mathcal{I} \to (\mathrm{Cat}_\infty)_{/\mathrm{Fin}_*}$ which lands in $\mathrm{Cat}_\infty^\otimes$. We refer to $\mathcal{C}_\bullet$ as the family of symmetric monoidal ∞-categories classifying $\mathcal{C}^\otimes$.

Proposition 6.4.5. *The previous construction furnishes an equivalence between the ∞-category of $\mathcal{I}^{\amalg}$-monoidal categories and* $\mathrm{Fun}(\mathcal{I}, \mathrm{Cat}_\infty^\otimes)$.

Proof. This is [DG22, Corollary A.12]. □

Definition 6.4.6. Consider a map of ∞-operads $p : \mathcal{O}^\otimes \to \mathcal{I}^{\amalg}$. Any object $i \in \mathcal{I}$ induces a functor

$$\{i\} \times \mathrm{Fin}_* \hookrightarrow \mathcal{I} \times \mathrm{Fin}_* \xrightarrow{c} \mathcal{I}^{\amalg},$$

see Notation 6.4.3. Equivalently, the map above can be obtained by applying $(-)^{\amalg}$ to the map $\Delta^0 \to \mathcal{I}$ defined by $i \in \mathcal{I}$. Inspired by the equivalence of Proposition 6.4.5 we will refer to the pullback $\mathcal{O}^\otimes \times_{\mathcal{I}^{\amalg}} \mathrm{Fin}_*$ as the *operadic fibre* of p at $i \in \mathcal{I}$. If p is an $\mathcal{I}^{\amalg}$-monoidal ∞-category, then its operadic fibre at i is a symmetric monoidal ∞-category, and corresponds to the value of the functor $\mathcal{C}_\bullet$ at i.

The following example will be crucial for later applications.

Example 6.4.7. Let $p\colon \mathcal{C}^{\otimes} \to \mathcal{I}^{\amalg}$ be a $\mathcal{I}^{\amalg}$-promonoidal ∞-category and let $\mathcal{D}^{\otimes} \to \mathcal{I}^{\amalg}$ be a map of ∞-operads which is compatible with colimits. Then the operadic fibre of the Day convolution $\mathrm{Fun}_{\mathcal{I}}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})^{Day}$ over $i \in \mathcal{I}$ is given by the symmetric monoidal ∞-category $\mathcal{C}_i$–$\mathcal{D}_i$, where $\mathcal{C}_i, \mathcal{D}_i$ are the operadic fibres over i of $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ respectively. To see this, first recall that $\mathcal{C}_i$–$\mathcal{D}_i$ is defined to be $\mathrm{Fun}(\mathcal{C}_i, \mathcal{D}_i)$ with the Day convolution symmetric monoidal structure. Then the claim follows from the following computation using Lemma 6.2.9

$$(N_p p^*\mathcal{D}^{\otimes}) \times_{\mathcal{I}^{\amalg}} \mathrm{Fin}_* \simeq N_{p_i}(p^*\mathcal{D}^{\otimes} \times_{\mathcal{C}^{\otimes}} \mathcal{C}_i^{\otimes}) \simeq N_{p_i} p_i^* \mathcal{D}_i^{\otimes} = \mathcal{C}_i\text{–}\mathcal{D}_i.$$

Recall that the lax limit of a diagram of ∞-categories was calculated by taking sections of the associated cocartesian fibration. Similarly, we can describe the lax limit of $\mathcal{C}_\bullet$ in terms of (suitable) sections of the ∞-operad $\mathcal{C}^{\otimes}$.

Proposition 6.4.8. *Let $\mathcal{C}^{\otimes} \to \mathcal{I}^{\amalg}$ be a $\mathcal{I}^{\amalg}$-monoidal ∞-category, and denote by $\mathcal{C}_\bullet\colon \mathcal{I} \to \mathrm{Cat}_\infty^{\otimes}$ the associated diagram of symmetric monoidal ∞-categories. Then there is a natural equivalence of symmetric monoidal ∞-categories*

$$\mathrm{laxlim}\, \mathcal{C}_\bullet \simeq N_{\mathcal{I}^{\amalg}} \mathcal{C}^{\otimes}$$

where the right hand side is the norm along $\mathcal{I}^{\amalg} \to \mathrm{Fin}_*$, which is well defined by Example 6.2.6.

Proof. We will show that the right hand side has the universal property of the lax limit. By the universal property of the norm, for any ∞-operad $\mathcal{P}^{\otimes}$ we have an equivalence

$$\mathrm{Alg}_{\mathcal{P}^{\otimes}}(N_{\mathcal{I}^{\amalg}}\mathcal{C}^{\otimes}) \simeq \mathrm{Alg}_{\mathcal{P}^{\otimes} \times_{\mathrm{Fin}_*} \mathcal{I}^{\amalg}/\mathcal{I}^{\amalg}}(\mathcal{C}^{\otimes})\,.$$

By [Lur16, Theorem 2.4.3.18], we can write

$$\begin{aligned}\mathrm{Alg}_{\mathcal{P}^{\otimes} \times_{\mathrm{Fin}_*} \mathcal{I}^{\amalg}/\mathcal{I}^{\amalg}}(\mathcal{C}^{\otimes}) &\simeq \mathrm{Alg}_{\mathcal{P}^{\otimes} \times_{\mathrm{Fin}_*} \mathcal{I}^{\amalg}}(\mathcal{C}^{\otimes}) \times_{\mathrm{Alg}_{\mathcal{P}^{\otimes} \times_{\mathrm{Fin}_*} \mathcal{I}^{\amalg}}(\mathcal{I}^{\amalg})} \{\mathrm{pr}_2\} \\ &\simeq \mathrm{Fun}(\mathcal{I}, \mathrm{Alg}_{\mathcal{P}^{\otimes}}(\mathcal{C}^{\otimes})) \times_{\mathrm{Fun}(\mathcal{I}, \mathrm{Alg}_{\mathcal{P}^{\otimes}}(\mathcal{I}^{\amalg}))} \{\mathrm{pr}_2\}\,.\end{aligned}$$

where $\mathrm{pr}_2\colon \mathcal{P}^{\otimes} \times_{\mathrm{Fin}_*} \mathcal{I}^{\amalg} \to \mathcal{I}^{\amalg}$ is the projection. In other words, we have shown that $\mathrm{Alg}_{\mathcal{P}^{\otimes}}(N_{\mathcal{I}^{\amalg}}\mathcal{C}^{\otimes})$ is the ∞-category of sections of the functor

$$\mathrm{Alg}_{\mathcal{P}^{\otimes}}(\mathcal{C}^{\otimes}) \times_{\mathrm{Alg}_{\mathcal{P}^{\otimes}}(\mathcal{I}^{\amalg})} \mathcal{I} \to \mathcal{I}$$

which is exactly the cocartesian fibration classified by $i \mapsto \mathrm{Alg}_{\mathcal{P}^{\otimes}}(\mathcal{C}_i^{\otimes})$. Our thesis then follows from Theorem 6.3.9. $\square$

Remark 6.4.9. Let $p\colon \mathcal{C}^\otimes \to \mathcal{I}^\amalg$ be an $\mathcal{I}^\amalg$-monoidal ∞-category, and write $\mathcal{C}_\bullet\colon \mathcal{I} \to \mathrm{Cat}_\infty^\otimes$ for the associated diagram of symmetric monoidal ∞-categories. Then by the discussion in Remark 6.2.11, the underlying category of $N_{\mathcal{I}^\amalg}\mathcal{C}^\otimes$ is given by $\mathrm{Fun}_{/\mathcal{I}}(\mathcal{I}, \mathcal{C})$. Therefore the proposition above is an operadic analogue of Theorem 6.3.9(b). Since we know that the partially lax limit of a diagram of ∞-operads is a fully faithful suboperad of the lax limit, the previous result also allows us to calculate the partially lax limit of $\mathcal{C}_\bullet$. Namely it is the fully faithful symmetric monoidal subcategory of $N_{\mathcal{I}^\amalg}\mathcal{C}^\otimes$ determined by the fully faithful subcategory $\mathrm{laxlim}^\dagger \mathcal{C}_\bullet \subset \mathrm{laxlim}\,\mathcal{C}_\bullet$.

We finish this section by proving that the formation of (partially) lax limits of symmetric monoidal categories commutes with taking modules, in a precise sense. This will be a key observation for the second part of the paper, and crucially uses the equivalence $N_{\mathcal{I}^\amalg}\mathcal{C}^\otimes \simeq \mathrm{laxlim}\,\mathcal{C}_\bullet$.

Theorem 6.4.10. *Let $\mathcal{C}^\otimes \to \mathcal{I}^\amalg$ be a $\mathcal{I}^\amalg$-monoidal ∞-category which is compatible with colimits, and write $\mathcal{C}_\bullet\colon \mathcal{I} \to \mathrm{Cat}_\infty^\otimes$ for the associated diagram of symmetric monoidal ∞-categories. Let $S \in \mathrm{CAlg}(\mathrm{laxlim}\,\mathcal{C}_\bullet)$ be an algebra in the lax limit, which corresponds to a (partially lax) family of algebras $S_i \in \mathcal{C}_i$. Then there is a functor*

$$\mathrm{Mod}_{S_\bullet}(\mathcal{C}_\bullet)\colon \mathcal{I} \to \mathrm{Cat}_\infty^\otimes, \quad i \mapsto \mathrm{Mod}_{S_i}(\mathcal{C}_i)$$

and an equivalence of symmetric monoidal ∞-categories

$$\mathrm{laxlim}\,\mathrm{Mod}_{S_\bullet}(\mathcal{C}_\bullet) \simeq \mathrm{Mod}_S(\mathrm{laxlim}\,\mathcal{C}_\bullet)\,.$$

Moreover, there is a natural transformation $\mathcal{C}_\bullet \to \mathrm{Mod}_{S_\bullet}(\mathcal{C}_\bullet)$ sending $x \in \mathcal{C}_i$ to the free S_i-module $S_i \otimes x$, which induces the functor $S \otimes -$ on the lax limit.

The proof of the previous result will require some preparation and some results from the following section. For this reason we recommend the reader to skip this part on a first reading.

We start our journey by studying how the lax limit interacts with the tensor product of algebras.

Construction 6.4.11. By [Lur16, Proposition 3.2.4.6] there is an equivalence of ∞-operads $\mathcal{I}^\amalg \otimes_{BV} \mathrm{Fin}_* \simeq \mathcal{I}^\amalg$, where $\otimes_{BV}$ is the Boardmann-Vogt tensor product, and so there exists a unique bifunctor of ∞-operads $\mathcal{I}^\amalg \times \mathrm{Fin}_* \to \mathcal{I}^\amalg$. For any ∞-operad $\mathcal{O}^\otimes$ we obtain a bifunctor of ∞-operads $m_{\mathcal{O}}$, which is given by the composition

$$\mathcal{I}^\amalg \times \mathcal{O}^\otimes \to \mathcal{I}^\amalg \times \mathrm{Fin}_* \to \mathcal{I}^\amalg\,.$$

Thus, for every map of ∞-operad $\mathcal{C}^\otimes \to \mathcal{I}^\amalg$ [Lur16, Construction 3.2.4.1] produces a map of ∞-operads

$$\mathrm{Alg}_{\mathcal{O}^\otimes/\mathcal{I}^\amalg}(\mathcal{C})^\otimes \to \mathcal{I}^\amalg$$

whose operadic fibre over $i \in I$ is given by $\mathrm{Alg}_{\mathcal{O}^\otimes}(\mathcal{C}_i)^\otimes$. Suppose that $\mathcal{C}^\otimes$ is a $\mathcal{I}^{\amalg}$-monoidal category. Then by [Lur16, Proposition 3.2.4.3.(3)] $\mathrm{Alg}_{\mathcal{O}^\otimes}(\mathcal{C}_i)^\otimes$ is also a $\mathcal{I}^{\amalg}$-monoidal ∞-category. In this case, Proposition 6.4.5 gives a functor $\mathcal{I} \to \mathrm{Cat}_\infty^\otimes$ sending $i \in \mathcal{I}$ to $\mathrm{Alg}_{\mathcal{O}^\otimes}(\mathcal{C}_i)^\otimes$. We will now compute the lax limit of this functor.

Lemma 6.4.12. *Let $\mathcal{I}$ be an ∞-category, $\mathcal{C}^\otimes \to \mathcal{I}^{\amalg}$ a map of ∞-operads and $\mathcal{O}^\otimes$ an ∞-operad. Then there is a natural equivalence of ∞-operads*

$$\mathrm{Alg}_{\mathcal{O}^\otimes}(N_{\mathcal{I}^{\amalg}}\mathcal{C}^\otimes)^\otimes \simeq N_{\mathcal{I}^{\amalg}}\,\mathrm{Alg}_{\mathcal{O}^\otimes/\mathcal{I}^{\amalg}}(\mathcal{C})^\otimes\,.$$

In particular if $\mathcal{C}^\otimes$ is $\mathcal{I}^{\amalg}$-symmetric monoidal we have a natural equivalence of ∞-operads

$$\mathrm{Alg}_{\mathcal{O}^\otimes}(\operatorname*{laxlim}_{i\in\mathcal{I}}\mathcal{C}_i)^\otimes \simeq \operatorname*{laxlim}_{i\in\mathcal{I}}\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}_i)^\otimes\,.$$

Proof. We will prove that both sides represent the same functor in the ∞-category of ∞-operads. Let $\mathcal{P}^\otimes$ an ∞-operad. Then

$$\begin{aligned}
\mathrm{Alg}_{\mathcal{P}^\otimes}\, N_{\mathcal{I}^{\amalg}}\,\mathrm{Alg}_{\mathcal{O}^\otimes/\mathcal{I}^{\amalg}}\,\mathcal{C}^\otimes &\simeq \mathrm{Alg}_{\mathcal{P}^\otimes\times_{\mathrm{Fin}_*}\mathcal{I}^{\amalg}/I^{\amalg}}\left(\mathrm{Alg}_{\mathcal{O}^\otimes}(\mathcal{C})^\otimes \times_{\mathrm{Alg}_{\mathcal{O}^\otimes}(\mathcal{I})^\otimes}\mathcal{I}^{\amalg}\right)\\
&\simeq \mathrm{Alg}_{\mathcal{P}^\otimes\times_{\mathrm{Fin}_*}\mathcal{I}^{\amalg}/\mathrm{Alg}_{\mathcal{O}^\otimes}(\mathcal{I})^\otimes}(\mathrm{Alg}_{\mathcal{O}^\otimes}(\mathcal{C})^\otimes)\\
&\simeq \mathrm{Alg}_{\mathcal{P}^\otimes\times_{\mathrm{Fin}_*}\mathcal{I}^{\amalg}}(\mathrm{Alg}_{\mathcal{O}^\otimes}(\mathcal{C})^\otimes)\times_{\mathrm{Alg}_{\mathcal{P}^\otimes\times_{\mathrm{Fin}_*}\mathcal{I}^{\amalg}}(\mathrm{Alg}_{\mathcal{O}^\otimes}(\mathcal{I})^\otimes)}\{pr_2\}\\
&\simeq \mathrm{Alg}_{(\mathcal{P}^\otimes\otimes_{BV}\mathcal{O}^\otimes)\times_{\mathrm{Fin}_*}\mathcal{I}^{\amalg}}(\mathcal{C}^\otimes)\times_{\mathrm{Alg}_{(\mathcal{P}^\otimes\otimes_{BV}\mathcal{O}^\otimes)\times_{\mathrm{Fin}_*}\mathcal{I}^{\amalg}}(\mathcal{I}^{\amalg})}\{pr_2\}\\
&\simeq \mathrm{Alg}_{(\mathcal{P}^\otimes\otimes_{BV}\mathcal{O}^\otimes)\times_{\mathrm{Fin}_*}\mathcal{I}^{\amalg}/\mathcal{I}^{\amalg}}(\mathcal{C})^\otimes\\
&\simeq \mathrm{Alg}_{\mathcal{P}^\otimes}\left(\mathrm{Alg}_{\mathcal{O}^\otimes}(N_{\mathcal{I}^{\amalg}}\mathcal{C})^\otimes\right)\,.
\end{aligned}$$

Here $\otimes_{BV}$ denotes the Boardman-Vogt tensor product of ∞-operads constructed in [Lur16, Section 2.2.5]. □

We are ready to prove the main result of this section.

Proof of Theorem 6.4.10. Note that by the definition of the norm we have an equivalence

$$\mathrm{CAlg}(N_{\mathcal{I}^{\amalg}}\mathcal{C}^\otimes) \simeq \mathrm{Alg}_{\mathcal{I}^{\amalg}/\mathcal{I}^{\amalg}}(\mathcal{C}^\otimes) \simeq \mathrm{Alg}_{\mathcal{I}^{\amalg}}(\mathrm{Alg}_{\mathrm{Fin}_*/\mathcal{I}^{\amalg}}(\mathcal{C})^\otimes)$$

therefore we can also consider S as a section of $\mathrm{Alg}_{\mathrm{Fin}_*/\mathcal{I}^{\amalg}}(\mathcal{C})^\otimes \to \mathcal{I}^{\amalg}$ in Op_∞. By Theorem 6.5.10 and Lemma 6.4.12 there is an equivalence

$$\begin{aligned}
\mathrm{Mod}_S(N_{\mathcal{I}^{\amalg}}\mathcal{C}^\otimes)^\otimes &\simeq \mathrm{Alg}_{\mathcal{CM}}(N_{\mathcal{I}^{\amalg}}\mathcal{C})^\otimes \times_{\mathrm{CAlg}(N_{\mathcal{I}^{\amalg}}\mathcal{C})^\otimes}\mathrm{Fin}_*\\
&\simeq N_{\mathcal{I}^{\amalg}}\left(\mathrm{Alg}_{\mathcal{CM}/\mathcal{I}^{\amalg}}(\mathcal{C})^\otimes \times_{\mathrm{Alg}_{\mathrm{Fin}_*/\mathcal{I}^{\amalg}}(\mathcal{C})^\otimes}\mathcal{I}^{\amalg}\right)
\end{aligned}$$

where $\mathcal{I}^{\amalg} \to \mathrm{Alg}_{\mathrm{Fin}_*/\mathcal{I}^{\amalg}}(\mathcal{C})^{\otimes}$ is the section corresponding to S. Moreover note that by Lemma 6.5.9

$$\mathrm{Alg}_{\mathcal{CM}/\mathcal{I}^{\amalg}}(\mathcal{C})^{\otimes} \times_{\mathrm{Alg}_{\mathrm{Fin}_*/\mathcal{I}^{\amalg}}(\mathcal{C})^{\otimes}} \mathcal{I}^{\amalg} \to \mathcal{I}^{\amalg}$$

is an $\mathcal{I}^{\amalg}$-monoidal ∞-category. Then Theorem 6.5.10 shows that the corresponding family of symmetric monoidal ∞-categories is exactly

$$i \mapsto \mathrm{Mod}_{S_i}(\mathcal{C}_i)\,,$$

and so our thesis follows from Proposition 6.4.8.

Finally let us construct the symmetric monoidal functor $\mathcal{C}_i^{\otimes} \to \mathrm{Mod}_{S_i}(\mathcal{C}_i)^{\otimes}$. There is a map of $\mathcal{I}^{\amalg}$-monoidal ∞-categories

$$\mathrm{Alg}_{\mathcal{CM}^{\otimes}/\mathcal{I}^{\amalg}}(\mathcal{C})^{\otimes} \to \mathrm{Alg}_{\mathrm{Fin}_*/\mathcal{I}^{\amalg}}(\mathcal{C})^{\otimes} \times_{\mathcal{I}^{\amalg}} \mathcal{C}^{\otimes}$$

induced by the map of ∞-operads $\mathrm{Fin}_* \boxplus \mathrm{Triv}^{\otimes} \to \mathcal{CM}^{\otimes}$ picking the algebra and the underlying object of the module. By [Lur16, Corollary 4.2.4.4] this has a left adjoint on every fibre which is compatible with the pushforwards by [Lur16, Corollary 4.2.4.8], and so by [Lur16, Corollary 7.3.2.12] it has a relative left adjoint F which is an $\mathcal{I}^{\amalg}$-monoidal functor. Then the functor we want is the composite

$$\mathcal{C}^{\otimes} \xrightarrow{(S,\mathrm{id})} \mathrm{Alg}_{\mathrm{Fin}_*/\mathcal{I}^{\amalg}}(\mathcal{C})^{\otimes} \times_{\mathcal{I}^{\amalg}} \mathcal{C}^{\otimes} \xrightarrow{F} \mathrm{Alg}_{\mathcal{CM}^{\otimes}/\mathcal{I}^{\amalg}}(\mathcal{C})^{\otimes}\,.$$

This induces the desired functor on the lax limit since applying $N_{\mathcal{I}^{\amalg}}$ preserve operadic adjunctions. □

6.5 TENSOR PRODUCT OF MODULES IN AN ∞-CATEGORY

The goal of this section is to provide a proof of Theorem 6.5.10 below, which will be useful when studying lax limits of ∞-categories of modules. This section uses some technical results about the theory of ∞-operads as developed in [Lur16] and so it should be skipped on a first reading.

Definition 6.5.1. We define $\mathcal{CM}^{\otimes}$ to be the ∞-operad corresponding to the symmetric multicategory with two objects a and m with

$$\mathrm{Mul}(\{x_i\}, a) = \begin{cases} * & \text{if } \forall i,\ x_i = a \\ \varnothing & \text{otherwise} \end{cases} \qquad \mathrm{Mul}(\{x_i\}, m) = \begin{cases} * & \text{if } |\{i \mid x_i = m\}| = 1 \\ \varnothing & \text{otherwise.} \end{cases}$$

We know by [Gla14, Proposition 7] or [Hin15, Lemma B.1.1] that for every ∞-operad $\mathcal{C}^{\otimes}$ there is a natural equivalence of ∞-categories

$$\mathrm{Mod}^{\mathrm{Fin}_*}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathcal{CM}^{\otimes}}(\mathcal{C})\,.$$

Our goal is to give a similar description of the tensor product of modules over a commutative algebra, that is of the family of ∞-operads $\mathrm{Mod}(\mathcal{C})^{\otimes}$. In order to do so we will introduce a variant of $\mathcal{CM}^{\otimes}$ which parametrizes finite sets of modules.

Construction 6.5.2. Let $\widetilde{\mathcal{CM}}^{\otimes}$ be the category whose objects consist of triples $(\langle n\rangle, \langle m\rangle, \{S_i\}_{i=1,\dots,n})$, where $\langle n\rangle, \langle m\rangle \in \mathrm{Fin}_*$ and $\{S_i\}$ is a family of pairwise disjoint subsets of $\langle m\rangle$. A map $(\langle n\rangle, \langle m\rangle, \{S_i\}) \to (\langle n'\rangle, \langle m'\rangle, \{S'_i\})$ is a pair of maps $f : \langle n\rangle \to \langle n'\rangle$ and $g : \langle m\rangle \to \langle m'\rangle$ in Fin_* such that

- for every $i \in \langle n\rangle^{\circ}$, we have $g(S_i) \subseteq S'_{f(i)} \cup \{*\}$ (where $S'_* = \varnothing$)
- for every $i \in f^{-1}\langle n'\rangle^{\circ}$ and every $s' \in S'_{f(i)}$ there is exactly one $s \in S_i$ such that $g(s) = s'$.

Lemma 6.5.3. *The projection $\widetilde{\mathcal{CM}}^{\otimes} \to \mathrm{Fin}_* \times \mathrm{Fin}_*$ that forgets the subsets $\{S_i\}$ is a Fin_*-family of ∞-operads in the sense of [Lur16, Definition 2.3.2.10], with inert arrows exactly those arrows that are sent to an equivalence by the first projection and to an inert arrow by the second projection.*

Proof. The inert arrows are the arrows

$$(\mathrm{id}_{\langle n\rangle}, f) : (\langle n\rangle, \langle m\rangle, \{S_i\}) \to (\langle n\rangle, \langle m'\rangle, \{f(S_i) \cap \langle m'\rangle^{\circ}\})$$

where $f : \langle m\rangle \to \langle m'\rangle$ is an inert arrow in Fin_*. It is easy to check that they satisfy all necessary properties. $\square$

Notation 6.5.4. For every ∞-category $X \to \mathrm{Fin}_*$ with a functor to Fin_* we will write $\widetilde{\mathcal{CM}}^{\otimes}_X$ for the X-family of ∞-operads $X \times_{\mathrm{Fin}_*} \widetilde{\mathcal{CM}}^{\otimes}$, where we pullback along the composite

$$\widetilde{\mathcal{CM}}^{\otimes} \to \mathrm{Fin}_* \times \mathrm{Fin}_* \xrightarrow{\mathrm{pr}_1} \mathrm{Fin}_*.$$

Note that $\widetilde{\mathcal{CM}}^{\otimes}_{\langle 1\rangle}$ is equivalent to $\mathcal{CM}^{\otimes}$. Intuitively the fibre $\widetilde{\mathcal{CM}}^{\otimes}_{\langle n\rangle}$ is the ∞-operad controlling pairs $(A, \{M_i\})$ where A is a commutative algebra and $\{M_i\}$ is an n-tuple of A-modules.

Write a_n for the object $(\langle n\rangle, \langle 1\rangle, \{\varnothing\})$ and $m_{j,n}$ for the object $(\langle n\rangle, \langle 1\rangle, \{S_i\})$ where $S_i = \varnothing$ for $i \neq j$ and $S_j = \{1\}$. It's easy to see these are all the objects of the underlying category of the generalized operad

$$\widetilde{\mathcal{CM}}^{\otimes} \to \mathrm{Fin}_* \times \mathrm{Fin}_* \xrightarrow{\mathrm{pr}_2} \mathrm{Fin}_*$$

First we will prove a generalization of [Gla14, Proposition 7] that shows how $\widetilde{\mathcal{CM}}^{\otimes}$ controls the tensor product of modules over commutative algebras.

Proposition 6.5.5. *Let* $X \in (\mathrm{Cat}_\infty)_{/\mathrm{Fin}_*}$ *be an* ∞*-category over* Fin_* *and let* $\mathcal{C}^\otimes \in \mathrm{Op}_\infty$ *be an* ∞*-operad. Then there is a natural equivalence*

$$\mathrm{Alg}_{\widetilde{\mathcal{CM}}_X}(\mathcal{C}^\otimes) \simeq \mathrm{Fun}_{/\mathrm{Fin}_*}(X, \mathrm{Mod}^{\mathrm{Fin}_*}(\mathcal{C})^\otimes).$$

Proof. Let $\mathcal{K} \subseteq \mathrm{Ar}(\mathrm{Fin}_*)$ be the full subcategory of semi-inert arrows [Lur16, Notation 3.3.2.1]. Consider the pullback

$$\begin{tikzcd} X \times_{\mathrm{Fin}_*} \mathcal{K} \arrow[r] \arrow[d] & X \arrow[d] \\ \mathcal{K} \arrow[r, "s"] \arrow[d, "t"] & \mathrm{Fin}_* \\ \mathrm{Fin}_*. & \end{tikzcd}$$

We will say that an arrow (f, g) in $X \times_{\mathrm{Fin}_*} \mathcal{K}$ is inert if f is an equivalence and $t(g)$ is an inert edge of Fin_*(this is different from the convention in [Lur16], but it is more suited to the current proof). Then recall that by [Lur16, Construction 3.3.3.1] the ∞-category $\mathrm{Mod}(\mathcal{C})^\otimes$ is defined so that there is a natural fully faithful inclusion

$$\mathrm{Fun}_{/\mathrm{Fin}_*}(X, \mathrm{Mod}^{\mathrm{Fin}_*}(\mathcal{C})^\otimes) \to \mathrm{Fun}_{/\mathrm{Fin}_*}(X \times_{\mathrm{Fin}_*} \mathcal{K}, \mathcal{C}^\otimes),$$

where $X \times_{\mathrm{Fin}_*} \mathcal{K}$ lives over Fin_* by the vertical composite in the diagram above, with essential image those functors sending inert arrows of $X \times_{\mathrm{Fin}_*} \mathcal{K}$ to inert arrows.

There is a functor $\mathcal{K} \to \widetilde{\mathcal{CM}}$ sending a semi-inert arrow $[s \colon \langle n\rangle \to \langle m\rangle]$ to $(\langle n\rangle, \langle m\rangle, \{\{s(i)\} \cap \langle m\rangle^\circ\}_i)$. It identifies $\mathcal{K}$ with the full subcategory of $\widetilde{\mathcal{CM}}$ spanned by those triples $(\langle n\rangle, \langle m\rangle, \{S_i\})$ where $|S_i| \le 1$ for every $i \in \langle n\rangle^\circ$. Moreover an arrow in $X \times_{\mathrm{Fin}_*} \mathcal{K}$ is inert if and only if its image in $\widetilde{\mathcal{CM}}_X$ is inert. Therefore restricting along this inclusion induces a natural transformation

$$\mathrm{Alg}_{\widetilde{\mathcal{CM}}_X}(\mathcal{C}^\otimes) \to \mathrm{Fun}_{/\mathrm{Fin}_*}(X, \mathrm{Mod}^{\mathrm{Fin}_*}(\mathcal{C})^\otimes)\,.$$

Our goal now is to prove this is an equivalence of ∞-categories. This follows from [Lur09, Proposition 4.3.2.15] together with the following two statements, where we write $p \colon \mathcal{C}^\otimes \to \mathrm{Fin}_*$ for the structure map of $\mathcal{C}^\otimes$:

1. Every map $F : X \times_{\mathrm{Fin}_*} \mathcal{K} \to \mathcal{C}^\otimes$ over Fin_* that sends inert arrows to inert arrows admits a right p-Kan extension to $\widetilde{\mathcal{CM}}_X$ that sends inert arrows to inert arrows;

2. A functor $F : \widetilde{\mathcal{CM}}_X \to \mathcal{C}^\otimes$ which sends inert arrows to inert arrows is the right p-Kan extension of its restriction to $X \times_{\mathrm{Fin}_*} \mathcal{K}$.

Let $(x, \langle m\rangle, \{S_i\})$ be an object of $\widetilde{\mathcal{CM}}_X$ and write $S = \coprod_i S_i \subseteq \langle m\rangle^\circ$. Let us consider the functor

$$\mathcal{P}(S)^{\mathrm{op}} \to \widetilde{\mathcal{CM}}_X$$

sending a subset $A \subseteq S$ to $(x, \langle m\rangle/(S \smallsetminus A), \{A \cap S_i\})$ and all arrows to inert arrows. This induces a functor

$$\mathcal{P}(S)^{\mathrm{op}} \to (\widetilde{\mathcal{CM}}_X)_{(x,\langle m\rangle,\{S_i\})/}\,,$$

which sends A to the inert morphism collapsing all elements of S not in A to the basepoint. If we let $\mathcal{Q}(S) \subseteq \mathcal{P}(S)$ be the subposet of those elements A such that $|A \cap S_i| \leq 1$ for every i we obtain a functor

$$\mathcal{Q}(S)^{\mathrm{op}} \to (X \times_{\mathrm{Fin}_*} \mathcal{K})_{(x,\langle m\rangle,\{S_i\})/}$$

to the comma category, which has a right adjoint given by

$$\big[(f, g)\colon (x, \langle m\rangle, \{S_i\}) \to (x', \langle m'\rangle, \{S'_i\})\big] \mapsto g^{-1}\left(\coprod_i S'_i\right) \cap S\,,$$

and therefore is coinitial. Thus, by [Lur09, Proposition 4.3.1.7] and [Lur09, Lemma 4.3.2.13] it suffices to show the following two conditions

1. Let $F : X \times_{\mathrm{Fin}_*} \mathcal{K} \to \mathcal{C}^\otimes$ sending inert arrows to inert arrows, then the composition
$$\mathcal{Q}(S)^{\mathrm{op}} \to X \times_{\mathrm{Fin}_*} \mathcal{K} \to \mathcal{C}^\otimes$$
has a p-limit diagram sending all edges to inert edges.

2. Let $F : \widetilde{\mathcal{CM}}_X \to \mathcal{C}^\otimes$ sending inert arrows to inert arrows, then the composition
$$(\mathcal{Q}(S)^{\mathrm{op}})^{\triangleleft} \to \mathcal{P}(S)^{\mathrm{op}} \to \widetilde{\mathcal{CM}}_X \to \mathcal{C}^\otimes$$
is a p-limit diagram, where the first functor sends the cone point to $S \subseteq S$.

Both of them are now an immediate consequence of the characterization of p-limit diagrams in terms of mapping spaces [Lur09, Remark 4.3.1.2] and the definition of ∞-operads. □

Now we will identify the inert and cocartesian arrows of $\mathrm{Mod}^{\mathrm{Fin}_*}(\mathcal{C})^\otimes$ in terms of the model of Proposition 6.5.5.

Construction 6.5.6 (Bar construction)**.** There is a functor

$$B : (\Delta^{\mathrm{op}})^{\triangleright} \to \widetilde{\mathcal{CM}}^\otimes$$

sending $[n]$ to $(\langle 2\rangle, \mathrm{Hom}_\Delta([n],[1])_+, \{\{r_0\},\{r_1\}\})$ where r_i is the constant arrow at i, and the point at ∞ to $m_{1,1} = (\langle 1\rangle, \langle 1\rangle, \{1\})$. Concretely this sends $[n]$ to the object $(m_{2,1}, a, \ldots, a, m_{2,2})$ in the fibre over $\langle n+2\rangle$ of the ∞-operad $\widetilde{\mathcal{CM}}^{\otimes}_{\langle 2\rangle}$ (and so it encodes the bar construction in $\widetilde{\mathcal{CM}}^{\otimes}_{\langle 2\rangle}$).

Lemma 6.5.7. *Let $e\colon \Delta^1 \to \mathrm{Mod}^{\mathrm{Fin}_*}(\mathcal{C})^{\otimes}$ be an arrow, and let $e_0\colon \langle n\rangle \to \langle n'\rangle$ be the image of e in* Fin_*. *Write*

$$F_e\colon \widetilde{\mathcal{CM}}^{\otimes}_{\Delta^1} \to \mathcal{C}^{\otimes}$$

for the functor corresponding to e via the isomorphism of Proposition 6.5.5.

1. *The arrow e is inert if and only if e_0 is inert and F_e sends the arrows $a_n \to a_{n'}$ and $m_{i,n} \to m_{e_0 i,n'}$ to cocartesian arrows.*
2. *Suppose that $\mathcal{C}^{\otimes}$ is a symmetric monoidal ∞-category compatible with geometric realizations, and that e_0 is the unique active arrow from $\langle 2\rangle$ to $\langle 1\rangle$. Then e is cocartesian if and only if F_e sends the arrow $a_2 \to a_1$ to a cocartesian arrow and the composition*
$$(\Delta^{\mathrm{op}})^{\triangleright} \xrightarrow{B} \widetilde{\mathcal{CM}}^{\otimes}_{\Delta^1} \xrightarrow{F_e} \mathcal{C}^{\otimes}$$
is an operadic colimit diagram.

Proof. This is immediate from the proofs of [Lur16, Proposition 3.3.3.10 and Theorem 4.5.2.1] and the identification of Proposition 6.5.5. □

Construction 6.5.8. There is a square of ∞-categories

$$\begin{tikzcd} \mathrm{Fin}_* \times \mathrm{Fin}_* \arrow[r, "{(1,\wedge)}"] \arrow[d, "j_1"] & \mathrm{Fin}_* \times \mathrm{Fin}_* \arrow[d, "j_2"] \\ \mathrm{Fin}_* \times \mathcal{CM}^{\otimes} \arrow[r, "\phi"] & \widetilde{\mathcal{CM}}^{\otimes} \end{tikzcd}$$

where

- The top horizontal arrow sends $(\langle n\rangle, \langle m\rangle)$ to $(\langle n\rangle, \langle n\rangle \wedge \langle m\rangle)$;
- The arrow j_1 sends $(\langle n\rangle, \langle m\rangle)$ to $(\langle n\rangle, (\langle m\rangle, \varnothing)) \in \mathrm{Fin}_* \times \mathcal{CM}^{\otimes}$;
- The arrow j_2 sends $(\langle n\rangle, \langle m\rangle)$ to $(\langle n\rangle, \langle m\rangle, \{\varnothing\}) \in \widetilde{\mathcal{CM}}^{\otimes}$;
- The arrow ϕ sends $(\langle n\rangle, (\langle m\rangle, S)) \in \mathrm{Fin}_* \times \mathcal{CM}^{\otimes}$ to $(\langle n\rangle, \langle n\rangle \wedge \langle m\rangle, \{\{i\} \times S\})$.

Since each of these functors sends inert arrows to inert arrows, it induces for every $X \in (\mathrm{Cat}_\infty)_{/\mathrm{Fin}_*}$ a natural square

$$\begin{tikzcd}
\mathrm{Fun}_{/\mathrm{Fin}_*}(X, \mathrm{Mod}^{\mathrm{Fin}_*}(\mathcal{C})^\otimes) \simeq \mathrm{Alg}_{\widetilde{\mathcal{CM}}^\otimes_X}(\mathcal{C}^\otimes) \arrow[r] \arrow[d] & \mathrm{Fun}_{/\mathrm{Fin}_*}(X, \mathrm{Alg}_{\mathcal{CM}^\otimes}(\mathcal{C})^\otimes) \arrow[d] \\
\mathrm{Fun}_{/\mathrm{Fin}_*}(X, \mathrm{Fin}_* \times \mathrm{CAlg}(\mathcal{C})) \simeq \mathrm{Alg}_{X \times \mathrm{Fin}_*}(\mathcal{C}^\otimes) \arrow[r] & \mathrm{Fun}_{/\mathrm{Fin}_*}(X, \mathrm{CAlg}(\mathcal{C})^\otimes)
\end{tikzcd}$$

and therefore a natural square of ∞-categories over Fin_*

$$\begin{tikzcd}
\mathrm{Mod}^{\mathrm{Fin}_*}(\mathcal{C})^\otimes \arrow[r] \arrow[d] & \mathrm{Alg}_{\mathcal{CM}^\otimes}(\mathcal{C})^\otimes \arrow[d] \\
\mathrm{Fin}_* \times \mathrm{CAlg}(\mathcal{C}) \arrow[r] & \mathrm{CAlg}(\mathcal{C})^\otimes
\end{tikzcd} \tag{6.5.8.1}$$

Our goal now is to show that the square (6.5.8.1) is cartesian. To do so we will show that the right vertical arrow is a cocartesian fibration in favourable situations.

Lemma 6.5.9. *Let $\mathcal{I}$ be an ∞-category and $\mathcal{C}^\otimes \to \mathcal{I}^\amalg$ be an $\mathcal{I}^\amalg$-monoidal ∞-category compatible with geometric realizations. Then the map of ∞-operads*

$$p_{\mathcal{I}} : \mathrm{Alg}_{\mathcal{CM}^\otimes/\mathcal{I}^\amalg}(\mathcal{C})^\otimes \to \mathrm{Alg}_{\mathrm{Fin}_*/\mathcal{I}^\amalg}(\mathcal{C})^\otimes$$

is a cocartesian fibration.

Proof. Note that by [Lur16, Proposition 3.2.4.3.(3)] this is a map of cocartesian fibrations over $\mathcal{I}^\amalg$. Moreover the fibre over $\{x_j\}_{j\in J} \in \mathcal{I}^\amalg$ is given by

$$\prod_{j\in J} \mathrm{Mod}(\mathcal{C}_{x_j}) \to \prod_{j \in J} \mathrm{CAlg}(\mathcal{C}_{x_j})$$

and therefore it is a cocartesian fibration by [Lur16, Theorem 4.5.3.1]. Therefore by [Lur09, Proposition 2.4.2.11] $p_{\mathcal{I}}$ is a locally cocartesian fibration with locally cocartesian arrows those given by the composition of a fibrewise cocartesian arrow and a cocartesian arrow over $\mathcal{I}^\amalg$. In order to prove it is a cocartesian fibration it suffices to show then that the composition of two locally cocartesian arrow is locally cartesian, that is that fibrewise cocartesian arrows are stable under pushforward along arrows in $\mathcal{I}^\amalg$. Unwrapping the various cases it suffices to show that for every $x, y \in \mathcal{I}$ and arrow $f : x \to y$ the squares

$$\begin{tikzcd}
\mathrm{Mod}(\mathcal{C}_x) \times \mathrm{Mod}(\mathcal{C}_x) \arrow[r, "\otimes"] \arrow[d] & \mathrm{Mod}(\mathcal{C}_x) \arrow[d] \\
\mathrm{CAlg}(\mathcal{C}_x) \times \mathrm{CAlg}(\mathcal{C}_x) \arrow[r, "\otimes"] & \mathrm{CAlg}(\mathcal{C}_x)
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
\mathrm{Mod}(\mathcal{C}_x) \arrow[r, "f_*"] \arrow[d] & \mathrm{Mod}(\mathcal{C}_y) \arrow[d] \\
\mathrm{CAlg}(\mathcal{C}_x) \arrow[r, "f_*"] & \mathrm{CAlg}(\mathcal{C}_y)
\end{tikzcd}$$

are maps of cocartesian fibrations. That is that for every two maps of commutative algebras $A \to A'$, $B \to B'$, A-module M, and B-module N, the canonical maps

$$(M \otimes N) \otimes_{A \otimes B} (A' \otimes B') \simeq (M \otimes_A A') \otimes (N \otimes_B B') \text{ and } f_*(M \otimes_A B) \simeq f_* M \otimes_{f_* A} f_* B\,.$$

are equivalences. This is easily seen to be true since f_* is symmetric monoidal and commutes with geometric realization, and the tensor product commutes with geometric realization in each variable. □

Finally we arrive at the main result of this section.

Theorem 6.5.10. *The square* (6.5.8.1) *is cartesian for every* ∞*-operad* $\mathcal{C}^{\otimes}$*.*

Proof. Let us do first the case where $\mathcal{C}^{\otimes}$ is a symmetric monoidal ∞-category compatible with geometric realizations. Then both vertical arrows are cocartesian fibrations by [Lur16, Theorem 4.5.3.1] and Lemma 6.5.9. Moreover the description of cocartesian arrows in Lemma 6.5.7 and [Lur16, Proposition 3.2.4.3.(4)] shows that

$$\mathrm{Mod}^{\mathrm{Fin}_*}(\mathcal{C})^{\otimes} \to (\mathrm{Fin}_* \times \mathrm{CAlg}(\mathcal{C})) \times_{\mathrm{CAlg}(\mathcal{C})^{\otimes}} \mathrm{Alg}_{\mathcal{CM}^{\otimes}}(\mathcal{C})^{\otimes}$$

is a map of cocartesian fibrations over Fin_*. So it suffices to show that it induces an equivalence on fibres. Since it is a map of generalized operads, it suffices to show it induces an equivalence on the fibres over $\langle 0 \rangle$ and $\langle 1 \rangle$. But this is immediate by Proposition 6.5.5.

Now let us show the result for small ∞-operads $\mathcal{C}$. Indeed, it is clear by inspection that if the square (6.5.8.1) is cartesian for an ∞-operad, then it is cartesian for any full suboperad. But every small ∞-operad embeds as a full suboperad of a symmetric monoidal ∞-category compatible with small colimits. Indeed this is just the composition $\mathcal{C}^{\otimes} \to \mathrm{Env}\,\mathcal{C}^{\otimes} \to \mathcal{P}(\mathrm{Env}\,\mathcal{C})^{\otimes}$ where $\mathrm{Env}\,\mathcal{C}^{\otimes}$ is the symmetric monoidal envelope of $\mathcal{C}^{\otimes}$, and the second arrow is the Yoneda embedding.

Finally, since every ∞-operad is a sufficiently filtered union of small suboperads, the thesis is true for any ∞-operad. □

7. ∞-categories of global objects as partially lax limits

In this second part of the paper we prove that various ∞-categories of global objects admit a description using (partially lax) limits. In Theorem 7.1.17, we show that the ∞-category of global spaces is equivalent to the partially lax limit of the functor sending a compact Lie group G to the ∞-category of G-spaces. Our main result is Theorem 7.6.10 which describes the ∞-category of global spectra as a partially lax limit of G-spectra where G runs over all compact Lie groups G. Finally, the techniques employed in the previous cases allow us to prove that for any Lie group G, the ∞-category of proper G-spectra is a limit of H-spectra for H running over all compact subgroups of G. The precise statement can be found in Theorem 7.7.11.

Remark. To not burden the notation even more, we have decided to state Theorem 7.1.17 and Theorem 7.6.10 for the family of all compact Lie groups. However, the proofs hold verbatim for any family of compact Lie groups which is closed under isomorphisms, finite products, passage to subgroups and passage to quotients (i.e., any multiplicative global family in the language of [Sch18]). If the family is not closed under finite products, then the equivalences of the two theorems still hold without symmetric monoidal structures. This is due to the fact that the model structure constructed in [Sch18] is only shown to be symmetric monoidal for a multiplicative global family. We note that our result in fact allows us to define a symmetric monoidal structure on global spectra with respect to any global family, as a partially lax limit of symmetric monoidal categories is automatically symmetric monoidal.

7.1 GLOBAL SPACES AS A PARTIALLY LAX LIMIT

In this section we show that the ∞-category of global spaces is equivalent to a certain partially lax limit of the functor which sends a group G to the ∞-category of G-spaces $\mathcal{S}_G$. This is an unstable version of our main result, and serves as a warm up for the considerable more details involved in that proof. We start off by recalling a few relevant definitions.

Definition 7.1.1. The *global category* Glo is the ∞-category associated to the topological category whose objects are compact Lie groups and whose mapping spaces are given by

$$\mathrm{Map}_{\mathrm{Glo}}(H,G) := |\operatorname{Hom}(H,G) /\!\!/ G|$$

the geometric realization of the action groupoid of G acting on the space of continuous group homomorphisms $\operatorname{Hom}(H,G)$ by conjugation. Composition is induced by the composition of group homomorphisms.

We define Orb and $\mathrm{Glo}^{\mathrm{sur}}$ to be the wide subcategory of Glo whose hom-spaces are given by those path-components of $\mathrm{Map}_{\mathrm{Glo}}(H,G)$ spanned by the injective and surjective group homomorphisms respectively. For later applications it will be convenient to mark all the edges in the full subcategory Orb $\subseteq$ Glo; we denote this marking by $\mathrm{Glo}^{\dagger}$. Finally, we let Rep denote the homotopy category of Glo, that is the category whose objects are compact Lie groups and whose morphisms are given by conjugacy classes of continuous group homomorphisms.

Remark 7.1.2. The definition of Glo agrees with the definition given in Section 4 of [GH07] restricted to compact Lie groups, up to one difference. We apply thin geometric realization to the action groupoids to obtain a topologically enriched category, while the original definition uses fat geometric realization. Up to a technical condition, the two conventions define Dwyer-Kan equivalent topological categories. See [K18̈, Remark 3.10] for a more detailed discussion. Note as well that [GH07] uses the name Orb for both Glo and what we call Orb.

Key to the main properties of Glo is the following description of the mapping spaces.

Proposition 7.1.3. *Let G,H be two compact Lie groups. Then*

$$\operatorname{Hom}(H,G) \simeq \coprod_{[\alpha]\in\mathrm{Rep}(H,G)} \alpha G \quad \textit{and} \quad \mathrm{Glo}(H,G) \simeq \coprod_{[\alpha]\in\mathrm{Rep}(H,G)} BC(\alpha)$$

where αG denotes the orbit of α under the G-conjugation action, and $C(\alpha)$ denotes the centraliser of the image of α.

Proof. See [K18̈, Proposition 2.4, 2.5] for a proof of the first and second statement respectively. □

Proposition 7.1.4. *Let $f\colon H \to G$ be a map in* Glo. *The induced map on mapping spaces $f_*\colon \mathrm{Glo}(K,H) \to \mathrm{Glo}(K,G)$ correspond under the equivalences of Proposition 7.1.3 to the composite of the map*

$$\coprod_{[\alpha]\in\mathrm{Rep}(H,G)} Bf\colon \coprod_{[\alpha]\in\mathrm{Rep}(K,H)} BC(\alpha) \to \coprod_{[\alpha]\in\mathrm{Rep}(K,H)} BC(f\alpha)$$

with the map

$$\coprod_{[\alpha]\in \mathrm{Rep}(K,H)} BC(f\alpha) \to \coprod_{[\beta]\in \mathrm{Rep}(K,G)} BC(\beta)$$

which is the identity on individual path-components, and acts on π_0 by the map $f_*\colon \mathrm{Rep}(K,H) \to \mathrm{Rep}(K,G)$.

Proof. The statement on π_0 follows from the fact that Rep is the homotopy category of Glo. Therefore, it suffices to restrict to one path component, and analyse the effect of f. The relationship $f_*(c_h\alpha) = c_{f(h)} f\alpha$ implies that f_* acts as f when restricted to a map $\alpha H \to f\alpha G$. This implies that the induced map $BC(\alpha) \to BC(f\alpha)$ equals Bf. □

Definition 7.1.5. The ∞-category of *global spaces* $\mathcal{S}_{\mathrm{gl}}$ is the category of functors from $\mathrm{Glo}^{\mathrm{op}}$ to $\mathcal{S}$. This admits a symmetric monoidal structure by pointwise product. This is equivalent to the symmetric monoidal category $(\mathrm{Glo}^{\mathrm{op}})^{\amalg}$–$\mathcal{S}$.

Remark 7.1.6. In [Sch20], the author proves that the underlying ∞-category of orthogonal spaces equipped with the positive global model structure of [Sch18, Proposition 1.2.23] is equivalent to presheafs on a topologically enriched category $\mathbf{O}_{gl}$. Furthermore, in [KÏ8] it is shown that $\mathbf{O}_{gl}$ is Dwyer-Kan equivalent to Glo. Therefore the two models of global spaces define the same ∞-category. In fact, the two ∞-categories are symmetric monoidal equivalent since they are both endowed with the cartesian monoidal structure, see [Sch18, Theorem 1.3.2].

Before stating and proving the main result of this section, we need some preparation. In the following we fix an ∞-category $\mathcal{C}$ with an orthogonal factorization system $(\mathcal{C}^L, \mathcal{C}^R)$. For a detailed discussion and a definition of orthogonal factorization systems on ∞-categories, the reader may consult [Lur09, Section 5.2.8]. We write $\mathcal{C}^L$ for the left class of maps and $\mathcal{C}^R$ for the right class. We will denote edges in $\mathcal{C}^L$ by $\twoheadrightarrow$ and edges in $\mathcal{C}^R$ by $\rightarrowtail$.

Proposition 7.1.7. *Let $\mathcal{C}$ be an ∞-category equipped with an orthogonal factorization $(\mathcal{C}^L, \mathcal{C}^R)$. Write $\mathrm{Ar}_R(\mathcal{C})$ for the full subcategory of the arrow category of $\mathcal{C}$ spanned by the edges in $\mathcal{C}^R$. Then the target projection $t\colon \mathrm{Ar}_R(\mathcal{C}) \to \mathcal{C}$ is a cocartesian fibration. Furthermore an edge in $\mathrm{Ar}_R(\mathcal{C})$ is t-cocartesian if and only if it is of the form*

$$\begin{tikzcd} X \arrow[r, two heads] \arrow[d, tail] & Y \arrow[d, tail] \\ X' \arrow[r] & Y'. \end{tikzcd} \tag{7.1.7.1}$$

Proof. Consider an edge in $\mathrm{Ar}_R(\mathcal{C})$:

$$\begin{tikzcd} X \arrow[r] \arrow[d, tail] & Y \arrow[d, tail] \\ X' \arrow[r] & Y'. \end{tikzcd}$$

This is cocartesian if and only if, given a 2-simplex in $\mathcal{C}$ and a (2,0)-horn in $\mathrm{Ar}_R(\mathcal{C})$, there is a contractible choice of extensions. This corresponds to showing that given a diagram in $\mathcal{C}$

$$\begin{tikzcd} X \arrow[r] \arrow[d, tail] \arrow[rr, bend left] & Y \arrow[d, tail] \arrow[dr] & Z \arrow[d, tail] \\ X' \arrow[r] & Y' \arrow[r] & Z', \end{tikzcd}$$

its extensions to a 2-simplex in $\mathrm{Ar}_R(\mathcal{C})$ form a contractible space. However, completing this diagram is equivalent to supplying an edge $Y \to Z$ which makes the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Z \arrow[d, tail] \\ Y \arrow[r] \arrow[ur, dotted] & Z' \end{tikzcd}$$

commute. There is a contractible choice of such factorizations if and only if $X \to Y$ is in $\mathcal{C}^L$. This shows that an edge is t-cocartesian if and only if it is of the form of Equation (7.1.7.1). Next, fix an edge in $\mathcal{C}$ and a lift of its source in $\mathrm{Ar}_R(\mathcal{C})$. This corresponds to a diagram

$$\begin{tikzcd} X \arrow[d, tail] \arrow[dr] & \\ X' \arrow[r] & Y'. \end{tikzcd}$$

Factorizing the composite $X \to Y'$ extends this to an edge

$$\begin{tikzcd} X \arrow[r, two heads] \arrow[d, tail] \arrow[dr] & Y \arrow[d, tail] \\ X' \arrow[r] & Y' \end{tikzcd}$$

in $\mathrm{Ar}_R(\mathcal{C})$, which is t-cocartesian. □

We record the following fact for later reference.

Lemma 7.1.8. *The constant functor $s_0 \colon \mathcal{C} \to \mathrm{Ar}_R(\mathcal{C})$ is a fully faithful left adjoint to the source functor $s \colon \mathrm{Ar}_R(\mathcal{C}) \to \mathcal{C}$.*

Construction 7.1.9. Suppose we are in the setting of Proposition 7.1.7. Straightening the cocartesian fibration $t\colon \mathrm{Ar}_R(\mathcal{C}) \to \mathcal{C}$ gives a functor

$$\mathcal{C}^R_{/-}\colon \mathcal{C} \to \mathrm{Cat}_\infty.$$

To justify our notation let us unravel the effect of this functor. By definition, the evaluation of $\mathcal{C}^R_{/-}$ at an object $X \in \mathcal{C}$ is given by $\mathrm{Ar}_R(\mathcal{C})_X$; the fibre of t at X. By construction this is the full subcategory of $\mathcal{C}_{/X}$ on the objects $C \rightarrowtail X$ in $\mathcal{C}^R$. A priori an edge in this full subcategory is given by a diagram

$$\begin{tikzcd} X \arrow[r] \arrow[dr, tail] & X' \arrow[d, tail] \\ & Y. \end{tikzcd}$$

However the edge $X \to X'$ is necessarily also in $\mathcal{C}^R$ by [Lur09, Proposition 5.2.8.6(3)], and therefore $\mathrm{Ar}_R(\mathcal{C})_X$ is in fact equivalent to $\mathcal{C}^R_{/X}$. Next consider an edge $f\colon Y \to Y'$. Then the induced map $f_*\colon \mathcal{C}^R_{/Y} \to \mathcal{C}^R_{/Y'}$ sends an object $X \rightarrowtail Y$ to an object $X' \rightarrowtail Y'$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, two heads] \arrow[d, tail] & X' \arrow[d, tail] \\ Y \arrow[r, "f"] & Y'. \end{tikzcd}$$

In particular if $f \in \mathcal{C}^R$ this is nothing but the standard functoriality of the slices $\mathcal{C}^R_{/-}$. Therefore the functor $\mathcal{C}^R_{/-}\colon \mathcal{C} \to \mathrm{Cat}_\infty$ extends the functoriality of the slices of $\mathcal{C}^R$ to all of $\mathcal{C}$.

Proposition 7.1.10. *Let $\mathcal{C}$ be an ∞-category equipped with a factorization system $(\mathcal{C}^L, \mathcal{C}^R)$. The partially lax colimit of $(-)^{\mathrm{op}} \circ \mathcal{C}^R_{/-}\colon \mathcal{C} \to \mathrm{Cat}_\infty$ with respect to the marking $\mathcal{C}^R \subset \mathcal{C}$ is equivalent to $\mathcal{C}^{\mathrm{op}}$.*

Proof. Recall that the partially lax colimit of a functor $F\colon \mathcal{C} \to \mathrm{Cat}_\infty$ is the localization of $\mathrm{Un}^{\mathrm{ct}}(F)$ at the cartesian edges which live above marked edges, see Theorem 6.3.9(b). In the case $F = (-)^{\mathrm{op}} \circ \mathcal{C}^R_{/-}$, we observe that $\mathrm{Un}^{\mathrm{ct}}(F) \simeq \mathrm{Un}^{\mathrm{co}}(\mathcal{C}^R_{/-})^{\mathrm{op}}$ and so we conclude that the partially lax colimit of F is equal to the opposite of $\mathrm{Ar}_R(\mathcal{C})$ localized at the edges of the form

$$\begin{tikzcd} X \arrow[r, two heads] \arrow[d, tail] & X' \arrow[d, tail] \\ Y \arrow[r, tail] & Y'. \end{tikzcd}$$

However note that because edges in $\mathcal{C}^R$ are left cancellable, $X \to X'$ is not only in $\mathcal{C}^L$ but also in $\mathcal{C}^R$. Therefore $X \to X'$ is in fact an equivalence. We will write M for this collection of edges. We claim that localizing at the edges of M is equivalent to localizing at the larger class of edges M' of the form

$$\begin{tikzcd} X \arrow[r, "\sim"] \arrow[d, tail] & X' \arrow[d, tail] \\ Y \arrow[r] & Y', \end{tikzcd}$$

where we do not impose any conditions on the edge $Y \to Y'$. To see this note that such an edge in M' fits into the following diagram:

$$\begin{tikzcd} X \arrow[r, "\sim"] \arrow[d, "\sim"'] & X' \arrow[r, equal] \arrow[d, tail] & X' \arrow[d, tail] \\ X' \arrow[r, tail] \arrow[rr, tail, bend right] & Y \arrow[r] & Y'. \end{tikzcd}$$

Both the first edge and the composite are in M, and so therefore M' is contained in the two-out-of-three closure of M. So it is enough to calculate the localization of $\mathrm{Ar}_R(\mathcal{C})$ at M'. Note that the source functor $s\colon \mathrm{Ar}_R(\mathcal{C}) \to \mathcal{C}$ sends an edge to an equivalence if and only if it is in M'. Then Lemma 7.1.8 implies that $\mathcal{C}$ is a Bousfield colocalization of $\mathrm{Ar}_R(\mathcal{C})$ at M'. So we conclude that the partially lax colimit of ${}^{\mathrm{op}} \circ \mathcal{C}^R_{/-}$ is equivalent to $\mathcal{C}^{\mathrm{op}}$, finishing the proof. □

Example 7.1.11. There are two extreme cases of the previous result. If $\mathcal{C}^R = \mathcal{C}$, $\mathcal{C}^L = \iota\mathcal{C}$, then

$$\mathrm{colim}((\mathcal{C}_{/-})^{\mathrm{op}}\colon \mathcal{C} \to \mathrm{Cat}_\infty) \cong \mathcal{C}^{\mathrm{op}}$$

If $\mathcal{C}^R = \iota\mathcal{C}$, $\mathcal{C}^L = \mathcal{C}$, then

$$\mathrm{laxcolim}(\iota\mathcal{C}_{/-}\colon \mathcal{C} \to \mathrm{Cat}_\infty) \cong \mathcal{C}^{\mathrm{op}}.$$

Now that we have introduced the main tools we need, we can build our functor and compute its partially lax limit. This relies on two important observations. The first key insight is the following, which was first stated in [GH07] and originally proven as [Rez14, Example 3.5.1].

Lemma 7.1.12. *For all compact Lie group G, the assignment $G/K \mapsto (K \hookrightarrow G)$ defines an equivalence $\mathbf{O}_G \simeq \mathrm{Orb}_{/G}$.*

Proof. We observe that the spaces $\mathbf{O}_G(G/H, G/K)$ are homeomorphic to the space $\{g \in G \mid c_g(H) \subseteq K\}/K$. The latter space is equivalent to the homotopy orbits $\{g \in G \mid c_g(H) \subseteq K\}_{hK}$ as the K-space is free, see for example [Kö18, Theorem A.7]. Therefore we can define a functor $F'\colon \mathbf{O}_G \to \mathrm{Glo}$, which sends G/H

to H, and on mapping spaces acts as homotopy orbits of the K-equivariant inclusion

$$\{g \in G \mid c_g(H) \subseteq K\} \to \hom(H,K), \quad g \mapsto [c_g \colon H \to K].$$

Note that the ∞-category $\mathbf{O}_G$ has a final object G/G, and therefore F' induces a functor $\mathbf{O}_G \to \mathrm{Glo}_{/G}$, which in fact factors through $\mathrm{Orb}_{/G}$. We claim that the induced functor $F \colon \mathbf{O}_G \to \mathrm{Orb}_{/G}$ is an equivalence of ∞-categories. First note that F is clearly essentially surjective. To deduce that the functor is fully faithful pick two objects G/H and G/K which we identify with inclusions $i \colon H \hookrightarrow G$ and $j \colon K \hookrightarrow G$. Recall that the mapping space between G/H and G/K is empty if and only if H is not subconjugate to K. In this case the mapping space in Orb/G between i and j is also empty. Now suppose that this is not the case. Consider the square

$$\begin{tikzcd}
\{g \in G \mid c_g(H) \subseteq K\}_{\mathrm{h}K} \arrow[r] \arrow[d] & \mathrm{Hom}(H,K)_{\mathrm{h}K} \arrow[d] \\
* \arrow[r] & \mathrm{Hom}(H,G)_{\mathrm{h}G}.
\end{tikzcd}$$

To prove F is fully faithful it suffices to prove that this square is homotopy cartesian. For every K-space X, $(G \times_K X)_{\mathrm{h}G} \simeq X_{\mathrm{h}K}$, so that the above square is equivalent to

$$\begin{tikzcd}
\left(G \times_K \{g \in G \mid c_g(H) \subseteq K\}\right)_{\mathrm{h}G} \arrow[r] \arrow[d] & (G \times_K \mathrm{Hom}(H,K))_{\mathrm{h}G} \arrow[d] \\
G_{\mathrm{h}G} \arrow[r] & \mathrm{Hom}(H,G)_{\mathrm{h}G}.
\end{tikzcd}$$

Because taking homotopy orbits preserves homotopy pullback diagrams, it suffices to show that the square

$$\begin{tikzcd}
G \times_K \{g \in G \mid c_g(H) \subseteq K\} \arrow[r] \arrow[d] & G \times_K \mathrm{Hom}(H,K) \arrow[d] \\
G \arrow[r] & \mathrm{Hom}(H,G)
\end{tikzcd}$$

is homotopy cartesian. In fact it is easily shown to be a pullback square of topological spaces, and the bottom horizontal arrow is a Serre fibration. To see this we note that the map $G \to \mathrm{Hom}(H,G)$ factors through one component of the decomposition of Proposition 7.1.3, and therefore is equivalent to the quotient map $G \to G/C(H)$ which is a fibration by [KÏ8, Theorem A.9]. $\square$

The second insight is the following, which was also observed in [Rez14].

Proposition 7.1.13. *The subcategories* $\mathrm{Glo}^{\mathrm{sur}}$ *and* Orb *are the left and right classes respectively of an orthogonal factorization system on* Glo.

Proof. We will apply [Lur09, Proposition 5.2.8.17] to the subcategories $\mathrm{Glo}^{\mathrm{sur}}$ and Orb. Clearly these subcategories contain all the equivalences and are closed under equivalences in Ar(Glo). Therefore it suffices to prove that given a diagram:

$$\begin{tikzcd} H \arrow[r] \arrow[d, two heads, "f"'] & J \arrow[d, hook, "g"] \\ G \arrow[r] \arrow[ur, dotted] & K, \end{tikzcd}$$

the space of dotted diagonal fillers is contractible. As noted in [Lur09, Remark 5.2.8.3], this is equivalent to the map

$$\mathrm{Map}_{\mathrm{Glo}_{H/}}(H \xrightarrow{f} G, H \to J) \xrightarrow{g} \mathrm{Map}_{\mathrm{Glo}_{H/}}(H \xrightarrow{f} G, H \to K)$$

being a weak homotopy equivalence for every lift of g to a map in $\mathrm{Glo}_{H/}$ from $H \to J$ to $H \to K$. Proposition 7.1.4 shows that when f is surjective the map

$$\mathrm{Map}_{\mathrm{Glo}}(G, J) \xrightarrow{f^*} \mathrm{Map}_{\mathrm{Glo}}(H, J)$$

is an inclusion of path-components for every J.

Therefore the space $\mathrm{Map}_{\mathrm{Glo}_{H/}}(H \xrightarrow{f} G, H \to J)$, being the homotopy fibre of this map, is either empty or contractible. Translating back this reduces our task to simply proving the existence of a lift in the square above. This is a simple exercise in group theory. □

Remark 7.1.14. When we restrict to finite groups, Glo is equivalent to the full subcategory of $\mathcal{S}$ given by the connected 1-truncated spaces. In this case the orthogonal factorization system constructed above is a restriction of the standard mono/epi factorization system of any ∞-topos. However in the generality of compact Lie groups no such description applies.

We are finally ready to construct the functor.

Construction 7.1.15. We apply Construction 7.1.9 to the orthogonal factorization system $(\mathrm{Glo}^{\mathrm{sur}}, \mathrm{Orb})$ to obtain a functor $\mathrm{Orb}_{/-}\colon \mathrm{Glo} \to \mathrm{Cat}_\infty$. Postcomposing the opposite of this functor with $\mathrm{Fun}((-)^{\mathrm{op}}, \mathcal{S})\colon \mathrm{Cat}_\infty^{\mathrm{op}} \to \mathrm{Cat}_\infty$ gives the desired functor

$$\mathcal{S}_\bullet\colon \mathrm{Glo}^{\mathrm{op}} \to \mathrm{Cat}_\infty.$$

Also note that $\mathcal{S}_\bullet$ clearly factors through product preserving functors, and so enhances to a functor

$$\mathcal{S}_\bullet \colon \mathrm{Glo}^{\mathrm{op}} \to \mathrm{Cat}_\infty^{\otimes},$$

where each category $(\mathrm{Orb}_{/G})^{\mathrm{op}}\text{–}\mathcal{S}$ is given the cartesian monoidal structure.

Lemma 7.1.12 and Elmendorf's theorem for G-spaces, see Example 6.2.40, imply that the value of $\mathcal{S}_\bullet$ at the object G is equivalent to the ∞-category of G-spaces $\mathcal{S}_G$. However we owe the reader the following consistency check, which implies that the functor $\mathcal{S}_\bullet$ also has the expected functoriality.

Proposition 7.1.16. *Let $\alpha \colon H \to G$ be a continuous group homomorphism. Then the following diagram commutes:*

$$\begin{tikzcd} \mathrm{Fun}((\mathrm{Orb}_{/G})^{\mathrm{op}}, \mathcal{S}) \arrow[r, "\simeq"] \arrow[d, "\mathcal{S}_\alpha"'] & \mathcal{S}_G \arrow[d, "\alpha^*"] \\ \mathrm{Fun}((\mathrm{Orb}_{/H})^{\mathrm{op}}, \mathcal{S}) \arrow[r, "\simeq"] & \mathcal{S}_H. \end{tikzcd}$$

Here the horizontal equivalences are obtained by applying Lemma 7.1.12 and Example 6.2.40.

Proof. It is enough to check that the analogous diagram where the vertical maps are replaced with left adjoints commutes. For this, let us denote by L_α and $\alpha_!$ the left adjoints of $\mathcal{S}_\alpha$ and α^* respectively. Note that the inclusion $\iota_H \colon \mathrm{Orb}_{/H} \hookrightarrow \mathrm{Glo}_{/H}$ has a left adjoint L_H which on objects sends $K \xrightarrow{\beta} H$ to $\beta(K) \hookrightarrow H$. By the universal property of the presheaf categories there exists a unique cocontinuous functor (the left Kan extension along ι_H)

$$(\iota_H)_! \colon \mathrm{Fun}((\mathrm{Orb}_{/H})^{\mathrm{op}}, \mathcal{S}) \to \mathrm{Fun}((\mathrm{Glo}_{/H})^{\mathrm{op}}, \mathcal{S})$$

which agrees with ι_H on representables. In a similar fashion, we define functors $(L_G)_!$ and $(\alpha_*)_!$ where $\alpha_* \colon \mathrm{Glo}_{/H} \to \mathrm{Glo}_{/G}$ is postcomposition by α. We claim that the following diagram commutes:

$$\begin{tikzcd} \mathrm{Fun}((\mathrm{Orb}_{/H})^{\mathrm{op}}, \mathcal{S}) \arrow[r, hook, "(\iota_H)_!"] \arrow[d, "L_\alpha"'] & \mathrm{Fun}((\mathrm{Glo}_{/H})^{\mathrm{op}}, \mathcal{S}) \arrow[d, "(\alpha_*)_!"] \\ \mathrm{Fun}((\mathrm{Orb}_{/G})^{\mathrm{op}}, \mathcal{S}) & \mathrm{Fun}((\mathrm{Glo}_{/G})^{\mathrm{op}}, \mathcal{S}). \arrow[l, "(L_G)_!"] \end{tikzcd}$$

This is easily seen by comparing the result on generators, and using that all the functors in the diagram commute with all colimits. Using this diagram we can reduce to a statement on the level of model categories. Namely all three functors which make up the long way around in the diagram above can be modelled by left Quillen functors between enriched functor categories with the projective model structure. Indeed, the right adjoint of $(\iota_H)_!$ is given by restriction along ι_H which is clearly a right Quillen functor. A similar argument also works for $(L_G)_!$ and $(\alpha_*)_!$. After pre-composing and post-composing with the equivalences

$$\mathcal{T}_H \simeq \mathrm{Fun}^{\mathrm{top}}((\mathrm{Orb}_{/H})^{\mathrm{op}}, \mathcal{T}) \quad \text{and} \quad \mathrm{Fun}^{\mathrm{top}}((\mathrm{Orb}_{/G})^{\mathrm{op}}, \mathcal{T}) \simeq \mathcal{T}_G$$

constructed in [Rez14, Proposition 3.5.1] (which agree with the equivalences constructed by [GM23] by inspection), we can apply the explicit description for $(L_G)_!$ and $(\iota_H)_!$ given in [Rez14, Section 5.3] (where $(L_G)_!$ is denoted by Π_G and $(\iota_H)_!$ by Δ_H) to deduce that the functor $L_\alpha \colon \mathcal{T}_H \to \mathcal{T}_G$ is equivalent to induction of H-spaces. □

We have now constructed our functor. Therefore we are left to prove that the partially lax limit is given by the ∞-category of global spaces.

Theorem 7.1.17. *Let* $\mathrm{Glo}^\dagger$ *denote the marked ∞-category from Definition 7.1.1. Then the partially lax limit over* $(\mathrm{Glo}^\dagger)^{\mathrm{op}}$ *of the diagram from Construction 7.1.15*

$$\mathrm{Glo}^{\mathrm{op}} \to \mathrm{Cat}_\infty^{\otimes}, \quad G \mapsto \mathcal{S}_G$$

is equivalent to the ∞-category of global spaces, equipped with the cartesian monoidal structure.

Proof. Recall that $\mathcal{S}_G = \mathrm{Fun}(\mathbf{O}_G^{\mathrm{op}}, \mathcal{S})$ and that $\mathbf{O}_G \simeq \mathrm{Orb}_{/G}$. First we prove the result on underlying categories. Proposition 6.3.11 implies that it suffices to prove an equivalence between the partially lax colimit of $(\mathrm{Orb}_{/-})^{\mathrm{op}}$ and $\mathrm{Glo}^{\mathrm{op}}$. However this follows from Proposition 7.1.10 applied to the factorization system $(\mathrm{Glo}^{\mathrm{sur}}, \mathrm{Orb})$ on Glo. Now we deduce the symmetric monoidal statement. First observe that the equivalence constructed before trivially lifts to a symmetric monoidal equivalence, where both sides are given the cartesian symmetric monoidal structure. Then note that the subcategory of Op_∞ spanned by the cartesian operads is closed under partially lax limits. This implies that $\mathcal{S}_{gl}$ is equivalent to the partially lax limit of the diagram $\mathcal{S}^\bullet \colon \mathrm{Glo}^{\mathrm{op}} \to \mathrm{Cat}_\infty^{\otimes}$, but now taken in symmetric monoidal ∞-categories. □

7.2 ∞-CATEGORIES OF EQUIVARIANT PRESPECTRA

In this section we define the ∞-categories of G-(pre)spectra for a Lie group G, and we introduce the ∞-category of global (pre)spectra. We will do this by first defining the relevant level model structures, which present the ∞-categories of prespectra objects, and then defining the stable model category as a Bousfield localization. This will then present the ∞-categories of spectra objects. The material in this section is classical, and largely well-known. Nevertheless we include the details of the model structures, mainly to emphasize that the level model structure on Sp_G^O is induced formally from the level model structure on $\mathcal{I}-G\mathcal{T}$. While not a deep statement, it is crucial to our proof strategy. In particular this observation will allow us to interpret the construction of the level model structure ∞-categorically, as will be explained in this section.

Definition 7.2.1. Let $\mathcal{I}$ denote the topological category whose objects are finite dimensional inner product spaces V, and morphisms space $\mathcal{I}(V, W)$ given by the space of linear isometric isomorphisms from V to W.

Definition 7.2.2. Let G be a Lie group (not necessarily compact). We write $\mathcal{I}-G\mathcal{T}$ for the enriched category of continuous functors from $\mathcal{I}$ into G-spaces, and call this the category of $\mathcal{I}$-G-spaces. When G is the trivial group, we simply write $\mathcal{I}-\mathcal{T}$ and refer to it as the category of $\mathcal{I}$-spaces.

Remark 7.2.3. As discussed in [Boh14, Section 5], the category of $\mathcal{I}$-G-spaces (as defined above) is equivalent as a topological category to the category of $\mathcal{I}_G$-spaces as defined by Mandell-May in [MM02, Chapter II, Definition 2.3].

Remark 7.2.4. The category $\mathcal{I}-G\mathcal{T}$ has a symmetric monoidal structures given by enriched Day convolution, see [MM02, Chapter II, Proposition 3.7]. Given $X, Y \in \mathcal{I}-G\mathcal{T}$ we have the formula

$$(X \otimes Y)(V) := \int^{(W,W')\in\mathcal{I}\times\mathcal{I}} \mathcal{I}(W \oplus W', V) \times X(W) \times Y(W').$$

Remark 7.2.5. Given any $\mathcal{I}$-G-space X and an inner product space V, the value $X(V)$ admits a $G \times O(V)$-action. If V is given the structure of an H-representation $\rho\colon H \to O(V)$, then we can equip $X(V)$ with an H-action by restricting along

$$H \xrightarrow{\Delta} H \times H \xrightarrow{i\times\rho} G \times O(V).$$

We will always consider the value $X(V)$ with this H-action in the following.

Construction 7.2.6 (Free $\mathcal{I}$-G-space)**.** For every H-representation V, there is an evaluation functor

$$\mathrm{ev}_V : \mathcal{I}-G\mathcal{T} \to H\mathcal{T}, \qquad X \mapsto X(V).$$

This functor admits a left adjoint $G \times_H \mathcal{I}_V$, given by the formula

$$G \times_H \mathcal{I}_V A = G \times_H (\mathcal{I}(V, -) \times A).$$

When $A = *$, we simply write $G \times_H \mathcal{I}_V$ and when $G = H$, we write $\mathcal{I}_V(-)$. By construction, the $\mathcal{I}$-G-space $G \times_H \mathcal{I}_V$ corepresents the functor $X \mapsto X(V)^H$.

For all compact subgroups H and K of G, all H-representations V and all K-representations W, there is an isomorphism of $\mathcal{I}$-G-spaces

$$(G \times_H \mathcal{I}_V) \otimes (G \times_K \mathcal{I}_W) \cong \Delta^*(G \times G \times_{H \times K} \mathcal{I}_{V \oplus W}) \tag{7.2.6.1}$$

where $\Delta\colon G \to G \times G$ is the diagonal embedding. This can be checked directly by applying the formula of the Day convolution product from Remark 7.2.4 and using that induction commutes with colimits.

We will now proceed to equip the category of $\mathcal{I}$-G-spaces with the level model structure. The following will be the weak equivalences, fibrations and cofibrations of this model structure.

Definition 7.2.7. Let G be a Lie group and let $f\colon X \to Y$ be a morphism in $\mathcal{I}{-}G\mathcal{T}$.

(a) We say f is a *level equivalence* if for any compact subgroup $H \leq G$ and any H-representation V, the map $f(V)^H\colon X(V)^H \to Y(V)^H$ is a weak homotopy equivalence of spaces.

(b) We say f is a *level fibration* if for any compact subgroup $H \leq G$ and any H-representation V, the map $f(V)^H\colon X(V)^H \to Y(V)^H$ is a Serre fibration.

(c) We say f is a *level cofibration* if for every $m \geq 0$, the map $f(\mathbb{R}^m)\colon X(\mathbb{R}^m) \to Y(\mathbb{R}^m)$ is a Com-cofibration of $(G \times O(m))$-spaces, see [Deg+23, Definition 1.1.2], and moreover the $O(m)$-action is free away from the image of $f(\mathbb{R}^m)$.

For all $m \geq 0$, we let $\mathcal{C}_G(m)$ denote the family of compact subgroups Γ of $G \times O(m)$ such that $\Gamma \cap (1 \times O(m))$ consists only of the neutral element. These are precisely the graph subgroups of a continuous homomorphism to $O(m)$ defined on some compact subgroup of G. The category of $G \times O(m)$-spaces admits a $\mathcal{C}_G(m)$-projective model structure by [Sch18, Proposition B.7]. We have the following useful characterization of the level equivalences, cofibrations and fibrations.

Lemma 7.2.8. *Let G be a Lie group and let $f\colon X \to Y$ be a morphism in $\mathcal{I}{-}G\mathcal{T}$. The following are equivalent:*

(a) the map $f\colon X \to Y$ is a level equivalence (resp., level fibration);

(b) the map $f(\mathbb{R}^m)\colon X(\mathbb{R}^m) \to Y(\mathbb{R}^m)$ is a weak equivalence (resp., fibration) in the $\mathcal{C}_G(m)$-projective model structure for all $m \geq 0$.

Furthermore, the following are equivalent:

(c) the map $f : X \to Y$ is a level cofibration;

(d) the map $f(\mathbb{R}^m)\colon X(\mathbb{R}^m) \to Y(\mathbb{R}^m)$ is a cofibration in the $\mathcal{C}_G(m)$-projective model structure for all $m \geq 0$.

Proof. Let $H \leq G$ be a compact subgroup and let V be an H-representation. Choose a linear isometric isomorphism $\varphi\colon V \cong \mathbb{R}^m$ and define a group homomorphism

$$\rho\colon G \to O(m), \qquad g \mapsto \varphi \circ (g \cdot -) \circ \varphi^{-1}.$$

The homeomorphism $X(\varphi)\colon X(V) \simeq X(\mathbb{R}^m)$ restricts to a homeomorphism

$$X(V)^H \simeq X(\mathbb{R}^m)^{\Gamma(\rho)}$$

where $\Gamma(\rho) = \{(h, \rho(h)) \in H \times O(m)\}$ by the definition of the H-action given in Remark 7.2.5. From this description, it is clear that (b) implies (a). Conversely given $\Gamma \in \mathcal{C}_G(m)$, we can always find a continuous group homomorphism $\alpha\colon H \to O(m)$ for $H \leq G$ compact such that $\Gamma = \Gamma(\alpha)$. By definition of the H-action, we have $X(\mathbb{R}^m)^H = X(\mathbb{R}^m)^\Gamma$ showing that (a) implies (b). Finally, that (c) and (d) are equivalent follows from (the topological version of) [Ste16, Proposition 2.16].

□

Theorem 7.2.9. *Let G be a Lie group. The category $\mathcal{I}{-}G\mathcal{T}$ admits a cofibrantly generated and topological model structure in which the weak equivalences are the level equivalences, the fibrations are the level fibrations and the cofibrations are the level cofibrations. The set of generating cofibrations I_G and acyclic cofibrations J_G are given by*

$$I_G = \{G \times_H \mathcal{I}_V \partial D^n \to G \times_H \mathcal{I}_V D^n \mid H \leq G, \ \ n \geq 0\}$$
$$J_G = \{G \times_H \mathcal{I}_V (D^n \times \{0\}) \to G \times_H \mathcal{I}_V (D^n \times [0,1]) \mid H \leq G, \ \ n \geq 0\}$$

where H runs over all compact subgroups of G and V runs over all H-representations. We call this the (proper) level model structure.

Proof. We observe that the category $\mathcal{I}{-}G\mathcal{T}$ is equivalent to $\prod_{m\geq 0}(G\times O(m))\mathcal{T}$. We can endow this latter category with the product of the $\mathcal{C}_G(m)$-projective model structures on $G \times O(m)$-spaces. By Lemma 7.2.8, the induced model structure on $\mathcal{I}{-}G\mathcal{T}$ has weak equivalences, fibrations and cofibrations as in the theorem. Also we note that the right lifting property against the sets I_G and J_G detect the level fibrations and level acyclic fibrations respectively, by the adjunction isomorphism

$$\mathrm{Hom}_{\mathcal{I}-G\mathcal{T}}(G \times_H \mathcal{I}_V A, X) \simeq \mathrm{Hom}_{\mathcal{T}}(A, X(V)^H)$$

for A a non-equivariant space. Finally we observe that resulting model structure is again topological by [Sch18, Proposition B.5]. □

As discussed in [Deg+23, Proposition 1.1.6], a continuous homomorphism $\alpha\colon K \to G$ between Lie groups gives rise to adjoint functors between the associated category of equivariant spaces

$$\begin{tikzcd} G\mathcal{T} \arrow[r, "\alpha^*"] & K\mathcal{T} \arrow[l, bend right, swap, "G\times_\alpha -"] \arrow[l, bend left, "\mathrm{Map}^\alpha(G,-)"] \end{tikzcd}$$

which by levelwise application gives rise to an adjoint triple

$$\begin{tikzcd} \mathcal{I}{-}G\mathcal{T} \arrow[r, "\alpha^*"] & \mathcal{I}{-}K\mathcal{T}. \arrow[l, bend right, swap, "G\times_\alpha -"] \arrow[l, bend left, "\mathrm{Map}^\alpha(G,-)"] \end{tikzcd}$$

Proposition 7.2.10. *Let $\alpha\colon K \to G$ be a continuous group homomorphism between Lie groups.*

(a) Then α^* preserves level fibrations and level equivalences. Thus the adjoint pair $(G \times_\alpha -, \alpha^*)$ is Quillen.

(b) If α has closed image and compact kernel, then the adjoint pair $(\alpha^*, \mathrm{Map}^\alpha(G,-))$ is also Quillen with respect to the level model structure.

Proof. Part (a) follows from [Deg+23, Proposition 1.1.6(ii)]. Suppose that α has closed image and compact kernel and note that by (a), it suffices to check that α^* preserves level cofibrations. We start by noting that the image of $\alpha \times O(m)$ is closed in $G\times O(m)$ since the image of α is closed in G. Moreover, the kernel of $\alpha\times O(m)$ is $\ker(\alpha)\times 1$, which is compact by hypothesis. So restriction along $\alpha \times O(m)$ takes Com-cofibrations of $(G \times O(m))$-spaces to Com-cofibrations of $(K \times O(m))$-spaces by [Deg+23, Proposition 1.1.6(iii)]. Now let $i\colon A \to B$ be a level cofibration of $\mathcal{I}$-G-spaces so that $i(\mathbb{R}^m)$ is a Com-cofibration of $(G\times O(m))$-spaces. By the previous discussion, $\alpha^*(i(\mathbb{R}^m))$ is a Com-cofibration of $(K \times O(m))$-spaces. Moreover, the $O(m)$-action is unchanged, so it still acts freely off the image of $\alpha^* i$. This shows that α^* preserves cofibrations as required. □

Proposition 7.2.11. *The level model structures on $\mathcal{I}{-}G\mathcal{T}$ is symmetric monoidal with cofibrant unit object.*

Proof. Let us show that the pushout-product axiom holds. By a standard reduction [Hov99, Corollary 4.2.5], it suffices to check that the pushout product $f\square g$ is

(i) a cofibration if f and g belong to the set of generating cofibrations;

(ii) an acyclic cofibration if furthermore f or g is a generating acyclic cofibration.

In this case we may assume $f = G \times_H \mathcal{I}_V f'$ and $g = G \times_K \mathcal{I}_W g'$ and so $f \square g = \Delta^*(G \times G \times_{H \times K} \mathcal{I}_{V \oplus W} f' \square g')$ by Equation (7.2.6.1). Since $\mathcal{T}$ is a symmetric monoidal model category, the pushout-product $f' \square g'$ satisfies conditions (i) and (ii) above. By Proposition 7.2.10 we see that the functors

$$\Delta^* \colon \mathcal{I}\text{–}(G \times G)\mathcal{T} \to \mathcal{I}\text{–}G\mathcal{T}$$

are left Quillen. Moreover, it is clear from the definition of the model structures that $\mathrm{ev}_{V \oplus W} \colon \mathcal{I} - (G \times G)\mathcal{T} \to (H \times K)\mathcal{T}$ is right Quillen, and therefore $(G \times G) \times_{H \times K} \mathcal{I}_{V \oplus W}$ is left Quillen. From these observations it follows that the pushout-product axiom holds for $\mathcal{I}\text{–}G\mathcal{T}$ too. Finally, the unit axiom holds since the unit object $* = G \times_G \mathcal{I}_0$ is cofibrant. □

In Section 6.1.3 we discussed how to induce a model structure on pointed objects. We will apply these results to the category $\mathcal{I}\text{–}G\mathcal{T}$ with the level model structure. Note first that the category of pointed objects in $\mathcal{I}\text{–}G\mathcal{T}$ is equivalent to $\mathcal{I}\text{–}G\mathcal{T}_*$, the category of continuous functors from $\mathcal{I}$ to $G\mathcal{T}_*$, the category of based G-spaces.

Proposition 7.2.12. *Let G be a Lie group. The category $\mathcal{I}\text{–}G\mathcal{T}_*$ admits a* proper level model structure *in which the weak equivalences, fibrations and cofibrations are detected by the forgetful functor $\mathcal{I}\text{–}G\mathcal{T}_* \to \mathcal{I}\text{–}G\mathcal{T}$. This model structure is topological, cofibrantly generated by the sets $(I_G)_+$ and $(J_G)_+$, symmetric monoidal and the unit object is cofibrant. Moreover, there exists a symmetric monoidal equivalence of ∞-categories*

$$\mathcal{I}\text{–}G\mathcal{T}_*[W_{lvl}^{-1}] \simeq (\mathcal{I}\text{–}G\mathcal{T}[W_{lvl}^{-1}])_*.$$

Proof. The first part follows from the discussion in Section 6.1.3 and [Sch18, Proposition B.5]. For the final claim apply Proposition 6.1.3 together with the fact that $\mathcal{I}\text{–}G\mathcal{T}[W_{lvl}^{-1}]$ is presentable by Theorem 7.3.9. □

We now change gears and consider the global analogue of the previous discussion. Recall that for any G-representation V and $\mathcal{I}$-space X, the value $X(V)$ admits a natural G-action by restricting along the canonical morphism $G \to O(V)$, see Remark 7.2.5.

Definition 7.2.13. Let $f \colon X \to Y$ be a morphism in $\mathcal{I}\text{–}\mathcal{T}$.

(a) We say f is a *faithful level equivalence* if for every compact Lie group G and every faithful G-representation V, the map $f(V) \colon X(V) \to Y(V)$ is a G-weak equivalence: for all closed subgroups $H \leq G$, the induced map $f(V)^H \colon X(V)^H \to Y(V)^H$ is a weak homotopy equivalence of spaces.

(b) We say f is a *faithful level fibration* if for every compact Lie group G and every faithful G-representation V, the map $f(V)\colon X(V) \to Y(V)$ is a fibration in the projective model structure of G-spaces.

The following result is a reformulation of [Sch18, Lemmas 1.2.7, 1.2.8] to our context.

Lemma 7.2.14. *Let $f\colon X \to Y$ be a morphism in $\mathcal{I}\text{–}\mathcal{T}$. Then the following are equivalent:*

(a) the map $f(V)\colon X(V)^G \to Y(V)^G$ is a weak homotopy equivalence (resp., Serre fibration) for every compact Lie group G and every G-representation V;

(b) the map $f\colon X \to Y$ is a faithful level equivalence (resp., faithful level fibration);

(c) the map $f(\mathbb{R}^m)\colon X(\mathbb{R}^m) \to Y(\mathbb{R}^m)$ is a $O(m)$-weak equivalence (resp., $O(m)$-fibration) for every $m \geq 0$.

Proof. It is clear that (a) implies (b), which implies (c). Suppose that (c) holds and let V be a G-representation. As in the proof of Lemma 7.2.8 we can choose a linear isometric isomorphism $\varphi\colon V \simeq \mathbb{R}^m$ and define a group homomorphism $\rho\colon G \to O(m)$ such that

$$X(V)^G \simeq X(\mathbb{R}^m)^{\rho(G)}$$

showing that (c) implies (a). □

Construction 7.2.15 (semifree $\mathcal{I}$-space)**.** For every G-representation V, there is an evaluation functor

$$\mathrm{ev}_{G,V}\colon \mathcal{I}\text{–}\mathcal{T} \to G\mathcal{T}, \qquad X \mapsto X(V)$$

which admits a left adjoint $\mathcal{I}_{G,V}$ given by the formula $\mathcal{I}_{G,V}(A) = \mathcal{I}(V, -) \times_G A$. When $A = *$, we simply write $\mathcal{I}_{G,V}$. For all H-representations V and K-representations W, there is an isomorphism of $\mathcal{I}$-G-spaces

$$\mathcal{I}_{H,V} \otimes \mathcal{I}_{K,W} \cong \mathcal{I}_{H\times K, V\oplus W}. \tag{7.2.15.1}$$

One can check this using the formula in Remark 7.2.4 or by mimicking the proof of [Sch18, Example 1.3.3].

The next result is an analogue of [Sch18, Proposition 1.2.10], adapted to our context.

Theorem 7.2.16. *The category $\mathcal{I}\text{–}\mathcal{T}$ admits a topological, cofibrantly generated model structure in which the weak equivalences are the faithful level equivalences*

W_{f-lvl} and the fibrations are the faithful level fibrations. The set of generating cofibrations I and acyclic cofibrations J are given by

$$\begin{aligned} I =& \{\mathcal{I}_{G,V}(\partial D^n) \to \mathcal{I}_{G,V}(D^n)\} \\ J =& \{\mathcal{I}_{G,V}(D^n \times \{0\}) \to \mathcal{I}_{G,V}(D^n \times [0,1])\} \end{aligned}$$

where G runs over all compact Lie groups, V over all faithful G-representations and $n \geq 0$. This is a symmetric monoidal model category with cofibrant unit object. We call this the faithful level model structure.

Proof. We can identify $\mathcal{I}{-}\mathcal{T}$ with the category $\prod_{m\geq 0} O(m)\mathcal{T}$ and endow the latter category with the product of the standard model structures on $O(m)$-spaces. The induced model structure on $\mathcal{I}{-}\mathcal{T}$ has weak equivalences and fibrations as in the theorem by Lemma 7.2.14. We note that the right lifting property against the sets I and J detect the level fibrations and level acyclic fibrations respectively, by the adjunction isomorphism

$$\mathrm{Hom}_{\mathcal{I}-\mathcal{T}}(\mathcal{I}_{H,V}A, X) \simeq \mathrm{Hom}_{\mathcal{T}}(A, X(V)^H)$$

for A a non-equivariant space. Let us next show that the pushout-product axiom holds. As explained in the proof of Proposition 7.2.11, it suffices to check that the pushout product $f \square g$ is an (acyclic) cofibration if f and g belong to the set of generating (acyclic) cofibrations. In any case we have $f = \mathcal{I}_{G,V} f'$ and $g = \mathcal{I}_{H,W} g'$. But then $f \square g = \mathcal{I}_{G\times H, V\oplus W} f' \square g'$ by Equation (7.2.15.1). Since $G\mathcal{T}$ is a symmetric monoidal model category, it suffices to check that the functor $\mathcal{I}_{G\times H,V\oplus W}$ is left Quillen. This is clear since $\mathrm{ev}_{G\times H,V\oplus W}$ is right Quillen by definition of the faithful level model structure. The pushout-product axiom then follows. Finally, the unit axiom holds since the unit object $* = \mathcal{I}_{e,0}$ is cofibrant and the model structure is topological by [Sch18, Proposition B.5]. $\square$

As before we obtain an induced model structured on pointed objects.

Proposition 7.2.17. *The category $\mathcal{I}{-}\mathcal{T}_*$ admits a faithful level model structure in which the weak equivalences, fibrations and cofibrations are detected by the forgetful functor $\mathcal{I}{-}\mathcal{T}_* \to \mathcal{I}{-}\mathcal{T}$. This model structure is topological, cofibrantly generated by the set I_+ and J_+, symmetric monoidal and the unit object is cofibrant. Finally, there exists a symmetric monoidal equivalence of ∞-categories*

$$\mathcal{I}{-}\mathcal{T}_*[W_{f-lvl}^{-1}] \simeq (\mathcal{I}{-}\mathcal{T}[W_{f-lvl}^{-1}])_*.$$

Proof. The first two claims follows from the discussion in Section 6.1.3 and [Sch18, Proposition B.5]. For the final claim apply Proposition 6.1.3, using the fact that $\mathcal{I}{-}\mathcal{T}[W_{f-lvl}^{-1}]$ is presentable. We will show this in Theorem 7.3.19. $\square$

We now pass from pointed objects to pre-spectrum objects. Observe that the category of pointed $\mathcal{I}$-G-spaces has a commutative algebra object S_G given by the functor sending V to its one-point compactification S^V equipped with the trivial G-action. If we are thinking of the category of $\mathcal{I}$-spaces with the faithful level model structure, we will write S_{fgl} for S_e, to emphasize that the sphere should be thought of as evaluated on all faithful representations of all groups (fgl stands for faithful global).

Definition 7.2.18. Let G be a Lie group. Following [MM02, Chapter II, Proposition 3.8], we define the topological category Sp_G^O of orthogonal G-spectra to be the category of S_G-modules in $\mathcal{I}-G\mathcal{T}_*$. These categories inherit induced model structures:

(a) The category of orthogonal G-spectra admits a *(proper) level model structure* whose weak equivalences and fibrations are created by the forgetful functor $\mathrm{Sp}_G^O \to \mathcal{I}-G\mathcal{T}_*$ where the target is endowed with the level model structure. This is a cofibrantly generated, proper, topological model category, see the proof of [Deg+23, Theorem 1.2.22]. We also obtain that a set of generating cofibrations and acyclic cofibrations are given by the maps $S_G \otimes I_G$ and $S_G \otimes J_G$ where $S_G \otimes -$ denotes the left adjoint to the forgetful functor $\mathrm{Sp}_G^O \to \mathcal{I}-G\mathcal{T}_*$.

(b) The category of orthogonal spectra admits a *faithful level model structure* whose weak equivalences and fibrations are created by the forgetful functor $\mathrm{Sp}^O \to \mathcal{I}-\mathcal{T}_*$ where the target is endowed with the faithful level model structure, see [Sch18, Propositions 4.3.5]. From this result we obtain that the faithful level model structure is cofibrantly generated and topological, with a set of generating cofibrations and acyclic cofibrations given by $S_{\mathrm{fgl}} \otimes I$ and $S_{\mathrm{fgl}} \otimes J$, where $S_{\mathrm{fgl}} \otimes -$ denotes the left adjoint to the forgetful functor $\mathrm{Sp}^O \to \mathcal{I}-\mathcal{T}_*$.

Remark 7.2.19. By combining straightforward generalizations of [MM02, Theorem 4.3] and [Sch18, Remark 3.1.8] to Lie groups, we conclude that Sp_G^O is equivalent to the category of orthogonal spectra defined in [Deg+23, Definition 1.1.9].

As discussed in [MM02, Chapter II Section 3], the category of orthogonal G-spectra admits a closed symmetric monoidal structure.

Proposition 7.2.20. *Let G be a Lie group.*

(a) The level model structure on Sp_G^O *is symmetric monoidal.*

(b) The faithful level model structure on Sp^O *is symmetric monoidal.*

Proof. The proof that the pushout product axiom holds for Sp_G^O is similar to that given in Proposition 7.2.11 for $\mathcal{I}$-G-spaces. The explicit argument for cofibrations can be found in [Deg+23, Proposition 1.2.28(i)] and we note that a slight modification of that argument then also gives the statement for acyclic cofibrations. The argument that the faithful level model structure satisfies the pushout-product axiom is similar to that given in Theorem 7.2.16. The argument for cofibrations can also be found in [Sch18, Proposition 4.3.23] and a slight modification of that argument also gives the statement for acyclic cofibrations. □

Definition 7.2.21. We define the ∞-category PSp_G of *G-prespectra* to be the symmetric monoidal ∞-category associated to the symmetric monoidal model category Sp_G^O with the level model structure. Similarly, we define $\mathrm{PSp}_{\mathrm{fgl}}$ of *faithful global prespectra* to be the symmetric monoidal ∞-category associated to the symmetric monoidal model category Sp^O with the faithful level model structure.

We have emphasized how the level model structures on Sp_G^O and Sp^O are induced by the level model structure on $\mathcal{I}{-}G\mathcal{T}_*$ and $\mathcal{I}{-}\mathcal{T}_*$ respectively by taking modules. This allows us to reinterpret the passage to modules internally to ∞-categories.

Proposition 7.2.22. *There are symmetric monoidal equivalences*

$$\mathrm{PSp}_G \simeq \mathrm{Mod}_{S_G}(\mathcal{I}{-}G\mathcal{T}[W_{lvl}^{-1}]_*) \quad \text{and} \quad \mathrm{PSp}_{\mathrm{fgl}} \simeq \mathrm{Mod}_{S_{\mathrm{fgl}}}(\mathcal{I}{-}\mathcal{T}[W_{f-lvl}^{-1}]_*).$$

Proof. Apply Proposition 6.1.4. □

Finally we pass from the level model structure to the stable model structure, which will present the categories of global and genuine G-spectra. Fix a complete G-universe $\mathcal{U}_G$ and write $s(\mathcal{U}_G)$ for the poset, under inclusion, of finite dimensional G-subrepresentations of $\mathcal{U}_G$. The G-equivariant homotopy groups of an orthogonal G-spectrum X are given by

$$\pi_k^G(X) = \begin{cases} \operatorname*{colim}_{V\in s(\mathcal{U}_G)} \, [S^{k+V}, X(V)]_*^G & \text{for } k \geq 0 \\ \operatorname*{colim}_{V\in s(\mathcal{U}_G)} \, [S^{V}, X(\mathbb{R}^{-k} \oplus V)]_*^G & \text{for } k \leq 0 \end{cases}$$

where the connecting maps in the colimit system are induced by the structure maps, and $[-,-]_*^G$ means G-equivariant homotopy classes of based G-maps. Note that the same definition works even if X is an orthogonal spectrum since the value $X(V)$ admits a G-action as discussed before Definition 7.2.13. Moreover, everything is functorial with respect to morphisms of orthogonal (G-)spectra. We finally note that the definition above a priori depends on a choice of complete G-universe. However the functors associated to different complete G-universes are naturally isomorphic, and so the choice is immaterial.

Definition 7.2.23. Let G be a Lie group.

- A morphism $f\colon X \to Y$ of orthogonal G-spectra is a *$\underline{\pi}_*$-isomorphism* if $\pi_*^H(f)\colon \pi_*^H(X) \to \pi_*^H(Y)$ is an isomorphism for all compact subgroups $H \leq G$. The $\underline{\pi}_*$-isomorphisms are part of a cofibrantly generated, topological, stable and symmetric monoidal model structure on the category of orthogonal G-spectra [Deg+23, Theorem 1.2.22], called the *G-stable model structure*.
- A morphism $f\colon X \to Y$ of orthogonal spectra is a *global equivalence* if $\pi_*^H(f)\colon \pi_*^H(X) \to \pi_*^H(Y)$ is an isomorphism for all compact Lie groups H. The global equivalences are part of a cofibrantly generated, topological, proper, stable and symmetric monoidal model structure on the category of orthogonal spectra [Sch18, Theorem 4.3.17, Proposition 4.3.24], called the *global model structure*.

Definition 7.2.24. We define the symmetric monoidal ∞-category Sp_G of *G-spectra* to be the underlying ∞-category of orthogonal G-spectra with the G-stable model structure. Similarly, we define the symmetric monoidal ∞-category $\mathrm{Sp}_{\mathrm{gl}}$ of *global spectra* to be the underlying ∞-category of orthogonal spectra with the global model structure.

We now make precise the observation that Sp_G and $\mathrm{Sp}_{\mathrm{gl}}$ are Bousfield localizations of PSp_G and $\mathrm{PSp}_{\mathrm{fgl}}$ respectively at an explicit collection of weak equivalences. We begin with global spectra.

Construction 7.2.25. Given a compact Lie group G and a G-representation V, we can consider the adjoint pairs

$$\mathrm{Sp}^O \underset{S_{gl}\otimes -}{\overset{\text{forget}}{\rightleftarrows}} \mathcal{I}\text{–}\mathcal{T}_* \underset{\mathcal{I}_{G,V}}{\overset{\mathrm{ev}_{G,V}}{\rightleftarrows}} G\mathcal{T}_*\,.$$

Following [Sch18, Construction 4.1.23], we denote the composite $S_{gl}\otimes\mathcal{I}_{G,V}$ by $F_{G,V}$. Note that the adjoint pairs above are Quillen with respect to the global level structure and so they yield corresponding adjoint pairs of underlying ∞-categories. As discussed before [Sch18, Theorem 4.1.29], there are maps in Sp^O

$$\lambda_{G,V,W}\colon F_{G,V\oplus W}S^V \to F_{G,W}S^0$$

for all compact Lie groupw G and G-representations V and W. Note that we can view these maps in PSp_{gl} since the domain and codomain of $\lambda_{G,V,W}$ are bifibrant. Consider the following diagram

$$\begin{tikzcd}
G\mathcal{T}_*(S^0, X(W)) \arrow[r, "\tilde{\sigma}_{G,V,W}"] & G\mathcal{T}_*(S^V, X(V \oplus W)) \arrow[d, "\sim"] \\
\mathrm{Sp}^O(F_{G,W}S^0, X) \arrow[u, "\sim"] \arrow[r] & \mathrm{Sp}^O(F_{G,V\oplus W}S^V, X),
\end{tikzcd}$$

where the vertical maps are the adjunction isomorphisms and the top map is the adjoint structure map of X. The bottom map is equal to precomposition by $\lambda_{G,V,W}$. In particular, taking $X = F_{G,W}S^0$, we may define $\lambda_{G,V,W}$ as the image of the identity of $F_{G,W}S^0$ under the bottom map. Note also that $\lambda_{G,V,W}$ is equivalent to $F_{G,W}S^0 \otimes \lambda_{G,V,0}$, and that $\lambda_{G,V,0}$ is adjoint to the identity.

Remark 7.2.26. Observe that both characterizations of $\lambda_{G,V,W}$ given above also uniquely specify the map on the level of ∞-categories.

Proposition 7.2.27. $\mathrm{Sp}_{\mathrm{gl}}$ *is a Bousfield localization of* $\mathrm{PSp}_{\mathrm{fgl}}$. *Furthermore, an object in* $\mathrm{PSp}_{\mathrm{fgl}}$ *lies in* $\mathrm{Sp}_{\mathrm{gl}}$ *if and only if it is local with respect to the morphisms* $\{\lambda_{G,V,W}\}$ *for all compact Lie groups G and G-representations V and W with W faithful.*

Proof. Let Λ denote the set of maps $\lambda_{G,V,W}$ for G, V and W as in the proposition. We write Sp^O_{lvl} and Sp^O_{gl} for the category of orthogonal spectra endowed with the faithful level model structure and the global stable model structure respectively. We will show that Sp^O_{gl} is a left Bousfield localization (in the model categorical sense) of Sp^O_{lvl} at the set Λ, that is $L_\Lambda \mathrm{Sp}^O_{lvl} = \mathrm{Sp}^O_{gl}$. Because both can be checked on underlying homotopy categories, Bousfield localizations of model categories present Bousfield localizations of ∞-categories. Therefore the claim in the proposition will follow by passing to underlying ∞-categories. By definition $X \in \mathrm{Sp}^O_{lvl}$ is Λ-local (and so fibrant in the Bousfield localization) if and only if X is fibrant in Sp^O_{lvl} (which always holds in this case), and the canonical map of homotopy function complexes

$$\lambda^*_{G,V,W} \colon \mathrm{Map}(F_{G,W}S^0, X) \to \mathrm{Map}(F_{G,V\oplus W}S^V, X)$$

is an equivalence for all $\lambda_{G,V,W} \in \Lambda$. By adjunction this is equivalent to asking that $X(W)^G \to \Omega^V(X(V \oplus W))^G$ is an equivalence for all G, V and W as in the proposition. In other words X is a global Ω-spectrum, see [Sch18, Definition 4.3.8]. By [Sch18, Theorem 4.3.17] these are precisely the fibrant objects Sp^O_{gl}. Since $L_\Lambda \mathrm{Sp}^O_{lvl}$ and Sp^O_{gl} have the same cofibrations and fibrant objects, the two model structure coincide by [Joy08, Proposition E.1.10]. $\square$

We repeat this analysis for Sp_G and PSp_G.

Construction 7.2.28. Let H be a compact subgroup of a Lie group G, and let V be an H-representation. We have a sequence of adjoint pairs

$$\begin{tikzcd} \mathrm{Sp}_G^O \arrow[r, shift left, "\text{forget}"] & \mathcal{I}\text{–}G\mathcal{T}_* \arrow[l, shift left, "S_G\otimes -"] \arrow[r, shift left, "\mathrm{ev}_V"] & H\mathcal{T}_* \arrow[l, shift left, "G_+\wedge_H \mathcal{I}_V"] \end{tikzcd}$$

which are Quillen with respect to the proper level model structure, and so they define adjoint pairs at the level of underlying ∞-categories. The composite $S_G \otimes (G_+ \wedge_H \mathcal{I}_V)$ will also be denoted by $G \ltimes_H F_V$ following [Deg+23, Example 1.1.15]. This notation is justified by the fact that $G \ltimes_H F_V$ is also equivalent to the induction of the H-prespectrum F_V as one can easily verify. For all pairs of H-representations V and W, there are maps in Sp_G^O

$$G \ltimes_H \lambda_{V,W} \colon G \ltimes_H F_{V\oplus W} S^V \to G \ltimes_H F_W,$$

see [Deg+23, Equation 1.2.19]. We can view these maps in PSp_G as the domains and codomains are bifibrant. Similarly to before, $G \ltimes_H \lambda_{V,W}$ is determined by the property that the map

$$\mathrm{Sp}_G^O(G \ltimes_H F_W, X) \to \mathrm{Sp}_G^O(G \ltimes_H F_{V\oplus W} S^V, X),$$

defined so that the diagram

$$\begin{tikzcd} H\mathcal{T}_*(S^0, X(W)) \arrow[r, "\mathrm{res}^G_H(\tilde{\sigma}_{V,W})"] & H\mathcal{T}_*(S^V, X(V\oplus W)) \arrow[d, "\sim"] \\ \mathrm{Sp}_G^O(G \ltimes_H F_W, X) \arrow[u, "\sim"] \arrow[r] & \mathrm{Sp}_G^O(G \ltimes_H F_{V\oplus W} S^V, X) \end{tikzcd}$$

commutes, is equal to precomposition by $G \ltimes_H \lambda_{H,V,W}$. Also note that $G \ltimes \lambda_{V,W}$ is equal to $G \ltimes_H F_W S^0 \otimes \lambda_{V,0}$ and that $\lambda_{V,0}$ is adjoint to the identity on S^V.

Remark 7.2.29. Once again, observe that the characterizations of $G \ltimes_H \lambda_{V,W}$ given above also uniquely specify the map on the level of ∞-categories.

Proposition 7.2.30. *Let G be a Lie group. Then Sp_G is a Bousfield localization of PSp_G. Furthermore, an object in PSp_G lies in Sp_G if and only if it is local with respect to the morphisms $\{G \ltimes_H \lambda_{V,W}\}$ for all compact subgroups $H \le G$ and H-representations V and W. Equivalently, $X \in \mathrm{PSp}_G$ lies in Sp_G if and only if for all compact subgroups $H \le G$, the object $\mathrm{res}^G_H X \in \mathrm{PSp}_H$ is local with respect to morphisms $\{\lambda_{V,W}\}$ for all H-representations V and W.*

Proof. The proof is similar to that of Proposition 7.2.27 but now we use the characterization of fibrant objects in the proper stable model structure given in [Deg+23, Theorem 1.2.22 (v)]. The second claim follows from the first one by adjunction. □

7.3 MODELS FOR ∞-CATEGORIES OF EQUIVARIANT PRESPECTRA

In the previous section we introduced the ∞-categories of equivariant and global (pre)spectra, and exhibited the spectrum objects as local objects in the relevant category of prespectra with respect to an explicit class of weak equivalences. Furthermore, we observed that the construction of PSp_G admitted a reinterpretation internal to ∞-categories, by first passing to pointed objects in $\mathcal{I}{-}G\mathcal{T}[W_{lvl}^{-1}]$ and then taking modules over S_G. Similarly, we observed that

$$\mathrm{PSp}_{\mathrm{fgl}} \simeq \mathrm{Mod}_{S_{\mathrm{fgl}}}(\mathcal{I}{-}\mathcal{T}[W_{f-lvl}^{-1}]_*).$$

Furthermore these equivalences were symmetric monoidal.

However this is only part of the story, because the ∞-categories $\mathcal{I}{-}G\mathcal{T}[W_{lvl}^{-1}]$ and $\mathcal{I}{-}\mathcal{T}[W_{f-lvl}^{-1}]$ are still too inexplicit for our arguments. Luckily we can give explicit models of these ∞-categories. Consider the case of $\mathcal{I}{-}G\mathcal{T}[W_{lvl}^{-1}]$. By construction this ∞-category records the fixed point spaces $X(V)^H$ for every (compact) subgroup H of G and every H-representation V of an $\mathcal{I}$-G-space X. By functoriality, these different fixed point spaces are related by subconjugacy relationships in H and equivariant linear isometries in V. We will prove that the ∞-category $\mathcal{I}{-}G\mathcal{T}$ is in fact freely generated under these properties. More precisely, we will exhibit an equivalence

$$\mathcal{I}{-}G\mathcal{T}[W_{lvl}^{-1}] \simeq \mathbf{OR}_G{-}\mathcal{S},$$

where the ∞-category $\mathbf{OR}_G$ indexes pairs (H, V), each one of which records one of the fixed point spaces $X(V)^H$ of an $\mathcal{I}$-G-space X. Similarly we will prove that

$$\mathcal{I}{-}\mathcal{T}[W_{f-lvl}^{-1}] \simeq \mathbf{OR}_{\mathrm{fgl}}{-}\mathcal{S},$$

where the ∞-category $\mathbf{OR}_{\mathrm{fgl}}$ indexes pairs (G, V), where G is a compact Lie group and V is a faithful G-representation.

In total we will obtain equivalences

$$\mathrm{PSp}_G \simeq \mathrm{Mod}_{S_G}(\mathbf{OR}_G{-}\mathcal{S}_*) \quad \text{and} \quad \mathrm{PSp}_{\mathrm{fgl}} \simeq \mathrm{Mod}_{S_{\mathrm{fgl}}}(\mathbf{OR}_{\mathrm{fgl}}{-}\mathcal{S}_*).$$

It will be in this guise that we will think of the ∞-category of G-prespectra and global prespectra for the remainder of the paper.

Finally, to make future constructions symmetric monoidal it will be important to understand how the symmetric monoidal structures transfer under the equivalences

$$\mathcal{I}{-}G\mathcal{T}[W_{lvl}^{-1}] \simeq \mathbf{OR}_G{-}\mathcal{S} \quad \text{and} \quad \mathcal{I}{-}\mathcal{T}[W_{f-lvl}^{-1}] \simeq \mathbf{OR}_{\mathrm{fgl}}{-}\mathcal{S}.$$

We may immediately apply Theorem 6.2.37 to conclude that the monoidal structure on $\mathcal{I}{-}G\mathcal{T}[W_{lvl}^{-1}]$ and $\mathcal{I}{-}\mathcal{T}[W_{f-lvl}^{-1}]$ are induced by Day convolution from the restricted promonoidal structure on $\mathbf{OR}_G$. We will make these promonoidal structures explicit.

To show that $\mathcal{I}\text{–}G\mathcal{T}[W_{lvl}^{-1}]$ and $\mathcal{I}\text{–}\mathcal{T}[W_{lvl}^{-1}]$ are equivalent to categories of copresheafs on an explicit set of generators, we will apply a version of Elmendorf's theorem, see Corollary 6.2.41. The application of this theorem to $\mathcal{I}\text{–}G\mathcal{T}[W_{lvl}^{-1}]$ and $\mathcal{I}\text{–}\mathcal{T}[W_{f-lvl}^{-1}]$ has a similar flavour, but are logically distinct. Therefore we treat each case separately.

7.3.1 $\mathcal{I}$-G-spaces and $\mathbf{OR}_G$-spaces

We begin with $\mathcal{I}\text{–}G\mathcal{T}[W_{lvl}^{-1}]$.

Remark 7.3.1. Let G be a Lie group and consider a map $\varphi\colon G/K \to G/H$ in the orbit category $\mathbf{O}_G$. Giving φ is equivalent to giving $gH \in (G/H)^K$, that is an element $gH \in G/H$ such that $c_g(K) = g^{-1}Kg \subseteq H$. When we need to emphasize this correspondence between gH and φ we will use subscripts φ_g and g_φ. Note that $g_{\psi\circ\varphi}H = g_\varphi g_\psi H$ so composition of maps corresponds to multiplication with reverse order.

Definition 7.3.2. For a Lie group G, the *proper G-orbit category* $\mathbf{O}_{G,\mathrm{pr}}$ is the full subcategory of $\mathbf{O}_G$ spanned by those cosets G/H with $H \leq G$ compact.

Let G be a Lie group and $H, K \leq G$ be compact subgroups. Given an H-representation V and a K-representation W, we can consider the space $G \times_H \mathcal{I}(V, W)$ where H acts on G by right translation, and on $\mathcal{I}(V, W)$ via $h.\varphi = \varphi h^{-1}$. Note that K acts diagonally on $G \times_H \mathcal{I}(V, W)$ via G and W. We have the following helpful criterion.

Lemma 7.3.3. *An element $[g, \varphi] \in G \times_H \mathcal{I}(V, W)$ is K-fixed if and only if $c_g(K) \subseteq H$ and $k.\varphi(v) = \varphi(c_g(k)v)$ for all $k \in K$ and $v \in V$.*

Proof. An element $[g, \varphi] \in G \times_H \mathcal{I}(V, W)$ is K-fixed if and only if $[kg, k.\varphi] = [g, \varphi]$ for all $k \in K$. This means that there exists $h \in H$ such that $kg = gh$ and $k.\varphi = \varphi h$ for all $k \in K$. In other words g is such that $c_g(K) \subseteq H$ and φ is K-equivariant in the sense that $k.\varphi = \varphi c_g(k)$ for all $k \in K$. □

Lemma 7.3.4. *Let G be a Lie group and $H, K, L \leq G$ be compact subgroups. Let V be an H-representation, W a K-representation and U an L-representation. Then the map*

$$\circ\colon (G \times_K \mathcal{I}(W, U))^L \times (G \times_H \mathcal{I}(V, W))^K \to (G \times_H \mathcal{I}(V, U))^L$$

given by $([g', \psi], [g, \varphi]) \mapsto [g'g, \psi\varphi]$ is well-defined and continuous. Furthermore, upon varying the objects, the collection of maps so obtained is associative and unital.

Proof. Let us first show that the map does not depend on the chosen representatives. For $h \in H$ and $k \in K$ we have $[g, \varphi] = [gh, \varphi h]$ and $[g', \psi] = [g'k, \psi k]$

so we ought to check that $[g'g, \psi\varphi] = [g'kgh, \psi k\varphi h]$. Using that $c_g(K) \subseteq H$ and φ is K-equivariant with respect to the c_g-twisted action, we can write

$$\begin{aligned}[g'kgh, \psi k\varphi h] = [g'g\, \underbrace{c_g(k)h}_{\in H}, \psi k\varphi h] &= [g'g, \psi k\varphi h(c_g(k)h)^{-1}] \\ &= [g'g, \psi k\varphi c_g(k^{-1})] = [g'g, \psi\varphi]\end{aligned}$$

as required. We verify that $[g'g, \psi\varphi]$ is K-fixed using the criterion from Lemma 7.3.3. Using that $c_{g'}(L) \subseteq K$ and $c_g(K) \subseteq H$ we immediately see that $c_{g'g}(L) \subseteq H$. Using the twisted equivariance of ψ and φ we see that

$$l.\psi\varphi = \psi\, \underbrace{c_{g'}(l)}_{\in K}\, \varphi = \psi\varphi c_g(c_{g'}(l)) = \psi\varphi c_{g'g}(l)$$

for all $l \in L$. Therefore $\psi\varphi$ is twisted equivariant and $[g'g, \psi\varphi]$ is indeed K-fixed. Finally the map is associative, unital and continuous since multiplication and composition maps are so. $\square$

We now formally define the ∞-category $\mathbf{OR}_G$.

Definition 7.3.5. Let G be a Lie group. We define a topological category $\mathbf{OR}_G$ whose objects are pairs (H, V) of a compact subgroup $H \leq G$ and an H-representation V. The morphism spaces are given by

$$\mathbf{OR}_G((H, V), (K, W)) = (G \times_H \mathcal{I}(V, W))^K.$$

Composition is given by the maps

$$\circ\colon \mathbf{OR}_G((K, W), (L, U)) \times \mathbf{OR}_G((H, V), (K, W)) \to \mathbf{OR}_G((H, V), (L, U))$$

defined in Lemma 7.3.4. Note that there is a projection map

$$\mathbf{OR}_G((H, V), (K, W)) \to (G/H)^K = \mathbf{O}_{G,\mathrm{pr}}(G/K, G/H), \qquad [g, \varphi] \mapsto [gH].$$

which extends to a functor $\pi_G\colon \mathbf{OR}_G \to \mathbf{O}_{G,\mathrm{pr}}^{\mathrm{op}}$.

Example 7.3.6. Let $G = e$ be the trivial group. Then the topological category $\mathbf{OR}_G$ is equivalent to $\mathcal{I}$.

Example 7.3.7. By definition $\mathbf{OR}_G((H, V), (e, W)) = G \times_H \mathcal{I}(V, W)$, which is a space with an action of

$$\mathbf{OR}_G((e, W), (e, W)) = G \times O(W).$$

One can identify the functor $\mathbf{OR}_G((H, V), (e, -))\colon \mathcal{I} \to G\mathcal{T}$ with the free $\mathcal{I}$-G-space $G \times_H \mathcal{I}_V$.

Definition 7.3.8. We let $\mathbf{OR}_G{-}\mathcal{S}$ denote the ∞-category of $\mathbf{OR}_G$-spaces, given by the ∞-category of functors $\mathbf{OR}_G \to \mathcal{S}$.

We are finally ready to prove the main result of this subsection.

Theorem 7.3.9. *Let G be a Lie group. Then there is an equivalence of ∞-categories*

$$\mathcal{I}{-}G\mathcal{T}[W_{lvl}^{-1}] \simeq \mathbf{OR}_G{-}\mathcal{S}.$$

Proof. The discussion in Example 7.3.7 shows that there exists a functor of topological categories (and so of ∞-categories)

$$\mathbf{OR}_G^{\mathrm{op}} \to \mathcal{I}{-}G\mathcal{T}, \qquad (H,V) \mapsto \mathbf{OR}_G((H,V),(e,-)) = G \times_H \mathcal{I}_V.$$

This is fully faithful by definition of $\mathbf{OR}_G$. Since the $\mathcal{I}$-G-spaces $G \times_H \mathcal{I}_V$ are bifibrant in the level model structure, the composite

$$L\colon \mathbf{OR}_G^{\mathrm{op}} \to \mathcal{I}{-}G\mathcal{T} \to \mathcal{I}{-}G\mathcal{T}[W_{lvl}^{-1}], \quad (H,V) \mapsto G \times_H \mathcal{I}_V$$

is also fully faithful. We apply Theorem 6.2.39 to the functor L. We note that the $\mathcal{I}$-G-space $G \times_H \mathcal{I}_V$ corepresents the functor $X \mapsto X(V)^H$. This functor commutes with small homotopy colimits since:

- the H-fixed points functor preserves small homotopy colimits as discussed in Example 6.2.40;
- and the evaluation functor $X \mapsto X(V)$ preserves small homotopy colimits. Indeed this functor preserves all colimits (as they are calculated pointwise), level equivalences by definition, and (acyclic) cofibrations (as one can verify by checking on the generating (acyclic) cofibrations).

Finally, the collection of objects $\{G \times_H \mathcal{I}_V \mid (H,V) \in \mathbf{OR}_G\}$ is jointly conservative by definition of the level equivalences. Thus the required equivalence follows from Theorem 6.2.39. $\square$

Next we explain how to upgrade the equivalence above to an equivalence of symmetric monoidal ∞-categories.

Construction 7.3.10. We enhance the topological category $\mathbf{OR}_G$ to a topological coloured operad as follows. The colours are simply the objects of $\mathbf{OR}_G$, and the space of multi-morphisms from $\{(H_i,V_i)\}_{i\in I}$ to (K,W) is given by

$$\mathbf{OR}_G(\{(H_i,V_i)\}_{i\in I},(K,W)) = \left(\left(\prod_{i\in I} G\right) \times_{(\prod_{i\in I} H_i)} \mathcal{I}\left(\bigoplus_{i\in I} V_i, W\right)\right)^K.$$

By Lemma 7.3.3, a point of this space is equivalent to the following data:

- For all $i \in I$, an element $g_i H_i \in G/H_i$ such that $c_{g_i}(K) \subseteq H_i$;
- A linear isometry $\varphi = \sum_i \varphi_i \colon \bigoplus_i V_i \to W$ such that for all $v \in V_i, k \in K$ and $i \in I$, $k.\varphi_i(v) = \varphi_i(c_{g_i}(k)v)$.

For every map $I \to J$ of finite sets with fibres $\{I_j\}_{j\in J}$, every finite collections of objects $\{(H_i, V_i)\}_{i\in I}$ and $\{(K_j, W_j)\}_{j\in J}$, and every $(L, U) \in \mathbf{OR}_G$ we have a composition map

$$\prod_{j\in J} \mathbf{OR}_G(\{(H_i, V_i)\}_{i\in I_j}, (K_j, W_j)) \times \mathbf{OR}_G(\{(K_j, W_j)\}_{j\in J}, (L, U)) \to \mathbf{OR}_G(\{(H_i, V_i)\}_{i\in I}, (L, U))$$

which is defined by the formulas

$$(\bigoplus_{i\in I_j} V_i \to W_j, \bigoplus_{j\in J} W_j \to U) \mapsto (\bigoplus_{i\in I} V_i = \bigoplus_{j\in J}\bigoplus_{i\in I_j} V_i \to \bigoplus_{j\in J} W_j \to U)$$

and

$$((g_i H_i)_{i\in I_j}, (g_j K_j)_{j\in J}) \mapsto (g_j g_i H_i)_{j\in J, i\in I_j}.$$

Note that for any colour $(H, V) \in \mathbf{OR}_G$, there is an identity element $[eH, 1_V] \in \mathbf{OR}_G((H, V), (H, V))$. Using Lemma 7.3.3 one can check that this composition is continuous, associative and unital and so that $\mathbf{OR}_G$ is indeed a topological coloured operad. We leave the details to the interested reader.

Remark 7.3.11. We can endow the topological category $\mathbf{O}^{\mathrm{op}}_{G,\mathrm{pr}}$ with a topological coloured operad structure whose colours are the objects of $\mathbf{O}_{G,\mathrm{pr}}$, and whose multimorphism spaces are given by

$$\mathbf{O}_{G,\mathrm{pr}}(\{G/H_i\}_{i\in I}, G/K) = \mathbf{O}_{G,\mathrm{pr}}(G/K, \prod_{i\in I} G/H_i) = (\prod_{i\in I} G/H_i)^K$$

with composition defined in the obvious way. The associated ∞-operad models the cocartesian monoidal structure. There is a canonical projection functor of topological coloured operads

$$\pi_G \colon \mathbf{OR}_G \to \mathbf{O}^{\mathrm{op}}_{G,\mathrm{pr}}.$$

By Lemma 6.1.1, we can lift π_G to a map of ∞-operads $\mathbf{OR}^{\otimes}_G \to (\mathbf{O}^{\mathrm{op}}_{G,\mathrm{pr}})^{\amalg}$, which by abuse of notation we still denote by π_G.

Recall that because $\mathcal{I}{-}G\mathcal{T}$ is a symmetric monoidal topological model category, we can construct a topological coloured operad whose colors are given by the bifibrant objects of $\mathcal{I}{-}G\mathcal{T}$ and the multimorphism spaces are given by

$$\mathrm{Mul}_{N^{\otimes}((\mathcal{I}-G\mathcal{T}^{\circ})^{\mathrm{op}})}(\{X_i\}, Y) = \mathcal{I}{-}G\mathcal{T}(Y, \bigotimes_{i\in I} X_i).$$

Furthermore the associated ∞-operad models the symmetric monoidal structure on $(\mathcal{I}{-}G\mathcal{T}[W_{lvl}^{-1}])^{\mathrm{op}}$.

Lemma 7.3.12. *The functor L of Theorem 7.3.9 lifts to a fully faithful functor of topological coloured operads.*

Proof. We define a functor between coloured operads by

$$\mathbf{OR}_G \to (\mathcal{I}{-}G\mathcal{T}^{\circ})^{\mathrm{op}}, \qquad \{(H_i, V_i)\} \mapsto \mathbf{OR}_G(\bigotimes(H_i, V_i), (e, -)).$$

Using Equation 7.2.6.1 we can rewrite this functor in more familiar terms as

$$\mathbf{OR}_G(\{(H_i, V_i)\}, (e, -)) = (\prod_i G) \times_{(\prod_i H_i)} \mathcal{I}(\bigoplus_i V_i, -) \simeq \bigotimes_i (G \times_{H_i} \mathcal{I}_{V_i}).$$

By construction this functor defines a coloured operad map which lifts L. Using this description of the functor and the fact that $G \times_H \mathcal{I}_W$ corepresents the functor $X \mapsto X(W)^K$, we also see that the map induced on multimorphism spaces

$$\mathbf{OR}_G(\{(H_i, V_i)\}_{i\in I}, (K, W)) \to \mathcal{I}{-}G\mathcal{T}(G \times_K \mathcal{I}_W, \bigotimes_{i\in I} G \times_{H_i} \mathcal{I}_{V_i})$$

is a homeomorphism. Therefore the functor of coloured operads is fully faithful. $\square$

The map L of topological coloured operads constructed above induces a functor $L\colon \mathbf{OR}_G^{\otimes} \to (\mathcal{I}{-}G\mathcal{T}[W_{lvl}^{-1}]^{\otimes})^{\mathrm{op}}$ of ∞-operads. Furthermore this functor is again fully faithful.

Corollary 7.3.13. *The functor $L\colon \mathbf{OR}_G^{\otimes} \to (\mathcal{I}{-}G\mathcal{T}_*[W_{lvl}^{-1}]^{\otimes})^{\mathrm{op}}$ induces a symmetric monoidal equivalence*

$$\mathcal{I}{-}G\mathcal{T}[W_{lvl}^{-1}] \simeq \mathbf{OR}_G{-}\mathcal{S},$$

where the right hand side is equipped with the Day convolution product.

Proof. This follows from Corollary 6.2.41, where we argue as in Theorem 7.3.9 and use Lemma 7.3.12. $\square$

As a convenient reference, let us summarize the final description of G-prespectrum objects obtained by combining all of the identifications obtained.

Corollary 7.3.14. *Let G be a Lie group. Then there is a symmetric monoidal equivalence*

$$\mathrm{PSp}_G \simeq \mathrm{Mod}_{\mathbb{S}_G}(\mathbf{OR}_G{-}\mathcal{S}_*).$$

Proof. Combine Theorem 7.3.13, Corollary 7.2.22 and Proposition 6.2.38. $\square$

Remark 7.3.15. We will often implicitly identify PSp_G with $\mathrm{Mod}_{\mathbb{S}_G}(\mathbf{OR}_G{-}\mathcal{S}_*)$ for the remainder of the paper.

7.3.2 $\mathcal{I}$-spaces and $\mathbf{OR}_{\mathrm{fgl}}$-spaces.

We now undertake a similar analysis for the ∞-category of $\mathcal{I}$-spaces localized at the faithful level equivalences. Because many of the details are similar, we will be briefer in this section than the previous one.

Definition 7.3.16. We define a topological category $\mathbf{OR}_{\mathrm{fgl}}$ whose objects are pairs (G, V) where G is a compact Lie group and V is a faithful G-representation. The morphism spaces are given by

$$\mathbf{OR}_{\mathrm{fgl}}((G, V), (H, W)) = (\mathcal{I}(V, W)/G)^H.$$

There is a composition map

$$\circ\colon \mathbf{OR}_{\mathrm{fgl}}((H, W), (L, U)) \times \mathbf{OR}_{\mathrm{fgl}}((G, V), (H, W)) \to \mathbf{OR}_{\mathrm{fgl}}((G, V), (L, U))$$

given by $([\psi], [\varphi]) \mapsto [\psi \circ \varphi]$. Similarly to Lemma 7.3.4, one may verify this composition is well-defined, associative, unital and continuous.

Example 7.3.17. By definition $\mathbf{OR}_{\mathrm{fgl}}((G, V), (e, W)) = \mathcal{I}(V, W)/G$. Thus we can identify the functor

$$\mathbf{OR}_{\mathrm{fgl}}((G, V), (e, -))\colon \mathcal{I} \to \mathcal{T}$$

with the semifree $\mathcal{I}$-space $\mathcal{I}_{G,V}$ from Construction 7.2.15. Recall this $\mathcal{I}$-space corepresents the functor $X \mapsto X(V)^G$.

Definition 7.3.18. We let $\mathbf{OR}_{\mathrm{fgl}}\text{–}\mathcal{S}$ denote the ∞-category of $\mathbf{OR}_{\mathrm{fgl}}$-spaces which is the ∞-category of functors $\mathbf{OR}_{\mathrm{fgl}} \to \mathcal{S}$. We also write $\mathbf{OR}_{\mathrm{fgl}}\text{–}\mathcal{S}_*$ for the ∞-category of functors $\mathbf{OR}_{\mathrm{fgl}} \to \mathcal{S}_*$.

We now prove the main result of this subsection.

Theorem 7.3.19. *There is an equivalences of ∞-categories*

$$\mathcal{I}\text{–}\mathcal{T}[W_{f-lvl}^{-1}] \simeq \mathbf{OR}_{\mathrm{fgl}}\text{–}\mathcal{S}.$$

Proof. The discussion in Example 7.3.17 shows that there exists a functor of topological categories (and so of ∞-categories)

$$(\mathbf{OR}_{\mathrm{fgl}})^{\mathrm{op}} \to \mathcal{I}\text{–}\mathcal{T}, \qquad (G, V) \mapsto \mathbf{OR}_{\mathrm{fgl}}((G, V), (e, -)) = \mathcal{I}_{G,V}.$$

This is fully faithful by definition of $\mathbf{OR}_{\mathrm{fgl}}$. Since the $\mathcal{I}$-spaces $\mathcal{I}_{G,V}$ are bifibrant in the faithful level model structure, the composite

$$(\mathbf{OR}_{\mathrm{fgl}})^{\mathrm{op}} \to \mathcal{I}\text{–}\mathcal{T} \to \mathcal{I}\text{–}\mathcal{T}[W_{f-lvl}^{-1}]$$

is also fully faithful. We note that the semifree $\mathcal{I}$-space $\mathcal{I}_{G,V}$ corepresents the functor $X \mapsto X(V)^G$, which commutes with small homotopy colimits.

Indeed the G-fixed points functor commutes will small homotopy colimits by the discussion in Example 6.2.40, and so does the evaluation functor $X \mapsto X(V)$ since it preserves all colimits (as they are calculated pointwise), faithful level equivalences by definitions and cofibrations (as one can verify by checking on the set of generating cofibrations). Finally, the collection of objects $\{\mathcal{I}_{G,V} \mid (G,V) \in \mathbf{OR}_{\mathrm{fgl}}\}$ is jointly conservative by definition of the faithful level equivalences. Thus the claimed equivalence follows by applying Theorem 6.2.39. $\square$

We now discuss how the symmetric monoidal structure on $\mathcal{I}{-}\mathcal{T}^c[W^{-1}_{f-lvl}]$ translates to $\mathbf{OR}_{\mathrm{fgl}}{-}\mathcal{S}_*$.

Lemma 7.3.20. *The topological category $\mathbf{OR}_{\mathrm{fgl}}$ is symmetric monoidal with unit object $(e,0)$ and tensor product given by $(G,V) \otimes (H,W) = (G \times H, V \oplus W)$. In particular, the ∞-category of $\mathbf{OR}_{\mathrm{fgl}}$-spaces admits a symmetric monoidal structure given by Day convolution.*

Proof. The first claim is a straightforward verification. The second claim then follows from Corollary 6.2.29. $\square$

Write $\mathbf{OR}^{\otimes}_{\mathrm{fgl}}$ for the ∞-operad associated to symmetric monoidal topological category $\mathbf{OR}_{\mathrm{fgl}}$.

Lemma 7.3.21. *The functor $L_{gl} \colon \mathbf{OR}_{\mathrm{fgl}} \to (\mathcal{I}{-}\mathcal{T}[W^{-1}_{f-lvl}])^{\mathrm{op}}$ given by $(G,V) \mapsto \mathcal{I}_{G,V}$ lifts to a fully faithful symmetric monoidal functor*

$$L_{gl} \colon \mathbf{OR}_{\mathrm{fgl}} \to (\mathcal{I}{-}\mathcal{T}[W^{-1}_{f-lvl}])^{\mathrm{op}}$$

of ∞-categories.

Proof. It suffices to observe that Equation (7.2.15.1) implies that $L_{gl} \colon \mathbf{OR}_{\mathrm{fgl}} \to \mathcal{I}{-}\mathcal{T}$ is a strong monoidal functor. $\square$

Corollary 7.3.22. *There is a symmetric monoidal equivalence*

$$\mathcal{I}{-}\mathcal{T}[W^{-1}_{lvl}] \simeq \mathbf{OR}_{\mathrm{fgl}}{-}\mathcal{S},$$

where the right hand side is symmetric monoidal via Day convolution.

Proof. This follows from Corollary 6.2.41, where we argue as in Theorem 7.3.19 and use Lemma 7.3.21. $\square$

Summarizing all of the identifications made, we have the following description of the symmetric monoidal ∞-category of faithful global prespectra.

Corollary 7.3.23. *Let G be a Lie group. Then there is a symmetric monoidal equivalence*

$$\mathrm{PSp}_{\mathrm{fgl}} \simeq \mathrm{Mod}_{S_{\mathrm{fgl}}}(\mathbf{OR}_{\mathrm{fgl}}\text{–}\mathcal{S}_*).$$

Proof. Combine Corollary 7.2.22, Theorem 7.3.22 and Proposition 6.2.38. □

Remark 7.3.24. We will often implicitly identify $\mathrm{PSp}_{\mathrm{fgl}}$ with $\mathrm{Mod}_{S_{\mathrm{fgl}}}(\mathbf{OR}_{\mathrm{fgl}}\text{–}\mathcal{S}_*)$.

7.4 FUNCTORIALITY OF EQUIVARIANT PRESPECTRA

The goal of this section is to construct a functor $\mathrm{PSp}_\bullet\colon \mathrm{Glo}^{\mathrm{op}} \to \mathrm{Cat}_\infty^\otimes$ sending a compact Lie group G to the symmetric monoidal ∞-category of G-prespectra of Definition 7.2.18, and to compute its (partially) lax limit. By Corollary 7.3.14, the ∞-category of G-prespectra can be identified with the category of modules over a certain object S_G in $\mathbf{OR}_G\text{–}\mathcal{S}_*$. Therefore our first step is to construct a functor sending a compact Lie group G to the ∞-category $\mathbf{OR}_G\text{–}\mathcal{S}_*$.

In the unstable case we observed that the relevant functoriality was induced by the functoriality of the partial slices $\mathrm{Orb}_{/G}$ in Glo. Formally, the functoriality of the categories $\mathbf{OR}_G\text{–}\mathcal{S}_*$ is induced by a (pro)functoriality of the categories $\mathbf{OR}_G$, and we will see that this is once again given by "passing to the slices" of a global analogue $\mathbf{OR}_{\mathrm{gl}}$ of the individual equivariant categories $\mathbf{OR}_G$. The category $\mathbf{OR}_{\mathrm{gl}}$ will be fibred over Glo and its objects will consist of pairs (G, V), where G is a compact Lie group and V is an arbitrary G-representation. Furthermore we will see that restricting to faithful representations, we recover $\mathbf{OR}_{\mathrm{fgl}}$.

Construction 7.4.1. Let G, H be compact Lie groups and V and W be orthogonal G and H-representations respectively. We equip the topological space

$$\mathrm{Hom}(H, G) \times \mathcal{I}(V, W)$$

with the right G-action and the left H-action given by

$$(\alpha, \varphi)\cdot g = (c_g\alpha, \varphi g^{-1}) \quad \text{and} \quad h\cdot(\alpha,\varphi) = (\alpha, h\varphi\alpha(h)^{-1})\,.$$

Since the G and H-actions commute, there is a residual G-action on the fixed points $(\mathrm{Hom}(H, G)\times\mathcal{I}(V, W))^H$. By definition, the fixed points space can be characterized as the space of pairs (α, φ) where $\alpha\colon H \to G$ is a Lie group homomorphism and $\varphi\colon V \to W$ is an H-equivariant isometry (where H acts on V via α). If K is another compact Lie group and U is an orthogonal K-representation, we define a composition map

$$(\mathrm{Hom}(H, G)\times\mathcal{I}(V, W))^H\times(\mathrm{Hom}(K, H)\times\mathcal{I}(W, U))^K \to (\mathrm{Hom}(K, G)\times\mathcal{I}(V, U))^K$$

via the assignment $(\alpha, \varphi)\cdot(\beta, \psi) = (\alpha\beta, \varphi\psi)$. This is compatible with the various actions, so that it induces an associative and unital composition map on the respective action groupoids:

$$\mathrm{Hom}(H, G)\times\mathcal{I}(V, W))^H /\!\!/ G\times(\mathrm{Hom}(K, H)\times\mathcal{I}(W, U))^K /\!\!/ H \to (\mathrm{Hom}(K, G)\times\mathcal{I}(V, U))^K /\!\!/ G\,.$$

Definition 7.4.2. Let $\mathbf{OR}_{\mathrm{gl}}$ be the topological category whose objects are pairs (G, V) where G is a compact Lie group and V is an orthogonal G-representation. Its morphism spaces are defined to be

$$\mathbf{OR}_{\mathrm{gl}}((G, V), (H, W)) = |(\mathrm{Hom}(H, G) \times \mathcal{I}(V, W))^H /\!\!/ G|$$

where $| - /\!\!/ G|$ is the geometric realization of the action groupoid of G on $\mathcal{I}(V, W)$ (as in Definition 7.1.1). As in Lemma 7.3.20, one sees that $\mathbf{OR}_{\mathrm{gl}}$ admits a symmetric monoidal structure given by $(G, V) \otimes (H, W) \simeq (G \times H, V \oplus W)$. We write $\mathbf{OR}^{\otimes}_{\mathrm{gl}}$ for the associated ∞-operad.

The next result tells us that the ∞-category $\mathbf{OR}_{\mathrm{fgl}}$ from Definition 7.3.16 is equivalent to the subcategory of $\mathbf{OR}_{\mathrm{gl}}$ spanned by the faithful representations.

Lemma 7.4.3. *Let $\mathcal{C}$ be the symmetric monoidal subcategory of $\mathbf{OR}_{\mathrm{gl}}$ spanned by (G, V) where V is a faithful G-representation. Then there is a symmetric monoidal functor of topological categories $\mathcal{C} \to \mathbf{OR}_{\mathrm{fgl}}$ sending (G, V) to (G, V), which induces a homotopy equivalence on mapping spaces (and so it is an equivalence of the underlying ∞-categories).*

Proof. The functor is the identity on objects, so it suffices to define it on mapping spaces. For any $(G, V), (H, W) \in \mathcal{C}$, let us consider the map

$$p\colon (\mathrm{Hom}(H, G) \times \mathcal{I}(V, W))^H \to (\mathcal{I}(V, W)/G)^H$$

sending (α, φ) to $[\varphi]$. We claim that this map exhibits the target as the quotient of the source by G. Firstly, note that the map is G-equivariant. Let us show that its fibres are exactly the G-orbits. Suppose we have a point $[\varphi]$ in the target and let us choose a representative $\varphi\colon V \to W$. Then we know that for every $h \in H$ $h \cdot [\varphi] = [h\varphi] = [\varphi]$. Then necessarily there exists $\alpha(h) \in G$ such that $h\varphi = \varphi\alpha(h)^{-1}$. Note that the element $\alpha(h)$ is unique since V is a faithful G-representation. Then the map $h \mapsto \alpha(h)$ is a Lie group homomorphism and its graph is closed in $H \times G$ (since it is a fibre of the continuous map $H \times G \to \mathcal{I}(V, W)$ sending (h, g) to $h\varphi g^{-1}$), so it is continuous. Then it is clear that (α, φ) is a preimage of $[\varphi]$, and so p is surjective.

On the other hand, if (α, φ) and (α', φ') have the same image under p, then there is some $g \in G$ so that $\varphi' = \varphi g$. A simple computation as before shows that this forces $\alpha' = c_g\alpha$ (since the G-action on $\mathcal{I}(V, W)$ is faithful, α' is determined by φ'). Moreover, the action of G on $(\mathrm{Hom}(H, G) \times \mathcal{I}(V, W))^H$ is free and proper, and so p is a principal G-bundle. In particular it induces a natural equivalence of topological groupoids

$$(\mathrm{Hom}(H, G) \times \mathcal{I}(V, W))^H /\!\!/ G \simeq (\mathcal{I}(V, W)/G)^H$$

and so a homotopy equivalence

$$\left|(\mathrm{Hom}(H, G) \times \mathcal{I}(V, W))^H /\!\!/ G\right| \simeq (\mathcal{I}(V, W)/G)^H$$

Finally, it is easy to check that p is compatible with composition and sends the identity to the identity. Therefore it induces an equivalence of ∞-categories $\mathcal{C} \to \mathbf{OR}_{\mathrm{fgl}}$. We leave to the reader to check that the above can be given the structure of a symmetric monoidal equivalence. □

Remark 7.4.4. There is a pair of functors of topological categories

$$s_0\colon \mathrm{Glo}^{\mathrm{op}} \to \mathbf{OR}_{\mathrm{gl}}\,,\ \pi_{gl}\colon \mathbf{OR}_{\mathrm{gl}} \to \mathrm{Glo}^{\mathrm{op}}$$

given by $s_0(G) = (G, 0)$ and $\pi_{gl}(G, V) = G$ on objects. Note that s_0 and π_{gl} are both symmetric monoidal, where Glo is symmetric monoidal under the cartesian product (and therefore $\mathrm{Glo}^{\mathrm{op}}$ is equipped with the cocartesian symmetric monoidal structure). This implies that the functors π_{gl} and s_0 lift to maps of ∞-operads $\pi_{gl}\colon \mathbf{OR}^{\otimes}_{\mathrm{gl}} \to (\mathrm{Glo}^{\mathrm{op}})^{\amalg}$ and $s_0\colon (\mathrm{Glo}^{\mathrm{op}})^{\amalg} \to \mathbf{OR}^{\otimes}_{\mathrm{gl}}$ respectively.

Lemma 7.4.5. *Let $\{(G_i, V_i)\}, (H, W)$ be objects of $\mathbf{OR}^{\otimes}_{\mathrm{gl}}$, and consider the map*

$$\pi_{gl}\colon \mathrm{Mul}_{\mathbf{OR}_{\mathrm{gl}}}(\{(G_i, V_i)\}, (H, W)) \to \mathrm{Mul}_{\mathrm{Glo}^{\mathrm{op}}}(\{G_i\}, H).$$

The homotopy fibre of this map over a group homomorphism $\alpha\colon H \to \prod_i G_i \in (\mathrm{Glo}^{\mathrm{op}})^{\amalg}$ is equivalent to the space of H-equivariant isometries $\bigoplus_i V_i \to W$ where H acts on $\bigoplus_i V_i$ via α.

Proof. Put $V = \bigoplus_i V_i$ and $G = \prod_i G_i$ so that $\alpha\colon H \to G$ and we can rewrite the map induced by π_{gl} as

$$\mathrm{Map}_{\mathbf{OR}_{\mathrm{gl}}}((G, V), (H, W)) \to \mathrm{Map}_{\mathrm{Glo}^{\mathrm{op}}}(G, H) = \mathrm{Map}_{\mathrm{Glo}}(H, G).$$

We recall from Proposition 7.1.3 that the G-space $\mathrm{Hom}(H, G)$ decomposes as a disjoint union of orbits

$$\mathrm{Hom}(H, G) \simeq \coprod_{(\alpha)} G/C(\alpha),$$

where α is a conjugacy class of homomorphisms and $C(\alpha)$ is the centralizer of the image of α. Therefore we have a decomposition

$$\mathrm{Map}_{\mathbf{OR}_{\mathrm{gl}}}((G, V), (H, W)) \simeq \big((\mathrm{Hom}(H, G) \times \mathcal{I}(V, W))^H\big)_{hG} \simeq \coprod_{(\alpha)} \mathcal{I}(V, W)^H_{hC(\alpha)},$$

depending on the choice of an α in each conjugacy class. This lies above the decomposition

$$\mathrm{Map}_{\mathrm{Glo}}(H, G) \simeq \coprod_{(\alpha)} BC(\alpha)$$

from Proposition 7.1.3 via the canonical maps $\mathcal{I}(V, W)^H \to *$. Therefore the homotopy fibre over α is precisely $\mathcal{I}(V, W)^H$. □

Lemma 7.4.6. *The functor* $\pi_{gl}\colon \mathbf{OR}_{\mathrm{gl}}^{\otimes} \to (\mathrm{Glo}^{\mathrm{op}})^{\amalg}$ *is a cocartesian fibration, and therefore exhibits* $\mathbf{OR}_{\mathrm{gl}}^{\otimes}$ *as a* $(\mathrm{Glo}^{\mathrm{op}})^{\amalg}$*-monoidal* ∞*-category.*

Proof. Consider $\{(G_i, V_i)\}_{i\in I} \in \mathbf{OR}_{\mathrm{gl}}^{\otimes}$, and let us set $V = \bigoplus_i V_i$ and $G = \prod_i G_i$ so that V is naturally a G-representation. Since π_{gl} is a map of ∞-operads, it is enough to find cocartesian lifts over active morphisms whose target is in $\mathrm{Glo}^{\mathrm{op}}$. A multimorphism from $\{G_i\}$ to H in $(\mathrm{Glo}^{\mathrm{op}})^{\amalg}$ is the datum of a continuous group homomorphism $\alpha\colon H \to G$. Consider the multimorphism $f \in \mathbf{OR}_{\mathrm{gl}}^{\otimes}(\{(G_i, V_i)\}, (H, \alpha^*V))$ lying over the map α which is represented by the element

$$[\alpha, 1_V] \in |(\mathrm{Hom}(H, G) \times \mathcal{I}(V, \alpha^*V))^H /\!\!/ G|.$$

We claim that this is a cocartesian edge. This follows from the fact that for all $(L, W) \in \mathbf{OR}_{\mathrm{gl}}^{\otimes}$, the square

$$\begin{tikzcd}
\mathrm{Mul}_{\mathbf{OR}_{\mathrm{gl}}}((H, \alpha^*V), (L, W)) \arrow[r, "f^*"] \arrow[d, "\pi_{gl}"] & \mathrm{Mul}_{\mathbf{OR}_{\mathrm{gl}}}(\{(G_i, V_i)\}, (L, W)) \arrow[d, "\pi_{gl}"] \\
\mathrm{Mul}_{\mathrm{Glo}^{\mathrm{op}}}(H, L) \arrow[r, "\alpha^*"] & \mathrm{Mul}_{\mathrm{Glo}^{\mathrm{op}}}(\{G_i\}, L)
\end{tikzcd}$$

is a homotopy pullback of spaces. We can verify this by checking that the vertical fibres are equivalent. This is now a consequence of Lemma 7.4.5. □

Definition 7.4.7. We define $\mathrm{Rep}\colon \mathrm{Glo}^{\mathrm{op}} \to \mathrm{Cat}_{\infty}^{\otimes}$ to be the functor corresponding to $\mathbf{OR}_{\mathrm{gl}}^{\otimes}$ under the equivalence of Proposition 6.4.5.

Remark 7.4.8. $\mathrm{Rep}(G)$ is the ∞-category corresponding to the topologically enriched category with objects V a G-representation, and morphism spaces $\mathrm{Rep}(V, W) = \mathcal{I}(V, W)^G$, the space of G-equivariant linear isometries from V to W. This is a symmetric monoidal category via direct sum. The functoriality in Glo is given by restriction of representations along group homomorphisms.

Recall from Remark 7.3.11 that there is a map of ∞-operads $\pi_G\colon \mathbf{OR}_G^{\otimes} \to (\mathbf{O}_{G,\mathrm{pr}}^{\mathrm{op}})^{\amalg}$. Also note that there is a canonical functor $\mathbf{O}_{G,\mathrm{pr}} \to \mathrm{Glo}$ which sends an object G/H to H and acts as

$$\mathbf{O}_{G,\mathrm{pr}}(G/H, G/K) \simeq \{g \in G \mid c_g(H) \subseteq K\}_{\mathrm{h}K} \to \hom(H, K)_{\mathrm{h}K}, \quad g \mapsto [c_g\colon H \to K].$$

This is an immediate generalization of the functor used in Lemma 7.1.12 to (not necessarily compact) Lie groups. We denote the opposite of this functor by ι_G. It induces a map of cocartesian ∞-operads which we denote by $\iota_G^{\amalg}$. We are now ready to state the next result.

Lemma 7.4.9. *Let G be a Lie group. Then there is a canonical map of ∞-operads $\nu_G\colon \mathbf{OR}_G^{\otimes} \to \mathbf{OR}_{\mathrm{gl}}^{\otimes}$ and a cartesian square of ∞-operads*

$$\begin{tikzcd} \mathbf{OR}_G^{\otimes} \arrow[r, "\nu_G"] \arrow[d, "\pi_G"'] & \mathbf{OR}_{gl}^{\otimes} \arrow[d, "\pi_{gl}"] \\ (\mathbf{O}_{G,\mathrm{pr}}^{\mathrm{op}})^{\amalg} \arrow[r, "\iota_G^{\amalg}"] & (\mathrm{Glo}^{\mathrm{op}})^{\amalg}. \end{tikzcd}$$

Proof. It will suffice to construct the map ν_G at the level of topological coloured operads and then apply Lemma 6.1.1. Recall from Definition 7.3.5 that

$$\mathbf{OR}_G((H,V),(K,W)) = (G \times_H \mathcal{I}(V,W))^K$$

where $G \times_H \mathcal{I}(V,W)$ is the quotient of $G \times \mathcal{I}(V,W)$ by the right H-action $(g,\varphi)\cdot h = (gh, \varphi h)$. Since the H-action is free, we can identify the quotient with the homotopy quotient (see [KÏ8, Theorem A.7] for example) and so there is a canonical identification

$$\mathbf{OR}_G((H,V),(K,W)) = |(G \times \mathcal{I}(V,W))^K /\!\!/ H|$$

that respects composition. Moreover under this identification, the multilinear spaces of the coloured operad structure are given by

$$\mathbf{OR}_G(\{(H_i,V_i)\}_i,(K,W)) = |(\prod_i G \times \mathcal{I}(\bigoplus_i V_i, W))^K /\!\!/ \prod H_i|.$$

Therefore we may define a functor of topological coloured operads $\mathbf{OR}_G \to \mathbf{OR}_{\mathrm{gl}}$ by sending (H,V) to (H,V) and on the multimorphism spaces we take the map which is induced by the map of topological groupoids

$$(\prod_i G \times \mathcal{I}(\bigoplus_i V_i, W))^K /\!\!/ \prod_i H_i \to (\mathrm{Hom}(K, \prod_i H_i) \times \mathcal{I}(\bigoplus_i V_i, W))^K /\!\!/ \prod_i H_i$$

which sends $(\{g_i\},\varphi)$ to $((c_{g_i}|_K)_i, \varphi)$. A tedious but simple calculation shows that these maps respect composition. This defines a map $\nu_G\colon \mathbf{OR}_G^{\otimes} \to \mathbf{OR}_{\mathrm{gl}}^{\otimes}$ as required.

Another tedious calculation shows that the square in the lemma commutes (already as a square of topological operads) and that it is a pullback on 0-vertices. Therefore it is enough to show that every induced square

$$\begin{tikzcd} \mathrm{Mul}_{\mathbf{OR}_G}(\{(H_i,V_i)\},(K,W)) \arrow[r, "\nu_G"] \arrow[d, "\pi_G"] & \mathrm{Mul}_{\mathbf{OR}_{\mathrm{gl}}}(\{(H_i,V_i)\},(K,W)) \arrow[d, "\pi_{gl}"] \\ \mathrm{Mul}_{\mathbf{O}_{G,\mathrm{pr}}^{\mathrm{op}}}(\{G/H_i\}, G/K) \arrow[r, "\iota_G"] & \mathrm{Mul}_{\mathrm{Glo}^{\mathrm{op}}}(\{H_i\},K) \end{tikzcd}$$

of multimorphism spaces is a homotopy pullback. It suffices to check that the vertical homotopy fibres are equivalent. A morphism $\varphi\colon G/K \to \prod G/H_i$ in $\mathbf{O}_{G,\mathrm{pr}}$ amounts to giving elements $g_i \in G$ such that $c_{g_i}(K) \subseteq H_i$. The homotopy fibre of π_G over φ is given by the space of K-equivariant isometries $\bigoplus_i V_i \to W$ where K acts on each V_i via c_{g_i}. The map ι_G sends φ to $(c_{g_i}\colon K \to H_i)$ and the homotopy fibre over this is again the space of K-equivariant isometries as above by Lemma 7.4.5. As the vertical homotopy fibres are equivalent, the square is a pullback of ∞-operads. □

We write $\mathrm{Ar}_{\mathrm{inj}}(\mathrm{Glo})$ for the *full* subcategory of $\mathrm{Ar}(\mathrm{Glo})$ spanned by the injective group homomorphisms.

Definition 7.4.10. We define $\mathbf{OR}^{\otimes}$ via the following pullback of ∞-operads

$$\begin{tikzcd} \mathbf{OR}^{\otimes} \arrow[r] \arrow[d, "\pi_{\mathrm{inj}}"'] & \mathbf{OR}^{\otimes}_{\mathrm{gl}} \arrow[d, "\pi_{gl}"] \\ (\mathrm{Ar}_{\mathrm{inj}}(\mathrm{Glo})^{\mathrm{op}})^{\amalg} \arrow[r, "s^{\mathrm{op}}"] & (\mathrm{Glo}^{\mathrm{op}})^{\amalg}. \end{tikzcd}$$

Thus an object of $\mathbf{OR}$, the underlying ∞-category of $\mathbf{OR}^{\otimes}$, is a pair $(\alpha\colon H \to G, V)$ where α is injective and V is a H-representation.

Lemma 7.4.11. *The composition*

$$\pi\colon \mathbf{OR}^{\otimes} \xrightarrow{\pi_{\mathrm{inj}}} (\mathrm{Ar}_{\mathrm{inj}}(\mathrm{Glo})^{\mathrm{op}})^{\amalg} \xrightarrow{t^{\mathrm{op}}} (\mathrm{Glo}^{\mathrm{op}})^{\amalg}$$

gives $\mathbf{OR}^{\otimes}$ *the structure of a* $(\mathrm{Glo}^{\mathrm{op}})^{\amalg}$*-promonoidal ∞-category, whose operadic fibre over G is exactly* $\mathbf{OR}^{\otimes}_G$.

Proof. We will show that each of the two maps in the defining composite is promonoidal in turn. Note that both are maps of ∞-operads. The map π_{inj} is a pullback of a cocartesian fibration, and therefore again cocartesian. The second map is then promonoidal by Example 6.2.7.

Finally we note that the operadic fibre of t^{op} over G is $(\mathbf{O}_G^{\mathrm{op}})^{\amalg}$ by Lemma 7.1.12 and the observation that $(-)^{\amalg}$ preserves pullbacks. Therefore, the calculation of the operadic fibre follows from Lemma 7.4.9 and the observation that the composite $(\mathbf{O}_G^{\mathrm{op}})^{\amalg} \to (\mathrm{Ar}_{\mathrm{inj}}(\mathrm{Glo})^{\mathrm{op}})^{\amalg} \xrightarrow{t^{\mathrm{op}}} (\mathrm{Glo}^{\mathrm{op}})^{\amalg}$ is equivalent to $\iota_G^{\amalg}$. □

Because π is a promonoidal category over $(\mathrm{Glo}^{\mathrm{op}})^{\amalg}$ with operadic fibre $\mathbf{OR}^{\otimes}_G$, morally it represents a profunctor of promonoidal ∞-categories. Therefore we can extract an honest symmetric monoidal functor by taking copresheafs. This will be the functor $\mathrm{Glo}^{\mathrm{op}} \to \mathrm{Cat}^{\otimes}_{\infty}$ sending G to $\mathbf{OR}_G\text{–}\mathcal{S}_*$.

Definition 7.4.12. The Day convolution $\mathrm{Fun}_{\mathrm{Glo}^{\mathrm{op}}}(\mathbf{OR}^{\otimes}, \mathcal{S}_*^{\wedge} \times (\mathrm{Glo}^{\mathrm{op}})^{\amalg})^{Day}$ is a $(\mathrm{Glo}^{\mathrm{op}})^{\amalg}$-monoidal ∞-category, whose operadic fibre over $G \in \mathrm{Glo}$ equals

$$\begin{aligned}\mathrm{Fun}_{\mathrm{Glo}^{\mathrm{op}}}(\mathbf{OR}^{\otimes}, \mathcal{S}_*^{\wedge} \times (\mathrm{Glo}^{\mathrm{op}})^{\amalg})^{Day} \times_{(\mathrm{Glo}^{\mathrm{op}})^{\amalg}} \mathrm{Fin}_* &\simeq \mathrm{Fun}(\mathbf{OR}^{\otimes} \times_{(\mathrm{Glo}^{\mathrm{op}})^{\amalg}} \mathrm{Fin}_*, \mathcal{S}_*^{\wedge})^{Day} \\ &\simeq \mathbf{OR}_G\text{–}\mathcal{S}_*\end{aligned}$$

by Example 6.4.7 and Lemma 7.4.11. We define $\mathbf{OR}_\bullet\text{–}\mathcal{S}_*\colon \mathrm{Glo}^{\mathrm{op}} \to \mathrm{Cat}_\infty^{\otimes}$ to be the functor associated to it under the equivalence of Proposition 6.4.5.

Lemma 7.4.13. *Let* $\mathbf{OR}$ *be the underlying category of the* ∞*-operad* $\mathbf{OR}^{\otimes}$*. Then the projection map*

$$\pi\colon \mathbf{OR} \to \mathrm{Glo}^{\mathrm{op}}$$

is cartesian over $\mathrm{Orb}^{\mathrm{op}}$*, and an edge* $(\sigma, \phi) \in \mathbf{OR}$ *is* π*-cartesian if and only if* $s^{\mathrm{op}}(\sigma)$ *and* ϕ *are equivalences.*

Proof. Suppose we have an injection $\alpha\colon H \to G$, and an object $(\beta\colon K \to H, V) \in \mathbf{OR}$. As noted before, the map $t^{\mathrm{op}}\colon \mathrm{Ar}_{\mathrm{inj}}(\mathrm{Glo})^{\mathrm{op}} \to \mathrm{Glo}^{\mathrm{op}}$ is a cartesian fibration. Furthermore over an injection $\alpha\colon H \to G$, cartesian lifts with target $\beta\colon K \to H$ are given by squares σ

$$\begin{tikzcd} K \arrow[d, "\alpha\beta"'] & K \arrow[l, "\sim"'] \arrow[d, "\beta"] \\ G & H. \arrow[l, "\alpha"] \end{tikzcd}$$

In particular, we note that cartesian lifts of injections are sent to equivalences by the source functor $s^{\mathrm{op}}\colon \mathrm{Ar}_{\mathrm{inj}}(\mathrm{Glo})^{\mathrm{op}} \to \mathrm{Glo}^{\mathrm{op}}$. Lifting $s^{\mathrm{op}}(\sigma)$ to an equivalence $\phi \in \mathbf{OR}_{\mathrm{gl}}$ with target (K, V), we obtain an edge (σ, ϕ) which lies over α and ends at (β, V). Because both components of the edge (σ, ϕ) in $\mathbf{OR}$ are π-cartesian, the edge (σ, ϕ) is itself π-cartesian. This shows that there are enough cartesian edges in $\mathbf{OR}$ over injections, and that they are exactly of the form claimed. □

Lemma 7.4.14. *The projection map*

$$\mathbf{OR}^{\otimes} \to \mathbf{OR}_{\mathrm{gl}}^{\otimes}$$

induces a fully faithful symmetric monoidal functor

$$\mathbf{OR}_{\mathrm{gl}}\text{–}\mathcal{S}_* \to \mathbf{OR}\text{–}\mathcal{S}_*$$

via restriction, with essential image those functors $F\colon \mathbf{OR} \to \mathcal{S}_*$ *that send cartesian arrows over* $\mathrm{Orb}^{\mathrm{op}}$ *to equivalences.*

Proof. Recall from Lemma 7.1.8 that the source projection $\mathrm{Ar}_{\mathrm{inj}}(\mathrm{Glo}) \to \mathrm{Glo}$ has a fully faithful left adjoint $\mathrm{Glo} \to \mathrm{Ar}_{\mathrm{inj}}(\mathrm{Glo})$ given by the diagonal embedding. Therefore, by the functoriality of the cocartesian operad [Lur16, Proposition 2.4.3.16], it follows that the source projection

$$(\mathrm{Ar}_{\mathrm{inj}}(\mathrm{Glo})^{\mathrm{op}})^{\amalg} \to (\mathrm{Glo}^{\mathrm{op}})^{\amalg}$$

has a fully faithful operadic right adjoint. Since Bousfield localizations are stable under basechange, it follows that the projection

$$\mathbf{OR}^{\otimes} \to \mathbf{OR}_{\mathrm{gl}}^{\otimes}$$

again has a fully faithful operadic right adjoint. Therefore $\mathbf{OR} \to \mathbf{OR}_{\mathrm{gl}}$ is a Bousfield localization on underlying ∞-categories and moreover the fully faithful functor

$$\mathbf{OR}_{\mathrm{gl}}\text{–}\mathcal{S} \to \mathbf{OR}\text{–}\mathcal{S}_*$$

is symmetric monoidal by Proposition 6.2.34(b). Finally, because $\mathbf{OR} \to \mathbf{OR}_{\mathrm{gl}}$ is a Bousfield localization, the essential image of the functor $\mathrm{Fun}(\mathbf{OR}_{\mathrm{gl}}, \mathcal{S}_*) \to \mathrm{Fun}(\mathbf{OR}, \mathcal{S}_*)$ is given by those functors which send the edges inverted by the map $\mathbf{OR} \to \mathbf{OR}_{\mathrm{gl}}$ to equivalences. But these are exactly the cartesian arrows over the injections by Lemma 7.4.13. □

Lemma 7.4.15. *There are symmetric monoidal equivalences*

$$\operatorname*{laxlim}_{G\in\mathrm{Glo}^{\mathrm{op}}} \mathbf{OR}_G\text{–}\mathcal{S}_* \simeq \mathbf{OR}\text{–}\mathcal{S} \qquad \textit{and} \qquad \operatorname*{laxlim}^{\dagger}_{G\in\mathrm{Glo}^{\mathrm{op}}} \mathbf{OR}_G\text{–}\mathcal{S}_* \simeq \mathbf{OR}_{\mathrm{gl}}\text{–}\mathcal{S}_*$$

where the lax limit is marked over the subcategory $\mathrm{Orb} \subseteq \mathrm{Glo}$ *of all objects and injective maps.*

Proof. By Proposition 6.4.8 there is a symmetric monoidal equivalence

$$\operatorname*{laxlim}_{G\in\mathrm{Glo}^{\mathrm{op}}} \mathbf{OR}_G\text{–}\mathcal{S}_* \simeq N_p\,\mathrm{Fun}_{\mathrm{Glo}^{\mathrm{op}}}(\mathbf{OR}^{\otimes}, \mathcal{S}_*^{\wedge} \times (\mathrm{Glo}^{\mathrm{op}})^{\amalg})^{Day}$$

where $p : (\mathrm{Glo}^{\mathrm{op}})^{\amalg} \to \mathrm{Fin}_*$ is the structure morphism of $(\mathrm{Glo}^{\mathrm{op}})^{\amalg}$. Applying the formula of Day convolution twice (see Definition 6.2.12), and the transitivity of norms of operads, we obtain

$$\begin{aligned} \operatorname{laxlim} \mathbf{OR}_{\bullet}\text{–}\mathcal{S}_* &\simeq N_p N_\pi \pi^*(\mathcal{S}_*^{\wedge} \times (\mathrm{Glo}^{\mathrm{op}})^{\amalg}) \\ &\simeq N_{\pi p}(\pi^* p^* \mathcal{S}_*^{\wedge}) \\ &\simeq \mathrm{Fun}(\mathbf{OR}^{\otimes}, \mathcal{S}_*^{\wedge})^{Day} \\ &= \mathbf{OR}\text{–}\mathcal{S}_* \end{aligned}$$

To compute the partially lax limit we appeal to Remark 6.4.2 to reduce to a statement on underlying categories. Combining Remarks 6.2.11 and 6.4.9, we conclude that the underlying ∞-category of the ∞-operad $N_p\,\mathrm{Fun}_{\mathrm{Glo}^{\mathrm{op}}}(\mathbf{OR}^{\otimes}, \mathcal{S}_*^{\wedge} \times$

$(\mathrm{Glo}^{\mathrm{op}})^{\amalg})^{Day}$ is given by sections of the cocartesian fibration $\pi_*\pi^*(\mathcal{S}_* \times \mathrm{Glo}^{\mathrm{op}})$, where by slight abuse of notation we write $\pi = U(\pi)$. Therefore we may calculate

$$\mathrm{Fun}_{/\mathrm{Glo}^{\mathrm{op}}}(\mathrm{Glo}^{\mathrm{op}}, \pi_*\pi^*(\mathcal{S}_* \times \mathrm{Glo}^{\mathrm{op}})) \simeq \mathrm{Fun}_{/\mathrm{Glo}^{\mathrm{op}}}(\mathbf{OR}, \mathcal{S}_* \times \mathrm{Glo}^{\mathrm{op}}) \simeq \mathrm{Fun}(\mathbf{OR}, \mathcal{S}_*)$$

using the definition of the left adjoints π^* and $\pi_!$. Now by Theorem 6.3.9 the partially lax limit of the diagram in question is given by the full subcategory of the left-most category spanned by those sections which map edges in $\mathrm{Orb}^{\mathrm{op}}$ to cocartesian arrows. We now apply [Lur09, Corollary 3.2.2.13] (with $p\colon \mathbf{OR} \times_{\mathrm{Glo}^{\mathrm{op}}} \mathrm{Orb}^{\mathrm{op}} \to \mathrm{Orb}^{\mathrm{op}}$, $q\colon \mathcal{S}_* \times \mathrm{Orb}^{\mathrm{op}} \to \mathrm{Orb}^{\mathrm{op}}$ and $T = \pi_*\pi^*(\mathcal{S}_* \times \mathrm{Glo}^{\mathrm{op}}) \times_{\mathrm{Glo}^{\mathrm{op}}} \mathrm{Orb}^{\mathrm{op}})$ together with Lemma 7.4.13, to see that these sections corresponds to those functors in $\mathrm{Fun}_{/\mathrm{Glo}^{\mathrm{op}}}(\mathbf{OR}, \mathcal{S}_* \times \mathrm{Glo}^{\mathrm{op}})$ which send cartesian edges over $\mathrm{Orb}^{\mathrm{op}}$ to cocartesian edges of $\mathcal{S}_* \times \mathrm{Orb}^{\mathrm{op}} \to \mathrm{Orb}^{\mathrm{op}}$. These are exactly those maps which are equivalences in the first component, and therefore such sections corresponds to functors $F\colon \mathbf{OR} \to \mathcal{S}_*$ which map cartesian edges over Orb to equivalences. Therefore we conclude by applying Lemma 7.4.14. □

Proposition 7.4.16. *There exists a functor* $\mathrm{PSp}_\bullet : \mathrm{Glo}^{\mathrm{op}} \to \mathrm{Cat}_\infty^{\otimes}$ *sending* G *to* PSp_G. *Moreover, there is a symmetric monoidal equivalence*

$$\operatorname*{laxlim}^{\dagger}_{G\in\mathrm{Glo}^{\mathrm{op}}} \mathrm{PSp}_G \simeq \mathrm{Mod}_{S_{gl}}(\mathbf{OR}_{\mathrm{gl}}\text{–}\mathcal{S}_*).$$

Proof. There is a lax symmetric monoidal topologically enriched functor $S_{gl}\colon \mathbf{OR}_{\mathrm{gl}} \to \mathcal{S}_*$ sending (G,V) to $(S^V)^G$. This induces a lax symmetric monoidal functor of ∞-operads, which uniquely specifies a commutative algebra in $\mathbf{OR}\text{–}\mathcal{S}_*$ by [Lur16, Example 2.2.6.9], where we view $\mathbf{OR}_{\mathrm{gl}}\text{–}\mathcal{S}_*$ as a symmetric monoidal subcategory of $\mathbf{OR}\text{–}\mathcal{S}_*$ using Lemma 7.4.14. Applying Theorem 6.4.10 to the lax limit of Lemma 7.4.15 shows that there is a functor sending G to $\mathrm{Mod}_{S_G}(\mathbf{OR}_G\text{–}\mathcal{S}_*) \simeq \mathrm{PSp}_G$ (see Corollary 7.3.14) whose lax limit is $\mathrm{Mod}_{S_{gl}}(\mathbf{OR}\text{–}\mathcal{S}_*)$.

Finally, we have to calculate the subcategory corresponding to the partially lax limit. Because the natural transformation $\mathrm{PSp}_G \to \mathbf{OR}_G\text{–}\mathcal{S}_*$ is point-wise conservative, we can check that an object lies in the partial lax limit of PSp_G by checking that its image lies in the partially lax limit of $\mathbf{OR}_G\text{–}\mathcal{S}_*$. In other words, we have a pullback square of symmetric monoidal ∞-categories

$$\begin{tikzcd}
\mathrm{laxlim}^{\dagger}_G \mathrm{PSp}_G \arrow[r] \arrow[d] & \mathrm{laxlim}_G \mathrm{PSp}_G \arrow[d] \\
\mathrm{laxlim}^{\dagger}_G \mathbf{OR}_G\text{–}\mathcal{S}_* \arrow[r] & \mathrm{laxlim}_G \mathbf{OR}_G\text{–}\mathcal{S}_* .
\end{tikzcd}$$

Therefore, by Lemma 7.4.15 and the previous paragraph we have a symmetric monoidal equivalence

$$\operatorname*{laxlim}^{\dagger}_{G\in\mathrm{Glo}^{\mathrm{op}}} \mathrm{PSp}_G \simeq \mathrm{Mod}_{S_{gl}}(\mathrm{Fun}(\mathbf{OR}, \mathcal{S}_*)) \times_{\mathrm{Fun}(\mathbf{OR},\mathcal{S}_*)} \mathrm{Fun}(\mathbf{OR}_{\mathrm{gl}}, \mathcal{S}_*)$$

Finally, since $S_{gl} \in \mathrm{Fun}(\mathbf{OR}_{\mathrm{gl}}, \mathcal{S}_*)$ this implies that

$$\operatorname*{laxlim}^{\dagger}_{G\in\mathrm{Glo}^{\mathrm{op}}} \mathrm{PSp}_G \simeq \mathrm{Mod}_{S_{gl}}(\mathbf{OR}_{\mathrm{gl}}\text{–}\mathcal{S}_*)\,. \qquad \square$$

Notation 7.4.17. We write $\mathrm{PSp}^{\dagger}_{\mathrm{gl}}$ for the ∞-category $\mathrm{Mod}_{S_{gl}}(\mathbf{OR}_{\mathrm{gl}}\text{–}\mathcal{S}_*)$, and identify it with $\mathrm{laxlim}^{\dagger}\,\mathrm{PSp}_{\bullet}$.

Recall the definition of the diagram $\mathcal{S}_{\bullet}\colon \mathrm{Glo}^{\mathrm{op}} \to \mathrm{Cat}^{\otimes}_{\infty}$ given in Construction 7.1.15, which sends a group G to the ∞-category of G-spaces. We would like to construct a natural transformation $\Sigma^{\infty}\colon \mathcal{S}_{\bullet} \to \mathrm{PSp}_{\bullet}$, whose component at G is given by an analogue of the suspension prespectrum functor. Morally, this sends a G-space X to the S_G-module $(H, V) \mapsto (X \wedge S^V)^H$. We make this precise in the next construction. Let us first fix some notation; we write $\mathcal{S}_{\bullet,*}$ for the composite $(-)_* \circ \mathcal{S}_{\bullet}$ of $\mathcal{S}_{\bullet}$ with the functor which sends a presentably symmetric monoidal category to the ∞-category of pointed objects.

Construction 7.4.18. We will construct natural transformations of functors $\mathrm{Glo}^{\mathrm{op}} \to \mathrm{Cat}^{\otimes}_{\infty}$

$$\mathcal{S}_{\bullet} \to \mathcal{S}_{\bullet,*} \to \mathrm{PSp}_{\bullet}$$

The first natural transformation is simply given by postcomposing $\mathcal{S}_{\bullet}$ with the natural transformation $(-)_{+}\colon \mathrm{id} \to (-)_*$ of functors $(\mathrm{Pr}^{\mathrm{L}})^{\otimes} \to (\mathrm{Pr}^{\mathrm{L}})^{\otimes}$.

For the second natural transformation, we will construct it as a composite

$$\mathcal{S}_{\bullet,*} \to \mathbf{OR}_{\bullet}\text{–}\mathcal{S}_* \to \mathrm{PSp}_{\bullet}.$$

For the latter transformation $\mathbf{OR}_{\bullet}\text{–}\mathcal{S}_* \to \mathrm{PSp}_{\bullet}$, we simply note that the free module functors

$$S_G \otimes -\colon \mathbf{OR}_G\text{–}\mathcal{S}_* \to \mathrm{Mod}_{S_G}(\mathbf{OR}_G\text{–}\mathcal{S}_*) \simeq \mathrm{PSp}_G$$

are symmetric monoidal and fit into a natural transformation by the second half of Theorem 6.4.10.

For the first, it will be technically convenient to construct the natural transformation $\mathcal{S}^{\wedge}_{\bullet,*} \to \mathbf{OR}_{\bullet}\text{–}\mathcal{S}_*$ as a map of $(\mathrm{Glo}^{\mathrm{op}})^{\amalg}$-monoidal ∞-categories and then to use Proposition 6.4.5.

For this, we need to pin down the $(\mathrm{Glo}^{\mathrm{op}})^{\amalg}$-monoidal ∞-category which corresponds to $\mathcal{S}_{\bullet,*}$ under Proposition 6.4.5. Note that the map

$$t^{\mathrm{op}}\colon (\mathrm{Ar}_{\mathrm{inj}}(\mathrm{Glo})^{\mathrm{op}})^{\amalg} \to (\mathrm{Glo}^{\mathrm{op}})^{\amalg}$$

exhibits $\mathrm{Ar}_{\mathrm{inj}}(\mathrm{Glo})^{\mathrm{op}}$ as a $(\mathrm{Glo}^{\mathrm{op}})^{\amalg}$-monoidal category, see Example 6.2.7. We claim that $\mathcal{S}_{\bullet,*}$ corresponds to the Day convolution

$$\mathrm{Fun}_{\mathrm{Glo}^{\mathrm{op}}}((\mathrm{Ar}_{\mathrm{inj}}(\mathrm{Glo})^{\mathrm{op}})^{\amalg}, \mathcal{S}_*^{\wedge} \times (\mathrm{Glo}^{\mathrm{op}})^{\amalg})^{Day}.$$

To see this, we first note that

$$\mathrm{Fun}_{\mathrm{Glo}^{\mathrm{op}}}((\mathrm{Ar}_{\mathrm{inj}}(\mathrm{Glo})^{\mathrm{op}})^{\amalg}, \mathcal{S}^{\times} \times (\mathrm{Glo}^{\mathrm{op}})^{\amalg})^{Day}$$

classifies $\mathcal{S}_{\bullet}^{\times}$, because it does so on underlying categories (combine Remark 6.2.11 and [GHN17, Proposition 7.3]) and the forgetful functor $\mathrm{Cat}_{\infty}^{\otimes} \to \mathrm{Cat}_{\infty}$ is faithful when restricted to cartesian monoidal ∞-categories. Now we observe that the $(\mathrm{Glo}^{\mathrm{op}})^{\amalg}$-monoidal functor

$$((-)_+)_*\colon \mathrm{Fun}_{\mathrm{Glo}^{\mathrm{op}}}(\mathrm{Ar}_{\mathrm{inj}}((\mathrm{Glo}^{\mathrm{op}})^{\amalg}, \mathcal{S}^{\times}\times(\mathrm{Glo}^{\mathrm{op}})^{\amalg})^{Day} \to \mathrm{Fun}_{\mathrm{Glo}^{\mathrm{op}}}(\mathrm{Ar}_{\mathrm{inj}}((\mathrm{Glo}^{\mathrm{op}})^{\amalg}, \mathcal{S}_*^{\wedge}\times(\mathrm{Glo}^{\mathrm{op}})^{\amalg})^{Day}$$

agrees pointwise with $(-)_+$, and therefore by the universal property of taking pointed objects (see [Lur16, Proposition 4.8.2.11]) $\mathrm{Fun}_{\mathrm{Glo}^{\mathrm{op}}}(\mathrm{Ar}_{\mathrm{inj}}((\mathrm{Glo}^{\mathrm{op}})^{\amalg}, \mathcal{S}_*^{\wedge}\times(\mathrm{Glo}^{\mathrm{op}})^{\amalg})^{Day}$ must classify $\mathcal{S}_{\bullet,*}$.

Now we can construct the $(\mathrm{Glo}^{\mathrm{op}})^{\amalg}$-monoidal functor which will induce $\mathcal{S}_{\bullet,*} \to \mathbf{OR}_{\bullet}\text{–}\mathcal{S}_*$. Pulling back the functor s_0 of Remark 7.4.4 along t^{op} we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{Ar}_{\mathrm{inj}}(\mathrm{Glo})^{\mathrm{op}})^{\amalg} \arrow[rr, "s_{0,\mathrm{inj}}"] \arrow[dr, "t^{\mathrm{op}}"'] & & \mathbf{OR}^{\otimes} \arrow[dl, "\pi"] \\
& (\mathrm{Glo}^{\mathrm{op}})^{\amalg} &
\end{tikzcd}$$

where t^{op} and π exhibit the sources as $(\mathrm{Glo}^{\mathrm{op}})^{\amalg}$-promonoidal ∞-categories by Lemma 7.4.11, so that $s_{0,\mathrm{inj}}$ is a map of $(\mathrm{Glo}^{\mathrm{op}})^{\amalg}$-promonoidal ∞-categories. One can then verify that $s_{0,\mathrm{inj}}$ satisfies the hypotheses of Proposition 6.2.34(a), and there exists a $(\mathrm{Glo}^{\mathrm{op}})^{\amalg}$-monoidal functor

$$(s_{0,\mathrm{inj}})_!\colon \mathrm{Fun}_{\mathrm{Glo}^{\mathrm{op}}}((\mathrm{Ar}_{\mathrm{inj}}(\mathrm{Glo})^{\mathrm{op}})^{\amalg}, \mathcal{S}_*^{\wedge}\times(\mathrm{Glo}^{\mathrm{op}})^{\amalg})^{Day} \to \mathrm{Fun}_{\mathrm{Glo}^{\mathrm{op}}}(\mathbf{OR}^{\otimes}, \mathcal{S}_*^{\wedge}\times(\mathrm{Glo}^{\mathrm{op}})^{\amalg})^{Day}$$

which then induces the required natural transformation. This description shows as well that the component at G coincides with $\mathcal{I}_0$, and so the composite functor $\mathcal{S}_{G,*} \to \mathrm{PSp}_G$ is analogous to the usual suspension prespectrum functor $F_0(-)$. We will formulate a precise statement to this effect as Proposition 7.5.5.

7.5 FUNCTORIALITY OF EQUIVARIANT SPECTRA

In the previous section we have constructed the functor

$$\mathrm{PSp}_{\bullet}\colon \mathrm{Glo}^{\mathrm{op}} \to \mathrm{Cat}_{\infty}^{\otimes},$$

and calculated its partially lax limit. In this section we will show that this functor descends to a diagram $\mathrm{Sp}_\bullet$ where on every level we restrict to the subcategory of spectrum objects. Furthermore, we will prove that the functoriality obtained in this way agrees with the standard functoriality of equivariant spectra under the restriction-inflation functors. Finally, we will compute the partially lax limit of $\mathrm{Sp}_\bullet$ as a Bousfield localization of $\mathrm{PSp}_{\mathrm{gl}}^\dagger = \mathrm{laxlim}^\dagger \mathrm{PSp}_\bullet$.

Given a continuous group homomorphism $\alpha\colon H \to G$ between compact Lie groups, we write

$$\alpha^*\colon \mathbf{OR}_G\text{–}\mathcal{S}_* \to \mathbf{OR}_H\text{–}\mathcal{S}_*$$

for the symmetric monoidal functor induced by α. Our goals require a better understanding of α^*. We start by studying the interaction between α^* and the Quillen adjunction of Construction 7.2.6

$$\mathcal{I}_V\colon G\mathcal{T} \leftrightarrows \mathcal{I}\text{–}G\mathcal{T} :\mathrm{ev}_V$$

for a given G-representation V. However before we do this, we first need to understand how these adjunctions manifest themselves under the equivalences

$$\mathcal{S}_G \simeq \mathbf{O}_G\text{–}\mathcal{S} \quad \text{and} \quad \mathbf{OR}_G\text{–}\mathcal{S} \simeq \mathcal{I}\text{–}G\mathcal{T}$$

of Example 6.2.40 and Theorem 7.3.9.

Remark 7.5.1. Consider $X \in \mathcal{I}\text{–}G\mathcal{T}$ and a G-representation V. Then the G-space $X(V)$ corresponds to the presheaf

$$G/H \mapsto X(V)^H \simeq \mathrm{Map}_{\mathcal{I}\text{–}G\mathcal{T}}(G \times_H \mathcal{I}_{V|_H}, X)\,.$$

Note that $G \times_H \mathcal{I}_{V|_H}$ is the image of $(H, V|_H)$ under the embedding L of Theorem 7.3.9. Therefore, if we let $s_V\colon \mathbf{O}_G^{\mathrm{op}} \to \mathbf{OR}_G$ be the cocartesian section of π_G sending G/G to (G, V), we have $s(G/H) \simeq (H, V|_H)$, so we can identify ev_V with

$$s_V^*\colon \mathbf{OR}_G\text{–}\mathcal{S} \to \mathcal{S}_G \qquad X \mapsto X \circ s_V$$

and similarly for the pointed version. It follows that the derived functor associated to $\mathcal{I}_V$ is given by the left Kan extension functor $(s_V)_!$. Finally, we can compute that this is given by

$$(\mathcal{I}_V X)(H, W) \simeq \mathcal{I}(V, W)^H \times X^H\,,$$

by the following Lemma.

Lemma 7.5.2. *Let $\pi\colon \mathcal{E} \to \mathcal{B}$ be a cocartesian fibration of ∞-categories and $s\colon \mathcal{B} \to \mathcal{E}$ be a cocartesian section. For every functor $F\colon \mathcal{B} \to \mathcal{C}$ where $\mathcal{C}$ is a cocomplete ∞-category, we can compute the left Kan extension along s by*

$$(s_! F)(e) \simeq \mathrm{Map}_{\pi^{-1}(\pi e)}(s\pi(e), e) \times F(\pi(e))$$

for all $e \in E$.

Proof. By the usual formula for left Kan extensions we have that

$$(s_!F)(e) \simeq \operatorname*{colim}_{b \in \mathcal{B} \times_{\mathcal{E}} \mathcal{E}_{/e}} F(b)\,.$$

We claim that the projection $\mathcal{B} \times_{\mathcal{E}} \mathcal{E}_{/e} \to \mathcal{B}_{/\pi e}$ is a left fibration with fibre over $f\colon b \to \pi e$ given by $\mathrm{Map}^f_{\mathcal{E}}(s(b), e)$. In particular, since F is constant along the fibres of this fibration and $\mathcal{B}_{/\pi e}$ has a final object, we have

$$\begin{aligned}\operatorname*{colim}_{b \in \mathcal{B} \times_{\mathcal{E}} \mathcal{E}_{/e}} F(b) &\simeq \operatorname*{colim}_{[f\colon b \to \pi e] \in \mathcal{B}_{/\pi e}} \mathrm{Map}^f_{\mathcal{E}}(s(b), e) \times F(b) \\ &\simeq \mathrm{Map}_{\pi^{-1}(\pi e)}(s\pi(e), e) \times F(\pi(e))\,.\end{aligned}$$

It only remains to prove that the functor $\mathcal{B} \times_{\mathcal{E}} \mathcal{E}_{/e} \to \mathcal{B}_{/\pi e}$ is a left fibration. That is, we need to show that for every diagram

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & \mathcal{B} \times_{\mathcal{E}} \mathcal{E}_{/e} \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & \mathcal{B}_{/\pi e} \end{tikzcd}$$

with $0 \leq i < n$ there exists a dotted arrow completing the diagram. Using the definition of slice ∞-categories, this is equivalent to finding a dotted arrow completing the dotted diagram

$$\begin{tikzcd} \Lambda^n_i \star \Delta^0 \arrow[r, "F"] \arrow[d] & \mathcal{E} \arrow[d, "\pi"] \\ \Delta^n \star \Delta^0 \simeq \Delta^{n+1} \arrow[r, "G"] \arrow[ur, dashed] & \mathcal{B} \end{tikzcd}$$

where F restricted to $\Lambda^n_i \subseteq \Lambda^{n+1}_i$ is given by the restriction of sG. This diagram is a diagram of marked simplicial sets when we give $\mathcal{B}$ the total marking, $\mathcal{E}$ the cocartesian marking and on the left column the marking $(\Lambda^n_i)^\sharp \star \Delta^0 \to (\Delta^n)^\sharp \star \Delta^0$. Since the left vertical arrow is left marked anodyne by [Sha18, Lemma 4.10], the lift exists. $\square$

Having understood the adjunction $\mathcal{I}_V \dashv \mathrm{ev}_V$, we now discuss how this interacts with the functor α^*.

Proposition 7.5.3. *Let us fix an arrow $\alpha\colon H \to G$ in* Glo*.*

1. *Given a pointed G-space X, there is a natural equivalence*

$$\alpha^* \mathcal{I}_V X \simeq \mathcal{I}_{\alpha^* V}(\alpha^* X)$$

2. *Given a pointed $\mathbf{OR}_G$-space Y, there is a natural equivalence*

$$\alpha^*\mathrm{ev}_V Y \simeq \mathrm{ev}_{\alpha^*V}\alpha^*Y$$

3. *Under the two previous identifications, the counit natural transformation*

$$\mathcal{I}_V\mathrm{ev}_V X \to X$$

is sent by α^* to

$$\mathcal{I}_{\alpha^*V}\mathrm{ev}_{\alpha^*}V(\alpha^*X) \to \alpha^*X$$

the counit natural transformation for α^*V applied to α^*X.

Proof. Write $\mathbf{O}_\alpha \simeq \mathrm{Ar}_{\mathrm{inj}}(\mathrm{Glo}) \times_{\mathrm{Glo}} [1]$ (using the target map $t\colon \mathrm{Ar}_{\mathrm{inj}}(\mathrm{Glo}) \to \mathrm{Glo}$) and let $i_0\colon \mathbf{O}_H \to \mathbf{O}_\alpha$, $i_1\colon \mathbf{O}_G \to \mathbf{O}_\alpha$ be the inclusions of the fibres over 0 and 1 respectively. Similarly, write $\mathbf{OR}_\alpha$ for the pullback $\mathbf{OR}_{\mathrm{gl}} \times_{\mathrm{Glo}^{\mathrm{op}}} \mathbf{O}_\alpha^{\mathrm{op}}$ and $j_0, j_1\colon \mathbf{OR}_H, \mathbf{OR}_G \to \mathbf{OR}_\alpha$ for the inclusion of the fibre of $\mathbf{OR}_\alpha \to [1]^{\mathrm{op}}$ over 0 and 1 respectively.

By Remark 6.2.23 we can identify

$$\alpha^* \simeq i_0^*(i_1)_!\colon \mathcal{S}_{G,*} \to \mathcal{S}_{H,*} \qquad \alpha^* \simeq j_0^*(j_1)_!\colon \mathbf{OR}_G\text{–}\mathcal{S}_* \to \mathbf{OR}_H\text{–}\mathcal{S}_*\,.$$

Let $s_V\colon \mathbf{O}_G^{\mathrm{op}} \to \mathbf{OR}_G$ be the cocartesian section of $\pi_G\colon \mathbf{OR}_G \to \mathbf{O}_G^{\mathrm{op}}$ which sends G/G to (G, V). Similarly let $s\colon \mathbf{O}_\alpha^{\mathrm{op}} \to \mathbf{OR}_\alpha$ be the cocartesian section sending the initial object $i_1(G/G)$ of $\mathbf{O}_\alpha^{\mathrm{op}}$ to $j_1(G, V)$. Then s restricts to s_V on $\mathbf{O}_G^{\mathrm{op}}$ and to s_{α^*V} on $\mathbf{O}_H^{\mathrm{op}}$, since a cocartesian section is determined by where it sends the initial object. Therefore by Remark 7.5.1 we obtain

$$\alpha^*\mathcal{I}_V X \simeq \alpha^*(s_V)_!X \simeq j_0^*(j_1 s_V)_!X \simeq j_0^* s_!(i_1)_!X$$

for every pointed G-space X. Using the formula for $s_!$ described in Lemma 7.5.2 we see that the above can be identified with $(s_{\alpha^*V})_! i_0^*(i_1)_!X$, thus proving the first statement.

Now let Y be an $\mathbf{OR}_G$-space. Then we claim that $s^*(j_1)_!Y$ is left Kan extended from $\mathbf{O}_G^{\mathrm{op}}$. In fact this happens if and only if $s^*(j_1)_!Y$ sends the arrows $(G, \alpha L) \to (H, L)$ in $\mathbf{O}_\alpha^{\mathrm{op}}$ to equivalences. But the arrow

$$[s(G, \alpha L) \to s(H, L)] \simeq [(G, \alpha L, V) \to (H, L, \alpha^*V)]$$

is a terminal object of $\mathbf{OR}_G \times_{\mathbf{OR}_\alpha} (\mathbf{OR}_\alpha)_{/(H,L,\alpha^*V)}$ and so it is sent to an equivalence by $(j_1)_!Y$. This implies that

$$\mathrm{ev}_{\alpha^*V}\alpha^*Y \simeq s_{\alpha^*V}^*(j_1)_!Y \simeq j_0^* s^*(j_1)_!Y \simeq j_0^*(j_1)_!(s_V)^*Y \simeq \alpha^*\mathrm{ev}_V Y,$$

proving the second statement.

Finally we consider for every $\mathbf{OR}_G$-space Y, the natural transformation

$$s_! s^*(j_1)_! Y \to (j_1)_! Y,$$

and note that this is a natural transformation of functors left Kan extended from $\mathbf{OR}_G$, which restricts to

$$(s_V)_! s_V^* Y \to Y \text{ and } (s_{\alpha^* V})_! s_{\alpha^* V}^* \alpha^* Y \to \alpha^* Y$$

on the fibres over 0 and 1 respectively. Thus α^* sends the former to the latter, showing the third statement. □

With this result we can show that $\mathrm{PSp}_\bullet$ restricts to a functor on spectrum objects.

Proposition 7.5.4. *There exists a functor* $\mathrm{Sp}_\bullet \colon \mathrm{Glo}^{\mathrm{op}} \to \mathrm{Cat}_\infty^\otimes$ *and a natural transformation of functors*

$$L_\bullet \colon \mathrm{PSp}_\bullet \to \mathrm{Sp}_\bullet$$

whose component for a fixed G is the spectrification functor $L_G \colon \mathrm{PSp}_G \to \mathrm{Sp}_G$.

Proof. Consider a group homomorphism $\alpha \colon H \to G$. We claim that the functor $\mathrm{PSp}_\alpha \colon \mathrm{PSp}_G \to \mathrm{PSp}_H$ preserves stable equivalences. It suffices to show that it preserves the generating equivalences $G \ltimes_K \lambda_{V,W}$ of Proposition 7.2.30. Moreover, since G is compact, we can restrict to the cofinal set W of K-representations that are extended from G.

First note that $\lambda_{V,W} \simeq (G \ltimes_K F_V(S^0)) \otimes \lambda_{0,W}$. Since PSp_α is symmetric monoidal by construction and stable equivalences are stable under tensor product, it suffices to show that $\mathrm{PSp}_\alpha(\lambda_{0,W})$ is a stable equivalence. We claim it is equivalent to $\lambda_{0,\alpha^* W}$. In fact $\lambda_{0,W}$ is exactly the counit of the adjunction $F_W \dashv \mathrm{ev}_W$ of Construction 7.2.28 applied to S_G. Therefore we can factor it as

$$(F_W \mathrm{ev}_W) S_G \simeq (S_G \otimes -)\mathcal{I}_W \mathrm{ev}_W U S_G \to (S_G \otimes -) U S_G \to S_G$$

where $(S_G \otimes -) \dashv U$ is the free-forgetful adjunction between PSp_G and $\mathbf{OR}_G\text{–}\mathcal{S}_*$, and the arrows are the counits of the respective adjunctions. Then our claim follows from Theorem 6.4.10 and Proposition 7.5.3.

Knowing that PSp_α preserves stable equivalences, we can combine Construction 7.4.18 and Corollary 6.3.14, to obtain $\mathrm{Sp}_\bullet$ and the natural transformation $L_\bullet \colon \mathrm{PSp}_\bullet \to \mathrm{Sp}_\bullet$. □

Recall that we constructed a natural transformation $\Sigma^\infty_\bullet \colon \mathcal{S}_{\bullet,*} \to \mathrm{PSp}_\bullet$ in Construction 7.4.18, which pointwise was our analogue of the suspension prespectrum functor. We may compose this with the natural transformation $L_\bullet$ to obtain a new natural transformation, which we again denote by $\Sigma^\infty_\bullet$.

Proposition 7.5.5. *The component of* $\Sigma^\infty_\bullet\colon \mathcal{S}_{\bullet,*} \to \mathrm{Sp}_\bullet$ *at the group* G *is equivalent to the standard suspension spectrum functor.*

Proof. Considering the component at G, we observe that the functor Σ^∞_G is defined as the composition

$$\mathcal{S}_{G,*} \to \mathbf{OR}_G\text{–}\mathcal{S}_* \to \mathrm{Mod}_{\mathbb{S}_G}(\mathbf{OR}_G\text{–}\mathcal{S}_*) \simeq \mathrm{PSp}_G \to \mathrm{Sp}_G$$

where the first functor is $\mathcal{I}_0$ (i.e. precomposition along $\mathbf{OR}_G \to \mathbf{O}_G^{\mathrm{op}}$), the second functor is the free $\mathbb{S}_G$-module functor ($\mathbb{S}_G \otimes -$) and the third functor is the localization functor. These functors are all modeled by left Quillen functors

$$G\mathcal{T}_* \to \mathcal{I}\text{–}G\mathcal{T}_* \to \mathrm{Sp}^O_G \to \mathrm{Sp}^O_G$$

given by the constant $\mathcal{I}$-G-space, the free $\mathbb{S}_G$-module and the identity respectively. Therefore Σ^∞_G is modelled by their composition, which is exactly the suspension spectrum functor constructed in [MM02]. $\square$

This suffices for us to conclude that the functoriality of $\mathrm{Sp}_\bullet$ agrees morphism-wise with the functoriality of equivariant spectra in restriction, by the universal property of G-spectra.

Corollary 7.5.6. *The functor* $\mathrm{Sp}_\bullet\colon \mathrm{Glo}^{\mathrm{op}} \to \mathrm{Cat}^{\otimes}_\infty$ *sends a compact Lie group* G *to* Sp_G *and a continuous group homomorphism* $\alpha\colon H \to G$ *to the restriction functor* $\alpha^*\colon \mathrm{Sp}_G \to \mathrm{Sp}_H$.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Sp}_G \arrow[r, "\mathrm{Sp}_\alpha"] & \mathrm{Sp}_H \\
\mathcal{S}_{G,*} \arrow[u, "\Sigma^\infty_G"] \arrow[r, "\mathcal{S}^\alpha_*"'] & \mathcal{S}_{H,*} \arrow[u, "\Sigma^\infty_H"'] \\
\mathcal{S}_G \arrow[u, "(-)_+"] \arrow[r, "\mathcal{S}^\alpha"'] & \mathcal{S}_H \arrow[u, "(-)_+"']
\end{tikzcd}$$

of symmetric monoidal functors. By the universal property of G-spectra given in [GM23, Corollary C.7], the functor Sp_α is uniquely determined by $\mathcal{S}_{\alpha,*}$, and this is completely determined by $\mathcal{S}_\alpha$ by [Lur16, Proposition 4.8.2.11]. Finally, Proposition 7.1.16 identifies the functor $\mathcal{S}^\alpha$ with α^*. $\square$

Remark 7.5.7. Note that the argument of Corollary 7.5.6 in facts shows that the natural transformation $\Sigma_{\bullet,*}\colon \mathcal{S}_{\bullet,*} \to \mathrm{Sp}_\bullet$ admits a universal property. This forces $\mathrm{Sp}_\bullet$ to coincide with the construction of [BH21, Section 9] on the subcategory of Glo spanned by finite groups. This suggests a possible comparison between ultracommutative Fin-global ring spectra in the sense of [Sch18] and normed spectra in the sense of [BH21].

We have now constructed $\mathrm{Sp}_\bullet$ and shown that it agrees with the standard functoriality of equivariant spectra. We will write $\mathrm{Sp}_{\mathrm{gl}}^\dagger$ for the partially lax limit $\operatorname{laxlim}^\dagger \mathrm{Sp}_\bullet$. We would like to describe $\mathrm{Sp}_{\mathrm{gl}}^\dagger$ as a Bousfield localization of $\mathrm{PSp}_{\mathrm{gl}}^\dagger$ by applying Lemma 6.3.13. To do this requires the following two lemmata.

Proposition 7.5.8. *Let $\alpha\colon H \to G$ be an injective group homomorphism. Then the functor $\alpha^*\colon \mathbf{OR}_G\text{–}\mathcal{S} \to \mathbf{OR}_H\text{–}\mathcal{S}$ has a left adjoint $\alpha_!$. Moreover, under the identification of Theorem 7.3.9 the adjunction $\alpha_! \dashv \alpha^*$ corresponds to the Quillen adjunction $G \times_H - \dashv \alpha^*$ of Proposition 7.2.10.*

In particular for $X \in \mathbf{OR}_H\text{–}\mathcal{S}$ and $Y \in \mathbf{OR}_G\text{–}\mathcal{S}$ the comparison map

$$\alpha_!(X \otimes \alpha^* Y) \to \alpha_! X \otimes Y$$

adjoint to $X \otimes \alpha^* Y \to \alpha^*\alpha_! X \otimes \alpha^* Y$ is an equivalence.

Proof. From the description of Remark 6.2.23 and Lemma 7.4.13 it follows that the functor $\alpha^*\colon \mathbf{OR}_H\text{–}\mathcal{S} \to \mathbf{OR}_G\text{–}\mathcal{S}$ is given by precomposition along the functor $p_\alpha\colon \mathbf{OR}_H \to \mathbf{OR}_G$ obtained by basechange from $\mathbf{O}_H^{\mathrm{op}} \to \mathbf{O}_G^{\mathrm{op}}$. In particular it has a left adjoint $\alpha_!$ given by left Kan extension along p_α.

In the proof of Theorem 7.3.9 we have constructed a functor $L_H\colon \mathbf{OR}_H^{\mathrm{op}} \to \mathcal{I}\text{–}H\mathcal{T}[W_{lvl}^{-1}]$ sending (K, W) to $H \times_K \mathcal{I}_W$. We claim there is a commutative diagram

$$\begin{tikzcd}
\mathbf{OR}_H^{\mathrm{op}} \arrow[r, "\text{Yoneda}"] \arrow[d, "p_\alpha"'] \arrow[rr, bend left, "L_H"] & \mathbf{OR}_H\text{–}\mathcal{S} \arrow[r, "\sim"] \arrow[d, "\alpha_!"] & \mathcal{I}\text{–}H\mathcal{T}[W_{lvl}^{-1}] \arrow[d, "G\times_H -"] \\
\mathbf{OR}_G^{\mathrm{op}} \arrow[r, "\text{Yoneda}"'] \arrow[rr, bend right, "L_G"'] & \mathbf{OR}_G\text{–}\mathcal{S} \arrow[r, "\sim"] & \mathcal{I}\text{–}G\mathcal{T}[W_{lvl}^{-1}]
\end{tikzcd}$$

where the horizontal equivalences are given by Theorem 7.3.9. The diagram on the left commutes by the universal property of presheaf categories and the outer square commutes by direct verification using the formulas of L_G and L_H. Therefore a generation argument using that all the functors preserve colimits, shows that the rightmost diagram commutes too. The right most vertical functor can be modelled by a left Quillen functor by Proposition 7.2.10 so the first claim follows.

Finally, since the map

$$G \times_H (X \otimes Y) \to (G \times_H X) \otimes Y$$

is an isomorphism in $\mathcal{I}\text{–}G\mathcal{T}$, it follows that the derived formula holds as well. $\square$

Lemma 7.5.9. *Let $\alpha\colon H \to G$ be an injective homomorphism of compact Lie groups. Then $\mathrm{PSp}_\alpha\colon \mathrm{PSp}_G \to \mathrm{PSp}_H$ sends Sp_G into Sp_H.*

Proof. Note that PSp_α sends X to $S_H \otimes_{\alpha^* S_G} \alpha^* X \simeq \alpha^* X$, since α is injective. Therefore PSp_α preserves all small limits and colimits, since α^* does, and so it has a left adjoint L_α. Moreover, by Lemma 7.5.8 there is an equivalence

$$L_\alpha(X \otimes \mathrm{PSp}_\alpha Y) \simeq L_\alpha(X) \otimes Y\,.$$

To prove that $\alpha^*(\mathrm{Sp}_G) \subseteq \mathrm{Sp}_H$ it suffices to show that L_α preserves stable equivalences. By confinality the stable equivalences in Sp_H are generated by those of the form $H \times_M \lambda_{V,W|_M}$ where $M < H$ is a closed subgroup, V is an M-representation and W is a G-representation. But then

$$L_\alpha(H \times_M \lambda_{V,W|_M}) \simeq L_\alpha((H \times_M F_V S^0) \otimes \alpha^* \lambda_{0,W|_H}) \simeq L_\alpha(H \times_M F_V S^0) \otimes \lambda_{0,W}\,.$$

Since stable equivalences are stable under tensoring and $\lambda_{0,W}$ is a stable equivalence, this proves the thesis. □

Given a compact Lie group $G \in \mathrm{Glo}$, we denote by $U^{\mathrm{gl}}_G\colon \mathrm{PSp}^{\dagger}_{\mathrm{gl}} \to \mathrm{PSp}_G$ the canonical functors associated to the universal cone.

Proposition 7.5.10. *The ∞-category $\mathrm{Sp}^{\dagger}_{\mathrm{gl}}$ is a Bousfield localization of $\mathrm{PSp}^{\dagger}_{\mathrm{gl}}$. We denote the associated left adjoint by $L_{\mathrm{gl}}\colon \mathrm{PSp}^{\dagger}_{\mathrm{gl}} \to \mathrm{Sp}^{\dagger}_{\mathrm{gl}}$. Furthermore, the following conditions are equivalent for an object $X \in \mathrm{PSp}^{\dagger}_{\mathrm{gl}}$:*

(a) X is in $\mathrm{Sp}^{\dagger}_{\mathrm{gl}}$;

(b) for every compact Lie group G, the G-prespectrum $U^{\mathrm{gl}}_G(X)$ is in Sp_G;

(c) for every compact Lie group G, the G-prespectrum $U^{\mathrm{gl}}_G X$ is local with respect to the maps $\lambda_{V,W}$ defined in Construction 7.2.28 for any G-representations V and W.

Proof. Recall that $\mathrm{Sp}_\bullet$ was constructed in Proposition 7.5.4 by localizing the functor $\mathrm{PSp}_\bullet$ using Lemma 6.3.13. Combining this with Lemma 7.5.9, we conclude that $\mathrm{Sp}^{\dagger}_{\mathrm{gl}}$ is a Bousfield localization and that conditions (a) and (b) are equivalent. By Proposition 7.2.30, condition (b) is equivalent to the condition that for every compact Lie group G and closed subgroup $H \leq G$, the H-prespectrum $\mathrm{res}^G_H U^{\mathrm{gl}}_G X$ is local with respect to the maps $\{\lambda_{V,W}\}$ where V and W over all H-representations. By construction we have $U^{\mathrm{gl}}_H = \mathrm{res}^G_H \circ U^{\mathrm{gl}}_G$ so (b) and (c) are equivalent. □

7.6 GLOBAL SPECTRA AS A PARTIALLY LAX LIMIT

Recall the two functors $\mathrm{PSp}_\bullet, \mathrm{Sp}_\bullet \colon \mathrm{Glo}^{\mathrm{op}} \to \mathrm{Cat}_\infty^\otimes$. These were constructed in Propositions 7.4.16 and 7.5.4 respectively. We also defined

$$\mathrm{PSp}_{\mathrm{gl}}^\dagger := \operatorname*{laxlim}^\dagger_{\mathrm{Glo}^{\mathrm{op}}} \mathrm{PSp}_G \quad \text{and} \quad \mathrm{Sp}_{\mathrm{gl}}^\dagger := \operatorname*{laxlim}^\dagger_{\mathrm{Glo}^{\mathrm{op}}} \mathrm{Sp}_G.$$

The goal of this section is to show that $\mathrm{Sp}_{\mathrm{gl}}^\dagger$ is symmetric monoidally equivalent to Schwede's ∞-category of global spectra $\mathrm{Sp}_{\mathrm{gl}}$, whose definition is recalled in Definition 7.2.23. Our proof will go roughly as follows:

- We will first construct a symmetric monoidal adjunction

  $$j_! \colon \mathrm{PSp}_{\mathrm{fgl}} \simeq \mathrm{Mod}_{\mathcal{S}_{\mathrm{fgl}}}(\mathbf{OR}_{\mathrm{fgl}}\text{–}\mathcal{S}_*) \leftrightarrows \mathrm{Mod}_{\mathcal{S}_{\mathrm{gl}}}(\mathbf{OR}_{\mathrm{gl}}\text{–}\mathcal{S}_*) \simeq \mathrm{PSp}_{\mathrm{gl}}^\dagger : j^*$$

  between prespectra objects, where the equivalences are given by Proposition 7.4.16 and Corollary 7.3.23.

- We note that there are Bousfield localizations $\mathrm{Sp}_{\mathrm{gl}} \subset \mathrm{PSp}_{\mathrm{fgl}}$ and $\mathrm{Sp}_{\mathrm{gl}}^\dagger \subset \mathrm{PSp}_{\mathrm{gl}}^\dagger$. We denote by $L_{\mathrm{gl}} \colon \mathrm{PSp}_{\mathrm{gl}}^\dagger \to \mathrm{Sp}_{\mathrm{gl}}^\dagger$ the localization functor.

- We will then check that j^* preserves spectrum objects, and therefore obtain an induced adjunction

  $$L_{gl} \circ j_! \colon \mathrm{Sp}_{\mathrm{gl}} \leftrightarrows \mathrm{Sp}_{\mathrm{gl}}^\dagger : j^*$$

  between the respective localizations.

- We will show that this adjunction is in fact an equivalence, by showing that j^* is conservative on spectrum objects, and that the unit of the adjunction $(L_{gl} \circ j_!, j^*)$ is an equivalence.

We start by constructing an adjunction between prespectrum objects. By Lemma 7.4.3 we can identify $\mathbf{OR}_{\mathrm{fgl}}$ with the full subcategory of $\mathbf{OR}_{\mathrm{gl}}$ spanned by (G, V), where V is a faithful G-representation. Then the canonical inclusion $j \colon \mathbf{OR}_{\mathrm{fgl}} \hookrightarrow \mathbf{OR}_{\mathrm{gl}}$ induces an adjunction

$$j_! \colon \mathbf{OR}_{\mathrm{fgl}}\text{–}\mathcal{S}_* \leftrightarrows \mathbf{OR}_{\mathrm{gl}}\text{–}\mathcal{S}_* : j^*.$$

Note that $j_!$ is fully faithful as it is given as a left Kan extension along a fully faithful functor. Moreover the functor $j_!$ is strong monoidal by Proposition 6.2.34.

Proposition 7.6.1. *The inclusion $j \colon \mathbf{OR}_{\mathrm{fgl}} \hookrightarrow \mathbf{OR}_{\mathrm{gl}}$ admits a right adjoint q, which is given on objects by*

$$(G, V) \mapsto (G/\ker(V), V),$$

where $\ker(V) < G$ is the subgroup of $g \in G$ acting trivially on V. In particular the left Kan extension $j_!$ is equivalent to the functor q^* given by precomposition by q.

Proof. The $G/\ker(V)$-representation V is clearly faithful, so to prove the thesis it is enough to show that for every $(H, W) \in \mathbf{OR}_{\mathrm{fgl}}$ the map $(G/\ker(V), V) \to (G, V)$ induces an equivalence on mapping spaces

$$\mathrm{Map}_{\mathbf{OR}_{\mathrm{gl}}}((H, W), (G/\ker V)) \xrightarrow{\sim} \mathrm{Map}_{\mathbf{OR}_{\mathrm{gl}}}((H, W), (G, V)) \ .$$

By Definition 7.4.2, this means we need to show that the map

$$(\mathrm{Hom}(G/\ker V, H) \times \mathcal{I}(W, V))^{G/\ker V}_{hH} \to (\mathrm{Hom}(G, H) \times \mathcal{I}(W, V))^{G}_{hH}$$

given by precomposition with $G \to G/\ker V$ on the first coordinate, is a homotopy equivalence. In fact we will show that

$$(\mathrm{Hom}(G/\ker V, H) \times \mathcal{I}(W, V))^{G/\ker V} \to (\mathrm{Hom}(G, H) \times \mathcal{I}(W, V))^{G}$$

is a homeomorphism. Since it is a continuous map of compact Hausdorff topological spaces, it suffices to show that it is bijective. As $\mathrm{Hom}(G/\ker V, H) \to \mathrm{Hom}(G, H)$ is injective, so is the above map. Therefore to conclude we need to show it is surjective.

Concretely this means that if we have a map $\alpha\colon G \to H$ and an isometry $\varphi\colon W \to V$ that is G-equivariant, we need to show that α is trivial when restricted to $\ker V$. But if $g \in \ker V$, then g acts as the identity on V, and therefore $\alpha(g)$ acts as the identity on W (since φ is G-equivariant). Since W is a faithful H-representation this implies that $\alpha(g) = 1$, as required. □

Note that it is clear from the definitions that $j^*S_{\mathrm{gl}} \simeq S_{\mathrm{fgl}}$ as commutative algebra objects. As an application of the previous proposition we find:

Corollary 7.6.2. *The counit map $\epsilon\colon j_!S_{\mathrm{fgl}} \to S_{\mathrm{gl}}$ is an equivalence of commutative algebra objects. In particular the functors $j_! \dashv j^*$ induce an adjunction*

$$j_!\colon \mathrm{PSp}_{\mathrm{fgl}} \simeq \mathrm{Mod}_{S_{\mathrm{fgl}}}(\mathbf{OR}_{\mathrm{fgl}}\text{–}\mathcal{S}_*) \leftrightarrows \mathrm{Mod}_{S_{\mathrm{gl}}}(\mathbf{OR}_{\mathrm{gl}}\text{–}\mathcal{S}_*) \simeq \mathrm{PSp}^{\dagger}_{\mathrm{gl}} : j^*$$

Proof. Because j is strong monoidal, the counit is canonically a map of commutative algebra objects. Therefore for all $(G, V) \in \mathbf{OR}_{\mathrm{gl}}$ we compute

$$j_!(S_{\mathrm{fgl}})(G, V) \simeq S_{\mathrm{fgl}}(q(G, V)) = (S^V)^{G/\ker(V)} \simeq (S^V)^G = S_{\mathrm{gl}}(G, V).$$

Because $j_!$ and j^* are strong and lax monoidal respectively and they swap the two algebra objects, they induce functors as in the statement which are evidently adjoint. □

We will now use the adjunction

$$j_!\colon \mathrm{PSp}_{\mathrm{fgl}} \leftrightarrows \mathrm{PSp}^{\dagger}_{\mathrm{gl}} : j^*$$

to induce an adjunction at the level of spectrum objects. To do this we need to see how the adjunction $(j_!, j^*)$ interacts with the full subcategories of spectrum objects. To this end we briefly rephrase the discussion of local objects in $\mathrm{PSp}_{\mathrm{fgl}}$ given at the end of Section 7.2.

Remark 7.6.3. Recall from Proposition 7.2.27 that $\mathrm{Sp}_{\mathrm{gl}}$ is a Bousfield localization of $\mathrm{PSp}_{\mathrm{fgl}}$ at the morphisms $\{\lambda_{G,V,W}\}$ where G is a compact Lie group and V and W are G-representations with W faithful. Because $j_!\colon \mathrm{PSp}_{\mathrm{fgl}} \to \mathrm{PSp}^{\dagger}_{\mathrm{gl}}$ is fully faithful, we can equivalently require that $j_!X$ is local with respect to the maps $j_!(\lambda_{G,V,W})$, where W is a faithful representation. These maps again corepresent the G-fixed points of the adjoint structure map $\tilde{\sigma}_{G,V,W}$, and therefore we will denote them by $\lambda^{\dagger}_{G,V,W}$, and similarly we will write $F^{\dagger}_{G,V}$ for $j_!F_{G,V}$.

We have seen in Construction 7.2.25 that for any compact Lie group G and G-representation V, there is a functor $\mathrm{ev}_{G,V}\colon \mathrm{PSp}_{\mathrm{fgl}} \to \mathcal{S}_{G,*}$ that sends a faithful global prespectrum X to the G-space $X(V)$. Under the equivalence

$$\mathrm{PSp}_{\mathrm{fgl}} \simeq \mathrm{Mod}_{\mathbb{S}_{\mathrm{fgl}}}(\mathbf{OR}_{\mathrm{fgl}}\text{–}\mathcal{S}_*)$$

this functor can be modelled as follows. Consider the cocartesian section $s_V\colon \mathbf{O}_G^{\mathrm{op}} \to \mathbf{OR}_G$ which is determined by the object $(G,V) \in \mathbf{OR}_G$ and write k_V for the composite $\mathbf{O}_G^{\mathrm{op}} \xrightarrow{s_V} \mathbf{OR}_G \xrightarrow{\nu_G} \mathbf{OR}_{\mathrm{gl}}$. If V is faithful then k_V lands in $\mathbf{OR}_{\mathrm{fgl}}$ and so we can define $\mathrm{ev}_{G,V}$ as the following composite of right adjoints

$$\mathrm{Mod}_{\mathbb{S}_{\mathrm{fgl}}}(\mathbf{OR}_{\mathrm{fgl}}\text{–}\mathcal{S}_*) \xrightarrow{\mathrm{fgt}} \mathbf{OR}_{\mathrm{fgl}}\text{–}\mathcal{S}_* \xrightarrow{k_V^*} \mathcal{S}_{G,*}.$$

Similarly, as discussed in Construction 7.2.28, there is a functor $\mathrm{ev}_V\colon \mathrm{PSp}_G \to \mathcal{S}_{G,*}$ sending a G-prespectrum X to the G-space $X(V)$. Under the equivalence

$$\mathrm{PSp}_G \simeq \mathrm{Mod}_{\mathbb{S}_G}(\mathbf{OR}_G\text{–}\mathcal{S}_*)$$

this functor is modelled by the composite

$$\mathrm{Mod}_{\mathbb{S}_G}(\mathbf{OR}_G\text{–}\mathcal{S}_*) \xrightarrow{\mathrm{fgt}} \mathbf{OR}_G\text{–}\mathcal{S}_* \xrightarrow{s_V^*} \mathcal{S}_{G,*}\,,$$

see also Remark 7.5.1.

Remark 7.6.4. From the previous discussion we conclude that there is a commutative diagram of right adjoints:

$$\begin{tikzcd}
\mathrm{PSp}_{\mathrm{fgl}} \arrow[d, "\sim"'] & & \mathrm{PSp}^{\dagger}_{\mathrm{gl}} \arrow[ll, "j^*"'] \arrow[rr, "U^{gl}_G"] \arrow[d, "\sim"] & & \mathrm{PSp}_G \arrow[d, "\sim"] \\
\mathrm{Mod}_{\mathcal{S}_{fgl}}(\mathbf{OR}_{\mathrm{fgl}}\text{–}\mathcal{S}_*) \arrow[d, "\mathrm{fgt}"'] & & \mathrm{Mod}_{\mathcal{S}^{\dagger}_{gl}}(\mathbf{OR}_{\mathrm{gl}}\text{–}\mathcal{S}_*) \arrow[ll, "j^*"'] \arrow[rr, "v^*_G"] \arrow[d, "\mathrm{fgt}"] & & \mathrm{Mod}_{\mathcal{S}_G}(\mathbf{OR}_G\text{–}\mathcal{S}_*) \arrow[d, "\mathrm{fgt}"] \\
\mathbf{OR}_{\mathrm{fgl}}\text{–}\mathcal{S}_* \arrow[drr, "k^*_W"'] & & \mathbf{OR}_{\mathrm{gl}}\text{–}\mathcal{S}_* \arrow[ll, "j^*"'] \arrow[rr, "v^*_G"] \arrow[d, "k^*_W"] & & \mathbf{OR}_G - \mathcal{S}_* \arrow[dll, "s^*_W"] \\
& & \mathcal{S}_{G,*} & &
\end{tikzcd}$$

.

Using that the corresponding diagram of left adjoints commute, we see that for all $X \in \mathrm{PSp}^{\dagger}_{\mathrm{gl}}$ and G-representations V and W with W faithful, the following diagram commutes

$$\begin{tikzcd}
\mathcal{S}_{G,*}(S^0, X(W)) \arrow[r, "\tilde{\sigma}_{V,W}"] & \mathcal{S}_{G,*}(S^V, X(V \oplus W)) \arrow[d, "\sim"] \\
\mathrm{PSp}_G(F_W S^0, U^{\mathrm{gl}}_G(X)) \arrow[u, "\sim"] \arrow[r, "\lambda^*_{V,W}"] & \mathrm{PSp}_G(F_{V\oplus W} S^V, U^{\mathrm{gl}}_G(X)) \arrow[d, "\sim"] \\
\mathrm{PSp}_{\mathrm{fgl}}(F_{G,W} S^0, j^*X) \arrow[u, "\sim"] \arrow[r, "\lambda^*_{G,V,W}"] & \mathrm{PSp}_{\mathrm{fgl}}(F_{G,V\oplus W} S^V, j^*X) \arrow[d, "\sim"'] \\
\mathrm{PSp}^{\dagger}_{\mathrm{gl}}(F^{\dagger}_{G,W} S^0, X) \arrow[u, "\sim"] \arrow[r, "(\lambda^{\dagger}_{G,V,W})^*"] & \mathrm{PSp}^{\dagger}_{\mathrm{gl}}(F^{\dagger}_{G,V\oplus W} S^V, X)
\end{tikzcd} \tag{7.6.4.1}$$

so all the various λ-maps correspond to each other under the various adjunctions.

Given any compact Lie group G and any faithful G-representation W, we define a functor

$$U^{\mathrm{fgl}}_{G,W} \colon \mathrm{PSp}_{\mathrm{fgl}} \to \mathrm{PSp}_G$$

as the composite

$$\mathrm{PSp}_{\mathrm{fgl}} \xrightarrow{j_!} \mathrm{PSp}^{\dagger}_{\mathrm{gl}} \xrightarrow{U^{\mathrm{gl}}_G} \mathrm{PSp}_G \xrightarrow{\mathrm{sh}_W} \mathrm{PSp}_G$$

where sh_W denotes the shift W-functor, given by cotensoring by $F_W S^0$.

Theorem 7.6.5. *An object $X \in \mathrm{PSp}_{\mathrm{fgl}}$ is in $\mathrm{Sp}_{\mathrm{gl}}$ if and only if for every compact Lie group G and faithful G-representation W, the object $U^{\mathrm{fgl}}_{G,W}(X)$ is in Sp_G. Furthermore, the functors $\{U^{\mathrm{fgl}}_{G,W}\}_{(G,W)}$ are also jointly conservative.*

Proof. By Remark 7.6.3, we know that $X \in \mathrm{PSp}_{\mathrm{fgl}}$ is in $\mathrm{Sp}_{\mathrm{gl}}$ if and only if $j_! X \in \mathrm{PSp}^{\dagger}_{\mathrm{gl}}$ is local with respect to the set of maps $\{\lambda^{\dagger}_{G,V,W}\}$ where G runs over all compact Lie groups and V and W are G-representations with W faithful. The commutative diagram (7.6.4.1) (together with the fact that $j^* j_! X \simeq X$) shows that this is equivalent to asking that for all compact Lie groups G, the object $U^{gl}_G(j_! X)$ is local with respect to $\{\lambda_{G,V,W}\}$ where V and W are as above. We next note that by definition, given an abitrary G prespectrum Y, the map

$$\lambda^*_{U,V}\colon \mathrm{PSp}_G(F_V S^0, \mathrm{sh}_W Y) \to \mathrm{PSp}_G(F_{U\oplus V} S^U, \mathrm{sh}_W Y)$$

is equivalent to $\lambda^*_{U,V\oplus W}$. Also recall that given a faithful G-representation W, $W \oplus U$ is also faithful for any G-representation U.

These two observations combine to imply that $U^{gl}_G(j_! X)$ is local with respect to $\{\lambda_{V,W}\}$ for G, V and W as above if and only if for all compact Lie groups G and faithful G-representations W, the object $\mathrm{sh}_W U^{gl}_G j_!(X) = U^{fgl}_{G,W} X$ is local with respect to $\{\lambda_{V,U}\}$ for arbitrary G-representations V and U.

On the other hand by Proposition 7.2.30, $U^{fgl}_{G,W} X$ is in Sp_G if and only if for all closed subgroups $H \leq G$, the H-prespectrum $\mathrm{res}^G_H U^{fgl}_{G,W} X = U^{fgl}_{H,\mathrm{res}^G_H W} X$ is local with respect to $\{\lambda_{V,U}\}$ for arbitrary H-representations V and U, and W a faithful G-representation. Varying these statements over all compact Lie groups, we find that $U^{\mathrm{fgl}}_{G,W} X$ is in Sp_G for all compact Lie groups G and all faithful G-representations W if and only if for all G and all faithful G-representations W, the G-prespectrum $U^{\mathrm{fgl}}_{G,W} X$ is $\{\lambda_{V,U}\}$-local for arbitrary G-representation V and U. This is identical to the condition of the previous paragraph, and so we obtain the first claim in the theorem. For the second statement, note that after forgetting module structures, the functor $U^{\mathrm{fgl}}_{G,W}$ is given by restriction along the functor

$$\mathrm{sh}_W\colon \mathbf{OR}_G \to \mathbf{OR}_{\mathrm{fgl}}, (G/H, U) \mapsto (H, U \oplus \mathrm{res}^G_H(W)).$$

The claim then follows from the fact that the functors $\{\mathrm{sh}_W\}_{(G,W)}$ where G runs over all compact Lie groups and W all faithful G-representations, are jointly essentially surjective. ☐

The following is the key fact about the right adjoint j^*.

Proposition 7.6.6. *Let G be a compact Lie group and let W be a faithful G-representation. Then the following square commutes:*

$$\begin{tikzcd}
\mathrm{PSp}_G \arrow[d, "\mathrm{sh}_W"'] & \mathrm{PSp}_{\mathrm{gl}}^{\dagger} \arrow[l, "U_G^{\mathrm{gl}}"'] \arrow[d, "j^*"] \\
\mathrm{PSp}_G & \mathrm{PSp}_{\mathrm{fgl}}. \arrow[l, "U_{G,W}^{\mathrm{fgl}}"']
\end{tikzcd}$$

Proof. The unit of the adjunction $j_! \dashv j^*$ provides a natural transformation

$$U_{G,W}^{\mathrm{fgl}} j^* = \mathrm{sh}_W U_G^{\mathrm{gl}} j_! j^* \to \mathrm{sh}_W U_G^{\mathrm{gl}}$$

which we claim is a natural equivalence. This follows from the fact that on underlying objects $\mathrm{sh}_W U_G^{gl}$ is given by restriction along the functor $\mathbf{OR}_G \to \mathbf{OR}_{\mathrm{gl}}, (H, V) \mapsto (H, \mathrm{res}_H^G(W) \oplus V)$. This only sees levels in the image of $\mathbf{OR}_{\mathrm{fgl}}$, where the unit is an equivalence. □

Corollary 7.6.7. *Suppose $X \in \mathrm{Sp}_{\mathrm{gl}}^{\dagger}$. Then $j^*(X) \in \mathrm{Sp}_{\mathrm{gl}}$. In particular we obtain a functor*

$$j^* : \mathrm{Sp}_{\mathrm{gl}}^{\dagger} \to \mathrm{Sp}_{\mathrm{gl}},$$

which admits a left adjoint given by $L_{\mathrm{gl}} \circ j_!$.

Proof. Because X is in $\mathrm{Sp}_{\mathrm{gl}}^{\dagger}$, we obtain that $U_G^{\mathrm{gl}}(X)$ is a G-spectrum by Proposition 7.5.10. Note that the functor sh_W preserves G-spectra for every G-representation W. We deduce using Proposition 7.6.6 that $U_{G,W}^{\mathrm{fgl}} j^*(X)$ is a G-spectrum for every G and W faithful. Therefore by Theorem 7.6.5 $j^*(X)$ is contained in $\mathrm{Sp}_{\mathrm{gl}}$. □

Proposition 7.6.8. *$j^* : \mathrm{Sp}_{\mathrm{gl}}^{\dagger} \to \mathrm{Sp}_{\mathrm{gl}}$ is conservative.*

Proof. Let $f : X \to Y$ be a map in $\mathrm{PSp}_{\mathrm{gl}}^{\dagger}$ such that $j^*(f)$ is an equivalence. This implies that $f_{(G,W)}$ is an equivalence of spaces for every faithful G-representation W. We finish the argument by proving that if f is in fact a map between objects in $\mathrm{Sp}_{\mathrm{gl}}^{\dagger}$, then $f_{(G,V)}$ is an equivalence for *every* G-representation V if and only if it is an equivalence for *faithful* G-representations. The forward direction is trivial. For the converse, note that because $\mathrm{PSp}_{\mathrm{gl}}^{\dagger}$ is a partially lax limit, the collection of functors $\{U_G^{\mathrm{gl}}\}_G$ is jointly conservative. Now our assumptions tell us that $U_G^{\mathrm{gl}}(f)_{(G,W)}$ is an equivalence for every faithful G-representation W. But because f is in fact in $\mathrm{Sp}_{\mathrm{gl}}^{\dagger}$, both the source and target of $U_G^{\mathrm{gl}}(f)$ are G-spectra. Therefore our claim reduces to the fact that a map between G-spectra which is an equivalence on faithful levels, is already an equivalence. The collection of faithful representations is cofinal in all representations, and so this is clear. □

Theorem 7.6.9. *The unit of the adjunction*

$$L_{\mathrm{gl}} \circ j_! \colon \mathrm{Sp}_{\mathrm{gl}} \leftrightarrows \mathrm{Sp}_{\mathrm{gl}}^{\dagger} : j^*$$

is an equivalence.

Proof. Consider $X \in \mathrm{Sp}_{\mathrm{gl}}$. Let $\eta_X \colon X \to j^* L_{\mathrm{gl}} j_! X$ be the unit of the adjunction $L_{\mathrm{gl}} \circ j_! \leftrightarrows j^*$ evaluated at X. This adjunction is given as a composite of two adjunctions and so the unit is given by the composite

$$X \xrightarrow{\eta'} j^* j_! X \xrightarrow{j^*(\gamma)} j^* L_{\mathrm{gl}} j_! X,$$

where η' is the unit of the adjunction $j_! \dashv j^*$ and γ exhibits $L_{\mathrm{gl}} j_! X$ as the localization of $j_! X$ in $\mathrm{PSp}_{\mathrm{gl}}^{\dagger}$. However recall that $j_!$ is fully faithful and therefore the first of the two maps is an equivalence. So it suffices to prove that the second map is also an equivalence.

The functors $U_{G,W}^{\mathrm{fgl}}$ are jointly conservative, and so it suffices to prove that $U_{G,W}^{\mathrm{fgl}}(j^*(\gamma))$ is an equivalence for every (G, W), where W is faithful. Applying Proposition 7.6.6 we conclude that $U_{G,W}^{\mathrm{fgl}}(j^*(\gamma))$ is equivalent to

$$\mathrm{sh}^W U_G^{\mathrm{gl}}(\gamma) : \mathrm{sh}^W U_G^{\mathrm{gl}} j_! X \to \mathrm{sh}^W U_G^{\mathrm{gl}} L_{\mathrm{gl}} j_! X.$$

By Proposition 7.5.10, $U_G^{\mathrm{gl}}(\gamma)$ is equivalent to

$$\gamma_G \colon U_G^{\mathrm{gl}} j_! X \to L_G U_G^{\mathrm{gl}} j_! X,$$

where γ_G exhibits $L_G U_G^{\mathrm{gl}} j_! X$ as the localization of $U_G^{\mathrm{gl}} j_! X$ in PSp_G. Spectrification of G-prespectra commutes with sh^W, and therefore $\mathrm{sh}^W(\gamma_G)$ gives the localization of $U_{G,W}^{\mathrm{fgl}}(X) = \mathrm{sh}^W U_G^{\mathrm{gl}} j_! X$ in PSp_G. Recall that $X \in \mathrm{Sp}_{\mathrm{gl}}$, and so $U_{G,W}^{\mathrm{fgl}}(X)$ is a G-Ω-spectrum by Theorem 7.6.5. Therefore $\mathrm{sh}^W(\gamma_G)$ is an equivalence, concluding the proof. □

Theorem 7.6.10. *There is a symmetric monoidal equivalence*

$$j^* \colon \mathrm{Sp}_{\mathrm{gl}}^{\dagger} := \underset{G}{\mathrm{laxlim}}^{\dagger}\, \mathrm{Sp}_G \to \mathrm{Sp}_{\mathrm{gl}}.$$

Proof. We have proven that $j_! \dashv j^*$ is an adjunction in which the right adjoint is conservative, and the unit is a natural equivalence. Therefore the functors are an adjoint equivalence. Moreover $j_!$ is strong monoidal, which implies that j^*, as its inverse, is also strong monoidal. □

7.7 PROPER EQUIVARIANT SPECTRA AS A LIMIT

The goal of this section is to exhibit the ∞-category of genuine proper G spectra Sp_G as a limit over the proper orbit category $\mathbf{O}_{G,\mathrm{pr}}^{\mathrm{op}}$ of a diagram

$$\mathrm{Sp}_{(-)}\colon \mathbf{O}_{G,\mathrm{pr}}^{\mathrm{op}} \to \mathrm{Cat}_\infty, \qquad G/H \to \mathrm{Sp}_H.$$

In contrast to the case of global spectra, once the diagram has been constructed, the identification of the limit will be almost immediate. In fact even the general strategy for constructing the diagram is essentially identical. For this reason we will be brief and refer to Section 7.4 for the relevant details.

Recall from Lemma 7.4.9 that the ∞-operad $\mathbf{OR}_G^{\otimes}$ fits into a pullback

$$\begin{tikzcd}
\mathbf{OR}_G^{\otimes} \arrow[r, "\nu_G"] \arrow[d, "\pi_G"'] & \mathbf{OR}_{gl}^{\otimes} \arrow[d, "\pi_{gl}"] \\
(\mathbf{O}_{G,\mathrm{pr}}^{\mathrm{op}})^{\amalg} \arrow[r, "\iota_G^{\amalg}"] & (\mathrm{Glo}^{\mathrm{op}})^{\amalg}.
\end{tikzcd}$$

Because $\mathbf{OR}_{gl}^{\otimes} \to (\mathrm{Glo}^{\mathrm{op}})^{\amalg}$ is a cocartesian fibration which by definition classifies the functor $\mathrm{Rep}(-)$, we immediately obtain:

Proposition 7.7.1. *For every Lie group G, the forgetful functor $\pi_G\colon \mathbf{OR}_G^{\otimes} \to (\mathbf{O}_{G,\mathrm{pr}}^{\mathrm{op}})^{\amalg}$ is a cocartesian fibration which classifies the functor*

$$\mathbf{O}_{G,\mathrm{pr}}^{\mathrm{op}} \to \mathrm{Cat}_\infty^{\otimes}, \quad G/H \mapsto \mathrm{Rep}(H).$$

Definition 7.7.2. We define $\widetilde{\mathbf{OR}}_G^{\otimes}$ via the following pullback of operads:

$$\begin{tikzcd}
\widetilde{\mathbf{OR}}_G^{\otimes} \arrow[r] \arrow[d, "\pi_{\mathrm{Ar}}"] & \mathbf{OR}_G^{\otimes} \arrow[d] \\
(\mathrm{Ar}(\mathbf{O}_{G,\mathrm{pr}})^{\mathrm{op}})^{\amalg} \arrow[r, "s^{\mathrm{op}}"] & (\mathbf{O}_{G,\mathrm{pr}}^{\mathrm{op}})^{\amalg}
\end{tikzcd}$$

We consider $\widetilde{\mathbf{OR}}_G^{\otimes}$ as living over $\mathbf{O}_{G,\mathrm{pr}}$ via the composite

$$\pi\colon \widetilde{\mathbf{OR}}_G^{\otimes} \xrightarrow{\pi_{\mathrm{Ar}}} (\mathrm{Ar}(\mathbf{O}_{G,\mathrm{pr}})^{\mathrm{op}})^{\amalg} \xrightarrow{t^{\mathrm{op}}} (\mathbf{O}_{G,\mathrm{pr}}^{\mathrm{op}})^{\amalg}.$$

Just as in Lemma 7.4.11, we can show that $\widetilde{\mathbf{OR}}_G^{\otimes}$ is a pro-$(\mathbf{O}_G)^{\amalg}$-monoidal category.

Proposition 7.7.3. *The functor $\pi\colon \widetilde{\mathbf{OR}}_G \to (\mathbf{O}_{G,\mathrm{pr}}^{\mathrm{op}})$, given by restricting π to underlying categories, is a cartesian fibration. Furthermore an edge $(f,g) \in \widetilde{\mathbf{OR}}_G$ is cartesian if and only if $s^{\mathrm{op}}(f)$ and g are equivalences.*

Proof. The proof is analogous to Lemma 7.4.13. □

Proposition 7.7.4. $\widetilde{\mathbf{OR}}_G^{\otimes} \times_{(\mathbf{O}_G^{\mathrm{op}})^{\amalg}} \{G/H\} \simeq \mathbf{OR}_H^{\otimes}$.

Proof. The pullback $P = \widetilde{\mathbf{OR}}_G^{\otimes} \times_{(\mathbf{O}_G^{\mathrm{op}})^{\amalg}} \{G/H\}$ fits into the following diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & \widetilde{\mathbf{OR}}_G \arrow[r] \arrow[d] & \mathbf{OR}_G^{\otimes} \arrow[r] \arrow[d] & \mathbf{OR}_{\mathrm{gl}}^{\otimes} \arrow[d] \\ (\mathbf{O}_H^{\mathrm{op}})^{\amalg} \arrow[r] \arrow[d] & (\mathrm{Ar}(\mathbf{O}_{G,\mathrm{pr}})^{\mathrm{op}})^{\amalg} \arrow[r] \arrow[d] & (\mathbf{O}_{G,\mathrm{pr}}^{\mathrm{op}})^{\amalg} \arrow[r] & (\mathrm{Glo}^{\mathrm{op}})^{\amalg} \\ \{G/H\} \arrow[r] & (\mathbf{O}_{G,\mathrm{pr}}^{\mathrm{op}})^{\amalg} & & \end{tikzcd}$$

in which every square is a pullback. One can show by direct computation that the middle composite $(\mathbf{O}_H^{\mathrm{op}})^{\amalg} \to (\mathrm{Glo}^{\mathrm{op}})^{\amalg}$ is equivalent to $\iota_H^{\amalg}$. Therefore the result follows from Lemma 7.4.9. □

Definition 7.7.5. Consider the Day convolution operad

$$\mathrm{Fun}_{\mathbf{O}_{G,\mathrm{pr}}}(\widetilde{\mathbf{OR}}_G^{\otimes}, \mathcal{S}_*^{\wedge} \times (\mathbf{O}_{G,\mathrm{pr}}^{\mathrm{op}})^{\amalg})^{Day}.$$

Just as in Section 7.4, this is an $(\mathbf{O}_{G,\mathrm{pr}}^{\mathrm{op}})^{\amalg}$-monoidal category. We define

$$\mathbf{OR}_{\bullet}\text{–}\mathcal{S}_* \colon \mathbf{O}_{G,\mathrm{pr}}^{\mathrm{op}} \to \mathrm{Cat}_{\infty}^{\otimes}$$

to be the functor associated to it by the equivalence of Proposition 6.4.5. By Proposition 7.7.4, the value of $\mathbf{OR}_{\bullet}\text{–}\mathcal{S}_*$ at G/H is equivalent to $\mathbf{OR}_H\text{–}\mathcal{S}_*$.

Proposition 7.7.6. *The projection map* $\widetilde{\mathbf{OR}}_G^{\otimes} \to \mathbf{OR}_G^{\otimes}$ *induces a fully faithful symmetric monoidal functor* $\mathbf{OR}_G\text{–}\mathcal{S}_* \to \widetilde{\mathbf{OR}}_G\text{–}\mathcal{S}_*$*, given by restriction. A functor* $F \colon \widetilde{\mathbf{OR}}_G \to \mathcal{S}_*$ *is in its essential image if and only if* F *sends* π*-cartesian edges to equivalences.*

Proof. The argument is identical to that of Lemma 7.4.14. □

Lemma 7.7.7. *There is a symmetric monoidal equivalence*

$$\lim_{\mathbf{O}_{G,\mathrm{pr}}^{\mathrm{op}}} \mathbf{OR}_{\bullet}\text{–}\mathcal{S}_* \simeq \mathbf{OR}_G\text{–}\mathcal{S}_*.$$

Proof. The calculation at the beginning of the proof of Lemma 7.4.15 shows that the lax limit of the diagram $\mathbf{OR}_{\bullet}\text{–}\mathcal{S}_*$ is equivalent to the symmetric monoidal category $\widetilde{\mathbf{OR}}_G\text{-}\mathcal{S}_*$ To compute the actual limit, we can once again argue on underlying categories by appealing to Remark 6.4.2. Note that by Remark 6.2.11, the underlying category of $\widetilde{\mathbf{OR}}_G\text{-}\mathcal{S}_*$ is equivalent to $\mathrm{Fun}(\widetilde{\mathbf{OR}}_G, \mathcal{S}_*)$. The analysis of the second half of the proof of Lemma 7.4.15 implies that the limit is equivalent to the full subcategory spanned by the functors which send π-cartesian edges to equivalences. By Proposition 7.7.6 this subcategory is equivalent to $\mathrm{Fun}(\mathbf{OR}_G, \mathcal{S}_*)$. □

Recall from Definition 7.2.18 that $\mathbf{OR}_G$-spaces admit an algebra object S_G, whose restriction to $\mathbf{OR}_H$-spaces for H a compact subgroup of G is equivalent to S_H.

Corollary 7.7.8. *There exists a functor* $\mathrm{PSp}_\bullet \colon \mathbf{O}_{G,\mathrm{pr}}^{\mathrm{op}} \to \mathrm{Cat}_\infty^\otimes$*, and one calculates*

$$\lim_{\mathbf{O}_{G,\mathrm{pr}}^{\mathrm{op}}} \mathrm{PSp}_\bullet \simeq \mathrm{Mod}_{S_G}(\mathbf{OR}_G\text{-}\mathcal{S})$$

Proof. Once again $\mathrm{PSp}_\bullet$ is defined as $\mathrm{Mod}_{S^\bullet}(\mathbf{OR}_\bullet\text{-}\mathcal{S}_*)$, using Theorem 6.4.10. An argument as in Proposition 7.4.16 allows us to calculate the limit. □

So far we have constructed and computed the limit of the diagram $\mathrm{PSp}_\bullet$. Given a map $\alpha\colon H \hookrightarrow K \subset G$ in $\mathbf{O}_{G,\mathrm{pr}}$, the induced map $\mathrm{PSp}_K \to \mathrm{PSp}_H$ is by construction equivalent to the global functoriality constructed in Section 7.4 evaluated at α. Therefore the results there imply that PSp_α preserves spectrum objects, and so we obtain a diagram $\mathrm{Sp}_\bullet \colon \mathbf{O}_{G,\mathrm{pr}}^{\mathrm{op}} \to \mathrm{Cat}_\infty^\otimes$. Furthermore, Corollary 7.5.6 implies that $\mathrm{Sp}_\alpha \colon \mathrm{Sp}_K \to \mathrm{Sp}_H$ agrees with the standard restriction functor between equivariant spectra. To calculate the limit of $\mathrm{Sp}_\bullet$, we apply Lemma 6.3.13 to conclude:

Corollary 7.7.9. $\lim_{\mathbf{O}_{G,\mathrm{pr}}^{\mathrm{op}}} \mathrm{Sp}_\bullet$ *is a Bousfield localization of* $\mathrm{Mod}_{S_G}(\mathbf{OR}_G\text{-}\mathcal{S}_*)$ *at the objects X whose restriction to* $\mathrm{Mod}_{S_H}(\mathbf{OR}_H\text{-}\mathcal{S}_*)$ *is an H-spectrum for every compact subgroup H of G.*

Recall from Section 7.3 that the category of genuine proper G-spectra is also a Bousfield localization of $\mathrm{Mod}_{S_G}(\mathbf{OR}_G\text{-}\mathcal{S}_*)$. Therefore it remains to show that the two subcategories agree.

Proposition 7.7.10. *An object* $X \in \mathrm{Mod}_{S_G}(\mathbf{OR}_G\text{-}\mathcal{S}_*)$ *is a G-spectrum if and only if for every compact subgroup $H \leq G$, the restriction of X to* $\mathrm{Mod}_{S_H}(\mathbf{OR}_H\text{-}\mathcal{S}_*)$ *is a H-spectrum.*

Proof. Recall from Proposition 7.2.30 that an object $X \in \mathrm{PSp}_G$ is a G-spectrum if and only if for all compact subgroups $H \leq G$, the object $\mathrm{res}^G_H X$ is local with respect to $\lambda_{H,V,W}$. Now by definition $\mathrm{res}^G_H X$ is a G-spectrum if and only if $\mathrm{res}^H_K \mathrm{res}^G_H X$ is local with respect to $\lambda_{K,V,W}$. However because $\mathrm{res}^H_K \mathrm{res}^G_H = \mathrm{res}^G_K$, we conclude that the two conditions of the theorem agree. □

Thus we can conclude the main theorem of this section:

Theorem 7.7.11. *The category of proper G-spectra is equivalent to the limit of the diagram* $\mathrm{Sp}_\bullet \colon \mathbf{O}_{G,\mathrm{pr}}^{\mathrm{op}} \to \mathrm{Cat}_\infty^\otimes$*, in symbols*

$$\mathrm{Sp}_G \simeq \lim_{\mathbf{O}_{G,\mathrm{pr}}^{\mathrm{op}}} \mathrm{Sp}_\bullet.$$

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