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Consensus income distribution

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Abstract

In determining the optimal redistribution of a given population's income, we ask which factor is more important: the social planner's aversion to inequality, embedded in an isoelastic social welfare function indexed by a parameter alpha, or the individuals' concern at having a low relative income, indexed by a parameter beta in a utility function that is a convex combination of (absolute) income and low relative income. Assuming that the redistribution comes at a cost (because only a fraction of a taxed income can be transferred), we find that there exists a critical level of beta below which different isoelastic social planners choose different optimal allocations of incomes. However, if beta is above that critical level, all isoelastic social planners choose the same allocation of incomes because they then find that an equal distribution of incomes maximizes social welfare regardless of the magnitude of alpha.

Keywords: Maximization of social welfare, Isoelastic social welfare functions, Deadweight loss of tax and transfer, Concern at having a low relative income, Social planners' aversion to inequality

JEL Codes: D31, D60, D63, H21, I38

1. Introduction

The fundamental tension between different social planners with regard to the income allocation rule under a deadweight loss of tax and transfer is easily understood, and has been alluded to for many years. For example, Tullock (1975) and Sen (1982) already grappled with the assumptions or conditions necessary to render equal division the optimal distributional rule for a given total income. However, neither of them enlisted individuals' concern at having a low relative income as a conciliator. In this note we bring together under the same isoelastic roof all the pivotal social planners, we incorporate a deadweight loss of tax and transfer, we display the received tension between the different social planners, and we ask what strength of the individuals' concern at having a low relative income will cause all the social planners to choose the same - equal - distribution of income.[1]

The class of isoelastic social welfare functions (Atkinson, 1970) enables us to represent the varying degrees of the social planners' aversion to inequality in the population's distribution of income as special cases. Due to its appealing axiomatic foundation and flexibility in embracing basic equality criteria,[2] the function has become a popular measure of social welfare in a variety of fields, ranging from optimal taxation (Atkinson and Stiglitz, 1976; Stern, 1976; Slemrod *et al*., 1994) to health economics (Abasolo and Tsuchiya, 2004, and references cited therein) and environmental economics (Shiell, 2003).

Our aim is to uncover a condition under which all the pivotal "isoelastic social planners" - a utilitarian, a Rawlsian, a Bernoulli-Nash, or any planner "in-between" - will come up with the same optimal income distribution when a tax and transfer procedure is subject to a deadweight loss. We obtain a strong congruence result: when the individuals' utility functions exhibit a sufficiently high concern at having a low relative income, the optimal tax policies of all the social planners align: this unanimity holds for the entire class of isoelastic social welfare functions with a parameter of inequality aversion, α, (defined in (1) below) spanning from zero (the case of a utilitarian social function) to infinity (the case of a

[1] Rich evidence from econometric studies, experimental economics, social psychology, and neuroscience confirms that individuals routinely engage in, and are affected by, interpersonal comparisons. In particular, people are dissatisfied when their consumption or income levels are lower than those of others who constitute their "comparison group." Studies that recognize such discontent are, among others, Stark and Taylor (1991), Zizzo and Oswald (2001), Luttmer (2005), Fliessbach *et al*. (2007), Blanchflower and Oswald (2008), Takahashi *et al*. (2009), Stark and Fan (2011), Stark and Hyll (2011), Fan and Stark (2011), Stark *et al*. (2012), and Card *et al*. (2012). Additionally, the comparisons that affect the sense of wellbeing significantly are those made by looking "up" the hierarchy, whereas the possibility that individuals derive satisfaction from looking "down" is not supported by studies of this subject. For example, Andolfatto (2002) demonstrates that individuals are adversely affected by the material wellbeing of others in their reference group when this wellbeing is far enough below theirs. See also Frey and Stutzer (2002) and Walker and Smith (2002) for a large body of evidence that supports the "upward comparison" hypothesis.

[2] The isoelastic social welfare function satisfies the criteria of unrestricted domain, independence of irrelevant alternatives, anonymity, separability, and the weak Pareto criterion (Roberts, 1980).

Rawlsian social function). We characterize the consensus optimal income distribution - which is a distribution of equal incomes - and we find that the intensity of the individuals' concern at having a low relative income crowds out the preferences over income distribution harbored by particular social planners. Moreover (and unless we begin with equal incomes), we identify the critical intensity of the individuals' concern at having a low relative income below which every isoelastic social planner other than the Rawlsian will choose a different, and particularly a non-equal, distribution. In other words, we formulate a necessary and sufficient condition for reconciliation of all the isoelastic social planners.

We proceed in two steps. First, we show that when the individuals' preferences do not exhibit a strong enough concern at having a low relative income, a deadweight loss of tax and transfer impedes equalization of incomes, and entails an optimal allocation in which, except for the Rawlsian social planner, all social planners end up with an *unequal* distribution of the available income. Second, we show that if the individuals' preferences exhibit a strong concern at having a low relative income, *any* isoelastic social planner who acknowledges this concern will end up equalizing incomes, no matter what is his degree of inequality aversion. Put more starkly, although the higher α, the more the isoelastic social welfare function tilts in favor of income equalization, a deadweight loss of tax and transfer interferes with this "proclivity" for any $\alpha < \infty$. Incorporation of the individuals' concern at having a low relative income restores the "power" (or reinvigorates the mandate) of the social welfare planner to equalize incomes.

Our analysis unravels an interesting distinction between the social planners' aversion to inequality (represented by the parameter α in the isoelastic social welfare function) and the individuals' concern at having a low relative income (represented by the parameter β in the individuals' utility functions). We find that when an "isoelastic social planner" faces a population characterized by an intensity of concern at having a low relative income that is higher than a critical value, the planner will choose to equalize incomes. There exists a critical level of the individuals' concern at having a low relative income that makes all the isoelastic social planners choose an equal income distribution, but there does not exist a (finite) level of the social planner's aversion to inequality that leads him to choose equal income distribution for any β. That is, the β preference of the individuals dominates the α taste of the social planner. Essentially, we show that inequality aversion formed "top-down" via the social welfare function is not a substitute for the intensity of a distaste percolating "bottom-up" from the preferences of the individuals.[3] In a sense, this finding is in conflict with intuition that has gained analytical support in the optimal taxation models of Stern

[3] Kaplow (2010) already noted asymmetry in the effect of the concavity of individuals' utility functions and the effect of the concavity of the social welfare function on the desirability of redistribution, finding that the former has more significant influence. In our model, a strong concern at having a low relative income embodied in the individuals' utility functions, namely, a high enough β, translates into a high marginal utility of individuals whose incomes are low, whereas the parameter α defines the concavity of the social welfare function.

(1976), Slemrod *et al*. (1994), and others, who show that embedding inequality aversion in the social welfare function suffices to render taxation more progressive, and the distribution of income more equal.

The plan of the remainder of this note is as follows. In Section 2 we introduce the class of isoelastic social welfare functions. In Section 3, which serves as a benchmark, we consider a population of two individuals. Under a deadweight loss of tax and transfer, we find that if the individuals' concern at having a low relative income, β, is not strong enough (in a sense made precise), then (i) the optimal income distributions chosen by different social planners differ from one another, depending on the social planners' parameter of inequality aversion, α; and (ii) only a Rawlsian social planner chooses an equal income distribution. In Section 4 we consider a population of any size. We show that there exists a critical level of the intensity of the concern at having a low relative income, β^*, which compels all isoelastic social planners to choose an equal income distribution, regardless of the value of their parameter of inequality aversion, α. Section 5 concludes.

2. The isoelastic social welfare function

Let there be a population of $n \geq 2$ individuals (where *n* is a natural number), and let the isoelastic social welfare function, *SWF*, be defined as

$$SWF_\alpha(\mathbf{x}) = \begin{cases} \left[\dfrac{1}{n} \displaystyle\sum_{i=1}^{n} u^{1-\alpha}(x_i, \mathbf{x}) \right]^{\frac{1}{1-\alpha}} & \text{for } \alpha \geq 0, \ \alpha \neq 1, \\ \sqrt[n]{\displaystyle\prod_{i=1}^{n} u^{1-\alpha}(x_i, \mathbf{x})} & \text{for } \alpha = 1, \end{cases} \tag{1}$$

where $\mathbf{x} = (x_1, \ldots, x_n)$, and $x_i \geq 0$ is the income of individual $i = 1, \ldots, n$; $u(x_i, \mathbf{x})$ is the utility function of individual *i*, which depends on the individual's income x_i and, possibly, on the incomes of other individuals in the population, such that $u(x_i, \mathbf{x}) > 0$ for any $\mathbf{x} \in \Omega$, where Ω is the set of possible income distributions; and $\alpha \in [0, \infty)$ is the social planner's parameter of inequality aversion.[4,5] The social welfare function in (1) is a discrete equally-distributed-equivalent utility, as is often defined in social choice theory.

As already noted in the Introduction, varying the parameter α allows us to represent the maximization problem of the social planners who differ from one another in the degree of their inequality aversion.

When $\alpha = 0$ (no preference for equality), we have that (1) reduces to

$$SWF_0(\mathbf{x}) = \frac{1}{n} \sum_{i=1}^{n} u(x_i, \mathbf{x}),$$

which is equivalent to a standard utilitarian social welfare function, $SWF_U(\mathbf{x}) = \sum_{i=1}^{n} u(x_i, \mathbf{x})$.

Using a well-known property of a generalized mean (Bullen, 2003, Theorem III.1.2), it follows that, when $\alpha \to 1$, we have that

[4] The literature offers several definitions or representations of an isoelastic social welfare function with $\alpha \geq 0$, $\alpha \neq 1$. Probably the one that is most commonly used is $S_\alpha(\mathbf{x}) = \frac{1}{1-\alpha} \sum_{i=1}^{n} u^{1-\alpha}(x_i, \mathbf{x})$. However, as noted by Iritani and Miyakawa (2002), this form of isoelastic function does not converge point-wise to a Rawlsian maximin social welfare function for $\alpha \to \infty$ (although the preference relations described by $S_\alpha(\cdot)$ converge to those of maximin function for $\alpha \to \infty$). For precision's sake, we resort throughout this note to the formulation of *SWF* given in (1), which has the advantages of converging to a Rawls maximin function for $\alpha \to \infty$ without additional transformations and, as a monotone transformation of $S_\alpha(\cdot)$, it is equivalent to $S_\alpha(\cdot)$ in terms of maxima for any $\alpha \geq 0$, $\alpha \neq 1$.

[5] The isoelastic function can be linked to the so-called Box-Cox transformation of population incomes, used often in statistics and in econometrics in order to render the data resemble a pattern akin to the normal distribution (Salas and Rodríguez, 2013).

$$\lim_{\alpha\to 1} SWF_{\alpha}(\mathbf{x}) = \sqrt[n]{\prod_{i=1}^{n} u(x_i, \mathbf{x})} = SWF_1(\mathbf{x}),$$

namely, the limit function for α approaching 1 is equivalent to the Bernoulli-Nash social welfare function, $SWF_{BN}(\mathbf{x}) = \prod_{i=1}^{n} u(x_i, \mathbf{x})$.

In addition, it is easy to show that when $\alpha \to \infty$, we have that

$$\lim_{\alpha\to\infty} SWF_{\alpha}(\mathbf{x}) = \min_{i\in\{1,\dots,n\}} \{u(x_i, \mathbf{x})\} = SWF_R(\mathbf{x}),$$

which implies that the Rawlsian maximin social welfare function $SWF_R(\mathbf{x})$ represents the extreme case of a social planner's inequality aversion.[6]

Thus, and as is already well known, by varying the coefficient $\alpha \geq 0$ and, additionally, by analyzing the limit case $\alpha \to \infty$, we can use the isoelastic social welfare function defined in (1) to represent the preferences of the most "prominent" social planners: utilitarian, Bernoulli-Nash, and Rawlsian.

We now formulate the social planner's optimization problem. Let the vector of initial incomes of the *n* individuals be $\mathbf{e} = (e_1, \dots, e_n)$ such that $0 < e_1 \leq \dots \leq e_n$. A social planner can transfer income from one individual to another in order to obtain what he considers to constitute the population's optimal income distribution. Let x_i denote the possible post-transfer (or post-tax) income of individual *i*, and let $t = \sum_{i=1}^{n} \max\{e_i - x_i, 0\}$ denote the total income that the social planner takes away from the individuals (henceforth "the tax"). Due to a deadweight loss of tax and transfer, only a fraction of the tax ends up being transferred.[7] We denote this fraction by $\lambda \in (0,1]$. Consequently, the set on which we search for the solution of the social planner's problem is

$$\Omega(\mathbf{e}, \lambda) = \left\{ \mathbf{x} = (x_1, \dots, x_n) : x_i \geq 0 \text{ for all } i, \text{ and } \lambda \sum_{i=1}^{n} \max\{e_i - x_i, 0\} = \sum_{i=1}^{n} \max\{x_i - e_i, 0\} \right\},$$

namely, we search over the set of incomes that can be attained from the initial allocation $\mathbf{e}$ by taxing some individuals; we thereby obtain the sum $t = \sum_{i=1}^{n} \max\{e_i - x_i, 0\}$; and we distribute λt between the remaining individuals such that the transfer amounts to $\lambda t = \sum_{i=1}^{n} \max\{x_i - e_i, 0\}$.

6 In what follows, we denote by $SWF_{\alpha}(\mathbf{x}) = SWF_R(\mathbf{x})$ the case when the social welfare function is Rawlsian, even though the parameter α is then not set.

7 The loss incurred in the course of the process of tax and transfer is in the spirit of Okun's (1975) concept of "leaky bucket."

Let the utility function of an individual whose income is x_i be

$$u(x_i, \mathbf{x}) \equiv (1-\beta) f(x_i) - \beta RI(x_i, \mathbf{x}) + E, \tag{2}$$

where $f(\cdot)$ is a strictly increasing, strictly concave function with $f(0) \geq 0$; $\beta \in [0,1)$ measures the intensity of the individual's distaste at having a low relative income, while the taste for having an (absolute) income is accorded the complementary weight $1-\beta$; RI is the index of low relative income, defined as

$$RI(x_i, \mathbf{x}) \equiv \frac{1}{n} \sum_{j=1}^{n} \max\{x_j - x_i, 0\},$$

namely, we operationalize the concern for low relative income by the index of relative deprivation;[8] and where *E* is a constant such that $E \geq \overline{E} \equiv \frac{1}{n}\sum_{k=1}^{n} e_k$, which we introduce in order to ensure that for each *i* and $\mathbf{x} \in \Omega(\mathbf{e}, \lambda)$ we will have that $u(x_i, \mathbf{x}) > 0$.[9]

[8] The index of relative deprivation, based on the seminal work of Runciman (1966), was proposed by Yitzhaki (1979), and axiomatized by Ebert and Moyes (2000) and Bossert and D'Ambrosio (2006). A detailed account of the background, rationale, and logic for this index is in Stark (2013). The index can be shown (see, for example, Stark, 2013) to be equal to the fraction of the individuals in the population whose incomes are higher than the income of the individual, times their mean excess income.

[9] From the definition of the set $\Omega(\mathbf{e}, \lambda)$ we have that $\sum_{i=1}^{n} e_i \geq \sum_{i=1}^{n} x_i$ for any $\mathbf{x} \in \Omega(\mathbf{e}, \lambda)$. Thus, $\beta RI(x_i, \mathbf{x}) = \beta \sum_{j=1}^{n} \max\{x_j - x_i, 0\} / n \leq \beta \sum_{j=1}^{n} x_j / n < \sum_{j=1}^{n} e_j / n = \overline{E} \leq E$ and, therefore, because $f(x_i) \geq 0$, $u(x_i, \mathbf{x}) > 0$ for any $i \in \{1, ..., n\}$ and $\mathbf{x} \in \Omega(\mathbf{e}, \lambda)$.

3. A population of two individuals

In this section we study a population of two individuals. We find that if the concern for a low relative income is not strong enough, an isoelastic social planner who faces a deadweight loss of tax and transfer will not, in general, choose to equalize incomes. The analysis of the case of two individuals serves as a foundation for analyzing the case of a population of any size, conducted in Section 4. The case of two individuals, and in particular the part leading to inequality (7) below, is significant in that it unravels the intuition of our main result, presented in the form of a proposition, in Section 4.

We consider a two-person population, $i=1,2$, in which individual $i=1$, the "poor," has an initial income e_1, and individual $i=2$, the "rich," has an initial income e_2, $0<e_1<e_2$.

Let there be an isoelastic social planner who can revise the prevailing income distribution by transferring an amount t from one individual to another, taking into account the deadweight loss as defined in Section 2. It is easy to verify that if we were to tax the income of the "poor" individual and transfer income to the "rich" individual, social welfare would decline. Thus, the only way in which the social planner could try to improve social welfare is to tax the "rich" individual, and make a transfer to the "poor" individual. In the case of two individuals, it is convenient to rewrite the utility levels in (2) as functions of the tax amount *t*, $v_i(t)$, $i=1,2$. The post-transfer utility of the "poor" individual is then

$$v_1(t) \equiv u\left(e_1+\lambda t,(e_1+\lambda t, e_2-t)\right) = (1-\beta) f(e_1+\lambda t) - \frac{\beta}{2}\max\{(e_2-t)-(e_1+\lambda t),0\}+E,$$

and the post-transfer utility of the "rich" individual is

$$v_2(t) \equiv u\left(e_2-t,(e_1+\lambda t, e_2-t)\right) = (1-\beta) f(e_2-t) - \frac{\beta}{2}\max\{(e_1+\lambda t)-(e_2-t),0\}+E.$$

Which *t* will an isoelastic social planner choose? To find out, we rewrite the social welfare function in (1) for a population of two individuals and for a transfer from the "rich" individual to the "poor" individual as

$$swf_\alpha(t) = \begin{cases} 2^{\frac{1}{\alpha-1}}\left[v_1^{1-\alpha}(t)+v_2^{1-\alpha}(t)\right]^{\frac{1}{1-\alpha}} & \text{for } \alpha \geq 0,\ \alpha \neq 1, \\ \sqrt{v_1(t)v_2(t)} & \text{for } \alpha = 1, \end{cases} \tag{3}$$

which is to be maximized over $t\in[0,e_2]$. We note that the amount *t* which equalizes the incomes of the two individuals is $\bar{t} = \dfrac{e_2-e_1}{1+\lambda}$, in which case the post-transfer income of each of the individuals' is $x^* = \dfrac{\lambda e_2+e_1}{1+\lambda}$. It is easy to verify that for $t\in(\bar{t},e_2)$ and any $\alpha \geq 0$ we have that $swf_\alpha{}'(t) < 0$ and, thus, the isoelastic social planner will surely not choose a tax level higher than $\bar{t}$.

For $t \in [0, \bar{t}]$, we have that $e_2 - t \geq e_1 + \lambda t$. Thus, while the "poor" individual experiences low relative income (that is, except when $t = \bar{t}$), the "rich" individual does not, so

$$\begin{aligned} v_1(t) &= (1-\beta) f(e_1 + \lambda t) - \frac{\beta}{2}\left[(e_2 - t) - (e_1 + \lambda t)\right] + E \\ &= (1-\beta) f(e_1 + \lambda t) - \frac{\beta}{2}\left[e_2 - e_1 - (1+\lambda)t\right] + E, \\ v_2(t) &= (1-\beta) f(e_2 - t) + E, \end{aligned}$$

and for any $t \in (0, \bar{t})$ and $\alpha \geq 0$, $\alpha \neq 1$ we have that

$$\begin{aligned} swf_\alpha'(t) &= 2^{\frac{1}{\alpha-1}}\left[v_1^{1-\alpha}(t) + v_2^{1-\alpha}(t)\right]^{\frac{\alpha}{1-\alpha}} \\ &\cdot \left[\frac{(1+\lambda)\beta/2 + \lambda(1-\beta) f'(e_1 + \lambda t)}{v_1^\alpha(t)} - \frac{(1-\beta) f'(e_2 - t)}{v_2^\alpha(t)}\right]. \end{aligned} \tag{4}$$

Because $2^{\frac{1}{\alpha-1}}\left[v_1^{1-\alpha}(t) + v_2^{1-\alpha}(t)\right]^{\frac{\alpha}{1-\alpha}} > 0$ for any $t \in (0, \bar{t})$, the sign of $swf_\alpha'(t)$ depends only on the sign of the term

$$s_\alpha(t) \equiv \frac{(1+\lambda)\beta/2 + \lambda(1-\beta) f'(e_1 + \lambda t)}{v_1^\alpha(t)} - \frac{(1-\beta) f'(e_2 - t)}{v_2^\alpha(t)}. \tag{5}$$

Strict concavity and monotonicity of $f(\cdot)$ imply that

$$\begin{aligned} s_\alpha'(t) &= \frac{v_1(t)\lambda^2(1-\beta) f''(e_1 + \lambda t) - \alpha\left[(1+\lambda)\beta/2 + \lambda(1-\beta) f'(e_1 + \lambda t)\right]^2}{v_1^{\alpha+1}(t)} \\ &+ \frac{v_2(t)(1-\beta) f''(e_2 - t) - \alpha\left[(1-\beta) f'(e_2 - t)\right]^2}{v_2^{\alpha+1}(t)} < 0 \end{aligned}$$

for any $t \in (0, \bar{t})$; namely, $s_\alpha(t)$ is a strictly decreasing function. Therefore, in order for $t = \bar{t}$ to constitute the optimal tax level, we must have that

$$\lim_{t \to \bar{t}} s_\alpha(t) \geq 0,$$

because this condition ensures that social welfare increases all the way up to $t = \bar{t}$. This limit condition is equivalent to requiring that

$$\frac{(1+\lambda)\beta/2 - (1-\lambda)(1-\beta) f'(x^*)}{v_1^\alpha(\bar{t})} \geq 0, \tag{6}$$

which, due to $v_1^\alpha(\bar{t}) > 0$, is equivalent to requiring that

$$\beta \geq \frac{(1-\lambda) f'(x^*)}{(1-\lambda) f'(x^*) + (1+\lambda)/2} \equiv \beta^*(\mathbf{e}, \lambda), \tag{7}$$

where, obviously, $0 \leq \beta^*(\mathbf{e}, \lambda) < 1$.

We see that for $\beta < \beta^*(\mathbf{e}, \lambda)$, the optimal tax lies somewhere in the range $t \in [0, \bar{t})$. Specifically, if $s_\alpha(0) \leq 0$, we have that $t = 0$, namely, the social planner will not choose to transfer any income, whereas if $s_\alpha(0) > 0$, the optimal level of the tax is given by setting (5) equal to zero:

$$\frac{(1+\lambda)\beta/2 + \lambda(1-\beta) f'(e_1 + \lambda t)}{v_1^\alpha(t)} = \frac{(1-\beta) f'(e_2 - t)}{v_2^\alpha(t)}. \tag{8}$$

For $\alpha = 1$, an analysis analogous to the one undertaken above shows that we get exactly the same condition as (7) for $t = \bar{t}$ to constitute the optimal level of the tax.

Two observations are worth making: (i) condition (6) is satisfied for any $\beta \geq 0$ if and only if $\lambda = 1$; that is, if and only if the transfer is perfectly costless; (ii) the critical level of β that is necessary for the equalization of incomes, $\beta^*(\mathbf{e}, \lambda)$ in (7), does not depend on the parameter α. We summarize the preceding analysis in the following lemma.

Lemma 1. In the case of two individuals with initial incomes $0 < e_1 < e_2$, an isoelastic social planner who is maximizing the function $swf_\alpha(t)$ over $t \in [0, e_2]$ with $\alpha \in [0, \infty)$, and who is facing a deadweight loss of tax and transfer $\lambda \in (0,1]$, will choose the optimal tax which equalizes incomes, $t^* = \bar{t}$, if and only if:

(a) $\lambda = 1$

or

(b) $\lambda < 1$ and $\beta \geq \beta^*(\mathbf{e}, \lambda)$.

Additionally, noting that the post-transfer level of income at the point of equality treated as a function of λ for given initial incomes $0 < e_1 < e_2$, namely, $x^*(\lambda) = \dfrac{\lambda e_2 + e_1}{1+\lambda}$, is increasing,[10] we obtain from (7) that

[10] We have that $\dfrac{dx^*(\lambda)}{d\lambda} = \dfrac{e_2 - e_1}{(1+\lambda)^2} > 0$.

$$\frac{d\beta^*(\mathbf{e},\lambda)}{d\lambda} = \frac{d}{d\lambda}\left[\frac{(1-\lambda)f'\left(x^*(\lambda)\right)}{(1-\lambda)f'\left(x^*(\lambda)\right)+(1+\lambda)/2}\right]$$

$$= \frac{2(1-\lambda^2)f''\left(x^*(\lambda)\right)\frac{dx^*(\lambda)}{d\lambda} - 4f'\left(x^*(\lambda)\right)}{\left[1+\lambda+2(1-\lambda)f'\left(x^*(\lambda)\right)\right]^2} < 0.$$

Thus, and aligning with intuition, the critical level $\beta^*(\mathbf{e},\lambda)$ increases when the deadweight loss becomes more onerous: when more is being lost in the tax and transfer procedure, following the procedure will be justified only if the weight attached to low relative income is higher.

Regarding the Rawlsian social welfare function, which is equivalent to the limit case of (3) with $\alpha \to \infty$, namely, to

$$swf_R(t) = \min\left\{v_1(t), v_2(t)\right\},$$

it is easy to see that as long as $e_1 + \lambda t \le e_2 - t,$ the Rawlsian social planner will find it optimal to increase the income of the "poor" individual at the expense of the "rich" individual, and that he will so act until reaching the point $e_1 + \lambda t = e_2 - t$, namely, until $t = \bar{t}$. Thereafter, transfers cannot anymore increase social welfare because they will render the "rich" "poor." We thus have the following lemma.

Lemma 2. In the case of two individuals, a Rawlsian social planner who is facing a deadweight loss of tax and transfer $\lambda \in (0,1]$ will choose to equalize incomes, that is, to set $t^{R*} = \bar{t}$ for any $\lambda \in (0,1]$ and any $\beta \in [0,1)$.

From a comparison of Lemma 1 with Lemma 2, we see that in the presence of a deadweight loss of tax and transfer, the optimal choices of isoelastic social planners (including a utilitarian social planner and a Bernoulli-Nash social planner) differ from the choice of a Rawlsian social planner; only the latter chooses to distribute incomes equally. Moreover, because the solution to (8) depends on the value of the parameter α, we get that the choices of isoelastic social planners for $\beta < \beta^*(\mathbf{e},\lambda)$ differ for different levels of inequality aversion. In Example 1, where we shorten the notation $\beta^*(\mathbf{e},\lambda)$ to β^*, we present this divergence diagrammatically for a chosen utility specification and specific values of the parameters.

Example 1. Consider the following preferences and initial endowments of the two individuals: $f(x) = \ln(x+1)$, $e_1 = 4$, $e_2 = 13$ and, $\lambda = 1/2$. Then $\bar{t} = 6$, and $\beta^* \approx 0.08$. In Figure 1 we depict the optimal tax t^* as a function of the social planner's inequality aversion parameter α for three different values of β: $0,\ 0.9\beta^*$, and β^*. As α increases, the optimal tax for $\beta = 0$ and for $\beta = 0.9\,\beta^*$ asymptotically approaches $\bar{t}$, although it does not reach $\bar{t}$ even for large values of α and for $\beta = 0.9\,\beta^*$. Other than the Rawlsian social planner, represented by the limit case $\alpha \to \infty$, no social planner chooses to equalize incomes. However, when $\beta = \beta^*$ (or, for that matter, when $\beta \geq \beta^*$) all the social planners, regardless of their α, pursue a redistribution policy that equalizes incomes.

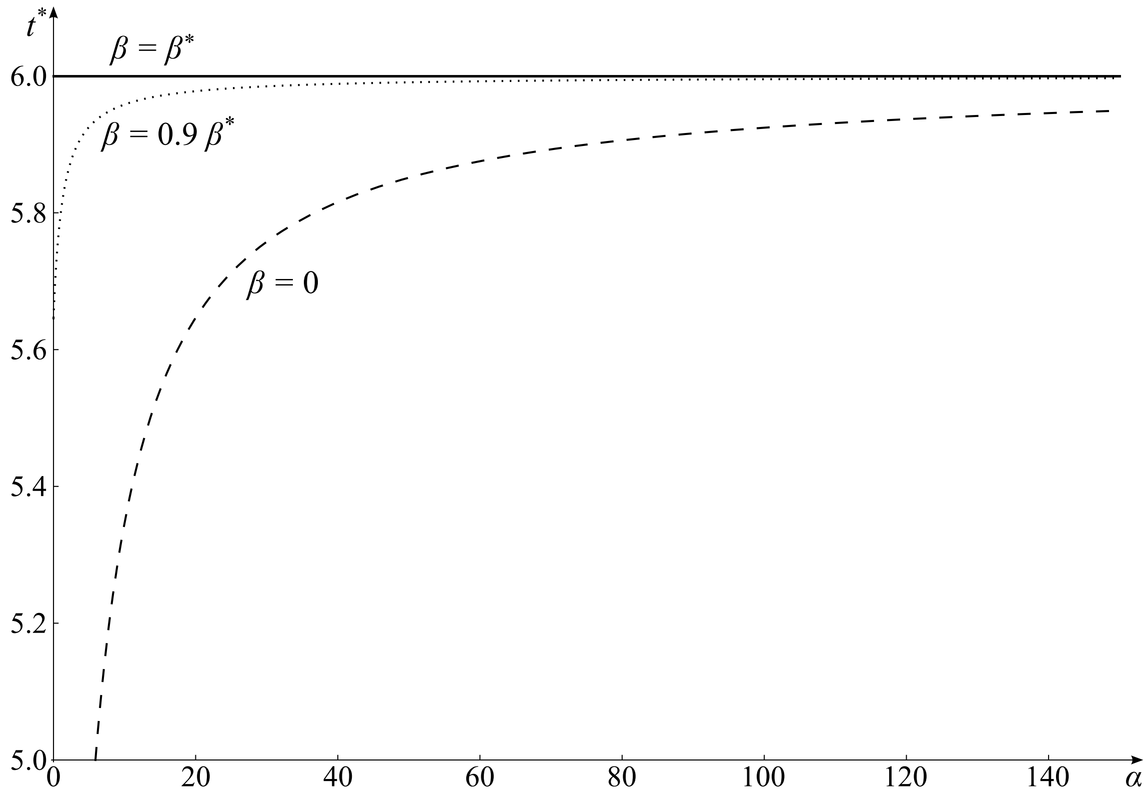

Figure 1: The optimal tax t^* as a function of α for three different values of β: $\beta = 0$ (dashed line), $\beta = 0.9\,\beta^*$ (dotted line), and $\beta = \beta^*$ (solid line), for $f(x_i) = \ln(x_i + 1)$, $i = 1,2$, $e_1 = 4$, $e_2 = 13$, and $\lambda = 1/2$.

In the case of a population of two individuals, we see that if an individual's concern at having a low relative income is not strong enough (in the sense of not exceeding the critical level β^*), then even a social planner's high level of aversion to inequality, α, does not bring about equalization of incomes. Drawing on this insight, in the next section we present this note's main result. We show that for a population of any size, the choices of all the social planners are identical when the individuals' concern at having a low relative income is strong enough, and that the congruence obtained is that all the social planners divide the available income equally. Moreover, acknowledgment by the social planners of the individuals' distaste for low relative income overrides the social planners' own preference for equality.

4. A population of any size

We next show that for a population of any size, the optimal choices of the isoelastic social planners (including the utilitarian and the Bernoulli-Nash) along with the optimal choice of a Rawlsian social planner are all the same if the individuals' concern at having a low relative income is acknowledged, and if this concern is strong enough.

As a preliminary, we establish that in the set $\Omega(\mathbf{e},\lambda)$ there exists a unique point of equal incomes.

Lemma 3. There exists a unique $(x_1,\ldots,x_n)\in\Omega(\mathbf{e},\lambda)$ such that $x_1=\ldots=x_n$.

Proof. Assuming that $e_1<e_n$,[11] we let

$$g(x)=\frac{\sum_{i=1}^{n}\max\{x-e_i,0\}}{\sum_{i=1}^{n}\max\{e_i-x,0\}}$$

for $x\in[e_1,e_n)$. Then, as a ratio of a continuous, strictly increasing, and positive function, and a continuous, strictly decreasing, and positive function, $g(\cdot)$ is a continuous and strictly increasing function, $g(e_1)=0,$ and $\lim_{x\to e_n} g(x)=\infty$. Thus, there exists a unique $x^*\in(e_1,e_n)$ such that $g(x^*)=\lambda$, which is the solution of the equation $\lambda\sum_{i=1}^{n}\max\{e_i-x,0\}$ $=\sum_{i=1}^{n}\max\{x-e_i,0\}$, and we set $x_1=\ldots=x_n=x^*$.

Henceforth, we will denote the unique point of equal incomes in $\Omega(\mathbf{e},\lambda)$, shown to exist in Lemma 3, by $\mathbf{x}^*=(x^*,\ldots,x^*)$.

The proposition that follows is this note's main result. It is helpful to highlight the essence of the proposition. First, if we begin with an income distribution in which all the incomes are equal, then no social planner chooses to interfere with the distribution. Second, if the social planner is a Rawlsian, then he always chooses to equalize the incomes. Third, there exists a critical level of the individuals' concern at having a low relative income which renders the choices of the utilitarian, the Bernoulli-Nash, and, for that matter, any isoelastic social planner perfectly congruent with the choice of the Rawlsian; namely, they all choose equal income distribution.

Proposition 1. Let the social welfare function $SWF_\alpha(\cdot)$ be defined on $\Omega(\mathbf{e},\lambda)$. Then:

[11] Otherwise, the distribution is equal at the outset, and any transfer permitted by the definition of the set $\Omega(\mathbf{e},\lambda)$ would make it unequal, and (as is easy to check) would decrease social welfare for any value of α.

(a) the solution of the social planner's problem is to divide incomes equally, namely,

$$\max_{\mathbf{x}\in\Omega(\mathbf{e};\lambda)} SWF_\alpha(\mathbf{x}) = SWF_\alpha(\mathbf{x}^*),$$

if and only if at least one of the following conditions holds:

(b) the incomes are equal to begin with, namely, $e_1 = ... = e_n = x^*$;

(c) the social planner is a Rawlsian, namely, $SWF_\alpha(\mathbf{x}) = SWF_R(\mathbf{x})$;

(d) $\beta \geq \beta^*(\mathbf{e},\lambda)$, where the critical level of the concern at having a low relative income, $\beta^*(\mathbf{e},\lambda)$, is such that $\beta^*(\mathbf{e},\lambda) \leq \frac{f'(x^*)(1-\lambda)}{f'(x^*)(1-\lambda)+\lambda+(1-\lambda)/n} < 1$ and it does not depend on $\alpha \in [0,\infty)$.

Proof. The proof is in the Appendix.

As long as $\lambda < 1$, the upper bound on $\beta^*(\mathbf{e},\lambda)$ in part (d) of Proposition 1 can be rewritten as

$$\frac{f'(x^*)}{f'(x^*)+\frac{\lambda}{1-\lambda}+\frac{1}{n}} < 1.$$

This formulation leads to two observations. First, because $\frac{\lambda}{1-\lambda}$ is increasing in λ, the smaller the leak incurred in the transfer, the smaller the upper bound on $\beta^*(\mathbf{e},\lambda)$; when the social planner sacrifices less income in the transfer process, then a smaller level of β suffices to entice him to equalize incomes. Moreover, when λ tends (from below) to 1, $\frac{\lambda}{1-\lambda}$ converges to infinity and, thus, $\beta^*(\mathbf{e},\lambda)$ converges to 0. This is in correspondence with the case of no leakage ($\lambda = 1$) in which independently of the magnitude of β all the isoelastic social planners choose an equal income distribution. Second, although the bigger the population the lower the upper bound on $\beta^*(\mathbf{e},\lambda)$, for a constant $\lambda \in (0,1)$ this bound is always (namely for any *n*) smaller than $\frac{f'(x^*)}{f'(x^*)+\lambda/(1-\lambda)}$ and, thus, for any population size this bound is essentially distanced from one.

5. Conclusion

We find that for the entire class of isoelastic social welfare functions, there exists a single critical level of intensity of the individuals’ concern at having a low relative income which leads to equal distribution of incomes being the optimum for any extent of the social planners’ inequality aversion.

The result that the critical level of intensity of the individuals’ concern at having a low relative income is the same for all levels of the isoelastic social planners’ parameter of inequality aversion questions the robustness of modeling the equality desired by the society by means of the parameter of the isoelastic social welfare function. In other words, it appears that the degree of the individuals’ concern at having a low relative income plays a distinct and more important role in the formation of the optimal redistribution policy than the intensity of the social planner’s inequality aversion. Further research on this issue will enrich our understanding of the role and relevance of the social planners’ distaste for inequality in shaping social preferences, and in guiding the search for a socially optimal income distribution.

Appendix

Proof of Proposition 1

We show that $\text{(a)} \Leftrightarrow (\text{(b)} \vee \text{(c)} \vee \text{(d)})$. We proceed in three steps. First, we remark that $\text{(b)} \Rightarrow \text{(a)}$. Second, we show that $\text{(c)} \Rightarrow \text{(a)}$. Third, we show that $(\neg\text{(b)} \wedge \neg\text{(c)}) \Rightarrow (\text{(d)} \Leftrightarrow \text{(a)})$.

Step 1. That $\text{(b)} \Rightarrow \text{(a)}$ is obvious.

Step 2. We next show that $\text{(c)} \Rightarrow \text{(a)}$, namely, that the unique solution to the Rawlsian social planner's problem,

$$\max_{\mathbf{x}\in\Omega(\mathbf{e},\lambda)} SWF_R(\mathbf{x}) = \max_{\mathbf{x}\in\Omega(\mathbf{e},\lambda)} \left\{\min\left\{u(x_1,\mathbf{x}),...,u(x_n,\mathbf{x})\right\}\right\}, \quad \text{(A1)}$$

is the equal income distribution $\mathbf{x}^* = (x^*,\ldots,x^*)$ for any $\beta \geq 0$. The proof is by contradiction. We assume that $\arg\max_{\mathbf{x}\in\Omega(\mathbf{e},\lambda)} SWF_R(\mathbf{x}) = \mathbf{z}$, where $\mathbf{z} = (z_1,...,z_n)$ is such that $\underline{z} = \min\{z_1,...,z_n\}$ $< \max\{z_1,...,z_n\}$, and we show that it is possible to construct a transfer from an individual with income higher than $\underline{z}$ to individual(s) with income $\underline{z}$ and obtain a $\mathbf{y}\in\Omega(\mathbf{e},\lambda)$ such that $SWF_R(\mathbf{y}) > SWF_R(\mathbf{z})$. Therefore, we will conclude that $\mathbf{z}$ cannot constitute a maximum.

Let $I_+ = \{i\in\{1,...,n\} : z_i = \underline{z} \wedge z_i \geq e_i\}$, $I_- = \{i\in\{1,...,n\} : z_i = \underline{z} \wedge z_i < e_i\}$, $\overline{z} = \min\{z_i : i \notin I_+ \cup I_-\}$, $k = \min\{i\in\{1,...,n\} : z_i = \overline{z}\}$, $J = I_+ \cup I_- \cup \{k\}$, and $h = |I_+ \cup I_-|$, where the notation $|A|$ stands for cardinality of the set *A*. Obviously, from the characteristics of the point $\mathbf{z}$, it follows that $I_+ \cup I_- \neq \varnothing$, and that $h \geq 1$. Let δ be such that $0 < \delta < \min\left\{\lambda(\overline{z} - \underline{z})/2, \min_{i\in J: e_i \neq z_i}\{|e_i - z_i|\}\right\}$. We define the coordinates of the point $\mathbf{y} = (y_1,\ldots,y_n)$ as

$$y_i = \begin{cases} z_i + \delta/h & \text{for } i\in I_+ \cup I_-, \\ z_i - \delta_k & \text{for } i = k, \\ z_i & \text{for } i\in\{1,\ldots,n\}\setminus J, \end{cases}$$

where $\delta_k = \delta(|I_+| + \lambda|I_-|)/(\lambda h)$ if $z_k \leq e_k$, and $\delta_k = \delta(|I_+| + \lambda|I_-|)/h$ otherwise. It is easy to verify that, indeed, $\mathbf{y}\in\Omega(\mathbf{e},\lambda)$.

We note that if $x_1 \leq ... \leq x_n$, then $f(x_1) \leq ... \leq f(x_n)$, and $RI(x_1,\mathbf{x}) \geq ... \geq RI(x_n,\mathbf{x})$. Therefore, $u(x_1,\mathbf{x}) \leq ... \leq u(x_n,\mathbf{x})$. Hence, for any $\mathbf{x} = (x_1,...,x_n)\in\Omega(\mathbf{e},\lambda)$ and any $k\in\{1,\ldots,n\}$ such that $x_k = \min\{x_1,...,x_n\}$, we have that $SWF_R(\mathbf{x}) = u_k(\mathbf{x})$. Because $f(\cdot)$ is an increasing function, and because a smaller difference between incomes implies a smaller value of the index of low relative income, it follows that for any $i \in I_+ \cup I_-$

$$\begin{aligned} & SWF_R(\mathbf{y}) - SWF_R(\mathbf{z}) \\ & = u(y_i, \mathbf{y}) - u(z_i, \mathbf{z}) \\ & = (1-\beta)\left[f(z_i + \delta / h) - f(z_i) \right] \\ & \quad -\beta\left[RI(z_i + \delta / h, \mathbf{y}) - RI(z_i, \mathbf{z}) \right] > 0 \end{aligned}$$

for any $\beta \in [0,1)$ and $0 < \lambda \leq 1$. Therefore, $SWF_R(\mathbf{y}) > SWF_R(\mathbf{z})$, which contradicts the presumption that $SWF_R(\cdot)$ attains a global maximum at $\mathbf{z}$. Additionally, because the function $SWF_R(\cdot)$ is continuous, it attains a global maximum on the compact set $\Omega(\mathbf{e}, \lambda)$. Thus, the solution of the problem of a Rawlsian social planner, (A1), has to be a transfer such that the post-transfer incomes are all equal and, as shown in Lemma 3, $\mathbf{x}^*$ is the unique point in $\Omega(\mathbf{e}, \lambda)$ such that all the incomes are equal. This completes the proof that $(\mathrm{c}) \Rightarrow (\mathrm{a})$ by contradiction.

Step 3. We next show that if neither (b) nor (c) holds, then the solution of the isoelastic social planner's maximization problem is an equal division of incomes if and only if (d) holds, that is, $(\neg(\mathrm{b}) \wedge \neg(\mathrm{c})) \Rightarrow ((\mathrm{d}) \Leftrightarrow (\mathrm{a}))$.

For the sake of comprehensibility, we begin this step by presenting a simple "organizational" remark.

Remark A1. Let $f : X \mapsto Y$, $X \subset \mathbf{R}^n$, $Y \subset \mathbf{R}$, and let $g : Y \mapsto \mathbf{R}$. We have that:

1. If g is increasing, then

$$\arg\max_{\mathbf{x} \in X} g(f(\mathbf{x})) = \arg\max_{\mathbf{x} \in X} f(\mathbf{x});$$

2. If g is decreasing, then

$$\arg\min_{\mathbf{x} \in X} g(f(\mathbf{x})) = \arg\max_{\mathbf{x} \in X} f(\mathbf{x})$$

and

$$\arg\max_{\mathbf{x} \in X} g(f(\mathbf{x})) = \arg\min_{\mathbf{x} \in X} f(\mathbf{x}).$$

Proof. The proof follows straightforwardly from the properties of the maxima and the minima of monotonous functions.

Proceeding with the proof of step 3 of the proposition, we assume that $e_1 < e_n$ (namely, $\neg(b)$) and that $\alpha \in [0, \infty)$ (namely, $\neg(c)$). We present a detailed proof for the case $\alpha \neq 1$; the proof for the case of the Bernoulli-Nash social planner ($\alpha = 1$) is analogous, and will be discussed briefly at the end.

For $\alpha \geq 0$, $\alpha \neq 1$, we proceed as follows. First, we will show that Remark A1 guarantees that the maximization problem of $SWF_\alpha(\cdot)$ on $\Omega(\mathbf{e},\lambda)$ is equivalent to the maximization problem of

$$F_\alpha(\mathbf{x}) = \sum_{i=1}^{n} \frac{u^{1-\alpha}(x_i,\mathbf{x})}{1-\alpha}$$

on $\Omega(\mathbf{e},\lambda)$. Second, we will show that for a sufficiently large β, namely, higher than or equal to a certain critical level denoted by $\beta^*(\mathbf{e},\lambda) < 1$, the point $\mathbf{x}^* = (x^*,\ldots,x^*) \in \Omega(\mathbf{e},\lambda)$ is a global maximum of $F_\alpha(\cdot)$ on $\Omega(\mathbf{e},\lambda)$. Therefore, $\mathbf{x}^* \in \Omega(\mathbf{e},\lambda)$ is also a global maximum of $SWF_\alpha(\cdot)$ on $\Omega(\mathbf{e},\lambda)$ for $\beta \geq \beta^*(\mathbf{e},\lambda)$ and any $\alpha \geq 0$, $\alpha \neq 1$, which yields the implication $(\mathrm{d}) \Rightarrow (\mathrm{a})$. We complete the proof by noting that if $\beta < \beta^*(\mathbf{e},\lambda)$, then the point $\mathbf{x}^*$ ceases to be an optimum of $SWF_\alpha(\cdot)$ on $\Omega(\mathbf{e},\lambda)$, which is equivalent to $(\mathrm{a}) \Rightarrow (\mathrm{d})$.

With $\alpha \geq 0$, $\alpha \neq 1$, we consider the functions $g_1(x) = x^{1-\alpha}$ and $g_2(x) = \dfrac{n}{1-\alpha}x$ for $x \geq 0$. Obviously, $g_2(g_1(SWF_\alpha(\mathbf{x}))) = F_\alpha(\mathbf{x})$. Because for $a < 1$ both $g_1(\cdot)$ and $g_2(\cdot)$ are increasing, whereas for $\alpha > 1$ both $g_1(\cdot)$ and $g_2(\cdot)$ are decreasing then, on applying Remark A1 twice, we get for any $X \subset \mathbf{R}^n_{\geq 0}$ that

$$\arg\max_{\mathbf{x}\in X} F_\alpha(\mathbf{x}) = \arg\max_{\mathbf{x}\in X} g_2(g_1(SWF_\alpha(\mathbf{x}))) = \arg\max_{\mathbf{x}\in X} g_1(SWF_\alpha(\mathbf{x})) = \arg\max_{\mathbf{x}\in X} SWF_\alpha(\mathbf{x})$$

when $a < 1$; and that

$$\arg\max_{\mathbf{x}\in X} F_\alpha(\mathbf{x}) = \arg\max_{\mathbf{x}\in X} g_2(g_1(SWF_\alpha(\mathbf{x}))) = \arg\min_{\mathbf{x}\in X} g_1(SWF_\alpha(\mathbf{x})) = \arg\max_{\mathbf{x}\in X} SWF_\alpha(\mathbf{x})$$

when $a > 1$. Thus, the global maxima on any $X \subset \mathbf{R}^n_{\geq 0}$ of $F_\alpha(\cdot)$ and $SWF_\alpha(\cdot)$ coincide, namely,

$$\arg\max_{\mathbf{x}\in X} F_\alpha(\mathbf{x}) = \arg\max_{\mathbf{x}\in X} SWF_\alpha(\mathbf{x}).$$

Our next task is to find the global maxima of $F_\alpha(\cdot)$ on $\Omega(\mathbf{e},\lambda)$.

To this end, we show that for a sufficiently large β, $\mathbf{x}^* = (x^*,\ldots,x^*)$ is the maximum of $F_\alpha(\cdot)$ on $\Omega(\mathbf{e},\lambda)$. We start with yet another remark.

Remark A2. A local maximum of $F_\alpha(\cdot)$ on $\Omega(\mathbf{e},\lambda)$ is a global maximum on $\Omega(\mathbf{e},\lambda)$.

Proof. Let

$$\Pi(\mathbf{e},\lambda) = \left\{ \mathbf{x} = (x_1,\ldots,x_n) : x_i \geq 0 \text{ for all } i, \text{ and } \lambda \sum_{i=1}^{n} \max\{e_i - x_i, 0\} \geq \sum_{i=1}^{n} \max\{x_i - e_i, 0\} \right\}.$$

We are interested in the maximization problem $\max_{\mathbf{x}\in\Pi(\mathbf{e},\lambda)} F_\alpha(\mathbf{x})$. Obviously, $\Omega(\mathbf{e},\lambda)\subset\Pi(\mathbf{e},\lambda)$. We seek to show that if $\mathbf{y}^*$ is a maximum of $F_\alpha(\cdot)$ on $\Pi(\mathbf{e},\lambda)$, then we must have that $\mathbf{y}^*\in\Omega(\mathbf{e},\lambda)$.

To achieve this, we first show that for every $\mathbf{x}\in\Pi(\mathbf{e},\lambda)\setminus\Omega(\mathbf{e},\lambda)$ there exists $\varepsilon>0$ such that for $\mathbf{x}_\varepsilon\equiv(x_1+\varepsilon/n,\ldots,x_n+\varepsilon/n)$ we have that $\mathbf{x}_\varepsilon\in\Pi(\mathbf{e},\lambda)$ and that $F_\alpha(\mathbf{x}_\varepsilon)>F_\alpha(\mathbf{x})$.

We fix $\mathbf{x}\in\Pi(\mathbf{e},\lambda)\setminus\Omega(\mathbf{e},\lambda)$. Because $\mathbf{x}$ does not belong to $\Omega(\mathbf{e},\lambda)$, we know that

$$\lambda\sum\max\{e_i-x_i,0\}>\sum\max\{x_i-e_i,0\}.$$

We take $\varepsilon<\mathrm{dist}(\mathbf{x},\Omega(\mathbf{e},\lambda))$, where $\mathrm{dist}(\mathbf{x},X)$ is the distance between point $\mathbf{x}$ and the set X.[12] We define

$$\mathbf{x}_\varepsilon=(x_{\varepsilon,1},\ldots,x_{\varepsilon,n})=(x_1+\varepsilon/n,\ldots,x_n+\varepsilon/n).$$

Because $\varepsilon<\mathrm{dist}(\mathbf{x},\Omega(\mathbf{e},\lambda))$, we know that

$$\lambda\sum\max\{e_i-x_{\varepsilon,i},0\}>\sum\max\{x_{\varepsilon,i}-e_i,0\}$$

holds. Thus, $\mathbf{x}_\varepsilon\in\Pi(\mathbf{e},\lambda)$. Moreover, because the relative incomes did not change (RI is translation invariant) and f is an increasing function, the utility levels of all the individuals will increase. Thus, $F_\alpha(\mathbf{x}_\varepsilon)>F_\alpha(\mathbf{x})$. This completes the proof that for every $\mathbf{x}\in\Pi(\mathbf{e},\lambda)\setminus\Omega(\mathbf{e},\lambda)$ there exists $\varepsilon>0$ such that $\mathbf{x}_\varepsilon\in\Pi(\mathbf{e},\lambda)$, and that $F_\alpha(\mathbf{x}_\varepsilon)>F_\alpha(\mathbf{x})$.

This last result implies that $F_\alpha(\cdot)$ cannot have a maximum on $\Pi(\mathbf{e},\lambda)\setminus\Omega(\mathbf{e},\lambda)$ because for any point $\mathbf{x}\in\Pi(\mathbf{e},\lambda)\setminus\Omega(\mathbf{e},\lambda)$ in its every neighborhood in $\Pi(\mathbf{e},\lambda)$, there exists a point at which the function $F_\alpha(\cdot)$ attains a higher value. On the other hand, because $F_\alpha(\cdot)$ is a strictly concave function - it is the sum of strictly concave functions $u(\cdot)$ raised to the power $1-a$ and divided by $1-a$ - maximized on a closed subset $\Pi(\mathbf{e},\lambda)$ characterized by a concave constraint function,[13] then, if a local maximum on $\Pi(\mathbf{e},\lambda)$ exists, then that maximum is also a global maximum on $\Pi(\mathbf{e},\lambda)$. This implies that the global maximum of

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[12] $\mathrm{dist}(\mathbf{x},X)=\inf_{\mathbf{y}\in X} d(\mathbf{x},\mathbf{y})$ where $d(\cdot,\cdot)$ is the Euclidean distance between two points.

[13] In other words, the problem

$$\min_{\mathbf{x}\in\mathbf{R}^n_{\geq 0}}\left\{-F_\alpha(\mathbf{x}):-g(\mathbf{x})\leq 0\right\},$$

where $g(\mathbf{x})=\lambda\sum_{i=1}^n\max\{e_i-x_i,0\}-\sum_{i=1}^n\max\{x_i-e_i,0\}$, is a convex optimization problem because as a sum of concave functions of the form $\xi_i(x_i)=\lambda\max\{e_i-x_i,0\}-\max\{x_i-e_i,0\}$, the function $g(\cdot)$ is concave.

$F_\alpha(\cdot)$ on $\Pi(\mathbf{e},\lambda)$ has to be realized on $\Omega(\mathbf{e},\lambda)$ and, thus, any local maximum of $F_\alpha(\cdot)$ on $\Omega(\mathbf{e},\lambda)$ is also a global maximum. This completes the proof of Remark A2.

Continuing with the proof of step 3 of Proposition 1, we assume that the point $\mathbf{x}^*$ was obtained by means of redistribution, that is, by means of a transfer enacted to obtain the allocation $\mathbf{x}^*$ from the initial allocation of incomes $\mathbf{e}$, as dictated by the constraints defining the set $\Omega(\mathbf{e},\lambda)$. We refer to this initial redistribution as the $\mathbf{e}\to\mathbf{x}^*$ transfer. Then, starting from $\mathbf{x}^*$, we consider a second redistribution, namely, a marginal transfer of $t>0$ between the individuals, subject to the conditions of the set $\Omega(\mathbf{e},\lambda)$, by which an allocation $\mathbf{y}\in\Omega(\mathbf{e},\lambda)$ is to be obtained. We refer to this second redistribution as the $\mathbf{x}^*\to\mathbf{y}$ transfer. The interdependence between the initial incomes $(e_1,\ldots,e_n)$ and x^* mandates division of $I=\{1,\ldots,n\}$ into three pairwise disjoint sets:

$$I_+=\{i\in I: e_i<x^*\};$$
$$I_0=\{i\in I: e_i=x^*\};$$
$$I_-=\{i\in I: e_i>x^*\}.$$

In the case of individual *i* who received income in the $\mathbf{e}\to\mathbf{x}^*$ transfer, that is, for $i\in I_+$, in the $\mathbf{x}^*\to\mathbf{y}$ transfer we can give to this individual either a share, denoted by $\tau_i\in[0,1]$, of the amount *t* (that which we can give is reduced by the deadweight loss), and increase his income by $\tau_i\lambda t$, or we can take from him a share, denoted by $\varphi_i\in[0,1]$, of the amount *t* (also reduced by the deadweight loss), namely, take back part of what he gained, in which case his income will be reduced by $\varphi_i\lambda t$. The set of the individuals from I_+ who receive a share will be denoted by $I_{+,+}$, and the set of the individuals from whom we take a share will be denoted by $I_{+,-}$, with $I_{+,+}\cap I_{+,-}=\varnothing$ and $I_+=I_{+,+}\cup I_{+,-}$.

In the case of an individual from whom income was taken in the $\mathbf{e}\to\mathbf{x}^*$ transfer, that is for $i\in I_-$, in the $\mathbf{x}^*\to\mathbf{y}$ transfer we can either give him a share, denoted by $\omega_i\in[0,1]$, of the amount *t*, which represents giving back part of what was taken from him, and thus his income will be raised by $\omega_i t$, or take from this individual a share of the amount *t*, denoted by $\nu_i\in[0,1]$, and decrease his income by $\nu_i t$. Similarly as for I_+, this procedure imposes division of a set I_- into two pairwise disjoint subsets $I_{-,+}$ and $I_{-,-}$.

Completing the mapping out of the possibilities, in the case of an individual whose income did not change ($i\in I_0$) in the $\mathbf{e}\to\mathbf{x}^*$ transfer, in the $\mathbf{x}^*\to\mathbf{y}$ transfer we can either give this individual a share, denoted by $\mu_i \in [0,1]$, of the amount *t* (that which we can give is reduced by the deadweight loss), in which case his income will be raised by $\mu_i\lambda t$, or we can take away a share, denoted by $\rho_i \in [0,1]$, in which case his income will be lowered by $\rho_i t$. These transfers define the division of I_0 into pairwise disjoint sets $I_{0,+}$ and $I_{0,-}$.

Obviously, we have that $I = I_{+,+} \cup I_{+,-} \cup I_{0,+} \cup I_{0,-} \cup I_{-,+} \cup I_{-,-}$. Thus, the coordinates of the point $\mathbf{y} = (y_1, ..., y_n)$ which is obtained by any given marginal transfer that starts from $\mathbf{x}^*$ and that does not violate the conditions of the set $\Omega(\mathbf{e}, \lambda)$, are characterized by

$$y_i = \begin{cases} x^* + \tau_i \lambda t & \text{for } i \in I_{+,+}; \\ x^* - \varphi_i \lambda t & \text{for } i \in I_{+,-}; \\ x^* + \mu_i \lambda t & \text{for } i \in I_{0,+}; \\ x^* - \rho_i t & \text{for } i \in I_{0,-}; \\ x^* + \omega_i t & \text{for } i \in I_{-,+}; \\ x^* - \nu_i t & \text{for } i \in I_{-,-}, \end{cases} \tag{A2}$$

where $t > 0$ is small enough so that the incomes y_i still satisfy inequalities that are analogous to the ones that define the sets $I_{+,-}$ and $I_{-,+}$; that is, $x^* - \varphi_i \lambda t > e_i$ for $i \in I_{+,-}$, and $e_j > x^* + \omega_j t$ for $j \in I_{+,-}$.

Because the $\mathbf{x}^* \to \mathbf{y}$ transfer characterized above must not violate the conditions of the set $\Omega(\mathbf{e}, \lambda)$, we have that

$$\lambda \sum_{i \in I_{-,-}} \left[e_i - (x^* - \nu_i t) \right] + \lambda \sum_{i \in I_{-,+}} \left[e_i - (x^* + \omega_i t) \right] + \lambda \sum_{i \in I_{0,-}} \left[e_i - (x^* - \rho_i t) \right]$$
$$= \sum_{i \in I_{+,+}} \left[(x^* + \tau_i \lambda t) - e_i \right] + \sum_{i \in I_{+,-}} \left[(x^* - \varphi_i \lambda t) - e_i \right] + \sum_{i \in I_{0,+}} \left[(x^* + \mu_i \lambda t) - e_i \right],$$

which, from $\lambda \sum_{i=1}^{n} \max\{e_i - x^*, 0\} = \sum_{i=1}^{n} \max\{x^* - e_i, 0\}$ and the characterization of I, reduces to

$$T \equiv \sum_{i \in I_{-,-}} \nu_i - \sum_{i \in I_{-,+}} \omega_i + \sum_{i \in I_{0,-}} \rho_i = \sum_{i \in I_{+,+}} \tau_i - \sum_{i \in I_{+,-}} \varphi_i + \sum_{i \in I_{0,+}} \mu_i \,. \tag{A3}$$

Let

$$\nu = \sum_{i \in I_{-,-}} \nu_i,\ \omega = \sum_{i \in I_{-,+}} \omega_i,\ \rho = \sum_{i \in I_{0,-}} \rho_i,\ \tau = \sum_{i \in I_{+,+}} \tau_i,\ \varphi = \sum_{i \in I_{+,-}} \varphi_i,\ \mu = \sum_{i \in I_{0,+}} \mu_i\,, \tag{A4}$$

so that condition (A3) simplifies to

$$T = \nu - \omega + \rho = \tau - \varphi + \mu.$$

The $\mathbf{x}^* \to \mathbf{y}$ transfer yields the following change in social welfare:

$$\eta_\alpha(t) \equiv F_\alpha(\mathbf{y}) - F_\alpha(\mathbf{x}^*)$$

$$
\begin{aligned}
&= \frac{1}{1-\alpha}\Bigg\{ \sum_{i\in I_{s,p}} \Big[(1-\beta)\, f(x^* + \lambda\tau_i t) - \beta RI(x^* + \lambda\tau_i t, y) + E \Big]^{1-\alpha} \\
&+ \sum_{i\in I_{-,-}} \Big[(1-\beta)\, f(x^* - \lambda\varphi_i t) - \beta RI(x^* - \lambda\varphi_i t, y) + E \Big]^{1-\alpha} \\
&+ \sum_{i\in I_{0,+}} \Big[(1-\beta)\, f(x^* + \lambda\mu_i t) - \beta RI(x^* + \lambda\mu_i t, y) + E \Big]^{1-\alpha} \\
&+ \sum_{i\in I_{0,-}} \Big[(1-\beta)\, f(x^* - \rho_i t) - \beta RI(x^* - \rho_i t, y) + E \Big]^{1-\alpha} \\
&+ \sum_{i\in I_{+,+}} \Big[(1-\beta)\, f(x^* + \omega_i t) - \beta RI(x^* + \omega_i t, y) + E \Big]^{1-\alpha} \\
&+ \sum_{i\in I_{+,-}} \Big[(1-\beta)\, f(x^* - \nu_i t) - \beta RI(x^* - \nu_i t, y) + E \Big]^{1-\alpha} \\
&- n \cdot \Big[(1-\beta)\, f(x^*) + E \Big]^{1-\alpha} \Bigg\}.
\end{aligned}
\tag{A5}
$$

Taking the right-hand derivative of $\eta_\alpha(t)$ and evaluating it at $t = 0$ yields

$$
\begin{aligned}
&\Big[(1-\beta)\, f(x^*) + E\Big]^{\alpha} \eta'_{\alpha+}(0) \\
&= (1-\beta)\, f'(x^*) \left[\lambda \left(\sum_{i\in I_{+,+}} \tau_i - \sum_{i\in I_{+,-}} \varphi_i + \sum_{i\in I_{0,-}} \mu_i \right) - \left(\sum_{i\in I_{+,-}} \nu_i - \sum_{i\in I_{-,+}} \omega_i + \sum_{i\in I_{0,-}} \rho_i \right) \right] \\
&- \beta \lim_{t\to 0^+} \frac{d}{dt} \sum_{i=1}^{n} RI(y_i, \mathbf{y}) \qquad\qquad (A6) \\
&= (1-\beta)\, f'(x^*)(\lambda T - T) - \beta \lim_{t\to 0^+} \frac{d}{dt} \sum_{i=1}^{n} RI(y_i, \mathbf{y}) \\
&= (1-\beta)\, f'(x^*)(\lambda - 1) T - \beta \lim_{t\to 0^+} \frac{d}{dt} \sum_{i=1}^{n} RI(y_i, \mathbf{y}).
\end{aligned}
$$

Because for a given set of the weights $\nu_i, \varphi_i, \rho_i, \tau_i, \omega_i, \mu_i$ the $RI(y_i, \mathbf{y})$ function is linear, we can simplify the notation, drawing on the fact that

$$\lim_{t\to 0^+} \frac{d}{dt} RI(y_i, \mathbf{y}) = RI(y_i, \mathbf{y})\big|_{t=1}.$$

Using the definition of the change in social welfare, $\eta_\alpha(t)$ in (A5), we first note that in the case $T \geq 0$, we have that $\eta'_{\alpha+}(0) \leq 0$ for any $\beta \geq 0$ and, thus, we construct the $\mathbf{x}^* \to \mathbf{y}$ transfer such that $T < 0$. We note that for any non-equal initial allocation of incomes, that is, for any $\mathbf{e} = (e_1, \ldots, e_n)$ such that $e_1 < e_n$, we can construct an $\mathbf{x}^* \to \mathbf{y}$ transfer such that $T < 0$: from Lemma 3 we infer that we must have that $e_1 < x^* < e_n$ and, thus, we can take $I_{+,-} = \{1\}$, $I_{-,+} = \{n\}$ and $\varphi_1 = \omega_n > 0$, with all other incomes remaining at x^*.

Without loss of generality (the change can be accommodated by choice of the magnitude of *t*), we can set $T=-1$ and, thus, in order for the incomes $y_1,...,y_n$ to remain in $\Omega(\mathbf{e},\lambda)$, we must have that

$$-1=\nu-\omega+\rho=\tau-\varphi+\mu\,, \tag{A7}$$

namely, that

$$\omega-1=\nu+\rho \text{ and } \varphi-1=\mu+\tau\,.$$

We, therefore, consider the minimization of the function $\sum_{i=1}^{n} RI(y_i,\mathbf{y})\Big|_{t=1}$, where $\mathbf{y}$ is defined by (A2), over the set

$$D=\{\nu_i,\varphi_i,\rho_i,\tau_i,\omega_i,\mu_i\in[0,1]:\omega-1=\nu+\rho\wedge\varphi-1=\mu+\tau\},$$

namely, we seek to solve

$$\min_D \sum_{i=1}^{n} RI(y_i,\mathbf{y})\Big|_{t=1}\,. \tag{A8}$$

We note that for each given division $\mathcal{I}$ of the set *I* into sets $I_{+,+}\cup I_{+,-}\cup I_{0,+}\cup I_{0,-}\cup I_{-,+}\cup I_{-,-}$, the sub-problem

$$\min_D\left\{\left[\sum_{i\in I_{+,+}} RI(y_i,\mathbf{y})+\sum_{i\in I_{+,-}} RI(y_i,\mathbf{y})+\sum_{i\in I_{-,-}} RI(y_i,\mathbf{y})\right.\right.$$
$$\left.\left.+\sum_{i\in I_{-,+}} RI(y_i,\mathbf{y})+\sum_{i\in I_{0,+}} RI(y_i,\mathbf{y})+\sum_{i\in I_{0,-}} RI(y_i,\mathbf{y})\right]\Big|_{t=1}\right\},$$

is a problem of the minimization of a continuous function over a closed and bounded set $D\subset\mathbf{R}^n$ and, thereby, over a compact set. Hence, a minimum exists, and we denote it by RI_{I}. The number of possible divisions I is finite; in particular, it is smaller than 6^n. In order to obtain a solution of (A8), it suffices then to take the minimum RI_{I} over all possible divisions, which we denote by RI^*.

Consequently, an equation analogous to (A6) for the $\mathbf{x}^*\to\mathbf{y}$ transfer in which $\sum_{i=1}^{n} RI(y_i,\mathbf{y})\Big|_{t=1}=RI^*$ is

$$
\begin{aligned}
&\left[(1-\beta) f(x^*)+E\right]^{\alpha} \eta'_{\alpha+}(0) \\
&=(1-\beta) f'(x^*)\left[\lambda\left(\sum_{i\in I_{+,+}} \tau_i-\sum_{i\in I_{+,-}} \varphi_i+\sum_{i\in I_{0,+}} \mu_i\right)-\left(\sum_{i\in I_{+,-}} \nu_i-\sum_{i\in I_{-,+}} \omega_i+\sum_{i\in I_{0,-}} \rho_i\right)\right] \\
&-\beta \sum_{i=1}^{n} RI(y_i,\mathbf{y})\Big|_{t=1} \\
&=(1-\beta) f'(x^*)(1-\lambda)-\beta RI^*=\min_{D}\left\{(1-\beta) f'(x^*)(1-\lambda)-\sum_{i=1}^{n} RI(y_i,\mathbf{y})\Big|_{t=1}\right\}.
\end{aligned} \tag{A9}
$$

Hence, we conclude that if $(1-\beta) f'(x^*)(1-\lambda)-\beta RI^* \le 0$, which is equivalent to

$$\beta \ge \frac{f'(x^*)(1-\lambda)}{f'(x^*)(1-\lambda)+RI^*},$$

we have that $\eta'_{\alpha+}(0) \le 0$, that is, as follows from Remark A2, $\mathbf{x}^*$ is a global maximum of $SWF_{\alpha}(\cdot)$ on $\Omega(\mathbf{e},\lambda)$, whereas for any $\beta < \frac{f'(x^*)(1-\lambda)}{f'(x^*)(1-\lambda)+RI^*}$ there exists a marginal $\mathbf{x}^* \to \mathbf{y}$ transfer such that $\eta'_{\alpha+}(0) > 0$, that is, $\mathbf{x}^*$ is not a global maximum of $SWF_{\alpha}(\cdot)$ on $\Omega(\mathbf{e},\lambda)$. We, thus, define

$$\beta^*(\mathbf{e},\lambda)=\frac{f'(x^*)(1-\lambda)}{f'(x^*)(1-\lambda)+RI^*}, \tag{A10}$$

which is the critical value that we searched for. Therefore, if the incomes $e_1,\ldots,e_n$ are not equal, the choice of any isoelastic social planner with $\alpha \ne 1$ is to divide the incomes equally if and only if $\beta \ge \beta^*(\mathbf{e},\lambda)$. In addition, we obtain that the magnitude of $\beta^*(\mathbf{e},\lambda)$ is not related to the degree of the social planner's aversion to inequality, α.

For $\alpha = 1$, that is, for a Bernoulli-Nash social planner, we replicate the preceding steps of the proof for the function $F_1(\mathbf{x}) = \ln SWF_{BN}(\mathbf{x}) = \sum_{i=1}^{n} \ln u(x_i,\mathbf{x})$ which, in view of Remark A1 with $g(x) = \ln x$, has the same maxima as that of the function $SWF_{BN}(\cdot)$. As an inequality analogous to (A9), we obtain

$$\left[(1-\beta) f(x^*)+E\right]\eta'_{1+}(0)=(1-\beta) f'(x^*)(\lambda-1)-\beta RI^*,$$

which yields $\eta'_{1+}(0) \le 0$ for any $\beta \ge \beta^*(\mathbf{e},\lambda)$, rendering the point $\mathbf{x}^*$ a global maximum of $SWF_{BN}(\cdot)$ on $\Omega(\mathbf{e},\lambda)$, whereas for $\beta < \beta^*(\mathbf{e},\lambda)$, the global maximum of $SWF_{BN}(\cdot)$ on $\Omega(\mathbf{e},\lambda)$ is attained at a point in which not all the incomes are equal.

To complete the proof of step 3 of the proposition, we next present a technical remark. We subsequently draw on this remark to characterize the critical level $\beta^*(\mathbf{e},\lambda)$.

Remark A3. For $h=\left|I_{+,-}\cup I_{0,-}\cup I_{-,-}\right|$ and $H=\left|I_{+,+}\cup I_{0,+}\cup I_{-,+}\right|$ we have that $\lim_{t\to 0^+}\frac{d}{dt}\sum_{i=1}^{n} RI(y_i,\mathbf{y})\ge\frac{1}{n}\left[h(\lambda\tau+\omega+\lambda\mu)+H(\lambda\varphi+\nu+\rho)\right]$.

Proof. At the point $\mathbf{y}$, we have that for any $i\in I_{+,-}$

$$\begin{aligned}RI(y_i,\mathbf{y})=\frac{1}{n}\Bigg[&\sum_{j\in I_{+,+}}(\tau_j\lambda t+\varphi_i\lambda t)+\sum_{j\in I_{-,+}}(\omega_j t+\varphi_i\lambda t)\\&+\sum_{j\in I_{0,+}}(\mu_j\lambda t+\varphi_i\lambda t)+\sum_{j\in I_{+,-}}\max\{\varphi_j\lambda t-\varphi_i\lambda t,0\}\\&+\sum_{j\in I_{-,-}}\max\{\nu_j t-\varphi_i\lambda t,0\}+\sum_{j\in I_{0,-}}\max\{\rho_j t-\varphi_i\lambda t,0\}\Bigg]\\&\ge\frac{1}{n}\left(\lambda\tau t+\omega t+\lambda\mu t+\lambda H\varphi_i t\right);\end{aligned}$$

for any $i\in I_{-,-}$ we have that

$$\begin{aligned}RI(y_i,\mathbf{y})=\frac{1}{n}\Bigg[&\sum_{j\in I_{+,+}}(\tau_j\lambda t+\nu_i t)+\sum_{j\in I_{-,+}}(\omega_j t+\nu_i t)\\&+\sum_{j\in I_{0,+}}(\mu_j\lambda t+\nu_i t)+\sum_{j\in I_{+,-}}\max\{\varphi_j\lambda t-\nu_i t,0\}\\&+\sum_{j\in I_{-,-}}\max\{\nu_j t-\nu_i t,0\}+\sum_{j\in I_{0,-}}\max\{\rho_j t-\nu_i t,0\}\Bigg]\\&\ge\frac{1}{n}\left(\lambda\tau t+\omega t+\lambda\mu t+H\nu_i t\right);\end{aligned}$$

for $i\in I_{0,-}$ we have that

$$\begin{aligned}RI(y_i,\mathbf{y})=\frac{1}{n}\Bigg[&\sum_{j\in I_{+,+}}(\tau_j\lambda t+\rho_i t)+\sum_{j\in I_{-,+}}(\omega_j t+\rho_i t)\\&+\sum_{j\in I_{0,+}}(\mu_j\lambda t+\rho_i t)+\sum_{j\in I_{+,-}}\max\{\varphi_j\lambda t-\rho_i t,0\}\\&+\sum_{j\in I_{-,-}}\max\{\nu_j t-\rho_i t,0\}+\sum_{j\in I_{0,-}}\max\{\rho_j t-\rho_i t,0\}\Bigg]\\&\ge\frac{1}{n}\left(\lambda\tau t+\omega t+\lambda\mu t+H\rho_i t\right);\end{aligned}$$

for $i\in I_{+,+}$ we have that

$$\begin{aligned}RI(y_i,\mathbf{y})=\frac{1}{n}\Bigg[&\sum_{j\in I_{+,+}}\max\{\tau_j\lambda t-\tau_i\lambda t,0\}+\sum_{j\in I_{-,+}}\max\{\omega_j t-\tau_i\lambda t,0\}\\&+\sum_{j\in I_{0,+}}\max\{\mu_j\lambda t-\tau_i\lambda t,0\}\Bigg]\\&\ge 0;\end{aligned}$$

for $i \in I_{-,+}$ we have that

$$\begin{aligned} RI(y_i, \mathbf{y}) &= \frac{1}{n}\Bigg[\sum_{j\in I_{+,+}} \max\{\tau_j \lambda t - \omega_i t, 0\} + \sum_{j\in I_{-,+}} \max\{\omega_j t - \omega_i t, 0\} \\ &\quad + \sum_{j\in I_{0,+}} \max\{\mu_j \lambda t - \omega_i t, 0\}\Bigg] \\ &\geq 0; \end{aligned}$$

and for $i \in I_{0,+}$ we have that

$$\begin{aligned} RI(y_i, \mathbf{y}) &= \frac{1}{n}\Bigg[\sum_{j\in I_{+,+}} \max\{\tau_j \lambda t - \mu_i \lambda t, 0\} + \sum_{j\in I_{-,+}} \max\{\omega_j t - \mu_i \lambda t, 0\} \\ &\quad + \sum_{j\in I_{0,+}} \max\{\mu_j \lambda t - \mu_i \lambda t, 0\}\Bigg] \\ &\geq 0. \end{aligned}$$

Thus,

$$\sum_{i=1}^{n} RI(y_i, \mathbf{y}) \geq \frac{1}{n}\left[h(\lambda\tau + \omega + \lambda\mu)t + H(\lambda\varphi + \nu + \rho)t \right],$$

and

$$\lim_{t\to 0^+} \frac{d}{dt} \sum_{i=1}^{n} RI(y_i, \mathbf{y}) \geq \frac{1}{n}\left[h(\lambda\tau + \omega + \lambda\mu) + H(\lambda\varphi + \nu + \rho) \right]. \qquad \text{(A11)}$$

This completes the proof of Remark A3.

Returning to the characterization of the critical level $\beta^*(\mathbf{e}, \lambda)$, on using (A6) we note that for $T \geq 0$, we surely have that $\eta'_{\alpha+}(0) \leq 0$. Therefore, we study the case $T < 0$, normalizing $T = -1$. From (A11) it follows that

$$RI^* \geq \frac{1}{n}\left[h(\lambda\tau + \omega + \lambda\mu) + H(\lambda\varphi + \nu + \rho) \right]. \qquad \text{(A12)}$$

Recalling (A7), we have that

$$\rho = \omega - \nu - 1 \text{ and } \mu = \varphi - \tau - 1, \qquad \text{(A13)}$$

and, from the definitions of *h* and *H*, we have that

$$H = n - h. \qquad \text{(A14)}$$

Using (A13) and (A14), (A12) simplifies to

$$
\begin{aligned}
RI^* &\geq \frac{1}{n}\left\{h\left[\lambda\tau+\omega+\lambda(\varphi-\tau-1)\right]+H(\lambda\varphi+\nu+\omega-\nu-1)\right\} \\
&= \frac{1}{n}\left\{h\left[\lambda(\varphi-1)+\omega\right]+(n-h)(\lambda\varphi+\omega-1)\right\} \\
&= \frac{1}{n}\left[h(1-\lambda)+n(\lambda\varphi+\omega-1)\right] \\
&= \frac{h(1-\lambda)}{n}+(\lambda\varphi+\omega-1).
\end{aligned}
$$

Because $\rho, \mu \geq 0$ we have that $\omega \geq \nu+1 \geq 1$ and that $\varphi \geq \tau+1 \geq 1$. Therefore, $\omega, \varphi, h \geq 1$ and, hence,

$$
RI^* \geq \frac{1-\lambda}{n}+\lambda.
$$

Thus, returning to (A10), we have that

$$
\beta^*(\mathbf{e},\lambda) \leq \frac{f'(x^*)(1-\lambda)}{f'(x^*)(1-\lambda)+RI^*} \leq \frac{f'(x^*)(1-\lambda)}{f'(x^*)(1-\lambda)+\lambda+\frac{1-\lambda}{n}} < 1.
$$

This completes the proof of the proposition.

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