On the Hodge conjecture for hyperkähler manifolds

Dissertation
zur
Erlangung des Doktorgrades (Dr. rer. nat.)
der
Mathematisch-Naturwissenschaftlichen Fakultät
der
Rheinischen Friedrich-Wilhelms-Universität Bonn

vorgelegt von
Mauro Varesco
aus
Cavalese, Italien

Bonn, 2024

Angefertigt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät der Rheinischen Friedrich-Wilhelms-Universität Bonn

Gutachter/Betreuer: Prof. Dr. Daniel Huybrechts
Gutachter: Prof. Dr. Philip Engel

Tag der Promotion: 12. September
Erscheinungsjahr: 2024

Alla mia famiglia

Preface

The Hodge conjecture predicts a deep connection between topology, complex geometry, and algebraic geometry. It asserts that any Hodge class on a smooth complex projective variety is algebraic, meaning it is a linear combination with rational coefficients of fundamental classes of subvarieties. If proven true, this conjecture would characterize algebraic classes as Betti cohomology classes that correspond, via the Betti-de Rham isomorphism, to de Rham cohomology classes of type (k, k) for some k.

For powers of a K3 surface, the Hodge conjecture is equivalent to the algebraicity of the Hodge classes in the tensor algebra of its *transcendental lattice*, which is the orthogonal complement of the Néron–Severi group in the second rational cohomology. In Appendix B, using invariant theory, we determine generators for this algebra of Hodge classes and we show that the Hodge conjecture for the powers of a K3 surface does not always follow from the Hodge conjecture for its square. Specifically, we prove that if the endomorphism field of the K3 surface is totally real, there are exceptional Hodge classes which do not appear on the square of the K3 surface. Additionally, we extend a result by Schlickewei [81] and establish the Hodge conjecture for all powers of certain K3 surfaces of Picard number 16.

The *Kuga–Satake* construction, reviewed in collaboration with Claire Voisin in Appendix A, provides an embedding of the transcendental lattice $T(X)$ of a projective K3 surface or hyperkähler manifold X into the square of the first cohomology of an abelian variety. If the Kuga–Satake Hodge conjecture holds, meaning this embedding is algebraic, the algebraicity of the Hodge classes in the tensor algebra of $T(X)$ follows if one proves that their image via this embedding is algebraic on the powers of the abelian variety. Note that to deduce this, one must assume that the Lefschetz standard conjecture in degree two holds for the variety X. This has been proven, in the cases we consider, by Charles–Markman [14] and Foster [24].

This observation is particularly relevant for Hodge similarities between the transcendental lattices of projective hyperkähler manifolds (or K3 surfaces). A *Hodge similarity* is a Hodge morphism which preserves, up to a positive scalar, the Beauville–Bogomolov forms (or the intersection products). In Appendix C, we prove that a Hodge similarity induces an isogeny between the Kuga–Satake varieties, and thus is algebraic on the product of the two Kuga–Satake varieties. If the Kuga–Satake Hodge conjecture holds, then the Hodge similarity is algebraic on the product of the two hyperkähler manifolds (or K3 surfaces). Voisin [98] has shown that the Kuga–Satake Hodge conjecture holds for hyperkähler manifolds of generalized Kummer type. Therefore, we deduce the algebraicity of any Hodge similarity between two such manifolds. This complements previous results on the algebraicity of Hodge isometries between K3 surfaces by Buskin in [11] and again by Huybrechts [45], and between $K3^{[n]}$-type varieties by Markman [62].

In Appendix D, a joint work with Salvatore Floccari, we study the Hodge conjecture in the special case of hyperkähler manifolds of generalized Kummer type. We prove that any Hodge class in the algebra generated by their second cohomology is algebraic. We deduce this from three key results: the algebraicity of Hodge similarities between hyperkähler manifolds of generalized Kummer type, a construction by Floccari [21] providing an algebraic Hodge similarity between a Kum^3-type variety and a K3 surface, and the fact that, by Appendix B, the Hodge conjecture holds for the K3 surfaces that appear in this construction. Remarkably, this implies the Hodge conjecture for projective Kum^2-type varieties, thereby providing the first instance of complete families of projective four-dimensional manifolds for which the Hodge conjecture holds.

The thesis is structured as follows: Chapter 1 revisits the properties of the objects under study, presenting proofs of some selected results that have motivated and inspired our research. Chapters 2–5 serve as an introduction to the four appended papers (Appendices A–D). Finally, Chapter 6 summarizes the results obtained in this thesis supplementing them with additional remarks.

Acknowledgements

My deepest and most sincere gratitude goes to Prof. Daniel Huybrechts, whose expertise and wisdom guided me through my Ph.D. studies. Not only has he been an extraordinary advisor, but he has also provided unwavering support throughout my doctoral journey. I am honored to have had the opportunity to be mentored by such a brilliant mathematician and a kind, caring person.

I am greatly indebted to Prof. Philip Engel for being the second reviewer of this thesis and for the valuable mathematical discussions we shared. His love for mathematics is an endless source of inspiration.

I would like to express my deepest appreciation to Prof. Dr. Carsten Burstedde and to Priv.-Doz. Dr. Stefan Förste for agreeing to be members of my defence committee.

I am profoundly thankful to Dr. Ulrike Rieß for supporting me during my Master's and for encouraging me to pursue my Ph.D. in Bonn. This turned out to be excellent advice.

I would like to express my gratitude to Prof. Claudio Fontanari. He is the reason I chose the fascinating subject of algebraic geometry. His guidance has been pivotal in my mathematical journey, providing me with invaluable and enriching experiences, both academic and personal. I will always deeply appreciate and remember the significant impact his mentorship had on my life.

I gratefully acknowledge the assistance of Ute Sachinidis with all the bureaucratic procedures over the past years. Additionally, I am thankful to the University of Bonn, the Hausdorff Center for Mathematics, and the Bonn International Graduate School of Mathematics for the opportunity to pursue my doctoral degree in Mathematics.

My gratitude also goes to Prof. Claire Voisin for hosting me in Paris in the spring of 2022. This experience and our discussions while preparing for our seminar talk for the Bonn–Paris seminar have deeply influenced my mathematical research.

I am thankful to my collaborator and friend Salvatore Floccari for the enriching conversations we have shared over the years. Discussing with you has been extremely stimulating and enjoyable. I only wish we worked at the same university.

I am grateful to my Bonn group: Dominique, Evgeny, Gebhard, Giacomo, Léonor, Reinder, Suzan, Yagna, and Yoon-Joo. Thank you for making these years so enjoyable. I will never forget Nennillo's pizza, not just for its guanciale, but for the great evenings we spent together. Special thanks to Reinder. Your friendship and our endless conversations have been fundamental, especially in this last year. Additionally, I would like to thank all the great people I met over these years, especially my office mates: Denis, Jiexiang, Maximilian, Moritz, and Thorsten.

I would like to thank the people I had the good fortune to meet during my three months in Paris. I will always remember exploring Oslo with Fabrizio, the warm welcome in Paris from Chenyu, the dinners at Pietro's house with his wonderful family, and the time at Climb Up with Francesca. I would also like to thank Corinne for organizing the workshop in Lille in the spring of 2022. This not only created an enriching working group but also helped build a strong circle of friends.

Many thanks to the amazing flatmates I had the luck to share an apartment with. Thanks to Mattia for welcoming me to Bonn, for all the time we spent together, for sharing a passion for cooking, and for filling our conversations with AGG references. And thank you, Camilla, for always including me in your activities and for supporting me during this last year. I will always remember the hockey match we watched in Köln.

To my friends Angelo and Iulia. I consider myself so lucky to have met you both and to have built such a strong friendship. Thanks to Angelo for including me in the EMA group, for your kindness and generosity. I will never forget the Yellow Wall in Dortmund we saw together with Ben and Jan. And thanks to Iulia, for being there through the ups and downs. I am glad I could share this Ph.D. journey with you.

I would like to extend my heartfelt thanks to the friends who provided invaluable support from afar throughout my journey: Giulio and Simone, who have always given me the certainty that I have friends at home; Dario, Magda, and Sofia, who have always been there to provide helpful advice; and Daniel and Marco, who have continuously encouraged me over the years.

Infine, un ringraziamento speciale ai miei genitori, Renzo e Amanda, e a mio fratello Claudio. Grazie per le chiamate giornaliere e per avermi sostenuto durante tutto il mio percorso accademico. Ovunque io vada, mi date la certezza di avere una casa e una famiglia da cui tornare. Grazie di cuore.

Bonn, *Mauro.*

Contents

Chapter 1

Introduction

In this chapter, we introduce the objects that are studied in the four appended papers [23, 89, 88, 90]. In addition, we review some results that have inspired our research.

1.1 The Hodge conjecture

We begin by recalling the statement and and some example applications of the Hodge conjecture, which is the central focus of this thesis. As a reference for this section we use [96].

Let X be a smooth complex projective variety, and let $\iota\colon Z \hookrightarrow X$ be the inclusion morphism of a smooth subvariety Z. Note that Z can be viewed as a real orientable manifold of dimension $2n - 2k$, where $n := \dim X$ and $k := \operatorname{codim}_X Z$. In particular, the singular homology $H_{2n-2k}(Z, \mathbb{Q})$ is one-dimensional and it is spanned by $[Z]$, the *fundamental class* of Z. Note that $[Z]$ defines a cohomology class in $H^{2k}(X, \mathbb{Q})$ via the composition

$$H_{2n-2k}(Z, \mathbb{Q}) \xrightarrow{\iota_*} H_{2n-2k}(X, \mathbb{Q}) \simeq H^{2k}(X, \mathbb{Q}).$$

In the case where Z is singular, by Hironaka [41], there exists a smooth projective variety $\widetilde{Z}$ with a degree-one morphism $\tau\colon \widetilde{Z} \to Z$. Then, one defines $[Z]$ as

$$[Z] := \widetilde{\iota}_*[\widetilde{Z}] \in H^{2k}(X, \mathbb{Q}),$$

where $\widetilde{\iota} := \iota \circ \tau$. As one shows, the class $[Z] \in H^{2k}(X, \mathbb{Q})$ is always a *Hodge class*, meaning an element of

$$H^{k,k}(X, \mathbb{Q}) := H^{2k}(X, \mathbb{Q}) \cap H^{k,k}(X),$$

i.e., a rational class of type (k, k). A Hodge class in $H^{k,k}(X, \mathbb{Q})$ is called *algebraic* if it is a linear combination (with rational coefficients) of fundamental classes of subvarieties of X of codimension k. The Hodge conjecture predicts that every Hodge class is algebraic:

Conjecture 1.1.1 (Hodge conjecture)**.** *Let X be a smooth projective variety. Then, for every non-negative integer k, every Hodge class in $H^{k,k}(X, \mathbb{Q})$ is algebraic.*

In codimension one, the Hodge conjecture is known to hold. This follows from the Lefschetz theorem on $(1,1)$-classes which states that the image of the first Chern map

$$c_1\colon \mathrm{Pic}(X)_{\mathbb{Q}} \to H^2(X,\mathbb{Q}),$$

surjects onto the space of Hodge classes $H^{1,1}(X,\mathbb{Q})$.

Hodge classes naturally appear when considering morphisms of Hodge structures. Let V and W be Hodge structures of weight k and $k' = k+2r$ respectively, and let $\psi\colon V \to W$ be a morphism of Hodge structures. This means that the extension $\psi_{\mathbb{C}}$ satisfies

$$\psi_{\mathbb{C}}(V^{p,q}) \subseteq W^{p+r,q+r},$$

for all $p+q=k$. By linear algebra, we can view ψ as an element $[\psi] \in V^* \otimes W$. From the fact ψ is a morphism of Hodge structures, one sees that $[\psi]$ is a Hodge class, where $V^* \otimes W$ is endowed with the Hodge structure induced by the Hodge structures on V and W. Let now $V = H^{2k}(X,\mathbb{Q})$ and $W = H^{2k'}(Y,\mathbb{Q})$ be cohomology groups of two smooth complex projective varieties X and Y. By Poincaré duality and the Künneth decomposition, there is an embedding of Hodge structures

$$H^{2k}(X,\mathbb{Q})^* \otimes H^{2k'}(Y,\mathbb{Q}) \simeq H^{2n-2k}(X,\mathbb{Q}) \otimes H^{2k'}(Y,\mathbb{Q}) \hookrightarrow H^{2n-2k+2k'}(X \times Y,\mathbb{Q}).$$

Therefore, the morphism of Hodge structures $\psi\colon H^{2k}(X,\mathbb{Q})^* \to H^{2k'}(Y,\mathbb{Q})$ determines a Hodge class $[\psi] \in H^{2n-2k+2k'}(X \times Y,\mathbb{Q})$. The Hodge conjecture then predicts that $[\psi]$ is a linear combination of fundamental classes of subvarieties of $X \times Y$. If this is the case, we say that the morphism ψ is algebraic.

As an example, consider the cup product with a line bundle. Let X be a complex projective variety, and let $h \in H^2(X,\mathbb{Q})$ be the fundamental class of an effective divisor D on X. As h is a Hodge class, cup product with h gives a morphism of Hodge structures

$$h \cup \bullet\colon H^*(X,\mathbb{Q}) \to H^{*+2}(X,\mathbb{Q}), \quad x \mapsto h \cup x.$$

Note that this morphism is algebraic since it is induced by the cohomology class of the divisor D embedded in $X \times X$ via the diagonal map.

In the case, where h is the cohomology class of an ample divisor, the hard Lefschetz theorem [94, Thm. 6.2.3] says that the morphism of Hodge structures

$$h^{n-k} \cup \bullet\colon H^k(X,\mathbb{Q}) \to H^{2n-k}(X,\mathbb{Q}), \quad n \coloneqq \dim X,$$

is an isomorphism for all $k \leq n$. In particular, one can consider its inverse

$$\delta_k \coloneqq (h^{n-k} \cup \bullet)^{-1}\colon H^{2n-k}(X,\mathbb{Q}) \to H^k(X,\mathbb{Q}).$$

The Hodge conjecture applied to the isomorphism of Hodge structures δ_k is also known as *Lefschetz standard conjecture* in degree k. As one shows, the algebraicity of δ_k does not depend on the choice of the ample class h. There are positive evidences for this conjecture. For an abelian variety A, the existence of the Poincaré bundle on $A \times \hat{A}$ can be used to prove the Lefschetz standard conjecture in all degrees. In the case of hyperkähler manifolds of $\mathrm{K3}^{[n]}$-type this conjecture has been proven by Charles and Markman [14]. More recently, Foster in [24] shows that the Lefschetz standard conjecture holds in low degrees also for hyperkähler manifolds of generalized Kummer type.

1.2 K3 surfaces and hyperkähler manifolds

The main geometrical objects we study in this thesis are K3 surfaces and hyperkähler manifolds. We recall here their definitions and examples, focusing on the properties of their second rational cohomology. In this section, we follow [44] while discussing K3 surfaces and [38] for the part on hyperkähler manifolds.

Definition 1.2.1. A *K3 surface* is a smooth compact connected complex surface S such that

$$\omega_S := \Omega_S^2 \cong \mathscr{O}_S \quad \text{and} \quad H^1(S, \mathscr{O}_S) = 0.$$

Let S be a K3 surface and consider the exponential sequence

$$0 \to \mathbb{Z} \to \mathscr{O}_S \to \mathscr{O}_S^* \to 0.$$

The induced long exact sequence in cohomology and the vanishing of $H^1(S, \mathscr{O}_S)$ imply that $H^1(S, \mathbb{Z}) = 0$. By extension of scalars and Poincaré duality, also

$$H^1(S, \mathbb{Q}) = H^3(S, \mathbb{Q}) = 0.$$

As $H^0(S, \mathbb{Q}) \simeq H^4(S, \mathbb{Q}) \simeq \mathbb{Q}$, the only other non-trivial cohomology group is $H^2(S, \mathbb{Q})$. By the Noether formula, one sees that $\dim H^2(S, \mathbb{Q}) = 22$.

Also the Hodge numbers of the K3 surface are easily computed. In degree zero and four, the Hodge structures $H^1(S, \mathbb{Q})$ and $H^4(S, \mathbb{Q})$ are of Tate-type since they are one-dimensional. As $h^1(S) = h^3(S) = 0$, the only Hodge structure we need to describe is $H^2(S, \mathbb{Q})$. By hypothesis, Ω_S^2 is trivial, so in particular $H^0(S, \Omega_S^2) \simeq \mathbb{C}$ is one-dimensional. This implies that $h^{2,0}(S) = 1$ and, by symmetry, $h^{0,2}(S) = 1$. As the second cohomology of a K3 surface is 22-dimensional, we conclude that $h^{1,1}(S) = 20$.

As S is a surface, the intersection pairing q_S induces a lattice structure on $H^2(S, \mathbb{Z})$. It can be shown that there is an isometry of lattices

$$(H^2(S, \mathbb{Z}), q_S) \simeq E_8(-1)^{\oplus 2} \oplus U^{\oplus 3} =: \Lambda_{\mathrm{K3}},$$

where U is the hyperbolic lattice and E_8 is the unique positive definite, even, unimodular lattice of rank eight. Extending scalars to $\mathbb{Q}$, we can also view q_S as a morphism of Hodge structures

$$q_S \colon H^2(S, \mathbb{Q}) \otimes H^2(S, \mathbb{Q}) \to H^4(S, \mathbb{Q}) \simeq \mathbb{Q}(-2),$$

where $\mathbb{Q}(-2)$ is the Tate Hodge structure.

The Hodge structure on the second cohomology of K3 surface leads to the following definition. A rational Hodge structure of weight two V is called of *K3-type* if it is effective and $\dim V^{2,0} = 1$. Here, effective means that $V^{p,q} = 0$ if p or q is negative. Note that these Hodge structures are sometimes also called of *hyperkähler-type*.

We now end our discussion on K3 surfaces by recalling two examples of these surfaces.

Example 1.2.2 (Quartics in $\mathbb{P}^3$)**.** The easiest example of K3 surface is a smooth quartic $S \subset \mathbb{P}^3$. From the vanishings $H^1(\mathbb{P}^3, \mathscr{O}) = H^2(\mathbb{P}^3, \mathscr{O}(-4)) = 0$ and the short exact sequence

$$0 \to \mathscr{O}(-4) \to \mathscr{O} \to \mathscr{O}_S \to 0,$$

one deduces that $H^1(S, \mathscr{O}_S) = 0$. Triviality of the canonical bundle of S is shown by taking the determinant of the conormal bundle sequence

$$0 \to \mathscr{O}(-4)|_S \to \Omega_{\mathbb{P}^3}|_S \to \Omega_S \to 0.$$

Example 1.2.3 (Kummer surfaces)**.** Let A be an abelian surface, and denote by $\iota\colon A \to A$ the involution $x \mapsto -x$. The quotient A/ι is singular and has 16 simple nodal points corresponding to the fixed points of the involution ι. Blowing up these singularities, we obtain a smooth surface $\mathrm{Kum}(A)$ called the *Kummer surface* of A. The fact that $\mathrm{Kum}(A)$ is a K3 surface can be deduced from the commutative diagram

$$\begin{array}{ccc} \widetilde{\iota} \circlearrowright \widetilde{A} & \longrightarrow & A \\ \downarrow & & \downarrow \\ \mathrm{Kum}(A) & \longrightarrow & A/\iota \end{array},$$

where $\widetilde{A}$ is the blowup of A at the fixed locus of ι, and $\widetilde{A} \to \mathrm{Kum}(A)$ is the quotient of morphism of $\widetilde{A}$ by the involution $\widetilde{\iota}$ induced by ι.

As mentioned at the beginning of this section, the other class of manifolds we focus our attention on is the class of hyperkähler manifolds:

Definition 1.2.4. A *hyperkähler manifold* is an irreducible simply-connected compact Kähler manifold X, such that $H^0(X, \Omega_X^2)$ is generated by an everywhere non-degenerate holomorphic two-form.

Hyperkähler manifolds can be considered as high-dimensional generalization of K3 surfaces. Indeed, the condition on $H^0(S, \Omega_S^2)$ for a K3 surface S follows from the definition, and any K3 surface is Kähler by Siu [84].

Note that, given a hyperkähler manifold X, the second cohomology group $H^2(X, \mathbb{Q})$ is a Hodge structure of K3-type. Furthermore, similarly to the case of K3 surfaces, $H^2(X, \mathbb{Z})$

has a lattice structure with signature $(3, b_2(X) - 3)$. This follows from the existence of the *Beauville–Bogomolov form* q_X, which is defined as follows. Let $2n := \dim X$, and let σ be a holomorphic two-form such that $\int_X (\sigma\overline{\sigma})^n = 1$. If $\alpha = \lambda\sigma + \beta + \mu\overline{\sigma} \in H^2(X, \mathbb{C})$ with $\beta \in H^{1,1}(X)$, define

$$q_X(\alpha) := \lambda\mu + \frac{n}{2} \int_X \beta^2 (\sigma\overline{\sigma})^{n-1}.$$

By the Beauville–Fujiki relation there exists a positive constant $c_X \in \mathbb{R}$, depending only on the deformation class of X, such that

$$q_X(\alpha)^n = c_X \int_X \alpha^{2n},$$

for all $\alpha \in H^2(X, \mathbb{C})$. This can be used to prove that q_X can be renormalized such that it restricts to a primitive integral quadratic form on $H^2(X, \mathbb{Z})$. The fact that the signature of q_X is $(3, b_2(X) - 3)$ is proven by showing that, if $[\omega] \in H^{1,1}(X)$ is a Kähler class, then q_X is positive definite on $(H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{R}} \oplus \mathbb{R}[\omega]$ and negative definite on the primitive part $H^{1,1}(X)_\omega$.

Beauville in [7] provided the first two examples of hyperkähler manifolds which are not K3 surfaces: Hilbert schemes of points on a K3 surface, and generalized Kummer varieties.

Example 1.2.5 (Hilbert schemes of points on a K3 surface)**.** Let $X := S^{[n]}$ be the Hilbert scheme of n points on a K3 surface S, i.e., the moduli space of zero-dimensional subschemes of S of length n. Then, X is a hyperkähler manifold of dimension $2n$, and there is an isometry

$$(H^2(X, \mathbb{Z}), q_X) \simeq (H^2(S, \mathbb{Z}), q_S) \oplus (-2(n-1))\mathbb{Z},$$

where q_S is the intersection pairing on $H^2(S, \mathbb{Z})$. A hyperkähler manifold which is derived equivalent to $S^{[n]}$ for some K3 surface S is called of $\mathrm{K3}^{[n]}$-type.

Example 1.2.6 (Generalized Kummer varieties)**.** Let T be a complex torus of dimension two, let $n \geq 2$, and let $T^{[n+1]}$ be the Hilbert scheme of $(n+1)$ points on T. In this case, $T^{[n+1]}$ is not a hyperkähler manifold since it is not simply-connected and its symplectic structure is not unique. The natural map

$$T^{[n+1]} \to T^{(n+1)} \to T,$$

induced by the summation map on T is an isotrivial fibration. One then checks that the fiber is a hyperkähler manifold of dimension $2n$. It is denoted by $\mathrm{Kum}^n(T)$, and it is called *generalized Kummer variety.* The lattice structure of $X := \mathrm{Kum}^n(T)$ on its second cohomology satisfies

$$(H^2(X, \mathbb{Z}), q_X) \simeq (H^2(T, \mathbb{Z}), q_T) \oplus (-2(n+1))\mathbb{Z},$$

where q_T is the natural intersection pairing on T. A hyperkähler manifold which is deformation equivalent to $\mathrm{Kum}^n(T)$ for some complex torus T is called of *generalized Kummer type* or, in short, of Kum^n*-type*.

1.3 Transcendental lattices and endomorphism fields

Let X be a projective hyperkähler manifold or a projective K3 surface, and let $\mathrm{NS}(X)$ be its Néron–Severi group, that is the image of the first Chern class map

$$c_1\colon \mathrm{Pic}(X) \to H^2(X, \mathbb{Z}).$$

The *transcendental lattice* $T(X)$ of X is defined as the orthogonal complement of $\mathrm{NS}(X)_{\mathbb{Q}}$ in $H^2(X, \mathbb{Q})$ with respect to the Beauville–Bogomolov form q_X or the intersection product if X is a K3 surface. By the Lefschetz theorem on $(1,1)$-classes, $\mathrm{NS}(X)_{\mathbb{Q}}$ coincides with $H^{1,1}(X, \mathbb{Q})$, so $T(X)$ is a Hodge structure of K3-type. Moreover, as X is projective by assumption, the restriction of $-q_X$ polarizes $T(X)$ in the following sense:

Definition 1.3.1. A pair (V, q) is a *polarized Hodge structure of K3-type* if V is a Hodge structure of K3-type, and $q\colon V \otimes V \to \mathbb{Q}(-2)$ is a morphism of Hodge structures on V whose real extension is negative definite on $(V^{2,0} \oplus V^{0,2})$ and has signature $(\dim V - 2, 2)$.

As we are assuming that X is projective, $T(X)$ can be equivalently defined as the smallest sub-Hodge structure $\widetilde{T}$ of $H^2(X, \mathbb{Q})$ for which $H^{2,0}(X) \subseteq \widetilde{T}_{\mathbb{C}}$. From this definition, one sees that $T(X)$ is an irreducible Hodge structure.

Given a Hodge structure V one can define its Hodge group as follows. Let $\rho\colon \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$ be the representation of the Deligne torus $\mathbb{S} \coloneqq \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}}$ defining the Hodge structure on V. The *Hodge group* $\mathrm{Hdg}(V)$ of V is the smallest algebraic subgroup of $\mathrm{GL}(V)$ defined over $\mathbb{Q}$ such that $\rho(\mathbb{U}(\mathbb{R})) \subseteq \mathrm{Hdg}(V)(\mathbb{R})$, where $\mathbb{U}(\mathbb{R}) = \{z | z\overline{z} = 1\} \subseteq \mathbb{S}(\mathbb{R})$.

The relevance of this notion in the context of the Hodge conjecture follows from the fact that a class $v \in \bigoplus (V^{\otimes n_i} \otimes V^{*\otimes m_i})$ is a Hodge class if and only if it is invariant under the natural action of $\mathrm{Hdg}(V)$.

For an irreducible polarized Hodge structure of K3-type (V, q), the Hodge group has a simple description. Let $E \coloneqq \mathrm{End}_{\mathrm{Hdg}}(V)$ be the ring of Hodge endomorphisms of V. As V is irreducible by assumption, E is a field. Let $F \subseteq E$ be the fixed locus of the *Rosati involution*, which is the natural involution on E which sends an element e to the element e' for which

$$q(ev, w) = q(v, e'w), \quad \forall v, w \in V.$$

As one checks, F is a totally real field, and either $F = E$ or E is a CM field with maximal totally real subfield F. Let $\Psi\colon V \times V \to E$ be the pairing defined by the condition

$$q(ev, w) = \mathrm{Tr}_{E/\mathbb{Q}}(e\Psi(v, w))$$

for all $e \in E$ and all $v, w \in V$. As shown in [101], the map Ψ is symmetric if E is totally real and E-hermitian in the CM case, and the Hodge group of V satisfies

$$\mathrm{Hdg}(V) = \begin{cases} \mathrm{SO}(V, \Psi) & \text{if } E \text{ is totally real} \\ \mathrm{U}(V, \Psi) & \text{if } E \text{ is a CM field.} \end{cases}$$

As Hodge classes can be characterized as invariant classes under the action of the Hodge group, knowing $\mathrm{Hdg}(V)$ one can use invariant theory to describe the algebra of Hodge classes in $\bigotimes^\bullet V$. This is especially relevant in the case where $V = T(X)$ is the transcendental lattice of a projective K3 surface. In fact, the Hodge conjecture for the powers of X is equivalent to the algebraicity of the Hodge classes in $\bigotimes^\bullet T(X)$.

In Appendix B, we correct a result by Ramon-Marí [78], and we describe a set of generators for the algebra of Hodge classes in $\bigotimes^\bullet T(X)$. We then use this description to prove the Hodge conjecture for the powers of projective K3 surfaces of Picard number 16 if an additional condition holds: the algebraicity of the Kuga–Satake correspondence. We recall the construction of Kuga–Satake varieties in Section 1.5.

To end this section, we recall that in a family of projective hyperkähler manifolds or K3 surfaces $\mathscr{X} \to B$, the endomorphism field of the general fibre is contained in the endomorphism field of any fibre. To fix the notation, let $\widetilde{T}$ be the transcendental lattice of the general fibre $\mathscr{X}_b$, and let $E \simeq \mathbb{Q}(\phi)$ be its endomorphism field. Let $0 \in B$ be any point and let $X := \mathscr{X}_0$. Note that there is a natural inclusion of quadratic spaces $T(X) \subseteq \widetilde{T}$. The quadratic form on X then induces an orthogonal decomposition

$$\widetilde{T} \simeq T(X) \oplus N,$$

for some $N \subseteq \mathrm{NS}(X)_{\mathbb{Q}}$. As the ϕ is a Hodge morphism on the general fibre and the Hodge locus of a cohomology class is closed, the morphism ϕ specializes to a morphism of Hodge structures

$$\phi \colon T(X) \oplus N \to T(X) \oplus N.$$

Since there is no non-trivial Hodge morphism between $T(X)$ and N, we see that ϕ can be written as a sum of two automorphisms $\phi_{T(X)} \colon T(X) \to T(X)$ and $\phi_N \colon N \to N$. This shows that elements of E define Hodge endomorphisms of $T(X)$ and of N. In particular, we conclude that $E \simeq \mathbb{Q}(\phi_{T(X)})$ is contained in the endomorphism field of X as claimed.

1.4 Hodge isometries

As we have recalled, the Hodge conjecture applies also to Hodge morphisms between cohomology groups of smooth projective varieties. In the case of Hodge isometries between hyperkähler manifolds or K3 surfaces X and Y there have been positive evidences. Recall that an isomorphisms of Hodge structures

$$\varphi \colon H^2(X, \mathbb{Q}) \to H^2(Y, \mathbb{Q})$$

is called a *Hodge isometry* if it is compatible with q_X and q_Y, meaning

$$q_Y(\varphi v, \varphi w) = q_X(v, w), \quad \forall v, w \in H^2(X, \mathbb{Q}).$$

If X and Y are K3 surfaces, the algebraicity of Hodge isometries has been proven by Buskin [11] and again by Huybrechts [45]. Before Buskin's paper, the algebraicity of Hodge isometries of K3 surfaces was known only in the case of Picard number bigger than five by Nikulin [75], which was an extension of the result by Mukai [72] on Hodge isometries between K3 surfaces of Picard number at least eleven. In the following example, we sketch the proof given in [45].

Example 1.4.1 (Hodge isometries for K3 surfaces)**.** Let S and S' be two K3 surfaces, and let

$$\varphi \colon H^2(S, \mathbb{Q}) \to H^2(S', \mathbb{Q})$$

be a Hodge isometry between their second cohomology. By lattice theory, any isometry $\phi \colon \Lambda_{\mathbb{Q}} \to \Lambda_{\mathbb{Q}}$ of a lattice Λ can be expressed as composition of reflections. I.e., there exist primitive elements $b_i \in \Lambda$, with $(b_i)^2 \neq 0$ such that

$$\phi = s_{b_1} \circ \cdots \circ s_{b_k},$$

where s_{b_i} is the reflection $x \mapsto x - \frac{2(x.b_i)}{(b_i)^2} b_i$. This observation, together with the surjectivity of the period map for K3 surfaces, shows that it suffices to prove the algebraicity of *reflective Hodge isometries*: I.e., such that after choosing markings $\Lambda \simeq H^2(S, \mathbb{Z})$ and $\Lambda \simeq H^2(S', \mathbb{Z})$, they are of the form s_b for some $b \in H^2(S, \mathbb{Z})$.

Let us now assume that $\varphi \colon H^2(S, \mathbb{Q}) \xrightarrow{\simeq} H^2(S', \mathbb{Q})$ is the reflective Hodge isometry induced by $b \in H^2(S, \mathbb{Z})$. For $n := (b)^2/2$, set $B := (1/n)b \in H^2(S, \mathbb{Q})$ and define

$$H^2(S, \mathbb{Z})_B := \{x \in H^2(S, \mathbb{Z}) \mid (x.B) \in \mathbb{Z}\}.$$

Let $\widetilde{H}(S, \mathbb{Z})$ be the Mukai lattice of S, which is $H^*(S, \mathbb{Z})$ with the sign change in the pairing of H^0 and H^4, and consider the following primitive embedding of lattices

$$\exp(B) \colon H^2(S, \mathbb{Z})_B \hookrightarrow \widetilde{H}(S, \mathbb{Z}), \quad x \mapsto x + x \wedge B.$$

Denote by $\widetilde{H}(S, B, \mathbb{Z})$ the space $\widetilde{H}(S, \mathbb{Z})$ with the natural Hodge structure of K3-type induced by $H^2(S, \mathbb{Z})_B$. I.e., the $(2,0)$-part of $\widetilde{H}(S, B, \mathbb{Z})$ is spanned by $\exp(B)(\sigma)$, where σ is any generator of $H^{2,0}(S)$. In the same way, define $H^2(S', \mathbb{Z})_{B'}$ and $\widetilde{H}(S', B', \mathbb{Z})$ with $B' := -\varphi(B)$.

As one checks, the rational Hodge isometry φ restricts to an isometry of integral Hodge structures $H^2(S, \mathbb{Z})_B \to H^2(S', \mathbb{Z})_{B'}$ which then extends to a Hodge isometry $\widetilde{\varphi} \colon \widetilde{H}(S, B, \mathbb{Z}) \xrightarrow{\simeq} \widetilde{H}(S', B', \mathbb{Z})$. In other words, there exists a commutative diagram of

Hodge morphisms

$$\begin{array}{ccc} H^2(S,\mathbb{Z})_B & \xhookrightarrow{\exp(B)} & \widetilde{H}(S,B,\mathbb{Z}) \\ \varphi\big\downarrow & & \big\downarrow\widetilde{\varphi} \\ H^2(S',\mathbb{Z})_{B'} & \xhookrightarrow[\exp(B')]{} & \widetilde{H}(S',B',\mathbb{Z}). \end{array} \tag{1.1}$$

Up to changing φ by a sign, we may assume that $\widetilde{\varphi}$ preserves the natural orientation of the four positive directions in the Mukai lattice. Therefore, by [48], the Hodge isometry $\widetilde{\varphi}$ lifts to an exact equivalence

$$\Phi\colon D^b(S,\alpha) \xrightarrow{\simeq} D^b(S',\alpha') \tag{1.2}$$

between derived category of twisted coherent sheaves, where α and α' are the Brauer classes on S and S' induced by B and B' via the exponential sequence. In particular, since any exact linear equivalence as in (1.2) is of Fourier–Mukai type by [12], the morphism Φ is induced by some $\mathcal{E} \in D^b(S \times S', \alpha^{-1} \boxtimes \alpha')$. This, together with the commutativity of (1.1), implies that φ is algebraic.

In the case of projective hyperkähler manifolds of $\mathrm{K3}^{[n]}$-type, the algebraicity of Hodge isometries is due to Markman in [62]. We recall now the idea of the proof.

Example 1.4.2 (Hodge isometries and $\mathrm{K3}^{[n]}$-type varieties)**.** Let X and Y be projective-hyperkähler manifolds of $\mathrm{K3}^{[n]}$-type, and consider a Hodge isometry

$$f\colon H^2(X,\mathbb{Q}) \xrightarrow{\simeq} H^2(Y,\mathbb{Q}).$$

The first step is to reduce to the case of isometries of cyclic type. In [62], a Hodge isometry $H^2(X,\mathbb{Q}) \xrightarrow{\simeq} H^2(Y,\mathbb{Q})$ is called of *r-cyclic type* for some positive integer r, if it is equal to $-g\rho_u$ where $g\colon H^2(X,\mathbb{Z}) \xrightarrow{\simeq} H^2(Y,\mathbb{Z})$ is a parallel-transport operator and $\rho_u \in \mathrm{O}(H^2(X,\mathbb{Q}))$ is the reflection $x \mapsto x - \frac{2(x.u)}{(u)^2}u$ for some primitive class $u \in H^2(X,\mathbb{Z})$ such that $(u)^2 = 2r$. Similarly to the case of K3 surfaces, this reduction step is done via lattice theory and the surjectivity of the period map: Let Λ be the $\mathrm{K3}^{[n]}$-lattice, and let $\mathrm{Mon}(\Lambda) \subseteq \mathrm{O}(\Lambda)$ be the subgroup corresponding to the action of the monodromy group of X on $H^2(X,\mathbb{Z})$. The fact that $\mathrm{Mon}(\Lambda)$ is well-defined is a result by Markman in [59, Thm. 1.6, Lem. 4.10]. Considering the orthogonal direct sum decomposition

$$\Lambda = L \oplus (2-2n)\mathbb{Z},$$

where L is the K3 lattice, one shows that the group of rational isometries $\mathrm{O}(\Lambda_{\mathbb{Q}})$ is generated by $\mathrm{Mon}(\Lambda)$ and $\mathrm{O}(L_{\mathbb{Q}})$. In particular, this implies that any isometry $\psi \in \mathrm{O}(\Lambda_{\mathbb{Q}})$ can be written as a composition $\psi = \psi_k \circ \cdots \circ \psi_1$, where each ψ_i is in the double orbit $\mathrm{Mon}(\Lambda)(-\rho_u)\mathrm{Mon}(\Lambda)$ for some primitive integral class $u \in L$ with $(u)^2 > 0$. Note that, considering the monodromy action, makes it possible to choose u in the K3-lattice L and not in the full lattice Λ. Fix two isometries

$$\eta_X\colon H^2(X,\mathbb{Z}) \xrightarrow{\simeq} \Lambda \quad \text{and} \quad \eta_Y\colon H^2(Y,\mathbb{Z}) \xrightarrow{\simeq} \Lambda,$$

such that the pairs (X, η_X) and (Y, η_Y) belong to the same connected component $\mathfrak{M}^\circ_\Lambda$ of the moduli space of marked hyperkähler manifolds of $\mathrm{K3}^{[n]}$-type. As f is an isometry by assumption, the composition $\psi \coloneqq \eta_Y \circ f \circ \eta_X^{-1}$ defines an element of $\mathrm{O}(\Lambda_{\mathbb{Q}})$. It can then be written as a composition $\psi = \psi_k \circ \cdots \circ \psi_1$ as above. Up to changing the sign of f, we may assume that ψ preserves the orientation of the positive cone.

In the case where ψ is the identity, the Hodge isometry $f = \eta_Y^{-1} \circ \eta_X$ is a parallel transport operator. Indeed, the isometries η_X and η_Y were chosen to ensure that (X, η_X) and (Y, η_Y) are in the same connected component $\mathfrak{M}^\circ_\Lambda$. By the Verbitsky's Torelli theorem [43, 92, 93] parallel transport operators which are Hodge isometries are algebraic. In particular, f is algebraic.

Let us now deal with the case in which ψ is not the identity. For all $i = 1, \dots, k$ by the surjectivity of the period map, there exist pairs $(X_i, \eta_i) \in \mathfrak{M}^\circ_\Lambda$ such that the isometry

$$f_i \coloneqq \eta_{X_i}^{-1} \circ \psi_i \circ \eta_{X_{i-1}} \colon H^2(X_{i-1}, \mathbb{Q}) \xrightarrow{\simeq} H^2(X_i, \mathbb{Q})$$

is a Hodge morphism. As $f = f_k \circ \ldots \circ f_1$, it suffices to prove the algebraicity of f_i. Assume now that $f \colon H^2(X, \mathbb{Q}) \to H^2(Y, \mathbb{Q})$ is in a fixed double orbit of a reflection in $u \in L$ and preserves the orientation of the positive cone. To show that f is algebraic, one considers the following construction.

For an object F of positive rank r in the derived category of (twisted) coherent sheaves on $X \times Y$, define $\kappa(F)$ as the r-th root with degree-zero summand equal to r of the Chern character of the untwisted object $F^{\otimes r} \otimes \det(F)^{-1}$. If F has negative rank, define $\kappa(F) \coloneqq -\kappa(F[1])$.

Let E be a (twisted) locally free coherent sheaf on $X \times Y$ such the pair $(X \times Y, \mathscr{E}nd(E))$ is deformation of $(X_0 \times Y_0, \mathscr{E}nd(E_0))$, where E_0 is a locally free untwisted sheaf for which the Fourier–Mukai transformation

$$\Phi_{E_0} \colon D^b(X_0) \to D^b(Y_0)$$

is an equivalence. As one checks, the Hodge morphism

$$\phi \coloneqq [\kappa(E)\sqrt{td_{X \times Y}}]_* \colon H^*(X, \mathbb{Q}) \xrightarrow{\simeq} H^*(Y, \mathbb{Q})$$

is a Hodge isometry with respect to the Mukai pairing. By a result of Taelman [86], the Hodge isometry ϕ induces a Hodge isometries of the rational LLV lattices

$$\widetilde{H}(\phi) \colon \widetilde{H}^*(X, \mathbb{Q}) \to \widetilde{H}^*(Y, \mathbb{Q}),$$

where $\widetilde{H}(X, \mathbb{Q})$ is the lattice $H^2(X, \mathbb{Q}) \oplus U_{\mathbb{Q}}$. Using a result proven independently by Beckmann [9] and Markman [61], the author shows that $\widetilde{H}(\phi)$ restricts to an isometry $H^2(X, \mathbb{Q}) \to H^2(Y, \mathbb{Q})$. More precisely, he shows that $\widetilde{H}(\phi)$ is degree-reversing in the following sense. Write

$$\widetilde{H}(X, \mathbb{Q}) = \alpha\mathbb{Q} \oplus H^2(X, \mathbb{Q}) \oplus \beta\mathbb{Q},$$

with α of degree -2, β of degree 2 and $H^2(X, \mathbb{Q})$ of degree 0. Then, $\widetilde{H}(\phi)$ preserves the degree-zero part and sends the degree -2 part to the degree 2 part and vice versa.

Finally, the author shows that any Hodge isometry which is in a double orbit of a reflection in some u and that preserves the orientation of the positive cone is the restriction to the H^2 part of a morphism $\widetilde{H}(\phi)$ for some twisted locally free coherent sheaf E on $X \times Y$ as above. In particular, this applies to the Hodge isometry f and shows that it is algebraic.

Note that in [62], the varieties X and Y are not assumed to be projective. In this case, the proof above shows that the Hodge isometry f is analytic. Which is a generalization of algebraicity to the non-projective setting.

In Appendix C, we study the case of hyperkähler manifolds of generalized Kummer type. For these manifolds, using the algebraicity of the Kuga–Satake correspondence, we prove that Hodge similarities are algebraic. Note that this in particular proves the case of Hodge isometries. See Chapter 4 for an introduction to this result.

1.5 Kuga–Satake varieties

A useful construction in the study of the Hodge conjecture for hyperkähler manifolds and powers of K3 surfaces is the Kuga–Satake construction. Since it is central in all the four appended paper, let us shortly recall it. See [44] or Appendix A for more details.

Let (V, q) be a polarized Hodge structure of K3-type, and denote by

$$\mathrm{Cl}(V) \coloneqq \left(\bigotimes^\bullet V\right)/I_q$$

the *Clifford algebra* of V, where I_q is the two-sided ideal generated by elements of the form $v \otimes v - q(v)$ for $v \in V$. As I_q is generated by even-degree elements, there is a natural $\mathbb{Z}/2\mathbb{Z}$-grading on $\mathrm{Cl}(V)$. The *even Clifford algebra* $\mathrm{Cl}^+(V)$ of V is the even-degree part of $\mathrm{Cl}(V)$.

Let $J \coloneqq e_1 \cdot e_2 \in \mathrm{Cl}^+(V_\mathbb{R})$, where $e_1 + ie_2 \in V^{2,0}$ is a generator such that $q(e_1) = q(e_2) = -1$. By the property of multiplication on $\mathrm{Cl}^+(V_\mathbb{R})$, left-multiplication by J induces a complex structure on $\mathrm{Cl}^+(V_\mathbb{R})$ which does not depend on the choice of the generator $e_1 + ie_2$.

For $x \in \mathrm{Cl}^*(V)$, let x^* be the image of x under the involution on $\mathrm{Cl}^+(V)$ induced by

$$v_1 \otimes \cdots \otimes v_{2m} \mapsto v_{2m} \otimes \cdots \otimes v_1, \quad \text{on } V^{\otimes 2m}.$$

Taking two orthogonal elements $f_1, f_2 \in V$ with positive square, one checks that the pairing

$$\mathrm{Cl}^+(V) \times \mathrm{Cl}^+(V) \to \mathbb{Q}, \quad (v, w) \mapsto \mathrm{tr}(f_1 \times f_2 \times v^* \times w)$$

defines, up to a sign, a polarization on $\mathrm{Cl}^+(V)$.

In conclusion, we see that $\mathrm{KS}(V) \coloneqq \mathrm{Cl}^+(V_\mathbb{R})/\mathrm{Cl}^+(V)$ defines the isogeny class of an abelian variety called the *Kuga–Satake variety* of V.

The usefulness of the Kuga–Satake construction lies in the fact that it allows to realize the original weight-two Hodge structure V as a sub-Hodge structure of the tensor product of two Hodge structures of weight one. In fact, one shows that there is an embedding of Hodge structures

$$\kappa_V \colon V \hookrightarrow \mathrm{Cl}^+(V) \otimes \mathrm{Cl}^+(V).$$

Let X be a projective hyperkähler manifold or a K3 surface, and denote by $\mathrm{KS}(X)$ the Kuga–Satake variety of $V = T(X)$ endowed with the Beauville–Bogomolov form (or the intersection product) with the sign changed. In this case, the above morphism is

$$\kappa_X \colon T(X) \hookrightarrow H^1(\mathrm{KS}(X), \mathbb{Q}) \otimes H^1(\mathrm{KS}(X), \mathbb{Q}).$$

The Hodge conjecture applied to κ_X gives then the following:

Conjecture 1.5.1 (Kuga–Satake Hodge conjecture). *Let X be a projective hyperkähler or a projective K3 surface, then the morphism of Hodge structures κ_X is algebraic, i.e., it is induced by the linear combination of fundamental classes of subvarieties of $X \times \mathrm{KS}(X)^2$.*

This conjecture is not known in general. Only in the case of hyperkähler manifolds of generalized Kummer type, it has been proven in full generality. Voisin in [98] deduces the algebraicity of the Kuga–Satake correspondence from two results by Markman [60] and O'Grady [76]. See Chapter 2 and Appendix A for a review of this proof.

In the case of K3 surfaces, there are just some examples for which the Kuga–Satake Hodge conjecture has been proven: The family of Kummer surfaces and K3 surfaces with a Shioda–Inose structure of Example 1.5.2, the family of double covers of $\mathbb{P}^2$ branched along six lines of Example 1.5.3, and the family of desingularizations of a singular K3 surfaces in $\mathbb{P}^4$ with 15 simple nodes of Example 1.5.4. Finally, there are other countably many four-dimensional families of K3 surfaces for which the Kuga–Satake Hodge conjecture has been proven. These families, studied by Floccari in [21], naturally appear when considering symplectic involutions on Kum^3-type varieties. We review this construction in depth in Example 1.5.5 since it is a central ingredient for the results in Appendix D.

Example 1.5.2 (Kummer surfaces and Shioda–Inose structures). Let $X \coloneqq \mathrm{Kum}(A)$ be the Kummer surface of a general abelian variety A. By construction, X is the minimal resolution of the quotient of A by the natural involution ι sending x to $-x$. As one checks, the involution ι is symplectic, i.e., it acts trivially on $H^{2,0}(A)$. This implies that the rational two-to-one map $\psi \colon A \dashrightarrow X$ induces an algebraic isomorphism between transcendental lattices $\psi^* \colon T(X) \xrightarrow{\cong} T(A)$. In particular, to prove the Kuga–Satake Hodge conjecture for X, it suffices to prove that the composition

$$T(A) \xrightarrow{(\psi^*)^\vee} T(X) \overset{\kappa_X}{\hookrightarrow} H^1(\mathrm{KS}(X), \mathbb{Q})^{\otimes 2} \tag{1.3}$$

is algebraic, where κ_X is the Kuga–Satake correspondence and $(\psi^*)^\vee$ denotes the transpose of the pullback map ψ^*. In [71], Morrison shows that the Kuga–Satake variety of a Kummer surface X is isogenous to a power of the original abelian surface. More precisely, that it satisfies

$$\mathrm{KS}(X) \sim A^8.$$

The morphism (1.3) is then induced by a Hodge class α in the cohomology of A^9. As shown by Ribet in [79], a result of Mumford proves the Hodge conjecture for all powers of abelian surfaces. In particular, this implies that α is algebraic. Hence, the Kuga–Satake Hodge conjecture holds for the K3 surface X.

This result for Kummer surfaces easily extends to prove the Kuga–Satake Hodge conjecture for K3 surfaces with a Shioda–Inose structure. Recall that a K3 surface X has a Shioda–Inose structure if it admits a symplectic involution σ such that the minimal resolution Y of X/σ is a Kummer surface. In this case the rational quotient map $\pi\colon X \dashrightarrow Y$ induces an algebraic isomorphism of Hodge structures $\pi_*\colon T(Y) \simeq T(X)$. As the Kuga–Satake Hodge conjecture holds for Kummer surfaces, we see that this conjecture also holds for K3 surfaces with a Shioda–Inose structure.

Example 1.5.3 (Double covers of $\mathbb{P}^2$ branched along six lines)**.** As one checks, a double cover of $\mathbb{P}^2$ branched along a general sextic is a K3 surface. In the case where this sextic is the union of six lines in a general position, the double cover $\pi\colon Y \to \mathbb{P}^2$ has simple nodes over the 15 points of intersection of the lines. Blowing up these nodes on Y one gets a smooth K3 surface $Y1$. In [77], Paranjape gives an alternative description for this four-dimensional family of K3 surfaces, and deduces from it the Kuga–Satake Hodge conjecture for these K3 surfaces. In recalling this construction, we use the same notation as in the reference.

Let C be a general genus five curve with an automorphism J of order four, such that the quotient $E \coloneqq C/J$ is an elliptic curve. Let G be the group acting on $C \times C$ generated by the automorphism (J, J^{-1}) and the involution swapping the two factors. Blowing up the twelve ordinary double points of $(C \times C)/G$, one gets a smooth surface $W1$ with a natural morphism

$$W1 \to (C \times C)/G \to \mathrm{Sym}^2(E) \to E,$$

where $\mathrm{Sym}^2(E)$ is the symmetric square of E and the last morphism is the summation map. One then checks that the involution $x \mapsto -x$ on E lifts to an involution on $W1$ with eight isolated fixed points and two fixed fibres. Let $W2$ be the blow up of $W1$ at these fixed points and denote by $Y2$ the quotient of $W2$ by the induced involution. Then $Y2$ is a K3 surface and is isomorphic to a double cover of $\mathbb{P}^2$ branched along six lines $Y1$. As the author shows, any such double cover can be obtained this way.

Let $Y1$ be a double cover of $\mathbb{P}^2$ branched along six lines, let C be the genus five curve with an automorphism J of order four given by the construction above, and let E be the quotient elliptic curve. Let $\mathrm{Jac}^0(C)$ be the degree-zero Jacobian variety of C, and let

$$\mathrm{Prym}(C/E) := (\mathrm{Id} - J^*)(\mathrm{Jac}^0(C))$$

be the Prym variety of the cover $C \to E$. Recall that $\mathrm{Prym}(C/E)$ is a polarized abelian variety of dimension four, and that there is a natural morphism of abelian varieties $\mathrm{Jac}^0(C) \to \mathrm{Prym}(C/E)$ inducing

$$H^1(\mathrm{Jac}^0(C), \mathbb{Q}) \simeq H^1(E, \mathbb{Q}) \oplus H^1(\mathrm{Prym}(C/E), \mathbb{Q}).$$

Let $\pi\colon \mathrm{Sym}^2(C) \dashrightarrow Y1$ be the dominant rational map given by the construction above, and consider the following diagram

$$\begin{array}{ccccc} \mathrm{Sym}^2(C) \times \mathrm{Sym}^2(C) & \longrightarrow & \mathrm{Jac}^0(C) & \longrightarrow & \mathrm{Prym}(C/E) \\ \downarrow{\scriptstyle p_1} & & & & \\ \mathrm{Sym}^2(C) & & & & \\ \downarrow & & & & \\ Y1 & & & & \end{array}$$

where p_1 is the first projection. On the level of the second cohomology, the diagram gives an algebraic morphism $H^2(Y1, \mathbb{Q}) \to H^2(\mathrm{Prym}(C/E), \mathbb{Q})$ which induces an algebraic embedding of Hodge structures

$$\kappa_{Y1}\colon T(Y1) \hookrightarrow H^2(\mathrm{Prym}(C/E), \mathbb{Q}).$$

By representation theory ([16] and [71]), one sees that κ_{Y1} is in fact the embedding given by the Kuga–Satake construction and that $\mathrm{Prym}(C/E)$ is a simple factor of the Kuga–Satake variety of $Y1$. In conclusion, this construction proves that the Kuga–Satake Hodge conjecture holds for $Y1$.

Example 1.5.4 (Desingularization of K3 surfaces in $\mathbb{P}^4$ with 15 simple nodes)**.** The family of singular K3 surfaces in $\mathbb{P}^m$ with 15 simple nodes has been first introduced and characterized in [27] by Garbagnati and Sarti. In [49], Ingalls, Logan, and Patashnick consider the case of $m = 4$. They prove that, up to isogeny, these K3 surfaces can be covered by the square of a genus seven curve C_3 with an automorphism of order three such that the quotient D_3 is a smooth curve of genus three. They then show that a variant of Paranjape's method generalizes to this case. That is, they prove that the Prym variety of $C_3 \to D_3$ is a simple factor of the Kuga–Satake of the K3 surface, and that this construction proves that the Kuga–Satake Hodge conjecture holds for this four-dimensional family of K3 surfaces.

We conclude this section by reviewing the construction studied by Floccari in [21] which associates to any given hyperkähler manifold of Kum^3-type a K3 surface. As a result, one can deduce the Kuga–Satake Hodge conjecture for these K3 surfaces from the same conjecture for hyperkähler manifolds of generalized Kummer type.

Example 1.5.5 (K3 surfaces associated to Kum^3-type varieties)**.** Let K be a variety of Kum^3-type, and let $\mathrm{Aut}_0(K)$ be the group of automorphisms of K which act trivially on the second cohomology. As proven by Hassett and Tschinkel [40], these automorphisms are deformation invariant. In particular, the group $\mathrm{Aut}_0(K)$ does not depend on K and can be described assuming $K = \mathrm{Kum}^3(A)$, where A is an abelian surface. This is done by Boissière, Nieper-Wißkirchen, and Sarti in [10]. They prove that

$$\mathrm{Aut}_0(K) \simeq A_4 \rtimes \langle -1 \rangle,$$

where A_4 is the group of points in A of order four. In particular, the subgroup of $\mathrm{Aut}_0(K)$ of automorphisms of order two is

$$G \simeq (\mathbb{Z}/2\mathbb{Z})^5.$$

As studied by Fujiki [25] and Menet [63], the quotient K/G is primitive symplectic. This means that there exists a non-degenerate holomorphic two form σ on the smooth locus of K/G and $h^{2,0}(K/G) = 1$. Furthermore, one shows that K/G is a *singular symplectic variety* in the sense of [6, Def. 3.1], i.e., that $H^1(K/G, \mathscr{O}_{K/G}) = 0$, and that there exists a resolution of singularities $r\colon Y \to K/G$ such that $r^*(\sigma)$ extends to a holomorphic from on Y. This last notion originates from the article [8] by Beauville.

In [21], an explicit description of the resolution is provided: The singular locus of K/G is the union of 16 hyperkähler manifolds of $\mathrm{K3}^{[2]}$-type. The resolution obtained by blow up is a hyperkähler manifold Y_K of $\mathrm{K3}^{[3]}$-type. The author then shows that there is an algebraic Hodge isometry

$$t_2\colon T(Y_K) \to T(K)(2),$$

where $T(K)(2)$ denotes the transcendental lattice of K with the quadratic form multiplied by two. As we remark at the end of Section 6.2, this can be directly deduced from the existence of the rational dominant map $K \dashrightarrow Y_K$ of degree 2^5. In fact, we show that rational dominant maps between hyperkähler manifolds induce Hodge similarities between their transcendental lattice. See Appendix C, for the definition of Hodge similarity.

If K is projective, the transcendental lattice of Y_K is at most six-dimensional and there exists a K3 surface S_K such that there exists a Hodge isometry

$$t_1\colon T(S_K) \to T(Y_K).$$

By Morrison [70], the K3 surface S_K can be chosen such that the above Hodge isometry holds already with integer coefficients. In this case, by a criterion due to Mongardi–Wandel [65] and independently Addington [2], Y_K is birational to a moduli space of stable sheaves $M_{S_K,H}(v)$ on S_K, and by Markman [58], there exists a quasi-tautological sheaf over $S_K \times M_{S_K,H}(v)$. From this, one deduces that the Hodge isometry t_1 is algebraic. Composing t_1 with t_2 one then gets an algebraic Hodge isometry

$$t_2 \circ t_1\colon T(S_K) \xrightarrow{\simeq} T(K)(2).$$

From this isometry, one deduces that the Kuga–Satake variety of S_K and of K are isogenous, and that the composition

$$T(S_K) \xrightarrow{t_2 \circ t_1} T(K) \xhookrightarrow{\kappa_K} H^1(\mathrm{KS}(K), \mathbb{Q})^{\otimes 2}$$

is the Kuga–Satake correspondence for S_K, where κ_K is the Kuga–Satake correspondence for K. As the Kuga–Satake Hodge conjecture is known for hyperkähler manifolds of generalized Kummer type this implies the Kuga–Satake Hodge conjecture for the K3 surface S_K.

As Hodge isometries between K3 surfaces are algebraic by the result of Buskin [11] and again Huybrechts [45], the above construction proves the Kuga–Satake Hodge conjecture for any K3 surface S such that there is a hyperkähler manifold of generalized Kummer type K with associated K3 surface S_K for which $T(S)$ and $T(S_K)$ are Hodge isometric. These K3 surfaces come in four-dimensional families, and can be characterized as the K3 surfaces S such that there is an isometric embedding

$$T(S) \hookrightarrow \Lambda_{\mathrm{Kum}^3}(2) \otimes_{\mathbb{Z}} \mathbb{Q},$$

where $\Lambda_{\mathrm{Kum}^3} = U^{\oplus 3} \oplus \langle -8 \rangle$ is the lattice $H^2(K, \mathbb{Z})$.

We end this section and the chapter by providing another example of a construction of an algebraic correspondence between some hyperkähler manifolds of $\mathrm{K3}^{[2]}$-type and an abelian variety. As we remark at the end of the example, the abelian variety appearing is not necessarily the Kuga–Satake variety so the correspondence constructed will be called *a* Kuga–Satake correspondence and not *the* Kuga–Satake correspondence. This is based on the result by van Geemen and Izadi in [33]. A similar remark can be found in [55] by Laterveer. For a general introduction to cubic hypersurfaces and more details about the following construction we refer to [46].

Example 1.5.6 (Cycic cubic fourfolds)**.** Let $X_4 \coloneqq V(F(x_0, \ldots, x_4) + x_5^3) \subseteq \mathbb{P}^5$ be the smooth cubic fourfold, which is the triple cover of $\mathbb{P}^3$ branched along a smooth cubic threefold $X_3 \coloneqq V(F(x_0, \ldots, x_4))$. The deck transformation group for this cover is generated by the order-three automorphism of X_4

$$\rho_4 \colon [x_0 : \ldots : x_4 : x_5] \mapsto [x_0 : \ldots : x_4 : \xi x_5],$$

where ξ is a third root of unity. Similar, let $X_5 \coloneqq V(F(x_0, \ldots, x_4) + x_5^3 + x_6^3) \subseteq \mathbb{P}^6$ be the triple cover of $\mathbb{P}^5$ branched along X_4. Finally, let $X_1 \coloneqq V(y_0^3 + y_1^3 + y_2^3) \subseteq \mathbb{P}^2$ be the Fermat elliptic curve with its order-three automorphism $\rho_1 \colon [y_0 : y_1 : y_2] \mapsto [y_0 : y_1 : \xi y_2]$. Consider the dominant rational map

$$\phi \colon X_4 \times X_1 \dashrightarrow X_5, \quad ([x_0 : \ldots : x_5], [y_0 : y_1 : y_2]) \mapsto [y_2 x_0 : \ldots : y_2 x_4 : y_0 x_5 : y_1 x_5].$$

As one checks, the map ϕ induces a morphism of Hodge structures

$$\phi^* \colon H^5(X_5, \mathbb{Q})_p \twoheadrightarrow \big(H^4(X_4, \mathbb{Q})_p \otimes H^1(X_1, \mathbb{Q})\big)^{\rho_4^* \otimes \rho_1^*} \subseteq H^4(X_4, \mathbb{Q})_p \otimes H^1(X_1, \mathbb{Q}),$$

where $H^5(X_5,\mathbb{Q})_p$ and $H^4(X_4,\mathbb{Q})_p$ denote the primitive cohomologies.

The morphism ϕ^* can be used to produce a Kuga–Satake correspondence for the Hodge structure of K3-type $H^4(X_4,\mathbb{Q})_p$ as follows. Tensoring ϕ^* by $H^1(X_1,\mathbb{Q})^*$ and composing it with the trace morphism

$$H^1(X_1,\mathbb{Q})\otimes H^1(X_1,\mathbb{Q})^*\to\mathbb{Q},$$

we get the following (algebraic) morphism of Hodge structures

$$\varphi\colon H^5(X_5,\mathbb{Q})_p\otimes H^1(X_1,\mathbb{Q})^*\to H^4(X_4,\mathbb{Q})_p.$$

Note that φ is surjective. To see this, assume that X_4 is a general, so that the Hodge structure $H^4(X_4,\mathbb{Q})_p$ is irreducible. In this case, it suffices to show that φ is not identically zero, and this follows from the non-degeneracy of the trace morphism.

Let $F(X_5)$ be the Fano surface of planes on X_5, and let $\mathrm{Alb}(F(X_5))$ be its Albanese variety. Then, there is an algebraic isomorphism

$$H^1(\mathrm{Alb}(F(X_5)),\mathbb{Q})\xrightarrow{\mathrm{Alb}^*}H^1(F(X_5),\mathbb{Q})\xrightarrow{h\cup\bullet}H^3(F(X_5),\mathbb{Q})\xrightarrow{\simeq}H^5(X_5,\mathbb{Q})_p,$$

where $h\in H^2(F(X_5),\mathbb{Q})$ is an ample class, and the last morphism is the Fano correspondence. The fact that the Fano correspondence is an isomorphism for cubic fivefolds has been proven by Collino in [15]. Similarly, the Fano correspondence between X_4 and its Fano variety of lines $F(X_4)$ gives an algebraic isomorphism

$$H^4(X_4,\mathbb{Q})_p\xrightarrow{\simeq}H^2(F(X_4),\mathbb{Q})_p.$$

Putting this all together, gives an algebraic morphism

$$g\colon H^1(\mathrm{Alb}(F(X_5)),\mathbb{Q})\otimes H^1(X_1,\mathbb{Q})^*\to H^2(F(X_4),\mathbb{Q})_p.$$

Let A be the dual abelian variety of $\mathrm{Alb}(F(X_5))\times\hat{X}_1$. Then, the dual morphism of g gives an algebraic embedding of Hodge structures

$$\kappa_{F(X_4)}\colon H^2(F(X_4),\mathbb{Q})_p\hookrightarrow H^1(\mathrm{Alb}(F(X_5)),\mathbb{Q})^*\otimes H^1(X_1,\mathbb{Q})\hookrightarrow H^1(A,\mathbb{Q})\otimes H^1(A,\mathbb{Q}).$$

Note that $\kappa_{F(X_4)}$ is an embedding of Hodge structures of the primitive second cohomology of the hyperkähler manifold $F(X_4)$ into the square of the first cohomology of the abelian variety A. In this sense, $\kappa_{F(X_4)}$ is a Kuga–Satake correspondence. However, since $H^2(F(X_4),\mathbb{Q})_p$ is not Mumford–Tate general (since its endomorphism is a CM field), one cannot use the universal property of the Kuga–Satake construction to conclude that A is a factor of the Kuga–Satake variety of $F(X_4)$ and that $\kappa_{F(X_4)}$ is the embedding given by the Kuga–Satake construction.

Chapter 2

Introduction to Hyper-Kähler manifolds of generalized Kummer type and the Kuga–Satake correspondence

In this chapter, we give an overview of the content of Appendix A, which has appeared in [90] and is a joint work with Claire Voisin. In the first part article, we review the Kuga–Satake construction and the classical results related to it. We then focus on the case of hyperkähler manifolds of generalized Kummer type. This is the only class of hyperkähler manifolds for which the Kuga–Satake Hodge conjecture has been proven in full generality. This is a result by Voisin [98], based on two previous results by Markman [60] and O'Grady [76].

2.1 Kuga–Satake varieties

As we have recalled in Section 1.5, given a polarized Hodge structure of K3-type (V, q), one can associate to it an abelian variety $\mathrm{KS}(V)$ called its *Kuga–Satake variety.* This abelian variety has the property that there exists an embedding of Hodge structures

$$V \hookrightarrow H^1(\mathrm{KS}(V), \mathbb{Q}) \times H^1(\mathrm{KS}(V), \mathbb{Q}).$$

In general, the abelian variety $\mathrm{KS}(V)$ is not simple, and the Hodge structure $H^1(\mathrm{KS}(V), \mathbb{Q})$ is not irreducible but is a power of a simple Hodge structure $H^1(\mathrm{KS}(V), \mathbb{Q})_c$. This is related with the fact that $\mathrm{End}(\mathrm{KS}(V)) \otimes \mathbb{Q}$ is in general very large: Consider the action of $\mathrm{Cl}^+(V)$ on itself by right-multiplication. As one checks, this action is compatible with the Kuga–Satake Hodge structure, and defines an embedding

$$\mathrm{Cl}^+(V) \hookrightarrow \mathrm{End}_{\mathrm{Hdg}}(\mathrm{Cl}^+(V)) \simeq \mathrm{End}(\mathrm{KS}(V)) \otimes \mathbb{Q}.$$

As $\dim \mathrm{Cl}^+(V) = 2^{\dim V - 1}$, we deduce that the endomorphism algebra of $\mathrm{KS}(V)$ is in general very big. As an example, consider the four-dimensional spaces of Hodge structures of K3-type (V, q) for which there is an isometric embedding of quadratic spaces

$$\iota \colon V \hookrightarrow U_{\mathbb{Q}}^2 \oplus \langle a \rangle \oplus \langle b \rangle,$$

for some negative numbers a and b. For the general such Hodge structure, the embedding ι is an isometry and by [56] the Kuga–Satake variety of (V,q) is isogenous to the fourth power of an abelian fourfold of $\mathbb{Q}(-\sqrt{ab})$ with discriminant one. Our interest in these Hodge structures lies in the fact that they appear as transcendental lattice of projective hyperkähler manifolds of generalized Kummer type.

Remarkably, $\mathrm{KS}(V)$ satisfies a universal property if (V,q) is Mumford–Tate general, i.e., if $\mathrm{Hdg}(V) \simeq \mathrm{SO}(V,q)$. Indeed, the following uniqueness result by Charles [13] holds.

Theorem 2.1.1 (Theorem A.4.1)**.** *Let (V,q) be a polarized Hodge structure of K3-type. Assume that the Hodge group of the Hodge structure on V is maximal. Let H be a simple effective weight-one Hodge structure, such that there exists an injective morphism of Hodge structures*

$$V \hookrightarrow \mathrm{End}(H).$$

Then, H is a direct summand of the Kuga–Satake Hodge structure $H^1(\mathrm{KS}(V),\mathbb{Q})$.

The Kuga–Satake construction satisfies other similar universal properties. Note however that the Mumford–Tate generality assumption appears in all of them. As explained in [35, Sec. 3], given a K3 surface S with a non-symplectic automorphism of order five, there is a positive-dimensional family of weight one Hodge structure H_1 such that, for some weight-one Hodge structure H_2, there is an embedding of Hodge structures

$$T(S) \hookrightarrow H_1 \otimes H_2.$$

This shows that if a Hodge structure of K3-type is not Mumford–Tate general, its Kuga–Satake variety does not need to satisfy a universal property as above.

2.2 Intermediate Jacobians and the Kuga–Satake Hodge conjecture

Let X be a projective hyperkähler manifold of generalized Kummer type. By Göttsche [36], the third cohomology $H^3(X,\mathbb{Q})$ is eight-dimensional and satisfies $H^{3,0}(X)=0$. In particular, $H^3(X,\mathbb{Q})$ is a Hodge structure of abelian-type and

$$J^3(X) \coloneqq H^{1,2}(X)/H^3(X,\mathbb{Z})$$

defines a complex torus. It is called the *intermediate Jacobian* of X, and by the projectivity of X, is an abelian variety. Naturally, there is an isomorphism

$$H_1(J^3(X),\mathbb{Q}) \simeq H^3(X,\mathbb{Q}).$$

In [60], Markman proves the Hodge conjecture for this morphism, i.e., that it is induced by a cycle $\mathcal{Z} \in \mathrm{CH}^2(X \times J^3(X))_{\mathbb{Q}}$.

The existence of intermediate Jacobians and the algebraicity of the above correspondence can be used to prove the Kuga–Satake Hodge conjecture for projective hyperkähler manifolds of generalized Kummer type as follows.

The first step is to show that $J^3(X)$ is a simple factor of the Kuga–Satake variety of X. Let $c_X \in \mathrm{Sym}^2 H^2(X,\mathbb{Q})$ be the dual of the Beauville–Bogomolov form on $H^2(X,\mathbb{Q})$. By Verbitsky [91], the cup product defines an embedding

$$\mathrm{Sym}^2 H^2(X,\mathbb{Q}) \hookrightarrow H^4(X,\mathbb{Q}).$$

Therefore, c_X can be viewed as a cohomology class in $H^4(X,\mathbb{Q})$, and one can define the morphism

$$\phi\colon \textstyle\bigwedge^2 H^3(X,\mathbb{Q}) \longrightarrow H^{4n-2}(X,\mathbb{Q}), \quad \alpha\wedge\beta \longmapsto c_X^{n-2}\cup\alpha\cup\beta.$$

The morphism ϕ is called the *O'Grady map* since it has been first introduced by O'Grady in [76]. There, the author shows that ϕ is surjective in the case where X is a hyperkähler manifold of generalized Kummer type. Note however that ϕ can be more generally defined for any hyperkähler manifold X for which $H^3(X,\mathbb{Q}) \neq 0$. By a result of Voisin [98], the morphism ϕ is surjective also in this more general case.

If X is Mumford–Tate general, we can apply the universal property of the Kuga–Satake variety of X to deduce to deduce that $H^3(X,\mathbb{Q})$ contains a direct summand of the Kuga–Satake Hodge structure $H^1(\mathrm{KS}(X),\mathbb{Q})$. As we mentioned above, for general X, there is an isomorphism of quadratic spaces

$$T(X) \simeq U_{\mathbb{Q}}^2 \oplus \langle a\rangle \oplus \langle b\rangle$$

for some negative number a and b. Therefore, $H^1(\mathrm{KS}(X),\mathbb{Q})$ is the fourth power of a simple weight-one Hodge structure $H^1(\mathrm{KS}(X),\mathbb{Q})_c$ of dimension eight. By dimension reasons, we conclude that there is an isomorphism

$$H^3(X,\mathbb{Q}) \simeq H^1(\mathrm{KS}(X),\mathbb{Q})_c.$$

The algebraicity of the Kuga–Satake Hodge correspondence then can be seen as follows: Let $C_X \in \mathrm{CH}^2(X)_{\mathbb{Q}}$ be a cycle whose cohomology class is c_X, which exists by [60], and let $\mathcal{Z} \in \mathrm{CH}^2(X \times J^3(X))_{\mathbb{Q}}$ be the Markman cycle above. Using the fact that $J^3(X)$ is a simple abelian subvariety of the Kuga–Satake variety of X, one then checks that the cycle

$$\Gamma = \mathcal{Z}^2 \cdot \mathrm{pr}_X^* C_X^{n-2} \in \mathrm{CH}^{2n}(J^3(X)\times X)_{\mathbb{Q}}$$

induces the Kuga–Satake correspondence

$$[\Gamma]_*\colon H_2(\mathrm{KS}(X)_c,\mathbb{Q}) \longrightarrow H_2(X,\mathbb{Q}).$$

Contribution by the author of the thesis

Appendix A, written by the author of the thesis in collaboration with Claire Voisin, originates from a presentation delivered during the Bonn–Paris seminar in the summer term of 2021. We do not claim mathematical originality in this appendix.

Notation

Appedix A follows mostly the notation we have introduced and used so far. We briefly highlight the differences that occur. The transcendental lattice of a hyperkähler manifold or a K3 surface X is denoted by $H^2(X,\mathbb{Q})_{\mathrm{tr}}$. Furthermore, the first cohomology of the Kuga–Satake variety of a polarized Hodge structure (V,q_V) is denoted by $H^1_{\mathrm{KS}}(V)$. Finally, in the paper, we refer to Hodge structures of K3-type as *Hodge structures of hyper-Kähler-type*, and to the Hodge group as the *Mumford–Tate group*.

Chapter 3

Introduction to the Hodge conjecture for powers of K3 surfaces of Picard number 16

In this chapter, we give an overview of the content of Appendix B, which will appear in [89]. In the article, we study the Hodge conjecture for powers of K3 surfaces. In particular, in the case of K3 surfaces with a totally real endomorphism field, we prove the existence of *exceptional Hodge classes*. This shows that the Hodge conjecture for the square of a K3 surface does not necessarily imply the same conjecture for all its powers. We then extend a result in [81] and prove the Hodge holds for all powers of a K3 surface of Picard number 16 if the Kuga–Satake Hodge conjecture holds.

3.1 The Hodge conjecture for powers of K3 surfaces

In the first half of the paper, we correct a result in [78] using techniques which were first introduced by Ribet [79] to study the Hodge conjecture for abelian varieties.

Let X be a projective K3 surface, and let $T(X)$ be its transcendental lattice. By the Künneth decomposition and the triviality of the first and third cohomology groups, one sees that the Hodge conjecture for the powers of X follows if one proves that the Hodge classes in the tensor algebra $\bigotimes^\bullet T(X)$ are algebraic. As recalled in Section 1.3, Hodge classes in this algebra are the invariant classes under the natural action of the Hodge group $\mathrm{Hdg}(X)$ of $T(X)$. By Zarhin [101], the group $\mathrm{Hdg}(X)$ is either a special orthogonal group or a unitary group depending on whether the endomorphism field

$$E \coloneqq \mathrm{End}_{\mathrm{Hdg}}(T(X))$$

is totally real or a CM field.

In particular, one can describe a set of generators of the algebra of Hodge classes in $\bigotimes^\bullet T(X)$ using invariant theory. Ramon-Marí in [78] shows that, if E is a CM field, this algebra is generated by degree-two elements. In particular, this implies that the Hodge conjecture for X^2 implies the Hodge conjecture for all powers of X.

In the totally real case, the situation is different and it is not true that the algebra of Hodge classes in $\bigotimes^\bullet T(X)$ is generated in degree-two. In fact, we prove the following:

Theorem 3.1.1 (Theorem B.1.1)**.** *Let X be a K3 surface with totally real endomorphism field E. Then, any Hodge class in $\bigotimes^{\bullet} T(X)$ can be expressed in terms of Hodge classes of degree two and the exceptional Hodge classes in $T(X)^{\otimes r}$, where $r := \dim_E T(X)$.*

We show that there is a natural embedding

$$\det_E T(X) := \textstyle\bigwedge_E^r T(X) \hookrightarrow \bigwedge_{\mathbb{Q}}^r T(X) \hookrightarrow T(X)^{\otimes r},$$

whose image consists of Hodge classes. We call them *exceptional Hodge classes* in analogy with the exceptional Hodge classes in the case of abelian varieties of Weil type, see [99] and [28]. The fact that the image of above embedding consists of Hodge classes follows from the fact that $\det_E T(X)$ is invariant under the action of the special orthogonal group of the E-vector space $T(X)$. Note however that, whereas degree-two Hodge classes $T(X)^{\otimes 2}$ are invariant under the action of the full orthogonal group of $T(X)$ as an E-vector space, classes in $\det_E T(X)$ are not. From this observation, one deduces that the exceptional Hodge classes are indeed not in the algebra generated by the Hodge classes in $T(X)^{\otimes 2}$.

The results above can be used to prove that the Hodge conjecture for all powers of a K3 surface specializes in families in the case where the endomorphism field of the special K3 surface is equal to the endomorphism field of the general element of the family. See Corollary B.4.2 for the case in which the endomorphism field of the special fiber is $\mathbb{Q}$. In this case, to prove the Hodge conjecture for the special fiber it suffices to prove that the determinant of its transcendental lattice is algebraic. As we show, this follows from the algebraicity of the determinant of the transcendental lattice of the general fibre. See Chapter 6 for a discussion on how to adapt the argument to the case where the endomorphism field of the special fibre is totally real but different from $\mathbb{Q}$. Note that we still assume that the endomorphism field of the special fibre is equal to the endomorphism field of the general fibre.

3.2 Families of K3 surfaces of Picard number 16

In [81], Schlickewei proves the Hodge conjecture for the square of the K3 surfaces in the family of double covers of $\mathbb{P}^2$ branched along six lines. The author deduces this result from the Kuga–Satake Hodge conjecture, which holds for these K3 surfaces by Paranjape [77] as we recalled in Example 1.5.3. The main result of our paper is an extension of this proof to show that the Hodge conjecture holds for all powers of these K3 surfaces. In particular, we produce in this way the first examples of K3 surfaces with algebraic exceptional Hodge classes. Furthermore, we show that these techniques extend to other families of K3 surfaces of Picard number 16.

Theorem 3.2.1 (Theorem B.1.2)**.** *Let $\mathscr{X} \to S$ be a four-dimensional family of K3 surfaces whose general fibre is of Picard number* 16 *with an isometry*

$$T(\mathscr{X}_s) \simeq U_{\mathbb{Q}}^2 \oplus \langle a \rangle \oplus \langle b \rangle,$$

for some negative integers a and b. If the Kuga–Satake correspondence is algebraic for the fibres of this family, then the Hodge conjecture holds for all powers of every K3 surface in this family.

The first step is to prove the Hodge conjecture for the powers of the general K3 surface X of the family. In this case, the endomorphism field of X is $\mathbb{Q}$. By a result of Lombardo [56], the Kuga–Satake variety of X is a power of an abelian fourfold of Weil type with discriminant one. The Hodge conjecture for these fourfolds holds by Markman in [60]. In Section B.5, we review a proof of a result of Abdulali [1] that shows that this implies the Hodge conjecture for all powers of these abelian fourfolds. In particular, we see that the image via the Kuga–Satake correspondence of every Hodge class in the tensor algebra of $T(X)$ is algebraic. Via the transposed morphism of the Kuga–Satake correspondence, we see that this implies that every Hodge class in $\bigotimes^* T(X)$ is algebraic. This concludes the proof for the general K3 surface of the family.

A similar proof works also for the K3 surfaces of the family with totally real multiplication different from $\mathbb{Q}$. In this case, the Hodge conjecture for the powers of the Kuga–Satake variety holds by another result of Abdulali [1], using the fact that by [81], the Kuga–Satake variety is a power of an abelian fourfold with quaternionic multiplication.

By [78], the Hodge conjecture holds for all powers of K3 surfaces with a CM endomorphism field. Therefore, to conclude the proof of Theorem 3.2.1, it only remains to study the case in which the K3 surface has with endomorphism field $\mathbb{Q}$ and Picard number higher than 16. In this case, the endomorphism field of the K3 surface is equal to the endomorphism field of the general K3 surface of the family. Therefore, as remarked above, the Hodge conjecture for the powers of these K3 surfaces can be deduced from the same conjecture for the general K3 surface of the family.

In Chapter 6, we slightly improve this result, showing the following: Let X be a K3 surface such that there is an isometrical embedding $T(X) \hookrightarrow U_{\mathbb{Q}}^2 \oplus \langle a \rangle \oplus \langle b \rangle$ for some negative numbers a and b. Then, if the Kuga–Satake Hodge conjecture for X implies the Hodge conjecture for all powers of X. Note that in Theorem 3.2.1, to deduce the Hodge conjecture for the powers of the K3 surfaces of Picard number higher than 16 we had to assume that the Kuga–Satake Hodge conjecture holds for all the K3 surfaces of the four-dimensional family.

Chapter 4

Introduction to Hodge similarities, algebraic classes, and Kuga–Satake varieties

In this chapter, we give an overview of the content of Appendix C, which has appeared in [88]. In the paper, we introduce and study the notion of Hodge similarity, and we prove the Hodge conjecture for some of these morphisms using symplectic automorphisms of K3 surfaces or the Kuga–Satake correspondence.

4.1 Hodge similarities

Let X and Y be projective hyperkähler manifolds or K3 surfaces, and let $(T(X), q_X)$ and $(T(Y), q_Y)$ be their transcendental lattices, where q_X and q_Y are the respective Beauville–Bogomolov forms (or intersection products). As in Section 1.1, a morphism of Hodge structure $\psi\colon T(X) \to T(Y)$ determines a Hodge class in $H^{2n,2n}(X \times Y, \mathbb{Q})$, where $\dim X = 2n$. As we have recalled in Example 1.4.1 and Example 1.4.2, if ψ is a Hodge isometry, i.e.,

$$q_Y(\psi v, \psi w) = q_X(v, w), \quad \forall v, w \in T(X)$$

the algebraicity of ψ is known in the case where X and Y are both K3 surfaces, or hyperkähler manifold of $\mathrm{K3}^{[n]}$-type of the same dimension. In Appendix C, we investigate the Hodge conjecture for Hodge similarities, which are Hodge morphisms that preserve the quadratic forms up to a scalar.

Definition 4.1.1 (Definition C.2.2)**.** Let (V, q_V) and $(V', q_{V'})$ be polarized Hodge structures of K3-type, and let $\psi\colon V \to V'$ be a Hodge isomorphism. We say that ψ is a *Hodge similarity* if there exists a positive rational number $\lambda_\psi \in \mathbb{Q}$ such that

$$q_{V'}(\psi v, \psi w) = \lambda_\psi q_V(v, w), \quad \forall v, w \in V.$$

We call λ_ψ the *multiplier* of ψ. A *Hodge isometry* is a Hodge similarity ψ of multiplier $\lambda_\psi = 1$.

The easiest example of Hodge similarity is the following: Let (V, q_V) be a polarized Hodge structure of K3-type and consider the polarized Hodge structure of K3-type

$(V, \lambda q_V)$, where the quadratic form is multiplied by a positive scalar $\lambda \in \mathbb{Q}$. The identity morphism

$$\mathrm{Id}_\lambda\colon (V, q_V) \to (V, \lambda q_V)$$

is then a Hodge similarity of multiplier λ. In the case where λ is not a square and the dimension of V is odd, the Hodge structures (V, q_V) and $(V, \lambda q_V)$ are not isometric so Id_λ is not a homothety (i.e., an isometry composed by multiplication by a scalar).

Consider now the case in which the endomorphism field of (V, q_V) is a totally real field E of degree two. Then, E is isomorphic to $\mathbb{Q}(\sqrt{d})$ for some positive square-free integer d. As totally real morphisms are fixed by the Rosati involution, we see that the morphism

$$\sqrt{d}\colon V \to V$$

is a Hodge similarity of multiplier d. This gives another example of Hodge similarity.

4.2 Symplectic automorphsims on K3 surfaces and Hodge similarities

Let X be a K3 surface with a symplectic automorphism σ_p of order p, and let Y be the K3 surface which is the minimal resolution of the quotient X/σ_p. We show in Section C.3 that the natural rational dominant map $X \dashrightarrow Y$ induces an algebraic Hodge similarity

$$\varphi\colon T(Y) \to T(X)$$

of multiplier p. See also Section 6.2 for more detail.

In the case where the endomorphism field of $T(X)$ is $\mathbb{Q}(\sqrt{p})$, the algebraicity of the morphism φ can be used to prove that $\psi := \sqrt{p}$ is an algebraic automorphism of $T(X)$: Consider the morphism $\varphi^{-1} \circ \psi\colon T(X) \to T(Y)$. By [11] and [45], it is algebraic since it is a Hodge isometry. Therefore, also ψ is algebraic since it can be written as the composition of two algebraic morphisms:

$$\psi = \varphi \circ (\varphi^{-1} \circ \psi).$$

In short, this proves the following.

Theorem 4.2.1 (Theorem C.3.1)**.** *Let X be a K3 surface Hodge isometric to a K3 surface with a symplectic automorphism of prime order p. Assume furthermore that $\mathbb{Q}(\sqrt{p}) \subseteq \mathrm{End}_{\mathrm{Hdg}}(T(X))$. Then, the Hodge similarity $\sqrt{p}$ is algebraic.*

In the remainder of Section C.3, we show that the family of K3 surfaces satisfying the hypotheses of Theorem 4.2.1 are four-dimensional for $p = 2$ and two-dimensional for $p = 3$. In particular, this result provides the first example of four-dimensional family of K3 surfaces with totally real multiplication of degree two for which the Hodge conjecture holds for the square of its general member.

4.3 Functoriality of Kuga–Satake varieties

The main result of Appendix C follows from the observation that the Kuga–Satake construction is functorial with respect to Hodge similarities in the following sense. Let $\psi\colon (V, q) \to (V', q')$ be a Hodge similarity of polarized Hodge structures of K3-type. In Proposition C.4.1, we show that there is an isogeny of abelian varieties $\psi_{\mathrm{KS}}\colon \mathrm{KS}(V) \to \mathrm{KS}(V')$ making the following diagram commute

$$\begin{array}{ccc} V & \xrightarrow{\quad\psi\quad} & V' \\ \big\downarrow & & \big\downarrow \\ H^1(\mathrm{KS}(V), \mathbb{Q})^{\otimes 2} & \xrightarrow{(\psi_{\mathrm{KS}})_*^{\otimes 2}} & H^1(\mathrm{KS}(V'), \mathbb{Q})^{\otimes 2} \end{array},$$

where the vertical arrows are the Kuga–Satake correspondences.

In the case where the Hodge structures V and V' are transcendental lattices of two projective hyperkähler manifolds or K3 surfaces X and X' for which the Kuga–Satake Hodge conjecture holds, the above functoriality almost implies the algebraicity of any Hodge similarity $\psi\colon T(X) \to T(X')$. Indeed, the following holds:

Theorem 4.3.1 (Theorem C.5.5). *Let X and X' be two hyperkähler manifolds for which the Kuga–Satake Hodge conjecture holds. Then, for every Hodge similarity $\psi\colon T(X) \to T(X')$, the composition*

$$T(X) \xrightarrow{\psi} T(X') \xrightarrow{h_{X'}^{2n-2}\cup\bullet} H^{4n-2}(X', \mathbb{Q})$$

is algebraic, where $2n \coloneqq \dim X'$.

If moreover X' satisfies the Lefschetz standard conjecture in degree two, i.e., the inverse of the map $T(X') \xrightarrow{h_{X'}^{2n-2}\cup\bullet} H^{4n-2}(X', \mathbb{Q})$ is algebraic, this result implies that any Hodge similarity $\psi\colon T(X) \to T(X')$ is algebraic.

The family of hyperkähler manifolds of generalized Kummer type gives the main examples of varieties for which the Kuga–Satake Hodge conjecture is known to hold. As we review in [90], this has been deduced by Voisin [98] based on previous results by Markman [60] and O'Grady [76]. As the Lefschetz standard conjecture in degree two holds for these manifolds by Foster [24], we conclude that Hodge similarities between the transcendental lattices of two projective hyperkähler manifolds of generalized Kummer type are algebraic. Using the fact that the endomorphism field of these varieties is always generated by Hodge similarities, we then conclude the following:

Theorem 4.3.2 (Theorem C.1.4). *Let X and X' be hyperkähler manifolds of generalized Kummer type such that $T(X)$ and $T(X')$ are Hodge similar. Then, every Hodge morphism between $T(X)$ and $T(X')$ is algebraic.*

Chapter 5

Introduction to algebraic cycles on hyper-Kähler varieties of generalized Kummer type

In this chapter, we give an overview of the content of Appendix D which is a joint work with Salvatore Floccari appeared in [23]. In the paper, we prove the Hodge conjecture for the Hodge classes in the algebra generated by the second cohomology of a hyperkähler manifold of generalized Kummer type. From this we deduce the Hodge conjecture for projective Kum^2-type manifolds.

5.1 An algebraic correspondence with a K3 surface

The second cohomology of a Kum^n-type variety X is seven-dimensional by the result of Beauville in [7]. In the case where X is projective, the transcendental part $H^2(X,\mathbb{Z})_{\mathrm{tr}}$ of its second integral cohomology is then at most six-dimensional. Given a positive integer k, let $H^2(X,\mathbb{Z})_{\mathrm{tr}}(k)$ be the Hodge structure $H^2(X,\mathbb{Z})_{\mathrm{tr}}$ with the quadratic form multiplied by k. By dimension and signature reasons, there exists by Nikulin [74, Thm. 1.14.4] a primitive embedding of lattices

$$H^2(X,\mathbb{Z})_{\mathrm{tr}}(k) \hookrightarrow \Lambda_{K3},$$

where $\Lambda_{K3} = E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$ is the K3-lattice. In particular, by the surjectivity of the period morphism for K3 surfaces, there exists a K3 surface S such that there is a Hodge isometry

$$\varphi\colon T(X)(k) \longrightarrow T(S). \tag{5.1}$$

Note that, even though this Hodge isometry holds already with integer coefficients, we extended scalars to $\mathbb{Q}$ since we are concerned with the rational Hodge conjecture for $S \times X$, which predicts that φ is algebraic. In the special case where X is six-dimensional and $k = 2$, the algebraicity of φ follows from the work [21]: As recalled in Example 1.5.5, given a projective Kum^3-type variety K there is a dominant rational map of degree 2^5

$$r\colon K \dashrightarrow Y_K$$

for some $K3^{[3]}$-type variety Y_K. By degree reasons and the fact that Y_K is birational to a moduli of sheaves on a K3 surface S_K, the pushforward by r induces an algebraic Hodge isometry

$$\frac{1}{16}r_*\colon T(K)(2) \longrightarrow T(S_K).$$

In the paper, we prove that for any dimension n and any projective Kum^n-type variety X, there exists an integer k and a K3 surface S for which a morphism φ as in (5.1) is algebraic. To do this, we first prove in Lemma D.4.3 that, given X, there exists a projective Kum^3-type variety K for which there is a Hodge similarity

$$T(X) \longrightarrow T(K).$$

Recall that Hodge similarities of hyperkähler manifolds of generalized Kummer type are algebraic by [88]. By the above construction, we know that there exists a K3 surface S_K with an algebraic Hodge isometry

$$T(K)(2) \longrightarrow T(S_K).$$

Taking the composition, we get an algebraic Hodge similarity $\varphi\colon T(X) \longrightarrow T(S_K)$ as claimed.

5.2 Algebraic cycles

The above construction is used to prove the following:

Theorem 5.2.1 (Theorem D.1.1)**.** *Let X be a projective manifold of* Kum^n*-type, $n \geq 2$. Denote by $A_2^\bullet(X) \subset H^\bullet(X,\mathbb{Q})$ the subalgebra of the rational cohomology generated by $H^2(X,\mathbb{Q})$. Then any Hodge class in $A_2^{2j}(X) \cap H^{j,j}(X)$ is algebraic, for any j.*

Note that by the Lefschetz theorem, it suffices to prove the algebraicity of the Hodge classes in $A_2^{2j}(X)$ for $j \leq n$. By [91], there is an isomorphism

$$A_2^{2j}(X) \simeq \mathrm{Sym}^j(H^2(X,\mathbb{Q})), \quad \forall j \leq n.$$

Using the decomposition $H^2(X,\mathbb{Q}) \simeq \mathrm{NS}(X)_{\mathbb{Q}} \oplus T(X)$, we see that to prove the theorem, we have to prove the algebraicity of the Hodge classes in $\mathrm{Sym}^j(T(X))$. This can be seen as follows.

As observed in [21], the four-dimensional families of K3 surfaces appearing in these construction satisfy the hypotheses of Theorem B.1.2: The general transcendental lattice is isometric to $U_{\mathbb{Q}}^2 \oplus \langle a \rangle \oplus \langle b \rangle$ for some negative numbers a and b, and the Kuga–Satake Hodge conjecture holds for these K3 surfaces has been deduced in [21] from the algebraicity of the Kuga–Satake correspondence for the associated Kum^3-type variety. In particular, the Hodge conjecture holds for all powers of these K3 surfaces.

Then, using the inverse of the algebraic morphism φ constructed above, the algebraicity of the Hodge classes in $\mathrm{Sym}^j(T(X))$ follows from the algebraicity of the Hodge classes in the tensor algebra of transcendental lattice of the associated K3 surface. Note that, to deduce that the inverse of φ is algebraic, one uses the fact that the Lefschetz standard conjecture in degree-two holds for hyperkähler manifolds of generalized Kummer type by [24].

In Theorem 5.2.1, we study only the subalgebra generated by the second cohomology. In general, this does not imply the Hodge conjecture for the variety X. Nonetheless, in low-dimension, the result above is in fact sufficient: In dimension four the complement of $A_2^\bullet(X)$ in $H^\bullet(X, \mathbb{Q})$ consists of the odd cohomology and of an 80-dimensional space of Hodge classes in $H^4(X, \mathbb{Q})$, by [57, Example 4.6]. The classes in this 80-dimensional space have the property that they are of Hodge-type on all deformations of X. This has been used by Hasset and Tschinkel in [40], to prove that they are algebraic. In particular, their result together with Theorem 5.2.1 implies the following.

Corollary 5.2.2 (D.1.2)**.** *Let X be a projective manifold of* Kum^2*-type. Then, the Hodge conjecture holds for X.*

Similarly, one can deduce the Hodge conjecture for Kum^3-type varieties. This has been done in [20], but requires considerably more work since one has to show that the Hodge classes in the complement of $A_2^\bullet(X)$ are of Hodge-type on all deformations of X and that this implies that they are algebraic.

Contribution by the author of the thesis

Appendix D is the product of a joint work with Salvatore Floccari. The project originated from a Zoom call in which we discussed the results in the papers [21, 88]. We then recognized that the findings in the two papers could be utilized to establish the aforementioned results. The ownership of the results are shared equally between both authors.

Notation

Appendix D follows mostly the notation we have introduced and used so far. The main difference is that we adopt in the paper a motivic approach. To maintain coherence, we have chosen to align this introduction with the style and terminology employed throughout the rest of the thesis.

Chapter 6

Summary and conclusive remarks

In this final chapter, we give an overview and complement the results produced in the four appended papers.

6.1 The Hodge conjecture for products of K3 surfaces

As we have seen, already in the case of powers and products of K3 surfaces, the Hodge conjecture is open and has been proved only in some special cases.

Thanks to the description of the Hodge group of a polarized Hodge structure of K3-type provided by Zarhin [101], we study in Appendix B the algebra of Hodge classes in the tensor algebra of the transcendental lattice $T(X)$ of a K3 surface X. Denoting by E the endomorphism field of $T(X)$, we prove the following. If E is a CM field, then the Hodge conjecture for X^2 is equivalent to the Hodge conjecture for all its powers. If E is totally real, to prove the Hodge conjecture for the powers of X, one has to prove the Hodge conjecture for X^2 and that the (exceptional) Hodge classes in $\det_E T(X)$ are algebraic, where $\det_E T(X)$ is defined as the image of the natural map

$$\bigwedge\nolimits_E^{\dim_E T(X)} T(X) \hookrightarrow \bigwedge\nolimits^{\dim_E T(X)} T(X) \hookrightarrow T(X)^{\otimes \dim_E T(X)}.$$

Note that the CM case was already studied in [78] and implies the Hodge conjecture for the powers of these K3 surfaces: Considering the isomorphism

$$(T(X) \otimes T(X))^{\mathrm{Hdg}(X)} \simeq (T(X) \otimes T(X)^*)^{\mathrm{Hdg}(X)} \simeq \mathrm{End}_{\mathrm{Hdg}}(T(X)) =: E,$$

we see that the Hodge conjecture for X^2 is equivalent to prove the algebraicity of the elements in E. As CM fields are generated by isometries, this follows from the fact that Hodge isometries between K3 surfaces are algebraic by [11] and [45].

The above description of the algebra of Hodge classes on the powers of K3 surfaces, allowed us to address the following question: *Does the Hodge conjecture for the powers of a K3 surface specialize in families?* That is, given a family of K3 surfaces $\mathscr{X} \to B$ such that the Hodge conjecture holds for all powers of its general elements, is it true that the

Hodge conjecture then holds for all powers of all K3 surfaces of the family? If not, by how much this fails to be true?

To answer this, let $X \coloneqq \mathscr{X}_0$ for some fixed element $0 \in B$. Comparing $T(X)$ with $T(\mathscr{X}_b)$ for a general $b \in B$, we see that two things might happen

(i) $\mathrm{End}_{\mathrm{Hdg}}(T(\mathscr{X}_b)) \subseteq \mathrm{End}_{\mathrm{Hdg}}(T(X))$ is not an equality;

(ii) $\dim T(\mathscr{X}_b) > \dim T(X)$.

The former case shows that the answer to the above question is negative. Indeed, if $\mathrm{End}_{\mathrm{Hdg}}(T(\mathscr{X}_b)) \subseteq \mathrm{End}_{\mathrm{Hdg}}(T(X))$ is not an equality, there exists a Hodge class $\alpha \in T(X)^{\otimes 2}$ which is not the specialization of a Hodge class in $T(\mathscr{X}_b)^{\otimes 2}$. Thus, the algebraicity of α cannot be deduced from the Hodge conjecture for the powers of the general K3 surface in the family.

On the other hand, if the endomorphism field of the special fibre is equal to the endomorphism field of the general fibre, the Hodge conjecture does specialize even if the dimension of the transcendental lattice of the special K3 surface drops as in (ii). In Corollary B.4.2, we prove the above statement in the case where

$$\mathrm{End}_{\mathrm{Hdg}}(T(X)) \simeq \mathrm{End}_{\mathrm{Hdg}}(T(\mathscr{X}_b)) \simeq \mathbb{Q}$$

by showing that $\det T(X)$ is algebraic: Let $\widetilde{T}$ be the transcendental lattice of the general fibre $\mathscr{X}_b$. By hypothesis, the Hodge class $\det \widetilde{T}$ is algebraic on the powers of $\mathscr{X}_b$. Therefore, it specializes to an algebraic class on the powers of X. Note that there is an orthogonal decomposition

$$\widetilde{T} \simeq T(X) \oplus N$$

for some $N \subseteq \mathrm{NS}(X)_{\mathbb{Q}}$. As $\det \widetilde{T}$ is algebraic and all classes in N are algebraic, the class

$$\alpha \coloneqq \det \widetilde{T} \otimes \det N,$$

is algebraic on X^{n+2k}, where $n \coloneqq \dim T(X)$ and $k \coloneqq \dim N$. Fix now a point $x \in X$ and consider the natural morphism

$$\varphi\colon X^n \times \{x\}^k \hookrightarrow X^n \times X^k \xhookrightarrow{\mathrm{Id}_{X^n} \times \Delta_{X^k}} X^{n+2k}.$$

As one checks, the pullback $\varphi^*(\alpha)$ is equal to a multiple of $\det T(X)$, so in particular $\det T(X)$ is algebraic. This follows from the fact that $T(X)$ and N are orthogonal (with respect to the intersection product), and that there is an equality

$$\det \widetilde{T} = \sum \pm q_I^* \det T(X) \otimes p_I^* \det N,$$

where the sum runs over all subsets $I \subseteq \{1, \ldots, n+k\}$ of length k, p_I is the projection from X^{n+k} onto the I-th factors, and q_I is the projection onto the remaining factors.

A similar proof works also in the case where $\mathrm{End}_{\mathrm{Hdg}}(T(X)) \simeq \mathrm{End}_{\mathrm{Hdg}}(T(\mathscr{X}_b))$ is any totally real field E. As recalled at the end of Section 1.3, the decomposition $\widetilde{T} \simeq T(X) \oplus N$ is a decomposition of E-vector spaces, and the algebraicity of the Hodge classes in $\det_E T(X)$ can be deduced from the algebraicity of the Hodge classes in $\det_E \widetilde{T}$. This then implies the Hodge conjecture for the powers of the K3 surface X.

As an application, we show the following in Theorem B.1.2. Given a four-dimensional family of K3 surfaces of generic Picard number 16 for which the Kuga–Satake Hodge conjecture holds, then the Hodge conjecture holds for all powers of the K3 surfaces of the family. Note that by [21], this hypothesis is satisfied by countably many such four-dimensional families. In particular, we get the Hodge conjecture for all powers of K3 surfaces in countably many four-dimensional families.

In the following, we refine the above statement. We prove that the Hodge conjecture for the powers of a K3 surface as in the theorem can be deduced directly from the algebraicity of the Kuga–Satake correspondence and does not rely on the Kuga–Satake Hodge conjecture for the nearby fibres.

Theorem 6.1.1. *Let X be a K3 surface such that there exists an embedding of quadratic spaces*

$$T(X) \hookrightarrow U_{\mathbb{Q}}^2 \oplus \langle a \rangle \oplus \langle b \rangle,$$

for some negative integers a and b. If the Kuga–Satake Hodge conjecture holds for X, then the Hodge conjecture holds for all powers of X.

Proof. Let E be the endomorphism field of $T(X)$. As $T(X)$ is at most six-dimensional, one of the following holds:

(i) $\dim T(X) = 6, 4$, or 2 and E is a CM field;

(ii) $\dim T(X) = 6$ and $E = \mathbb{Q}$;

(iii) $\dim T(X) = 6$ and E is a totally real field of degree two;

(iv) $\dim T(X) \leq 5$ and $E \simeq \mathbb{Q}$.

In the case (i), the Hodge conjecture for the powers of X holds by [78]. In the cases (ii) and (iii), we see from the proof of Theorem B.1.2 that the Hodge conjecture for the powers of X follows from the fact that the Hodge conjecture holds for all powers of $\mathrm{KS}(X)$ and the fact that the Kuga–Satake correspondence is algebraic for X by assumption.

Only in the case (iv), we used a different argument and we deduced the Hodge conjecture for the powers of X from the same conjecture for the powers of the nearby fibres of the family $\mathscr{X}$. This can be avoided as follows. As the endomorphism field of X is $\mathbb{Q}$ by assumption, to prove the Hodge conjecture for the powers of X, it suffices to show that $\det T(X)$ is algebraic. By hypothesis, $T(X)$ is a quadratic subspace of

$$\widetilde{T} \coloneqq U_{\mathbb{Q}}^2 \oplus \langle a \rangle \oplus \langle b \rangle.$$

Up to an orthogonal transformation of the K3-lattice, we may assume that the embedding $T(X) \hookrightarrow H^2(X,\mathbb{Q})$ factors through $\widetilde{T} \hookrightarrow H^2(X,\mathbb{Q})$. As we recalled earlier in this section, to prove that $\det T(X)$ is algebraic, it suffices to show that $\det \widetilde{T}$ is algebraic. As $\widetilde{T} \subseteq H^2(X,\mathbb{Q})$ decomposes as $\widetilde{T} = T(X) \oplus N$, for some $N \subseteq \mathrm{NS}(X)_{\mathbb{Q}}$, we see that $\mathrm{KS}(\widetilde{T})$ is isogenous to $\mathrm{KS}(X)^{2^{\dim N}}$. By assumption, the Kuga–Satake Hodge conjecture holds for X, and the morphism

$$H^2(X,\mathbb{Q}) \twoheadrightarrow \widetilde{T} \hookrightarrow H^1(\mathrm{KS}(\widetilde{T}),\mathbb{Q})^{\otimes 2}$$

is algebraic. Therefore, to show the algebraicity of $\det \widetilde{T}$ on the powers of X it suffices to prove that $\det \widetilde{T}$ is algebraic on some power of the abelian variety $\mathrm{KS}(\widetilde{T})$. Denote by $\widetilde{T}_\sigma$ the quadratic space $\widetilde{T}$ endowed with a very general Hodge structure of K3-type determined by some $\sigma \in \widetilde{T}$. By [56], the abelian variety $\mathrm{KS}(\widetilde{T}_\sigma)$ is the fourth power of a (general) abelian fourfold of Weil type with discriminant one. By Theorem B.1.3, the Hodge conjecture holds for all powers of $\mathrm{KS}(\widetilde{T}_\sigma)$. In particular, the Hodge class $\det \widetilde{T}_\sigma$ is algebraic on the powers of $\mathrm{KS}(\widetilde{T}_\sigma)$. As the Kuga–Satake construction works in families, we see that $\det \widetilde{T}_\sigma$ specializes to the Hodge class $\det \widetilde{T}$ on some power of $\mathrm{KS}(\widetilde{T})$. Therefore, $\det \widetilde{T}$ is algebraic on the powers of X. As remarked above, this implies that $\det T(X)$ is algebraic and concludes the proof. □

6.2 Hodge similarities and a bound on the degree of correspondences

In Appendix C, we introduce and study the concept of *Hodge similarity*. These are morphisms of Hodge structures of K3-type $\psi\colon V \to W$ such that there exists a positive $\lambda_\psi \in \mathbb{Q}$ for which

$$q_W(\psi v_1, \psi v_2) = \lambda_\psi q_V(v_1, v_2) \quad \forall v_1, v_2 \in V.$$

The number λ_ψ is called the *multiplier* of ψ. Clearly, Hodge isometries are special cases of Hodge similarities. Let us now recall two contexts in which Hodge similarities naturally appear.

Let E be the endomorphism field of a Hodge structure of K3-type. If E is a totally real field of degree two, then $E \simeq \mathbb{Q}(\sqrt{d})$ for some positive square-free $d \in \mathbb{Z}$. From the fact that totally real endomorphisms are fixed by the Rosati involution, we see that the endomorphism $\sqrt{d}$ is a Hodge similarity of multiplier d.

A second, more geometrical, instance in which Hodge similarities appear is the following. Let $f\colon Y \dashrightarrow X$ be a dominant rational map between two K3 surfaces. Then, the pullback via f induces a Hodge similarity of ratio d between $T(X)$ and $T(Y)$, where d is the degree of f: Resolving the rational map f, we obtain a diagram

$$\begin{array}{ccc} Z & \xrightarrow{f_2} & X \\ {\scriptstyle f_1}\downarrow & \overset{f}{\nearrow} & \\ Y & & \end{array},$$

where $f_1\colon Z \to Y$ is birational and $f_2\colon Z \to X$ is dominant and generically finite of order d. By the compatibility of the pullback and product, we see that f_2^* induces a Hodge similarity of ratio d between $T(X)$ and $T(Z)$, and that f_1^* gives a Hodge isometry between $T(Z)$ and $T(Y)$.

In Appendix C, we consider K3 surfaces with a symplectic automorphism of order p and totally real endomorphism field $\mathbb{Q}(\sqrt{p})$. We prove in Theorem C.1.1 that the existence of the symplectic automorphism implies the algebraicity of $\sqrt{p}$. Note that this is not obvious since the symplectic automorphism acts trivially on the transcendental lattice of the K3 surface. In particular, for $p = 2$, we prove the Hodge conjecture for the square of the general member of the first four-dimensional families of K3 surfaces with totally real multiplication of degree two.

Also in the case of hyperkähler manifolds, a rational dominant map $f\colon Y \dashrightarrow X$ induces a Hodge similarity $f^*\colon T(X) \to T(Y)$. To show this, one proves using the Fujiki relations, that f^* is a Hodge similarity of multiplier $(c_X \deg f / c_Y)^{1/n}$, where c_X and c_Y are the Fujiki constants of X and Y, and $2n$ is the dimension of X. Note in particular, that this implies that $(c_X \deg f / c_Y)^{1/n}$ is a rational number. Therefore, it imposes a constraint on the degree of rational maps between hyperkähler manifolds. In the special case of a rational dominant self-map $f\colon X \dashrightarrow X$, we see that $\deg f$ has to be an n-th power. If in addition we assume that X is Mumford–Tate general, then $\deg f$ has to be a $2n$-th power. This follows from the fact that in this case, every endomorphism of $T(X)$ is just scalar multiplication by some $d \in \mathbb{Q}$, so it is a Hodge similarity of degree d^2. These observations are product of a joint work with Evgeny Shinder.

6.3 The case of hyperkähler manifolds of generalized Kummer type

The class of Kum^n-type manifolds is the only class of hyperkähler manifolds for which the Kuga–Satake Hodge conjecture has been proven. We exploit this result multiple times in the thesis.

In Appendix C, we prove that the Kuga–Satake construction is functorial with respect to Hodge similarities. As a result we get that, if the Kuga–Satake Hodge conjecture holds for two projective hyperkähler manifolds X and Y and the Lefschetz conjecture in degree two holds for Y, then any Hodge similarity between $T(X)$ and $T(Y)$ is algebraic. In the case of Kum^n-type varieties both hypotheses are satisfied. We then deduce that Hodge similarities between projective hyperkähler manifolds of generalized Kummer type are algebraic.

In Appendix D, we study the Hodge conjecture for these manifolds and we prove the following. Let X be a Kum^n-type variety, and denote by $A_2^\bullet(X)$ the subalgebra of $H^\bullet(X, \mathbb{Q})$ generated by the second cohomology. Then, every Hodge class in $A_2^\bullet(X)$ is algebraic. In the case where X is four-dimensional, the Hodge classes in the complement of $A_2^\bullet(X)$ in $H^\bullet(X,\mathbb{Q})$ are algebraic by [40]. Therefore, we conclude that the Hodge conjecture holds for hyperkähler manifolds of Kum^2-type.

To prove this, we first construct an algebraic Hodge similarity between $T(X)$ and $T(S)$ for some K3 surface S: By lattice theory and the surjectivity of the period map, there exists a Kum^3-type variety K such that $T(K)$ is Hodge similar to $T(X)$. The construction in [21] that we recalled in Example 1.5.5 shows that there exists an algebraic Hodge similarity between the transcendental lattice of K and a K3 surface S. As we showed that Hodge similarities between hyperkähler manifolds of generalized Kummer type are algebraic, we get that also the composition

$$T(X) \xrightarrow{\simeq} T(K) \xrightarrow{\simeq} T(S)$$

is algebraic. As remarked in [21], the hypotheses of Theorem B.1.2 are satisfied and therefore the Hodge conjecture holds for the powers of the K3 surface S. This, together with the algebraicity of the Hodge morphism above, implies that every Hodge class in $A_2^\bullet(X)$ is algebraic as required.

We conclude this section by providing a more direct proof of the above result that only relies on the algebraicity of Hodge similarities for Kum^n-type varieties and the algebraicity of the Lefschetz isomorphism in degree two proven by Foster [24].

Let X be a projective Kum^n-type variety. By the Lefschetz isomorphism, to prove the Hodge conjecture for the Hodge classes in $A_2^\bullet(X)$, it suffices to show it for the Hodge classes in $A_2^j(X)$ for $j \leq n$. By [91], there is an isomorphism

$$A_2^j(X) \simeq \mathrm{Sym}^j(H^2(X,\mathbb{Q}))$$

for $j \leq n$. As all Hodge classes in $\mathrm{NS}(X)_\mathbb{Q}$ are algebraic, we just have to prove the Hodge conjecture for the Hodge classes in

$$\mathrm{Sym}^j(T(X)) \subseteq H^{2j}(X,\mathbb{Q})$$

for $j \leq n$. Considering the pullback via the diagonal embedding $X \hookrightarrow X^j$, we see that the statement follows if we prove that any symmetric Hodge class in $T(X)^{\otimes j} \subseteq H^{2j}(X^j,\mathbb{Q})$ is algebraic.

By the results of Appendix B recalled in Section 6.1, any symmetric Hodge class in $T(X)^{\otimes j}$ is in the tensor algebra generated by Hodge classes in $T(X)^{\otimes 2}$. Indeed, if the endomorphism field E of X is a CM field, the algebra of Hodge classes in $\bigotimes^\bullet T(X)$ is generated by Hodge classes in $T(X)^{\otimes 2}$, and if E is totally real, the algebra of Hodge classes in $\bigotimes^\bullet T(X)$ is generated by $T(X)^{\otimes 2}$ together with the exceptional Hodge classes in $\det_E T(X)$. As these exceptional Hodge classes are anti-symmetric, we conclude that the claim holds also in this case.

To conclude the proof, we then have to show that any Hodge class in $T(X)^{\otimes 2}$ is algebraic. By [24], the Lefschetz conjecture in degree two holds for X, and there is an algebraic isomorphism

$$T(X) \otimes (h_X^{2n-2} \cup T(X)) \xrightarrow{\simeq} T(X)^{\otimes 2}, \tag{6.1}$$

where $h_X \in H^2(X, \mathbb{Q})$ denotes the cohomology class of an ample line bundle on X. It then suffices to prove that any Hodge class on the left-hand-side of (6.1) is algebraic. As

$$E \simeq (T(X) \otimes T(X)^*)^{\mathrm{Hdg}(X)} \simeq \big(T(X) \otimes (h_X^{2n-2} \cup T(X))\big)^{\mathrm{Hdg}(X)},$$

we just have to show that any Hodge morphism in E is algebraic. This then follows from the algebraicity of Hodge similarities between Kum^n-type varieties and the fact that E is always generated by Hodge similarities. Indeed, as $\dim T(X) \leq 6$, the field E is either a CM field, $\mathbb{Q}$, or a totally real degree-two extension of $\mathbb{Q}$.

Appendix A

Hyper-Kähler manifolds of generalized Kummer type and the Kuga–Satake correspondence

M. Varesco and C. Voisin[1]

Abstract. We first describe the construction of the Kuga–Satake variety associated to a (polarized) weight-two Hodge structure of hyper-Kähler type. We describe the classical cases where the Kuga–Satake correspondence between a hyper-Kähler manifold and its Kuga–Satake variety has been proved to be algebraic. We then turn to recent work of O'Grady and Markman which we combine to prove that the Kuga–Satake correspondence is algebraic for projective hyper-Kähler manifolds of generalized Kummer deformation type.

A.1 Introduction

The Kuga–Satake construction associates to any K3 surface, and more generally to any weight-two Hodge structure of hyper-Kähler type a complex torus which is an abelian variety when the Hodge structure is polarized. This construction allows to realize the Hodge structure on degree-two cohomology of a projective hyper-Kähler manifold as a direct summand of the H^2 of an abelian variety. Although the construction is formal and not known to be motivic, it has been used by Deligne in [16] to prove deep results of a motivic nature, for example the Weil conjecture for K3 surfaces can be deduced from the Weil conjectures for abelian varieties.

Section A.2 of the notes is devoted to the description of the original construction and the presentation of a few classical examples where the Kuga–Satake correspondence is known to be algebraic, i.e., realized by a correspondence between the hyper-Kähler manifold and its Kuga–Satake variety. In Section A.3, we focus on the case of hyper-Kähler manifolds of a generalized Kummer type and present a few recent results. If X is a (very

[1] Mauro Varesco and Claire Voisin. Hyper-Kähler manifolds of generalized Kummer type and the Kuga–Satake correspondence. *Milan J. Math.*, pages 433-444, 2022. https://doi.org/10.1007/s00032-022-00369-8.

This review was prepared in the context of the seminar organized by the ERC Synergy Grant HyperK, Grant agreement ID 854361. The talk was delivered on May 14, 2021.

general) projective hyper-Kähler manifold of generalized Kummer type, the Kuga–Satake variety $\mathrm{KS}(X)$ built on $H^2(X,\mathbb{Z})_{\mathrm{tr}}$ is a sum of copies of an abelian fourfold $\mathrm{KS}(X)_c$ of Weil type. There is another abelian fourfold associated to X, namely the intermediate Jacobian $J^3(X)$ which is defined as the complex torus

$$J^3(X) = H^{1,2}(X)/H^3(X,\mathbb{Z})$$

where $b_3(X) = 8$. Here we use the fact that $H^{3,0}(X) = 0$ and the projectivity of X guarantees that $J^3(X)$ is an abelian variety. O'Grady [76] proves the following result.

Theorem A.1.1. *The two abelian varieties $J^3(X)$ and $\mathrm{KS}(X)_c$ are isogenous.*

We also prove in Section A.4 a more general statement concerning hyper-Kähler manifolds with $b_3(X) \neq 0$. Section A.5 is devoted to the question of the algebraicity of the Kuga–Satake correspondence. Following [98], we prove, using a theorem of Markman and Theorem A.1.1 above that the Kuga–Satake correspondence is algebraic for hyper-Kähler manifolds of generalized Kummer type.

Theorem A.1.2. *There exists a codimension-$2n$ cycle $\mathcal{Z} \in \mathrm{CH}^{2n}(\mathrm{KS}(X)_c \times X)_{\mathbb{Q}}$ such that*

$$[\mathcal{Z}]_* : H_2(\mathrm{KS}(X)_c, \mathbb{Q}) \to H_2(X,\mathbb{Q}) \tag{A.1}$$

is surjective.

A.2 The Kuga–Satake construction

A.2.1 Main Construction

In this section, we recall the construction and some properties of the Kuga–Satake variety associated to a Hodge structure of *hyper-Kähler type*. This construction is due to Kuga and Satake in [54]. For a complete introduction see [44, Ch. 4] and [30].

Definition A.2.1. A pair (V,q) is a Hodge structure of *hyper-Kähler type* if the following conditions hold: V is a rational level-two Hodge structure with $\dim V^{2,0} = 1$, and $q\colon V \otimes V \to \mathbb{Q}(-2)$ is a morphism of Hodge structures whose real extension is negative definite on $(V^{2,0} \oplus V^{0,2}) \cap V_{\mathbb{R}}$.

Remark A.2.2. Note that if X is a hyper-Käler manifold and q_X is the Beauville-Bogomolov quadratic form, the pair $(H^2(X,\mathbb{Q}), -q_X)$ is indeed a Hodge structure of *hyper-Kähler type*.

Let (V,q) be a Hodge structure of hyper-Kähler type, and let $T(V)$ be the tensor algebra of the underlying rational vector space V:

$$T(V) := \bigoplus_{i\geq 0} V^{\otimes i}$$

where $V^{\otimes 0} := \mathbb{Q}$. The *Clifford algebra* of (V, q) is the quotient algebra

$$\mathrm{Cl}(V) := \mathrm{Cl}(V, q) := T(V)/I(q),$$

where $I(q)$ is the two-sided ideal of $T(V)$ generated by elements of the form $v \otimes v - q(v)$ for $v \in V$. Since $I(q)$ is generated by elements of even degree, the natural $\mathbb{Z}/2\mathbb{Z}$-grading on $T(V)$ induces a $\mathbb{Z}/2\mathbb{Z}$-grading on $\mathrm{Cl}(V)$. Write

$$\mathrm{Cl}(V) = \mathrm{Cl}^+(V) \oplus \mathrm{Cl}^-(V),$$

where $\mathrm{Cl}^+(V)$ is the even part and $\mathrm{Cl}^-(V)$ is the odd part. Note that $\mathrm{Cl}^+(V)$ is still a $\mathbb{Q}$-algebra, it is called *the even Clifford algebra*.

We now use the assumption that (V, q) is a Hodge structure of hyper-Kähler type to define a complex structure on $\mathrm{Cl}^+(V)_{\mathbb{R}}$. Consider the decomposition of the real vector space $V_{\mathbb{R}} = V_1 \oplus V_2$, with

$$V_1 := V^{1,1} \cap V_{\mathbb{R}}, \quad V_2 := \{V^{2,0} \oplus V^{0,2}\} \cap V_{\mathbb{R}}.$$

The $\mathbb{C}$-linear span of V_2 is the two-dimensonal vector space $V^{2,0} \oplus V^{0,2}$. Therefore, q is negative definite on V_2. Pick a generator $\sigma = e_1 + ie_2$ of $V^{2,0}$ with $e_1, e_2 \in V_2$ and $q(e_1) = -1$. Since $q(\sigma) = 0$, we deduce that $q(e_1, e_2) = 0$ and $q(e_2) = -1$, i.e., $\{e_1, e_2\}$ is an orthonormal basis of V_2. From this, it is straightforward to check that $e_1 \cdot e_2 = -e_2 \cdot e_1$ in $\mathrm{Cl}(V)_{\mathbb{R}}$. Therefore, $J := e_1 \cdot e_2$ satisfies the equation $J^2 = -1$ and left multiplication by J induces a complex structure on the real vector space $\mathrm{Cl}(V)_{\mathbb{R}}$ which preserves the real subspaces $\mathrm{Cl}^+(V)_{\mathbb{R}}$ and $\mathrm{Cl}^-(V)_{\mathbb{R}}$. Giving a complex structure on a real vector space is equivalent to giving a Hodge structure of level one on the rational vector space:

Definition A.2.3. The *Kuga–Satake Hodge structure* $H^1_{\mathrm{KS}}(V)$ is the Hodge structure of level one on $\mathrm{Cl}^+(V)$ given by

$$\rho\colon \mathbb{C}^* \to \mathrm{GL}(\mathrm{Cl}^+(V)_{\mathbb{R}}), \quad x + yi \to x + yJ,$$

where $x + yJ$ acts on $\mathrm{Cl}^+(V)_{\mathbb{R}}$ via left multiplication.

Therefore, starting from a rational level-two Hodge structure of hyper-Kähler type (V, q), we constructed a rational Hodge structure of level one on $\mathrm{Cl}^+(V)$. This determines naturally a complex torus up to isogeny: Let $\Gamma \subseteq \mathrm{Cl}^+(V)$ be a lattice in the rational vector space $\mathrm{Cl}^+(V)$. Then, the *Kuga–Satake variety* associated to (V, q) is the (isogeny class of) the complex torus

$$\mathrm{KS}(X) := \mathrm{Cl}^+(V)_{\mathbb{R}}/\Gamma,$$

where $\mathrm{Cl}^+(V)_{\mathbb{R}}$ is endowed with the complex structure induced by left multiplication by J. Note that if (V, q) is an integral Hodge structure of hyper-Kähler type, then V can be viewed as a lattice in $\mathrm{Cl}^+(V_{\mathbb{Q}})$. Thus, the natural choice $\Gamma := V$ determines the complex torus $\mathrm{KS}(V)$, and not just up to isogeny.

By construction, one has the following:

$$H^1_{\mathrm{KS}}(V) := H^1(\mathrm{KS}(V), \mathbb{Q}) \simeq \mathrm{Cl}^+(V)^* \simeq \mathrm{Cl}^+(V),$$

where the isomorphism between $\mathrm{Cl}^+(V)$ and its dual is induced by the nondegenerate form q.

Remark A.2.4. Consider the case where V can be written as a direct sum of Hodge structures $V = V_1 \oplus V_2$. Since $\dim V^{2,0} = 1$, either V_1 or V_2 has to be pure of type $(1,1)$. We may then assume that $V_2^{2,0} = 0$. In this case, one checks that the Kuga–Satake Hodge structure $\mathrm{Cl}^+(V)$ is isomorphic to the product of 2^{n_2-1} copies of $\mathrm{Cl}^+(V_1) \oplus \mathrm{Cl}^-(V_1)$, with $n_2 := \dim V_2$. In particular:

$$\mathrm{KS}(V_1 \oplus V_2) \sim \mathrm{KS}(V_1)^{2^{n_2}}.$$

Remark A.2.5. For any element $w \in \mathrm{Cl}^+(V)$, the right-multiplication morphism

$$r_w \colon \mathrm{Cl}^+(V) \to \mathrm{Cl}^+(V), \quad r_w(x) := x \cdot w$$

is a morphism of Hodge structures. This follows from the fact that the Kuga–Satake Hodge structure on $\mathrm{Cl}^+(V)$ is induced by left multiplication by $J \in \mathrm{Cl}^+(V)$ which commutes with right multiplication by elements of $\mathrm{Cl}^+(V)$. Therefore, there is an embedding

$$\mathrm{Cl}^+(V) \hookrightarrow \mathrm{End}_{\mathrm{Hdg}}(\mathrm{Cl}^+(V)) \simeq \mathrm{End}(\mathrm{KS}(V)) \otimes \mathbb{Q}.$$

Since the dimension of $\mathrm{Cl}^+(V)$ is $2^{\dim V - 1}$, we deduce that the endomorphism algebra of $\mathrm{KS}(V)$ is in general big. This is related with the fact that the Kuga–Satake variety of a Hodge structure of hyper-Kähler type is in general not simple, but it is isogenous to the power of a smaller-dimensional torus.

Remarkably, the Kuga–Satake construction realizes the starting level-two Hodge structure as a Hodge substructure of the tensor product of two Hodge structures of level one:

Theorem A.2.6. *Let (V, q) be a Hodge structure of hyper-Kähler type. Then, there is an embedding of Hodge structures:*

$$V \hookrightarrow \mathrm{Cl}^+(V) \otimes \mathrm{Cl}^+(V),$$

where $\mathrm{Cl}^+(V)$ is endowed with the level-one Hodge structure of Definition A.2.3.

Proof. We recall here just the definition of the desired map, for more details we refer to [44, Prop. 3.2.6]. Fix an element $v_0 \in V$ such that $q(v_0) \neq 0$ and consider the following left multiplication map:

$$\varphi \colon V \to \mathrm{End}(\mathrm{Cl}^+(V)), \quad v \mapsto f_v,$$

where $f_v(w) \coloneqq v{\cdot}w{\cdot}v_0$. The injectivity of φ follows from the equality $f_v(v'{\cdot}v_0) = q(v_0)(v{\cdot}v')$ for any $v' \in V$. See the reference above for the proof of the fact that φ is a morphism of Hodge structures. Finally, the desired embedding is given by the composition of ϕ and the isomorphisms

$$\mathrm{End}(\mathrm{Cl}^+(V)) \simeq \mathrm{Cl}^+(V)^* \otimes \mathrm{Cl}^+(V) \simeq \mathrm{Cl}^+(V) \otimes \mathrm{Cl}^+(V),$$

where the isomorphism $\mathrm{Cl}^+(V)^* \simeq \mathrm{Cl}^+(V)$ is induced by q. □

Remark A.2.7. Note that the embedding of Theorem A.2.6 depends on the choice of $v_0 \in V$. Nevertheless, choosing another $v_0' \in V$ changes the embedding by the automorphism of $\mathrm{Cl}^+(V)$ which sends w to $\frac{w\cdot v_0\cdot v_0'}{q(v_0)}$.

Theorem A.2.6 shows that any Hodge structure of hyper-Kähler type can be realized as a Hodge substructure of $W \otimes W$ for some level-one Hodge structure W. On the other hand, in [16, Sec. 7], Deligne proves that the same conclusion does not hold for a very general level-two Hodge structure. We recall here a version of this fact as presented in [30, Prop. 4.2].

Theorem A.2.8. *Let (V, q) be a polarized level-two Hodge structure whose Mumford–Tate group $\mathrm{MT}(V)$ is maximal, that is, equal to $\mathrm{SO}(q)$. If $\dim V^{2,0} > 1$, then V cannot be realized as a Hodge substructure of $W \otimes W$ for any level-one Hodge structure W.*

Remark A.2.9. One can show in some cases that the technical condition $\mathrm{MT}(V) = \mathrm{SO}(q)$ of Theorem A.2.8 is satisfied for a very general Hodge structure, see [16, Sec. 7] and [95, Cor. 4.12]. The proof goes as follows: Given a smooth projective morphism $\pi\colon \mathcal{X} \to B$, one shows that for very general $t \in B$, the Mumford–Tate group $\mathrm{MT}(\mathcal{X}_t)$ contains a finite index subgroup of the monodromy group of the base. Already in the case of hypersurfaces in a $(2r + 1)$-dimensional projective space, this shows that for a very general hypersurface X_s, the Mumford–Tate group of $H^{2r}(X, \mathbb{Q})$ is maximal in the above sense. Applying Theorem A.2.8, one then sees that the second cohomology of a very general surface X in $\mathbb{P}^3$ of degree ≥ 5 cannot be realized as a Hodge substructure of $W \otimes W$ for any level-one Hodge structure W.

To conclude this section, we recall the fact that if the Hodge structure of hyper-Kähler type is polarized, the resulting Kuga–Satake Hodge structure on the even Clifford algebra is naturally polarized.

Theorem A.2.10. *If (V, q) is a Hodge structure of hyper-Kähler type such that q is a polarization for V, then the Kuga–Satake Hodge structure on $\mathrm{Cl}^+(V)$ has a natural polarization. In particular, the Kuga–Satake torus $\mathrm{KS}(V)$ is an abelian variety.*

A.2.2 Some examples

Let X be a hyper-Kähler variety (resp. a two-dimensional complex torus). The pair $(H^2(X, \mathbb{Q}), -q_X)$ where q_X is the Beauville-Bogomolov form (resp. the intersection pairing) is a Hodge structure of hyper-Kähler type. Therefore, we can apply the Kuga–Satake construction to it and we get the *Kuga–Satake variety* of X:

$$\mathrm{KS}(X) \coloneqq \mathrm{KS}(H^2(X,\mathbb{Q})).$$

Since $-q_X$ is not a polarization on the whole $H^2(X,\mathbb{Q})$, the variety $\mathrm{KS}(X)$ is not necessarily an abelian variety, but it is just a complex torus. On the other hand, if X is projective and l is an ample class on X, the primitive part

$$H^2(X,\mathbb{Q})_p \coloneqq l^{\perp} \subseteq H^2(X,\mathbb{Q})$$

is a Hodge substructure which is polarized by the restriction of the form $-q_X$. Therefore, by Theorem A.2.10, the Kuga–Satake variety of $H^2(X,\mathbb{Q})_p$ is an abelian variety. Moreover, by Remark A.2.4, we have

$$\mathrm{KS}(X) \coloneqq \mathrm{KS}(H^2(X,\mathbb{Q})) \sim \mathrm{KS}(H^2(X,\mathbb{Q})_p)^2.$$

In particular, in the projective case, $\mathrm{KS}(X)$ is an abelian variety. A similar argument can be applied to $H^2(X,\mathbb{Q})_{\mathrm{tr}} \subseteq H^2(X,\mathbb{Q})$, the transcendental lattice of a projective K3 surface, to deduce that $\mathrm{KS}(X)$ is isogenous to some power of the abelian variety $\mathrm{KS}(H^2(X,\mathbb{Q})_{\mathrm{tr}})$. On the other hand, if X is not projective, the torus $\mathrm{KS}(X)$ need not be polarized.

Theorem A.2.11. *[71] Let A be a complex torus of dimension two. Then, there exists an isogeny*

$$\mathrm{KS}(A) \sim (A \times \hat{A})^4,$$

where $\hat{A}$ is the dual complex torus. In particular, if A is an abelian surface

$$\mathrm{KS}(A) \sim A^8 \quad \textit{and} \quad \mathrm{KS}(\mathrm{Kum}(A)) \sim A^{2^{19}},$$

where $\mathrm{Kum}(A)$ *is the Kummer surface associated to A.*

Definition A.2.12. Let A be an abelian variety of dimension $2n$ and let d be a positive real number. Then, A is called of $\mathbb{Q}(\sqrt{-d})$-*Weil type* if $\mathbb{Q}(\sqrt{-d}) \subseteq \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ and if the action of $\sqrt{-d}$ on the tangent space at the origin of A has eigenvalues $\sqrt{-d}$ and $-\sqrt{-d}$ both with multiplicity n.

Given an abelian of $\mathbb{Q}(\sqrt{-d})$-Weil type A, then one can associate naturally an element $\delta \in \mathbb{Q}/\mathrm{N}(\mathbb{Q}(\sqrt{-d}))$, where $\mathrm{N}(\mathbb{Q}(\sqrt{-d}))$ is the set of norms of $\mathbb{Q}(\sqrt{-d})$. The element δ is called the *discriminant* of A. Abelian varieties of Weil type appear often as simple factors of Kuga–Satake varieties; the next result due to Lombardo [56] gives an example of this fact. We recall here the version presented in [30, Thm. 9.2]. In the following, U denotes the hyperbolic plane.

Theorem A.2.13. *Let d be a positive real number and let A be an abelian fourfold of $\mathbb{Q}(\sqrt{-d})$-Weil type of discriminant $\delta = 1$. Then, A^4 is the Kuga–Satake variety of a polarized Hodge structure of hyper-Kähler type of dimension six (V, q), such that*

$$V \simeq U^{\oplus 2} \oplus \langle -1 \rangle \oplus \langle -d \rangle$$

as quadratic spaces. Conversely, if (V, q) is a Hodge structure of hyper-Kähler type of dimension six as above, its Kuga–Satake variety is isogenous to A^4 for some abelian fourfold of $\mathbb{Q}(\sqrt{-d})$-Weil type.

A.2.3 Kuga–Satake Hodge conjecture

In this section, we analyze the morphism of Hodge structures

$$V \hookrightarrow \mathrm{Cl}^+(V) \otimes \mathrm{Cl}^+(V)$$

of Theorem A.2.6, in the case where $V = H^2(X, \mathbb{Q})_{\mathrm{tr}}$, the transcendental lattice of a projective hyper-Kähler variety X. Using the isomorphism $\mathrm{Cl}^+(H^2(X, \mathbb{Q})_{\mathrm{tr}}) \simeq H^1_{\mathrm{KS}}(H^2(X, \mathbb{Q})_{\mathrm{tr}})$, we apply the Künneth decomposition and obtain an embedding

$$H^1_{\mathrm{KS}}(H^2(X, \mathbb{Q})_{\mathrm{tr}}) \otimes H^1_{\mathrm{KS}}(H^2(X, \mathbb{Q})_{\mathrm{tr}}) \hookrightarrow H^2(\mathrm{KS}(H^2(X, \mathbb{Q})_{\mathrm{tr}})^2, \mathbb{Q}).$$

On the other hand, since we the variety X is projective there is a natural projection map $H^2(X, \mathbb{Q}) \to H^2(X, \mathbb{Q})_{\mathrm{tr}}$. Composing these morphisms, we obtain the morphism of Hodge structures

$$H^2(X, \mathbb{Q}) \to H^2(\mathrm{KS}(H^2(X, \mathbb{Q})_{\mathrm{tr}})^2, \mathbb{Q}),$$

which is called the *Kuga–Satake correspondence*. This morphism corresponds via Poincaré duality to a Hodge class

$$\kappa \in H^{2n,2n}(X \times \mathrm{KS}(H^2(X, \mathbb{Q})_{\mathrm{tr}}) \times \mathrm{KS}(H^2(X, \mathbb{Q})_{\mathrm{tr}})),$$

where $2n = \dim X$. The Hodge conjecture applied to this special case gives us the following:

Conjecture A.2.14 (Kuga–Satake Hodge conjecture)**.** *Let X be a projective hyper-Kähler variety or a complex projective surface with $h^{2,0} = 1$. Then, the class κ is algebraic.*

Remark A.2.15. In the case where X is an abelian surface or a Kummer surface, the Kuga–Satake Hodge conjecture can be deduced from Theorem A.2.11, using the fact that the Hodge conjecture is known for self-products of any given abelian surface [68].

The Kuga–Satake Hodge conjecture is not known in most cases, already in the case of K3 surfaces. One of the very few examples for which it has been proved is the family of K3 surfaces studied by Paranjape in [77]: Let $L_1, \ldots, L_6$ be six lines in $\mathbb{P}^2$ no three of which intersect in one point, and let $\pi \colon Y \to \mathbb{P}^2$ be the double cover of $\mathbb{P}^2$ branched along the six lines. Then, Y is a singular surface with simple nodes in the preimages of the intersection points of the lines L_i. Resolving the singularities of π by blowing up the nodes one obtains a K3 surface X. For a general choice of the six lines, the Picard number of X is equal to 16, where a basis of the Néron–Severi group is given by the 15 exceptional divisors over the singular points of Y, together with the pullback of the ample line of $\mathbb{P}^2$ via the map $X \to \mathbb{P}^2$. In particular, the transcendental lattice of X is six-dimensional, and satisfies the hypotheses of Theorem A.2.13. Its Kuga–Satake variety is therefore isogenous to the fourth power of some abelian fourfold. In [77], the author shows that this abelian fourfold is the Prym variety of some $4:1$ cover $C \to E$ where C is a genus 5 curve and E is an elliptic curve, and finds a cycle in the product of X and the Prym variety which realizes the Kuga–Satake correspondence.

The fact that the Kuga–Satake correspondence is algebraic for the family described above has been used by Schlickewei to prove the Hodge conjecture for the square of those K3 surfaces:

Theorem A.2.16. *[81, Thm. 2] Let X be a K3 surface which is the desingularization of a double cover of $\mathbb{P}^2$ branched along six lines no three of which intersect in one point. Then, the Hodge conjecture is true for X^2.*

In [49], the Kuga–Satake Hodge conjecture is proved for K3 surfaces which are desingularization of singular K3 surfaces in $\mathbb{P}^4$ with 15 nodal points. The authors then show that the same techniques as in Theorem A.2.16 can be used to prove the Hodge conjecture for the square of these K3 surfaces.

Theorem A.2.17. *[49] Let X be a K3 surface which is the desingularization of a singular K3 surface in $\mathbb{P}^4$ with 15 nodal points. Then, the Kuga–Satake Hodge conjecture holds for X and the Hodge conjecture is true for X^2.*

As a part of its PhD thesis, the first author of these notes generalize these two results and proves the following:

Theorem A.2.18. *[89, Thm. 4.3] Let $\mathscr{X} \to S$ be a four-dimensional family of K3 surfaces whose general fibre is of Picard number 16 with an isometry*

$$T(\mathscr{X}_s) \simeq U^2 \oplus \langle a \rangle \oplus \langle b \rangle,$$

for some negative integers a and b. If the Kuga–Satake correspondence is algebraic for the fibres of this family, then the Hodge conjecture holds for all powers of every K3 surface in this family.

The families of K3 surfaces studied in [81] and in [49] satisfy the hypotheses of Theorem A.2.18. Therefore, Theorem A.2.18 shows in particular that the Hodge conjecture holds for all powers of the K3 surfaces in the two families considered in [81] and [49]. The techniques applied are similar to the one introduced in [81] with the addition of a deformation argument which allows to prove the Hodge conjecture for all powers of the K3 surfaces of higher Picard number in the family.

In the next section, we review another type of polarized hyper-Kähler manifolds for which the Kuga–Satake Hodge conjecture can be proved: The family of hyper-Kähler manifolds of generalized Kummer type.

A.3 The case of hyper-Kähler manifolds of generalized Kummer type

A.3.1 Cup-product: generalization of a result of O'Grady

Let X be a hyper-Kähler manifold of dimension $2n$ with $n \geq 2$. The Beauville-Bogomolov quadratic form q_X is a nondegenerate quadratic form on $H^2(X, \mathbb{Q})$, whose inverse gives an element of $\mathrm{Sym}^2 H^2(X, \mathbb{Q})$. By Verbitsky [91], the later space imbeds by cup-product in $H^4(X, \mathbb{Q})$, hence we get a class

$$c_X \in H^4(X, \mathbb{Q}). \tag{A.2}$$

The O'Grady map $\phi\colon \bigwedge^2 H^3(X, \mathbb{Q}) \to H^{4n-2}(X, \mathbb{Q})$ is defined by

$$\phi(\alpha \wedge \beta) = c_X^{n-2} \cup \alpha \cup \beta. \tag{A.3}$$

The following result was first proved by O'Grady [76] in the case of a hyper-Kähler manifold of generalized Kummer deformation type.

Theorem A.3.1. *([76], [98]) Let X be a hyper-Kähler manifold of dimension $2n$. Assume that $H^3(X, \mathbb{Q}) \neq 0$. Then, the O'Grady map map $\phi\colon \bigwedge^2 H^3(X, \mathbb{Q}) \to H^{4n-2}(X, \mathbb{Q})$ is surjective.*

Proof. We can choose the complex structure on X to be general, so that $\rho(X) = 0$. This implies that the Hodge structure on $H^2(X, \mathbb{Q})$ (or equivalently $H^{4n-2}(X, \mathbb{Q})$ as they are isomorphic by combining Poincaré duality and the Beauville-Bogomolov form) is simple. As the morphism ϕ is a morphism of Hodge structures, its image is a Hodge substructure of $H^{4n-2}(X, \mathbb{Q})$, hence either ϕ is surjective, or it is 0. Theorem A.3.1 thus follows from the next proposition. □

Proposition A.3.2. *The map ϕ is not identically* 0.

Sketch of proof. Let $\omega \in H^2(X, \mathbb{R})$ be a Kähler class. Then, we know that the ω-Lefschetz intersection pairing $\langle\, ,\, \rangle_\omega$ on $H^3(X, \mathbb{R})$, defined by

$$\langle \alpha, \beta \rangle_\omega := \int_X \omega^{2n-3} \cup \alpha \cup \beta$$

is nondegenerate. This implies that the image of the map

$$\psi\colon \bigwedge^2 H^3(X, \mathbb{Q}) \to H^6(X, \mathbb{Q})$$

pairs nontrivially with the image of $\mathrm{Sym}^{2n-3}H^2(X,\mathbb{Q})$ in $H^{4n-6}(X,\mathbb{Q})$. Note that the Hodge structure on $H^3(X,\mathbb{Q})$ has Hodge level one, so that the Hodge structure on the image of $\operatorname{Im}\psi$ in $\mathrm{Sym}^{2n-3}H^2(X,\mathbb{Q})^*$ is a Hodge structure of level at most two. One checks by a Mumford–Tate group argument (see [98] for more details) that, for a very general complex structure on X, the only level-two Hodge substructure of $\mathrm{Sym}^{2n-3}H^2(X,\mathbb{Q})$ is $c_X^{n-2}H^2(X,\mathbb{Q})$, where we see here c_X as an element of $\mathrm{Sym}^2H^2(X,\mathbb{Q})$. It follows that the image of $\operatorname{Im}\psi$ in $\mathrm{Sym}^{2n-3}H^2(X,\mathbb{Q})^*$ pairs non-trivially with $c_X^{n-2}H^2(X,\mathbb{Q})$, which concludes the proof. □

A.4 Intermediate Jacobian and the Kuga–Satake variety

A.4.1 Universal property of the Kuga–Satake construction

The following result is proved in [13]. Using the Mumford–Tate group, this is a statement in representation theory of the orthogonal group.

Theorem A.4.1. *Let (H^2,q) be a polarized Hodge structure of hyper-Kähler type. Assume that the Mumford–Tate group of the Hodge structure on H^2 is maximal (that is, equal to the orthogonal group of q). Let H be a simple effective weight-one Hodge structure, such that there exists an injective morphism of Hodge structures of bidegree $(-1,-1)$*

$$H^2 \hookrightarrow \mathrm{End}(H).$$

Then, H is a direct summand of the Kuga–Satake Hodge structure $H^1_{\mathrm{KS}}(H^2,\mathbb{Q})$.

Idea of the proof. Let $G := \mathrm{MT}(\mathrm{End}(H))$ and denote by $\mathfrak{g}$ its Lie algebra. Note that the group G acts on H^2, since H^2 is a Hodge substructure of $\mathrm{End}(H)$. Using the fact that the action of G preserves the polarization on H^2 and the hypothesis $\mathrm{MT}(H^2) = \mathrm{SO}(H^2)$, one sees that the image of G in $\mathrm{GL}(H^2)$ is $\mathrm{SO}(H^2)$. As $\mathfrak{so}(H^2)$ is a simple Lie algebra, we conclude that there exists a simple factor $\mathfrak{g}_0$ of the Lie algebra $\mathfrak{g}$ that maps isomorphically onto $\mathfrak{so}(H^2)$. Note that G is naturally a subgroup of $\mathrm{MT}(H)$, which is contained in CSp(H), the group generated by the symplectic group and the homotheties of H. In particular, there is a morphism of Lie algebras:

$$\mathfrak{so}(H^2) \simeq \mathfrak{g}_0 \hookrightarrow \mathfrak{sp}(H). \tag{A.4}$$

By the classification result presented in [80] and explained in [17, 1.3.5-1.3.9], one concludes that the only embeddings as in (A.4) which correspond to irreducible representations of $\mathrm{SO}(H)$ are the spin representations. This proves that H is a direct summand of $H^1_{\mathrm{KS}}(H^2,q)$. □

Charles' theorem is in fact stronger, as it proves a similar statement for all tensor powers $H^{\otimes k} \otimes (H^*)^{\otimes k+2r}$. It also addresses the nonpolarized case that appears when dealing with nonprojective hyper-Kähler manifolds. In [35], another version of the universality property is proved. Namely

Theorem A.4.2. *Let (H^2, q) be a polarized Hodge structure of hyper-Kähler type. Assume that $\dim H^2 \geq 5$ and that the Mumford–Tate group of the Hodge structure on H^2 is maximal. Let H be a simple effective weight-one Hodge structure, such that there exists an injective morphism of Hodge structures of bidegree $(-1, -1)$*

$$H^2 \hookrightarrow \mathrm{Hom}(H, A),$$

for some weight-one Hodge structure A. Then, H is a direct summand of the Kuga–Satake Hodge structure $H^1_{\mathrm{KS}}(H^2, q)$.

Coming back to Theorem A.4.1, under the same assumption on the Mumford–Tate group, one knows that the Kuga–Satake weight-one Hodge structure is a power of a simple weight-one Hodge structure of dimension $\geq 2^{\lfloor \frac{b_2-1}{2} \rfloor}$, where $b_2 = \dim H^2$, hence one gets as a consequence an inequality (see [13] for a more precise estimate)

$$\dim H \geq 2^{\lfloor \frac{b_2-1}{2} \rfloor}.$$

Proof of Theorem A.1.1. Let X be a very general projective hyper-Kähler manifold of generalized Kummer type of dimension ≥ 4. We apply Theorem A.4.1 to the O'Grady map (A.3) that we know to be a surjective morphism of Hodge structures by Theorem A.3.1, or rather to its dual. We then conclude that $H^3(X, \mathbb{Q})$ contains a direct summand of $H^1_{\mathrm{KS}}(H^2(X, \mathbb{Q})_{\mathrm{tr}})$. As $H^1_{\mathrm{KS}}(H^2(X, \mathbb{Q})_{\mathrm{tr}})$ is a power of a simple weight-one Hodge structure $H^1_{\mathrm{KS}}(H^2(X, \mathbb{Q})_{\mathrm{tr}})_c$ of dimension 8, and $b_3(X) = 8$, we conclude that $H^3(X, \mathbb{Q}) \cong H^1_{\mathrm{KS}}(H^2(X, \mathbb{Q})_{\mathrm{tr}})_c$ as rational Hodge structures. □

A.5 Algebraicity of the Kuga–Satake correspondence for hyper-Kähler manifolds of generalized Kummer type

A.5.1 Markman's result

For a projective manifold X with $h^{3,0}(X) = 0$, it is expected from the Hodge conjecture that there exists a cycle $\mathcal{Z} \in \mathrm{CH}^2(J^3(X) \times X)_{\mathbb{Q}}$ such that $[\mathcal{Z}]_* \colon H_1(J^3(X), \mathbb{Q}) \to H^3(X, \mathbb{Q})$ is the natural isomorphism. Indeed, the map $[\mathcal{Z}]_*$ is an isomorphism of Hodge structures, hence provides a degree-4 Hodge class on $J^3(X) \times X$. Equivalently, after replacing $\mathcal{Z}$ by a multiple that makes it integral, the Abel–Jacobi map

$$\Phi_{\mathcal{Z}} \colon J^3(X) \to J^3(X), \quad \Phi_{\mathcal{Z}} := \Phi_X \circ \mathcal{Z}_*,$$

is a multiple of the identity and in particular Φ_X is surjective.

Theorem A.5.1. *(Markman [60]) Let X be a projective hyper-Kähler manifold of generalized Kummer type. Then, there exists a codimension-two cycle $\mathcal{Z} \in \mathrm{CH}^2(J^3(X) \times X)_{\mathbb{Q}}$ satisfying the property above.*

The proof of this theorem uses a deformation argument starting from a generalized Kummer manifold, using the fact that $J^3(X)$ can be realized as a moduli space of sheaves on X in that case.

We now use Markman's result to prove Theorem A.1.2.

Proof of Theorem A.1.2. Let $\mathcal{Z}$ be the Markman codimension-two cycle of Theorem A.5.1. We choose a cycle $C_X \in \mathrm{CH}^2(X)_{\mathbb{Q}}$ of class $[C_X] = c_X$ (it exists by results of Markman [60]). We now consider the cycle

$$\Gamma = \mathcal{Z}^2 \cdot \mathrm{pr}_X^* C_X^{n-2} \in \mathrm{CH}^{2n}(J^3(X) \times X)_{\mathbb{Q}}.$$

One checks using the Künneth decomposition (see [98] for more details) that

$$[\Gamma]_* \colon H_2(J^3(X), \mathbb{Q}) \to H_2(X, \mathbb{Q})$$

is the O'Grady map. By Theorem A.1.1, this is also the surjective morphism of Hodge structures (A.1). □

Appendix B

The Hodge conjecture for powers of K3 surfaces of Picard number 16

M. Varesco[1]

Abstract. We study the Hodge conjecture for powers of K3 surfaces and show that if the Kuga–Satake correspondence is algebraic for a family of K3 surfaces of generic Picard number 16, then the Hodge conjecture holds for all powers of any K3 surface in that family.

B.1 Introduction

B.1.1 The determinant and the exceptional Hodge classes

Let X be a K3 surface, and denote by $T(X)$ its transcendental lattice with its induced Hodge structure of weight two. In the study of the Hodge conjecture for powers of X a central role is played by the endomorphism field E of $T(X)$. If E is a CM field, then the Hodge conjecture for all powers of X is known to hold: Buskin [11] and, again, Huybrechts [45], using derived categories, prove the Hodge conjecture for the square of these K3 surfaces. Ramón–Marí [78] then shows that for K3 surfaces with a CM endomorphism field, the Hodge conjecture for the square implies the Hodge conjecture for all powers of these K3 surfaces. On the other hand, if E is a totally real field, the situation is more difficult. Unlike the CM case, the algebra of Hodge classes in $\bigotimes^\bullet T(X)$ is never generated by degree-two elements: The determinant of the transcendental lattice $\det T(X) \in T(X)^{\otimes \dim T(X)}$ gives an example of an *exceptional* Hodge class, i.e., a Hodge class in the tensor algebra of $T(X)$ that cannot be expressed in terms of Hodge classes in $T(X)^{\otimes 2}$. In particular, we see that it is not sufficient to prove the Hodge conjecture for the square of these surfaces. Moreover, if the endomorphism field of X is not $\mathbb{Q}$, the determinant is not the unique exceptional Hodge class. Indeed, we prove the following:

[1]Mauro Varesco. The Hodge conjecture for powers of K3 surfaces of Picard number 16. *Mich. Math. J.*, Advance Publication 1-32, 2024, https://doi.org/10.1307/mmj/20236349.

Theorem B.1.1 (Theorem B.3.16). *Let X be a K3 surface with totally real endomorphism field E. Then, any Hodge class in $\bigotimes^\bullet T(X)$ can be expressed in terms of Hodge classes of degree two and the exceptional Hodge classes in $T(X)^{\otimes r}$, where $r := \dim_E T(X)$.*

In particular, one sees that $\det T(X)$ decomposes as tensor product of exceptional Hodge classes of lower degree. To see what these exceptional classes are, consider the decomposition $T(X)_\mathbb{C} = \bigoplus_\sigma V_\sigma$ of the complex vector space $T(X)_\mathbb{C}$ into eigenspaces for the action of the field E, where the direct sum runs over all embeddings $\sigma\colon E \hookrightarrow \mathbb{C}$. The classes $\det V_\sigma \in (T(X)_\mathbb{C})^{\otimes r}$ are of type (r, r), but are not rational. However, it can be shown that the space $\langle \det V_\sigma \rangle_\sigma$ is indeed rational, i.e., spanned by Hodge classes. These are the exceptional Hodge classes appearing in the theorem. In particular, to prove the Hodge conjecture for all powers of a K3 surface X with a totally real endomorphism field, one needs to show the Hodge conjecture for the square X^2, i.e., prove that all classes in $\mathrm{End}_{\mathrm{Hdg}(X)}(T(X))$ are algebraic, and then show that one of the exceptional classes in $T(X)^{\otimes r}$ is algebraic on X^r. Indeed, if one of these exceptional classes is algebraic, then, applying the endomorphisms in $\mathrm{End}_{\mathrm{Hdg}(X)}(T(X))$, we see that all of them are algebraic.

The second implication of our theorem is the fact that the Hodge conjecture does not specialize in families: In a family of K3 surfaces the dimension of the endomorphism field may jump and the rank of the transcendental lattice may go down, therefore, the Hodge conjecture for powers of the general member does not imply the Hodge conjecture for powers of all other fibres. However, we prove in Proposition B.4.1 that the algebraicity of $\det T(X)$ specializes also when the dimension of $T(X)$ drops. A similar statement can be proven for the exceptional Hodge classes, this allows us to conclude that the Hodge conjecture specializes in families if the endomorphism field of the special fibre is equal to the endomorphism field of the general fibre.

B.1.2 The main theorem

A possible approach to tackle the Hodge conjecture for powers of K3 surfaces is via the Kuga–Satake construction, which gives a correspondence between the K3 surface and an abelian variety. When this correspondence is known to be algebraic, it is possible to produce algebraic cycles on powers of the K3 surface from algebraic cycles on powers of the abelian variety. For example, this shows the Hodge conjecture for powers of K3 surfaces that are Kummer surfaces: By Morrison [70], the Kuga–Satake variety of a Kummer surface is a power of the starting abelian surface, and by Ribet [79], the Hodge conjecture holds for powers of abelian surfaces. This same approach has been employed by Schlickewei [81] to prove the Hodge conjecture for the square of K3 surfaces which are double covers of the projective plane branched along six lines using a result by Paranjape [77]. Inspired by these techniques, we analyze in depth the Kuga–Satake correspondence for K3 surfaces of Picard number 16, and we prove the following:

Theorem B.1.2 (Theorem B.6.3). *Let $\mathscr{X} \longrightarrow S$ be a four-dimensional family of K3 surfaces whose general fibre is of Picard number* 16 *with an isometry*

$$T(\mathscr{X}_s) \simeq U_{\mathbb{Q}}^2 \oplus \langle a \rangle \oplus \langle b \rangle,$$

for some negative integers a and b. If the Kuga–Satake correspondence is algebraic for the fibres of this family, then the Hodge conjecture holds for all powers of every K3 surface in this family.

At the time of us writing this article, there are two families of K3 surfaces which satisfy the hypotheses of Theorem B.1.2: The family of double covers of $\mathbb{P}^2$ branched along six lines studied in [81] and the family of K3 surfaces which are desingularization of singular K3 surfaces in $\mathbb{P}^4$ with 15 simple nodes studied in [49]. In particular, Theorem B.1.2 proves the Hodge conjecture for all powers of the K3 surfaces in these families.

B.1.3 Outline of the proof

The first step of the proof is a study of the Hodge conjecture for powers of the Kuga–Satake varieties of the K3 surfaces of Theorem B.1.2. By Lombardo [56], these abelian varieties are powers of abelian fourfolds of Weil type with discriminant one. We correct in Section B.5 the proof of the following theorem.

Theorem B.1.3 (Theorem B.5.16). *[1] Let A be a general abelian variety of Weil type. Then, the Hodge conjecture for A implies the Hodge conjecture for all powers A^k.*

In a recent preprint, Milne [64] gives an alternative proof of Theorem B.1.3 using an equality between the Mumford–Tate group and the algebraic group which preserves the algebraic classes on the powers of the abelian variety[2]. Our proof has the advantage of giving a concrete description of all Hodge classes on the powers of a general abelian variety of Weil type. For example this allows us to show that if the Kuga–Satake correspondence for the general K3 surface of Picard number 16 with transcendental lattice as in Theorem B.1.2 is algebraic, then the Hodge conjecture holds for the resulting abelian fourfold of Weil type with discriminant one. Indeed we prove the following:

Proposition B.1.4 (Prop. B.6.2). *Let X be a general K3 surface of Picard number* 16 *as in Theorem B.1.2, and denote by A the abelian fourfold of Weil type with discriminant one appearing as simple factor of the Kuga–Satake variety of X. Then, if the Kuga–Satake correspondence is algebraic for X, the Weil classes on A are algebraic. Thus, the Hodge conjecture holds for A and, hence, for all powers A^k.*

The Hodge conjecture for general abelian fourfolds of Weil type with discriminant one has already been proven by Markman [60] using generalized Kummer varieties. Our

[2]Thanks to Bert van Geemen for the reference.

approach is different but no new case of the Hodge conjecture is established. Proposition B.1.4 is included to highlight the strong link between algebraicity of the Kuga—Satake correspondence and the Hodge conjecture for the Kuga–Satake variety. Assuming the algebraicity of Kuga–Satake correspondence, the Hodge conjecture for powers of the Kuga–Satake variety implies the Hodge conjecture for the K3 surface. This allows us deduce the Hodge conjecture the general K3 surface of the family of Theorem B.1.2. Similarly, we prove the Hodge conjecture for all powers of K3 surfaces of Picard number 16 and a totally real endomorphism field of degree two. To establish the same for the K3 surfaces of higher Picard number with totally real endomorphism field, we rely on the aforementioned result stating that the Hodge conjecture specializes in families if the endomorphism field of the special fibre is equal to the endomorphism field of the general fibre. This last hypothesis is satisfied, as the endomorphism field of K3 surfaces of Picard number higher than 16 is $\mathbb{Q}$ if it is not a CM field.

Acknowledgments

I would like to thank my PhD. supervisor Daniel Huybrechts for suggesting me this topic and for his continuous advice and support. I am also grateful to Ben Moonen and Bert van Geemen for reviewing a preliminary version of this article. Our discussion of the role of $\det T(X)$ was influenced by Claire Voisin and Qizheng Yin. This research was funded by ERC Synergy Grant HyperK, Grant agreement ID 854361.

B.2 The Hodge conjecture and Kuga–Satake varieties

In this section, we recall the Hodge conjecture in the special case of powers of K3 surfaces and review the construction of the Kuga–Satake varieties. For a complete introduction we refer to [44].

B.2.1 The Hodge conjecture

Let X be a smooth complex projective variety. For a non-negative integer k denote by $H^{k,k}(X,\mathbb{Q}) \coloneqq H^{k,k}(X,\mathbb{C}) \cap H^{2k}(X,\mathbb{Q})$ the set of Hodge classes of degree k.

Conjecture B.2.1 (Hodge conjecture for powers of K3 surfaces)**.** *Let X be a K3 surface, and let k and n be positive integers. Then, the $\mathbb{Q}$-algebra of Hodge classes in $H^{k,k}(X^n,\mathbb{Q})$ is generated by cohomology classes of algebraic cycles on X^n.*

As the K3 surface X is projective by assumption, the intersection pairing on X induces the direct sum decomposition

$$H^2(X,\mathbb{Q}) = \mathrm{NS}(X)_{\mathbb{Q}} \oplus T(X),$$

where $\mathrm{NS}(X)$ is the Néron–Severi group of X, and $T(X)$ is its transcendental lattice, i.e., the smallest rational sub-Hodge structure T of $H^2(X,\mathbb{Q})$ such that $T^{2,0} = H^{2,0}(X)$.

Lemma B.2.2. *The Hodge conjecture holds for all powers of a K3 surface X if and only if all Hodge classes in the tensor algebra of $T(X)$ are algebraic.*

Proof. Let n and k be positive integers. By Künneth decomposition, we have that

$$H^{2k}(X^n, \mathbb{Q}) \cong \bigoplus \left(T(X)^{\otimes a} \otimes \mathrm{NS}(X)_{\mathbb{Q}}^{\otimes b} \otimes H^0(X, \mathbb{Q})^{\otimes c} \otimes H^4(X, \mathbb{Q})^{\otimes d}\right),$$

where the direct sum runs over all non-negative integers a, b, c, and d satisfying $2k = 2a + 2b + 4d$ and $a + b + c + d = n$. Since all elements in $\mathrm{NS}(X)_{\mathbb{Q}}$, $H^0(X, \mathbb{Q})$, and $H^4(X, \mathbb{Q})$ are obviously algebraic, to prove the Hodge conjecture for the powers of X, it suffices to show that the Hodge classes in the algebra $\bigotimes^\bullet T(X)$ are algebraic. □

Let us recall the notion of Hodge group of $T(X)$, which we use to study the algebra of Hodge classes in $\bigotimes^\bullet T(X)$. Let V be a rational Hodge structure, and denote by $\rho \colon \mathbb{C}^* \longrightarrow \mathrm{GL}(V_{\mathbb{R}})$ the morphism defining the Hodge structure on V. The *Hodge group* of V is defined as the smallest algebraic sub-group $\mathrm{Hdg}(V)$ of $\mathrm{GL}(V)$ defined over $\mathbb{Q}$ such that

$$\rho(S^1) \subseteq \mathrm{Hdg}(V)(\mathbb{R}),$$

where $S^1 \subseteq \mathbb{C}^*$ is the unit circle. Hodge classes can be equivalently defined as the classes that are invariant under he action of the Hodge group, indeed, the following holds:

Lemma B.2.3. *[79, Sec. 2] Let a and b be non-negative integers. An element $v \in V^{\otimes a} \otimes (V^*)^{\otimes b}$ is a Hodge class if and only if it is invariant under the action of* $\mathrm{Hdg}(V)$. □

Given a K3 surface X, the algebra of Hodge classes in $\bigotimes^\bullet T(X)$ is then the invariant algebra $(\bigotimes^\bullet T(X))^{\mathrm{Hdg}(X)}$, where $\mathrm{Hdg}(X)$ is the Hodge group of $T(X)$. Similarly, given an abelian variety A, denote by $\mathrm{Hdg}(A)$ the Hodge group of $H^1(A, \mathbb{Q})$. Considering the natural embedding of Hodge structures $\bigwedge^\bullet H^1(A, \mathbb{Q}) \hookrightarrow \bigotimes^\bullet H^1(A, \mathbb{Q})$, we see that $(\bigwedge^\bullet H^1(A, \mathbb{Q}))^{\mathrm{Hdg}(A)}$ is the algebra of Hodge classes in $\bigwedge^\bullet H^1(A, \mathbb{Q})$.

B.2.2 Kuga–Satake varieties

We shortly recall the Kuga–Satake construction following [30] and [44, Ch. 4].

Let (V, q) be a polarized rational Hodge structure of weight two of K3-type, i.e., $\dim V^{2,0} = 1$. The *Clifford algebra* of V is the quotient of the tensor algebra of V by the two-sided ideal generated by elements of the form $v \otimes v - q(v)$ for $v \in V$:

$$\mathrm{Cl}(V) \coloneqq \bigotimes\nolimits^\bullet V / \langle v \otimes v - q(v) \rangle.$$

Denote by $\mathrm{Cl}^+(V)$ the subalgebra of $\mathrm{Cl}(V)$ generated by the elements of even degree. As shown in [44, Prop. 2.6], the Hodge structure on V induces a Hodge structure of weight one on $\mathrm{Cl}^+(V)$ for which there exists an embedding of Hodge structures $V \hookrightarrow \mathrm{Cl}^+(V) \otimes \mathrm{Cl}^+(V)$.

Definition B.2.4. The *Kuga–Satake variety* of (V, q) is an abelian variety $\mathrm{KS}(V)$ such that there is an isomorphism of Hodge structures $H^1(\mathrm{KS}(V), \mathbb{Q}) \simeq \mathrm{Cl}^+(V)$.

Note that the abelian variety $\mathrm{KS}(V)$ is determined only up to isogeny. For our purposes, this description of the Kuga–Satake variety up to isogeny is sufficient.

Let X be a K3 surface, and denote by $\mathrm{KS}(X)$ the Kuga–Satake variety of $T(X)$. By construction, there is an embedding of Hodge structures:

$$T(X) \hookrightarrow H^1(\mathrm{KS}(X), \mathbb{Q}) \otimes H^1(\mathrm{KS}(X), \mathbb{Q}) \subseteq H^2(\mathrm{KS}(X)^2, \mathbb{Q}).$$

Composing this map with the natural projection $H^2(X, \mathbb{Q}) \longrightarrow T(X)$ induced by the polarization on X, we obtain the following morphism of Hodge structures:

$$H^2(X, \mathbb{Q}) \longrightarrow H^2(\mathrm{KS}(X)^2, \mathbb{Q}).$$

This morphism is called *Kuga–Satake correspondence.* By Poincaré duality, this map is induced by a Hodge class $\kappa \in H^{2,2}(X \times \mathrm{KS}(X)^2, \mathbb{Q})$. The Hodge conjecture then predicts the following:

Conjecture B.2.5 (Kuga–Satake Hodge conjecture for K3 surfaces)**.** *Let X be a K3 surface. Then, the Hodge class κ is algebraic.*

If the Kuga–Satake correspondence is algebraic, it is possible reduce the study of the Hodge conjecture on powers of K3 surfaces to the study of the Hodge conjecture for powers of abelian varieties. Indeed, as in the proof of [81, Thm. 2], the following holds:

Lemma B.2.6. *Let X be a K3 surface for which the Kuga–Satake correspondence is algebraic. Then, a Hodge class in the tensor algebra of $T(X)$ is algebraic if and only if its image via the Kuga–Satake correspondence is algebraic.*

Proof. Let α be a Hodge class in $T(X)^{\otimes k}$ for some k. If α is algebraic on X^k then also its image via the Kuga–Satake correspondence is algebraic as we are assuming that the Kuga–Satake Hodge conjecture for X. Conversely, applying [51, Cor. 3.14] to the Kuga–Satake correspondence, we see that there is an algebraic projection

$$H^2(\mathrm{KS}(X)^2, \mathbb{Q}) \longrightarrow T(X) \subseteq H^2(X, \mathbb{Q}).$$

Therefore, if the image of α is algebraic on $\mathrm{KS}(X)^{2k}$ also α is algebraic on X^k. □

B.3 Generators of the algebra of Hodge classes

In this section, using the techniques introduced by Ribet [79], we study the algebra of Hodge classes for the powers of K3 surfaces. A similar study has been done in [78]. However, there is a mistake in the proof of [78, Prop. 5.2] that leads to a wrong conclusion in the case of K3 surfaces with totally real multiplication.

By Lemma B.2.2 and Lemma B.2.3, to study the Hodge conjecture for powers of X it suffices to investigate it for the algebra $(\bigotimes^\bullet T(X))^{\mathrm{Hdg}(X)}$. To ease the exposition, let us introduce some terminology we will use in the following sections. An element $f \in \bigotimes^\bullet T(X)$ is called *homogeneous of degree d* if $f \in T(X)^{\otimes d}$. Note that the group of permutations $\mathfrak{S}^d$ naturally acts on $T(X)^{\otimes d}$.

Definition B.3.1. We say that homogeneous elements $e_1, \ldots, e_r \in (\bigotimes^\bullet T(X))^{\mathrm{Hdg}(X)}$ generate $(\bigotimes^\bullet T(X))^{\mathrm{Hdg}(X)}$ if any element $f \in (\bigotimes^\bullet T(X))^{\mathrm{Hdg}(X)}$ can be written as a sum $f = \sum_i f_i$, where each f_i is homogeneous and, up to permutation and up to a scalar, it is a tensor product of elements in $\{e_1, \ldots, e_k\}$. In this case, we say that any element in $(\bigotimes^\bullet T(X))^{\mathrm{Hdg}(X)}$ can be expressed in terms of $e_1, \ldots, e_k$.

Note that if $\{e_1, \ldots, e_r\}$ is a set of generators of $(\bigotimes^\bullet T(X))^{\mathrm{Hdg}(X)}$, to prove the Hodge conjecture for the powers of X, it suffices to show that the classes e_i are algebraic.

Let $E \coloneqq \mathrm{End}_{\mathrm{Hdg}(X)}(T(X))$ be the endomorphism algebra of $T(X)$. As $T(X)$ is an irreducible Hodge structure, E is a field, cf. [44, Sec. 3.2]. We call it the *endomorphism field* of X. Let ψ be the polarization on $T(X)$ induced by the intersection form on $H^2(X, \mathbb{Q})$. Note that ψ is symmetric, since the Hodge structure on $T(X)$ has weight two. The *Rosati involution* is the involution on E sending an element e to the element e' for which

$$\psi(e(x), y) = \psi(x, e'(y)), \quad \forall x, y \in T(X).$$

As one checks, $F \coloneqq \{e \in E \,|\, e' = e\} \subseteq E$ is a totally real field, and either $E = F$ or E is a CM field with maximal totally real sub-field F, see [44, Thm. 3.3.7]. We treat these two cases separately.

B.3.1 K3 surfaces with a CM endomorphism field

In this section, we prove the following:

Theorem B.3.2. *[78, Prop. 5.2] Let X be a K3 surface whose endomorphism field is a CM field. Then, any Hodge class $\bigotimes^\bullet T(X)$ can be expressed in terms of Hodge classes in $T(X)^{\otimes 2}$.*

Note that in the reference it is stated that the same holds also in the case of K3 surfaces with totally real multiplication. In the next section, we show that this is not true.

Before proving Theorem B.3.2, let us deduce from it the Hodge conjecture for all powers of the K3 surface X:

Corollary B.3.3. *[78, Thm. 5.4] Let X be a complex, projective K3 surface whose endomorphism field is a CM field. Then, the Hodge conjecture holds for all powers of X.*

Proof. By Lemma B.2.2, in order to prove the Hodge conjecture for all powers of X, we need to show that the Hodge classes in $\bigotimes^\bullet T(X)$ are algebraic. By Theorem B.3.2, to show this, it suffices to show that every Hodge class in $T(X)^{\otimes 2}$ is algebraic. This has been proven in [11] and again in [45, Cor. 0.4.ii], where the authors prove the Hodge conjecture for the square of a K3 surface with CM endomorphism field. □

The remainder of this section is dedicated to the proof of Theorem B.3.2. Let us start by recalling the following result from linear algebra:

Lemma B.3.4. *[18, Lem. 4.3] Let k be a field and let V be a vector space of finite dimension over a finite separable field extensions k' of k. Then, the map*

$$\mathrm{Hom}_{k'}(V, k') \longrightarrow \mathrm{Hom}_k(V, k), \quad f \longmapsto \mathrm{Tr}_{k'/k} \circ f,$$

is an isomorphism of k-vector spaces.

Lemma B.3.5. *[101, Sec. 2.1] Let X be a K3 surface with a CM endomorphism field E and let ψ be the polarization on $T(X)$ induced by the intersection pairing. Then, there exists a unique non-degenerate E-Hermitian map $\varphi\colon T(X) \times T(X) \longrightarrow E$ which satisfies $\psi(v, w) = \mathrm{Tr}_{E/\mathbb{Q}}(\varphi(v, w))$ for every $v, w \in T(X)$.*

Proof. Denote by $T(X)^\dagger$ the space $T(X)$ with E acting on it via complex conjugation. The polarization ψ can be seen as a $\mathbb{Q}$-linear morphism $T(X) \otimes_E T(X)^\dagger \longrightarrow \mathbb{Q}$. Lemma B.3.4 then says that there exists a unique E-linear map $\varphi\colon T(X) \otimes_E T(X)^\dagger \longrightarrow E$ such that $\psi(v, w) = \mathrm{Tr}_{E/\mathbb{Q}}(\varphi(v, w))$ for every $v, w \in T(X)$. Viewing φ as a map $T(X) \times T(X) \longrightarrow E$, we see that it is E-Hermitian and satisfies all the required proprieties. □

Let $\mathrm{Hdg}(X)$ be the Hodge group of the transcendental lattice of X. As the polarization ψ is a morphism of Hodge structures

$$\psi\colon T(X) \otimes T(X) \longrightarrow \mathbb{Q}(-2),$$

the action of $\mathrm{Hdg}(X)$ on $T(X)$ preserves ψ, i.e., $\psi(Av, Aw) = \psi(v, w)$ for every $v, w \in T(X)$ and every $A \in \mathrm{Hdg}(X)$. By construction of φ, we conclude that $\mathrm{Hdg}(X) \subseteq \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{U}(T(X), \varphi))$, where $\mathrm{U}(T(X), \varphi)$ is the unitary group with respect to the E-Hermitian form φ, and F is the maximal totally real sub-field of E. In [101, Thm. 2.3.1], it is shown that this inclusion is always an equality, i.e., that the Hodge group of X satisfies $\mathrm{Hdg}(X) = \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{U}(T(X), \varphi))$. By Lemma B.2.3, the ring of Hodge classes in $\bigotimes^\bullet T(X)$ is then equal to

$$(\textstyle\bigotimes^\bullet T(X))^{\mathrm{Hdg}(X)} = (\bigotimes^\bullet T(X))^{\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{U}(T(X),\varphi))}.$$

Let $\mathrm{Hdg}(X)(\mathbb{C})$ be the group of $\mathbb{C}$-valued points of $\mathrm{Hdg}(X)$. As in [28, Sec. 6.7], there is an isomorphism of graded algebras

$$(\textstyle\bigotimes^\bullet T(X))^{\mathrm{Hdg}(X)} \otimes_{\mathbb{Q}} \mathbb{C} \simeq (\bigotimes^\bullet T(X)_{\mathbb{C}})^{\mathrm{Hdg}(X)(\mathbb{C})}. \tag{B.1}$$

From this, using the terminology of Definition B.3.1, we deduce the following:

Lemma B.3.6. *If $\widetilde{e}_1, \ldots, \widetilde{e}_r$ are homogeneous generators of $(\bigotimes^\bullet T(X)_{\mathbb{C}})^{\mathrm{Hdg}(X)(\mathbb{C})}$ over $\mathbb{C}$, then there exist generators $e_1, \ldots, e_r$ of $(\bigotimes^\bullet T(X))^{\mathrm{Hdg}(X)}$ over $\mathbb{Q}$, such that e_i is homogeneous with $\deg e_i = \deg \widetilde{e}_i$ for every i.*

In particular, to show that any element in $(\bigotimes^\bullet T(X))^{\mathrm{Hdg}(X)}$ can be expressed in terms of degree-two elements, it suffices to show that any element in $(\bigotimes^\bullet T(X)_{\mathbb{C}})^{\mathrm{Hdg}(X)(\mathbb{C})}$ can be expressed in terms of degree-two elements. Let F be the maximal totally real subfield of E. Extending scalars to $\mathbb{R}$, we get the following well known fact:

Lemma B.3.7. *The real vector space $T(X)_{\mathbb{R}}$ decomposes as a direct sum*

$$T(X)_{\mathbb{R}} = \bigoplus_{\sigma\colon F \hookrightarrow \mathbb{R}} V_\sigma,$$

where V_σ are real vector spaces with an E-action. Moreover, this decomposition is φ-orthogonal and φ induces a non-degenerate $\mathbb{C}$-Hermitian form on V_σ for every σ.

Proof. As $T(X)$ is a free F-module, $T(X) \otimes_{\mathbb{Q}} \mathbb{R}$ is a free $F \otimes_{\mathbb{Q}} \mathbb{R}$-module. Using the isomorphism

$$F \otimes_{\mathbb{Q}} \mathbb{R} \simeq \prod_{\sigma\colon F \hookrightarrow \mathbb{R}} \mathbb{R}, \quad e \otimes r \longmapsto (\sigma(e) r)_\sigma,$$

we see that

$$T(X)_{\mathbb{R}} = \bigoplus_{\sigma\colon F \hookrightarrow \mathbb{R}} V_\sigma,$$

where $V_\sigma \coloneqq T(X) \otimes_{F,\sigma} \mathbb{R} \simeq \{v \in T(X) \otimes_{\mathbb{Q}} \mathbb{R} \mid f(v) = \sigma(f) v \quad \forall f \in F\}$. Let us now show that E acts on V_σ. Let $\sigma\colon F \hookrightarrow \mathbb{R}$ be an embedding, and let $v \in V_\sigma$ be any element. For every $e \in E$ and every $f \in F$, we have that

$$f(e(v)) = e(f(v)) = e(\sigma(f) v) = \sigma(f) e(v),$$

where the last equality follows from the fact that the action of E on $T(X)_{\mathbb{R}}$ is the $\mathbb{R}$-linear extension of the action of E on $T(X)$. This shows that $e(v) \in V_\sigma$ and that the action of E on $T(X)_{\mathbb{R}}$ induces an action on V_σ. To see that this decomposition is φ-orthogonal, let $\sigma, \widetilde{\sigma}\colon F \hookrightarrow \mathbb{R}$ be two different embeddings and choose $f \in F$ such that $\sigma(f) \neq \widetilde{\sigma}(f)$. Since φ is E-Hermitian and since the elements of F are fixed by the Rosati involution, we have that

$$\sigma(f)\varphi(v, w) = \varphi(f(v), w) = \varphi(v, f(w)) = \widetilde{\sigma}(f)\varphi(v, w),$$

for every $v \in V_\sigma$ and $w \in V_{\widetilde{\sigma}}$. From this, we deduce that $\varphi(v, w) = 0$, i.e., $\varphi|_{V_\sigma \times V_{\widetilde{\sigma}}} = 0$. To define the $\mathbb{C}$-Hermitian form φ_σ on V_σ, extend the embedding $\sigma\colon F \hookrightarrow \mathbb{R}$ to an embedding $\tau\colon E \hookrightarrow \mathbb{C}$, which can be done since E is a totally imaginary quadratic extension of F. Let φ_σ be the composition

$$\varphi_\sigma\colon V_\sigma \times V_\sigma \xrightarrow{\varphi \otimes_{F,\sigma} \mathbb{R}} E \otimes_{F,\sigma} \mathbb{R} \xhookrightarrow{\tau \otimes_{F,\sigma} \mathbb{R}} \mathbb{C}.$$

From the fact that the decomposition $T(X)_{\mathbb{R}} = \bigoplus_\sigma V_\sigma$ is φ-orthogonal, it follows that φ_σ is a non-degenerate $\mathbb{C}$-Hermitian form on V_σ. This concludes the proof. □

Remark B.3.8. Since the action of $\mathrm{Hdg}(X)$ on $T(X)$ is E-linear, it preserves the eigendecomposition $T(X)_{\mathbb{R}} = \bigoplus_\sigma V_\sigma$ of Lemma B.3.7. From this, one deduces that the group of real valued points of $\mathrm{Hdg}(X)$ decomposes as a product

$$\mathrm{Hdg}(X)(\mathbb{R}) = \prod_{\sigma\colon F \hookrightarrow \mathbb{R}} U_\sigma(\mathbb{R}),$$

where $U_\sigma \coloneqq \mathrm{U}(V_\sigma, \varphi_\sigma)$. Therefore, the invariants in $(\bigotimes^\bullet T(X)_{\mathbb{R}})^{\mathrm{Hdg}(X)(\mathbb{R})}$ can be expressed in terms of invariants in the spaces $(\bigotimes^\bullet V_\sigma)^{U_\sigma(\mathbb{R})}$.

By definition of CM field, $E = F(\rho)$ with $\rho^2 \in F$ such that $\tilde{\sigma}(\rho^2)$ is negative for any embedding $\tilde{\sigma}\colon F \hookrightarrow \mathbb{R}$. Fix $\sigma\colon F \hookrightarrow \mathbb{R}$ an embedding, and let $\lambda \in \mathbb{R}$ such that $\sigma(\rho^2) = -\lambda^2$. As we have seen in Lemma B.3.7, the field E acts on V_σ. The action of $\frac{\rho}{\lambda}$ then induces a $\mathbb{C}$-vector space structure on V_σ. In particular, V_σ is even-dimensional and there is a decomposition $(V_\sigma)_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$ with $V^{1,0} \coloneqq \{v \in (V_\sigma)_{\mathbb{C}} \,|\, \rho(v) = i\lambda v\}$ and $V^{0,1} \coloneqq \{v \in (V_\sigma)_{\mathbb{C}} \,|\, \rho(v) = -i\lambda v\}$. Let ω_σ be the $\mathbb{C}$-linear extension of the map $\mathrm{Im}\varphi_\sigma\colon V_\sigma \times V_\sigma \longrightarrow \mathbb{R}$. Since φ_σ is non-degenerate and E-Hermitian, ω_σ is a non-degenerate symplectic form on $(V_\sigma)_{\mathbb{C}}$.

Lemma B.3.9. *The decomposition $(V_\sigma)_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$ of the $\mathbb{C}$-vector space $(V_\sigma)_{\mathbb{C}}$ is Lagrangian with respect to the symplectic form ω_σ, i.e.,*

$$\omega_\sigma|_{V^{1,0}\times V^{1,0}} = 0 \quad \text{and} \quad \omega_\sigma|_{V^{0,1}\times V^{0,1}} = 0.$$

Furthermore, ω_σ induces an isomorphism of complex vector spaces $V^{0,1} \simeq (V^{1,0})^*$.

Proof. The space $V^{1,0}$ is Lagrangian since for $v, v' \in V^{1,0}$ the following holds:

$$\omega_\sigma(v, v') = \frac{1}{\lambda^2}\omega_\sigma(\rho(v), \rho(v')) = \frac{1}{\lambda^2}\omega_\sigma(i\lambda v, i\lambda v') = -\omega_\sigma(v, v'),$$

where the first equality follows from the fact that φ is E-Hermitian. A similar argument proves that $V^{0,1}$ is Lagrangian. The second assertion then follows from the fact that ω_σ is non-degenerate. □

Remark B.3.10. As Hermitian forms are determined by their symplectic imaginary part, we deduce that $U_\sigma(\mathbb{C}) \simeq \mathrm{Sp}((V_\sigma)_{\mathbb{C}}, \mathbb{C})$. Identifying these two groups, we can consider $U_\sigma(\mathbb{C})$ as a subgroup of $\mathrm{GL}((V_\sigma)_{\mathbb{C}}, \mathbb{C})$. On the other hand, we can consider $\mathrm{GL}(V^{1,0}, \mathbb{C})$ as a subgroup of $\mathrm{GL}((V_\sigma)_{\mathbb{C}}, \mathbb{C})$ via the embedding

$$A \longmapsto \begin{pmatrix} A & 0 \\ 0 & A^* \end{pmatrix} \in \mathrm{GL}(V^{1,0} \oplus (V^{1,0})^*, \mathbb{C}) \simeq \mathrm{GL}((V_\sigma)_{\mathbb{C}}, \mathbb{C}),$$

where A^* denotes the natural action of A on $(V^{1,0})^*$.

Lemma B.3.11. *The groups $U_\sigma(\mathbb{C})$ and $\mathrm{GL}(V^{1,0}, \mathbb{C})$ coincide when viewed as subgroups of $\mathrm{GL}((V_\sigma)_{\mathbb{C}}, \mathbb{C})$ as in Remark B.3.10.*

Proof. Let $A \in \mathrm{GL}(V^{1,0}, \mathbb{C})$. By construction, A sends an element $v \in V^{0,1}$ to the element of $V^{0,1}$ corresponding to the element $\omega_\sigma(A^{-1}(\cdot), v) \in (V^{1,0})^*$ via the isomorphism of Lemma B.3.9. This implies that the action of A on $(V_\sigma)_{\mathbb{C}}$ preserves the symplectic form ω_σ, i.e., A belongs to $U_\sigma(\mathbb{C})$. To prove the other inclusion, note that the action of $U_\sigma(\mathbb{C})$ is E-linear and hence preserves the direct sum decomposition $(V_\sigma)_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$. Therefore, if A is an element of $U_\sigma(\mathbb{C})$, its restriction $A|_{V^{1,0}}$ is an invertible matrix on $V^{1,0}$. Then, via the isomorphism $\mathrm{GL}(V_\sigma, \mathbb{C}) \simeq \mathrm{GL}(V^{1,0} \oplus (V^{1,0})^*, \mathbb{C})$, the matrix A is sent to $\begin{pmatrix} A|_{V^{1,0}} & 0 \\ 0 & (A|_{V^{1,0}})^* \end{pmatrix}$. This shows that $A \in \mathrm{GL}(V^{1,0}, \mathbb{C})$. As we have shown the double inclusion, we conclude that the two groups are the same. □

We are now able to prove Theorem B.3.2:

Proof of Theorem B.3.2. By the previous discussion and Remark B.3.8, we just need to prove that any element in $(\bigotimes^\bullet (V_\sigma)_{\mathbb{C}})^{U_\sigma(\mathbb{C})}$ can be expressed in terms of elements of degree two. By Lemma B.3.9, this algebra can be identified with $(\bigotimes^\bullet (V^{1,0} \oplus (V^{1,0})^*))^{\mathrm{GL}(V^{1,0},\mathbb{C})}$. Given $f \in \big(\bigotimes^p (V^{1,0} \oplus (V^{1,0})^*)\big)^{\mathrm{GL}(V^{1,0},\mathbb{C})}$ an invariant element of degree p, view it as a map

$$f\colon ((V^{1,0})^* \oplus V^{1,0}) \otimes \cdots \otimes ((V^{1,0})^* \oplus V^{1,0}) \longrightarrow \mathbb{C}.$$

Decomposing f as a sum $\sum_i f_i$ where each f_i is a function $W_1 \otimes \cdots \otimes W_p \longrightarrow \mathbb{C}$, with $W_j = V^{1,0}$ or $(V^{1,0})^*$ for all j, we see that it suffices to show that each f_i can be expressed in terms of invariant elements of degree two. By construction, up to permuting its factors,

$$f_i \in \big(\mathrm{Hom}((V^{1,0})^{\otimes r} \otimes ((V^{1,0})^*)^{\otimes s}, \mathbb{C})\big)^{\mathrm{GL}(V^{1,0},\mathbb{C})}$$

for some r and s with $r + s = p$. According to the fundamental theorem of invariants for $\mathrm{GL}(V^{1,0}, \mathbb{C})$ as presented in [52, Thm. 4.2], the ring

$$\big(\mathrm{Hom}((V^{1,0})^{\otimes r} \otimes ((V^{1,0})^*)^{\otimes s}, \mathbb{C})\big)^{\mathrm{GL}(V^{1,0},\mathbb{C})}$$

is non-trivial if and only if $r = s$ and in this case it is generated by complete contractions, i.e., maps of the form

$$(V^{1,0})^{\otimes s} \otimes ((V^{1,0})^*)^{\otimes s} \longrightarrow \mathbb{C}, \quad v_1 \otimes \cdots \otimes v_s \otimes \mu_1 \otimes \cdots \otimes \mu_s \longmapsto \textstyle\prod_i \mu_i(v_{\sigma(i)}),$$

for some $\sigma \in \mathfrak{S}_s$. This statement of invariant theory can be found also in [99, Ch. III]. This concludes the proof of Theorem B.3.2: Indeed, any complete contraction can be written (up to permuting its factors) as a tensor product of complete contractions of degree two:

$$V^{1,0} \otimes (V^{1,0})^* \longrightarrow \mathbb{C}, \quad v \otimes \mu \longmapsto \mu(v),$$

which are invariant. □

B.3.2 K3 surfaces with a totally real endomorphism field

Let X be a complex, projective K3 surface with totally real endomorphism field $E = \mathrm{End}_{\mathrm{Hdg}(X)}(T(X))$. In this case, the Rosati involution is the identity, and, for every $e \in E$ and every $v, w \in T(X)$, we have that

$$\psi(ev, w) = \psi(v, ew).$$

As in Lemma B.3.5, one shows that there exists a non-degenerate symmetric E-bilinear map $\varphi\colon T(X) \times T(X) \longrightarrow E$ such that $\psi(v, w) = \mathrm{Tr}_{E/\mathbb{Q}}(\varphi(v, w))$. Note that this time φ is E-bilinear since E is totally real. Moreover, by [101, Thm. 2.2.1], the following equality holds

$$\mathrm{Hdg}(X) = \mathrm{Res}_{E/\mathbb{Q}}(\mathrm{SO}(T(X), \varphi)).$$

Similarly to Lemma B.3.7, we have the following result:

Lemma B.3.12. *The complex vector space $T(X)_{\mathbb{C}} := T(X) \otimes_{\mathbb{Q}} \mathbb{C}$ decomposes as a direct sum*

$$T(X)_{\mathbb{C}} = \bigoplus_{\sigma\colon E \hookrightarrow \mathbb{C}} V_\sigma,$$

where V_σ are complex vector spaces with an E-action. Moreover, the action of $\mathrm{Hdg}(X)$ *preserves this decomposition, and the E-bilinear symmetric form φ induces a non-degenerate $\mathbb{C}$-bilinear symmetric form φ_σ on V_σ for any σ.* □

Remark B.3.13. Note that the decomposition of the transcendental lattice into E-eigenspaces of Lemma B.3.12 already holds over $\mathbb{R}$ as E is totally real. We extended scalars directly to $\mathbb{C}$ to introduce as little notation as necessary.

Similarly to the CM case, we deduce from the isomorphsim

$$(\bigotimes\nolimits^\bullet T(X))^{\mathrm{Hdg}(X)} \otimes_{\mathbb{Q}} \mathbb{C} \simeq (\bigotimes\nolimits^\bullet T(X)_{\mathbb{C}})^{\mathrm{Hdg}(X)(\mathbb{C})}$$

that the invariants in $(\bigotimes^\bullet T(X)_{\mathbb{C}})^{\mathrm{Hdg}(X)(\mathbb{C})}$ can be expressed in terms of invariants in the spaces $(\bigotimes^\bullet V_\sigma)^{\mathrm{SO(V_\sigma,\mathbb{C})}}$. The complex vector space $\bigwedge^{\dim V_\sigma} V_\sigma \subseteq V_\sigma^{\otimes \dim V_\sigma}$ is one-dimensional and thus determines a unique element up to a complex scalar, denote it by $\det V_\sigma$. Note that $\det V_\sigma$ is invariant under the action of $\mathrm{SO}(V_\sigma, \mathbb{C})$. With this notation and with the convention of Definition B.3.1, the following holds:

Theorem B.3.14. *[52, Thm. 10.2] Let V_σ be a complex vector space endowed with a non-degenerate symmetric $\mathbb{C}$-bilinear form. Then, the* $\mathrm{SO}(V_\sigma, \mathbb{C})$*-invariants in $\bigotimes^\bullet V_\sigma$ can be expressed in terms of* $\mathrm{SO}(V_\sigma, \mathbb{C})$*-invariants of degree two and* $\det V_\sigma$.

Proof. To deduce this statement from the one in the reference, just note that, for any positive integer k, any element in $V_\sigma^{\otimes k}$ can be viewed as a homogeneous polynomial function $(V_\sigma^*)^{\oplus k} \longrightarrow \mathbb{C}$ of degree k: For example, the map $(V_\sigma^*)^{\oplus k} \longrightarrow \mathbb{C}$ which corresponding to a decomposable element $v_1 \otimes \cdots \otimes v_k \in V_\sigma^{\otimes k}$ is $\mu_1 \otimes \cdots \otimes \mu_n \longmapsto \prod \mu_i(v_i)$. □

By Theorem B.3.14, we see that, unlike the CM case, if the endomorphism field is totally real, there are Hodge classes in $\bigotimes^\bullet T(X)$ which cannot be expressed in terms of Hodge classes of degree two. To describe the additional classes needed to generate $(\bigotimes^\bullet T(X))^{\mathrm{Hdg}(X)}$, let us recall the following result from linear algebra:

Lemma B.3.15. *[18, Lem. 4.3] Let k be a field, and let V be a vector space of finite dimension over a finite separable field extension k' of k. Then, for any integer r, there is a natural embedding of k vector spaces:*

$$\textstyle\bigwedge_{k'}^r V \hookrightarrow \bigwedge_k^r V.$$

Proof. For later use, we recall the construction of the embedding. By Lemma B.3.4, the trace map induces an isomorphism of k-vector spaces

$$\mathrm{Hom}_{k'}(V, k') \simeq \mathrm{Hom}_k(V, k), \quad f \longrightarrow \mathrm{Tr}_{k'/k} \circ f.$$

This induces a natural map

$$\kappa\colon \textstyle\bigwedge_k^\bullet \mathrm{Hom}_k(V, k) \longrightarrow \bigwedge_{k'}^\bullet \mathrm{Hom}_{k'}(V, k').$$

The dual of κ as a map of k-vector spaces gives a map

$$\kappa^*\colon \mathrm{Hom}_k\left(\textstyle\bigwedge_{k'}^\bullet \mathrm{Hom}_{k'}(V, k'), k\right) \longrightarrow \mathrm{Hom}_k\left(\textstyle\bigwedge_k^\bullet \mathrm{Hom}_k(V, k), k\right).$$

The desired embedding is then the composition

$$\begin{aligned}\textstyle\bigwedge_{k'}^\bullet V &\xrightarrow{\simeq} \mathrm{Hom}_{k'}\left(\textstyle\bigwedge_{k'}^\bullet \mathrm{Hom}_{k'}(V, k'), k'\right) \xrightarrow{\mathrm{Tr}_{k'/k}} \mathrm{Hom}_k\left(\textstyle\bigwedge_{k'}^\bullet \mathrm{Hom}_{k'}(V, k'), k\right)\\ &\xrightarrow{\kappa^*} \mathrm{Hom}_k\left(\textstyle\bigwedge_k^\bullet \mathrm{Hom}_k(V, k), k\right) \xrightarrow{\simeq} \textstyle\bigwedge_k^\bullet V.\end{aligned}$$

□

Let $r := \dim_E T(X) = \dim V_\sigma$. Applying Lemma B.3.15 with $V = T(X)$, $k' = E$, and $k = \mathbb{Q}$, we get the following embedding

$$\textstyle\bigwedge_E^r T(X) \hookrightarrow \bigwedge^r T(X).$$

Since the right-hand side embeds naturally into $T(X)^{\otimes r}$, we can consider $\bigwedge_E^r T(X)$ as a vector subspace of $T(X)^{\otimes r}$. Then, using the isomorphism $E \otimes_{\mathbb{Q}} \mathbb{C} \simeq \prod_\sigma \mathbb{C}$, one sees that the $\mathbb{C}$-linear span of this vector subspace is

$$\left(\textstyle\bigwedge_E^r T(X)\right) \otimes_{\mathbb{Q}} \mathbb{C} \simeq \textstyle\bigwedge_{E\otimes_{\mathbb{Q}}\mathbb{C}}^r T(X)_{\mathbb{C}} \simeq \bigoplus_\sigma \left(\bigwedge_{\mathbb{C}}^r V_\sigma\right) \simeq \langle \det V_\sigma \rangle_\sigma,$$

where $\det V_\sigma$ are as in Theorem B.3.14. For a similar computation, see [18, Prop. 4.4]. Since $\det V_\sigma$ are invariant under the action of $\mathrm{Hdg}(X)(\mathbb{C})$, we deduce that the vector subspace $\bigwedge_E^r T(X)$ consists of Hodge classes, we call them *the exceptional Hodge classes*. As a consequence of Theorem B.3.14, we conclude the following:

Theorem B.3.16. *Let X be a K3 surface with totally real endomorphism field E. Then, any Hodge class in $\bigotimes^\bullet T(X)$ can be expressed in terms of Hodge classes of degree two and the exceptional Hodge classes in $T(X)^{\otimes r}$, where $r := \dim_E T(X)$.*

The name *exceptional classes* alludes to the fact that those classes cannot be expressed in terms of Hodge classes in the square of the transcendental lattice. This is a consequence of the following fact: Let V_σ be as in Theorem B.3.14. The $\mathrm{SO}(V_\sigma, \mathbb{C})$-invariant elements in $V_\sigma \otimes V_\sigma$ are also $\mathrm{O}(V_\sigma, \mathbb{C})$-invariant, as they are complete contractions. On the other hand, the class $\det V_\sigma$ is just $\mathrm{SO}(V_\sigma, \mathbb{C})$-invariant and thus cannot be expressed in terms of degree-two elements. In the literature, the name *exceptional classes* is used in the context of abelian varieties and denotes Hodge classes which are not in the algebra generated by (intersection of) divisor classes.

Remark B.3.17. The mistake in the proof of [78, Prop. 5.2] in the case of K3 surfaces with totally real multiplication is the wrong assumption that any invariant under the action of the special orthogonal group can be expressed in terms of invariants of degree two.

In conclusion, if the endomorphism field E of the K3 surface is totally real, then there are $|E : \mathbb{Q}|$ Hodge classes in $T(X)^{\otimes r}$, with $r = \dim_E T(X)$, which cannot be expressed in terms of Hodge classes in $T(X)^{\otimes 2}$. Thus, in this case, it is not true that the Hodge conjecture for the second power of the K3 surface implies the Hodge conjecture for all its powers.

B.3.3 A motivic approach

Inspired by the techniques in the paper [64], we give a different proof of the results of the previous section.

Let X be a K3 surface and denote by E its endomorphism field. Similarly to the case of abelian varieties, we give the following definition:

Definition B.3.18. The *Lefschetz group* of X is the biggest algebraic subgroup of $\mathrm{GL}(T(X))$ which preserves the Hodge classes in $T(X) \otimes T(X)$. We denote it by $\mathrm{L}(X)$.

Assume that E is a CM field and let F be its maximal totally real subfield. Denoting by $\varphi\colon T(X) \times T(X) \longrightarrow E$ the E-Hermitian map of Lemma B.3.5, we see that

$$L(X) \simeq \mathrm{Res}_{F/\mathbb{Q}} \mathrm{U}(T(X), \varphi),$$

where we regard $\mathrm{U}(T(X), \varphi)$ as an algebraic group over $\mathbb{Q}$ using the Weil restriction. Note that in this case, $\mathrm{L}(X)$ and $\mathrm{Hdg}(X)$ coincide. Similarly, if E is a totally real field and $\varphi\colon T(X) \times T(X) \longrightarrow E$ is the E-bilinear form on $T(X)$, the Lefschetz group of X is

$$\mathrm{L}(X) \simeq \mathrm{Res}_{E/\mathbb{Q}} \mathrm{O}(T(X), \varphi).$$

By the invariant theory results of previous section, the algebra $\left(\bigotimes^\bullet T(X)\right)^{\mathrm{L}(X)}$ is generated by its degree two elements both in the totally real and in the CM case.

Definition B.3.19. The *motivic group* $\mathrm{M}(X)$ is the biggest algebraic subgroup of $\mathrm{GL}(T(X))$ which preserves all the algebraic classes in $\bigotimes^\bullet T(X)$.

In this case, the fact that only the algebraic classes in $\bigotimes^\bullet T(X)$ are invariant under the action of $\mathrm{M}(X)$ is non-trivial: As in the case of abelian varieties [64, App. A], it follows from the fact that the category of motives on a variety is a Tannakian category if the Künneth components of the diagonal are algebraic [50, Cor. 3].

Since all algebraic classes are Hodge classes, there is an inclusion $\mathrm{Hdg}(X) \subseteq \mathrm{M}(X)$. Then, proving the Hodge conjecture for the powers of X is equivalent to prove that this inclusion is an equality.

If we assume the Hodge conjecture for X^2 holds, i.e., that the classes in $E \simeq (T(X) \otimes T(X))^{\mathrm{Hdg}(X)}$ are algebraic, we have the following chain of inclusions:

$$\mathrm{Hdg}(X) \subseteq \mathrm{M}(X) \subseteq \mathrm{L}(X). \tag{B.2}$$

In the case of K3 surfaces with CM endomorphism field, the Hodge group and the Lefschetz group are equal, all the inclusions in $(B.2)$ are therefore equalities. This gives another proof of Theorem B.3.2, and shows that, in the case of CM endomorphism field, the Hodge conjecture for X^2 implies the Hodge conjecture for all powers of X.

As we saw in the previous section, in the case of K3 surfaces with a totally real endomorphism field, the situation is different: Even if we assume the Hodge conjecture for the square we cannot conclude the Hodge conjecture for all powers of the given K3 surface due to the presence of the exceptional Hodge classes. From the group perspective, this is a consequence of the fact that the Hodge group and the Lefschetz group are not equal, so $(B.2)$ does not imply the equality between the Hodge and the motivic group. Assuming the Hodge conjecture for the square of the K3 surface, let us find which are the possibilities for the motivic group: Let X be a K3 surface with totally real endomorphism field E and let us take the complex points in $(B.2)$

$$\prod_{\sigma\colon E \hookrightarrow \mathbb{C}} \mathrm{SO}(V_\sigma, \mathbb{C}) \subseteq \mathrm{M}(X)(\mathbb{C}) \subseteq \prod_{\sigma\colon E \hookrightarrow \mathbb{C}} \mathrm{O}(V_\sigma, \mathbb{C}).$$

We see that $\mathrm{M}(X)(\mathbb{C})$ corresponds to a subgroup of the quotient

$$\left(\prod_{\sigma\colon E \hookrightarrow \mathbb{C}} \mathrm{O}(V_\sigma, \mathbb{C})\right) / \left(\prod_{\sigma\colon E \hookrightarrow \mathbb{C}} \mathrm{SO}(V_\sigma, \mathbb{C})\right) \simeq (\mathbb{Z}/2\mathbb{Z})^{|E:\mathbb{Q}|}.$$

From the fact that $\mathrm{M}(X)$ is defined over $\mathbb{Q}$, there are constraints on the possibilities for $\mathrm{M}(X)(\mathbb{C})$: Let $\bar{E}$ be the Galois closure of the field E and let $G := \mathrm{Gal}(\bar{E}/\mathbb{Q})$ be the Galois group of $\bar{E}$ over $\mathbb{Q}$, then G acts on $(\mathbb{Z}/2\mathbb{Z})^{|E:\mathbb{Q}|}$ by changing the factors. As $\mathrm{M}(X)$ is defined over $\mathbb{Q}$, the group $\mathrm{M}(X)(\mathbb{C})$ corresponds to a subgroup of $(\mathbb{Z}/2\mathbb{Z})^{|E:\mathbb{Q}|}$ which is preserved under the G-action. This allows us to conclude that

$$\mathrm{M}(X)(\mathbb{C}) / \left(\textstyle\prod_\sigma \mathrm{SO}(V_\sigma, \mathbb{C})\right)$$

is either trivial, the full group $(\mathbb{Z}/2\mathbb{Z})^{|E:\mathbb{Q}|}$, or the subgroup of $(\mathbb{Z}/2\mathbb{Z})^{|E:\mathbb{Q}|}$ generated by $(1,\dots,1)$.

In the first case, $\mathrm{M}(X)$ coincides with $\mathrm{Hdg}(X)$ and hence the Hodge conjecture holds for all powers of X. In the second case, $\mathrm{M}(X)$ coincides with $\mathrm{L}(X)$ and only the classes in the algebra generated by the Hodge classes in $T(X)\otimes T(X)$ are algebraic. Finally, in the latter case,

$$\mathrm{M}(X)(\mathbb{C}) = \{A_1\times\cdots\times A_k \in \textstyle\prod_\sigma \mathrm{O}(V_\sigma,\mathbb{C}) \mid \prod_i \det(A_i)=1\}.$$

This corresponds to the case where the class $\det T(X)$ is algebraic but the exceptional classes in $\bigwedge_E^r T(X)$ are not. Showing the Hodge conjecture for all powers of X is then equivalent to exclude the two latter cases. In particular, note that to prove the Hodge conjecture for all powers of a K3 surface X it suffices to prove it for X^2 and then show that there exists an algebraic class in $\bigwedge_E^r T(X)$. Indeed, this would force the motivic group to be equal to the Hodge group by the above discussion.

Remark B.3.20. The fact that we can exclude some possibilities for $\mathrm{M}(X)(\mathbb{C})$ is linked with the fact that $(\bigotimes^\bullet T(X)_\mathbb{C})^{\mathrm{M}(X)(\mathbb{C})}$ is defined over $\mathbb{Q}$: For example, assume that $\mathrm{M}(X)(\mathbb{C})$ corresponds to the subgroup of $(\mathbb{Z}/2\mathbb{Z})^{|E:\mathbb{Q}|}$ generated by $(0,1,\dots,1)$, i.e.

$$\mathrm{M}(X)(\mathbb{C}) \simeq \mathrm{SO}(V_{\sigma_1},\mathbb{C})\times \mathrm{O}(V_{\sigma_2},\mathbb{C})\times\cdots\times \mathrm{O}(V_{\sigma_k},\mathbb{C}).$$

Then $(\bigwedge_E^r T(X))^{\mathrm{M}(X)}\otimes_\mathbb{Q}\mathbb{C}$ is one-dimensional and generated by $\det V_{\sigma_1}$. In particular, up to complex scalar, $\det V_{\sigma_1}$ is a rational cohomology class. To see that this cannot happen, consider the action of E on the first factor of $T(X)^{\otimes r}$. As any element of E acts on $\det V_{\sigma_1}$ by multiplication by its image via $\sigma_1\colon E\longrightarrow\mathbb{C}$, the class $\det V_{\sigma_1}$ cannot be rational.

Remark B.3.21. Note that this motivic approach is equivalent to the approach we presented. It might seem easier due to the fact that we implicitly used in more places the results proved above. For example, to show that the algebra of invariant classes under the Lefschetz group is generated by degree-two elements, we used the invariant theory of the previous section.

B.4 The Hodge conjecture and deformations of K3 surfaces

Let X be a K3 surface and let $n := \dim_\mathbb{Q} T(X)$. Let $\det T(X)$ be a generator of the one-dimensional vector space

$$\textstyle\bigwedge^n T(X)\subseteq T(X)^{\otimes n}$$

Note that $\det T(X)$ is a Hodge class for every K3 surface X. If the endomorphism field of X is a CM field, by Theorem B.3.2, the class $\det T(X)$ can be expressed in terms of $(2,2)$-classes. On the other hand, if the endomorphism field is $\mathbb{Q}$ or any other totally real field, $\det T(X)$ cannot be expressed in terms of Hodge classes of degree two.

In this section, we prove that the property *the determinant of the transcendental lattice is algebraic* is a closed property in families of K3 surfaces:

Proposition B.4.1. *Let $\mathscr{X} \longrightarrow S$ be a family of K3 surfaces such that $\det T(\mathscr{X}_s)$ is algebraic for general $s \in S$. Then, the same holds for all $s \in S$.*

Proof. We may assume that the transcendental lattices of the general fibres of $\mathscr{X} \longrightarrow S$ are isometric to a given quadratic space $\tilde{T}$. Write $n := \dim_{\mathbb{Q}} \tilde{T}$. Since $\det T(\mathscr{X}_s)$ is algebraic for general $s \in S$ by assumption, we conclude that $\det \tilde{T}$ is algebraic when seen as a class of $H^{r,r}(\mathscr{X}_s^r, \mathbb{Q})$ for all $s \in S$. Note that this does not conclude the proof, since $\det \tilde{T}$ does not necessarily specialize to $\det T(\mathscr{X}_s)$ for every $s \in S$, since it may happen that $\dim T(\mathscr{X}_s) < n$ for some $s \in S$. Fix an element $0 \in S$ and let $X := \mathscr{X}_0$ be the corresponding K3 surface. By construction, the transcendental lattice of X is naturally a subspace of $\tilde{T}$. Hence, if $\dim_{\mathbb{Q}} T(X) = n$ then $T(X) = \tilde{T}$ (as rational quadratic spaces) and $\det T(X)$ is algebraic so there is nothing to prove. Let assume that $\dim_{\mathbb{Q}} \tilde{T} - \dim_{\mathbb{Q}} T(X) = c > 0$ and denote by $\tilde{T}/T(X)$ the quotient vector space. Up to a rational coefficient, we have the following equality:

$$\det \tilde{T} = \sum \pm q_I^* \det T(X) \otimes p_I^* \det(\tilde{T}/T(X)), \tag{B.3}$$

where the sum runs over all subsets $I \subseteq \{1, \dots, n\}$ of length c, p_I is the projection from X^n onto the I-th factors and q_I is the projection onto the remaining factors.

Let Z_1 be an algebraic class representing $\det \tilde{T}$ on X^n. Then, Z_1 defines the following algebraic map:

$$\begin{aligned} \varphi_1 \colon H^{1,1}(X, \mathbb{Q}) &\longrightarrow H^{n-1,n-1}(X^{n-1}, \mathbb{Q}), \\ y &\longmapsto q_{1*}(Z_1 \cap p_1^* y) \end{aligned}$$

where p_1 is the projection from X^n onto the first factor and q_1 is the projection onto the remaining factors. Let $x_1 \in H^{1,1}(X, \mathbb{Q})$ be a fixed element and denote by Z_2 the algebraic class $\varphi_1(x_1) \in H^{n-1,n-1}(X^{n-1}, \mathbb{Q})$. Then, similarly to the previous case, Z_2 defines an algebraic map

$$\begin{aligned} \varphi_2 \colon H^{1,1}(X, \mathbb{Q}) &\longrightarrow H^{n-2,n-2}(X^{n-2}, \mathbb{Q}). \\ y &\longmapsto q_{2*}(Z_1 \cap p_2^* y) \end{aligned}$$

Repeating this procedure c times we get an algebraic class Z_c in $H^{n-c,n-c}(X^{n-c}, \mathbb{Q})$. Note that the only summand of (B.3) contributing to the resulting class Z_c is the one for which $I = \{1, \dots, c\}$. This follows from the fact that all the elements of $T(X)$ are orthogonal to $\mathrm{NS}(X)$ with respect to the canonical pairing on X. Using a basis of the rational quotient space $\tilde{T}/T(X)$ consisting of algebraic classes, we see that it is possible to choose a sequence $x_1, \dots, x_c \in H^{1,1}(X, \mathbb{Q})$ for which the resulting Z_c is non-zero and represents the class $\det T(X) \in H^{n-c,n-c}(X^{n-c}, \mathbb{Q})$. This shows that $\det T(X)$ is algebraic and concludes the proof. □

As an immediate corollary, we have the following:

Corollary B.4.2. *Let* $\mathscr{X} \longrightarrow S$ *be a family of K3 surfaces such that the Hodge conjecture for all powers of* $\mathscr{X}_s$ *holds for general* $s \in S$. *If* $0 \in S$ *is an element such that* $\mathrm{End}_{\mathrm{Hdg}(\mathscr{X}_0)}(T(\mathscr{X}_0)) = \mathbb{Q}$, *then the Hodge conjecture holds for all powers of* $\mathscr{X}_0$.

Proof. Since we are assuming that the endomorphism field of $\mathscr{X}_0$ is $\mathbb{Q}$, by Theorem B.3.16, the algebra of Hodge classes in $\bigotimes^\bullet T(\mathscr{X}_0)$ is generated by

$$\mathrm{End}_{\mathrm{Hdg}(\mathscr{X}_0)}(T(\mathscr{X}_0)) \simeq \mathbb{Q} \quad \text{and} \quad \det T(\mathscr{X}_0).$$

This implies the statement, indeed, the generator of $\mathrm{End}_{\mathrm{Hdg}(\mathscr{X}_0)}(T(\mathscr{X}_0))$ is algebraic, since it corresponds to a component of the class of the diagonal $\Delta \subseteq \mathscr{X}_0 \times \mathscr{X}_0$, and $\det T(\mathscr{X}_0)$ is algebraic by Proposition B.4.1. □

Remark B.4.3. With a little bit of work, one can prove the same statement of Proposition B.4.1 for the exceptional Hodge classes. Note that the determinant of the transcendental lattice is the unique exceptional Hodge class in the case $E = \mathbb{Q}$. This in particular shows that, in the statement of Corollary B.4.2, the assumption $\mathrm{End}_{\mathrm{Hdg}(\mathscr{X}_0)}(T(\mathscr{X}_0)) = \mathbb{Q}$ can be weakened to

$$\mathrm{End}_{\mathrm{Hdg}(\mathscr{X}_0)}(T(\mathscr{X}_0)) = \mathrm{End}_{\mathrm{Hdg}(\mathscr{X}_s)}(T(\mathscr{X}_s)).$$

Note that the inclusion "$\supseteq$" always holds true. We do not give here the detailed proof since this extended result is not needed in the remainder of this paper.

B.5 Abelian varieties of Weil type

In this section, we recall the definition of abelian varieties of Weil type and we study the Hodge conjecture for their powers. Our interest in these varieties lies in the fact that, as we recall in the next section, the Kuga–Satake varieties of K3 surfaces of Picard number 16 are powers of abelian fourfolds of Weil type. These abelian varieties have been first introduced and studied by Weil [99]. We refer to [28] for a complete introduction. For a survey on the Hodge conjecture for abelian varieties we refer to [68] and to [69].

In this section, we denote by K the CM field $\mathbb{Q}(\sqrt{-d})$, where d is a positive rational number.

Definition B.5.1. Let A be an abelian variety of dimension $2n$ such that $K \subseteq \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. Then, A is an abelian variety of K*-Weil type* if the action of $\sqrt{-d}$ on the tangent space at the origin of A has eigenvalues $\sqrt{-d}$ and $-\sqrt{-d}$ both with multiplicity n.

Given an abelian variety A of K-Weil type, there exists a polarization H on A such that $(\sqrt{-d})^* H = dH$. This polarization induces the following K-Hermitian map on $H^1(A, \mathbb{Q})$:

$$\tilde{H} \colon H^1(A,\mathbb{Q}) \times H^1(A,\mathbb{Q}) \longrightarrow K, \quad \tilde{H}(x,y) := H(x, \sqrt{-d}y) + \sqrt{-d}H(x,y). \qquad \text{(B.4)}$$

Let $\mathrm{N}(K)$ be the set of norms of K, and let $\det(\tilde{H})$ be the determinant of $\tilde{H}$ with respect to any K-basis of $H^1(A,\mathbb{Q})$. Then, one checks that $\det(\tilde{H})$ belongs to $\mathbb{Q}^*$ and that its image δ in $\mathbb{Q}/\mathrm{N}(K)$ does not depend on the choice of the basis. The class δ is called the *discriminant* of A. We say that an abelian variety A of K-Weil type is *general* if it has maximal Hodge group. By [28, Thm. 6.11], this means that $\mathrm{Hdg}(A) = \mathrm{SU}(\tilde{H})$ and that $\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q} = K$.

Note that there are similarities between this situation and the case of K3 surfaces with CM endomorphism field. Let E be a CM field with maximal totally real subfield $\mathbb{Q}$, and let X be a K3 surface with endomorphism field E and let $2n := \dim T(X)$. By [101, Thm. 2.3.1], the Hodge group of X is $U(\varphi)$, where φ is E-Hermitian form on $T(X)$. The discussion of Section B.3.1 and in particular Lemma B.3.11 show that $U(\varphi)(\mathbb{C}) \simeq \mathrm{GL}(n,\mathbb{C})$ and that the representation of $\mathrm{Hdg}(X)(\mathbb{C})$ on $T(X)_{\mathbb{C}}$ is the direct sum of the standard representation of $\mathrm{GL}(n,\mathbb{C})$ and its dual. In the case of a general abelian variety A of Weil type of dimension $2n$, the Hodge group is $\mathrm{SU}(\tilde{H})$. Then, a similar argument shows the following:

Proposition B.5.2. *[28, Lem. 6.10] Let A be a general abelian variety of Weil type of dimension $2n$. Then,* $\mathrm{Hdg}(A)(\mathbb{C})$ *is isomorphic to* $\mathrm{SL}(2n,\mathbb{C})$*, and the representation* $H^1(A,\mathbb{C})$ *of* $\mathrm{Hdg}(A)(\mathbb{C})$ *is the direct sum of the standard representation of* $\mathrm{SL}(2n,\mathbb{C})$ *and its dual.*

Abelian varieties of Weil type are characterized by the existence of Hodge classes which do not belong to the algebra generated by divisors:

Remark B.5.3. Let A be an abelian variety of K-Weil type of dimension $2n$. By Lemma B.3.15, there exists an embedding

$$\epsilon\colon \textstyle\bigwedge_K^{2n} H^1(A,\mathbb{Q}) \hookrightarrow \bigwedge^{2n} H^1(A,\mathbb{Q}) \simeq H^{2n}(A,\mathbb{Q}).$$

The condition on the action of the field K on $H^1(A,\mathbb{Q})$ implies that the image of ϵ consists of Hodge classes, see [18, Prop. 4.4]. These classes are called *exceptional classes* or *Weil classes*. From now on, we identify $\bigwedge_K^{2n} H^1(A,\mathbb{Q})$ with its image via ϵ in $\bigwedge^{2n} H^1(A,\mathbb{Q})$.

By Proposition B.5.2, the algebra of Hodge classes on a general abelian variety of Weil type satisfies the following

$$\left(\textstyle\bigwedge^{\bullet} H^1(A,\mathbb{Q})\right)^{\mathrm{Hdg}(A)} \otimes \mathbb{C} \simeq \left(\textstyle\bigwedge^{\bullet}(W \oplus W^*)\right)^{\mathrm{SL}(W,\mathbb{C})},$$

where W is a complex vector space of dimension $2n$. Complete contractions are natural examples of invariant elements of the right-hand side. Let us recall their definitions.

Definition B.5.4. Let W be a $2n$-dimensional vector space and let s be a positive integer. Then, an element in $W^{\otimes s} \otimes (W^*)^{\otimes s}$ is called a *complete contraction* of degree $2s$ if its image under the natural $\mathrm{SL}(W,\mathbb{C})$-invariant isomorphism

$$W^{\otimes s} \otimes (W^*)^{\otimes s} \simeq \mathrm{Hom}((W^*)^{\otimes s} \otimes W^{\otimes s}, \mathbb{C})$$

is equal to

$$\mu_1 \otimes \cdots \otimes \mu_s \otimes v_1 \otimes \cdots \otimes v_s \longmapsto \prod_i \mu_i(v_{\sigma(i)}),$$

for some permutation $\sigma \in \mathfrak{S}^s$

Remark B.5.5. From the definition, it is immediate to see that complete contractions are invariant under the action of $\mathrm{SL}(W,\mathbb{C})$ and that any complete contraction can be expressed in terms of complete contractions of degree two.

Let s and s' be non-negative integers, and let $I = (i_1, \dots, i_k)$ and $I' = (i'_1, \dots, i'_k)$ be partitions of s and s' respectively, i.e., $i_1 + \cdots + i_k = s$ and $i'_1 + \cdots + i'_k = s'$. Considering the natural action of $\mathfrak{S}^s \times \mathfrak{S}^{s'}$ on $W^{\otimes s} \otimes (W^*)^{\otimes s'}$, we introduce following definition.

Definition B.5.6. An element $\alpha \in W^{\otimes s} \otimes (W^*)^{\otimes s'}$ is (I, I')-*alternating* if it satisfies

$$(\sigma, \sigma')(\alpha) = \mathrm{sgn}(\sigma)\mathrm{sgn}(\sigma')\alpha,$$

for any pair of permutations $\sigma \in \mathfrak{S}^{i_1} \times \cdots \times \mathfrak{S}^{i_k} \subseteq \mathfrak{S}^s$ and $\sigma' \in \mathfrak{S}^{i'_1} \times \cdots \times \mathfrak{S}^{i'_k} \subseteq \mathfrak{S}^{s'}$.

In the case where $s = s'$ and $I = I' = (s)$, the vector space of (I, I')-alternating elements in $W^{\otimes s} \otimes (W^*)^{\otimes s}$ is equal to $\bigwedge^s W \otimes \bigwedge^s W^*$.

We are now able to state the following theorem from invariant theory.

Theorem B.5.7. *[52, Thm. 8.4] Let W be a $2n$-dimensional complex vector space and denote by W^* its dual. Then, any element of*

$$\left(\bigotimes\nolimits^{\bullet}(W \oplus W^*)\right)^{\mathrm{SL}(W,\mathbb{C})}$$

can be expressed in terms of complete contractions and the two determinants $\det W$ *and* $\det W^*$ *in* $(W \oplus W^*)^{\otimes 2n}$.

Proof. See Definition B.3.1 for formal meaning of the expression "can be expressed in terms of". To deduce this statement from the one in the reference, just note that, for any pair of non-negative integers p and q, any element of $W^{\otimes p} \otimes (W^*)^{\otimes q} \subseteq (W \oplus W^*)^{\otimes(p+q)}$ can be viewed as a homogeneous polynomial function $(W^*)^{\oplus p} \oplus W^{\oplus q} \longrightarrow \mathbb{C}$ of degree $p + q$. □

Applying Theorem B.5.7 to a general abelian variety of Weil type, one can compute the dimension of the vector space of Hodge classes. Indeed, the following holds:

Theorem B.5.8. *[99, Thm. 6.12] Let A be a general abelian variety of K-Weil type of dimension $2n$. Then,*

$$\dim(H^{s,s}(A,\mathbb{Q})) = \begin{cases} 3, & \text{if } s = n \\ 1, & \text{if } s \neq n. \end{cases}$$

In particular, the Picard number of A is one.

Proof. We just give a sketch of the proof since the same techniques will be used with much more detail later in this section: By assumption, A is a general abelian variety of Weil type. Then, as before, we have the following isomorphism:

$$\left(\bigwedge^{\bullet} H^1(A,\mathbb{Q})\right)^{\mathrm{Hdg}(A)} \otimes_{\mathbb{Q}} \mathbb{C} \simeq \left(\bigwedge^{\bullet}(W \oplus W^*)\right)^{\mathrm{SL}(W,\mathbb{C})}, \tag{B.5}$$

where W is a $2n$-dimensional complex vector space. After considering the natural $\mathrm{SL}(W,\mathbb{C})$-invariant embedding

$$\bigwedge^{\bullet}(W \oplus W^*) \hookrightarrow \bigotimes^{\bullet}(W \oplus W^*),$$

we can apply Theorem B.5.7 to deduce that the $\mathbb{C}$-algebra $\left(\bigwedge^{\bullet}(W \oplus W^*)\right)^{\mathrm{SL}(W,\mathbb{C})}$ is generated by (s,s)-alternating linear combinations of complete contractions in $\bigotimes^{2s}(W \oplus W^*)$ for all $1 \leq s \leq 2n$ together with the two determinants $\det W, \det W^* \in \bigotimes^{2n}(W \oplus W^*)$. Note that for every s there exists a unique linear combination of complete contractions in $W^{\otimes s} \otimes (W^*)^{\otimes s}$ which is (s,s)-alternating: It corresponds to the $\mathrm{SL}(W)$-invariant map

$$\begin{aligned} \bigwedge^{s} W^* \otimes \bigwedge^{s} W &\longrightarrow \mathbb{C} \\ \mu_1 \wedge \cdots \wedge \mu_s \otimes v_1 \wedge \cdots \wedge v_s &\longmapsto \det(\mu_i(v_j))_{i,j} \end{aligned}$$

This, together with the isomorphism in (B.5), proves the statement. □

Remark B.5.9. As an immediate consequence of Theorem B.5.8, we deduce that the algebra of Hodge classes on A is generated by the unique class in $\mathrm{NS}(A)_{\mathbb{Q}}$ and the exceptional classes in $\bigwedge_K^{2n} H^1(A,\mathbb{Q})$. As in the case of K3 surfaces with totally real multiplication, this follows from the isomorphism

$$\left(\bigwedge_K^{2n} H^1(A,\mathbb{Q})\right) \otimes_{\mathbb{Q}} \mathbb{C} \simeq \langle \det W, \det W^* \rangle_{\mathbb{C}},$$

cf. [18, Prop. 4.4].

In general, it is not known whether the Weil classes are algebraic, but in the case of abelian fourfolds of Weil type with discriminant one, the following holds:

Theorem B.5.10. *[60, Thm. 1.5] Let A be abelian fourfold of Weil type with discriminant one. Then, the Weil classes on A are algebraic. In particular, the Hodge conjecture holds for the general abelian fourfold of Weil type with discriminant one.*

The same result was previously proven by Schoen [82] in the case of abelian fourfolds $\mathbb{Q}(\sqrt{-3})$-Weil type for arbitrary discriminant and by Schoen [82] and independently by van Geemen [29] in the case of abelian fourfolds $\mathbb{Q}(i)$-Weil type with discriminant one.

In the remainder of this section, we prove that the Hodge conjecture for a general abelian variety of Weil type implies the Hodge conjecture for all its powers. Our strategy is the following: We first prove an extension of Theorem B.5.7 which allows us to find a set of generators for the algebra of Hodge classes on the powers of A. Then, we show that there are relations between these generators and we conclude that these generators are algebraic if the Hodge conjecture holds for A. For a comparison with the work of Abdulali [1], see Remark B.5.17.

Let A be a general abelian variety of Weil type and let k be a positive integer. By Theorem B.5.7, the ring of Hodge classes on A^k satisfies

$$\left(\bigwedge^\bullet(H^1(A,\mathbb{Q})^{\oplus k})\right)^{\mathrm{Hdg}(A)} \otimes \mathbb{C} \simeq \left(\bigwedge^\bullet((W \oplus W^*)^{\oplus k})\right)^{\mathrm{SL}(W,\mathbb{C})},$$

where W is a $2n$-dimensional complex vector space. To study this ring, let us introduce the notion of *realizations* of $\det W$ and $\det W^*$.

Remark B.5.11. Let W be a $2n$-dimensional complex vector space and let k be a positive integer. Consider the following canonical decomposition:

$$\bigwedge^{2n}\left((W \oplus W^*)^{\oplus k}\right) = \bigoplus \left(\bigwedge^{i_1} W_1 \otimes \cdots \otimes \bigwedge^{i_k} W_k \otimes \bigwedge^{i_{k+1}} W_1^* \otimes \cdots \otimes \bigwedge^{i_{2k}} W_k^*\right),$$

where the sum runs over all $2k$-partitions I of $2n$ and $W_j = W$ for all j. We introduced the W_j to be able to distinguish between $\bigwedge^{2n} W_1$ and $\bigwedge^{2n} W_2$, etc. Note that if I is a k-partition of $2n$, i.e., $i_{k+1} = \ldots = i_{2n} = 0$, there is a natural embedding

$$\iota_I \colon \bigwedge^{2n} W \hookrightarrow \bigwedge^{i_1} W_1 \otimes \cdots \otimes \bigwedge^{i_k} W_k.$$

This follows from the fact that the image of $\bigwedge^{2n} W \hookrightarrow W^{\bigotimes 2n}$ is contained in the image of the natural embedding $\bigwedge^{i_1} W_1 \otimes \cdots \otimes \bigwedge^{i_k} W_k \hookrightarrow W^{\bigotimes 2n}$. As one sees, ι_I is compatible with the natural action of $\mathrm{SL}(W,\mathbb{C})$. Hence, its image determines (up to a complex scalar) an $\mathrm{SL}(W,\mathbb{C})$-invariant class. Denote this class as $(\det W)_I$ and consider it as an element in $\bigwedge^{2n}\left((W \oplus W^*)^{\oplus k}\right)$. We call it a *realization* of $\det W$. We then call

$$\{(\det W)_I \mid I \text{ is a } k\text{-partition of } 2n\} \subseteq \bigwedge^{2n}\left((W \oplus W^*)^{\oplus k}\right)$$

the *set of all realizations* of $\det W$ in $\bigwedge^{2n}\left((W \oplus W^*)^{\oplus k}\right)$. Similarly, we introduce the notion of *realizations* of $\det W^*$.

To give an example, let $k = 2$, $2n = 4$, and consider the 2-partition of 4 given by $I = (2,2)$. Then, denoting by $v_1, \ldots, v_4$ a basis of W, the realization $(\det W)_I$ in $\bigwedge^4((W \oplus W^*))^{\oplus 2}$ is given by the image of

$$\begin{aligned} \bigwedge^4 W &\hookrightarrow \bigwedge^2 W_1 \otimes \bigwedge^2 W_2 \\ v_1 \wedge \cdots \wedge v_4 &\longmapsto \sum \pm (v_i \wedge v_j) \otimes (v_k \wedge v_l), \end{aligned}$$

where the sum runs over all $i < j, k < l$ such that $\{i,j,k,l\} = \{1,2,3,4\}$.

With this notion, we can now state and prove the following extension of Theorem B.5.7.

Corollary B.5.12. *Let W be a $2n$-dimensional complex vector space and denote by W^* its dual. Then, for every positive integer k, any invariant in*

$$\left(\bigwedge^\bullet((W \oplus W^*)^{\oplus k})\right)^{\mathrm{SL}(W,\mathbb{C})}$$

can be expressed in terms of invariants of degree two and all realizations of $\det W$ *and* $\det W^*$ *in* $\bigwedge^* \left((W \oplus W^*)^{\oplus k}\right)$ *defined in Remark B.5.11.*

Proof. Let s and k be two positive integers. As in Remark B.5.11, we decompose the space $\bigwedge^s \left((W \oplus W^*)^{\oplus k}\right)$ as

$$\bigwedge^s \left((W \oplus W^*)^{\oplus k}\right) = \bigoplus \left(\bigwedge^{i_1} W_1 \otimes \cdots \otimes \bigwedge^{i_k} W_k \otimes \bigwedge^{i_{k+1}} W_1^* \otimes \cdots \otimes \bigwedge^{i_{2k}} W_k^*\right),$$

where the direct sum runs over all $2k$-partitions of s, and $W_j = W$ for all j. For any $2k$-partition I of s, denote by $Z_{I,s}$ the corresponding direct summand of the decomposition. Since the action of $\mathrm{SL}(W,\mathbb{C})$ preserves this decomposition, an element in $\bigwedge^s \left((W \oplus W^*)^{\oplus k}\right)$ is $\mathrm{SL}(W,\mathbb{C})$-invariant if and only if its component in $Z_{I,s}$ is invariant for every $2k$-partition I of s. Therefore, it suffices to study the invariants in each $Z_{I,s}$. To this end, we first embed each $Z_{I,s}$ individually into $\bigotimes^s(W \oplus W^*)$ and then apply Theorem B.5.7:

Let $I = (i_1, \ldots, i_{2k})$ be a fixed $2k$-partition of s. Let $s' := i_1 + \cdots + i_k$, and consider the natural $\mathrm{SL}(W,\mathbb{C})$-invariant embedding of $Z_{I,s}$

$$Z_{I,s} = \bigwedge^{i_1} W_1 \otimes \cdots \otimes \bigwedge^{i_k} W_k \otimes \bigwedge^{i_{k+1}} W_1^* \otimes \cdots \otimes \bigwedge^{i_{2k}} W_k^* \hookrightarrow W^{\otimes s'} \otimes (W^*)^{\otimes(s-s')} \quad \text{(B.6)}$$

given by the tensor product of the canonical embeddings

$$\bigwedge^{i_j} W \hookrightarrow W^{\otimes i_j} \text{ and } \bigwedge^{i_j} W^* \hookrightarrow (W^*)^{\otimes i_j}.$$

From now on, we identify $Z_{I,s}$ and its image. Note that an element of $W^{\otimes s'} \otimes (W^*)^{\otimes(s-s')}$ belongs to $Z_{I,s}$ if and only if it is I-alternating.

Let us first deal with the case where $s < 2n$: Since $\det W$ and $\det W^*$ are elements of $(W \oplus W^*)^{\otimes 2n}$, Theorem B.5.7 shows that the vector space of invariants in $W^{\otimes s'} \otimes (W^*)^{\otimes s-s'}$ is zero if $s - s' \neq s'$, and it is generated by complete contractions if $s - s' = s'$. Therefore, we conclude that $Z_{I,s}$ does not contain any non-trivial invariant if $s - s' \neq s'$ and it is generated by linear combinations of complete contractions which are I-alternating if $s - s' = s'$.

For $s = 2n$, Theorem B.5.7 shows that the vector space of invariants in $W^{\otimes s'} \otimes (W^*)^{\otimes(2n-s')}$ is generated by $\det W$ if $s' = 2n$, by $\det W^*$ if $s' = 0$, by complete contractions of degree $2n$ if $s' = n$, and it is zero in all other cases. In the case $s' = 2n$, we have

$$Z_{I,2n} = \bigwedge^{i_1} W_1 \otimes \cdots \otimes \bigwedge^{i_k} W_k \subseteq W^{\otimes 2n}.$$

Since $\det W$ is I-invariant, we conclude that $\det W \in Z_{I,2n}$ and that the vector space of invariants in $Z_{I,2n}$ is generated by $\det W$. Similarly, if $s' = 0$, one sees that $\det W^*$ generate the ring of invariants in $Z_{I,2n}$. Finally, if $s' = n$, one sees that the ring of invariants in $Z_{I,2n}$ is generated by I-alternating linear combinations of complete contractions, as in the previous case.

For $s > 2n$, the invariants in $Z_{I,s}$ can be expressed in terms of invariants of $Z_{\tilde{I},\tilde{s}}$ for $\tilde{s} \leq 2n$. This follows from the fact that invariants in $(\bigotimes^{\bullet}(W \oplus W^*))^{\mathrm{SL}(W,\mathbb{C})}$ can be expressed in terms of invariants of degree $\leq 2n$.

To sum up, we proved that invariants in $\left(\bigwedge^{\bullet}((W \oplus W^*)^{\oplus k})\right)^{\mathrm{SL}(W,\mathbb{C})}$ can be expressed in terms of linear combinations of complete contractions and the images the maps $\bigwedge^{2n} W \longhookrightarrow Z_{I,2n}$ for all $I = (i_1, \ldots, i_{2k})$ such that $i_1 + \cdots + i_k = 2n$ and $\bigwedge^{2n} W^* \longhookrightarrow Z_{I,2n}$ for all $I = (i_1, \ldots, i_{2k})$ such that $i_1 + \cdots + i_k = 0$.

Similarly to Remark B.5.5, any linear combination of complete contractions can be written as a linear combination of wedge products of degree-two complete contractions in $\bigwedge^2 \left((W \oplus W^*)^{\oplus k}\right)$. Therefore, to conclude the proof, it suffices to note that the set of all images of the maps $\bigwedge^{2n} W \longhookrightarrow Z_{I,2n}$ for all $I = (i_1, \ldots, i_{2k})$ such that $i_1 + \cdots + i_k = 2n$ is the set of all realizations of $\det W$ defined in Remark B.5.11, and similarly for $\det W^*$. □

Before applying Corollary B.5.12 to study Hodge classes for powers of abelian varieties of Weil type, let us define the set of realizations of exceptional classes:

Remark B.5.13. Let A be an abelian variety of K-Weil type of dimension $2n$. As in Remark B.5.3, we identify $\bigwedge_K^{2n} H^1(A,\mathbb{Q})$ with the set of exceptional classes on A via the natural embedding

$$\epsilon\colon \bigwedge_K^{2n} H^1(A,\mathbb{Q}) \longhookrightarrow \bigwedge^{2n} H^1(A,\mathbb{Q}) \simeq H^{2n}(A,\mathbb{Q}).$$

For any integer $k > 1$, similarly to Remark B.5.11, consider various embeddings of $\bigwedge^{2n} H^1(A,\mathbb{Q})$ into $\bigwedge^{2n}(H^1(A,\mathbb{Q})^{\oplus k}) \simeq H^{2n}(A^k,\mathbb{Q})$: If $I = (i_1, \ldots, i_k)$ is a k-partition of $2n$, denote by

$$\iota_I\colon \bigwedge^{2n} H^1(A,\mathbb{Q}) \longhookrightarrow \bigwedge^{i_1} H^1(A,\mathbb{Q}) \otimes \cdots \otimes \bigwedge^{i_k} H^1(A,\mathbb{Q}) \subseteq H^{2n}(A^k,\mathbb{Q}),$$

the corresponding embedding as in Remark B.5.11. As ι_I is a morphisms of Hodge structures, $\iota_I(\alpha)$ is a Hodge class for every $\alpha \in \bigwedge_K^{2n} H^1(A,\mathbb{Q})$. We call $\iota_I(\alpha)$ *a realization* of α on A^k. Finally, the set of *all realizations* on A^k of the exceptional classes is the set

$$\{\iota_I(\alpha) \mid \alpha \in \textstyle\bigwedge_K^{2n} H^1(A,\mathbb{Q}), \text{and } I \text{ is a } k\text{-partition of } 2n\} \subseteq H^{2n}(A^k,\mathbb{Q}).$$

The following result describes the set of Hodge classes on the powers of A in terms of realizations of exceptional classes of A.

Lemma B.5.14. *Let A be a general abelian variety of K-Weil type of dimension $2n$. Then, for every positive integer k, any Hodge class on A^k can be expressed in terms of rational $(1,1)$-classes and all realizations of the exceptional classes on A^k.*

Proof. Let k be a positive integer. As A is a general abelian variety of Weil type, the algebra of Hodge classes on A^k satisfies the following:

$$\left(\bigwedge^\bullet(H^1(A,\mathbb{Q})^{\oplus k})\right)^{\mathrm{Hdg}(A)} \otimes_{\mathbb{Q}} \mathbb{C} \simeq \left(\bigwedge^\bullet((W \oplus W^*)^{\oplus k})\right)^{\mathrm{SL}(W,\mathbb{C})}. \tag{B.7}$$

By Corollary B.5.12, any invariant in this algebra can be expressed in terms of invariants of degree two and realizations of $\det W$ and $\det W^*$. The $\mathbb{C}$-linear span of invariants of degree two is equal to the $\mathbb{C}$-linear span of rational $(1,1)$-classes on A^k, hence, it suffices to show that the $\mathbb{C}$-linear span of the set of all realizations of $\det W$ and $\det W^*$ in $\bigwedge^\bullet((W \oplus W^*)^{\oplus k})$ is equal to the $\mathbb{C}$-linear span of the set of all realizations of the exceptional classes on A^k defined in Remark B.5.13. This follows from the fact that, if I is any k-partition of $2n$, the isomorphism in (B.7) sends the $\mathbb{C}$-linear span of $\iota_I\left(\bigwedge_K^{2n} H^1(A,\mathbb{Q})\right) \subseteq H^{2n}(A^k,\mathbb{Q})$ onto the $\mathbb{C}$-linear span of

$$\iota_I\left(\langle \bigwedge^{2n} W, \bigwedge^{2n} W^* \rangle\right) \subseteq \bigwedge^{2n}\left((W \otimes W^*)^{\oplus k}\right). \qquad \square$$

In particular, to show that the Hodge conjecture for A implies the Hodge conjecture for all powers of A, it suffices to show that if an exceptional class is algebraic on A then all its realizations are algebraic. The following lemma gives some relations between the realizations of the exceptional classes:

Lemma B.5.15. *Let A be a general abelian variety of K-Weil type of dimension $2n$, and let α be an exceptional class on A. Denote by $\widetilde{\alpha}$ the realization of α on A^{2n} via the embedding*

$$\bigwedge_K^{2n} H^1(A,\mathbb{Q}) \hookrightarrow H^1(A,\mathbb{Q}) \otimes \cdots \otimes H^1(A,\mathbb{Q}) \subseteq H^{2n}(A^{2n},\mathbb{Q}).$$

Then, for any positive integer k, and any k-partition I of $2n$, the realization α_I on A^k is the pullback of $\widetilde{\alpha}$, via an algebraic map $A^k \longrightarrow A^{2n}$. In particular, if $\widetilde{\alpha}$ is algebraic on A^{2n}, then any realization of α on any power of A is algebraic.

Proof. Writing α as a sum of decomposable elements of $\bigwedge^{2n} H^1(A,\mathbb{Q})$, we see that it suffices to show that, for any decomposable element $\beta := v_1 \wedge \ldots \wedge v_{2n} \in \bigwedge^{2n} H^1(A,\mathbb{Q})$, any realization of β on A^k is the pullback via an algebraic map $A^k \longrightarrow A^{2n}$ of

$$\widetilde{\beta} = \textstyle\sum_\sigma \pm v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(2n)},$$

where the sum runs over all permutations σ of $\{1,\ldots,2n\}$. Let $I = (i_1,\ldots,i_k)$ be a k-partition of $2n$, and let β_I be the realization of β on A^k via the embedding

$$\bigwedge_K^{2n} H^1(A,\mathbb{Q}) \hookrightarrow \bigwedge^{i_1} H^1(A,\mathbb{Q}) \otimes \cdots \otimes \bigwedge^{i_k} H^1(A,\mathbb{Q}) \subseteq H^{2n}(A^k,\mathbb{Q}).$$

Then, one sees that a multiple of β_I is equal to the pullback of $\widetilde{\beta}$ via the map

$$\Delta_{i_1} \times \cdots \times \Delta_{i_k} \colon A^k \longrightarrow A^{2n},$$

where Δ_{i_j} is the diagonal map $A \longrightarrow A^{i_j}$. $\square$

We have now everything needed to prove that the Hodge conjecture for a general abelian variety of Weil type implies the Hodge conjecture for all its powers.

Theorem B.5.16. *[1] Let A be a general abelian variety of K-Weil type. Then, the Hodge conjecture for A implies the Hodge conjecture for all powers A^k.*

Proof. By Lemma B.5.14, for any positive integer k, any Hodge class on A^k can be expressed in terms of rational $(1,1)$-classes and realizations of the exceptional classes on A^k. Since rational $(1,1)$-classes are algebraic by the Lefschetz $(1,1)$ theorem, to prove the Hodge conjecture for all powers of A, we need to show that all realizations of the exceptional classes on the powers of A are algebraic. By Lemma B.5.15, it suffices to show that for any exceptional class α on A, its realization $\widetilde{\alpha}$ via the map

$$\textstyle\bigwedge_K^{2n} H^1(A,\mathbb{Q}) \hookrightarrow H^1(A,\mathbb{Q})^{\otimes 2n} \subseteq H^{2n}(A^{2n},\mathbb{Q})$$

is algebraic.

To do this, consider the following maps: For $J \subseteq \{1,\dots,2n\}$, let $p_J\colon A^{2n} \longrightarrow A^{|J|}$ be the projection from A^{2n} onto the J-th components, and, for $i \geq 1$, let $\Sigma_i\colon A^i \longrightarrow A$ be the summation map. In cohomology, the pullback via Σ_i is

$$\Sigma_i^*\colon H^1(A,\mathbb{Q}) \longrightarrow H^1(A,\mathbb{Q})^{\oplus i}, \quad v \longmapsto (v,\dots,v).$$

With this notation, we will prove that, for every $\beta \in \bigwedge^{2n} H^1(A,\mathbb{Q})$, the following equality holds:

$$\Sigma_{2n}^*(\beta) = \sum_{\emptyset \neq J \subsetneq \{1,\dots,2n\}} (-1)^{|J|-1} p_J^*(\Sigma_{|J|}^*(\beta)) + \widetilde{\beta} \subseteq H^{2n}(A^{2n},\mathbb{Q}), \tag{B.8}$$

where $\widetilde{\beta}$ is the image of β via the natural map $\bigwedge^{2n} H^1(A,\mathbb{Q}) \hookrightarrow H^1(A,\mathbb{Q})^{\otimes 2n} \subset H^{2n}(A^{2n},\mathbb{Q})$. For $\beta = \alpha$, we will then conclude from (B.8) that $\widetilde{\alpha}$ is algebraic since α is assumed to be algebraic and all the maps involved are algebraic.

Note that by linearity, it suffices to prove (B.8) in the case where β is decomposable, i.e., $\beta = v_1 \wedge \cdots \wedge v_{2n} \in \bigwedge^{2n} H^1(A,\mathbb{Q})$. By the commutativity of pullbacks and cup products, we have the following equalities:

$$\begin{aligned}\Sigma_{2n}^*(\beta) =& \Sigma_{2n}^*(v_1) \wedge \cdots \wedge \Sigma_{2n}^*(v_{2n}) = (v_1,\dots,v_1) \wedge \cdots \wedge (v_{2n},\dots,v_{2n})\\ =& \sum (p_{i_1}^*(v_1) \wedge \ldots \wedge p_{i_{2n}}^*(v_{2n})),\end{aligned}$$

where the sum runs over all $i_1,\dots,i_{2n} \in \{1,\dots,2n\}$. To prove (B.8), we show that each of these summands appears exactly once on the right-hand-side of (B.8) after simplifying it. Let us start with the summands for which all i_j are different. These terms do not appear in the sum on the right-hand side of the equality, indeed they do not come from the pullback under the projection onto some lower power of A, but they appear exactly once in $\widetilde{\beta}$ since by construction

$$\widetilde{\beta} = \textstyle\sum_\sigma \pm v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(2n)} = \sum_\sigma \pm p_1^*(v_{\sigma(1)}) \wedge \cdots \wedge p_{2n}^*(v_{\sigma(2n)}).$$

If i_j are not all different, let $I := \{i_1, \ldots, i_{2n}\}$. The term $p_{i_1}^*(v_1) \wedge \ldots \wedge p_{i_{2n}}^*(v_{2n})$ appears on the right-hand side of (B.8) once for every J such that $I \subseteq J$. Therefore, taking into account the sign $(-1)^{|J|-1}$ the term $p_{i_1}^*(v_1) \wedge \ldots \wedge p_{i_{2n}}^*(v_{2n})$ appears exactly once, since

$$\sum_{i=|I|}^{2n-1} (-1)^{i-1} \binom{2n-|I|}{i-|I|} = 1.$$

This concludes the proof of (B.8). □

Remark B.5.17. As mentioned in the introduction, Theorem B.5.16 can already be found in [1]. However, in the proof, the author applied an incomplete version of Corollary B.5.12 which does not mention the different realizations of the two determinants $\det W$ and $\det W^*$. As we have seen, the different realizations of the two determinants are linked to the existence of the different realizations of the exceptional classes on the powers of the abelian variety. A priori, the fact that if one realization of an exceptional class is algebraic then all the realizations of it are algebraic was not clear to us. For a motivic proof of Theorem B.5.14 see [64].

A result similar to Theorem B.5.16 holds also for abelian varieties of Weil type with definite quaternionic multiplication. Note that one needs to add to the proof the same modifications as we introduced in the proof of Theorem B.5.16.

Theorem B.5.18. *[1, Thm. 4.1],[81, Thm. 4.2.1] Let A be a general abelian variety of K-Weil type with definite quaternionic multiplication. Then, if the Weil classes on A are algebraic, the Hodge conjecture holds for every power A^k.*

Theorem B.5.16 and Theorem B.5.18 allow us to deduce the Hodge conjecture for all powers A^k, where A is a general abelian variety of Weil type or a general abelian variety of Weil type with definite quaternionic multiplication whenever the Weil classes on A are algebraic.

B.6 Families of K3 surfaces of general Picard number 16

In this section, we prove the Hodge conjecture for all powers of the K3 surfaces belonging to families whose general element has Picard number 16 assuming the Kuga–Satake Hodge conjecture.

The relation between K3 surfaces of Picard number 16 and abelian fourfolds of Weil type is given by the Kuga–Satake construction due to the following result:

Theorem B.6.1. *[56],[30, Thm. 9.2] Let (V, q) be a polarized rational Hodge structure of K3-type such that there exists an isomorphism of quadratic spaces*

$$(V, q) \simeq U_{\mathbb{Q}}^2 \oplus \langle a \rangle \oplus \langle b \rangle,$$

where a and b are negative integers, and U is the hyperbolic lattice. Then, the Kuga–Satake variety of (V,q) is isogenous to A^4 for an abelian fourfold of $\mathbb{Q}(\sqrt{-ab})$-Weil type with discriminant one. Moreover, for general (V,q), there is an isomorphism $\operatorname{End}(A)\otimes_{\mathbb{Z}}\mathbb{Q}\simeq\mathbb{Q}(\sqrt{-ab})$. Conversely, if A is an abelian fourfold of Weil type with discriminant one, then A^4 is the Kuga–Satake variety of a polarized rational Hodge structure of K3-type of dimension six as above.

By Theorem B.5.10, the Hodge conjecture holds for the general abelian fourfold of Weil type with discriminant one. The following proposition shows that the same result follows in the cases where Kuga–Satake correspondence is known to be algebraic. As mentioned in the introduction, this does not prove the Hodge conjecture for any new abelian variety, it just shows the strong relation between the algebraicity Kuga–Satake correspondence and the Hodge conjecture for the Kuga–Satake variety in this special case.

Proposition B.6.2. *Let A be a general abelian fourfold of K-Weil type with discriminant one. If the Kuga–Satake Hodge conjecture holds for the corresponding K3 surface, then the Weil classes on A are algebraic. Thus, the Hodge conjecture holds for A and, hence, for all powers A^k.*

Proof. Let X be a K3 surface such that $\mathrm{KS}(X)\sim A^4$ which exists by Theorem B.6.1. By [56, Thm. 3.8], we have the following isomorphism of Hodge structures:

$$\textstyle\bigwedge_K^2 H^1(A,\mathbb{Q})\simeq T(X)\oplus\phi(T(X)), \tag{B.9}$$

where ϕ denotes the natural action of $\sqrt{-d}$ on $\bigwedge_K^2 H^1(A,\mathbb{Q})$ sending $v\wedge_K w$ to $\sqrt{-d}v\wedge_K w$. In particular, there is an embedding of Hodge structures

$$\kappa\colon T(X)\hookrightarrow \textstyle\bigwedge_K^2 H^1(A,\mathbb{Q})\hookrightarrow\bigwedge^2 H^1(A,\mathbb{Q}).$$

By generality assumption, the K3 surface X is Mumford–Tate general. In particular, [13, Thm. 4.8] and its proof show that κ is induced by the Kuga–Satake correspondence. As we are assuming the Kuga–Satake Hodge conjecture for X we deduce that κ is algebraic. Tensoring (B.9) by $\mathbb{C}$, we get as in [18, Prop. 4.4] the following isomorphism

$$\textstyle\bigwedge^2 W\oplus\bigwedge^2 W^*\simeq\left(\bigwedge_K^2 H^1(A,\mathbb{Q})\right)\otimes_{\mathbb{Q}}\mathbb{C}\simeq T(X)_{\mathbb{C}}\oplus\phi(T(X))_{\mathbb{C}}, \tag{B.10}$$

where, as usual, W is a complex vector space of dimension $2n=4$ such that $\mathrm{Hdg}(A)(\mathbb{C})\simeq\mathrm{SL}(W,\mathbb{C})$. Using the left-hand side of (B.10), we compute the ring of Hodge classes in

$$S\coloneqq\left(\textstyle\bigwedge_K^2 H^1(A,\mathbb{Q})\right)_{\mathbb{C}}\otimes\left(\textstyle\bigwedge_K^2 H^1(A,\mathbb{Q})\right)_{\mathbb{C}}.$$

We see that it is four-dimensional and is spanned by two linear combinations of complete contractions together with two realizations on A^2 of the Weil classes. On the other hand, computing the same ring of Hodge classes using the right-hand side of (B.10), we conclude for dimension reasons that there is a unique Hodge class in $T(X) \otimes T(X)$, i.e., that the endomorphism field of X is $\mathbb{Q}$. Denoting by α this unique Hodge class, we see that the ring of Hodge classes in S is spanned by

$$\alpha, (\mathrm{Id} \otimes \phi)\alpha, (\phi \otimes \mathrm{Id})\alpha, \text{ and } (\phi \otimes \phi)\alpha.$$

Since we are assuming the Kuga–Satake Hodge conjecture, the class α is algebraic on A^2 as it is represented on X^2 by a component of the class of the diagonal. Moreover, note that ϕ is algebraic as it is the restriction to $\bigwedge_K^2 H^1(A, \mathbb{Q})$ of the algebraic morphism

$$\sqrt{-d} \otimes \mathrm{Id} \colon H^1(A, \mathbb{Q})^{\otimes 2} \longrightarrow H^1(A, \mathbb{Q})^{\otimes 2}.$$

We then conclude that every class in S is algebraic. In particular, we see that the realizations of the Weil classes in S are algebraic. Using a similar argument to the one of Lemma B.5.15, we conclude that the Weil classes on A are algebraic. This implies by Theorem B.5.16 the Hodge conjecture for all powers A^k. □

We are finally able to state and prove our main theorem which extends [81, Thm. 2].

Theorem B.6.3. *Let $\mathscr{X} \longrightarrow S$ be a four-dimensional family of K3 surfaces whose general fibre is of Picard number* 16 *with an isometry*

$$T(\mathscr{X}_s) \simeq U_{\mathbb{Q}}^2 \oplus \langle a \rangle \oplus \langle b \rangle,$$

for some negative integers a and b. If the Kuga–Satake correspondence is algebraic for the fibres of this family, then the Hodge conjecture holds for all powers of every K3 surface in this family.

Proof. Recall that, if a K3 surface X has totally real endomorphism field E, the dimension of $T(X)$ as an E-vector space is at least three. Using this observation, together with the assumption that the general K3 surface of this family has Picard number 16, we see that, for any $s \in S$ the pair $(\dim T(\mathscr{X}_s), E \coloneqq \mathrm{End}_{\mathrm{Hdg}(\mathscr{X}_s)}(T(\mathscr{X}_s))$ satisfies one of the following:

(i) $\dim T(\mathscr{X}_s) = 6, 4$, or 2 and E is a CM field;

(ii) $\dim T(\mathscr{X}_s) = 6$ and $E = \mathbb{Q}$;

(iii) $\dim T(\mathscr{X}_s) = 6$ and E is a totally real field of degree two;

(iv) $\dim T(\mathscr{X}_s) \leq 5$ and $E \simeq \mathbb{Q}$.

In case (i), the endomorphism field of $T(\mathscr{X}_s)$ is a CM field. Therefore, we may apply Corollary B.3.3 to deduce the Hodge conjecture for all powers of $\mathscr{X}_s$.

In case (ii), the transcendental lattice is six-dimensional and by Theorem B.6.1, the Kuga–Satake variety of $\mathscr{X}_s$ is the fourth power of a general abelian fourfold A of Weil type. By Theorem B.5.10 and Theorem B.5.16, the Weil classes on A are algebraic and the Hodge conjecture holds for all powers of A. Using Lemma B.2.6, we conclude that all Hodge classes in the tensor algebra of $T(\mathscr{X}_s)$ are algebraic on the powers of $\mathscr{X}_s$.

In case (iii), one knows that the abelian fourfold A appearing in the decomposition of the Kuga–Satake variety of $\mathscr{X}_s$ is an abelian fourfold of Weil type whose endomorphism algebra is of definite quaternion type, see [30, Prop. 5.7]. By Theorem B.5.16 together with Theorem B.5.10, we see as in the previous case that the Hodge conjecture holds for all powers of the general A. As before, this is sufficient to conclude that all Hodge classes on the tensor algebra of $T(\mathscr{X}_s)$ are algebraic.

Finally in case (iv), the endomorphism field of $\mathscr{X}_s$ is $\mathbb{Q}$. Since we have already proven the Hodge conjecture for all powers of the K3 surfaces of Picard number 16 belonging to this family, we may apply our degeneration result of Corollary B.4.2 to conclude that the Hodge conjecture holds for all powers of the K3 surface $\mathscr{X}_s$.

This concludes the proof. □

We end this paper recalling two four-dimensional families of K3 surfaces satisfying the hypothesis of Theorem B.6.3.

Example B.6.4 (*Double covers of* $\mathbb{P}^2$ *branched along six lines*)**.** This family of K3 surfaces has been first studied by Paranjape [77] and it is the example studied by Schlickewei [81]: Let $\pi\colon Y \longrightarrow \mathbb{P}^2$ be a double cover branched along six lines no three of which intersect in one point. The surface Y has simple nodes in the 15 points of intersection of the lines. Blowing up these 15 points on Y we get a smooth K3 surface X. The 15 exceptional lines on X together with pullback of the ample line on $\mathbb{P}^2$ span a sublattice of $\mathrm{NS}(X)$ of rank 16. Since this family of K3 surfaces is four-dimensional, the Picard number of a general member is 16. The transcendental lattice of the general element of this family has been computed in [77, Lem. 1], where it is shown that it is isomorphic to $U_{\mathbb{Q}}^2 \oplus \langle -2 \rangle^2$. In particular, by Theorem B.6.1 the Kuga–Satake variety is the fourth power of an abelian fourfold of $\mathbb{Q}(i)$-Weil type. This was already known to Paranjape [77] where the author constructs the Kuga–Satake correspondence for this family, covering the general K3 surface by the square of a curve of genus five. Our Theorem B.6.3 then extends the result in [81], since it allows us to conclude that the Hodge conjecture holds for all powers of the K3 surfaces in this family and not just for their square.

Example B.6.5 (*Desingularization of K3 surfaces in* $\mathbb{P}^4$ *with* 15 *simple nodes*)**.** This family of K3 surfaces has been first introduced in [27]. In [49], the authors show that the same techniques as in [81] can be used to prove the Hodge conjecture for the square of these K3 surfaces: Let X be a general K3 surface which is the desingularization of a singular K3 surface in $\mathbb{P}^4$ with 15 nodal points. Then, the 15 rational lines on X together with the pullback of the ample line bundle on $\mathbb{P}^4$ span a sublattice of $\mathrm{NS}(X)$ of rank 16. Therefore, since the family of such K3 surfaces is four-dimensional, the Picard number of X is equal to 16. In [49, Rmk. 4.8], using elliptic fibrations, it is shown that the transcendental lattice of a general K3 surface is isomorphic to $U_{\mathbb{Q}}^{\oplus 2} \oplus \langle -6 \rangle \oplus \langle -2 \rangle$. In particular, applying Theorem B.6.1, the Kuga–Satake variety of X is isogenous to A^4 where A is an abelian fourfold of $\mathbb{Q}(\sqrt{-3})$-Weil type. Inspired by [77], Ingalls, Logan, and Patashnick [49] show that the Kuga–Satake correspondence is algebraic for these K3 surfaces. The authors then show that the proof of Schlickewei applies which shows the Hodge conjecture for the square of these K3 surfaces. As in the previous example, by Theorem B.6.3, we conclude that the Hodge conjecture holds for all powers of the K3 surfaces in this family.

Appendix C

Hodge similarities, algebraic classes, and Kuga–Satake varieties

M. VARESCO[1]

Abstract We introduce in this paper the notion of Hodge similarities of transcendental lattices of hyperkähler manifolds and investigate the Hodge conjecture for these Hodge morphisms. Studying K3 surfaces with a symplectic automorphism, we prove the Hodge conjecture for the square of the general member of the first four-dimensional families of K3 surfaces with totally real multiplication of degree two. We then show the functoriality of the Kuga–Satake construction with respect to Hodge similarities. This implies that, if the Kuga–Satake Hodge conjecture holds for two hyperkähler manifolds, then every Hodge similarity between their transcendental lattices is algebraic after composing it with the Lefschetz isomorphism. In particular, we deduce that Hodge similarities of transcendental lattices of hyperkähler manifolds of generalized Kummer deformation type are algebraic.

C.1 Introduction

C.1.1 Hyperkähler manifolds and the Hodge conjecture.

Let X be a hyperkähler manifold, and let $T(X) \subseteq H^2(X,\mathbb{Q})$ be its *transcendental lattice*, which is the orthogonal complement of the Néron–Severi group of X in $H^2(X,\mathbb{Q})$ with respect to the Beauville–Bogomolov quadratic form. The relevance of this notion in the context of the Hodge conjecture can be evinced from the following observation: let X and Y be hyperkähler manifolds. By Lefschetz $(1,1)$ theorem, a Hodge morphism $H^2(X,\mathbb{Q}) \to H^2(Y,\mathbb{Q})$ is algebraic if and only if the induced Hodge morphism $T(X) \to T(Y)$ is algebraic. Recall that a Hodge morphism $H^2(X,\mathbb{Q}) \to H^2(Y,\mathbb{Q})$ is said to be algebraic if the corresponding Hodge class in $H^{2n,2n}(X \times Y,\mathbb{Q})$ is algebraic, where $2n$ is the dimension of X.

In general, it is not known whether Hodge morphisms of transcendental lattices are algebraic or not. However, there have been promising results for the class of Hodge

[1]Mauro Varesco. Hodge similarities, algebraic classes, and Kuga–Satake varieties. *Math. Z.*, 305(4): 69, 2023. https://doi.org/10.1007/s00209-023-03390-8.

isometries. Recall that a Hodge isomorphism $T(X) \to T(Y)$ is called a Hodge isometry if it is an isometry with respect to the Beauville–Bogomolov quadratic forms on X and Y. A result by Buskin [11] reproved by Huybrechts [45] shows that Hodge isometries of transcendental lattices of projective K3 surfaces are algebraic. The same has been proven by Markman [62] for Hodge isometries of transcendental lattices of hyperkähler manifolds of $\mathrm{K3}^{[n]}$-type.

In this paper, we introduce a natural generalization of Hodge isometries which we call *Hodge similarities*: a Hodge isomorphism is a Hodge similarity if it multiplies the quadratic form by a non-zero scalar called *multiplier*, see Definition C.2.2. Note that Hodge isometries are Hodge similarities with multiplier one. There are two contexts where Hodge similarities naturally appear. The main instance is given by hyperkähler manifolds X whose endomorphism field $E \coloneqq \mathrm{End}_{\mathrm{Hdg}}(T(X))$ is a totally real field of degree two: indeed, every totally real field of degree two is isomorphic to $\mathbb{Q}(\sqrt{d})$ for some positive integer d. One then sees that $\sqrt{d}\colon T(X) \to T(X)$ is a Hodge similarity. This follows immediately from the fact that, as E is totally real, the Rosati involution is the identity. Note that in this case E is generated by Hodge similarities. A second source of examples of Hodge similarities is the following: given a hyperkähler manifold X, there might exist another hyperkähler manifold Y with transcendental lattice Hodge isometric to $T(X)(\lambda)$, for some $\lambda \in \mathbb{Q}_{>0}$, where (λ) indicates that the quadratic form is multiplied by λ. The identity of $T(X)$ then defines a natural Hodge morphism $T(Y) \to T(X)$ which is a Hodge similarity. At the time of writing this paper, there are very few examples of Hodge similarities that are not isometries which can be proven to be algebraic. For example, in the case of K3 surfaces with totally real endomorphism field $E = \mathbb{Q}(\sqrt{d})$, the algebraicity of $\sqrt{d}$ has been proven only for some one-dimensional families of such K3 surfaces. This is a result by Schlickewei [81] which has then been extended in [89]. Note that the proof in the references involves the study of the Hodge conjecture for Kuga–Satake variety of these K3 surfaces, and does not use the fact that E is in these cases generated by Hodge similarities.

C.1.2 Hodge similarities of K3 surfaces and symplectic automorphisms

Recall that the Hodge conjecture for the product of two K3 surfaces X and Y to the algebraicity of the elements of $\mathrm{Hom}_{\mathrm{Hdg}}(T(X), T(Y))$. This follows from the Künneth decomposition and the fact that the quadratic form q_X identifies $(T(X) \otimes T(Y))^{2,2} \cap (T(X) \otimes T(Y))$ with $\mathrm{Hom}_{\mathrm{Hdg}}(T(X), T(Y))$. As mentioned above, Hodge isometries between the transcendental lattices of two K3 surfaces are known to be algebraic. In particular, the Hodge conjecture holds for $X \times Y$ whenever $\mathrm{Hom}_{\mathrm{Hdg}}(T(X), T(Y))$ is generated by Hodge isometries. This is the case when $T(X)$ and $T(Y)$ are Hodge isometric and $\mathrm{Hom}_{\mathrm{Hdg}}(T(X), T(Y))$ is $\mathbb{Q}$ or a CM field.

The main result of Section C.3 is the proof of the algebraicity of some Hodge similarities for some families of K3 surfaces with totally real multiplication of degree two:

Theorem C.1.1 (Theorem C.3.1, C.3.9, and C.3.15)**.** *Let X be a K3 surface Hodge isometric to a K3 surface with a symplectic automorphism of order p with $p = 2, 3$. Assume furthermore that $\mathbb{Q}(\sqrt{p})$ is contained in the endomorphism field of X. Then, $\sqrt{p}\colon T(X) \to T(X)$ is algebraic. In particular, the Hodge conjecture for $X \times X$ holds if $\mathrm{End}_{\mathrm{Hdg}}(T(X)) \simeq \mathbb{Q}(\sqrt{p})$.*

The condition "X is Hodge isometric to a K3 surface with a symplectic automorphism of order p" is equivalent to $T(X) \hookrightarrow U_{\mathbb{Q}}^3 \oplus E_8(-2)_{\mathbb{Q}}$ for $p = 2$ and to $T(X) \hookrightarrow U_{\mathbb{Q}}^3 \oplus (A_2)_{\mathbb{Q}}^2$ for $p = 3$. This is deduced in Proposition C.3.5 and Proposition C.3.11 from the classical result by Nikulin [73], van Geemen and Sarti [34], and Garbagnati and Sarti [26]. Using these conditions on the transcendental lattice, we show that the families of K3 surfaces satisfying the hypotheses of Theorem C.1.1 are at most four-dimensional for $p = 2$ and two-dimensional for $p = 3$. We then produce examples of such maximal-dimensional families in Proposition C.3.6 and Proposition C.3.12. In particular, Theorem C.1.1 provides the first four-dimensional families of K3 surfaces with totally real multiplication of degree two for which the Hodge conjecture can be proven for the square of its general member and the first two-dimensional family of K3 surfaces with totally real multiplication of degree two for which the Hodge conjecture can be proven for the square of all its members.

C.1.3 Kuga–Satake varieties and Hodge similarities

In Section C.4, we prove that the functoriality of the Kuga–Satake construction with respect to Hodge isometries extends to Hodge similarities in the following sense:

Proposition C.1.2 (Proposition C.4.1)**.** *Let $\psi\colon (V, q) \to (V', q')$ be a Hodge similarity of polarized Hodge structures of K3-type. Then, there exists an isogeny of abelian varieties $\psi_{\mathrm{KS}}\colon \mathrm{KS}(V) \to \mathrm{KS}(V')$ making the following diagram commute*

$$\begin{array}{ccc} V & \xrightarrow{\quad\psi\quad} & V' \\ \big\downarrow & & \big\downarrow \\ H^1(\mathrm{KS}(V), \mathbb{Q})^{\otimes 2} & \xrightarrow{(\psi_{\mathrm{KS}})_*^{\otimes 2}} & H^1(\mathrm{KS}(V'), \mathbb{Q})^{\otimes 2} \end{array},$$

where the vertical arrows are the Kuga–Satake correspondence for V and V'.

In Section C.4.1, we exploit the observation that a similarity of quadratic spaces induces an isomorphism of even Clifford algebras to extend the result by Kreutz, Shen, and Vial [53] which shows that de Rham–Betti isometries between the second de Rham–Betti cohomology of two hyperkähler manifolds defined over $\overline{\mathbb{Q}}$ are motivated in the sense of André. We note in Proposition C.4.8 that the same proof as in the reference can be used to show that de Rham–Betti similarities are motivated.

In Section C.5, we use the functoriality property of the Kuga–Satake construction proven in Proposition C.1.2 to deduce the following:

Theorem C.1.3 (Theorem C.5.5). *Let X' and X be two hyperkähler manifolds for which the Kuga–Satake Hodge conjecture holds. Then, for every Hodge similarity $\psi\colon T(X') \to T(X)$, the composition*

$$T(X') \xrightarrow{\psi} T(X) \xrightarrow{h_X^{2n-2}\cup\bullet} H^{4n-2}(X,\mathbb{Q})$$

is algebraic, where $2n \coloneqq \dim X$ and h_X is the cohomology class of an ample divisor on X.

By a result of Voisin [98] based on previous results by Markman [60] and O'Grady [76], the Kuga–Satake Hodge conjecture holds for hyperkähler manifolds of generalized Kummer type. This is the main source of examples of manifolds which satisfy the hypotheses of Theorem C.1.3. As the Lefschetz standard conjecture in degree two for these manifolds is proved by Foster [24], Theorem C.1.3 shows that Hodge similarities between the transcendental lattices of two hyperkähler manifolds of generalized Kummer type are algebraic. Using the fact that the endomorphism field of these varieties is always generated by Hodge similarities, we then conclude the following:

Theorem C.1.4 (Theorem C.6.1). *Let X and X' be hyperkähler manifolds of generalized Kummer type such that $T(X)$ and $T(X')$ are Hodge similar. Then, every Hodge morphism between $T(X')$ and $T(X)$ is algebraic.*

Note that, opposed to the case of K3 surfaces and hyperkähler manifolds of $\mathrm{K3}^{[n]}$-type, already the algebraicity of Hodge isometries was not known in the case of hyperkähler manifolds of generalized Kummer type. Furthermore, note that Theorem C.1.4 also applies for hyperkähler manifolds of generalized Kummer type of different dimension.

In the case of K3 surfaces, the Lefschetz standard conjecture is trivially true. Hence, if the Kuga–Satake Hodge conjecture holds for two given K3 surfaces, Theorem C.1.3 shows that every Hodge similarity between their transcendental lattices is algebraic. In particular, this provides a more direct proof of the Hodge conjecture for the square of the K3 surfaces in the one-dimensional families of K3 surfaces with totally real field of degree two studied in [81, 89] that we mentioned above.

For hyperkähler manifolds of $\mathrm{K3}^{[n]}$-type, the Kuga–Satake Hodge conjecture is known only for certain families: the paper [21] proves this conjecture for countably many four-dimensional families of $\mathrm{K3}^{[3]}$-type hyperkähler manifolds. Recall that, for hyperkähler manifolds of $\mathrm{K3}^{[n]}$-type, the Lefschetz standard conjecture has been proven by Charles and Markman [14]. Therefore, we deduce the algebraicity of Hodge similarities for the hyperkähler manifolds of $\mathrm{K3}^{[3]}$-type appearing in [21].

As a final remark, note that the manifolds X and X' as in Theorem C.1.3 are neither assumed to be of the same deformation type nor of the same dimension.

Acknowledgements

I would like to thank my PhD. supervisor Daniel Huybrechts for the corrections and the numerous suggestions on this article. I am also grateful to Charles Vial for reviewing a preliminary version of this paper and for suggesting me the application of similarities in the context of De Rham–Betti classes. I am furthermore thankful to Alice Garbagnati and Giacomo Mezzedimi for their help on lattice theory and to Eyal Markman for referring me to the article [24]. This research was funded by ERC Synergy Grant HyperK, Grant agreement ID 854361.

C.2 Main definitions

In this paper, all varieties are assumed to be projective. Unless otherwise stated, the definition field of the varieties we consider is $\mathbb{C}$.

A *hyperkähler manifold* is a simply connected, projective, compact, Kähler manifold X such that $H^0(X, \Omega_X^2)$ is generated by a nowhere degenerate symplectic form. Denote by q_X the *Beauville–Bogomolov quadratic form*, which is a non-degenerate quadratic form on $H^2(X, \mathbb{Q})$. Recall that q_X induces the following direct sum decomposition

$$H^2(X, \mathbb{Q}) = \mathrm{NS}(X)_{\mathbb{Q}} \oplus T(X),$$

where $\mathrm{NS}(X)$ is the Néron–Severi group of X and $T(X)$ is the transcendental lattice of X. When talking about the transcendental lattice of a hyperkähler manifold X we will always refer to the rational quadratic subspace $T(X)$ of $H^2(X, \mathbb{Q})$. The pair $(T(X), -q_X)$ gives an example of polarized Hodge structures of K3-type:

Definition C.2.1. A rational Hodge structure V of weight two is called of *K3-type* if

$$\dim_{\mathbb{C}} V^{2,0} = 1, \text{ and } V^{p,q} = 0 \text{ for } |p - q| > 2.$$

Moreover, we say that a pair (V, q) is a *polarized Hodge structure of K3-type* if $q \colon V \otimes V \to \mathbb{Q}(-2)$ is a morphism of Hodge structures whose real extension is negative definite on $(V^{2,0} \oplus V^{0,2}) \cap V_{\mathbb{R}}$ and has signature $(\dim V - 2, 2)$.

Let $E \coloneqq \mathrm{End}_{\mathrm{Hdg}}(T(X))$ be the endomorphism algebra of the Hodge structure $T(X)$. As $T(X)$ is an irreducible Hodge structure, E is a field. As explained in [44, Thm. 3.3.7], E is either totally real or CM. Recall that a field extension E of $\mathbb{Q}$ is totally real if every embedding $E \hookrightarrow \mathbb{C}$ has image contained in $\mathbb{R}$, and it is CM if $E = F(\rho)$, where F is a totally real field and ρ satisfies the following:

$$\sigma(\rho)^2 \in \sigma(F) \cap \mathbb{R}_{<0}, \quad \forall \sigma \colon E \hookrightarrow \mathbb{C}.$$

These two cases can be distinguished by the action of the *Rosati involution*, which is the involution on E which sends an element $e \in E$ to the element $e' \in E$ such that

$$q_X(ev, w) = q_X(v, e'w), \quad \forall v, w \in T(X).$$

The Rosati involution is the identity if E is totally real, and it acts as complex conjugation if E is CM.

As mentioned in the introduction, we focus in this paper on the notion of Hodge similarities:

Definition C.2.2. Let (V, q_V) and $(V', q_{V'})$ be polarized Hodge structures of K3-type, and let $\psi\colon V \to V'$ be a Hodge isomorphism. We say that ψ is a *Hodge similarity* if there exists a non-zero $\lambda_\psi \in \mathbb{Q}$ such that

$$q_{V'}(\psi v, \psi w) = \lambda_\psi q_V(v, w), \quad \forall v, w \in V.$$

We call λ_ψ the *multiplier* of ψ. A *Hodge isometry* is a Hodge similarity ψ of multiplier $\lambda_\psi = 1$.

We say that two hyperkähler manifolds are *Hodge similar* (resp., *Hodge isometric*) if there exists a Hodge similarity (resp., a Hodge isometry) between their transcendental lattices. Note that the multiplier of a Hodge similarity is always a positive number.

C.3 Symplectic automorphisms and algebraic Hodge similarities

Let X be a K3 surface, and denote by q the polarization on $T(X)$ given by the negative of the intersection form. Identifying $T(X)$ with its dual via q, we see that

$$\mathrm{End}_{\mathrm{Hdg}}(T(X)) \simeq (T(X) \otimes T(X))^{2,2} \cap (T(X) \otimes T(X)).$$

This shows that proving the Hodge conjecture for X^2 is equivalent to showing that every element of $\mathrm{End}_{\mathrm{Hdg}}(T(X))$ is algebraic. In this section, considering K3 surfaces with a symplectic automorphism, we produce examples of K3 surfaces X with $\mathbb{Q}(\sqrt{p}) \subseteq \mathrm{End}_{\mathrm{Hdg}}(T(X))$ for which the Hodge similarity $\sqrt{p}$ can be shown to be algebraic.

The starting observation is the following: given a K3 surface X with a symplectic automorphism of order p, there exists a K3 surface Y and an algebraic Hodge similarity $\varphi\colon T(Y) \to T(X)$ of multiplier p. To show this, recall that, by [44, Prop. 15.3.11], the prime p is at most 7, the fixed locus of σ_p is a finite union of points, and the minimal resolution of X/σ_p is a K3 surface Y. Moreover, Y can also be obtained as follows: after a finite sequence of blowups of X at the fixed locus of σ_p, we get a variety $\widetilde{X}$ with a free action $\widetilde{\sigma}_p$ and $Y \simeq \widetilde{X}/\widetilde{\sigma}_p$. I.e., there is a commutative diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\beta} & X \\ \downarrow{\scriptstyle \pi} & & \downarrow \\ Y & \longrightarrow & X/\sigma_p \end{array}.$$

As $\pi\colon \widetilde{X} \to Y$ is a finite map of degree p and $\beta\colon \widetilde{X} \to X$ just contracts the exceptional divisors, we see that

$$\varphi := \beta_* \pi^* \colon T(Y) \to T(X)$$

is a Hodge similarity of multiplier p. Note that φ is algebraic. From this construction, we deduce the following:

Theorem C.3.1. *Let X be a K3 surface Hodge isometric to a K3 surface with a symplectic automorphism of prime order p. Assume furthermore that $\mathbb{Q}(\sqrt{p}) \subseteq \mathrm{End}_{\mathrm{Hdg}}(T(X))$. Then, the Hodge similarity $\sqrt{p}$ is algebraic.*

Proof. As Hodge isometries of K3 surfaces are algebraic by [11] and [45], we may assume that X admits a symplectic automorphism of order p. Let ψ be the Hodge similarity of multiplier p on $T(X)$, which exists since $\mathbb{Q}(\sqrt{p}) \subseteq \mathrm{End}_{\mathrm{Hdg}}(T(X))$ by assumption. As remarked above, denoting by Y the minimal resolution of the quotient X/σ_p, the map $\varphi := \beta_*\pi^*\colon T(Y) \to T(X)$ is a Hodge similarity of multiplier p. The composition $\varphi^{-1} \circ \psi\colon T(X) \to T(Y)$ is then a Hodge isometry. In particular, $\varphi^{-1} \circ \psi$ is algebraic by [11] and [45]. As φ is algebraic, we conclude that $\psi = \varphi \circ (\varphi^{-1} \circ \psi)$ is algebraic. This concludes the proof. ☐

Remark C.3.2. The two conditions "X is isometric to a K3 surface with a symplectic automorphisms of order p" and "the endomorphisms field of X contains $\mathbb{Q}(\sqrt{p})$" are not related. In fact, the general K3 surface with a symplectic automorphism of order p has endomorphism field equal to $\mathbb{Q}$. Moreover, note that the requirement "the endomorphisms field of X contains $\mathbb{Q}(\sqrt{p})$" is equivalent to the condition "X admits a Hodge similarity ψ of multiplier d which is fixed by the Rosati involution". Indeed, if ψ such a Hodge similarity, then $\mathbb{Q}(\psi)$ is a totally real subfield of the endomorphism field of X. Using the fact that totally real fields have no non-trivial isometry, we see that ψ^2/d is the identity, i.e., that $\mathbb{Q}(\psi) \simeq \mathbb{Q}(\sqrt{d})$.

In the remainder of this section, we construct families of K3 surfaces satisfying the hypotheses of Theorem C.3.1. To do this, we use the following result is adapted from [31, Sec. 3], we give here a detailed proof for later use.

Proposition C.3.3. *Let $d \in \mathbb{Z}$ be a positive integer which is not a square, and let (Λ, q) be a rational quadratic space of signature $(2, \Lambda - 2)$ with $\dim \Lambda > 4$. Let ψ be a similarity of Λ of multiplier d which is fixed by the Rosati involution. Then, Λ is even-dimensional, and the locus of Hodge structures of K3-type on Λ for which ψ defines a Hodge similarity is either empty or of dimension $(\dim \Lambda)/2 - 2$.*

Proof. The first statement is immediate from the fact that odd-dimensional quadratic spaces do not admit any similarity of multiplier d if d is not a square.

Let us assume that Λ is even-dimensional. As in Remark C.3.2, we see that, for every Hodge structure on Λ for which ψ is a Hodge morphism, $\mathbb{Q}(\psi) \simeq \mathbb{Q}(\sqrt{d})$ is a totally real subfield of the endomorphism field of Λ.

Note that Λ can be viewed as a $\mathbb{Q}(\psi)$-vector space, that is $\Lambda \simeq \mathbb{Q}(\psi)^{(\dim \Lambda)/2}$. The decomposition $\mathbb{Q}(\psi) \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathbb{R}_{\sqrt{d}} \oplus \mathbb{R}_{-\sqrt{d}}$ into eigenspaces for the action of ψ then induces a decomposition

$$\Lambda_{\mathbb{R}} \simeq \Lambda_{\sqrt{d}} \oplus \Lambda_{-\sqrt{d}},$$

where $\Lambda_{\sqrt{d}} := \{v \in \Lambda_{\mathbb{R}} \mid \psi v = \sqrt{d} v\}$ and similarly for $\Lambda_{-\sqrt{d}}$. Note that $\Lambda_{\sqrt{d}}$ and $\Lambda_{-\sqrt{d}}$ are both of dimension $(\dim \Lambda)/2$. From the fact that ψ is fixed by the Rosati involution, we deduce that this decomposition is orthogonal with respect to the quadratic form q on $\Lambda_{\mathbb{R}}$.

Recall that giving a Hodge structure of K3-type on Λ is equivalent to giving an element ω in the period domain

$$\Omega_{\Lambda} := \{\omega \in \mathbb{P}(\Lambda_{\mathbb{C}}) \mid q(\omega) = 0, q(\omega, \overline{\omega}) > 0\}.$$

Note that ψ defines a morphism of Hodge structures if and only if ω is an eigenvector. Therefore, as $\Lambda_{\mathbb{R}} \simeq \Lambda_{\sqrt{d}} \oplus \Lambda_{-\sqrt{d}}$ is orthogonal with respect to q and $(\Lambda^{2,0} \oplus \Lambda^{0,2}) \cap \Lambda_{\mathbb{R}}$ has to be positive definite, there exists a Hodge structure for which ψ is a Hodge morphism if and only if $\Lambda_{-\sqrt{d}}$ is negative definite and $\Lambda_{\sqrt{d}}$ has signature $(2, (\dim \Lambda)/2 - 2)$ or vice versa. Let us assume that ψ satisfy this hypothesis. Then, up to changing the sign of ψ, we may assume that $\Lambda_{\sqrt{d}}$ has signature $(2, (\dim \Lambda)/2 - 2)$. We conclude that ψ defines a Hodge automorphism if and only if the Hodge structure corresponds to an element in

$$\{\omega \in \mathbb{P}\left((\Lambda_{\sqrt{d}})_{\mathbb{C}}\right) \mid q(\omega) = 0, q(\omega, \overline{\omega}) > 0\}.$$

Therefore, the locus of Hodge structures on Λ for which ψ defines a Hodge morphism has dimension equal to $\dim \Lambda_{\sqrt{d}} - 2 = (\dim \Lambda)/2 - 2$. □

Remark C.3.4. Let ψ be a similarity of multiplier d as in Proposition C.3.3. From the proof of Proposition C.3.3, we see that locus of Hodge structures of K3-type on Λ for which ψ is a Hodge similarity is non-empty (hence, of dimension $(\dim \Lambda)/2 - 2$) if and only if either $\Lambda_{\sqrt{d}}$ or $\Lambda_{-\sqrt{d}}$ is negative definite.

We use Proposition C.3.3 to show that the families of K3 surfaces which satisfy the hypotheses of Theorem C.3.1 are at most four-dimensional for $p = 2$ and two-dimensional for $p = 3$. Moreover, we produce examples of such families with these maximal dimensions. As we will see, no K3 surface satisfies the hypotheses of Theorem C.3.1 for higher values of p.

Let us start from the case $p = 2$. Following [34], we call a symplectic involution on a K3 surface a *Nikulin involution*. By [34, Prop. 2.2, 2.3], a K3 surface X admits a Nikulin involution if and only if the lattice $E_8(-2)$ is primitively embedded in the Néron–Severi group of X. Note that, up to an automorphism of the K3-lattice, there exists a unique primitive embedding of $E_8(-2)$ in the K3-lattice. Therefore, we deduce from [34, Sec. 1.3] that $(E_8(-2))^{\perp} \simeq U^3 \oplus E_8(-2)$. From this fact, we get the following criterion in terms of the transcendental lattice of X:

Proposition C.3.5. *A K3 surface X is Hodge isometric to a K3 surface admitting a Nikulin involution if and only if $T(X) \subseteq U^3_{\mathbb{Q}} \oplus E_8(-2)_{\mathbb{Q}}$.*[2]

Proof. Let us first prove the "only if" part. Let X be a K3 surface such that $T(X)$ is Hodge isometric to $T(X')$ for some K3 surface X' admitting a Nikulin involution. By [34, Prop. 2.2, 2.3], the lattice $E_8(-2)$ is primitively embedded in $\mathrm{NS}(X')$. Therefore, $\mathrm{NS}(X')^{\perp} \hookrightarrow E_8(-2)^{\perp} \simeq U^3 \oplus E_8(-2)$. Over $\mathbb{Q}$, we conclude that

$$T(X) \simeq T(X') \hookrightarrow U^3_{\mathbb{Q}} \oplus E_8(-2)_{\mathbb{Q}}.$$

For the "if" part, let us assume that there is an embedding of quadratic spaces

$$T(X) \hookrightarrow U^3_{\mathbb{Q}} \oplus E_8(-2)_{\mathbb{Q}}.$$

Denote by $H^2(X,\mathbb{Z})_{\mathrm{tr}}$ the transcendental part of the second integral cohomology of X. Clearing the denominators, we find a positive integer $\lambda \in \mathbb{Z}$ such that the above embedding restricts to an embedding of lattices

$$j \colon \lambda H^2(X,\mathbb{Z})_{\mathrm{tr}} \hookrightarrow U^3 \oplus E_8(-2).$$

Fix a primitive embedding $\iota \colon U^3 \oplus E_8(-2) \hookrightarrow H^2(X,\mathbb{Z})$ such that

$$\iota(U^3 \oplus E_8(-2))^{\perp} \simeq E_8(-2).$$

Let T' be the saturation of the lattice $(\iota \circ j)(H^2(X,\mathbb{Z})_{\mathrm{tr}}) \subseteq H^2(X,\mathbb{Z})$. For any K3 surface X' such that $H^2(X',\mathbb{Z})_{\mathrm{tr}} \simeq T'$ we get an embedding

$$E_8(-2) \simeq (U^3 \oplus E_8(-2))^{\perp} \hookrightarrow (T')^{\perp} \simeq \mathrm{NS}(X').$$

This embedding is primitive, since $E_8(-2)$ is obtained as an orthogonal complement. Therefore, X' admits a Nikulin involution by [34, Prop. 2.2, 2.3]. Note that $T(X)$ and $T'_{\mathbb{Q}}$ are isometric quadratic spaces. Hence, by the surjectivity of the period map we can find a K3 surface X' with $H^2(X',\mathbb{Z})_{\mathrm{tr}} \simeq T'$ such that $T(X')$ is Hodge isometric to $T(X)$. This concludes the proof since the K3 surface X' is Hodge isometric to X and admits a Nikulin involution as required. □

In particular, we deduce that the transcendental lattice of a K3 surfaces which is Hodge isometric to a K3 surface with a Nikulin involution is at most 13-dimensional. Proposition C.3.3 then shows that the families of K3 surfaces satisfying the hypotheses of Theorem C.3.1 in the case $p = 2$ are at most four-dimensional. To prove the existence of such a four-dimensional family of K3 surfaces we consider a particular quadratic subspace of $U^3_{\mathbb{Q}} \oplus E_8(-2)_{\mathbb{Q}}$, and we show that it admits a similarity of multiplier 2.

[2]Thanks to G. Mezzedimi for the help with this argument.

Proposition C.3.6. *The locus of Hodge structures of K3-type on* $\Lambda \coloneqq U_{\mathbb{Q}}^2 \oplus E_8(-2)_{\mathbb{Q}}$ *which admit a Hodge similarity of multiplier 2 which is fixed by the Rosati involution is non-empty and has a four-dimensional component.*

Proof. As $\dim \Lambda = 12$, Proposition C.3.3 shows that the locus of Hodge structures of K3-type on Λ which admit a Hodge similarity of multiplier 2 which is fixed by the Rosati involution has a four-dimensional component if non-empty. By Remark C.3.4, we just need to produce a similarity ψ of Λ of multiplier 2 fixed by the Rosati involution such that $\Lambda_{\sqrt{2}}$ has signature $(2,4)$.

As the quadratic space $E_8(-2)_{\mathbb{Q}}$ is isometric to $\langle -2\rangle^8$, we can write $\Lambda = Q_1 \oplus Q_2 \oplus Q_3 \oplus Q_4 \oplus Q_5$, with

$$Q_1 = Q_2 \coloneqq \langle 1\rangle \oplus \langle -1\rangle, \quad Q_3 = Q_4 = Q_5 = Q_6 \coloneqq \langle -2\rangle \oplus \langle -2\rangle.$$

As in [31, Exmp. 3.4], we restrict to finding a similarity ψ which preserves the decomposition of Λ as above. I.e., we look for matrices $M_i \in \mathrm{GL}_2(\mathbb{Q})$ which satisfy the following: ${}^tM_iQ_i = Q_iM_i$ and $M_i^2 = 2\mathrm{Id}$ for $i = 1, \dots, 6$. Then, $\psi \coloneqq M_1 \oplus \dots \oplus M_6$ will be fixed by the Rosati involution by the first condition and will be a similarity of multiplier 2. A direct computation shows that the following matrices satisfy all the above conditions

$$M_1 = M_2 = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{3}{2} \end{pmatrix}, \quad M_3 = M_4 = M_5 = M_6 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

and that the signature of $\Lambda_{\sqrt{2}}$ is $(2,4)$. Thus, ψ satisfies the required properties. □

Example C.3.7. By [34, Sec.1 4], the family of elliptic K3 surfaces with a section and a two-torsion section provides an example of a ten-dimensional family of K3 surfaces with a Nikulin involution and general transcendental lattice $\Lambda = U_{\mathbb{Q}}^2 \oplus \langle -2\rangle^8$. By Proposition C.3.6, there exists a four-dimensional family of elliptic K3 surfaces with a two-torsion section with endomorphism field containing $\mathbb{Q}(\sqrt{2})$.

Remark C.3.8. One can produce other examples of quadratic subspace of $U_{\mathbb{Q}}^3 \oplus E_8(-2)_{\mathbb{Q}}^2$ which admit a similarity of multiplier 2 which is fixed by the Rosati involution. For example, if $d > 1$ is a square-free integer such that 2 is a quadratic residue modulo d, the space $U_{\mathbb{Q}}^2 \oplus \langle -2\rangle^7 \oplus \langle -2d\rangle$ admits a similarity of multiplier 2 and is not isometric to $U_{\mathbb{Q}}^2 \oplus E_8(-2)_{\mathbb{Q}}$. This provides other four-dimensional families of K3 surfaces satisfying the hypotheses of Theorem C.3.1 for $p = 2$.

To sum up, our discussion shows that Theorem C.3.1 in the case of Nikulin involutions gives the following:

Theorem C.3.9. *For every K3 surface in the four-dimensional families of K3 surfaces with endomorphism field containing* $\mathbb{Q}(\sqrt{2})$ *which are Hodge isometric to a K3 surface with a Nikulin involution, the endomorphism* $\sqrt{2}$ *is algebraic. In particular, the Hodge conjecture holds for the square of the general such K3 surface.*

Remark C.3.10. The only case where Theorem C.3.9 is not enough to prove the Hodge conjecture for the square of the K3 surfaces X as in the statement is when the endomorphism field E of X is totally real of degree four and $T(X)$ is twelve-dimensional: this follows from the well known fact that, if the endomorphism field E is totally real, the dimension of $T(X)$ as E-vector space is at least three. Recall that if E is a CM field, then the Hodge conjecture for X^2 follows from [11] and [45] using the fact that E is generated by Hodge isometries. Similarly to Proposition C.3.3, one sees that the families of K3 surfaces as in Theorem C.3.9 with totally real endomorphism field of degree four are one-dimensional.

Let us come to the case $p = 3$. Let X be a K3 surface with a symplectic automorphism of order 3, and let Y be the minimal resolution of the quotient. As above, we have an algebraic similarity $T(Y) \to T(X)$ of multiplier 3 and $T(Y)$ is Hodge isometric to $T(X)(\frac{1}{3})$. By [26, Thm. 4.1], a K3 surface X admits a symplectic automorphism of order 3 if and only if $K_{12}(-2)$ is primitively embedded in $\mathrm{NS}(X)$, where $K_{12}(-2)$ denotes the Coxeter–Todd lattice with the bilinear form multiplied by -2. With a similar proof as in Proposition C.3.5, we can reformulate this in terms of the transcendental lattice as follows:

Proposition C.3.11. *A K3 surface X is Hodge isometric to a K3 surface admitting a symplectic automorphism of order* 3 *if and only if* $T(X) \subseteq U_{\mathbb{Q}}^3 \oplus (A_2)_{\mathbb{Q}}^2$. □

Proposition C.3.3 shows that families of K3 surfaces X with $T(X) \subseteq U_{\mathbb{Q}}^3 \oplus (A_2)_{\mathbb{Q}}^2$ whose endomorphism field contains $\mathbb{Q}(\sqrt{3})$ are at most two-dimensional. As in the case of Nikulin involutions, we consider a particular quadratic subspace of $U_{\mathbb{Q}}^3 \oplus (A_2)_{\mathbb{Q}}^2$, and we show that it admits a similarity of multiplier 3:

Proposition C.3.12. *The locus of Hodge structures of K3-type on* $\Gamma := U_{\mathbb{Q}}^2 \oplus (A_2)_{\mathbb{Q}}^2$ *which admit a Hodge similarity of multiplier* 3 *which is fixed by the Rosati involution is non-empty and has a two-dimensional component.*

Proof. As in the proof of Proposition C.3.6, we will construct an explicit similarity ψ of Γ of multiplier 3 fixed by the Rosati involution such that $\Gamma_{\sqrt{3}}$ has signature $(2, 2)$. Then, by Proposition C.3.3 and Remark C.3.4, the locus of Hodge structures on Γ for which ψ is a Hodge morphism has a two-dimensional component.

Diagonalizing the quadratic space $(A_2)_{\mathbb{Q}}$, we see that there is an isometry

$$\Gamma \simeq Q_1 \oplus Q_2 \oplus Q_3 \oplus Q_4,$$

with $Q_1 = Q_2 = \langle 1 \rangle \oplus \langle -1 \rangle$ and $Q_3 = Q_4 = \langle -2 \rangle \oplus \langle -3/2 \rangle$. We provide now matrices $M_1, M_2, M_3, M_4 \in \mathrm{GL}_2(\mathbb{Q})$ such that ${}^tM_iQ_i = Q_iM_i$ and $M_i^2 = 3\mathrm{Id}$. As one checks, setting

$$M_1 = M_2 := \begin{pmatrix} 2 & -1 \\ 1 & -2 \end{pmatrix} \text{ and } M_3 = M_4 := \begin{pmatrix} 0 & \frac{3}{2} \\ 2 & 0 \end{pmatrix},$$

the map $\psi := M_1 \oplus M_2 \oplus M_3 \oplus M_4$ defines a similarity of multiplier 3 of Γ satisfying all the requirements. □

Example C.3.13. By [26, Prop. 4.2], the family of elliptic K3 surfaces with a section and a three-torsion section provides an example of a six-dimensional family of K3 surfaces with a symplectic automorphism of order 3 and general transcendental lattice isometric to $\Gamma = U_{\mathbb{Q}}^2 \oplus (A_2)_{\mathbb{Q}}^2$. By Proposition C.3.12, there is a two-dimensional subfamily of K3 surfaces with endomorphism field containing $\mathbb{Q}(\sqrt{3})$.

Remark C.3.14. As in the case of Nikulin involutions, one can construct other eight-dimensional quadratic subspaces of $U_{\mathbb{Q}}^3 \oplus (A_2)_{\mathbb{Q}}^2$ admitting a similarity of multiplier 3 as Γ of Propostion C.3.12.

Our discussion shows that Theorem C.3.1 in the case $p = 3$ gives the following:

Theorem C.3.15. *For every K3 surface in the two-dimensional families of K3 surfaces with endomorphism field containing* $\mathbb{Q}(\sqrt{3})$ *which are Hodge isometric to a K3 surface with a symplectic automorphism of order* 3*, the endomorphism* $\sqrt{3}$ *is algebraic. In particular, the Hodge conjecture holds for the square of every such K3 surface.*

Remark C.3.16. Note that in this case, Theorem C.3.15 proves the Hodge conjecture for the square of every K3 surfaces of these families. The reason for this lies in the fact that the transcendental lattice of these K3 surfaces is at most eight-dimensional. Therefore, by a similar argument as in Remark C.3.10, we see that the endomorphism field of such K3 surface is either $\mathbb{Q}(\sqrt{3})$ or a CM field. In the latter case, the Hodge conjecture for the square of the K3 surface follows from the fact that CM fields are generated by Hodge isometries.

In the case of symplectic automorphisms of order bigger than 3, the same procedure does not produce any K3 surface. In fact, the endomorphism of a K3 surface with a symplectic automorphism of order 5 or 7 is always $\mathbb{Q}$ or a CM field. This can be deduced from [26, Prop. 1.1]: indeed, the transcendental lattice of a K3 surface admitting a symplectic automorphism of order 5 is of dimension at most five, and for K3 surfaces with a symplectic automorphism of order 7 its dimension is at most three. As in Remark C.3.10, one sees that, in both cases, the endomorphism field of these K3 surfaces cannot be a totally real field different from $\mathbb{Q}$.

C.4 Kuga–Satake varieties and Hodge similarities

By a construction due to Kuga and Satake [54], given a polarized Hodge structure of K3-type (V, q), there exists an abelian variety $\mathrm{KS}(V)$, called the Kuga–Satake variety of (V, q), together with an embedding of Hodge structures $\kappa\colon V \hookrightarrow H^1(\mathrm{KS}(V), \mathbb{Q})^{\otimes 2}$. We refer the reader for this construction to [44, Ch. 4], [30], and [90]. In this section, we prove the functoriality of the Kuga–Satake construction with respect to Hodge similarities:

Proposition C.4.1. *Let $\psi\colon (V,q)\to(V',q')$ be a Hodge similarity of polarized Hodge structures of K3-type. Then, there exists an isogeny of abelian varieties $\psi_{\mathrm{KS}}\colon \mathrm{KS}(V)\to \mathrm{KS}(V')$ making the following diagram commute*

$$\begin{array}{ccc} V & \xrightarrow{\quad\psi\quad} & V' \\ \big\downarrow & & \big\downarrow \\ H^1(\mathrm{KS}(V),\mathbb{Q})^{\otimes 2} & \xrightarrow{(\psi_{\mathrm{KS}})_*^{\otimes 2}} & H^1(\mathrm{KS}(V'),\mathbb{Q})^{\otimes 2} \end{array},$$

where the vertical arrows are the Kuga–Satake correspondences for (V,q) and (V',q').

In the remainder of this section we prove Proposition C.4.1.

Let (V,q) and (V',q') be polarized Hodge structures of K3-type, and let $\psi\colon (V,q)\to(V',q')$ be a Hodge similarity of multiplier λ_ψ. The next lemma shows that ψ induces an isomorphism between the even Clifford algebras $\mathrm{Cl}^+(V)$ and $\mathrm{Cl}^+(V')$. Recall that $\mathrm{Cl}^+(V)$ is defined as the even degree part of $\bigotimes^* V/I_V$, where I_V is the two-sided ideal generated by elements of the form $v\otimes v - q(v)$, for $v\in V$.

Lemma C.4.2. *The isomorphism of graded rings*

$$\psi_\otimes\colon \bigotimes^{\mathrm{ev}} V\to \bigotimes^{\mathrm{ev}} V',\quad v_1\otimes\cdots\otimes v_{2m}\mapsto (1/\lambda_\psi)^m\psi v_1\otimes\cdots\otimes\psi v_{2m}$$

induces an isomorphism $\psi_{\mathrm{Cl}}\colon \mathrm{Cl}^+(V)\xrightarrow{\simeq}\mathrm{Cl}^+(V')$.

Proof. From the definition, it is immediate to see that the map $\psi_\otimes$ is an isomorphism of graded rings. Given $v\in V$, we have the following

$$\begin{aligned}\psi_\otimes(v\otimes v-q(v)) &= (1/\lambda_\psi)(\psi v\otimes\psi v)-q(v) = (1/\lambda_\psi)(\psi v\otimes \psi v-\lambda_\psi q(v))\\ &= (1/\lambda_\psi)(\psi v\otimes\psi v - q'(\psi v)),\end{aligned}$$

where in the last step we used that ψ is a similarity of multiplier λ_ψ. This equality shows that $\psi_\otimes(v\otimes v-q(v))$ belongs to the ideal of $\bigotimes^{\mathrm{ev}} V'$ generated by $w\otimes w-q'(w)$ for $w\in V'$. Hence, the isomorphism $\psi_\otimes$ descends to an isomorphism $\psi_{\mathrm{Cl}}\colon \mathrm{Cl}^+(V)\to\mathrm{Cl}^+(V')$. □

In the construction of the Kuga–Satake variety associated to a polarized Hodge structure of K3-type, the complex Hodge structure on $\mathrm{Cl}^+(V)_{\mathbb{R}}$ is given by left multiplication by $J := e_1\cdot e_2$, with $\{e_1,e_2\}$ an orthogonal basis of $V_{\mathbb{R}}\cap(V^{2,0}\oplus V^{0,2})$ satisfying $q(e_1)=q(e_2)=-1$. As one checks, this complex structure does not depend on the choice of the basis.

Lemma C.4.3. *The map $\psi_{\mathrm{Cl},\mathbb{R}}\colon \mathrm{Cl}^+(V)_{\mathbb{R}}\to\mathrm{Cl}^+(V')_{\mathbb{R}}$ is compatible with the natural complex structures on $\mathrm{Cl}^+(V)_{\mathbb{R}}$ and $\mathrm{Cl}^+(V')_{\mathbb{R}}$.*

Proof. Let $\{e_1, e_2\}$ be an orthogonal basis of $V_{\mathbb{R}} \cap (V^{2,0} \oplus V^{0,2})$ with $q(e_i) = -1$, and define

$$e_i' := \psi_{\mathbb{R}} e_i / \sqrt{\lambda_\psi} \in V_{\mathbb{R}}' \cap (V'^{2,0} \oplus V'^{0,2}), \quad \text{for } i = 1, 2.$$

As ψ is a Hodge similarity of multiplier λ_ψ, one sees that $\{e_1', e_2'\}$ is an orthogonal basis of $V_{\mathbb{R}}' \cap (V'^{2,0} \oplus V'^{0,2})$ such that $q'(e_i') = -1$. The complex structure on $\mathrm{Cl}^+(V)_{\mathbb{R}}$ (resp., on $\mathrm{Cl}^+(V')_{\mathbb{R}}$) is then induced by left multiplication by $J := e_1 \cdot e_2$ (resp., by $J' := e_1' \cdot e_2'$). Hence, the equality

$$\psi_{\mathrm{Cl},\mathbb{R}}(J \cdot x) = J' \cdot \psi_{\mathrm{Cl},\mathbb{R}}(x) \quad \forall x \in \mathrm{Cl}^+(V)$$

proves that $\psi_{\mathrm{Cl},\mathbb{R}}$ is a morphism of complex vector spaces. □

The Kuga–Satake variety of (V, q) is defined as (the isogeny class) of the complex torus $\mathrm{KS}(V) := \mathrm{Cl}^+(V)_{\mathbb{R}} / \mathrm{Cl}^+(V)$, where $\mathrm{Cl}^+(V)_{\mathbb{R}}$ is endowed with the complex structure we recalled above. Lemma C.4.3 then shows that $\psi_{\mathrm{Cl}} \colon \mathrm{Cl}^+(V) \to \mathrm{Cl}^+(V')$ induces an isogeny of complex tori

$$\psi_{\mathrm{KS}} \colon \mathrm{KS}(V) \to \mathrm{KS}(V').$$

Recall that Kuga–Satake varieties of polarized Hodge structures of K3-type are abelian varieties: let $(f_1, f_2) \in V \times V$ be a pair of orthogonal elements of V with positive square, and consider the pairing

$$Q \colon \mathrm{Cl}^+(V) \times \mathrm{Cl}^+(V) \to \mathbb{Q}, \quad (v, w) \to \mathrm{tr}(f_1 \cdot f_2 \cdot v^* \cdot w),$$

where $\mathrm{tr}(x)$ denotes the trace of the endomorphism of $\mathrm{Cl}^+(V)$ given by left multiplication by $x \in \mathrm{Cl}^+(V)$ and v^* denotes the image of v under the involution of $\mathrm{Cl}^+(V)$ induced by the involution $v_1 \otimes \cdots \otimes v_{2m} \mapsto v_{2m} \otimes \cdots \otimes v_1$ on $\bigotimes^{\mathrm{ev}} V$. Then, Q defines up to a sign a polarization for the weight-one Hodge structure $\mathrm{Cl}^+(V)$. Note that the pair $(\psi f_1/\lambda_\psi, \psi f_2) \in V' \times V'$ satisfies the same hypotheses. Hence, it defines a polarization Q' on $\mathrm{Cl}^+(V')$.

Lemma C.4.4. *The isomorphism $\psi_{\mathrm{Cl}} \colon \mathrm{Cl}^+(V) \to \mathrm{Cl}^+(V')$ is compatible with the polarizations Q and Q' defined above. Hence, $\psi_{\mathrm{KS}} \colon \mathrm{KS}(V) \to \mathrm{KS}(V')$ is an isogeny of abelian varieties.*

Proof. To prove the lemma, we need to show that

$$Q(v, w) = Q'(\psi_{\mathrm{Cl}} v, \psi_{\mathrm{Cl}} w),$$

for all $v, w \in \mathrm{Cl}^+(V)$. By definition of Q and Q', this is equivalent to prove that

$$\mathrm{tr}(f_1 \cdot f_2 \cdot v^* \cdot w) = \mathrm{tr}(\psi f_1/\lambda_\psi \cdot \psi f_2 \cdot (\psi_{\mathrm{Cl}} v)^* \cdot \psi_{\mathrm{Cl}} w).$$

Note that, by definition of ψ_{Cl}, the following holds

$$\psi f_1/\lambda_\psi \cdot \psi f_2 \cdot (\psi_{\mathrm{Cl}} v)^* \cdot \psi_{\mathrm{Cl}} w = \psi_{\mathrm{Cl}}(f_1 \cdot f_2 \cdot v^* \cdot w).$$

We then need to prove that, for every $x \in \mathrm{Cl}^+(V)$, the left multiplication by x on $\mathrm{Cl}^+(V)$ has the same trace as the left multiplication by $\psi_{\mathrm{Cl}}x$ on $\mathrm{Cl}^+(V')$. This can be checked as follows: let $\{b_i\}_i$ be a basis of $\mathrm{Cl}^+(V)$ with dual basis $\{b^i\}_i$. By definition, we have that

$$\mathrm{tr}(x) = \Sigma_i b^i(x \cdot b_i)$$

As a basis of $\mathrm{Cl}^+(V')$, consider $\{\psi_{\mathrm{Cl}} b_i\}_i$. Its dual basis is $\{\psi_{\mathrm{Cl}}^\vee b^i\}_i$, where $\psi_{\mathrm{Cl}}^\vee$ is the dual action of ψ_{Cl}. The trace of the left multiplication by $\psi_{\mathrm{Cl}}x$ is then

$$\mathrm{tr}(\psi_{\mathrm{Cl}}x) = \Sigma_i(\psi_{\mathrm{Cl}}^\vee b^i)(\psi_{\mathrm{Cl}}x \cdot \psi_{\mathrm{Cl}} b_i) = \Sigma_i b^i(x \cdot b_i) = \mathrm{tr}(x).$$

This concludes the proof. □

The last ingredient for the proof of Proposition C.4.1 is the compatibility of the isomorphism ψ_{Cl} with the embedding $\varphi_V \colon V \hookrightarrow \mathrm{End}(\mathrm{Cl}^+(V))$ given by the Kuga–Satake construction. Recall that φ_V is given as follows: let $v_0 \in V$ be an element with $q(v_0) \neq 0$, then $\varphi_V(v) := f_v \in \mathrm{End}(\mathrm{Cl}^+(V))$ where $f_v(w) := v \cdot w \cdot v_0$. Similarly, let $\varphi_{V'} \colon V' \hookrightarrow \mathrm{End}(\mathrm{Cl}^+(V'))$ be the embedding corresponding to the element $\psi(v_0)/\lambda_\psi \in V'$.

Lemma C.4.5. *With the previous notation, the following diagram commutes*

$$\begin{array}{ccc} V & \xrightarrow{\quad\psi\quad} & V' \\ \big\downarrow{\scriptstyle \varphi_V} & & \big\downarrow{\scriptstyle \varphi_{V'}} \\ \mathrm{End}(\mathrm{Cl}^+(V)) & \xrightarrow{\mathrm{End}(\psi_{\mathrm{Cl}})} & \mathrm{End}(\mathrm{Cl}^+(V')) \end{array},$$

where $\mathrm{End}(\psi_{\mathrm{Cl}})$ *is the map* $f \mapsto \psi_{\mathrm{Cl}} \circ f \circ \psi_{\mathrm{Cl}}^{-1}$.

Proof. By definition, the composition of φ_V with $\mathrm{End}(\psi_{\mathrm{Cl}})$ is the map

$$V \to \mathrm{End}(\mathrm{Cl}^+(V')), \quad v \mapsto (w' \mapsto \psi(v) \cdot w' \cdot \psi(v_0)/\lambda_\psi).$$

This shows that the above square is commutative. Indeed, $\varphi_{V'}$ is the map

$$V' \hookrightarrow \mathrm{End}(\mathrm{Cl}^+(V')), \quad v' \mapsto (w' \mapsto v' \cdot w' \cdot \psi(v_0)/\lambda_\psi).$$ □

Proof of Propostion C.4.1. Lemma C.4.4 shows that there exists an isogeny of abelian varieties $\psi_{\mathrm{KS}} \colon \mathrm{KS}(V) \to \mathrm{KS}(V')$ such that $(\psi_{\mathrm{KS}})_* = \psi_{\mathrm{Cl}} \colon \mathrm{Cl}^+(V) \to \mathrm{Cl}^+(V')$. Recall that the Kuga–Satake embedding is the composition

$$V \overset{\varphi_V}{\hookrightarrow} \mathrm{End}(\mathrm{Cl}^+(V)) \simeq \mathrm{Cl}^+(V) \otimes \mathrm{Cl}^+(V),$$

where the isomorphism is given by the polarization Q on $\mathrm{Cl}^+(V)$ which induces an isomorphism between $\mathrm{Cl}^+(V)$ and $\mathrm{Cl}^+(V)^*$. Note that the commutativity of the square in the theorem follows from the commutativity of the square of Lemma C.5.4 by the compatibility of ψ_{Cl} with the polarizations Q and Q'. □

C.4.1 De Rham–Betti similarities

In [53, Thm. 9.5], the authors prove that de Rham–Betti isometries between the second de Rham–Betti cohomology groups of two hyperkähler manifolds defined over $\overline{\mathbb{Q}}$ are motivated in the sense of André using the fact that the Kuga–Satake correspondence is motivated. We note here that the observation that similarities between two quadratic spaces induce isomorphisms between the respective even Clifford algebras shows that the result in [53] can be extended to de Rham–Betti similarities.

Let us briefly recall the notions of de Rham–Betti morphism and of motivated cycles as presented in [53]. To simplify the exposition, we avoid going into too much detail of the Tannakian formalism.

Definition C.4.6. Let X be a smooth projective variety over $\overline{\mathbb{Q}}$, and let $\mathbb{Q}(k) := (2\pi i)^k\mathbb{Q} \subseteq \mathbb{C}$. The *de Rham–Betti cohomology groups* of X are the triples

$$H^n_{\mathrm{dRB}}(X, \mathbb{Q}(k)) := (H^n_{\mathrm{dR}}(X/\overline{\mathbb{Q}}), H^n_{\mathrm{B}}(X_{\mathbb{C}}, \mathbb{Q}(k)), c_X),$$

where

$$c_X \colon H^n_{\mathrm{dR}}(X/\overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \xrightarrow{\simeq} H^n_{\mathrm{B}}(X_{\mathbb{C}}, \mathbb{Q}(k)) \otimes_{\mathbb{Q}} \mathbb{C}$$

is the Grothendieck's period comparison isomorphism. Given X' another smooth projective variety over $\overline{\mathbb{Q}}$, a *de Rham–Betti morphism* between $H^n_{\mathrm{dRB}}(X, \mathbb{Q}(k))$ and $H^n_{\mathrm{dRB}}(X', \mathbb{Q}(k))$ consists of a pair of morphisms

$$f_{\mathrm{dR}} \colon H^n_{\mathrm{dR}}, (X/\overline{\mathbb{Q}}) \to H^n_{\mathrm{dR}}(X'/\overline{\mathbb{Q}}) \quad \text{and} \quad f_B \colon H^n_{\mathrm{B}}(X_{\mathbb{C}}, \mathbb{Q}(k)) \to H^n_{\mathrm{B}}(X'_{\mathbb{C}}, \mathbb{Q}(k)),$$

where f_{dR} is $\overline{\mathbb{Q}}$-linear and f_B is $\mathbb{Q}$-linear, and their $\mathbb{C}$-linear extensions are compatible with c_X and $c_{X'}$.

Definition C.4.7. Let X be a smooth projective variety over $\overline{\mathbb{Q}}$. A *motivated cycle* on X is an element of $H^{2r}_{\mathrm{B}}(X_{\mathbb{C}}, \mathbb{Q}(r))$ of the form $p_{X,*}(\alpha \cup *_L\beta)$, where α and β are algebraic cycles on $X \times_{\overline{\mathbb{Q}}} Y$ for some smooth projective variety Y over $\overline{\mathbb{Q}}$, $*_L$ is the (inverse of the) Lefschetz isomorphism, and $p_{X,*}$ is the first projection. Note that, given a motivated cycle α_{B} in $H^{2k}_{\mathrm{B}}(X, \mathbb{Q}(k))$, there exists a de Rham cohomology class α_{dR} in $H^{2k}_{\mathrm{dR}}(X/\overline{\mathbb{Q}})$ such that $c_X(\alpha_{\mathrm{dR}}) = \alpha_{\mathrm{B}}$. In particular, we see that a motivated cycle on $X \times_{\overline{\mathbb{Q}}} Y$ induces a de Rham–Betti morphism between the de Rham–Betti cohomologies of X and Y. One says that a de Rham–Betti morphism between cohomology groups of two smooth projective varieties X and Y over $\overline{\mathbb{Q}}$ is *motivated* if it is induced by a motivated cycle on $X \times_{\overline{\mathbb{Q}}} Y$. For a complete introduction on this subject we refer the reader to [4].

Following [53], a variety over $\overline{\mathbb{Q}}$ is called a *hyperkähler manifold over* $\overline{\mathbb{Q}}$ if its base-change to $\mathbb{C}$ is a hyperkähler manifold with second Betti number at least three. This last assumption is needed to ensure that the Kuga–Satake correspondence is motivated as proved by André [3]. The following proposition extends the result of [53, Thm. 9.5].

Proposition C.4.8. *Let X and X' be hyperkähler manifolds over $\overline{\mathbb{Q}}$. Then, any de Rham–Betti similarity $H^2_{\mathrm{dRB}}(X,\mathbb{Q}) \xrightarrow{\simeq} H^2_{\mathrm{dRB}}(X',\mathbb{Q})$ is motivated.*

Proof. The proof of this theorem is exactly the same as the one in the reference with the only addition that similarities (and not just isometries) induce isomorphisms between the even Clifford algebras. We give here just a sketch of the proof.

Let $H^2_{\mathrm{dRB}}(X,\overline{\mathbb{Q}})$ be the second $\overline{\mathbb{Q}}$-de Rham–Betti cohomology group of X. That is

$$H^2_{\mathrm{dRB}}(X,\overline{\mathbb{Q}}) \coloneqq (H^2_{\mathrm{dR}}(X/\overline{\mathbb{Q}}), H^2_{\mathrm{B}}(X_{\mathbb{C}},\overline{\mathbb{Q}}), c_X).$$

Similarly define $H^2_{\mathrm{dRB}}(X',\overline{\mathbb{Q}})$. By [53, Lem. 6.2, 6.17], to prove the theorem, it suffices to prove the same statement over $\overline{\mathbb{Q}}$. I.e., that every $\overline{\mathbb{Q}}$-de Rham–Betti similarity between $H^2_{\mathrm{dRB}}(X,\overline{\mathbb{Q}})$ and $H^2_{\mathrm{dRB}}(X',\overline{\mathbb{Q}})$ is a $\overline{\mathbb{Q}}$-linear combination of motivated cycles on $X \times_{\overline{\mathbb{Q}}} X'$.

Let $T^2_{\mathrm{dRB}}(X,\overline{\mathbb{Q}})$ be the orthogonal complement of the subspace of $H^2_{\mathrm{dRB}}(X,\overline{\mathbb{Q}})$ spanned by divisor classes, and similarly define $T^2_{\mathrm{dRB}}(X',\overline{\mathbb{Q}})$. As in the reference, one shows that, to prove the result, it suffices to show that every de Rham–Betti similarity between $T^2_{\mathrm{dRB}}(X,\overline{\mathbb{Q}})$ and $T^2_{\mathrm{dRB}}(X',\overline{\mathbb{Q}})$ is $\overline{\mathbb{Q}}$-motivated.

Consider the $\overline{\mathbb{Q}}$-linear category $\mathscr{C}_{\overline{\mathbb{Q}}-\mathrm{dRB}}$ whose objects are triples $(M_{\mathrm{dR}}, M_{\mathrm{B}}, c_M)$, where M_{dR} and M_{B} are finite dimensional $\overline{\mathbb{Q}}$-vector spaces and

$$c_M \colon M_{\mathrm{dR}} \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \to M_{\mathrm{B}} \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$$

is a $\mathbb{C}$-linear isomorphism. As in [53, Sec. 4.3], one sees that $\mathscr{C}_{\overline{\mathbb{Q}}-\mathrm{dRB}}$ is a neutral Tannakian category. Denote by $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}$ its Tannakian fundamental group. Note that $T^2_{\mathrm{dRB}}(X,\overline{\mathbb{Q}})$ and $T^2_{\mathrm{dRB}}(X',\overline{\mathbb{Q}})$ are objects in $\mathscr{C}_{\overline{\mathbb{Q}}-\mathrm{dRB}}$.

Denote by $V \coloneqq T^2_{\mathrm{B}}(X,\mathbb{Q}(1))$ and $V' \coloneqq T^2_{\mathrm{B}}(X',\mathbb{Q}(1))$ the transcendental Betti cohomologies of X and X'. Given a $\overline{\mathbb{Q}}$-de Rham–Betti similarity $\psi_{\mathrm{dRB}} \colon T^2_{\mathrm{dRB}}(X,\overline{\mathbb{Q}}) \xrightarrow{\simeq} T^2_{\mathrm{dRB}}(X',\overline{\mathbb{Q}})$, it induces a similarity

$$\psi \colon V \otimes \overline{\mathbb{Q}} \to V' \otimes \overline{\mathbb{Q}}.$$

As ψ_{dRB} is a morphism in $\mathscr{C}_{\overline{\mathbb{Q}}-\mathrm{dRB}}$, the morphism ψ is $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}$-invariant by the Tannakian formalism. With the same definition as in Lemma C.4.2, we see that ψ induces a $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}$-invariant isomorphism of algebras

$$\psi_{\mathrm{Cl}} \colon \mathrm{Cl}^+(V \otimes \overline{\mathbb{Q}}) \to \mathrm{Cl}^+(V' \otimes \overline{\mathbb{Q}}).$$

As in the reference, one then shows that this induces a $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}$-invariant isomorphism of algebras $J \colon \mathrm{End}(\mathrm{Cl}(V) \otimes \overline{\mathbb{Q}}) \to \mathrm{End}(\mathrm{Cl}(V') \otimes \overline{\mathbb{Q}})$. One then shows that J is $\overline{\mathbb{Q}}$-motivated. This in turn implies that ψ is $\overline{\mathbb{Q}}$-motivated using the fact that the Kuga–Satake correspondence is $\overline{\mathbb{Q}}$-motivated as proven in [53, Prop. 8.5]. This concludes the proof. □

C.5 Hodge similarities and algebraic classes

We now go back to the case of hyperkähler manifolds defined over $\mathbb{C}$ and study the consequences of the functoriality of the Kuga–Satake construction relative to Hodge similarities in the case where the Hodge structure (V, q) is geometrical. In other words, we assume that there is a hyperkähler manifold X for which $V = T(X)$ or $V = H^2(X, \mathbb{Q})$ and q is the Beauville–Bogomolov quadratic form with the sign changed.

Remark C.5.1. In Section C.4, we studied the Kuga–Satake construction for polarized Hodge structures of K3-type. The same construction also works for the second cohomology group of a hyperkähler manifold X even though it is not polarized by the Beauville–Bogomolov quadratic form. Indeed, using the direct sum decomposition $H^2(X, \mathbb{Q}) \simeq T(X) \oplus \mathrm{NS}(X)_{\mathbb{Q}}$, one sees that the even Clifford algebra of $H^2(X, \mathbb{Q})$ is a power of the even Clifford algebra of $T(X)$. Thus, the Kuga–Satake variety $\mathrm{KS}(H^2(X, \mathbb{Q}))$ is an abelian variety isogenous to a power of $\mathrm{KS}(T(X))$.

Let $\mathrm{KS}(X)$ be the Kuga–Satake variety of $H^2(X, \mathbb{Q})$. The Kuga–Satake correspondence gives an embedding of Hodge structures:

$$\kappa_X \colon H^2(X, \mathbb{Q}) \hookrightarrow H^1(\mathrm{KS}(X), \mathbb{Q})^{\otimes 2} \subseteq H^2(\mathrm{KS}(X)^2, \mathbb{Q}).$$

The Hodge conjecture predicts that κ_X is algebraic:

Conjecture C.5.2 (Kuga–Satake Hodge conjecture)**.** *Let X be a hyperkähler manifold, then, the Kuga–Satake correspondence κ_X is algebraic.*

Remark C.5.3. Note that the Kuga–Satake correspondence depends on the choice of the three elements $v_0, f_1, f_2 \in T(X)$ as in Section C.4. Choosing a different $\widetilde{v}_0 \in T(X)$ changes the embedding by the automorphism of $\mathrm{Cl}^+(H^2(X, \mathbb{Q}))$ which sends w to $\frac{w \cdot v_0 \cdot \widetilde{v}_0}{q(v_0)}$, and choosing a different pair $\widetilde{f}_1, \widetilde{f}_2 \in T(X)$ corresponds to changing the polarization on the complex torus $\mathrm{KS}(X)$. However, neither of these two operations affects the algebraicity of κ_X. Hence, Conjecture C.5.2 does not depend on the choices made in the definition of κ_X.

Let $2n \coloneqq \dim X$ and $N \coloneqq \dim \mathrm{KS}(X)$. The transpose $\kappa_X^\vee$ of κ_X is the surjection

$$\kappa_X^\vee \colon H^{4N-2}(\mathrm{KS}(X)^2, \mathbb{Q}) \twoheadrightarrow H^{4n-2}(X, \mathbb{Q}).$$

Note that, as κ_X and $\kappa_X^\vee$ are transpose of each other, κ_X is algebraic if and only if $\kappa_X^\vee$ is algebraic. Let $h_X \in H^2(X, \mathbb{Q})$ be the cohomology class of an ample divisor on X. By the strong Lefschetz theorem, the cup product with h_X^{2n-1} induces an isomorphism of Hodge structures

$$h_X^{2n-1} \cup \bullet \colon H^2(X, \mathbb{Q}) \to H^{4n-2}(X, \mathbb{Q})(2n-2),$$

where $(2n-2)$ denotes the Tate twist by $\mathbb{Q}(2n-2)$. Let

$$\delta_X \coloneqq (h_X^{2n-1} \cup \bullet)^{-1} \colon H^{4n-2}(X, \mathbb{Q})(2n-2) \to H^2(X, \mathbb{Q})$$

be the inverse map. As $h_X^{2n-1} \cup \bullet$ is an isomorphism of Hodge structures, also δ_X is an isomorphism of Hodge structures. Note that it is in general not known whether δ_X is algebraic or not. Adapting the proof of [97, Lem. 3.4], we show that κ_X and $\kappa_X^\vee$ satisfy the following:

Lemma C.5.4. *Let X be a hyperkähler manifold of dimension $2n$, and let $h_{\mathrm{KS}} \in H^2(\mathrm{KS}(X), \mathbb{Q})$ be the class of an ample divisor on $\mathrm{KS}(X)$. Denote by φ the restriction to $T(X)$ of the composition*

$$\delta_X \circ \kappa_X^\vee \circ (h_{\mathrm{KS}}^{2N-2} \cup \bullet) \circ \kappa_X \colon H^2(X, \mathbb{Q}) \to H^2(X, \mathbb{Q}).$$

Then, φ is a non-zero rational multiple of the identity $\mathrm{Id}_{T(X)} \colon T(X) \to T(X)$.

Proof. Let us begin by showing that φ is a Hodge automorphism. Note that φ is by construction a morphism of Hodge structures. It is then an element of the endomorphism field of $T(X)$. As $T(X)$ is an irreducible Hodge structure, every non-trivial endomorphism is an automorphism. In particular, we just need to show that φ is non-zero. As $\delta_X \colon H^{4n-2}(X, \mathbb{Q})(2n-2) \to H^2(X, \mathbb{Q})$ is an isomorphism of Hodge structures, it suffices to show that the map

$$\left(\kappa_X^\vee \circ (h_{\mathrm{KS}}^{2N-2} \cup \bullet) \circ \kappa_X\right)\Big|_{T(X)} \colon T(X) \to H^{4n-2}(X, \mathbb{Q})$$

is non-zero. Let $\omega \in H^{2,0}(X)$ be the class of a symplectic form. As κ_X and $\kappa_X^\vee$ are adjoint with respect to the Hodge–Riemann pairing, we have the following equality

$$\langle \omega, \kappa_X^\vee \circ (h_{\mathrm{KS}}^{2N-2} \cup \bullet) \circ \kappa_X(\overline{\omega}) \rangle_X = \langle \kappa_X(\omega), (h_{\mathrm{KS}}^{2N-2} \cup \bullet) \circ \kappa_X(\overline{\omega}) \rangle_{\mathrm{KS}(X)}.$$

The right-hand side is non-zero by the Hodge–Riemann relations, as $0 \neq \kappa_X(\omega) \in H^{2,0}(\mathrm{KS}(X))$ by the injectivity of κ_X. In particular, we conclude that $\kappa_X^\vee \circ (h_{\mathrm{KS}}^{2N-2} \cup \bullet) \circ \kappa_X(\overline{\omega}) \neq 0$. This implies that $\kappa_X^\vee \circ (h_{\mathrm{KS}}^{2N-2} \cup \bullet) \circ \kappa_X$ restricted to $T(X)$ is non-zero and proves the first statement of the lemma.

To prove that φ is a rational multiple of the identity, let us first assume that X is Mumford–Tate general. In this case, $\mathrm{End}_{\mathrm{Hdg}}(T(X)) = \mathbb{Q}$. Hence, the statement is obvious since every Hodge automorphism is a rational multiple of the identity. For the special case, just note that it is possible to deform in the moduli space of polarized hyperkähler manifolds the pair (X, h_X) to a pair $(X', h_{X'})$, where X' is Mumford–Tate general. Then, as all the maps involved in the definition of φ deform in families, the statement for $(X', h_{X'})$ readily implies the statement for (X, h_X). □

We finally have all tools to show the main theorem of this section:

Theorem C.5.5. *Let X' and X be two hyperkähler manifolds for which the Kuga–Satake Hodge conjecture holds. Then, for every Hodge similarity $\psi \colon T(X') \to T(X)$, the composition*

$$T(X') \xrightarrow{\psi} T(X) \xrightarrow{h_X^{2n-2} \cup \bullet} H^{4n-2}(X, \mathbb{Q})$$

is algebraic, where $2n := \dim X$.

Proof. By the functoriality of the Kuga–Satake correspondence of Propositon C.4.1, the similarity $\psi\colon T(X') \to T(X)$ induces an isogeny $\psi_{\mathrm{KS}}\colon \mathrm{KS}(X') \to \mathrm{KS}(X)$ such that the induced isomorphism

$$(\psi_{\mathrm{KS}})_*^{\otimes 2}\colon H^1(\mathrm{KS}(X'),\mathbb{Q})^{\otimes 2} \to H^1(\mathrm{KS}(X),\mathbb{Q})^{\otimes 2}$$

makes the following diagram commute:

$$\begin{array}{ccc} T(X') & \xrightarrow{\quad\psi\quad} & T(X) \\ \downarrow{\scriptstyle \kappa_{X'}} & & \downarrow{\scriptstyle \kappa_X} \\ H^1(\mathrm{KS}(X'),\mathbb{Q})^{\otimes 2} & \xrightarrow{(\psi_{\mathrm{KS}})_*^{\otimes 2}} & H^1(\mathrm{KS}(X),\mathbb{Q})^{\otimes 2} \end{array}, \tag{C.1}$$

where $\kappa_{X'}$ (resp., κ_X) is the Kuga–Satake correspondence for $T(X')$ (resp., $T(X)$). By Lemma C.5.4, the automorphism φ of $T(X)$ makes the following diagram commute

$$\begin{array}{ccc} T(X) & \xrightarrow{(h_X^{2n-2}\cup\bullet)\circ\varphi} & H^{4n-2}(X,\mathbb{Q}) \\ \downarrow{\scriptstyle \kappa_X} & & \uparrow{\scriptstyle \kappa_X^\vee} \\ H^2(\mathrm{KS}(X)^2,\mathbb{Q}) & \xrightarrow{h_{\mathrm{KS}}^{2N-2}\cup\bullet} & H^{2N-2}(\mathrm{KS}(X)^2,\mathbb{Q}) \end{array}. \tag{C.2}$$

The commutativity of the squares (C.1) and (C.2) implies that the following equality holds

$$(h_X^{2n-2}\cup\bullet)\circ\varphi\circ\psi = \kappa_X^\vee\circ(h_{\mathrm{KS}}^{N-2}\cup\bullet)\circ(\psi_{\mathrm{KS}})_*^{\otimes 2}\circ\kappa_{X'}.$$

Note that the right-hand side of the equality is algebraic: $(\psi_{\mathrm{KS}})_*^{\otimes 2}$ is induced by an isogeny of abelian varieties, and $\kappa_X^\vee$ and $\kappa_{X'}$ are algebraic by assumption. We then conclude that the composition

$$(h_X^{2n-2}\cup\bullet)\circ\varphi\circ\psi\colon T(X') \to H^{4n-2}(X,\mathbb{Q})$$

is algebraic as well. I.e., there exists a cycle $\Gamma \in \mathrm{CH}^*(X'\times X)$ inducing the morphism

$$[\Gamma]_* = (h_X^{2n-2}\cup\bullet)\circ\varphi\circ\psi\colon T(X') \to H^{4n-2}(X,\mathbb{Q}).$$

By Lemma C.5.4, the automorphism φ is by equal to $\lambda\mathrm{Id}_{T(X)}$ for some non-zero $\lambda \in \mathbb{Q}$. Therefore, the class $\Gamma/\lambda \in \mathrm{CH}^*(X'\times X)$ induces the morphism

$$T(X') \xrightarrow{\psi} T(X) \xrightarrow{h_X^{2n-2}\cup\bullet} H^{4n-2}(X,\mathbb{Q}).$$

This concludes the proof. □

This shows that, if the Kuga–Satake correspondence is algebraic, then every Hodge similarity is algebraic after composing it with the Lefschetz isomorphism. The Lefschetz standard conjecture in degree two for X predicts that the inverse of $h_X^{2n-2}\cup\bullet\colon H^2(X,\mathbb{Q}) \to H^{4n-2}(X,\mathbb{Q})$ is algebraic. If X satisfies this conjecture, Theorem C.5.5 gives the following:

Corollary C.5.6. *Let X and X' be hyperkähler manifolds satisfying the Kuga–Satake Hodge conjecture. Assume moreover that X satisfies the Lefschetz standard conjecture in degree two. Then, every Hodge similarity $\psi\colon T(X') \to T(X)$ is algebraic.* □

C.6 Applications

In this section, we recall the main cases where the Kuga–Satake Hodge conjecture has been proven and the cases in which the Lefschetz standard conjecture in degree two is known to hold. This way, we describe examples of applications of Theorem C.5.5 and Corollary C.5.6.

As mentioned in the introduction, Hodge similarities between transcendental lattices of hyperkähler manifolds appear naturally in two cases: as elements of totally real endomorphism fields of degree two, and as Hodge isomorphisms $T(Y) \to T(X)$, where X and Y are hyperkähler manifolds with $T(Y)$ Hodge isometric to $T(X)(\lambda)$ for some $\lambda \in \mathbb{Q}_{>0}$.

Let us start from the case of hyperkähler manifolds of generalized Kummer type. For these varieties the Kuga–Satake Hodge conjecture is proven in [98] and the Lefschetz standard conjecture in degree two has been proven in [24]. We thus get our main families of examples of varieties satisfying the hypotheses of Corollary C.5.6, and we conclude that every Hodge similarity between the transcendental lattices of two hyperkähler manifolds of generalized Kummer type is algebraic. Note that the dimension of the transcendental lattice of a hyperkähler manifold of generalized Kummer type is at most six-dimensional. Therefore, its endomorphism field is either a CM field or a totally real field of degree one or two. In all cases, we see that it is always generated by Hodge similarities. We therefore deduce the following:

Theorem C.6.1. *Let X and X' be hyperkähler manifolds of generalized Kummer type such that $T(X)$ and $T(X')$ are Hodge similar. Then, every Hodge morphism between $T(X')$ and $T(X)$ is algebraic.*

Remark C.6.2. Taking $X = X'$ in Theorem C.6.1, we see that every Hodge morphism in $E \coloneqq \mathrm{End}_{\mathrm{Hdg}}(T(X))$ is algebraic. Note that, Theorem C.6.1 also covers the case where X and X' are hyperkähler manifolds of generalized Kummer type with Hodge similar transcendental lattice but of different dimension. Let us briefly recall why this happens: recall that the second cohomology group of a hyperkähler manifold X of generalized Kummer type of dimension $2n$ satisfies

$$(H^2(X,\mathbb{Q}), q_X) \simeq U_{\mathbb{Q}}^{\oplus 3} \oplus \mathbb{Q}\delta_n,$$

where $(\delta_n)^2 = -2(n+1)$. Let k be a positive integer. Using the fact that $U_{\mathbb{Q}}$ is isometric with $U_{\mathbb{Q}}(k)$, one sees that there is an isometry

$$U_{\mathbb{Q}}^{\oplus 3} \oplus \mathbb{Q}\delta_{n'} \to (U_{\mathbb{Q}}^{\oplus 3} \oplus \mathbb{Q}\delta_n)(k),$$

for $n' := k(n+1)-1$. In other words, there is a similarity

$$\psi\colon U_{\mathbb{Q}}^{\oplus 3} \oplus \mathbb{Q}\delta_{n'} \to U_{\mathbb{Q}}^{\oplus 3} \oplus \mathbb{Q}\delta_{n}.$$

Let $[\sigma'] \in \mathbb{P}((U_{\mathbb{Q}}^{\oplus 3} \oplus \mathbb{Q}\delta_{n'}) \otimes_{\mathbb{Q}} \mathbb{C})$ be a class satisfying $(\sigma')^2 = 0$ and $(\sigma', \overline{\sigma'}) > 0$. Then, $[\sigma']$ determines a Hodge structure on the quadratic space $U_{\mathbb{Q}}^{\oplus 3} \oplus \mathbb{Q}\delta_{n'}$. Similarly the class $[\psi(\sigma')]$ satisfies the same hypotheses of σ', hence, it defines a Hodge structure on $U_{\mathbb{Q}}^{\oplus 3} \oplus \mathbb{Q}\delta_{n}$. By the surjectivity of the period map, we obtain a hyperkähler manifold X' of $\mathrm{Kum}^{n'}$-type whose symplectic form is given by σ' and a hyperkähler manifold X of Kum^{n}-type whose symplectic form is given by $\psi(\sigma)$. By construction, the morphism ψ defines a Hodge similarity between $T(X')$ and $T(X)$. Thus, X and X' satisfy the hypotheses for Theorem C.6.1, and we conclude that any Hodge morphism $T(X') \to T(X)$ is algebraic.

Let us briefly comment on the application of our result to the case of K3 surfaces. In this case, the Lefschetz standard conjecture is trivially true. Hence, applying Corollary C.5.6, we get the following:

Theorem C.6.3. *Let S and S' be K3 surfaces for which the Kuga–Satake Hodge conjecture holds. Then, every Hodge similarity between $T(S)$ and $T(S')$ is algebraic.* □

For K3 surfaces, the Kuga–Satake Hodge conjecture is in general not known. However, in [21], the author proves it for the (countably many) four-dimensional families of K3 surfaces with transcendental lattice isometric to $T(K)(2)$ for a hyperkähler manifold K of generalized Kummer type of dimension six. By [31, Sec. 3], there are one-dimensional subfamilies in these four-dimensional families of K3 surfaces which have totally real endomorphism field of degree two. As totally real fields of degree two are generated by Hodge similarities, Theorem C.6.3 proves the Hodge conjecture for the square of these K3 surfaces. As mentioned in the introduction, the Hodge conjecture for all powers of these particular K3 surfaces has been proven in [89] by extending the techniques introduced in [81]. The proof we provided here for the square of these K3 surfaces is however more direct since it does not involve the study of the Hodge conjecture for the Kuga–Satake varieties.

Finally, let us come to the case of hyperkähler manifolds of $\mathrm{K3}^{[n]}$-type. As mentioned in the introduction, the Lefschetz standard conjecture holds for these manifolds by [14]. Therefore, applying Corollary C.5.6, we get the following:

Theorem C.6.4. *Let X and X' be hyperkähler manifolds of $\mathrm{K3}^{[n]}$- and $\mathrm{K3}^{[n']}$-type for which the Kuga–Satake Hodge conjecture holds. Then, every Hodge similarity between $T(X')$ and $T(X)$ is algebraic.* □

The Kuga–Satake Hodge conjecture has not been proven for hyperkähler manifolds of $\mathrm{K3}^{[n]}$-type. In dimension six, this conjecture follows from the construction in [21] for the families of hyperkähler manifolds of $K3^{[3]}$-type which are resolution of the quotient of a hyperkähler manifold of generalized Kummer type of dimension six by a symplectic group $G \simeq (\mathbb{Z}/2\mathbb{Z})^5$. This way, we obtain four-dimensional families of hyperkähler manifolds of $K3^{[3]}$-type which satisfy the hypotheses of Theorem C.6.4.

Remark C.6.5. Note that Theorem C.5.5 and Corollary C.5.6 provide algebraic classes on $X' \times X$ whenever the Kuga–Satake correspondence is algebraic for X and X' and the transcendental lattices are Hodge similar. This also works when X and X' are not of the same deformation type, and when the dimensions of X and X' are not the same.

Appendix D

Algebraic cycles on hyper-Kähler varieties of generalized Kummer type

S. Floccari and M. Varesco[1]

Abstract. We prove the conjectures of Hodge and Tate for any four-dimensional hyper-Kähler variety of generalized Kummer type. For an arbitrary variety X of generalized Kummer type, we show that all Hodge classes in the subalgebra of the rational cohomology generated by $H^2(X,\mathbb{Q})$ are algebraic.

D.1 Introduction

Despite many efforts, the Hodge conjecture remains widely open. Much work has been devoted to the study of algebraic cycles on abelian varieties. In spite of several positive results, already in this setting the Hodge conjecture proved to be a rather formidable problem. It is in general open for abelian varieties of dimension at least 4 (see [68, 69]).

Another interesting class of varieties with trivial canonical bundle is that of hyper-Kähler varieties. See the articles [7] and [42] for general information. In the present paper we study the Hodge conjecture for hyper-Kähler varieties of generalized Kummer type (Kum^n-varieties for short). By definition, these are deformations of Beauville's generalized Kummer varieties ([7]) constructed from abelian surfaces.

Important progress on Kum^n-varieties came from the works of O'Grady [76] and Markman [60]. They uncovered the relation between a Kum^n-variety X and its intermediate Jacobian $J^3(X)$, which is shown to be an abelian fourfold of Weil type. Their results lead to a Torelli theorem for Kum^n-varieties in terms of the Hodge structure on $H^3(X,\mathbb{Q})$. Markman further constructs an algebraic cycle on $X \times J^3(X)$ realizing the canonical isomorphism $H^3(X,\mathbb{Q}) \cong H^1(J^3(X),\mathbb{Q})(-1)$ of Hodge structures, for any variety X of Kum^n-type. It follows that the Kuga–Satake correspondence ([54]) is algebraic for these varieties, as shown by Voisin [98]. See also [90] and [32] for an account of these results.

In the recent article [20], the first author proved the Hodge conjecture for any six-

[1]Salvatore Floccari and Mauro Varesco. Algebraic cycles on hyper-Kähler varieties of generalized Kummer type. arXiv preprint https://arxiv.org/abs/2308.04865, 2023.

dimensional variety K of Kum^3-type. No locally complete family of hyper-Kähler or abelian varieties (of dimension at least 4) satisfying the Hodge conjecture was previously known. An important ingredient in the proof is the construction from [21] of a K3 surface S_K naturally associated with K, and the fact that the Hodge conjecture holds for any power of this K3 surface. This was proven in [21, Corollary 5.8], using a theorem of the second author [88] and the aforementioned results of O'Grady, Markman and Voisin.

In the present article we obtain some results on the Hodge conjecture for varieties of generalized Kummer type of arbitrary dimension.

Theorem D.1.1. *Let X be a projective manifold of Kum^n-type, $n \geq 2$. Denote by $A_2^\bullet(X) \subset H^\bullet(X, \mathbb{Q})$ the subalgebra of the rational cohomology generated by $H^2(X, \mathbb{Q})$. Then any Hodge class in $A_2^{2j}(X) \cap H^{j,j}(X)$ is algebraic, for any j.*

To prove this result we use the works of Foster [24] and of the second author [88] to show that an arbitrary variety X of Kum^n-type is related via an algebraic correspondence to the K3 surface S_K associated to some Kum^3-variety K. Theorem D.1.1 is then deduced from the Hodge conjecture for the powers of S_K. Our proof leads to the suggestive expectation that any variety of Kum^n-type is naturally associated with a K3 surface, generalizing the construction given in [21] for the six-dimensional case. We remark that it should also be possible to obtain the theorem via the representation-theoretic methods of [88].

For $j \leq n$, cup-product induces an isomorphism $A_2^{2j}(X) \cong \mathrm{Sym}^j(H^2(X, \mathbb{Q}))$ of Hodge structures, by a theorem of Verbitsky [91]. However, Theorem D.1.1 is not sufficient to prove the Hodge conjecture for X (see [37]). For $n = 3$, the full Hodge conjecture proven in [20] is a stronger result and requires considerably more work. For $n = 2$, the Hodge classes in the complement of $A_2^\bullet(X)$ form an 80-dimensional subspace of the middle cohomology; Hassett and Tschinkel have shown in [40] that these Hodge classes are algebraic, for any X of Kum^2-type. Hence, Theorem D.1.1 yields the following.

Corollary D.1.2. *Let X be a projective manifold of Kum^2-type. Then the Hodge conjecture holds for X, i.e., $H^{j,j}(X) \cap H^{2j}(X, \mathbb{Q})$ consists of algebraic classes for any j.*

Let now k be a finitely generated field of characteristic 0, with algebraic closure $\bar{k}$, and let X/k be a smooth and projective variety over k. Given a prime number ℓ, the absolute Galois group of k acts on the ℓ-adic étale cohomology of $X_{\bar{k}}$. In analogy with the Hodge conjecture, the Tate conjecture predicts that the subspace of Galois invariants in $H^{2j}_{\text{ét}}(X_{\bar{k}}, \mathbb{Q}_\ell(j))$ is spanned by the fundamental classes of k-subvarieties of X. See [87] for general information on the Tate conjecture.

Corollary D.1.3. *Let $k \subset \mathbb{C}$ be a finitely generated field and let X/k be a smooth and projective variety such that $X_{\mathbb{C}}$ is of Kum^2-type. Then, for any prime number ℓ, the strong Tate conjecture holds for X, i.e., the Galois representations $H^j_{ét}(X_{\bar{k}}, \mathbb{Q}_\ell)$ are semisimple* *and the subspace of Galois invariants in $H^{2j}_{ét}(X_{\bar{k}}, \mathbb{Q}_\ell(j))$ is the $\mathbb{Q}_\ell$-span of fundamental classes of k-subvarieties of X, for any j.*

A third conjecture, the Mumford-Tate conjecture, connects those of Hodge and Tate; see [67, §2.1] for its statement. While this is a hard open problem in itself, the Mumford-Tate conjecture has been proven for any hyper-Kähler variety of known deformation type in [19], [85] and [22]. As a consequence, the conjectures of Hodge and Tate are equivalent for such a variety. It follows that Corollary D.1.3 is equivalent to Corollary D.1.2.

Aknowledgements

We are grateful to Stefan Schreieder for many suggestions and remarks. We thank Ben Moonen for his comments on a first version of this draft. The second author was funded by ERC Synergy Grant HyperK, Grant agreement ID 854361.

D.2 Motives

Grothendieck's theory of motives provides a useful framework to study the Hodge conjecture. We will work with the category of homological motives with rational coefficients over $\mathbb{C}$, which we denote by Mot; see [83] or [4] for its construction. The objects of Mot are triples (X, p, n) where X is a smooth and projective complex variety, p is an idempotent correspondence given by an algebraic class in $H^{2\dim X}(X \times X, \mathbb{Q})$, and n is an integer. Morphisms are given by algebraic cycles modulo homological equivalence via the formalism of correspondences; more precisely, morphisms from (X, p, n) to (Y, q, m) are by definition the algebraic classes $\gamma \in H^{2\dim X - 2n + 2m}(X \times Y, \mathbb{Q})$ such that $\gamma \circ p = q \circ \gamma$.

The category Mot is a pseudo-abelian tensor category. The unit object for the tensor product is denoted by Q; the Tate motives (resp., the Tate twists of a motive m) will be denoted by $\mathsf{Q}(i)$ (resp., by $\mathsf{m}(i)$). Given a motive m, we let $\langle \mathsf{m} \rangle_{\mathsf{Mot}}$ be the pseudo-abelian tensor subcategory of Mot generated by m, i.e., the smallest thick and full such subcategory containing m and closed under direct sums, tensor products, duals and subobjects.

There is a natural contravariant functor $\mathsf{h} \colon \mathrm{SmProj}_{\mathbb{C}} \to \mathsf{Mot}$, associating to a variety X its motive $\mathsf{h}(X) \coloneqq (X, \Delta, 0)$. Here, Δ is the cohomology class of the diagonal in $X \times X$.

Remark D.2.1. Let X be a smooth and projective variety. A polarization on X gives a split inclusion of $\mathsf{Q}(-1)$ into $\mathsf{h}(X)$. It follows that $\langle \mathsf{h}(X) \rangle_{\mathsf{Mot}}$ contains all Tate motives. This category consists of the motives (Y, q, m) such that Y is a power of X or $\mathrm{Spec}(\mathbb{C})$.

The functor associating to X its Hodge structure $H^\bullet(X, \mathbb{Q})$ factors as the composition of h and the realization functor $R \colon \mathsf{Mot} \to \mathsf{HS}$ to the category HS of polarizable $\mathbb{Q}$-Hodge structures. By definition, $R(X, p, n) \coloneqq p_*(H^\bullet(X)(n))$, and R is faithful. The Hodge conjecture is equivalent to the fullness of the realization functor R.

Remark D.2.2. Let X be a smooth and projective variety. Then the Hodge conjecture holds for X and all of its powers if and only if the restriction of R to $\langle \mathsf{h}(X) \rangle_{\mathsf{Mot}}$ is full.

Grothendieck's standard conjectures [39] would ensure that the category Mot has much better properties than just being a pseudo-abelian tensor category. We recall the following theorem due to Jannsen [50] and André [4]; see [5, Theorem 4.1].

Theorem D.2.3. *Let X be a smooth and projective complex variety. Then the following are equivalent:*

- *Grothendieck's standard conjectures hold for X;*
- *$\langle \mathsf{h}(X)\rangle_{\mathsf{Mot}}$ is a semisimple abelian category.*

In light of the above theorem, we will say that the standard conjectures hold for a motive $\mathsf{m} \in \mathsf{Mot}$ if the category $\langle \mathsf{m}\rangle_{\mathsf{Mot}}$ is semisimple and abelian.

The standard conjectures are known to hold for curves and surfaces, and for abelian varieties [51]. If they hold for varieties X and Y, then they hold for $X \times Y$ as well. We collect below some consequences of these conjectures.

Remark D.2.4. Assume that the standard conjectures hold for $\mathsf{m} \in \mathsf{Mot}$. Then, by [4, Corollaire 5.1.3.3], the restriction of the realization functor R to $\langle \mathsf{m}\rangle_{\mathsf{Mot}}$ is conservative, which means that a morphism f in this category is an isomorphism if and only if its realization $R(f)$ is an isomorphism of Hodge structures. Moreover, m (as well as any object in $\langle \mathsf{m}\rangle_{\mathsf{Mot}}$) admits a canonical weight decomposition $\mathsf{m} = \bigoplus_i \mathsf{m}^i$ such that $R(\mathsf{m}^i)$ is a pure Hodge structure of weight i; see [4, §5.1.2]. If the standard conjectures hold for the smooth and projective variety X, we shall thus write $\mathsf{h}(X) = \bigoplus_i \mathsf{h}^i(X)$.

We will also use the following easy fact from the theory of motives.

Remark D.2.5. Assume that Γ is a finite group acting on a smooth and projective variety X. Then the Γ-invariant part $\mathsf{h}(X)^\Gamma := (X, p^\Gamma, 0)$ is a direct summand of the motive of X, cut out by the projector $p^\Gamma := \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}[\mathrm{graph}(\gamma)] \in H^{2\dim X}(X \times X, \mathbb{Q})$. Moreover, if the quotient X/Γ is smooth, then $\mathsf{h}(X)^\Gamma$ equals the motive $\mathsf{h}(X/\Gamma)$.

D.3 Some recent result

Let X be a Kum^n-variety, $n \geq 2$. In [60], Markman constructs a four-dimensional abelian variety T_X associated to X, which is isogenous to the intermediate Jacobian $J^3(X)$. The automorphisms of X which act trivially on its second and third cohomology groups form a group $\Gamma_n \cong (\frac{\mathbb{Z}}{(n+1)\mathbb{Z}})^4$, by [10, 40]. Markman proves that Γ_n acts on T_X via translations and that the quotient $M_X := (X \times T_X)/\Gamma_n$ by the anti-diagonal action is a smooth holomorphic symplectic variety deformation equivalent to a smooth and projective moduli space of stable sheaves on an abelian surface ([47, 100]). Building on Markman's results and the strategy used by Charles and Markman in [14] to prove the standard conjectures for $\mathrm{K3}^{[n]}$-varieties, Foster obtaines the following theorem in [24].

Theorem D.3.1 ([24])**.** *Let X be a variety of* Kum^n*-type. Then the standard conjectures hold for M_X. The Künneth projector $H^\bullet(X, \mathbb{Q}) \to H^2(X, \mathbb{Q})$ is algebraic.*

Remark D.3.2. The cohomology of M_X is naturally identified with the Γ_n-invariants in $H^\bullet(X \times T_X, \mathbb{Q})$. Since Γ_n acts trivially on the cohomology of T_X, we have

$$H^\bullet(M_X, \mathbb{Q}) = H^\bullet(X, \mathbb{Q})^{\Gamma_n} \otimes H^\bullet(T_X, \mathbb{Q}).$$

The standard conjectures holds for the abelian variety T_X, and $\mathsf{h}^0(T_X) = \mathsf{Q}$. Therefore, the motive $\mathsf{h}(X)^{\Gamma_n}$ is a direct summand of $\mathsf{h}(M_X)$. By Foster's Theorem D.3.1, the standard conjectures hold for $\mathsf{h}(X)^{\Gamma_n}$.

The following result of the second author [88] is crucial for our proof of Theorem D.1.1. It allows us to relate varieties of generalized Kummer type of different dimensions via algebraic correspondences. Let $r \neq 0$ be a rational number. A similarity of multiplier r between two quadratic spaces V_1 and V_2 is a linear isomorphism $t\colon V_1 \to V_2$ which multiplies the form by a factor r, i. e., such that $(t(u), t(w))_2 = r \cdot (u, w)_1$ for all $u, w \in V_1$.

Theorem D.3.3 ([88, Theorem 0.4])**.** *Let X_1, X_2 be varieties of generalized Kummer type (not necessarily of the same dimension). Assume that*

$$\phi\colon H^2(X_1, \mathbb{Q}) \xrightarrow{\sim} H^2(X_2, \mathbb{Q})$$

is a rational Hodge similarity. Then ϕ is induced by an algebraic cycle on $X_1 \times X_2$.

The main observation behind this result is that the similarity ϕ induces an isogeny between the Kuga–Satake varieties of X_1 and X_2. The above statement is then deduced using the algebraicity of the Kuga–Satake correspondence and Foster's Theorem D.3.1.

D.4 Conclusion

As mentioned in the introduction, we will use the construction of a K3 surface S_K associated to any Kum^3-variety K, given by the first author in [21]. We will only need the following result which is deduced from the construction, see [21, Proof of Theorem 5.7 and Corollary 5.8].

Theorem D.4.1 ([21])**.** *Let K be a projective manifold of* Kum^3*-type, with associated K3 surface S_K. The Hodge conjecture holds for any power of S_K, and there exists an algebraic cycle on $S_K \times K$ which induces a Hodge similarity of multiplier 2 of transcendental lattices*

$$\psi\colon H^2_{\mathrm{tr}}(S_K, \mathbb{Q}) \xrightarrow{\sim} H^2_{\mathrm{tr}}(K, \mathbb{Q}).$$

Let X be a Kum^n-variety, $n \geq 2$, and denote by $A^\bullet_2(X)$ the subalgebra of the rational cohomology generated by $H^2(X, \mathbb{Q})$. By Foster's Theorem D.3.1, the degree 2 component $\mathsf{h}^2(X)$ of $\mathsf{h}(X)$ is well-defined. Since Γ_n acts trivially on the second cohomology of X, the motive $\mathsf{h}^2(X)$ is a direct summand of the Γ_n-invariant part $\mathsf{h}(X)^{\Gamma_n}$ of $\mathsf{h}(X)$.

Lemma D.4.2. *The subalgebra $A^\bullet_2(X) \subset H^\bullet(X, \mathbb{Q})$ is the realization of a submotive* $\mathsf{a}_2(X)$ *of* $\mathsf{h}(X)$*. Moreover,* $\mathsf{a}_2(X) \in \langle \mathsf{h}^2(X) \rangle_{\mathsf{Mot}}$.

Proof. For $i > 0$, we let $\delta_{i+1} \in H^{2i \dim X}(X^{i+1}, \mathbb{Q})$ denote the class of the small diagonal $\{(x, \dots, x)\} \subset X^{i+1}$. This class induces the cup-product morphism $\mathsf{h}(X)^{\otimes i} \to \mathsf{h}(X)$, which restricts to a morphism $\big(\mathsf{h}(X)^{\Gamma_n}\big)^{\otimes i} \to \mathsf{h}(X)^{\Gamma_n}$ in Mot. Since $\mathsf{h}^2(X)$ is a direct summand of $\mathsf{h}(X)^{\Gamma_n}$, the cup-product induces a morphism of motives

$$\beta\colon \bigoplus_i \mathsf{h}^2(X)^{\otimes i} \to \mathsf{h}(X)^{\Gamma_n}.$$

Note that β is a morphism in the category $\langle \mathsf{h}(X)^{\Gamma_n} \rangle_{\mathsf{Mot}}$, which is abelian and semisimple by Foster's Theorem D.3.1. Therefore, the image of β is a submotive $\mathsf{a}_2(X) \subset \mathsf{h}(X)$, whose realization is $A_2^\bullet(X) \subset H^\bullet(X, \mathbb{Q})$. By semisimplicity, $\mathsf{a}_2(X)$ is a direct summand of $\bigoplus_i \mathsf{h}^2(X)^{\otimes i}$, and hence $\mathsf{a}_2(X) \in \langle \mathsf{h}^2(X) \rangle_{\mathsf{Mot}}$. □

The next lemma shows that X is Hodge similar to a variety of Kum^3-type.

Lemma D.4.3. *For any projective variety X of Kum^n-type, there exists a projective variety K of Kum^3-type and a Hodge similarity of multiplier $n + 1$*

$$\phi\colon H^2(K, \mathbb{Q}) \xrightarrow{\sim} H^2(X, \mathbb{Q})$$

with respect to the Beauville-Bogomolov pairings.

Proof. By [7], the integral second cohomology group of a Kum^n-variety is identified with the lattice $\Lambda_{\mathrm{Kum}^n} = \mathrm{U}^{\oplus 3} \oplus \langle -2n - 2 \rangle$, where U is a hyperbolic plane. It is easy to define a rational similarity

$$\phi_n\colon \Lambda_{\mathrm{Kum}^3} \otimes \mathbb{Q} \xrightarrow{\sim} \Lambda_{\mathrm{Kum}^n} \otimes \mathbb{Q}$$

of multiplier $n + 1$. Explicitly, let e_i^n, f_i^n, $i = 1, 2, 3$ and ξ^n be a basis of Λ_{Kum^n}, where: e_i^n, f_i^n are isotropic and $(e_i^n, f_i^n) = 1$, the planes $\langle e_i^n, f_i^n \rangle$ and $\langle e_j^n, f_j^n \rangle$ are orthogonal for $i \neq j$, ξ^n has square $-2n - 2$ and it is orthogonal to each e_i^n and f_i^n. Then ϕ_n is defined via

$$e_i^3 \mapsto e_i^n, \quad f_i^3 \mapsto (n + 1) f_i^n, \quad \text{for } i = 1, 2, 3, \quad \xi^3 \mapsto \frac{1}{4} \xi^n.$$

Let X be a Kum^n-variety, and let $\eta\colon H^2(X, \mathbb{Z}) \xrightarrow{\sim} \Lambda_{\mathrm{Kum}^n}$ be an isometry. The Hodge structure on the left hand side is determined by its period $[\sigma] = \eta(H^{2,0}(X))$, which is an isotropic line in $\Lambda_{\mathrm{Kum}^n} \otimes \mathbb{C}$ such that $(\sigma, \bar{\sigma}) > 0$. Via the similarity ϕ_n^{-1}, we obtain the isotropic line $[\theta] := [\phi_n^{-1}(\sigma)]$ in $\Lambda_{\mathrm{Kum}^3} \otimes \mathbb{C}$, such that $(\theta, \bar{\theta}) > 0$. By the surjectivity of the period map [42], there exists a manifold K of Kum^3-type with period $[\theta]$, which means that there is an isometry $\eta'\colon H^2(K, \mathbb{Z}) \xrightarrow{\sim} \Lambda_{\mathrm{Kum}^3}$ mapping $H^{2,0}(K)$ to $[\theta]$. By construction, the composition $\eta^{-1} \circ \phi_n \circ \eta'$ gives an isomorphism of rational Hodge structures

$$\phi\colon H^2(K, \mathbb{Q}) \xrightarrow{\sim} H^2(X, \mathbb{Q}),$$

and hence ϕ is a rational Hodge similarity. Since X is projective, K is projective as well, thanks to Huybrechts' projectivity criterion [42]. □

We can now complete the proofs of our results.

Proof of Theorem D.1.1. Given the projective Kum^n-variety X, let K be the Kum^3-variety with a Hodge similarity $\phi\colon H^2(K,\mathbb{Q}) \xrightarrow{\sim} H^2(X,\mathbb{Q})$ given by Lemma D.4.3. By Theorem D.3.3, the morphism ψ is induced by an algebraic cycle on $K \times X$. Applying Theorem D.4.1 to K, we obtain the associated K3 surface S_K and a Hodge similarity $\psi\colon H^2_{\mathrm{tr}}(S_K,\mathbb{Q}) \xrightarrow{\sim} H^2_{\mathrm{tr}}(K,\mathbb{Q})$ induced by an algebraic cycle on $S_K \times K$. The composition of ψ with ϕ thus gives a rational Hodge similarity of multiplier $2n+2$

$$\Psi\colon H^2_{\mathrm{tr}}(S_K,\mathbb{Q}) \xrightarrow{\sim} H^2_{\mathrm{tr}}(X,\mathbb{Q})$$

of transcendental lattices, which is induced by an algebraic cycle on $S_K \times X$.

Consider the submotive $\mathsf{h}^2(X)$ of $\mathsf{h}(X)$. By Lefschetz (1,1) theorem, we have the decomposition $\mathsf{h}^2(X) = \mathsf{h}^2_{\mathrm{tr}}(X) \oplus \mathsf{h}^2_{\mathrm{alg}}(X)$ into transcendental and algebraic part (and similarly for S_K); the algebraic part is a sum of Tate motives $\mathsf{Q}(-1)$. Since Ψ is induced by an algebraic cycle on $S_K \times K$, it is the realization of a morphism

$$\widetilde{\Psi}\colon\ \mathsf{h}^2_{\mathrm{tr}}(S_K) \longrightarrow \mathsf{h}^2_{\mathrm{tr}}(X)$$

of motives. Recall that $\mathsf{h}^2(X)$ is a direct summand of $\mathsf{h}(X)^{\Gamma_n}$, and hence $\mathsf{h}^2(X)$ belongs to $\langle \mathsf{h}(M_X)\rangle_{\mathsf{Mot}}$. Therefore, $\widetilde{\Psi}$ is a morphism in the subcategory $\langle \mathsf{h}(S_K \times M_X)\rangle_{\mathsf{Mot}}$ of Mot. The standard conjectures hold for $S_K \times M_X$ by Foster's Theorem D.3.1. By conservativity (see Remark D.2.4), it follows that $\widetilde{\Psi}$ is an isomorphism of motives, since its realization Ψ is an isomorphism of Hodge structures. As $\mathsf{h}^2(X)$ is the sum of $\mathsf{h}^2_{\mathrm{tr}}(X)$ and Tate motives, we conclude that $\mathsf{h}^2(X) \in \langle \mathsf{h}(S_K)\rangle_{\mathsf{Mot}}$.

Consider now the submotive $\mathsf{a}_2(X) \subset \mathsf{h}(X)$ constructed in Lemma D.4.2. By the above, $\mathsf{a}_2(X)$ belongs to $\langle \mathsf{h}(S_K)\rangle_{\mathsf{Mot}}$. Since, by Theorem D.4.1, the Hodge conjecture holds for all powers of S_K, the realization functor R is full when restricted to $\langle \mathsf{h}(S_K)\rangle_{\mathsf{Mot}}$, and we deduce that any Hodge class in $A^\bullet_2(X) \subset H^\bullet(X,\mathbb{Q})$ is algebraic. □

Remark D.4.4. Let X be a Kum^n-variety as above and consider any power $Z = X^r$. Denote by $A^\bullet_2(Z)$ the subalgebra of $H^\bullet(Z,\mathbb{Q})$ generated by $H^2(Z,\mathbb{Q})$. Then our argument implies that all Hodge classes in $A^\bullet_2(Z)$ are algebraic. In fact, note that $A^\bullet_2(Z)$ is the graded tensor product $A^\bullet_2(X)^{\otimes r}$, because $H^1(X,\mathbb{Q})$ is zero. With notation as in the above proof, the argument given implies that $A^\bullet_2(Z)$ is the realization of a submotive $\mathsf{a}_2(Z)$ of $\mathsf{h}(Z)$, and moreover that $\mathsf{a}_2(Z) = \mathsf{a}_2(X)^{\otimes r}$ belongs to $\langle \mathsf{h}(S_K)\rangle_{\mathsf{Mot}}$. As the Hodge conjecture holds for all powers of S_K, it follows that any Hodge class in $A^\bullet_2(Z)$ is algebraic.

Proof of Corollary D.1.2. If X is of Kum^2-type, the complement of $A^\bullet_2(X)$ in $H^\bullet(X,\mathbb{Q})$ consists of the odd cohomology and of an 80-dimensional space of Hodge classes in $H^4(X,\mathbb{Q})$, by [57, Example 4.6]. The classes in this 80-dimensional subspace of the middle cohomology have the special property of remaining Hodge on any deformation of X, and Hassett and Tschinkel have shown in [40] that they are algebraic. Together with Theorem D.1.1, this implies that the Hodge conjecture holds for any X of Kum^2-type. □

Proof of Corollary D.1.3. As mentioned in the introduction, the Mumford-Tate conjecture has been proven for any hyper-Kähler variety X/k of known deformation type, by work of the first author [19], Soldatenkov [85] and of the first author with Fu and Zhang [22]. The final result may be found in [22, Theorem 1.18]. As a consequence, the Galois representations on $H^j_{\text{ét}}(X_{\bar{k}}, \mathbb{Q}_\ell)$ are semisimple, and the Tate conjecture for X/k is equivalent to the Hodge conjecture for $X_{\mathbb{C}}$ (see [66, Proposition 2.3.2]). Therefore, Corollary D.1.3 follows from Corollary D.1.2. □

References

[1] Salman Abdulali. Abelian varieties of type III and the Hodge conjecture. *Int. J. Math.*, 10(06):667–675, 1999.

[2] Nicolas Addington. On two rationality conjectures for cubic fourfolds. *Math. Res. Lett.*, 23(1):1–13, 2016.

[3] Yves André. On the Shafarevich and Tate conjectures for hyperkähler varieties. *Math. Ann.*, 305:205–248, 1996.

[4] Yves André. *Une introduction aux motifs (motifs purs, motifs mixtes, périodes)*. Number 17 in Panoramas et Synthèses. Société Mathématique de France, Paris, 2004.

[5] Donu Arapura. Motivation for Hodge cycles. *Adv. Math.*, 207(2):762–781, 2006.

[6] Benjamin Bakker and Christian Lehn. The global moduli theory of symplectic varieties. *J. reine angew. Math.*, 2022(790):223–265, 2022.

[7] Arnaud Beauville. Variétés Kähleriennes dont la premiere classe de Chern est nulle. *J. Differ. Geom.*, 18(4):755–782, 1983.

[8] Arnaud Beauville. Symplectic singularities. *Invent. Math.*, 139:541–549, 2000.

[9] Thorsten Beckmann. Derived categories of hyper-Kähler manifolds: extended Mukai vector and integral structure. *Compos. Math.*, 159(1):109–152, 2023.

[10] Samuel Boissière, Marc Nieper-Wißkirchen, and Alessandra Sarti. Higher dimensional Enriques varieties and automorphisms of generalized Kummer varieties. *J. Math. Pures Appl.*, 95(5):553–563, 2011.

[11] Nikolay Buskin. Every rational Hodge isometry between two K3 surfaces is algebraic. *J. reine angew. Math.*, 2019(755):127–150, 2019.

[12] Alberto Canonaco and Paolo Stellari. Twisted Fourier–Mukai functors. *Adv. Math.*, 212(2):484–503, 2007.

[13] François Charles. Two results on the Hodge structure of complex tori. *Math. Z.*, 300(4):3623–3643, 2022.

[14] François Charles and Eyal Markman. The standard conjectures for holomorphic symplectic varieties deformation equivalent to Hilbert schemes of K3 surfaces. *Compos. Math.*, 149(3):481–494, 2013.

[15] Alberto Collino. The Abel–Jacobi isomorphism for the cubic fivefold. *Pac. J. Math.*, 122(1):43–55, 1986.

[16] Pierre Deligne. La conjecture de Weil pour les surfaces K3. *Invent. Math.*, 15(3):206–226, 1971.

[17] Pierre Deligne. Variétés de Shimura: interprétation modulaire, et techniques de construction de modeles canoniques. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*, pages 247–289, 1979.

[18] Pierre Deligne. Hodge cycles on abelian varieties. In *Hodge cycles, motives, and Shimura varieties*, pages 9–100. Springer, 1982.

[19] Salvatore Floccari. On the Mumford–Tate conjecture for hyper-Kähler varieties. *Manuscripta Math.*, 168:893–916, 2022.

[20] Salvatore Floccari. The Hodge and Tate conjectures for hyper-Kähler sixfolds of generalized Kummer type. *arXiv preprint: 2308.02267*, 2023.

[21] Salvatore Floccari. Sixfolds of generalized Kummer type and K3 surfaces. *Compos. Math.*, 160(2):388–410, 2024.

[22] Salvatore Floccari, Lie Fu, and Ziyu Zhang. On the motive of O'Grady's ten-dimensional hyper-Kähler varieties. *Commun. Contemp. Math.*, 23(4), 2021.

[23] Salvatore Floccari and Mauro Varesco. Algebraic cycles on hyper-Kähler varieties of generalized Kummer type. *arXiv preprint arXiv:2308.04865*, 2023.

[24] Josiah Foster. The Lefschetz standard conjectures for IHSMs of generalized Kummer deformation type in certain degrees. *arXiv preprint arXiv:2303.14327*, 2023.

[25] Akira Fujiki. On Primitively Symplectic Compact Kähler V-Manifolds of Dimension Four. In Kenji Ueno, editor, *Classification of algebraic and analytic manifolds*, volume 39 of *Progr. Math.*, pages 71–250, 1983.

[26] Alice Garbagnati and Alessandra Sarti. Symplectic automorphisms of prime order on K3 surfaces. *J. Algebra*, 318(1):323–350, 2007.

[27] Alice Garbagnati and Alessandra Sarti. Kummer surfaces and K3 surfaces with $(\mathbb{Z}/2\mathbb{Z})^4$ symplectic action. *Rocky Mt. J. Math.*, 46(4):1141–1205, 2016.

[28] Bert van Geemen. An introduction to the Hodge conjecture for abelian varieties. In *Algebraic Cycles and Hodge Theory*, pages 233–252. Springer, 1994.

[29] Bert van Geemen. Theta functions and cycles on some abelian fourfolds. *Math. Z*, 221(4):617–631, 1996.

[30] Bert van Geemen. Kuga–Satake varieties and the Hodge conjecture. In *The arithmetic and geometry of algebraic cycles*, pages 51–82. Springer, 2000.

[31] Bert van Geemen. Real multiplication on K3 surfaces and Kuga–Satake varieties. *Mich. Math. J.*, 56(2):375–399, 2008.

[32] Bert van Geemen. Fourfolds of Weil type and the spinor map. *Expo. Math.*, 41(2):418–447, 2023.

[33] Bert van Geemen and Elham Izadi. Half twists and the cohomology of hypersurfaces. *Math. Z.*, 242(2):279–301, 2002.

[34] Bert van Geemen and Alessandra Sarti. Nikulin involutions on K3 surfaces. *Math. Z.*, 255(4):731–753, 2007.

[35] Bert van Geemen and Claire Voisin. On a conjecture of Matsushita. *Int. Math. Res. Not.*, 2016(10):3111–3123, 2016.

[36] Lothar Göttsche. *Hilbert schemes of zero-dimensional subschemes of smooth varieties*. Springer, 2006.

[37] Mark Green, Yoon-Joo Kim, Radu Laza, and Colleen Robles. The LLV decomposition of hyperkähler cohomology (the known cases and the conjectural behaviour). *Math. Ann.*, 382:1517–1590, 2021.

[38] Mark Gross, Daniel Huybrechts, and Dominic Joyce. *Calabi–Yau manifolds and related geometries: lectures at a summer school in Nordfjordeid, Norway, June 2001*. Springer Science & Business Media, 2012.

[39] Alexander Grothendieck. Standard conjectures on algebraic cycles. In *Algebraic geometry, Int. Colloq., Tata Inst. Res. Fund. (Bombay 1968)*, pages 193–199. Oxford Univ. Press, London, 1969.

[40] Brendan Hassett and Yuri Tschinkel. Hodge theory and Lagrangian planes on generalized Kummer fourfolds. *Mosc. Math. J.*, 13(1):33–56, 2013.

[41] Heisuke Hironaka. Resolution of singularities of an algebraic variety over a field of characteristic zero: II. *Ann. Math.*, pages 205–326, 1964.

[42] Daniel Huybrechts. Compact hyper-Kähler manifolds: basic results. *Invent. Math.*, 135(1):63–113, 1999.

[43] Daniel Huybrechts. A global Torelli Theorem for hyper-Kähler manifolds [after M. Verbitsky]. *Séminaire Bourbaki Juin 2011, Astérisque*, 348(1040):375–403, 2012.

[44] Daniel Huybrechts. *Lectures on K3 surfaces*, volume 158. Cambridge University Press, 2016.

[45] Daniel Huybrechts. Motives of isogenous K3 surfaces. *Comment. Math. Helv.*, 94(3):445–458, 2019.

[46] Daniel Huybrechts. *The geometry of cubic hypersurfaces*, volume 206. Cambridge University Press, 2023.

[47] Daniel Huybrechts and Manfred Lehn. *The Geometry of Moduli Spaces of Sheaves.* Cambridge University Press, 2^{nd} edition, 2010.

[48] Daniel Huybrechts and Paolo Stellari. Proof of Căldăraru's conjecture. *Moduli spaces and arithmetic geometry*, pages 31–42, 2006.

[49] Colin Ingalls, Adam Logan, and Owen Patashnick. Explicit coverings of families of elliptic surfaces by squares of curves. *Math. Z.*, 302(2):1191–1238, 2022.

[50] Uwe Jannsen. Motives, numerical equivalence, and semi-simplicity. *Invent. Math.*, 107(3):447–452, 1992.

[51] Steven L. Kleiman. *Algebraic cycles and the Weil conjectures.* Columbia Univ., Department of mathematics, 1968.

[52] Hanspeter Kraft and Claudio Procesi. Classical invariant theory, a primer. *Lecture Notes. Preliminary version*, 1996.

[53] Tobias Kreutz, Mingmin Shen, and Charles Vial. Around the de Rham–Betti conjecture. *arXiv preprint arXiv:2206.08618*, 2022.

[54] Michio Kuga and Ichirô Satake. Abelian varieties attached to polarized K3-surfaces. *Math. Ann.*, 169(1):239–242, 1967.

[55] Robert Laterveer. A family of cubic fourfolds with finite-dimensional motive. *J. Math. Soc. Jpn*, 70(4):1453–1473, 2018.

[56] Giuseppe Lombardo. Abelian varieties of Weil type and Kuga–Satake varieties. *Tohoku Math. J., Second Series*, 53(3):453–466, 2001.

[57] Eduard Looijenga and Valery A. Lunts. A Lie algebra attached to a projective variety. *Invent. Math.*, 129(2):361–412, 1997.

[58] Eyal Markman. Integral generators for the cohomology ring of moduli spaces of sheaves over Poisson surfaces. *Adv. Math.*, 208:622–646, 2007.

[59] Eyal Markman. On the monodromy of moduli spaces of sheaves on K3 surfaces. *J. Algebraic Geom.*, 17(1):29–99, 2008.

[60] Eyal Markman. The monodromy of generalized Kummer varieties and algebraic cycles on their intermediate Jacobians. *J. Eur. Math. Soc.*, 2022.

[61] Eyal Markman. Stable vector bundles on a hyper-Kahler manifold with a rank 1 obstruction map are modular. *arXiv preprint: 2107.13991*, 2022.

[62] Eyal Markman. Rational Hodge isometries of hyper-Kähler varieties of $K3^{[n]}$-type are algebraic. *Compos. Math.*, 160(6):1261–1303, 2024.

[63] Grégoire Menet. Global Torelli theorem for irreducible symplectic orbifolds. *J. Math. Pures Appl.*, 137:213–237, 2020.

[64] James S. Milne. The Tate and Standard Conjectures for Certain Abelian Varieties. *arXiv preprint arXiv:2112.12815*, 2021.

[65] Giovanni Mongardi and Malte Wandel. Induced automorphisms on irreducible symplectic manifolds. *J. Lond. Math. Soc.*, 92:123–143, 2015.

[66] Ben Moonen. Families of Motives and the Mumford-Tate Conjecture. *Milan J. Math.*, 85(2):257–307, 2017.

[67] Ben Moonen. On the Tate and Mumford–Tate conjectures in codimension 1 for varieties with $h^{2,0} = 1$. *Duke Math J.*, 166(4):739–799, 2017.

[68] Ben Moonen and Yuri G. Zarhin. Hodge classes and Tate classes on simple abelian fourfolds. *Duke Math. J.*, 77(3):553–581, 1995.

[69] Ben Moonen and Yuri G. Zarhin. Hodge classes on abelian varieties of low dimension. *Math. Ann.*, 4(315):711–733, 1999.

[70] David R. Morrison. On K3 surfaces with large Picard number. *Invent. Math*, 75:105–121, 1984.

[71] David R. Morrison. The Kuga–Satake variety of an abelian surface. *J. Algebra*, 92(2):454–476, 1985.

[72] Shigeru Mukai. On the moduli space of bundles on K3 surfaces. I. *Vector bundles on algebraic varieties (Bombay, 1984)*, 11:341–413, 1984.

[73] Viacheslav V. Nikulin. Finite groups of automorphisms of Kählerian K3 surfaces. *Trudy Moskov. Mat. Obshch.*, 38:75–137, 1979.

[74] Viacheslav V. Nikulin. Integral symmetric bilinear forms and some of their applications. *Math. USSR-Izv.*, 14(1):103, 1980.

[75] Viacheslav V. Nikulin. On correspondences between K3 surfaces. *Math. USSR-Izv.*, 30(2):375, 1988.

[76] Kieran O'Grady. Compact tori associated to hyperkähler manifolds of Kummer type. *Int. Math. Res. Not.*, 2021(16):12356–12419, 2021.

[77] Kapil Paranjape. Abelian varieties associated to certain K3 surfaces. *Compos. Math.*, 68(1):11–22, 1988.

[78] José Ramón-Marí. On the Hodge conjecture for products of certain surfaces. *Collect. Math.*, 59(1):1–26, 2008.

[79] Kenneth A. Ribet. Hodge classes on certain types of abelian varieties. *Am. J. Math.*, 105(2):523–538, 1983.

[80] Ichirô Satake. Holomorphic imbeddings of symmetric domains into a Siegel space. In *Proceedings of the Conference on Complex Analysis*, pages 40–48. Springer, 1965.

[81] Ulrich Schlickewei. The Hodge conjecture for self-products of certain K3 surfaces. *J. Algebra*, 324(3):507–529, 2010.

[82] Chad Schoen. Hodge classes on self-products of a variety with an automorphism. *Compos. Math.*, 65(1):3–32, 1988.

[83] Anthony J. Scholl. Classical motives. In *Motives (Seattle 1991), part 1, Proc. Symp. Pure Math*, volume 55, pages 163–187. American Mathematical Society, 1994.

[84] Yum-Tong Siu. Every K3 surface is Kähler. *Invent. Math.*, 73(1):139–150, 1983.

[85] Andrey Soldatenkov. Deformation principle and André motives of projective hyper-Kähler manifolds. *Int. Math. Res. Not.*, 2022(21):16814–16843, 2021.

[86] Lenny Taelman. Derived equivalences of hyperkähler varieties. *Geom. Topol.*, 27(7):2649–2693, 2023.

[87] Burt Totaro. Recent progress on the Tate conjecture. *Bull. Americ. Math. Soc.*, 54:575–590, 2017.

[88] Mauro Varesco. Hodge similarities, algebraic classes, and Kuga–Satake varieties. *Math. Z.*, 305(4):69, 2023.

[89] Mauro Varesco. The Hodge conjecture for powers of K3 surfaces of Picard number 16. *Mich. Math. J.*, Advance Publication:1–32, 2024.

[90] Mauro Varesco and Claire Voisin. Hyper-Kähler manifolds of generalized Kummer type and the Kuga–Satake correspondence. *Milan J. Math.*, pages 433–444, 2022.

[91] Misha Verbitsky. Cohomology of compact hyperkähler manifolds and its applications. *Geom. Func. Anal.*, 6(4):601–611, 1996.

[92] Misha Verbitsky. Mapping class group and a global Torelli theorem for hyper-Kähler manifolds. *Duke Math. J.*, 162(15):2929–2986, 12 2013.

[93] Misha Verbitsky. Errata for "Mapping class group and a global Torelli theorem for hyper-Kähler manifolds" by Misha Verbitsky. *Duke Math. J.*, 169(5):1037–1038, 04 2020.

[94] Claire Voisin. *Hodge Theory and Complex Algebraic Geometry I: Volume 1*, volume 76. Cambridge University Press, 2002.

[95] Claire Voisin. Hodge loci. *Handbook of moduli*, 3:507–546, 2010.

[96] Claire Voisin. Lectures on the Hodge and Grothendieck–Hodge conjectures. *Rend. Semin. Mat. Univ. Politec. Torino*, 69(2):149–198, 2011.

[97] Claire Voisin. Bloch's conjecture for Catanese and Barlow surfaces. *J. Differ. Geom.*, 97(1):149–175, 2014.

[98] Claire Voisin. Footnotes to papers of O'Grady and Markman. *Math. Z.*, 300(4):3405–3416, 2022.

[99] André Weil. Abelian varieties and the Hodge ring. In *Collected Papers*, volume III, pages 9–100. Springer, 1982.

[100] Kōta Yoshioka. Moduli spaces of stable sheaves on abelian surfaces. *Math. Ann.*, 321(4):817–884, 2001.

[101] Yuri G. Zarhin. Hodge group of K3 surfaces. *J. reine angew. Math*, 391:193–220, 1983.