Non-equilibrium situations and shear flows in kinetic theory and fluid mechanics

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Abstract

In this thesis two models are studied: the Boltzmann equation and the incompressible Euler-Poisson equation. Concerning the Boltzmann equation we analyze the longtime behavior of so-called homoenergetic solutions in specific situations. In addition, we prove the limit of the homogeneous Boltzmann equation with inverse power law interactions to the equation with hard spheres interactions. On the other hand, we construct rotating solutions to the incompressible Euler-Poisson equation.

In Chapter 1 we give an introduction into both models under consideration. We first give an introduction into the Boltzmann equation. We focus on previous mathematical results in particular concerning the Cauchy problem in various settings as well as on homoenergetic solutions. Then, we introduce the incompressible Euler-Poisson equation and give an overview of previous results on ellipsoidal figures of equilibrium. Furthermore, we summarize related results on models for stars and galaxies described by the compressible Euler-Poisson equation and the Vlasov-Poisson equation, respectively. In addition, we review some standard methods used in fluid mechanics, in particular in the study of stationary solutions to the Euler equations. Finally, we end the introduction with a summary of the main results of the thesis.

In Chapter 2 we give an overview of the results in the work (I) given in Appendix A. In this work the homoenergetic solutions to the Boltzmann equation for Maxwell molecules and shear is studied. We prove that solutions converge to a self-similar solution as time goes to infinity. In comparison with previous results we also cover the case of non-cutoff kernels.

In Chapter 3 we give a summary of the work (II) reproduced in Appendix B. Here we are concerned with the longtime behavior of homoenergetic solutions for hard potentials and shear. We prove that solutions close to equilibrium and with initially high enough temperature behave like a Maxwellian distribution with a time-dependent temperature. Furthermore, we also prove an asymptotics for the temperature as time goes to infinity.

In Chapter 4 we give an overview of the work (III) reproduced in Appendix C. This is joint work with Jin Woo Jang, Alessia Nota, and Juan J. L. Velázquez. We prove that the collision kernel for inverse power law potentials $1/r^{s-1}$ converges to the collision kernel for hard spheres interactions when $s \to \infty$. Furthermore, we show that solutions to the homogeneous Boltzmann equation with inverse power law interaction converge to solutions to the homogeneous Boltzmann equation with hard spheres interaction.

In Chapter 5 we summarize the results in the article (IV) given in Appendix D. It is joint work with Diego Alonso-Orán and Juan J. L. Velázquez. In this work we prove that the incompressible Euler-Poisson equation admits stationary solutions in a rotating frame of reference in two dimensions. More precisely, we consider a self-interaction fluid body which is perturbed by an external particle with small mass. The fluid body is close to the unit disk and contains a non-trivial velocity field. The velocity field is construct as a perturbation of a shear flow in the unperturbed domain, the unit disk.

Finally, in Chapter 6 we give some conclusive remarks as well as several open problems related to homoenergetic solutions and the incompressible Euler-Poisson equation.

Appendices A, B, C and D contain the accepted manuscript of the published version of the articles included in the thesis.

List of articles

The articles included in this thesis have been published in peer-reviewed journals:

(I) Bernhard Kepka. Self-similar Profiles for Homoenergetic Solutions of the Boltzmann Equation for Non-cutoff Maxwell Molecules, Journal of Statistical Physics, 190(2):27, 2022. DOI: 10.1007/s10955-022-03034-x

The accepted manuscript to the published version [105] is reproduced in Appendix A of the thesis. The preprint is available on arXiv:2103.10744.

(II) Bernhard Kepka. Longtime behavior of homoenergetic solutions in the collision dominated regime for hard potentials, Pure and Applied Analysis 6-2 (2024), 415–454. DOI: 10.2140/paa.2024.6.415

The accepted manuscript to the published version [106] is reproduced in Appendix B of the thesis. The preprint is available on arXiv.2202.09074.

(III) Jin Woo Jang, Bernhard Kepka, Alessia Nota, and Juan J. L. Velázquez. Vanishing Angular Singularity Limit to the Hard-Sphere Boltzmann Equation, Journal of Statistical Physics, 190(4):77, 2023. DOI: 10.1007/s10955-023-03089-4

The accepted manuscript to the published version [103] is reproduced in Appendix C of the thesis. The preprint is available on arXiv:2209.14075.

(IV) Diego Alonso-Orán, Bernhard Kepka and Juan J. L. Velázquez. Rotating solutions to the incompressible Euler-Poisson equation with external particle, Annales de l'Institut Henri Poincaré C, to appear – DOI: 10.4171/AIHPC/130, Preprint: arXiv.2302.01146.

The accepted manuscript of this article is reproduced in Appendix D of the thesis. The preprint to [9] is available on arXiv.2302.01146.

The following articles are not included in the thesis, but have been completed during the PhD study.

(V) Eugenia Franco, Bernhard Kepka, J. J. L. Velázquez. Description of Chemical Systems by means of Response Functions (2023),
Preprint: arXiv.2309.02021.

(VI) Eugenia Franco, Bernhard Kepka, J. J. L. Velázquez. Characterizing the detailed balance property by means of measurements in chemical networks (2024),
Preprint: arXiv.2402.12935.

(VII) Peter Gladbach, Bernhard Kepka. Variational interacting particle systems and Vlasov equations (2024),
Preprint: arXiv:2404.04350.

Contents

Chapter 1

Introduction

In this chapter we give an introduction of the two main models studied in this thesis: (1) the Boltzmann equation for dilute gases of kinetic theory and (2) the incompressible Euler-Poisson equation of fluid mechanics. In Section 1.1 we consider the first while in Section 1.2 the second model.

More precisely, in Section 1.1 we introduce the Boltzmann equation as well as basic properties. Furthermore, we give an overview of known results concerning both the Cauchy problem as well as the longtime behavior. In addition, we discuss the modeling of collisions, in particular the hard-sphere model and inverse power law models. The relation between these modeling approaches was studied in (III). Then, we introduce in detail the class of so-called *homoenergetic solutions*, which were studied in the papers (I), (II). We give an overview of previous results, in particular on the different possible longtime behaviors of homoenergetic solutions.

In Section 1.2 we introduce the incompressible Euler-Poisson equation and give an overview of rotating self-gravitating fluid bodies. Such a model in two dimension including an external particle was studied in (IV). Then, we draw the connection to models for stars and galaxies, i.e. to the compressible Euler-Poisson and Vlasov-Poisson equation. Here, we state previous results on the existence of steady states and rotating solutions. Finally, we give an overview of known techniques which have been applied to the incompressible Euler and Euler-Poisson equation, in particular those which have been used in this thesis.

1.1 Introduction to the Boltzmann equation

A fundamental model in collision kinetic theory of dilute gases is given by the Boltzmann equation. More precisely, particles in a gas are described in a statistical manner using a distribution function $f = f(t,x,v) \geq 0$. The only features in the gas captured by this description is the statistical distribution of particles at a certain time $t \geq 0$ at a given space $x \in \Omega$ ($\Omega \subset \mathbb{R}^3$ some domain) with a given velocity $v \in \mathbb{R}^3$. The only effects taken into account in the dynamics of the gas are collisions between the particles (or collisions with the boundary of the domain). The Boltzmann equation describes the evolution in time of the distribution function subject to some given initial distribution f_0, that is

$$\partial_t f + v \cdot \nabla_x f = Q(f,f), \quad f(0,x,v) = f_0(x,v). \tag{1.1.1}$$

The term $Q(f,f)$ accounts for the interaction due to collisions between the particles and is called collision operator. If one studies a gas in a domain then certain boundary conditions have to be taken into account as well.

The above equation has been introduced and studied by Boltzmann in [35], see also the treatise [37]. An equivalent form was already derived by Maxwell in [128]. The main assumptions in the derivation are

(i) *Free streaming:* Between collisions particles are assumed to move at constant speed. More precisely, the trajectory of a particle in phase space is given by $(x(t), v(t)) = (x(0) + v(0)t, v(0))$.

(ii) *Locality in space and time:* Collision events between particles are assumed to happen instantaneously and at one point in space. In particular, only the velocities are changed due to a collision event.

(iii) *Diluteness of gas:* Collision events are assumed to take place only between two particles. This reflects that the gas is rarefied / dilute enough, such that the probability of three or more particles coming close to each other at the same time is negligible.

(iv) *Molecular chaos:* Two particles colliding are assumed to be uncorrelated (more precisely, particles are independent and identically distributed with respect to f).

A non-rigorous derivation in the case of hard spheres (i.e. particles elastically collide like billiard balls) can be found for instance in [47, Section II.3]. It uses the fact that the N-particle distribution function is preserved along the Hamiltonian dynamics, which is typically referred to as Liouville's theorem. This entails the following general form of the collision operator

$$Q(f,f)(v) = \int_{\mathbb{R}^3}\int_{S^2} B(|v-v_*|, n\cdot\sigma)\left(f(v')f(v'_*) - f(v)f(v_*)\right) d\sigma dv_*, \quad n = \frac{v-v_*}{|v-v_*|}. \tag{1.1.2}$$

Here, we omitted the dependence on (t,x) in f since collision events are local in space and time. Furthermore, (v', v'_*) are the velocities after a collision of two particles with velocities (v, v_*). Assuming that collisions are elastic (i.e. the kinetic energy is preserved) and using the conservation of momentum one can derive the following representation of possible post-collisional velocities

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \quad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma, \quad \sigma \in S^2.$$

Let us mention that this representation is referred to as σ-representation. A different representation is given by the ω-representation, see [47]. Note that indeed we have conservation of momentum and kinetic energy

$$v + v_* = v' + v'_*, \quad \frac{1}{2}|v|^2 + \frac{1}{2}|v_*|^2 = \frac{1}{2}|v'|^2 + \frac{1}{2}|v'_*|^2. \tag{1.1.3}$$

In addition, in (1.1.2) the function $B(|v-v_*|, n\cdot\sigma) \geq 0$, termed *collision kernel*, depends on the modeling of collision events. It is a function of the relative velocity $|v-v_*|$ and the deviation angle θ via $n\cdot\sigma = \cos\theta$. We give more details on this function later on.

Let us mention that $Q(f,f)$ is quadratic in f due to the diluteness of the gas (only two particles are engaged in a collision event). Moreover it depends only on the tensor product $f(v)\otimes f(v_*)$, rather than the two-particle distribution function, due to the assumption of molecular chaos. The collision operator naturally splits into a gain and a loss term

$$Q_{\text{gain}}(f,f)(v) = \int_{\mathbb{R}^3}\int_{S^2} B(|v-v_*|, n\cdot\sigma) f(v')f(v'_*)\, d\sigma dv_*,$$
$$Q_{\text{loss}}(f,f)(v) = \int_{\mathbb{R}^3}\int_{S^2} B(|v-v_*|, n\cdot\sigma) f(v)f(v_*)\, d\sigma dv_*$$

taking into account the gain and loss of particles with velocity v due to collision events. Furthermore, the fact that on the microscopic level particles behave according to the laws of classical mechanics can be seen from the appearance of the free streaming operator $v \cdot \nabla_x$ and the σ-representation of post-collisional velocities. In particular, both are time-reversible, whereas the dynamics of the Boltzmann equation is not time-reversible.

Let us mention that the conservation of kinetic energy (1.1.3) underlies the assumption that collisions are elastic. However, it is possible to take into account the loss of kinetic energy due to for instance friction. The second equation in (1.1.3) is then relaxed to

$$\frac{1}{2}|v|^2 + \frac{1}{2}|v_*|^2 - \frac{1}{2}|v'|^2 - \frac{1}{2}|v_*'|^2 = \frac{1-e^2}{4}\frac{n \cdot \sigma - 1}{2} \leq 0.$$

Here, the parameter $e \in [0,1]$ is the restitution coefficient and measures the in-elasticity of collisions. We refer to the survey [159] and references therein for applications and the mathematical study of the inelastic Boltzmann equation. Inelastic collisions were also considered for homoenergetic solutions. However, in this thesis collisions are always elastic.

Collision kernel. The collision kernel $B(|v-v_*|, n \cdot \sigma)$ appearing in the collision operator depends on the model used to describe collision events. In this sense, it allows to tune which collisions are more probably than others.

There are two type of common models in the mathematical literature, see [47, 156]:

(i) *Hard spheres:* particles collide like billiard balls and the collision operator has the form

$$B(|v-v_*|, n \cdot \sigma) = |v-v_*|.$$

(ii) *Inverse power law interactions:* two particles interact with potential $1/r^{s-1}$, $s \in (2,\infty)$, and the collision operator has the form

$$B(|v-v_*|, n \cdot \sigma) = |v-v_*|^\gamma b_s(\cos\theta), \quad \gamma = \frac{s-5}{s-1}.$$

Here, we used spherical coordinates for $\sigma \in S^2$ with north pole n to write $n \cdot \sigma = \cos\theta$. The quantity $\theta \in [0,\pi]$ is called the deviation angle. A formal derivation of this formula from the study of the Hamiltonian dynamics induced by the inverse power law potential can be found in Section C.1.2, see also [47, Section II.5]. Due to the fact that the interaction potential $1/r^{s-1}$ is long-ranged the function $b_s(\cos\theta)$ has a singularity of the form

$$\sin\theta\, b_s(\cos\theta) \sim \theta^{-1-2/(s-1)}, \quad \theta \to 0. \tag{1.1.4}$$

The term $\sin\theta$ is included as it appears as a Jacobian when using spherical coordinates. In fact, since the potential $1/r^{s-1}$ is small, but does not vanish for large distances, two particles interact weakly in a collision event. Weak interactions correspond to small deviation angles $\theta \approx 0$.

In particular, due to the singularity the collision rate is infinite, i.e.

$$\int_{S^2} B(|v-v_*|, n \cdot \sigma) d\sigma = 2\pi |v-v_*|^\gamma \int_0^\pi b_s(\cos\theta) \sin\theta\, d\theta = \infty. \tag{1.1.5}$$

Let us mention that the case $s=2$, i.e. Coulomb interactions, is excluded. In fact, in this case the collision kernel is too singular. In this situation a different model, the so-called Landau equation, was proposed by Landau, see e.g. [114, Chapter IV]. Finally, as is proven in the work (III) when $s \to \infty$ the collision kernel converges to the hard sphere kernel.

Due to the above models it is customary in the mathematical literature to assume that the collision operator has the form

$$B(|v-v_*|, n\cdot\sigma) = |v-v_*|^\gamma b(\cos\theta), \quad n\cdot\sigma = \cos\theta,$$

for some non-negative function b and $\gamma \in (-3,1]$. The function b is often referred to as *angular part*, while $|v-v_*|^\gamma$ is called the *kinetic part* of the collision kernel. Furthermore, the following nomenclature is used.

(i) *Cutoff vs. non-cutoff:* if the function $\sin\theta\, b(\cos\theta)$ has a non-integrable singularity (like in (1.1.5)) one refers to non-cutoff kernels. Otherwise one refers to a cutoff kernel. Cutoff kernels are also said to satisfy the Grad's cutoff assumption. In this case, one assumes $b(n\cdot\sigma) \in L^1(S^2)$ or even $b(n\cdot\sigma)\in L^\infty(S^2)$.

(ii) *Homogeneity γ:* depending on γ one says that the underlying interaction potential is hard if $\gamma > 0$ or soft if $\gamma < 0$. Moreover, if $\gamma = 0$ one refers to Maxwellian molecules. Note that for inverse power laws $1/r^{s-1}$ we have the cases: $s > 5$ (hard potentials), $s = 5$ (Maxwellian molecules) and $s < 5$ (soft potentials).

Let us mention that the mathematical analysis strongly varies in each cases. In general, non-cutoff kernels are more involved due to the singular behavior. Furthermore, hard potentials turn out to be simpler than soft potentials. On the other hand, the fact that the collision kernel does not depend on $|v-v_*|$ for Maxwell molecules ($\gamma = 0$) leads to several simplifications in the analysis.

Observables. In statistical physics the most relevant objects are given by macroscopic, physically measurable observables. In terms of the distribution function these are integrals of the form

$$\int_{\mathbb{R}^3} \varphi(t,x,v) f(t,x,v)\, dv$$

for some test function $\varphi(t,x,v)$. Using the fact that the mapping $(\sigma, v, v_*) \to (n, v', v'_*)$ is bijective with unit Jacobian and $|v-v_*| = |v'-v'_*|$, $(v-v_*)\cdot\sigma = (v'-v'_*)\cdot n$ one obtains

$$\int_{\mathbb{R}^3} \varphi(v)\, Q(f,f)(v)\, dv = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} B(|v-v_*|, n\cdot\sigma) f f_* \left(\varphi' - \varphi\right) d\sigma dv_* dv$$

The above change of variables is usually referred to as pre-post-collisional change of variables. Here, we use the common abbreviation $\varphi' = \varphi(v')$ and $f_* = f(v_*)$. Taking also into account the change of variables $(v, v_*) \mapsto (v_*, v)$ in combination with the pre-post-collisional change of variables one obtains

$$\int_{\mathbb{R}^3} \varphi(v)\, Q(f,f)(v)\, dv = \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} B(|v-v_*|, n\cdot\sigma) f f_* \left(\varphi' + \varphi'_* - \varphi_* - \varphi\right) d\sigma dv_* dv.$$

As a consequence of (1.1.3) we have

$$\int_{\mathbb{R}^3} \varphi(v)\, Q(f,f)(v)\, dv = 0, \quad \text{for} \quad \varphi(v) = 1, v_1, v_2, v_3, \frac{1}{2}|v|^2.$$

Observables $\varphi = \varphi(v)$ satisfying the above equation are called *collision invariants*. In fact, all collision invariants are a linear combination of $1, v_1, v_2, v_3, |v|^2$, see e.g. [51, Section 3.1].

Let us mention that these collision invariants allow to derive macroscopic conservation laws for the local density $\rho(t,x)$, local momentum $V(t,x)$ and local energy $e(t,x)$, see [47, Section II.8], where

$$\begin{aligned}\rho(t,x) &= \int_{\mathbb{R}^3} f(t,x,v)\,dv, \quad \rho(t,x)V(t,x) = \int_{\mathbb{R}^3} v f(t,x,v)\,dv,\\ \rho(t,x)e(t,x) &= \frac{1}{2}\int_{\mathbb{R}^3} |v-V(t,x)|^2 f(t,x,v)\,dv.\end{aligned} \tag{1.1.6}$$

Let us recall that the local energy $e(t,x)$ is related to the local temperature $T(t,x)$ via the formula $e(t,x) = \frac{3}{2}k_B T(t,x)$, where $k_B = 1.380649 \cdot 10^{-23}\,\mathrm{J}\mathrm{K}^{-1}$ is the Boltzmann constant.

***H*-Theorem.** An important result of Boltzmann's analysis in [35, 36] was the derivation of the second law of thermodynamics, that is the entropy in a closed thermodynamical system is increasing in time. Moreover, the system evolves towards a Maxwellian equilibrium distribution. This is now referred to as H-Theorem. A mathematical derivation of all equilibrium distributions to the Boltzmann equation is given in [61]. Let us mention only the main steps.

The negative entropy is given by

$$H(f(t)) = \int_\Omega \int_{\mathbb{R}^3} f(t,x,v)\ln f(t,x,v)\,dvdx.$$

One can then deduce

$$\frac{d}{dt}H(f(t)) = \int_\Omega \int_{\mathbb{R}^3} Q(f,f)(t,x,v)\ln f(t,x,v)\,dvdx.$$

Exploiting the symmetry of the collision operator and applying the pre-post-collisional change of variables one obtains

$$\begin{aligned}&\frac{d}{dt}H(f(t)) = -D(f(t)),\\ &D(f(t)) = \frac{1}{4}\int_\Omega \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} B(|v-v_*|, n\cdot\sigma)\left(f'f'_* - ff_*\right)\left(\ln(f'f'_*) - \ln(ff_*)\right)dv_*dvdx.\end{aligned}$$

Note that the so-called entropy dissipation $D(f(t))$ is non-negative, since $x \mapsto \ln x$ is an increasing function. Thus, the negative entropy is non-increasing. Furthermore, f_0 is a time-independent solution if and only if $D(f) = 0$. This implies that $\ln f_0(x,v)$ is a collision invariant for (almost all) $x \in \Omega$ and thus

$$f_0(x,v) = \frac{\rho(x)}{(2\pi T(x))^{3/2}}\exp\left(-\frac{1}{2}\frac{|v-V(x)|^2}{T(x)}\right).$$

Functions of this form are called *local Maxwellian* and satisfy $Q(f_0,f_0) = 0$. On the other hand, in order for f_0 to be a stationary solution to the Boltzmann equation (1.1.1) one needs $v\cdot\nabla_x f_0 = 0$ including appropriate boundary conditions. One can deduce that this implies the functional form

$$f_0(x,v) = \frac{\rho_0}{(2\pi T_0)^{3/2}}e^{-|v|^2/2T_0 - 2\Lambda_0 x\cdot v}$$

with constants $\rho_0, T_0 > 0$ and a skew-symmetric matrix $\Lambda_0 \in \mathbb{R}^{3\times 3}$. Let us mention that the appearance of Λ_0 depends on the boundary conditions and most importantly on the symmetry properties of the boundary $\partial\Omega$. It takes into account for a non-zero angular momentum around an axis of symmetry of Ω. If Ω has no axis of symmetry $\Lambda_0 = 0$ and one recovers a *homogeneous* Maxwellian distribution.

1.1.1 Overview of mathematical results

In this subsection we give an overview of results in the mathematical literature. We refer to [51] and [156] for a more detailed overview of the mathematical theory. We will split this exposition in the study of homogeneous solutions, i.e. x-independent solutions, and general inhomogeneous solutions.

Cauchy problem. In the case of the homogeneous Boltzmann equation, that is

$$\partial_t f = Q(f,f), \quad f(0,v) = f_0(v), \tag{1.1.7}$$

a well-established formulation of weak solutions is the following. We say that $f = f(t,v)$ is a solution to the homogeneous Boltzmann equation if for any regular enough test function $\varphi : \mathbb{R}^3 \to \mathbb{R}$ and $t \geq 0$ we have

$$\begin{aligned}\int_{\mathbb{R}^3} \varphi(v) f(t,v)\, dv = &\int_{\mathbb{R}^3} \varphi(v) f_0(v)\, dv \\ &+ \frac{1}{2}\int_0^t \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} |v-v_*|^\gamma b(n\cdot\sigma)\left(\varphi' + \varphi'_* - \varphi_* - \varphi\right) f(s,v) f(s,v_*)\, d\sigma dv_* dv\, ds.\end{aligned}$$

The weak form can be derived from the symmetry properties of the collision kernel, i.e. applying pre-post-collisional change of variables as well as the transformation $(v,v_*) \mapsto (v_*,v)$. This weak form can be used for any collision kernel. However it is in particular convenient for non-cutoff kernels, since it allows to deal with the angular singularity. More precisely, for non-cutoff kernels one assumes

$$\Lambda := \int_0^\pi \theta^2 b(\cos\theta)\sin\theta\, d\theta < \infty. \tag{1.1.8}$$

Note that this is satisfied for inverse power law potentials, see (1.1.4). Under this condition and assuming that $f(t,v)$ has finite moments of order $|v|^{2+\gamma}$, the last integral in the weak formulation is well-defined due to the estimate

$$\left|\int_{S^2} b(n\cdot\sigma)\left\{\varphi'_* + \varphi' - \varphi * - \varphi\right\} d\sigma\right| \leq C \left\|D^2\varphi\right\|_\infty \Lambda |v-v_*|^2. \tag{1.1.9}$$

Let us give a proof of this bound. To this end, we use spherical coordinates based on the orthonormal basis (e_1, e_2, n), where $n = (v-v_*)/|v-v_*|$. In particular,

$$\sigma = (\cos\phi\sin\theta, \cos\phi\sin\theta, \cos\theta).$$

By a Taylor expansion we obtain

$$\begin{aligned}\varphi(v'_*) - \varphi(v_*) + \varphi(v') - \varphi(v) =& [\nabla\varphi(v_*) - \nabla\varphi(v)] \cdot \left(\frac{v-v_*}{2} - \frac{|v-v_*|}{2}\sigma\right) \\ &+ \mathcal{O}\left(\left\|D^2\varphi\right\|_\infty |v-v_*|^2\theta^2\right), \quad \theta \to 0.\end{aligned} \tag{1.1.10}$$

Here, we used

$$\begin{aligned}\left|\frac{v-v_*}{2} - \frac{|v-v_*|}{2}\sigma\right|^2 &= |v-v_*|^2 \left|\frac{1-\cos\theta}{2} n + \sin\theta\frac{\sin\phi}{2} e_1 + \sin\theta\frac{\cos\phi}{2} e_2\right|^2 \\ &= |v-v_*|^2 \mathcal{O}(\theta^2), \quad \theta \to 0,\end{aligned}$$

with $1-\cos\theta = \mathcal{O}(\theta^2)$ as $\theta \to 0$. Note that the first order term in (1.1.10) has the form

$$\frac{|v-v_*|}{2}\left[(1-\cos\theta)n + \sin\theta\sin\phi e_1 + \sin\theta\cos\phi e_2\right].$$

In particular, when integrating in ϕ over $(0,2\pi)$ the last two terms vanish, i.e. we have

$$\left|\int_0^{2\pi}\{\varphi(v'_*)-\varphi(v_*)+\varphi(v')-\varphi(v)\}\,d\phi\right| \le C\left\|D^2\varphi\right\|_\infty |v-v_*|^2\theta^2.$$

This is combined with the assumption (1.1.8) to yield the asserted estimate (1.1.9). Let us remark here that for non-cutoff kernels it is essential to combine both the gain term Q_{gain} and the loss term Q_{loss} of the collision operator to make use of cancellations. Each term on its own would not be well-defined. This is one major difficulty in the analysis of non-cutoff kernels. For cutoff kernels one can treat gain and loss term separately without any ambiguities.

The general approach is then to use the conservation of mass and energy, yielding bounds in the weighted spaces $L^1(1+|v|^2)$ as well as the entropy dissipation estimate. The latter allows to prove that any approximation sequence is compact in L^1 due to the Dunford-Pettis theorem. One can then pass to the limit in the weak formulation. Indeed, the weak formulation is very robust even if merely weak convergence is available. This also allows to treat non-cutoff kernels by approximating such kernels by cutoff ones.

Alternatively, one can construct measure-valued solutions. To this end, the compactness is not due to the Dunford-Pettis theorem but due to moment bounds and the Prokhorov theorem (a sequence of tight probability measures is compact with respect to weak convergence). Again the weak formulation allows to pass to the limit.

Let us mention the following results which strongly depend on the assumptions on the collision kernel.

(i) *Hard potentials:* The Cauchy problem of weak solutions was first studied by [14, 15]. This includes the study of non-cutoff kernels by making use of cancellations of $\varphi' + \varphi'_* - \varphi_* - \varphi$ when the deviation angle goes to zero. This can be used to construct also measure-valued solutions, see [121]. The study of moments as well as uniqueness results are established in [129]. The given conditions for uniqueness are sharp due to the results in [122, 161]. Corresponding uniqueness results for non-cutoff kernels can be found in [62]. The study of the moments uses the so-called Povzner estimates. These allow to show that solutions have finite moments of any order for positive times, even if they are infinite initially. This is due to the strong interaction for large velocities, since the collision kernel scales like $|v|^\gamma$ and $\gamma > 0$ for hard potentials.

(ii) *Soft potentials:* For $\gamma \ge -2$ the study of [14, 15] can be adapted to yield weak solutions. However, in the case $\gamma \in (-3,-2)$ this is no longer the case due to the singularity with respect to $|v-v_*|$. A different formulation of so-called H-solutions was introduced in [155] covering situations in which $\gamma \in (-4,-2)$. This also allowed to treat the limit $s \to 2$ for inverse power law potentials in order to make the limit towards the Landau equation rigorous (grazing collision limit). In comparison to the hard potential case, moment estimates are still available by means of the Povzner estimates, see e.g. [45, 151]. However, the creation of moments are no longer valid. Finally, conditional uniqueness results are provided in [62] using analytical methods and in [67] using probabilistic methods.

(iii) *Maxwell molecules:* As was observed first in [154] it is possible to write a closed system of ODEs for the moments of the solution to the homogeneous Boltzmann equation with

Maxwell molecules. This allowed for an implicit study of the corresponding distribution. In particular, moments are finite for $t>0$ if and only if they are finite initially at time $t=0$, in contrast to the hard potential case. On the other hand, a successful tool in the analysis of the equation is the Fourier transform with respect to the velocity variable, which was first introduced by Bobylev in [27, 28]. Uniqueness results have been proved in [150] via the Fourier transform approach for non-cutoff kernels. Concerning the longtime behavior one remarkable feature of the collision operator is that it is contractive with respect to several metrics. In this direction Fourier-based metrics are studied in [68, 150]. On the other hand, a probabilistic treatment showing the contraction property with respect to the 2-Wasserstein distance was done in [149], see also [158, Section 7.5] for a proof from the point of view of optimal transport.

We remark that a lot of the properties of the Boltzmann equation for Maxwell molecules mentioned here have been exploited in the work (I).

Let us mention that regularity properties of solutions strongly differ for cutoff and non-cutoff kernels. For cutoff kernels there is no regularization effect in time. Nevertheless, the gain term $Q_+(f,f)$ of the collision operator turns out to be regularizing, as was first observed by Lions in [118, 119, 120]. Roughly speaking, the operator $f\mapsto Q_+(f,f)$ acts like a convolution. The regularity of solutions for hard potentials was then studied in [137]. Regularity results for (smoothed) soft potentials are given in [151].

On the other hand, for non-cutoff kernels the collision operator has a regularizing effect similar to a fractional diffusion as was observed first in [3]. Corresponding regularity results for hard and soft potentials are given in [54, 86].

Let us now mention results concerning the Cauchy problem of the inhomogeneous Boltzmann equation. First of all, solutions for short times have been constructed under a variety of assumptions on the collision kernel, see e.g. [4, 90, 104]. Short time solutions are also provided by the derivation of the Boltzmann equation due to Lanford [109], see also [73]. In the perturbative regime two cases have been studied.

(i) *Close to vacuum:* Perturbations of the zero solution have been constructed in [88, 93], see also [47].

(ii) *Close to equilibrium:* Solutions close to global equilibrium $\mu(v)=e^{-|v|^2/2}/(2\pi)^{3/2}$ have been studied under very general assumptions on the collision kernel. Two types of perturbations have been considered.

   (a) Solutions of the form $f=\mu+\sqrt{\mu}h$, where h lies in some functional space with polynomial decay. Thus the solution has Maxwellian decay at infinity. As is common in perturbative studies the most important operator is the linearization. In this case the linearized collision operator has the form

$$Lh=-\frac{1}{\sqrt{\mu}}\left(Q(\mu,\sqrt{\mu}h)+Q(\sqrt{\mu}h,\mu)\right).$$

   This operator is mostly considered on the space $L^2(\mathbb{R}^3)$. It has the kernel

$$\ker L=\operatorname{span}\left\{\sqrt{\mu},v_1\sqrt{\mu},v_2\sqrt{\mu},v_3\sqrt{\mu},|v|^2\sqrt{\mu}\right\}.$$

   This follows from the collision invariants (mass, momentum and energy) for the nonlinear collision operator. Furthermore, one can show that this operator is non-negative and self-adjoint. The operator L has been studied in [21, 87, 107, 133, 136, 138], in particular under which conditions on the singularity of the collision kernel and its homogeneity γ it has a spectral gap. Sharp regularity estimates in combination with corresponding nonlinear estimates of the collision operator allowed to construct solutions close to equilibrium, see e.g. [5, 78].

(b) Solutions of the form $f = \mu + h$ with h in some functions space with polynomial decay have been constructed only recently. In comparison to (a) the linearization now takes the form $\mathscr{L}h = -(Q(\mu,h) + Q(h,\mu))$. The kernel is given by

$$\ker \mathscr{L} = \operatorname{span}\left\{\mu,\, v_1\mu,\, v_2\sqrt{\mu},\, v_3\mu, |v|^2\mu\right\}.$$

If considered on the space $L^2(\mu^{-1/2})$ it coincides with the operator L. However, in the perturbative setting the error term h has merely polynomial decay, so that the operator is defined on some larger space, say $L^1(|v|^p)$ for $p > 2$. On this space the operator is no longer self-adjoint and hence the study of a spectral gap becomes ambiguous. Instead the decay properties of $e^{-\mathscr{L}t}$ for $t \to \infty$ was studied based on the behavior of e^{-Lt} for $t \to \infty$ in [79, 134, 152] . For hard potentials this was used to construct solutions in [91]. Soft potentials have been considered in [43]. These results have been applied in the work (II).

Let us mention that in the study of the linearized operator a general method introduced in [79] was used. In fact, the main problem is to extend decay properties of a linear semigroup from a small functional space (say $L^2(\mu^{-1/2})$) to a larger one (say $L^1(|v|^p)$). The generator of the semigroup is typically not self-adjoint on the larger space. This extension is then possible under certain conditions on this linear operator (factorization condition), see Section 3.2 for a more detailed discussion as well as [79].

(iii) *Weakly inhomogeneous solutions:* Perturbations of homogeneous solutions have been studied first in [16] on the whole space and more recently in [80] on bounded domains.

Let us now mention that large data solutions to the inhomogeneous Boltzmann equation have been constructed by diPerna and Lions in [64]. To this end, the concept of renormalized solutions was introduced. Extensions to non-cutoff collision kernels are given in [7]. This also allowed to derive the Landau equation from the Boltzmann equation in the inhomogeneous setting [8].

Finally, the smoothness of solutions under certain regularity conditions on the macroscopic quantities (local density, local energy and local entropy) was established recently, see [94] and references therein.

Trend to equilibrium. The H-Theorem provides a formal proof for the trend to equilibrium and therefore was used as a basis for most studies of the qualitative behavior for large times. A first step was the quantification of the entropy dissipation with respect to the entropy itself, i.e. the study of entropy-entropy-dissipation inequalities, see [157] and references therein. For the homogeneous Boltzmann equation this allowed to prove the trend to equilibrium, see e.g. [45, 137, 151]. In the case of the inhomogeneous Boltzmann equation this was established under conditional regularity bounds in [63]. This study evolved into the general theory of hypercoercivity, see e.g. [160].

Let us mention that in the case of the homogeneous Boltzmann equation with Maxwell molecules other proofs of the trend to equilibrium were provided using the Wasserstein metric [139, 149] and Fourier metrics [68, 150].

1.1.2 Homoenergetic solutions

In this section we give an introduction to homoenergetic solutions and an overview of known results. To this end, let us consider as in [98] solutions of the form

$$f(t,x,v) = g(t,w), \quad w = v - \xi(t,x) \tag{1.1.11}$$

with $\xi : [0,\infty) \times \mathbb{R}^3 \to \mathbb{R}^3$. Plugging this into the Boltzmann equation and assuming that g belongs to a large class of functions one necessarily obtains

$$D_x \xi = 0, \quad \partial_t \xi + (\xi \cdot \nabla)\xi = 0.$$

In particular, $\xi(t,x) = L(t)x + v_0(t)$ with

$$\frac{d}{dt}L(t) + L(t)^2 = 0, \quad \frac{d}{dt}v_0(t) + L(t)v_0(t) = 0. \tag{1.1.12}$$

The first equation has the solution $L(t) = L(0)(I + tL(0))^{-1}$. Furthermore, the function g satisfies the equation

$$\partial_t g - L(t)w \cdot \nabla_w g = Q(g,g)(w), \tag{1.1.13}$$

where the Boltzmann collision operator only acts in the variable w. Let us mention that one can assume with out loss of generality that $v_0(t) \equiv 0$ by using the change of variables $w \mapsto w - v_0(t)$, which leaves the equation (1.1.13) invariant.

Solutions to the Boltzmann equation (1.1.1) of the functional form (1.1.11) are called *homoenergetic solutions*. In [98] also the name *equidispersive solutions* was used. Note that the following properties hold.

(i) The density $\rho(t,x)$, internal energy per unit mass $e(t,x)$, the stress tensor $p(t,x)$ and the heat flux $q(t,x)$ do depend only on time but not on the position variable. Recall the definition in (1.1.6) as well as

$$p(t,x) = \int_{\mathbb{R}^3} (v - V(t,x))^\top (v - V(t,x)) f(t,x,v)\, dv,$$
$$q(t,x) = \int_{\mathbb{R}^3} (v - V(t,x)) |v - V(t,x)|^2 f(t,x,v)\, dv.$$

(ii) The bulk velocity is given by $V(t,x) = \xi(t,x)$.

In [47, Section VIII.8] the above observations are given as the defining properties of homoenergetic solutions. In fact, extending property (i) by assuming that all moments in $v - V(t,x)$ only depend on time suggest the functional form (1.1.11). Let us mention that another interpretation based on symmetry properties on the level of molecular dynamics was introduced in [58, 59], see also [98].

Let us now give an overview of the literature on homoenergetic solutions. First of all, they were introduced by Truesdell [153] and Galkin [71]. They have been studied further by Galkin in [69, 70, 72]. Furthermore, for Maxwell molecules the system of ODEs of the moments were analyzed in [154]. Later the particular case of shear flow was studied in [74] also for the so-called BGK approximation and mixtures of gases.

Let us now turn to the more recent mathematical literature concerning the Cauchy problem as well as the qualitative behavior for large times.

Cauchy problem. As one can see from (1.1.13) the equation satisfied by the profile g is very much reminiscent of the homogeneous Boltzmann equation up to the drift term on the left hand side. In fact, this is the reason why the Cauchy problem is similar to the homogeneous Boltzmann equation. In particular, the existence of solutions can be proven using the a priori bounds given by the moments as well as the entropy. However, note that the entropy is no longer monotone in time. Nevertheless, it gives an a priori bound yielding compactness in L^1 by means of the Dunford-Pettis theorem. Following the general idea used for the homogeneous equation the first well-posedness result was given in [48]. Measure-valued solutions were constructed in [98], see also [26] and (I) for the case of Maxwell molecules. The well-posedness as well as the regularity for non-cutoff hard potentials was provided in (II).

Longtime behavior. Concerning the longtime behavior the situation is very much different from the homogeneous equation. In fact, the dynamics in the homogeneous equation approaches the equilibrium due to the entropy dissipation. More generally, the existence of an entropy is a consequence of the fact that at equilibrium every collision process is balanced. This property is also referred to as detailed balance. This is no longer the case for homoenergetic solutions as the system has non-zero fluxes of energy due to the forces acting on the gas described by the matrix $L(t)$.

A systematical study of the qualitative behavior for large times was initiated in the works [97, 98, 99]. First of all, in [98, Section 3] the asymptotics of $L(t)$ as $t\to\infty$ was classified using the Jordan normal form and orthonormal changes of variables under the assumption $\det(I+tL(0))>0$ for all times $t\geq 0$. For convenience let us state here the corresponding result, cf. [98, Theorem 3.1]: assuming $L(0)\neq 0$ the matrix $L(t)=L(0)(I+tL(0))^{-1}$ can admit the following forms as $t\to\infty$.

(i) Homogeneous dilation:

$$L(t)=\frac{1}{t}I+\mathcal{O}\left(\frac{1}{t^2}\right).$$

Looking at the characteristic equation of $-L(t)w\cdot\nabla_w$, shows that in this case velocities of the particle are dilated, i.e. the characteristics have the form $v(t)=v(1)/t$, for $t\geq 1$, up to lower order terms.

(ii) Cylindrical dilation:

$$L(t)=\frac{1}{t}\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}+\mathcal{O}\left(\frac{1}{t^2}\right).$$

From the characteristics one can see that here velocities are dilated only in the v_1v_2-plane.

(iii) Cylindrical dilatation with shear:

$$L(t)=\frac{1}{t}\begin{pmatrix}1&0&K\\0&1&0\\0&0&0\end{pmatrix}+\mathcal{O}\left(\frac{1}{t^2}\right),\quad K\neq 0.$$

In this case, velocities are dilated in the v_1v_2-plane and a shear in the v_1v_3-plane is present as follows from the solution to the characteristic equations

$$\begin{pmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{pmatrix} = \begin{pmatrix} v_1(1)/t - Kv_3(1)(t-1)^2/2t \\ v_2(1)/t \\ v_3(1) \end{pmatrix}, \quad t \geq 1,$$

up to lower order terms.

(iv) Planar shear:

$$L(t) = \frac{1}{t}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & K \\ 0 & 0 & 1 \end{pmatrix} + \mathcal{O}\left(\frac{1}{t^2}\right), \quad K \in \mathbb{R}$$

Here, a dilation in the v_3-direction together with a shear in the v_2v_3-plane occurs.

(v) Simple shear:

$$L(t) = \begin{pmatrix} 0 & K & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad K \neq 0.$$

In this case, a shear in the v_1v_2-plane occurs.

(vi) Simple shear with decaying planar dilatation/shear:

$$L(t) = \begin{pmatrix} 0 & K_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{t}\begin{pmatrix} 0 & K_1K_3 & K_1 \\ 0 & 0 & 0 \\ 0 & K_3 & 1 \end{pmatrix} + \mathcal{O}\left(\frac{1}{t^2}\right), \quad K_2 \neq 0, \quad K_1, K_3 \in \mathbb{R}$$

Here, to highest order a shear occurs, while on the next order a planar dilation or a shear appears.

(vii) Combined orthogonal shear:

$$L(t) = \begin{pmatrix} 0 & K_3 & K_2 - tK_1K_3 \\ 0 & 0 & K_1 \\ 0 & 0 & 0 \end{pmatrix}, \quad K_1K_3 \neq 0, \quad K_1, K_2, K_3 \in \mathbb{R}.$$

In this case the characteristics system of $-L(t)w \cdot \nabla_w$ have the form

$$\begin{pmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{pmatrix} = \begin{pmatrix} v_1(0) - K_2v_3(0)t - K_3(v_2(0) - K_1v_3(0)t)t \\ v_2(0) - K_1v_3(0)t \\ v_3(0) \end{pmatrix}, \quad t \geq 0,$$

Here, two simple shears are combined. One is appearing in the v_1v_2-plane and the other in the v_1v_3-plane for $v_2(0) - K_1K_3v_3(0)t = 0$.

Besides the asymptotic form of $L(t)$ there is also the homogeneity γ of the collision kernel that plays a crucial role. Roughly speaking, the reason for this is the scaling behavior of the collision operator $Q(g,g)(w)$ with respect to w. To be more precise, let us rescale the mass $\rho(t) = \int g(t,w)dw$ of the solution to one, so that we obtain for $G = g/\rho$ the equation

$$\partial_t G = \mathrm{div}_w(L(t)wG) + \rho(t)\,Q(G,G), \quad \rho(t) = \rho(0)\exp\left(-\int_0^t \mathrm{tr}\,L(s)ds\right). \tag{1.1.14}$$

Here, $\mathrm{tr}\,L$ denotes the trace of the matrix L. In terms of physical dimensions the drift term and the collision term have the scaling behavior $L(t)[G]$ and $\rho(t)[G][w]^\gamma$, respectively.

Let us now give an overview of the three different regimes that have been identified.

(i) *Self-similar asymptotics*: In the case of Maxwell molecules, that is $\gamma = 0$, for simple and planar shear one can see that $L(t)[G]$ and $\rho(t)[G][w]^\gamma$ have the same order. This suggest to look for profiles balancing both terms.

Indeed, it was proved in [26, 98] that there exists a self-similar solution to the following model equation

$$\partial_t G = \mathrm{div}_w(AwG) + Q(G,G), \quad A \in \mathbb{R}^{3\times 3} \tag{1.1.15}$$

Here, the matrix A is time-independent, in contrast to the matrix $L(t)$ above. Furthermore, the self-similar profile is unique (up to scaling of the energy) and gives the longtime asymptotics of all measure-valued weak solutions of (1.1.15) in self-similar variables. A crucial assumption was the smallness of A in order to consider the hyperbolic term as a perturbation of the collision operator. In addition, they restricted to kernels satisfying Grad's angular cutoff assumption. In [66], under similar assumptions the existence and uniqueness of this self-similar solution of (1.1.15) has been proved in a smooth setting close to Maxwellian for uniform shear flow (i.e. simple shear in the terminology of [98]). In particular, the self-similar profile is smooth. Furthermore, in the work (I) these results are extended to non-cutoff kernels. In this case, the smoothness of the self-similar profiles is a consequence of the regularizing behavior of the collision kernel. Finally, this analysis was then applied to prove the longtime asymptotics for simple and planar shear in (I). Let us mention that the analysis of the works [26] and (I) strongly rely on the structure of the Boltzmann collision operator for Maxwell molecules and in particular on the usage of the Fourier transform method. Furthermore, let us remark that the necessity to use self-similar variables comes from the fact that the temperature is preserved by the collision operator, but not by the drift term. In fact, the kinetic energy behaves like $e^{\beta t}$, $\beta \in \mathbb{R}$, so that the self-similar solution has the form

$$G(t,w) = e^{-3\beta t} G_{st}\left(\frac{w}{e^{\beta t}}\right).$$

Here, G_{st} is the self-similar profile. The parameter β is now used to balance the change of temperature, more precisely the change of the stress tensor. To this end, β is chosen as an eigenvalue of the second order moment equations for the stress tensor. Thus, we construct a self-similar solution of the second kind. We refer to [22, Chapter 4] for the definition of self-similar solutions of the first and second kind.

Finally, let us mention that self-similar solutions are constructed in [49, 50] in the case of shear flow for the inelastic Boltzmann equation, see also the survey [159].

(ii) *Collision-dominated behavior:* As in the previous case a crucial quantity for the longtime asymptotics is the temperature $T(t)$ of the gas. Thus, we rescale the function

$G(t,w) = F(t,\beta(t)^{1/2}w)\beta(t)^{3/2}$ where $\beta(t) = T(t)^{-1}$ is the inverse temperature. This yields the equation

$$\partial_t F = \operatorname{div}_w \left(\left(L(t)w - \frac{\beta'(t)}{2\beta(t)} w \right) F \right) + \rho(t)\beta(t)^{-\gamma/2} Q(F,F).$$

In the regime described now the collision operator is the dominant term compared to the drift term, that is $\rho(t)\beta(t)^{-\gamma/2}$ is much larger than $L(t)$ as $t \to \infty$. Since collisions are the most important effect, the profile approaches equilibrium. However, the corresponding Maxwellian distribution has a time-dependent density and most importantly a time-dependent temperature. In order to give an asymptotics of the temperature a Hilbert-type expansion was used in [97], that is the solutions is written as a perturbation of the Maxwellian. Using the linearization of the collision operator it is possible to compute higher order perturbations explicitly to yield a self-consistent asymptotics for the temperature under the assumption that $\rho(t)\beta(t)^{-\gamma/2}$ is larger than $L(t)$ as $t \to \infty$. Here, depending on the matrix $L(t)$ both $\gamma > 0$ and $\gamma < 0$ are possible. In the work [97] only the Hilbert-type expansion was computed to yield a formal asymptotics. The rigorous proof in the case of hard potentials for simple shear, simple shear with decaying planar dilatation/shear and combined orthogonal shear was provided in the work (II).

Let us mention that the shear flow in the gas is essential, since this leads to an increase of the velocity in the gas and thus the temperature. In contrast to this, if there is a dilation in the gas, the temperature decreases. Nevertheless, for $\gamma < 0$ (soft potentials) we have $\beta(t)^{-\gamma/2} \to \infty$ and a similar asymptotics can be obtained as for the hard potential case. However, the case of soft potentials was not yet analyzed rigorously.

(iii) *Hyperbolic-dominated behavior*: Here, a similar rescaling as in (ii) is used. However, now the choice of the matrix $L(t)$ yields that the divergence term is more dominant compared to the collision operator. In this regime several cases have to be treated independently and only vague conjectures have been formulated in [99]. There is no rigorous study available at the moment.

To end this section, let us stress that the ramification of the large time behvior is a consequence of the absence of an entropy dissipation formula (H-theorem). This is due to the energy fluxes induced into the system by the mechanical forces acting on the gas. In this way, the above behavior describes non-equilibrium situations.

1.2 Introduction to the Euler-Poisson equation

In this section we introduce the second model studied in this thesis, that is the incompressible Euler-Poisson equations. In the context studied here this equation is used as an oversimplified model for a self-gravitating body, e.g. a star or a planet. The body is assumed to be composed of an incompressible fluid. The body itself is surrounded by vacuum. Since the fluid can move freely the boundary of the body is in principle unknown. Thus the problem is a free-boundary problem. We give more information on this in Section 1.2.1.

The main goal of the study here is to consider a fluid body close to the unit ball and perturb it by an external particle. The whole system, fluid body and external particle, is assumed to rotate around the common center of mass. However, due to the gravitational force between the body and the particle the free-boundary changes. The main issue is to prove that such a configuration exists, i.e. there is a periodic solution to the Euler-Poisson equation. The solution is periodic in time in the sense that the whole configuration is rotating. More precisely, we consider solutions which are stationary in a rotating frame of reference.

As mentioned previously the incompressible Euler-Poisson equation can be viewed as an oversimplified model of a star. On the other hand, an established model for stars is given by the compressible Euler-Poisson equation. A kinetic model related to this model used for galaxies, that is the Vlasov-Poisson equation. In fact, the study of stationary and periodic solutions to these two models is well-established. Periodic solutions are studied in a similar way as explained above. The star or the galaxy is stationary in a rotating frame of reference. However, as we will see in the overview below these solutions have a very simple structure, more precisely in a certain frame of reference (in the rotating frame for periodic solutions) the velocity field of the star and the bulk velocity of the galaxy is zero, respectively. Thus, there is no (macroscopic) movement in the star or galaxy.

The goal in the study of the incompressible Euler-Poisson equation is to construct more general solutions with internal motion of the fluid.

1.2.1 Incompressible Euler-Poisson equation

Let us introduce the general problem considered here. As mentioned above we study a fluid body $E \subset \mathbb{R}^n$ for $n = 2$ or $n = 3$. The fluid is described by the velocity field $v = v(t,x) \in \mathbb{R}^n$. The equations are a combination of the incompressible Euler equation and the Poisson equation yielding the gravitational field. More precisely, we have

$$\begin{cases} \partial_t v + (v \cdot \nabla)v = -\nabla p - \nabla U_{E(t)}, & \text{in } E(t), \\ \nabla \cdot v = 0, & \text{in } E(t), \\ n_{\partial E(t)} \cdot v = V_N(t), & \text{on } \partial E(t), \\ p = 0 & \text{on } \partial E(t), \\ \Delta U_{E(t)} = \mathbb{1}_{E(t)}, & \text{in } \mathbb{R}^n. \end{cases} \tag{1.2.1}$$

Let us give some comments on the equations. The first equation is the Euler equation containing the so-called material derivative $\partial_t v + (v \cdot \nabla)v$, the pressure $p = p(t,x) \in \mathbb{R}$ and the interaction potential $U_{E(t)}$. The pressure can be viewed as a Lagrange parameter due to the divergence-free constraint $\nabla \cdot v = 0$. The interaction potential $U_{E(t)}$ is self-induced by the body $E(t)$ and solves the Poisson equation $\Delta U_{E(t)} = \mathbb{1}_{E(t)}$. In fact, one can write the solution using the fundamental solution in the form

$$\begin{aligned} U_{E(t)}(x) &= \frac{1}{2\pi} \int_{E(t)} \ln|x-y| \, dy, \quad \text{for } n = 2, \\ U_{E(t)}(x) &= -\frac{1}{4\pi} \int_{E(t)} \frac{dy}{|x-y|}, \quad \text{for } n = 3. \end{aligned}$$

Finally, in the equation $n_{\partial E(t)} \cdot v = V_N(t)$ the outer unit normal vector $n_{\partial E(t)}$ of the boundary $\partial E(t)$ together with the normal velocity $V_N(t)$ of the boundary $\partial E(t)$ appears. This constraint ensures that the fluid remains inside the domain. The normal velocity of the boundary can be computed using for instance local charts of the boundary, computing the time-derivative and taking the projection onto the normal vector $n_{\partial E(t)}$. Since a change of the domain corresponds to fluid particles moving across the domain this should coincide with the projection of the velocity field onto the normal vector, that is $n_{\partial E(t)} \cdot v$. Finally, the condition $p = 0$ ensures that the pressure is constant along the boundary $\partial E(t)$. Since the body is surrounded by vacuum with constant pressure (and without loss of generality equal to zero), this ensures that p is continuous.

Ellipsoidal figures of equilibrium. The model of self-gravitating fluid bodies, described by (1.2.1), has been extensively studied during several centuries, in particular by Newton, Maclaurin, Jacobi, Dedekind, Riemann and Poincaré. An overview of these results can be found in [52]. Let us give some details concerning these results. First of all, the three-dimensional case $n=3$ was studied. The main problem is to find rotating solutions to the equations (1.2.1). More precisely, in a rotating frame of reference the solution is stationary. Furthermore, the main restriction concerns the domain, which is assumed to be ellipsoidal.

To give an idea let us indicate the solutions constructed by Maclaurin. To this end, we let $\Omega=\Omega_0(0,0,1)^\top$, $\Omega_0\in\mathbb{R}$, be the axis of rotation and write (1.2.1) in a rotating frame of reference at angular speed Ω_0. We obtain the equations

$$\begin{cases} \partial_t u+(u\cdot\nabla)u+2\Omega\times u+\Omega\times(\Omega\times x)=-\nabla p-\nabla U_{E(t)}, & \text{in } E(t),\\ \nabla\cdot u=0, & \text{in } E(t),\\ n_{\partial E(t)}\cdot u=V_N(t), & \text{on } \partial E(t),\\ p=0 & \text{on } \partial E(t),\\ \Delta U_{E(t)}=\mathbb{1}_{E(t)}, & \text{in } \mathbb{R}^n. \end{cases} \tag{1.2.2}$$

The velocity in the rotating frame of reference is given by u. Note that the term $2\Omega\times u$ is the Coriolis force whereas the term $\Omega\times(\Omega\times x)$ is the centrifugal force. We now look for solutions which are stationary in such a frame of reference, that is, we look for solutions to

$$\begin{cases} (u\cdot\nabla)u+2\Omega\times u+\Omega\times(\Omega\times x)=-\nabla p-\nabla U_E, & \text{in } E,\\ \nabla\cdot u=0, & \text{in } E,\\ n_{\partial E}\cdot u=0, & \text{on } \partial E,\\ p=0 & \text{on } \partial E,\\ \Delta U_E=\mathbb{1}_E, & \text{in } \mathbb{R}^n. \end{cases} \tag{1.2.3}$$

Note that since the domain E is now time-independent the normal velocity of the domain vanishes, i.e. $V_N(t)\equiv 0$.

The first simplification of Maclaurin is to assume that there is no internal motion that is $u=0$. Furthermore, as mentioned above the domain is assumed to be ellipsoidal, i.e.

$$E=\left\{x\in\mathbb{R}^3\ :\ \sum_{i=1}^3\frac{x_i^2}{a_i^2}=1\right\}.$$

Here, $a_1,a_2,a_3>0$ are the semi-axes. The mass is assumed to be fixed and hence $a_1a_2a_3=\mathit{const}.$. The above equations reduce to

$$\begin{cases} \Omega\times(\Omega\times x)=-\nabla p-\nabla U_E, & \text{in } E,\\ p=0 & \text{on } \partial E,\\ \Delta U_E=\mathbb{1}_E, & \text{in } E. \end{cases}$$

It follows from the Poisson equation that $\nabla U_E(x)=-\mathcal{G}x$ when $x\in E$ for some matrix $\mathcal{G}\in\mathbb{R}^{3\times3}$ which depends on a_1,a_2,a_3. Using $\Omega\times(\Omega\times x)=\Omega_0^2Px$, $P=I-e_3\otimes e_3$, we have

$$\nabla p=\mathcal{G}x+\Omega_0^2Px.$$

This yields together with $p=0$

$$p(x)=p_0\left(1-\sum_{i=1}^{3}\frac{x_i^2}{a_i^2}\right)$$

for $x\in E$ and some $p_0\in\mathbb{R}$. In total we have the system of equations

$$\mathcal{G}+\Omega_0^2P=-2p_0A^{-2},\quad a_1a_2a_3=const.,\quad A=\begin{pmatrix}a_1&0&0\\0&a_2&0\\0&0&a_3\end{pmatrix}.$$

Assuming that the ellipsoid is symmetric, i.e. $a_1=a_2$, one can show that these equations are satisfied if the angular velocity Ω_0 is chosen appropriately. More precisely, the angular velocity is given via the eccentricity of the ellipsoid $e=1-a_3^2/a_1^2$

$$\Omega_0^2=G(e),$$

for some function G, as was proved by Maclaurin. It turns out that Ω_0 is incresaing with e until a maximal value is reached and then it decreases to zero as $e\to 1$.

As was proved by Jacobi, there are non-symmetric ellipsoidal solutions (i.e. $a_1\neq a_2$), which bifurcate from the ones by Maclaurin. On the other hand, the above solutions can also be constructed assuming a constant vorticity, but removing the rotation $\Omega=0$, as was done by Dedekind. Let us finally mention that Riemann construed solutions including both a constant vorticity $\omega\in\mathbb{R}^3$ and a rigid rotation. Due to the constraint that the body is ellipsoidal one can show that

(i) either ω and Ω are parallel and they lie on one coordinate axis,

(ii) or they are both in a principal plane, i.e. in the x_1x_2- or x_2x_3- or x_1x_3-plane.

Furthermore, as was shown by Poincaré, when the angular velocity Ω_0 increases bifurcations to pear-shaped figures occur. We refer to [52] for the connection of all mentioned solutions and corresponding bifurcations. Further studies were done by Lichtenstein, see [112].

Finally, let us mention that, except the solutions of Riemann, the above constructed rotating figures have zero internal motion either in the rotating or non-rotating coordinate system.

Including external particle. We now turn to the model considered in the work (IV). We study the problem (1.2.1) including an external particle $X(t)$ with mass $m>0$. The equations in a rotating frame of reference have the form

$$\begin{cases}\partial_t u+(u\cdot\nabla)u+2\Omega\times u+\Omega\times(\Omega\times x)=-\nabla p-\nabla U_{E(t)}-m\nabla U_{X(t)}, & \text{in } E(t),\\ \nabla\cdot u=0, & \text{in } E(t),\\ n_{\partial E(t)}\cdot u=V_N(t), & \text{on } \partial E(t),\\ p=0 & \text{on } \partial E(t),\\ \Delta U_{E(t)}=\mathbb{1}_{E(t)}, & \text{in } \mathbb{R}^n,\\ \dfrac{d^2X}{dt^2}+2\Omega\times\dfrac{dX}{dt}+\Omega\times(\Omega\times X)=-\nabla U_{E(t)}(X). & \end{cases}\tag{1.2.4}$$

The first equation includes the gravitational force acting on the fluid body due to the particle $X(t)$. This is given by

$$\begin{aligned} U_{X(t)}(x) &= \frac{1}{2\pi}\ln|x-X(t)|, \quad \text{for } n=2,\\ U_{X(t)}(x) &= -\frac{1}{4\pi}\frac{1}{|x-X(t)|}, \quad \text{for } n=3. \end{aligned}$$

The last equation in (1.2.4) is Newton's equation of motion for the external particle in the gravitational field of the fluid body in a rotating frame of reference.

We now assume that the whole configuration, fluid body and particle, rotate around their common center of mass and is stationary in a rotating frame of reference. Thus, we look for solutions to the equations

$$\begin{cases} (u\cdot\nabla)u + 2\Omega\times u + \Omega\times(\Omega\times x) = -\nabla p - \nabla U_E - m\nabla U_X, & \text{in } E,\\ \nabla\cdot u = 0, & \text{in } E,\\ n_{\partial E}\cdot u = 0, & \text{on } \partial E,\\ p = 0 & \text{on } \partial E,\\ \Delta U_E = \mathbb{1}_E, & \text{in } \mathbb{R}^n,\\ \Omega\times(\Omega\times X) = -\nabla U_E(X). \end{cases} \tag{1.2.5}$$

Furthermore, we impose that the center of mass is at the origin, that is

$$\int_E x\,dx + mX = 0.$$

In addition, one needs to give a constraint on the total mass of the fluid body $|E|$.

In the work (IV) the two-dimensional case $n=2$ was studied. More precisely, under certain assumptions the uniqueness and existence of solutions close to the configuration $E=\mathbb{D}$ was proved. Moreover, more general internal motions of the fluid body are considered in comparison with the above discussed ellipsoidal figures. In fact, we construct a velocity field close to a solution for $E=\mathbb{D}$. More precisely, the unperturbed velocity field v_0 is a shear flow with circular flow lines, i.e.

$$v_0(x) = f_0(|x|)\begin{pmatrix}-x_2\\ x_1\end{pmatrix}$$

for some function $f_0:\mathbb{R}\to\mathbb{R}$. Furthermore, the solutions constructed are classical. We refer to Chapter 5 for more details.

1.2.2 Models for stars and galaxies

In this section we discussed related models to the one presented in the previous section. Here, we give a small overview of the compressible Euler-Poisson equation and the Vlasov-Poisson equation. The first one is used as a model for stars while the other is used to model galaxies. We refer to [25] for more information concerning the modeling in astrophysics.

The Euler-Poisson equation describes the behavior of a star in terms of the density $\rho = \rho(t,x)\geq 0$ and the velocity field $v=v(t,x)\in\mathbb{R}^3$. We stick here to the three-dimensional case.

The equations are given by

$$\begin{aligned} \partial_t \rho + \operatorname{div}(\rho v) &= 0, \\ \rho \partial_t v + \rho (v \cdot \nabla) v &= -\nabla [p(\rho)] - \rho \nabla U_\rho, \\ U_\rho(x) &= -\int_{\mathbb{R}^3} \frac{\rho(y)\, dy}{|x-y|}. \end{aligned} \tag{1.2.6}$$

In contrast to the incompressible Euler-Poisson equation (1.2.1) the compressible Euler-Poisson equation contains also the density ρ. Accordingly, $E(t) = \{\rho(t,\cdot) > 0\}$ is the domain of the fluid. Thus, the density also describes the free-boundary which is given by $\partial\{\rho(t,\cdot) > 0\}$. The first equation in (1.2.6) is the continuity equation. The density is transported along the flow of the velocity field v. The second equation is the Euler equation coupled with the gravitational potential U_ρ induced by the body. Here, we used the fundamental solution to write the gravitational potential. Let us mention that with the constant chosen above the potential solves $\Delta U_\rho = 4\pi\rho$. Furthermore, the pressure $p = p(\rho)$ is assumed to be given via the density. The function $\rho \mapsto p(\rho)$ is typically chosen as a power law of the form $p(\rho) = \rho^s$ for $s > 1$. Note that the velocity field is no longer assumed to be incompressible. Hence, the pressure does not appear as a Lagrange parameter.

In contrast to the previous model the Vlasov-Poisson equation is a kinetic model. As in the case of the Boltzmann equation (Section 1.1) a statistical description via a probability density $f = f(t,x,v) \geq 0$ is used. The particles however are now for instance stars in a galaxy, which interact via gravitational forces. The equation has the form

$$\begin{aligned} \partial_t f + v \cdot \nabla_x f - \nabla_x U_\rho \cdot \nabla_v f &= 0 \\ U_\rho(t,x) = -\int_{\mathbb{R}^3} \frac{\rho(t,y)\, dy}{|x-y|}, \quad \rho(t,x) &:= \int f(t,x,v)\, dv. \end{aligned} \tag{1.2.7}$$

Here, the density ρ is given via the distribution f by integrating with respect to the velocity variable. In contrast to the Boltzmann equation, here no collision are present. In fact, a formal estimation shows that collision events can be neglected due to the Newtonian interaction, see [25, Section 1.2]. The Vlasov-Poisson equation is a transport equation. In particular, the density f is transported along the flow of the characteristic system

$$\frac{dX}{dt} = V, \quad \frac{dV}{dt} = -\nabla U_\rho(X).$$

The characteristic system is in fact a Hamiltonian system with respect to the Hamiltonian $H(x,v) = \frac{1}{2}|v|^2 + U_\rho(x)$. The interaction of the particles is contained in the self-induced potential U_ρ. Since the particles do not interact with each particle individually, but only through the averaged field induced by the potential U_ρ, the field $-\nabla U_\rho$ is referred to as mean-field. For an introduction to the mathematical theory of the Vlasov-Poisson equation we refer to [143]. Let us mention that the Vlasov-Poisson equation is also used to model collisionless plasmas. To this end, the sign of the potential U_ρ needs to be changed, see [114, Chapter III]

Steady states. In the following we give a short overview of the large classes of steady states which have been constructed for both the compressible Euler-Poisson and the Vlasov-Poisson equation. As explained below the steady states of the Vlasov-Poisson equation mentioned here are very much related to steady states of the Euler-Poisson equation. Thus, we will discuss first the case of the Vlasov-Poisson equation.

In the study of gravitational systems the most relevant solutions have compact support, i.e. galaxies are confined in a certain region. Furthermore, due to the fact that the characteristic system is the Hamiltonian flow to the particle energy $E(x,v) = \frac{1}{2}|v|^2 + U_\rho(x)$ functions of the form

$$f(x,v) = F(E(x,v))$$

yield stationary solutions to (1.2.7). Here, the function $F : \mathbb{R} \to \mathbb{R}$ needs to be chosen such that the potential U_ρ coincides with the potential induced by the density, i.e. we need to ensure that

$$\Delta U_\rho(x) = 4\pi\rho(x) = 4\pi \int F(E(x,v))\, dv = 16\pi^2 \int_{U_\rho(x)}^{\infty} F(E)\sqrt{2(E - U_\rho(x))}\, dE.$$

Here, we used the change of variables $|v| = \sqrt{2(E - U_\rho(x))}$.

In order to solve the above problem we give an overview of a variational technique introduced by Guo and Rein [82], see also [143]. Since the Vlasov-Poisson equation is a transport equation, functionals of the form

$$\mathcal{C}(f) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \Phi(f(t,x,v))\, dv dx$$

are conserved over time. Furthermore, the total energy

$$\mathcal{E}(f(t)) = \mathcal{E}_{\mathrm{kin}}(f(t)) + \mathcal{E}_{\mathrm{pot}}(f(t)) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{1}{2}|v|^2 f(t,x,v)\, dv dx + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{1}{2} U_\rho(t,x) f(t,x,v)\, dv dx$$

is conserved. Note that $\mathcal{E}_{\mathrm{kin}}(f(t))$ is the kinetic energy and $\mathcal{E}_{\mathrm{pot}}(f(t))$ the potential energy. Using the Poisson equation $\Delta U_\rho = 4\pi\rho$ one can rewrite

$$\mathcal{E}_{\mathrm{pot}}(f(t)) = \frac{1}{2}\int_{\mathbb{R}^3} U_\rho(t,x)\rho(t,x)\, dx = -\frac{1}{8\pi}\int_{\mathbb{R}^3} |\nabla U_\rho(t,x)|^2\, dx.$$

The main idea of the method in [143] is to look for minimizers of the functional

$$\begin{aligned}\mathcal{H}(f) &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \Phi(f(t,x,v))\, dv dx + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{1}{2}|v|^2 f(t,x,v)\, dv dx - \frac{1}{8\pi}\int_{\mathbb{R}^3} |\nabla U_\rho(t,x)|^2\, dx \\ &= \mathcal{C}(f) + \mathcal{E}_{\mathrm{kin}}(f) + \mathcal{E}_{\mathrm{pot}}(f).\end{aligned}$$

The function $\Phi : \mathbb{R} \to \mathbb{R}$ is chosen such that methods from The Calculus of Variations can be applied, most importantly the Direct Method. This function is also referred to as Casimir function. For instance the following assumptions appear, cf. [143, Section 2.1.4],

(i) $\Phi \in C^1([0,\infty))$ with $\Phi(0) = \Phi'(0) = 0$,

(ii) Φ is strictly convex,

(iii) $\Phi(f) \geq C f^{1+1/k}$ for $f \geq 0$ large, where $k \in (0,3/2)$,

(iv) $\Phi(f) \leq C f^{1+1/k'}$ for $f \geq 0$ small, where $k' \in (0,3/2)$.

An example is given by $\Phi(f) = f^{1+1/k}$ for $k \in (0,3/2)$.

The minimization of $\mathcal{H}$ has to be performed under the constraints

$$f \geq 0, \quad \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(x,v)\, dx dv = M.$$

Here, $M>0$ is the total mass of the distribution. In fact, the Euler-Lagrange equation yields

$$f_0(x,v)=(\Phi')^{-1}\left(E_0-E(x,v)\right)_+. \tag{1.2.8}$$

Here, the positive part $()_+$ appears due to the non-negativity constraint. Furthermore, the constant $E_0<0$ is given via the mass constraint. Since $E(x,v)$ are positive for $|x|, |v|\to\infty$ the function f_0 is compactly supported. Furthermore, one can prove the following properties, cf. [143, Proposition 2.7] :

(i) The induced density ρ_0 and the corresponding potential $U_{\rho_0}=U_0$ are spherically symmetric and ρ_0 is a decreasing function of $|x|$.

(ii) $U_0\in C^2(\mathbb{R}^3)$ with $U(x)\to 0$ as $|x|\to\infty$ and $\rho_0\in C^1_c(\mathbb{R}^3)$.

(iii) The function f_0 is spherically symmetric with respect to both x and v.

Let us now discuss the proof of the existence of minimizers to the problem

$$I_M:=\inf\left\{\mathcal{H}(f)\,:\,f\in L^1(R^6),\,f\geq 0,\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}f(x,v)\,dxdv=M,\,\mathcal{E}_{\text{kin}}(f)+\mathcal{C}(f)<\infty\right\}.$$

Note that in the functional $\mathcal{H}$ the term due to the potential energy has a negative sign. In particular, cancellations might occur with other terms. However, due to the generalized Young convolution inequality we obtain

$$|\mathcal{E}_{\text{pot}}(f)|\leq C\,\|\rho\|_{L^{6/5}}\leq C\,\|\rho\|^{\alpha}_{L^1}\,\|\rho\|^{1-\alpha}_{L^{1+1/k}}\leq CM\left(\mathcal{C}(f)\right)^{1-\alpha},$$

for some $\alpha=\alpha(k)\in(0,1)$. In particular, the functional is bounded from below and any minimizing sequence is bounded in $L^{1+1/k}(\mathbb{R}^6)\cap L^1(\mathbb{R}^6)$.

The next step is a reduction to a problem without any velocity dependence. To this end, we define a new Casimir function through

$$\Psi(r)=\inf_{f\in\mathcal{G}_r}\left\{\int_{\mathbb{R}^3}\Phi(f(v))\,dv+\int_{\mathbb{R}^3}\frac{1}{2}|v|^2f(v)\,dv\right\},$$

where the constraints are given by

$$\mathcal{G}_r=\left\{f\in L^1(\mathbb{R}^3)\;:\;f\geq 0,\int_{\mathbb{R}^3}f(v)=r,\int_{\mathbb{R}^3}\Phi(f(v))\,dv+\int_{\mathbb{R}^3}\frac{1}{2}|v|^2f(v)\,dv<\infty\right\}.$$

We then have the reduced functional for the densities $\rho=\rho(x)$

$$\mathcal{H}_r(\rho):=\int_{\mathbb{R}^3}\Psi(\rho(x))\,dx+\mathcal{E}_{\text{pot}}(\rho),\quad \mathcal{E}_{\text{pot}}(\rho)=-\frac{1}{8\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\rho(x)\rho(y)}{|x-y|}\,dxdy,$$

and the reduced problem

$$R_M:=\inf\left\{\mathcal{H}_r(\rho)\;:\;\rho\in L^1(\mathbb{R}^3),\,\rho\geq 0,\int_{\mathbb{R}^3}\rho(x)\,dx=M,\int_{\mathbb{R}^3}\Psi(\rho(x))\,dx<\infty\right\}.$$

One can see from the definitions that $I_M\geq R_M$. In addition, one can show the following statements, see [143, Theorem 2.2]:

(i) If f_0 is a minimizer of $\mathcal{H}$ then we have $I_M=R_M$.

(ii) If ρ_0 is a minimizer to $\mathcal{H}_r$ then

$$\rho(x) = (\Psi')^{-1}(E_0 - U_{\rho_0}(x))_+$$

and f_0 given by (1.2.8) is a minimizer to $\mathcal{H}$.

In particular, by (ii) we can reduce the problem I_M to R_M. The above reduction allows to choose for any concentration $\rho(x)$ the energetically most convenient distribution in velocity space, i.e. we have

$$\int_{\mathbb{R}^3} \left\{ \Phi(f(v)) + \frac{1}{2}|v|^2 f(v) \right\} dv \geq \Psi(\rho(x))$$

for $\int_{\mathbb{R}^3} f(v)\, dv = \rho(x)$. In fact, for given $\rho(x)$ minimizers have the form

$$f_0(v) = (\Phi')^{-1} \left(\lambda(x) - \frac{1}{2}|v|^2 \right)_+ . \tag{1.2.9}$$

Here, the parameter $\lambda(x)$ is chosen to yield the density $\rho(x)$. One can interpret this type of distributions as Gibbs distributions. These are reminiscent of the Gaussian/Maxwellian distributions which in fact appear in the case $\Phi(f) = f \ln f - f + 1$ of the classical Boltzmann entropy.

Finally, we need to show the existence of minimizers of the reduced problem R_M. To this end, we note that the function Ψ satisfies similar properties as Φ. Most importantly the growth condition at infinity $\Psi(f) \geq C f^{1+1/n}$, $n = k + 3/2$, see [143, Lemma 2.3]. This again allows to show that R_M is bounded from below and any minimizing sequence is bounded in $L^{1+1/n} \cap L^1$. However, due the translation invariance of the problem and the fact that the domain is the whole space $\mathbb{R}^3$ one cannot infer any useful compactness result from this alone. At this stage one can use the so-called *concentration compactness method* due to Lions, see [116, 117]. This method allows to show that a subsequence (ρ_n) of a minimizing sequence remains concentrated (tight) up to a translation, that is there are $x_n \in \mathbb{R}^3$ with

$$\lim_{R \to \infty} \sup_n \int_{B_R(x_n)} \rho_n(x)\, dx = M.$$

Here $B_R(x_n)$ denotes the ball around x_n of radius R. Due to the translation invariance of the problem we can shift the densities $\tilde{\rho}_n(x) = \rho(x - x_n)$ to obtain a compact sequence, $\tilde{\rho} \to \rho_0$. Finally, using the bound in $L^{1+1/n} \cap L^1$ it is possible to show that also the potential energies converge

$$\mathcal{E}_{\text{pot}}(\tilde{\rho}_n) \to \mathcal{E}_{\text{pot}}(\rho_0).$$

On the other hand, the functional $\int_{\mathbb{R}^3} \Psi(\rho)$ is lower-semicontinuous due to the convexity of Ψ inherited from the convexity of Φ. This shows the existence of a minimizer ρ_0.

Let us mention that the main assumption in the concentration compactness method is a strict convexity property of the function $M \to R_M$. More precisely, one requires $R_M < 0$ and

$$R_M < R_{M-\alpha} + R_\alpha, \quad \text{for all } \alpha \in (0, M).$$

In the above case this can be proved using the scaling properties of the functional, i.e.

$$R_{\bar{M}} \leq \left(\frac{\bar{M}}{M} \right)^{5/3} R_M, \quad 0 < M \leq \bar{M}.$$

The above reduction allows to prove the existence of minimizers to I_M and thus steady states to the Vlasov-Poisson equation. Let us now show that the above reduction problem in fact shows the existence of corresponding steady states to the compressible Euler-Poisson equation. To this end, we note that Ψ and Φ are related by the equation, cf. [143, Lemma 2.3]

$$\Psi^*(\lambda) = \int_{\mathbb{R}^3} \Phi^* \left(\lambda - \frac{1}{2}|v|^2\right) dv,$$

where Ψ^* and Φ^* are the Legendre transforms of the convex functions Ψ and Φ, respectively. Recall that the Legendre transform is given by (we extend Φ by zero for negative values)

$$\Phi^*(x) = \sup_{y\in\mathbb{R}} \{xy - \Phi(y)\}.$$

Let us also recall the property $(\Phi')^{-1} = (\Phi^*)'$. Thus, we have for minimizers f_0, using (1.2.8),

$$\begin{aligned}\int_{\mathbb{R}^3} f_0(x,v)\,dv &= \int_{\mathbb{R}^3} (\Phi')^{-1}\left(E_0 - \frac{1}{2}|v|^2 - U_0(x)\right)_+ dv = \int_{\mathbb{R}^3} (\Phi^*)'\left(E_0 - \frac{1}{2}|v|^2 - U_0(x)\right)_+ dv \\ &= (\Psi^*)'(E_0 - U_0(x))_+ = (\Psi')^{-1}(E_0 - U_0(x))_+ = \rho_0.\end{aligned}$$

The latter function is a minimizer to the reduced problem R_M. Furthermore, due to the rotational symmetry the macroscopic velocity V_0 is zero, i.e.

$$\rho_0(x)V_0(x) = \int_{\mathbb{R}^3} v f_0(x,v)\,dv = 0.$$

We can now integrate the Vlasov-Poisson equation

$$v\cdot\nabla_x f_0 - \nabla_x U_0 \cdot \nabla_v f_0 = 0$$

with respect to v after testing with $1, v$. This shows that $(\rho_0, V_0) = (\rho_0, 0)$ is a stationary solution to the Euler-Poisson equation with pressure

$$\begin{aligned}p(\rho(x)) &= \frac{1}{3}\int_{\mathbb{R}^3} |v|^2 f_0(x,v)\,dv = \frac{1}{3}\int_{\mathbb{R}^3} |v|^2 (\Phi^*)'\left(E_0 - \frac{1}{2}|v|^2 - U_0(x)\right)_+ dv \\ &= \int_{\mathbb{R}^3} \Phi^*\left(E_0 - \frac{1}{2}|v|^2 - U_0(x)\right)_+ dv = \Psi^*(E_0 - U_0(x))_+ = \Psi^*\left((\Psi^*)'(\rho_0(x))\right).\end{aligned}$$

To summarize, minimizers ρ_0 to the reduced problem R_M are steady states to the Euler-Poisson equation. Furthermore, they yield minimizers f_0 to I_M in the form of Gibbs states (1.2.8). Minimizers f_0 of I_M are steady states to the Vlasov-Poisson equation. Let us mention that the above reduction procedure works under the conditions on Φ given above. The restriction $k < 3/2$ can be relaxed in the case of the Vlasov-Poisson equation to $k < 7/2$, see [143]. When $k = 7/2$ for $\Phi(f) = f^{1+1/k}$ the corresponding steady state have infinity support and are also referred to as Plummer spheres, see [25, Sections 2.2, 4.3]. For $k < 7/2$ when $\Phi(f) = f^{1+1/k}$ solutions are refereed to as polytropic galaxies respectively stars. Also the name Lane-Emden star is used, see [25, Section 4.3].

The above minimization method also allows to show the stability of the corresponding steady states, see for instance [141, 142]. This result can be extended under certain conditions to critical points of the functional $\mathcal{H}$, see [110]. Such results are referred to as orbital stability and yield results in strong norms, i.e. with respect to the L^1-norm. However, these results do not yield asymptotic stability.

Rotating solutions. In addition to the above steady states rotating configurations have been constructed in perturbative settings. The aim is to construct steady states in a rotating frame of reference at angular speed Ω_0. Two main techniques have been used.

(i) Variational methods: Similarly to the energy minimization technique one can aim to construct extremal points to the functional

$$\mathcal{H}_{\Omega_0}(f) = \mathcal{H}(f) - \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{1}{2}\Omega_0^2|x|^2 f(x,v)\, dvdx.$$

The term added accounts for the angular momentum of the system. Since this functional is no longer bounded from below only the existence of local minimizers can be established. To this end, a constraint minimization problem is considered (more precisely assuming that the support of f lies in some large ball). This problem admits a global minimizer. Assuming that the angular speed is small enough one can show that such global minimizers are also local minimizers to the unconstrained problem.

This was done in the case of flat galaxies in [65]. Similarly, one can construct two galaxies rotating around each other, see [102]. In addition, this technique can be applied as well to the Euler-Poisson equation, see [18, 19, 40, 111, 123, 124].

(ii) Implicit function theorem: A different method to construct rotating solutions is the application of the implicit function theorem with parameter Ω_0. For $\Omega_0 = 0$ the above mentioned steady states yield solutions. Looking for deformations of these one can construct solutions to the problem with $\Omega_0 \neq 0$ small. This approach was introduced by Lichtenstein in [113] and further developed by Heilig [89]. More recent results are available for both the Euler-Poisson equation as well as the Vlasov-Poisson equation, see [100, 147]. By the uniqueness of the implicit function theorem and the symmetry of the problem these solutions are axisymmetric.

In the above works the perturbation parameter is the angular velocity Ω_0. In particular, in the case of the implicit function theorem this parameter is small. For large Ω_0 global bifurcation theorems are used, see e.g. [148].

In most of the above results the rotating star or galaxy has trivial internal motion in the rotating frame of reference. More precisely, in the case of the Euler-Poisson equation the velocity field vanishes in the rotating coordinate system, whereas in the case of the Vlasov-Poisson equation the mean velocity is zero. In contrast, we studied more general internal motion in the work (IV). However, we restricted to the case of the incompressible Euler-Poisson equation in two dimensions, see Chapter 5. In this way one can consider the problem studied in (IV) as an oversimplified model for a star or even a galaxy.

1.2.3 Overview of mathematical results on the incompressible Euler and Euler-Poisson equation

Here we give a short overview of methods and results used to construct stationary solutions to the incompressible Euler and Euler-Poisson equation. We also briefly mention some related results for the time-dependent problem.

Stationary solutions. The construction of steady states to the Euler equations

$$\partial_t v + (v \cdot \nabla)v = -\nabla p, \quad \nabla \cdot v = 0.$$

strongly depends on the dimension.

In the two-dimensional case a very flexible approach is the usage of the stream function. It allows to reduce the problem to an elliptic equation. This method is also referred to as Grad-Shafranov approach [77, 146]. Let us indicate the main steps of this method.

Since the velocity field $v = (v_1, v_2)^\top$ is divergence-free, the vector field Jv is irrotational, where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

In particular, on simply connected domains we can write $Jv = -\nabla\psi$ or $v = J\nabla\psi$ for some real-valued function ψ. The vorticity $\omega = \operatorname{curl} v = \partial_{x_1} v_2 - \partial_{x_2} v_1$ of the velocity field is then related to the stream function by $\omega = \Delta\psi$. In the case that we consider the problem in a bounded domain for instance the zero flux boundary condition $n \cdot v = 0$ is added, where n is the outer unit normal vector. This yields then the condition $\psi = const.$ at the boundary. Since the stream function is only given up to a constant we can set $\psi = 0$ at the boundary.

On the other hand, using the vorticity we can rewrite the stationary Euler equations as follows

$$-\omega J v = \nabla H, \quad H = p + \frac{1}{2}|v|^2.$$

Let us mention that the function H is usually referred to as Bernoulli head. In particular, applying the curl-operator yields

$$(v \cdot \nabla)\omega = 0.$$

Thus, $\nabla\psi$ is orthogonal to $\nabla\omega$. In particular, ω is constant on the level sets of ψ. This suggest an ansatz of the form $\omega = G(\psi)$ for some function $G : \mathbb{R} \to \mathbb{R}$. In total, we reduced the problem to the elliptic equation

$$\Delta\psi = G(\psi)$$

with zero boundary conditions. As a result, given a function G and solving the above elliptic equation yields a stationary solution to the Euler equation. In order to solve the elliptic equation one can use methods from The Calculus of Variations. In fact, such energy minimization techniques also allow to obtain stability results for steady states, see [17, Section II.4]. The same method can also be applied to the Euler-Poisson equation, since the attractive forces enter the Euler equation in terms of the gradient ∇U which can be absorbed into the Bernoulli head. Furthermore, it can be applied to the magneto-hydrostatic equation as well, see e.g. [55, 56, 83]. In particular, different boundary conditions were studied in [10].

In the three-dimensional case two methods can be used. First of all, one can construct irrortational flows, i.e. the vorticity $\omega = \operatorname{curl} v = 0$ vanishes. In this case $v = \nabla\phi$, for some scalar-valued function. Again the problem reduces to an elliptic equation. Due to the divergence-free condition we have $\Delta\phi = 0$. On the other hand, the boundary condition $n \cdot v = 0$ yields $n \cdot \nabla\phi = 0$.

A different method is given by the vorticity transport method introduced in [2]. This allows to construct solutions with non-zero vorticity. The main idea is to use the vorticity transport equation (obtained by applying the curl-operator to the Euler equation)

$$(v \cdot \nabla)\omega = (\omega \cdot \nabla)v.$$

Given a velocity field v one can solve the above transport equation for ω in the domain (taking into account in-flux and out-flux boundary conditions). Using the Biot-Savart-law one can reconstruct the velocity field from the vorticity field. This circular construction was turned into a rigorous fixed point argument in a perturbative framework in [2]. This allowed to construct solutions close to a given solution with well-behaved flow lines from the in-flux to the out-flux boundary of the domain.

Time-dependent problems. At the end of this section we give a few results for the time-dependent case. The time-dependent problem of the two-dimensional Euler-Poisson equation was studied in [24]. Here, it was shown that solutions with initial datum close to the unit disk configuration exist for times of order $1/\varepsilon^2$, where ε is the size of the initial perturbation. Local well-posedness results (also for higher dimensions) were obtained in [115].

The free-boundary problem studied in (IV) is very much related to the theory of water waves, see [85] for a recent overview. The time-dependent problem was studied in particular in [162, 163].

Furthermore, problems including rotations for the Euler-equation have been studied as well. Let us mention in particular the work [81], in which global axisymmetric solutions close to a rigid rotation $v(x) = \Omega \times x$ are constructed on the whole space. To this end, stabilizing effects of the rotation were used.

Finally, stabilizing effects for the Euler equation were also identified for shear flows in [23]. This allowed to show that a profile close to the shear flow configuration $v(x,y) = (y,0)^\top$ on the domain $\mathbb{T} \times \mathbb{R}$ converges to a shear flow configuration of the form $v_\infty(x,y) = (y + u_\infty(y), 0)$ for some small $u_\infty : \mathbb{R} \to \mathbb{R}$ as $t \to \infty$.

1.3 Main results of the thesis

In this section we summarize the main result of the works included in the thesis.

(I) In (I) we study homoenergetic solutions (1.1.13) for Maxwell molecules in the case of simple and planar shear. Assuming that the shear is small we show that there is a unique and asymptotically stable self-similar solution. Although the self-similar profile is close to the Maxwellian distribution, it is not a thermodynamic equilibrium. In fact, the system does not satisfy the H-Theorem since the shear flow induces a flux of energy into the system.

(II) In (II) we consider homoenergetic solutions (1.1.13) for hard potentials with shear. More precisely, we prove that the formal asymptotics given by the Hilbert-type expansion, see Section 1.1.2, describes the longtime behavior of the system. The presence of the shear flow leads to an increase of the temperature. Again this is an non-equilibrium situation.

(III) Differently from the previous two works we studied the homogeneous Boltzmann equation in (III). The attention here is the modeling of the collision events via either inverse power laws $1/r^{s-1}$ of hard spheres. In fact, for $s \to \infty$ the potential of the inverse power law converges to the hard spheres potential. We show that accordingly the collision operators for inverse power laws converge to the hard sphere kernel. Moreover, we prove that solutions to the homogeneous Boltzmann equation (1.1.7) with inverse power law kernel tend to solutions to the homogeneous Boltzmann equation with the hard sphere kernel.

(IV) Finally, in (IV) we study rotating solutions to the two-dimensional, incompressible Euler-Poisson equation with the presence of an external particle with small mass, see (1.2.5). More precisely, we show the existence of stationary solutions close to the unit disk. Due to the interaction with the external particle tidal waves appear and the domain deforms to a different configuration. Furthermore, we include general internal velocity profiles. The unperturbed velocity profile has the form of a general shear flow with circular flow lines. In particular, in our perturbation framework we obtain stationary velocity fields close to these shear flows. In contrast, these non-trivial velocity fields do not appear in the historical works on ellipsoidal figures and the works on the Vlasov-Poisson respectively compressible Euler-Poisson equation. In fact, rotating solutions to the two latter equations are often obtained by perturbing stationary solutions resulting from a minimization problem. These minimizers can be interpreted as (thermodynamical) equilibrium solutions. In particular, such minimizers do not contain any non-trivial average motion. Consequently, perturbations of such solutions studied for the Vlasov-Poisson equation and Euler-Poisson equation are close to equilibrium up to small rotations. In this way, the above stationary solutions constructed in (IV) are different from such equilibrium situations as already the unperturbed configuration contains non-zero internal motion.

Chapter 2

Summary of article (I). Self-similar profiles for homoenergetic solutions of the Boltzmann equation for non-cutoff Maxwell molecules

In this chapter we give an overview of the result of the work (I) as well as the methods used. The accepted manuscript of its published version [105] is reproduced in Appendix A.

In this work the longtime asymptotics of homoenergetic solutions ot the Boltzmann equation with non-cutoff Maxwellian molecules was studied. More precisely, the existence, uniqueness and stability of a self-similar solution was proved.

2.1 Main results on self-similar asymptotics

The equation under study has the form

$$\partial_t f = \operatorname{div}(Av\, f) + Q(f,f), \tag{2.1.1}$$

for a constant matrix $A \in \mathbb{R}^{3\times 3}$. Furthermore, we consider Maxwell molecules, i.e. the collision operator takes the form

$$Q(f,f)(v) = \int_{\mathbb{R}^3}\int_{S^2} b(n\cdot\sigma)\left(f'_*f' - f_*f\right) d\sigma dv_*.$$

Here, the angular part $b: [-1,1) \to [0,\infty)$ is assumed to have a singular behavior

$$\lim_{\theta\to 0} \sin\theta\, b(\cos\theta)\theta^{1+2s} = K_b > 0, \quad \text{as } \theta \to 0, \tag{2.1.2}$$

for some $s \in (0,1)$ and $K_b > 0$. In particular, we have

$$\Lambda = \int_0^\pi \sin\theta\, b(\cos\theta)\theta^2\, d\theta < \infty.$$

As already mentioned in the introduction, Section 1.1.2, in this case the equation (2.1.1) admits a self-similar asymptotics. Let us now give the heuristics of such a behavior. First of all, the drift term and the collision operator have the same scaling. Thus, if there is a limiting profile f_{st} both terms have to balance. However, one has to take into account the conserved quantities of the collision operator, that is, the conservation of mass, momentum and energy. Mass is preserved by the drift term as well. Furthermore, the momentum $V(t) = \int_{\mathbb{R}^3} v f(t,v)\, dv$ of the system is given by $V(t) = e^{-tA}V(0)$. Using the change of variables $v \mapsto v - e^{tA}V(0)$, which leaves the equation (2.1.1) invariant, one can assume that $V(t) \equiv 0$. In particular, any limiting profile can be assumed to have total mass one and zero momentum. Finally, the kinetic energy is preserved by the collision operator too. However, the drift term has an essential effect on this quantity. In fact, if f_{st} is a limiting profile then the change of the kinetic energy has to be taken into account as well, since one cannot expect to find a (non-trivial) steady state to the equation (2.1.1), i.e.

$$\operatorname{div}(Av\, f_{st}) + Q(f_{st}, f_{st}) = 0.$$

Consequently, a self-similar change of variables was introduced in [98]. More precisely, one makes the ansatz

$$f(t,v) = f_{st}\left(\frac{v}{e^{\beta t}}\right) e^{-3\beta t}, \quad \beta \in \mathbb{R},$$

where f_{st} is the self-similar profile to be found. Note that $f(t,v)$ has mass one and zero momentum if this is the case for f_{st}. The equation solved by the above ansatz is

$$\operatorname{div}((A+\beta I)v\, f_{st}) + Q(f_{st}, f_{st}) = 0 \tag{2.1.3}$$

Note that the assumption $\gamma = 0$, i.e. Maxwell molecules, on the collision kernel is essential. Otherwise the term $e^{\gamma\beta t}$ would appear in front of the collision operator after the self-similar change of variables. The parameter β can now be used in order to find a non-trivial solution to (2.1.3). We give further explanations on how to choose β in the next section. Nevertheless, let us mention that the smallness of A (with respect to some matrix norm) is crucial here. As was proved first in [98] for cutoff kernels there is a solution to equation (2.1.3) in the space of probability measures $\mathscr{P}_p(\mathbb{R}^3)$ for $p > 2$ fixed and A sufficiently small. In fact, this induces a whole family of solutions by the rescaling $f_{st}(v/K)K^{-3}$, $K > 0$. The uniqueness and the stability of this profile was proved in [26] for cutoff kernels. Most importantly, it was proved that any solution to (2.1.3) has a self-similar asymptotics, i.e. for any weak measure-valued solution $f \in C([0,\infty); \mathscr{P}_p)$ with $p > 2$ we have for some $K > 0$

$$f\left(t, ve^{\beta t}\right) e^{3\beta t} \to f_{st}(v/K)K^{-3}, \quad t \to \infty.$$

Here, the topology is induced by the weak convergence in the sense of probability measures. These results were then extended to non-cutoff kernels in the work (I), i.e. assuming (2.1.2). We refer to Theorem A.1.3 for the precise formulation of this result. Furthermore, due to the singular behavior of the collision kernel, the collision operator has a smoothing effect. This allows to obtain the smoothness of the self-similar profile, i.e.

$$f_{st} \in L^1(\mathbb{R}^3) \cap \bigcap_{k\in\mathbb{N}} H^k(\mathbb{R}^3),$$

unless f_{st} is a Dirac in zero. Furthermore, it was proved in (I) that the profile f_{st} has certain decay properties depending on the smallness of A. More precisely, for any $p > 2$ one can choose A small enough such that $f_{st} \in \mathscr{P}_p$.

Finally, let us mention that the results on the self-similar asymptotics to (2.1.1) can be applied to homoenergetic solutions in the case of planar and simple shear. The corresponding self-similar asymptotics was formulated in the work (I), see Theorem A.4.1.

In the next sections we give an overview of the methods used for the existence and the stability of the self-similar profile.

Let us finally give some comments on previous studies on self-similar solutions to the *homogeneous* Boltzmann equation, i.e. $A = 0$ in (2.1.1). Self-similar profiles were considered for elastic and inelastic collisions with infinite and finite energy, respectively, in [29, 30, 32] for Maxwell molecules and in [31] for non-Maxwellian molecules. They proved stability of those self-similar solutions in [34] for inelastic collisions (with cutoff) and in [30] for elastic collisions (without cutoff), see also [32]. Furthermore, self-similar solutions were analyzed in a general framework of Maxwell models in [33]. The case of non-cutoff Maxwell molecules was also discussed in [41, 42]. In particular, they proved smoothness based on a regularity result of the homogeneous Boltzmann equation for measure-valued solutions [132].

2.2 Existence of self-similar solution

In this section we give an overview of the proof of the existence of a self-similar solution, i.e. a solution to (2.1.3). To this end, we give some comments on the well-posedness of (2.1.1) in the space of probability measures.

Well-posedness of measure-valued solutions. As mentioned in the introduction, see Section 1.1.1, one can define a weak formulation to the homogeneous Boltzmann equation. Accordingly, this can be done for the equation (2.1.1) as well, cf. Definition (A.1.1). More precisely, we say that a family of probability measures $(f_t)_{t\geq 0} \subset \mathscr{P}_p$ with $p \geq 2$ is a weak solution to (2.1.1) if for any $\psi \in C_b^2$ and all $0 \leq t < \infty$ we have

$$\begin{aligned}\langle \psi, f_t\rangle = \langle \psi, f_0\rangle - \int_0^t \langle Av\cdot\nabla\psi, f_r\rangle\, dr \\ + \frac{1}{2}\int_0^t \int_{\mathbb{R}^3\times\mathbb{R}^3}\int_{S^2} b(n\cdot\sigma)\left\{\psi_*' + \psi' - \psi_* - \psi\right\} d\sigma f_r(dv) f_r(dv_*) dr.\end{aligned}$$

As mentioned already in the introduction, see Section 1.1.1, the weak formulation turns out to be robust concerning approximations of solutions. This allows to prove the existence of weak solutions to (2.1.1), see Proposition A.2.1 (i). To be more precise, one constructs a sequence of solutions $(f^n)_n$ to the equation with a truncated collision kernel. In this case, the well-posedness can be proved using a fixed point argument, see e.g. [98]. One then proves that moments of order $p > 2$ of f^n are uniformly bounded in $n \in \mathbb{N}$. This follows from an application of the so-called Povzner estimates, see e.g. [129]. The moment bound allows to show that the sequence of solutions $(f^n)_n$ is compact in the space of probability measures, thus up to a subsequence $f^n \to f$. Finally, the weak convergence in the sense of measures is sufficient to pass to the limit in the weak formulation.

On the other hand, the uniqueness can be proved using the Fourier transform method, cf. Proposition A.2.1 (ii). We give an overview of this method in the next section. Let us mention that the proof of uniqueness follows the argument in [150] used in the case of the homogeneous Boltzmann equation.

Finally, one can prove that any weak solution to (2.1.1) is smooth for positive times $t > 0$, unless it is a Dirac initially, cf. Proposition A.2.1 (iii). This is a consequence of the singular behavior of the collision kernel (2.1.2). In the case of the homogeneous Boltzmann equation this was proved in [132] making again use of the Fourier transform method. The proof in case of homoenergetic solutions follows the same lines of reasoning.

Perturbation of eigenvalue problem. An essential step towards the proof of the existence of a self-similar solution was the study of the second moment equation. Recalling the equation (2.1.3), the goal is to find a parameter $\beta = \bar{\beta}$ such that (2.1.3) has a non-trivial solution. In fact, this choice will depend on A. This parameter appears through an eigenvalue problem. This eigenvalue problem appears through the study of the second moment equation. In fact, for Maxwell molecules it is possible to obtain a closed system of equations for any moments of order $p \in \mathbb{N}$. In the case of the second moments

$$M_{ij} = \int_{\mathbb{R}^3} v_i v_j f(t,v)\, dv$$

one obtains from (2.1.1), we refer to Lemma A.3.2,

$$\frac{dM}{dt} = -AM - (AM)^\top - 2\bar{b}\left(M - \frac{\operatorname{tr}(M)}{3} I\right).$$

with the constant

$$\bar{b} = \frac{3\pi}{4}\int_0^\pi b(\cos\theta)\sin^3\theta d\theta.$$

This equation has to be considered on the space of symmetric matrices. Similarly, from (2.1.3) we get the equation

$$-AM - (AM)^\top - 2\bar{b}\left(M - \frac{\operatorname{tr}(M)}{3} I\right) = 2\beta M. \tag{2.2.1}$$

Note that the trace of the last term is zero (as expected from the invariance of kinetic energy of the collision operator). Furthermore, for $A = 0$, corresponding to the classical homogeneous Boltzmann equation, one solution is given by $M = I$, $\beta = 0$, i.e. the second moments of the Maxwellian distribution. This eigenvalue is in fact simple and thus for A sufficiently small there is a simple eigenvalue $\beta = \bar{\beta}(A)$ and an eigenvector $M = \bar{N}$ satisfying (2.2.1). Note that the symmetric matrix $\bar{N}$ is close to I and thus positive definite for A sufficiently small. As a consequence any solution to (2.1.3) which is close to the Maxwellian distribution has to have second moments $\bar{N}$ up to rescaling.

Compactness argument. Another step in order to prove the existence of a self-similar solution is a compactness argument. Let us indicate the main steps which was used first in [98] and then in the work (I). To this end, the time-dependent problem of (2.1.3) is considered, that is

$$\partial_t f = \operatorname{div}\left((A + \bar{\beta} I) v f\right) + Q(f,f), \quad f\,|_{t=0} = f_0. \tag{2.2.2}$$

As in the case of (2.1.1) this equation is well-posed. In particular, it induces a nonlinear semigroup $\mathcal{P}_t : \mathscr{P}_p \to \mathscr{P}_p$ for $p > 2$. Using Povzner estimates it is possible to show the following, see Lemma A.3.4. If A is sufficiently small there is a constant C_* such that for any solution satisfying

$$\int_{\mathbb{R}^3} v f(t,dv) = 0, \quad \int_{\mathbb{R}^3} v_i v_j f(t,dv) = \bar{N}_{ij}$$

we have for all $t \geq 0$

$$\int_{\mathbb{R}^3} |v|^p f_0(dv) \leq C_* \implies \int_{\mathbb{R}^3} |v|^p f(t,dv) \leq C_*.$$

In other words, the set

$$\left\{ f \in \mathscr{P}_p \ : \ \int_{\mathbb{R}^3} v f(dv) = 0, \int_{\mathbb{R}^3} v_i v_j f(dv) = \bar{N}_{ij}, \int_{\mathbb{R}^3} |v|^p f(dv) \leq C_* \right\}$$

is invariant under the nonlinear semigroup $\mathcal{P}_t$. The invariance of the second order moments is a consequence of the choice of $\bar{\beta} = \bar{\beta}(A)$ and $\bar{N} = \bar{N}(A)$.

On the other hand, this set is compact with respect to weak convergence of probability measures due to the Prokhorov theorem (or alternatively due to the fact that the space of measures is a dual space). Thus, for any $t \geq 0$ the map $\mathcal{P}_t$ has a fixed point. In particular, $\mathcal{P}_{1/n}$ has a fixed point, say f^n. Using another compactness argument and the equicontinuity of the semigroup with respect to time one obtains a fixed point f_{st} of $\mathcal{P}_t$ for all $t \geq 0$. However, this means f_{st} is a stationary solution to (2.2.2), i.e. it is a solution to (2.1.3). This concludes the proof of the existence of a self-similar solution.

Let us mention that the above way of reasoning is used to prove existence of stationary or self-similar solutions in many other problems of PDEs. One example we want to mention here are coagulation-fragmentation equations, describing the coagulation of particles, droplets, aerosols, etc. For these type of kinetic equations the above procedure was used to construct self-similar profiles, see e.g. [20, Chapter 10].

Although the above method is very flexible, it has two main disadvantages. First of all, it is non-constructive as it uses the compactness in an essential way. Thus, further qualitative properties of the self-similar profiles have to be proved by other means. Secondly, the construction does not provide uniqueness of the profile. Furthermore, it remains unclear if the solution is asymptotically approached by time-dependent solutions. In the problem studied here all these issues can be answered affirmatively as will be discussed in the next section. Finally, let us mention that in the case of cutoff kernels the existence and uniqueness of a self-similar profile close to a Maxwellian distribution was proved in [66]. In contrast to the above argument, a perturbative framework (reminiscent of an implicit function theorem with small parameter A) was used. In particular, the argument was constructive.

2.3 Fourier transform method and stability of self-similar profile

In this section we give an overview of the proof of the stability of the self-similar profile. The main tool is the Fourier transform, which is suited to study Maxwell gases as will be explained now.

Fourier transform method. As was observed by Bobylev in [27] the Boltzmann collision operator takes a much simpler form when applying the Fourier transform in the velocity variable. For convenience we replicate the computation here. For the exposition we assume that the collision kernel satisfies Grad's cutoff condition, i.e. the angular part is bounded $b \in L^\infty$.

First, we consider the Fourier transform of the loss term. We have for any $k \in \mathbb{R}^3$, which is the Fourier variable,

$$\begin{aligned}\int_{\mathbb{R}^3} Q_{\text{loss}}(f,f)(v)e^{-iv\cdot k}\,dv &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} b(n\cdot\sigma)e^{-iv\cdot k} ff_*\,d\sigma dv dv_* \\ &= \left(\int_{S^2} b(n\cdot\sigma)\,d\sigma\right)\hat{f}(k)\hat{f}(0).\end{aligned}$$

Let us recall that

$$n = \frac{v-v_*}{|v-v_*|}.$$

For the gain term we have via the pre-post-collisional change of variables

$$\begin{aligned}&\int_{\mathbb{R}^3} Q_{\text{gain}}(f,f)(v)e^{-iv\cdot k}\,dv = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} b(n\cdot\sigma)e^{-iv'\cdot k} ff_*\,d\sigma dv dv_* \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} b(n\cdot\sigma)\exp\left(-i\left(\frac{v-v_*}{2}+\frac{|v-v_*|}{2}\sigma\right)\cdot k\right)e^{-iv_*\cdot k} ff_*\,d\sigma dv dv_* \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} b(n\cdot\sigma)\exp\left(-i|v-v_*||k|\frac{n\cdot\hat{k}+\sigma\cdot\hat{k}}{2}\right)e^{-iv_*\cdot k} ff_*\,d\sigma dv dv_*.\end{aligned}$$

Here, we defined $\hat{k} = k/|k|$. Let us now do a change of variables in σ. We use a rotation $R \in \mathrm{SO}(3)$ such that $Rn = \hat{k}$ and transform $\sigma \mapsto R^\top \sigma$. This yields

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} b(\hat{k}\cdot\sigma)\exp\left(-i|v-v_*||k|\frac{n\cdot\hat{k}+\sigma\cdot n}{2}\right)e^{-iv_*\cdot k} ff_*\,d\sigma dv dv_*.$$

Defining $k_\pm = (k \pm |k|\sigma)/2$ we can rearrange this to

$$\begin{aligned}&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} b(\hat{k}\cdot\sigma)\exp\left(-i(v-v_*)\cdot\frac{k+|k|\sigma}{2}\right)e^{-iv_*\cdot k} ff_*\,d\sigma dv dv_* \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} b(\hat{k}\cdot\sigma)e^{-i(v-v_*)\cdot k_+}e^{-iv_*\cdot k} ff_*\,d\sigma dv dv_* \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} b(\hat{k}\cdot\sigma)e^{-iv\cdot k_+}e^{-iv_*\cdot k_-} ff_*\,d\sigma dv dv_* \\ &= \int_{S^2} b(\hat{k}\cdot\sigma)\hat{f}(k_+)\hat{f}(k_-)\,d\sigma.\end{aligned}$$

Thus, combining this with the loss term, noting that

$$\int_{S^2} b(n\cdot\sigma)\,d\sigma = \int_{S^2} b(k\cdot\sigma)\,d\sigma$$

we have

$$\int_{\mathbb{R}^3} Q(f,f)(v)e^{-iv\cdot k}\,dv = \int_{S^2} b(\hat{k}\cdot\sigma)\left(\hat{f}(k_+)\hat{f}(k_-)-\hat{f}(k)\hat{f}(0)\right)d\sigma.$$

Finally, note that $\hat{f}(0) = 1$ since f is assumed to be a probability measure and thus

$$\int_{\mathbb{R}^3} Q(f,f)(v)e^{-iv\cdot k}\,dv = \int_{S^2} b(\hat{k}\cdot\sigma)\left(\hat{f}(k_+)\hat{f}(k_-)-\hat{f}(k)\right)d\sigma, \quad k_\pm = \frac{k\pm|k|\sigma}{2}. \tag{2.3.1}$$

Let us observe the two essential simplifications of (2.3.1) compared to the collision operator. First of all, the loss term is no longer nonlinear. Second, the dimension of the integral in the gain term reduces and there is only an integral on the sphere appearing. Note that the above reasoning is also valid in the case of non-cutoff kernels, since the test function used is $v \mapsto e^{-iv\cdot k}$ for $k \in \mathbb{R}^3$. On the other hand, one can see that (2.3.1) is well-defined by making use of the regularity $\hat{f} \in C_b^2$ for $f \in \mathscr{P}_2$ and cancellations.

In total after applying the Fourier transform the homogeneous Boltzmann equation takes the form

$$\partial_t \hat{f}(t,k) = \int_{S^2} b(\hat{k}\cdot\sigma)\left(\hat{f}(t,k_+)\hat{f}(t,k_-) - \hat{f}(t,k)\right) d\sigma.$$

On the other, for homoenergetic solutions we obtain from (2.1.1) the equation

$$\partial_t \hat{f}(t,k) = -A^\top k \cdot \nabla_k \hat{f}(t,k) + \int_{S^2} b(\hat{k}\cdot\sigma)\left(\hat{f}(t,k_+)\hat{f}(t,k_-) - \hat{f}(t,k)\right) d\sigma.$$

Contraction properties. As we will explain now, the remarkable property of the collision operator for Maxwell molecules is the fact that it is contractive with respect to some Fourier-based metrics. The metric we consider here has the form, cf. Definition A.1.2,

$$d_2(f,g) = \sup_{k\in\mathbb{R}^3} \frac{|\hat{f}(k) - \hat{g}(k)|}{|k|^2}, \quad f,g \in \mathscr{P}_2.$$

In order that $d_2(f,g) < \infty$ it is necessary that f,g have the same moments of order one. This metric is also termed Toscani's distance, see [156, Section 4.2]. As was shown in [150] for the homogeneous Boltzmann equation (without cutoff) this metric is contractive, i.e.

$$d_2(f(t),g(t)) \le d_2(f(0),g(0))$$

for weak solutions f,g to the homogeneous Boltzmann equation. An extension of this result to homoenergetic solutions is given in the work (I), see Theorem A.2.1 (ii). It takes the form

$$d_2(f(t),g(t)) \le e^{2\|A\|t} d_2(f(0),g(0))$$

for weak solutions f,g to (2.1.1).

It was observed in [68] using metrics of the form

$$d_p(f,g) = \sup_{k\in\mathbb{R}^3} \frac{|\hat{f}(k) - \hat{g}(k)|}{|k|^p}, \quad f,g \in \mathscr{P}_2$$

for $p > 2$, the exponent in the estimate of the contraction property increases with $p > 2$. In fact, one can prove

$$d_p(f(t),g(t)) \le e^{-\lambda_p t} d_p(f(0),g(0)), \tag{2.3.2}$$

where the constant λ_p is given by

$$\lambda_p = \int_{S^2} b(e\cdot\sigma)\, w_p(e\cdot\sigma) d\sigma, \quad w_p(s) = 1 - \left(\frac{1+s}{2}\right)^{p/2} - \left(\frac{1-s}{2}\right)^{p/2}.$$

One can see that $\lambda_p > 0$ for $p > 2$, while $\lambda_2 = 0$. Furthermore, $p \mapsto \lambda_p$ is increasing. However, for the above estimate it is required that f_0, g_0 have finite moments of order $p > 2$ and more importantly the same moments up to order $p-1$ (otherwise $d_p(f_0, g_0) = \infty$).

Let us give the main reasoning towards the estimate (2.3.2). To this end, we consider a cutoff kernel b. We have the following bound for the gain term

$$\begin{aligned}
&\frac{1}{|k|^p}\left|\int_{S^2} b(\hat{k}\cdot\sigma)\hat{f}(k_+)\hat{f}(k_-)\,d\sigma - \int_{S^2} b(\hat{k}\cdot\sigma)\hat{g}(k_+)\hat{g}(k_-)\,d\sigma\right| \\
&\quad\le \frac{1}{|k|^p}\int_{S^2} b(\hat{k}\cdot\sigma)\left[|\hat{f}-\hat{g}|(k_+) + |\hat{f}-\hat{g}|(k_-)\right] d\sigma \\
&\le d_p(f,g)\int_{S^2} b(\hat{k}\cdot\sigma)\left[\frac{|k_+|^p}{|k|^p} + \frac{|k_-|^p}{|k|^p}\right] d\sigma = \mu_p d_p(f,g)
\end{aligned}$$

where

$$\mu_p = \int_{S^2} b(e\cdot\sigma)\left[1 - w_p(e\cdot\sigma)\right] d\sigma.$$

Compare this with the constant λ_p above. Making use of the loss term and applying a Gronwall argument yields the bound (2.3.2) for solutions to the homogeneous Boltzmann equation. This reasoning can then also be extended to non-cutoff kernels and in particular to solutions to (2.2.2).

Stability of self-similar profile. In order to prove the stability of the self-similar solution an extension to (2.3.2) for solutions to (2.2.2) can be proved. However, as mentioned above this requires that the second moments of the initial data are the same. To overcome this limitation, the longtime dynamics of the second order moment equation is used. In fact, the linear system of ODEs

$$\frac{dM}{dt} = -2\bar{\beta}M - AM - (AM)^\top - 2\bar{b}\left(M - \frac{\mathrm{tr}\,(M)}{3}I\right).$$

can be shown to converge to the eigenspace to the eigenvector $\bar{N}$ obtained before. This is due to the fact that for $A = 0$, $\bar{\beta} = \bar{\beta}(A) = 0$ the solution to this linear system of ODEs converges to the eigenspace to the eigenvector I close by $\bar{N}$.

Combining this together with the contraction property above one can show that the solution f to (2.2.2) satisfy

$$d_2\left(f(t), f_{st}^K\right) \le e^{-\theta t} d_2\left(f_0, f_{st}^K\right), \quad f_{st}^K(v) = K^{-3/2} f_{st}(v/K^{1/2}).$$

for some $K \ge 0$. The parameter K is determined by the eigenvector $K\bar{N}$ approached by the second moments of the solution f. Furthermore, the parameter $\theta = \theta(p, A) > 0$ is close to $\lambda_p > 0$ for A small enough.

Chapter 3

Summary of article (II). Longtime behavior of homoenergetic solutions in the collision dominated regime for hard potentials

In this chapter we give an overview of the results in the work II, which is reproduced in Appendix (B) and has been published in [106]. In this work the longtime behavior of homoenergetic solutions in the collision-dominated case for hard potentials was studied.

3.1 Hilbert-type expansion and longtime asymptotics

As was mentioned in the introduction, see Section 1.1.2, in the collision-dominated case homoenergetic solutions converge towards a Maxwellian distribution. However, due to the effect of the mechanical forces acting on the gas the temperature (as well as the density) depends on time. The longtime behavior was studied formally in [97] and proved rigorously for hard potentials in the work (II).

In this section we give the formal explanation for the longtime asymptotics as derived in [97]. The main tool is a so-called Hilbert-type expansion. The equation under study has the form

$$\partial_t f = Av \cdot \nabla f + Q(f,f), \quad A = \begin{pmatrix} 0 & K & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Recall that in this case the form of the matrix A corresponds to simple shear, see Section 1.1.2. The study of simple shear with decaying planar dilation/shear and combined orthogonal shear can be studied analogously. In this exposition we consider merely the case of simple shear.

Since A is trace-free we have

$$\partial_t f = \operatorname{div}(Av\, f) + Q(f,f),$$

in particular the density is preserved. Since the drift term has an effect on the temperature it is convenient to rescale the temperature to one, i.e. we define

$$F(t,v) = \beta(t)^{-3/2} f\left(t, \frac{v}{\beta(t)^{1/2}}\right), \quad \beta(t)^{-1} = T(t) = \frac{1}{3}\int_{\mathbb{R}^3} |v|^2 f(t,v)\, dv.$$

The quantity β is the inverse temperature. Note that strictly speaking the quantity T defined above is 2/3 times the kinetic energy. Up to the factor k_B (Boltzmann constant) this is the temperature of the gas.

We then obtain the equation

$$\partial_t F = \operatorname{div}\left(\left(A - \frac{\beta'}{2\beta}\right) v\, F\right) + \beta^{-\gamma/2} Q(F,F), \quad \frac{\beta'}{2\beta} = \alpha_F := \frac{1}{3}\int_{\mathbb{R}^3} v \cdot Av\, F(t,v)\, dv. \tag{3.1.1}$$

We now make an ansatz for the longtime asymptotics and show that it is self-consistent. More precisely, we assume now that $\beta(t)^{-\gamma/2} \to \infty$ as $t \to \infty$. In this case, we expect that F converges towards a Maxwellian distribution. Thus, we furthermore assume that F admits the expansion

$$F(t,v) = \mu(v) + \mu^{(1)}(t,v) + \mu^{(2)}(t,v) + \cdots,$$

where for $k \geq 1$ and $t \to \infty$

$$\mu^{(k)}(t,v) \ll \mu^{(k-1)}(t,v), \quad \mu^{(0)}(t,v) = \mu(v). \tag{3.1.2}$$

We now plug the above expansion into (3.1.1) and collect terms of same order, taking into account also the fact that $\beta(t)^{-\gamma/2} \to \infty$ for $t \to \infty$. First of all, we obtain the equation

$$\operatorname{div}\left((A - \alpha_\mu I)\, v\, \mu\right) - \beta^{-\gamma/2} \mathscr{L} \mu^{(1)} = 0, \quad \alpha_\mu = \frac{1}{3}\int_{\mathbb{R}^3} v \cdot Av\, \mu(v)\, dv = 0. \tag{3.1.3}$$

One can see that

$$\operatorname{div}\left((A - \alpha_\mu I)\, v\, \mu\right) = -v \cdot Av\, \mu(v) \in (\ker \mathscr{L})^\perp,$$

with respect to the $L^2(\mu^{-1/2})$ scalar product, that is

$$\int_{\mathbb{R}^3} (-v \cdot Av\, \mu(v))\, \varphi(v)\, \mu(v)^{-1}\, dv = 0, \quad \text{for} \quad \varphi \in \left\{1, v_1\, \mu(v), v_2\, \mu(v), v_3\, \mu(v), |v|^2 \mu(v)\right\}.$$

In particular, one can invert the equation (3.1.3) on $L^2(\mu^{-1/2})$ yielding

$$\mu^{(1)}(t,v) = -\beta(t)^{\gamma/2} \mathscr{L}^{-1}\left[v \cdot Av\, \mu\right].$$

Note that due to $\beta(t)^{-\gamma/2} \to \infty$ we obtain $\mu^{(1)}(t,v) \to 0$ as $t \to \infty$, which is consistent with the assumption (3.1.2) for $k = 1$. Similarly, one can show formally that $\mu^{(k)}(t,v) = \mathcal{O}(\beta^{k\gamma/2}(t))$ as $t \to \infty$. With this let us now turn to the asymptotics of the temperature

$$\frac{\beta'}{2\beta} = \alpha_F = \frac{1}{3}\int_{\mathbb{R}^3} v \cdot Av\, \mu^{(1)}(t,v)\, dv + \mathcal{O}(\beta^\gamma(t)). \tag{3.1.4}$$

First, let us ignore the last term, yielding the equation

$$\begin{aligned}\left(\beta^{-\gamma/2}(t)\right)' &= -\gamma \beta(t)^{-\gamma/2} \frac{1}{3}\int_{\mathbb{R}^3} v \cdot Av\, \mu^{(1)}(t,v)\, dv = \frac{\gamma}{3}\int_{\mathbb{R}^3} v \cdot Av\, \mathscr{L}^{-1}\left[v \cdot Av\mu\right] dv \\ &= \frac{\gamma a_0}{3}, \quad a_0 := \left\langle v \cdot Av\mu, \mathscr{L}^{-1}\left[v \cdot Av\mu\right]\right\rangle_{L^2(\mu^{-1/2})}.\end{aligned} \tag{3.1.5}$$

Note that $a_0 > 0$, since $\mathscr{L}$ is a positive operator on $(\ker \mathscr{L})^{\perp}$. In particular, we have

$$\beta^{-\gamma/2}(t) \sim t, \quad t \to \infty.$$

Furthermore, terms of order $\beta(t)^{k\gamma/2} \sim t^{-k}$ can be omitted for $k \geq 2$ as they are integrable in time and thus have no essential effect on the asymptotics. This was in particular done in (3.1.4) yielding the asymptotics for the temperature. Observe that all in all the ansatz used above is consistent. Furthermore, observe that the above computation yields $T(t) \sim t^{2/\gamma}$ as $t \to \infty$, i.e. due to the effect of the shear the temperature is increasing in time.

The physical interpretation of the above formal asymptotics is as follows. Due to the action of the shear the velocities in the gas become larger (at least if the gas is already close to equilibrium). Higher velocities lead to more collisions, since the collision operator scales like $B(v - v_*, n \cdot \sigma) \sim |v - v_*|^{\gamma}$ as $|v - v_*| \to \infty$ with $\gamma > 0$ (hard potential case). Thus, the mixing properties of the collision operator and hence the relaxation towards the equilibrium is enforced. This leads again to the growth of the temperature.

The above asymptotics can be carried out also in the case of shear with decaying planar dilation/shear as well as combined orthogonal shear, see Section 1.1.2. In this case the shear is the most relevant effect compared to the dilation. On the other hand, dilation leads to a decrease of temperature. If accordingly the Boltzmann operator scales like $B(v - v_*, n \cdot \sigma) \sim |v - v_*|^{\gamma}$ as $|v - v_*| \to \infty$ with $\gamma < 0$ (soft potential case), then one can use the Hilbert-type expansion to obtain a similar behavior. The ranges of the parameter $\gamma < 0$ for which this can be done have been identified in [97].

In order to make the above asymptotics rigorous, the following has to be taken into account:

(i) One has to show that the solution indeed takes the form

$$F(t,v) = \mu(v) + \mu^{(1)}(t,v) + h(t,v). \tag{3.1.6}$$

Here, the error term h is expected to be integrable in time. In the above case we expect $h = \mathcal{O}(t^{-2})$ as $t \to \infty$.

(ii) The function $\beta^{-\gamma/2}$ behaves like t for $t \to \infty$.

To this end, two main assumptions have been used in the work (II):

(i) The error term $h\,|_{t=0}$ is initially small (in appropriate norms).

(ii) The initial temperature $\beta(0)^{-1} = T(0)$ is sufficiently large.

The first assumption ensures that the Hilbert expansion (3.1.6) is valid not only asymptotically when $t \to \infty$ (as expected from the formal study), but also for times of order one. In fact, one shows that the smallness $h\,|_{t=0}$ is propagated in time. The second assumption ensures that the collision operator is always dominant compared to the drift term and not only asymptotically as $t \to \infty$. Here one shows that if $\beta(0)^{-1}$ is large then also $\beta(t)^{-1}$ is large for all times $t \geq 0$.

Under these assumptions the above asymptotics can be proved rigorously, see Theorem B.1.1 (i) and Theorem B.1.2 (i) in the work (II). The above asymptotics can also be proved in the case of simple shear with decaying planar dilation/shear and combined orthogonal shear. Let us mention that the precise statements in particular the norms used depend on the linearized collision operator, in particular if the kernel has an angular cutoff or not.

3.2 Methods and main strategy of the proof

Let us now give the main ideas towards the proof of the previously discussed asymptotics. We will again stick to the case of simple shear for simplicity. We consider a solution to (3.1.1) of the form

$$F(t,v) = \mu(v) + \bar{\mu}(t,v) + h(t,v)$$

where h is some error term and

$$\bar{\mu}(t,v) = \mu^{(1)}(t,v) = -\beta(t)^{\gamma/2}\mathscr{L}^{-1}\left[v\cdot Av\,\mu\right]$$

is the first order approximation according to the Hilbert-type expansion. Let us mention that one can prove the well-posedness and regularity to (3.1.1) using tools known for the homogeneous Boltzmann equation, see Propositions B.2.2 and B.2.6.

The equation solved by the error term h has the form

$$\begin{aligned}\partial_t h &= S(t) + \mathscr{R}(t)h - \alpha_h(t)\operatorname{div}(v\,h) + \beta(t)^{-\gamma/2}Q(h,h) - \beta(t)^{-\gamma/2}\mathscr{L}h,\\ \alpha_h(t) &= \frac{1}{3}\int_{\mathbb{R}^3} v\cdot Av\,h(t,v)\,dv.\end{aligned} \tag{3.2.1}$$

Here, $S(t)$ is a source term, while $\mathscr{R}(t)$ is a linear operator acting on h. Both depend on $\bar{\mu}$. Furthermore, we have from the formula of $\bar{\mu}$ that

$$S(t) = \mathcal{O}\left(\frac{1}{\beta(t)^{-\gamma/2}}\right). \tag{3.2.2}$$

See Section B.3.1 for the precise formulas of $S(t)$ and $\mathscr{R}(t)$. The only nonlinear terms present are given by

$$\alpha_h(t)\operatorname{div}(v\,h) \quad \text{and} \quad \beta(t)^{-\gamma/2}Q(h,h).$$

On the other hand, the most important term in (3.2.1) is the linear one given by $\beta(t)^{-\gamma/2}\mathscr{L}h$.

Continuation argument. The main idea in the study of the asymptotics of solutions to (3.2.1) are the following two implications for some $T\geq 0$.

(i) As long as $\beta(t)^{-\gamma/2}\approx\varepsilon^{-1}+t$ for $t\in[0,T]$ we have

$$\|h(t)\| \lesssim \frac{\varepsilon}{(1+t)^2} + \frac{1}{(\varepsilon^{-1}+t)^2}$$

for $t\in[0,T]$.

(ii) As long as

$$\|h(t)\| \lesssim \frac{\varepsilon}{(1+t)^2} + \frac{1}{(\varepsilon^{-1}+t)^2}$$

for $t\in[0,T]$ we have $\beta(t)^{-\gamma/2}\approx\varepsilon^{-1}+t$ for $t\in[0,T]$.

Note that the implications (i) and (ii) are the reverse of the other. The parameter $\varepsilon > 0$ is then chosen sufficiently small in order to show that both statements are true for all times. This is needed as the constants appearing in the estimates (suppressed with the symbols $\approx$ and $\lesssim$) might differ in (i) and (ii). Furthermore, note that the above estimates are true for $t = 0$ by assuming that $\beta(0) = T(0)^{-1}$ and $\|h(0)\|$ are sufficiently small. Let us mention that the particular form of the estimate for $\|h\|$ is due to the effect of the initial datum and the source term as will be seen below.

Let us give the main arguments for (i) and (ii). Concerning (i) the idea is as follows. As the error term is small one can neglect the nonlinear terms in (3.2.1). Furthermore, since $\beta(t)^{-\gamma/2}$ is large and grows linearly in time, we expect that the semigroup generated by the operator $\mathscr{R}(t) - \beta(t)^{-\gamma/2}\mathscr{L}$ behaves as the one generated by $-\beta(t)^{-\gamma/2}\mathscr{L}$. We thus need to consider

$$\partial_t h = -\beta(t)^{-\gamma/2}\mathscr{L}h + S(t).$$

We can solve this using Duhamel's formula yielding

$$h(t) \approx \mathcal{P}_{0,t}h_0 + \int_0^t \mathcal{P}_{r,t}S(r)\,dr, \quad \mathcal{P}_{r,t} = \exp\left(-\int_r^t \beta(s)^{-\gamma/2}\,ds\,\mathscr{L}\right).$$

We now use the fact that the semigroup $e^{-t\mathscr{L}}$ has an exponential decay in appropriate spaces (this is discussed below), i.e. we have

$$\left\|e^{-t\mathscr{L}}f\right\| \lesssim e^{-\kappa t}\|f\|, \quad f \in (\ker\mathscr{L})^{\perp}.$$

We thus obtain, together with $S(t) \in (\ker\mathscr{L})^{\perp}$ for all $t \geq 0$,

$$\|h(t)\| \lesssim \exp\left(-\kappa\int_r^t \beta(s)^{-\gamma/2}\,ds\right)\|h_0\| + \int_0^t \exp\left(-\kappa\int_r^t \beta(s)^{-\gamma/2}\,ds\right)\frac{1}{\beta(r)^{-\gamma/2}}\,dr.$$

Here, we used (3.2.2) and $\|h_0\| \leq \varepsilon$. We now use the assumption $\beta(t)^{-\gamma/2} \approx \varepsilon^{-1} + t$ in (i) and use partial integration to obtain

$$\begin{aligned}\|h(t)\| &\lesssim \frac{\varepsilon}{(1+t)^2} + \int_0^t e^{-\kappa\int_r^t(\varepsilon^{-1}+s)\,ds}\frac{1}{\varepsilon^{-1}+r}\,dr \\ &\lesssim \frac{\varepsilon}{(1+t)^2} + \frac{1}{(\varepsilon^{-1}+t)^2} + \int_0^t e^{-\kappa\int_r^t(\varepsilon^{-1}+s)\,ds}\frac{1}{(\varepsilon^{-1}+r)^3}\,dr.\end{aligned}$$

The last integral can now be estimated by splitting into $r \in [0, t/2]$ and $r \in [t/2, t]$. This yields the bound asserted in (i).

On the other hand, concerning (ii) the argument follows as in the previous subsection. We have the ODE, see (3.1.1),

$$\frac{\beta'(t)}{2\beta(t)} = \alpha_F(t) = a_0 + \mathcal{O}(\|h(t)\|),$$

where a_0 is given in (3.1.5). From the assumption on $\|h(t)\|$ in (ii) and $\beta(0)$ small enough it then follows

$$\beta(t)^{-\gamma/2} \approx \varepsilon^{-1} + t.$$

The main difficulty in the proof is to track carefully the constants in the estimates to make sure that the above statements in (i) and (ii) are true for all times. To this end, the smallness on β_0 and $\|h_0\|$ is essential. This allows for some extra room in the estimates to propagate the bounds in (i) and (ii) for all times (continuation argument).

Decay of semigroup. As mentioned above the most important term is given by the linear operator $\mathscr{L}$. As is well-known this operator is non-negative and self-adjoint on $L^2(\mu^{-1/2})$. It has a spectral gap in the case of hard potentials $\gamma > 0$. Quantitative estimates of this spectral gap are provided in [133, 136]. Let us denote by Π_0 the orthogonal projection onto the kernel $\ker \mathscr{L}$ and $\Pi_1 = I - \Pi_0$. The spectral gap implies

$$\left\| e^{-t\mathscr{L}} \Pi_1 \right\|_{L^2(\mu^{-1/2})} \leq e^{-\theta t}$$

for some $\theta > 0$.

However, the space $L^2(\mu^{-1/2})$ is inconvenient for the study of homoenergetic solutions. In fact, this space forces $h \in L^2(\mu^{-1/2})$ to have Maxwellian decay at infinity. However, due to the drift term in (3.1.1) such a Maxwellian decay is not preserved over time. Thus, we need to provide estimates of the semigroup $e^{-t\mathscr{L}}$ in more convenient spaces such as L^1_p. Here, we denote by L^1_p the space $L^1(\mathbb{R}^3)$ with weight $\langle v \rangle^p$, where $\langle v \rangle = \sqrt{1+|v|^2}$. This problem was studied in [134] and a general method was provided in [79]. The result in the case of non-cutoff kernels was then studied in [152]. More precisely, it was proved that

$$\left\| e^{-t\mathscr{L}} \Pi_1 \right\|_{L^1_p} \leq C e^{-\kappa t}$$

for $p > 2$ and some $\kappa = \kappa(p, \theta) > 0$.

Let us give the main idea of the method provided in [79]. We consider an operator $\mathscr{L}$ on Banach spaces $E \subset \mathcal{E}$, e.g. $E = L^2(\mu^{-1/2})$ and $\mathcal{E} = L^1_p$ as above.

We have the following assumptions.

(i) We have $\ker \mathscr{L} \subset E$ and for some $\theta > 0$

$$\left\| e^{-t\mathscr{L}} \Pi_1 \right\|_E \leq e^{-\theta t}.$$

(ii) There is a decomposition of the form

$$\mathscr{L} = \mathcal{A} + \mathcal{B}$$

on $\mathcal{E}$ with the following properties. The operator $\mathcal{A} : \mathcal{E} \to E$ is bounded and the operator $\mathcal{B} : \mathcal{E} \to \mathcal{E}$ generates a semigroup with

$$\left\| e^{-t\mathcal{B}} \right\|_{\mathcal{E}} \leq e^{-\omega t}.$$

Then, the semigroup $e^{-t\mathscr{L}}$ satisfies

$$\left\| e^{-t\mathscr{L}} \Pi_1 \right\|_{\mathcal{E}} \leq C e^{-\kappa t} \tag{3.2.3}$$

for some $C > 0$ and $\kappa > 0$. This is a simplified version of the extension theorem in [79, Theorem 2.13]. The proof below is inspired by the proof in [46, Section 3.3], which allows to cover also cases with merely fractional exponential or polynomial decay of the semigroup.

Let us give some remarks. First of all, (i) gives the decay of the semigroup on the smaller space E. This is provided by knowing for instance a spectral gap as in the case of the linearized Boltzmann operator. Assumption (ii) provides a decomposition of $\mathscr{L}$ on $\mathcal{E}$ into a regularizing operator $\mathcal{A}$ and an operator $\mathcal{B}$ generating a decaying semigroup on $\mathcal{E}$. Note that $\mathcal{A}$ maps the space $\mathcal{E}$ into the smaller space E. Let us mention that $\mathscr{L}$ is in general not self-adjoint on $\mathcal{E}$ as in the case of the linearized Boltzmann operator.

In the case of the linearized Boltzmann operator the decomposition is found as follows. The operator splits into four parts

$$\begin{aligned}\mathscr{L}h &= -Q(h,\mu)-Q(\mu,h) = -\int_{\mathbb{R}^3}\int_{S^2} B(|v-v_*|, n\cdot\sigma)\left[\mu'_* h' + h'_*\mu' - \mu h_* - \mu_* h\right] d\sigma dv_* \\ &= T_1 h + T_2 h - T_3 h - T_4 h.\end{aligned}$$

The decomposition is essentially given as follows

$$\mathcal{A}h = \mathbb{1}_{|v|\le R}(T_1 h + T_2 h - T_3 h), \quad \mathcal{B}h = \mathbb{1}_{|v|>R}(T_1 h + T_2 h - T_3 h) - T_4 h$$

for cutoff kernels. For non-cutoff kernels one has to take into account the singularity of the collision kernel, see [152]. One can show, using Povzner estimates of the moments of order $p>2$, that $\mathcal{B}$ is hypodissipative on L^1_p, $p>2$ for R sufficiently large. More precisely, hypodissipative means that the following estimate holds

$$-\langle \mathcal{B}h, \mathrm{sgn} h\rangle_{L^1_p} = -\int_{\mathbb{R}^3} \mathcal{B}h \sin h \langle v\rangle^p \, dv \le -c_0 \|h\|_{L^1_p}.$$

In fact, this shows that $\mathscr{L}$ is hypodissipative for large velocities. The above estimate implies the exponential decay of the semigroup $e^{-t\mathcal{B}}$. On the other hand, the operator $\mathcal{A}$ maps L^1_p functions to bounded, compactly supported functions. In particular, they are in $L^2(\mu^{-1/2})$. All in all, assumption (ii) is satisfied.

Let us now give the main idea of the proof of the above extension theorem. We consider the equation

$$\partial_t h = -\mathscr{L}h = -\mathcal{A}h - \mathcal{B}h, \quad h(0) = h_0, \quad \Pi_0 h_0 = 0.$$

We now use the semigroup generated by $\mathcal{B}$ and write

$$h(t) = f(t) + g(t), \quad f(t) = e^{-t\mathcal{B}}h_0, \quad g(0) = 0.$$

We get

$$\partial_t g = -\partial_t(h-f) = -\mathcal{A}h - \mathcal{B}(h-f) = -\mathscr{L}g + \mathcal{A}f.$$

We apply Duhamel's formula to obtain

$$g(t) = e^{-t\mathscr{L}}g(0) + \int_0^t e^{-(t-s)\mathscr{L}}\mathcal{A}f(s)\, ds = \int_0^t e^{-(t-s)\mathscr{L}}\mathcal{A}e^{-s\mathcal{B}}h_0 \, ds.$$

We thus obtain

$$h(t) = f(t) + g(t) = e^{-t\mathcal{B}}h_0 + \int_0^t e^{-(t-s)\mathscr{L}}\mathcal{A}e^{-s\mathcal{B}}h_0\, ds.$$

Applying the operator Π_1 yields, as $\Pi_0 h(t) = 0$ for all times $t\ge 0$,

$$h(t) = \Pi_1[f(t)+g(t)] = \Pi_1 e^{-t\mathcal{B}}h_0 + \int_0^t e^{-(t-s)\mathscr{L}}\Pi_1\mathcal{A}e^{-s\mathcal{B}}h_0\, ds.$$

Note that $e^{-t\mathscr{L}}$ commutes with Π_1. We now use assumption (i) and (ii) to obtain

$$\begin{aligned}\|h(t)\|_{\mathcal{E}} &\leq e^{-\omega t}\|h_0\|_{\mathcal{E}} + \int_0^t e^{-\theta(t-s)}\left\|\mathcal{A}e^{-s\mathcal{B}}h_0\right\|_E ds \leq e^{-\omega t}\|h_0\|_{\mathcal{E}} + C\int_0^t e^{-\theta(t-s)}\left\|e^{-s\mathcal{B}}h_0\right\|_{\mathcal{E}} ds \\ &\leq e^{-\omega t}\|h_0\|_{\mathcal{E}} + C\int_0^t e^{-\theta(t-s)}e^{-\omega s}\|h_0\|_{\mathcal{E}}\, ds \leq Ce^{-\kappa t}\|h_0\|_{\mathcal{E}},\end{aligned}$$

with $\kappa = \min(\theta,\omega)$ if $\theta \neq \omega$ or any $\kappa < \theta = \omega$ otherwise. Note that the essential term is $e^{-(t-s)\mathscr{L}}\Pi_1\mathcal{A}e^{-s\mathcal{B}}h_0$. The operator $\mathcal{A}$ regularizes the function $e^{-s\mathcal{B}}h_0$ such that the semigroup $e^{-t\mathscr{L}}$ can dissipate it. This proves (3.2.3).

Nonlinear estimates. The last ingredient of the proof of the longtime behavior for the homoenergetic solutions are the nonlinear estimates. As is typical in perturbative frameworks the norms used to deal with nonlinear terms, such as $Q(h,h)$, also need to be appropriate for the linear operator, that is $\mathscr{L}h$. In the case of the Boltzmann equation without cutoff anisotropic, fractional Sobolev spaces with appropriate polynomial weights are used. We refer to Section B.1.3 and B.3.1 for the precise definitions. As can be proved, see Lemma B.3.8, the operator $\mathscr{L}$ is hypodissipative up to terms which can be bounded via L^1_p-norms. For these terms we use the decay properties of the semigroup in L^1_p. This allows to provide the needed estimates of the error h in (3.2.1).

Chapter 4

Summary of article (III). Vanishing angular singularity limit to the hard-sphere Boltzmann equation

In this chapter we give an overview of the work (III). The accepted manuscript to the published version [103] is reproduced in Appendix C.

In this work the family of collision kernels to the Boltzmann equation derived from inverse power law interactions is studied. In particular, the limit to the hard spheres model is proved on the level of the collision kernels as well as for solutions to the homogeneous Boltzmann equation.

4.1 Limit towards hard spheres collision kernel

Here, we give an overview of the connection between the two models of inverse power law potentials and the hard spheres potential.

Inverse power law interactions. As mentioned in the introduction, Section 1.1, these interactions yield collision kernels of the form

$$B_s(|v-v_*|, n\cdot\sigma) = |v-v_*|^\gamma b_s(n\cdot\sigma), \quad \gamma = \gamma(s) = \frac{s-5}{s-1}, \quad n = \frac{v-v_*}{|v-v_*|}.$$

Here, the index $s \in (2,\infty)$ refers to the power law potential $1/r^{s-1}$. Furthermore, the angular part $b_s : [-1,1) \to [0,\infty)$ can be computed implicitly by solving the two-body Hamiltonian equation with interaction potential $1/r^{s-1}$. We refer to Section C.1.2 for a formal derivation of the above collision kernels. It is customary to use spherical coordinates for σ with North Pole n. In this way we have $n\cdot\sigma = \cos\theta$. The angle θ is also referred to as the deviation angle of the corresponding collision event.

Let us note that one important feature of the angular part b_s is its singular behavior close to $n\cdot\sigma = 1$, i.e. θ close to zero. Such collisions are also referred to as grazing collisions. Recalling the post-collisional velocities

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \quad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma, \quad \sigma \in S^2$$

we have

$$\frac{v'-v'_*}{|v'-v'_*|}\cdot\frac{v-v_*}{|v-v*|}=\frac{v'-v'_*}{|v-v_*|}\cdot n=n\cdot\sigma.$$

Thus, in the reference frame moving at velocity $(v+v_*)/2=(v'+v'_*)/2$ the so-called deviation angle θ, where $\cos\theta=n\cdot\sigma$, is very small for grazing collisions. This is due to the fact that power law potentials are long-ranged, i.e. particles interact weakly at large distances.

On the level of the angular part the singular behavior of the form

$$\sin\theta\, b_s(\cos\theta)\sim\theta^{-1-2/(s-1)},\quad \theta\to 0,$$

was already computed in [47, Section II.5]. The precise asymptotics is given by, cf. Theorem C.2.1 (ii) in (III),

$$\lim_{\theta\to 0}\theta^{1+2/(s-1)}b_s(\cos\theta)\sin\theta=C_s,\quad C_s:=\frac{2^{4/(s-1)}}{s-1}\left(\frac{\sqrt{\pi}\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s-1}{2}\right)}\right)^{2/(s-1)}. \tag{4.1.1}$$

In particular, we have $b_s(\cos\theta)\sin\theta\sim\theta^{-1-2/(s-1)}$ as $\theta\to 0$.

Hard spheres interaction. A different model is given by hard spheres inducing the collision kernel $B(|v-v_*|,n\cdot\sigma)=\frac{1}{4}|v-v_*|$. The corresponding interaction potential has the form

$$r\mapsto\begin{cases}0 & r>1,\\ \infty & r\le 1.\end{cases} \tag{4.1.2}$$

Let us mention that the factor $1/4$ in the collision kernel is in general irrelevant and is customarily assumed to be one. This factor depends on the range of the hard spheres potential. However, for the study here the precise formula is used.

Limit behavior for $s\to\infty$. Note that the hard spheres potential (4.1.2) is the limit of the inverse power law potentials $1/r^{s-1}$ when $s\to\infty$ for $r\neq 1$. In the work (III) this limit is proved (cf. Theorem C.2.1 (i)) for the collision kernels, i.e. we have

$$B_s(|v-v_*|,n\cdot\sigma)=|v-v_*|^{\gamma(s)}b_s(n\cdot\sigma)\to\frac{1}{4}|v-v_*|,\quad s\to\infty.$$

Moreover, note that the singular behavior of the angular part vanishes in the limit $s\to\infty$. This is due to $C_s\sim 1/s\to 0$ as $s\to\infty$, where C_s is given in (4.1.1).

Hence, in order to describe the vanishing of the singular layer we make the following observation: as stated in (4.1.1), we have for $\theta\to 0$

$$b_s(\cos\theta)\sim C_s\theta^{-2-2/(s-1)}$$

while for $s\to\infty$

$$C_s\theta^{-2-2/(s-1)}\sim\frac{\theta^{-2}}{s}. \tag{4.1.3}$$

Thus, in order to capture the behavior of the singular layer of $b_s(\cos\theta)$ for $\theta \to 0$ when $s \to \infty$ it is convenient to introduce the variable $\theta = \psi/\sqrt{s}$. In the limit $s \to \infty$ we then obtain, cf. Theorem C.3.1 in (III),

$$\lim_{s\to\infty} b_s\left(\cos\left(\frac{\psi}{\sqrt{s}}\right)\right) = \Phi(\psi),$$

for a real analytic function $\Phi : (0,\infty) \to \mathbb{R}$. The function Φ connects the singular layer of the angular part b_s for $s \to \infty$ with the angular part of the hard spheres kernel (which is constant 1/4). More precisely, we have

$$\lim_{\psi\to\infty} \Phi(\psi) = \frac{1}{4}, \quad \Phi(\psi) = \frac{1}{\psi^2} + \frac{1}{\sqrt{\pi}}\frac{1}{\psi} + \Phi_0(\psi),$$

where $\Phi_0 : [0,\infty) \to \mathbb{R}$ is continuous, in particular finite when $\psi \to 0$. The singularity of order ψ^{-2} is expected from (4.1.3).

4.2 Limit towards Boltzmann equation with hard spheres

In the previous section we discussed the limit of the collision kernel for inverse power laws $1/r^{s-1}$ towards the hard spheres collision kernel when $s \to \infty$. Based on this it was proved in (III), see Theorem C.4.4, that also the solutions f^s to the homogeneous Boltzmann equation with collision kernel B_s

$$\partial_t f^s = Q_s(f^s, f^s), \quad f^s\,|_{t=0} = f_0 \tag{4.2.1}$$

converge weakly in L^1 to the solution f^∞ of the Boltzmann equation with hard spheres kernel

$$\partial_t f^\infty = Q_\infty(f^\infty, f^\infty), \quad f^\infty\,|_{t=0} = f_0 \tag{4.2.2}$$

when $s \to \infty$. Here, Q_s denotes the collision operator with kernel $B_s(|v-v_*|, n\cdot\sigma)$ while Q_∞ denotes the operator with kernel $B(|v-v_*|, n\cdot\sigma) = \frac{1}{4}|v-v_*|$. Note that all solutions have the same initial datum f_0.

Let us comment further on the methods used in this result. One crucial observation is that for $s > 5$ the collision kernel falls into the class of hard potential kernels. As mentioned in the introduction, see Section 1.1.1, such kernels are easier to tread, even in the case of non-cutoff kernels (as is the situation for inverse power law potentials). In particular, the concept of weak solutions can be used without any ambiguities. There are three main ingredients used in the proof:

(i) *H*-Theorem: assuming that the entropy of the initial datum is finite, i.e.

$$H(f_0) = \int_{\mathbb{R}^3} f_0(v) \ln f_0(v)\, dv < \infty$$

we have $H(f^s(t)) \le H(f_0)$. This bound is uniform in $s > 5$.

(ii) Moment estimates: in the case of hard potentials one can apply variants of the Povzner estimates to obtain the bound for $p > 2$

$$\sup_{t\ge 0} \|f^s(t)\|_{L^1_p} \le C_p.$$

This bound is again uniform in $s > 5$ and depends only on $\|f_0\|_{L^1_p}$.

(iii) Uniqueness for hard spheres kernel: as was shown in [129], in the case of hard spheres weak solutions f^∞ to the homogeneous Boltzmann equation (4.2.2) which preserve the total energy are unique. Conservation of the kinetic energy means that for all $t \geq 0$

$$\int_{\mathbb{R}^3} |v|^2 f^\infty(t,v)\,dv = \int_{\mathbb{R}^3} |v|^2 f^\infty(t,v)\,dv.$$

In fact, this result turns out to be sharp. In [122, 161] solutions with increasing energy have been constructed. These solutions are constructed as the limit of solutions f^n with an initial profile of the form

$$f^n(0,v) = f_0(v) + \frac{1}{4\pi n}|v|^{-5-1/n}\mathbb{1}_{\{|v|\geq 1\}}.$$

Here, $f_0 \in L^1_2$ with finite entropy is arbitrary. In the limit, the solution f with initial condition f_0 has an energy jump at $t = 0$. This is due to the effect of the second term which only decays like $|v|^{-5-1/n}$. In particular, it is close to the threshold of integrability with respect to the weight $|v|^2$. The argument can be extended to yield jumps in the energy for positive times, see [122].

These ingredients can be then combined to yield the result as follows: first of all for any initial datum $f_0 \in L^1_p$, $p > 2$, one can construct a sequence of weak solutions to (4.2.1) which preserve the kinetic energy. Due to the bound in (i) a sequence of weak solutions f^s to (4.2.1) have a converging subsequence in L^1. Passing to the limit in the weak formulation entails that any limit f^∞ is a weak solution to (4.2.2). Assuming $p > 2$ and using the bound in (ii) also all the moments of order $q < p$ of the weak solutions along the subsequence converge. In particular, the limit point f^∞ is a weak solution to (4.2.2) satisfying the conservation of kinetic energy. Thus, by (iii) it is unique. Consequently, the whole sequence f^s, and not merely a subsequence, converges towards the same limit point f^∞. We refer to Appendix C for the full proof.

Chapter 5

Summary of article (IV). Rotating solutions to the incompressible Euler-Poisson equation with external particle

In this chapter we give an overview of the work (IV), which is reproduced in Appendix D.

In this work we consider a two-dimensional, self-interacting, incompressible fluid body $E(t) \subset \mathbb{R}^2$ which is perturbed by an external particle $X(t)$ with small mass $m > 0$. The shape of the fluid body is assumed to be closed to the unit disk $\mathbb{D}$ and is deformed due to the interaction with the particle. Under certain assumptions we prove the existence and uniqueness of stationary solutions to the Euler-Poisson equation in a rotating frame of reference. In addition, differently from the results on self-gravitating, ellipsoidal figures reviewed in [52], see Section 1.2.1, we consider solutions which contain non-trivial internal motion $v \neq 0$ in any coordinate system.

5.1 Main Model

The equations under study is the incompressible Euler-Poisson equation with an external particle see (1.2.4). We are looking for stationary solutions in a rotating frame of reference, that is, to the equation (1.2.5). Here, we restrict our attention to the two-dimensional case. To this end, we identify $v = (v_1, v_2)^\top$ with the vector $(v_1, v_2, 0)^\top$ and similarly for $x = (x_1, x_2)$. One can rewrite the vector product with $\Omega = \Omega_0(0,0,1)^\top$ as $\Omega \times v = \Omega_0 Jv$, $Jv = (-v_2, v_1)^\top$. Furthermore we have $\Omega \times (\Omega \times x) = -\Omega_0^2 x$. Thus, from the system (1.2.5) we obtain the equations

$$\begin{cases} (v\cdot\nabla)v + 2\Omega_0 Jv - \Omega_0^2 x = -\nabla p - \nabla U_E - m\nabla U_X & \text{in } E \\ \nabla\cdot v = 0 & \text{in } E \\ n\cdot v = 0 & \text{on } \partial E \\ p = 0 & \text{on } \partial E \\ \Omega_0^2 X = \nabla U_E(X) & \\ |E| = \pi & \\ \int_E x\,dx + mX = 0. & \end{cases} \tag{5.1.1}$$

In the above system we also included the constraint $|E| = |\mathbb{D}| = \pi$, which fixes the mass of the fluid body. Furthermore, we also included

$$\int_E x\,dx + mX = 0$$

so that the center of mass of the system is at the origin. One can show that in fact this condition is a consequence of the other equations in (5.1.1). Indeed, the center of mass of a system without external forces moves at constant speed. However, due to the rotating frame of reference it has to be at rest and thus in the origin.

Let us mention that the above two-dimensional model cannot be viewed as a flat or planar solution to the three-dimensional case. In this case the pressure would also act in the vertical direction. However, in the model above the pressure is only acting in the plane.

In the two-dimensional case the attractive force is given via the fundamental solution of the two-dimensional Laplace equation, that is

$$U_{X(t)}(x) := \ln|x - X(t)|, \quad U_{E(t)}(x) := \int_{E(t)} \ln|x-y|\,dy.$$

However, in the work (IV) we can treat a family of power law potentials as well. More precisely for $\nu \in (0,1]$ have

$$U_{X(t)}(x) := -\frac{1}{|x - X(t)|^\nu}, \quad U_{E(t)}(x) := -\int_{E(t)} \frac{dy}{|x-y|^\nu}.$$

This includes also the standard Newtonian gravitational potential for $\nu = 1$. However, in this case the gradient $\mathrm{grad} U_E$ is not well-defined due to the singularity. In order to solve this problem one absorbs this singular term in the pressure by writing $p = P - U_E - mU_X$. Here, P is the so-called non-hydrostatic pressure. We obtain then the equation

$$\begin{cases} (v\cdot\nabla)v + 2\Omega_0 Jv - \Omega_0^2 x = -\nabla P & \text{in } E \\ \nabla\cdot v = 0 & \text{in } E \\ n\cdot v = 0 & \text{on } \partial E \\ P = U_E + mU_X & \text{on } \partial E \\ \Omega_0^2 X = \nabla U_E(X) \\ |E| = \pi \\ \int_E x\,dx + mX = 0. \end{cases} \tag{5.1.2}$$

Here, the term U_E is now well-defined.

Let us mention that the system (5.1.2) is invariant under rotations around the origin. Hence, we can assume without loss of generality that the particle $X = (a,0)$ is located on the x_1-axis. In particular, a solution to (5.1.2) yields a family of solutions by rotating the solution around the origin. As a result of the choice $X = (a,0)$, the equations are invariant under reflections $x_2 \mapsto -x_2$. Consequently, since our perturbative framework allows the construction of unique solutions, the domain E is symmetric w.r.t. the x_1-axis.

The perturbation parameter is $m > 0$. In the case that $m = 0$ the shape of the body is given by the unit disk. As a consequence the corresponding interaction potential U_0 is radial. Furthermore, the position of the particle $X = (a,0)$ is close to $X_0 = (a_0,0)$ satisfying $\Omega_0^2 X_0 = \nabla U_0(X_0)$, that is $a_0\Omega_0^2 = U_0'(a_0)$. In fact, this yields a one-to-one correspondence between $a_0 \geq 1$ and $0 < \Omega_0 \leq U_0(1)^{1/2}$. We stress that we studied only the case in which the body and the particle are separated. In particular, we assume (say) $a_0 \geq 2$, i.e. $0 < \Omega_0 \leq U_0(2)^{1/2}$

5.2 Reformulation of the problem

In order to study the above free-boundary problem we make use of conformal mappings which are well-suited for two-dimensional problems. More precisely, the shape of the fluid body E is given via a conformal, i.e. an injective and analytic function $f : \mathbb{D} \to \mathbb{C}$, $f(\mathbb{D}) =: E$. Here, we identify $\mathbb{R}^2 \simeq \mathbb{C}$. For the perturbative framework we make the ansatz $f_h(z) = z + h(z)$ for some small analytic function h and set $E_h := f_h(\mathbb{D})$. Using a re-parametrization of $\mathbb{D}$ by the so-called Blaschke factors we can ensure $h(0) = 0$, $h'(0) \in \mathbb{R}$.

On the other hand in order to rewrite the Euler-equations for the velocity field we use the Grad-Shafranov method [77, 146]. More precisely, we write the velocity field v via a stream function ψ, i.e. $v = J\nabla\psi$. In two space dimensions the corresponding vorticity $(0,0,\omega) = \nabla \times v$ satisfies $v \cdot \nabla(\omega + 2\Omega_0) = 0$ in E. This can be seen by applying $\nabla\times$ to the Euler-Poisson equation in (5.1.1) and using $\nabla \cdot v = 0$. In particular, ω is constant along stream lines and hence on level sets of ψ. Hence, we can assume that $\omega = G(\psi)$ for some (regular) function $G : \mathbb{R} \to \mathbb{R}$. Since $\Delta\psi = \omega$ the stream function is chosen to satisfy

$$\begin{cases} \Delta\psi = G(\psi) & \text{in } E_h, \\ \psi = 0 & \text{on } \partial E_h, \end{cases} \tag{5.2.1}$$

The above boundary condition follows from $v \cdot n = 0$ on ∂E_h together with the fact that ψ is given by v only up to some constants (modifying G if necessary).

In our study we assume that G is given and fixed in order to describe the internal motion of the fluid. Furthermore, we assume that the function G is non-decreasing. This ensures the existence and uniqueness of solutions to (5.2.1). Let us emphasize that $\psi = \psi_h$ depends on the shape of the body E_h, in particular on the function h. The unperturbed stream function ψ_0 is accordingly given by (5.2.1) for $h \equiv 0$. Due to the rotational symmetry ψ_0 is rotationally symmetric.

We can now reformulate the problem (5.1.2) using the fact that the pressure is constant on the free-boundary. Thus, we obtain that the so-called Bernoulli head $H = p + \frac{1}{2}|v|^2 - \frac{1}{2}\Omega_0^2|x|^2 + U_E + mU_X$ is constant on ∂E_h, i.e. $H \equiv \lambda$, for some $\lambda \in \mathbb{R}$. With this the system (5.1.2) can be reduced to

$$\begin{cases} \frac{1}{2}|\nabla\psi_h|^2 - \frac{\Omega_0^2}{2}|x|^2 + U_{E_h} + mU_X = \lambda & \text{on } \partial E_h \\ \Omega_0^2 a = \partial_{x_1} U_{E_h}(a,0) \\ |E_h| = \pi. \end{cases} \tag{5.2.2}$$

The unknowns are h, $X = (a,0)$, λ. Furthermore, the function ψ_h is given via (5.2.1).

Note that in (5.2.2) we only included the first component of the equation $\Omega_0^2 X = \nabla U_{E_h}(X)$ since the second component is satisfied by the symmetry of the domain with respect to the x_1-axis.

5.3 Main result

The main result concerns the existence and uniqueness of solutions to (5.2.2). To this end, we apply a version of the implicit function theorem, see Lemma D.3.3. The perturbation parameter is the mass m of the external particle. This allows to construct solutions to (5.2.2) with $h \approx 0$, $X \approx X_0$, $\lambda \approx \lambda_0 = \frac{1}{2}\psi_0'(1)^2 - \frac{1}{2}\Omega_0^2 + U_0(1)$. Concerning the function h we use the Banach space

$H(\mathbb{D})\cap C^{k+2,\alpha}(\overline{\mathbb{D}})$, where $H(\mathbb{D})$ is the space of analytic functions on $\mathbb{D}$ and $k\in\mathbb{N}_0$, $\alpha\in(0,1)$. We equip it with the standard Hölder norm $\|\cdot\|_{k+2,\alpha}$.

An important part in the application of the implicit function theorem is the study of the invertibility of the linearized operator at the unperturbed solution. The main part of this operator acts on h, restricted to the boundary $\partial\mathbb{D}$ of the unit disk (i.e. the one-dimensional torus), via Fourier multipliers given by

$$\omega_n = -\frac{1}{2}\Omega_0^2 - \frac{1}{2}\psi_0'(1)^2(|n|+1) + \psi_0'(1)A_{|n|}'(1)(|n|+1) + c_{|n|}. \tag{5.3.1}$$

The coefficients c_n enter the linearization through the linearization of the interaction potential U_{E_h}. Furthermore, $A_n'(1)$ is implicitly given by the functions $A_n:(0,1)\to\mathbb{R}$ solving the ODE

$$\frac{1}{r}(rA_n')' - \frac{n^2}{r^2}A_n - G'(\psi_0(r))A_n = r^{|n|}G(\psi_0(r)), \quad A_n(1)=0. \tag{5.3.2}$$

They appear through the linearization of the stream function $h\mapsto\psi_h\circ f_h$. We refer to Sections D.4 and D.5 for the derivation of the above formulas.

The main result can now be summarized as follows, see Theorem D.2.1.

Theorem 5.3.1. *Let $k\in\mathbb{N}_0$, $\alpha\in(0,1)$ and choose $a_0(\Omega_0)\geq 2$ with $a_0\Omega_0^2 = U_0'(a_0)$. In addition, assume that*

(i) $G\in C^{k+3}(\mathbb{R};\mathbb{R})$ is non-decreasing;

(ii) $\psi_0'(1)\neq 0$;

(iii) $\omega_n\neq 0$ for all $n\neq 0$.

Then, there are $\varepsilon,\delta>0$ such that: for any $m\in(0,\delta)$ there is a unique solution h, $X=(a,0)$, λ to (5.2.2) with

$$\|h\|_{k+2,\alpha} + |a-a_0| + |\lambda-\lambda_0| < \varepsilon.$$

Finally, the domain $E_h = f_h(\mathbb{D})$ is symmetric w.r.t. the x_1-axis and the corresponding velocity field $v=\nabla^\perp\psi_h$ together with the position of the particle $X=(a,0)$ yield a solution to (5.1.1).

Let us note that the condition (i) is used in order to ensure well-posedness of (5.2.1). Condition (ii) on the other hand is used to avoid the appearance of local extrema of ψ_0 at the boundary. Perturbations of such extrema lead generically to saddle points, which correspond to formations of vortices. Finally, (iii) ensures that the linearized operator is invertible. The condition (iii) can be interpreted as a non-resonance condition on the angular velocity Ω_0. For resonant values of Ω_0 bifurcations to other shapes might occur. It is shown in the work (IV) that in fact, this condition is satisfied for $|n|$ large. Finally, some numerical studies can be found after Theorem D.2.1.

Chapter 6

Conclusion and open problems

In this thesis we studied both the Boltzmann equation and incompressible Euler-Poisson equation. The main focus in the case of the Boltzmann equation were solutions that are out of equilibrium due to the effect of a shear flow in the gas. This lead to either a self-similar longtime behavior, cf. (I), or to the approach of the equilibrium distribution with variable temperature, cf. (II). Furthermore, in (III) we studied the collision kernels as well as solutions to the homogeneous Boltzmann equation when passing from inverse power law interactions to the hard spheres interaction. Finally, in (IV) we proved the existence of a rotating solutions to the two-dimensional, incompressible Euler-Poisson equation with a small external particle. The configuration of the fluid body is constructed as a perturbation of the disk. Furthermore, we included non-trivial internal motions close to general shear flows leaving the disk configuration invariant.

In the following we give a list of open questions and possible future projects arising from the study in this thesis.

6.1 Self-similar behavior for Maxwell molecules

The study of the self-similar solution in (I) made essential use of the smallness of the shear. In particular, this condition appeared

(i) in the study of the eigenvalue problem for the second order moment equations, see (2.2.1), so that the spectrum remains close to the situation without shear, in which the spectrum can be computed explicitly;

(ii) in the proof of uniform bounds of higher moments using Povzner estimates;

(iii) in the proof of the stability of the self-similar solution.

However, for large shears the longtime behavior of solutions remains in general unclear. A first step would be the study of the second order moment equations. In the case of simple shear and planar shear this was done in [98, Section 5]. They prove (under certain conditions on the collision kernel in the case of planar shear) that the spectrum shares the same structure as in the study of (I). Furthermore, for simple shear they formulated a condition yielding also uniform moment bounds as in (ii). This allows to prove the existence of a self-similar solution. The condition formulated can be verified using numerical methods.

However, the stability of this solution, point (iii) above, needs to be studied independently. The first step would be to study the stability of the eigenvector $\bar{N} \in \mathbb{R}^{3\times 3}$ in the ODE system of the second moments, i.e. the second moments of the self-similar profile. Furthermore, the study of the higher moments using numerical methods might be useful.

On the other hand, it might be that the self-similar behavior (or even the existence of a self-similar profile) is broken for certain values of the shear. In this case, one might again gain some insights through the ODE systems for the moments.

Another open question concerns qualitative properties of the self-similar solution constructed in (I). It was shown there that the profile has moments of order p if the shear is chosen small enough, depending on p. It remains unclear if the profile for a fixed (but still small) value of the shear has only power law decay. For shear flows this is suggested by numerical experiments, see [74].

6.2 Collision-dominated behavior for soft potentials

In the work (II) we studied the collision-dominated behavior for hard potentials. As stated in [97] the Hilbert-type expansion can also be used to yield a similar longtime behavior for soft potentials. In contrast to the hard potential case the shear of the mechanical work is not the decisive mechanism but the dilation. This leads to a decrease of the temperature. Rescaling the temperature to one, one can see that for certain values of $\gamma < 0$, depending on the choice of the matrix $L(t)$, the Hilbert-type expansion yields a consistent asymptotics close to a Maxwellian distribution. In fact, in this case the expansion has the simple form $f = \mu + h$ and there is no need to include the first order approximation $\mu^{(1)}$ as in (3.1.6), see [97]. Following the analysis of the hard potential case in Chapter 3 the most important part is the study of the linearized problem. In this case, the linear equation has the form

$$\partial_t h = \operatorname{div}\left(\left(L(t) - \frac{\beta'(t)}{2\beta(t)} I\right) v\, h\right) - \beta(t)^{-\gamma/2} \mathscr{L} h.$$

It is expected from the Hilbert-type expansion that $\beta(t)^{-\gamma/2}$ grows exponentially in time.

The main difference to the hard potential case is twofold.

(i) The linearized collision operator $\mathscr{L}$ has no longer a spectral gap on the space $L^2(\mu^{-1/2})$. It is proved in [78], in the case of the inhomogeneous Boltzmann equation close to equilibrium, that one can recast at least polynomial decay in $L^2(\mu^{-1/2})$ of the semigroup. However, to this end an a priori bound on a $L^2(\mu^{-1/2})$-norm with polynomial weight is needed. On the other hand, as in (II) the space $L^2(\mu^{-1/2})$ is inconvenient for homoenergetic solutions since the flow of the equation does not preserve the Maxwellian decay at infinity. Thus, extension results on larger function spaces, like weighted L^1-spaces, have to be established again.

(ii) The drift term is more dominant for $|v| \to \infty$ for times of order one. In fact, the drift term scales like $\mathcal{O}(1)$ while the collision operator scales like $\mathcal{O}(|v|^\gamma)$ for $|v| \to \infty$. In particular, moment bounds cannot be established without using the specific structure of the drift term.

In fact, the method in [79] and the variant of the proof presented in Chapter 3 is flexible enough to prove a solution of point (i). More precisely, one can obtain the following result.

Let $q > p > |\gamma| + 2$. Then the semigroup $e^{-t\mathscr{L}}$ of the operator $\mathscr{L}$ satisfies the following estimate for $t \geq 0$

$$\left\| e^{-t\mathscr{L}} \Pi_1 h \right\|_{L^1_p} \leq \frac{C}{(1+t)^{q/(q+|\gamma|)}} \|h\|_{L^1_q}.$$

Compare this with the exponential decay in Chapter 3 or in Lemma B.3.12.

This yields polynomial decay in time for the linear semigroup. However, an estimate of the L^1_q-norm is needed. In particular, this requires moment bounds of order q, which requires a resolution of point (ii) above. Up to now a solution to the issue in (ii) remains to be found.

6.3 Hyperbolic dominated behavior

As mentioned in Section 1.1.2 there is a different regime for the longtime behavior for homoenergetic solutions which has not been studied rigorously so far. In this case the drift term (hyperbolic term) is more important than the collision operator. Conjectures have been stated in [99].

One particular case appears for instance for simple shear with homogeneity $\gamma < -1$. Here, the collision rate is of the order e^{-t} for $t \to \infty$ as long as the solutions follows the flow of the drift term. In particular, the effect of the collision operator is expected to be integrable in time. This situation was termed *frozen collisions* in [99]. The mathematical result one could aim for is that the solution approaches some profile moving along the characteristics of the drift term. This profile would depend on the initial datum and one might not be able to construct it explicitly.

6.4 Shear flows for mixture of gases

The studies on homoenergetic solutions mentioned so far and studied in this thesis concern a single gas subject to mechanical forces. When considering a multi-component gas further non-equilibrium situations can occur. Let us mention here a particular situation studied in [74, Section 4.4]. We consider a mixture of two gases, which are statistically described by two distribution functions $f_1(t,x,v)$, $f_2(t,x,v)$. The distribution functions satisfy two coupled Boltzmann equations. The coupling appears through the collision of particles from species 1 with particles from species 2. We assume that the particles in species 1 and 2 have respectively mass m_1 and m_2. Furthermore, we assume the presence of a simple shear in the gas. In this case one can write the distribution functions as

$$f_1(t,x,v) = g_1(t,v-\xi(t,x)), \quad f_2(t,x,v) = g_2(t,v-\xi(t,x))$$

as in Section 1.1.2. For simple shear we have

$$\xi(t,x) = Lx, \quad L = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad a \neq 0.$$

Assuming that both species are composed of Maxwell molecules one can compute a closed system of ODEs for the moments of the gas mixture. Again the most important quantities are the temperatures T_i, $i = 1,2$, of the two gases, i.e.

$$T_i(t) = \frac{1}{3n_i} \int_{\mathbb{R}^3} |w|^2 g_i(t,w)\, dw, \quad n_i = \int_{\mathbb{R}^3} g_i(t,w)\, dw.$$

Here, n_i is the density, which is time-independent. Studying the eigenvalues of the ODE system yields the asymptotics for the temperatures

$$T_i(t) \sim e^{-\alpha_i t}, \quad t \to \infty,$$

for some $\alpha_i \in \mathbb{R}$, $i = 1, 2$.

An important regime is the so-called tracer limit. In this case, one sends the mole fraction $x_1 = n_1/n$, $n = n_1 + n_2$, to zero. Thus, one assumes that the density of the first species (the tracer particles) is much smaller than the density of the second species (the excess component). In particular, the effect of the tracer particles on the excess component is expected to be negligible. Such *tracer problems* arise in radiation-transfer and rarefied-gas dynamics when tagged particles are diffused in a concentrated colloidal suspension, see [74, Section 4.4] and references therein.

In this regime, it is of interest to look at the relative energy of the tracer particles to the total energy, i.e.

$$\frac{E_1(t,x_1)}{E(t,x_1)} = x_1 \frac{T_1(t,x_1)}{T(t,x_1)},$$
$$E_i(t,x_1) = n_i T_i(t,x_1), T(t,x_1) = T_1(t,x_1) + T_2(t,x_1), E(t,x_1) = E_i(t,x_1) + E_2(t,x_1).$$

Note that all quantities $E_i(t,x_1), T_i(t,x_1)$ depend on the parameter x_1.

In thermodynamical equilibrium (i.e. $a = 0$ for $t \to \infty$) one expects the equipartition of energy, that is $\lim_{t\to\infty} E_i(t,x_1) = \lim_{t\to\infty} n_i E(t,x_1)$, so that $\lim_{t\to\infty} E_1(t,x_1)/E(t,x_1) \to 0$ as $x_1 \to 0$. However, for shear flows the behavior can be much different. First of all, one can show using the system of ODEs for the second moments that for $x_1 \ll 1$ the behavior of the relative temperature is given by

$$\frac{T_1(t)}{T(t)} \sim \frac{T_1(t)}{T_2(t)} \sim e^{-\Omega t} + const.$$

with $\Omega = \alpha_1 - \alpha_2$. In fact, Ω depends on the relative mass $\mu = m_1/m_2$ and the shear rate $a > 0$ (as well as on the collision kernel). One can show that there is a critical value μ_c such that for $\mu > \mu_c$ the function $\Omega(\mu, a)$ remains positive for all $a > 0$. On the other hand, for $\mu < \mu_c$ this function can become negative for $a > a_c(\mu)$, for a critical value $a_c(\mu)$. In the first case the ratio of the temperature of the tracer particles to the total temperature approaches a finite value. However, in the second case the temperature of the gas of the trace particles goes to infinity. It is now of interest to look at the relative energy. In fact, one can show that

$$\lim_{x_1\to 0}\left\{\lim_{t\to\infty}\frac{E_1(t,x_1)}{E(t,x_1)}\right\} = \begin{cases} F(a,\mu) > 0 & \text{if } \mu < \mu_c,\, a > a_c(\mu), \\ 0 & \text{otherwise.} \end{cases}$$

The function $F(a,\mu)$ depends also on the collision kernel chosen.

We thus observe that the equipartition of energy is broken above the critical value $a > a_c(\mu)$, as long as $\mu < \mu_c$. In other words we observe a non-equilibrium phase transition in the tracer limit $x_1 \to 0$. Below the critical regime the energy is distributed according to the thermodynamical equilibrium (disordered state) while above the critical value the energy distribution breaks the law of equipartition of energy (ordered state).

The reason for this behavior can be explained as follows. The total temperature behaves as $T(t) = T_2(t) \sim e^{-\alpha_2 t}$ for $x_1 \ll 1$. We can now put the gas mixture into a heat bath, so that $T(t)$ approaches a finite value as $t \to \infty$. In this situation the tracer particles experience two competing effects. First, the thermostat is acting on the tracer particles. Second, tracer particles also collide with particles in the excess component (species 2). For $\mu < \mu_c$ this leads to an increase of the temperature (viscous heating). Note that the masses m_i of the particles, in particular the mass ratio μ, appear in the collision rule for the post-collisional velocities. Thus, on one hand the heat bath controls the temperature, while collisions with the second gas increases the temperature of the tracer particles. In the regime $a > a_c(\mu)$ the heat bath is not able to cool the viscous heating.

Numerical examples show that for $x_1 \approx 0.01$, $\mu = 0.1$ and $a = 2a_c(\mu)$ more than 60% of the energy is stored in the gas of the tracer particles. In strong contrast to the equilibrium situation in which the energy ratio would be of order $x_1 \approx 0.01$, see [74, Section 4.4].

The above behavior was studied in [74, Section 4.4] for Maxwell molecules using merely the second order moment equations. It would be of interest to study the regime $x_1 \approx 0$ for the full system of Boltzmann equations of the two species and in particular obtain the above results in the limits $t \to \infty$ and $x_1 \to 0$. A similar behavior is also expected for non-Maxwell molecules, see [74, Section 4.4] for the case of the BGK model.

6.5 3D incompressible Euler-Poisson equation

A natural extension of the work (IV) discussed in Chapter 5 on steady states of the two-dimensional, incompressible Euler-Poisson equation is the corresponding three-dimensional problem. Here, we again consider the gravitational interaction of a fluid body with shape $E \subset \mathbb{R}^3$ and an external particle $X \in \mathbb{R}^3 \setminus \overline{E}$ with small mass $m > 0$. Furthermore, the fluid experiences the force due to the gravitational self-interaction induced by its density $\rho = \mathbb{1}_E$. In addition, the whole configuration is assumed to rotate around the common center of mass with angular velocity $\Omega = (0,0,\Omega_0)$, $\Omega_0 > 0$.

The goal is to construct stationary solutions in a rotating frame of reference with angular velocity Ω. The system of equations then takes the form

$$\begin{cases} (v\cdot\nabla)v + 2\Omega\times v + \Omega\times(\Omega\times x) = \nabla\left[-p - U_E - mU_X\right], & \text{in } E_h, \\ \nabla\cdot v = 0, & \text{in } E_h, \\ n\cdot v = 0, & \text{on } \partial E_h, \\ p = 0, & \text{on } \partial E_h, \\ \Omega\times(\Omega\times X) = -\nabla U_E(X), & \\ |E| = \frac{4\pi}{3}, & \\ \int_E x\,dx + mX = 0. & \end{cases} \tag{6.5.1}$$

Here, we denoted

$$U_X(x) = -\frac{1}{|x-X|}, \quad U_E(x) = -\int_E \frac{dy}{|x-y|}$$

for the potentials due to the gravitational interaction. Note that the equations (6.5.1) are a tree-dimensional variant of (5.1.1).

As in the work (IV) we study a perturbative situation in which the shape of the fluid is close to the ball $B_1(0)$ while $m > 0$ is small. For the shape of the body we use a parametrization of the form $E_h = \{x \in \mathbb{R}^3 \ : \ |x| \le 1 + h(x/|x|)\}$, where $h : S^2 \to \mathbb{R}$ is to be found. Accordingly, we abbreviate $U_h := U_{E_h}$. The corresponding velocity field in the non-rotating frame of reference is close to be zero, i.e. the fluid has no internal motion to first order. In the rotating coordinate system the velocity is thus close to $V_0 = -\Omega\times x$. Different from the two-dimensional case we make an additional assumption on the perturbation of the fluid velocity. More precisely, the perturbation is assumed to be a potential flow, i.e. $v = V_0 + \nabla\phi$. The function $\phi : E_h \to \mathbb{R}$

satisfies the equation

$$\begin{cases} \Delta\phi = 0 & \text{in } E_h, \\ n_h \cdot \nabla\phi = -V_0 \cdot n_h & \text{on } \partial E_h. \end{cases}$$

Here, we denoted by n_h the unit outer normal vector of ∂E_h. These equations are a consequence of the divergence-free condition $\nabla \cdot v = 0$ and the zero flux condition $n_h \cdot v = 0$.

Using the vorticity $\omega = \nabla \times v$ we can rewrite the Euler equations in the form

$$v \times \omega - 2\Omega \times v = \nabla H$$

where the Bernoulli head H is defined by

$$H := p + \frac{1}{2}|v|^2 + U_h + U_X - \frac{1}{2}|\Omega \times x|^2.$$

Since $\omega = -2\Omega$ we obtain $H = \mathit{const.}$ on E_h. In particular, restricting on the boundary gives with $p = 0$ on ∂E_h

$$\begin{cases} \frac{1}{2}|\nabla\phi|^2 - (\Omega \times x)\cdot\nabla\phi + U_h + mU_X = \lambda, & \text{on } \partial E_h, \\ \Delta\phi = 0, & \text{in } E_h, \\ n_h \cdot \nabla\phi = -V_0 \cdot n_h, & \text{on } \partial E_h, \\ \Omega \times (\Omega \times X) = -\nabla U_h(X), \\ |E_h| = \frac{4\pi}{3}. \end{cases} \tag{6.5.2}$$

Here, λ is a constant. The unknowns are then ϕ, h, λ and X. Let us mention that the unknown λ is chosen to fix the total mass of the fluid body. In particular, we can obtain $|E_h| = 4\pi/3$. Furthermore, due to the invariance under rotation, we can assume without loss of generality that $X = (P,0,0)$ for some $P > 0$. Nevertheless, we consider the case $X \in \mathbb{R}^3 \setminus \overline{E_h}$.

Unperturbed solution. For a fixed angular velocity $\Omega_0 > 0$ a solution for $m = 0$ is given by $E_0 = B_1$ and $v = -\Omega \times x$ that is $h \equiv 0$, $\phi \equiv 0$. Accordingly, we have

$$U_0(x) = \begin{cases} -\frac{4\pi}{3|x|} & |x| \geq 1, \\ -\frac{2\pi}{3}(1-|x|^2) - \frac{4\pi}{3} & |x| \leq 1. \end{cases}$$

This yields $\lambda_0 = U_0(1) = -4\pi/3$. On the other hand, the unperturbed position of the particle $X_0 = (P_0,0,0)$ satisfies the last equation in (6.5.2), that is,

$$\Omega_0^2 P_0 = \frac{4\pi}{3P_0^2}, \quad P_0 = \left(\frac{4\pi}{3}\right)^{1/3} \Omega_0^{-2/3}.$$

Furthermore, we assume that Ω_0 is small enough such that $1 \lesssim P_0$ ensuring $X \in \mathbb{R}^3 \setminus \overline{E_h}$.

Linearization and small divisors. As it turns out the above problem contains small divisors, more precisely the linearized operator is not continuously invertible on any, say, L^2 Sobolev space, due to a loss of regularity. Thus, a standard implicit function theorem is not applicable.

We give here a formal derivation of the linearization at the unperturbed solution for $m=0$. In order to linearize $\phi=\phi(h)$ we linearize the boundary condition

$$\sigma\cdot\nabla\phi(\sigma)\approx n_h\cdot\nabla\phi=-n_h\cdot V_0\approx-(\Omega\times\sigma)\cdot\nabla_{S^2}h.$$

Here, we use the sign $\approx$ to indicate that equality holds when ignoring quadratic terms in h. Furthermore, in the equation for ϕ we can restrict to the unit ball yielding

$$\begin{cases}\Delta\phi=0, & \text{on } B_1\\ \dfrac{\partial\phi}{\partial n}=-(\Omega\times\sigma)\cdot\nabla_{S^2}h & \text{on } S^2.\end{cases}$$

The solution of this equation is abbreviated by

$$\phi=\Delta_N^{-1}\left[-(\Omega\times\sigma)\cdot\nabla_{S^2}h\right],$$

where the index N stands for Neumann problem.

Now, we linearize the Bernoulli function, that is the first equation in (6.5.2). The first term is quadratic in ϕ and thus can be neglected. Concerning the gravitational potential, the equation is evaluated at a point $X(\sigma)=(1+h(\sigma))\sigma$ on the boundary ∂E_h. We thus have

$$U_h(X(\sigma))=U_0(X(\sigma))+(U_h-U_0)(X(\sigma)).$$

The first term is

$$\begin{aligned}U_0(X(\sigma))&=U_0(\sigma)+h(\sigma)\nabla U_0(\sigma)\cdot\sigma+\mathcal{O}(\|h\|^2)=\\&=-\frac{4\pi}{3}+\frac{4\pi}{3}h(\sigma)+\mathcal{O}(\|h\|^2).\end{aligned}$$

The second term is

$$-\int_{S^2}\int_1^{1+h(\tau)}\frac{r^2\,dr d\tau}{|X(\sigma)-r\tau|}=-\int_{S^2}\int_{1/(1+h(\sigma))}^{1+\varepsilon[h](\sigma,\tau)}(1+h(\sigma))^2\frac{s^2\,dr d\tau}{|\sigma-s\tau|},$$

where we used $r=(1+h(\sigma))s$ and

$$\begin{aligned}1+\varepsilon[h](\sigma,\tau)&=\frac{1+h(\tau)}{1+h(\sigma)}=1+\frac{h(\tau)-h(\sigma)}{1+h(\sigma)}=1+h(\tau)-h(\sigma)+\mathcal{O}(\|h\|^2),\\ \frac{1}{1+h(\sigma)}&=1-\frac{h(\sigma)}{1+h(\sigma)}=1-h(\sigma)+\mathcal{O}(\|h\|^2).\end{aligned}$$

We conclude (note that $1/|\sigma-\tau|$ is integrable on the sphere)

$$\begin{aligned}-(1+h(\sigma))^2\int_{S^2}\int_{1/(1+h(\sigma))}^{1+\varepsilon[h](\sigma,\tau)}\frac{s^2\,dr d\tau}{|\sigma-s\tau|}&=-\int_{S^2}\frac{h(\tau)}{|\sigma-\tau|}d\tau+o(\|h\|)\\&=-\int_{S^2}\frac{h(\tau)}{|\sigma-\tau|}d\tau+o(\|h\|).\end{aligned}$$

For the second term in the first equation in (6.5.2) we use spherical coordinates. We have

$$V_0(\sigma)=-\Omega\times\sigma=-\Omega_0\sin\theta\,\mathbf{e}_\varphi.$$

Here, $\mathbf{e}_\varphi$ is the normalized vector field parallel to ∂_φ. We hence have

$$V_0(\sigma)\cdot\nabla = -\Omega_0\partial_\varphi.$$

This yields

$$V_0\cdot\nabla\phi = -\Omega_0\partial_\varphi\phi = -\Omega_0\partial_\varphi\left[\Delta_N^{-1}\left[-\Omega_0\partial_\varphi h\right]\right] = \Omega_0^2\partial_\varphi\left[\Delta_N^{-1}\left[\partial_\varphi h\right]\right].$$

In total the first equation in (6.5.2) gives the linearized operator for $m=0$

$$(h,\lambda)\mapsto \mathscr{L}h-\lambda$$

where

$$\mathscr{L}h := \Omega_0^2\,\partial_\varphi\left[\Delta_N^{-1}\left[\partial_\varphi h\right]\right] + \frac{4\pi}{3}h(\sigma) - \int_{S^2}\frac{h(\tau)}{|\sigma-\tau|}\,d\tau.$$

Diagonalizing the linearized operator. In order to invert the linearized operator we use spherical harmonics $Y_{\ell,m}(\theta,\varphi)=e^{im\varphi}P_\ell(\cos\theta)$. This allows to give an explicit expression for Δ_N^{-1}. In fact, the solution to the Neumann problem has the form

$$\phi(r,\theta,\varphi)=\sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell}\frac{f_{\ell,m}}{\ell}\,r^\ell\,Y_{\ell,m}(\theta,\varphi),\quad f(\sigma)=-\Omega_0\partial_\varphi h(\sigma),\quad f_{\ell,m}=-\Omega_0\,imh_{\ell,m}.$$

Here, we use the notation

$$f_{\ell,m}=\int_{S^2}f(\sigma)\,Y_{\ell,m}(\sigma)\,d\sigma$$

We then obtain

$$\Omega_0^2\,\partial_\varphi\left[\Delta_N^{-1}\left[\partial_\varphi h\right]\right] = -\Omega_0^2\sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell}\frac{m^2\,h_{\ell,m}}{\ell}\,Y_{\ell,m}(\theta,\varphi).$$

We diagonalize the spherical integral

$$-\int_{S^2}\frac{h(\tau)}{|\sigma-\tau|}\,d\tau.$$

This is the gravitational potential of a mass density $h(\sigma)$ concentrated on the sphere S^2. Such a potential can be compute with spherical harmonics by means of the multipole decomposition, see [25, Section 2.4], yielding

$$-\int_{S^2}\frac{h(\tau)}{|\sigma-\tau|}\,d\tau = -4\pi\sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell}\frac{h_{\ell,m}}{2\ell+1}\,Y_{\ell,m}(\theta,\varphi).$$

We summarize

$$\mathscr{L}h(\theta,\varphi)=\sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell}\lambda_{\ell,m}(\Omega_0)h_{\ell,m}\,Y_{\ell,m}(\theta,\varphi),\quad \lambda_{\ell,m}(\Omega_0):=-\frac{\Omega_0^2m^2}{\ell}+\frac{4\pi}{3}-\frac{4\pi}{2\ell+1}.$$

In order to invert $\mathscr{L}-\lambda$ we have to consider two cases.

1. Case: $\ell = m = 0$. Here we can choose λ accordingly. On the other hand, the value of $h_{0,0}$ is determined by the condition $|E_h| = 4\pi/3$ as one can see from its linearization with respect to h.

2. Case: $\ell > 0$, $m = 0$. The eigenvalues $\lambda_{\ell,0} = \frac{4\pi}{3} - \frac{4\pi}{2\ell+1}$ are independent of Ω_0. We see that $\lambda_{\ell,0} > 0$ for $\ell > 1$, $\lambda_{\ell,0} \nearrow \frac{4\pi}{3}$. However, we have $\lambda_{1,0} = 0$, since the whole problem is invariant under translation along the z-axis. This symmetry has to be taken into account when inverting the operator. (Note that for $\Omega_0 = 0$ we have $\lambda_{1,-1} = \lambda_{1,0} = \lambda_{1,1} = 0$ due to the translation invariance in all three coordinate directions, as one would expect in the absence of rotation.)

3. Case: $\ell > 0$, $m \neq 0$. In order to invert the operator we have to assume the following non-resonance condition
$$\lambda_{\ell,m}(\Omega_0) \neq 0 \iff \Omega_0^2 \neq \frac{\ell}{m^2}\left(\frac{16\pi}{3} - \frac{4\pi}{2\ell+1}\right).$$

In all cases under the non-resonance condition we can determine the values of $h_{\ell,m}$ and λ, i.e. we can invert the linearized operator.

Nevertheless, note that even under the non-resonance condition $\lambda_{\ell,m}(\Omega_0) \neq 0$ the operator cannot be inverted continuously. In fact, the eigenvalues $\lambda_{\ell,m}(\Omega_0)$ are arbitrarily close to zero for suitable choices of ℓ, m for any fixed Ω_0.

This phenomenon is typically referred to as small divisors and leads to a loss of regularity when inverting the linearized operator. One way to control this loss of regularity is to assume a Diophantine condition on Ω_0. We do not go into further details here, but refer to the work [96]. In fact, in this work a very similar problem on three-dimensional traveling water waves was studied, which also contains small divisors. The general strategy is to use a Diophantine condition on Ω_0 to control the smallness of the eigenvalues. The smallness of the eigenvalues lead to a loss of regularity, i.e. the Fourier frequencies of the function h in terms of the basis of the spherical harmonics have worse decay after inversion of the operator $\mathscr{L}$. In order to compensate for such a loss of the derivatives a Nash-Moser implicit function theorem is used. The main difference compared to a standard implicit function theorem is the application of a Newton scheme. We refer to the work by Zehnder [164, 165] which describes variants of such a scheme suitable for small divisor problems. However, in comparison with the implicit function theorem the Newton scheme requires the study of the linearized operator in a neighborhood of the unperturbed solution. As was worked out in [96] for the case of traveling water waves this study requires the transformation of the linearized operator to a normal form, in order to control the small divisors, i.e. the spectrum of the linearized operator.

Let us mention that the main difference of [96] and other previous works on small divisor problems for PDEs, is the appearance of the non-trivial geometry of the sphere S^2. In fact, in [96] unknowns are defined on the torus $\mathbb{T}^2$. In particular, standard methods of pseudo-differential operators can be used to transform the linearized operator to a normal form. In the problem described here one has to use pseudo-differential operators on the sphere S^2. More precisely, instead of the functions of the form $e^{ik\cdot x}$ on $\mathbb{T}^2$ one has to use spherical harmonics $Y_{\ell,m}$ to decompose functions on S^2.

In fact, the study of pseudo-differential operators on the sphere leads to further difficulties. Most importantly, pseudo-differential symbols are no longer complex-valued functions but matrix-valued. This is a consequence of the non-commutativity of the action of the group of rotations on the sphere. In comparison, the group of translations acting on the torus is commutative. We refer to the work [145] for an introduction into pseudo-differential operators on Lie groups and their quotient spaces.

We aim to solve the above described problem with the mentioned methods in near future.

6.6 Steady states and rotating solutions for star and galaxy models

We constructed rotating solutions to the incompressible Euler-Poisson equation in (IV). One major difference in this work is the appearance of general internal motions. On the other hand, as mentioned in the introduction, see Section 1.2.2, steady states and rotating solutions to the compressible Euler-Poisson and Vlasov-Poisson equation do not contain non-trivial internal respectively macroscopic velocity fields. Thus, one might look for more general steady states or rotating solutions to these equations.

Appendix A

Self-similar profiles for homoenergetic solutions of the Boltzmann equation for non-cutoff Maxwell molecules

Abstract

We consider a modified Boltzmann equation which contains, together with the collision operator, an additional drift term which is characterized by a matrix A. Furthermore, we consider a Maxwell gas, where the collision kernel has an angular singularity. Such an equation is used in the study of homoenergetic solutions to the Boltzmann equation. Under smallness assumptions on the drift term, we prove that the longtime asymptotics is given by self-similar solutions. We work in the framework of measure-valued solutions with finite moments of order $p > 2$ and show existence, uniqueness and stability of these self-similar solutions for sufficiently small A. Furthermore, we prove that they have finite moments of arbitrary order if A is small enough. In addition, the singular collision operator allows to prove smoothness of these self-similar solutions. Finally, we study the asymptotics of particular homoenergetic solutions. This extends previous results from the cutoff case to non-cutoff Maxwell gases.

A.1 Introduction

The inhomogeneous Boltzmann equation is given by

$$\partial_t f + v \cdot \nabla_x f = Q(f, f), \tag{A.1.1}$$

where $f = f(t, x, v) : [0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty)$ is the one-particle distribution of a dilute gas in whole space. In this paper, we restrict ourselves to the physically most relevant case of three dimensions, although our study can be extended to dimensions $d \geq 3$ without any additional difficulties.

On the right-hand side we have Boltzmann's collision kernel

$$Q(f, f) = \int_{\mathbb{R}^3} \int_{S^2} B(|v - v_*|, n \cdot \sigma)(f'_* f' - f_* f) d\sigma dv_*,$$

where $n = (v - v_*)/|v - v_*|$ and $f'_* = f(v'_*)$, $f' = f(v')$, $f_* = f(v_*)$, with the pre-collisional velocities (v, v_*) resp. post-collisional velocities (v', v'_*). One parameterization of the post-collisional velocities is given by the σ-representation, i.e. for $\sigma \in S^2$

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma, \qquad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma.$$

For an introduction into the physical and mathematical theory of the Boltzmann equation (A.1.1) see for instance [47, 156].

The collision kernel is given by $B(|v - v_*|, n \cdot \sigma)$ and it can be obtained from an analysis of the binary collisions of the gas molecules. For instance, power-law potentials $1/r^{q-1}$ with $q > 2$ lead to (see e.g. [47, Section II.5])

$$B(|v - v_*|, n \cdot \sigma) = |v - v_*|^\gamma b(n \cdot \sigma), \quad \gamma = (q-5)/(q-1), \tag{A.1.2}$$

where $b : [-1, 1) \to [0, \infty)$ has a non-integrable singularity of the form

$$\sin\theta \, b(\cos\theta) \sim \theta^{-1-2/(q-1)}, \quad \text{as } \theta \to 0, \tag{A.1.3}$$

where $\cos\theta = n \cdot \sigma$, with θ being the deviation angle. It is customary to classify the collision kernels according to their homogeneity γ with respect to $|v - v_*|$. There are three cases: *hard potentials* ($\gamma > 0$), *Maxwell molecules* ($\gamma = 0$) and *soft potentials* ($\gamma < 0$). In this paper, we consider the case of Maxwell molecules, hence B does not depend on $|v - v_*|$, cf. (A.1.2). This corresponds to $q = 5$ for power-law interactions.

Collision kernels with an angular singularity of the form (A.1.3) are called non-cutoff kernels. When $\gamma = 0$, one refers to non-cutoff or true Maxwell molecules. This singularity reflects the fact that for power-law interactions the average number of *grazing collisions*, i.e. collisions with $v \approx v'$, diverges. In kinetic theory the Boltzmann equation has often been studied assuming that the collision kernel B is integrable in the angular variable (Grad's cutoff assumption), since the mathematical analysis is usually simpler.

In this paper, we analyze a particular class of solutions to (A.1.1) namely the so-called *homoenergetic solutions*, which have been studied in particular in [26, 98] in the case of cutoff Maxwell molecules. We show that the results obtain in their papers extend to non-cutoff Maxwell molecules.

A.1.1 Homoenergetic solutions and existing results

Our study concerns solutions to (A.1.1) of the form

$$f(t, x, v) = g(t, v - L(t)x), \quad w = v - L(t)x, \tag{A.1.4}$$

for $L(t) \in \mathbb{R}^{3\times 3}$ and a function $g = g(t, w) : [0, \infty) \times \mathbb{R}^3 \to [0, \infty)$ to be determined. One can check that solutions to (A.1.1) of the form (A.1.4) for large classes of functions g exist if and only if g and L satisfy

$$\partial_t g - L(t)w \cdot \nabla_w g = Q(g, g), \quad \frac{d}{dt}L(t) + L(t)^2 = 0. \tag{A.1.5}$$

The second equation allows the reduction to the variable w. In particular, the collision operator acts on g through the variable w. The second equation can be solved explicitly $L(t) = L(0)(I +$

$tL(0))^{-1}$. Note that the inverse matrix might not be defined for all times, although this situation will not be considered here.

Solutions to (A.1.5) are called *homoenergetic solutions* and were introduced by Truesdell [153] and Galkin [69]. They studied their properties via moment equations in the case of Maxwell molecules. As is known since the work by Truesdell and Muncaster [154], it is possible to write a closed systems of ordinary differential equations for the moments up to any arbitrary order for such interactions. This allows to derive properties about the solution to (A.1.5). In particular, this approach has been applied in [69, 70, 72, 153]. More recently, this method has also been used in [74] (and references therein) in order to obtain information on homoenergetic solutions to the Boltzmann equation, as well as other kinetic models like BGK. The case of mixtures of gases has been studied there as well. The well-posedness of (A.1.5) for a large class of initial data, was proved by Cercignani [48]. Furthermore, the shear flow of a granular material for Maxwell molecules was studied in [49, 50].

A systematic analysis of the longtime behavior of solutions to (A.1.5) for kernels with arbitrary homogeneities has been undertaken in [26, 97, 98, 99]. In [97] they discussed the case of dominant collision term, see also [106]. Furthermore, they proved the existence of a class of self-similar solutions in the case of cutoff Maxwell molecules in [98]. The uniqueness and stability of these self-similar solutions have been proved in [26] and the regularity has been obtained in [66]. Homoenergetic solutions for the two-dimensional Boltzmann equation with hard sphere interactions, as well as for a class of Fokker-Planck equations have been studied in [125].

It is worth mentioning that homoenergetic solutions to (A.1.1) can be interpreted in a wider framework introduced in [58, 59]. There the authors studied a formulation of the molecular dynamics of many interacting particle systems with symmetries. In particular, if the particles of the system of molecules of a gas interact by means of binary collisions one obtains the functional form (A.1.4) for the particle distribution.

In this paper, we extensively use the Fourier transform method, which was introduced by Bobylev [27, 28] to study the homogeneous Boltzmann equation for Maxwell gases. This method has also been applied in [26] for homoenergetic solutions with cutoff Maxwell molecules.

The main contribution of this paper is to adapt the techniques in [26, 98] and well established methods for the non-cutoff Boltzmann equation to extend the results to the case of non-cutoff Maxwell molecules. The main difficulty is the singular behavior of the collision kernel (A.1.3).

A.1.2 Overview and main results

Notation. We denote by $\mathscr{P}(\mathbb{R}^3)$ the set of Borel probability measures on $\mathbb{R}^3$ and by $\mathscr{P}_p(\mathbb{R}^3) \subset \mathscr{P}(\mathbb{R}^3)$ the set of those which have finite moments of order p, i.e. $\mu \in \mathscr{P}_p$ if

$$\|\mu\|_p = \int_{\mathbb{R}^3} |v|^p \mu(dv) < \infty.$$

The action of $\mu \in \mathscr{P}$ on a test function ψ via integration is abbreviated by $\langle \psi, \mu \rangle$. The Fourier transform or characteristic function of a probability measure $\mu \in \mathscr{P}$ is defined by

$$\varphi(k) = \mathcal{F}[\mu](k) = \int_{\mathbb{R}^3} e^{-ik\cdot x} d\mu(x).$$

We denote by $\mathcal{F}_p$ the set of all characteristic functions of probability measures $\mu \in \mathscr{P}_p$. Furthermore, we write $\psi \in C^k$ for k-times continuously differentiable functions and $\psi \in C_b^k$ if the standard norm $\|\psi\|_{C^k}$ is finite.

We also use the notation $\langle k \rangle := \sqrt{1+|k|^2}$ and denote the space of functions $h : \mathbb{R}^3 \to \mathbb{R}$ such that $\langle k \rangle^m h(k) \in L^2(\mathbb{R}^3)$ by $L^2_m(\mathbb{R}^3)$. For matrices $A \in \mathbb{R}^{3\times 3}$ we use the matrix norm $\|A\| = \sum_{ij} |A_{ij}|$. Finally, $\mathbb{1}_B$ is the indicator function for some set B.

Assumption on the kernel. We consider non-cutoff Maxwell molecules, i.e. the collision kernel has the form $B = b(n\cdot\sigma) = b(\cos\theta)$. The function $b : [-1,1) \to [0,\infty)$ is measurable, locally bounded and has the angular singularity

$$\sin\theta b(\cos\theta)\theta^{1+2s} \to K_b > 0, \quad \text{as } \theta \to 0 \tag{A.1.6}$$

for some $s \in (0,1)$ and $K_b > 0$. This implies

$$\Lambda = \int_0^\pi \sin\theta\, b(\cos\theta)\, \theta^2 d\theta < \infty. \tag{A.1.7}$$

In particular, this covers inverse power-law interactions with $q = 5$, cf. (A.1.2) and (A.1.3).

Main result. In our study we consider the following modified Boltzmann equation, which is a variant of equation (A.1.5),

$$\partial_t f = \operatorname{div}(Av\, f) + Q(f,f), \quad f(0,\cdot) = f_0(\cdot). \tag{A.1.8}$$

In contrast to the previous equation, $A \in \mathbb{R}^{3\times 3}$ is a time-independent matrix. However, the study of solutions to (A.1.5) can be reduced to this situation using a change of variables and perturbation arguments, see Section A.4. We work with weak solutions with finite energy.

Definition A.1.1. A family of probability measures $(f_t)_{t\geq 0} \subset \mathscr{P}_p$ with $p \geq 2$ is a weak solution to (A.1.8) if for all $\psi \in C_b^2$ and all $0 \leq t < \infty$ it holds

$$\begin{aligned} \langle \psi, f_t \rangle = {}& \langle \psi, f_0 \rangle - \int_0^t \langle Av \cdot \nabla\psi, f_r \rangle\, dr \\ & + \frac{1}{2} \int_0^t \int_{\mathbb{R}^3\times\mathbb{R}^3} \int_{S^2} b(n\cdot\sigma) \left\{ \psi'_* + \psi' - \psi_* - \psi \right\} d\sigma f_r(dv) f_r(dv_*) dr. \end{aligned} \tag{A.1.9}$$

Here, we also assume that the integrands in the time integrals are measurable with respect to the time variable.

Above we abbreviated $\psi'_* = \psi(v'_*)$, etc. This formulation is motivated by testing (A.1.8) with ψ and applying the usual pre-postcollisional change of variables $(v,v_*) \leftrightarrow (v',v'_*)$ as well as $v \leftrightarrow v_*$. See also e.g. [98, 121] concerning the above definition. For brevity we will sometimes denote the term involving the collision operator $\langle \psi, Q(f_r,f_r) \rangle$. Note that this is well-defined due to the moment assumption $f_t \in \mathscr{P}_p$, $p \geq 2$, in conjunction with the estimate (see e.g. [121, 155])

$$\left| \int_{S^2} b(n\cdot\sigma) \left\{ \psi'_* + \psi' - \psi_* - \psi \right\} d\sigma \right| \leq 2\pi\Lambda \left(\max_{|\xi| \leq \sqrt{|v|^2+|v_*|^2}} |D^2\psi(\xi)| \right) |v - v_*|^2. \tag{A.1.10}$$

Using this and an approximation one can also use test functions $\psi \in C^2$, which satisfy the condition $|D^2\psi(v)| \leq C(1+|v|^{p-2})$, in the weak formulation.

Let us mention that one can always consider, without loss of generality, the case of vanishing momentum/mean $\int_{\mathbb{R}^3} v f_0(dv) = 0$. To get a solution F with initial mean $U \in \mathbb{R}^3$ from f_t, one defines $F(t,v) = f_t(v - e^{tA}U)$ interpreted as a push-forward. However, as we will see, solutions with initial condition different from a Dirac measure are smooth for positive times due to the regularizing effect of the angular singularity.

Let us also define the following Fourier-based metric on probability measures.

Definition A.1.2. For two probability measures $\mu, \nu \in \mathscr{P}_p$ with finite moments of order $p \geq 2$ we define a distance using the Fourier transforms $\varphi = \mathcal{F}[\mu]$, $\psi = \mathcal{F}[\nu]$ via

$$d_2(\mu,\nu) := \sup_k \frac{|\varphi(k) - \psi(k)|}{|k|^2}.$$

Note that $d_2(\mu,\nu) < \infty$ is finite if μ,ν have equal first moments. We sometimes write $d_2(\varphi,\psi)$.

Theorem A.1.3. *Consider the equation* (A.1.8). *Let* $2 < p \leq 4$. *There is a constant* $\varepsilon_0 = \varepsilon_0(p,b) > 0$ *such that if* $\|A\| \leq \varepsilon_0$, *the following holds.*

(i) There is $\bar{\beta} = \bar{\beta}(A)$ *and* $f_{st} \in \mathscr{P}_p$ *so that* (A.1.8) *has a self-similar solution*

$$f(v,t) = e^{-3\bar{\beta}t} f_{st}\left(\frac{v - e^{-tA}U}{e^{\bar{\beta}t}}\right), \quad U \in \mathbb{R}^3,$$

where f_{st} *has moments*

$$\int_{\mathbb{R}^3} v f_{st}(dv) = 0, \quad \int_{\mathbb{R}^3} v_i v_j f_{st}(dv) = K\bar{N}_{ij}.$$

Here, $K \geq 0$ *and* $\bar{N} = \bar{N}(A) \in \mathbb{R}^{3\times 3}$ *is a uniquely given positive definite, symmetric matrix with* $\left\|\bar{N}\right\| = 1$. *For* $K = 0$, *we have* $f_{st} = \delta_0$, *a Dirac measure in zero.*

Furthermore, when $K > 0$ *the self-similar solutions are smooth*

$$f(t,\cdot) \in L^1(\mathbb{R}^3) \cap \bigcap_{k \in \mathbb{N}} H^k(\mathbb{R}^3).$$

(ii) Let $(f_t)_t \subset \mathscr{P}_p$ *be a weak solution to* (A.1.8) *with initial condition* $f_0 \in \mathscr{P}_p$ *and*

$$U = \int_{\mathbb{R}^3} v f_0(dv).$$

Then there is $\alpha = \alpha(f_0) \in \mathbb{R}$, $C = C(f_0,p) > 0$, $\theta = \theta(\varepsilon_0) > 0$ *such that the rescaled function*

$$\tilde{f}(t,v) := e^{3\bar{\beta}t} f\left(e^{\bar{\beta}t}v + e^{-At}U, t\right)$$

satisfies

$$d_2\left(\tilde{f}(t,\cdot), f_{st}(t,\cdot)\right) \leq Ce^{-\theta t},$$

where f_{st} *is given in (i) with second moments* $\alpha^2\bar{N}$, $K = \alpha^2$. *In particular, the self-similar solution in (i) is unique for given* $K \geq 0$.

(iii) In addition, for all $M \in \mathbb{N}$, $M \geq 3$ *there is* $\varepsilon_M \leq \varepsilon_0$ *such that the self-similar solution from (i) has finite moments of order* M *if* $\|A\| \leq \varepsilon_M$.

Remark A.1.4. Note that f_{st} in (i) solves

$$\operatorname{div}((A+\beta)v f_{st}) + Q(f_{st}, f_{st}) = 0, \quad \beta = \bar{\beta}(A). \tag{A.1.11}$$

Furthermore, $\bar{N}$ is a stationary solution to the second order moment equations ($\bar{b} \in \mathbb{R}$ depending only on the collision kernel, see Lemma A.3.2)

$$-A\bar{N} - (A\bar{N})^\top - 2\bar{b}\left(\bar{N} - \frac{\operatorname{tr}(\bar{N})}{3} I\right) = 2\bar{\beta}\bar{N}.$$

As we will see, $\bar{\beta} = \bar{\beta}(A)$ is chosen such that $2\bar{\beta} \in \mathbb{R}$ is the simple eigenvalue with largest real part. The corresponding eigenvector is given by $\bar{N} = \bar{N}(A)$.

The uniqueness result in (ii) can now be formulated in a more precise way: within the class of probability measures $\mathscr{P}_p$, $p > 2$, there is a unique solution $f_{st} \in \mathscr{P}_p$ to the stationary equation (A.1.11) with $\beta = \bar{\beta}(A)$ having moments

$$\int_{\mathbb{R}^3} v f_{st}(dv) = 0, \quad \int_{\mathbb{R}^3} v_i v_j f_{st}(dv) = \bar{N}_{ij}.$$

Since $f_{st}(K^{-1/2}v)K^{-3/2}$ solves (A.1.11) and has second moments $K\bar{N}_{ij}$, $K > 0$, it is the respective self-similar profile in (i). For $K = 0$ this is a Dirac in zero.

Remark A.1.5. The above theorem is similar to the results in [26, 98], where *cutoff* Maxwell molecules have been considered. A comparison with Theorem A.1.3, which covers the *non-cutoff* case, shows that all results hold true under the same assumptions. Here, the smoothness statement in (i) is a consequence of the regularizing effect of the non-cutoff collision kernel, in contrast to the cutoff case [66], where this has been obtained in a perturbative framework close to a Maxwellian.

Remark A.1.6. Regarding part (iii) in Theorem A.1.3 it might be that for small but fixed $A \neq 0$ the self-similar solutions do not have finite moments of arbitrary order, but that they have power-law tails. For shear flow this is suggested by numerical experiments, see [74].

Let us also mention that the smallness of $\|A\|$ is crucial for our perturbation arguments. The precise behavior of solutions to (A.1.8) for large values of A remains open (see also Remark A.4.2).

The paper is organized in the following way. In Section A.2 we discuss the well-posedness theory of equation (A.1.8) and in Section A.3 the proof of Theorem A.1.3. Finally, in Section A.4 we study the self-similar asymptotics of homoenergetic solution in the case of simple and planar shear.

A.2 Well-posedness of the modified Boltzmann equation

The following result summarizes the well-posedness theory of equation (A.1.8), needed in our study. The assumption $p > 2$ can be relaxed, however we only need this case in the sequel.

Proposition A.2.1. *Under our general assumptions, the following statements hold.*

(i) For all $f_0 \in \mathscr{P}_p$, $p > 2$, there is a weak measure-valued solution $(f_t)_t \subset \mathscr{P}_p$ to (A.1.8). *In addition, every weak solution has the property $t \mapsto \langle \psi, f_t \rangle \in C^1([0,\infty);\mathbb{R})$ for all test functions $\psi \in C^2$ with $\|D^2\psi\|_\infty < \infty$.*

(ii) For two weak solutions $(f_t)_t, (g_t)_t \subset \mathscr{P}_p$ *to* (A.1.8)*,* $p>2$*, such that* f_0, g_0 *have equal first moments, it holds*

$$d_2(f_t,g_t) \le e^{2\|A\|t} d_2(f_0,g_0). \tag{A.2.1}$$

In particular, solutions are unique.

(iii) If the initial datum $f_0 \in \mathscr{P}_p$*,* $p>2$*, is not a Dirac measure, the solution is smooth, i.e. for* $t>0$

$$f(t,\cdot) \in L^1(\mathbb{R}^3) \cap \bigcap_{k\in\mathbb{N}} H^k(\mathbb{R}^3).$$

Remark A.2.2. The setting of measure-valued solutions was also used in [98] for homoenergetic solutions. Measure-valued solutions to the homogeneous Boltzmann equation ($A=0$ in (A.1.8)) were considered in e.g. [121, 131] for both hard and soft potentials with homogeneity $\gamma \ge -2$. In [131] solutions with infinite energies are studied as well, see also [41, 130] for the case of Maxwell molecules.

The metric in Definition A.1.2 is also termed Toscani metric and appeared first in [68] for the study of convergence to equilibrium of the homogeneous Boltzmann equation with true Maxwell molecules. Furthermore, it was used to prove uniqueness of respective solutions in [150], by showing that solutions are contractive w.r.t. d_2. Inequality (A.2.1) is the extension of this Lipschitzianity to homoenergetic solutions.

A key ingredient in the proof of Theorem A.1.3 is the following comparison principle between solutions to (A.1.8). A similar result was used in [26, Section 5].

Proposition A.2.3. *Consider two weak solutions* $(f_t)_t, (g_t)_t \subset \mathscr{P}_p$*,* $p>2$*, to* (A.1.8) *with zero momentum. Let* $\varphi, \psi \in C([0,\infty);\mathcal{F}_p)$ *be the corresponding Fourier transforms. Suppose that*

$$|\varphi_0(k) - \psi_0(k)| \le C_1|k|^p + C_2|k|^2, \quad \forall k \in \mathbb{R}^3.$$

Then, we have for all $t \ge 0$ *and* $k \in \mathbb{R}^3$

$$|\varphi_t(k) - \psi_t(k)| \le C_1 e^{-(\lambda(p)-p\|A\|)t}|k|^p + C_2 e^{2\|A\|t}|k|^2.$$

Here, $\lambda(p)>0$ *is defined in Lemma A.2.5 and depends only on the collision kernel.*

In the proof of both propositions we use an approximation by the cutoff problem. To this end, let us introduce an arbitrary cutoff sequence $b_n : [-1,1) \to [0,\infty)$, $b_n \not\equiv 0$, with $b_n \nearrow b$, $\|b_n\|_\infty < \infty$, e.g. $b_n := \min(b,n)$ and denote the corresponding collision operators by Q_n. Furthermore, let $\Lambda_n \le \Lambda$ be the corresponding constant as defined in (A.1.7) with b_n replacing b.

Let us mention that (A.2.1) follows from Proposition A.2.3 for $C_1 = 0$. However, in the proof we rely on the uniqueness of solutions due to our approximation procedure.

Proof of Proposition A.2.1. (i). The proof follows well-known methods for the homogeneous Boltzmann equation (i.e. $A=0$). We only give the essential arguments.

First of all, for all $f_0 \in \mathscr{P}_p$, $p>2$ one can prove the existence of a unique weak solution $(f_t^n)_t \in C([0,\infty);\mathscr{P}_p)$ of the corresponding cutoff equation with collision kernel b_n using e.g. semigroup theory [98, Section 4.1].

To get a solution to the non-cutoff equation on $[0,T]$ we use a weak compactness argument, see e.g. [121]. One can obtain the a priori bound $\|f_t^n\|_p \leq Ce^{C(p,A)T}\|f_0\|_p$ via a Gronwall argument, which yields tightness of the sequence $(f_t^n)_n$ for all $t \in [0,T]$. Furthermore, the a priori bound and the weak formulation (A.1.9) imply the following continuity property independent of $n \in \mathbb{N}$: for any test function $\psi \in C^2$ with $\|D^2\psi\|_{C^2} < \infty$ and $0 \leq s < t \leq T$

$$|\langle \psi, f_t^n \rangle - \langle \psi, f_s^n \rangle| \leq (t-s) C\left(T, \left\|D^2\psi\right\|_\infty, A, \Lambda\right) \|f_0\|_2 .$$

Hence, we conclude that there is a weakly converging subsequence $f_t^{n_k} \rightharpoonup f_t$ for all $t \in [0,T]$. We pass to the limit in the weak formulation as in [121, Section 4].

Finally, the stated regularity property $t \mapsto \langle \psi, f_t \rangle \in C^1$ follows from the weak formulation. □

We give a proof of part (ii) of Proposition A.2.1 and Proposition A.2.3 in the next subsection using the Fourier transform method. Part (iii) of Proposition A.2.1 is proved in Subsection A.2.2.

A.2.1 The modified Boltzmann equation in Fourier space

We reformulate the problem (A.1.8) via the Fourier transform. Consider a weak solution $(f_t)_t \subset \mathscr{P}_p$, $p > 2$ and its Fourier transform $\varphi_t(k) = \mathcal{F}[f_t](k)$. For a fixed $k \in \mathbb{R}^3$, we use $\psi(v) = e^{-ik\cdot v}$ as a test function in the weak formulation of (A.1.8) yielding

$$\partial_t \varphi_t(k) + A^\top k \cdot \nabla \varphi_t(k) = \hat{Q}(\varphi_t, \varphi_t)(k). \tag{A.2.2}$$

Note that part (i) in Proposition A.2.1 implies that $t \mapsto \varphi_t(k) \in C^1$ for any $k \in \mathbb{R}^3$. The last term in (A.2.2) corresponds to the collision operator, which has the form (Bobylev's formula [27, 28])

$$\hat{Q}(\varphi,\varphi)(k) = \int_{S^2} b(\hat{k}\cdot\sigma)\left\{\varphi(k_+)\varphi(k_-) - \varphi(k)\varphi(0)\right\} d\sigma,$$

where $k_\pm = (k \pm |k|\sigma)/2$, $\hat{k} = k/|k|$. Let us write $\hat{Q}_n$ for the Fourier representation of the collision operator corresponding to a cutoff sequence $0 \leq b_n \nearrow b$. We will often consider a decomposition of it in a gain and loss term

$$\hat{Q}_n^+(\varphi,\varphi)(k) = \int_{S^2} b_n(\hat{k}\cdot\sigma)\varphi(k_+)\varphi(k_-) d\sigma, \quad \hat{Q}_n^-(\varphi,\varphi)(k) = S_n \varphi(k).$$

In the last equation, we used $\varphi(0) = 1$ for characteristic functions and the constant

$$S_n := \int_{S^2} b_n(e\cdot\sigma) d\sigma, \quad e \in S^2. \tag{A.2.3}$$

Observe that the integral does not depend on $e \in S^2$ by rotational invariance. This integral measures the average number of collisions and, since b is singular, we have $S_n \nearrow +\infty$ as $n \to \infty$.

Finally, let us recall the following property of characteristic functions.

Lemma A.2.4. *Consider $\mu \in \mathscr{P}_p$, $p > 0$, then its characteristic function satisfies $\varphi \in C_b^{\lfloor p \rfloor, p - \lfloor p \rfloor}$ if $p \notin \mathbb{N}$ and $\varphi \in C_b^p$ if $p \in \mathbb{N}$. Furthermore, $\|\varphi\|_C \leq 1$ and $\overline{\varphi(k)} = \varphi(-k)$.*

Linearization and Lipschitz property of the gain term

For the Fourier transform of the cutoff operator $\hat{Q}_n$ we introduce the linearization of $\hat{Q}_n^+$ defined by

$$\mathscr{L}_n(\varphi)(k) = \int_{S^2} b_n(\hat{k}\cdot\sigma)(\varphi(k_+)+\varphi(k_-))d\sigma, \tag{A.2.4}$$

where $\varphi \in C_b$, say. The following lemma can be proved as in [26, Theorem 5.8].

Lemma A.2.5. *Let us define*

$$\begin{aligned} w_p(s) &:= 1-\left(\frac{1+s}{2}\right)^{p/2}-\left(\frac{1-s}{2}\right)^{p/2}, \\ \lambda_n(p) &:= \int_{S^2} b_n(e\cdot\sigma)\, w_p(e\cdot\sigma)d\sigma, \quad \lambda(p) := \int_{S^2} b(e\cdot\sigma)\, w_p(e\cdot\sigma)d\sigma. \end{aligned} \tag{A.2.5}$$

Then, $\lambda(p)$ is well-defined for $p \geq 2$ and $\lambda_n(p) \to \lambda(p)$. Furthermore, $\lambda(p)$ is strictly increasing w.r.t. $p \geq 2$. In particular, we have $\lambda(p) > \lambda(2) = 0$ for $p > 2$.

Remark A.2.6. We remark that $|k|^p$, $p > 0$, can be interpreted as an eigenfunction of the operator $(\mathscr{L}_n - S_n I)$ w.r.t. the eigenvalue $-\lambda_n(p)$, since we have

$$(\mathscr{L}_n - S_n I)|k|^p = -\lambda_n(p)|k|^p.$$

The following result is an adaptation of [26, Lemma 3.1], where we made the dependence on the constant S_n explicit. Such an estimate was termed $\mathscr{L}$-Lipschitz in [33, Definition 3.1].

Lemma A.2.7. *Consider two characteristic functions $\varphi, \psi \in \mathcal{F}_p$, $p \geq 2$, and a cutoff sequence $b_n \nearrow b$. Then, we have with $\varphi = \mathcal{F}[f]$, $\psi = \mathcal{F}[g]$*

$$|\hat{Q}_n^+(\varphi,\varphi) - \hat{Q}_n^+(\psi,\psi)|(k) \leq \mathscr{L}_n(|\varphi-\psi|)(k) \leq S_n d_2(f,g)|k|^2. \tag{A.2.6}$$

Proof. The first inequality follows from

$$|\varphi(k_+)\varphi(k_-) - \psi(k_+)\psi(k_-)| \leq |\varphi(k_+)-\psi(k_+)| + |\varphi(k_-)-\psi(k_-)|$$

The second one is a consequence of a straightforward estimation and $|k_+|^2+|k_-|^2 = |k|^2$. □

Uniqueness of weak solutions

We turn to the proof of part (ii) of Proposition A.2.1. The argument is similar to the ones for the homogeneous Boltzmann equation in [150]. We only give the essential steps.

Proof of Proposition A.2.1. (ii). Let $(f_t)_t$, $(g_t)_t$ be two weak solutions and $\varphi_t(k) = \mathcal{F}[f_t](k)$, $\psi_t(k) = \mathcal{F}[g_t](k)$ be the corresponding Fourier transforms. Assuming $d_2(f_0,g_0) < \infty$, it follows that the first moments are equal initially and hence for all times. As a consequence $d_2(f_t,g_t) < \infty$ for all $t \geq 0$. Using a priori bounds of the moments of order $p \geq 2$ we get for $t \in [0,T]$, $T > 0$ arbitrary but fixed,

$$R_n(t,k) := \frac{1}{|k|^2}|(\hat{Q}-\hat{Q}_n)(\varphi_t,\varphi_t) - (\hat{Q}-\hat{Q}_n)(\psi_t,\psi_t)|(k) \leq C(T)r_n.$$

Here $r_n = \Lambda - \Lambda_n \to 0$ as $n \to \infty$.

Let us abbreviate $E_t = e^{tA^\top}$. A calculation shows that for $k \neq 0$

$$\frac{d}{dt}\left[\frac{e^{S_n t}(\varphi_t - \psi_t)(E_t k)}{|E_t k|^2}\right] = \frac{2\langle AE_t k, E_t k\rangle}{|E_t k|^2}\frac{e^{S_n t}(\varphi_t - \psi_t)(E_t k)}{|E_t k|^2} + \frac{e^{S_n t}}{|E_t k|^2}\left[\hat{Q}_n^+(\varphi_t,\varphi_t)(E_t k) - \hat{Q}_n^+(\psi_t,\psi_t)(E_t k)\right] + e^{S_n t} R_n.$$

Here, we used a splitting of $\hat{Q}_n$ into gain and loss part. We estimate this term by term, in particular using (A.2.6) in Lemma A.2.7 for the gain term. Abbreviating $h_t(k) := (\varphi - \psi)(t, e^{tA}k)/|e^{tA}k|^2$ and applying Gronwall's lemma yields

$$e^{S_n t}\|h_t\|_\infty \leq \|h_0\|_\infty e^{[2\|A\| + S_n]t} + C(T) r_n \int_0^t e^{S_n r} e^{[2\|A\|+S_n](t-r)} dr. \tag{A.2.7}$$

We divide by $e^{S_n t}$ and let $n \to \infty$. This concludes the proof since $\|h_t\|_\infty = d_2(f_t, g_t)$. □

Comparison principle in Fourier space

For the proof of Proposition A.2.3 we consider the linearization of the cutoff equation given by

$$\partial_t \varphi + A^\top k \cdot \nabla \varphi = (\mathscr{L}_n - S_n I)(\varphi)(k), \quad \varphi(0,\cdot) = \varphi_0(\cdot). \tag{A.2.8}$$

Recall that $\mathscr{L}_n$, S_n are defined in (A.2.4) and (A.2.3), respectively. As in [26], one can see that the operator $\mathscr{L}_n : C_p \to C_p$ is bounded, where

$$C_p(\mathbb{R}^3) := \left\{\varphi \in C(\mathbb{R}^3) \ : \ \|\varphi\|_{C_p} := \sup_k |\varphi(k)|/(1+|k|^p) < \infty\right\}$$

for $p \geq 2$. Hence, the equation (A.2.8) defines a semigroup $\mathcal{P}_t^n : C_p \to C_p$.

In the non-cutoff case, the linear semigroup $\mathcal{P}_t^n$ is in general not well-defined for arbitrary functions u_0 as $n \to \infty$. However, the term $(\mathscr{L}_n - S_n)u$ still makes sense for $n \to \infty$ when u satisfies $u(0) = 0$ and $u \in C_b^2$. Let us hence define $u_{n,p} \in C_p$ via

$$u_{n,p}(k,t) := |k|^p \exp(-(\lambda_n(p) - p\|A\|)t),$$

where $\lambda_n(p)$ is given in (A.2.5).

Proof of Proposition A.2.3. We approximate φ, ψ by solutions $\varphi^n, \psi^n \in C([0,\infty);\mathcal{F}_p)$ to equation (A.2.2) with cutoff kernel $0 \leq b_n$ and initial datum φ_0 resp. ψ_0. Let us define $U(k) := C_1|k|^p + C_2|k|^2$.

We can write in mild form

$$\varphi_t^n(k) - \psi_t^n(k) = \varphi_0(k) - \psi_0(k) + \int_0^t e^{-(t-r)(S_n + A^\top k\cdot\nabla)}\left[\hat{Q}_n^+(\varphi_r^n,\varphi_r^n) - \hat{Q}_n^+(\psi_r^n,\psi_r^n)\right](k) dr.$$

Here, we used the semigroup notation $e^{-tA^\top k\cdot\nabla}\varphi(k) = \varphi(e^{-tA^\top}k)$. Set $v_t^n(k) := \varphi_t^n(k) - \psi_t^n(k)$ and estimate using the $\mathscr{L}$-Lipschitz property in Lemma A.2.7 to get

$$|v_t^n(k)| \leq |v_0(k)| + \int_0^t e^{-(t-r)(S_n + A^\top k\cdot\nabla)} \mathscr{L}_n(|v_r^n|)(k) dr.$$

A comparison principle for the linear equation implies $|v_t^n(k)| \leq \mathcal{P}_t^n[|v_0|](k)$. Since $\mathscr{L}_n$ is positivity preserving, one can conclude that $\mathcal{P}_t^n$ is monotonicity preserving. Hence, $\mathcal{P}_t^n[|v_0|](k) \leq \mathcal{P}_t^n[U](k)$ due to our assumption $|v_0| \leq U$.

Now, we estimate $\mathcal{P}_t^n[U]$. It is straightforward to prove

$$u_{n,p}(k,t) \geq e^{-t(S_n + A^\top k \cdot \nabla)}|k|^p + \int_0^t e^{-(t-r)(S_n + A^\top k \cdot \nabla)} \mathscr{L}_n(u_{n,p}(\cdot, r))dr.$$

A comparison principle for the linear equation yields

$$\mathcal{P}_t^n[|\cdot|^p](k) \leq u_{n,p}(t,k)$$

and we infer

$$\mathcal{P}_t^n[U](k) \leq C_1 u_{n,p}(t,k) + C_2 u_{n,2}(t,k).$$

We combining all estimates to get

$$|\varphi_t^n(k) - \psi_t^n(k)| \leq C_1 u_{n,p}(t,k) + C_2 u_{n,2}(t,k).$$

Since weak convergence implies pointwise convergence of the characteristic function, we can pass to the limit in the preceding inequality. Recall also $\lambda_n(p) \to \lambda(p)$ from Lemma A.2.5. □

A.2.2 Regularity of weak solutions

We finally prove the regularity result in Proposition A.2.1 (iii). We sketch the arguments following [132], which covers the homogeneous Boltzmann equation, i.e. $A = 0$.

Proof of Proposition A.2.1. (iii). Let $(\psi_t)_t$ be the Fourier transform of a weak solution $(f_t)_t \subset \mathscr{P}_2$.

Step 1. Let us first state a coercivity estimate analogous to the one in [132, Lemma 1.4]. As in the original work, the non-cutoff assumption (A.1.6) is essential as well as the assumption that f_0 differs from a Dirac. There is $T_0 > 0$ and a constant $C > 0$, both depending on $f_0 \in \mathscr{P}_2$, such that for all $h \in L_2^2(\mathbb{R}^3)$ and all $t \in [0, T_0]$

$$t \int_{\mathbb{R}^3} \langle \xi \rangle^{2s} |h(\xi)|^2 d\xi \leq C \left\{ \int_{\mathbb{R}^3} \int_{S^2} b(\hat{\xi} \cdot \sigma)(1 - |\psi(t, \xi_-)|) d\sigma |h(\xi)|^2 d\xi + \int_{\mathbb{R}^3} |h(\xi)|^2 d\xi \right\}. \quad \text{(A.2.9)}$$

The constant $s \in (0,1)$ is given in (A.1.6).

The proof of this estimate in [132] still works in our case, since in most arguments only the continuity of ψ and $\partial_t \psi$ is used. Only in the case when f_0 is supported on a straight line, the equation (A.2.2) is used. However, the same arguments can be applied to the function $\psi(t, e^{-A^\top t}\xi)$ along the characteristics of the drift term. Since $t \leq T_0$ is chosen sufficiently small, $e^{-A^\top t}$ is close to the identity and the original line of reasoning works.

Step 2. As in [132, Proof of Thm. 1.3] we prove smoothness of the solutions for $0 < t \leq T_0/2$. To this end, we test equation (A.2.2) with $M_\delta^2 \overline{\psi}$, where

$$M_\delta(t, \xi) := \langle \xi \rangle^{Nt^2 - 4} \langle \delta \xi \rangle^{-NT_0^2 - 4}, \quad N \in \mathbb{N}.$$

Here, N is chosen large enough such that $M_\delta \psi \in L^2_2$ for $t \leq T_0/2$. We use straightforward estimates for the drift term and bounds from the original proof in [132], which rely on (A.2.9), to get

$$\frac{d}{dt}\int_{\mathbb{R}^3} |M_\delta(t,\xi)\psi(t,\xi)|^2\, d\xi \leq C(T_0,A)\int_{\mathbb{R}^3} |M_\delta(t,\xi)\,\psi(t,\xi)|^2 d\xi$$
$$+\,t\int_{\mathbb{R}^3}\left[4N\log\langle\xi\rangle - C_2\,\langle\xi\rangle^{2s}\right]|M_\delta(t,\xi)\,\psi(t,\xi)|^2 d\xi.$$

Since $\langle\xi\rangle^{2s}/\log\langle\xi\rangle \to \infty$ as $|\xi| \to \infty$, the last term on the right can be absorbed in the first term. Using Gronwall's lemma and letting $\delta \to 0$ one obtains

$$\int_{\mathbb{R}^3} |\,\langle\xi\rangle^{Nt^2-4}\,\psi(t,\xi)|^2\, d\xi \leq C\int_{\mathbb{R}^3} |\,\langle\xi\rangle^{-4}\,\psi_0(\xi)|^2\, d\xi.$$

Since this holds for all $N \in \mathbb{N}$, we have $f(t,\cdot) \in \bigcap_{k\in\mathbb{N}} H^k(\mathbb{R}^3)$ for $0 < t \leq T_0/2$.

Step 3. Here, we extend the smoothness to times $t \geq T_0/2$. By the smoothness we infer that f_{t_0} has finite entropy for $t_0 \in (0, T_0/2)$, i.e.

$$H(f_{t_0}) := \int_{\mathbb{R}^3} f(t_0,v)\log f(t_0,v)\, dv < \infty.$$

An a priori estimation yields for some arbitrary but fixed time $T' > t_0$

$$H(f_t) \leq H(f_{t_0}) + C(T',A), \quad t \in [t_0,T'].$$

To make this rigorous, we use a construction of weak solutions in L^1_2 with finite entropy initiating from f_{t_0}. Here, L^1_2 is the weighted L^1-space with weight $(1+|v|^2)$. Let us mention that this was done in [48] in the case of homoenergetic solutions for cutoff kernels. Using weak L^1-compactness arguments, following from the Dunford-Pettis theorem, yields solutions for the non-cutoff problem. See e.g. [155, Section 4] for such a construction in the case of the homogeneous Boltzmann equation. These solutions are unique by Proposition A.2.1.

As was noticed in [132], using the result [3, Lemma 3], the estimate (A.2.9) holds now without the condition of small times. Thus, as above we get $f(t,\cdot) \in \bigcap_{k\in\mathbb{N}} H^k(\mathbb{R}^3)$ for $t \geq t_0$. □

A.3 Self-similar solutions and self-similar asymptotics

In this section, we give the proof of Theorem A.1.3. Let us briefly summarize the strategy, which is partly guided by [26, 98]. We first study the linear equations satisfied by the second moments of a solution. Here, we use perturbation arguments to gain information of the eigenvalues and eigenvectors. Then, the existence of self-similar solutions follows from a fixed point argument.

The convergence to the self-similar solution in Theorem A.1.3 (ii), is a consequence of the comparison principle in Proposition A.2.3 and a longtime analysis of the second moments.

Finally, Theorem A.1.3 (iii), is a result of successive application of the Povzner estimate.

A.3.1 Existence of self-similar solutions

Let us recall the following version of the Povzner estimate due to Mischler and Wennberg [129, Section 2]. As was noticed e.g. in [155, Appendix], their calculation also works in the non-cutoff case.

Lemma A.3.1. *Let $\varphi(v) = |v|^{2+\delta}$ for $\delta > 0$. Then we have the following decomposition*

$$\int_{S^2} b(n\cdot\sigma)\{\varphi'_* + \varphi' - \varphi_* - \varphi\}\, d\sigma = G(v,v_*) - H(v,v_*)$$

with G,H satisfying

$$G(v,v_*) \le C\Lambda(|v||v_*|)^{1+\delta/2}, \quad H(v,v_*) \ge c\Lambda(|v|^{2+\delta} + |v_*|^{2+\delta})\left(1 - \mathbb{1}_{\{|v|/2<|v_*|<2|v|\}}\right). \tag{A.3.1}$$

Hence, for any $f \in \mathscr{P}_p$, with $2 < p \le 4$, $p = 2+\delta$ we have for some $C', c' > 0$

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} (G(v,v_*) - H(v,v_*)) f(dv) f(dv_*) \le C'\Lambda \|f\|_2^2 - c'\Lambda \|f\|_p. \tag{A.3.2}$$

Proof. The definition and estimates for G,H can be found in [129, Section 2], see also [155, Appendix]. To derive (A.3.2) note that $\delta = p-2 \le 2$, thus $1+\delta/2 \le 2$. We conclude by applying (A.3.1) and

$$(|v|^{2+\delta} + |v_*|^{2+\delta})\mathbb{1}_{\{|v|/2<|v_*|<2|v|\}} \le 8(|v||v_*|)^{1+\delta/2}.$$

□

The following result follows by choosing $\varphi_{jk}(v) = v_j v_k$ in the weak formulation (A.1.9), recalling that $t \mapsto \langle \varphi_{jk}, f_t\rangle$ is continuously differentiable, see [98, Prop. 4.10] or [26, Section 6].

Lemma A.3.2. *The second moments $M_{jk}(t) := \langle v_j v_k, f_t\rangle$ of a solution to* (A.1.8) *satisfy the equations*

$$\frac{dM_t}{dt} = -AM_t - (AM_t)^\top - 2\bar{b}\left(M_t - \frac{\mathrm{tr}\,(M_t)}{3} I\right) =: \mathcal{A}(\bar{b}, A) M_t \tag{A.3.3}$$

with the constant

$$\bar{b} = \frac{3\pi}{4}\int_0^\pi b(\cos\theta)\sin^3\theta d\theta. \tag{A.3.4}$$

Here, the linear operator $\mathcal{A}(\bar{b},A) : \mathbb{R}^{3\times 3}_{sym} \to \mathbb{R}^{3\times 3}_{sym}$ acts on symmetric 3×3 matrices. As noticed in Remark A.1.4, a self-similar solution f_{st} is a steady state of the equation (A.1.8) with A replaced by $A + \beta I$. Hence, as in the cutoff case [98, Lemma 4.16], we study the linear map $\mathcal{A}(\bar{b}, A+\beta I) = \mathcal{A}(\bar{b}, A) - 2\beta I$.

Lemma A.3.3. *Consider the linear operator $\mathcal{A}(\bar{b},A)$ from Lemma A.3.2. There is a sufficiently small constant $\varepsilon_0 = \varepsilon_0(b) > 0$ such that for all $A \in \mathbb{R}^3$ with $\|A\| \le \varepsilon_0$ the following holds.*

(i) The eigenvalue $2\bar{\beta} > 0$, $\bar{\beta} = \bar{\beta}(\bar{b},A)$, with largest real part is unique and simple. One can uniquely choose a corresponding eigenvector $\bar{N} = \bar{N}(\bar{b},A) \in \mathbb{R}^{3\times 3}_{sym}$ with $\|N\| = 1$ which is positive definite.

(ii) The nonzero eigenvalues of $\mathcal{A}(\bar{b},A) - 2\bar{\beta}I$ have real part less than $-\nu$, for some $\nu > 0$.

(iii) In addition, there is $c_0 > 0$ such that $|\bar{\beta}(\bar{b},A)| \le c_0\varepsilon_0$.

Proof. This is a perturbation argument noting that $\mathcal{A}(\bar{b},A):\mathbb{R}^{3\times 3}_{sym}\to\mathbb{R}^{3\times 3}_{sym}$ depends smoothly on A. For $A=0$ there are the eigenvalues $\lambda_1=0$ and $\lambda_2=-2\bar{b}$ with a one-dimensional subspace of eigenvectors given by $M=KI$, $K\in\mathbb{R}$, respectively, a five-dimensional subspace of eigenvectors defined by $\{\mathrm{tr}\,(M)=0\}$. The statement now follows by continuity results for eigenvalues when $\|A\|$ is small. We choose $2\bar{\beta}(\bar{b},A)$ to be the eigenvalue close to $\lambda_1=0$ and let $\bar{N}(\bar{b},A)\in\mathbb{R}^{3\times 3}_{sym}$ be the corresponding normalized eigenvector close to I. □

In the fixed point argument compactness is a consequence of the following estimate.

Lemma A.3.4. *Consider a weak solution $(f_t)_t\in C([0,\infty);\mathscr{P}_p)$ to (A.1.8), $2<p\le 4$, with matrix A replaced by $A+\bar{\beta}I$. Assume that $\|A\|\le\varepsilon_0$ with $\varepsilon_0>0$ from Lemma A.3.3 and that the initial condition has zero mean as well as second moments $K\bar{N}$. Then, we have for all $t\ge 0$*

$$\int_{\mathbb{R}^3} v f_t(dv)=0,\quad \int_{\mathbb{R}^3} v_i v_j f_t(dv)=K\bar{N}_{ij}. \tag{A.3.5}$$

Furthermore, by decreasing $\varepsilon_0=\varepsilon_0(b,p)>0$, if necessary, there is $C_*=C_*(K)$ such that for all $t\ge 0$

$$\|f_0\|_p\le C_* \implies \|f_t\|_p\le C_*.$$

Proof. As was mentioned already in the introduction, the first moment remains zero for all times. Since $\bar{N}$ is a stationary solution to the equation (A.3.3), we obtain (A.3.5). For the final statement, we use the Povzner estimate from Lemma A.3.1

$$\begin{aligned}\frac{d}{dt}\|f_t\|_p&=\frac{d}{dt}\langle|v|^p,f_t\rangle\le p\left\|A+\bar{\beta}I\right\|\|f_t\|_p+C'\Lambda\|f_t\|_2^2-c'\Lambda\|f_t\|_p\\&\le[p\varepsilon_0(1+c_0)-c'\Lambda]\|f_t\|_p+C'\Lambda K^2.\end{aligned}$$

For ε_0 sufficiently small we have $\delta:=c'\Lambda-p\varepsilon_0(1+c_0)>0$ and hence from a Gronwall type argument, in conjunction with $\|f_0\|_p\le C_*$,

$$\|f_t\|_p\le C_*e^{-\delta t}+\frac{C'\Lambda K^2}{\delta}=C_*+\left(1-e^{-\delta t}\right)\left(\frac{C'\Lambda K^2}{\delta}-C_*\right)$$

We conclude by choosing $C_*=C_*(K)$ sufficiently large. □

For convenience let us recall the following fact concerning the topology induced by the metric d_2, see e.g. [150, Lemma 1, Lemma 2].

Lemma A.3.5. *Define $D_e\subset\mathscr{P}_2$ by*

$$D_e=\left\{f\in\mathscr{P}_2\ :\ \int vf(dv)=0,\quad\int|v|^2f(dv)=e\right\},\quad e\ge 0.$$

Consider $f^n,f\in\mathscr{P}_2$ for $n\in\mathbb{N}$. Then, the following statements are equivalent:

(i) $f_n,f\in D_e$ *and* $f^n\rightharpoonup f$ *weakly, i.e.* $\langle\psi,f^n\rangle\to\langle\psi,f\rangle$ *as* $n\to\infty$ *for all* $\psi\in C_b$;

(ii) $d_2(f_n,f)\to 0$ *as* $n\to\infty$.

Proof of Theorem A.1.3. (i). We use similar arguments as in [98, Section 4.3]. Let us define the set $\mathscr{U} \subset \mathscr{P}_p$, $2 < p \leq 4$, consisting of measures $f \in \mathscr{P}_p$ with

$$\int_{\mathbb{R}^3} v f(dv) = 0, \quad \int_{\mathbb{R}^3} v_i v_j f(dv) = K\bar{N}_{ij}, \quad \|f\|_p \leq C_*.$$

Here, $\bar{N}$ is given in Lemma A.3.3 and we assume that $\|A\| \leq \varepsilon_0$ as in Lemmas A.3.3, A.3.4. Note that $\mathscr{U}$ is a convex, compact subset of the space $\mathscr{M}_f(\mathbb{R}^3)$ of signed Radon measures on $\mathbb{R}^3$ with finite total variation, equipped with the weak-$*$ topology. With this topology $\mathscr{M}_f(\mathbb{R}^3)$ is a locally convex space. Note that weak convergence within $\mathscr{U}$ implies convergence w.r.t. the metric d_2 by Lemma A.3.5.

Let us define the nonlinear semigroup $\mathscr{S}_t : \mathscr{P}_p \to \mathscr{P}_p$ mapping any f_0 to f_t, where $(f_t)_t$ is the unique solution to the equation (A.1.8) with matrix $A + \bar{\beta} I$ replacing A and initial condition f_0. By Lemma A.3.4 we have $\mathscr{S}_t : \mathscr{U} \to \mathscr{U}$. Furthermore, $f \mapsto \mathscr{S}_t f$ is continuous on $\mathscr{U}$ for each $t \geq 0$, as follows from (A.2.1). We can now apply Schauder's fixed point theorem to $\mathscr{S}_{1/n} : \mathscr{U} \to \mathscr{U}$ yielding a fixed point f_{st}^n. By compactness of $\mathscr{U}$ we have for a subsequence $f_{st}^{n_k} \to f_{st}$ as $k \to \infty$. As a consequence of the semigroup property, it holds $\mathscr{S}_{m/n_k} f_{st}^{n_k} = f_{st}^{n_k}$ for any $k, m \in \mathbb{N}$.

Now, let $t \geq 0$ be arbitrary. We can find a sequence of integers $m_k \in \mathbb{N}$ with $m_k/n_k \to t$ as $k \to \infty$ and write

$$f_{st} = \lim_{k\to\infty} f_{st}^{n_k} = \lim_{k\to\infty} \mathscr{S}_{m_k/n_k} f_{st}^{n_k} = \mathscr{S}_t f_{st}.$$

To verify the last equality, we use (A.2.1) and estimate

$$\begin{aligned} d_2\left(\mathscr{S}_{m_k/n_k} f_{st}^{n_k}, \mathscr{S}_t f_{st}\right) &\leq d_2\left(\mathscr{S}_{m_k/n_k} f_{st}^{n_k}, \mathscr{S}_{m_k/n_k} f_{st}\right) + d_2\left(\mathscr{S}_{m_k/n_k} f_{st}, \mathscr{S}_t f_{st}\right) \\ &\leq e^{2(t+1)\|A+\bar{\beta}I\|} d_2\left(f_{st}^{n_k}, f_{st}\right) + d_2\left(\mathscr{S}_{m_k/n_k} f_{st}, \mathscr{S}_t f_{st}\right). \end{aligned}$$

The first term goes to zero, since $f_{st}^{n_k} \to f_{st}$ in $\mathscr{U}$. By an approximation we obtain from Proposition A.2.1 (i) that $t \mapsto \langle \psi, f_t \rangle$ is continuous for any $\psi \in C_b$. Since the second moments are $K\bar{N}$, we conclude with Lemma A.3.5 that the last term goes to zero.

This yields a self-similar solution with zero momentum. To obtain mean $U \in \mathbb{R}^3$ we use the change of variables $v \mapsto v - e^{-tA} U$. For $K > 0$ any self-similar profile is smooth by Proposition A.2.1 (iii). Finally, one can see that the Dirac measure $f_{st} = \delta_0$ is a weak solution to (A.1.11), yielding a self-similar profile with $K = 0$. This concludes the existence proof. □

A.3.2 Uniqueness and stability of self-similar solutions

Here, we prove that any solution to (A.1.8) converges to a self-similar solution after a change of variables.

Proof of Theorem A.1.3 (ii). Let us denote by $\Psi = \mathcal{F}[f_{st}]$ the characteristic function of the profile $f_{st} \in \mathscr{P}_p$, $4 \geq p > 2$ with second moments $\bar{N}$. We assume $\|A\| \leq \varepsilon_0$, where $\varepsilon_0 > 0$ is chosen sufficiently small, such that part (i) of Theorem A.1.3 holds.

For a solution $(f_t)_t \subset \mathscr{P}_p$, $2 < p \leq 4$ to (A.1.8) we take $(\tilde{f}_t)_t$ as in Theorem A.1.3 (ii), which yields a solution to (A.1.8) with matrix $A + \bar{\beta} I$ and zero momentum. Let us denote the characteristic functions of $(\tilde{f}_t)_t$ by $(\varphi_t)_t$ and the second moments by $(M_t)_t$.

By Lemma A.3.2, $(M_t)_t$ satisfies the equation $M_t' = (\mathcal{A}(\bar{b}, A) - 2\bar{\beta} I) M_t$. Furthermore, by Lemma A.3.3 the nonzero eigenvalues of $\mathcal{A}(\bar{b}, A) - 2\bar{\beta} I$ have real part less than $-\nu < 0$. The steady states are given by the span of $\bar{N}$. Thus, there is $C = C(M_0) \geq 0$ and $\alpha = \alpha(M_0) \geq 0$ such that

$$\left\| M_t - \alpha^2 \bar{N} \right\| \leq C e^{-\nu t}. \tag{A.3.6}$$

Using a Povzner estimate as in the proof of Lemma A.3.4 we get $\sup_{t\geq 0} \left\| \tilde{f}_t \right\|_p < \infty$ as long as $\|A\| \leq \varepsilon_0$ is sufficiently small. Note that the second moments are uniformly bounded by (A.3.6). This yields a uniform estimate of $\|\varphi_t\|_{C^{2,p-2}}$.

Observe that $\Psi(\alpha\cdot)$ is the characteristic function of the steady state $\alpha^{-3} f_{st}(v/\alpha)$ with second moments $\alpha^2 \bar{N}$. We estimate the characteristic functions

$$|\varphi_t(k) - \Psi(\alpha k)| \leq \left| \varphi_t(k) - 1 + \frac{1}{2} M_t : k \otimes k \right| + \frac{1}{2} \left\| M_t - \alpha^2 \bar{N} \right\| |k|^2 + \left| 1 - \frac{1}{2} \alpha^2 \bar{N} : k \otimes k - \Psi(\alpha k) \right|.$$

For the first term we use a Taylor expansion, in conjunction with the fact that $D^2 \varphi_t$ is at least $(p-2)$-Hölder continuous with $\|D^2 \varphi_t\|_{C^{p-2}} \leq C_*$. We can assume here w.l.o.g $p < 3$. We have

$$\left| \varphi_t(k) - 1 + \frac{1}{2} M_t : k \otimes k \right| \leq C_* |k|^p.$$

The last term is treated similarly due to $\Psi \in \mathcal{F}_p$. For the second term we apply (A.3.6). This yields

$$|\varphi_t(k) - \Psi(\alpha k)| \leq C|k|^p + Ce^{-\nu t} |k|^2.$$

Now, we apply the comparison principle in Proposition A.2.3 starting at time T to obtain

$$|\varphi_{T+t}(k) - \Psi(\alpha k)| \leq Ce^{-(\lambda(p) - p\|A + \bar{\beta} I\|)t} |k|^p + Ce^{-\nu T + 2\|A + \bar{\beta} I\| t} |k|^2.$$

Now, we further assume that $\varepsilon_0 > 0$ is small enough to ensure

$$\left\| A + \bar{\beta} I \right\| \leq (1 + c_0) \|A\| \leq \min\left(\frac{\lambda(p)}{2p}, \frac{\nu}{4} \right).$$

Thus, we get for $t = T$ and $\theta' = \min(\frac{\lambda(p)}{2}, \frac{\nu}{2})$

$$|\varphi_{2T}(k) - \Psi(\alpha k)| \leq Ce^{-\theta' T} \left(|k|^p + |k|^2 \right), \tag{A.3.7}$$

where $C = C(\varphi_0, p)$. Now, we apply the following inequality valid for all $\varphi, \psi \in \mathcal{F}_p$

$$d_2(\varphi, \psi) \leq c_p(\gamma + \gamma^{2/p}), \quad \gamma := \sup_k \frac{|\varphi - \psi|(k)}{|k|^2 + |k|^p}. \tag{A.3.8}$$

This can be proved by splitting the supremum in $d_2(\varphi, \psi)$ into $|k| \leq R$ and $|k| \geq R$ and minimizing over R. Combining both (A.3.7) and (A.3.8) yields for some $\theta > 0$

$$d_2(\varphi_t, \Psi(\alpha\cdot)) \leq Ce^{-\theta t}.$$

This concludes the proof. $\square$

A.3.3 Finiteness of higher moments

To prove part (iii) of Theorem A.1.3, we need an extension of Lemma A.3.4.

Lemma A.3.6. *Let $M \in \mathbb{N}$, $M \geq 3$ and $p \geq M$. Consider a solution $(f_t)_t \in C([0,\infty);\mathscr{P}_p)$ to* (A.1.8) *with A replaced by $A + \bar{\beta} I$ satisfying* (A.3.5). *Let $\|A\| \leq \varepsilon_0$ and $\varepsilon_0 > 0$ from Lemma A.3.3.*

Then, there is $\varepsilon_M \leq \varepsilon_0$ and $C_* = C_*(K,M)$ such that: if $\|A\| \leq \varepsilon_M$ we have for all $t \geq 0$

$$\|f_0\|_M \leq C_* \implies \|f_t\|_M \leq C_*.$$

Proof. This can be proved by induction over M by applying repeatedly Lemma A.3.1. The case $M = 3,4$ is covered by Lemma A.3.4 and at each step one has to choose $\varepsilon_M \leq \varepsilon_{M-1}$ and $\|A\| \leq \varepsilon_M$ to absorb the drift term. □

Proof of Theorem A.1.3. (iii). We argue as for (i) of Theorem A.1.3. However, now we include the uniform bound $\|f\|_M \leq C_*(M,K)$ in the definition of the sets $\mathscr{U}$. The so constructed stationary solutions coincide with the ones in (i) by uniqueness. □

A.4 Application to simple and planar shear

In this section, we discuss the longtime behavior of homoenergetic solutions in the case of simple and planar shear. Recall that homoenergetic flows have the form $g(t,x,v) = f(t, v - L(t)x)$ and $f = f(t,v)$ satisfies

$$\partial_t f - L(t)v \cdot \nabla f = Q(f,f) \tag{A.4.1}$$

with the matrix $L(t) = (I + tL_0)^{-1} L_0$. Under the assumption $\det(I + tL_0) > 0$ for all $t \geq 0$, one can study the form of $L(t)$ as $t \to \infty$ (see [98, Section 3]). We consider the case of simple shear resp. planar shear ($K \neq 0$)

$$L(t) = \begin{pmatrix} 0 & K & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{resp.} \quad L(t) = \frac{1}{t} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & K \\ 0 & 0 & 1 \end{pmatrix} + \mathcal{O}\left(\frac{1}{t^2}\right) \quad (t \to \infty). \tag{A.4.2}$$

In the first case, (A.4.1) preserves mass, since $\operatorname{tr} L = 0$, and our study applies for K sufficiently small. Alternatively, one can assume a largeness condition on the kernel b, see the assumption below.

Let us now turn to planar shear and write $L(t) = A/(1+t) + \tilde{A}(t)$ with $\operatorname{tr} A = 1$, $\left\|\tilde{A}(t)\right\| \leq \mathcal{O}\left(1/(1+t)^2\right)$. First, let us introduce the time-change $\log(1+t) = \tau$ and set $f(t,v) = F(\tau,v)/(t+1)$ yielding the equation (after multiplying with $(1+t)^2$)

$$\partial_\tau F - \operatorname{div}((A + B(\tau))v \cdot F) + \operatorname{tr} B(\tau) F = Q(F,F) \tag{A.4.3}$$

where $B(\tau) = (1+t)\tilde{A}(t) = \mathcal{O}(1/(1+t)) = \mathcal{O}(e^{-\tau})$. The well-posedness theory of (A.4.3) does not change compared to (A.1.8) and so we omit further details about existence, uniqueness and regularity. We apply our results to (A.4.3) yielding a self-similar asymptotics. More precisely, we have the following result (note that we write t instead of τ in the theorem).

Theorem A.4.1. *Consider* (A.4.3) *with* $A \in \mathbb{R}^{3\times 3}$ *and* $B_t \in C([0,\infty);\mathbb{R}^{3\times 3})$ *such that* $\|B_t\| = \mathcal{O}(e^{-t})$*. Let* $(F_t)_t \subset \mathscr{P}_p$*,* $2 < p$*, be a weak solution to* (A.4.3) *with* $F_0 \in \mathscr{P}_p$ *and first moments*

$$\int_{\mathbb{R}^3} vF_0(dv) = U.$$

We define $m_t \in \mathbb{R}$*,* $E_t \in \mathbb{R}^{3\times 3}$ *as follows*

$$m_t = \int F_t(v)dv = \exp\left(-\int_0^t \operatorname{tr} B_s\, ds\right), \quad \lim_{t\to\infty} m_t = m_\infty, \quad E_t' = (A+B_t)E_t, \quad E_0 = I.$$

There is a constant $\varepsilon_0 = \varepsilon_0(m_\infty \bar{b}, p) > 0$ *such that for* $\|A\| \le \varepsilon_0$*, the following holds. Defining*

$$\tilde{F}_t := \frac{e^{3\bar{\beta}t}}{m_t} F_t\left(e^{\bar{\beta}t}v + E_t U\right), \quad \tilde{f}_{st}(v) = f_{st}(v\alpha_\infty^{-1})\alpha_\infty^{-3}$$

for a constant $\alpha_\infty = \alpha_\infty(F_0)$ *we have for* $\lambda > 0$

$$d_2(\tilde{F}_t, \tilde{f}_{st}) \le Ce^{-\lambda t}.$$

Here, $f_{st} \in \mathscr{P}_p$ *is the solution to*

$$\operatorname{div}((A+\bar{\beta}I)v \cdot f_{st}) + m_\infty Q(f_{st}, f_{st}) = 0, \qquad \int_{\mathbb{R}^3} v_i v_j\, f_{st}(v)\, dv = \bar{N}_{ij},$$

as in Theorem A.1.3 with the corresponding objects $\bar{\beta} = \bar{\beta}(m_\infty \bar{b}, A)$*,* $\bar{N} = \bar{N}(m_\infty \bar{b}, A)$*.*

With this let us now go back to solutions $(f_t)_t$ to equation (A.4.1) with $L(t) = A/(1+t) + \tilde{A}(t)$. To apply the previous result, we need $\|A\| \le \varepsilon_0$. This might not be true for A coming from the matrix $L(t)$ above. However, one can instead assume a largeness condition on the kernel b. To see this, let us rescale time $\tau \mapsto \tau M$ yielding

$$\partial_\tau F - \frac{1}{M}\operatorname{div}((A+B(\tau))v \cdot F) + \frac{1}{M}\operatorname{tr} B(\tau)F = \frac{1}{M}Q(F,F).$$

In particular, the collision kernel is given by b/M. We can hence consider the following assumption. A similar condition was also used in [98, Section 5.2].

Assumption. Assume that the kernel b is chosen such that

$$\|A/M\| \le \varepsilon_0(m_\infty \bar{b}/M)$$

is satisfied for some $M > 0$. Recall the definition of $\bar{b}$ in (A.3.4).

Under this assumption, we can apply Theorem A.4.1 to obtain the asymptotics in terms of $(f_t)_t$ solving (A.4.1). For this we undo the above transformations yielding

$$\frac{e^{t/M}e^{3\bar{\beta}t}}{m_t} f\left(e^{t/M} - 1, e^{\bar{\beta}t}v + E_t U\right) \to f_{st}(v\alpha^{-1})\alpha^{-3} \qquad \text{as } t \to \infty. \tag{A.4.4}$$

Here, $U \in \mathbb{R}^3$ is the mean of the initial condition $f_0 \in \mathscr{P}_p$ and $\alpha = \alpha(f_0)$ is as in Theorem A.4.1. For $B_\tau := e^\tau \tilde{A}(e^\tau - 1)$ we defined

$$m_t = \exp\left(-\frac{1}{M}\int_0^t \operatorname{tr} B_s\, ds\right), \quad E_t' = \frac{1}{M}(A+B_t)E_t, \quad E_0 = I.$$

The convergence in (A.4.4) appears with an order $\mathcal{O}(e^{-\lambda\tau}) = \mathcal{O}(t^{-\lambda M})$ w.r.t the metric d_2.

Finally, let us give the main arguments for the proof of Theorem A.4.1 following the analysis in Section A.3.

Proof of Theorem A.4.1. Preparation. We rescale the solution $G_t(v) = F_t(v + E_tU)/m_t$ so that the mass is one and the momentum is zero. This solves

$$\partial_t G - \operatorname{div}((A+B_t)v \cdot G) = m_t Q(G,G), \quad \int G_t(dv) = 1, \quad \int vG_t(dv) = 0.$$

The assumption $\|B_t\| = \mathcal{O}(e^{-t})$ implies $m_t \to m_\infty > 0$ and $|m_{T+t} - m_T| \le Ce^{-T}$. We introduce the self-similar variables $G_t(v) = f_t(ve^{-\bar{\beta}t})e^{-3\bar{\beta}t}$ and get

$$\partial_t f - \operatorname{div}((A + \bar{\beta}I + B_t)v \cdot f) = m_t Q(f,f). \tag{A.4.5}$$

where $\bar{\beta} = \bar{\beta}(A, m_\infty \bar{b})$ is as in Theorem A.1.3 or Lemma A.3.3 when considering the collision kernel $m_\infty \bar{b}$.

Now, the plan is as follows. First, we study the longtime behavior of the second moments M_t of f_t in Step 1. Then, in Step 2, we want to compare (A.4.5) to solutions $g^{(T)}$ to

$$\partial_t g^{(T)} = \operatorname{div}\left((A + \bar{\beta}I)v \cdot g^{(T)}\right) + m_\infty Q\left(g^{(T)}, g^{(T)}\right), \quad g_0^{(T)} = f_T. \tag{A.4.6}$$

This equation has the stationary solution f_{st}. In Step 3, we apply Theorem A.1.3 to $g^{(T)}$ to obtain $g^{(T)} \to f_{st}(\alpha_T^{-1}\cdot)\alpha_T^{-3}$. Altogether, we conclude $f_t \to f_{st}(\alpha_\infty^{-1}\cdot)\alpha_\infty^{-3}$. Here, α_T, α_∞ are constants, which precise values will be apparent below.

Step 1. Let M_t be the second moments of f_t, which satisfy (see also Lemma A.3.2)

$$\frac{dM_t}{dt} = \mathcal{A}(m_t\bar{b}, A + 2\bar{\beta}I)M_t + \mathcal{B}_t M_t,$$

where $\mathcal{A}(m_t\bar{b}, A + \bar{\beta}I)$, $\mathcal{B}_t$ are linear operators $\mathbb{R}^{3\times3}_{sym} \to \mathbb{R}^{3\times3}_{sym}$ and $\|\mathcal{B}_t\| \le Ce^{-t}$. The first operator corresponds to the drift term with matrix $A + \bar{\beta}I$ and the collision operator. The second operator captures the drift term with B_t. Due to the linear dependence of $\mathcal{A}$ w.r.t. $m_\infty \bar{b}$ we can write

$$\frac{dM_t}{dt} = \mathcal{A}(m_\infty\bar{b}, A + \bar{\beta}I)M_t + \mathcal{R}_t M_t.$$

Since $|m_t - m_\infty| \le Ce^{-t}$ we still have $\|\mathcal{R}_t\| \le Ce^{-t}$. The results of Lemma A.3.3 hold for the semigroup $e^{\mathcal{A}t}$ generated by $\mathcal{A} := \mathcal{A}(m_\infty\bar{b}, A + \bar{\beta}I)$. Using Duhamel's formula one can prove that

$$e^{\mathcal{A}t}M_T \to \alpha_T^2 \bar{N}, \quad M_t \to \alpha_\infty^2 \bar{N}$$

as $t \to \infty$ for all $T \ge 0$ with a convergence of order $Ce^{-\nu t}$. Furthermore, $|\alpha_\infty^2 - \alpha_T^2| \le Ce^{-T}$ where the constants $C > 0$ are always independent of T.

Step 2. Now, we compare f with $g^{(T)}$ satisfying (A.4.6) via the following estimate for all $t, T \ge 0$

$$d_2\left(f_{t+T}, g_t^{(T)}\right) \le Ct\, e^{-T + 2\|A + \bar{\beta}I\| t}. \tag{A.4.7}$$

Here, C is independent of t, T. This inequality can be proved as part (ii) in Proposition A.2.1. The difference here is the coefficient m_t in front of the collision operator, as well as the term due to B_t in (A.4.5). Both of them lead to a term of order e^{-T}. We get analogously to (A.2.7)

$$e^{m_\infty S_n t} d_2(\varphi_{t+T}, \psi_t) \le \left(r_n + Ce^{-T}\right) \int_0^t e^{m_\infty S_n r} e^{[2\|A + \bar{\beta}I\| + m_\infty S_n](t-r)} dr.$$

Dividing by $e^{m_\infty S_n t}$ and sending $n \to \infty$ yields (A.4.7).

Step 3. Now, we apply Theorem A.1.3 to the solutions $g^{(T)}$ to (A.4.6). For this, let f_{st} be the stationary solution to (A.4.6) with second moments $\bar{N}$ and $\Psi = \mathcal{F}[f_{st}]$. We get in Fourier space $\psi_t^{(T)} = \mathcal{F}[g^{(T)}]$, with α_T as in Step 1,

$$d_2\left(\psi_t^{(T)}, \Psi(\alpha_T \cdot)\right) \leq Ce^{-\theta t}.$$

The only problem now is that the constant C might depend on the initial condition f_T and thus on T. If we trace back the dependence of this constant in the proof of Theorem A.1.3 (ii), then two constants C_1, C_2 contribute. The first one satisfies

$$\left\| e^{\mathcal{A}_{\bar{\beta}} t} M_T - \alpha_T^2 \bar{N} \right\| \leq C_1 e^{-\nu t}$$

and depends only on M_T, which is uniformly bounded. The second constant is a uniform bound on the moments of order $4 \geq p > 2$, see Step 2 in the proof of Theorem A.1.3 (ii). Looking at the arguments there, we see that it suffices to show $\sup_t \|f_t\|_p < \infty$ in order to obtain $\sup_{t,T} \left\| g_t^{(T)} \right\|_p < \infty$. This can be proved again by an application of the Povzner estimate to the equation (A.4.5). The difference here is an additional term due to B_t. Since this is integrable in time one can choose $\varepsilon_0 > 0$ small enough in exactly the same way.

Conclusion. Let us combine all our estimates in Fourier space $\varphi_t = \mathcal{F}[f_t]$, $\psi_t^{(T)} = \mathcal{F}[g_t^{(T)}]$

$$\begin{aligned} d_2\left(\varphi_{t+T}, \Psi(\alpha_\infty \cdot)\right) &\leq d_2\left(\varphi_{t+T}, \psi_t^{(T)}\right) + d_2\left(\psi_t^{(T)}, \Psi(\alpha_T \cdot)\right) + d_2\left(\Psi(\alpha_T \cdot), \Psi(\alpha_\infty \cdot)\right) \\ &\leq Ct e^{-T + 2\|A + \bar{\beta} I\| t} + Ce^{-\theta t} + Ce^{-T}. \end{aligned}$$

The first two estimates follow from Step 2 and Step 3. The last one follows from a Taylor expansion and $|\alpha_\infty^2 - \alpha_T^2| \leq Ce^{-T}$. Let us now choose $t = T$ and ensure $2\left\| A + \bar{\beta} I \right\| \leq 2(1 + c_0)\|A\| \leq 1/2$, by choosing $\|A\|$ sufficiently small. This concludes the proof. □

Remark A.4.2. Let us comment on the smallness condition on A, which was used at three different points: (1) in Lemma A.3.3 when studying the eigenvalues resp. eigenvectors, (2) in Lemma A.3.4 for a uniform bound in time of moments of order $p > 2$ and (3) in the proof of the convergence to the self-similar profile. The first two incidences concerned the existence of self-similar solutions. In the case of simple shear, i.e. A is given by the first matrix in (A.4.2), Lemma A.3.3 has been extended for large values of K via explicit computations in [98, Section 5.1]. Furthermore, they formulated a condition to extend (2) for such matrices A. However, this condition has not been studied in further detail. Concerning the stability result, different convergence methods would be needed, which take into account the effect of the drift term.

Appendix B

Longtime behavior of homoenergetic solutions in the collision dominated regime for hard potentials

Abstract

We consider a particular class of solutions to the Boltzmann equation which are referred to as homoenergetic solutions. They describe the dynamics of a dilute gas due to collisions and the action of either a shear, a dilation or a combination of both. More precisely, we study the case in which the shear is dominant compared with the dilation and the collision operator has homogeneity $\gamma > 0$. We prove that solutions with initially high temperature remain close and converge to a Maxwellian distribution with temperature going to infinity. Furthermore, we give precise asymptotic formulas for the temperature. The proof relies on an ansatz which is motivated by a Hilbert-type expansion. We consider both non-cutoff and cutoff kernels.

B.1 Introduction

The inhomogeneous Boltzmann equation is given by

$$\partial_t f + v \cdot \nabla_x f = Q(f,f), \tag{B.1.1}$$

where $f = f(t,x,v) : [0,\infty) \times \mathbb{R}^3 \times \mathbb{R}^3 \to [0,\infty)$ denotes the one-particle distribution of a dilute gas in whole space. In this paper, we will restrict ourselves to the physically most relevant case of three dimensions, although our study can be extended to dimensions $N \geq 3$ without any additional difficulties.

In (B.1.1) the bilinear collision operator has the form

$$Q(f,g) = \int_{\mathbb{R}^3} \int_{S^2} B(|v-v_*|, n \cdot \sigma)(f'_* g' - f_* g) d\sigma dv_*,$$

where $n = (v-v_*)/|v-v_*|$ and $f'_* = f(v'_*)$, $g' = g(v')$, $f_* = f(v_*)$, with the pre-collisional velocities (v, v_*) and post-collisional velocities (v', v'_*). Here, we use the σ-representation of post-collisional velocities, i.e. for $\sigma \in S^2$

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \qquad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma.$$

Recall that the collision operator satisfies

$$\int_{\mathbb{R}^3} Q(f,f)\varphi(v)dv = 0, \quad \varphi(v) = 1, v_1, v_2, v_3, |v|^2$$

which correspond to the conservation of mass, momentum and energy. We refer to [47, 156] for an introduction into the physical and mathematical theory of the Boltzmann equation (B.1.1).

The collision kernel is given by $B(|v-v_*|, n\cdot\sigma)$ and it can be obtained from an analysis of the binary collisions of the gas molecules. For instance, power law potentials $1/r^{q-1}$ with $q > 2$ lead to (see e.g. [47, Sec. II.5])

$$B(|v-v_*|, n\cdot\sigma) = |v-v_*|^\gamma b(n\cdot\sigma), \quad \gamma = (q-5)/(q-1), \tag{B.1.2}$$

where $b : [-1,1) \to [0,\infty)$ has a non-integrable singularity of the form

$$\sin\theta\, b(\cos\theta) \sim \theta^{-1-2/(q-1)} = \theta^{-1-2s}, \quad \text{as } \theta \to 0, \quad s = \frac{1}{q-1}, \tag{B.1.3}$$

where $\cos\theta = n\cdot\sigma$, with θ being the deviation angle. It is customary to classify the collision kernels according to their homogeneity γ with respect to relative velocities $|v-v_*|$. There are three cases: *hard potentials* ($\gamma > 0$), *Maxwell molecules* ($\gamma = 0$) and *soft potentials* ($\gamma < 0$). Furthermore, collision kernels with an angular singularity of the form (B.1.3) are called non-cutoff kernels. This singularity reflects the fact that for power law interactions the average number of *grazing collisions*, i.e. collisions with $v \approx v'$, diverges. In the main part of the paper, we consider non-cutoff collision kernels for hard potentials $\gamma > 0$.

In this paper, we study a particular class of solutions to (B.1.1), namely the so-called *homoenergetic solutions*, which have been analyzed in particular in [97]. There conjectures for the longtime asymptotics of solutions have been formulated. Here, we give a rigorous proof of these conjectures in the case of hard potentials $\gamma > 0$.

B.1.1 Homoenergetic solutions

Our study concerns solutions to (B.1.1) with the form

$$f(t,x,v) = g(t, v - L(t)x), \quad w = v - L(t)x, \tag{B.1.4}$$

for $L(t) \in \mathbb{R}^{3\times 3}$ and a function $g = g(t,w) : [0,\infty) \times \mathbb{R}^3 \to [0,\infty)$ to be determined. One can check that in general, solutions to (B.1.1) with the form (B.1.4) exist for a large class of functions g if and only if g and L satisfy

$$\begin{aligned} \partial_t g - L(t)w\cdot\nabla_w g &= Q(g,g), \\ \frac{d}{dt}L(t) + L(t)^2 &= 0. \end{aligned} \tag{B.1.5}$$

The second equation is used to reduce the variables (t,x,v) to (t,w). In particular, the collision operator acts on g only through the variable w. The second equation can be solved explicitly $L(t) = L(0)(I + tL(0))^{-1}$ given an initial datum $L(0)$. Note that, depending on $L(0)$, the inverse matrix might not be defined for all times, although this situation will not be considered here.

Solutions to (B.1.5) are called *homoenergetic solutions* and were introduced by Galkin [71] and Truesdell [153]. They studied their properties in the case of Maxwell molecules via moment equations, a method known since the work by Truesdell and Muncaster [154], see also [69, 70, 71, 72, 153]. More recently, this method has also been used in [74] for homoenergetic solutions of the Boltzmann equation as well as other kinetic models like BGK. The case of mixtures of gases has been studied there as well.

The well-posedness of (B.1.5) in the case of cutoff hard potentials was proved by Cercignani [48]. See also [49, 50] for a study of shear flow for granular media. Homoenergetic solutions for the two-dimensional Boltzmann equation as well as for a class of Fokker-Planck equations have been studied in [125].

A systematic analysis of the large time behavior of solutions to (B.1.5) for kernels with arbitrary homogeneities has been undertaken in [26, 97, 98, 99]. In particular, self-similar solutions have been studied in [26, 66, 98] for cutoff Maxwell molecules, and in [105] for non-cutoff Maxwell molecules. On the other hand, in [97] the longtime behavior for non-Maxwellian molecules has been analyzed, when the collision operator is dominant over the drift term. This suggests that solutions approach the equilibrium distribution. However, since the temperature is not conserved, the equilibrium distribution has a temperature varying with time. The core of the analysis was an adaptation of a Hilbert expansion in order to determine the behavior of the temperature for large times. However, the arguments in [97] concentrated on the first two (highest order) terms in the adapted Hilbert expansion, leaving open a rigorous analysis of the approach.

The main contribution of this paper is to give rigorous proofs of the longtime behavior and the asymptotics of the temperature. Here, we study the case of non-cutoff interactions with hard potentials $\gamma > 0$. As a result, we verify the conjectures in [97] for such collision kernels.

B.1.2 Linearized collision operator and Hilbert-type expansion

Linearized collision operator. In order to recall the formal arguments in [97] let us introduce the linearized collision operator given by

$$\mathscr{L}h = -Q(h,\mu) - Q(\mu,h) = -\int_{\mathbb{R}^3}\int_{S^2} B(|v-v_*|, n\cdot\sigma)\left[\mu'_* h' + h'_*\mu' - \mu_* h - \mu h_*\right] d\sigma dv_*.$$

Here, μ denotes the Maxwellian

$$\mu(v) = \frac{1}{(2\pi)^{3/2}} e^{-|v|^2/2},$$

which is the (up to a change of the mass, momentum and energy) unique equilibrium solution of the homogeneous Boltzmann equation. The operator $\mathscr{L}$ on $L^2(\mu^{-1/2})$ or equivalently the operator $Lh = \mu^{-1/2}\mathscr{L}(\mu^{1/2}h)$ on L^2 has been extensively studied, see [5, 21, 78, 107, 133, 136, 138]. Here, we used the notation $L^2(\mu^{-1/2})$ for the weighted L^2-space with weight $\mu^{-1/2}$, see the notations in Subsection B.1.3. It is known that $\mathscr{L}$ is a non-negative, self-adjoint operator on $L^2(\mu^{-1/2})$. Furthermore, it has a spectral gap if and only if $\gamma + 2s \geq 0$, see [78]. Here, $s \in (0,1)$ is measuring the angular singularity as in (B.1.3) and γ is the homogeneity of the kernel (B.1.2) with respect to $|v - v_*|$. The case of cutoff kernels is included by setting $s = 0$. Let us recall that the kernel of $\mathscr{L}$ is given by

$$\ker \mathscr{L} = \operatorname{span}\left\{\mu, v_1\mu, v_2\mu, v_3\mu, |v|^2\mu\right\}.$$

This is related to the conservation of mass, momentum and energy of the collision operator. In the case $\gamma \in (0,1), s \in (0,1/2)$, one can show that $e^{-t\mathscr{L}}h$, with h in weighted L^1-spaces, approaches $\ker \mathscr{L}$ exponentially fast, see [152] or Lemma B.3.12 below. This relies on the more general framework in [79, 134].

For our analysis, the space $L^2(\mu^{-1/2})$, on which $\mathscr{L}$ is self-adjoint and has a spectral gap, is inconvenient, since the exponential decay at infinity $|v| \to \infty$ is a priori not preserved by (B.1.5). Thus, we make use of the decay estimates in weighted L^1-spaces from [152].

Hilbert-type expansion for homoenergetic solutions. Our goal is to give a rigorous proof of conjectures in [97] in the case $\gamma > 0$. For $\gamma > 0$ they considered matrices $L_t = L_0(I + tL_0)^{-1}$ which have one of the following asymptotic forms as $t \to \infty$, assuming $\det(I + tL(0)) > 0$ for all $t \geq 0$:

(i) Simple shear:

$$L_t = \begin{pmatrix} 0 & K & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad K \neq 0. \tag{B.1.6}$$

(ii) Simple shear with decaying planar dilatation/shear:

$$L_t = \begin{pmatrix} 0 & K_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{1+t} \begin{pmatrix} 0 & K_1K_3 & K_1 \\ 0 & 0 & 0 \\ 0 & K_3 & 1 \end{pmatrix} + \mathcal{O}\left(\frac{1}{(1+t)^2}\right), \quad K_2 \neq 0. \tag{B.1.7}$$

(iii) Combined orthogonal shear:

$$L_t = \begin{pmatrix} 0 & K_3 & K_2 - tK_1K_3 \\ 0 & 0 & K_1 \\ 0 & 0 & 0 \end{pmatrix}, \quad K_1K_3 \neq 0. \tag{B.1.8}$$

The above nomenclature was used in [98, Theorem 3.1], where one can find all possible asymptotic forms of L_t as $t \to \infty$ under the assumption $\det(I + tL_0) > 0$ for all $t \geq 0$. Note that we used the notation L_t to abbreviate the time-dependence $L(t)$ as will be done throughout the paper.

For convenience, let us recall the formal asymptotics in [97], which can be seen as a variation of a Hilbert expansion. This allows to construct solutions to (B.1.5) which, after some change of variables, remain close and converge to a Maxwellian distribution.

For the relevant rescaling let us recall the mass ρ_t, momentum V_t and temperature T_t of g_t, given by

$$\rho_t = \int_{\mathbb{R}^3} g_t(w)\, dw, \quad \rho_t V_t = \int_{\mathbb{R}^3} w g_t(w)\, dw, \quad \rho_t T_t = \frac{1}{3} \int_{\mathbb{R}^3} |w - V_t|^2 g_t(w)\, dw.$$

(Strictly speaking, the term defining T_t above is $2/3$ times the internal energy with mass set to one. However, this is related to the temperature via the Boltzmann constant.) Using (B.1.5) one can show that

$$\rho_t' = -\operatorname{tr} L_t \rho_t, \quad V_t' = -L_t V_t, \quad T_t' = -\frac{2}{3\rho_t} \int_{\mathbb{R}^3} (w - V_t) \cdot L_t(w - V_t)\, g_t(w)\, dw. \tag{B.1.9}$$

The first two equations can be solved explicitly. We then introduce a rescaling which sets mass to one, momentum to zero and temperature to one. Define $f_t(v) = g_t(v\beta_t^{-1/2} + V_t)\beta_t^{-3/2}\rho_t^{-1}$, where we used the inverse temperature $\beta_t = T_t^{-1}$. As a consequence we have

$$\int_{\mathbb{R}^3} f_t(v)\,dv = 1, \quad \int_{\mathbb{R}^3} v f_t(v)\,dv = 0, \quad \frac{1}{3}\int_{\mathbb{R}^3} |v|^2 f_t(v)\,dv = 1. \tag{B.1.10}$$

Furthermore, we obtain with (B.1.5) and (B.1.9) the equations

$$\begin{aligned} \partial_t f_t &= \operatorname{div}\left((L_t - \alpha_t)\, v\, f_t\right) + \rho_t\, \beta_t^{-\gamma/2} Q(f_t, f_t), \quad f(0,\cdot) = f_0(\cdot), \\ \beta_t &= \beta_0 \exp\left(2\int_0^t \alpha_s\, ds\right), \quad \alpha_t := \frac{1}{3}\int v\cdot L_t v\, f_t(v)\,dv, \\ \rho_t &= \exp\left(-\int_0^t \operatorname{tr} L_s\, ds\right). \end{aligned} \tag{B.1.11}$$

Note that we set $\rho_0 = 1$, whereas the initial inverse temperature is given by β_0. The equation for the inverse temperature is a consequence of (B.1.9) yielding

$$\frac{\beta_t'}{2\beta_t} = \alpha_t = \frac{1}{3}\int v\cdot L_t v\, f_t(v)\,dv. \tag{B.1.12}$$

Furthermore, observe that the momentum V_t does not appear in the evolution equation for f due to the translation invariance of (B.1.5).

Our analysis is concerned with the longtime behavior of f_t and we consider here the collision-dominated behavior. This is the case $\eta_t := \rho_t\, \beta_t^{-\gamma/2} \to \infty$ as $t \to \infty$ and the drift term is of lower order compared to the collision operator. (Note that this is an a priori assumption that has to be check a posteriori, since η_t depends on f_t.) This situation suggests that f_t remains close and converges to equilibrium μ. We hence use the following ansatz, which is the Hilbert-type expansion introduced in [97],

$$f_t(v) = \mu(v) + h_t^{(1)}(v) + \cdots + h_t^{(k)}(v) + \cdots.$$

We assume as $t \to \infty$

$$h^{(1)} \ll \mu, \quad h^{(k+1)} \ll h^{(k)}, \quad k \in \mathbb{N},$$

and we can decompose

$$\alpha_t = \frac{1}{3}\int v\cdot L_t v\, f_t(v)\,dv = \frac{\operatorname{tr} L_t}{3} + \alpha_t^{(1)} + \cdots, \quad \alpha_t^{(k)} := \frac{1}{3}\int_{\mathbb{R}^3} v\cdot L_t v\, h_t^{(k)}(v)\,dv. \tag{B.1.13}$$

As a consequence we observe $\alpha_t^{(k+1)} \ll \alpha_t^{(k)}$ as $t \to \infty$. We plug the above ansatz into (B.1.11) and collect terms of equal order (one has to take into account that $\eta_t = \rho_t\, \beta_t^{-\gamma/2} \to \infty$ as $t \to \infty$). The first order term $h^{(1)}$ satisfies

$$0 = \operatorname{div}\left(\left(L_t - \frac{1}{3}\operatorname{tr} L_t\, I\right) v\, \mu\right) - \eta_t \mathscr{L} h_t^{(1)}.$$

One can show that the first term on the right-hand side is orthogonal to $\ker \mathscr{L}$ with respect to the scalar product in $L^2(\mu^{-1/2})$. Hence, we can invert $\mathscr{L}$ and obtain

$$h_t^{(1)} = -\frac{1}{\eta_t}\mathscr{L}^{-1}\left[v\cdot A_t v\, \mu\right], \quad A_t := L_t - \frac{1}{3}\operatorname{tr} L_t\, I. \tag{B.1.14}$$

Furthermore, we have

$$\alpha_t^{(1)} = -\frac{a_t}{3\eta_t}, \quad a_t := \left\langle v \cdot A_t v\, \mu, \mathscr{L}^{-1}\left[v \cdot A_t v\, \mu\right]\right\rangle_{L^2(\mu^{-1/2})}. \tag{B.1.15}$$

Note that $a_t > 0$ for $A_t \neq 0$, since $\mathscr{L}$ is a positive operator on $(\ker \mathscr{L})^\perp$. We also observe that $h_t^{(1)} = \mathcal{O}(1/\eta_t)$, recalling $\eta_t \to \infty$, so that $h_t^{(1)} \ll \mu$ as $t \to \infty$. Similarly, one can formally solve the equations for $h^{(k)}$ and conclude $h_t^{(k)} = \mathcal{O}(1/\eta_t^k)$. In each equation the term $\alpha_t^{(k)}$ allows to invert the operator $\mathscr{L}$ on $(\ker \mathscr{L})^\perp$ as for $k = 0$ above. Hence, the term $\alpha_t^{(k)}$ can be interpreted as Lagrange multiplier.

Finally, we need to show a posteriori that $\eta_t = \rho_t \beta_t^{-\gamma/2}$ as $t \to \infty$. As was observed in [97] this is possible for L_t given by (B.1.6), (B.1.7) or (B.1.8). Let us consider here the case of simple shear (B.1.6), so that $L_t \equiv L_0$ is constant in time and $\rho_t \equiv 1$, due to $\operatorname{tr} L_t = 0$. We use (B.1.12) and (B.1.15) to obtain for $\eta_t = \beta_t^{-\gamma/2}$

$$\eta_t' = \frac{\gamma a_0}{3} + \mathcal{O}(1/\eta_t).$$

Here, we used that $\alpha_t^{(k)} = \mathcal{O}(1/\eta_t^k)$ for $k \geq 2$. Hence, we get

$$\eta_t = \frac{\gamma a_0}{3} t + o(t),$$

i.e. $\eta_t \to \infty$ as $t \to \infty$, and thus

$$\beta_t = \left[\frac{\gamma a_0}{3} t + o(t)\right]^{-2/\gamma}.$$

Consequently, the temperature T_t goes to infinity like $t^{2/\gamma}$.

Below we give a rigorous proof of the above longtime behavior $f_t \to \mu$ and exact asymptotic formulas for the temperature in all the cases (B.1.6), (B.1.7) and (B.1.8). There are two crucial assumptions that we use. We assume that initially f_0 is close enough to a Maxwellian and that the initial temperature $T_0 = \beta_0^{-1}$ is sufficiently large. The latter condition ensures that the collision operator is dominant for all times, due to the term $\rho_t \beta_t^{-\gamma/2}$.

Physical interpretation. Let us briefly comment on the physical picture of the formal asymptotic study. To clarify the effect of the drift term $L_t w \cdot \nabla_w g$ in (B.1.5) we consider the flow $U_t \in \mathbb{R}^{3\times 3}$ of $-L_t$, that is

$$U_t' = -L_t U_t, \quad U_0 = I, \quad \det U_t = \exp\left(-\int_0^t \operatorname{tr} L_s\, ds\right).$$

The sign of $\operatorname{tr} L_t$ determines whether we have expansion or dilatation in velocity space. In the case of homoenergetic solutions, we always have $\operatorname{tr} L_t \geq 0$ to highest order. Thus, the dilatation would lead to a decrease of velocities and hence the temperature. But there is also the shearing effect due to the trace-free part A_t of L_t. If the temperature is already high enough (β_0 small), most particles have very large velocities and the shear leads to an increase of them.

In total we have two competing effects. Both are present in the zeroth and first order terms in the formula for the inverse temperature (B.1.12). More precisely, we have with (B.1.13), (B.1.14) and (B.1.15)

$$\left(\beta_t^{-\gamma/2}\right)' = -\gamma\, \alpha_t \beta_t^{-\gamma/2} \approx -\gamma\left(\alpha_t^{(0)} + \alpha_t^{(1)}\right)\beta_t^{-\gamma/2} = -\frac{\gamma \operatorname{tr} L_t}{3}\beta_t^{-\gamma/2} + \frac{\gamma a_t}{3\rho_t}.$$

The shear is present through the term a_t given in (B.1.15), which only depends on A_t. The dilatation is visible through $\operatorname{tr} L_t \geq 0$. In the case that L_t is given by (B.1.7) the shear is always more dominant than the dilatation, while L_t in (B.1.6) and (B.1.8) contain no dilation. As a result we have $\beta_t^{-1} = T_t \to \infty$ as $t \to \infty$.

B.1.3 Main results

Notation. For a function $f : I \to X$, $I \subset \mathbb{R}$ some time interval, into a Banach space X we denote by $f_t \in X$ its evaluation at the time $t \in I$.

Let us define the weighted L^p-spaces $L^p(w)$ with a positive weight function $w : \mathbb{R}^3 \to (0,\infty)$ with norm

$$\|f\|_{L^p(w)} := \|w\, f\|_{L^p}.$$

For $p = 2$ the scalar product is given by

$$\langle f, g\rangle_{L^2(w)} = \int_{\mathbb{R}^3} f(v)\, g(v)\, w(v)^2\, dv.$$

In the case $w(v) = \langle v\rangle^m$, $m \geq 0$, we abbreviate them by L^p_m, where $\langle v\rangle := \sqrt{1+|v|^2}$. We also use the corresponding weighted Sobolev spaces $W^{k,p}_m(\mathbb{R}^3)$, $k \in \mathbb{N}_0$, $p \geq 1$. More precisely, $f \in W^{k,p}_m$ if and only if the weak derivatives of order less or equal $k \in \mathbb{N}_0$ exist $\partial^\alpha f$, for any multi-index $\alpha \in \mathbb{N}_0^3$, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 \leq k$ and

$$\|f\|^p_{W^{k,p}_m} := \sum_{|\alpha| \leq k} \|\partial^\alpha f\|^p_{L^p_m}$$

is finite. Let us also define the norm of the homogeneous fractional Sobolev space $\dot{H}^r$ for $r \geq 0$ by

$$\|f\|^2_{\dot{H}^r} := \int_{\mathbb{R}^3} |\xi|^{2r} |\mathscr{F}[f](\xi)|^2\, d\xi,$$

where $\mathscr{F}[f]$ is the Fourier transform. The norm of the inhomogeneous Sobolev space H^r for $r \geq 0$ is defined by

$$\|f\|^2_{H^r} = \|f\|^2_{L^2} + \|f\|^2_{\dot{H}^r}.$$

Recall that $W^{k,2} = H^k$ for $k \in \mathbb{N}$ and we use both definitions interchangeably. The corresponding spaces with weights $\langle v\rangle^m$ are denoted by $\dot{H}^r_m$ respectively H^r_m and the norms are given by

$$\|f\|_{\dot{H}^r_m} = \|\langle\cdot\rangle^m f\|_{\dot{H}^r}, \quad \|f\|_{H^r_m} = \|\langle\cdot\rangle^m f\|_{H^r}.$$

We also use the following weighted Sobolev space $\mathcal{H}^1_p$ with norm

$$\|f\|^2_{\mathcal{H}^1_p} := \|f\|^2_{L^2_p} + \sum_{|\alpha|=1} \|\partial^\alpha f\|^2_{L^2_{p-2s}},$$

where $s \in (0, 1/2)$ measures the singularity of the angular part in the collision kernel, see (B.1.16) below. Finally, we write $A \lesssim B$ resp. $A \gtrsim B$, if there is a positive constant $C > 0$ with $A \leq CB$ resp. $CA \geq B$. We write $A \approx B$ if both $A \lesssim B$ and $A \gtrsim B$.

Assumptions on the kernel. We make the following assumptions.

- (A-1) The collision kernel has the product form
$$B(v-v_*,\sigma)=b(n\cdot\sigma)|v-v_*|^\gamma.$$
- (A-2) The function $b:[-1,1)\to[0,\infty)$ is locally smooth and has the angular singularity
$$\sin\theta\, b(\cos\theta)\,\theta^{1+2s}\to K_b>0, \quad \text{as } \theta\to 0 \tag{B.1.16}$$
for some $s\in(0,1/2)$ and $K_b>0$.
- (A-3) The parameter γ satisfies $\gamma\in(0,1)$.

In particular, this implies
$$\Lambda=\int_0^\pi \sin\theta\, b(\cos\theta)\,\theta d\theta<\infty. \tag{B.1.17}$$
These assumptions cover inverse power law interactions with $q>5$, see (B.1.2) and (B.1.3). Finally, we also assume without loss of generality that $b(\cos\theta)$ is supported on $[0,\pi/2]$ by using the symmetrization
$$b(n\cdot\sigma)\mathbb{1}_{\{n\cdot\sigma\geq 0\}}+b(-n\cdot\sigma)\mathbb{1}_{\{n\cdot\sigma\geq 0\}}.$$
This does not change the collision operator $Q(f,f)$, since $f(v')f(v'_*)$ is invariant under the change of variables $\sigma\mapsto-\sigma$. Let us mention that assumption (A-3) very likely can be relaxed, in particular including the case $\gamma=1$. However, we assume $\gamma\in(0,1)$ as was done in most works we rely on in this paper. Nevertheless, in Section B.5, covering the cutoff case, we allow $\gamma\in(0,1]$. This includes in particular the hard sphere model $B(v-v_*,\sigma)=|v-v_*|$.

Results for homoenergetic solutions with collision-dominated behavior. We consider solutions g to (B.1.5) with initial mass $\rho_0=1$, momentum $V_0\in\mathbb{R}^3$ and inverse temperature $\beta_0>0$. They are related to solutions $f_t(v)=g_t(v\beta_t^{-1/2}+V_t)\beta_t^{-3/2}\rho_t^{-1}$ to the equations (B.1.11). Here, we used
$$\rho_t=\exp\left(-\int_0^t \operatorname{tr} L_s\,ds\right), \quad V_t=U_tV_0, \quad \frac{1}{\beta_t}=\frac{1}{3\rho_t}\int_{\mathbb{R}^3}|w-V_t|^2 g_t(w)\,dw \tag{B.1.18}$$
with $t\mapsto U_t\in\mathbb{R}^{3\times 3}$ satisfying $U_t'=-L_tU_t$, $U_0=I$. The well-posedness and regularity theory of these equations is discussed in Section B.2, see Propositions B.2.2 and B.2.6.

Theorem B.1.1. *Consider equation* (B.1.5) *with matrix $L_t=L_0(I+tL_0)^{-1}$ having the asymptotic form* (B.1.6)*,* (B.1.7) *or* (B.1.8)*. Let $p_0>4+4s+3/2$ be arbitrary and $g_0\in\mathcal{H}^1_{p_0}$. Consider the unique solution g to* (B.1.5)*. Define with* (B.1.18)
$$f_t(v):=g_t(v\beta_t^{-1/2}+V_t)\beta_t^{-3/2}\rho_t^{-1}, \quad \bar{\mu}_t:=\frac{1}{\rho_t\beta_t^{-\gamma/2}}\mathscr{L}^{-1}\left[-v\cdot A_tv\mu\right], \quad A_t:=L_t-\frac{\operatorname{tr} L_t}{3}I \tag{B.1.19}$$
and $h_t(v):=f_t(v)-\mu(v)-\bar{\mu}_t(v)$.

There is $\varepsilon_0\in(0,1)$ sufficiently small depending only on p_0, L and the collision kernel B, such that: If $\|h_0\|_{\mathcal{H}^1_{p_0}}=\varepsilon\leq\varepsilon_0$ and $\beta_0\leq\varepsilon_0$, we have
$$\|f_t-\mu\|_{\mathcal{H}^1_{p_0}}\to 0 \tag{B.1.20}$$
as $t\to\infty$. Furthermore, the inverse temperature has the following asymptotics in each case.

(i) *Simple shear, L_t given by* (B.1.6)*: We have*

$$\lim_{t\to\infty} \frac{\beta_t^{-\gamma/2}}{t} = \frac{\gamma\bar{a}}{3} \tag{B.1.21}$$

with the constant $\bar{a} > 0$ given by

$$\bar{a} := \left\langle v\cdot A^0 v\,\mu, \mathscr{L}^{-1}\left[v\cdot A^0 v\,\mu\right]\right\rangle_{L^2(\mu^{-1/2})}, \quad A^0 := \begin{pmatrix} 0 & K & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{B.1.22}$$

(ii) *Simple shear with decaying planar dilatation/shear, L_t given by* (B.1.7)*: We have*

$$\lim_{t\to\infty} \frac{\beta_t^{-\gamma/2}}{t^2} = \frac{\gamma\bar{a}}{\gamma+6}\exp\left(\int_0^\infty r_s\,ds\right), \tag{B.1.23}$$

where we defined

$$r_t := \operatorname{tr} L_t - \frac{1}{1+t} = \mathcal{O}\left(\frac{1}{(1+t)^2}\right), \quad t\to\infty, \tag{B.1.24}$$

and the constant $\bar{a} > 0$ is given by

$$\bar{a} := \left\langle v\cdot A^0 v\,\mu, \mathscr{L}^{-1}\left[v\cdot A^0 v\,\mu\right]\right\rangle_{L^2(\mu^{-1/2})}, \quad A^0 := \begin{pmatrix} 0 & K_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{B.1.25}$$

(iii) *Combined orthogonal shear, L_t given by* (B.1.8)*: We have*

$$\lim_{t\to\infty} \frac{\beta_t^{-\gamma/2}}{t^3} = \frac{\gamma\bar{a}}{9}, \tag{B.1.26}$$

where the constant $\bar{a} > 0$ is defined by

$$\bar{a} := \left\langle v\cdot A^0 v\,\mu, \mathscr{L}^{-1}\left[v\cdot A^0 v\mu\right]\right\rangle_{L^2(\mu^{-1/2})}, \quad A^0 := \begin{pmatrix} 0 & 0 & -K_1K_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{B.1.27}$$

In fact we can give a more quantitative statement than (B.1.20), which also verifies the formal Hilbert-type expansion discussed previously.

Theorem B.1.2. *Under the assumptions of Theorem B.1.1 the following statements hold for $h_t = f_t - \mu - \bar{\mu}_t$, where the constant $C' > 0$ depends only on p_0, L and the collision kernel B.*

(i) *Simple shear, L_t given by* (B.1.6)*: We have*

$$\|h_t\|_{\mathcal{H}^1_{p_0}} \le C'\left(\frac{\varepsilon}{(1+t)^2} + \frac{1}{\zeta_t^2}\right), \quad \left\|\frac{\bar{\mu}_t}{\sqrt{\mu}}\right\|_{H^k_p} \le \frac{C_{k,p}}{\zeta_t} \tag{B.1.28}$$

for all $t \ge 0$, $k, p \in \mathbb{N}$ and

$$\zeta_t := \beta_0^{-\gamma/2} + t.$$

(ii) *Simple shear with decaying planar dilatation/shear, L_t given by* (B.1.7)*: We have*

$$\|h_t\|_{\mathcal{H}^1_{p_0}} \leq C'\left(\frac{\varepsilon}{(1+t)^2}+\frac{1}{\zeta_t^2}\right), \quad \left\|\frac{\bar{\mu}_t}{\sqrt{\mu}}\right\|_{H^k_p} \leq \frac{C_{k,p}}{\zeta_t} \tag{B.1.29}$$

for all $t \geq 0$, $k,p \in \mathbb{N}$ and

$$\zeta_t := \beta_0^{-\gamma/2}(1+t)^{-1-\gamma/3}+t.$$

(iii) *Combined orthogonal shear, L_t given by* (B.1.8)*: We have*

$$\|h_t\|_{\mathcal{H}^1_{p_0}} \leq C'\left(\frac{\varepsilon}{(1+t)^4}+\frac{1}{\zeta_t^2}\right), \quad \left\|\frac{\bar{\mu}_t}{\sqrt{\mu}}\right\|_{H^k_p} \leq \frac{C_{k,p}}{\zeta_t} \tag{B.1.30}$$

for all $t \geq 0$, $k,p \in \mathbb{N}$ and

$$\zeta_t := \beta_0^{-\gamma/2}(1+t)^{-1}+t^2.$$

Remark B.1.3. Let us give a few comments.

(i) The estimates for h_t in (B.1.28) and (B.1.29) contain a quadratic decay whereas in (B.1.30) the decay is of order four. Similarly, for $\bar{\mu}_t$ the decay is faster in (B.1.30). The reason is that in this case the matrix L_t is given by (B.1.8), for which the shear is growing linearly in time. Since the shear is the driving mechanism for the temperature to grow, the process is accelerated.

(ii) As we will see in Section B.3, the equation satisfied by h contains a source term, see (B.3.11). This term leads to a decay of order $1/\zeta_t^2$. The other terms contain h and imply a decay of order $\varepsilon/(1+t)^2$. This yields the particular form of the above estimates. Note that for t of order $\beta_0^{-\gamma/2}$ the term $1/\zeta_t^2$ is larger than $\varepsilon/(1+t)^2$ and $\|h_t\| \lesssim 1/(1+t)^2$. For times of order one, it depends on the relative size of ε, $\beta_0^{\gamma/2}$ to see which term is larger.

(iii) Our estimates on the perturbation h, in particular the use of the norm $\|\cdot\|_{\mathcal{H}^1_{p_0}}$, rely on results in [91], where polynomially decaying solutions to the inhomogeneous Boltzmann equation close to equilibrium have been constructed. However, we combine them with the stability of $\mathscr{L}$ in L^1_m, for some $m>2$, proved in [152].

(iv) In the main part of the paper, we consider the non-cutoff case. In Section B.5 we discuss a variant of the above theorem in the cutoff case. The proof follows the main arguments in Section B.3, but uses less technical estimates on the collision operator.

The paper is organized in the following way. In Section B.2, we show well-posedness and regularity for equation (B.1.5) with general initial data. In Section B.3 we prove the more general Theorem B.3.1 based on regularity estimates on the level of the linearization h. In Section B.4, we conclude Theorem B.1.1 and Theorem B.1.2 as an application of Theorem B.3.1. Finally, in Section B.5 we discuss how the above theorems can be proved in the cutoff case.

B.2 Well-posedness and regularity for homoenergetic solutions

In this section, we study existence, uniqueness and regularity of solutions f to

$$\partial_t f_t = L_t v \cdot \nabla f_t + Q(f_t, f_t). \tag{B.2.1}$$

The matrix L_t is given, not necessarily defined by (B.1.6), (B.1.7) or (B.1.8).

Let us introduce the notion of weak solutions to (B.2.1), which is reminiscent of weak solutions to the homogeneous Boltzmann equation. Recall that the entropy $H(f)$ of some function $f \geq 0$ is given by

$$H(f) = \int_{\mathbb{R}^3} f(v) \log f(v)\, dv.$$

Definition B.2.1. Let $f_0 \in L^1_2$ with $H(f_0) < \infty$. We say that $f \in L^\infty_{\text{loc}}([0,\infty); L^1_2)$, $f \geq 0$ is a weak solution to (B.2.1) if for all $T \geq 0$ and all test functions $\varphi \in C^1_b([0,T] \times \mathbb{R}^3)$ we have

$$\begin{aligned} &\int_{\mathbb{R}^3} f_T(v)\varphi_T(v)\, dv - \int_{\mathbb{R}^3} f_0(v)\varphi_0(v)\, dv - \int_0^T \int_{\mathbb{R}^3} f_s\, \partial_s \varphi_s\, ds dv \\ &= -\int_0^T \int_{\mathbb{R}^3} \operatorname{div}(L_s v\, \varphi_s)\, f_s(v)\, dv ds + \int_0^T \langle Q(f_s, f_s), \varphi_s \rangle\, ds. \end{aligned} \tag{B.2.2}$$

Furthermore, f satisfies for all $t \geq 0$

$$H(f_t) \leq H(f_0) \exp\left(- \int_0^t \operatorname{tr} L_s\, ds \right). \tag{B.2.3}$$

Here, we interpret

$$\langle Q(f_t, f_t), \varphi_t \rangle = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2} |v - v_*|^\gamma b(n \cdot \sigma)\, f_t(v) f_t(v_*)\, (\varphi_t(v') - \varphi_t(v))\, d\sigma dv_* dv.$$

This is motivated by testing $Q(f_t, f_t)$ with φ and applying the pre-post-collisional change of variables. Note that

$$\int_{S^2} b(n \cdot \sigma)(\varphi(v') - \varphi(v))\, d\sigma \leq \Lambda \left(\sup_{|\xi| \leq \sqrt{|v|^2 + |v_*|^2}} |\nabla \varphi(\xi)| \right) |v - v_*|,$$

which follows from $|v' - v| = |v - v_*| \sin(\theta/2)$ and our assumptions on b, see (B.1.17). Hence, we have

$$|\langle Q(f_t, f_t), \varphi \rangle| \leq C\Lambda\, \|\nabla \varphi\|_\infty\, \|f_t\|^2_{L^1_{1+\gamma}}$$

and the weak formulation is well-defined due to $1 + \gamma \leq 2$. In order to motivate (B.2.3) it is convenient to integrate (B.2.1) by characteristics yielding

$$\partial_t [f_t(U_{0,t} v)] = Q(f_t, f_t)(U_{0,t} v),$$

where $U_{0,t} \in \mathbb{R}^{3\times 3}$ satisfies $U'_{0,t} = -L_t U_{0,t}$, $U_{0,0} = I$. We formally calculate

$$\frac{d}{dt}\left[\det(U_{0,t})^{-1} H(f_t) \right] = \int_{\mathbb{R}^3} Q(f_t, f_t)(U_{0,t} v) \log f_t(U_{0,t} v)\, dv \leq 0,$$

which implies (B.2.3).

Proposition B.2.2. *Consider* (B.2.1) *with* $L \in L^\infty_{\text{loc}}([0,\infty);\mathbb{R}^{3\times 3})$.

(i) Let $p>2$ be arbitrary. For any $f_0 \in L^1_p$ with $H(f_0)<\infty$ there is a weak solution $f \in L^\infty_{\text{loc}}([0,\infty);L^1_p)$ to (B.2.1). *Furthermore, for any $t_0>0$ and any $q\in\mathbb{N}$, $T\geq t_0$*

$$\sup_{t\in[t_0,T]} \|f_t\|_{L^1_q} \leq C.$$

Here, C depends on $t_0, q, \sup_{t\in[0,T]}\|L_t\|$ and T.

(ii) Let $q\geq 2$. Then, there is at most one weak solution $f \in L^\infty_{\text{loc}}([0,\infty);W^{1,1}_{q+1+\gamma})$ to (B.2.1).

To prove this proposition we recall the following version of the Povzner estimate proved in [129, Sect. 2]. As was noticed e.g. in [155, Appendix], their calculation also works in the non-cutoff case.

Lemma B.2.3. *Let $\varphi(v)=|v|^{2+\delta}$ for $\delta>0$. Then we have the decomposition*

$$\int_{S^2} b(n\cdot\sigma)\left\{\varphi'_* + \varphi' - \varphi_* - \varphi\right\} d\sigma = G(v,v_*) - H(v,v_*)$$

with G,H satisfying

$$\begin{aligned} &G(v,v_*) \leq C\Lambda(|v||v_*|)^{1+\delta/2}, \\ &H(v,v_*) \geq c\Lambda(|v|^{2+\delta}+|v_*|^{2+\delta})\left(1-\mathbb{1}_{\{|v|/2<|v_*|<2|v|\}}\right). \end{aligned} \tag{B.2.4}$$

for some $c,C>0$ depending on δ. Recall that Λ is defined in (B.1.17).

Proof of Proposition B.2.2. We split the proof into three steps. For item (i) we first show existence by reduction to the cutoff case (Step 1). Then, we apply the Povzner estimates in Lemma B.2.3 to obtain the gain of moments (Step 2). Finally, we conclude with the proof of (ii) using arguments from [62, Theorem 1, Proposition 1] developed for the homogeneous Boltzmann equation.

(i) To prove existence we first consider the case of angular cutoff, as in the analysis of the homogeneous Boltzmann equation, see e.g. [155].

Step 1: Let us consider a cutoff collision kernel, i.e. for $n\in\mathbb{N}$, $n\in\mathbb{N}$, $B_n := (|v-v_*|\wedge n)^\gamma\,[b(\cos\theta)\wedge n]$. The corresponding collision operator is denoted by Q_n. Solutions f^n to the corresponding problem with finite entropy were constructed by Cercignani in [48]. The main idea was to study the problem by integrating via characteristics and to adapt arguments in [14]. Furthermore, the mass, momentum and energy/temperature satisfy the a priori estimates in (B.1.9). In particular, $\|f^n_t\|_{L^1_2}$ is bounded uniformly in $n\in\mathbb{N}$, locally in time.

If $f_0\in L^1_p$ then we have $f^n\in L^\infty_{\text{loc}}([0,\infty);L^1_p)$. We use the above Povzner estimates to obtain bounds in L^1_p uniformly in $n\in\mathbb{N}$, $p>2$. More precisely, we have with Lemma B.2.3 for $M^n_p(t) := \int_{\mathbb{R}^3}|v|^p f^n_t\,dv$

$$\begin{aligned} \frac{d}{dt}M^n_p &\leq C(p,L)M^n_p + C\Lambda_n M^n_{\gamma+p/2}M^n_{p/2} \\ &\quad - c\Lambda_n \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}(|v-v_*|\wedge n)^\gamma |v|^p\, f^n_t(v) f^n_t(v_*)\,dv_* dv. \end{aligned}$$

Here, we used for the last term in $H(v,v_*)$ in (B.2.4), $|v|^p \mathbb{1}_{\{|v|/2<|v_*|<2|v|\}} \lesssim (|v||v_*|)^{p/2}$. Furthermore, similarly as Λ, Λ_n is given through the angular part $[b(\cos\theta)\wedge n]$ of the cutoff kernel in terms of (B.1.17). We apply

$$(|v-v_*|\wedge n)^\gamma \geq \frac{1}{4}(|v|\wedge n)^\gamma - |v_*|^\gamma$$

to get with $M_\gamma^n \leq \|f_t^n\|_{L^1_2}$

$$\frac{d}{dt}M_p^n \leq CM_p^n + C\Lambda_n M_{\gamma+p/2}^n M_{p/2}^n + C\Lambda_n M_p^n - \frac{c\Lambda_n}{4}\int_{\mathbb{R}^3}(|v|\wedge n)^\gamma |v|^p f_t^n(v)dv.$$

If $p\leq 4$ we have $M_{p/2}^n \leq \|f_t^n\|_{L^1_2}$ and we can use (recalling $p>2$, $0<\gamma\leq 1$ and hence $\gamma+p/2<p$)

$$M_{\gamma+p/2}^n \leq C_\varepsilon M_0^n + \varepsilon\int_{\mathbb{R}^3}(|v|\wedge n)^\gamma |v|^p f_t^n(v)dv$$

for all $\varepsilon>0$. Hence, a Gronwall argument applies and M_p^n is bounded locally in time. This is uniform in $n\in\mathbb{N}$, since $0<\Lambda/2\leq\Lambda_n\leq\Lambda$ for sufficiently large $n\in\mathbb{N}$. One can see, using the weak formulation, that $t\mapsto\int f_t^n\varphi\, dv$ is Lipschitz, uniformly in $n\in\mathbb{N}$, for any test function. The entropy bound (B.2.3) yields then weak L^1 compactness by the Dunford-Pettis theorem, $f^{n_k}\rightharpoonup f$ for a limit $f\in L^\infty_{\text{loc}}(L^1_p)$ and a subsequence $n_k\to\infty$. Furthermore, we can pass to the limit in the definition of the weak formulation (B.2.2), since $p>2$. In particular, f satisfies (B.1.9).

In the case $p>4$, one can use the above reasoning inductively, so that the term $M_{p/2}^n$ can be bounded by the previous inductive step.

Step 2: We now prove the gain of moments using the Povzner estimates similarly to the homogeneous Boltzmann equation. One argues again inductively. We obtain the estimate by testing with $|v|^p$

$$\frac{d}{dt}M_p \leq CM_p + C\Lambda M_{\gamma+p/2}M_{p/2} - c\Lambda M_{p+\gamma}.$$

From a previous inductive step we know that $M_{p/2}$ is bounded locally in time. We can use $M_p + M_{\gamma+p/2} \leq C_\varepsilon M_0 + \varepsilon M_{p+\gamma}$ to get

$$\frac{d}{dt}M_p \leq C_\varepsilon M_0 - \frac{c\Lambda}{2}M_{p+\gamma}.$$

An integration yields

$$M_p(T) + \frac{c\Lambda}{2}\int_0^T M_{p+\gamma}(s)\,ds \leq M_p(0) + C(T).$$

Hence, for any $t_1>0$ we can find $t_0\in(0,t_1)$ such that $M_{p+\gamma}(t_0)$ is finite. Thus, the solution gained a moment of order $\gamma>0$. One can then argue with $p+\gamma$ instead of p and starting from the time t_0. However, the preceding argument was formal, but can be made rigorous when using a cutoff of $|v|^p$ as a test function.

(ii) We prove uniqueness by adapting the strategy in [62, Theorem 1, Proposition 1].

Let $f,\tilde{f}$ be two weak solutions to (B.2.1), see Definition B.2.1. Due to the assumption $f,\tilde{f}\in L^\infty_{\text{loc}}([0,\infty);W^{1,1}_{q+1+\gamma})$ the right hand side in (B.2.1) is in $L^\infty_{\text{loc}}([0,\infty);L^1_q)$, see Lemma B.3.13

for the estimate of the collision operator. Thus $t \mapsto f_t, \tilde{f}_t \in L^1_q$ are Lipschitz. This allows to make the following reasoning rigorous. As in [62] set $D = f - \tilde{f}$ and $S = f + \tilde{f}$ to get

$$\partial_t D_t = L_t v \cdot \nabla D_t + \frac{1}{2}\left(Q(S_t, D_t) + Q(D_t, S_t)\right).$$

We then obtain (by formally testing this equation with $\operatorname{sgn}(D_t)\langle v\rangle^q$ and integrating in time)

$$\begin{aligned}\|D_T\|_{L^1_q} &= \int_0^T \int_{\mathbb{R}^3} \left[-\operatorname{div}\left(L_t v \langle v\rangle^q\right)|D_t| + \frac{1}{2}\left(Q(S_t, D_t) + Q(D_t, S_t)\right)\operatorname{sgn} D_t \langle v\rangle^q\right] dv dt \\ &\le C_q \|L\|_\infty \int_0^T \|D_t\|_{L^1_q} dt + \int_0^T \int_{\mathbb{R}^3} \frac{1}{2}\left(Q(S_t, D_t) + Q(D_t, S_t)\right)\operatorname{sgn} D_t \langle v\rangle^q dv dt.\end{aligned}$$

We use the same estimates as in [62] for the collision operator. The idea is to split into a cutoff and non-cutoff part $Q = Q_{c,\varepsilon} + Q_{nc,\varepsilon}$ with respect to the angular part $b = b_{c,\varepsilon} + b_{nc,\varepsilon}$ for a parameter $\varepsilon > 0$. For the cutoff part, a variant of the Povzner estimate is used (see [122, Lemma 1]) to get

$$\begin{aligned}&\langle Q_{c,\varepsilon}(S_t, D_t) + Q_{c,\varepsilon}(D_t, S_t), \operatorname{sgn} D_t \langle v\rangle^q\rangle \\ &\le \langle Q_{c,\varepsilon}(S_t, |D_t|) + Q_{c,\varepsilon}(|D_t|, S_t), \langle v\rangle^q\rangle + 2\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} |v - v_*|^\gamma b_{c,\varepsilon}(n\cdot\sigma)\,|(D_t)_*|\, S_t \langle v\rangle^q \, d\sigma dv_* dv \\ &\le C_\varepsilon \|D_t\|_{L^1_q} - K\|D_t\|_{L^1_{q+\gamma}}.\end{aligned}$$

For the non-cutoff part, we have

$$\begin{aligned}&\langle Q_{nc,\varepsilon}(S_t, D_t), \operatorname{sgn} D_t \langle v\rangle^q\rangle \\ &\le \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} |v - v_*|^\gamma b_{nc,\varepsilon}(n\cdot\sigma)\left((S_t)'_*|(D_t)'| - (S_t)_*|D_t|\right)\langle v\rangle^q \, d\sigma dv_* dv.\end{aligned}$$

We can now use the pre-post collisional change of variables and the fact that $|\langle v'\rangle^q - \langle v\rangle^q| \lesssim \sin(\theta/2)\langle v\rangle^q \langle v_*\rangle^q$ to get

$$\langle Q_{nc,\varepsilon}(S_t, D_t), \operatorname{sgn} D_t \langle v\rangle^q\rangle \le c_\varepsilon \|S_t\|_{L^1_{q+\gamma}} \|D_t\|_{L^1_{q+\gamma}}.$$

Finally, using the estimate for the collision operator in Lemma B.3.13 below we have

$$\|Q_{nc,\varepsilon}(D_t, S_t)\|_{L^1_q} \le c_\varepsilon \|D_t\|_{L^1_{q+\gamma}} \|S_t\|_{W^{1,1}_{q+\gamma+1}}.$$

Note that $c_\varepsilon \to 0$ as $\varepsilon \to 0$. For ε small enough we get in total

$$\|D_T\|_{L^1_q} \le \left(C_q \|L\|_\infty + C_\varepsilon\right)\int_0^T \|D_t\|_{L^1_q} dt$$

for all $T \ge 0$. Hence, we conclude $D \equiv 0$. □

Now we want to prove that the weak solutions constructed in Proposition B.2.2 are smooth for positive times. This is analogous to the homogeneous Boltzmann equation and follows from the singular behavior of the angular part of the collision kernel, see [3]. Here, we follow the treatment in [6] and show how to adapt the arguments for equation (B.2.1) containing also a drift term. To this end, we use two lemmas proved in [6]. For items (i) and (ii) of Lemma 2.4 we refer to Propositions 2.1 and 3.8 of [6], respectively.

Lemma B.2.4. *The following statements hold.*

(i) For any $g \in L^1_2$ *with* $g \geq 0$, $\|g\|_{L^1_2} + H(g) \leq E_0$, $\|g\|_{L^1} \geq D_0$ *for* $D_0, E_0 > 0$ *we have*

$$-\langle Q(g,f), f\rangle_{L^2} \geq c_0 \|f\|^2_{H^s_{\gamma/2}} - C\|f\|^2_{L^2}.$$

Here, the constants c_0, C *only depend on* D_0, E_0. *Recall that* $s \in (0,1/2)$ *is given in* (B.1.16).

(ii) For any $r \in [2s-1, 2s]$, $\ell \in [0, \gamma+2s]$ *we have*

$$|\langle Q(f,g), h\rangle_{L^2}| \lesssim \|f\|_{L^1_{\gamma+2s}} \|g\|_{H^r_{\gamma+2s-\ell}} \|h\|_{H^{2s-r}_{\ell}}.$$

Since we want to use estimates in Sobolev spaces, we need to regularize the (a priori not smooth enough) solution. As in [6] we define the mollifier in Fourier space via

$$M^\delta_\lambda(\xi) = \frac{\langle\xi\rangle^\lambda}{(1+\delta\langle\xi\rangle^{N_0})}$$

for $\delta > 0$, λ, $N_0 \in \mathbb{R}$. This is a pseudo-differential symbol $M^\delta_\lambda \in S^{\lambda-N_0}_{1,0}$ and we define accordingly

$$M^\delta_\lambda(D_v)f = \mathscr{F}^{-1}\left[M^\delta_\lambda \mathscr{F}[f]\right],$$

where $\mathscr{F}[f]$ denotes the Fourier transform of f. We also abbreviate $M^\delta_\lambda f$. The next lemma is a commutator estimate, see [6, Theorem 3.6]. Let us recall that we assume $s \in (0,1/2)$.

Lemma B.2.5. *Let* $s' \in (0,s)$ *and assume that* λ, N_0 *satisfy*

$$5+\gamma \geq 2(N_0-\lambda). \tag{B.2.5}$$

(i) If $s'+\lambda < 3/2$ *we have*

$$\left|\left\langle M^\delta_\lambda Q(f,g) - Q(f, M^\delta_\lambda g), h\right\rangle_{L^2}\right| \lesssim \|f\|_{L^1_\gamma} \left\|M^\delta_\lambda g\right\|_{H^{s'}_{\gamma/2}} \|h\|_{H^{s'}_{\gamma/2}}.$$

(ii) If $s'+\lambda \geq 3/2$ *we have*

$$\left|\left\langle M^\delta_\lambda Q(f,g) - Q(f, M^\delta_\lambda g), h\right\rangle_{L^2}\right| \lesssim \left(\|f\|_{L^1_\gamma} + \|f\|_{H^{(\lambda+s'-3)_+}}\right) \left\|M^\delta_\lambda g\right\|_{H^{s'}_{\gamma/2}} \|h\|_{H^{s'}_{\gamma/2}}.$$

Proposition B.2.6. *Any weak solution* f *to* (B.2.1) *with* $f \in L^\infty([0,T]; L^1_p)$ *for all* $p \geq 0$ *satisfies for all* $k, p \geq 0$ *and any* $t_0 > 0$

$$f \in L^\infty([t_0,T]; H^k_p).$$

Note that due to Proposition B.2.2 (i) the above assumptions are satisfied for positive times.

Proof of Proposition B.2.6. The proof is similar to the original one in [6, Theorem 4.1, Theorem 5.1].

Step 1: First we prove that $f \in L^\infty([T_0, T]; L^2_\ell)$ for all $\ell \geq 0$ and some $T_0 \geq 0$ implies the claim. We do this by induction and indicate the induction step. Accordingly, let us assume without loss of generality that for some $a \geq 0$ and any $\ell \geq 0$ we have

$$f \in L^\infty([0,T]; H^a_\ell).$$

Choose $T_1 > 0$ arbitrary. We define $\lambda(t) := Nt + a$ for $N > 0$ with $NT_1 = (1-s)$ and $N_0 := a + (5+\gamma)/2$. In particular, (B.2.5) holds. For any $t \in [0, T_1]$ we have

$$\lambda(t) - N_0 - a \leq \lambda(T_1) - N_0 = 1 - s - (5+\gamma)/2 < -3/2.$$

Hence, we have for all $p \geq 0$

$$M^\delta_{\lambda(t)} f_{t'} \in L^\infty([0,T] \times [0,T]; H^{3/2}_p \cap L^\infty). \tag{B.2.6}$$

In Step 3, we show that this implies

$$M^\delta_{\lambda(t)} f_t \in C([0,T]; L^2) \tag{B.2.7}$$

and that the following formal argument can be made rigorous. We use $(M^\delta_{\lambda(t)})^2 f_t$ as a test function to get

$$\begin{aligned}\frac{1}{2}\left\|M^\delta_{\lambda(t)} f_t\right\|^2_{L^2} = \frac{1}{2}\|f_0\|^2_{H^a} + \frac{1}{2}\int_0^t \operatorname{tr} L_\tau \left\|M^\delta_{\lambda(\tau)} f_\tau\right\|^2_{L^2} d\tau + \frac{1}{2}\int_0^t \int_{\mathbb{R}^3} f_\tau \, \partial_\tau \left[(M^\delta_{\lambda(\tau)})^2\right] f_\tau \, dv d\tau \\ - \int_0^t \int_{\mathbb{R}^3} f_\tau \, L_\tau v \cdot \nabla (M^\delta_{\lambda(\tau)})^2 f_\tau \, dv d\tau + \int_0^t \int_{\mathbb{R}^3} Q(f_\tau, f_\tau)(M^\delta_{\lambda(\tau)})^2 f_\tau \, dv d\tau.\end{aligned}$$

The last two terms make sense when we use commutators. This leads to

$$\begin{aligned}\frac{1}{2}\left\|M^\delta_{\lambda(t)} f_t\right\|^2_{L^2} &= \frac{1}{2}\|f_0\|^2_{H^a} + N\int_0^t \int_{\mathbb{R}^3} \left(\sqrt{\log\langle D_v\rangle} M^\delta_{\lambda(\tau)} f_\tau\right)^2 dv d\tau \\ &\quad + \int_0^t \int_{\mathbb{R}^3} (C_\tau f_\tau)\left(M^\delta_{\lambda(\tau)} f_\tau\right) dv d\tau + \int_0^t \left\langle Q(f_\tau, M^\delta_{\lambda(\tau)} f_\tau), M^\delta_{\lambda(\tau)} f_\tau \right\rangle d\tau \\ &\quad + \int_0^t \left\langle M^\delta_{\lambda(\tau)} Q(f_\tau, f_\tau) - Q(f_\tau, M^\delta_{\lambda(\tau)} f_\tau), M^\delta_{\lambda(\tau)} f_\tau \right\rangle d\tau.\end{aligned} \tag{B.2.8}$$

Here, we introduced the commutator

$$C_t(D_v) := L_t v \cdot \nabla M^\delta_{\lambda(t)}(D_v) - \nabla M^\delta_{\lambda(t)}(D_v) \cdot L_t v.$$

For (B.2.8) we used several observations. First of all, we applied

$$\partial_\tau M^\delta_{\lambda(\tau)} = N \log\langle\xi\rangle \, M^\delta_{\lambda(\tau)}.$$

For the drift term we used

$$\begin{aligned}&\frac{1}{2}\int_0^t \operatorname{tr} L_\tau \left\|M^\delta_{\lambda(\tau)} f_\tau\right\|^2_{L^2} d\tau - \int_0^t \int_{\mathbb{R}^3} f_\tau \, L_\tau v \cdot \nabla (M^\delta_{\lambda(\tau)})^2 f_\tau \, dv d\tau \\ &= \frac{1}{2}\int_0^t \operatorname{tr} L_\tau \left\|M^\delta_{\lambda(\tau)} f_\tau\right\|^2_{L^2} d\tau + \int_0^t \int_{\mathbb{R}^3} (C_\tau f_\tau)\left(M^\delta_{\lambda(\tau)} f_\tau\right) dv d\tau \\ &\quad + \int_0^t \int_{\mathbb{R}^3} \left[M^\delta_{\lambda(\tau)} f_\tau\right] L_\tau v \cdot \nabla \left[M^\delta_{\lambda(\tau)} f_\tau\right] dv d\tau.\end{aligned}$$

In the last expression we apply partial integration so that the term involving $\operatorname{tr} L_t$ cancels.

In formula (B.2.8) all terms make sense due to (B.2.6). In fact, using the Fourier transform one can show that the symbol of C_t is given by $-L_t^\top \xi \cdot \nabla_\xi M^\delta_{\lambda(t)}$, which is bounded by $C\,\|L_t\|\, M^\delta_{\lambda(t)}$ uniformly in $\delta > 0$. The terms involving the collision operator make sense due to Lemma B.2.4 (ii) and Lemma B.2.5 in conjunction with (B.2.6).

In (B.2.8) we use the previous observations and Lemma B.2.4 (i) to obtain

$$\begin{aligned}\frac{1}{2}\left\|M^\delta_{\lambda(t)} f_t\right\|^2_{L^2} \le \frac{1}{2}\|f_0\|^2_{H^a} + C\|L\|_{L^\infty}\int_0^t \left\|M^\delta_{\lambda(\tau)} f_\tau\right\|^2_{L^2} d\tau + N\int_0^t \left\|\sqrt{\log\langle D_v\rangle} M^\delta_{\lambda(\tau)} f_\tau\right\|^2_{L^2} d\tau \\ + C_f\int_0^t \left\|M^\delta_{\lambda(\tau)} f_\tau\right\|^2_{H^{s'}_{\gamma/2}} d\tau - c_f\int_0^t \left\|M^\delta_{\lambda(\tau)} f_\tau\right\|^2_{H^{s}_{\gamma/2}} d\tau + C_f\int_0^t \left\|M^\delta_{\lambda(\tau)} f_\tau\right\|^2_{L^2} d\tau.\end{aligned}$$

For the second, third, fourth and last term we use interpolation in Sobolev spaces to get

$$\sup_{t\in[0,T_1]} \left\|M^\delta_{\lambda(t)} f_t\right\|^2_{L^2} \le C(T_1).$$

This implies, with interpolation in weighted Sobolev spaces, see [6, Lemma 3.9], for $s_0 := (1-s)/3$

$$f \in L^\infty([T_1/2, T_1], H^{s_0+a}_\ell),$$

since $0 < s_0 < \lambda(T_1/2) = (1-s)/2 + a$. Since $T_1 > 0$ was arbitrary we conclude for any $t_0 > 0$ and $\ell \ge 0$

$$f \in L^\infty([t_0, T], H^{s_0+a}_\ell).$$

The regularity improved by the fixed amount $s_0 > 0$ so that we can repeat the above reasoning inductively, starting at some new time $t_0 > 0$. Let us comment here on the use of Lemma B.2.5. Here, we needed to distinguish two cases. In the second case, when $\lambda(t) + s' \ge 3/2$ the constant C_f above includes the norm $\|f_t\|_{H^{(\lambda(t)+s'-3)_+}}$. Let us verify that this is bounded in the induction. In the k-th step we have $\lambda(t) := Nt + ks_0$ for $t \le T_1$, $NT_1 = 1-s$. Hence, we get

$$\lambda(t) + s' - 3 \le ks_0 - 2 - s + s' \le ks_0 - 2.$$

Thus, this term is bounded due to the $(k-1)$-th step.

Step 2: We assumed $f \in L^\infty([T_0, T]; L^2_p)$ for all $p \ge 0$ and any $T_0 \ge 0$ in Step 1. To prove that our assumption $f \in L^\infty([0,T]; L^1_p)$ for all $p \ge 0$ implies this, one can follow the arguments in [6, Theorem 5.1]. Here, one starts the induction with the regularity

$$f \in L^\infty([T_0, T]; H_p^{-3/2-\varepsilon})$$

for any $p \ge 0$, $\varepsilon > 0$ due to the embedding $L^1_p \subset H_p^{-3/2-\varepsilon}$. Furthermore, one chooses $\lambda(t)$ and N_0 such that for any $p \ge 0$

$$M^\delta_{\lambda(t)} f_{t'} \in L^\infty([0,T]\times[0,T]; H^{s_1}_p)$$

and some $s_1 > s$. This regularity allows to make the corresponding computations rigorous, see Step 2. For more details see [6, Theorem 5.1].

Step 3: Finally, we show that the above formal computations can be made rigorous. This corresponds to [6, Lemma 4.3]. More precisely, we prove that, with the notation as in Step 1,

$$M^\delta_{\lambda(t)} f_{t'} \in L^\infty([0,T]\times[0,T]; H^{s_1}_\ell) \tag{B.2.9}$$

for all $\ell \geq 0$ and some $s_1 > s$ implies (B.2.7) and (B.2.8). We first show (B.2.7) and due to the drift term a regularization is necessary. Let us define for $\kappa > 0$

$$M^{\delta,\kappa}_{\lambda(t)}(D_v) = \frac{1}{1+\kappa\langle D_v\rangle} M^{\delta}_{\lambda(t)}(D_v).$$

For $t' < t$ we choose $(M^{\delta,\kappa}_{\lambda(\bar{t})})^2 f_{\bar{t}}$ with $\bar{t} = t', t$ as time-independent test function. We can do this by an approximation $(M^{\delta,\kappa}_{\lambda(\bar{t})})^{-1}\psi_j \to M^{\delta,\kappa}_{\lambda(\bar{t})} f_{\bar{t}}$ in $H^s_{\ell_0}$ for $\psi_j \in C^\infty_0$. In the corresponding expressions we use then again commutator estimates. For the collision operator, we apply Lemma B.2.5 and Lemma B.2.4 (ii). For the drift term, we use the commutator

$$L_t v \cdot \nabla M^{\delta,\kappa}_{\lambda(\bar{t})}(D_v) - \nabla M^{\delta,\kappa}_{\lambda(\bar{t})}(D_v) \cdot L_t v.$$

As in Step 1, the corresponding symbol is bounded by $CM^{\delta,\kappa}_{\lambda(\bar{t})}$ uniformly in $\kappa, \delta > 0$. The two results for $\bar{t} = t', t$ are added to yield

$$\left\| M^{\delta,\kappa}_{\lambda(t)} f_t \right\|^2_{L^2} - \left\| M^{\delta,\kappa}_{\lambda(t')} f_{t'} \right\|^2_{L^2} = \int_{\mathbb{R}^3} f_t \left((M^{\delta,\kappa}_{\lambda(t)})^2 - (M^{\delta,\kappa}_{\lambda(t')})^2 \right) f_{t'}\, dv + \mathcal{O}(|t-t'|).$$

The last part contains all the remaining terms, which are time-integrals with bounded integrand. The first term on the right-hand is seen to be also of order $\mathcal{O}(|t-t'|)$. All of the terms are uniformly bounded in $\kappa > 0$, so we let $\kappa \to 0$. This shows

$$\lim_{t' \to t} \left\| M^{\delta}_{\lambda(t')} f_{t'} \right\|^2_{L^2} = \left\| M^{\delta}_{\lambda(t)} f_t \right\|^2_{L^2}.$$

If we take the differences of the expressions for $\bar{t} = t', t$ we get

$$\lim_{t' \to t} \int_{\mathbb{R}^3} \left(M^{\delta}_{\lambda(t')} f_{t'} \right) \left(M^{\delta}_{\lambda(t)} f_t \right) dv = \left\| M^{\delta}_{\lambda(t)} f_t \right\|^2_{L^2}.$$

This implies (B.2.7).

To prove that (B.2.8) is rigorous, we divide $[0,t]$ into time steps $t_j = jt/k$, $j = 0, \ldots, k$. As above we want to use $(M^{\delta,\kappa}_{\lambda(\bar{t})})^2 f_{\bar{t}}$ with $\bar{t} = t_{j-1}, t_j$ as time-independent test function. Subtracting the resulting expressions and adding in $j = 0, \ldots, k$, we aim to letting $k \to \infty$. This would lead to (B.2.8). To this end, the integrands in the time-integrals have to be continuous in t. In fact, we have

$$M^{\delta}_{\lambda(t)} f_t \in C([0,T]; H^s_\ell)$$

as a consequence of (B.2.7) and interpolation with the estimate (B.2.9). This yields (B.2.8), concluding the proof. □

B.3 Collision dominated behavior for a model equation

In this section, we study a rescaling of solutions g to (B.1.5). As in the introduction we consider $f_t(v) = g_t(v\beta_t^{-1/2} + V_t)\beta_t^{-3/2}\rho_t^{-1}$ with ρ_t, V_t, β_t given in (B.1.18). This yields a solution to (B.1.11). However, the matrix L might be unbounded in time, as is the case for combined orthogonal shear (B.1.8). In the analysis here, it will be convenient to reduce it to the case of bounded matrices L. To this end, we use a change of time, see the proof of Theorem B.1.1 in Section B.4. Such a transformation yields a solution f to equations of the form (see (B.1.11))

$$\begin{aligned}\partial_t f_t &= \operatorname{div}((L_t - \alpha_t)\, v\, f_t) + \nu_t \beta_t^{-\gamma/2} Q(f_t, f_t)\\ \beta_t &= \beta_0 \exp\left(2\int_0^t \alpha_s\, ds\right) \quad \alpha_t := \frac{1}{3}\int_{\mathbb{R}^3} v\cdot L_t v\, f_t(v)\, dv.\end{aligned} \tag{B.3.1}$$

In these equations, the positive function ν and the matrix-valued function L are given. In the sequel, we study solutions to (B.3.1). Note that our investigations in the last section yields corresponding well-posedness and regularity results for (B.3.1).

Let us introduce the decomposition $L_t = A_t + b_t I$ into the trace-free and trace part, $b_t := \operatorname{tr} L_t/3$. We study solutions to (B.3.1) of the form

$$f_t(v) = \mu(v) + \bar{\mu}_t(v) + h_t(v). \tag{B.3.2}$$

The term $\bar{\mu}$ corresponds to the first order term in the Hilbert-type expansion, see (B.1.14) and (B.1.19), and is defined by

$$\bar{\mu}_t := \frac{1}{\eta_t}\mathscr{L}^{-1}[-v\cdot A_t v\mu], \quad \eta_t := \nu_t\, \beta_t^{-\gamma/2}. \tag{B.3.3}$$

Compare the definition of η_t with the definition $\eta_t = \rho_t\, \beta_t^{-\gamma/2}$ in the introduction concerning (B.1.11) instead of (B.3.1). Here, the given function ν instead of ρ appears. As in the introduction, let us determine the asymptotics of the inverse temperature β_t, which yields the behavior of $\eta_t = \nu_t \beta_t^{-\gamma/2}$. The inverse temperature β_t satisfies the equation, see (B.3.1),

$$\frac{\beta_t'}{2\beta_t} = \alpha_t = \frac{1}{3}\int_{\mathbb{R}^3} v\cdot L_t v\cdot f_t(v)\, dv.$$

If we consider only the first two terms in (B.3.2) we obtain

$$\left(\beta_t^{-\gamma/2}\right)' = -\gamma\, b_t\, \beta_t^{-\gamma/2} + \frac{\gamma\, a_t}{3\nu_t}, \tag{B.3.4}$$

where we defined

$$a_t := \left\langle v\cdot A_t v\, \mu, \mathscr{L}^{-1}[v\cdot A_t v\, \mu]\right\rangle_{L^2(\mu^{-1/2})} > 0. \tag{B.3.5}$$

The solution to (B.3.4) is given by

$$B_t(\beta_0) := \beta_0^{-\gamma/2}\, e^{-\gamma\int_0^t b_s\, ds} + \int_0^t \frac{\gamma\, a_s}{3\nu_s}\, e^{-\gamma\int_s^t b_r\, dr}\, ds.$$

If h amounts to an error which is integrable in time, the behavior of $\eta_t = \nu_t \beta_t^{-\gamma/2}$ is determined by the function

$$Z_t(\beta_0) := \nu_t B_t(\beta_0) = \beta_0^{-\gamma/2}\, \nu_t e^{-\gamma\int_0^t b_s\, ds} + \frac{\gamma}{3}\int_0^t \frac{\nu_t\, a_s}{\nu_s}\, e^{-\gamma\int_s^t b_r\, dr}\, ds. \tag{B.3.6}$$

One crucial assumption in the theorem below is a growth condition on $Z_t(\beta_0)$. As we will see in Section B.4, this condition is always satisfied for homoenergetic solutions.

Theorem B.3.1. *Consider equations* (B.3.1) *under the following structural assumptions.*

(I) The matrix $L \in C^1([0,\infty);\mathbb{R}^{3\times 3})$ *satisfies* $\sup_{t\geq 0}\{\|L_t\| + \|L_t'\|\} < \infty$. *Furthermore, for* $\nu \in C^1([0,\infty);(0,\infty))$ *we have* $\sup_{t\geq 0}|\nu_t'/\nu_t| < \infty$.

(II) We assume that $Z_t(1) \approx 1+t$ *for all* $t \geq 0$*, where* Z_t *is given in* (B.3.6).

Let $p_0 > 4+4s+3/2$ *be arbitrary and* $f_0 \in \mathcal{H}^1_{p_0}$ *satisfy the normalization* (B.1.10). *Consider the unique solution* f *to* (B.3.1) *and define* $h_t := f_t - \mu - \bar{\mu}_t$ *with* $\bar{\mu}_t$ *given in* (B.3.3).

There are $\varepsilon_0 \in (0,1)$ *sufficiently small and a constant* $C' > 0$*, depending only on* p_0*,* L, ν *and the collision kernel* B*, such that the following holds.*

If $\|h_0\|_{\mathcal{H}^1_{p_0}} = \varepsilon \leq \varepsilon_0$ *and* $\beta_0 \leq \varepsilon_0$*, then we have for all* $t \geq 0$

$$\|h_t\|_{\mathcal{H}^1_{p_0}} \leq C'\left(\frac{\varepsilon}{(1+t)^2} + \frac{1}{Z_t(\beta_0)^2}\right). \tag{B.3.7}$$

In addition, for any $k, p \in \mathbb{N}$ *and* $t \geq 0$ *we have*

$$\left\|\frac{\bar{\mu}_t}{\sqrt{\mu}}\right\|_{H^k_p} \leq \frac{C_{k,p}}{Z_t(\beta_0)}. \tag{B.3.8}$$

Finally, we have

$$\frac{1}{4}Z_t(\beta_0) \leq \eta_t = \nu_t \beta_t^{-\gamma/2} \leq 4Z_t(\beta_0). \tag{B.3.9}$$

Remark B.3.2. In the above theorem, we merely obtain (B.3.9) concerning the inverse temperature. The precise asymptotics of β_t can be calculated with equation (B.3.4), when more information (than just assumption (II)) on b_t, a_t and ν_t in (B.3.6) is available. We do this in the proof of Theorem B.1.1 in Section B.4. As we will see, the equation satisfied by h contains a source term, see (B.3.11). This source term leads to a decay of order $1/Z_t(\beta_0)^2$. The other terms lead to a decay of order $\varepsilon/(1+t)^2$. Hence, the combination of both yields (B.3.7).

Remark B.3.3. In assumption (II) one can also assume $Z_t(1) \approx (1+t)^r$ with $r > 1/2$. The reason for the latter condition is that h_t will then be of order $\mathcal{O}(t^{-2r})$, which is then integrable in time.

B.3.1 Proof of Theorem B.3.1

In this section we prove Theorem B.3.1 and for this reason prove several estimates.

Preparation. Due to $f_0 \in \mathcal{H}^1_{p_0}$ there is a unique solution f to (B.3.1) by Proposition B.2.2, which is smooth for positive times by Proposition B.2.6. Moreover, note that the norm $t \mapsto \|f_t\|_{H^1_{p_0}}$ is continuous for $t \geq 0$. As in Theorem B.3.1 we set $h_t = f_t - \mu - \bar{\mu}_t$. Correspondingly, let us decompose

$$\begin{aligned}
&\alpha_t = \alpha_t^0 + \alpha_t^1 + \alpha_t^2, \\
&\alpha_t^0 = \frac{1}{3}\int_{\mathbb{R}^3} v \cdot L_t v \, \mu \, dv = b_t, \quad \alpha_t^1 = \frac{1}{3}\int_{\mathbb{R}^3} v \cdot L_t v \, \bar{\mu}_t \, dv, \quad \alpha_t^2 = \frac{1}{3}\int_{\mathbb{R}^3} v \cdot L_t v \, h_t \, dv.
\end{aligned} \tag{B.3.10}$$

In order to obtain the equation solved by h, we plug the expansion $f_t = \mu + \bar{\mu}_t + h_t$ into (B.3.1). Recall that $\eta_t = \nu_t \beta_t^{-\gamma/2}$. We obtain with

$$0 = \operatorname{div}\Big((L_t - \alpha_t^0) v\, \mu\Big) - \eta_t \mathscr{L} \bar{\mu}_t = \operatorname{div}(A_t v\, \mu) - \eta_t \mathscr{L} \bar{\mu}_t,$$

by definition of $\bar{\mu}$, see (B.3.3), the equation

$$\begin{aligned} \partial_t h_t =& \big[-\partial_t \bar{\mu}_t + \operatorname{div}(A_t v\, \bar{\mu}_t) - \alpha_t^1 \operatorname{div}(v\,(\mu + \bar{\mu}_t)) + \eta_t Q(\bar{\mu}_t, \bar{\mu}_t) \big] \\ &+ \big[-\alpha_t^2 \operatorname{div}(v\,(\mu + \bar{\mu}_t)) + \operatorname{div}(A_t v\ h_t) - \alpha_t^1 \operatorname{div}(v\, h_t) + \eta_t\, (Q(\bar{\mu}_t, h_t) + Q(h_t, \bar{\mu}_t)) \big] \\ &- \alpha_t^2 \operatorname{div}(v\, h_t) + \eta_t Q(h_t, h_t) - \eta_t \mathscr{L} h_t \\ =&: S_t + (\mathscr{R} h)_t - \alpha_t^2 \operatorname{div}(v\, h_t) + \eta_t Q(h_t, h_t) - \eta_t \mathscr{L} h_t. \end{aligned} \tag{B.3.11}$$

Observe that α_t^2 in the definition of $\mathscr{R}$ depends linearly on h_t and that S, $\mathscr{R}$ are time-dependent. Note also that $h_t \in (\ker \mathscr{L})^\perp$, since $\bar{\mu}_t \in (\ker \mathscr{L})^\perp$ and the fact that f_t has the same mass, momentum and energy as the Maxwellian μ. In particular, this implies that $Q(h_t, h_t) \in (\ker \mathscr{L})^\perp$ and

$$S_t + (\mathscr{R} h)_t - \alpha_t^2 \operatorname{div}(v\, h_t) \in (\ker \mathscr{L})^\perp$$

for all $t \geq 0$.

Strategy. The proof relies on the following two estimates:

$$\frac{1}{4} Z_t(\beta_0) \leq \eta_t = \nu_t \beta_t^{-\gamma/2} \leq 4 Z_t(\beta_0), \tag{B.3.12}$$

$$\|h_t\|_{\mathcal{H}^1_{p_0}} \leq \Omega \left(\frac{\varepsilon}{(1+t)^2} + \frac{1}{\eta_t^2} \right) \tag{B.3.13}$$

for some constant $\Omega > 0$. Note that both imply (B.3.7).

Note that (B.3.12) together with assumption (II) in Theorem B.3.1 implies for $\beta_0 \leq 1$

$$\eta_t \geq \frac{1}{4} Z_t(\beta_0) \geq \frac{1}{8} \bar{\eta}(\beta_0) + \frac{1}{8} Z_t(1) \geq c_0 (\bar{\eta}(\beta_0) + t). \tag{B.3.14}$$

Here, the constant $c_0 > 0$ does not depend on β_0 and we defined

$$\bar{\eta}(\beta_0) := \min_{t \geq 0} Z_t(\beta_0).$$

In fact, we assume $\beta_0 \leq \varepsilon_0 \in (0,1)$ sufficiently small as in Theorem B.3.1, so that $\beta_0 \leq 1$ is always satisfied. Furthermore, it holds $\bar{\eta}(\beta_0) \to \infty$ as $\beta_0 \to 0$. To see this, note that by assumption (I) in Theorem B.3.1 we have $\nu_t \geq \nu_0 e^{-ct}$ for some $c \geq 0$ and thus for $\beta_0 \leq 1$

$$Z_t(\beta_0) \geq \frac{1}{2} Z_t(\beta_0) + \frac{1}{2} Z_t(1) \geq \beta_0^{-\gamma/2} \nu_0 e^{-ct} e^{-\gamma \|L\| t} + C(1+t).$$

Here, we also used assumption (II) in Theorem B.3.1. One can see that the minimum of the last term behaves like $\ln\left(\beta_0^{-\gamma/2}\right) \to \infty$ as $\beta_0 \to 0$. This yields $\bar{\eta}(\beta_0) \to \infty$ as $\beta_0 \to 0$. Consequently, choosing β_0 small ensures that the factor η_t in front of the collision operator is large for all times. As a consequence the collision operator is always the dominant term.

We will prove that (B.3.12) implies (B.3.13) and the other way round. However, in order to choose the constant Ω such that this loop can be closed, we need to take ε_0 sufficiently small. Recall that $\|h_0\|_{\mathcal{H}^1_{p_0}} = \varepsilon \leq \varepsilon_0$ and $\beta_0 \leq \varepsilon_0$.

Estimates on $\bar{\mu}$ and η.

Let us first give the following regularity properties for the first order approximation $\bar{\mu}$ defined in (B.3.3), which implies (B.3.8).

Lemma B.3.4. *The function $\bar{\mu}_t$ defined above satisfies for all $q,k \in \mathbb{N}$*

$$\|\bar{\mu}_t/\sqrt{\mu}\|_{H^k_q} \le C_{q,k}\frac{\|A\|}{\eta_t} \le \frac{C_{q,k}}{\eta_t}, \tag{B.3.15}$$

$$\|\partial_t\bar{\mu}_t/\sqrt{\mu}\|_{H^k_q} \le C_{q,k}\frac{1}{\eta_t}\left(\frac{\|A\|\,|\eta_t'|}{\eta_t} + \|\partial_t A\|\right) \le \frac{C_{q,k}}{\eta_t}. \tag{B.3.16}$$

Proof. First of all, note that $\tilde{\mu}_t := \bar{\mu}_t/\sqrt{\mu}$ satisfies the equation

$$L\tilde{\mu}_t = \frac{1}{\eta_t} v\cdot A_t v\,\sqrt{\mu} \tag{B.3.17}$$

due to (B.3.3). Here, we used the notation $Lg = \mu^{-1/2}\mathscr{L}[\sqrt{\mu}g]$. As mentioned in the introduction, the operator L is non-negative, self-adjoint on $L^2(\mathbb{R}^3)$ with spectral gap (since here $\gamma > 0$). It coincides with the operator $\mathscr{L}$ on $L^2(\mu^{-1/2})$. Corresponding coercivity estimates are available in [5, 78], in particular see [78, Lemma 2.6]. (In fact, in the case of $\gamma > 0$ the operator L has a regularizing effect both in terms of weights and Sobolev regularity.) Since $v\cdot A_t v\,\sqrt{\mu} \in H^k_p$ for all $k,p \in \mathbb{N}$, these coercivity estimates allow to prove that $\tilde{\mu}_t \in H^k_p$ for all $k,p \in \mathbb{N}$ with the asserted bound in (B.3.15).

The estimate (B.3.16) follows from differentiating equation (B.3.17) with respect to time. Note that $\partial_t[v\cdot A_t v\,\sqrt{\mu}/\eta_t] \in (\ker L)^\perp$, since for all $t \ge 0$ we have $v\cdot A_t v\,\sqrt{\mu}/\eta_t \in (\ker L)^\perp$. Hence, the corresponding equation has a unique solution. The coercivity estimates mentioned before allow to prove the asserted bounds. Finally, note that, due to assumption (I) in Theorem B.3.1, we have $\|A\|_{C^1(0,\infty)} < \infty$ and

$$\left|\frac{\eta_t'}{\eta_t}\right| \le \left|\frac{\nu_t'}{\nu_t}\right| + \gamma\left|\frac{\beta_t'}{2\beta_t}\right| \le C(1+|\alpha_t|) \le C(1+\|f_t\|_{L^1_2}) \le C. \tag{B.3.18}$$

This concludes the proof. ◻

Let us now prove that (B.3.13) implies (B.3.12).

Lemma B.3.5. *Assume that h_t satisfies* (B.3.13) *for a constant $\Omega > 0$ on some interval $[0,T]$. Then, we have on $[0,T]$*

$$\exp\left(-c\gamma\Omega\, R_T(\varepsilon,\beta_0)\right) Z_t(\beta_0) \le \eta_t \le \exp\left(c\gamma\Omega\, R_T(\varepsilon,\beta_0)\right) Z_t(\beta_0),$$
$$R_T(\varepsilon,\beta_0) := \int_0^T \left(\frac{\varepsilon}{(1+t)^2} + \frac{1}{\eta_t^2}\right) dt$$

for some constant $c > 0$.

In the final step, we will choose $\varepsilon, \beta_0 \le \varepsilon_0$ and hence $R_T(\varepsilon,\beta_0)$ small enough to close the continuation argument.

Proof of Lemma B.3.5. Due to the equation

$$\frac{\beta_t'}{2\beta_t} = \alpha_t = \alpha_t^0 + \alpha_t^1 + \alpha_t^2$$

we conclude with (B.3.10)

$$\frac{d}{dt}\beta_t^{-\gamma/2} = -\gamma\beta_t^{-\gamma/2} + \frac{\gamma\, a_t}{3\nu_t} - \gamma\,\alpha_t^2\,\beta_t^{-\gamma/2}. \tag{B.3.19}$$

Recall the definition of a_t in (B.3.5). By assumption, h_t satisfies (B.3.13) and we obtain

$$\alpha_t^2 = \frac{1}{3}\int_{\mathbb{R}^3} v\cdot L_t v\, h_t\, dv \le \|L\|_{L^\infty}\|h_t\|_{L^1_2} \le c\|h_t\|_{\mathcal{H}^1_{p_0}}.$$

Consequently, we have

$$\int_0^T \alpha_t^2\, dt \le c\Omega\, R_T(\varepsilon,\beta_0).$$

We integrate (B.3.19) to obtain the claim. □

Estimate on the error term

Here, we prove that (B.3.12) implies (B.3.13). This is more involved and relies on several estimates and known results. We split the analysis in the estimates in the L^2-framework and the estimates in the L^1-framework. To this end, let us define

$$m := p_0 - 2 - 4s - 3/2 > 2, \tag{B.3.20}$$

which will be used as a weight in the L^1 estimates. Let us note that $\|h_0\|_{L^1_m} \le C_*\varepsilon$ for some constant C_* by our assumption on h_0 in Theorem B.3.1.

Estimates in L^2-framework. Here we discuss the estimates of solutions h to (B.3.11) in the space $\mathcal{H}^1_{p_0}$. Due to the angular singularity in the collision operator, we are led to use the (homogeneous) anisotropic norm

$$\|g\|^2_{\dot{H}^{s,*}} := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} b_\delta(n\cdot\sigma)\mu_*\langle v_*\rangle^{-\gamma}\left(g'\langle v'\rangle^{\gamma/2} - g\langle v\rangle^{\gamma/2}\right)^2 d\sigma dv_* dv.$$

Here, we defined $b_\delta(\cos\theta) = \chi(\theta/\delta)b(\cos\theta)$ for a smooth function χ with $\mathbb{1}_{[-1,1]} \le \chi \le \mathbb{1}_{[-2,2]}$. The parameter $\delta > 0$ will be fixed such that Lemma B.3.8 holds. Finally, we also define the weighted anisotropic norm

$$\|g\|^2_{H^{s,*}_p} = \|g\|^2_{L^2_{p+\gamma/2}} + \|g\langle\cdot\rangle^p\|^2_{\dot{H}^{s,*}}.$$

Remark B.3.6. Let us mention that the above anisotropic norm was introduced in [91] following works by [5]. A different anisotropic norm was used in [78]. We refer to [87] for a discussion and further estimates of the Boltzmann collision operator in anisotropic spaces.

The following estimate relates the space $H^{s,*}_p$ to the standard fractional Sobolev spaces, see [91, Lemma 2.1].

Lemma B.3.7. *For any $k \geq 0$ and all $g \in H^s_{k+\gamma/2+s}$ we have*

$$\delta^{2-2s} \|g\|_{H^s_{k+\gamma/2}} \lesssim \|g\|_{H^{s,*}_k} \lesssim \|g\|_{H^s_{k+\gamma/2+s}}.$$

We fix $\delta > 0$ in the definition of the norm of $\dot{H}^{s,*}$ such that the following lemma holds, see [91, Lemma 4.2].

Lemma B.3.8. *Let $k \geq \gamma/2+3+2s$, then for sufficiently small $\delta > 0$ we have*

$$-\langle \mathscr{L} h, h\rangle \leq -c_\delta \|h\|^2_{H^{s,*}_k} + C_\delta \|h\|^2_{L^2}$$

for some constants $c_\delta, C_\delta > 0$ depending on $\delta > 0$.

Let us also recall the following estimates on the collision operator, see [91, Lemma 2.3].

Lemma B.3.9. *Let $k > \gamma/2+2+2s$.*

(i) If $\ell > \gamma+1+3/2$ we have

$$|\langle Q(f,g), h\rangle_{L^2_k}| \lesssim \|f\|_{L^2_\ell} \|g\|_{H^{s_1}_{N_1+k}} \|h\|_{H^{s_2}_{N_2+k}} + \|f\|_{L^2_{\gamma/2+k}} \|g\|_{L^2_\ell} \|h\|_{L^2_{\gamma/2+k}},$$

where s_1, $s_2 \in [0,2s]$, $s_1+s_2 = 2s$ and $N_1 \geq \gamma/2$, $N_2 \geq 0$, $N_1+N_2 = \gamma+2s$.

(ii) If $\ell > 4+3/2$ we have

$$|\langle Q(f,g), g\rangle_{L^2_k}| \lesssim \|f\|_{L^2_\ell} \|g\|^2_{H^{s,*}_k} + \|f\|_{L^2_{\gamma/2+k}} \|g\|_{L^2_\ell} \|g\|_{L^2_{\gamma/2+k}}.$$

Finally, let us recall the following interpolation estimate. It can be proved using Fourier transform and a splitting in small respectively large frequencies.

Lemma B.3.10. *For any $s > r \geq 0$ we have*

$$\|g\|_{\dot{H}^r} \lesssim \|g\|^\theta_{L^1} \|g\|^{1-\theta}_{\dot{H}^s}$$

with $\theta = (2s-2r)/(2s+3)$.

Let us now give the first conditional estimate on the error term h.

Proposition B.3.11. *Under the assumptions of Theorem B.3.1 there is a constant C' such that the following holds. Assuming that* (B.3.12) *and*

$$\|h\|_{L^1_m} \leq \Omega' \left(\frac{\varepsilon}{(1+t)^2} + \frac{1}{\eta_t^2} \right). \tag{B.3.21}$$

hold on some interval $t \in [0,T]$ for some constant Ω', we can find a sufficiently small constant $\varepsilon_0' \in (0,1)$ so that: if also $\varepsilon \leq \varepsilon_0'$ and $\beta_0 \leq \varepsilon_0'$ we have the estimate

$$\|h_t\|_{\mathcal{H}^1_{p_0}} \leq C'(\Omega'+1) \left(\frac{\varepsilon}{(1+t)^2} + \frac{1}{\eta_t^2} \right) \tag{B.3.22}$$

for all $t \in [0,T]$. Here, ε_0' depends on Ω'.

Proof. We split the proof into several steps. We first derive the necessary a priori estimates. Let us write for brevity p instead of p_0. Furthermore, it is convenient to use the following norm, which is equivalent to $\|\cdot\|_{\mathcal{H}^1_p}$,

$$|||h|||^2_{\mathcal{H}^1_p} = \|h\|^2_{L^2_p} + \kappa \sum_{|\alpha|=1} \|\partial^\alpha h\|^2_{L^2_{p-2s}}.$$

Here, $\kappa \in (0,1)$ will be chosen small enough, but fixed.

Step 1: Using (B.3.11) we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|h_t\|^2_{L^2_p} &= \langle S_t, h_t\rangle_{L^2_p} + \langle (\mathscr{R}h)_t, h_t\rangle_{L^2_p} - \alpha_t^2 \langle \operatorname{div}(vh_t), h_t\rangle_{L^2_p}\\ &\quad + \eta_t \langle Q(h_t,h_t), h_t\rangle_{L^2_p} - \eta_t \langle \mathscr{L}h_t, h_t\rangle_{L^2_p}.\end{aligned}$$

We estimate term by term.

(i) With Lemma B.3.4 and Lemma B.3.9 we obtain

$$\begin{aligned}\langle S_t, h_t\rangle_{L^2_p} &= \left\langle -\partial_t\bar{\mu}_t + \operatorname{div}(A_t v\,\bar{\mu}_t) - \alpha_t^1 \operatorname{div}(v(\mu+\bar{\mu}_t)) + \eta_t Q(\bar{\mu}_t,\bar{\mu}_t), h_t\right\rangle_{L^2_p}\\ &\le C_1\left(\frac{1}{\eta_t} + \frac{1}{\eta_t^2}\right)\|h_t\|_{L^2} \le \frac{C_1}{\eta_t}\|h_t\|_{L^2}.\end{aligned}$$

Recall that $t \mapsto \|A_t\|$ is uniformly bounded in time and $|\alpha_1| \lesssim 1/\eta_t$. Furthermore, we used $\eta_t \ge c_0\bar{\eta}(\beta_0) \ge 1$ for $\beta_0 \le \varepsilon_0'$ small, see (B.3.14). This will be used repeatedly in the sequel.

(ii) By definition of $\mathscr{R}h$ in (B.3.11) we have

$$\begin{aligned}&\langle (\mathscr{R}h)_t, h_t\rangle_{L^2_p}\\ &= \left\langle -\alpha_t^2 \operatorname{div}(v(\mu+\bar{\mu}_t)) + \operatorname{div}(A_t v\ h_t) - \alpha_t^1 \operatorname{div}(v\,h_t) + \eta_t\left(Q(\bar{\mu}_t,h_t) + Q(h_t,\bar{\mu}_t)\right), h_t\right\rangle_{L^2_p}.\end{aligned}$$

The first term can be estimated using Lemma B.3.4 and the second and third via partial integration. This yields an upper bound of the form $C_2\|h_t\|^2_{L^2_p}$. Note that we used $|\alpha_t^2| \lesssim \|h_t\|_{L^2_p}$. Using Lemma B.3.4 and Lemma B.3.9 (i) with $s_1 = 2s$, $s_2 = 0$, $N_1 = \gamma + 2s$, $N_2 = 0$ we get

$$\eta_t \langle Q(h_t,\bar{\mu}_t), h_t\rangle_{L^2_p} \lesssim \|h_t\|^2_{L^2_{p+\gamma/2}}.$$

Note that by $s \in (0,1/2)$ and our choice of $p = p_0 > 4 + 4s + 3/2$ the Lemma applies. Furthermore, we can choose ℓ with $p \ge \ell > \gamma + 1 + 3/2$. For the last term we use Lemma B.3.4 and Lemma B.3.9 (ii) to get

$$\eta_t \langle Q(\bar{\mu}_t,h_t), h_t\rangle_{L^2_p} \lesssim \|h_t\|_{L^2_p}\|h_t\|_{H^{s,*}_p}.$$

Estimating $\alpha_t^2 \langle \operatorname{div}(vh_t), h_t\rangle_{L^2_p}$ via partial integration gives

$$\langle (\mathscr{R}h)_t, h_t\rangle_{L^2_p} - \alpha_t^2 \langle \operatorname{div}(vh_t), h_t\rangle_{L^2_p} \le C_2\|h_t\|^2_{H^{s,*}_p}.$$

We also used that $|\alpha_t^2| \le |\alpha_t - \alpha_t^0 - \alpha_t^1| \le C_2$.

(iii) For the collision operator we apply Lemma B.3.9 (ii) to get

$$\langle Q(h,h),h\rangle_{L^2_p} \le C_3 \|h\|_{L^2_p} \|h\|^2_{H^{s,*}_p},$$

where we chose ℓ such that $p \ge \ell > 4+3/2$.

(iv) Finally, using Lemma B.3.8 we obtain

$$-\langle \mathscr{L}h,h\rangle_{L^2_p} \le -c_\delta \|h\|^2_{H^{s,*}_p} + C_\delta \|h\|^2_{L^2}.$$

We now apply Lemma B.3.10 and Young's inequality to the last term to get

$$-\langle \mathscr{L}h,h\rangle_{L^2_p} \le -\frac{c_\delta}{2} \|h\|^2_{H^{s,*}_p} + C'_\delta \|h\|^2_{L^1}.$$

We summarize the preceding estimates yielding

$$\frac{1}{2}\frac{d}{dt}\|h_t\|^2_{L^2_p} \le -\left(\frac{\eta_t c_\delta}{2} - C_2 - \eta_t C_3 \|h_t\|_{L^2_p}\right)\|h_t\|^2_{H^{s,*}_p} + \frac{C_1}{\eta_t}\|h_t\|_{L^2} + \eta_t C'_\delta \|h_t\|^2_{L^1}. \tag{B.3.23}$$

Let us now turn to the estimates for $g^i := \partial_i h$. We abbreviate $q := p-2s$. Differentiating equation (B.3.11) yields (we denote by A^i_t the i-th column of the matrix A_t)

$$\begin{aligned}\partial_t g^i_t =& \partial_i S_t - \alpha^2_t \partial_i\left[\operatorname{div}(v(\mu+\bar\mu_t))\right] + A^i_t\cdot\nabla h_t - \alpha^1_t g^i_t + \operatorname{div}\left((A_t-\alpha^1_t)v\, g^i_t\right)\\ &+ \eta_t\left(Q(\partial_i\bar\mu_t,h_t) + Q(h_t,\partial_i\bar\mu_t)\right) + \eta_t\left(Q(\bar\mu_t,g^i_t) + Q(g^i_t,\bar\mu_t)\right)\\ &- \alpha^2_t g^i_t - \alpha^2_t \operatorname{div}(v\, g^i_t) + \eta_t\left(Q(h_t,g^i_t)+Q(g^i_t,h_t)\right) + \eta_t\left(Q(\partial_i\mu_t,h_t)+Q(h_t,\partial_i\mu_t)\right) - \eta_t\mathscr{L}g^i_t.\end{aligned}$$

Here we used the well-known identity $\partial_i Q(u,v) = Q(\partial_i u,v) + Q(u,\partial_i v)$ for functions u, v. We now estimate term by term in

$$\frac{1}{2}\frac{d}{dt}\left\|g^i_t\right\|^2_{L^2_q} = \left\langle \partial_t g^i_t, g^i_t\right\rangle_{L^2_q}.$$

(i) Using Lemma B.3.4 and either $|\alpha^2_t| \le \|h_t\|_{L^2_p}$ or $|\alpha^2_t| \le C_4$ we obtain

$$\begin{aligned}&\left\langle \partial_i S_t - \alpha^2_t\partial_i\left[\operatorname{div}(v(\mu+\bar\mu_t))\right] + A^i_t\cdot\nabla h_t - \alpha^1_t g^i_t + \operatorname{div}\left((A_t-\alpha^1_t)v\,g^i_t\right), g^i_t\right\rangle_{L^2_q}\\ &\qquad \le \frac{C_4}{\eta_t}\left\|g^i_t\right\|_{L^2} + C_4\left(\|h_t\|_{L^2_p}\left\|g^i_t\right\|_{L^2_q} + \|\nabla h_t\|_{L^2_q}\left\|g^i_t\right\|_{L^2_q} + \left\|g^i_t\right\|^2_{L^2_q}\right).\end{aligned}$$

(ii) We apply Lemma B.3.9 (i) with $s_1 = s_2 = s$, $N_1 = \gamma/2+2s$, $N_2 = \gamma/2$ to obtain

$$\begin{aligned}&\eta_t\left\langle Q(\partial_i\bar\mu_t+\partial_i\mu,h_t) + Q(h_t,\partial_i\bar\mu_t+\partial_i\mu), g^i_t\right\rangle_{L^2_q}\\ &\le \eta_t C_5\left(\|h_t\|_{H^s_{q+\gamma/2+2s}}\left\|g^i_t\right\|_{H^s_{q+\gamma/2}} + \|h_t\|_{L^2_{q+\gamma/2}}\left\|g^i_t\right\|_{L^2_{q+\gamma/2}}\right)\\ &\le \eta_t C_5\left(\|h_t\|_{H^s_{p+\gamma/2}}\left\|g^i_t\right\|_{H^s_{q+\gamma/2}} + \|h_t\|_{L^2_{p+\gamma/2}}\left\|g^i_t\right\|_{L^2_{q+\gamma/2}}\right).\end{aligned}$$

In the last inequality we used $q+2s = p$.

(iii) With Lemma B.3.9 we can estimate

$$\eta_t \left\langle Q(\bar{\mu}_t, g_t^i) + Q(g_t^i, \bar{\mu}_t), g_t^i \right\rangle_{L_q^2} \le C_6 \left\| g_t^i \right\|_{H_q^{s,*}}^2 .$$

(iv) We also have with $|\alpha_t^2| \le C_7$

$$\left\langle -\alpha_t^2 g_t^i - \alpha_t^2 \mathrm{div}(v\, g_t^i), g_t^i \right\rangle_{L_q^2} \le C_7 \left\| g_t^i \right\|_{L_q^2}^2 .$$

(v) For the mixed terms $Q(h_t, g_t^i) + Q(g_t^i, h_t)$ we use Lemma B.3.9 (ii) to get

$$\left\langle Q(h, g^i), g^i \right\rangle_{L_q^2} \lesssim \|h\|_{L_p^2} \left\| g^i \right\|_{H_q^{s,*}}^2 + \left\| g^i \right\|_{L_q^2} \|h\|_{L^2_{q+\gamma/2}} \left\| g^i \right\|_{L^2_{q+\gamma/2}} .$$

Here, we chose ℓ such that $q \ge \ell > 4 + 3/2$. Applying Lemma B.3.9 (i) with $s_1 = s_2 = s$, $N_1 = \gamma/2 + 2s$, $N_2 = \gamma/2$ yields

$$\left\langle Q(g^i, h), g^i \right\rangle_{L_q^2} \lesssim \left\| g^i \right\|_{L_q^2} \|h\|_{H^s_{q+\gamma+2s}} \left\| g^i \right\|_{H^s_{q+\gamma/2}} + \|h\|_{L_p^2} \left\| g^i \right\|_{L^2_{q+\gamma/2}}^2 .$$

In total we get

$$\left\langle Q(h, g^i) + Q(g^i, h), g^i \right\rangle_{L_q^2} \le C_8 \left(\|h\|_{L_p^2} \left\| g^i \right\|_{H_q^{s,*}}^2 + \left\| g^i \right\|_{L_q^2} \|h\|_{H_p^{s,*}} \left\| g^i \right\|_{H_q^{s,*}} \right) .$$

(vi) Applying Lemma B.3.8 gives

$$-\left\langle \mathscr{L} g^i, g^i \right\rangle \le -c_\delta \left\| g^i \right\|_{H_q^{s,*}}^2 + C_\delta \left\| g^i \right\|_{L^2}^2 .$$

For the last term we use Lemma B.3.10 to get

$$\left\| g^i \right\|_{L^2} \le \|h\|_{\dot{H}^1} \lesssim \|h\|_{L^1}^{\theta} \|\nabla h\|_{\dot{H}^s}^{1-\theta}$$

with $\theta \in (0,1)$ and apply Young's inequality yielding

$$-\left\langle \mathscr{L} g^i, g^i \right\rangle \le -c_\delta \left\| g^i \right\|_{H_q^{s,*}}^2 + \frac{c_\delta}{2} \|\nabla h\|_{H_q^{s,*}}^2 + C_\delta'' \|h\|_{L^1}^2 .$$

We now sum the estimates for $i = 1,2,3$ to get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt} \|\nabla h_t\|_{L_q^2}^2 \le & (C_4 + C_7) \left(\|\nabla h_t\|_{L_q^2}^2 + \|h_t\|_{L_p^2} \|\nabla h_t\|_{L_q^2} \right) \\
& + \eta_t C_5 \left(\|h_t\|_{H^s_{p+\gamma/2}} \|\nabla h_t\|_{H^s_{q+\gamma/2}} + \|h_t\|_{L^2_{p+\gamma/2}} \|\nabla h_t\|_{L^2_{q+\gamma/2}} \right) \\
& + \eta_t C_8 \|\nabla h_t\|_{L_q^2} \|h_t\|_{H_p^{s,*}} \|\nabla h_t\|_{H_q^{s,*}} \\
& - \left(\frac{\eta_t\, c_\delta}{2} - C_6 - \eta_t C_8 \|h_t\|_{L_p^2} \right) \|\nabla h_t\|_{H_q^{s,*}}^2 + \frac{C_4}{\eta_t} \|\nabla h_t\|_{L^2} + \eta_t C_\delta'' \|h_t\|_{L^1}^2 .
\end{aligned}$$

Let us apply rough estimates in $H^{s,*}$ to the first line and Young's inequality to the third line. Furthermore, for the second line we use Young's inequality to absorb both norms with ∇h in the first term in the last line. We thus obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla h_t\|_{L^2_q}^2 \leq &\left(C_4+C_7+\eta_t C_5'+\eta_t C_8\|\nabla h_t\|_{L^2_q}\right)\|h_t\|_{H^{s,*}_p}^2 \\ &-\left(\frac{\eta_t c_\delta}{4}-C_4-C_6-C_7-\eta_t C_8\|h_t\|_{L^2_p}-\eta_t C_8\|\nabla h_t\|_{L^2_q}\right)\|\nabla h_t\|_{H^{s,*}_q}^2 \\ &+\frac{C_4}{\eta_t}\|\nabla h_t\|_{L^2}+\eta_t C_\delta''\|h_t\|_{L^1}^2 .\end{aligned} \tag{B.3.24}$$

Next, due to (B.3.12), i.e. $\eta_t \geq c_0\bar{\eta}(\beta_0)$ by (B.3.14), and $\bar{\eta}(\beta_0)\to\infty$ as $\beta_0\to 0$, we can find ε_0' such that the constants C_2 resp. C_4, C_6, C_7 in (B.3.23) resp. (B.3.24) can be absorbed in the term involving c_δ. We then obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|||h_t|||_{\mathcal{H}^1_p}^2 \leq &-\left(\frac{\eta_t c_\delta}{4}-\eta_t C_3\|h_t\|_{L^2_p}-\kappa\eta_t C_5'-\eta_t C_8\,\kappa\|\nabla h_t\|_{L^2_q}\right)\|h_t\|_{H^{s,*}_p}^2 \\ &-\left(\frac{\eta_t c_\delta}{8}-\eta_t C_8\|h_t\|_{L^2_p}-\eta_t C_8\|\nabla h_t\|_{L^2_q}\right)\kappa\|\nabla h_t\|_{H^{s,*}_q}^2 \\ &+\frac{C_1+C_4}{\eta_t}|||h_t|||_{\mathcal{H}^1_p}+\eta_t\left(C_\delta'+\kappa C_\delta''\right)\|h_t\|_{L^1}^2 .\end{aligned}$$

Now, let us choose $\kappa\in(0,1)$ small enough such that C_5' can be absorbed into the term involving c_δ, yielding (recall that $q=p-2s$)

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|||h_t|||_{\mathcal{H}^1_p}^2 \leq &-\eta_t\left(\frac{c_\delta}{8}-C_9|||h_t|||_{\mathcal{H}^1_p}\right)\left(\|h_t\|_{H^{s,*}_p}^2+\kappa\|\nabla h_t\|_{H^{s,*}_{p-2s}}^2\right) \\ &+\frac{C_9}{\eta_t}|||h_t|||_{\mathcal{H}^1_p}+\eta_t C_\delta'''\|h_t\|_{L^1}^2 .\end{aligned}$$

Finally, let us apply Young's inequality for the second term

$$\frac{1}{\eta_t}|||h_t|||_{\mathcal{H}^1_p}=\frac{1}{\eta_t^{3/2}}\eta_t^{1/2}|||h_t|||_{\mathcal{H}^1_p}\leq\frac{c_\delta\eta_t}{16}|||h_t|||_{\mathcal{H}^1_p}^2+\frac{C}{\eta_t^3}$$

to obtain

$$\frac{1}{2}\frac{d}{dt}|||h_t|||_{\mathcal{H}^1_p}^2\leq-\eta_t\left(\frac{c_\delta}{16}-C_9|||h_t|||_{\mathcal{H}^1_p}\right)|||h_t|||_{\mathcal{H}^1_p}^2+\frac{C_{10}}{\eta_t^3}+\eta_t C_\delta'''\|h_t\|_{L^1}^2 . \tag{B.3.25}$$

The constants depend on κ, which is a fixed numerical constant.

Step 2: Let us derive an a priori bound on $|||h_t|||_{\mathcal{H}^1_p}$, which is used in the next step for a continuation argument. For this let us assume that

$$|||h_t|||_{\mathcal{H}^1_p}\leq\frac{c_\delta}{32C_9} \tag{B.3.26}$$

holds on $[0,T']$ for some $0<T'\leq T$. As a consequence of (B.3.25) we have

$$\frac{1}{2}\frac{d}{dt}|||h_t|||_{\mathcal{H}^1_p}^2\leq-\eta_t c_\delta'|||h_t|||_{\mathcal{H}^1_p}^2+\frac{C_{10}}{\eta_t^3}+\eta_t C_\delta'''\|h_t\|_{L^1}^2 .$$

We apply Gronwall's inequality, (B.3.21) and (B.3.26) to get

$$|||h_t|||^2_{\mathcal{H}^1_p} \le e^{-E_t}\varepsilon^2 + \int_0^t e^{-E_t+E_s}\left(\frac{2C'''_\delta(\Omega')^2\varepsilon^2\eta_s}{(1+s)^4} + \frac{2C'''_\delta(\Omega')^2 + C_{10}}{\eta_s^3}\right) ds, \tag{B.3.27}$$

where $E_t := c'_\delta \int_0^t \eta_s\, ds$. By (B.3.12) and hence (B.3.14) we have

$$e^{-E_t} \le \frac{C_{11}}{(1+t)^4}.$$

Note that this does not depend on β_0, since we can assume $\bar\eta(\beta_0) \ge 1$, due to the smallness $\beta_0 \le \varepsilon'_0$. Let us now estimate the time integrals.

(i) We use partial integration to get

$$\int_0^t e^{-E_t+E_s}\eta_s \frac{1}{(1+s)^4}\, ds \lesssim \frac{1}{(1+t)^4} + \int_0^t e^{-E_t+E_s}\frac{1}{(1+s)^5}\, ds.$$

We estimate the last integral by considering the case $t \le 1$ and $t \ge 1$. In the second case, we split $[0,t]$ into $[0,t/2]$ and $[t/2,t]$. This yields with (B.3.12) and hence (B.3.14)

$$\begin{aligned}\int_0^t e^{-E_t+E_s}\frac{1}{(1+s)^5}\, ds &\le \mathbb{1}_{\{t\le 1\}}\frac{C}{(1+t)^4} + \mathbb{1}_{\{t\ge 1\}}\left(e^{-c'_\delta\int_{t/2}^t \eta_r\, dr} + \frac{C}{(1+t)^4}\right)\\ &\le \frac{C_{12}}{(1+t)^4}.\end{aligned}$$

Note that C_{12} does not depend on β_0, since $\bar\eta(\beta_0) \ge 1$.

(ii) Finally, we have with partial integration

$$\int_0^t e^{-E_t+E_s}\frac{1}{\eta_s^3}\, ds \le \frac{1}{c'_\delta\eta_t^4} + \frac{4}{c'_\delta}\int_0^t e^{-E_t+E_s}\frac{\eta'_s}{\eta_s^5}\, ds.$$

We now use that $|\eta'_t/\eta_t| \le C$, see (B.3.18), and $\eta_t \ge c_0\bar\eta$, see (B.3.14), to get

$$\int_0^t e^{-E_t+E_s}\frac{1}{\eta_s^3}\, ds \le \frac{1}{c'_\delta\,\eta_t^4} + \frac{1}{c_0 c'_\delta\,\bar\eta(\beta_0)}\int_0^t e^{-E_t+E_s}\frac{1}{\eta_s^3}\, ds.$$

If $\beta_0 \le \varepsilon'_0$ is small enough, i.e. $\bar\eta(\beta_0)$ large enough, we can absorb the last term into the left-hand side yielding

$$\int_0^t e^{-E_t+E_s}\frac{1}{\eta_s^3}\, ds \le \frac{C_{13}}{\eta_t^4}.$$

Note that in this argument the smallness of ε'_0 depends only on numerical constants.

Hence, we obtain with these estimates from (B.3.27)

$$|||h_t|||_{\mathcal{H}^1_p} \le C'\,(\Omega'+1)\left(\frac{\varepsilon}{(1+t)^2} + \frac{1}{\eta_t^2}\right). \tag{B.3.28}$$

for some constant $C' \ge 1$. Recall that for this to hold we needed (B.3.26) and $\beta_0 \le \varepsilon'_0$.

Step 3: Continuation argument. We now use a continuation argument to show that the a priori estimate (B.3.26) in Step 2 can be justified rigorously. To this end, we choose $\varepsilon_0' = \varepsilon_0'(\Omega') > 0$, by reducing it further if necessary, such that

$$C'(\Omega'+1)\left(\frac{\varepsilon}{(1+t)^2}+\frac{1}{\eta_t^2}\right) \le \frac{c_\delta}{64C_9} \tag{B.3.29}$$

for $\varepsilon \le \varepsilon_0'$ and $\beta_0 \le \varepsilon_0'$. This in particular implies $\varepsilon \le c_\delta/64C_9$.

The continuation argument starts as follows. Since $|||h_0|||_{\mathcal{H}^1_p} \le \varepsilon \le c_\delta/64C_9$ and the continuity of the norm, the bound (B.3.26) holds on some small interval $[0,t_0]$, $t_0>0$. Let us now assume (B.3.26) holds on some interval $[0,t_1]$, $t_1 \le T$. By the arguments in Step 2 we obtain (B.3.28) on $[0,t_1]$. Due to (B.3.29) and the continuity of the norm, the estimate (B.3.26) is then also valid on some larger interval $[0,t_2]$, $t_2>t_1$.

This shows that the set of times for which (B.3.26) is valid, is both open and closed. Hence, (B.3.26) holds for all $t\in[0,T]$. In particular, by Step 2 also (B.3.28) is valid on $[0,T]$. Finally, note that the bound (B.3.28) implies (B.3.22) up to the numerical factor $\kappa\in(0,1)$. This concludes the proof. ☐

Estimates in L^1-framework. Based on the estimates in Proposition B.3.11 we prove that (B.3.12) implies (B.3.13). For this we need the following result from [152] for the linearized collision operator.

Lemma B.3.12. *For any $m>2$ the semigroup generated by $-\mathscr{L}$, $e^{-t\mathscr{L}}: L^1_m \to L^1_m$, has the following property: there is $C_m,\lambda_m>0$ such that for all $t\ge 0$*

$$\left\| e^{-t\mathscr{L}}g - \Pi_0 g\right\|_{L^1_m} \le C_m e^{-\lambda_m t}\|g\|_{L^1_m}.$$

Here, Π_0 denotes the projection onto $\ker\mathscr{L}$ in L^1_m.

The next lemma gives an estimate of the collision operator in L^1. It is a bilinear variant of [152, Proposition 3.1] and can be proved along the same lines.

Lemma B.3.13. *For any two functions f,g we have*

$$\|Q(f,g)\|_{L^1_m} \le C\left(\|f\|_{L^1_{m+\gamma}}\|g\|_{L^1_{m+\gamma}} + \|f\|_{L^1_{\gamma+1}}\|g\|_{W^{1,1}_{m+\gamma+1}}\right).$$

Proposition B.3.14. *Under the assumptions of Theorem B.3.1 there are a constant Ω and a sufficiently small constant $\varepsilon_0''\in(0,1)$ such that the following holds. Assuming that* (B.3.12) *is true on some interval $[0,T]$ and $\varepsilon\le\varepsilon_0''$ and $\beta_0\le\varepsilon_0''$ we have for all $t\in[0,T]$*

$$\|h_t\|_{\mathcal{H}^1_{p_0}} \le \Omega\left(\frac{\varepsilon}{(1+t)^2}+\frac{1}{\eta_t^2}\right). \tag{B.3.30}$$

Proof. We use Duhamel's formula together with the properties of the (non-autonomous) semigroup $P_{0,t}$ generated by $\eta_t\mathscr{L}$ to obtain from (B.3.11)

$$\begin{aligned}\|h_t\|_{L^1_m} \le \|P_{0,t}h_0\|_{L^1_m} + \int_0^t \Big[&\left\|P_{r,t}(S_r + (\mathscr{R}h)_r - \alpha_r^2 \mathrm{div}(vh_r))\right\|_{L^1_m} \\ &+ \eta_r \|P_{r,t}Q(h_r,h_r)\|_{L^1_m}\Big]dr.\end{aligned} \tag{B.3.31}$$

Step 1: Let us first estimate (B.3.31). As can be checked by the time-change $\tau(t) = \int_0^t \eta_s \, ds$, Lemma B.3.12 yields

$$\|P_{s,t}g - \Pi_0 g\|_{L^1_m} \leq C_m e^{-E_t + E_s} \|g\|_{L^1_m},$$

where $E_t := \lambda_m \int_0^t \eta_s ds$ and $\lambda_m > 0$ is defined by $\mathscr{L}$, see Lemma B.3.12. Since $S_r + (\mathscr{R}h)_r - \alpha_r^2 \mathrm{div}(vh_r)$ and $Q(h_r, h_r)$ are in $(\ker \mathscr{L})^\perp$ we obtain

$$\|h_t\|_{L^1_m} \leq e^{-E_t} \|h_0\|_{L^1_m} + \int_0^t e^{-E_t + E_r} \Big[\|S_r\|_{L^1_m} + \|(\mathscr{R}h)_r\|_{L^1_m} + \left|\alpha_r^2\right| \|\mathrm{div}(vh_r)\|_{L^1_m} + \eta_r \|Q(h_r, h_r)\|_{L^1_m} \Big] dr.$$

We recall that with $m > 2$

$$|\alpha_t^1| \leq \frac{C}{\eta_t}, \quad |\alpha_t^2| \leq C \|h_t\|_{L^1_m}. \tag{B.3.32}$$

Let us now estimate term by term.

(i) We obtain from Lemma B.3.4 and Lemma B.3.13

$$\|S_t\|_{L^1_m} \lesssim \frac{1}{\eta_t} + \frac{1}{\eta_t^2} \lesssim \frac{1}{\eta_t}.$$

(ii) For the term $\mathscr{R}h$ we obtain similarly

$$\begin{aligned}\|(\mathscr{R}h)_t\|_{L^1_m} &\lesssim \left(|\alpha_t^2| + \|h_t\|_{W^{1,1}_{m+1}} + |\alpha_t^1| \|h_t\|_{W^{1,1}_{m+1}} + \eta_t \|\bar{\mu}_t\|_{W^{1,1}_{\gamma+m+1}} \|h_t\|_{W^{1,1}_{\gamma+m+1}} \right) \\ &\lesssim \|h_t\|_{W^{1,1}_{\gamma+m+1}} \lesssim \|h_t\|_{H^1_{\gamma+m+1+2s+3/2}} \lesssim \|h_t\|_{\mathcal{H}^1_{p_0}},\end{aligned}$$

where we used $p_0 - 2s \geq m + 2 + 2s + 3/2 \geq m + \gamma + 1 + 2s + 3/2$, $\gamma \leq 1$.

(iii) In addition, we have with (B.3.32)

$$|\alpha_t^2| \|\mathrm{div}(vh_t)\|_{L^1_m} \lesssim \|h_t\|_{W^{1,1}_{m+1}} \|h_t\|_{L^1_m} \lesssim \|h_t\|_{\mathcal{H}^1_{p_0}} \|h_t\|_{L^1_m} \lesssim \|h_t\|^2_{\mathcal{H}^1_{p_0}}.$$

(iv) Finally, we apply Lemma B.3.13 to get

$$\|Q(h,h)\|_{L^1_m} \lesssim \left(\|h\|^2_{L^1_{m+\gamma}} + \|h\|_{L^1_{\gamma+1}} \|h\|_{W^{1,1}_{m+\gamma+1}} \right) \lesssim \|h\|^2_{\mathcal{H}^1_{p_0}}.$$

Putting all bounds together yields

$$\|h_t\|_{L^1_m} \leq C_m e^{-E_t} \|h_0\|_{L^1_m} + C \int_0^t e^{-E_t + E_s} \left(\frac{1}{\eta_s} + \|h_s\|_{\mathcal{H}^1_{p_0}} + \|h_t\|^2_{\mathcal{H}^1_{p_0}} + \eta_s \|h_s\|^2_{\mathcal{H}^1_{p_0}} \right) ds. \tag{B.3.33}$$

Step 2: Let us now give an a priori estimate assuming

$$\|h_t\|_{L^1_m} \leq \Omega' \left(\frac{\varepsilon}{(1+t)^2} + \frac{1}{\eta_t^2} \right), \quad \|h_t\|_{\mathcal{H}^1_{p_0}} \leq C'(\Omega' + 1) \left(\frac{\varepsilon}{(1+t)^2} + \frac{1}{\eta_t^2} \right). \tag{B.3.34}$$

Here, C' is the constant determined in Proposition B.3.11 in (B.3.22). Let us define for brevity $\Omega := C'(\Omega' + 1)$. We estimate the time integrals in (B.3.33) using (B.3.34) and obtain

$$\|h_t\|_{L^1_m} \leq C_m e^{-E_t} \varepsilon + C \int_0^t e^{-E_t + E_s} \left(\frac{1}{\eta_s} + \frac{\Omega \varepsilon}{(1+s)^2} + \frac{\Omega}{\eta_s^2} + \frac{\eta_s \Omega^2 \varepsilon^2}{(1+s)^4} + \frac{\Omega^2}{\eta_s^3} \right) ds. \tag{B.3.35}$$

Let us estimate term by term.

(i) As in the proof of Proposition B.3.11, Step 2, (ii) we can use partial integration and $\bar{\eta}(\beta_0)$ large enough to get

$$\int_0^t e^{-E_t+E_s}\frac{1}{\eta_s}\,ds \le \frac{C_1}{\eta_t^2}.$$

(ii) For the second and third term in (B.3.35) we estimate similarly

$$\int_0^t e^{-E_t+E_s}\left(\frac{\varepsilon}{(1+s)^2}+\frac{1}{\eta_s^2}\right)ds \le \frac{C_2\varepsilon}{\eta_t(1+t)^2}+\frac{C_2}{\eta_t^3}+C_2\int_0^t e^{-E_t+E_s}\left(\frac{\varepsilon}{\eta_s(1+s)^2}+\frac{1}{\eta_s^3}\right)ds.$$

For $c_0\bar{\eta}(\beta_0)\le \eta_t$ large enough, i.e. $\beta_0\le\varepsilon_0''$ small enough, we can absorb the last term into the left-hand side.

(iii) Finally, we treat the last two terms in (B.3.35) at once. We obtain as in the proof of Proposition B.3.11 in Step 2, (i) and (ii)

$$\int_0^t e^{-E_t+E_s}\left(\frac{\varepsilon^2\eta_s}{(1+s)^4}+\frac{1}{\eta_s^3}\right)ds \le \frac{C_3\varepsilon^2}{(1+t)^4}+\frac{C_3}{\eta_t^4}.$$

Combining the above estimates leads to

$$\|h_t\|_{L^1_m} \le \left[C_m+C_1+\frac{C_2\Omega}{\bar{\eta}(\beta_0)}+C_3\Omega^2\left(\varepsilon+\frac{1}{\bar{\eta}(\beta_0)}\right)\right]\left(\frac{\varepsilon}{(1+t)^2}+\frac{1}{\eta_t^2}\right).$$

By the definition of $\Omega = C'(\Omega'+1)$ we get for some C''

$$\|h_t\|_{L^1_m} \le C''\left[1+\frac{\Omega'}{\bar{\eta}(\beta_0)}+(\Omega')^2\left(\varepsilon+\frac{1}{\bar{\eta}(\beta_0)}\right)\right]\left(\frac{\varepsilon}{(1+t)^2}+\frac{1}{\eta_t^2}\right).$$

If we would have $\Omega' = 6\max\{C'',1\}$ and ε, $\beta_0\le\varepsilon_0''$ are sufficiently small, such that

$$\frac{\Omega'}{\bar{\eta}(\beta_0)}\le 1, \quad (\Omega')^2\left(\varepsilon+\frac{1}{\bar{\eta}(\beta_0)}\right)\le 1 \tag{B.3.36}$$

holds, then we would obtain

$$\|h_t\|_{L^1_m} \le \frac{\Omega'}{2}\left(\frac{\varepsilon}{(1+t)^2}+\frac{1}{\eta_t^2}\right). \tag{B.3.37}$$

In the next step, we show that this particular choice enables us to conclude the proof.

Step 3: Continuation argument. We now prove that (B.3.34) holds using a continuation argument. Here, we define $\Omega' = 6\max\{C'',1\}$ as in the end of the last step. We can assume without loss of generality that $\Omega' > C_*$, where the constant $C_*>0$ satisfies

$$\|g\|_{L^1_m} \le C_*\|g\|_{\mathcal{H}^1_{p_0}}.$$

Let us also recall that C' is the constant in Proposition B.3.11. Furthermore, we choose $\varepsilon_0''\le\varepsilon_0'$ such that (B.3.36) is valid for $\varepsilon\le\varepsilon_0''$, $\beta_0\le\varepsilon_0''$. Note that here ε_0' is the constant in Proposition B.3.11, which depends our choice of Ω'. However, Ω' is now fixed. Let us now proceed with the continuation argument.

First, the estimates (B.3.34) are true on some small interval $[0,t_0]$, since by assumption $\|h_0\|_{\mathcal{H}^1_{p_0}} \le \varepsilon$ and $\|h_0\|_{L^1_m} \le C_*\varepsilon$. Let us now assume that (B.3.34) hold on some interval $[0,t_1]$. We want to extend it by continuity to some larger interval. For $t \in [0,t_1]$ we obtain from the previous step that (B.3.37) is valid. Hence, we can extend this bound on some larger interval $[0,t_2]$, $t_2 > t_1$. Using now Proposition B.3.11 (noting that all the assumptions are satisfied) on the interval $[0,t_2]$ we get the second estimate in (B.3.34). As a consequence (B.3.34) is valid on the whole interval $[0,T]$. This yields (B.3.30) by defining the numerical constant $\Omega := C'(\Omega'+1)$ and concludes the proof. □

Conclusion of proof

With this preparation we can give the proof of Theorem B.3.1.

Proof of Theorem B.3.1. We select the constant Ω as well as ε_0 to ensure that (B.3.12), (B.3.13) hold for all times. We make the following choices.

(i) Define Ω as in Proposition B.3.14.

(ii) Select $\varepsilon_0 \in (0,1)$ such that $\varepsilon_0 \le \varepsilon_0''$, the constant ε_0'' given in Proposition B.3.14, and

$$\exp\left(c\gamma\Omega \int_0^\infty \left(\frac{\varepsilon}{(1+t)^2} + \frac{1}{c_0^2(\bar\eta(\beta_0)+t)^2}\right) dt\right) \le 2 \tag{B.3.38}$$

holds for all ε, $\beta_0 \le \varepsilon_0$. The integral on the left-hand side is motivated by the term $R_T(\varepsilon,\beta_0)$ in Lemma B.3.5. Using formally $\eta_t \ge c_0(\bar\eta(\beta_0)+t)$, according to (B.3.14), gives the above integral for $T=\infty$.

Finally, we use again a continuation argument to prove (B.3.12) and (B.3.13). First of all, by continuity the estimate (B.3.12) holds on some small interval $[0,t_0]$, $t_0>0$, since $\eta_0 = \nu_0\beta_0^{-\gamma/2} = Z_0(\beta_0)$. Recall the definition of $Z_t(\beta_0)$ in (B.3.6). The assumptions of Proposition B.3.14 are satisfied on $[0,t_0]$, so that (B.3.13) is valid on $[0,t_0]$ as well.

Let us assume that (B.3.12) and (B.3.13) hold on some interval $[0,t_1]$. This interval can be assumed to be closed by continuity. Lemma B.3.5 yields for $t \in [0,t_1]$

$$\exp\left(-c\gamma\Omega_2 R_{t_1}(\varepsilon,\beta_0)\right) Z_t(\beta_0) \le \eta_t \le \exp\left(c\gamma\Omega_2 R_{t_1}(\varepsilon,\beta_0)\right) Z_t(\beta_0).$$

Using $\eta_t \ge c_0(\bar\eta(\beta_0)+t)$ for $t\in[0,t_1]$, we get with (B.3.38)

$$\frac{1}{2} Z_t(\beta_0) \le \eta_t \le 2Z_t(\beta_0).$$

Hence, we can extend (B.3.12) on some larger interval $[0,t_2]$, $t_2>t_1$. On this interval we can apply Proposition B.3.14, which yields also (B.3.13) on $[0,t_2]$. Thus, both (B.3.12) and (B.3.13) hold for all times. These estimates imply (B.3.7) and (B.3.9). Finally, let us note that (B.3.8) is a consequence of Lemma B.3.4. □

B.4 Application to homoenergetic solutions

In this section, we apply Theorem B.3.1 to homoenergetic solutions in the case of *simple shear*, *simple shear with decaying planar dilatation/shear* and *combined orthogonal shear* to conclude Theorem B.1.1 and Theorem B.1.2. To this end, let us first give a lemma that allows to verify assumption (II) in Theorem B.3.1.

Lemma B.4.1. *Let $L \in C^1([0,\infty);\mathbb{R}^{3\times 3})$ and $\nu \in C^1([0,\infty);(0,\infty))$ satisfy assumption (I) in Theorem B.3.1. Consider the decomposition of L into its trace-free and trace part $L_t = A_t + b_t I$. Assume that*

(i) $A_t = A^0 + A^1_t$ *with* $\operatorname{tr} A^0 = 0$, $A^0 \neq 0$, $A^1_t \to 0$ *as* $t \to \infty$ *and* A^0 *is time-independent;*

(ii) $b_t = b^0_t + b^1_t$ *with* $b^0_t \geq 0$ *and* $|b^1_t| \leq C/(1+t)^2$ *for all* $t \geq 0$ *with some constant* $C > 0$*;*

(iii) $t \mapsto \nu_t$ *is bounded on* $[0,\infty)$*;*

(iv) we have for all $t \geq 1$

$$N_t := \int_0^t \frac{\nu_t}{\nu_s} e^{-\int_s^t b_r\,dr}\,ds \approx t.$$

Then, assumption (II) in Theorem B.3.1 is satisfied.

Note that for the matrix L given by (B.1.6), (B.1.7) or (B.1.8) we have $\operatorname{tr} L_t = 3b_t \geq 0$ up to terms of order $\mathcal{O}(1/t^2)$. This motivates assumption (ii) in the lemma.

Proof of Lemma B.4.1. First of all, let us note that

$$a_t = \left\langle v\cdot A_t v\,\mu, \mathscr{L}^{-1}\left[v\cdot A_t v\,\mu\right]\right\rangle_{L^2(\mu^{-1/2})} \to \left\langle v\cdot A^0 v\,\mu, \mathscr{L}^{-1}\left[v\cdot A^0 v\,\mu\right]\right\rangle_{L^2(\mu^{-1/2})} > 0.$$

Recall that $v\cdot A_t v\,\mu \in (\ker\mathscr{L})^\perp$ and that $\mathscr{L}$ is a positive operator on $(\ker\mathscr{L})^\perp$. Thus, we have $\lim_{t\to\infty} a_t > 0$. Since $t \mapsto a_t > 0$ is continuous, we have $0 < c_0 \leq a_t \leq C_0$ for all $t \geq 0$ and some constants $c_0, C_0 > 0$.

Due to assumptions (ii) and (iii) the first term in $Z_t(1)$ in (B.3.6) is bounded. With the above observation, the second term is equivalent to

$$\int_0^t \frac{\nu_t}{\nu_s} e^{-\int_s^t b_r\,dr}\,ds \approx t.$$

This implies $Z_t(1) \approx 1+t$ for $t \geq 0$. □

Let us now give the proof of Theorem B.1.1 and Theorem B.1.2.

Proof of Theorem B.1.1 and Theorem B.1.2. First of all, let us recall that a solution g to (B.1.5) is related to a solution f to, see (B.1.11),

$$\begin{aligned} &\partial_t f_t = \operatorname{div}\left((L_t - \alpha_t)\,v\,f_t\right) + \rho_t\,\beta_t^{-\gamma/2} Q(f_t,f_t), \quad f(0,\cdot) = f_0(\cdot),\\ &\beta_t = \beta_0 \exp\left(2\int_0^t \alpha_s\,ds\right), \quad \alpha_t := \frac{1}{3}\int v\cdot L_t v\,f_t(v)\,dv,\\ &\rho_t = \exp\left(-\int_0^t \operatorname{tr} L_s\,ds\right) \end{aligned} \tag{B.4.1}$$

via the rescaling $f_t(v) = g_t(v\beta_t^{-1/2} + V_t)\beta_t^{-3/2}\rho_t^{-1}$, see Theorem B.1.1. Also recall that with this rescaling f satisfies the normalization (B.1.10). As already indicated earlier we use equation (B.3.1) for a specific choice of ν to draw conclusions for solutions to (B.4.1). We discuss this reduction in each case of *simple shear*, *simple shear with decaying planar dilatation/shear* and *combined orthogonal shear* separately. Let us recall that the inverse temperature β_t in (B.3.1) satisfies the following equation

$$\frac{\beta_t'}{2\beta_t} = \alpha_t = \frac{1}{3}\int v \cdot L_t v\, f_t\, dv. \tag{B.4.2}$$

Simple Shear: The matrix L_t is given by (B.1.6), in particular it is constant in time and $L = A$ is trace-free. This implies that the density is constant $\rho_t = \rho_0$. We set $\nu_t \equiv \rho_0$ and equation (B.4.1) reduces to (B.3.1). Let us now check the structural conditions of Theorem B.3.1. Assumption (I) is satisfied. Furthermore, Lemma B.4.1 applies with $N_t = t$, yielding assumption (II). Note that the definition of $\bar{\mu}_t$ in (B.3.3) is the same as in Theorem B.1.1, formula (B.1.19). The smallness assumptions on h_0, β_0 as well as the considered spaces coincide with the conditions in Theorem B.1.1. Hence, Theorem B.3.1 applies.

As a consequence of (B.3.7) and (B.3.9) we get $\|h_t\|_{L^1_2} = \mathcal{O}((1+t)^{-2})$. In addition, (B.3.9) implies $\eta_t = \beta_t^{-\gamma/2} \approx 1+t$. Let us now compute the asymptotics of the inverse temperature (B.1.26). We plug the decomposition $f_t = \mu + \bar{\mu}_t + h_t$ into (B.4.2), yielding

$$\frac{d}{dt}\left(\beta_t^{-\gamma/2}\right) = \frac{\gamma\bar{a}}{3} - \gamma\, \alpha_t^2\, \beta_t^{-\gamma/2}, \quad \alpha_t^2 := \frac{1}{3}\int v \cdot Av\, h_t(v)\, dv,$$

where $\bar{a}$ is given in Theorem B.1.1 in (B.1.22). We know already that $\beta_t^{-\gamma/2} = \mathcal{O}(t)$ and $\alpha_t^{(2)} = \mathcal{O}((1+t)^{-2})$ as $t \to \infty$. Hence, we have for some remainder R_t

$$\frac{d}{dt}\left(\beta_t^{-\gamma/2}\right) = \frac{\gamma\bar{a}}{3} + R_t, \quad |R_t| \le \frac{C}{1+t}.$$

We integrate this in time to get (B.1.21). Finally, (B.1.21) implies $\eta_t = \beta_t \approx \beta_0^{-\gamma/2} + t =: \zeta_t$. As a consequence, we obtain (B.1.28) from (B.3.7) and (B.3.8).

Simple shear with decaying planar dilatation/shear: In this case, L_t is given by (B.1.7). We define also $\nu_t := \rho_t$, where the density is given in (B.4.1). We again check the structural conditions in Theorem B.3.1. Assumption (I) is satisfied, since $\sup_{t\ge 0}\|L_t\| < \infty$, $\nu_t'/\nu_t = \operatorname{tr} L_t$ and $L_t' = -L_t^2$. The latter equation is part of the ansatz of homoenergetic solutions in (B.1.5). For assumption (II) we apply Lemma B.4.1. To this end, we use the decomposition of $L_t = A_t + b_t I$ into its trace-free and trace part. Here, we have

$$A_t = A^0 + A_t^1 = \begin{pmatrix} 0 & K_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{1+t}\begin{pmatrix} 0 & K_1K_3 & K_1 \\ 0 & 0 & 0 \\ 0 & K_3 & 0 \end{pmatrix} + \mathcal{O}\left(\frac{1}{(1+t)^2}\right),$$
$$b_t = \operatorname{tr} L_t = \frac{1}{1+t} + b_t^2, \quad b_t^2 = \mathcal{O}\left(\frac{1}{(1+t)^2}\right).$$

Note that $A^0 \neq 0$ due to $K_2 \neq 0$. Furthermore, we have

$$\nu_t = \rho_t = \exp\left(-\int_0^t \operatorname{tr} L_s\, ds\right) \approx (1+t)^{-1},$$
$$N_t \approx \int_0^t \left(\frac{1+s}{1+t}\right)^{\gamma/3+1} ds = \frac{3}{\gamma+6}\,\frac{(1+t)^{\gamma/3+2}-1}{(1+t)^{\gamma/3+1}} \approx t.$$

Hence, Lemma B.4.1 implies assumption (II) in Theorem B.3.1. Note again that the definition of $\bar{\mu}_t$ in (B.3.3) is the same as in Theorem B.1.1, formula (B.1.19).

We can now apply Theorem B.3.1. Again, (B.3.7) and (B.3.9) yields $\|h_t\|_{L^1_2} = \mathcal{O}((1+t)^{-2})$. We now calculate the asymptotics for β_t. We again have with (B.4.2) and the decomposition $f_t = \mu + \bar{\mu}_t + h_t$

$$\frac{d}{dt}\left(\beta_t^{-\gamma/2}\right) = -\frac{\gamma \operatorname{tr} L_t}{3}\beta_t^{-\gamma/2} + \frac{\gamma\, a_t}{3\rho_t} - \gamma \alpha_t^2 \beta_t^{-\gamma/2}, \quad \alpha_t^2 := \int v \cdot L_t v\, h_t(v)\, dv.$$

Here, we used

$$a_t := \left\langle v \cdot A_t v\, \mu, \mathscr{L}^{-1}[v \cdot A_t v\, \mu]\right\rangle_{L^2(\mu^{-1/2})}.$$

One can see that

$$\rho_t^{-1} = (1+t)\exp\left(\int_0^t r_s\, ds\right) = (1+t)\exp\left(\int_0^\infty r_s\, ds\right) + \mathcal{O}(1),$$
$$a_t = \bar{a} + \mathcal{O}\left(\frac{1}{1+t}\right).$$

Here, $\bar{a}$ and r_t are given in Theorem B.1.1, see (B.1.24) and (B.1.25). From (B.1.23) and $\nu_t^{-1} = \rho_t^{-1}$ we get $\beta^{-\gamma/2} = \mathcal{O}(t^2)$. Using this and $|\alpha_t^2| \lesssim \|h_t\|_{L^1_2} = \mathcal{O}((1+t)^{-2})$ we obtain

$$\frac{d}{dt}\left(\beta_t^{-\gamma/2}\right) = -\frac{\gamma}{3(1+t)}\beta_t^{-\gamma/2} + \frac{\gamma \bar{a}}{3}(1+t)\exp\left(\int_0^\infty r_s\, ds\right) + R_t, \quad |R_t| \le C.$$

We integrate this ODE yielding

$$\beta_t^{-\gamma/2} = \beta_0^{-\gamma/2}(1+t)^{-\gamma/3} + \frac{\gamma \bar{a}}{\gamma+6}\exp\left(\int_0^\infty r_s\, ds\right)\left((1+t)^2 - 1\right) + \tilde{R}_t. \tag{B.4.3}$$

Here, we have the lower order term

$$|\tilde{R}_t| \le C\int_0^t \left(\frac{1+t}{1+s}\right)^{-\gamma/3} ds \le C(1+t).$$

Hence, we get (B.1.23). The formula (B.4.3), $\nu_t = \rho_t \approx (1+t)^{-1}$ and (B.1.23) yields

$$\eta_t = \nu_t \beta_t^{-\gamma/2} \approx \beta_0^{-\gamma/2}(1+t)^{-1-\gamma/3} + t =: \zeta_t.$$

Thus, by (B.3.7) and (B.3.8) we obtain (B.1.29).

Combined orthogonal shear: Here, L_t is given by (B.1.8). In this case, the matrix L_t is not the matrix in (B.3.1). The reason is that we need to take care of the linear growth of L_t, so that assumption (I) in Theorem B.3.1 is not satisfied.

We have first of all $\operatorname{tr} L_t = 0$ so that $\rho_t \equiv \rho_0 = 1$. In order to apply Theorem B.3.1, let us introduce the time-change $\tau = (t+1)^2/2 - 1/2$, i.e. $1+t = \sqrt{2\tau+1}$ in equation (B.4.1). Hence, $F(\tau, v) := f(t(\tau), v)$ solves

$$\partial_\tau F = \operatorname{div}\left(\left(\tilde{L}_\tau - \alpha_\tau\right) v\, F\right) + \nu_\tau \beta_\tau^{-\gamma/2} Q(F,F)$$

where we defined

$$\tilde{L}_\tau = \frac{1}{\sqrt{2\tau+1}}L(t(\tau)) = A^0 + A^1_\tau = \begin{pmatrix} 0 & 0 & -K_1K_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{\sqrt{2\tau+1}}\begin{pmatrix} 0 & K_3 & K_2 + K_1K_2 \\ 0 & 0 & K_1 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\nu_\tau = \frac{1}{\sqrt{2\tau+1}}.$$

The above equation is of the form (B.3.1) and we aim to apply Theorem B.3.1. Note that the first order approximation $\bar{\mu}_t$ in (B.1.19) can be written as

$$\bar{\mu}_{t(\tau)} = \frac{1}{\beta_{t(\tau)}^{-\gamma/2}} \mathscr{L}^{-1}\left[-v \cdot L_{t(\tau)} v\mu\right] = \frac{1}{\nu_\tau \beta_{t(\tau)}^{-\gamma/2}} \mathscr{L}^{-1}\left[-v \cdot A_\tau v\mu\right].$$

Hence, we have (B.3.3) in terms of the time τ, which is used in Theorem B.3.1.

Let us check now the assumptions in Theorem B.3.1. First of all, $\tilde{L}$ and ν satisfy assumption (I) in Theorem B.3.1. Moreover, $\operatorname{tr}\tilde{L}_\tau = 3b_\tau = 0$ and the formula for $\tilde{L}_\tau = A_\tau$ yields the decomposition $A^0 + A^1_\tau$ as in Lemma B.4.1 (i). Furthermore, ν satisfies (iii) in Lemma B.4.1. With

$$N_\tau = \int_0^\tau \frac{\nu_\tau}{\nu_\sigma}\, d\sigma = \frac{2\tau+1}{3} - \frac{1}{3\sqrt{2\tau+1}} \approx \tau$$

we can apply Lemma B.4.1 and assumption (II) in Theorem B.3.1 holds.

We have the decomposition $F_\tau = f_{t(\tau)} = \mu + \bar{\mu}_{t(\tau)} + h_{t(\tau)}$. From (B.3.7) and (B.3.9) we have $\beta_{t(\tau)}^{-\gamma/2} = \mathcal{O}(\tau^{3/2})$ and $\left\| h_{t(\tau)} \right\|_{L^1_2} = \mathcal{O}((1+\tau)^{-2})$. Hence, with respect to the original time $1+t = \sqrt{1+2\tau}$ we get $\|h_t\|_{L^1_2} = \mathcal{O}((1+t)^{-4})$ and $\beta_t^{-\gamma/2} = \mathcal{O}(t^3)$. Let us now compute the asymptotics of the inverse temperature β_t. We use (B.4.2) and the decomposition $f_t = \mu + \bar{\mu}_t + h_t$. This leads to

$$\frac{d}{dt}\left(\beta_t^{-\gamma/2}\right) = \frac{\gamma\, a_t}{3} - \gamma\, \alpha_t^2\, \beta_t^{-\gamma/2}.$$

Here, we abbreviated

$$\begin{aligned} \alpha_t^2 &= \frac{1}{3}\int v \cdot L_t v\, h_t\, dv = \mathcal{O}((1+t)^{-3}), \\ a_t &= \left\langle v \cdot L_t v\mu, \mathscr{L}^{-1}\left[v \cdot L_t v\mu\right]\right\rangle_{L^2(\mu^{-1/2})} = \bar{a}\, t^2 + \mathcal{O}(t). \end{aligned}$$

Note that we used the form of the matrix L_t in (B.1.8) for the last equality. The constant $\bar{a}$ is given in (B.1.27). Together with $\beta_t^{-\gamma/2} = \mathcal{O}(t^3)$ we obtain the equation

$$\frac{d}{dt}\left(\beta_t^{-\gamma/2}\right) = \frac{\gamma\, \bar{a}\, t^2}{3} + R_t, \quad |R_t| \le C(1+t).$$

Integrating this equation yields (B.1.26). Also we obtain $\beta_t^{-\gamma/2} \approx \beta_0^{-\gamma/2} + t^3$. Thus, we get

$$\eta_{t(\tau)} = \nu_\tau\, \beta_{t(\tau)}^{-\gamma/2} \approx \left(\beta_0^{-\gamma/2} + t(\tau)^3\right)(1+t(\tau))^{-1}$$

and with respect to the original time t

$$\eta_t \approx \beta_0^{-\gamma/2}(1+t)^{-1} + t^2 =: \zeta_t.$$

Finally, as a consequence of (B.3.7) ad (B.3.9) we have

$$\left\| h_{t(\tau)} \right\|_{\mathcal{H}^1_{p_0}} \le C' \left(\frac{\varepsilon}{(1+\tau)^2} + \frac{16}{\eta_{t(\tau)}^2} \right).$$

Using the previous estimate for η_t and writing this with respect to the original time $1+t = \sqrt{1+2\tau}$ yields (B.1.30). The same can be done using (B.3.8) to get the second estimate in (B.1.30). $\square$

B.5 Collision dominated behavior for cutoff kernels

In this final section, we indicate an extension of the previous analysis to cutoff kernels. In particular, we consider the following assumptions.

Assumptions on the kernel. The collision kernel has the product form $B(v-v_*,\sigma)=b(n\cdot\sigma)|v-v_*|^\gamma$, where $b:[-1,1]\to[0,\infty)$ is a smooth function and γ satisfies $\gamma\in(0,1]$.

The most prominent application is the case of hard spheres interactions $B(v-v_*,\sigma)=|v-v_*|$. Let us now give the following L^1-variant of Theorem B.1.1 and Theorem B.1.2.

Theorem B.5.1. *Consider equation* (B.1.5) *with matrix* $L_t=L_0(I+tL_0)^{-1}$ *having the asymptotic form* (B.1.6)*,* (B.1.7) *or* (B.1.8)*. Let* $p_0>3$ *be arbitrary and* $g_0\in W^{1,1}_{p_0}$*. Consider the unique solution* g *to* (B.1.5)*. Define* $f,\bar{\mu}$ *as in* (B.1.19) *and* $h_t(v):=f_t(v)-\mu(v)-\bar{\mu}_t(v)$*.*

There are $\varepsilon_0\in(0,1)$ *sufficiently small and a constant* $C'>0$*, depending only on* p_0*,* L_t *and the collision kernel* B*, such that: If* $\|h_0\|_{W^{1,1}_{p_0}}=\varepsilon\leq\varepsilon_0$ *and* $\beta_0\leq\varepsilon_0$*, we have in each case the asymptotics* (B.1.26)*,* (B.1.21) *and* (B.1.23)*. Finally, the bounds in* (B.1.28)*,* (B.1.29) *and* (B.1.30) *are true when replacing* $\|h_t\|_{\mathcal{H}^1_{p_0}}$ *by* $\|h_t\|_{W^{1,1}_{p_0}}$*.*

Let us mention that existence of (weak) solutions to (B.1.5) are known by the work of Cercignani [48]. Uniqueness can be proved as in Proposition B.2.2, which amounts to an application of a Povzner estimate. Furthermore, Povzner estimates allow to show the gain of moments as in Proposition B.2.2 (i). The propagation of regularity estimates can be proved by arguments used for the homogeneous Boltzmann equation, see e.g. [137].

In order to prove this theorem, we show that the corresponding L^1-variant of Theorem B.3.1 is valid. To this end, we follow the strategy in Subsection B.3.1.

B.5.1 Proof of Theorem B.5.1

The proof is analogous to the one discussed in Subsection B.3.1 and the goal is to prove (B.3.12) and the variant of (B.3.13), namely,

$$\|h_t\|_{W^{1,1}_{p_0}}\leq\Omega\left(\frac{\varepsilon}{(1+t)^2}+\frac{1}{\eta_t^2}\right).$$

The four main ingredients in Subsection B.3.1 are Lemma B.3.5, Lemma B.3.4 and Proposition B.3.11, Proposition B.3.14. The first lemma extends without any changes. Let us discuss the other three preparatory results.

Estimate on $\bar{\mu}$ for cutoff kernels

The result in Lemma B.3.4 relies on corresponding coercivity estimates for the operator $Lg=\mu^{-1/2}\mathscr{L}[\sqrt{\mu}g]$ in the cutoff case. Such estimates in Sobolev spaces H^k were discussed in [135]. (They considered general operators L which satisfy the hypothesis H1' and H2' therein. These assumptions are exactly the needed coercivity estimates.) To prove bounds including higher moments, it suffices by interpolation to prove corresponding coercivity estimates in L^2_p, i.e. for all $p\geq 0$

$$\langle Lg,g\rangle_{L^2_p}\geq c_0\|g\|^2_{L^2_{p+\gamma/2}}-C\|g\|^2_{L^2}. \tag{B.5.1}$$

For this one uses the commutator estimate

$$|\langle\langle v\rangle^p Lg, \langle v\rangle^p g\rangle_{L^2} - \langle L[\langle v\rangle^p g], \langle v\rangle^p g\rangle_{L^2}| \le \varepsilon \|g\|^2_{L^2_{p+\gamma/2}} + C_\varepsilon \|g\|^2_{L^2_{p-1+\gamma/2}}$$

holding for all $\varepsilon > 0$. Furthermore, we have by the spectral gap inequality, see e.g. [135],

$$\langle L[\langle v\rangle^p g], \langle v\rangle^p g\rangle_{L^2} \ge c_0 \|(I - \Pi_0)\langle v\rangle^p g\|^2_{L^2_{\gamma/2}} \ge \frac{c_0}{2}\|g\|^2_{L^2_{p+\gamma/2}} - C\|g\|^2_{L^2}.$$

Recall that Π_0 is the projection onto $\ker L$, which is spanned by the functions $\varphi\sqrt{\mu}$ with $\varphi(v) = 1, v_1, v_2, v_3, |v|^2$. Combining the previous estimates, using interpolation and choosing ε small yields (B.5.1).

Estimate on the error term for cutoff kernels

Instead of the estimates in L^2 leading to Proposition B.3.11 we use the following bounds, which replace Lemma B.3.8 and Lemma B.3.9. However, due to the fact that there is no regularizing effect, we need a second estimate for the linearized collision operator, which takes into account first order derivatives.

Lemma B.5.2. *We have the following estimates.*

(i) For $p \ge \gamma$ we have

$$\|Q(f,g)\|_{L^1_p} \lesssim \|f\|_{L^1_p}\|g\|_{L^1_{p+\gamma}} + \|f\|_{L^1_{p+\gamma}}\|g\|_{L^1_p}.$$

(ii) For $p > 2$ we have

$$-\int_{\mathbb{R}^3} \mathscr{L}h \operatorname{sgn}(h)\langle v\rangle^p\, dv \le -c_0\|h\|_{L^1_{p+\gamma}} + C\|h\|_{L^1}.$$

(iii) Let $i = 1, 2, 3$ and $p > 2$, then we have

$$-\int_{\mathbb{R}^3} \partial_i[\mathscr{L}h] \operatorname{sgn}(\partial_i h)\langle v\rangle^p\, dv \le -c_0\|\partial_i h\|_{L^1_{p+\gamma}} + C\|h\|_{L^1_{p+\gamma}}.$$

Proof. The first bound can be proved via straightforward estimates and the second one uses a variant of the Povzner estimate. For the last inequality, we use the decomposition of $\mathscr{L} = \mathscr{B}_\varepsilon + \mathscr{A}_\varepsilon$ in [152], which is defined as follows. Let $0 \le \Theta_\varepsilon \le 1$ be smooth, equal to one on the set

$$\{|v| \le 1/\varepsilon,\ 2\varepsilon \le |v - v_*| \le 1/\varepsilon,\ |\cos\theta| \le 1 - 2\varepsilon\}$$

and supported in the set

$$\{|v| \le 1/2\varepsilon,\ \varepsilon \le |v - v_*| \le 2/\varepsilon,\ |\cos\theta| \le 1 - \varepsilon\}.$$

One can choose it in the form $\Theta_\varepsilon(v, v_*, \theta) = \Theta^1_\varepsilon(v)\Theta^2_\varepsilon(v - v_*)\Theta^3_\varepsilon(\theta)$. Then, we define

$$\mathscr{B}_\varepsilon h = \int_{\mathbb{R}^3}\int_{S^2}(1 - \Theta_\varepsilon)|v - v_*|^\gamma b(\cos\theta)(\mu' h'_* + \mu'_* h' - \mu h_*)\, d\sigma dv_* - \|b\|_{L^1(S^2)}(|\cdot|^\gamma * \mu)\, h,$$

$$\mathscr{A}_\varepsilon h = \int_{\mathbb{R}^3}\int_{S^2}\Theta_\varepsilon|v - v_*|^\gamma b(\cos\theta)(\mu' h'_* + \mu'_* h' - \mu h_*)\, d\sigma dv_*.$$

To calculate the derivative with respect to v_i we use the change of variables $w = v - v_*$ in the v_*-integration, perform the differentiation and undo the transformation again. We obtain in this way

$$\begin{aligned}\partial_i [\mathscr{L}h] = \partial_i [\mathscr{B}_\varepsilon h] + \partial_i [\mathscr{A}_\varepsilon h] &= \mathscr{B}_\varepsilon \partial_i h + \partial_i [\mathscr{A}_\varepsilon h] - \|b\|_{L^1(S^2)} \left(|\cdot|^\gamma * \partial_i \mu\right) h \\ &+ \int_{\mathbb{R}^3} \int_{S^2} (1-\Theta_\varepsilon) |v - v_*|^\gamma b(\cos\theta) \left((\partial_i \mu)' h_*' + (\partial_i \mu)_*' h' - (\partial_i \mu) h_*\right) d\sigma dv_* \\ &- \int_{\mathbb{R}^3} \int_{S^2} \left[\partial_i \Theta_\varepsilon^1\right] \Theta_\varepsilon^2 \Theta_\varepsilon^3 |v - v_*|^\gamma b(\cos\theta) \left((\partial_i \mu)' h_*' + (\partial_i \mu)_*' h' - \partial_i \mu h_*\right) d\sigma dv_*.\end{aligned}$$

The last three terms can be estimated in L^1_p by $C_\varepsilon \|h\|_{L^1_{p+\gamma}}$. The first term is strongly dissipative in L^1_p, $p > 2$, for $\varepsilon > 0$ sufficiently small. For this see the proof of [152, Lemma 2.6], which covers the non-cutoff case and uses a variant of the Povzner estimate. In the cutoff case, there is no need to split b into a cutoff and non-cutoff part. We then obtain

$$\int_{\mathbb{R}^3} \mathscr{B}_\varepsilon \partial_i h \operatorname{sgn}(\partial_i h) \langle v \rangle^p \, dv \le -c_0 \|\partial_i h\|_{L^1_{p+\gamma}}.$$

Finally, the operator $\mathscr{A}_\varepsilon$ is regularizing, in the sense that it maps L^1_1 to compactly supported functions and

$$\|\mathscr{A}_\varepsilon h\|_{H^1} \le C_\varepsilon \|h\|_{L^1_1}.$$

For this result see [79, Lemma 4.16]. This relies on the regularizing effect of the gain term, see [137, Theorem 3.1]. Putting all estimates together yields the result. □

Sketch of estimates. We state here the estimates for the corresponding L^1-variant of Proposition B.3.11 and Proposition B.3.14. Let us mention that compared to Subsection B.3.1, see (B.3.20), we define here $m := p_0 - 1 > 2$. Furthermore, we give here only a priori estimates. In particular, when it comes to the continuation argument the continuity of $t \mapsto \|h_t\|_{W^{1,1}_{p_0}}$ is crucial. A way to ensure this is to regularize the initial data h_0^n with $\|h_0^n\|_{W^{1,1}_{p_0}} \le 2\varepsilon$. For the corresponding solution we can prove the L^1-variant of (B.3.7) as well as (B.3.9) and pass to the limit.

Concerning the proof of Proposition B.3.11 one uses again an equivalent norm (we again abbreviate $p = p_0$)

$$|||h|||_{W^{1,1}_p} := \|h\|_{L^1_p} + \kappa \sum_{|\alpha|=1} \|\partial^\alpha h\|_{L^1_p}$$

for $\kappa \in (0,1)$. We estimate

$$\frac{d}{dt} |||h|||_{W^{1,1}_p} = \int_{\mathbb{R}^3} \partial_t h \operatorname{sgn}(h) \langle v \rangle^p \, dv + \kappa \sum_{i=1}^3 \int_{\mathbb{R}^3} \partial_i \partial_t h \operatorname{sgn}(\partial_i h) \langle v \rangle^p \, dv.$$

The first term can be treated similar as in the proof of Proposition B.3.11 using Lemma B.5.2 (i), (ii). For the derivatives, the most important change appears for the linearized collision operator (compare Step 1 (vi) in the proof of Proposition B.3.11), where we use Lemma B.5.2

(iii). Here, the lower order term $\kappa\eta_t C \|h_t\|_{L^1_{p+\gamma}}$ can be absorbed into the term $-c_0\eta_t \|h_t\|_{L^1_{p+\gamma}}$, resulting from Lemma B.5.2 (ii), for $\kappa > 0$ small enough. All in all, one obtains

$$\frac{d}{dt}|||h_t|||_{W^{1,1}_p} = -\left(\frac{c_0\eta_t}{2} - C - \eta_t C|||h_t|||_{W^{1,1}_p}\right)|||h_t|||_{W^{1,1}_p} + \frac{C'}{\eta_t} + \eta_t C' \|h_t\|_{L^1} .$$

The second term is a consequence of the source and the last term is the remaining term in the estimate of Lemma B.5.2 (ii). We can again choose $\beta_0 \le \varepsilon_0$ small enough, so that $\eta_t \gtrsim \bar{\eta}(\beta_0)$ is large enough to absorb the constant C. Following Step 2 and Step 3 in the proof of Proposition B.3.11, we can conclude the corresponding L^1-variant of Proposition B.3.11.

Concerning Proposition B.3.14 we can use the arguments without any changes, noting that the collision operator can be bounded via Lemma B.5.2 (i). The drift term is estimated via the $W^{1,1}_{p_0}$-norm, recalling $p_0 = m+1$. A key ingredient is Lemma B.3.12, which was proved in [152] for the non-cutoff case. The same proof can be used to show the result for cutoff kernels. In fact, the proof simplifies, since a decomposition $b = b_\delta + b_\delta^c$ into a cutoff and non-cutoff part is not needed.

Finally, the conclusion based on a continuation argument does not change. All in all, the corresponding variant of Theorem B.3.1 holds true and the same arguments as in Section B.4 conclude the proof of Theorem B.5.1.

Appendix C

Vanishing angular singularity limit to the hard-sphere Boltzmann equation

Abstract

In this note we study Boltzmann's collision kernel for inverse power law interactions $U_s(r) = 1/r^{s-1}$ for $s > 2$ in dimension $d = 3$. We prove the limit of the non-cutoff kernel to the hard-sphere kernel and give precise asymptotic formulas of the singular layer near $\theta \simeq 0$ in the limit $s \to \infty$. Consequently, we show that solutions to the homogeneous Boltzmann equation converge to the respective solutions.

C.1 Introduction

The Boltzmann equation reads as

$$\partial_t f + v \cdot \nabla_x f = Q(f,f)(v), \tag{C.1.1}$$

where $f = f(t,x,v)$ is the velocity distribution of particles with position $x \in \Omega \subset \mathbb{R}^3$ and velocity $v \in \mathbb{R}^3$ at time $t \in [0,\infty)$.

The equation has been considered as a fundamental model for the collisional gases that interact either under the hard-sphere potential $U_s(r) = \infty$ for $r \leq 2\epsilon$ and $= 0$ for $r \geq 2\epsilon$, or under the long-range potential $U_s(r) \simeq \frac{1}{r^{s-1}}$ for $s > 2$. Here ϵ is the radius of each hard-sphere. The prototype of the model was suggested by Maxwell [126, 127] and Boltzmann [35].

In this note we consider the particular case of inverse power law interactions $U_s(r) = 1/r^{s-1}$ leading to non-cutoff kernels (cf. formula (C.1.3))

$$B_s(|v - v_*|, \cos\theta) = |v - v_*|^\gamma b_s(\cos\theta), \quad \gamma = \frac{s-5}{s-1}.$$

Here, b_s is the so-called angular part. We prove that the function B_s converges to the hard-sphere kernel in the limit $s \to \infty$. We give a precise study of the singularity as $\theta \to 0$ when $s \to \infty$. Finally, we show that solutions to the homogeneous Boltzmann equation with collision kernel B_s converge to the solution to the equation for hard-spheres. Such a limit result was suggested to exist in [73, Remark 1.0.1].

C.1.1 Boltzmann collision operator

The Boltzmann collision operator Q takes the form

$$Q(f,f)(v) = \int_{\mathbb{R}^3}\int_{S^2} B(|v-v_*|, n\cdot\sigma)(f'f'_* - ff_*)\,d\sigma dv_*, \quad n := \frac{v-v_*}{|v-v_*|},$$

where we used the standard notation $f' = f(v')$, $f'_* = f(v'_*)$, $f_* = f(v_*)$. Also (v', v'_*) are the post-collisional velocities and (v, v_*) the pre-collisional velocities. The function B is Boltzmann's collision kernel and strongly depends on the microscopic interaction of two particles in the course of a collision. It only depends on the length of relative velocities $|v-v_*|$ and the so-called deviation angle $\theta \in [0,\pi]$ through $n\cdot\sigma = \cos\theta$.

It is customary to distinguish two main classes of kernels, namely angular cutoff and non-cutoff kernels. This refers to a possible singularity of the kernel when $\theta \to 0$. Such deviation angles correspond to grazing collisions, i.e. collisions such that $v \approx v'$. They appear only for long-range or weak interactions.

C.1.2 Derivation of Boltzmann's collision kernel for long-range interactions

Let us give here a derivation of the collision kernel for inverse power law interactions. We consider the collision of two particles (x,v), (x_*,v_*) with equal mass $m=1$. Due to conservation of momentum and conservation of energy, both $v_c = (v+v_*)/2$ and $|v-v_*|$ are conserved. Here, v_c is the velocity of the center of mass $x_c = (x+x_*)/2$. It is convenient to use the coordinate system $(\bar{x},\bar{v}) = (x-x_*, v-v_*)$, in which the center of mass is zero and at rest. In this coordinate system, the velocities after the collisions have equal lengths but opposite directions due to the conservation of momentum and energy. Hence, they are given by $|\bar{v}|\sigma/2$ and $-|\bar{v}|\sigma/2$, respectively, for $\sigma \in S^2$. In the original coordinate system, we thus get

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \quad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma.$$

In order to derive the distribution of σ in the scattering problem, we need to consider the interaction of both particles via the potential U. As is well-known we can reduce it to a single particle problem in the center of mass coordinate system $(\bar{x},\bar{v})$ with (reduced) mass $\mu = 1/2$, see e.g. [108, Section 13]. The motion is planar and we can use polar coordinates. The Hamiltonian reads,

$$H(r,\varphi,\dot{r},\dot{\varphi}) = \frac{\mu}{2}\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) + U(r),$$

where $\dot{r}$, $\dot{\varphi}$ denote the velocity variables (i.e., derivatives with respect to the time variable t) corresponding to r, φ, respectively. Both energy $E = H(r,\varphi,\dot{r},\dot{\varphi})$ and angular momentum $L = \mu r^2\dot{\varphi}$ are conserved.

For the collision process we consider the particle $(\bar{x},\bar{v})(t)$ passing the center of the potential with asymptotic velocity $v - v_*$ as $t \to -\infty$, $r \to \infty$. The particle is scattered and moves away from the center with asymptotic velocity $v' - v'_*$ as $t \to \infty$, $r \to \infty$. The turning point ($\dot{r} = 0$) is given at distance r_m, which is the largest root of

$$E - \frac{L^2}{r_m^2} - U(r_m) = 0.$$

We can determine E and L by considering the asymptotic value $t \to -\infty$. This yields

$$E = \frac{|v - v_*|^2}{4} \text{ and } L = \mu|\bar{x} \times \bar{v}| = \frac{|\bar{x}||\bar{v}|\sin(\psi)}{2} = \frac{|v - v_*|\rho}{2},$$

where ψ is the angle between $\bar{x}$ and $\bar{v}$. Furthermore, ρ is the impact parameter, which is the distance of the closest approach if the particle is passing the center without the presence of an interaction, see Figure C.1. The formula for L can be obtained by a geometric argument.

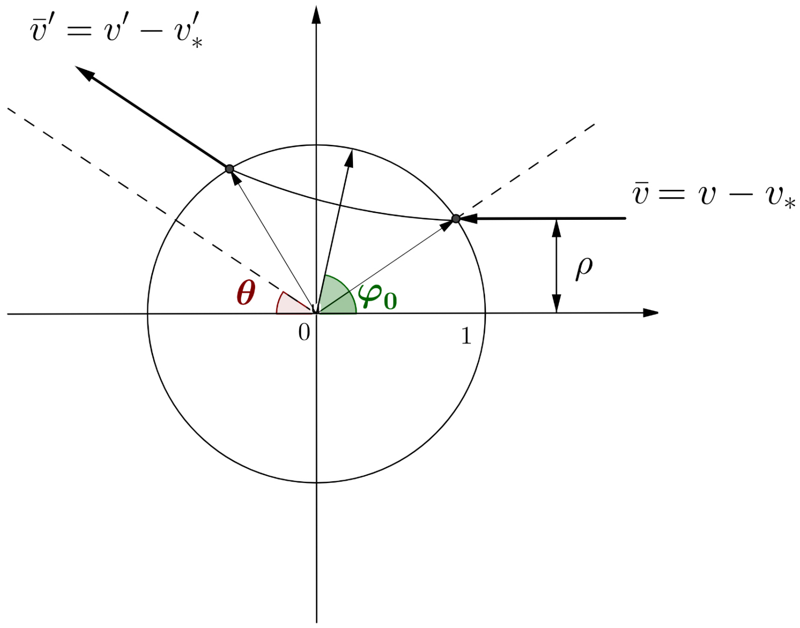

Figure C.1: Two-body scattering process: ρ is the impact parameter, θ the deviation angle and φ_0 the angle of the axis of symmetry.

The solution to the above problem is implicitly given by, see e.g. [108, Section 14],

$$\varphi = \text{const.} + \int_{r_m}^{r} \frac{L/r_*^2\, dr_*}{\sqrt{E - U(r_*) - \frac{L^2}{r_*^2}}}, \quad t = \text{const.} + \int_{r_m}^{r} \frac{dr_*}{2\sqrt{E - U(r_*) - \frac{L^2}{r_*^2}}}.$$

In the limit $t \to -\infty$ the angle φ is zero. By a symmetry argument, one can see that the angle φ_0 of the line through the center and the point of closest approach is given by (see Figure C.1)

$$\varphi_0 = \int_{r_m}^{\infty} \frac{L/r_*^2\, dr_*}{\sqrt{E - U(r_*) - \frac{L^2}{r_*^2}}}.$$

Now, we plug in the values for E, L and use the change of variables $y = \rho/r_*$. Furthermore, we use $U(r) = r^{-(s-1)}$ and define $\beta = \rho(|v - v_*|/2)^{2/(s-1)}$ to get, cf. [47, page 69-71],

$$\varphi_0 = \int_0^{x_0} \frac{dy}{\sqrt{1 - y^2 - (y/\beta)^{s-1}}}, \quad x_0 = \rho/r_m. \tag{C.1.2}$$

The deviation angle is given by $\theta = \pi - 2\varphi_0$ for a given impact parameter ρ.

The number of particles scattered with deviation angle close to θ is proportional to $|v-v_*|$ and the corresponding cross-section, that is $2\pi\rho d\rho = 2\pi\rho(\theta)|\rho'(\theta)|\,d\theta$. Changing to the variable β and integrating via the solid angle yields the formula

$$B_s(|v-v_*|,\cos\theta)\,d\sigma = 2^{\frac{4}{s-1}}|v-v_*|^{\frac{s-5}{s-1}}\frac{\beta(\theta)}{\sin\theta}\beta'(\theta)\,d\sigma. \tag{C.1.3}$$

Let us note that $\beta'(\theta) > 0$. This completes the formal derivation of the Boltzmann collision operator for the long-range interactions.

C.1.3 Outline of the article

We now provide a brief outline of the rest of the article. In Section C.2, we give a proof of the limit of the non-cutoff kernel to the hard-sphere kernel as $s \to \infty$. Then in Section C.3, we study the asymptotics of the singular layer near $\theta \simeq 0$ as $s \to \infty$. Finally, in Section C.4, we prove the convergence of the solution to the spatially homogeneous Boltzmann equation without angular cutoff to the solution to the hard-sphere Boltzmann equation as $s \to \infty$.

C.2 Limit of the non-cutoff collision kernel

In this section, we study the limit of the kernel (C.1.3) as $s \to \infty$. Our first result contains the limit of the kernel as $s \to \infty$ as well as some uniform estimates. These estimates together with the ones in Section C.3 play a crucial role for the proof of the rigorous limit of a weak solution to the spatially homogeneous Boltzmann equation without angular cutoff to the one for the hard-sphere interaction, see Section C.4.

Theorem C.2.1. *Let us define the angular part of the collision kernel via*

$$b_s(\cos\theta) = 2^{4/(s-1)}\frac{\beta(\theta)}{\sin\theta}\beta'(\theta),\ \ s \geq 2.$$

(i) We have as $s \to \infty$

$$b_s(\cos\theta) \to \frac{1}{4}$$

locally uniformly for $\theta \in (0,\pi]$.

(ii) The following asymptotics holds

$$\lim_{\theta\to 0}\theta^{1+2/(s-1)}b_s(\cos\theta)\sin\theta = C_s, \quad C_s := \frac{2^{4/(s-1)}}{s-1}\left(\frac{\sqrt{\pi}\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s-1}{2}\right)}\right)^{2/(s-1)}.$$

(iii) Finally, we have the uniform bound

$$\sup_{s\geq 3}\sup_{\theta\in(0,\pi]}\theta^{1+2/(s-1)}b_s(\cos\theta)\sin\theta < \infty.$$

Remark C.2.2. Note that in (i) the limiting collision kernel corresponds to hard-sphere interactions. Writing the kernel (C.1.3) in terms of the angle $\varphi = (\pi-\theta)/2$ we get $|v-v_*|\cos\varphi\,\mathbb{1}_{\cos\varphi\geq 0}$ as $s \to \infty$.

Furthermore, in (ii) we have $C_s \to 0$ as $s \to \infty$. In fact,

$$\frac{\sqrt{\pi}\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s-1}{2}\right)} = \frac{s-1}{2}\frac{\sqrt{\pi}\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)} = \frac{s-1}{2}B\left(\frac{s}{2},\frac{1}{2}\right) = (s-1)W_{s-1},$$

where W_{s-1} is the Wallis integral. It is known that $\lim_{s\to\infty}\sqrt{s}W_{s-1} = \sqrt{\pi/2}$.

Finally, compare (iii) with [108, Section 20].

C.2.1 Rearrangement of the deviation angle

It is convenient to rearrange (C.1.2)

$$\varphi = x\int_0^1 \frac{dz}{\sqrt{1-z^{s-1}-x^2(z^2-z^{s-1})}}. \tag{C.2.1}$$

Here, we dropped the index zero in φ_0, x_0, used the change of variables $z = y/x_0$ and the fact that $x_0 = x$ is the positive root of

$$1 - x^2 - \frac{x^{s-1}}{\beta^{s-1}} = 0. \tag{C.2.2}$$

We recall that the deviation angle $\theta = \pi - 2\varphi$. One can see that the mappings $\beta \mapsto x$, $x \mapsto \varphi$ are strictly increasing and real analytic functions $[0,\infty) \to [0,1) \to [0,\pi/2)$ for each $s \geq 2$. We will use the index s to indicate that we consider the variable as a function.

C.2.2 Proof of Theorem C.2.1

Proof of Theorem C.2.1 (i). We first study the function $\varphi_s(x)$. The integrand can be written

$$\frac{1}{\sqrt{1-z^{s-1}-x^2(z^2-z^{s-1})}} = \frac{1}{\sqrt{1-z^{s-1}}\sqrt{1-x^2z^2+x^2z^2\left(1-\frac{1-z^{s-3}}{1-z^{s-1}}\right)}}$$
$$\leq \frac{1}{\sqrt{1-z}}\frac{1}{\sqrt{1-x^2z^2}}.$$

Here, we used that

$$1 - \frac{1-z^{s-3}}{1-z^{s-1}} \geq 0, \text{ for } s \geq 3.$$

This yields for any $x \in \mathbb{C}$ with $|x| \in [0, 1-\varepsilon]$, $\varepsilon > 0$ a uniform majorant, entailing locally uniform convergence,

$$s \to \infty, \ \varphi_s(x) \to \arcsin x.$$

As a consequence of the analyticity we have (x_s is the inverse of φ_s)

$$x_s(\varphi) \to \sin\varphi \text{ and } x_s'(\varphi) \to \cos\varphi$$

locally uniformly for $\varphi \in [0,\pi/2)$.

Next, we look at the functions (see (C.2.2))

$$\beta_s(x) = \frac{x}{(1-x^2)^{1/(s-1)}}, \quad \beta_s'(x) = \frac{2}{s-1}\frac{1}{(1-x^2)^{s/(s-1)}} + \frac{s-3}{s-1}\frac{1}{(1-x^2)^{1/(s-1)}}.$$

Hence, we have the locally uniform convergence for $x \in [0,1)$ as $s \to \infty$

$$\beta_s(x) \to x, \quad \beta_s'(x) \to 1.$$

We conclude with the above analysis

$$\begin{aligned} b_s(\cos\theta) &= \frac{1}{2}\,\frac{2^{4/(s-1)}}{\sin\theta}\,\beta_s\left(x_s\left(\frac{\pi-\theta}{2}\right)\right)\beta_s'\left(x_s\left(\frac{\pi-\theta}{2}\right)\right)x_s'\left(\frac{\pi-\theta}{2}\right) \\ &\to \frac{1}{2\sin\theta}\sin\left(\frac{\pi-\theta}{2}\right)\cos\left(\frac{\pi-\theta}{2}\right) = \frac{1}{4} \end{aligned} \tag{C.2.3}$$

locally uniformly for $\theta \in (0,\pi]$ as $s \to \infty$. Notice that $\varphi = (\pi-\theta)/2$ and the extra factor $1/2$ results from $d\varphi/d\theta = -1/2$.

Proof of Theorem C.2.1 (ii). We have the following equalities for $\varphi \in [0,\pi/2)$ and some $\psi \in (\varphi,\pi/2)$

$$\begin{aligned} 1 - x_s(\varphi) &= \varphi_s'(x_s(\psi))^{-1}\left(\frac{\pi}{2}-\varphi\right), \\ \beta_s(x) &= \frac{x}{(1+x)^{1/(s-1)}}\,(1-x)^{-1/(s-1)}, \\ \beta_s'(x) &= \frac{2}{(s-1)(1+x)^{s/(s-1)}}\,(1-x)^{-s/(s-1)} + \frac{s-3}{s-1}\,\frac{(1-x)^{-1/(s-1)}}{(1+x)^{1/(s-1)}}. \end{aligned} \tag{C.2.4}$$

Combining them yields

$$\begin{aligned} &\lim_{\varphi\to\pi/2}\left(\frac{\pi}{2}-\varphi\right)^{(s+1)/(s-1)}\beta_s(x_s(\varphi))\,\beta_s'(x_s(\varphi))\,x_s'(\varphi) \\ &\qquad = \frac{1}{2^{1/(s-1)}}\varphi_s'(1)^{1/(s-1)}\,\frac{2}{s-1}\,\frac{1}{2^{s/(s-1)}}\,\varphi_s'(1)^{s/(s-1)}\varphi_s'(1)^{-1} \\ &\qquad = \frac{1}{2^{2/(s-1)}}\,\frac{\varphi_s'(1)^{2/(s-1)}}{s-1}. \end{aligned}$$

Let us note that

$$\varphi_s'(x) = \int_0^1 \frac{1-z^{s-1}}{(1-z^{s-1}-x^2(z^2-z^{s-1}))^{3/2}}\,dz.$$

and as a consequence we have

$$\varphi_s'(1) = \int_0^1 \frac{1-z^{s-1}}{(1-z^2)^{3/2}}\,dz = \frac{\sqrt{\pi}\,\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s-1}{2}\right)}.$$

Using a similar expression as in (C.2.3) we get the asserted asymptotics.

Proof of Theorem C.2.1 (iii). For the last estimate we use (C.2.4). Note that φ_s' is increasing for $s \geq 3$, so that

$$\sup_{\varphi\in[0,\pi/2)} x_s'(\varphi) = \varphi_s'(0)^{-1}.$$

Note that

$$\varphi_s'(0) = \int_0^1 \frac{dz}{\sqrt{1-z^{s-1}}} \geq 1.$$

The last inequality follows from the fact that $s \mapsto \varphi_s'(0)$ is a decreasing function and $\varphi_s'(0) \to 1$ as $s \to \infty$. This implies $x_s' \le 1$. Using (C.2.4) for $x \in [0,1)$ we obtain

$$\beta_s(x)\beta_s'(x) \le \frac{2}{s-1}(1-x)^{(s+1)/(s-1)} + \frac{s-3}{s-1}(1-x)^{-2/(s-1)}.$$

Since φ_s' is increasing for $s \ge 3$ we have

$$(1-x_s(\varphi))^{-1} \le \varphi_s'(1)\left(\frac{\pi}{2}-\varphi\right)^{-1}.$$

We then obtain with the previous estimates

$$\begin{aligned}\left(\frac{\pi}{2}-\varphi\right)^{(s+1)/(s-1)} & \beta_s(x_s(\varphi))\beta_s'(x_s(\varphi))x_s'(\varphi) \\ & \le \frac{2}{s-1}\varphi_s'(1)^{(s+1)/(s-1)} + \frac{s-3}{s-1}\left(\frac{\pi}{2}-\varphi\right)\varphi_s'(1)^{2/(s-1)}. \qquad \text{(C.2.5)}\end{aligned}$$

One can see that

$$\varphi_s'(1) \le c(s-1),$$

for some constant $c > 0$. All in all, the right hand side in (C.2.5) is uniformly bounded in $s \ge 3$ and $\varphi \in [0, \pi/2]$. This implies the uniform bound. $\square$

This completes the proof of the limit of the non-cutoff collision kernel to the hard-sphere kernel. In the next section, we further study the behavior of b_s for $\theta \to 0$ when $s \to \infty$.

C.3 Asymptotics of the non-cutoff collision kernel

We now study the asymptotics of the singular layer of $b_s(\cos\theta)$ near $\theta \simeq 0$ when $s \to \infty$. To this end, we note that Theorem C.2.1 (ii) in combination with Remark C.2.2 yields

$$b_s(\cos\theta) \sim \frac{1}{s-1}\theta^{-2-2/(s-1)} \sim \frac{\theta^{-2}}{s} \qquad \text{as } s \to \infty.$$

Thus, we need to look at the scaled function

$$\psi \mapsto b_s(\cos(\psi/\sqrt{s})),$$

with $\theta = \psi/\sqrt{s}$. In the following, we use this scaling to compute the limit $s \to \infty$. First, we derive a similar formula to (C.2.1). Note that

$$\varphi = \frac{\pi}{2} - \frac{\theta}{2} = \frac{\pi}{2} - \frac{\psi}{2\sqrt{s}}.$$

Let us define

$$\frac{\xi_s(\psi)}{2s} := 1 - x_s\left(\frac{\pi}{2} - \frac{\psi}{2\sqrt{s}}\right), \qquad \text{(C.3.1)}$$

where ξ_s is defined for $\psi \in [0, \pi\sqrt{s}]$. The inverse function for $\xi \in [0, 2s]$ is given by

$$\begin{aligned}
\psi_s(\xi) &= 2\sqrt{s}\left[\frac{\pi}{2} - \varphi_s\left(1 - \frac{\xi}{2s}\right)\right] \\
&= 2\sqrt{s}\int_0^1 \left(\frac{1}{\sqrt{1-z^2}} - \frac{1}{\sqrt{1 - z^{s-1} - \left(1 - \frac{\xi}{2s}\right)^2 (z^2 - z^{s-1})}}\right) dz \\
&\qquad + \frac{\xi}{\sqrt{s}} \int_0^1 \frac{1}{\sqrt{1 - z^{s-1} - \left(1 - \frac{\xi}{2s}\right)^2 (z^2 - z^{s-1})}} dz. \qquad \text{(C.3.2)}
\end{aligned}$$

Notice that in the last equality we used the definition of φ_s in (C.2.1). Note that ψ_s is an analytic function on $(0, 2s)$. With this we can state the asymptotic behavior.

Theorem C.3.1. *The angular part $b_s(\cos\theta)$, $s \geq 2$, satisfies the following asymptotic limit*

$$\lim_{s\to\infty} b_s\left(\cos\left(\frac{\psi}{\sqrt{s}}\right)\right) = \Phi(\psi),$$

which holds locally uniformly for $\psi \in (0,\infty)$. Here, $\Phi : (0,\infty) \to \mathbb{R}$ is real analytic satisfying

$$\lim_{\psi\to\infty} \Phi(\psi) = \frac{1}{4}. \qquad \text{(C.3.3)}$$

Furthermore, we have

$$\Phi(\psi) = \frac{1}{\psi^2} + \frac{1}{\sqrt{\pi}}\frac{1}{\psi} + \Phi_0(\psi), \qquad \text{(C.3.4)}$$

where $\Phi_0 : [0,\infty) \to \mathbb{R}$ is continuous.

Remark C.3.2. Note that the singularity $1/\psi^2$ of Φ for $\psi \to 0$ is consistent with the asymptotics in Theorem C.2.1 (ii), since $sC_s \to 1$ as $s \to \infty$. Furthermore, the result of the limit $\psi \to \infty$ coincides with Theorem C.2.1 (i).

Proof of Theorem C.3.1. The proof consists of the following 4 steps.
Step 1. We first derive the limits

$$\lim_{s\to\infty} \psi_s(\xi) = \psi_\infty(\xi) = 2\xi \int_0^\infty \frac{1 - e^{-\zeta}}{\sqrt{2\zeta}\sqrt{h(\zeta,\xi)}(\sqrt{2\zeta} + \sqrt{h(\zeta,\xi)})} d\zeta, \qquad \text{(C.3.5)}$$

$$\lim_{s\to\infty} \psi_s'(\xi) = \psi_\infty'(\xi) = \int_0^\infty \frac{1 - e^{-\zeta}}{h(\zeta,\xi)^{3/2}} d\zeta, \qquad \text{(C.3.6)}$$

where

$$h(\zeta,\xi) = 2\zeta + \xi\left(1 - e^{-\zeta}\right).$$

To this end we choose $\xi \in [0,\infty)$ and assume s large enough such that $\xi \in [0, 2s]$. Let us write

$$1 - z^{s-1} - \left(1 - \frac{\xi}{2s}\right)^2 (z^2 - z^{s-1}) = 1 - z^2 + \left(\frac{\xi}{s} - \frac{\xi^2}{4s^2}\right)(z^2 - z^{s-1}) =: g_s(z,\xi).$$

Since $g_s \geq 1-z^2$ the second integral in (C.3.2) goes to zero as $s \to \infty$. The first term in (C.3.2) can be rearranged to get

$$2\sqrt{s}\int_0^1 \frac{(\xi/s-\xi^2/4s^2)(z^2-z^{s-1})}{\sqrt{1-z^2}\sqrt{g_s(z,\xi)}(\sqrt{1-z^2}+\sqrt{g_s(z,\xi)})}\,dz =: I_s(\xi)$$

We now perform the change of variables $z = 1-\zeta/s$ to get with

$$\begin{aligned}
1-\left(1-\frac{\zeta}{s}\right)^2 &= \frac{1}{s}\left(2\zeta-\frac{\zeta^2}{s}\right),\\
g_s\left(1-\frac{\zeta}{s},\xi\right) &= \frac{1}{s}\left(2\zeta-\frac{\zeta^2}{s}\right)+\frac{1}{s}\left(\xi-\frac{\xi^2}{4s}\right)\left(\left(1-\frac{\zeta}{s}\right)^2-\left(1-\frac{\zeta}{s}\right)^{s-1}\right)\\
&=: \frac{1}{s}h_s(\zeta,\xi),
\end{aligned}$$

and the formula

$$I_s(\xi) = \left(2\xi-\frac{\xi^2}{2s}\right)\int_0^s \frac{(1-\zeta/s)^2-(1-\zeta/s)^{s-1}}{\sqrt{2\zeta-\zeta^2/s}\sqrt{h_s(\zeta,\xi)}(\sqrt{2\zeta-\zeta^2/s}+\sqrt{h_s(\zeta,\xi)})}\,d\zeta. \tag{C.3.7}$$

Using that $\zeta \leq s$ and $\xi \leq 2s$ we can obtain

$$2\zeta-\frac{\zeta^2}{s} \geq \zeta,$$

and

$$\left(1-\frac{\zeta}{s}\right)^2-\left(1-\frac{\zeta}{s}\right)^{s-1} \geq 0.$$

Hence, we have $h_s(\zeta,\xi) \geq \zeta$. In addition, we also have

$$\left(1-\frac{\zeta}{s}\right)^2-\left(1-\frac{\zeta}{s}\right)^{s-1} \leq \min\left\{1, \frac{s-3}{s}\zeta\left(1-\frac{\zeta}{s}\right)^2\right\} \leq \min\{1,\zeta\}.$$

Thus, the integrand in (C.3.7) can be estimated by

$$\min\left\{\frac{1}{2\sqrt{\zeta}}, \frac{1}{2\zeta^{3/2}}\right\}.$$

In conjunction with

$$\lim_{s\to\infty} h_s(\zeta,\xi) = 2\zeta+\xi\left(1-e^{-\zeta}\right) = h(\zeta,\xi)$$

we conclude the locally uniform convergence

$$\lim_{s\to\infty} \psi_s(\xi) = \psi_\infty(\xi),$$

where ψ_∞ is given in (C.3.5). Since the above estimates also hold in a neighborhood of $\xi \in (0,\infty)$ in the complex plane, the limit is real analytic. A calculation allows to derive the formula (C.3.6). Alternatively, one can compute the derivative of (C.3.2) and mimic the preceding computation.

Step 2. Since $\psi'_\infty > 0$ we also have from the analyticity and the locally uniform convergence

$$\xi_s(\psi) \to \xi_\infty(\psi) = \psi_\infty^{-1}(\psi), \quad \xi'_s(\psi) \to \xi'_\infty(\psi) = \frac{1}{\psi'_\infty(\xi_\infty(\psi))},$$

locally uniformly for $\psi \in (0,\infty)$. Furthermore, by (C.3.1)

$$\lim_{s\to\infty} x_s\left(\frac{\pi}{2} - \frac{\psi}{2\sqrt{s}}\right) = \lim_{s\to\infty} 1 - \frac{\xi_s(\psi)}{2s} = 1.$$

This yields with the definition of $b_s(\cos(\psi/s))$, cf. (C.2.3) and formulas (C.2.4),

$$\begin{aligned}
&\lim_{s\to\infty} b_s\left(\cos\left(\frac{\psi}{\sqrt{s}}\right)\right)\\
&\quad= \lim_{s\to\infty} \frac{1}{2}\frac{1}{\sin(\psi/\sqrt{s})}\frac{2}{(s-1)}\frac{1}{2}\left(1 - x_s\left(\frac{\pi}{2} - \frac{\psi}{2\sqrt{s}}\right)\right)^{-(s+1)/(s-1)} x'_s\left(\frac{\pi}{2} - \frac{\psi}{2\sqrt{s}}\right)\\
&\qquad\qquad + \lim_{s\to\infty} \frac{1}{2}\frac{1}{\sin(\psi/\sqrt{s})}\left(1 - x_s\left(\frac{\pi}{2} - \frac{\psi}{2\sqrt{s}}\right)\right)^{-2/(s-1)} x'_s\left(\frac{\pi}{2} - \frac{\psi}{2\sqrt{s}}\right).
\end{aligned}$$

Using a Taylor expansion we can replace $\sin(\psi/\sqrt{s})$ by $\psi/\sqrt{s}$ without modifying the value of the limit. We use (C.3.1) and

$$x'_s\left(\frac{\pi}{2} - \frac{\psi}{2\sqrt{s}}\right) = \frac{1}{\sqrt{s}}\xi'_s(\psi),$$

which is a consequence of (C.3.1), to obtain

$$\lim_{s\to\infty} b_s\left(\cos\left(\frac{\psi}{\sqrt{s}}\right)\right) = \frac{\xi'_\infty(\psi)}{\xi_\infty(\psi)\psi} + \frac{\xi'_\infty(\psi)}{2\psi} =: \Phi(\psi). \tag{C.3.8}$$

Step 3. We now use a Taylor approximation for (C.3.8). It is convenient to define

$$\psi_\infty(\xi) = 2\xi J(\xi), \quad f(\psi) := 2\,\xi'_\infty(\psi)\,J(\xi_\infty(\psi)).$$

Here, $J(\xi)$ is the integral in (C.3.5). This yields

$$\xi_\infty(\psi) = \frac{\psi}{2J(\xi_\infty(\psi))}, \quad \Phi(\psi) = \frac{f(\psi)}{\psi^2} + \frac{\xi'_\infty(\psi)}{2\psi}.$$

We then have

$$\Phi(\psi) = \frac{f(0)}{\psi^2} + \frac{f'(0) + \xi'_\infty(0)/2}{\psi} + \frac{1}{\psi}\left(\frac{f(\psi) - f(0) - f'(0)\psi}{\psi} + \frac{\xi'_\infty(\psi) - \xi'_\infty(0)}{2}\right),$$

which defines Φ_0. The following formulas hold

$$\xi'_\infty(0) = \sqrt{\frac{2}{\pi}}, \quad f(0) = 1, \quad f'(0) = \frac{\sqrt{2}-1}{\sqrt{2\pi}}. \tag{C.3.9}$$

With this we derive

$$f'(0) + \frac{\xi'_\infty(0)}{2} = \frac{1}{\sqrt{\pi}},$$

which yields the expression in (C.3.4).

The formulas (C.3.9) can be calculated without difficulty, since the integrals are well-defined. For instance,

$$2J(0) = \psi'_\infty(0) = \frac{1}{\xi'_\infty(0)} = \int_0^\infty \frac{1-e^{-\zeta}}{(2\zeta)^{3/2}}\,d\zeta = \int_0^\infty \frac{e^{-\zeta}}{\sqrt{2\zeta}}\,d\zeta = \sqrt{\frac{\pi}{2}}.$$

Step 4. Finally, for the limit in (C.3.3) we have with (C.3.8)

$$\lim_{\psi\to\infty} \Phi(\psi) = \lim_{\xi\to\infty}\left(\frac{1}{2\xi^2\,J(\xi)\,\psi'_\infty(\xi)} + \frac{1}{4\xi\,J(\xi)\,\psi'_\infty(\xi)}\right).$$

We prove below that

$$\lim_{\xi\to\infty} \sqrt{\xi}\psi'_\infty(\xi) = \lim_{\xi\to\infty} \sqrt{\xi}J(\xi) = 1,$$

which implies the assertion. For the preceding two limits we use the change of variables $\zeta = \xi z$ to get

$$\sqrt{\xi}\,\psi'_\infty(\xi) = \int_0^\infty \frac{1-e^{-\xi z}}{(2z+1-e^{-\xi z})^{3/2}}\,dz.$$

The integrand can be estimated by (we use here $\xi \geq 1$ say)

$$\min\left\{\frac{1}{z^{3/2}}, \frac{1}{\sqrt{1-e^{-\xi z}}}\right\} \leq \min\left\{\frac{1}{z^{3/2}}, \frac{1}{\sqrt{1-e^{-z}}}\right\}.$$

Hence, we can use the dominated convergence theorem to obtain the stated limit. A similar computation applies to $\sqrt{\xi}J(\xi)$. This concludes the proof. □

This completes the proof of the asymptotics of the singularity for $\theta \simeq 0$ as $s \to \infty$. In the next section, we provide a proof of the limit of solutions to the spatially homogeneous Boltzmann equation without cutoff to solutions of the homogeneous Boltzmann equation for hard-spheres using the estimates in Sections C.2 and C.3.

C.4 Convergence of the solution for the homogeneous Boltzmann equation

In this section, we consider the spatially homogeneous Boltzmann equation

$$\partial_t f = Q(f,f), \quad f(0,\cdot) = f_0(\cdot) \tag{C.4.1}$$

with collision kernel $B_s(|v-v_*|, n\cdot\sigma)$, $s>2$, given in (C.1.3). Let us first recall the following well-posedness result for cutoff kernels with hard potentials $\gamma \in (0,1]$ (e.g. hard-sphere corresponding to $s=\infty$), see [129, Theorem 1.1] and [156, Section 3.7, Theorem 3]. The first well-posedness results are due to Arkeryd [14, 15]. We use here the weighted spaces L^1_p with weight function $(1+|v|^2)^{p/2}$.

Lemma C.4.1. *Let $f_0 \in L^1_2$, then there is a unique solution $f \in C([0,\infty); L^1_2)$ to* (C.4.1) *which preserves energy, i.e. for all $t \geq 0$*

$$\int_{\mathbb{R}^3} |v|^2 f(t,v)\,dv = \int_{\mathbb{R}^3} |v|^2 f_0(v)\,dv.$$

Remark C.4.2. Let us mention that the condition of the energy conservation is essential for uniqueness [122, 161].

Next, we consider the non-cutoff kernel B_s. Since we are interested in the limit $s \to \infty$, we can assume $s > 5$ so that

$$\gamma(s) = \frac{s-5}{s-1} > 0, \quad \int_0^\pi \theta \, b_s(\cos\theta) \sin\theta \, d\theta \leq c_0, \tag{C.4.2}$$

where the constant c_0 is independent of $s > 5$, see Theorem C.2.1 (iii). In this case, we can use the weak formulation of (C.4.1) by testing with functions $\psi \in C_b^1([0,\infty) \times \mathbb{R}^3)$, see e.g. [156, Section 4.1]. The collision operator can be define by means of the pre-postcollisional change of variables

$$\int_{\mathbb{R}^3} Q_s(f,f)(v)\,\psi(v)\,dv = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |v-v_*|^\gamma f f_* \int_{S^2} b_s(\cos\theta)\,(\psi' - \psi)\,d\sigma dv_* dv.$$

For the integral on the sphere we have, via a Taylor approximation,

$$\left| \int_{S^2} b_s(\cos\theta)\,(\psi' - \psi)\,d\sigma \right| \leq C_0 \,\|\psi\|_{C^1(\mathbb{R}^3)}\, |v - v_*|,$$

for some constant $C_0 > 0$ independent of $s > 5$. Let us also define the entropy of f

$$H(f) = \int_{\mathbb{R}^3} f \ln f \, dv.$$

We also recall the existence of weak solutions to the homogeneous Boltzmann equation, which is the content of the following lemma, see e.g. [155, Section 4] and [156, Section 4.7, Theorem 9 (ii)]. With a slight abuse of notation we write $f^s(t,v)$ and $f^\infty(t,v)$ to describe the solutions to the Boltzmann equations with kernels B_s and B_∞, respectively.

Lemma C.4.3. *Let $f_0 \in L^1_{1+\gamma+\delta}$, for $\delta > 0$ arbitrary, with finite entropy. Under the conditions* (C.4.2) *there is a weak solution $f^s \in L^\infty([0,\infty); L^1_{1+\gamma+\delta})$ to* (C.4.1) *which preserves energy. Furthermore, we have $H(f^s(t)) \leq H(f_0)$ for all $t \geq 0$.*

We finally have the following convergence result.

Theorem C.4.4. *Let $f_0 \in L^1_p$ with finite entropy and arbitrary $p > 2$. Consider a sequence of weak solutions f^s to* (C.4.1) *as in Lemma C.4.3 with collision kernel B_s, $s > 5$. Then, $f^s(t) \rightharpoonup f^\infty(t)$ weakly in L^1 for all $t \geq 0$ as $s \to \infty$, where f^∞ is the unique solution to* (C.4.1) *for hard-sphere interactions.*

Proof of Theorem C.4.4. First of all, applying a version of the Povzner estimate (see [129, Lemma 2.2] which is also applicable for non-cutoff kernels, cf. [156, Appendix]) we have

$$\sup_{t\in[0,\infty)} \|f^s(t)\|_{L^1_p} \leq C(\|f_0\|_{L^1_p}) =: C_p. \tag{C.4.3}$$

This estimate is independent of s as long as s is sufficiently large. Assume for example $s > 6$. In fact, in the Povzner estimate we only need a uniform lower and upper bound on the angular part $b_s(\cos\theta)$. This is ensured by Theorem C.2.1 items (i) and (iii). Also note that for, say, $s > 6$ we have $\gamma(s) \geq 1/5$. Furthermore, from the weak formulation we also obtain

$$\left| \int_{\mathbb{R}^3} \psi(v) f^s(t_1,v)\,dv - \int_{\mathbb{R}^3} \psi(v) f^s(t_2,v)\,dv \right| \leq C\,\|\psi\|_{C^1}\,|t_1 - t_2|,$$

for all $t_1, t_2 \geq 0$. Here, the constant C is independent of $s > 6$ due to (C.4.2) and (C.4.3). By the uniform entropy bound

$$H(f^s(t)) \leq H(f_0),$$

and the previous weak equicontinuity property we can apply the Dunford-Pettis theorem yielding

$$f^{s_n}(t) \rightharpoonup f^{\infty}(t),$$

weakly in L^1 for all $t \geq 0$ for a subsequence $s_n \to \infty$.

Using Theorem C.2.1, items (i) and (iii), we can pass to the limit in the weak formulation. Hence, f^{∞} is a weak solution to (C.4.1) for hard-sphere interactions. Since there is no angular singularity, one can infer

$$f^{\infty} \in C([0,\infty), L^1_2).$$

By the uniform moment bound (C.4.3), the second moments also converge for all $t \geq 0$ as $s_n \to \infty$. As a consequence f^{∞} preserves energy and thus f^{∞} is the unique solution in Lemma C.4.1. This implies that the whole sequence converges $f^s(t) \rightharpoonup f^{\infty}(t)$ as $s \to \infty$. □

C.4.1 Conclusion

We proved the convergence of the collision kernel for inverse power law interactions $1/r^{s-1}$ to the hard-sphere kernel as $s \to \infty$. We furthermore studied the asymptotics of the angular singularity $\theta \to 0$. Finally, solutions to the homogeneous Boltzmann equation converge respectively.

Appendix D

Rotating solutions to the incompressible Euler-Poisson equation with external particle

Abstract

We consider a two-dimensional, incompressible fluid body, together with self-induced interactions. The body is perturbed by an external particle with small mass. The whole configuration rotates uniformly around the common center of mass. We construct solutions, which are stationary in a rotating coordinate system, using perturbative methods. In addition, we consider a large class of internal motions of the fluid. The angular velocity is related to the position of the external particle and is chosen to satisfy a non-resonance condition.

D.1 Introduction and previous results

The shape of fluid objects due to the combination of rotational and self-gravitating forces is a classical research field which has been extensively considered for different fluid models. In particular, a detailed description of the historical evolution of the field can be found [52] for the (three-dimensional) incompressible Euler equations. Further results were established by Lichtenstein [112]. For the case of compressible fluids we refer to the works [18, 53, 89, 100, 101, 102, 111, 113, 123, 124, 147, 148] and references therein. A kinetic model, namely the Vlasov-Poisson equation, has been studied as well, see e.g. [65, 102]. In fact, there is a relation between steady states of the Vlasov-Poisson equation and the compressible Euler equation, see [143] and references therein for an overview of the variational methods used in these problems.

In this paper, we consider a two-dimensional, self-interacting, incompressible fluid body modeled by the Euler equations. Furthermore, we study the problem of deformations of the geometry when it is perturbed by some external particle. The fluid body and the external particle are assumed to rotate around their center of mass. This problem (adding a small particle) can be seen as a test of stability of the rotating solutions and also as a simple model of tides. Furthermore, differently from the results reviewed in [52] (excluding the figures studied by Riemann), we construct solutions of the Euler-Poisson equation for which the fluid velocity is in general different from zero in any coordinate system. Recently, in [24], the authors studied the stability of solutions for long times in suitable functional spaces close to the equilibrium states of an inviscid, incompressible, and irrotational fluid, subject to the self-gravitational force.

In this work, we study a family of interaction potentials including the classical (Newtonian) gravitational forces. The latter can be interpreted as an extremely simplified model for galaxies. However, this does not correspond to a three-dimensional problem restricted to planar geometries. The reason being that the pressure would necessary act only in the plane which contains the fluid body as well as the external particle. Nevertheless, such a model can be considered in the case of the Vlasov-Poisson equation, assuming that the velocities of the particles are contained only in the same plane as the fluid. In this situation the tensor describing the pressure is anisotropic and it yields zero forces in the direction perpendicular to the plane but not in the horizontal direction, cf. [140].

Since we consider a two-dimensional fluid body we can apply two tools that cannot be employed in three dimensional problems. Specifically, we use conformal mappings as well as the Grad–Shafranov method [77, 146]. We restrict ourselves to the two-dimensional setting since the corresponding three-dimensional version requires to understand some small denominator problem which cannot be tackled with the methods employed in this article.

Beside the problem treated here, a variety of different free-boundary problems arising in fluid mechanics have been studied in the last decades. For instance, the problem of jets and cavities with or without gravity has been studied in [11, 12, 13] and the theory of gravity water waves has been developed in several works, cf. [95, 162, 163]. Let us also highlight the recent survey [85] that covers the mathematical theory of the steady water waves problem. A question that has been discussed in [85, Section 6.2] is the effect on the free-boundary of the presence of point-vortices. This question is different from the one treated in this article but has some mathematical analogies.

An important difference between the previous free-boundary problems and the one studied in this paper is that the interacting force (e.g. gravity) is due to the fluid itself. Another type of problems that have some similarities with the one considered in this article are those related to the theory of rotating vortex patches. The first rigorous result was shown by Burbea [38] where he constructed rotating vortex patches close to the disk by means of the classical Crandall-Rabinowitz bifurcation approach. A more thorough study of rotating vortex patches can be found in [84, 92] and the references therein.

D.1.1 Setting of the problem

We are concerned with a flat incompressible fluid body with density $\rho = \mathbb{1}_E$. Here, $\mathbb{1}_E$ denotes the indicator function of the set E. The shape of the body $E(t) \subset \mathbb{R}^2$ has a smooth boundary, is simple connected and close to a disk, see below for the precise meaning of this. We also include a particle $X = X(t) \in \mathbb{R}^2$ with small mass m. However, we consider only situations in which the particle and the fluid body are at a positive distance. The velocity field v of the fluid body then satisfies the following free-boundary problem for the Euler-Poisson system

$$\begin{cases} \partial_t v + (v\cdot\nabla)v = -\nabla p - \nabla U_{E(t)} - m\nabla U_{X(t)} & \text{in } E(t), \\ \nabla\cdot v = 0 & \text{in } E(t), \\ n\cdot v = V_N & \text{on } \partial E(t), \end{cases} \tag{D.1.1}$$

where V_N is the normal velocity of the interface $\partial E(t)$, n the outer unit normal vector of $\partial E(t)$ and ∇ denotes the classical gradient operator in two-dimensions, namely, $\nabla = (\partial_{x_1}, \partial_{x_2})$. Here, $U_{E(t)}$ and $U_{X(t)}$ are the gravitational potentials, see below for the precise definitions.

Furthermore, $p = p(t,x)$ is the scalar pressure which describes the internal pressure of the body for $x \in E(t)$ and the external pressure of the surrounding space for $x \in \mathbb{R}^2 \backslash E(t)$. We assume the external pressure to be constant on $\mathbb{R}^2 \backslash E(t)$ and without of generality we can take this constant to be zero. This reflects that the configuration is surrounded by a uniform medium. Therefore, the continuity of the pressure at the interface that separates the liquid from the exterior implies that

$$p = 0 \quad \text{on } \partial E(t). \tag{D.1.2}$$

Since there are no external forces acting on the configuration described by the fluid body and the external particle, their common center of mass moves at constant speed. Consequently, we can assume without loss of generality (using a change of the coordinate system) that the center of mass is at zero, i.e.

$$\int_{E(t)} x\, dx + mX(t) = 0. \tag{D.1.3}$$

As mentioned in the introduction we study two cases for the potentials $U_{E(t)}$ and $U_{X(t)}$ in (D.1.1).

(A) We consider a family of power law potentials, more precisely for $\nu \in (0,1]$ we define

$$U_{X(t)}(x) := -\frac{1}{|x - X(t)|^\nu}, \quad U_{E(t)}(x) := -\int_{E(t)} \frac{dy}{|x-y|^\nu}. \tag{D.1.4}$$

(B) We consider potentials given via the fundamental solution of the (two-dimensional) Laplace operator, i.e.

$$U_{X(t)}(x) := \ln|x - X(t)|, \quad U_{E(t)}(x) := \int_{E(t)} \ln|x-y|\, dy. \tag{D.1.5}$$

Note that in both cases the signs are chosen to yield attractive forces. Furthermore, Case (A) with $\nu = 1$ can be interpreted as Newtonian gravitational interactions.

Let us mention here that in Case (A) with $\nu = 1$ some care is needed in order to define a solution to (D.1.1) since the gradient $\nabla U_{E(t)}$ is not well-defined due to the onset of a singularity. However, this does not suppose a problem since the pressure gradient ∇p has also a similar singularity with a reverse sign that compensates the singularity of $\nabla U_{E(t)}$. In order to avoid this singular terms, it is convenient to rewrite the problem (D.1.1) substracting the hydrostatic pressure. To this end, we define $p = P - U_{E(t)} - mU_{X(t)}$ where P is the non-hydrostatic pressure. Then the system (D.1.1) turns into

$$\begin{cases} \partial_t v + (v\cdot\nabla)v = -\nabla P & \text{in } E(t), \\ \nabla\cdot v = 0 & \text{in } E(t), \\ n\cdot v = V_N & \text{on } \partial E(t), \\ P = U_{E(t)} + mU_{X(t)} & \text{on } \partial E(t), \end{cases} \tag{D.1.6}$$

where the last equation follows from (D.1.2). Now, these equations do not contain singular terms.

The solutions to (D.1.6) studied in this paper are classical solutions, i.e. $v : E(t) \to \mathbb{R}^2$ and $\partial E(t)$ are regular. However, the function $P : \overline{E(t)} \to \mathbb{R}$ is in general only continuous, i.e. in

Case (A) the gradient ∇P is not defined on $\partial E(t)$. As we will see in the next section, this condition of continuity of the pressure and the last equation in (D.1.6) yields an equation for the free-boundary.

Furthermore, the solutions constructed in this paper occur as perturbations of solutions to the time-independent equation with $m = 0$, that is

$$\begin{cases} (v \cdot \nabla)v = -\nabla P & \text{in } E, \\ \nabla \cdot v = 0 & \text{in } E, \\ n \cdot v = 0 & \text{on } \partial E, \\ P = U_E & \text{on } \partial E. \end{cases} \tag{D.1.7}$$

One particular solution we consider is given by the unit disk $E = \mathbb{D}$ together with a corresponding velocity field v and the non-hydrostatic pressure P.

In addition, we assume that the perturbed fluid body and the external particle solving (D.1.1) rotate around their center of mass with angular speed of rotation $\Omega_0 > 0$. Furthermore, we look for configurations which are time-independent in a rotating frame at angular speed Ω_0, see Figure D.1. Changing to such a rotating coordinate system we obtain the equations

$$\begin{cases} (v \cdot \nabla)v + 2\Omega_0 J v - \Omega_0^2 x = -\nabla P & \text{in } E, \\ \nabla \cdot v = 0 & \text{in } E, \\ n \cdot v = 0 & \text{on } \partial E, \\ P = U_E + m U_X & \text{on } \partial E, \\ \Omega_0^2 X = \nabla U_E(X) & \\ |E| = \pi & \\ \int_E x \, dx + mX = 0. & \end{cases} \tag{D.1.8}$$

In equations (D.1.8) we used the matrix J defined by

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{D.1.9}$$

which encodes the action of the vector product in the two-dimensional case.

Notice that in this setting, the shape of the body E, the velocity field v and the position of the particle X do not depend on time. Furthermore, we construct solutions $v \neq 0$, which can be interpreted as some type of tidal waves induced by the gravity of the external particle as well as the velocity of the unperturbed fluid.

We briefly comment on the system of equations (D.1.8). First, note that the terms $2\Omega_0 J v$ and $-\Omega_0^2 x$ represent the Coriolis and the centrifugal forces, respectively, which appear in the rotating frame of reference. The third equation in (D.1.8) ensures that the free-boundary is stationary, i.e. the fluid inside the body does not move across the boundary. As stated above, the external pressure is assumed to be constant outside the body. The equation $\Omega_0^2 X = \nabla U_E(X)$ follows from Newton's law and ensures that the external particle is at rest. Note that $\nabla U_E(X)$ is now well-defined also in Case (A), since we consider only cases with X separated from E. The centrifugal force acting on X balances with the gravitational force of the fluid body. In addition, for definiteness, we assume that the total mass of the fluid is $\pi = |\mathbb{D}|$. The last equation in (D.1.8) ensures that the center of mass is at the origin. In fact, as we will see in the proof of

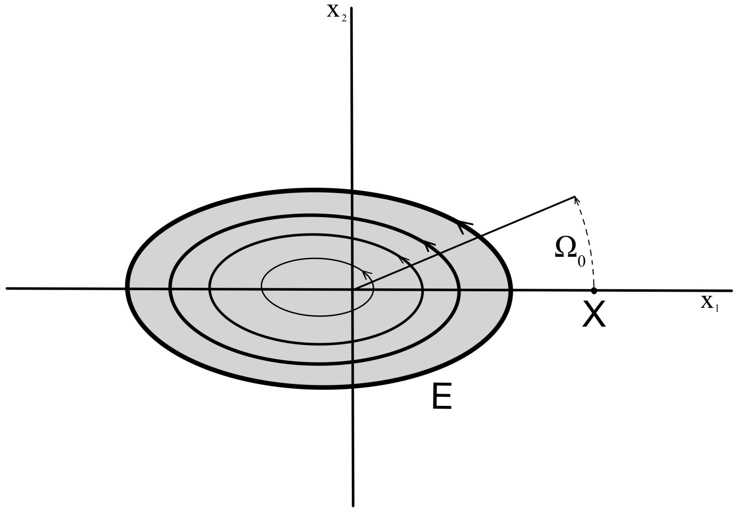

Figure D.1: Configuration of the fluid body E and the external particle X. Both rotate around their common center of mass (at the origin) with angular speed Ω_0.

our main result (see Section D.6) this last equation in (D.1.8) follows from the other equations in (D.1.8).

Finally, let us mention that equations (D.1.8) are invariant under rotations around the origin. Hence, we can assume w.l.o.g. that the particle $X = (a, 0)$ is located on the x_1-axis. In particular, a solution to (D.1.8) yields a family of solutions by applying rotations.

In this paper, we construct solutions to (D.1.8) obtained as perturbation of solutions to (D.1.7) with $E = \mathbb{D}$ by means of an implicit function theorem in Hölder spaces. We require a non-resonance condition on Ω_0 and a non-degeneracy condition on the unperturbed velocity field solving (D.1.7), see Theorem D.2.1 and Corollary D.2.2.

The paper is organized as follows. In Section D.2 we reformulate the problem using Grad–Shafranov, the Bernoulli equation and conformal mappings to derive a reduced system of equations that will be more amenable to mathematical analysis. These new system is solved using an implicit function theorem. To this end, we provide some preliminary results concerning conformal mapping properties, estimates for elliptic equations, as well as suitable representations of the gravitational potentials in Section D.3. In Section D.4 we prove the Fréchet differentiability of the reduced system of equations w.r.t. the unknowns of the problem. Furthermore, we prove the invertibility of the Fréchet derivative at the unperturbed solution in Section D.5. Finally, we conclude the article with the proof of the main results in Section D.6.

D.2 Reformulation of the problem and main result

In this section, we reduce the problem (D.1.8) to a set of equations that will be studied in the main part of the paper. To this end, we apply in particular conformal mappings as well as the Grad–Shafranov method.

Conformal mappings

We use conformal mappings, i.e. bijective analytic functions, to parameterize the domain of the fluid. Recall that by the Riemann mapping theorem for any simply connected domain $E \subset \mathbb{C}$ one can find a conformal mapping $f : \mathbb{D} \to E$. Here, we identify $\mathbb{C}$ with $\mathbb{R}^2$ via $z = x_1 + ix_2$. In the case of smooth domains the mapping extends conformally to $\overline{\mathbb{D}} \to \overline{E}$.

In our study, we consider conformal mappings of the form $f_h : \mathbb{D} \to \mathbb{R}^2$, $f_h(z) = z + h(z)$, where h is small such that the domain is close to the disk. Let us mention that under a general smallness condition on some arbitrary analytic function $h : \mathbb{D} \to \mathbb{C}$ the mapping f_h is conformal, see Lemma D.3.1. We denote the corresponding domain by $E_h = f_h(\mathbb{D})$ to emphasize the dependence on h. Accordingly, we use the notation $U_h = U_{E_h}$. Furthermore, we denote by f_h' the complex derivative, i.e. understanding f_h as a mapping $\mathbb{D} \subset \mathbb{C} \to \mathbb{C}$.

Let us also introduce the so-called Blaschke factors, see [144], defined by

$$b_{c,d}(z) = d\frac{z-c}{1-\bar{c}z}, \quad c \in \mathbb{D}, d \in \mathbb{C}, |d| = 1. \tag{D.2.1}$$

These factors are the only conformal mappings $\mathbb{D} \to \mathbb{D}$. Choosing c, d accordingly allows to set $h(0) = 0$ and $h'(0) \in \mathbb{R}$ by replacing f_h by $f_h \circ b_{c,d}$. This defines the conformal mapping f_h and hence also h uniquely.

Grad–Shafranov method

In order to construct the velocity field v solving (D.1.8) we use the Grad–Shafranov method. Roughly speaking, the Grad-Shafranov approach allows to transform the problem (D.1.8) to an elliptic problem for the stream function. These ideas have been very useful for constructing solutions in different problems arising in plasma physics. For instance, to prove flexibility and rigidity results in magneto-hydrostatics (cf. [55, 56, 83]) or studying boundary value problems (cf. [10]). In the next paragraphs, we will recall the key ideas of this approach.

In this paper we are interested only in two dimensional vector fields $v = (v_1, v_2)$. However, in order to use classical formulas for fluid mechanics in three dimensions it is convenient to think in those vector fields as three dimensional fields with zero third component, namely, $\tilde{v} = (v_1, v_2, 0)$. Therefore, the vorticity associated to this vector field $\tilde{v}$ is denoted by $\tilde{\omega}$, i.e, $\tilde{\omega} = \tilde{\nabla} \times \tilde{v}$. Here we use the notation $\tilde{\nabla} = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3}) = (\nabla, \partial_{x_3})$. Due to the form of the vector field $\tilde{v}$, it turns out that $\tilde{\omega} = (0, 0, \omega(x))$ with $x = (x_1, x_2)$. Similarly, the vector angular velocity is denoted by $\tilde{\Omega} = (0, 0, \Omega_0)$. We denote as $\mathcal{P} : \mathbb{R}^3 \to \mathbb{R}^2$ the projector given by

$$\mathcal{P}(y_1, y_2, y_3) = (y_1, y_2), \ \forall y = (y_1, y_2, y_3) \in \mathbb{R}^3.$$

Using the classical formula

$$\tilde{v} \times \tilde{\omega} = -(\tilde{v} \cdot \tilde{\nabla})\tilde{v} + \frac{1}{2}\tilde{\nabla}(|\tilde{v}|^2),$$

as well as

$$\mathcal{P}(\tilde{v} \times \tilde{\omega}) = -\omega J v,$$

we infer that

$$-vJ\omega = -(v \cdot \nabla)v + \frac{1}{2}\nabla(|v|^2).$$

Recall that J is the matrix defined in (D.1.9).

Hence, the first three equations in (D.1.8) can be written as

$$\begin{cases} -(\omega+2\Omega_0)Jv = \nabla H & \text{in } E_h, \\ \nabla\cdot v = 0 & \text{in } E_h, \\ n_h\cdot v = 0 & \text{on } \partial E_h. \end{cases} \tag{D.2.2}$$

Here, H is called the Bernoulli head and is defined by

$$H := P + \frac{1}{2}|v|^2 - \frac{\Omega_0^2}{2}|x|^2.$$

The term $2\Omega_0$ can be interpreted as the third component vorticity of the velocity field $\mathcal{P}(\tilde{\Omega}\times(x_1,x_2,x_3))$ which occurs in terms of the Coriolis force due to the rotating frame of reference. Applying the operator $\nabla^{\perp}\cdot = (-\partial_{x_2},\partial_{x_1})\cdot$ to the first equation in (D.2.2) and using that $\nabla\cdot v = 0$ yields

$$(v\cdot\nabla)(\omega+2\Omega_0) = 0.$$

Let us remark that this identity holds in general only in two dimensions, which restricts the Grad–Shafranov method to these situations. As a corollary of the above identity we obtain that $\omega+2\Omega_0$ and thus ω is constant along stream lines (characteristics) of v.

The main object in the Grad–Shafranov approach is the stream function $\psi: E_h\to\mathbb{R}$ satisfying $v = \nabla^{\perp}\psi := J\nabla\psi = (-\partial_{x_2}\psi,\partial_{x_1}\psi)$. Let us mention that in general, in order to guarantee the existence of a stream function ψ we need to work with a simply connected domain. However, since the boundary conditions (cf. second equation in (D.2.2)) implies that ψ is a constant in each of the connected components of the boundary of ∂E_h as well as the fact that the divergence free condition on v implies that ψ is harmonic, it then follows that the function ψ is well defined for arbitrary domains, not necessarily simply connected. However, during this work E_h is simply connected.

Now, with the stream function at hand, we can write $\omega = \Delta\psi$. Since ψ is also constant along the characteristics of $v = J\nabla\psi$, one might conclude the existence of a function $G:\mathbb{R}\to\mathbb{R}$ such that $\Delta\psi = G(\psi)$. Let us remark here that in general the existence of G can be concluded only locally when $\nabla\psi\neq 0$. Furthermore the function G might be multi-valued, a situation, although interesting we will not consider in this paper. In addition, we require $n_h\cdot v = 0$ on ∂E_h, and thus

$$0 = n_h\cdot J\nabla\psi = \tau_h\cdot\nabla\psi,$$

where τ_h is the positively-oriented tangential vector on ∂E_h. We integrate along the boundary to get for $x\in\partial E_h$ that $\psi(x) = c_0$ for some constant $c_0\in\mathbb{R}$. Note that the potential ψ is given up to a constant, so we can choose $c_0 = 0$ by adapting the function G if needed. Thus, the stream function solves the equation

$$\begin{cases} \Delta\psi = G(\psi) & \text{in } E_h, \\ \psi = 0 & \text{on } \partial E_h. \end{cases} \tag{D.2.3}$$

In the Grad–Shafranov approach the above reasoning is reversed in the sense that we are given some (regular enough) function G and we construct the stream function (hence also the velocity field) by solving equation (D.2.3).

Note that $\psi : E_h \to \mathbb{R}$ is a function of h, so that we sometimes write ψ_h if we want to emphasize the dependence in h. Let us also remark that the existence and uniqueness of solutions to (D.2.3) is ensured in general by assuming that G is non-decreasing, see Lemma D.3.4.

We now use the conformal mapping in order to reduce equation (D.2.3) to the domain $\mathbb{D}$. We set $\phi_h := \psi_h \circ f_h$, which is now defined on the disk $\phi_h : \mathbb{D} \to \mathbb{R}$. The corresponding equation reads

$$\begin{cases} \Delta\phi_h = |f_h'|^2 G(\phi_h) & \text{in } \mathbb{D}, \\ \phi_h = 0 & \text{on } \partial\mathbb{D}. \end{cases} \tag{D.2.4}$$

It must be stressed that the function G only depends on the stream function ψ_h and the conformal mapping but not on the external mass particle m.

Equation of the free-boundary

The equation determining the free-boundary can be derived from the fact that the non-hydrostatic pressure P is continuous along the free-boundary. We can write using the stream function $v = \nabla^\perp \psi_h$

$$\mathcal{P}(\tilde{v} \times (\tilde{\omega} + 2\tilde{\Omega})) = -(G(\psi_h) + 2\Omega_0) J \nabla^\perp \psi_h = (G(\psi_h) + 2\Omega_0)\nabla\psi_h, \quad \text{in } E_h.$$

We conclude that

$$\mathcal{P}(\tilde{v} \times (\tilde{\omega} + 2\tilde{\Omega})) = \nabla\left[F(\psi_h)\right], \quad F(\psi_h)\,|_{\partial E_h} = 0,$$

where $F' = G + 2\Omega_0$ is a primitive with $F(0) = 0$. Consequently, in order to ensure equality in the first equation in (D.2.2) the non-hydrostatic pressure is given (up to a constant λ) by

$$P = F(\psi_h) - \frac{1}{2}|\nabla\psi_h|^2 + \frac{\Omega_0^2}{2}|x|^2 + \lambda \quad \text{in } E_h. \tag{D.2.5}$$

The condition that P is continuous along the free-boundary yields with $P = U_h + mU_X$ on ∂E_h and $F(\psi_h)\,|_{\partial E_h} = 0$ the equation

$$\frac{1}{2}|\nabla\psi_h|^2 - \frac{\Omega_0^2}{2}|x|^2 + U_h + mU_X = \lambda \quad \text{on } \partial E_h. \tag{D.2.6}$$

The evaluation at the boundary $\partial E_h = f_h(\partial\mathbb{D})$ in (D.2.6) can be performed using the conformal mapping f_h. We now summarize the reduced system that we aim to solve in our study

$$\begin{cases} \dfrac{1}{2}\dfrac{|\nabla\phi_h|^2}{|f_h'|^2} - \dfrac{\Omega_0^2}{2}|f_h|^2 + U_h \circ f_h + mU_X \circ f_h = \lambda & \text{on } \partial\mathbb{D}, \\ \Delta\phi_h = |f_h'|^2 G(\phi_h) & \text{in } \mathbb{D}, \\ \phi_h = 0 & \text{on } \partial\mathbb{D}. \\ \Omega_0^2 a = \partial_{x_1} U_h(X) & \\ |E_h| = \pi. & \end{cases} \tag{D.2.7}$$

Recall that the position of the particle is chosen as $X = (a, 0)$. The unknown triplet is (h, a, λ). As we will see, cf. Corollary D.2.2, solutions of (D.2.7) constructed in this paper yield solutions to (D.1.8). Let us mention that the fourth equation in (D.2.7) is the x_1-component of Newton equation for the particle X, see also the fifth equation in (D.1.8). The other component follows, as we will see in Corollary D.2.2, by the symmetry of the domain E w.r.t. the x_1-axis.

Solution for $m=0$

In the case when no external particle is present, i.e. $m=0$, we assume that the fluid body has the shape of a disk $\mathbb{D}$. Furthermore, we consider a velocity field on $\mathbb{D}$ with stream function ϕ_0 solving

$$\begin{cases}\Delta\phi_0 = G(\phi_0) & \text{in } \mathbb{D},\\ \phi_0 = 0 & \text{on } \partial\mathbb{D}.\end{cases} \tag{D.2.8}$$

Note that this coincides with (D.2.4) for $h=0$. Observe that due to the rotational invariance $\phi_0=\phi_0(|x|)$ the equation reduces to the ODE

$$\frac{1}{r}\left(r\phi_0'\right)' = G(\phi_0(r)), \quad \phi_0(1)=0.$$

This ODE is complemented with the condition that $\lim_{r\to 0}\phi_0(r)$ exists. Therefore, the velocity field becomes $v(x)=\frac{\phi_0'(|x|)}{|x|}Jx$. It describes a non-uniform rotation with angular speed depending on the distance to the center. Since the velocity field is rotationally symmetric, the velocity in the non-rotating coordinate system is given by $(\frac{\phi_0'(|x|)}{|x|}+\Omega_0)Jx$. Furthermore, note that the function ϕ_0 can be extended to $r>1$. This is necessary, for instance, when evaluating ϕ_0 on the boundary ∂E_h, which is close to $\partial\mathbb{D}$.

The position of the unperturbed particle is chosen of the form $X_0=(a_0,0)$. Since we consider only cases for which the fluid body and the external particle are strictly separated, we assume say $a_0\geq 2$. Hence, E_h does not contain $X\approx X_0$ for small enough h. The Newton equation for the particle requires that

$$\Omega_0^2 X_0 = \nabla U_0(X_0).$$

Further information of the potentials U_0 of the disk in both Case (A) and Case (B) are given in Lemma D.3.5 and D.3.6. For $a_0>1$ we have $U_0'(a_0)>0$ and furthermore $U_0'(a_0)/a_0\to 0$ as $a_0\to\infty$. In addition $a_0\mapsto U_0'(a_0)/a_0$ is strictly decreasing for $a_0>1$. Hence, there is a one-to-one correspondence between $\Omega_0\in(0,\sqrt{U_0'(1)}]$ and $a_0\geq 1$ via

$$\Omega_0 = \sqrt{\frac{U_0'(a_0)}{a_0}}. \tag{D.2.9}$$

All in all, this defines a map $\Omega_0\mapsto a_0(\Omega_0)$. Finally, the constant in (D.2.6) is given by $\lambda_0 = \frac{1}{2}\phi_0'(1)^2-\frac{1}{2}\Omega_0^2+U_0(1)$.

D.2.1 Notation

We will use the following notation throughout the manuscript.

- We use $\mathbb{D}$ to denote the unit disk with boundary $\partial\mathbb{D}$ and $\mathbb{T}=[0,2\pi]$ the 2π-periodic torus with endpoints identified.
- The Hölder seminorm of a function $u:\mathbb{T}\to\mathbb{R}$ or $u:\mathbb{D}\to\mathbb{R}$ is defined by

$$[u]_{k,\alpha} = \sup_{x_1\neq x_2}\frac{|u^{(k)}(x_2)-u^{(k)}(x_1)|}{|x_2-x_1|^\alpha}, \quad \alpha\in(0,1),$$
$$[u]_{k,0} = \left\|u^{(k)}\right\|_\infty, \quad \alpha=0.$$

- We abbreviate $H^{k,\alpha} := H^{k,\alpha}(\mathbb{D}) := H(\mathbb{D}) \cap C^{k,\alpha}(\overline{\mathbb{D}})$, where $H(\mathbb{D})$ is the space of analytic functions on $\mathbb{D}$ and $k \in \mathbb{N}_0$, $\alpha \in (0,1)$. We equip it with the standard Hölder norm $\|\cdot\|_{k,\alpha}$.

- We denote by $H_0^{k,\alpha} \subset H^{k,\alpha}$ the subspace of analytic functions h such that $h(0) = 0$ and $h'(0) \in \mathbb{R}$.

- Furthermore, the Fourier coefficients of a function $g : \mathbb{T} \to \mathbb{R}$ are given by

$$\hat{g}_n = \frac{1}{2\pi} \int_0^{2\pi} g(\varphi) e^{-in\varphi} \, d\varphi.$$

  Recall that $\hat{g}_n = \overline{\hat{g}_{-n}}$, since g is real-valued.

- We denote by $C_0^{k,\alpha}(\mathbb{T}) \subset C^{k,\alpha}(\mathbb{T})$ those functions g with zero average, i.e. $\hat{g}_0 = 0$.

- Let us abbreviate with $B_r = B_r(0) \subset H_0^{k,\alpha}$ the ball of radius r around zero.

- We will denote with C a positive generic constant that depends only on fixed parameters including Ω_0 and norms of the function G in (D.2.7). Note also that this constant might differ from line to line.

D.2.2 Main result and strategy towards the proof

In order to construct the desired solution, we make use of the implicit function theorem, cf. Lemma D.3.3. To do so, let us introduce the following functional spaces

$$\mathbb{X}^{k+2,\alpha} := H_0^{k+2,\alpha}(\mathbb{D}) \times \mathbb{R} \times \mathbb{R}, \quad \mathbb{Z}^{k+1,\alpha} := C^{k+1,\alpha}(\mathbb{T}) \times \mathbb{R} \times \mathbb{R}. \tag{D.2.10}$$

We define the following function related to the system (D.2.7). Define the map $\mathcal{F} : U \times V \to \mathbb{Z}^{k+1,\alpha}$, where $U \subset H_0^{k+2,\alpha}(\mathbb{D}) \times \mathbb{R} \times \mathbb{R}$, $V \subset \mathbb{R}$, with $X = (a,0)$, by

$$\mathcal{F}(h,a,\lambda,m) = \begin{pmatrix} \left[\frac{1}{2} \frac{|\nabla \phi_h|^2}{|f_h'|^2} - \frac{\Omega_0^2}{2} |f_h|^2 + U_h \circ f_h + m U_X \circ f_h - \lambda \right] \Big|_{z=e^{i\varphi}} \\ \Omega_0^2 a - \partial_{x_1} U_h(X) \\ |f_h(\mathbb{D})| - \pi \end{pmatrix}. \tag{D.2.11}$$

The subset U is a sufficiently small neighborhood of $(0, a_0, \lambda_0)$. In particular, it ensures that h defines a conformal mapping $f_h(z) = z + h(z)$, see Lemma D.3.1.

Our goal is to solve the equation $\mathcal{F}(h,a,\lambda,m) = 0$ via the implicit function theorem. To this end, we study the Fréchet derivative at the point $(0, a_0, \lambda_0, 0)$. We will apply a Fourier decomposition for the first component of $\mathcal{F}$, which is a function on the torus $\mathbb{T}$. As we will see, cf. Lemma D.5.7, the corresponding linear operator can be diagonalized and the Fourier multipliers have the form

$$\omega_n = -\frac{1}{2}\Omega_0^2 - \frac{1}{2}\phi_0'(1)^2(|n|+1) + \phi_0'(1) A_{|n|}'(1)(|n|+1) + c_{|n|}. \tag{D.2.12}$$

The coefficients ω_n are visible in a non-resonance condition for Ω_0 in our main result, cf. Theorem D.2.1. In the definition of ω_n the function ϕ_0 is the unperturbed stream function for $m = 0$. The coefficients c_n enter through the interaction potential $h \mapsto (U_h \circ f_h)(e^{i\varphi})$. In Case (A), they are given by (note we identify again $\mathbb{R}^2 \backsimeq \mathbb{C}$)

$$c_n = \frac{1}{2}\int_{\mathbb{D}} \left(\nu \frac{1-y^{n+1}}{1-y} - 2(n+1)y^n \right) \frac{dy}{|1-y|^{\nu}}, \tag{D.2.13}$$

and in Case (B) by

$$c_n = \begin{cases} \frac{\pi}{2}\left(1-\frac{1}{n}\right) & n \geq 1, \\ \frac{\pi}{2} & n = 0. \end{cases} \tag{D.2.14}$$

Let us note that the integral in (D.2.13) defines a real quantity. Note also that only the first term in (D.2.12) depends on Ω_0, whereas all the other terms depend on either the function G or the choice of the interaction.

Finally, the numbers $A_n'(1)$ are computed by means of the functions $A_n : (0,1) \to \mathbb{R}$ solving the ODE

$$\frac{1}{r}(rA_n')' - \frac{n^2}{r^2}A_n - G'(\phi_0(r))A_n = r^{|n|}G(\phi_0(r)), \quad A_n(1) = 0. \tag{D.2.15}$$

They appear in the Fréchet derivative of the stream function $h \mapsto \phi_h$, cf. Section D.5.

The main result of this work reads as follows.

Theorem D.2.1. *Let $k \in \mathbb{N}_0$, $\alpha \in (0,1)$ and $a_0 \geq 2$. Assume $G \in C^{k+3}(\mathbb{R};\mathbb{R})$ to be non-decreasing. Let $\Omega_0 \geq 0$ be related to $X_0 = (a_0(\Omega_0), 0)$ as stated in* (D.2.9) *and let the non-resonance condition*

$$\forall n \in \mathbb{N} : \omega_n \neq 0, \tag{D.2.16}$$

be satisfied for ω_n given in (D.2.12). *Furthermore, we assume for the unperturbed stream function ϕ_0 that*

$$\phi_0'(1) \neq 0. \tag{D.2.17}$$

Then, there are $\delta > 0$, $\varepsilon > 0$ such that for any $m \in [0,\delta)$ there is a unique solution $(h,a,\lambda) \in \mathbb{X}^{k+2,\alpha}$ of the equation $\mathcal{F}(h,a,\lambda,m) = 0$ satisfying

$$\|h\|_{k+2,\alpha} + |a-a_0| + |\lambda - \lambda_0| < \varepsilon.$$

Furthermore, the dependence $m \mapsto (h,a,\lambda)(m)$ is continuous.

As a corollary we obtain that a solution to $\mathcal{F}(h,a,\lambda,m) = 0$ yields a solution to our original problem (D.1.8).

Corollary D.2.2. *Under the assumption of Theorem D.2.1, the domain $E_h = f_h(\mathbb{D})$ in Theorem D.2.1 is symmetric w.r.t. the x_1-axis. Finally, the corresponding velocity field $v = \nabla^{\perp}\psi_h$ together with the position of the particle $X = (a,0)$ and the non-hydrostatic pressure P yield a solution to* (D.1.8).

Remark D.2.3. Let us comment on the non-resonance condition (D.2.16).

(i) It ensures that the linearized operator can be inverted in order to apply the implicit function theorem. In the case that (D.2.16) is not satisfied bifurcations to other shapes might occur.

(ii) As mentioned before the quantities ω_n in (D.2.12) contain a term only depending on Ω_0, while the other terms depend only on the choice of the function G and the interaction. Thus, the non-resonance condition (D.2.16) is a condition on Ω_0. Furthermore, note that this condition (D.2.16) is needed only for $n \in \mathbb{N}$, since ω_n only depends on $|n|$. Furthermore, as we will see in Lemma D.5.3 and Lemma D.5.5 the leading order term on the right hand side of (D.2.16) is given by $-\phi_0'(1)^2(|n|+1)$, whereas the other terms are at most of order $\mathcal{O}(\ln n)$ as $n \to \infty$. In particular, the condition (D.2.16) is automatically satisfied for sufficiently large n. Hence, it is possible to verify the condition numerically.

(iii) In the particular case that the fluid has no internal motion in the non-rotating coordinate system for $m = 0$ we have $v(x) = -\Omega_0 Jx$ and thus $\phi_0(x) = -\Omega_0(|x|^2 - 1)/2$. This corresponds to the choice $G = -2\Omega_0$. Then, we can readily check that solutions to (D.2.15) have the form

$$A_n(r) = -2\Omega_0 \frac{r^n(r^2-1)}{4n+4}, \quad A_n'(1) = -\frac{\Omega_0}{n+1}.$$

Hence, the condition (D.2.16) reduces to

$$\omega_n = -\frac{|n|}{2}\Omega_0^2 + c_{|n|} \neq 0.$$

Remark D.2.4. Let us mention that the assumption (D.2.17) in Theorem D.2.1 is also needed to prove the invertibility of the Fréchet derivative in order to apply the implicit function theorem. This condition implies that the function ϕ_0 has no local extremum at the boundary. When perturbing such extrema, saddle points are created generically. Consequently, vortices would appear. Furthermore, let us comment on the assumption that G is non-decreasing. This condition crucially implies that the stream function is well-defined and regular enough (see Lemma D.3.4). It might be possible to relax this assumption and instead assume that in a neighborhood of an initially chosen solution ϕ_0 to (D.2.3) for $h = 0$ one can uniquely solve equation (D.2.3). This could be achieved using an auxiliary implicit function theorem. However, we do not pursue this here.

Remark D.2.5. We are assuming in Theorem D.2.1 that $m \geq 0$ since it is the most natural setting from the physical point of view. However, the proof of Theorem D.2.1 is also valid for the case $m \in (-\delta, \delta)$.

Remark D.2.6. In this paper we restricted ourselves to interaction potentials defined in Case (A) and Case (B). The study of more general interactions would require further modifications. In particular, a better understanding of results like Lemma D.5.8 on pseudo-differential operators on the torus.

Remark D.2.7. Finally, let us mention that the corresponding three-dimensional problem of (D.1.8) requires a different approach, since conformal mappings and the Grad–Shafranov method are restricted to two-dimensional problems. Furthermore, the study of the eigenvalues of the linearization involves several technical complications due to instabilities. Let us mention that the presence of the external particle does not allow to constructing axisymmetric configurations, since the interaction with the particle breaks this symmetry.

We conclude this section with the discussion of the particular case of constant $G \equiv K \in \mathbb{R} \setminus \{0\}$. This corresponds to an unperturbed velocity field $v_0(x) = KJx/2$ in the rotating and

$V_0(x) = (K+2\Omega_0)Jx/2$ in the non-rotating frame of reference. In this case, one can do a formal linearization using the ansatz $\partial E_h = T_\eta(\mathbb{T})$, $T_\eta(\theta) = 1+\varepsilon\eta(\theta)$, $\theta \in \mathbb{T}$ for the free boundary. Here, $\eta \in \mathbb{T} \to \mathbb{R}$ allows to change the boundary of the fluid body and $\varepsilon = m$ is the mass of the particle. More precisely, one can linearize the system (compare with (D.2.7))

$$\begin{cases} \frac{1}{2}|\nabla\psi_\eta|^2 - \frac{\Omega_0^2}{2} + U_h + mU_X = \lambda & \text{on } \partial E_\eta, \\ \Delta\psi_\eta = K & \text{in } E_\eta, \\ \psi_h = 0 & \text{on } \partial E_\eta. \\ |E_h| = \pi. \end{cases}$$

The linearization yields the following formula for η in terms of Fourier series

$$\eta(\theta) = \sum_{n\in\mathbb{Z}} \hat{\eta}_n e^{i\theta n}, \quad \hat{\eta}_n = -\frac{\hat{S}_n}{\tilde{\omega}_n}, \quad \tilde{\omega}_n := \frac{K^2}{4} - \frac{K^2}{4}|n| - \Omega_0^2 + \pi - \frac{\pi}{|n|}$$

The terms $\hat{S}_n$ are the Fourier coefficients of the perturbation, that is

$$\hat{S}_n = \frac{1}{2\pi}\int_0^{2\pi} U_X(\cos\theta, \sin\theta)e^{-in\theta}\,d\theta.$$

Here, $X = (a_0, 0)$ is the unperturbed position of the external particle, cf. (D.2.9).

Let us mention that the mass constraint $|E_\eta| = \pi$ imposes $\hat{\eta}_0 = 0$. Furthermore, one obtains $\hat{\eta}_n = \hat{\eta}_{-n} \in \mathbb{R}$. In particular, the function η is invariant under reflection $(x_1, x_2) \to (x_1, -x_2)$. In addition, the condition that the center of mass is at zero yields (after linearizing) $\hat{\eta}_1 = -a_0/2\pi$, which in fact can be shown to match with the above formula $\hat{\eta}_1 = -\hat{S}_1/\tilde{\omega}_1 = \hat{S}_1/\Omega_0^2$. Furthermore, note that the Fourier coefficients $\hat{\eta}_n$ do not depend on the sign of K or Ω_0.

Let us mention that the non-resonance condition (D.2.16) is equivalent to $\tilde{\omega}_n \neq 0$.

In Figure D.2 we plot the function η for the values $\Omega_0 = 1$, $K = -2, 0.1, 10$ and for an interaction potential U_X as in Case (B). In the plot the zero level line is shown. Outside this circle the function is positive whereas inside it is negative. Let us recall that the particular case $K = -2 = -2\Omega_0$ corresponds to the situation in which the unperturbed fluid body has no internal motion in the non-rotating coordinate system. Furthermore, in Figure D.3 we plotted the function η in a situation close to resonance due to the mode $n = 8$. In fact, for $\Omega_0 = 1$, $K = 1$ we have $\omega_8 \approx 10^{-3}$ so that the largest contribution to the Fourier series of η is due to the two coefficients $\hat{\eta}_8 = \hat{\eta}_{-8}$.

D.3 Preliminary results

We collect here some auxiliary results that will be used in the subsequent sections. Let us start with a well-known result in complex analysis regarding analytic functions.

Lemma D.3.1. *Consider the analytic function $f_h(z) = z + h(z)$ with $\|h\|_{C^1(\overline{\mathbb{D}})} < 1/\sqrt{2}$. Then, $f_h : \mathbb{D} \to f_h(\mathbb{D})$ is conformal.*

Proof. We prove that f_h is injective. Define the function $\zeta(\varphi) = f_h(e^{i\varphi})$, $\varphi \in \mathbb{T}$. Let $\varphi_1, \varphi_2 \in \mathbb{T}$. We can assume $|\varphi_1 - \varphi_2| \leq \pi$. If $|\varphi_1 - \varphi_2| \geq \pi/2$ we have

$$|\zeta(\varphi_2) - \zeta(\varphi_1)| \geq \left|e^{i\varphi_2} - e^{i\varphi_1}\right| - 2\|h\|_{C(\overline{\mathbb{D}})} = 2\left|\sin\left(\frac{\varphi_2 - \varphi_1}{2}\right)\right| - 2\|h\|_{C(\overline{\mathbb{D}})} > 0.$$

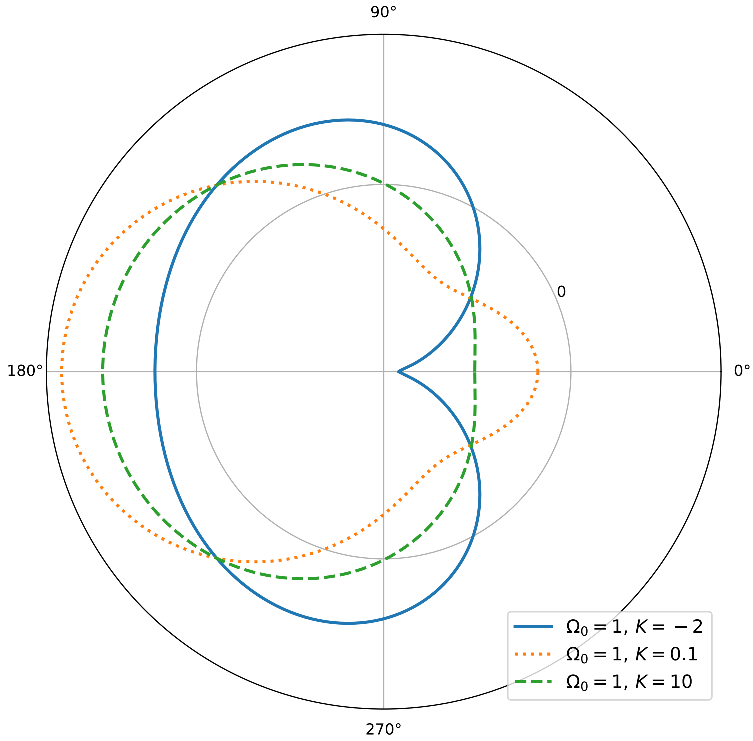

Figure D.2: Plot of the function η for $\Omega_0 = 1$ and different values of K. The interaction is given as in Case (B). Furthermore, the particle is to the leading order at position $X = (a_0, 0)$, $a_0 = \sqrt{\pi}$, cf. (D.2.9).

On the other hand, if $|\varphi_1 - \varphi_2| < \pi/2$ we estimate

$$\begin{aligned} |\zeta(\varphi_2) - \zeta(\varphi_1)| &= \left| \int_{\varphi_1}^{\varphi_2} f_h'(e^{i\psi}) ie^{i\psi}\, d\psi \right| \geq \left| e^{i\varphi_2} - e^{i\varphi_1} \right| - |\varphi_2 - \varphi_1| \, \|h\|_{C^1(\overline{\mathbb{D}})} \\ &= 2 \left| \sin\left(\frac{\varphi_2 - \varphi_1}{2} \right) \right| - |\varphi_2 - \varphi_1| \, \|h\|_{C^1(\overline{\mathbb{D}})} \geq |\varphi_2 - \varphi_1| \left(\frac{1}{\sqrt{2}} - \|h\|_{C^1(\overline{\mathbb{D}})} \right). \end{aligned}$$

Hence, f_h is one-to-one on the boundary. As a consequence of the Darboux-Picard theorem, see [39, Thm. 9.16], f_h is injective on $\overline{\mathbb{D}}$. □

Lemma D.3.2 (Faà di Bruno formula [57]). *For any $n \in \mathbb{N}$ and two functions $f, g \in C^n(\mathbb{R}; \mathbb{R})$ we have the formula*

$$\frac{d^n}{dx^n}(f \circ g)(x) = \sum_{\substack{\ell_1, \dots, \ell_n \\ 1 \cdot \ell_1 + \dots + n \cdot \ell_n = n}} n! \left[\frac{d^{\ell_1 + \dots + \ell_n}}{dx^{\ell_1 + \dots + \ell_n}} f \right] (g(x)) \prod_{j=1}^{n} \left(\frac{1}{\ell_j ! \, j!} \frac{d^j}{dx^j} g(x) \right)^{\ell_j}.$$

We recall the following version of the implicit function theorem.

Lemma D.3.3 (Implicit function theorem, [60]). *Let $\mathsf{X}, \mathsf{Y}, \mathsf{Z}$ be Banach spaces and $U \subset \mathsf{X}$, $V \subset \mathsf{Y}$ be neighborhoods of x_0, y_0, respectively, where $\mathcal{F}(x_0, y_0) = 0$. Suppose that $\mathcal{F} : U \times V \to \mathsf{Z}$ is continuous, continuously differentiable with respect to $x \in U$ and $D_x \mathcal{F}(x_0, y_0) \in \mathcal{L}(\mathsf{X}, \mathsf{Z})$ is invertible. Then, there are balls $B_\varepsilon(x_0) \subset U$, $B_\delta(y_0) \subset V$ and a unique map $\xi : B_\delta(y_0) \to B_\varepsilon(x_0)$ with $\mathcal{F}(\xi(y), y) = 0$ for all $y \in B_\delta(y_0)$. Furthermore, ξ is continuous.*

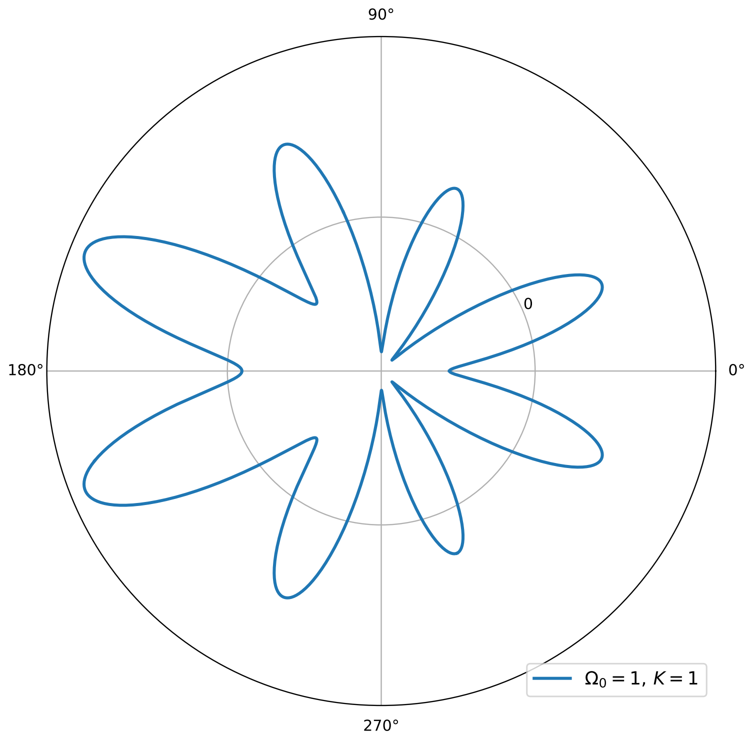

Figure D.3: Plot of the function η for $\Omega_0 = 1$ and $K = 1$. In this case there is almost resonance since $\tilde{\omega}_8 \approx 10^{-3}$. In particular, the biggest contribution to the Fourier series of η is due to the Fourier coefficients $\hat{\eta}_8 = \hat{\eta}_{-8}$.

Here, we denote by $\mathcal{L}(\mathsf{X},\mathsf{Z})$ the space of bounded linear operators $\mathsf{X} \to \mathsf{Z}$. Furthermore, $D_x\mathcal{F}(x_0,y_0) \in \mathcal{L}(\mathsf{X},\mathsf{Z})$ is the Fréchet derivative w.r.t. the first variable, i.e. we have

$$\mathcal{F}(x_0+\xi,y_0) = \mathcal{F}(x_0,y_0) + D_x\mathcal{F}(x_0,y_0)[\xi] + o(\|\xi\|_{\mathsf{X}}),$$

as $\|\xi\|_{\mathsf{X}} \to 0$.

Let us also give an existence and uniqueness result for the equation (D.2.4). Such elliptic equations have been studied extensively both in Hölder and Sobolev spaces, see e.g. [75, 76].

Lemma D.3.4. *Let $h \in B_{1/2} \subset H_0^{k+2,\alpha}$ and assume $G \in C^{k+3}(\mathbb{R};\mathbb{R})$ to be non-decreasing. Then there is a unique solution $\phi_h \in C^{k+2,\alpha}(\overline{\mathbb{D}})$ to* (D.2.4)*. Furthermore, there exists a constant $C > 0$ independent of h such that*

$$\|\phi_h\|_{C^{k+2,\alpha}(\overline{\mathbb{D}})} \leq C. \tag{D.3.1}$$

Proof. We prove the assertion in terms of $\psi_h = \phi_h \circ f_h^{-1}$. The existence follows from standard methods of calculus of variations applied to the functional

$$\psi \mapsto \int_{E_h} |\nabla\psi|^2\, dx + \int_{E_h} F(\psi)\, dx,$$

where $F' = G$ is a primitive. Note that F is convex, since G is non-decreasing. The regularity follows via a bootstrapping argument, recalling that $G \in C^{k+3}(\mathbb{R};\mathbb{R})$. Observe that due to

$f_h \in H^{k+2,\alpha}$, the boundary ∂E_h is sufficiently regular. The uniqueness can be proved using a comparison principle, since G is non-decreasing.

The estimate (D.3.1) is a consequence of the maximum principle and Schauder estimates. Indeed, this will be done by separating two cases.

Case 1. We assume that there is $y_0 \in \mathbb{R}$ with $G(y_0)=0$. Since G is non-decreasing, we can find $N>0$ sufficiently large such that $G(-N) \le 0 \le G(N)$. We conclude from a comparison principle that $\|\phi_h\|_\infty \le N$. Hence, the right-hand side in (D.2.4) is uniformly bounded in h. We apply regularity theory in Sobolev spaces to conclude that $\phi_h \in W^{2,2}$ with a bound independent of $h \in B_{1/2}$. Hence, by Sobolev embedding we obtain $\phi_h \in C^\alpha$. Now, the right-hand side in (D.2.4) is uniformly bounded in C^α. We hence apply repeatedly Schauder estimates to yield the result.

Case 2. If G is always non-zero, we can assume w.l.o.g. that $G>0$. In this case, we infer $\phi_h \le 0$ by the maximum principle. Thus, the right-hand side in (D.2.4) is uniformly bounded. We can now argue as in Case 1. $\square$

Next, we prove a formula for the unperturbed interaction potential of $\mathbb{D}$, i.e. of the unperturbed solution for $m=0$, in Case (A) with $\nu=1$ and Case (B).

Lemma D.3.5. *The following formulas hold in Case* (A) *with* $\nu=1$

$$\begin{aligned} U_0(r) &= -\frac{4}{\pi^2}\sum_{k\ge 0} W_{2k}^2\left(\frac{r}{2k+2}+\frac{r}{2k-1}-\frac{r^{2k}}{2k-1}\right), \quad 0\le r\le 1,\\ U_0(r) &= -\frac{4}{\pi^2}\sum_{k\ge 0}\frac{W_{2k}^2}{2k+2}\frac{1}{r^{2k+1}}, \quad r\ge 1. \end{aligned} \tag{D.3.2}$$

Here, $W_\ell = \frac{\pi}{2}\frac{(\ell-1)!!}{\ell!!}$ *is Wallis' formula, see [1, Formula 6.1.49]. In Case* (B) *we have that*

$$U_0(r) = \begin{cases} -\dfrac{\pi}{2}(1-r^2) & r\le 1,\\ \pi\ln r & r\ge 1. \end{cases} \tag{D.3.3}$$

Let us recall that $\lim_{\ell\to\infty}\sqrt{\ell}W_\ell = \sqrt{\pi/2}$. Consequently, the series in (D.3.2) converges also for the critical value $r=1$.

Proof. To this end, we use a multipole expansion for $x=x(r,\theta,\varphi)$, $y=y(s,\theta',\varphi')\in\mathbb{R}^3$

$$\begin{aligned} \frac{1}{|x-y|} &= \frac{1}{\sqrt{r^2+s^2-2rs(\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\varphi-\varphi'))}}\\ &= \sum_{\ell\ge 0}\sum_{|m|\le\ell}\frac{4\pi}{2\ell+1}\frac{(r\wedge s)^\ell}{(r\vee s)^{\ell+1}}Y_{\ell,m}(\theta,\varphi)Y_{\ell,m}(\theta',\varphi')^*, \end{aligned}$$

where the spherical harmonics are given by

$$Y_{\ell,m}(\theta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}e^{im\varphi}P_\ell^m(\cos\theta).$$

Here, P_ℓ^m are the associated Legendre polynomials. We have for $\theta=\theta'=\pi/2$

$$U_0(r) = -\sum_{\ell\ge 0}\frac{4\pi c_\ell^2}{2\ell+1}\int_0^1\frac{(r\wedge s)^\ell\, s ds}{(r\vee s)^{\ell+1}}, \quad c_\ell := Y_{\ell 0}(\pi/2,0) = \sqrt{\frac{2\ell+1}{4\pi}}P_\ell(0).$$

A computation shows that

$$P_\ell(0) = \begin{cases} (-1)^{\ell/2}\dfrac{(\ell-1)!!}{\ell!!} & \ell \text{ even},\\ 0 & \ell \text{ odd}.\end{cases}$$

Rewriting the coefficients of the series in terms of the Wallis' formula and choosing $\ell = 2k$ yields both formulas in (D.3.2). The formula in (D.3.3) follows by solving the Poisson equation $\Delta U_0 = \mathbb{1}_{\mathbb{D}}$. $\square$

The following lemma contains information on the unperturbed potential U_0 in all cases considered.

Lemma D.3.6. *In both Case* (A) *and* (B) *the potential U_0 satisfies*

(i) $U_0'(r) > 0$ *for* $r > 1$,

(ii) $\frac{d}{dr}[U_0'(r)/r] < 0$ *for* $r > 1$, *in particular* $r \mapsto U_0'(r)/r$ *is strictly decreasing for* $r \geq 1$. *Furthermore,* $\lim_{r\to\infty} U_0'(r)/r = 0$.

Proof. In Case (A), we first use equation (D.1.4) as well as polar coordinates to derive the explicit formula

$$U_0(r) = -\int_0^1\int_0^{2\pi} \frac{s\,ds d\varphi}{(r^2+s^2-2rs\cos\varphi)^{\nu/2}}.$$

Then claim (i) follows by differentiating the previous formula and using that $r > 1 \geq s$. Indeed,

$$U_0'(r) = \nu\int_0^1\int_0^{2\pi} \frac{(r-s\cos\varphi)s\,ds d\varphi}{(r^2+s^2-2rs\cos\varphi)^{1+\nu/2}} > 0.$$

Next, let us show claim (ii). To that purpose, we use the change of variable $s \mapsto \eta r$ yielding

$$\frac{U_0'(r)}{r} = \nu r^{-\nu}\int_0^{1/r}\int_0^{2\pi} \frac{(1-\eta\cos\varphi)\eta\,d\eta d\varphi}{(1+\eta^2-2\eta\cos\varphi)^{1+\nu/2}}.$$

Differentiating this formula with respect to r shows that $\frac{d}{dr}[U_0'(r)/r] < 0$ for $r > 1$, since the integrand is non-negative and $\nu > 0$. Furthermore, $U_0'(r)/r \to 0$ as $r \to \infty$ also follows from the previous formula. In Case (B) both claims (i) and (ii) are a direct consequence of the explicit formula (D.3.3). $\square$

D.4 Fréchet derivative of the main problem

In this section we prove the Fréchet differentiability of the function $\mathcal{F}$. We consider separately the stream function ϕ_h and the interaction potential $U_h \circ f_h$.

D.4.1 Fréchet derivative of the stream function

In this subsection we derive the Fréchet differential of the function $h \mapsto \phi_h$.

Lemma D.4.1. *Let $k \in \mathbb{N}_0$ and $\alpha \in (0,1)$. There exists $\varepsilon_0 > 0$ sufficiently small such that for $B_{\varepsilon_0} \subset H_0^{k+2,\alpha}$*

$$h \mapsto \phi_h \in C^1(B_{\varepsilon_0}; C^{k+2,\alpha}(\overline{\mathbb{D}})).$$

More precisely, the linear operator $D_h\phi_h$ is defined by $g \mapsto D_h\phi_h[g] =: \bar{\phi}$ where

$$\begin{cases} \Delta\bar{\phi} = |f_h'|^2 G'(\phi_h)\bar{\phi} + 2\mathrm{Re}\left[(1+h')\overline{g'}\right] G(\phi_h) & \text{in } \mathbb{D}, \\ \bar{\phi} = 0 & \text{on } \partial\mathbb{D}. \end{cases} \tag{D.4.1}$$

Proof. First of all, the equation (D.4.1) has a unique solution $\bar{\phi}$, since $G' \geq 0$. We apply Schauder estimates for the Laplacian and absorb the term $|f_h'|^2 G'(\phi_h)\bar{\phi}$ into the left hand side by choosing $\|h\|_{k+2,\alpha} \leq \varepsilon_0$ sufficiently small. This yields

$$\left\|\bar{\phi}\right\|_{k+2,\alpha} \leq C \|g\|_{k+2,\alpha}, \tag{D.4.2}$$

where $C > 0$ is independent of $h \in B_{\varepsilon_0} \subset H_0^{k+2,\alpha}$ by Lemma D.3.4.

Furthermore, by taking the difference of the equations for ϕ_{h+g} and ϕ_h, given respectively via (D.2.4), we obtain

$$\left[-\Delta + \int_0^1 G'((1-t)\phi_h + t\phi_{h+g})dt\right](\phi_{h+g} - \phi_h) = -\left[|f_{h+g}'|^2 - |f_h'|^2\right] G(\phi_{h+g}).$$

Therefore, using Schauder estimates, we infer that

$$\|\phi_{h+g} - \phi_h\|_{k+2,\alpha} \leq C \|g\|_{k+2,\alpha}, \tag{D.4.3}$$

where $C > 0$ is independent of h.

Next we find that for $D_h\phi_h[g] = \bar{\phi}$ and denoting $R := \phi_{h+g} - \phi_h - \bar{\phi}$

$$\begin{aligned} \Delta R &= \left(|f_{h+g}'|^2 - |f_h'|^2 - 2\mathrm{Re}\left[(1+h')\overline{g'}\right]\right) G(\phi_h) \\ &\quad + \left(|f_{h+g}'|^2 - |f_h'|^2\right) G'(\phi_h)\bar{\phi} + |f_{h+g}'|^2 \left(G(\phi_{h+g}) - G(\phi_h) - G'(\phi_h)\bar{\phi}\right) \\ &= |g'|^2 G(\phi_h) + \left(|f_{h+g}'|^2 - |f_h'|^2\right) G'(\phi_h)\bar{\phi} + G'(\phi_h)R \\ &\quad + \int_0^1 G''((1-t)\phi_h + t\phi_{h+g})\, dt\, (\phi_{h+g} - \phi_h)^2. \end{aligned}$$

Similarly as above, invoking Schauder estimates and bounds (D.4.2)-(D.4.3) we obtain (note that $G \in C^{k+3}(\mathbb{R};\mathbb{R})$)

$$\|R\|_{k+2,\alpha} \leq C \|g\|_{k+2,\alpha}^2.$$

Here, the constant $C > 0$ is independent of h.

Finally, we need to prove that $h \mapsto D_h\phi_h \in \mathcal{L}(H_0^{k+2,\alpha}; C^{k+2,\alpha}(\overline{\mathbb{D}}))$ is continuous. To this end, one has to consider differences of solutions to (D.4.1) for $h_1, h_2 \in B_{\varepsilon_0}$. Applying Schauder estimates we find the bound

$$\|D_h\phi_{h_2}[g] - D_h\phi_{h_1}[g]\|_{k+2,\alpha} \leq C \|g\|_{k+2,\alpha} \left(\|h_1 - h_2\|_{k+2,\alpha} + \|\phi_{h_1} - \phi_{h_2}\|_{k+2,\alpha}\right).$$

which shows the continuity property. □

D.4.2 Fréchet derivative of the interaction potential

Here, we derive the Fréchet derivative of the mapping $h \mapsto (U_h \circ f_h)(e^{i\varphi})$. We give only the details of the proof of Case (A) with $\nu = 1$. The remaining cases can be shown in a similar way (and are in fact simpler since the integrals are less singular). We summarize the corresponding results for Case (B) at the end of this section.

First of all, it is convenient to apply a change of variables

$$(U_h \circ f_h)(e^{i\varphi}) = -\int_{\mathbb{D}} \frac{|f_h'(y)|^2}{|f_h(e^{i\varphi}) - f_h(y)|^\nu}\, dy = -\int_{\mathbb{D}} \frac{|f_h'(e^{i\varphi}y)|^2}{|f_h(e^{i\varphi}) - f_h(e^{i\varphi}y)|^\nu}\, dy.$$

We then have the following result.

Proposition D.4.2. *Let U_h be defined as in Case* (A) *and let $k \in \mathbb{N}_0$, $\alpha \in (0,1)$. There exists $\varepsilon_0 > 0$ sufficiently small such that for $h \in B_{\varepsilon_0} \subset H_0^{k+2,\alpha}$ we have that*

$$h \mapsto (U_h \circ f_h)(e^{i\varphi}) \in C^1(B_{\varepsilon_0}, C^{k+1,\alpha}(\mathbb{T})).$$

More precisely, for $h + g \in B_{\varepsilon_0}$ it holds that

$$D_h(U_h \circ f_h)[g](e^{i\varphi}) = -\int_{\mathbb{D}} \frac{\sigma_h^1[g](\varphi,y)}{d_h(\varphi,y)^\nu}\, dy + \nu \int_{\mathbb{D}} \frac{\sigma_h^2[g](\varphi,y)}{d_h(\varphi,y)^{\nu+2}} |f_h'(e^{i\varphi}y)|^2\, dy,$$

where we define

$$\begin{aligned} d_h(\varphi,y) &:= |f_h(e^{i\varphi}) - f_h(e^{i\varphi}y)|, \\ \sigma_h^1[g](\varphi,y) &:= 2\mathrm{Re}\left[(1+h'(e^{i\varphi}y))\overline{g'(e^{i\varphi}y)}\right], \\ \sigma_h^2[g](\varphi,y) &:= \mathrm{Re}\left[\left(e^{i\varphi}(1-y) + h(e^{i\varphi}) - h(e^{i\varphi}y)\right)\overline{(g(e^{i\varphi}) - g(e^{i\varphi}y))}\right]. \end{aligned} \tag{D.4.4}$$

From now on we restrict ourselves to the case $\nu = 1$. In order to prove Proposition D.4.2 it is convenient to introduce the following notation

$$e_h(\varphi,y) = d_h(\varphi,y)^2 = |f_h(e^{i\varphi}) - f_h(e^{i\varphi}y)|^2.$$

Furthermore, we need the following computation with $t \in [0,1]$

$$\begin{aligned} \frac{d^2}{dt^2}\left[-\frac{|f_{h+tg}'(e^{i\varphi}y)|^2}{d_{h+tg}(\varphi,y)}\right] &= T^0_{h+tg,g}(\varphi,y) + T^1_{h+tg,g}(\varphi,y) + T^2_{h+tg,g}(\varphi,y), \\ T^0_{h+tg,g}(\varphi,y) &:= -\frac{2|g'(e^{i\varphi}y)|^2}{d_{h+tg}(\varphi,y)}, \\ T^1_{h+tg,g}(\varphi,y) &:= \frac{\tau^1_{h+tg,g}(\varphi,y)}{d_{h+tg}(\varphi,y)^3}, \\ \tau^1_{h+tg,g}(\varphi,y) &:= 2\sigma^1_{h+tg}[g](\varphi,y)\,\sigma^2_{h+tg}[g](\varphi,y) + |f_{h+tg}'(e^{i\varphi}y)|^2|g(e^{i\varphi}) - g(e^{i\varphi}y)|^2, \\ T^2_{h+tg,g}(\varphi,y) &:= \frac{\tau^2_{h+tg,g}(\varphi,y)}{d_{h+tg}(\varphi,y)^5}, \\ \tau^2_{h+tg,g}(\varphi,y) &:= -3|f_{h+tg}'(e^{i\varphi}y)|^2(\sigma^2_{h+tg}(\varphi,y))^2. \end{aligned} \tag{D.4.5}$$

Lemma D.4.3. *Let $k \in \mathbb{N}_0$, $\alpha \in [0,1)$. For $\varepsilon_0 > 0$ sufficiently small and $g, h \in B_{\varepsilon_0} \subset H_0^{k+2,\alpha}$ the following estimates hold*

(i) For $\ell \in \mathbb{N}_0, \ell \leq k+1$, $y \in \mathbb{D}$ we have

$$\sup_{t\in[0,1]} \left[\sigma^2_{h+tg}[g](\cdot,y)\right]_{\ell,\alpha} \leq C\,\|g\|_{k+2,\alpha}\,|1-y|^{2-\alpha}.$$

(ii) For $\ell \in \mathbb{N}_0, \ell \leq k+1$, $m = 1,2$, $y \in \mathbb{D}$ we have

$$\sup_{t\in[0,1]} \left[\tau^m_{h+tg}[g](\cdot,y)\right]_{\ell,\alpha} \leq C\,\|g\|^2_{k+2,,\alpha}\,|1-y|^{2m-\alpha}.$$

(iii) For $\ell \in \mathbb{N}_0, \ell \leq k+1$, $y \in \mathbb{D}$ we have

$$\sup_{t\in[0,1]} [e_{h+tg}[g](\cdot,y)]_{\ell,\alpha} \leq C|1-y|^{2-\alpha}.$$

Proof. The proof of this lemma is straightforward. Indeed, one can readily check the bounds by means of the following general estimate. For any $u \in H^1(\mathbb{D})$ we have that

$$\begin{aligned}\frac{1}{|\varphi_2-\varphi_1|^\alpha}&\left|\left[u(e^{i\varphi_1}y)-u(e^{i\varphi_1})\right]-\left[u(e^{i\varphi_2}y)-u(e^{i\varphi_2})\right]\right|\\ &\leq \left(2\,\|u'\|_\infty\right)^\alpha\left(2|1-y|\,\|u'\|_\infty\right)^{1-\alpha} \leq 2\,\|u\|_{C^1}\,|1-y|^{1-\alpha}.\end{aligned}$$

□

The next lemma is also useful.

Lemma D.4.4. *Let $k \in \mathbb{N}_0$, $\alpha \in [0,1)$. There is a sufficiently small $\varepsilon_0 > 0$ such that for any $h \in B_{\varepsilon_0} \subset H_0^{k+2,\alpha}$ and $y \in \mathbb{D}$ we have that*

$$d_h(\varphi,y) \geq c|1-y|. \tag{D.4.6}$$

Furthermore, for any $q \in \mathbb{N}$, $n \in \mathbb{N}_0, n \leq k+1$ and $y \in \mathbb{D}$ estimate

$$\left[\frac{1}{d_h(\cdot,y)^q}\right]_{n,\alpha} \leq \frac{C_{k,q}}{|1-y|^{q+\alpha}}.$$

holds. Both $c > 0$ and $C_{k,q>} > 0$ are independent of h.

Proof. The first assertion follows using the mean-value theorem and choosing $\varepsilon_0 > 0$ sufficiently small. To prove the second assertion we first consider $n = 0$. It then suffices to consider $q = 1$. We have

$$\begin{aligned}\frac{1}{|\varphi_1-\varphi_2|^\alpha}\left|\frac{1}{d_h(\varphi_1,y)}-\frac{1}{d_h(\varphi_2,y)}\right| &= \frac{|d_h(\varphi_1,y)-d_h(\varphi_2,y)|}{|\varphi_1-\varphi_2|^\alpha}\frac{1}{d_h(\varphi_1,y)d_h(\varphi_2,y)}\\ &\leq \frac{C}{|1-y|^2}\frac{|f_h(e^{i\varphi_1})-f_h(e^{i\varphi_2})+f_h(e^{i\varphi_1}y)-f_h(e^{i\varphi_2}y)|}{|\varphi_1-\varphi_2|^\alpha}\\ &\leq \frac{C}{|1-y|^{1+\alpha}}.\end{aligned}$$

We now compute the n-th order derivative, $n \le k+1$. By Faà di Bruno's formula, cf. Lemma D.3.2, applied to the composition of the functions $x \mapsto x^{-q/2}$ and $e_h = d_h^2$ the preceding expression is a sum of terms of the form

$$\frac{1}{d_h(\varphi,y)^{2(\ell_1+\dots+\ell_n)+q}} \prod_{j=1}^{n} \left(\frac{d^j}{d\varphi^j} e_h(\varphi,y) \right)^{\ell_j} = \frac{1}{d_h(\varphi,y)^q} \prod_{j=1}^{n} \left(\frac{1}{d_h(\varphi,y)^2} \frac{d^j}{d\varphi^j} e_h(\varphi,y) \right)^{\ell_j},$$

where $\ell_1, \ell_2, \dots, \ell_n \in \mathbb{N}_0$ satisfy $\ell_1 + 2\ell_2 + \dots + n\ell_n = n$. The supremum norm of each term in the product $j = 1, \dots, n$ is bounded due to Lemma D.4.3 (iii) and (D.4.6).

We estimate now the seminorm $[\cdot]_\alpha$ of this expression. Note that for products only one term is estimated in this seminorm while the other terms are estimated in the supremum norm. For the seminorm we apply Lemma D.4.3 (iii) and the case $n = 0$ we discussed above. Hence, the seminorm is bounded up to a constant by $|1-y|^{-q-\alpha}$. □

As a result of the previous lemmas we obtain the following estimates.

Lemma D.4.5. *Let $k \in \mathbb{N}_0$ and $\alpha \in (0,1)$. We have for sufficiently small $\varepsilon_0 > 0$, $g, h \in B_{\varepsilon_0} \subset H_0^{k+2,\alpha}$ and $m = 0,1,2$*

$$\left\| \frac{\sigma_h^1[g](\cdot,y)}{d_h(\cdot,y)} \right\|_{k+1,\alpha} + \left\| \frac{\sigma_h^2[g](\cdot,y)}{d_h(\cdot,y)^3} \right\|_{k+1,\alpha} \le C \|g\|_{k+2,\alpha} |1-y|^{-1-\alpha},$$

$$\left\| T^m_{h+tg,g}(\cdot,y) \right\|_{k+1,\alpha} \le C \|g\|^2_{k+2,\alpha} |1-y|^{-1-\alpha}.$$

The constant $C > 0$ is independent of h, g.

Proof. The previous lemmas can be applied without difficulty. Note that in the case of $T^2_{h+tg,g}$ there is the factor d^5_{h+tg} in the denominator. This is compensated by the extra factor in Lemma D.4.3 (ii) for $m = 2$. □

With the previous lemmas we can give the proof of Proposition D.4.2.

Proof of Proposition D.4.2. We consider only $\nu = 1$. For the sake of the exposition we divide the proof into two steps.

Step 1. We first show the Fréchet differentiability. We write

$$\begin{aligned} R(\varphi) :=& (U_{h+g} \circ f_{h+g})(e^{i\varphi}) - (U_h \circ f_h)(e^{i\varphi}) - D_h(U_h \circ f_h)[g](e^{i\varphi}) \\ =& \int_{\mathbb{D}} \int_0^1 (1-t) \frac{d^2}{dt^2} \left[-\frac{|f'_{h+tg}(e^{i\varphi}y)|^2}{d_{h+tg}(\varphi,y)} \right] dt dy \\ =& \int_{\mathbb{D}} \int_0^1 (1-t) \left[T^0_{h+tg,g}(\varphi,y) + T^1_{h+tg,g}(\varphi,y) + T^2_{h+tg,g}(\varphi,y) \right] dt dy, \end{aligned}$$

recalling (D.4.5). We apply Lemma D.4.5 to get (note that $-1-\alpha > -2$)

$$\|R\|_{k+1,\alpha} \le C \|g\|^2_{k+2,\alpha},$$

which implies the Fréchet differentiability.

Step 2. Now, we show that $h \mapsto D_h(U_h \circ f_h)[g]$ is continuous. First of all, one can estimate using Lemma D.4.5

$$\|D_h(U_h \circ Y_h)[g]\|_{k+1,\alpha} \le C_k \|g\|_{k+2,\alpha}.$$

where $C_k > 0$ is independent of $h, g \in B_{\varepsilon_0} \subset H_0^{k+2,\alpha}$. We can use these estimates to cut out the singularity in the integral uniformly in $h, g \in B_{\varepsilon_0}$. The remaining integrand is then a smooth function with respect to h. As a consequence it is continuous in h. The above bounds show that this is also uniform in $\|g\|_{k+2,\alpha}$, which ensures these estimates in the operator norm. Hence, $h \mapsto D_h(U_h \circ f_h)[\cdot](e^{i\varphi}) \in \mathcal{L}(H_0^{k+2,\alpha}; C^{k+1,\alpha}(\mathbb{T}))$ is continuous. $\square$

Let us give the corresponding result of Proposition D.4.2 in Case (B). To this end, we write

$$(U_h \circ f_h)(e^{i\varphi}) = \int_{\mathbb{D}} |f_h'(y)|^2 \ln|f_h(e^{i\varphi}) - f_h(y)|\, dy = \int_{\mathbb{D}} |f_h'(e^{i\varphi}y)|^2 \ln|f_h(e^{i\varphi}) - f_h(e^{i\varphi}y)|\, dy.$$

Proposition D.4.6. *Let U_h be defined as in Case* (B) *and let $k \in \mathbb{N}_0$, $\alpha \in (0,1)$. There exists $\varepsilon_0 > 0$ sufficiently small such that for $h \in B_{\varepsilon_0} \subset H_0^{k+2,\alpha}$ we have that*

$$h \mapsto (U_h \circ f_h)(e^{i\varphi}) \in C^1(B_{\varepsilon_0}, C^{k+1,\alpha}(\mathbb{T})).$$

More precisely, for $h + g \in B_{\varepsilon_0}$ it holds that

$$D_h(U_h \circ f_h)[g](e^{i\varphi}) = \int_{\mathbb{D}} \sigma_h^1[g](\varphi, y) \ln d_h(\varphi, y)\, dy + \int_{\mathbb{D}} \frac{\sigma_h^2[g](\varphi, y)}{d_h(\varphi, y)^2} |f_h'(e^{i\varphi}y)|^2\, dy,$$

where $\sigma_h^1[g], \sigma_h^2[g]$ and $d_h(\varphi, y)$ are given in (D.4.4).

D.4.3 Fréchet derivative of the full problem

Here, we give compute the Fréchet derivative of the second and third component of $\mathcal{F}$ in (D.2.11).

For the second component note that the continuous differentiability of $(h, X) \mapsto \nabla U_h(X)$ involves no complications since $X = (a, 0)$ is assumed to be close to $X_0 = (a_0, 0)$ with $a_0 \geq 2$. Hence, ∇U_h is smooth on a neighborhood of X_0 and

$$\nabla U_h(X) = \nu \int_{E_h} \frac{X - y}{|X - y|^{\nu+2}}\, dy = \nu \int_{\mathbb{D}} \frac{X - f_h(y)}{|X - f_h(y)|^{\nu+2}} |f_h'(y)|^2\, dy,$$

in Case (A) and

$$\nabla U_h(X) = \int_{\mathbb{D}} \frac{X - f_h(y)}{|X - f_h(y)|^2} |f_h'(y)|^2\, dy,$$

in Case (B). Here, we identify f_h with a function $\mathbb{D} \to \mathbb{R}^2$. We obtain the following result for the Fréchet derivative (we again identify $X = (a, 0) \in \mathbb{R}^2 \backsimeq \mathbb{C}$).

Lemma D.4.7. *Let U_h be defined as in Case* (A) *or Case* (B) *and $k \in \mathbb{N}_0$, $\alpha \in (0,1)$. The map $(h, a) \mapsto \partial_{x_1} U_h(a, 0)$ is continuously differentiable for $|a - a_0| \leq \varepsilon_0$, $h \in B_{\varepsilon_0} \subset H_0^{k+2,\alpha}$, $\varepsilon_0 > 0$ sufficiently small, with derivative*

$$D_{(h,a)}\left(\partial_{x_1} U_h(a, 0)\right)[g, b] = \partial_{x_1}^2 U_h(a, 0) b + W_{h,a}[g].$$

The function $W_{h,a}[g]$ is defined by

$$\begin{aligned} W_{h,a}[g] :=& \nu \int_{\mathbb{D}} \frac{-\mathrm{Re}[g(y)] |f_h'(y)|^2 + 2\mathrm{Re}[(1 + h'(y))\overline{g'(y)}]\, \mathrm{Re}[a - f_h(y)]}{|a - f_h(y)|^{\nu+2}}\, dy \\ &- \nu(\nu+2) \int_{\mathbb{D}} \frac{\mathrm{Re}[(a - f_h(y))\overline{g(y)}]\, \mathrm{Re}[a - f_h(y)]}{|a - f_h(y)|^{\nu+4}} |f_h'(y)|^2\, dy \end{aligned}$$

in Case (A) *and by*

$$W_{h,a}[g] := \int_{\mathbb{D}} \frac{-\mathrm{Re}[g(y)]|f_h'(y)|^2 + 2\mathrm{Re}[(1+h'(y))\overline{g'(y)}]\,\mathrm{Re}[a - f_h(y)]}{|a - f_h(y)|^2}\,dy$$
$$- \int_{\mathbb{D}} \frac{2\mathrm{Re}[(a - f_h(y))\overline{g(y)}]\,\mathrm{Re}[a - f_h(y)]}{|a - f_h(y)|^4}|f_h'(y)|^2\,dy$$

in Case (B)*, respectively.*

The mass constraint, i.e. the third component of $\mathcal{F}$, leads to the following Fréchet derivative.

Lemma D.4.8. *The Fréchet derivative of the map*

$$h \mapsto |E_h| = \int_{\mathbb{D}} |f_h'(x)|^2\,dx$$

is given by

$$g \mapsto 2\mathrm{Re}\int_{\mathbb{D}} (1+h')\overline{g'}\,dx.$$

We omitted the proof of the previous Lemmas since they can be easily checked. Finally, the following differentiability result follows from combining Lemma D.4.1, Proposition D.4.2, respectively Proposition D.4.6, and Lemmas D.4.7, D.4.8.

Proposition D.4.9. *Let $k \in \mathbb{N}_0$, $\alpha \in (0,1)$. There is $\varepsilon_0 > 0$ sufficiently small such that $\mathcal{F} \in C^1(U;\mathbb{Z}^{k+1,\alpha})$, where*

$$U = B_{\varepsilon_0}(0) \times (a_0 - \varepsilon_0, a_0 + \varepsilon_0) \times (\lambda_0 - \varepsilon_0, \lambda_0 + \varepsilon_0) \subset \mathbb{X}^{k+2,\alpha}.$$

D.5 Invertibility of the linearized operator

In order to apply the implicit function theorem we need to invert the linearized operator at the point $(0, X_0, \lambda_0, 0)$, i.e. the linear operator

$$\mathbb{X}^{k+2,\alpha} \to \mathbb{Z}^{k+1,\alpha} : (g,b,\mu) \mapsto D_{(h,a,\lambda)}\mathcal{F}(0,a_0,\lambda_0,0)[g,b,\mu]. \tag{D.5.1}$$

We recall that the functional spaces are defined in (D.2.10). It is convenient to write the function $g \in H_0^{k+2,\alpha}$ using power series of the form

$$g(z) = \sum_{n\geq 0} \hat{g}_n z^{n+1}. \tag{D.5.2}$$

Recall that $g(0) = 0$ and $g'(0) = \hat{g}_0 \in \mathbb{R}$ since $g \in H_0^{k+2,\alpha}$.

Remark D.5.1. Let us briefly comment on the form of the power series (D.5.2).

(i) We choose an index shift in the coefficients of (D.5.2), in order that the linearized operator (D.5.1) is diagonalized when using a Fourier decomposition, cf. Lemma D.5.7.

(ii) With this choice the coefficient $\hat{g}_0$ corresponds to a rescaling $z \mapsto (1+\hat{g}_0)z$. Consequently, it appears in the linearization of $h \mapsto |E_h|$, cf. Lemma D.5.7. In the Fourier series it appears as the zeroth coefficient.

(iii) Furthermore, infinitesimal translations are given by the conformal mappings $T_\varepsilon : z \mapsto z+\varepsilon$ for small $\varepsilon > 0$. In order to satisfy the conditions $T_\varepsilon(0) = 0$ and $T'_\varepsilon(0) \in \mathbb{R}$ we use a Blaschke factor, see (D.2.1), yielding the conformal mapping

$$z \mapsto \frac{z-\varepsilon}{1-\varepsilon z} + \varepsilon = \frac{z-\varepsilon^2 z}{1-\varepsilon z} = z + h_\varepsilon(z), \quad h_\varepsilon(z) = \frac{\varepsilon z^2 - \varepsilon^2 z}{1-\varepsilon z} = \varepsilon z^2 + \mathcal{O}(\varepsilon^2) \tag{D.5.3}$$

as $\varepsilon \to 0$. In particular, infinitesimal translations correspond to the coefficient of z^2, i.e. $\hat{g}_1$ in (D.5.2). In Fourier series they correspond to the coefficients for $e^{\pm i\varphi}$, which is $\hat{g}_1$ respectively $\overline{\hat{g}_1}$ in the linearization, cf. Lemma D.5.7.

The main result of this section is the following proposition.

Proposition D.5.2. *Let $k \in \mathbb{N}_0$, $\alpha \in (0,1)$. The operator* (D.5.1) *is an isomorphism under the assumptions of Theorem D.2.1.*

For the purpose of proving Proposition D.5.2 it is necessary to compute explicitly the form of the linear operator (D.5.1). This is done in the following subsections, which also contain further needed auxiliary results.

Linearization of the stream function.

For the proof we write the operator $\bar{\phi}[g] := D_h\phi_h(0)[g]$, i.e. the Fréchet derivative of ϕ_h at $h = 0$, more explicitly. Due to Lemma D.4.1 it solves the equation

$$\Delta\bar{\phi} = G'(\phi_0)\bar{\phi} + 2\mathrm{Re}\,[g']\,G(\phi_0), \quad \bar{\phi}\,|_{\partial\mathbb{D}} = 0. \tag{D.5.4}$$

Hence, using the expression (D.5.2) we find that

$$2\mathrm{Re}\left[g'(re^{i\varphi})\right] = 2\mathrm{Re}\left[\sum_{n\geq 0}(n+1)\hat{g}_n r^n e^{in\varphi}\right] = \sum_{n\in\mathbb{Z}}(|n|+1)\hat{\xi}_n[g]\,r^{|n|}e^{in\varphi}, \tag{D.5.5}$$

where the coefficients $\hat{\xi}_n[g]$ are given by

$$\hat{\xi}_n[g] := \begin{cases} \hat{g}_n & n \geq 1, \\ 2\hat{g}_0 & n = 0, \\ \overline{\hat{g}_n} & n \leq -1. \end{cases} \tag{D.5.6}$$

Recall that $g'(0) = \hat{g}_0 \in \mathbb{R}$ since $g \in H_0^{k+2,\alpha}$. We use a Fourier decomposition to obtain the formula

$$\bar{\phi}[h](r,\varphi) = \sum_{n\in\mathbb{Z}}(|n|+1)\hat{\xi}_n[g]\,A_n(r)e^{in\varphi},$$

where $A_n(r)$ solves the ordinary differential equation, see (D.2.15),

$$\frac{1}{r}(rA'_n)' - \frac{n^2}{r^2}A_n - G_1 A_n = r^{|n|}G_0, \quad A_n(1) = 0. \tag{D.5.7}$$

Above we used the shortcut notation $G_0(r) := G(\phi_0(r))$, $G_1(r) := G'(\phi_0(r))$. Moreover, notice that $A_n = A_{-n}$ by symmetry.

The function $\bar{\phi}$ enters the linearization of $\mathcal{F}$ in the following way

$$\left[\nabla\phi_0\cdot\nabla\bar{\phi}\right](e^{i\varphi}) = \phi_0'(1)\partial_r\bar{\phi}(1,\varphi) = \phi_0'(1)\sum_{n\in\mathbb{Z}}(|n|+1)\hat{\xi}_n[g]A_n'(1)e^{in\varphi}. \tag{D.5.8}$$

Recall that the unperturbed stream function ϕ_0 is radial. Hereafter we provide a crucial result concerning the asymptotics of $A_n'(1)$ as $n\to\infty$.

Lemma D.5.3. *Consider the solution $\bar{\phi}$ of* (D.5.4)*. Then, the coefficients A_n have the asymptotics*

$$\lim_{n\to\infty} nA_n'(1) = \frac{G(\phi_0(1))}{2}. \tag{D.5.9}$$

Remark D.5.4. Compare (D.5.9) with the explicit solutions for $G\equiv -2\Omega_0$ in Remark D.2.3 (iii).

Proof of Lemma D.5.3. Let $n\geq 1$ throughout the proof. Writing $\widetilde{\phi}_n(r,\varphi)=A_n(r)e^{in\varphi}$ we have

$$(\Delta - G_1)\widetilde{\phi}_n = r^n e^{in\varphi}G_0, \quad \widetilde{\phi}_n(1,\varphi)=0. \tag{D.5.10}$$

Since $G_1 = G'\circ\phi_0\geq 0$, the operator $\Delta - G_1$ with zero boundary conditions is invertible. Furthermore, since $G_0\in C^{k+3,\alpha}$ we have $\widetilde{\phi}_n\in C^{k+5,\alpha}(\overline{\mathbb{D}})$.

Let us look at the following auxiliary ODE

$$\frac{1}{r}(rb_n')' - \frac{n^2}{r^2}b_n = r^n G_0, \quad b_n(1)=0.$$

Note that comparing it with the ODE solved by A_n given in (D.5.7), only the term G_1 is removed, which is expected to be of lower order for $n\to\infty$. It is convenient to write $b_n(r)=r^n\beta_n(r)$ with

$$\frac{1}{r}(r\beta_n')' + \frac{2n}{r}\beta_n' = G_0, \quad \beta_n(1)=0.$$

We can find the solution explicitly up to a parameter

$$\beta_n(r) = -\frac{\beta_n'(1)}{2n}\frac{1-r^{2n}}{r^{2n}} - \frac{1}{2n}\int_r^1 G_0(s)s\,ds + \frac{1}{2n}\frac{1}{r^{2n}}\int_r^1 G_0(s)s^{2n+1}\,ds.$$

In order that β_n exists for $r\to 0$ we choose

$$B_n := \beta_n'(1) := \int_0^1 G_0(s)s^{2n+1}\,ds \tag{D.5.11}$$

yielding

$$\beta_n(r) = \frac{B_n}{2n} - \frac{1}{2n}\int_r^1 G_0(s)s\,ds - \frac{1}{2n}\frac{1}{r^{2n}}\int_0^r G_0(s)s^{2n+1}\,ds.$$

Observe that $|\beta_n(r)|\leq C/n$ for some constant $C>0$ independent of r and n.

Let us now decompose

$$\widetilde{\phi}_n(r,\varphi) = \widetilde{\phi}_n^1(r,\varphi) + \widetilde{\phi}_n^2(r,\varphi), \quad \widetilde{\phi}_n^1(r,\varphi) := r^n\beta_n(r)e^{in\varphi}.$$

Accordingly, we get the decomposition

$$A_n(r) = A_n^1(r) + A_n^2(r), \quad A_n^1(r) = r^n \beta_n(r). \tag{D.5.12}$$

Hence, with the above calculations we have

$$(\Delta - G_1)\widetilde{\phi}_n^2 = (\Delta - G_1)(\widetilde{\phi}_n - \widetilde{\phi}_n^1) = G_1 \widetilde{\phi}_n^1, \quad \widetilde{\phi}_n^2 \mid_{\partial\mathbb{D}} = 0.$$

We can apply regularity estimates in Sobolev spaces to obtain

$$\left\| \widetilde{\phi}_n^2 \right\|_{W^{2,2}(\mathbb{D})} \leq C \left\| g \widetilde{\phi}_n^1 \right\|_{L^2(\mathbb{D})} \leq \frac{C}{n} \left(\int_0^1 s^{2n+1}\, ds \right)^{1/2} \leq \frac{C}{n^{3/2}}.$$

Here, we used $|\widetilde{\phi}_n^1(r,\varphi)| \leq |\beta_n(r)| \leq C/n$. Applying the trace theorem (cf. [76]) gives

$$\left\| \partial_r \widetilde{\phi}_n^2(1,\cdot) \right\|_{L^2(\partial\mathbb{D})} \leq \frac{C}{n^{3/2}},$$

and hence $|(A_n^2)'(1)| \leq C/n^{3/2}$. In conclusion, combining (D.5.12) and (D.5.11) we find that

$$\begin{aligned}
\lim_{n\to\infty} n\, A_n'(1) = \lim_{n\to\infty} n\, (A_n^1)'(1) = \lim_{n\to\infty} n\, \beta_n'(1) &= \lim_{n\to\infty} n\, B_n \\
&= \lim_{n\to\infty} n \int_0^1 G_0(s) s^{2n+1}\, ds \\
&= \lim_{n\to\infty} n \left(\frac{G_0(1)}{2n+2} - \frac{1}{2n+2} \int_0^1 G_0'(s) s^{2n+2}\, ds \right) \\
&= \frac{G_0(1)}{2} = \frac{G(\phi_0(1))}{2},
\end{aligned}$$

showing the desired result. □

Linearization of the interaction potential, Case (A)**.**

Due to Proposition D.4.2 we have for $x = e^{i\varphi}$ (we use here the change of variables $e^{i\varphi} y \mapsto y$)

$$D_h(U_h \circ f_h) \mid_{h=0} [g](x) = -\int_{\mathbb{D}} \frac{2\mathrm{Re}[g'(y)]}{|x-y|^\nu}\, dy + \nu \int_{\mathbb{D}} \frac{\mathrm{Re}[(\overline{x-y})(g(x)-g(y))]}{|x-y|^{2+\nu}}\, dy.$$

As before we use the power series expansion for g given in (D.5.2). We have

$$D_h(U_h \circ f_h) \mid_{h=0} [g](x) = \sum_{n=0}^{\infty} \mathrm{Re} \left[\hat{g}_n \int_{\mathbb{D}} \left(\nu \frac{x^{n+1} - y^{n+1}}{x-y} - 2(n+1) y^n \right) \frac{dy}{|x-y|^\nu} \right].$$

Since $x = e^{i\varphi}$ we can use a rotation to obtain

$$\sum_{n=0}^{\infty} \mathrm{Re} \left[\hat{g}_n e^{in\varphi} \int_{\mathbb{D}} \left(\nu \frac{1 - y^{n+1}}{1-y} - 2(n+1) y^n \right) \frac{dy}{|1-y|^\nu} \right].$$

Recalling the definition of c_n in (D.2.13), the fact that c_n are real and the definition of $\hat{\xi}_n[g]$ in (D.5.5), we obtain

$$D_h(U_h \circ f_h) \mid_{h=0} [g](e^{i\varphi}) = \sum_{n\in\mathbb{Z}} c_n\, \hat{\xi}_n[g]\, e^{in\varphi}, \tag{D.5.13}$$

where we define $c_n = c_{|n|}$ for $n < 0$. The following result will be useful.

Lemma D.5.5. *The sequence c_n defined in* (D.2.13) *with $\nu = 1$ satisfies*

$$\lim_{n\to\infty} \frac{c_n}{\ln n} = \gamma_0 \quad \textit{with} \quad \gamma_0 := \int_0^\infty \int_0^\infty \frac{e^{-r}\zeta \sin(\zeta)}{(r^2+\zeta^2)^{3/2}}\, dr d\zeta. \tag{D.5.14}$$

For $\nu \in (0,1)$ we have $\sup_{n\geq 0} |c_n| < \infty$.

Proof. First of all, we have

$$c_n = \sum_{k=0}^{n} \frac{\nu}{2} \int_{\mathbb{D}} \frac{y^k}{|1-y|^\nu}\, dy - (n+1) \int_{\mathbb{D}} \frac{y^n}{|1-y|^\nu}\, dy.$$

Let us define

$$\widetilde{c}_k := \frac{1}{2} \int_{\mathbb{D}} \frac{y^k}{|1-y|^\nu}\, dy.$$

We show below that

$$\lim_{k\to\infty} k^{2-\nu}\, \widetilde{c}_k = \gamma_0^\nu, \quad \gamma_0^\nu := \nu \int_0^\infty \int_0^\infty \frac{e^{-r}\zeta \sin(\zeta)}{(r^2+\zeta^2)^{(2+\nu)/2}}\, dr d\zeta. \tag{D.5.15}$$

Note that $\gamma_0^1 = \gamma_0$ for $\nu = 1$. With this we infer for $\nu = 1$

$$\lim_{n\to\infty} \frac{c_n}{H_n} = \gamma_0, \quad H_n = \sum_{k=1}^{n} \frac{1}{k}.$$

Since $H_n = \ln(n)\,(1+o(1))$ as $n \to \infty$, this implies the asymptotics (D.5.14) for $\nu = 1$. The claim for $\nu \in (0,1)$ is also a consequence of the above asymptotics.

We now prove (D.5.15). The term $\widetilde{c}_k$ is real-valued so that

$$\widetilde{c}_k = \frac{1}{2} \int_0^1 \int_0^{2\pi} \frac{r^{k+1} \cos(k\varphi)}{(1+r^2-2r\cos\varphi)^{\nu/2}}\, d\varphi dr = I_k^1 + I_k^2. \tag{D.5.16}$$

The two terms I_k^1 and I_k^2 are defined by splitting the integral (D.5.16) with respect to r into the regions $(0,1/2)$ and $(1/2,1)$. We can readily check that

$$k^{2-\nu} |I_k^1| \leq \frac{C k^{2-\nu}}{2^{k+1}} \to 0. \tag{D.5.17}$$

To deal with the I_k^2 we notice that with the change of variables $r = 1-s$

$$\begin{aligned} k^{2-\nu} I_k^2 &= \frac{k^{2-\nu}}{2} \int_0^{1/2} \int_0^{2\pi} \frac{(1-s)^{k+1} \cos(k\varphi)}{\left(s^2+4(1-s)\sin^2(\varphi/2)\right)^{\nu/2}}\, d\varphi ds \\ &= \frac{k^{1-\nu}}{2} \int_0^{k/2} \int_0^{2\pi} \frac{\left(1-\dfrac{r}{k}\right)^{k+1} \cos(k\varphi)}{\left[\left(\dfrac{r}{k}\right)^2 + 4\left(1-\dfrac{r}{k}\right)\sin^2\left(\dfrac{\varphi}{2}\right)\right]^{\nu/2}}\, d\varphi dr. \end{aligned}$$

In the second equality we used the change of variables $ks = r$. Furthermore, writing $k\varphi = \psi$ we get

$$k^{2-\nu} I_k^2 = \frac{1}{2}\int_0^{k/2}\int_0^{2k\pi} \frac{\left(1-\frac{r}{k}\right)^{k+1}\cos(\psi)}{\left[r^2+4\left(1-\frac{r}{k}\right)k^2\sin^2\left(\frac{\psi}{2k}\right)\right]^{\nu/2}}d\psi dr$$
$$= \int_0^{k/2}\int_0^{k\pi} \frac{\left(1-\frac{r}{k}\right)^{k+1}\cos(\psi)}{\left[r^2+4\left(1-\frac{r}{k}\right)k^2\sin^2\left(\frac{\psi}{2k}\right)\right]^{\nu/2}}d\psi dr,$$

where we used the symmetry in the last equality. Let us now define the function $\zeta_k : (0,k\pi) \to (0,2k)$ by

$$\zeta_k(\psi) = 2k\sin\left(\frac{\psi}{2k}\right),$$

which is one-to-one and onto. Furthermore, by a Taylor expansion one can see that $\zeta_k(\psi) \to \psi$ for any $\psi \in (0,k\pi)$ as $k \to \infty$. Consequently, we have for the inverse function $\psi_k(\zeta) \to \zeta$ as $k \to \infty$. We obtain by the change of variables $\psi \mapsto \zeta$

$$k^{2-\nu} I_k^2 = \int_0^{k/2}\int_0^{2k} \frac{\left(1-\frac{r}{k}\right)^{k+1}}{\left[r^2+\left(1-\frac{r}{k}\right)\zeta^2\right]^{\nu/2}}\cos(\psi_k(\zeta))\psi_k'(\zeta)d\zeta dr.$$

We now use an integration by parts in ζ to obtain (note that the boundary terms vanish, due to $\psi_k(0) = 0, \psi_k(2k) = k\pi$)

$$k^{2-\nu} I_k^2 = \nu\int_0^{k/2}\int_0^{2k} \frac{\left(1-\frac{r}{k}\right)^{k+2}\zeta\sin(\psi_k(\zeta))}{\left[r^2+\left(1-\frac{r}{k}\right)\zeta^2\right]^{(\nu+2)/2}}d\zeta dr.$$

The integrand converges pointwise to

$$\frac{e^{-r}\zeta\sin(\zeta)}{(r^2+\zeta^2)^{(\nu+2)/2}}$$

as $k \to \infty$. Since

$$\left(1-\frac{r}{k}\right)^{k+2} = \exp\left((k+2)\ln\left(1-\frac{r}{k}\right)\right) \le e^{-r},$$
$$\psi_k(\zeta) = 2k\arcsin\left(\frac{\zeta}{2k}\right) \le C\zeta$$

for say $\zeta \in (0,1)$, a majorant is given by

$$\frac{e^{-r}\min(C\zeta^2,\zeta)}{(r^2+\zeta^2/2)^{(2+\nu)/2}}.$$

Hence, we get $k^{2-\nu}I_k^2 \to \gamma_0^\nu$. Combining this with (D.5.17) and (D.5.16) yields (D.5.15). □

Linearization of the interaction potential, Case (B)**.**

By Proposition D.4.6 we have for $x = e^{i\varphi}$

$$D_h(U_h \circ f_h)\,|_{h=0}\,[g](x) = 2\int_{\mathbb{D}} \ln|x-y|\,\mathrm{Re}[g'(y)]\,dy + \int_{\mathbb{D}} \frac{\mathrm{Re}[(\overline{x-y})(g(x)-g(y))]}{|x-y|^2}\,dy.$$

Again, we use the power series expansion for g, cf. (D.5.2), yielding

$$D_h(U_h \circ f_h)\,|_{h=0}\,[g]\left(e^{i\varphi}\right) = \sum_{n=0}^{\infty} 2\mathrm{Re}\left[\hat{g}_n \int_{\mathbb{D}} \left((n+1)y^n \ln|x-y| + \frac{1}{2}\frac{x^{n+1}-y^{n+1}}{x-y}\right) dy\right].$$

For $x = e^{i\varphi}$ and applying the change of variables $y \mapsto e^{i\varphi}y$ gives

$$\sum_{n=0}^{\infty} 2\mathrm{Re}\left[\hat{g}_n e^{in\varphi} \int_{\mathbb{D}} \left((n+1)y^n \ln|1-y| + \frac{1}{2}\frac{1-y^{n+1}}{1-y}\right) dy\right].$$

As we will see below we have, see (D.2.14),

$$c_n = \int_{\mathbb{D}} \left((n+1)y^n \ln|1-y| + \frac{1}{2}\frac{1-y^{n+1}}{1-y}\right) dy = \begin{cases} \frac{\pi}{2}\left(1-\frac{1}{n}\right) & n \geq 1, \\ \frac{\pi}{2} & n = 0. \end{cases} \tag{D.5.18}$$

Recalling the definition of $\hat{\xi}_n[g]$ in (D.5.5), we have (see (D.5.13))

$$D_h(U_h \circ f_h)\,|_{h=0}\,[g](e^{i\varphi}) = \sum_{n\in\mathbb{Z}} c_n\,\hat{\xi}_n[g]\,e^{in\varphi},$$

where we again define $c_n = c_{|n|}$ for $n < 0$.

Let us now prove (D.5.18). For $n = 0$ the integral reduces to $U_0(1) + \pi/2 = \pi/2$, cf. Lemma D.3.5. For the other cases let us first observe that

$$\frac{1}{2}\int_{\mathbb{D}} \frac{1-y^{n+1}}{1-y}\,dy = \frac{1}{2}\sum_{k=0}^{n}\int_{\mathbb{D}} y^k\,dy = \frac{1}{2}\sum_{k=0}^{n}\int_0^{2\pi}\int_0^1 r^k e^{ik\varphi}\,r\,dr\,d\varphi = \frac{\pi}{2}. \tag{D.5.19}$$

Moreover, we can also write

$$(n+1)\int_{\mathbb{D}} y^n \ln|1-y|\,dy = (n+1)\int_0^1\int_0^{2\pi} r^n e^{in\varphi} \ln|1-re^{i\varphi}|\,r\,dr\,d\varphi. \tag{D.5.20}$$

Hence using the following expansion

$$\begin{aligned}\ln|1-re^{i\varphi}| &= \frac{1}{2}\left(\ln(1-re^{i\varphi}) + \ln(1-re^{-i\varphi})\right) \\ &= -\frac{1}{2}\left(\sum_{k=0}^{\infty}\frac{r^{k+1}e^{i(k+1)\varphi}}{k+1} + \sum_{k=0}^{\infty}\frac{r^{k+1}e^{-i(k+1)\varphi}}{k+1}\right).\end{aligned}$$

and plugging it in (D.5.20) we find that

$$(n+1)\int_{\mathbb{D}} y^n \ln|1-y|\,dy = -2\pi\frac{n+1}{2n}\int_0^1 r^{2n+1}\,dr = -\frac{\pi}{2n}.$$

Thus, combining (D.5.19) and (D.5.20) we infer (D.5.18).

Remark D.5.6. Let us note that in both Case (A) and Case (B) we have $c_1 = 0$. This holds in general since $(U_{h_\varepsilon} \circ f_{h_\varepsilon})(1) = U_0(1)$, where $h_\varepsilon(z) = \varepsilon z^2 + \mathcal{O}(\varepsilon^2)$ is associated to translations, see formula (D.5.3) in Remark D.5.1. We hence obtain

$$c_1 = D_h(U_h \circ f_h)\,|_{h=0}\,[z^2](1) = \frac{d}{d\varepsilon}\,|_{\varepsilon=0}\,(U_{h_\varepsilon} \circ f_{h_\varepsilon})(1) = 0.$$

As mentioned in Remark D.5.1 the effects of conformal mappings due to translations appear to first order in the Fourier modes $n = \pm 1$ and thus in the coefficient c_1.

Linearization of the full problem.

We summarize the full linearized operator at $(h_0 \equiv 0, a_0, \lambda_0, m = 0)$ in the following lemma.

Lemma D.5.7. *The operator $D_{(h,a,\lambda)}\mathcal{F}(0, a_0, \lambda_0, 0)$ has the form*

$$(g, b, \mu) \mapsto \begin{pmatrix} \mathscr{L} g - \mu \\ \Omega_0^2 b - \partial_{x_1}^2 U_0(a_0, 0) b - W_{0,a_0}[g] \\ \pi \hat{h}_0 \end{pmatrix},$$

$$\mathscr{L} g(\varphi) := 2\omega_0 \hat{g}_0 + \sum_{n \geq 1} \omega_n \hat{g}_n e^{in\varphi} + \sum_{n \leq -1} \omega_n \overline{\hat{g}_{|n|}} e^{in\varphi},$$

$$\omega_n = -\frac{1}{2}\phi_0'(1)^2(|n|+1) + \phi_0'(1) A_n'(1)(|n|+1) - \frac{1}{2}\Omega_0^2 + c_{|n|},$$

Here, $W_{0,a_0}[g]$ is defined in Lemma D.4.7 in both Case (A) *and Case* (B).

Note that in the last component of the linearized operator we identify again $\mathbb{R}^2 \backsimeq \mathbb{C}$. Furthermore, the coefficients ω_n appeared already in (D.2.12).

Proof of Lemma D.5.7. The first component of $\mathcal{F}$ in (D.2.11) has the linearization at the point $(h = 0, X_0, \lambda_0, m = 0)$

$$\begin{aligned}(g, \mu) \mapsto & \phi_0'(1)\, \partial_r \bar{\phi}(1, \varphi) - \phi_0'(1)^2 \mathrm{Re}[g'(e^{i\varphi})] \\ & - \Omega_0^2 \,\mathrm{Re}\left[e^{-i\varphi} g(e^{i\varphi})\right] + D_h(U_h \circ f_h)\,|_{h=0}\,[g](e^{i\varphi}) - \mu.\end{aligned}$$

Using (D.5.2) and (D.5.5) we obtain

$$\mathrm{Re}\left[e^{-i\varphi} g(e^{i\varphi})\right] = \frac{1}{2}\sum_{n \in \mathbb{Z}} \hat{\xi}_n[g]\, e^{in\varphi}.$$

Using both (D.5.8) and (D.5.13) yields the expression of the first component. Applying the definition of $\hat{\xi}_n[g]$ in (D.5.5) yields the form of the operator $\mathscr{L} g$.

The linearization of the second component $\mathcal{F}$ in (D.2.11) is a consequence of Lemma D.4.7. For the last component, note that the linearization of the mass constraint in Lemma D.4.8 becomes $g \mapsto \pi\,\mathrm{Re}[\hat{g}_0] = \pi \hat{g}_0$, since $g \in H_0^{k+2,\alpha}$. This concludes the proof. □

Before providing the proof of Proposition D.5.2 we need to show the following result on Fourier multipliers on the torus in Hölder spaces.

Lemma D.5.8. *Let $k \in \mathbb{N}$ and $\alpha \in (0,1)$. Consider a sequence $\beta = (\beta_n)_n$ of the form $\beta_n = \kappa/(|n|+b_n)$, $\beta_0 = 0$, $n \in \mathbb{Z}$ with some real constant $\kappa \neq 0$. Assume that $b_n \neq -|n|$ is a sequence satisfying $\sup_{n\neq 0}|b_n||n|^{-\gamma} \leq C$ for some $0 \leq \gamma \leq 1/2$. Then, the periodic pseudodifferential operator*

$$\mathsf{OP}(\beta)\xi(\varphi) = \sum_{n\in\mathbb{Z}\setminus\{0\}} \beta_n \hat{\xi}_n e^{in\varphi}$$

defines a bounded map $C_0^{k,\alpha}(\mathbb{T}) \to C_0^{k+1,\alpha}(\mathbb{T})$.

Proof. Recall that the Hilbert transform $\mathscr{H}$ defined by the Fourier multipliers $-i\mathrm{sgn}(n)$ is a bounded map $C_0^{k,\alpha}(\mathbb{T}) \to C_0^{k,\alpha}(\mathbb{T})$ for all $k \in \mathbb{N}$, $\alpha \in (0,1)$. Since the operator with multiplier $1/in$ corresponds to integration, we conclude that the operator with multiplier $1/|n| = i\mathrm{sgn}(n)/in$ is a bounded map $C_0^{k,\alpha}(\mathbb{T}) \to C_0^{k+1,\alpha}(\mathbb{T})$.

We now write

$$\beta_n = \frac{\kappa}{|n|} - \frac{\kappa}{|n|}\cdot\frac{b_n}{(|n|+b_n)} = \frac{\kappa}{|n|}(1+r_n).$$

By assumption it holds $c_1 \leq |1+b_n/|n||$ for some constant $c_1 > 0$. Hence, we have

$$|r_n| \leq \frac{|b_n|}{c_1|n|} \leq \frac{C}{|n|^{1-\gamma}} \leq \frac{C}{|n|^{1/2}}.$$

Thus, the sequence $r = (r_n)_n$ satisfies the ρ-condition in [44, Theorem 3.1] with $\rho = 1/2$ and hence $\mathsf{OP}(r)$ constitutes a bounded map $C_0^{k,\alpha}(\mathbb{T}) \to C_0^{k,\alpha}(\mathbb{T})$ for all $k \in \mathbb{N}$, $\alpha \in (0,1)$. In the mentioned reference, periodic Besov space $B^s_{\infty,\infty}$ have been used. Recall that $B^s_{\infty,\infty}$ coincides with the classical Hölder space $C^{k,\alpha}(\mathbb{T})$ for $s = k+\alpha \notin \mathbb{N}$. This concludes the proof. $\square$

Proof of Proposition D.5.2

We consider Case (A) and Case (B) simultaneously, since the proof is the same. Given $(S,Z,M) \in \mathbb{Z}^{k+1,\alpha} = C^{k+1,\alpha}(\mathbb{T}) \times \mathbb{R} \times \mathbb{R}$ we want to solve for $(g,b,\mu) \in H_0^{k+2,\alpha} \times \mathbb{R} \times \mathbb{R}$ the equations

$$\begin{aligned} \mathscr{L}g - \mu &= S, \\ \Omega_0^2 b - \partial_{x_1}^2 U_0(a_0,0)b - W_{0,a_0}[g] &= Z, \\ \pi\hat{g}_0 &= M. \end{aligned} \tag{D.5.21}$$

First, we have $\hat{g}_0 = M/\pi$. For the first equation in (D.5.21) we decompose S in its Fourier coefficients $(\hat{S}_n)_{n\in\mathbb{Z}}$. Then, the first equation in (D.5.21) becomes

$$\sum_{n\geq 1}\omega_n \hat{g}_n e^{in\varphi} + \sum_{n\leq -1}\omega_n \overline{\hat{g}_{|n|}}e^{in\varphi} = \hat{S}_0 - \frac{2\omega_0 M}{\pi} + \mu + \sum_{n\geq 1}\hat{S}_n e^{in\varphi} + \sum_{n\leq -1}\overline{\hat{S}_{|n|}}e^{in\varphi}.$$

Recall that $\overline{\hat{S}_{-n}} = \hat{S}_n$ for $n \geq 0$ since S is a real-valued function. We then choose $\mu = 2\omega_0 M/\pi - \hat{S}_0$. Since the multipliers ω_n of $\mathscr{L}$ are non-zero by assumption (D.2.16), we can define $\mathscr{L}^{-1} = OP(\omega_n^{-1})$. By Lemmas D.5.3 and D.5.5 we can write

$$\omega_n = \frac{|n|+b_n}{\kappa}, \quad \kappa^{-1} := -\phi_0'(1)^2,$$

with $\sup_{n\neq 0}|b_n||n|^{-\gamma} \le C$ for any $\gamma > 0$. Note that by our assumption in Theorem D.2.1 we also have $\phi_0'(1) \neq 0$. We can hence apply Lemma D.5.8 yielding $F \in C_0^{k+2,\alpha}(\mathbb{T})$ defined by

$$F = OP(\omega_n^{-1})(S - \hat{S}_0).$$

Note that F is real-valued with $\hat{F}_n = \hat{S}_n/\omega_n$ for $n \ge 1$.

The function F is only defined on the torus. We now define the function g from F via

$$g(z) = \hat{g}_0 z + \sum_{n\ge 1} \hat{F}_n z^{n+1} = \frac{M}{\pi} z + \sum_{n\ge 1} \frac{\hat{S}_n}{\omega_n} z^{n+1}. \tag{D.5.22}$$

We need to show that $g \in H_0^{k+2,\alpha}$. To this end, define the function $\tilde{F} := \frac{1}{2}(I + \mathscr{H})F$, recalling that $\mathscr{H}$ denotes the Hilbert transform. The function $\tilde{F}$ has the Fourier decomposition

$$\tilde{F}(\varphi) = \sum_{n\ge 1} \hat{F}_n e^{in\varphi}, \quad \left\|\tilde{F}\right\|_{C^{k+2,\alpha}(\mathbb{T})} \le \|F\|_{C^{k+2,\alpha}(\mathbb{T})}.$$

The last inequality follows from the fact that $\mathscr{H} : C^{k+2,\alpha}(\mathbb{T}) \to C^{k+2,\alpha}(\mathbb{T})$ is bounded with $\|\mathscr{H}\| = 1$. Since $\tilde{F}$ contains only Fourier modes $n \ge 0$, there is a unique holomorphic extension in $C^{k+2,\alpha}(\overline{\mathbb{D}})$. This extension has the power series expansion

$$\tilde{F}(z) = \sum_{n\ge 1} \hat{F}_n z^{n+1}.$$

Consequently, the function $g(z) := \hat{g}_0 z + \tilde{F}(z) \in H_0^{k+2,\alpha}$ satisfies (D.5.22) and hence also (D.5.21).

Finally, we determine b in (D.5.21). To this end, we need to solve

$$\left(\Omega_0^2 - U_0''(a_0)\right) b = Z + W_{0,a_0}[g].$$

At this point $W_{0,a_0}[g]$ is a determined real number. We observe that due to (D.2.9) and Lemma D.3.6

$$\Omega_0^2 - U_0''(a_0) = \frac{U_0'(a_0)}{a_0} - U_0''(a_0) = -a_0\left(-\frac{U_0'(a_0)}{a_0^2} + \frac{U_0''(a_0)}{a_0}\right) = -a_0 \frac{d}{dr}\Big|_{r=a_0}\left[\frac{U_0'(r)}{r}\right] > 0.$$

Thus, we can invert the above equation in terms of b.

The above arguments show that $D_{(h,a,\lambda)}\mathcal{F}(0, a_0, \lambda_0, 0)$ is one-to-one and onto. Hence, it is an isomorphism which concludes the proof.

D.6 Proof of Theorem D.2.1 and consequences

In this last section, we first provide the proof of Theorem D.2.1. We also include the details towards Corollary D.2.2 which is a direct consequence of the previous main result.

Proof of Theorem D.2.1

Due to Proposition D.4.9 the function $\mathcal{F}$ is continuously differentiable. Under assumption (D.2.16) we can invert the linearized operator $D_{(h,a,\lambda)}\mathcal{F}(0, a_0, \lambda_0, 0)$ by Proposition D.5.2. Hence, we can apply the implicit function theorem, see Lemma D.3.3. This concludes the proof.

Proof of Corollary D.2.2

For the sake of clarity we divide the proof into three steps.

Step 1: Symmetry. We first prove the symmetry of the domain E_h. To this end, we show that the function $g(z) := \overline{h(\overline{z})} \in H_0^{k+2,\alpha}$ satisfies $\mathcal{F}(g,a,\lambda,m) = 0$. Note that g induces a conformal map f_g which parameterizes the domain $R(E_h)$, where $R(x_1,x_2) = (x_1,-x_2)$. As a consequence of the uniqueness of solutions to (D.2.3), the stream function satisfies $\psi_g(x) = \psi_h(Rx)$. Furthermore, we have, recalling $X = (a,0)$,

$$\begin{aligned} U_g(x) &= U_{R(E_h)}(x) = U_h(Rx), \\ U_X(x) &= U_X(Rx). \end{aligned}$$

Since (h,a,λ,m) is a solution, we obtain from (D.2.6), which is equivalent to the first component of $\mathcal{F}$, and application of $x \mapsto Rx$

$$\frac{1}{2}|\nabla^\perp \psi_g(x)|^2 - \frac{\Omega_0^2}{2}|x|^2 + U_g(x) + mU_X(x) = \lambda \quad x \in \partial E_g,$$

The other components of $\mathcal{F}(g,a,\lambda,m) = 0$ follow in the same manner. By the uniqueness statement of the implicit function theorem we have $f_h(z) = f_g(z) = \overline{f_h(\overline{z})}$, i.e. the domain E_h is symmetric.

Step 2: Solution. The symmetry of the domain E_h implies

$$\partial_{x_2} U_h(X) = 0 = \Omega_0^2 X_2.$$

We can now define the non-hydrostatic pressure P as in (D.2.5) and observe that all equations but the last one in the system (D.1.8) are satisfied for $v = \nabla^\perp \psi_h$, $X = (a,0)$ and P.

Step 3: Center of mass. We now show that the last equation in (D.1.8) is a consequence of the other equations in (D.1.8). More precisely, they imply that the center of mass is zero

$$X_c := \frac{1}{\pi + m}\left(\int_{E_h} x\,dx + mX\right) = 0.$$

Combining the first equation and the fifth equation in (D.1.8) gives

$$(\pi + m)\,\Omega_0^2 X_c = \int_{E_h} ((v\cdot\nabla)v + 2\Omega_0 Jv + \nabla P)\,dx + m\nabla U_h(X).$$

Since $(v\cdot\nabla)v = \operatorname{div}(v\otimes v)$ and $v\cdot n_h = 0$ on ∂E_h the first term is zero. Furthermore, due to $v = \nabla^\perp\psi_h = J\nabla\psi_h$ and $\psi_h = 0$ on ∂E_h we have

$$\int_{E_h} Jv\,dx = -\int_{E_h} \nabla\psi_h\,dx = -\int_{\partial E_h} \psi_h n_h\,dS = 0.$$

We have for the non-hydrostatic pressure

$$\int_{E_h} \nabla P\,dx = \int_{E_h} P n_h\,dS = \int_{\partial E_h} (U_h + mU_X)\,n_h\,dS,$$

where we used the fourth equation in (D.1.8). Furthermore, we have in Case (A)

$$m\nabla U_h(X) = -m\int_{E_h} \nabla_y \left[\frac{1}{|X-y|^\nu}\right] dy = -m\int_{\partial E_h} \frac{n_h}{|X-y|^\nu}\, dS(y) = -m\int_{\partial E_h} U_X\, n_h\, dS.$$

In Case (B) we get a corresponding equality. This yields

$$(\pi + m)\,\Omega_0^2\, X_c = \int_{\partial E_h} U_h\, n_h\, dS.$$

By symmetry of the interaction potential we obtain in Case (A)

$$\int_{\partial E_h} U_h\, n_h\, dS = \int_{E_h} \nabla U_h(x)\, dx = \nu \int_{E_h}\int_{E_h} \frac{x-y}{|x-y|^{\nu+2}}\, dxdy = 0.$$

However, this argument holds only for $\nu < 1$ due to the singularity. For $\nu = 1$ we use an approximation. The same conclusion holds in Case (B). This implies $X_c = 0$, since $\Omega_0 \neq 0$, which concludes the proof.

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