QUANTUM OPTIMAL TRANSPORT FOR AF-C^*-ALGEBRAS

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Abstract

We introduce quantum optimal transport of states on tracial AF-C^*-algebras to study non-spatial transport of quantum information, and view it as the pointwise case of a general parametrised one. We define quantum optimal transport distances as dynamic transport distances in a tracial but non-ergodic and infinite-dimensional quantum setting, called AF-C^*-setting, clearly motivated by Benamou-Brenier-type distances.

Pointwise division is replaced with inverses of evaluated operator means in the sense of Kubo and Ando, i.e. with noncommutative division operators. To this end, we initially extend quasi-entropies after Hiai and Petz to the AF-C^*-setting and use the latter to define energy functionals. We further extend foundational results of Carlen and Maas to the AF-C^*-setting and develop a theory of quantum optimal transport yielding non-spatial lower Ricci bounds suitable for meaningful geometric analysis. Essential for our discussion is a coarse graining process arising from the underlying metric geometry as encoding scheme of the given tracial AF-C^*-algebra. Since energy functionals are Γ-limits w.r.t. the coarse graining process, the latter reduces the AF-C^*-setting to the finite-dimensional one s.t. ergodicity is recovered up to a controlled remainder.

In the logarithmic mean setting, i.e. for all quantum L^2-Wasserstein distances, we apply the coarse graining process to all finitely supported accessibility components of a given state space simultaneously. We thereby show equivalence of the EVI_λ-gradient flow property for quantum relative entropy, its strong geodesic λ-convexity, a, possibly infinite-dimensional, Bakry-Émery condition, and a Hessian lower bound condition. We subsequently define lower Ricci bounds of our quantum gradients using any one of said equivalent conditions, give sufficient conditions for lower Ricci bounds of direct sum quantum gradients and, assuming lower Ricci bounds, derive functional inequalities HWI_λ, MLSI_λ and TW_λ in the AF-C^*-setting alongside their chain of implications.

Fundamental example classes give quantum optimal transport of normal states on hyperfinite factors of type I and II with both non-negative and strictly positive lower Ricci bounds. An application is given by first and second quantisation of spectral triples. Upon passing to second quantisation, we introduce gauge fields as spatial coordinates in a first effort to parametrise quantum optimal transport. This yields an ansatz to study noncommutative gauge theories through the dynamics of such generalised gauge fields described as gradient flows driven by a proposed internalisation of the spectral action on gauge fields. The latter action is known from the celebrated spectral action principle of Connes and Chamseddine.

Take it, brave York.
– Henry V

Preface

This work fully presents the author's doctoral thesis in mathematics at the Institute for Applied Mathematics of the University of Bonn under the supervision of Karl-Theodor Sturm starting in October 2016. It was and is motivated by the lack of a general notion of curvature in Connes' program of noncommutative geometry, sufficiency of lower Ricci bounds for meaningful geometric analysis in the classical case, and, at its inception still recent, work of Carlen and Maas for lower Ricci bounds in an ergodic finite-dimensional setting using a dynamic formulation of quantum optimal transport distances. Its main goal is to extend results of Carlen and Maas, in particular their notion of lower Ricci bound based on the first properly noncommutative analogue of a classical equivalence for EVI_λ-gradient flows of relative entropy, to a tracial infinite-dimensional setting in order to derive novel quantitative statements in noncommutative geometry.

The discussion given in this work includes such an extension to a well-behaved yet sufficiently general approximately finite-dimensional, or AF-C^*-setting. However, its exact nature, formulation and implications were not visible from the outset and thus underwent several iterations during two principal phases of work. From October 2016 to October 2020, the author worked as a member of the research group of Karl-Theodor Sturm and presented earlier versions of this work at seminars in Bonn, IST Austria, and the Oberwolfach Research Institute for Mathematics. From November 2020 onwards, he held two public-private research and development positions as applied mathematician in operations research for the German federal government and at IABG mbH.

The discussion which emerged during this time moves beyond the author's initial expectations and goal. The theory of quantum optimal transport as presented here lies in the intersection of noncommutative gauge theory, quantum statistical mechanics and quantum information theory. Whereas some technical effort ensures classical optimal transport theory is emulated successfully, its spatial interpretation as mass transport is invalidated by the simplest properly noncommutative example, i.e. transport of states on two-dimensional complex matrices encoding a single qubit, since spin and mass are independent intrinsic properties of elementary particles. Any reasonable extension to the infinite-dimensional quantum setting hence requires a non-spatial interpretation as transport of, suitably general, quantum information. The author hopes to have provided the latter in this discussion, with an eye towards future applications to noncommutative gauge theory upon explicit introduction of gauge fields as spatial coordinates acting as control parameters for varying encoding schemes. An account of relations to other work at the end of the introduction focuses on the foundational work of Carlen and Maas, as well as the work of Wirth and Zhang in the tracial infinite-dimensional setting.

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David Francis Hornshaw
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A.M.D.G.

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1 | Introduction

Connes' program of noncommutative geometry [67][69][137][138] unifies continuous and discrete geometries [114][197][198] using operator theory [29][192][193][194]. The program lacks a general notion of curvature [111][147] even as several exist for example classes such as noncommutative tori [70][98][110][146]. We instead study non-spatial lower Ricci bounds, rather than curvature directly, since these often suffice for classical geometric analysis [148][183]. Lower Ricci bounds [151][189][190] for optimal transport on continuous geometries [8][97][199] are displacement convexity [72][156] of relative entropy. In the infinitesimally Hilbertian setting, they act as limiting cases for Bochner inequalities [105] and imply a chain of functional inequalities [151][168] probing the underlying metric geometry. Maas [152] and Mielke [159] extended optimal transport to discrete geometries. Pointwise division is replaced with inverses of evaluated operator means in the sense of Kubo and Ando [13]. Erbar and Maas further extended lower Ricci bounds and functional inequalities [104][106][107]. Operator means let Carlen and Maas extend to an ergodic finite-dimensional quantum setting [48][49][50]. They allow for, possibly non-tracial, weights [193]. We in turn extend their results to a tracial but non-ergodic and infinite-dimensional quantum setting, called AF-C^*-setting, and develop a theory of quantum optimal transport yielding non-spatial lower Ricci bounds suitable for meaningful geometric analysis. We in fact study a non-spatial transport of quantum information [62] and view it as the pointwise case of a general parametrised one with an ansatz to study noncommutative gauge theories [51][54][55][197][198].

We emulate the classical case in the infinitesimally Hilbertian setting. Following work of Jordan, Kinderlehrer and Otto for Fokker-Planck equations [131], resp. Otto for porous medium equations [167][169], Ambrosio, Gigli and Savaré give EVI_λ-gradient flows of proper l.s.c. functionals defined on metric spaces [8] to study evolution partial differential equations using gradient flows absent differential structures [75][160]. If EVI_λ-gradient flow of relative entropy exists for L^2-Wasserstein distances determined by weak upper gradients [7][56] inducing Dirichlet forms [117], then it is heat flow [9] [10]. Existence is equivalent to λ-convexity of relative entropy [9][10] and Bakry-Émery conditions [19][20] linking heat flow to a weak Riemannian structure [8][103] for the given classical L^2-Wasserstein distance [11][12][105]. Sturm [189][190], as well as Lott and Villani [151], each established λ-convexity of relative entropy [72][156] as synthetic lower Ricci bounds [191]. The latter imply a HWI_λ-interpolation inequality, a modified logarithmic Sobolev inequality MLSI_λ, and a Talagrand inequality TW_λ [151][168].

Equivalent characterisation of heat flow as EVI_λ-gradient flow of relative entropy and functional inequalities are extended to the discrete cases [152][159] in [106], resp. to the ergodic finite-dimensional setting in [48][49][50]. Note Datta and Rouzé extended results as per [50] to the finite-dimensional Lindblad setting in [77]. In addition, see [21][164]. Equivalence in [50] uses arguments fully given by Erbar and Maas in [106] alone. The logarithmic operator mean yields analogues of L^2-Wasserstein distances and allows a Hessian lower bound condition crucial to show equivalence. In our logarithmic mean setting, which does assume the AF-C^*-setting, yet neither ergodicity nor finite trace, we extend results in [48][49][50] and [106]. This demands an involved technical discussion for which we summarise our twelve main contributions as follows:

A.1) We introduce noncommutative differential structures. They collect the data which define quantum optimal transport distances. Theorem 2.2.49 and Theorem 2.2.58 show they lets us define noncommutative division operators. They determine, and are in turn determined by, quasi-entropies in the sense of Hiai and Petz [127][128] extended to the AF-C^*-setting as per Theorem 2.2.29.

A.2) We define and discuss quantum optimal transport distances of states on tracial AF-C^*-algebras. These are dynamic transport distances in the AF-C^*-setting and motivated by Benamou-Brenier-type distances [24][97]. We thus define and use both quantum gradients and noncommutative division operators in our analogous constructions. Assuming traciality but allowing non-ergodicity, defined as complex kernel dimension larger than one for quantum Laplacians, we extend [152][159] and [48][49][50] to the AF-C^*-setting as discussed above.

A.3) Theorem 3.1.47 shows accessibility components of quantum optimal transport distances are complete geodesic length-metric spaces [8][40]. States at finite distance have identical fixed parts under noncommutative heat semigroups of quantum Laplacians. Non-ergodicity implies differing fixed parts. Assuming spectral gaps of quantum Laplacians and fixed parts, Theorem 3.2.65 classifies accessibility components of square integrable normal states using fixed parts.

A.4) We in turn use the above classification to formulate a coarse graining process as per Diagram 1.19. The latter reduces the AF-C^*-setting to the finite-dimensional one s.t. ergodicity is recovered up to a controlled remainder by reducing to accessibility components in the finite-dimensional setting. We take great care to show objects and properties are compatible with compression and finite-dimensional approximation, i.e. restrict suitably and are scaling limits as $j \uparrow \infty$ [122].

A.5) Theorem 3.1.31 shows energy functionals are Γ-limits [74] w.r.t. the coarse graining process as per Diagram 1.19. We formalise the latter as existence of sufficient minimising geodesics approximated in finite dimensions. Theorem 3.1.52 gives such existence. Using the latter, the coarse graining process lets us view quantum optimal transport as transport of, suitably general, quantum information. Upon allowing mixed states [116], we transport scaling limits of uniformly conditioned spin states encoding sequences of qubits [42][43][62][93][95].

B.1) We extend quantum relative entropy in the sense of Araki [16][17] and Umegaki [196] to the AF-C^*-setting. Specifically, we extend Kosaki's formula [163] in the second variable to, possibly non-finite, traces. We require properties of the strongly unital finite-trace case. We introduce finitely supported accessibility components to rectify this. Upon restriction, Theorem 4.1.29 shows we recover said case as per Theorem 4.1.25 depending on the given finitely supported fixed state.

B.2) Following a maximum entropy production principle [91][92][155], we view quantum Laplacians as generators of quantum noise evolution. Theorem 4.2.35 shows quantum Laplacians satisfy, up to sign, a quantum Fokker-Planck equation with vanishing drift term in scaling limit, i.e. only noise diffusion term.

B.3) Theorem 4.3.8 yields equivalence of EVI_λ-gradient flow, λ-convexity, Bakry-Émery and Hessian lower bound conditions by means of the coarse graining process as claimed above. We are motivated in our proof by analogous arguments in [50] and [106]. However, Theorem 4.2.22 replaces essential steps therein letting us argue using Riemannian metrics on relative interiors.

B.4) Lower Ricci bounds are given by λ-convexity of quantum information along minimising geodesics measured by quantum relative entropy. Their non-spatiality is further visible as follows. Assuming strictly positive lower Ricci bounds and finitely supported fixed part, Theorem 4.3.12 classifies accessibility components of normal states with finite quantum relative entropy using fixed parts. Using the latter, we show strictly positive lower Ricci bounds determine energy-information trade-offs parametrised by lower bounds on quantum noise.

B.5) Theorem 4.3.18 gives sufficient conditions for lower Ricci bounds of direct sum quantum gradients. In order to do so, we adapt the proof of Theorem 10.9 in [50] to the AF-C^*-setting by means of the coarse graining process. Lemma 4.3.15 provides detailed and, to our knowledge, initially lacking proof of a necessary extension of Theorem 5 in [127] to all finite-dimensional C^*-algebras.

B.6) Theorem 4.3.25 derives functional inequalities HWI_λ, MLSI_λ and TW_λ in the AF-C^*-setting. Non-ergodicity requires relative entropy of finitely supported fixed states in their formulation. We adapt the proofs of Theorem 11.3, Theorem 11.4 and Theorem 11.5 in [50] to the AF-C^*-setting by means of the coarse graining process. We introduce quantum Fisher information in the AF-C^*-setting.

C) We provide fundamental example classes. The latter yield quantum optimal transport of normal states on hyperfinite factors of type I and II [173]. An application is given by first and second quantisation of spectral triples [54][55][197][198]. This yields our ansatz to study noncommutative gauge theories based on a proposed internalised spectral action [51][52][53][197][198].

The remaining introduction details A.1) to A.5) as per Chapter 2 and Chapter 3, as well as B.1) to B.6) as per Chapter 4. We do not detail C) here. At the end, we explain use of notation, give structure of our discussion, and elaborate on relations to other work.

We summarise our discussion of noncommutative differential structures given in Chapter 2 and construction of quantum optimal transport distances as per Chapter 3. Noncommutative differential structures collect the data which define quantum optimal transport distances. Each consists of two components and one setting. Let $(\phi,\psi,\gamma,\nabla)$ be such noncommutative differential structure for tracial AF-C^*-algebras (A,τ) and (B,ω) in (f,θ)-setting. We briefly describe its components and setting necessary to establish our underlying noncommutative topology, measures and integrals.

The approximately finite-dimensional, or AF-C^*-algebras A and B are C^*-algebras s.t. A is norm closure of $A_0 = \bigcup_{j\in\mathbb{N}} A_j$ and B is that of $B_0 = \bigcup_{j\in\mathbb{N}} B_j$ for ascending chains $\{A_j\}_{j\in\mathbb{N}}$ and $\{B_j\}_{j\in\mathbb{N}}$ of finite-dimensional C^*-algebras [29][38][192]. Their f.s.n. traces $\tau : A_+ \longrightarrow [0,\infty]$ and $\omega : B_+ \longrightarrow [0,\infty]$ are finite on A_0, resp. B_0 [96][192][193]. For all $p \in [1,\infty]$, we define noncommutative L^p-spaces $L^p(A,\tau)$ and $L^p(B,\omega)$ of measurable operators equipped with L^p-norm [130][161]. They fulfil Hölder inequalities. We have a modified standard pairing encoding duality [193]. In particular, get $L^\infty(A,\tau) = L^1(A,\tau)^*$ and $L^\infty(B,\omega) = L^1(B,\omega)^*$. We have state space $\mathscr{S}(A) = \{\mu \in A^*_+ \mid \|\mu\|_A = 1\}$ and normal state space $\mathscr{S}^{\mathrm{N}}(A) = \mathscr{S}(A) \cap L^1(A,\tau)^\flat$ of A. We do not require state spaces of B in our discussion. We see τ and ω are, possibly unbounded [170][171], noncommutative Radon measures (cf. Example A.1.33). States on A are noncommutative probability measures. They are normal if they have noncommutative density in $L^1(A,\tau)$. Elements in B^* are noncommutative totally finite signed outer regular Radon measures [170][171].

We use two components in a single setting. First, we have AF-A-bimodule structure (ϕ,ψ,γ) on B given by local *-homomorphisms $\phi,\psi : A \longrightarrow B$ and anti-linear involution $\gamma : L^2(B,\omega) \longrightarrow L^2(B,\omega)$. AF-$C^*$-bimodules generalise the notion of tracial AF-*-algebras s.t. underlying noncommutative topologies, measures and integrals interact through local *-homomorphisms under anti-linear involutions. We have bounded A-bimodule action on B, called the (ϕ,ψ)-action, given by

$$xuy = L^\phi_x\Big(R^\psi_y(u)\Big) = \phi(x)u\psi(y) \tag{1.1}$$

for all $x,y \in A$ and $u \in B$. The (ϕ,ψ)-action satisfies γ-symmetry given by

$$\gamma\big(\phi(x)u\psi(y)\big) = \phi(y^*)\gamma(u)\psi(x^*) \tag{1.2}$$

in each case. Locality of ϕ and ψ lets us extend Equation 1.1 to a normal, unital and bounded $L^\infty(A,\tau)$-bimodule action on $L^2(B,\omega)$. Moreover, Equation 1.2 extends in turn. We thereby see $L^2(B,\omega)$ is a symmetric W^*-bimodule over $L^\infty(A,\tau)$. This establishes, in full, noncommutative topology, measures and integrals. Secondly, we have a quantum gradient $\nabla : A_0 \longrightarrow L^2(B,\omega)$. It satisfies its own locality condition. The latter shows ∇ is a symmetric W^*-derivation. These are noncommutative gradients with likewise chain rule. The relationship between gradients, heat semigroups and Dirichlet forms extends to the noncommutative setting [63][65]. We further know $\nabla(A_0) \subset B_0$ and $\nabla^*(B_0) \subset A_0$. Dualising $\nabla : A_0 \longrightarrow B_0$ provides the weak formulation of a continuity equation as per Equation 1.11. Elements in B^* serve as synthetic tangent vectors [8][97][103].

The data collected is, by definition or construction, compatible with compression and finite-dimensional approximation by their locality properties. These are two general operations we formalise in a coarse graining process as per Diagram 1.19. To this end, we give two classes of compression used throughout our discussion.

We use two classes of compression. First, we compress to induced AF-C^*-bimodules. For all $j \in \mathbb{N}$, we have induced AF-A_j-bimodule structure $(\phi_j, \psi_j, \gamma_j) = (\phi|_{A_j}, \psi|_{A_j}, \gamma|_{A_j})$ on B_j and j-th restricted quantum gradient $\nabla_j = \nabla|_{A_j} : A_j \longrightarrow B_j$. Finite-dimensional approximation is given by $j \uparrow \infty$ for suitable convergence. Secondly, we compress with projections. Let $p \in L^\infty(A, \tau)$ be a projection. We have tracial W^*-algebra $L^\infty(A[p], \tau) = pL^\infty(A, \tau)p$ and symmetric W^*-bimodule $L^2(B[p], \omega) = pL^2(B, \omega)p$ over the former. We thereby compress the extended (ϕ, ψ)-action with p as

$$xuy = L^\phi_{x,p}\left(R^\psi_{y,p}(u)\right) = \phi(pxp)u\psi(pyp) \tag{1.3}$$

for all $x, y \in L^\infty(A[p], \tau)$ and $u \in L^2(B[p], \omega)$. Locality lets us extend Equation 1.3 to a unital unbounded $L^0(A[p], \tau)$-bimodule action on $L^0(B[p], \omega)$, i.e. to their spaces of measurable operators. For all $x, y \in L^0(A[p], \tau)_h$, get joint spectral measure $E_{x,y,L^\infty(A[p],\tau)}$ and its domain set $\mathscr{S}_p(E_{x,y})$ of suitable $E_{x,y,L^\infty(A[p],\tau)}$-a.e. defined $g : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{C}$ satisfying strong resolvent convergence [88] as $\varepsilon \downarrow 0$ upon ε-perturbation. Each such joint spectral measure determines compressed pulled-back joint functional calculus

$$\Gamma^{L^\phi, R^\psi}_{x,y,p} : \mathscr{S}_p\left(E_{x,y}\right) \longrightarrow \mathscr{UB}\left(L^2(B[p], \omega)\right)_h \tag{1.4}$$

of extended AF-C^*-bimodule actions as per Equation 1.3. Note $\mathscr{UB}(L^2(B[p], \omega))$ is the set of all unbounded operators on $L^2(B[p], \omega)$ here. Let $A_{0,L^\infty(A[p],\tau)}$ be the *-subalgebra generated by pA_0p in $L^\infty(A[p], \tau)$. If p satisfies additional technical properties, then we have p-compressed quantum gradient $\nabla_p = \nabla|_{A_{0,L^\infty(A[p],\tau)}} : A_{0,L^\infty(A[p],\tau)} \longrightarrow L^2(B[p], \omega)$.

Finally, we have a representing function $f : (0, \infty) \longrightarrow (0, \infty)$ of an operator mean [13] together with an interpolation factor $\theta \in [0, 1]$ s.t. $\|\omega\|^{1-\theta} = \omega(1_B)^{1-\theta} < \infty$. We have mean $m_f : (0, \infty) \times (0, \infty) \longrightarrow (0, \infty)$ given by $m_f(t, s) = f(ts^{-1})s$ for all $t, s > 0$. For all $\varepsilon > 0$, we furthermore have mean $m_{f,\varepsilon} : [0, \infty) \longrightarrow (0, \infty)$ perturbed with ε given by $m_{f,\varepsilon}(t, s) = m_f(t + \varepsilon, s + \varepsilon)$ for all $t, s \geq 0$. For all $x, y \in L^0(A[p], \tau)_+$, we have the noncommutative division operators of x and y given by

$$\mathscr{D}^\theta_{x,y,p} = \Gamma^{L^\phi, R^\psi}_{x,y,p}\left(m_f^{-\theta}\right) = m_f^{-\theta}\left(L^\phi_{x,p}, R^\psi_{y,p}\right) \tag{1.5}$$

if $m_f^{-1} \in \mathscr{S}_p(E_{x,y})$. For all $x, y \in L^0(A[p], \tau)_+$ and $\varepsilon > 0$, we also have the noncommutative division operator of x and y perturbed with ε given by

$$\mathscr{D}^\theta_{x^\flat, y^\flat, \varepsilon} = \Gamma^{L^\phi, R^\psi}_{x,y,p}\left(m_{f,\varepsilon}^{-\theta}\right) = m_f^{-\theta}\left(L^\phi_{\mu,\varepsilon}, R^\psi_{\eta,\varepsilon}\right). \tag{1.6}$$

Strong resolvent convergence as $\varepsilon \downarrow 0$ upon ε-perturbation is given by

$$\mathscr{D}^{\theta}_{x,y,p} = \operatorname{sr-lim}_{\varepsilon\downarrow 0} \mathscr{D}^{\theta}_{x^\flat,y^\flat,\varepsilon} = \operatorname{sr-lim}_{\varepsilon\downarrow 0} \Gamma^{L^\phi,R^\psi}_{x,y,p}\left(m^{-\theta}_{f,\varepsilon}\right) = \operatorname{sr-lim}_{\varepsilon\downarrow 0} m^{-\theta}_{f,\varepsilon}\left(L^{\phi}_{x,p}, R^{\psi}_{y,p}\right) \tag{1.7}$$

if $m_f^{-1} \in \mathscr{S}_p(E_{x,y})$. This holds for applications of Equation 1.5 since, assuming fixed parts with integrable support, we show heat flow instantaneously regularises normal states on $A[p]$ to be, possibly unboundedly, invertible up to fixed part. States at finite distance have identical fixed parts under noncommutative heat semigroups of quantum Laplacians as per Equation 1.14. We show a technical but weaker assumption on majorants of local support as per Equation 1.24 is stable under heat flow and ensures integrable support. The latter in turn implies suitable compressibility.

Equation 1.7 itself extends to all states on A without any assumptions by means of quasi-entropies [127][128]. Note quasi-entropies generalise quantum f-divergences [125][126], a class of dissimilarity measures for information encoded in states of quantum systems [62][141]. We use the modified standard pairing, in particular their flat and sharp operators. For all $j \in \mathbb{N}$, we have quasi-entropy $\mathscr{I}^{f,\theta}_j : A^*_{j,+} \times A^*_{j,+} \times B^*_j \longrightarrow [0,\infty]$ in the finite-dimensional setting given by

$$\mathscr{I}^{f,\theta}_j\left(\mu_j,\eta_j,w_j\right) = \sup_{\varepsilon>0} \langle \mathscr{D}^{\theta}_{\mu_j,\eta_j,\varepsilon}\left(\sharp w_j\right), \sharp w_j \rangle_\omega \tag{1.8}$$

for all $\mu,\eta \in A^*_+$ and $w \in B^*$. Note subscripts $j \in \mathbb{N}$ in Equation 1.8 denote restriction to A_j, resp. B_j. Equation 1.8 uses the induced AF-A_j-bimodule structure (ϕ_j,ψ_j,γ_j) on B_j in each case. Monotonicity of quasi-entropies lets us extend Equation 1.8 as claimed to a quasi-entropy $\mathscr{I}^{f,\theta} : A^*_+ \times A^*_+ \times B^* \longrightarrow [0,\infty]$ given by

$$\mathscr{I}^{f,\theta}(\mu,\eta,w) = \sup_{j\in\mathbb{N}} \mathscr{I}^{f,\theta}_j\left(\mu_j,\eta_j,w_j\right) = \lim_{j\in\mathbb{N}} \mathscr{I}^{f,\theta}_j\left(\mu_j,\eta_j,w_j\right) \tag{1.9}$$

for all $\mu,\eta \in A^*_+$ and $w \in B^*$. Equation 1.9 gives quasi-entropies for AF-C^*-bimodules.

Moreover, Equation 1.8 implies Equation 1.9 decomposes as

$$\mathscr{I}^{f,\theta}(\mu,\eta,w) = \sup_{j\in\mathbb{N}} \sup_{\varepsilon>0} \langle \mathscr{D}^{\theta}_{\mu_j,\eta_j,\varepsilon}\left(\sharp w_j\right), \sharp w_j \rangle_\omega = \sup_{\varepsilon>0} \sup_{j\in\mathbb{N}} \langle \mathscr{D}^{\theta}_{\mu_j,\eta_j,\varepsilon}\left(\sharp w_j\right), \sharp w_j \rangle_\omega \tag{1.10}$$

in each case. Using monotonicity of nets in Equation 1.7 and the Kato-Robinson theorem [88], we kill both suprema in Equation 1.10 by taking limits. We consequently obtain closed positive unbounded quadratic forms on $L^2(B,\omega)$ represented uniquely by those positive unbounded operators which extend Equation 1.7 to all states. Quasi-entropies as per Equation 1.9 define energy functionals as per Equation 1.12 by integrating their own evaluation on admissible paths. Altogether, we extend the quasi-entropy approach for defining noncommutative division operators in [50] to AF-C^*-bimodules.

We construct quantum optimal transport distances using data as above. This follows the classical case [97]. Let $\overline{\mathscr{S}(A)}$ denote the w^*-closure of $\mathscr{S}(A) \subset A_+^*$. We metricise its w^*-topology and obtain a compact metric space. This uses separability of A. Note the Arzelà-Ascoli theorem applies to paths in compact metric spaces [136]. For all $I = [a,b] \subset \mathbb{R}$, we have the set $\mathrm{AC}(I, \mathscr{S}(A))$ of all weakly absolutely continuous $\mu : I \longrightarrow \overline{\mathscr{S}(A)}$ s.t. $\mathrm{im}\,\mu \subset \mathscr{S}(A)$. We say that $(\mu, w) \in \mathrm{AC}([a,b], \mathscr{S}(A)) \times L^2([a,b], B^*)_{\mathrm{w}}$ is an admissible path if (μ, w) satisfies

$$\frac{d}{dt}\mu(t)(x) = w(t)(\nabla x) = \lim_{j\in\mathbb{N}} w_j(t)\big(\nabla_j x_j\big) \tag{1.11}$$

for all $x \in A_0$ and a.e. $t \in [a,b]$. We call $\mu(a), \mu(b) \in \mathscr{S}(A)$ the marginals of (μ, w), resp. μ in this case. Note $L^2([a,b], B^*)_{\mathrm{w}}$ is the Banach dual space of the Bochner L^2-space $L^2([a,b], B)$, and the second identity in Equation 1.11 holds in general.

We require some bookkeeping. For all $\mu^0, \mu^1 \in \mathscr{S}(A)$, we have the set $\mathrm{Adm}^{[a,b]}(\mu^0, \mu^1)$ of all admissible paths defined on $[a,b] \subset \mathbb{R}$ with marginals μ^0 and μ^1. We further have the set $\mathrm{Adm}^{[a,b]}$ of all admissible paths defined on $[a,b] \subset \mathbb{R}$ regardless of marginals, as well as the set Adm of all admissible paths regardless of either definition intervals or marginals. We therefore have energy functional $E^{f,\theta} : \mathrm{Adm} \longrightarrow [0,\infty]$ given by

$$E^{f,\theta}(\mu, w) = \int_a^b \mathscr{I}^{f,\theta}\big(\mu(t), \mu(t), w(t)\big)dt = \lim_{j\in\mathbb{N}} \int_a^b \mathscr{I}_j^{f,\theta}\big(\bar{\mu}_j(t), \bar{\mu}_j(t), \bar{w}_j(t)\big)dt \tag{1.12}$$

for all $[a,b] \subset \mathbb{R}$ and $(\mu, w) \in \mathrm{Adm}^{[a,b]}$. Note, in contrast to Equation 1.11, subscripts $j \in \mathbb{N}$ in Equation 1.12 denote normalised restriction to A_j, resp. B_j via bars. We normalise to norm one in the first two variables, and in the third one s.t. Equation 1.11 remains satisfied. Since normalisation invalidates monotonicity of quasi-entropies, Equation 1.12 is not a supremum in general even as Equation 1.9 is. Upon restricting domains to sets of admissible paths with identical interval and marginals, we further show energy functionals as per Equation 1.12 are Γ-limits [74] of suitable restrictions.

We therefore have the quantum optimal transport distance of $(\phi, \psi, \gamma, \nabla)$ on $\mathscr{S}(A)$ in (f, θ)-setting given by

$$\mathscr{W}_{\nabla}^{f,\theta}\big(\mu^0, \mu^1\big) = \inf_{\mathrm{Adm}^{[0,1]}(\mu^0,\mu^1)} \sqrt{E^{f,\theta}(\mu, w)} \tag{1.13}$$

for all $\mu^0, \mu^1 \in \mathscr{S}(A)$. Accessibility components of quantum optimal transport distances are complete geodesic length-metric spaces. Metric geometry reduces to accessibility components. There may exist uncountable infinitely many since sets of states at finite distance have identical fixed parts under noncommutative heat semigroups of quantum Laplacians. Assuming spectral gaps of quantum Laplacians and fixed parts, we use such fixed parts to classify accessibility components of square integrable normal states. We in turn use the latter classification for the coarse graining process since its assumptions are satisfied for all accessibility components in the finite-dimensional setting.

Classification uses regularisation of normal states under heat flow as mentioned for Equation 1.7. We have heat semigroup $h : [0,\infty) \longrightarrow \mathscr{B}(L^2(A,\tau))$ of $\Delta = \nabla^*\nabla$ given by

$$h_t(u) = e^{-t\Delta}(u) \tag{1.14}$$

for all $t \geq 0$ and $u \in L^2(A,\tau)$. The heat semigroup of Δ extends as follows. For all $j \in \mathbb{N}$, we have symmetric C^*-derivation $\nabla_j : A_j \longrightarrow B_j$. We obtain C^*-Dirichlet form $u \mapsto \|\nabla_j u\|_\tau^2$ on A_j in each case [65]. Using the latter, we have completely Markovian semigroup $h^j : [0,\infty) \longrightarrow \mathscr{B}(A_j)$ as well [63]. Note completely Markovian semigroups [83][85][86] and their extensions to Banach dual spaces are given by completely positive dilations [63]. Iterated dualisation using the modified standard pairing extends Equation 1.14 accordingly. Altogether, we have noncommutative heat semigroup of Δ mapping to $\mathscr{B}(V)$ if $V = A^*$ or $V = L^p(A,\tau)$ for $p \in \{1,2,\infty\}$.

For all $\mu \in A^*$, $h(\mu) = h_\infty(\mu)$ is its fixed part and $h^\perp(\mu) = \mu - h(\mu)$ its image part. We call $\xi \in \mathscr{S}(A)$ a fixed state, or fixed if $h(\xi) = \xi$. For all fixed states $\xi \in \mathscr{S}(A)$, we have the set $\mathrm{Fix}_A(\xi) = \{\mu \in \mathscr{S}(A) \mid h(\mu) = \xi\}$ of states on A with fixed part ξ, as well as the set $\mathscr{C}_A(\xi) = \{\mu \in \mathscr{S}(A) \mid \mu \sim \xi\}$ of states on A at finite distance to ξ. Intersecting with $\mathscr{S}^{\mathrm{N}}(A)$ yields the set $\mathrm{Fix}_A^{\mathrm{N}}(\xi)$, resp. $\mathscr{C}_A^{\mathrm{N}}(\xi)$ of such normal states on A. These sets underpin both classification and regularisation. For all fixed states $\xi \in \mathscr{S}(A)$, we have $\mathscr{C}_A(\xi) \subset \mathrm{Fix}_A(\xi)$ and decomposition

$$\mathrm{Fix}_A(\xi) = \coprod_{\mathscr{C} \subset \mathrm{Fix}_A(\xi)} \mathscr{C} \tag{1.15}$$

into accessibility components. Let $\xi \in \mathscr{S}(A)$ be a fixed state. We say that an accessibility component $\mathscr{C} \subset (\mathscr{S}(A), \mathscr{W}_\nabla^{f,\theta})$ has fixed part ξ if $\mathscr{C} \subset \mathrm{Fix}_A(\xi)$.

Assume $\xi \in \mathscr{S}^{\mathrm{N}}(A)$ has integrable support. For all $\mu \in \mathrm{Fix}_A^{\mathrm{N}}(\xi)$, we have

$$h_t(\mu) \in \mathscr{S}_{>0}^{\mathrm{N}}\big(A[\mathrm{supp}\,\xi]\big) \tag{1.16}$$

for all $t \in (0,\infty]$. Note Equation 1.16 uses the support projection $\mathrm{supp}\,\xi \in L^\infty(A,\tau)$ of ξ. We have $\mathrm{supp}\,\xi$-compressibility and write $A_\xi = A[\mathrm{supp}\,\xi]$. As such, the subscript in Equation 1.16 denotes normal states on A_ξ s.t. densities are unboundedly invertible in $\mathscr{U}\mathscr{B}(L^2(A_\xi,\tau))$ under compressed canonical left- and right-action. If ξ, resp. its density is boundedly invertible in this sense and square integrable, then, assuming Δ has spectral gap, regularisation as per Equation 1.16 lets us show

$$\mathscr{C}_A^{\mathrm{N},2}(\xi) = \mathscr{C}_A(\xi) \cap \mathscr{S}^{\mathrm{N},2}(A) = \mathrm{Fix}_A(\xi) \cap \mathscr{S}^{\mathrm{N},2}(A). \tag{1.17}$$

Upon intersecting $\mathrm{Fix}_A(\xi)$ with the set $\mathscr{S}^{\mathrm{N},2}(A)$ of all square integrable normal states on A, we at once see Equation 1.15 and Equation 1.17 show we classify as claimed. In the finite-dimensional setting, assumptions as above are always satisfied and we therefore classify all accessibility components using fixed parts.

The coarse graining process as per Diagram 1.19 uses classification of accessibility components as per Equation 1.17 in the finite-dimensional setting and lets us view quantum optimal transport as transport of, suitably general, quantum information [43] [62][95]. We use compression for all its vertical chains of arrows and finite-dimensional approximation for its horizontal ones. The coarse graining process decomposes global pictures, objects and properties into sequences of local ones together with a uniformity condition ensuring convergence of limits.

For all $j \in \mathbb{N}$, we use induced AF-A_j-bimodule structure on B_j and j-th restricted quantum gradient $\nabla_j : A_j \longrightarrow B_j$. For all $\mu^0, \mu^1 \in \mathscr{S}(A)$, we have

$$\mathscr{W}_{\nabla}^{f,\theta}(\mu^0, \mu^1) = \lim_{j \in \mathbb{N}} \mathscr{W}_{\nabla_j}^{f,\theta}(\bar{\mu}_j^0, \bar{\mu}_j^1). \tag{1.18}$$

Note we do have a uniformity condition as required for Equation 1.18 because $\mathscr{W}_{\nabla}^{f,\theta}$ is l.s.c. in w^*-topology. In particular, we show, a priori, states are at finite distance if and only if the limit on the right-hand side of Equation 1.18 exists.

Diagram 1.19 itself expands the underlying process generating the limit on the right-hand side of Equation 1.18. Let $j_{\min} \in \mathbb{N}$ minimal among all $j \in \mathbb{N}$ s.t. $\xi_j \neq 0$. For all $j \geq j_{\min}$ in $\mathbb{N}$, we consider normalised restriction $\bar{\xi}_j \in \mathscr{S}(A_j)$, i.e. a fixed state, as well $\mathscr{F}_{A_j}(\bar{\xi}_j)$ and $\mathscr{C}_{A_j}(\bar{\xi}_j)$. We require convex subset $K \subset \mathscr{S}(A)$ to have lower left corner in

$$\begin{tikzcd}
A^* \arrow[r, two heads] & \cdots \arrow[r, two heads] & A_j^* \arrow[r, two heads] & \cdots \arrow[r, two heads] & A_{j_{\min}}^* \\
\mathscr{F}_A(\xi) \arrow[u, hook] \arrow[r, two heads] & \cdots \arrow[r, two heads] & \mathscr{F}_{A_j}(\bar{\xi}_j) \arrow[u, hook] \arrow[r, two heads] & \cdots \arrow[r, two heads] & \mathscr{F}_{A_{j_{\min}}}(\bar{\xi}_{j_{\min}}) \arrow[u, hook] \\
\mathscr{C} \cap K \arrow[u, hook] \arrow[r, two heads] & \cdots \arrow[r, two heads] & \mathscr{C}_{A_j}(\bar{\xi}_j) \arrow[u, hook] \arrow[r, two heads] & \cdots \arrow[r, two heads] & \mathscr{C}_{A_{j_{\min}}}(\bar{\xi}_{j_{\min}}) \arrow[u, hook]
\end{tikzcd} \tag{1.19}$$

We show the AF-C^*-setting yields noncommutative analogues of scaling limits [122]. As such, Diagram 1.19 lets us argue we transport scaling limits of uniformly conditioned spin states encoding sequences of qubits [42][43][62][93][95]. Non-ergodicity restricts information-bearing degrees of freedom. Since energy functionals are Γ-limits w.r.t. the coarse graining process, the latter reduces the AF-C^*-setting to the finite-dimensional one s.t. ergodicity is recovered up to a controlled remainder by reducing to accessibility components in the finite-dimensional setting. If K in Diagram 1.19 equals the domain of quantum relative entropy as per Equation 1.23, then we are able to apply the coarse graining process in Chapter 4. Altogether, we study a non-spatial transport of quantum information with restricted information-bearing degrees of freedom.

We describe our results in Chapter 4 for the logarithmic mean setting. We are in the latter setting if $\theta = 1$ and we use the unique symmetric representing function $f = f_{\log}$ of the logarithmic operator mean $m_{\log} = m_{f_{\log}} : (0,\infty) \times (0,\infty) \longrightarrow (0,\infty)$ given by

$$m_{\log}(t,s) = \frac{t-s}{\log t - \log s} = \int_0^1 t^\alpha s^{1-\alpha} d\alpha \tag{1.20}$$

for all $t,s > 0$. We consider fixed state $\xi \in \mathscr{S}(A)$ as above. We further suppress its, by assumption integrable, support projection $\operatorname{supp}\xi$ in all subscripts and write ξ instead. If $x > 0$ in $L^\infty(A_\xi,\tau)$, then Equation 1.20 implies the noncommutative division operator of $x = y$ as per Equation 1.5 acts by

$$\mathscr{D}_{x,\xi}(u) = \int_0^\infty \left(\alpha I + L^\phi_{x,p}\right)^{-1}\left(\left(\alpha I + R^\psi_{x,p}\right)^{-1}(u)\right) d\alpha \tag{1.21}$$

for all $u \in L^2(B_\xi,\omega)$. Note Equation 1.21 corresponds to multiplication with inverses of densities in the classical case [97], resp. use of the Kubo-Mori-Bogoliubov inner product [176] in [50]. As such, Equation 1.21 yields quantum L^2-Wasserstein distances in direct analogy to the classical case [97].

If $x \in L^\infty(A_\xi,\tau)_\nabla$ s.t. $x > 0$ in $L^\infty(A_\xi,\tau)$, i.e. a boundedly invertible element in the C^1-algebra of ∇ upon compressing the latter with $\operatorname{supp}\xi$, then $\log x \in L^\infty(A_\xi,\tau)_\nabla$ as well and the noncommutative chain rule shows we have

$$\nabla_\xi \log x = \mathscr{D}_{x,\xi} \nabla_\xi x. \tag{1.22}$$

Using results in [172], Equation 1.22 implies heat flow is, up to coarse graining, gradient flow of quantum relative entropy as per Equation 1.23 on relative interiors. Heat flow further satisfies a steepest entropy ascent property [25] by considering the steepest descent property of gradient flows in smooth Riemannian manifolds [144] and taking limits. We seek conditions s.t. steepest entropy ascent implies quantum noise evolution as per B.2). We accomplish this with our maximum entropy production principle [91] [92][155]. Applying heat flow to a state for $t > 0$ introduces quantum noise in B.4).

Umegaki defined relative entropy for semi-finite W^*-algebras [196]. Using relative modular operators, Araki generalised to all W^*-algebras [16][17]. We extend Kosaki's formula [163] in the second variable to get the relative entropy $\operatorname{Ent}^\tau : A^*_+ \longrightarrow [-\infty,\infty]$ w.r.t. τ, i.e. quantum relative entropy. It measures information required to discriminate a given state and, possibly non-finite, trace through observation. If $\mu \notin L^1(A,\tau)^\flat_+$, then $\mu \notin \operatorname{dom}\operatorname{Ent}^\tau$, i.e. $|\operatorname{Ent}(\mu,\tau)| = \infty$ as expected. If $\mu \in L^1(A,\tau)^\flat_+$ and $p \in L^1(A,\tau) \cap L^\infty(A,\tau)$ is a projection s.t. $\operatorname{supp}\mu \leq p$, then $\operatorname{Ent}(\mu,\tau) > -\infty$ and we have

$$\operatorname{Ent}(\mu,\tau) = \sup_{\substack{n\in\mathbb{N},\\ F\in\mathscr{T}^u_n(A[p])}} \left\{ \|\mu\|_{A[p]^*} \log n - \int_{n^{-1}}^\infty t^{-1}\left\|p - F(t)\right\|^2_\mu + t^{-2}\left\|F(t)\right\|^2_\tau dt \right\}, \tag{1.23}$$

where we take the supremum over all suitable step functions $F : (n^{-1},\infty) \longrightarrow A[p]$ and use the GNS-inner product $\|.\|_\mu$ of μ, resp. $\|.\|_\tau$ of τ [192][193].

The negative of Umegaki's definition is quantum entropy, i.e. von Neumann entropy (cf. p.17 in [163]). Equation 1.23 reduces to Umegaki's definition if $\tau < \infty$. We further know it is jointly convex, l.s.c. in w^*-topology of $L^\infty(A,\tau)$ and has restriction property in this case. Either may fail if (A,τ) is not strongly unital. Uniform majorisation of the local support of fixed parts suffices to prevent failure and recover a finite-dimensional approximation property. As such, we require l.s.c. in topology of the given quantum optimal transport distance on all accessibility components with suitable fixed part, as well as compatibility with compression and finite-dimensional approximation.

For this, we compress with projections as per Equation 1.23 in general. We say that $p \in L^1(A,\tau) \cap L^\infty(A,\tau)$ majorises the local support of ξ if

$$\operatorname{supp} \xi_j \le p \tag{1.24}$$

in $L^\infty(A,\tau)$ for a.e. $j \in \mathbb{N}$. We further call p a majorant of the local support of ξ. We say that ξ is finitely supported if $\xi \in \operatorname{dom}\operatorname{Ent}^\tau$ and there exists a majorant of its local support. Assume the latter. For all $\mu \in \operatorname{Fix}^{\mathrm{N}}_A(\xi)$, finite-dimensional approximation is

$$\operatorname{Ent}(\mu,\tau) = \lim_{j\in\mathbb{N}} \operatorname{Ent}\big(\mu_j,\tau\big) = \lim_{j\in\mathbb{N}} \operatorname{Ent}\big(\bar{\mu}_j,\tau\big). \tag{1.25}$$

Finally, we show $\operatorname{Ent}^\tau : \operatorname{Fix}^{\mathrm{N}}_A(\xi) \longrightarrow (-\infty,\infty]$ is l.s.c. in $\mathscr{W}^{f,\theta}_\nabla$-topology. We need not assume the logarithmic mean setting in our discussion of quantum relative entropy.

We use quantum relative entropy as measure of quantum information. Assume the logarithmic mean setting. We write $\mathscr{I}^{\log} := \mathscr{I}^{f,1}$, as well as $E^{\log} = E^{f,1}$ and $\mathscr{W}^{\log}_\nabla = \mathscr{W}^{f,1}_\nabla$. For all $\mu^0,\mu^1 \in \mathscr{S}(A)$, the set $\operatorname{Geo}(\mu^0,\mu^1)$ of all minimising geodesics with marginals μ^0 and μ^1 is non-empty if the latter are at finite distance. Lower Ricci bounds are given by λ-convexity of quantum information as per CNV_λ below along minimising geodesics measured by quantum relative entropy. Let $\xi \in \mathscr{S}(A)$ be a finitely supported fixed state. Let $\mathscr{C} \subset (\mathscr{S}(A), \mathscr{W}^{\log}_\nabla)$ be finitely supported with fixed part ξ s.t. $\mathscr{C} \cap \operatorname{dom}\operatorname{Ent}^\tau \neq \emptyset$. Let $\lambda \in \mathbb{R}$ here. We know Ent^τ is λ-convex in the sense of metric geometry [8][160] if for all $\mu^0,\mu^1 \in \mathscr{C} \cap \operatorname{dom}\operatorname{Ent}^\tau$ and $(\mu,w) \in \operatorname{Geo}(\mu^0,\mu^1)$ s.t. $\mu(t) \in \operatorname{dom}\operatorname{Ent}^\tau$ for all $t \ge 0$, we have

$$\operatorname{Ent}\big(\mu(t),\tau\big) \le (1-t)\operatorname{Ent}\big(\mu^0,\tau\big) + t\operatorname{Ent}\big(\mu^1,\tau\big) - \frac{\lambda}{2} t(1-t) \mathscr{W}^{\log}_\nabla\big(\mu^0,\mu^1\big)^2 \tag{CNV_λ}$$

for all $t \in [0,1]$. We follow [151] and [189][190], resp. [50][106] in our definition. We use CNV_λ to view lower Ricci bounds as measurement convexity of quantum information. If we have noncommutative analogues of displacement interpolations [72][156], then such measurement convexity in the Schrödinger picture is convexity under measurement of observables in the Heisenberg picture. Unfortunately, existence results are unknown to us. We instead show strictly positive lower Ricci bounds determine energy-information trade-offs parametrised by lower bounds on quantum noise. Lower resolution implies lower energy paths. We avoid spatial interpretations of the classical case [97][151].

Strictly speaking, we apply our equivalence theorem to define lower Ricci bounds of quantum gradients in direct analogy to the classical case [9][10][11][12][105], resp. as per [50][106] using CNV_λ together with all of the following equivalent conditions. We see $h:[0,\infty)\times\mathscr{C}\cap\operatorname{dom}\operatorname{Ent}^\tau\longrightarrow\mathscr{C}\cap\operatorname{dom}\operatorname{Ent}^\tau$ is EVI_λ-gradient flow of Ent^τ in $\mathscr{C}\cap\operatorname{dom}\operatorname{Ent}^\tau$ in the sense of metric geometry [8][160] if for all $\mu,\eta\in\mathscr{C}\cap\operatorname{dom}\operatorname{Ent}^\tau$, we have

$$\frac{e^{\lambda(t-s)}}{2}\mathscr{W}_\nabla^{\log}\big(h_t(\mu),\eta\big)^2-\frac{1}{2}\mathscr{W}_\nabla^{\log}\big(h_s(\mu),\eta\big)^2\leq\int_0^{t-s}e^{\lambda r}dr\cdot\Big(\operatorname{Ent}(\eta,\tau)-\operatorname{Ent}\big(h_t(\mu),\tau\big)\Big)\quad(\mathrm{EVI}_\lambda^\int)$$

for all $0<s<t<\infty$. Note $\mathrm{EVI}_\lambda^\int$ as above is the well-known integral characterisation of EVI_λ-gradient flows [8], denoted by EVI_λ throughout our discussion. If EVI_λ-gradient flow of relative entropy exists, then it is heat flow as above.

Equivalence of EVI_λ and CNV_λ is also well-known [160]. We have three equivalent global conditions. Upon ranging over all finitely supported accessibility components as above, the first one is EVI_λ and the second one is CNV_λ. The third one is a, possibly infinite-dimensional, Bakry-Émery condition [19][20] adapted to the logarithmic mean setting as per [50]. We say that h satisfies BE_λ if for all finitely supported fixed states $\xi\in\mathscr{S}(A)$ and $\mathscr{C}\subset(\mathscr{S}(A),\mathscr{W}_\nabla^{\log})$ with fixed part ξ, we have

$$\big\|\mathscr{M}_{\sharp\mu}^{\frac{1}{2}}\nabla h_t(u)\big\|_\omega^2\leq e^{-2\lambda t}\big\|\mathscr{M}_{h_t(\sharp\mu)}^{\frac{1}{2}}\nabla u\big\|_\omega^2\qquad(\mathrm{BE}_\lambda)$$

for all $\mu\in\mathscr{C}\cap L^{2,\infty}(A_\xi,\tau)^\flat$, $u\in\operatorname{dom}\nabla_\xi$ and $t\geq0$. Note BE_λ uses those noncommutative multiplication operators whose inverses are noncommutative division operators as per Equation 1.5. Compatibility with compression and finite-dimensional approximation of all objects involved, in particular but not only finite-dimensional approximation as per Equation 1.18 and Equation 1.25, ensure all three global conditions arise from and are equivalent to three local conditions mirroring the above in the finite-dimensional setting for a.e. induced noncommutative differential structure.

We therefore have EVI_λ-gradient flow, λ-convexity and Bakry-Émery conditions in global and local form. We cannot show their equivalence directly. For this, we consider a Hessian lower bound condition H_λ as per [50]. In the finite-dimensional logarithmic mean setting, we require such to show equivalence as claimed. We are motivated in our proof by analogous arguments in [50] and [106]. However, we must use two differential equations for Hessians of quantum relative entropy in order to replace essential steps therein letting us argue using Riemannian metrics on relative interiors induced by the given quasi-entropy. We say that $\operatorname{Hess}\operatorname{Ent}^\tau$ has lower bound λ if for all for all finitely supported fixed states $\xi\in\mathscr{S}(A)$ and a.e. $j\in\mathbb{N}$ in each case, we have

$$\operatorname{Hess}_\mu\operatorname{Ent}^\tau(\eta)\geq\lambda g_\mu^{\bar\xi_j}(\eta,\eta)\qquad(\mathrm{H}_\lambda)$$

for all $\mu\in\vartheta(\bar\xi_j)$ and $\eta\in I(\Delta_{\bar\xi_j})^\flat$. Each $\vartheta(\bar\xi_j)=\operatorname{relint}\mathscr{C}_{A_j}(\bar\xi_j)$ and $\vartheta(\bar\xi_j)\times I(\Delta_{\bar\xi_j})^\flat$ is a smooth Riemannian manifold, resp. its trivial tangent bundle plus Riemannian metric as per the right-hand side of H_λ above. Taking limits yields equivalence as claimed.

Following Diagram 1.19, it is H_λ which most clearly shows how underlying metric geometric properties such as lower Ricci bounds may be scaling limits of Riemannian ones up to heat flow regularised boundary. This requires suitable K in Diagram 1.19. Let $\xi \in \mathscr{S}(A)$ be a fixed state. If $\xi \in \mathscr{S}(A)$ is finitely supported fixed state, then, assuming strictly positive lower Ricci bounds, existence of a unique minimum for EVI_λ-gradient flows of l.s.c. functionals with complete sublevels [160] lets us show

$$\mathscr{C}_A^{\mathrm{Ent}}(\xi) = \mathscr{C}_A(\xi) \cap \operatorname{dom}\mathrm{Ent}^\tau = \mathrm{Fix}_A(\xi) \cap \operatorname{dom}\mathrm{Ent}^\tau \neq \emptyset. \tag{1.26}$$

As for Equation 1.17 and $K = \mathscr{S}^{\mathrm{N},2}(A)$, Equation 1.26 and $K = \operatorname{dom}\mathrm{Ent}^\tau$ readily show we classify accessibility components of normal states with finite quantum relative entropy using fixed parts. This yields suitable K, as is visible from our equivalent conditions above. Strictly lower Ricci bounds avoid assumptions on spectral gaps. Equation 1.26 lets us formulate energy-information trade-offs as claimed using Talagrand inequality TW_λ for $\lambda \geq 0$ as given below. It formulates an energy-information trade-off since lower energy paths are obtained by introducing quantum noise. The latter requires our view of quantum Laplacians as generators of quantum noise evolution as per B.2).

We then give sufficient conditions for strictly positive lower Ricci bounds of direct sum quantum gradients. We adapt the proof of Theorem 10.9 in [50] for λ-intertwining symmetric C^*-derivations to the AF-C^*-setting by means of the coarse graining process. We give an essential estimate for quasi-entropies evaluated on states under heat flow extending its analogue in [50] to the AF-C^*-setting. Our proof requires an extension of Theorem 5 in [127] to all finite-dimensional C^*-algebras. Examples for strictly positive lower Ricci bounds are twisted dynamic quantum gradients induced by intertwining sets of Clifford generators. This generalises [48] but needs detailed implementation of Bogoliubov automorphisms on anti-symmetric Fock space [114][177].

Assuming lower Ricci bounds, we derive functional inequalities HWI_λ, MLSI_λ and TW_λ for $\lambda \geq 0$, resp. $\lambda > 0$ as per [50]. Non-ergodicity requires relative entropy of finitely supported fixed states in their formulation. We introduce quantum Fisher information in the AF-C^*-setting. Its rôle mirrors the classical case [151][168]. We have quantum Fisher information $\mathrm{I}^{\log} : A_+^* \longrightarrow [0,\infty]$ given by

$$\mathrm{I}^{\log}(\mu) = \sup_{j \in \mathbb{N}} \mathscr{I}_j^{\log}\Big(\mu_j, \mu_j, \big(\nabla \sharp \mu_j\big)^\flat\Big) \tag{1.27}$$

for all $\mu \in A_+^*$. Equation 1.27 immediately shows quantum Fisher information inherits properties of quasi-entropies. For all finitely supported fixed states $\xi \in \mathscr{S}(A)$, we use the inherited properties and the gradient flow property to show

$$\mathrm{I}^{\log}(\mu) = -\frac{d}{dt}\bigg|_{t=0} \mathrm{Ent}^\tau\big(h_t(\mu)\big) \tag{1.28}$$

for all $\mu \in \mathrm{Fix}_A^{\mathrm{N}}(\xi) \cap \mathscr{S}^{\mathrm{N}}(A_\xi) \cap \mathrm{GL}(L^\infty(A_\xi,\tau)) \cap (\operatorname{dom}\Delta)^\flat$. Note $\mathrm{GL}(L^\infty(A_\xi,\tau))$ is the set of all boundedly invertible elements in $L^\infty(A_\xi,\tau)$. Equation 1.28 implies $\mathrm{I}^{\log}(\mu)$ is indeed a noncommutative analogue for parametrisations $\{h_t(\mu)\}_{t\geq 0}$ given $\mu \in \mathscr{S}^{\mathrm{N}}(A)$.

We adapt the proof of Proposition 11.2 in [50] to the AF-C^*-setting by means of the coarse graining process. For all $\mu, \eta \in \mathscr{S}(A)$, Equation 1.28 lets us show

$$\limsup_{j \in \mathbb{N}} \frac{d^+}{dt} \mathscr{W}_{\nabla}^{\log}\big(h_t(\bar{\mu}_j), \bar{\eta}_j\big) \leq \sqrt{\mathrm{I}^{\log}\big(h_t(\mu)\big)} \tag{1.29}$$

for all $t \geq 0$. Equation 1.29 in turn provides sufficient control of metric derivatives using quantum Fisher information. It is the crucial estimate allowing us to adapt the proofs of Theorem 11.3, Theorem 11.4 and Theorem 11.5 in [50] to the AF-C^*-setting by means of the coarse graining process.

We derive three functional inequalities. Let $\lambda \in \mathbb{R}$. We say that Ent^τ satisfies HWI_λ if for all finitely supported fixed states $\xi \in \mathscr{S}(A)$ and $\mathscr{C} \subset (\mathscr{S}(A), \mathscr{W}_{\nabla}^{\log})$ with fixed part ξ s.t. $\mathscr{C} \cap \operatorname{dom} \mathrm{Ent}^\tau \neq \emptyset$, we have

$$\mathrm{Ent}(\mu, \tau) \leq \mathscr{W}_{\nabla}^{\log}(\mu, \xi) \sqrt{\mathrm{I}^{\log}(\mu)} - \frac{\lambda}{2} \mathscr{W}_{\nabla}^{\log}(\mu, \xi)^2 + \mathrm{Ent}(\xi, \tau) \tag{HWI_λ}$$

for all $\mu \in \mathscr{C}$. Assume $\lambda > 0$. We say that Ent^τ satisfies MLSI_λ if for all finitely supported fixed states $\xi \in \mathscr{S}(A)$ and $\mathscr{C} \subset (\mathscr{S}(A), \mathscr{W}_{\nabla}^{\log})$ with fixed part ξ s.t. $\mathscr{C} \cap \operatorname{dom} \mathrm{Ent}^\tau \neq \emptyset$, we have

$$\mathrm{Ent}(\mu, \tau) \leq \frac{1}{2\lambda} \mathrm{I}^{\log}(\mu) + \mathrm{Ent}(\xi, \tau) \tag{MLSI_λ}$$

for all $\mu \in \mathscr{C}$. We further say that Ent^τ satisfies TW_λ if for all finitely supported fixed states $\xi \in \mathscr{S}(A)$ and $\mathscr{C} \subset (\mathscr{S}(A), \mathscr{W}_{\nabla}^{\log})$ with fixed part ξ s.t. $\mathscr{C} \cap \operatorname{dom} \mathrm{Ent}^\tau \neq \emptyset$, we have

$$\mathscr{W}_{\nabla}^{\log}(\mu, \xi) \leq \sqrt{\frac{2}{\lambda}\big(\mathrm{Ent}(\mu, \tau) - \mathrm{Ent}(\xi, \tau)\big)} \tag{TW_λ}$$

for all $\mu \in \mathscr{C}$. We obtain the following implications in direct analogy to the classical case [151][168] and extending results in [50] to the AF-C^*-setting as claimed. Analogous to our equivalent conditions for lower Ricci bounds, all three functional inequalities above are scaling limits w.r.t. the coarse graining process.

We have an expected chain of functional inequalities. If we do have lower Ricci bounds for $\lambda \in \mathbb{R}$, then Ent^τ satisfies HWI_λ. If Ent^τ satisfies HWI_λ for $\lambda > 0$, then Ent^τ in turn satisfies MLSI_λ. If Ent^τ satisfies MLSI_λ, then Ent^τ finally satisfies TW_λ. Their proofs pass through the finite-dimensional setting.

Notation. We follow notational conventions of our stated standard references whenever possible. However, we must tie together different ones tailored to our use. We establish a single coherent notation in our definitions and paragraphs marked as **Notation**. Unless stated otherwise, the latter are in force once stated. This includes notation given in the appendix. The latter are revisited in the main matter prior to first use.

Structure. We divide our discussion into main matter and its appendix. The latter gives auxiliary technical results. In Chapter 2, we discuss the data necessary to define quantum optimal transport distances and collect such initial data in noncommutative differential structures. This covers A.1). In Chapter 3, we define our quantum optimal transport distances, discuss fundamental properties and provide fundamental example classes. This covers A.2) to A.5), and C). In Chapter 4, we construct quantum relative entropy for, possibly non-finite, traces, discuss the logarithmic mean setting, and extend results in [48][49][50] and [106] to the AF-C^*-setting. This covers B.1) to B.6).

Relations to other work. We may categorise noncommutative optimal transport into dynamic [37][48][49][50][57][58][59][200] and static [14][90][99][112][66] formulations. As explained above, quantum optimal transport distances as per A.2) are dynamic transport distances motivated by Benamou-Brenier-type distances [24][97]. The latter is shared by all dynamic formulations. Following work of Maas and Mielke for the discrete cases [152][159], Carlen and Maas pioneered the dynamic formulation in [49][50] to study quantum Fokker-Planck equations [48]. Our discussion, resp. any of its prior versions, and independent but concurrent work of Wirth [200] together with Zhang [202] are the first infinite-dimensional dynamic formulations and extensions of results in [48][49][50]. Assuming regular operator mean and restricting to densities, i.e. normal states, the dynamic formulation in [200] and our discussion coincide. However, each has considerably different technical foundation, assumptions and applicability. Results and their proofs, as well as range of examples, differ accordingly. We closely examine these differences further below. In general terms, we see our bottom-up design yields flexible architecture for the AF-C^*-setting capable of stronger results therein.

All dynamic formulations avoid the lack of a natural noncommutative analogue of conditioning for couplings [47]. Recent static ones consider specific sets of couplings or quantum channels for trace-class operators on Hilbert spaces [14][90][66], balanced transport plans [99][101], or use entropic regularisation [112]. Noncommutative duality formulas remain difficult to find. Following the work of Erbar, Maas and Wirth [108] and Gangbo, Li and Mou [119] for discrete cases, Wirth gives such a duality formula [201] for quantum optimal transport distances in the finite-dimensional setting, resp. their entropic regularisations [22], via subsolutions of Hamilton-Jacobi-Bellmann equations [30][168]. We do not have an infinite-dimensional extension but consider finding one by means of the coarse graining process a test of our approach we defer to future work.

We focus on the relation of our main contributions to the two most related dynamic formulations [48][49][50] and [200]. Moreover, we consider the use of our discussion for studying noncommutative gauge theories [51][54][55][197][198] within Connes' program of noncommutative geometry [67][69][137][138]. This leads us to view quantum optimal transport as transport of quantum information [62] without considering spatial coordinates [68]. Furthermore, we view quantum Laplacians as generators of quantum noise evolution as per B.2) in order to have non-spatiality of lower Ricci bounds as per B.4) and associated energy-information trade-offs. Applications of other approaches to entropic inequalities [6], quantum channels [90][100][120], statistical learning [22] and variational algorithms [89], among several more [59][76][77], fit our point of view.

We explain similarities and differences of foundational work of Carlen and Maas [48][49][50], as well as related work of Wirth [200], resp. Wirth and Zhang [202], to our discussion. As explained in the introduction, we extend results in [48][49][50] and [106] to the AF-C^*-setting as per A.1) by means of the coarse graining process as per A.4). Assumptions differ from ours in two points apart from dimensionality. First, they allow for, possibly non-tracial, weights [193]. Of course, they are finite. We assume traciality but not finiteness. Traciality implies neither our discussion nor [200] fully subsumes [50]. Secondly, they assume ergodicity and we do not. Using our assumptions, which let us cover all fundamental example classes as per C), we extend the equivalence in [50] to that of the EVI_λ-gradient flow of quantum relative entropy as per B.1), its strong geodesic λ-convexity, a, possibly infinite-dimensional, Bakry-Émery condition, and a Hessian lower bound condition as per B.3). This is our equivalence theorem. We further obtain non-spatial lower Ricci bounds as per B.4), sufficient conditions for lower Ricci bounds of direct sum quantum gradients as per B.5), and derive functional inequalities HWI_λ, MLSI_λ and TW_λ as per B.6). Finite-dimensional cases are given in [50].

Yet we cannot naively extend results to the AF-C^*-setting by taking limits. We thus consider objects and properties compatible with compression and finite-dimensional approximation as per A.4). We explain such compatibility at the end of Chapter 2 and formalise it in the coarse graining process [122] in Chapter 3. This in turn demands an involved technical discussion culminating in our introduction of finitely supported accessibility components as per B.1) and our restriction of quantum relative entropy to the latter. An essential technique is compressed pulled-back joint functional calculus of extended AF-C^*-bimodule actions explained in Chapter 2 based on Appendix A and Appendix B. However, use of the coarse graining process requires us to adapt or even replace essential arguments in [50] and [106] as explained for our main contributions and throughout our discussion when proving suitable analogous results.

Wirth gives a dynamic formulation [200] in a tracial infinite-dimensional setting. Assuming energy dominant trace [132] but not ergodicity, these are noncommutative optimal transport distances of densities, i.e. normal states, in tracial W^*-algebras. They are determined by suitable symmetric C^*-derivations inducing C^*-Dirichlet forms on noncommutative L^2-spaces of tracial W^*-algebras [63][65]. Results in [200][202] often assume tracial state and may assume ergodicity. Note [202] is based on [200]. Assuming tracial state and ergodicity, Wirth shows a, possibly infinite-dimensional, Bakry-Émery condition [200] as per [50] implies heat flow is EVI_λ-gradient flow of relative entropy for W^*-algebras [163] and therefore, by standard arguments [160], λ-convexity of such relative entropy. This cannot satisfyingly define lower Ricci bounds since [200] lacks full equivalence as per B.3). Assuming tracial state, Wirth and Zhang give sufficient conditions for satisfying Bakry-Émery conditions [202] as per [50] using intertwining property for general families of bounded linear operators. They need not assume direct sum noncommutative gradients since they give an argument dual to the monotonicity argument in [50] we extend. Assuming tracial state, Wirth and Zhang obtain functional inequalities HWI_λ, MLSI_λ [202] and TW_λ [200] as per [50] using relative entropy for W^*-algebras conditioned to fixed-point subalgebras. Such a priori conditioning handles non-ergodicity but does not emerge from an underlying metric geometry.

Note [200][202] and our discussion share the tracial infinite-dimensional setting. Yet each approach has considerably different technical foundation, assumptions and applicability. Results and their proofs, as well as range of examples, differ accordingly. We examine these differences. Whereas noncommutative differential structures collect our initial data, [200] considers C^*-Dirichlet forms [1] in order to define test algebras of observables via Lipschitz seminorms using the induced noncommutative gradient [63] [65] and given operator mean [13]. They may equivalently assume a given symmetric C^*-derivation, i.e. noncommutative gradient, as we do using quantum gradients. If both approaches apply, then test algebras in [200] are larger and contain ours, i.e. unions of all generating C^*-subalgebras. Assuming regular operator mean and restricting to densities, the dynamic formulation in [200] and our discussion coincide for two further reasons. First, [200] assumes energy dominant trace in order to have σ-weak extensions of bimodule actions. We show a general extension of AF-C^*-bimodule actions to spaces of measurable operators using extendability of local *-homomorphisms. Note we thereby avoid use of C^*-Dirichlet forms as in [132][200]. Secondly, [200] uses noncommutative multiplication operators for densities. We construct noncommutative division operators for all states. We show both choices are equivalent in the finite-dimensional setting by considering vector fields along admissible paths minimising the given quasi-entropy at a.e. time. Taking limits shows both dynamic formulations coincide as claimed.

Assumptions and applicability differ from ours in several points. Results in [200] [202] often assume tracial state and may assume ergodicity. As stated above, we assume neither. We give three differences to [200] and two to [202] showing why their results are insufficient for our purposes. First, they have weaker results concerning existence of minimising geodesics. Assuming tracial state, the logarithmic mean setting and heat flow is EVI_λ-gradient flow of relative entropy for W^*-algebras, [200] shows existence of minimising geodesics for densities at finite distance in the domain of relative entropy for W^*-algebras. We show each accessibility component for any given symmetric operator mean is a geodesic length-metric space s.t. minimising geodesics approximated in finite dimensions as per A.5) exist between states at finite distance. This only requires our initial data. Secondly, they lack classification of accessibility components. We show two such classifications for varying assumptions on states as per A.3) and B.4). These coincide in the finite-dimensional logarithmic mean setting. Thirdly, they do not prove an equivalence theorem as per [50] or B.3). Assuming tracial state and ergodicity, [200] shows the chain of implications starting from a Bakry-Émery condition stated above.

We assume neither and prove full equivalence as per B.3). We state and prove such using existence of sufficient minimising geodesics approximated in finite dimensions and classification. We use the latter for the coarse graining process and our control of quantum relative entropy as per B.1) on finitely supported accessibility components. In contrast, tracial states are necessary in [200] for existence of geodesics and since it gives no extension of relative entropy for W^*-algebras as per B.1). The proof of equivalence in [50] uses direct calculations involving the Hessian of quantum relative entropy in the finite-dimensional Riemannian setting. We engage in our own and apply the coarse graining process. We see no substitute for this in [200], resp. its continued development in [202][203]. We expect alternatives to be new even in the finite-dimensional setting.

We turn to [202]. First, sufficient conditions for satisfying Bakry-Émery conditions as per [50] differ in applicability. Assuming tracial state, [202] gives sufficient ones as stated above using a novel intertwining property for general families of bounded linear operators. They do not assume direct sum noncommutative gradients and are therefore more general than us in the finite-trace case. This provides means to construct examples for complete gradient estimates stable under tensoring which otherwise appear difficult according to [202] itself. These do not cover crucial fundamental example classes as per C), resp. further iterations on the latter using standard constructions.

We assume direct sum noncommutative gradients, rather than complete gradient estimates, but not finite trace. We cover natural examples given by dynamic quantum gradients [133], e.g. intertwining sets of Clifford generators, which indeed have no finite trace. They use tensor product AF-C^*-bimodules in each summand and thus generalise [48] to infinite dimensions. In the logarithmic mean setting, we use sufficient conditions as per B.5) to show they have strictly positive lower Ricci bounds. Secondly, functional inequalities in [202] require use of relative entropy for W^*-algebras conditioned in the second variable to the given fixed-point subalgebra. Assuming tracial state, [202] gives functional inequalities as stated above. Recent work of Brannan, Gao and Junge [33][34] independently obtained similar results to Wirth and Zhang [202] for tracial states using likewise a priori conditioning of relative entropy for W^*-algebras. Their assumptions imply neither approach covers all fundamental example classes as per C), in particular our strictly positive case, nor considers its conditioning as determined by the underlying metric geometry. We do show restriction to finitely supported accessibility components is compression of quantum relative entropy with support projections of the given fixed part. We therefore have a conditioning determined by the underlying metric geometry as restriction to finitely supported accessibility components. This is used in the coarse graining process, necessary for our equivalence theorem, and yields non-spatial lower Ricci bounds plus functional inequalities using unconditioned quantum relative entropy as only functional - regardless of finiteness or ergodicity. We thereby ensure functional inequalities reveal properties of the given metric geometry.

As explained in the introduction, we study non-spatial lower Ricci bounds as per B.4) and apply functional inequalities as per B.6) to probe any underlying metric geometry arising from one of our fundamental example classes as per C), resp. an iteration using standard constructions. Following our examination of differences above, we see results in [200][202], as well as [33][34], are insufficient for our purposes. Conversely, allowing for non-traciality and non-ergodicity lets us cover quantum optimal transport of normal states on arbitrary hyperfinite factors and therefore our motivating application given by first and second quantisation of spectral triples. We appear to have comparatively higher control of fine-structures determined by our initial data, a control we ensure is inherited by all objects we consider through compatibility. We see this bottom-up design yields flexible architecture with the coarse graining process its step-by-step reduction process terminating in a well-behaved finite-dimensional Riemannian setting open to direct calculation. We apply the latter to show stronger results for a wider range of examples in the AF-C^*-setting covering common algebras of observables in quantum statistical mechanics [35][36][162]. We view our approach as complementary to [200].

This concludes our explanation of similarities and differences. We consider the use of our discussion for studying noncommutative gauge theories [51][54][55][197][198] within Connes' program of noncommutative geometry [67][69][137][138]. The program so far lacks a general notion of curvature [111][147] independent of a particular class of spectral triples [68][69][114][198]. Noncommutative tori are a challenge [70][98][110] [146]. We study the weaker notion of curvature bounds for EVI_λ-gradient flows driven by l.s.c. functionals for relevant metric geometries [8][160]. The spectral paradigm of noncommutative geometry [51][52][53][68][69] based on Gelfand duality [192] implies a suitable notion must cover continuous, discrete and finally mixed continuous-discrete noncommutative geometries [114][197][198]. Unfortunately, the AF-C^*-setting does not consider spatial coordinates, i.e. non-discrete geometries, unless we introduce them in form of parametrisations for continuous fields of AF-C^*-algebras [197]. First and second quantisation of spectral triples exemplify such lack of spatial coordinates.

First quantisation considers commutative spectral triples, i.e. first quantisation of compact spin manifolds [68]. We show quantum optimal transport is transversal to spatial optimal transport in this case. Second quantisation rectifies this by quantising all spatial coordinates. We apply a characterisation in [55] to obtain sufficient conditions s.t. the quantum gradients used are infinitesimal evolution of observables at thermal equilibrium determined by KMS-states [36]. Each assumes fixed gauge field [51][197] [198]. Varying von Neumann entropy [163] of such KMS-states w.r.t. the canonical trace yields description of the spectral action on gauge fields [51][52][53] in terms of quantum statistical mechanics using quantum relative entropy as per B.1) [55]. Upon passing to second quantisation, we introduce gauge fields as spatial coordinates. We consider all normalised Radon measures on finite-dimensional spaces of admissible gauge fields evaluating in CAR-algebras [162], i.e. states on continuous fields of AF-C^*-algebras. We thereby generalise to quantum optimal transport parametrised by gauge fields and give an internalised spectral action on the aforementioned states using relative entropy for W^*-algebras. This gives our ansatz as per C) in Chapter 3. If key technical challenges are solved in future work, then we hope to study the dynamics of such generalised gauge fields described as gradient flows driven by the internalised spectral action for the given parametrised quantum optimal transport. We are motivated by the classical approach of Jordan, Kinderlehrer and Otto for Fokker-Planck equations [131][167][169].

We may relax assumptions on fibres to cover disintegration of tracial W^*-algebras into direct integrals of hyperfinite factors according to the von Neumann disintegration theorem [192]. We see fundamental example classes using tracial AF-C^*-algebras generating hyperfinite factors of type I and II by σ-weak closure are of particular interest. We thereby define general parametrised quantum optimal transport. We view quantum optimal transport as its pointwise case. We explain states on CAR-algebras are scaling limits of spin states encoding qubits [42][43][62][93][95], but not necessarily pure [116]. Using noncommutative conditional expectations [192], we therefore consider states on tracial AF-C^*-algebras as scaling limit of uniformly conditioned spin states encoding a sequence of qubits without use of any spatial coordinates. We view quantum optimal transport as transport of quantum information, and the parametrised one as transport of densities of quantum information over encoding schemes, at the end of Chapter 3.

2 Noncommutative Differential Structures

Noncommutative differential structures collect the data which define quantum optimal transport distances. Each consists of two components and one setting. First, we have an AF-C^*-bimodule over a, possibly different, tracial AF-C^*-algebra. This establishes noncommutative topology, measures and integrals. Secondly, we have a quantum gradient for the given AF-C^*-bimodule. These are noncommutative gradients with likewise chain rule. The relationship between gradients, heat semigroups and Dirichlet forms extends to the noncommutative setting [63][65]. Finally, we have a representing function of an operator mean together with an interpolation factor. This lets us define noncommutative division operators. They determine, and are in turn determined by, quasi-entropies [127] [128] used to define energy functionals. In Chapter 3, we readily see our construction of quantum optimal transport distances follows the classical case [97] but using data as above. Thus Banach dual spaces of AF-C^*-bimodules serve as synthetic tangent spaces for the weak formulation of continuity equations in the AF-C^*-setting.

The data collected is, by definition or construction, compatible with compression and finite-dimensional approximation. These are two general operations we formalise in a coarse graining process. Compatibility transfers to quantum Laplacians, i.e. Laplacians of quantum gradients, their noncommutative heat semigroups, as well as continuity equations. Compatibility therefore transfers to quantum optimal transport. The coarse graining process formalising the latter is thereby essential for the majority of our results as it reduces the AF-C^*-setting to the finite-dimensional one s.t. ergodicity is recovered up to a controlled remainder.

Structure. In Section 2.1, we discuss AF-C^*-bimodules over tracial AF-C^*-algebras. In Section 2.2, we discuss noncommutative division operators. In Section 2.3, we discuss quantum gradients for AF-C^*-bimodules. We then define noncommutative differential structures, discuss compatibility and outline the coarse graining process.

2.1 The AF-C*-Setting

AF-C^*-bimodules over tracial AF-C^*-algebras are the setting for continuity equations of states compatible with compression and finite-dimensional approximation. Elements in Banach dual spaces of AF-C^*-bimodules serve as likewise compatible synthetic tangent vectors in our weak formulation. In particular, AF-C^*-bimodules have an extension of bimodule actions to spaces of measurable operators s.t. their noncommutative L^2-spaces are symmetric W^*-bimodules. The latter are Hilbert spaces on which noncommutative division operators act even upon compression. As such, they provide suitable setting for the Leibniz rule and serve as codomains of quantum gradients.

Structure. In Subsection 2.1.1, we study AF-C^*-bimodules over tracial AF-C^*-algebras and extensions of AF-C^*-bimodule actions. In Subsection 2.1.2, we discuss compressed pulled-back joint functional calculus of extended AF-C^*-bimodule actions.

2.1.1 AF-C*-bimodules over tracial AF-C*-algebras

AF-C^*-bimodules over tracial AF-C^*-algebras are defined using local *-homomorphisms of tracial AF-C^*-algebras. These are *-homomorphisms of C^*-algebras compatible with all AF-C^*-structures in use, further extending to spaces of measurable operators. Noncommutative L^2-spaces of AF-C^*-bimodules are symmetric W^*-bimodules.

Tracial C^*-algebras and spaces of measurable operators. Let (M,τ) be a tracial W^*-algebra, i.e. W^*-algebra M and f.s.n. trace $\tau : M_+ \longrightarrow [0,\infty]$ with definition domain $\mathfrak{m}_\tau$ (cf. Definition B.1.1 and Definition B.1.5). Uniform closure of M in measure topology is the space of measurable operators $L^0(M,\tau)$ (cf. Definition B.1.23). Algebra involution on M extends to $L^0(M,\tau)$. We obtain the space $L^0(M,\tau)_h$ of self-adjoint, as well as the space $L^0(M,\tau)_+$ of positive elements (cf. Definition B.1.33). Since M_+ generates the partial order on M (cf. Proposition A.1.23), note $L^0(M,\tau)_+$ generates the partial order on $L^0(M,\tau)$ by density in measure topology (cf. Proposition B.1.49). We extend the f.s.n. trace to $\tau : L^0(M,\tau)_+ \longrightarrow [0,\infty]$ (cf. Definition B.1.39). For details on C^*- and W^*-algebras, we refer to Subsection A.1.2. For details on tracial W^*-algebras and their spaces of measurable operators, we refer to Subsection B.1.1.

Let $p \in [1,\infty]$. Noncommutative L^p-space $(L^p(M,\tau), \|.\|_p) \subset L^0(M,\tau)$ is a Banach space (cf. Definition B.1.41). Algebra involution on M extends to $L^p(M,\tau)$. We obtain the space $L^p(M,\tau)_h$ of self-adjoint, as well as the space $L^p(M,\tau)_+$ of positive elements. We may decompose accordingly (cf. Proposition B.1.47). If $p=1$, then $\tau \in L^1(A,\tau)^*_+$ (cf. 3) in Proposition B.1.42). If $p=2$, then $(L^2(M,\tau), \|.\|_2)$ is a Hilbert space. If $p=\infty$, then $(L^\infty(M,\tau), \|.\|_\infty) = (M, \|.\|_M)$. Noncommutative L^p-spaces fulfil Hölder inequalities. Note Definition 2.1.1 uses the modified standard pairing as per Remark 2.1.2. For details on noncommutative integration, we refer to Subsection B.1.2.

Definition 2.1.1. For all $\mu \in L^1(M,\tau)^\flat$, let $\sharp\mu \in L^1(M,\tau)$ be unique s.t. $\mu = (\sharp\mu)^\flat$. If $p = q = 2$, then set $\sharp := \flat^{-1} \in \mathscr{B}(L^2(M,\tau))$ and call $(\flat,\sharp)$ musical isomorphisms on $L^2(M,\tau)$.

Remark 2.1.2. Let $p,q \in [1,\infty]$. If $1 = p^{-1} + q^{-1}$, then the modified standard pairing

$$(x,y) \mapsto x^\flat(y) = \tau(x^* y) \tag{2.1}$$

defined on $L^p(A,\tau) \times L^q(A,\tau)$ is bounded, anti-linear in the first and linear in the second variable, as well as non-degenerate (cf. Definition B.1.50 and Proposition B.1.51). For all $x \in L^p(M,\tau)$ and $y \in L^q(A,\tau)$, get $\tau(x^* y) = \tau(yx^*)$ and $\overline{\tau(x^* y)} = \tau(xy^*)$ by traciality.

If $p = 1$ and $q = \infty$, then $\flat : L^1(M,\tau) \longrightarrow M^*$ is positivity-preserving and anti-linear isometry onto the set $M_* \subset M^*$ of all normal bounded functional on M equipped with the dual space partial order (cf. Proposition B.1.51 and Remark B.1.52). If $A \subset M$ is a σ-weakly dense C^*-subalgebra, then $A \subset M$ is strongly dense. Normality therefore yields $L^1(A,\tau)^\flat \subset A^*$ as partially ordered Banach spaces. For all $\mu \in L^1(A,\tau)^\flat$, get unique $\sharp\mu \in L^1(A,\tau)$ s.t. $\mu = (\sharp\mu)^\flat$. If $p = q = 2$, then $\flat \in \mathrm{GL}(\mathscr{B}(L^2(M,\tau)))$.

Positive elements generate the partial order on C^*- and W^*-algebras, as well as their Banach dual spaces (cf. Definition A.1.15 and Proposition A.1.23). Definition 2.1.3 gives abstract tracial C^*-algebras (cf. Remark A.2.14). Following Remark 2.1.5, the latter extends Definition B.1.1 and subsumes the concrete case s.t. we have consistent use of canonical left- and right-actions for joint functional calculus of self-adjoint measurable operators. As consequence, compressing with projections as per Lemma 2.1.6 extends readily from one to two variables as special case of the tracial W^*-algebra setting.

Definition 2.1.3. Let $A \subset M$ be a σ-weakly dense C^*-subalgebra. We call (A,τ) a tracial C^*-algebra in M. Set $1_A := 1_M$.

1) Set $L^0(A,\tau) := L^0(M,\tau)$ and

$$L^0(A,\tau)_h := L^0(M,\tau)_h,\ L^0(A,\tau)_+ := L^0(M,\tau)_+. \tag{2.2}$$

2) For all $p \in [1,\infty]$, set $\big(L^p(A,\tau), \|.\|_p\big) := \big(L^p(M,\tau), \|.\|_p\big)$ and

$$L^p(A,\tau)_h := L^p(M,\tau)_h,\ L^p(A,\tau)_+ := L^p(M,\tau)_+. \tag{2.3}$$

3) For all $p,q \in [1,\infty]$, set $L^{p,q}(A,\tau) := L^p(A,\tau) \cap L^q(A,\tau)$ and

$$L^{p,q}(A,\tau)_h := L^p(A,\tau)_h \cap L^q(A,\tau)_h,\ L^{p,q}(A,\tau)_+ := L^p(A,\tau)_+ \cap L^q(A,\tau)_+. \tag{2.4}$$

Notation 2.1.4. Unless stated otherwise, we write (A,τ) and $\|.\|_\tau = \|.\|_2$ for all σ-weakly dense C^*-subalgebras $A \subset M$. This differs from the distinct notation $(\mathscr{H}(M,\tau), \|.\|_\tau)$ and $(L^2(M,\tau), \|.\|_2)$ used in the appendix. Notation remains unambiguous throughout since we only use $(L^2(A,\tau), \|.\|_\tau) = (L^2(M,\tau), \|.\|_2)$ in the main matter.

Remark 2.1.5. Let (A,τ) be a tracial C^*-algebra and $\mathscr{L} : A \longrightarrow \mathscr{B}(\mathscr{H}(A,\tau))$ canonical left-action of A on $\mathscr{H}(A,\tau)$ (cf. Definition B.1.1 and Definition B.1.3). Using normal extension (cf. Proposition A.1.34 and Proposition B.1.7), get tracial W^*-algebra $(\mathscr{L}(A)'',\tau)$ with σ-weakly dense C^*-subalgebra $\mathscr{L}(A) \subset \mathscr{L}(A)''$. We thereby construct the tracial C^*-algebra $(\mathscr{L}(A),\tau)$ in $\mathscr{L}(A)''$. This is the concrete case.

If (A,τ) is a tracial C^*-algebra in M, then the canonical left-action $\mathscr{L}$ of A on $\mathscr{H}(A,\tau)$ is not the canonical left-action L of M on $L^2(M,\tau)$ in general (cf. Definition B.1.55). Note L subsumes $\mathscr{L}$ by twisting with natural Hilbert space isometry $\mathscr{H}(A,\tau) \cong L^2(M,\tau)$ (cf. Proposition B.1.58), L is given by multiplication in $L^0(M,\tau)$, as well as inclusions $A \subset M \subset L^0(M,\tau)$ of *-subalgebras. The analogous holds for canonical right-actions and opposite algebras. Altogether, requiring M in Definition 2.1.3 avoids difficulties arising from identification of $A \cong \mathscr{L}(A)$, as is common yet implicit in the literature, while using canonical left- and right-actions for joint functional calculus of self-adjoint measurable operators (cf. Remark B.1.65)

Note suitable inclusion maps of Banach dual spaces arise from Banach duals of non-commutative conditional expectations. For abstract tracial C^*-algebras, Definition 2.1.7 gives inclusion maps obtained from compressing with projections in W^*-algebras. This uses abstract compression maps. Assuming positivity and fixed norm, get injectivity in the non-unital case as per 1) in Proposition 2.1.13. For tracial AF-C^*-algebras, further note Definition 2.1.27 gives inclusion maps obtained from Hilbert space projections to generating C^*-subalgebras. This additionally yields restriction maps.

We compress C^*-subalgebras with projections. Let $A \subset M$ be a C^*-subalgebra and $p \in M$ be a projection. We have compressed C^*-subalgebra $A[p] = pC^*(A,p)p \subset M$ (cf. 2) in Definition A.2.15). If $p = 1_M$, then we recover the unitalisation $A[1_M] = C^*(A,1_M)$ of A in M (cf. Definition A.1.64). If $A = M$, then $M[p] = pMp \subset (M,\tau)$ is a semi-finite W^*-subalgebra (cf. Remark A.2.16, Definition B.2.1 and 2) in Proposition B.2.13). We have tracial W^*-algebra $(M[p],\tau)$ (cf. 1) in Proposition B.2.3). Note $p^\perp = 1_M - p$ and $M[p][1_M] = M[p] \oplus \langle p^\perp\rangle_{\mathbb{C}}$ (cf. Proposition A.1.71). Assume $A \subset M$ is a σ-weakly dense C^*-subalgebra. The compressed C^*-subalgebra $A[p] \subset M[p]$ is σ-weakly dense itself in this case. We moreover have $A[p][1_M] = A[p] \oplus \langle p^\perp\rangle_{\mathbb{C}}$ (cf. Proposition A.1.65).

Lemma 2.1.6. *Let $A \subset M$ be a σ-weakly dense C^*-subalgebra. For all projections $p \in M$ and $q \in [1,\infty]$, we have*

1) *tracial C^*-algebra $(A[p],\tau)$ in $M[p]$,*
2) *$L^0(A[p],\tau) = pL^0(A,\tau)p$ and $L^q(A[p],\tau) = pL^q(A,\tau)p$.*

Proof. We know 1). Thus $L^\infty(A[p],\tau) = M[p]$, hence 2) follows by Proposition B.2.30. □

The abstract compression map $\mathrm{com}_p : A[1_M] \longrightarrow A[p]$ is given by $\mathrm{com}_p\, x = pxp$ for all $x \in A[1_M]$ (cf. Definition A.2.15). Note com_p is a completely positive, normal, unital and surjective bounded linear map (cf. Proposition A.2.17). If $A = M$, then we recover non-commutative conditional expectations as per Remark 2.1.8. For details on compressed C^*-subalgebras and their abstract compression maps, we refer to Subsection A.2.2. For details on semi-finite W^*-subalgebras, we refer to Subsection B.2.1.

We have positivity-preserving injective Banach dual $\mathrm{com}_p^* : A[p]^* \longrightarrow A[1_M]^*$. If we restrict to $A \subset A[1_M]$, then we further have positivity-preserving bounded linear map $\mathrm{com}_p^* : A[p]^* \longrightarrow A^*$. The latter is not injective in general. If $q \in L^\infty(A,\tau)$ is a projection s.t. $p \leq q$, then $pq = p$ implies $\mathrm{com}_p(A[q]) = A[p]$. Get positivity-preserving injective Banach dual $\mathrm{com}_p^* : A[p]^* \longrightarrow A[q]^* \subset A[1_M]^*$ in this case.

Definition 2.1.7. Let $A \subset M$ be a C^*-subalgebra. For all projections $p \in L^\infty(A,\tau)$, we define the p-th inclusion $\mathrm{inc}_p := \mathrm{com}_p^* : A[p]^* \longrightarrow A[1_M]^*$.

Remark 2.1.8. Semi-finite W^*-subalgebras have unique noncommutative conditional expectations (cf. Definition B.2.7, Remark B.2.8 and Definition B.2.9). For all projections $p \in M$, $\mathrm{com}_p : M \longrightarrow M[p]$ is the noncommutative conditional expectation $\pi^M_{M[p]}$ from M to $M[p]$ (cf. Proposition B.2.10 and 2) in Proposition B.2.13).

Proposition 2.1.9. *Let $A \subset M$ be a σ-weakly dense C^*-subalgebra. All inclusion maps in Definition 2.1.7 are bounded linear, positivity-preserving and injective. They furthermore satisfy the following.*

1) *All inclusion maps in Definition 2.1.7 are w^*-continuous.*
2) *For all projections $p \leq q$ in $L^\infty(A,\tau)$, we have $A[p]^* \subset A[q]^* \subset A[1_A]^*$ as partially ordered Banach spaces.*

Proof. Bounded linearity and 1) are immediate. Let $p \in L^\infty(A,\tau)$ be a projection. Since com_p is positivity-preserving (cf. Proposition A.2.17), inc_p is as well. If $q \in L^\infty(A,\tau)$ is a projection s.t. $p \leq q$, then $pq = p$ implies $\mathrm{com}_p \circ \mathrm{com}_q = \mathrm{com}_p$ and therefore 2). □

Notation 2.1.10. Let $A \subset M$ be a σ-weakly dense C^*-subalgebra. For all projections $p \leq q$ in $L^\infty(A,\tau)$, we suppress inc_p and inc_q on $A[p]^*$.

States on abstract tracial C^*-algebras are noncommutative probability measures. They are normal if they have noncommutative density. Equation 2.5 and Equation 2.6 use spectral measures and spectra, as well as bounded measurable functional calculus of self-adjoint measurable operators (cf. Definition B.1.69 and Lemma B.1.72).

Definition 2.1.11. Let $A \subset M$ be a σ-weakly dense C^*-subalgebra.

1) We define the state space $\mathscr{S}(A) := \{\mu \in A^*_+ \mid \|\mu\|_A = 1\}$ and the normal state space $\mathscr{S}^{\mathrm{N}}(A) := \mathscr{S}(A) \cap L^1(A,\tau)^\flat$ of A. Set

$$\mathscr{S}^{\mathrm{N}}_{>0}(A) := \mathscr{S}^{\mathrm{N}}(A) \cap \left\{ x \in L^1(A,\tau)_h \;\middle|\; \Gamma_{x,L^\infty(A,\tau)}(\delta_0) = 0 \right\}^\flat. \tag{2.5}$$

2) For all $p \in (1,\infty]$, set $\mathscr{S}^{\mathrm{N},p}(A) := \mathscr{S}(A) \cap L^{1,p}(A,\tau)^\flat$ and

$$\mathscr{S}^{\mathrm{N},p}_{-1}(A) := \mathscr{S}^{\mathrm{N},p}(A) \cap \left\{ x \in L^1(A,\tau)_h \;\middle|\; 0 \notin \mathrm{spec}_M x \right\}^\flat. \tag{2.6}$$

Remark 2.1.12. For all σ-weakly dense C^*-subalgebras $A \subset M$, our construction and Remark 2.1.2 shows $\mathscr{S}^{\mathrm{N}}(A) = \mathscr{S}(M) \cap L^1(M,\tau)^\flat = \mathscr{S}^{\mathrm{N}}(M)$ (cf. Definition B.1.53). For all σ-weakly dense C^*-subalgebras $A \subset M$, projections $p \in L^\infty(A,\tau)$ and $x \in L^1(A[p],\tau)$, we have $(\mathrm{inc}_p\, x^\flat)(y) = \tau(x^* y)$ for all $y \in A[p]$. Lemma 2.1.6 shows $x = xp = px$ in each case.

Proposition 2.1.13. *Let $A \subset M$ be a σ-weakly dense C^*-subalgebra. For all projections $p \leq q$ in $L^\infty(A,\tau)$ and $r \in (1,\infty]$, we have*

1) $\mathscr{S}(A[p]) \subset \mathscr{S}(A[q]) \subset \mathscr{S}(A)$ *and* $\mathscr{S}^{\mathrm{N}}(A[p]) \subset \mathscr{S}^{\mathrm{N}}(A[q]) \subset \mathscr{S}^{\mathrm{N}}(A)$,
2) $\mathscr{S}^{\mathrm{N},r}(A[p]) \subset \mathscr{S}^{\mathrm{N},r}(A[q]) \subset \mathscr{S}^{\mathrm{N},r}(A)$.

Proof. Proposition 2.1.9 shows positive elements are preserved. Let p in $L^\infty(A,\tau)$ be a projection. Since $\mu(1_A) = \mu(p) = \|\mu\|_{A[p]^*}$ for all $\mu \in A[p]^*_+ \subset A[1_A]^*$, fixed norm ensures injectivity upon restriction to $A \subset A[1_A]$. Get 1). Using the latter, get 2). ☐

Tracial AF-C^*-algebras. Approximately finite-dimensional, or AF-C^*-algebras are all C^*-algebras which are norm closures of ascending chains of finite-dimensional C^*-algebras. We index all AF-C^*-algebras over $\mathbb{N}$. This is equivalent to using countable directed sets by existence of cofinal subsets isomorphic to $\mathbb{N}$. Tracial AF-C^*-algebras are both AF-C^*-algebras and abstract tracial C^*-algebras.

Definition 2.1.14. Let A be a C^*-algebra.

1) A sequence $\{A_j\}_{j\in\mathbb{N}}$ of finite-dimensional C^*-algebras is ascending if $A_j \subset A_{j+1}$ is a C^*-subalgebra for all $j \in \mathbb{N}$. Set $A_0 := \bigcup_{j\in\mathbb{N}} A_j$.
2) We call A an AF-C^*-algebra if $A = \overline{A_0}^{\|.\|_A}$ for an ascending sequence $\{A_j\}_{j\in\mathbb{N}}$ of finite-dimensional C^*-subalgebras. We further call $\{A_j\}_{j\in\mathbb{N}}$ a generating sequence of A and say that A is generated by $\{A_j\}_{j\in\mathbb{N}}$.
3) If A is an AF-C^*-algebra generated by $\{A_j\}_{j\in\mathbb{N}}$, then we say that A is
   3.1) strongly unital if $1_{A_j} = 1_{A_k}$ for all $j,k \in \mathbb{N}$,
   3.2) finite-dimensional if $\dim_{\mathbb{C}} A < \infty$,
   3.3) finite if $A = A_j$ for all $j \in \mathbb{N}$.

Notation 2.1.15. For all $n \in \mathbb{N}$, $I_n \in M_n(\mathbb{C})$ denotes the unit and tr_n the non-normalised canonical trace. In infinite dimensions, i.e. $n = \infty$, we suppress the subscript and write I and tr. Up to C^*-isometries, finite-dimensional C^*-algebras are of form $\oplus_{l=1}^n M_{n_l}(\mathbb{C})$ for $n \in \mathbb{N}$ [38]. If A is a AF-C^*-algebra generated by $\{A_j\}_{j\in\mathbb{N}}$, then we fix C^*-isometries

$$r_A := \left\{ r_{A_j} : A_j \longrightarrow \oplus_{l=1}^{n_j} M_{n_{j,l}}(\mathbb{C}) \right\}_{j\in\mathbb{N}}. \tag{2.7}$$

If A is furthermore finite, then set $\{r_{A_j}\}_{j\in\mathbb{N}}$ to be constant unless stated otherwise.

Proposition 2.1.16. *Let A be an AF-C^*-algebra and M a W^*-algebra. If A is generated by $\{A_j\}_{j\in\mathbb{N}}$, then*

1) $\{1_{A_j}\}_{j\in\mathbb{N}} \subset A$ *is a left- and right-approximate identity in A,*

2) $1_M = \text{s-lim}_{j\in\mathbb{N}} 1_{A_j}$ *if $A \subset M$ is a σ-weakly dense C^*-subalgebra.*

Proof. Get 1) since $\bigcup_{j\in\mathbb{N}} A_j \subset A$ is $\|.\|_A$-dense and $1_{A_j}(1_{A_k} - 1_{A_j}) = (1_{A_k} - 1_{A_j})1_{A_j} = 0$ for all $j \leq k$ in $\mathbb{N}$. Get 2) by 1) and uniqueness of units in C^*-algebras. □

Remark 2.1.17. Note 1) in Proposition 2.1.16 shows strong unitality implies unitality. In the setting of 2) in Proposition 2.1.16, we have $1_M = 1_A$ if A is unital.

Definition 2.1.18 gives tracial AF-C^*-algebras using abstract formulation. Following Remark 2.1.5 and Remark 2.1.19, the latter therefore subsumes the concrete case in the AF-C^*-setting s.t. we have consistent use of canonical left- and right-actions for joint functional calculus of self-adjoint measurable operators.

Definition 2.1.18. Let A be an AF-C^*-algebra generated by $\{A_j\}_{j\in\mathbb{N}}$ and (M,τ) a tracial W^*-algebra. We call (A,τ) a tracial AF-C^*-algebra in M generated by $\{A_j\}_{j\in\mathbb{N}}$ if $A \subset M$ is a σ-weakly dense C^*-subalgebra and $A_0 \subset \mathfrak{m}_\tau$.

Remark 2.1.19. Let (A,τ) be a tracial C^*-algebra and AF-C^*-algebra generated by $\{A_j\}_{j\in\mathbb{N}}$ s.t. $A_0 \subset \mathfrak{m}_\tau$. Following construction in Remark 2.1.5, get tracial AF-C^*-algebra $(\mathscr{L}(A),\tau)$ in $\mathscr{L}(A)''$ generated by $\{\mathscr{L}(A)_j\}_{j\in\mathbb{N}} := \{\mathscr{L}(A_j)\}_{j\in\mathbb{N}}$. This is the concrete case of the AF-C^*-setting. Note this requires $\mathscr{L}$ to be a faithful *-representation.

Proposition 2.1.20. *For all tracial AF-C^*-algebras (A,τ), we have*

1) $A_0 \subset L^\infty(A,\tau)$ *is strongly dense,*

2) $A_0 \subset L^2(A,\tau)$ *is $\|.\|_\tau$-dense,*

3) $A_0 \subset L^1(A,\tau)$ *is $\|.\|_1$-dense.*

Proof. We show Proposition B.1.54 applies. We know $A_0 \subset A$ is $\|.\|_A$-dense. We show $A_0 \subset L^2(A,\tau)$ is $\|.\|_\tau$-dense. For all $j \in \mathbb{N}$, set $\mathscr{A}_j := 1_{A_j} A_0 1_{A_j} \subset A_0$ and note

$$\mathscr{M}_j := \overline{\mathscr{A}_j} = L^\infty(A,\tau)[1_{A_j}] = 1_{A_j} L^\infty(A,\tau) 1_{A_j} \subset L^\infty(A,\tau) \tag{2.8}$$

w.r.t. closure in strong operator topology (cf. 2) in Definition A.2.15). Equation 2.8 shows $L^2(\mathscr{M}_j,\tau) = 1_{A_j} L^2(A,\tau) 1_{A_j}$ by Lemma 2.1.6 in each case. Thus $\bigcup_{j\in\mathbb{N}} L^2(\mathscr{M}_j,\tau) \subset L^2(A,\tau)$ is $\|.\|_\tau$-dense by Proposition 2.1.16, hence

$$L^2(A,\tau) = \overline{\bigcup_{j\in\mathbb{N}} L^2(\mathscr{M}_j,\tau)}^{\|.\|_\tau} \subset \overline{\bigcup_{j\in\mathbb{N}} \mathscr{A}_j}^{\|.\|_\tau} \subset \overline{A_0}^{\|.\|_\tau} \subset L^2(A,\tau). \tag{2.9}$$

Equation 2.9 shows $A_0 \subset L^2(A,\tau)$ is $\|.\|_\tau$-dense. □

We consider the finite-dimensional setting. Example 2.1.21 provides finite case. We discuss the case of restricting to generating C^*-subalgebras. We use Notation A.1.2. For all AF-C^*-algebras A generated by $\{A_j\}_{j\in\mathbb{N}}$ and $j \in \mathbb{N}$, $A_{j,h}$ denotes the self-adjoint and $A_{j,+}$ the positive elements in A_j.

Example 2.1.21. Let (A,τ) be a finite-dimensional tracial C^*-algebra. Note $L^\infty(A,\tau) = A$ by σ-weak density. For all $j \in \mathbb{N}$, set $A_j = A$. This defines finite tracial AF-C^*-algebra (A,τ) in A. Finiteness does not hold in general.

Definition 2.1.22. Let (A,τ) be a tracial AF-C^*-algebra. For all $j \in \mathbb{N}$, set $\tau_j := \tau|_{A_j}$ and we define sequence of finite-dimensional C^*-algebras by setting

$$A_{j,l} := \begin{cases} A_l & \text{if } l < j \text{ in } \mathbb{N}, \\ A_j & \text{else}. \end{cases}$$

Remark 2.1.23. Unless stated otherwise, any finite-dimensional tracial C^*-algebra we consider alone is a finite tracial AF-C^*-algebras as per Example 2.1.21. For generating C^*-subalgebras, we instead restrict as per Definition 2.1.22.

Proposition 2.1.24. *Let (A,τ) be a tracial AF-C^*-algebra. For all $j \in \mathbb{N}$, we have*

1) *tracial AF-C^*-algebra $(A_j,\tau) = (A_j,\tau_j)$ in A_j generated by $\{A_{j,l}\}_{l\in\mathbb{N}}$,*

2) $\tau_j = \oplus_{l=1}^{n_j} C_{j,l} \operatorname{tr}_{n_{j,l}} \circ r_{A_j}$ *with $C_{j,l} > 0$ for all $l \in \{1,\dots,n_j\}$,*

3) $r_{A_j}(1_{A_j}) = \sum_{l=1}^{n_j} I_{n_{j,l}}$.

Proof. We have 1) since $A_0 \subset \mathfrak{m}_\tau$. Restricting to summands shows 2) by uniqueness of the normalised trace on full matrix algebras. Get 3) by unitality. □

Remark 2.1.25. For all $j \in \mathbb{N}$, $\langle .,. \rangle_{\tau|_{A_j}}$ equals $\sum_{l=1}^{n_j} C_{j,l} \langle .,. \rangle_{\operatorname{tr}_{n_{j,l}}}$ pulled back along $r_{A_j}^{-1}$.

Proposition 2.1.26. *Let (A,τ) be a tracial AF-C^*-algebra. For all $j \in \mathbb{N}$, we consider the Hilbert space projection $\pi_j^A : L^2(A,\tau) \longrightarrow A_j$. We have*

1) $\bigcup_{j\in\mathbb{N}} A_{j,+} \subset A_+$ *is $\|.\|_A$-dense,*

2) $\bigcup_{j\in\mathbb{N}} A_{j,+} \subset L^\infty(A,\tau)_+$ *is strongly dense,*

3) $I_{L^2(A,\tau)} = \text{s-lim}_{j\in\mathbb{N}} \pi_j^A$.

Proof. For all $j \in \mathbb{N}$, get $A_{j,+} \subset A_+ \subset L^\infty(A,\tau)$. If $\{x_n\}_{n\in\mathbb{N}} \subset A_0$ s.t. $\|.\|_A$-$\lim_{n\in\mathbb{N}} x_n = x \geq 0$ in A, then $\|.\|_A$-$\lim_{n\in\mathbb{N}} \max\{x_n, 0\} = x$. Using strong convergence, the analogous statement follows if $x \geq 0$ in $L^\infty(A,\tau)$. This shows 1) and 2). We know $A_0 \subset L^2(A,\tau)$ is $\|.\|_\tau$-dense by Proposition 2.1.20. Thus $\|.\|_\tau$-$\lim_{j\in\mathbb{N}} \pi_j^A(x) = x$ for all $x \in A_0$, hence 3) follows. □

Banach dual spaces of tracial AF-C^*-algebras. Inclusion and restriction maps of tracial AF-C^*-algebras in Definition 2.1.27 are used for bookkeeping. Notation 2.1.29 fixes conventions. We use the modified standard pairing, in particular their flat and sharp operators as per Definition 2.1.1 and Remark 2.1.2.

Let (A,τ) be a tracial AF-C^*-algebra. For all $j \in \mathbb{N}$, we have $A_j \cong A_j^*$ via musical isomorphisms. Let $\mathfrak{A}$ as per Definition 2.1.27. Note $A_0 \subset \mathfrak{A}$.

Definition 2.1.27. For all Hilbert subspaces $V \subset L^2(A,\tau)$, let $\pi_V^A : L^2(A,\tau) \longrightarrow V$ be the Hilbert space projection. Let $\mathfrak{A} = A$ or $\mathfrak{A} = L^p(A,\tau)$ for $p \in [1,\infty]$.

1) For all $j \in \mathbb{N}$, let $\iota_j^A : A_j \longrightarrow \mathfrak{A}$ be the inclusion and set $\pi_j^A := \pi_{A_j}^A$.

2) For all $j \le k$ in $\mathbb{N}$, let $\iota_{kj}^A : A_j \longrightarrow A_k$ be the inclusion and set $\pi_{jk}^A := \pi_{A_j}^{A_k}$.

3) For all $j \le k$ in $\mathbb{N}$, we define the j-th inclusion and j-th restriction

$$\mathrm{inc}_j := \flat \circ \iota_j^A \circ \sharp : A_j^* \longrightarrow \mathfrak{A}^*, \ \mathrm{res}_j := \left(\iota_j^A\right)^* : \mathfrak{A}^* \longrightarrow A_j^*, \tag{2.10}$$

as well as the kj-inclusion and jk-restriction

$$\mathrm{inc}_{kj} := \left(\pi_{jk}^A\right)^* : A_j^* \longrightarrow A_k^*, \ \mathrm{res}_{jk} := \left(\iota_{kj}^A\right)^* : A_k^* \longrightarrow A_j^*. \tag{2.11}$$

Proposition 2.1.28. *All inclusion and restriction maps in Definition 2.1.27 are bounded linear, positivity-preserving, as well as injective, resp. surjective. They furthermore satisfy the following.*

1) *All inclusion and restriction maps in Definition 2.1.27 are w^*-continuous.*

2) *For all indices,* $\mathrm{res} \circ \mathrm{inc} = \mathrm{id}$. *For all $j \le k$ in $\mathbb{N}$, we have $A_j^* \subset A_k^* \subset A^*$ as partially ordered Banach spaces and*

   2.1) $\mathrm{inc}_{kj} = \flat \circ \iota_{kj}^A \circ \sharp$ *and* $\mathrm{res}_{jk} = \flat \circ \pi_{jk}^A \circ \sharp$,

   2.2) $\mathrm{inc}_{kj} = \mathrm{res}_k \circ \mathrm{inc}_j$ *and* $\mathrm{res}_{jk} = \mathrm{res}_j \circ \mathrm{inc}_k$.

Proof. Bounded linearity is immediate. Since $A_0 \subset A$ is $\|.\|_A$-dense, testing on A_0 shows continuity in each case. We directly verify all remaining claims. □

Notation 2.1.29. For all $j \le k$ in $\mathbb{N}$, the following holds. We suppress inc_j and inc_{kj} on A_j^*. We neither distinguish π_j^A and π_{jk}^A on A_k, nor res_j and res_{jk} on A_k^*.

Definition 2.1.30. Let $j \in \mathbb{N}$ and $p \in [1,\infty]$.

1) For all $\mu \in A^*$, set $\mu_j := \operatorname{res}_j \mu \in A_j^*$.

2) For all $x \in L^p(A,\tau)$, set $x_j := \sharp \operatorname{res}_j x^\flat \in A_j$.

Proposition 2.1.31.

1) For all $\mu \in A^*$, we have

1.1) $\|\mu\|_{A^*} = \sup_{j\in\mathbb{N}} \|\mu_j\|_{A^*} = \lim_{j\in\mathbb{N}} \|\mu_j\|_{A^*}$,

1.2) $\mu = w^*\text{-}\lim_{j\in\mathbb{N}} \mu_j$.

2) Let $p \in [1,\infty]$. For all $x \in L^p(A,\tau)$, we have

2.1) $\|x\|_p = \sup_{j\in\mathbb{N}} \|x_j\|_p = \lim_{j\in\mathbb{N}} \|x_j\|_p$,

2.2) $x = w^*\text{-}\lim_{j\in\mathbb{N}} x_j$.

3) For all $x \in L^\infty(A,\tau)$, we have $x = \text{bds-}\lim_{j\in\mathbb{N}} x_j = \text{bdw-}\lim_{j\in\mathbb{N}} x_j$.

Proof. We directly verify 1.1) and 2.1). They ensure uniform boundedness upon testing for 1.2), 2.2) and 3) on A_0. We conclude by density in each case. □

Remark 2.1.32. Let $j \in \mathbb{N}$. For all $x \in L^2(A,\tau)$, we have $x_j = \pi_j^A(x)$. Note Theorem 2.2.53 furthermore generalises strong convergence as per 3) in Proposition 2.1.31 to strong resolvent convergence of positive and suitably integrable measurable operators under canonical left- and right-actions of AF-C^*-bimodules.

Following Notation 2.1.29, we treat restriction as single operation even if domains vary or identified with duals via musical isomorphisms. We use Notation A.1.2. For all $j \in \mathbb{N}$, $A_{j,h}^*$ denotes the real and $A_{j,+}^*$ the positive elements in A_j^*.

Proposition 2.1.33. *For all $j \le k$ in $\mathbb{N}$, we have*

1) $A_{j,+}^* \subset A_{k,+}^* \subset A_+^*$ *and* $\mathscr{S}(A_j) \subset \mathscr{S}(A_k) \subset \mathscr{S}^{\mathrm{N}}(A)$,

2) $\operatorname{res}_j\left(A_+^*\right) \subset A_{j,+}^*$ *and* $\operatorname{res}_j\left(A_{k,+}^*\right) \subset A_{j,+}^*$.

Proof. For all $j \in \mathbb{N}$ and $\mu \in A_j^*$, get $\lim_{k\in\mathbb{N}} \mu(1_{A_k}) = \|\mu\|_{A_j^*}$. Apply Proposition 2.1.28. □

Semi-finite W^*-subalgebras have unique noncommutative conditional expectations as per Remark 2.1.8. In the unital finite-dimensional case, they are averages of unitary conjugations [46][127][128] as per Proposition 2.1.34. Proposition 2.1.35 generalises to the non-unital finite-dimensional one. In Subsection 2.2.1, Lemma 2.2.22 moreover uses Proposition 2.1.35 to show monotonicity of quasi-entropies.

Assume A is finite-dimensional. Let $N \subset A$ be a C^*-subalgebra. The commutant $N' \subset A$ of N in A is a C^*-algebra. The unitaries $\mathscr{U}(A)$ of A are a compact group, hence $\mathscr{U}(N') = \mathscr{U}(A) \cap N'$ is one. We know $\mathscr{U}(N') = \mathscr{U}(N[1_A]')$ since $N' = N[1_A]'$. We therefore have $1_N^\perp = 1_A - 1_N$ and $N[1_A] = N \oplus \langle 1_N^\perp \rangle_{\mathbb{C}}$ using direct sum of C^*-algebras. Finally, we use the rescaling map $\kappa_N^A : A \longrightarrow \mathbb{C}$ (cf. Definition B.2.12).

Proposition 2.1.34. *Assume A is finite-dimensional. Let $N \subset A$ be a C^*-subalgebra and ν_N the Haar probability measure on $\mathscr{U}(N')$. The noncommutative conditional expectation from A to $N[1_A]$ is given by $\pi^A_{N[1_A]}(x) = \int_{\mathscr{U}(N')} uxu^* d\nu_N$ for all $x \in A$.*

Proof. We have $N[1_A] = \mathscr{U}(N[1_A]')' = \mathscr{U}(N')'$ (cf. Proposition A.1.37). For all $x \in A$, set $P(x) := \int_{\mathscr{U}(N')} uxu^* d\nu_N(u)$. Note transformation of Haar measures under group actions implies $P(x) \in \mathscr{U}(N')' = N[1_A]$. Using uniqueness of the noncommutative conditional expectation from A to $N[1_A]$ (cf. Definition B.2.7), we directly verify our claim. □

Proposition 2.1.35. *Assume A is finite-dimensional. Let $N \subset A$ be a C^*-subalgebra. We have $\pi^A_N = \pi^A_{N[1_A]} - \kappa^A_N 1^\perp_N$. For all $x \in A$, this Hilbert space projection is given by*

$$\pi^A_N(x) = \begin{cases} \int_{\mathscr{U}(N')} uxu^* d\nu_N + \tau(1_N^\perp)^{-1} \tau\left(\pi^A_{\langle 1_N^\perp \rangle_{\mathbb{C}}}(x)\right) \cdot 1_N^\perp & \text{if } 1_A \neq 1_N, \\ \int_{\mathscr{U}(N')} uxu^* d\nu_N & \text{else}. \end{cases} \tag{2.12}$$

Proof. Apply Proposition 2.1.34 and 1) in Proposition B.2.13. □

Definition using local *-homomorphisms. We use local *-homomorphisms to define AF-C^*-bimodule actions. In addition, Lemma 2.1.41 and Corollary 2.1.42 show local *-homomorphisms extend to a *-homomorphism of spaces of measurable operators s.t. L^p-norms are preserved. Definition 2.1.46 gives AF-C^*-bimodules.

Definition 2.1.36. Let (A,τ) and (B,ω) be tracial AF-C^*-algebras. Let $\phi : A \longrightarrow B$ be a *-homomorphism.

1) For all $j \in \mathbb{N}$ s.t. $\phi(A_j) \subset B_j$, set $\phi_j := \phi|_{A_j} : (A_j, \|.\|_\tau) \longrightarrow (B_j, \|.\|_\omega)$ and $\phi_j^* := (\phi_j)^*$ for its adjoint.

2) We say that ϕ satisfies

   2.1) local unitality if $\phi(1_{A_j}) = 1_{B_j}$ for all $j \in \mathbb{N}$,

   2.2) locality if $\phi(A_j) \subset B_j$ and $\phi_k^*(B_j) \subset A_j$ for all $j \leq k$ in $\mathbb{N}$,

   2.3) extendability if $\sup_{j\in\mathbb{N}} \|\phi_j^*(1_{B_j})\|_A, \sup_{j\in\mathbb{N}} \|\phi_j^*\|_{\mathscr{B}(B_j,A_j)} < \infty$.

3) We call ϕ local if it satisfies locality, local unitality and extendability.

Example 2.1.37. For all tracial AF-C^*-algebras (A,τ), its identity map id_A is local.

Proposition 2.1.38. *Let (A,τ) be a tracial AF-C^*-algebra s.t. $\tau<\infty$. If $T\in\mathscr{B}(L^2(A,\tau))$ s.t. $T(1_{A_j})=1_{A_j}$ for all $j\in\mathbb{N}$, then $T(1_A)=1_A$.*

Proof. Since $\tau<\infty$, get $1_A\in L^2(A,\tau)$ and $A_0\subset L^2(A,\tau)$. Thus 2) in Proposition 2.1.16 implies $1_A=\text{s-lim}_{j\in\mathbb{N}}1_{A_j}$, hence $1_A=\|.\|_\tau\text{-lim}_{j\in\mathbb{N}}1_{A_j}$. □

Let (A,τ) and (B,ω) be tracial AF-C^*-algebras. Note 2) in Proposition 2.1.20 shows $A_0\subset L^2(A,\tau)$ is $\|.\|_\tau$-dense and $B_0\subset L^2(B,\omega)$ is $\|.\|_\omega$-dense. We use such density for Equation 2.14. Let $\phi:A\longrightarrow B$ be a *-homomorphism. If ϕ satisfies locality, then

$$\phi_k^*|_{B_j}=\phi_j^* \tag{2.13}$$

for all $j\le k$ in $\mathbb{N}$. Assume ϕ is local. For all $x\in A\cap L^2(A,\tau)$ and $u\in L^2(B,\omega)$, we use density and extendability to get $\phi(x)\in L^2(B,\omega)$ and

$$\left\langle\phi(x),u\right\rangle_\omega\le\|x\|_\tau\cdot\sup_{j\in\mathbb{N}}\|\phi_j^*\|_{\mathscr{B}(B_j,A_j)}\|u\|_\omega<\infty. \tag{2.14}$$

Equation 2.14 yields extension $\phi^2\in\mathscr{B}(L^2(A,\tau),L^2(B,\omega))$ of ϕ with norm

$$\|\phi\|_2:=\|\phi^2\|_{\mathscr{B}(L^2(A,\tau),L^2(B,\omega))}\le\sup_{j\in\mathbb{N}}\|\phi_j^*\|_{\mathscr{B}(B_j,A_j)}. \tag{2.15}$$

Definition 2.1.39. Let (A,τ) and (B,ω) be tracial AF-C^*-algebras. Let $\phi:A\longrightarrow B$ be a local *-homomorphism. We call $\phi^2\in\mathscr{B}(L^2(A,\tau),L^2(B,\omega))$ the L^2-extension of ϕ. Let $\phi^{2,*}:=\left(\phi^2\right)^*$ be its adjoint.

Proposition 2.1.40. *Let (A,τ) and (B,ω) be tracial AF-C^*-algebras. Let $\phi:A\longrightarrow B$ be a local *-homomorphism.*

1) $\phi^*\circ\flat|_{B_0}=\flat\circ\left(\phi^{2,*}|_{B_0}\right)$ *using Banach dual* $\phi^*:B^*\longrightarrow A^*$.

2) $\phi^{2,*}$ *is positivity-preserving.*

3) For all $j\in\mathbb{N}$, we have

3.1) $\phi^{2,*}|_{B_j}=\phi_j^*$ *and* $[\pi_j^A,\phi^2]=0$,

3.2) $\|\phi^{2,*}(u)\|_A\le\|\phi_j^*(1_{B_j})\|_A\|u\|_B$ *for all* $u\in B_{j,h}$.

Proof. We have 3.1) by locality. Using 3.1), we directly verify 1) by testing on A_0 in each case. Then 1) shows 2) since ϕ is a *-homomorphism and $\flat$ is positivity-preserving. For all Hilbert spaces H, $T\in\mathscr{B}(H)_h$ and $C\ge0$, we have $\|T\|_{\mathscr{B}(H)}\le C$ if and only if $-CI\le T\le CI$. Using the latter, note 2) and 3.1) show 3.2) immediately. □

Lemma 2.1.41. *Let (A,τ) and (B,ω) be tracial AF-C*-algebras. Let $\phi : A \longrightarrow B$ be a local *-homomorphism.*

1) *There exists positivity-preserving w^*-continuous $\phi^1 \in \mathscr{B}\big(L^1(A,\tau),L^1(B,\omega)\big)$ with norm $\|\phi\|_1 := \|\phi^1\| \leq 2\sup_{j\in\mathbb{N}} \|\phi_j^*(1_{B_j})\|_A$ extending ϕ. Let $\phi^{1,*} := \big(\phi^1\big)^*$ be its Banach dual. We have*

   1.1) $\omega\big(\phi^1(x)^* u\big) = \tau\big(x^* \phi^{1,*}(u)\big)$ *for all $x \in L^1(A,\tau)$ and $u \in L^\infty(B,\omega)$,*

   1.2) $\phi_0^* := \phi^{1,*}\big|_{B_0} = \phi^{2,*}\big|_{B_0}$.

2) *There exists normal unital *-homomorphism $\phi^\infty : L^\infty(A,\tau) \longrightarrow L^\infty(B,\omega)$ with norm $\|\phi\|_\infty := \|\phi^\infty\| = 1$ extending ϕ. Let $\phi^{\infty,*} := \big(\phi^\infty\big)^*$ be its Banach dual. We have $\phi^{\infty,*} \circ \flat|_{B_0} = \flat \circ \phi_0^*$.*

3) *For all $j \in \mathbb{N}$, $\phi^1(x_j) = \phi^1(x)_j$ for all $x \in L^1(A,\tau)$.*

Proof. Note (σ-)weak- and w^*-convergence coincide on bounded sets (cf. Lemma II.2.5 in [192] and Proposition A.1.34). We use bounded strong and bounded weak convergence (cf. Definition A.1.39 and Remark A.1.41). In particular, multiplication in W^*-algebras is bounded strongly continuous (cf. Remark A.1.43). We know Proposition A.1.38 applies to $A_0 \subset L^\infty(A,\tau)$ and $B_0 \subset L^\infty(B,\omega)$ by σ-weak density.

We show 1). Let $x \in A_0$ and $u \in L^\infty(B,\omega)$. If $\|u\|_\infty = 1$, then Proposition A.1.38 yields $\{u_k\}_{k\in K} \subset B_0$ s.t. $\sup_{k\in K} \|u_k\|_B \leq 1$ and $u = w^*\text{-}\lim_{k\in K} u_k$. If we furthermore apply 3.2) in Proposition 2.1.40 to $\mathrm{Re}(u_k)$ and $\mathrm{Im}(u_k)$ for all $k \in K$ (cf. Proposition B.1.47), then we calculate

$$\begin{aligned} \big|\omega\big(\phi(x)^* u\big)\big| &= \limsup_{k\in K} \big|\omega\big(x^* \phi^{2,*}(u_k)\big)\big| \\ &\leq \limsup_{k\in K} \big|\omega\big(x^* \phi^{2,*}\big(\mathrm{Re}(u_k)\big)\big)\big| + \limsup_{k\in K} \big|\omega\big(x^* \phi^{2,*}\big(\mathrm{Im}(u_k)\big)\big)\big| \\ &\leq \|x\|_1 \cdot 2\sup_{j\in\mathbb{N}} \|\phi_j^*(1_{B_j})\|_A . \end{aligned}$$

Using the above calculation, linearity and extendability of ϕ let us estimate

$$\big|\omega\big(\phi(x)^* u\big)\big| \leq \|x\|_1 \cdot 2\sup_{j\in\mathbb{N}} \|\phi_j^*(1_{B_j})\|_A \|u\|_\infty < \infty. \tag{2.16}$$

Equation 2.16 yields extension $\phi^1 \in \mathscr{B}(L^1(A,\tau),L^1(B,\omega))$ of ϕ with norm estimate as claimed. Using boundedness and 3.1) in Proposition 2.1.40, we directly verify 1.1) by testing on A_0 and B_0. Note 1.1) implies 1.2). Using properties of the modified standard pairing (cf. Proposition B.1.51), we additionally see 1.1) implies positivity-preservation and w^*-continuity of ϕ^1. Altogether, get 1).

We show 2). Using 1.1), traciality lets us calculate

$$\omega\big(\big(\phi(x)-\phi(y)\big)^* u\big) = \tau\big(\big(x^* - y^*\big)\phi^{1,*}(u)\big) = \overline{\tau\big((x-y)\phi^{1,*}(u)^*\big)} \tag{2.17}$$

for all $x, y \in A_0$ and $u \in L^\infty(B,\omega)$. For all $x \in L^\infty(A,\tau)$, Proposition A.1.38 shows there exists bounded net $\{x_k\}_{k\in K} \subset A_0$ s.t. $x = w^*\text{-}\lim_{k\in K} x_k$. Using the latter in order to test on A_0, Equation 2.17 yields positivity-preserving and w^*-continuous linear extension $\phi^\infty : L^\infty(A,\tau) \longrightarrow L^\infty(B,\omega)$ of ϕ by boundedness. For all $u \in B_0$, 2) in Proposition 2.1.40 implies $\phi^{2,*}(uu^*) = a_u^2 \geq 0$ for a self-adjoint $a_u \in A_0$. For all $x, y \in A_0$ and $u \in B_0$, get

$$\big\|\big(\phi(x)-\phi(y)\big)u\big\|_\omega^2 = \big\langle \phi\big((x-y)^*(x-y)\big), uu^* \big\rangle_\omega = \big\|(x-y)a_u\big\|_\tau^2. \tag{2.18}$$

We know $B_0 \subset L^2(B,\omega)$ is $\|.\|_\omega$-dense. Thus Equation 2.18 shows ϕ^∞ is bounded strongly convergent, hence ϕ is a *-homomorphism. Ergo ϕ^∞ is normal by Proposition A.1.49, as well as unital by local unitality and 2) in Proposition 2.1.16. Get $\|\phi^\infty\| = 1$. Using 1) in Proposition 2.1.40, we directly verify $\phi^{\infty,*} \circ \flat|_{B_0} = \flat \circ \phi_0^*$. Altogether, get 2).

We show 3). Following Remark 2.1.2, noncommutative L^1-spaces are subsets of their Banach double dual spaces. Following Notation 2.1.29, get 3) if

$$\phi^1(x_j) = \phi^1\Big(\sharp\mathrm{res}_j x^\flat\Big) = \sharp\mathrm{res}_j \phi^1(x)^\flat = \phi^1(x)_j \tag{2.19}$$

for all $x \in L^1(A,\tau)$ and $j \in \mathbb{N}$. Using 1.1) and $\phi^{2,*}(B_0) \subset A_0$, get Equation 2.19 at once. □

Corollary 2.1.42. *Let (A,τ) and (B,ω) be tracial AF-C^*-algebras. Let $\phi : A \longrightarrow B$ be a local *-homomorphism. There exists unital *-homomorphism $\phi : L^0(A,\tau) \longrightarrow L^0(B,\omega)$ continuous in measure topologies extending ϕ.*

Proof. We use uniform structures (cf. Equation B.5). If $p \in L^\infty(A,\tau)$ is a projection, then $\phi(p) \in L^\infty(B,\omega)$ is a projection and $\phi(p^\perp) = \phi(p)^\perp$ by 2) in Lemma 2.1.41. If furthermore $p^\perp \in L^1(A,\tau)$, then $\phi(p)^\perp \in L^1(B,\omega)$ by 1) in Lemma 2.1.41. Let $\varepsilon, \delta > 0$. If $x \in L^\infty(A,\tau)$ and $p \in L^\infty(A,\tau)$ projection s.t. $\|xp\|_\infty < \varepsilon$ and $\tau(p^\perp) < \delta$, then $\|\phi(x)\phi(p)\|_\infty \leq \|xp\|_\infty < \varepsilon$ by 2) in Lemma 2.1.41 and $\tau(\phi(p)^\perp) \leq \|\phi\|_1 \tau(p^\perp) < \|\phi\|_1 \delta$ by 1) in Lemma 2.1.41.

For all $\varepsilon, \delta > 0$, get $\phi(N(\varepsilon,\delta)) \subset N(\varepsilon, \|\phi\|_1 \delta)$. Thus ϕ maps bounded Cauchy nets to bounded Cauchy nets in measure topologies, hence extends as claimed. For this, note algebra involution and multiplication in spaces of measurable operators are continuous in measure topology on bounded subsets (cf. Theorem IX.2.2 in [193] or [161]). □

Notation 2.1.43. All extensions of local *-homomorphisms as discussed above coincide on intersections of domains. Unless stated otherwise, we do not discern extensions. For all local *-homomorphisms ϕ, we write ϕ for extensions and ϕ^* for their adjoints.

Definition 2.1.46 gives AF-C^*-bimodules. Proposition 2.1.49 moreover shows they induce symmetric W^*-bimodules as per Definition 2.1.48, i.e. as per Definition 2.1.50 in all further use below. Let $\phi, \psi : A \longrightarrow B$ be local *-homomorphisms. We define bounded A-bimodule action on B by setting

$$xuy := \phi(x)u\psi(y) \tag{2.20}$$

for all $x, y \in A$ and $u \in B$. Applying 2) in Lemma 2.1.41, we extend Equation 2.20 to a normal, unital and bounded $L^\infty(A,\tau)$-bimodule action on $L^2(B,\omega)$. Symmetry requires anti-linear involution, with algebra involution the canonical example.

Definition 2.1.44. Let (A,τ) be a tracial AF-C^*-algebra. We call anti-linear isometric involution $\gamma : L^2(A,\tau) \longrightarrow L^2(A,\tau)$ local if $\gamma(A_j) \subset A_j$ and $\gamma(1_{A_j}) = 1_{A_j}$ for all $j \in \mathbb{N}$.

Example 2.1.45. For all tracial AF-C^*-algebras (A,τ), note the algebra involution on A itself extends to a local anti-linear isometric involution $\mathrm{Adj} : L^2(A,\tau) \longrightarrow L^2(A,\tau)$ since $A_0 \subset \mathfrak{m}_\tau$ (cf. Proposition B.1.42).

Definition 2.1.46. Let (A,τ) and (B,ω) be tracial AF-C^*-algebras. Let $\phi, \psi : A \longrightarrow B$ be local *-homomorphisms. Let $\gamma : L^2(B,\omega) \longrightarrow L^2(B,\gamma)$ be a local anti-linear isometric involution.

1) The AF-A-bimodule action given by Equation 2.20 is called the (ϕ,ψ)-action of A on B. Its extension to $L^\infty(A,\tau)$ acting on $L^2(B,\omega)$ is called normal extension.

2) We say that the (ϕ,ψ)-action satisfies γ-symmetry if

$$\gamma\big(\phi(x)u\psi(y)\big) = \phi(y^*)\gamma(u)\psi(x^*) \tag{2.21}$$

for all $x, y \in A$ and $u \in B$.

3) We call (ϕ,ψ,γ) an AF-A-bimodule structure on B, or AF-A-bimodule over B if the (ϕ,ψ)-action satisfies γ-symmetry. We further call (ϕ,ψ,γ) an AF-C^*-bimodule.

4) Let (ϕ,ψ,γ) be an AF-A-bimodule structure on B. For all $j \in \mathbb{N}$, we consider tracial AF-C^*-algebras (A_j,τ) and (B_j,ω) as per Definition 2.1.22. We furthermore call $(\phi_j,\psi_j,\gamma_j) := (\phi|_{A_j}, \psi|_{A_j}, \gamma|_{A_j})$ the induced AF-A_j-bimodule structure on B_j.

5) Assume $\phi = \psi = \mathrm{id}_A$ and further $\gamma = \mathrm{Adj}$ as per Example 2.1.45 for A as anti-linear involution. We call $(\mathrm{id}_A, \mathrm{id}_A, \mathrm{Adj})$ the canonical AF-A-bimodule structure on A.

Proposition 2.1.47. *Let (A,τ) and (B,ω) be tracial AF-C^*-algebras. If (ϕ,ψ,γ) is an AF-A-bimodule structure on B, then we have AF-A_j-bimodule structure (ϕ_j,ψ_j,γ_j) on B_j for all $j \in \mathbb{N}$. If $\phi = \psi = \mathrm{id}_A$, then we have AF-$A$-bimodule structure $(\mathrm{id}_A, \mathrm{id}_A, \mathrm{Adj})$ on A.*

Proof. By construction of either case. □

Definition 2.1.48. Let A be a C^*-algebra and $\phi, \psi : A \longrightarrow \mathscr{B}(H)$ *-homomorphisms. For all $x, y \in A$, let $[\phi(x), \psi(y)] = 0$. Let H be a Hilbert space and $\gamma : H \longrightarrow H$ an anti-linear isometric involution. We define bounded A-bimodule action by setting

$$xuy := \phi(x)\big(\psi(y)(u)\big) \tag{2.22}$$

for all $x, y \in A$ and $u \in H$. The A-bimodule action given by Equation 2.22 is called the (ϕ, ψ)-action of A on H. We say that the (ϕ, ψ)-action satisfies γ-symmetry if

$$\gamma(xuy) = y^* \gamma(u) x^* \tag{2.23}$$

for all $x, y \in A$ and $u \in H$. We call H a symmetric C^*-bimodule over A if the bounded A-bimodule action satisfies γ-symmetry. We call H a symmetric W^*-bimodule if $A = M$ is a W^*-algebra, H is a symmetric C^*-bimodule over M, and ϕ, ψ are normal unital.

Proposition 2.1.49. *Let (A, τ) and (B, ω) be tracial AF-C^*-algebras. If (ϕ, ψ, γ) is an AF-A-bimodule structure on B, then $L^2(B, \omega)$ equipped with the normal extension of the (ϕ, ψ)-action and γ is a symmetric W^*-bimodule over $L^\infty(A, \tau)$.*

Proof. Note (ϕ, ψ)-action as per Equation 2.20 is (L^ϕ, R^ψ)-action as per Definition 2.1.48 and Definition 2.1.51. Thus $L^2(B, \omega)$ is symmetric C^*-bimodule over A for γ anti-linear involution. We extend by 2) in Lemma 2.1.41 and bounded strong continuity of γ. □

Definition 2.1.50. Let (A, τ) and (B, ω) be tracial AF-C^*-algebras. Let (ϕ, ψ, γ) be an AF-A-bimodule structure on B. We equip $L^2(B, \omega)$ with the normal extension of the (ϕ, ψ)-action and γ. We call $L^2(B, \omega)$ the induced symmetric W^*-bimodule of (ϕ, ψ, γ).

2.1.2 Functional calculus for AF-C*-bimodules

We discuss canonical left- and right-actions of AF-C^*-bimodules. Theorem B.2.44 states sufficient conditions for compressing joint functional calculus pulled-back along such canonical left- and right-actions to joint functional calculus of self-adjoint measurable operators. This defines the compressed pulled-back joint functional calculus of extended AF-C^*-bimodule actions. In Subsection 2.2.2, we use the latter to construct and control noncommutative division operators of positive measurable operators.

Canonical left- and right-actions of AF-C^*-bimodules. Tracial W^*-algebras determine canonical left- and right-actions of their spaces of measurable operators on noncommutative L^2-space (cf. Definition B.1.55). Compression of AF-C^*-bimodules uses semi-finite W^*-algebras and canonical inclusions of spaces of measurable operators as per Theorem B.2.28 (cf. Definition B.2.1 and Remark B.2.29). For details on underlying compression maps, we refer to Subsection B.2.1.

Pulling back along AF-C^*-bimodule actions defines canonical left- and right-actions of AF-C^*-bimodules. We use the opposite algebra construction (cf. Definition B.1.15). Let (A,τ) and (B,ω) be tracial AF-C^*-algebras. Let (ϕ,ψ,γ) be an AF-A-bimodule structure on B. Corollary 2.1.42 lets us extend to unital *-homomorphisms $\phi : L^0(A,\tau) \longrightarrow L^0(B,\omega)$ and $\psi : L^0(A,\tau)^{\mathrm{op}} \longrightarrow L^0(B,\omega)^{\mathrm{op}}$. We have canonical left- and right-action

$$L_{L^\infty(B,\omega)} : L^0(B,\omega) \longrightarrow \mathscr{UB}\big(L^2(B,\omega)\big),\ R_{L^\infty(B,\omega)} : L^0(B,\omega)^{\mathrm{op}} \longrightarrow \mathscr{UB}\big(L^2(B,\omega)\big) \tag{2.24}$$

of $L^0(B,\omega)$ on $L^2(B,\omega)$ (cf. Definition B.1.55 and Definition B.1.56). Moreover, we know they are unbounded faithful unital *-representations (cf. Corollary B.1.64). For details on canonical left- and right-actions, we refer to Subsection B.1.3.

Definition 2.1.51. Set $L^\phi := L_{L^\infty(B,\omega)} \circ \phi$ and $R^\psi := R_{L^\infty(B,\omega)} \circ \psi$. We thereby define canonical left- and right-action

$$L^\phi : L^0(A,\tau) \longrightarrow \mathscr{UB}\big(L^2(B,\omega)\big),\ R^\psi : L^0(A,\tau)^{\mathrm{op}} \longrightarrow \mathscr{UB}\big(L^2(B,\omega)\big) \tag{2.25}$$

of $L^0(A,\tau)$ on $L^2(B,\omega)$.

Notation 2.1.52. For all $x \in L^0(A,\tau)$, we write $L^\phi_x := L^\phi(x)$ and $R^\psi_x := R^\psi(x)$. We suppress ϕ and ψ in Definition 2.1.51 if $\phi = \psi = \mathrm{id}_A$.

Example 2.1.53. In the setting of 5) in Definition 2.1.46, note Definition 2.1.51 is in fact canonical left- and right-action of $L^0(A,\tau)$ on $L^2(A,\tau)$.

Proposition 2.1.54 shows canonical left- and right-actions as per Definition 2.1.51 are unbounded faithful unital *-representations. Restriction to the bounded case yields induced symmetric W^*-bimodule actions. Proposition 2.1.55 uses bounded measurable functional calculus of self-adjoint measurable operator (cf. Definition B.1.73). The latter ensures positivity-preservation and shows parts of Lemma 2.1.59.

Proposition 2.1.54. *For all $x \in L^0(A,\tau)$, L^ϕ_x and R^ψ_x are densely defined closed operators on $L^2(M,\tau)$. For all $x, y \in L^0(B,\omega)$ and $\lambda \in \mathbb{C}$, we have*

1) $L^\phi_{\lambda_1 x + \lambda_2 y} = \overline{\lambda_1 L^\phi_x + \lambda_2 L^\phi_y}$ *and* $R^\psi_{\lambda_1 x + \lambda_2 y} = \overline{\lambda_1 R^\psi_x + \lambda_2 R^\psi_y}$,

2) $L^\phi_{xy} = \overline{L^\phi_x L^\phi_y}$ *and* $R^\psi_{xy} = \overline{R^\psi_y R^\psi_x}$,

3) $L^\phi_{x^*} = \big(L^\phi_x\big)^*$ *and* $R^\psi_{x^*} = \big(R^\psi_x\big)^*$.

Proof. Apply Corollary 2.1.42 and Corollary B.1.64. □

Proposition 2.1.55. *For all $x, y \in L^0(A,\tau)_h$, we have*

1) $L^\phi_x, R^\psi_y \in \mathscr{U}\mathscr{B}\big(L^2(B,\omega)\big)_+$ *commute strongly,*

2) $L^\phi\big(\Gamma_{x,L^\infty(A,\tau)}(R_{\pm i})\big) = R_{\pm i}\big(L^\phi_x\big)$ *and* $R^\psi\big(\Gamma_{y,L^\infty(A,\tau)}(R_{\pm i})\big) = R_{\pm i}\big(R^\psi_y\big)$.

Proof. Note $R_{\pm i}$ are resolvents in $\pm i$ (cf. Notation A.1.81). Let $x, y \in L^0(A,\tau)_+$. Then $\phi(x), \psi(y) \in L^0(B,\omega)_h$ by Corollary 2.1.42. Get $\Gamma_{x,L^\infty(A,\tau)}(R_{\pm i}), \Gamma_{y,L^\infty(A,\tau)}(R_{\pm i}) \in L^\infty(A,\tau)$ using their bounded measurable functional calculus. Moreover, Proposition A.1.96 and 2) in Lemma B.1.72 imply canonical left- and right-actions of self-adjoint measurable operators commute strongly. Yet $L^\phi = L_{L^\infty(B,\omega)} \circ \phi$ and $R^\psi = R_{L^\infty(B,\omega)} \circ \psi$. Thus 1) follows by Proposition 2.1.54. We have

$$\phi\big(\Gamma_{x,L^\infty(A,\tau)}(R_{\pm i})\big) = \Gamma_{\phi(x),L^\infty(B,\omega)}(R_{\pm i}),\ \psi\big(\Gamma_{y,L^\infty(A,\tau)}(R_{\pm i})\big) = \Gamma_{\psi(x),L^\infty(B,\omega)}(R_{\pm i}) \tag{2.26}$$

by the *-homomorphism property. Hence 2) follows by Equation 2.26. □

Definition 2.1.56 gives compression of AF-C^*-bimodules by compressing canonical left- and right-actions. We use the compressibility property in Definition B.2.43, itself based on Definition B.2.38, for the pair of normal unital *-homomorphisms

$$L^\phi : L^\infty(A,\tau) \longrightarrow \mathscr{B}\big(L^2(B,\omega)\big),\ R^\psi : L^\infty(A,\tau)^{\mathrm{op}} \longrightarrow \mathscr{B}\big(L^2(B,\omega)\big). \tag{2.27}$$

We give two classes of compression. First, we compress to induced AF-C^*-bimodules in Corollary 2.1.63. Secondly, we compress with projections in Corollary 2.1.65.

Definition 2.1.56. Let $N \subset (L^\infty(A,\tau),\tau)$ and $V \subset L^2(B,\omega)$ be a Hilbert subspace. We say that (ϕ,ψ,γ) is (N,V)-compressible, and call (N,V) a compression of (ϕ,ψ,γ), if (L^ϕ, R^ψ) is (N,V)-compressible as per Definition B.2.43 and $\gamma(V) \subset V$.

Remark 2.1.57. Let $N \subset (L^\infty(A,\tau),\tau)$ and $V \subset L^2(B,\omega)$ be a Hilbert subspace. Note $\pi^B_V : L^2(B,\omega) \longrightarrow V$ is the Hilbert space projection. Then (L^ϕ, R^ψ) is (N,V)-compressible if $L^\phi(L^\infty(A,\tau)), R^\psi(L^\infty(A,\tau)) \subset \mathscr{B}(V)$ and

$$\pi^B_V = L^\phi_{1_A}\pi^B_V = R^\psi_{1_A}\pi^B_V. \tag{2.28}$$

Proposition 2.1.58. *Let $N \subset (L^\infty(A,\tau),\tau)$ and $V \subset L^2(B,\omega)$ be a Hilbert subspace. If (ϕ,ψ,γ) is (N,V)-compressible, then*

1) $\phi(N)V \subset V$, $V\psi(N) \subset V$ *and* $\gamma(V) \subset V$,

2) *V equipped with the (ϕ,ψ)-action and γ is a symmetric W^*-bimodule over N.*

Proof. Following our discussion in Remark 2.1.57, we directly verify all claims. □

Following Lemma 2.1.59, note compressibility as per Definition 2.1.56 lets us apply Theorem B.2.44. We use reducing Hilbert subspaces and restriction to Hilbert subspaces given by concrete compression maps (cf. Definition A.2.18 and Definition A.2.20). Then Definition 2.1.60 gives compressed canonical left- and right-actions. Restriction to the bounded case yields compressed induced symmetric W^*-bimodule actions.

Lemma 2.1.59. *Let $N \subset (L^\infty(A,\tau),\tau)$ and $V \subset L^2(B,\omega)$ be a Hilbert subspace. Let (ϕ,ψ,γ) (N,V)-compressible. If $x,y \in L^0(N,\tau)_h$, then $L^\phi_x, R^\psi_y \in \mathscr{UB}_V(L^2(B,\omega))$ commute strongly and we have*

$$L^\phi\big(\Gamma_{x,L^\infty(A,\tau)}(R_{\pm i})\big) = R_{\pm i}\big(L^\phi_x\big),\ R^\psi\big(\Gamma_{y,L^\infty(A,\tau)}(R_{\pm i})\big) = R_{\pm i}\big(R^\psi_y\big). \tag{2.29}$$

Proof. Let $x,y \in L^0(N,\tau)_h$. Proposition 2.1.55 shows all claims except V-reducibility of L^ϕ_x and R^ψ_y. Set $\phi_L := L^\phi \circ L^{-1}_{L^\infty(A,\tau)}$ and $\psi_R := R^\psi \circ R^{-1}_{L^\infty(A,\tau)}$ on their respective images. Note resolvents are preserved under canonical left- and right-actions.

Arguing as in the proof of Lemma A.2.12, we know mapping C^*-generators as per Equation 2.29 and subsequent closing in σ-weak operator topology readily yields normal unital *-isomorphisms

$$\phi_L : W^*\big(L_{x,L^\infty(A,\tau)}\big) \longrightarrow W^*\big(L^\phi_x\big),\ \psi_R : W^*\big(R_{y,L^\infty(A,\tau)}\big) \longrightarrow W^*\big(R^\psi_y\big). \tag{2.30}$$

Moreover, said argument for the *-homomorphisms in Equation 2.30 shows

$$\phi_L\big(E_{L_{x,L^\infty(A,\tau)}}(Z)\big) = E_{L^\phi_x}(Z),\ \psi_R\Big(E_{R_{y,L^\infty(A,\tau)}}(Z)\Big) = E_{R^\psi_y}(Z). \tag{2.31}$$

for all $Z \in \mathfrak{B}(\mathbb{R})$. Using 2) in Lemma B.1.72, Equation 2.31 implies

$$\phi\big(E_{x,L^\infty(A,\tau)}(Z)\big) = E_{L^\phi_x}(Z),\ \psi\big(E_{L^\infty(A,\tau)}(Z)\big) = E_{R^\psi_y}(Z) \tag{2.32}$$

in each case. Using 1) in Proposition 2.1.58, Equation 2.32 in turn shows

$$\Big[E_{L^\phi_x}(Z),\pi^B_V\Big] = \Big[E_{R^\psi_y}(Z),\pi^B_V\Big] = 0 \tag{2.33}$$

for all $Z \in \mathfrak{B}(\mathbb{R})$. Corollary A.2.28 shows Equation 2.33 implies V-reducibility. □

The *-homomorphism property ensures ϕ and ψ preserve real and imaginary parts (cf. Proposition B.1.47). Following Proposition 2.1.54, note restriction viewed as concrete compression map shows unbounded operators as per Equation 2.34 are densely defined and closed (cf. Proposition A.2.24).

Definition 2.1.60. Let $N \subset (L^\infty(A,\tau),\tau)$ and $V \subset L^2(B,\omega)$ be a Hilbert subspace. Let (ϕ,ψ,γ) be (N,V)-compressible. For all $x,y \in L^0(A,\tau)$, set

$$L^{\phi}_{x,N} := L^{\phi}_{x}|_V,\ R^{\psi}_{y,N} := R^{\psi}_{y}|_V \tag{2.34}$$

Notation 2.1.61. We suppress ϕ and ψ in Definition 2.1.60 if $\phi = \psi = \mathrm{id}_A$. We further suppress N if $N = L^\infty(A,\tau)$. In particular, $L_{\phi(x)}$ and $R_{\psi(y)}$ denote evaluated canonical left- and right-actions of $L^0(B,\omega)$ on $L^2(B,\omega)$.

Lemma 2.1.62. *Let $N_A \subset (L^\infty(A,\tau),\tau)$ and $N_B \subset (L^\infty(B,\omega),\omega)$. If*

1) $\phi(N_A), \psi(N_A) \subset N_B$ and $\phi(1_{N_A}) = \psi(1_{N_A}) = 1_{N_B}$,

2) $\gamma\big(N_B \cap L^2(B,\omega)\big) \subset N_B \cap L^2(B,\omega)$,

then (ϕ,ψ,γ) is $(N_A, L^2(N_B,\omega))$-compressible.

Proof. Following our discussion in Remark 2.1.57, we directly verify all claims. □

Corollary 2.1.63. *For all $j \in \mathbb{N}$, (ϕ,ψ,γ) is (A_j,B_j)-compressible.*

Proof. Let $j \in \mathbb{N}$. Apply Lemma 2.1.62 to $N_A = A_j$ and $N_B = B_j$. □

Let $p \in L^\infty(A,\tau)$ be a projection. We know $L^\infty(A,\tau)[p] \subset (L^\infty(A,\tau),\tau)$. Lemma 2.1.6 shows $(A[p],\tau)$ is a tracial C^*-algebra in $L^\infty(A,\tau)[p]$. Note $L^\infty(A[p],\tau) = pL^\infty(A,\tau)p$.

Definition 2.1.64. Let $p \in L^\infty(A,\tau)$ be a projection. Set $L^2(B[p],\omega) := pL^2(B,\omega)p$. For all $u \in L^2(B,\omega)$, further set

$$\pi_p(u) := pup,\ \pi_p^\perp(u) := pup^\perp + p^\perp up + p^\perp up^\perp. \tag{2.35}$$

Corollary 2.1.65. *For all projections $p \in L^\infty(A,\tau)$, we have*

1) $L^2(B[p],\omega) \subset L^2(B,\omega)$ is a Hilbert subspace and $\pi^B_{L^2(B[p],\omega)} = \pi_p$,

2) (ϕ,ψ,γ) is $\big(L^\infty(A[p],\tau), L^2(B[p],\omega)\big)$-compressible.

Proof. Apply Lemma 2.1.6, 2) in Proposition B.2.13 and Equation 2.35. □

Functional calculus. Following Lemma 2.1.59, we apply Theorem B.2.44 to get compressed pulled-back joint functional calculus of extended AF-C^*-bimodule actions (cf. Definition B.2.46). For this, we compress joint functional calculus pulled-back along canonical left- and right-actions of AF-C^*-bimodules. We use joint functional calculus of strongly commuting self-adjoint unbounded operators (cf. Definition A.1.94), as well as bounded measurable joint functional calculus of self-adjoint measurable operators (cf. Definition B.1.78). For details on the former, we refer to Subsection A.1.3. For details on the latter, we refer to Subsection B.1.3.

Let (A,τ) and (B,ω) be tracial AF-C^*-algebras. Let (ϕ,ψ,γ) be an AF-A-bimodule structure on B. Let $N \subset (L^\infty(A,\tau),\tau)$ and $V \subset L^2(B,\omega)$ be a Hilbert subspace. Assume (ϕ,ψ,γ) is (N,V)-compressible. Let $x,y \in L^0(A,\tau)_h$. Lemma 2.1.59 shows Theorem B.2.44 applies to $T = L^\phi_{x,N}$ and $S = R^\psi_{y,N}$ using (L^ϕ,R^ψ) as (N,V)-compressible pair. Following 1) in Definition B.2.46, we have bounded measurable joint functional calculus

$$\Gamma^{L^\phi,R^\psi}_{x,y,N} : L^\infty\big(\mathrm{spec}_N x \times y, dE_{x,y,N}\big) \longrightarrow \mathscr{B}(V) \tag{2.36}$$

of $x \otimes y$ in $N \otimes N^{\mathrm{op}}$ under $L^\phi \otimes R^\psi$. Following 2) and 3) in Definition B.2.46, we have joint functional calculus

$$\Gamma^{L^\phi,R^\psi}_{x,y,N} : \mathscr{S}_V\big(E_{x,y,N}\big) \longrightarrow \mathscr{UB}(V)_h \tag{2.37}$$

of $x \otimes y$ in $N \otimes N^{\mathrm{op}}$ under $L^\phi \otimes R^\psi$ (cf. Corollary B.2.45).

Remark 2.1.66. In the setting of 5) in Definition 2.1.46, we have $\Gamma^{L^\phi,R^\psi}_{x,y,N} = \Gamma_{x,y,N}$.

Let H be a Hilbert space. If $V \subset H$ is a Hilbert subspace and $\pi_V : H \longrightarrow V$ its Hilbert space projection, then get positivity-preserving canonical inclusion $\mathscr{UB}(V) \subset \mathscr{UB}(H)$ by mapping $T \mapsto \mathrm{com}_V T = \pi_V T \pi_V$. This yields $\mathscr{B}(V) \oplus \mathscr{B}(V^\perp) \subset \mathscr{B}(H)$ (cf. Equation A.42).

Lemma 2.1.67. *Let (ϕ,ψ,γ) be (N,V)-compressible. We consider $L^2(B,\omega) = V \oplus V^\perp$.*

1) For all $x \in L^0(N,\tau)_h$, $L^\phi_x = L^\phi_{x,N} + L^\phi_x\big(I - \pi^B_V\big)$ and $R^\psi_x = R^\psi_{x,N} + R^\psi_x\big(I - \pi^B_V\big)$.

2) Let $x \in L^0(N,\tau)_h$. If $g \in L^\infty\big(\mathrm{spec}_{L^\infty(A,\tau)} x \times y, dE_{x,y,L^\infty(A,\tau)}\big)$, then

2.1) $g \in L^\infty\big(\mathrm{spec}_N x \times y, dE_{x,y,N}\big)$ by restricting to $\mathrm{spec}_N x \times y \subset \mathrm{spec}_{L^\infty(A,\tau)} x \times y$,

2.2) $g\Big(L^\phi_x, R^\psi_y\Big) \in \mathscr{B}\big(L^2(B,\omega)\big) \cap \mathscr{UB}_V\big(L^2(B,\omega)\big)$ and $g\Big(L^\phi_x, R^\psi_y\Big)\Big|_V = g\Big(L^\phi_{x,N}, R^\psi_{y,N}\Big)$,

2.3) $g\Big(L^\phi_x, R^\psi_y\Big) = g\Big(L^\phi_{x,N}, R^\psi_{y,N}\Big) \oplus g\Big(L^\phi_x, R^\psi_y\Big)\Big|_{V^\perp} \in \mathscr{B}(V) \oplus \mathscr{B}(V^\perp)$.

3) If $x,y \in L^0(N,\tau)_+$, $\alpha,\beta \geq 0$ and $g \in C_b([0,\infty) \times [0,\infty))$, then

$$g\Big(L^\phi_{x+\alpha 1^\perp_N}, R^\psi_{y+\beta 1^\perp_N}\Big)\Big|_V = g\Big(L^\phi_{x,N}, R^\psi_{y,N}\Big). \tag{2.38}$$

Proof. Let $x,y \in L^0(N,\tau)_h$. Lemma 2.1.59 shows $L^\phi_x, R^\psi_y \in \mathscr{UB}_V(L^2(B,\omega))$, i.e. each is a V-reducible self-adjoint unbounded operator. Get 1) by 1.3) in Proposition A.2.24. Note using abstract and concrete spectral measures yields identical commutative L^∞-spaces in the uncompressed, resp. compressed case.

We therefore have

$$\Gamma_{L_x^\phi,R_y^\psi}(g)=g\left(L_x^\phi,R_y^\psi\right),\ \Gamma_{L_{x,N}^\phi,R_{y,N}^\psi}(g)=g\left(L_{x,N}^\phi,R_{y,N}^\psi\right)=g\left(L_x^\phi|_V,R_y^\psi|_V\right) \tag{2.39}$$

in each case. Following Lemma 2.1.59, we apply Theorem B.2.44 as discussed above. Spectra restrict as claimed and have

$$g\left(L_x^\phi,R_y^\psi\right)\Big|_V=g\left(L_x^\phi|_V,R_y^\psi|_V\right). \tag{2.40}$$

Equation 2.39 and Equation 2.40 imply 2.1) and 2.2) at once. Using 2.1), get 2.3) by 1.3) in Proposition A.2.24. We have direct sum by boundedness. Altogether, get 2).

We show 3). Let $x,y\in L^0(N,\tau)_+$, $\alpha,\beta\ge 0$ and $g\in C_b([0,\infty)\times[0,\infty))$. Theorem B.2.44 implies Equation 2.38 is

$$\left(L^\phi\otimes_V R^\psi\right)\left(\mathrm{com}_{1_N\otimes 1_N}\left(\Gamma_{x+\alpha 1_N^\perp,y+\beta 1_N^\perp,L^\infty(A,\tau)}(g)\right)\right)=\left(L^\phi\otimes_V R^\psi\right)\left(\Gamma_{x,y,N}(g)\right). \tag{2.41}$$

Applying $L^\phi\otimes_V R^\psi$ to Equation B.126 in Corollary B.2.48 yields Equation 2.41. □

Notation 2.1.68. Assume $(N,V)=(L^\infty(A[p],\tau),L^2(B[p],\omega))$ for projection $p\in L^\infty(A,\tau)$. For all $x,y\in L^0(A,\tau)_h$, we write

1) $L_{x,p}^\phi:=L_{x,L^\infty(A[p],\tau)}^\phi$ and $R_{y,p}^\psi:=R_{y,L^\infty(A[p],\tau)}^\psi$,

2) $\Gamma_{x,y,p}^{\phi,\psi}:=\Gamma_{x,y,L^\infty(A[p],\tau)}^{L^\phi,R^\psi}$ and $\mathscr{S}_p\left(E_{x,y}\right):=\mathscr{S}_{L^2(B[p],\omega)}\left(E_{x,y,L^\infty(A[p],\tau)}\right)$.

We suppress ϕ and ψ if $\phi=\psi=\mathrm{id}_A$. We further suppress p if $p=1_A$.

Lemma 2.1.69. *Assume $(N,V)=(L^\infty(A[p],\tau),L^2(B[p],\omega))$ for projection $p\in L^\infty(A,\tau)$. If $x,y\in L^0(A[p],\tau)_+$ and $g\in C_b([0,\infty)\times[0,\infty))$, then*

1) $g\left(L_x^\phi,R_y^\psi\right)\pi_p=g\left(L_{x,p}^\phi,R_{y,p}^\psi\right)\pi_p$,

2) $g\left(L_x^\phi,R_y^\psi\right)\pi_p^\perp=g\left(L_x^\phi,0\right)L_p^\phi R_{p^\perp}^\psi+g\left(0,R_y^\psi\right)L_{p^\perp}^\phi R_p^\psi+g(0,0)\pi_{p^\perp}$.

Proof. For details on the tensor product of normal unital *-homomorphisms, we refer to Corollary A.1.53. By definition, we have

$$\Gamma_{x,y,L^\infty(A,\tau)}^{L^\phi,R^\psi}=\left(L^\phi\otimes R^\psi\right)\circ\Gamma_{x,y,L^\infty(A,\tau)}. \tag{2.42}$$

Following Notation 2.1.68, Equation 2.35 rewrites as

$$\pi_p = \pi^B_{L^2(B[p],\omega)} = L^\phi_p R^\psi_p,\ \pi^\perp_p = L^\phi_p R^\psi_{p^\perp} + L^\phi_{p^\perp} R^\psi_p + L^\phi_{p^\perp} R^\psi_{p^\perp}. \tag{2.43}$$

Using π_p itself as our Hilbert space projection, get 1) by 2) in Lemma 2.1.67 and 1.1) in Proposition A.2.24. We show 2) by arguing as in the proof of Corollary B.2.48. We assume $p, p^\perp \in W^*_{L^\infty(A,\tau)}(x) \cap W^*_{L^\infty(A,\tau)}(y)$ without loss of generality. Since $W^*_{L^\infty(A,\tau)}(x,y) = W^*_{L^\infty(A,\tau)}(x) \otimes W^*_{L^\infty(A,\tau)}(y)$ (cf. 2) in Definition B.1.75), Equation 2.42 and Equation 2.43 let us calculate

$$\left(L^\phi \otimes R^\psi\right)^{-1}\left(\Gamma^{\phi,\psi}_{x,y,L^\infty(A,\tau)}(g)\pi^\perp_p\right) = \Gamma_{x,y,L^\infty(A,\tau)}(g)\left(p \otimes p^\perp + p^\perp \otimes p + p^\perp \otimes p^\perp\right). \tag{2.44}$$

We write each summand in Equation 2.44 as element in $W^*_{L^\infty(A,\tau)}(x,y)$. Theorem B.2.44 then implies

$$\Gamma_{x,y,L^\infty(A,\tau)}(g)\left(p \otimes p^\perp\right) = \Gamma_{x,0,L^\infty(A,\tau)}(g)\left(p \otimes p^\perp\right), \tag{2.45}$$

$$\Gamma_{x,y,L^\infty(A,\tau)}(g)\left(p^\perp \otimes p\right) = \Gamma_{0,y,L^\infty(A,\tau)}(g)\left(p^\perp \otimes p\right), \tag{2.46}$$

$$\Gamma_{x,y,L^\infty(A,\tau)}(g)\left(p^\perp \otimes p^\perp\right) = \Gamma_{0,0,L^\infty(A,\tau)}(g)\left(p^\perp \otimes p^\perp\right). \tag{2.47}$$

Upon applying $L^\phi \otimes R^\psi$ to Equation 2.44, the above equations show 2) at once. □

2.2 Noncommutative division operators

Noncommutative division operators generalise division by densities in the classical case [97]. In the tracial finite-dimensional cases of [48][49][50], they determine, and are in turn determined by, quasi-entropies [127][128] used to define energy functionals. Note quasi-entropies generalise quantum f-divergences [125][126], a class of dissimilarity measures for information encoded in states of quantum systems [62][141]. Applying the Kato-Robinson theorem [88] lets us extend the approach in [50] to AF-C^*-bimodules.

Noncommutative division operators represent closed positive unbounded quadratic forms determined by quasi-entropies. Quasi-entropies are non-negative, jointly convex and w^*-l.s.c. functionals on Banach dual spaces of AF-C^*-bimodules. The Kato-Robinson theorem shows noncommutative division operators are strong resolvent limits of, upon suitable evaluation for each, perturbed inverses of operator means [13] as perturbation tends to zero. Such perturbed inverses are expressed using compressed pulled-back joint functional calculus of extended AF-C^*-bimodule actions. We recover noncommutative division operators of positive measurable operators if and only if the strong resolvent limit is likewise expressed using compressed pulled-back joint functional calculus.

Structure. In Subsection 2.2.1, we discuss operator means, noncommutative division operators of positive measurable operators and quasi-entropies for AF-C^*-bimodules. In Subsection 2.2.2, we represent closed positive unbounded quadratic forms determined by quasi-entropies using noncommutative division operators.

2.2.1 Quasi-entropies for AF-C*-bimodules

We define noncommutative division operators of positive measurable operators, as well as perturbed ones. Following Lemma 2.1.67 and Lemma 2.1.69, we have control as per Lemma 2.2.13. We define quasi-entropies in the finite-dimensional setting by letting perturbation tend to zero upon applying perturbed noncommutative division operators of positive measurable operators. Using monotonicity under restriction maps, we extend quasi-entropies to AF-C^*-bimodules. Theorem 2.2.29 collects fundamental properties.

Noncommutative division operators of positive measurable operators. Let (A,τ) and (B,ω) be tracial AF-C^*-algebras. Let (ϕ,ψ,γ) be an AF-A-bimodule structure on B. Let $N \subset (L^\infty(A,\tau),\tau)$ and $V \subset L^2(B,\omega)$ be a Hilbert subspace. Let f be representing function of an operator mean as per 2) in Definition 2.2.1 and $\theta \in [0,1]$.

Definition 2.2.1. Let $f:(0,\infty) \longrightarrow (0,\infty)$.

1) We call f symmetric if $f(t) = f(t^{-1})$ for all $t > 0$.

2) We call f representing function of an operator mean, or representing function if it is operator monotone and $f(1) = 1$. We define its mean $m_f:(0,\infty)\times(0,\infty) \longrightarrow (0,\infty)$ by setting $m_f(t,s) := f(ts^{-1})s$ for all $t,s > 0$. For all $\varepsilon > 0$, we furthermore define its mean $m_{f,\varepsilon}:[0,\infty) \longrightarrow (0,\infty)$ perturbed with ε by setting $m_{f,\varepsilon}(t,s) := m_f(t+\varepsilon, s+\varepsilon)$ for all $t,s \geq 0$.

3) Let $\mathscr{A}$ be a unital *-algebra equipped with partial order generated by positive elements. Set $\mathscr{A}_{>0} := \{x \in A_+ \mid \exists \varepsilon > 0 : \ x \geq \varepsilon 1_{\mathscr{A}}\}$. For all $x \in \mathscr{A}$, we say that $x > 0$ in $\mathscr{A}$ if $x \in \mathscr{A}_{>0}$.

Remark 2.2.2. If f is symmetric, then $m_f(t,s) = m_f(s,t)$ for all $t,s > 0$. If f is a representing function, then given separable Hilbert space H and letting $m_f(T,S)$ for all commuting $T,S > 0$ in $\mathscr{B}(H)$ defines operator mean following Kubo and Ando [13].

Proposition 2.2.3. *For all $t_1 \geq t_0 > 0$ and $s_1 \geq s_0 > 0$, get $m_f(t_1,s_1) \geq m_f(t_0,s_0)$. There exists unique continuous extension of m_f to $[0,\infty)\times[0,\infty)$.*

Proof. Let $\mathbb{C} \cong \langle I \rangle_{\mathbb{C}} \subset \mathscr{B}(H)$ for a separable Hilbert space H. For all $t,s > 0$, get $m_f(t,s) = f(tI \cdot s^{-1}I)\cdot sI$. Operator means are connections by Theorem 3.2 in [13]. We see our first claim follows by (I), and our second one by (III) on p.206 in [13]. $\square$

Remark 2.2.4. For all $\varepsilon > 0$, get $m_{f,\varepsilon}^{-1} \in C_b([\varepsilon,\infty)\times[\varepsilon,\infty))$ by Proposition 2.2.3.

Definition 2.2.6 uses joint functional calculus to give noncommutative multiplication and division operators of positive measurable operators. Proposition 2.2.5 ensures 2) and 3) in Definition 2.2.6 are well-defined.

Proposition 2.2.5. *Let (ϕ,ψ,γ) be (N,V)-compressible.*

1) *If $x,y \in L^0(N,\tau)_+$ s.t. $m_f^{-1} \in \mathscr{S}\big(E_{x,y,N}\big)$, then $m_f^{-\theta} \in \mathscr{S}_V\big(E_{x,y,N}\big)$.*

2) *If $x > 0$ in $L^0(N,\tau)$, then there exists $\varepsilon > 0$ s.t. $\operatorname{spec}_N x \subset [\varepsilon,\infty)$.*

Proof. Let $x,y \in L^0(N,\tau)_+$. Note $I_V \in \mathscr{B}(V)$ is the unit. By functional calculus, get

$$m_f^{-\theta}\Big(L^{\phi}_{x,N} + \varepsilon I_V, R^{\psi}_{y,N} + \varepsilon I_V\Big) = m_{f,\varepsilon}^{-\theta}\Big(L^{\phi}_{x,N}, R^{\psi}_{y,N}\Big). \tag{2.48}$$

Let $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset (0,\infty)$ be descending sequence converging to zero. Then Proposition 10.1.8 in [88] implies

$$L^{\phi}_{x,N} = \operatorname{sr-}\lim_{n\in\mathbb{N}} L^{\phi}_{x,N} + \varepsilon_n I_V,\ R^{\psi}_{y,N} = \operatorname{sr-}\lim_{n\in\mathbb{N}} R^{\psi}_{y,N} + \varepsilon_n I_V. \tag{2.49}$$

We show 1). Assume $m_f^{-1} \in \mathscr{S}(E_{x,y,N})$. Note, by definition, 1) holds if $m_f^{-\theta}$ satisfies 1) and 2) in Corollary B.2.45. Get 1) in the corollary by Remark 2.2.4. In order to get 2) in the corollary, we calculate strong convergence of resolvents. Using Equation 2.48 and Equation 2.49, we apply Lemma A.2.5 in the one-variable case to get

$$\begin{aligned} R_{\pm i}\Big(m_f^{-\theta}\Big(L^{\phi}_{x,N}, R^{\psi}_{y,N}\Big)\Big) &= \operatorname{s-}\lim_{\varepsilon\downarrow 0} R_{\pm i}\Big(m_f^{-\theta}\Big(L^{\phi}_{x,N} + \varepsilon I_V, R^{\psi}_{y,N} + \varepsilon I_V\Big)\Big) \\ &= \operatorname{s-}\lim_{\varepsilon\downarrow 0} R_{\pm i}\Big(m_{f,\varepsilon}^{-\theta}\Big(L^{\phi}_{x,N}, R^{\psi}_{y,N}\Big)\Big). \end{aligned}$$

Get 1). Using 2) in Lemma B.1.72 and Corollary B.1.64, we directly verify 2). □

Definition 2.2.6. Let (ϕ,ψ,γ) be (N,V)-compressible. For all $x,y \in L^0(N,\tau)_+$, we define

1) $\mathscr{M}_{x,y,N} := m_f\Big(L^{\phi}_{x,N}, R^{\psi}_{y,N}\Big)$ and $\mathscr{M}_{x,N} := \mathscr{M}_{x,x,N}$,

2) $\mathscr{D}_{x,y,N} := \mathscr{M}^{-1}_{x,y,N}$ and $\mathscr{D}_{x,N} := \mathscr{D}_{x,x,N}$ if $m_f^{-1} \in \mathscr{S}\big(E_{x,y,N}\big)$,

3) $\mathscr{D}_{x,y,N,\varepsilon} := \mathscr{D}_{x+\varepsilon 1_N, y+\varepsilon 1_N, N}$ for all $\varepsilon > 0$.

Notation 2.2.7. We suppress N in Definition 2.2.6 if $N = L^\infty(A,\tau)$.

Remark 2.2.8. All unbounded operators in Definition 2.2.6 are positive. If $x,y > 0$ in $L^0(N,\tau)$, then $\mathscr{D}_{x,y,N} \in \mathscr{B}(V)_+$ by 2) in Proposition 2.2.5. If further $x,y \in L^\infty(A,\tau)$, then $\mathscr{D}_{x,y,N} > 0$ in $\mathscr{B}(V)$ by construction.

Notation 2.2.9. Assume $(N,V) = (L^\infty(A[p],\tau), L^2(B[p],\omega))$ for projection $p \in L^\infty(A,\tau)$. For all $x,y \in L^0(A[p],\tau)_+$, we write

1) $\mathscr{M}_{x,y,p} := \mathscr{M}_{x,y,L^\infty(A[p],\tau)}$ and $\mathscr{M}_{x,p} := \mathscr{M}_{x,x,p}$,

2) $\mathscr{D}_{x,y,p} := \mathscr{D}_{x,y,L^\infty(A[p],\tau)}$ and $\mathscr{D}_{x,p} := \mathscr{D}_{x,x,L^\infty(A[p],\tau)}$ if $m_f^{-1} \in \mathscr{S}_p(E_{x,y})$,

3) $\mathscr{D}_{x,y,p,\varepsilon} := \mathscr{D}_{x,y,L^\infty(A[p],\tau),\varepsilon}$ for all $\varepsilon > 0$.

We suppress p if $p = 1_A$.

Proposition 2.2.10. *Let (ϕ,ψ,γ) be (N,V)-compressible. For all $x,y \in L^0(N,\tau)_+$, we have*

1) $\mathscr{D}^\theta_{x,y,N,\varepsilon} = m_f^{-\theta}\left(L^\phi_{x,N} + \varepsilon I_V, R^\psi_{y,N} + \varepsilon I_V\right) = m_{f,\varepsilon}^{-\theta}\left(L^\phi_{x,N}, R^\psi_{y,N}\right)$ *for all* $\varepsilon > 0$,

2) $\mathscr{D}^\theta_{x,y,N,\varepsilon_1} \le \mathscr{D}^\theta_{x,y,N,\varepsilon_0}$ *in* $\mathscr{B}(V)$ *for all* $\varepsilon_1 \ge \varepsilon_0 > 0$ *in* $\mathbb{R}$,

3) $\mathscr{D}^\theta_{x,y,N} = \text{sr-lim}_{\varepsilon\downarrow 0}\mathscr{D}^\theta_{x,y,N,\varepsilon}$ *if* $m_f^{-1} \in \mathscr{S}(E_{x,y,N})$.

Proof. Since $I_V = L^\phi_{1_N,N} = R^\psi_{1_N,N}$ by unitality as per 2) in Lemma 2.1.41, Equation 2.48 shows 1). Bounded measurable joint functional calculus is positivity-preserving since it is a normal unital *-homomorphism (cf. 1) in Proposition A.1.100). Thus 2) follows from 1) and Proposition 2.2.3. Proposition 2.2.10 shows 3) follows from Corollary B.2.45. □

Lemma 2.2.11. *Let (ϕ,ψ,γ) be (N,V)-compressible. Let $x \in N_h$ and $g \in C_b(\mathbb{R}\times\mathbb{R})$. If $K \subset \mathbb{R}$ is compact s.t.* $\operatorname{spec}_N x \subset K$ *and $g(t,s) = g(s,t)$ for all $t,s \in K$, then*

$$\left[\gamma, g\left(L^\phi_{x,N}, R^\psi_{x,N}\right)\right] = 0. \tag{2.50}$$

Proof. Since K is compact and $g(t,s) = g(s,t)$ for all $t,s \in K$, approximate g uniformly on $K\times K$ by symmetric polynomials. Thus reduce to g polynomial by $\operatorname{spec}_N x \subset K$. Apply γ-symmetry as per Equation 2.21. □

Corollary 2.2.12. *Let (ϕ,ψ,γ) be (N,V)-compressible, f symmetric. For all $x \in N_+$, get*

1) $\mathscr{M}^\theta_{x,N} \circ \gamma = \gamma \circ \mathscr{M}^\theta_{x,N}$,

2) $\mathscr{D}^\theta_{x,N} \circ \gamma = \gamma \circ \mathscr{D}^\theta_{x,N}$ *if* $x > 0$ *in* N,

3) $\mathscr{D}^\theta_{x,N,\varepsilon} \circ \gamma = \gamma \circ \mathscr{D}^\theta_{x,N,\varepsilon}$ *for all* $\varepsilon > 0$.

Proof. By symmetry, get $m_f(t,s)^\theta = m_f(s,t)^\theta$ for all $t,s \ge 0$. Apply Lemma 2.1.69. □

Lemma 2.2.13. *Let (ϕ,ψ,γ) be (N,V)-compressible.*

1) *For all $x,y\in L^0(N,\tau)_+$, we have $\mathscr{M}^\theta_{x,y}\in\mathscr{UB}_V\big(L^2(B,\omega)\big)$ and*

$$\mathscr{M}^\theta_{x,y}\big|_V=\mathscr{M}^\theta_{x,y,N}. \tag{2.51}$$

2) *For all $x,y>0$ in $L^0(N,\tau)$ and $\alpha,\beta>0$, we have $\mathscr{D}^\theta_{x,y,N}=\big(\mathscr{M}_{x,y}\big|_V\big)^{-\theta}$ and*

$$\mathscr{D}^\theta_{x+\alpha 1^\perp_N,y+\beta 1^\perp_N}\Big|_V=\mathscr{M}^{-\theta}_{x+\alpha 1^\perp_N,y+\beta 1^\perp_N}\Big|_V=\mathscr{M}^{-\theta}_{x,y,N}=\mathscr{D}^\theta_{x,y,N}. \tag{2.52}$$

3) *Let $N_A\subset(L^\infty(A,\tau),\tau)$ and $N_B\subset(L^\infty(B,\omega),\omega)$ be finite-dimensional s.t.* 1) *and* 2) *in Lemma 2.1.62 hold. For all $x,y>0$ in N_A and $\alpha,\beta>0$, we have*

$$\mathscr{D}^\theta_{x+\alpha 1^\perp_{N_A},y+\beta 1^\perp_{N_A}}=\mathscr{D}^\theta_{x,y,N_A}\oplus m_f\big(\alpha,\beta\big)^{-\theta}I_{\langle 1^\perp_{N_B}\rangle_{\mathbb{C}}} \tag{2.53}$$

w.r.t. $\mathscr{B}(N_B)\oplus\mathscr{B}(\langle 1^\perp_{N_B}\rangle_{\mathbb{C}})$.

4) *Assume $(N,V)=\big(L^\infty(A[p],\tau),L^2(B[p],\omega)\big)$ for a projection $p\in L^\infty(A,\tau)$. For all $x,y\in L^0(A[p],\tau)_+$ and $\varepsilon>0$, we have*

$$\mathscr{D}^\theta_{x,y,\varepsilon}=\mathscr{D}^\theta_{x,y,p,\varepsilon}\oplus\left(\mathscr{D}^\theta_{x,0,\varepsilon}L^\phi_pR^\psi_{p^\perp}+\mathscr{D}^\theta_{0,y,\varepsilon}L^\phi_{p^\perp}R^\psi_p+\varepsilon^{-\theta}\pi_{p^\perp}\right) \tag{2.54}$$

w.r.t. $\mathscr{B}\big(L^2(B[p],\omega)\big)\oplus\mathscr{B}\big(L^2(B[p],\omega)^\perp\big)$.

Proof. We have 1) by applying 2) in Lemma 2.1.67 to $g=m^\theta_f$. We use 1) to obtain 2) by likewise application of 3) in Lemma 2.1.67. This uses strict positivity since application demands, for g here, an extension from compact joint spectra to $[0,\infty)\times[0,\infty)$.

We show 3). Assume its setting. Let $x,y>0$ in N_A and $\alpha,\beta>0$. Using 2), we have

$$\mathscr{D}^\theta_{x+\alpha 1^\perp_{N_A},y+\beta 1^\perp_{N_A}}\pi^B_{N_B}=\mathscr{D}^\theta_{x,y,N_A}. \tag{2.55}$$

Note $N_B1^\perp_{N_B}=1^\perp_{N_B}N_B=0$ and $\phi(1^\perp_{N_A})=\psi(1^\perp_{N_A})=1^\perp_{N_B}$. Approximating $m^{-\theta}_{f,\varepsilon}$ uniformly using polynomials as in the proof of Lemma 2.2.11, we calculate

$$\mathscr{D}^\theta_{x+\alpha 1^\perp_{N_A},y+\beta 1^\perp_{N_A}}\left(1^\perp_{N_B}\right)=m^{-\theta}_f\left(\alpha 1^\perp_{N_B},\beta 1^\perp_{N_B}\right)=m^{-\theta}_f\big(\alpha,\beta\big)1^\perp_{N_B}. \tag{2.56}$$

Using 2.3) in Lemma 2.1.67, Equation 2.55 and Equation 2.56 imply Equation 2.53 at once. Get 3).

We show 4). Assume its setting. Let $x, y \in L^0(A[p], \tau)_+$ and $\varepsilon > 0$. Note $m_{f,\varepsilon}(\varepsilon, \varepsilon) = f(1)\varepsilon = \varepsilon$ since $f(1) = 1$. In addition, 1) in Proposition 2.2.10 implies

$$\mathscr{D}_{x,y,\varepsilon} = m_{f,\varepsilon}^{-\theta}\left(L_x^{\phi}, R_y^{\psi}\right), \ \mathscr{D}_{x,y,p,\varepsilon} = m_{f,\varepsilon}^{-\theta}\left(L_{x,p}^{\phi}, R_{y,p}^{\psi}\right). \tag{2.57}$$

Equation 2.57 shows 4) by Lemma 2.1.69 and 2.3) in Lemma 2.1.67 applied to $m_{f,\varepsilon}^{-\theta}$. $\square$

Assume A and B are finite-dimensional. Let $B = r_B^{-1}(\oplus_{l=1}^n M_{n_l}(\mathbb{C}))$ equipped with its canonical AF-B-bimodule structure. The latter uses Notation 2.1.15. Corollary A.1.93 implies normal unital *-homomorphisms preserve functional calculus. For all normal $z \in B$, get $\operatorname{spec}_B z = \bigcup_{l=1}^n \operatorname{spec} r_B(z)_l$. Thus z is positive, resp. strictly positive if and only if all $\{r_B(z)_l\}_{l=1}^n$ are. For all $x, y \in A_h$ and $g \in L^\infty(\operatorname{spec}_A x \times y, dE_{x,y,A})$, we obtain

$$g\left(L_x^{\phi}, R_y^{\psi}\right)(u) = r_B^{-1}\left(\oplus_{l=1}^n g\left(L_{r_B(\phi(x))_l}, R_{r_B(\psi(y))_l}\right)\left(r_B(u)_l\right)\right) \tag{2.58}$$

for all $u \in B$.

Proposition 2.2.14. *Assume A and B are finite-dimensional. Let $B = r_B^{-1}(\oplus_{l=1}^n M_{n_l}(\mathbb{C}))$ and equip B with its canonical AF-B-bimodule structure. For all $x, y \in A_+$, we have*

1) $\mathscr{M}_{x,y}^{\theta} = r_B^{-1} \circ \left(\oplus_{l=1}^n \mathscr{M}_{r_B(\phi(x))_l, r_B(\psi(y))_l}^{\theta}\right) \circ r_B$,
2) $\mathscr{D}_{x,y}^{\theta} = r_B^{-1} \circ \left(\oplus_{l=1}^n \mathscr{D}_{r_B(\phi(x))_l, r_B(\psi(y))_l}^{\theta}\right) \circ r_B$ *if $x, y > 0$ in A.*

Proof. Equation 2.58 for $g = m_f^{\theta}$, resp. $g = m_f^{-\theta}$. $\square$

Quasi-entropies in the finite-dimensional setting. Following the notion of monotone metric [175], quasi-entropies for full matrix algebras are given in [127][128]. These are used to define energy functionals in [48][49][50]. Quasi-entropies, elsewhere known as quasi-entropy type functions instead, generalise quantum f-divergences [125] [126]. We clarify and use terminology as per Remark 2.2.16.

Let (A, τ) and (B, ω) be finite-dimensional tracial AF-C^*-algebras. Let (ϕ, ψ, γ) be an AF-A-bimodule structure on B. Let $N \subset A$ and $V \subset B$ be a Hilbert subspace. Let f be representing function of an operator mean and $\theta \in [0, 1]$.

Definition 2.2.15. We define functional $\mathscr{I}_{A,B}^{f,\theta} : A_{>0} \times A_{>0} \times B \longrightarrow [0, \infty)$ by setting

$$\mathscr{I}_{A,B}^{f,\theta}(x, y, u) := \left\langle \mathscr{D}_{x,y}^{\theta}(u), u \right\rangle_{\omega} \tag{2.59}$$

for all $x, y > 0$ in A and $u \in B$.

Remark 2.2.16. In case of full matrix algebras, the terminology in both [127] and [128] is quasi-entropy type functions, rather than quasi-entropies. The latter are a special case for $\theta = -1$ fixed. We nevertheless use quasi-entropies consistent with [50].

Lemma 2.2.17. *$\mathscr{I}^{f,\theta}$ is jointly convex. For all $u \in B$, the map $(x,y) \mapsto \mathscr{I}^{f,\theta}(x,y,u)$ decreases in partial order on $A_{>0} \times A_{>0}$ induced by pairs of positive elements.*

Proof. Proposition 2.1.24 and 2) in Proposition 2.2.14 imply we have

$$\mathscr{I}^{f,\theta}_{A,B}(x,y,u) = \sum_{l=1}^{n} C_l \mathrm{tr}_{n_l}\Big(r_B(u)_l^* m_f^{-\theta}\Big(L_{r_B(\phi(x))_l}, R_{r_B(\psi(y))_l}\Big)\big(r_B(u)_l\big)\Big) \tag{2.60}$$

for all $x,y>0$ in A and $u \in B$. If each summand on the right-hand side of Equation 2.60 satisfies the claimed properties, then our claims follow. We therefore reduce to the case of full matrix algebras since ϕ,ψ and r_B are *-homomorphisms.

Assume $A = B = M_n(\mathbb{C})$ for $n \in \mathbb{N}$ and $\phi = \psi = \mathrm{id}_{M_n(\mathbb{C})}$ without loss of generality. Note γ is of no consequence here. Following [128], get the quasi-entropy type function

$$(X,Y,U) \mapsto \langle \mathscr{D}_{X,Y}(U), U\rangle_{\mathrm{tr}} = \mathrm{tr}\Big(U^* m_f^{-\theta}(L_X, R_Y)(U)\Big) \tag{2.61}$$

defined on $M_n(\mathbb{C})_{>0} \times M_n(\mathbb{C})_{>0} \times M_n(\mathbb{C})$. Theorem 2.1 in [128] gives joint convexity of such functionals since f is operator monotone and $\theta \in [0,1]$. We have operator mean $(X,Y) \mapsto \mathscr{M}_{X,Y} = m_f(L_X,R_Y)$. Operator means are monotonically increasing on positive bounded operators [13]. Since inversion additionally reverts partial order on strictly positive bounded operators (cf. Proposition A.2.30), the map $(X,Y) \mapsto \mathscr{D}_{X,Y}$ decreases in partial order. Exponentiation with $\theta \in [0,1]$ preserves order, hence we obtain the map $(X,Y) \mapsto \mathrm{tr}(U^* m_f^{-\theta}(L_X,R_Y)(U))$ decreases in partial order for all $U \in M_n(\mathbb{C})$. □

Identifying via musical isomorphisms, $A \cong A^*$ and $B \cong B^*$ as partially ordered vector spaces. Using 2) in Proposition 2.2.10, we extend Equation 2.59 and therefore $\mathscr{I}^{f,\theta}_{A,B}$ to $A_+ \cong A_+^*$ in the first two variables.

Definition 2.2.18. We define quasi-entropy $\mathscr{I}^{f,\theta}_{A,B} : A_+^* \times A_+^* \times B^* \longrightarrow [0,\infty]$ by setting

$$\mathscr{I}^{f,\theta}_{A,B}(\mu,\eta,w) := \sup_{\varepsilon>0} \langle \mathscr{D}^{\theta}_{\sharp\mu,\sharp\eta,\varepsilon}(\sharp w), \sharp w\rangle_{\omega} \tag{2.62}$$

for all $\mu,\eta \in A_+^*$ and $w \in B^*$.

Notation 2.2.19. Let $\mathscr{I}^{f,\theta}_{B,B}$ denote the quasi-entropy for B equipped with its canonical AF-B-bimodule structure.

Proposition 2.2.20. *$\mathscr{I}^{f,\theta}_{A,B}$ is jointly convex and l.s.c. in w^*-topology.*

Proof. Lemma 2.2.17 shows joint convexity. For all $\varepsilon > 0$, note $(x,y,u) \mapsto \langle \mathscr{D}^{\theta}_{x,y,\varepsilon}(u), u\rangle_{\omega}$ defined on $A_+ \times A_+ \times B$ is norm continuous. Equation 2.62 shows l.s.c. in w^*-topology by finite-dimensionality. □

Proposition 2.2.21. *For all $x, y \in A_+$ and $u \in B$, we have*

1) $\mathscr{D}^{\theta}_{x,y,\varepsilon} = \mathscr{D}^{\theta}_{\phi(x),\psi(y),\varepsilon}$,

2) $\mathscr{I}^{f,\theta}_{A,B}(x^{\flat}, y^{\flat}, u^{\flat}) = \mathscr{I}_{B,B}(\phi(x)^{\flat}, \psi(y)^{\flat}, u^{\flat})$.

Proof. Apply 1) in Proposition 2.2.10 to get 1). The latter yields 2) by construction. □

Lemma 2.2.22. *Let (ϕ, ψ, γ) be (N, V)-compressible.*

1) *For all $x, y \in N_+$ and $u \in V$, we have*

$$\mathscr{I}^{f,\theta}_{A,B}\left(x^{\flat}, y^{\flat}, u^{\flat}\right) = \sup_{\varepsilon>0} \langle \mathscr{D}^{\theta}_{x,y,N,\varepsilon}(u), u \rangle_{\omega}. \tag{2.63}$$

2) *Let $N_A \subset A$ and $N_B \subset B$ be C^*-subalgebras s.t.* 1) *and* 2) *in Lemma 2.1.62 hold. Let $\phi^*(N_B), \psi^*(N_B) \subset N_A$. For all $\mu, \eta \in A^*_+$ and $w \in B^*$, we have*

$$\mathscr{I}^{f,\theta}_{N_A,N_B}(\mu|_{N_A}, \eta|_{N_A}, w|_{N_B}) \leq \mathscr{I}^{f,\theta}_{A,B}(\mu, \eta, w). \tag{2.64}$$

Proof. In this proof, γ is of no consequence. We have 1) at once by 2) in Lemma 2.2.13. We show 2). Assume its setting. Note Remark 2.1.23. We therefore consider (N_A, τ) and (N_B, ω) to be finite tracial AF-C^*-algebras.

We know $\phi, \psi : N_A \longrightarrow N_B$ are local *-homomorphisms. We have AF-N_A-bimodule (ϕ, ψ, γ) on N_B. Set $\pi_A := \pi^A_{N_A}, \pi_{A,\mathrm{u}} := \pi^A_{N_A[1_A]}$ and $\pi_B := \pi^B_{N_B}, \pi_{B,\mathrm{u}} := \pi^B_{N_B[1_B]}$ here. Using 3.1) in Proposition 2.1.40, we have

$$\phi|_{N_A} \circ \pi_A = \pi_B \circ \phi|_{N_A}, \ \psi|_{N_A} \circ \pi_A = \pi_B \circ \psi|_{N_A}. \tag{2.65}$$

Arguing as for 2.1) in Proposition 2.1.28, note identifying $A^* \cong A$ and $B^* \cong B$ via musical isomorphisms yields

$$\mathrm{res}_{N_A} = \pi_A, \ \mathrm{res}_{N_B} = \pi_B \tag{2.66}$$

for restriction maps $\mathrm{res}_{N_A} : A^* \longrightarrow N_A$ and $\mathrm{res}_{N_B} : B^* \longrightarrow N_B$ obtained by dualising the given C^*-subalgebra inclusion maps. Finite-dimensionality shows we are in the setting of Proposition 2.1.35. Using Proposition 2.1.35, we see both noncommutative conditional expectations $\pi_A : A \longrightarrow N_A$ and $\pi_B : B \longrightarrow N_B$ decompose as

$$\pi_A = \pi_{A,\mathrm{u}} - \kappa^A_{N_A} 1^{\perp}_{N_A}, \ \pi_B = \pi_{B,\mathrm{u}} - \kappa^B_{N_B} 1^{\perp}_{N_B}. \tag{2.67}$$

Equation 2.65 and Equation 2.66 hold if we use $N_A[1_A]$ and $N_B[1_B]$ instead, i.e. $\pi_{A,\mathrm{u}}$ and $\pi_{B,\mathrm{u}}$. Let $\mu,\eta \in A_+^*$ and $w \in B^*$. Set $x := \sharp\mu$, $y := \sharp\eta$ and $z := \sharp w$. Equation 2.65 and Equation 2.66 show

$$\pi_B\big(\phi(x)\big) = \phi(\pi_A(x)) = \phi\big(\sharp\mu|_{N_A}\big),\ \pi_B\big(\psi(y)\big) = \psi(\pi_A(y)) = \psi\big(\sharp\eta|_{N_A}\big) \tag{2.68}$$

and

$$\pi_B(z) = \sharp w|_{N_B}. \tag{2.69}$$

We may use $N_A[1_A]$ and $N_B[1_B]$ instead. Using 1) in our setting, we see Equation 2.68 and Equation 2.69 show

$$\mathscr{I}^{f,\theta}_{A,B}\Big(\pi_A(x)^\flat, \pi_A(y)^\flat, \pi_B(z)^\flat\Big) = \mathscr{I}^{f,\theta}_{N_A,N_B}\big(\mu|_{N_A}, \eta|_{N_A}, w|_{N_B}\big). \tag{2.70}$$

Equation 2.70 and 2) in Proposition 2.2.21 imply Equation 2.64 if for all $\varepsilon > 0$, we have

$$\big\langle \mathscr{D}^\theta_{\pi_A(x),\pi_A(y),\varepsilon}(\pi_B(z)), \pi_B(z)\big\rangle_\omega \le \big\langle \mathscr{D}^\theta_{\pi_{A,\mathrm{u}}(x),\pi_{A,\mathrm{u}}(y),\varepsilon}\big(\pi_{B,\mathrm{u}}(z)\big), \pi_{B,\mathrm{u}}(z)\big\rangle_\omega \tag{2.71}$$

and

$$\big\langle \mathscr{D}^\theta_{\pi_{A,\mathrm{u}}(x),\pi_{A,\mathrm{u}}(y),\varepsilon}\big(\pi_{B,\mathrm{u}}(z)\big), \pi_{B,\mathrm{u}}(z)\big\rangle_\omega \le \big\langle \mathscr{D}^\theta_{x,y,\varepsilon}(z), z\big\rangle_\omega. \tag{2.72}$$

We show Equation 2.71. Let $\varepsilon > 0$. Using 3) in Lemma 2.2.13, Equation 2.67, as well as unitality of noncommutative conditional expectations for unital C^*-subalgebras, we see writing $\varepsilon 1_A = \varepsilon 1_N + \varepsilon 1_N^\perp$ yields

$$\mathscr{D}^\theta_{\pi_{A,\mathrm{u}}(x),\pi_{A,\mathrm{u}}(y),\varepsilon} = \mathscr{D}_{\pi_A(x),\pi_A(y),N_A,\varepsilon} \oplus m_f\Big(\varepsilon + \kappa^A_{N_A}(x), \varepsilon + \kappa^A_{N_A}(y)\Big)^{-\theta} I_{\langle 1^\perp_{N_B}\rangle_{\mathbb{C}}} \tag{2.73}$$

w.r.t. $\mathscr{B}(N_B) \oplus \mathscr{B}(\langle 1^\perp_{N_B}\rangle_{\mathbb{C}})$. Note $\mathscr{D}_{\pi_A(x),\pi_A(y),N_A,\varepsilon}(N_B) \subset N_B \subset \langle 1^\perp_{N_B}\rangle^\perp_{\mathbb{C}}$. We obtain

$$m_f\Big(\varepsilon + \kappa^A_{N_A}(x), \varepsilon + \kappa^A_{N_A}(y)\Big)^{-\theta} \kappa^B_{N_B}(z) \ge 0. \tag{2.74}$$

Using Equation 2.73 and Equation 2.74, we estimate

$$\begin{aligned}
&\big\langle \mathscr{D}^\theta_{\pi_{A,\mathrm{u}}(x),\pi_{A,\mathrm{u}}(y),\varepsilon}\big(\pi_{B,\mathrm{u}}(z)\big), \pi_{B,\mathrm{u}}(z)\big\rangle_\omega \\
&= \big\langle \mathscr{D}^\theta_{\pi_A(x),\pi_A(y),\varepsilon}(\pi_B(z)), \pi_B(z)\big\rangle_\omega + m_f\Big(\varepsilon + \kappa^A_{N_A}(x), \varepsilon + \kappa^A_{N_A}(y)\Big)^{-\theta} \kappa^B_{N_B}(z) \cdot \|1^\perp_N\|_\omega \\
&\ge \big\langle \mathscr{D}^\theta_{\pi_A(x),\pi_A(y),\varepsilon}(\pi_B(z)), \pi_B(z)\big\rangle_\omega.
\end{aligned}$$

The above calculation shows Equation 2.71.

We show Equation 2.72. Let $\varepsilon > 0$. Using Equation 2.68 and Equation 2.69 for $N_A[1_A]$ and $N_B[1_B]$, 1) in Proposition 2.2.21 lets us calculate

$$\begin{aligned}
&\left\langle \mathscr{D}^{\theta}_{\pi_{A,\mathrm{u}}(x),\pi_{A,\mathrm{u}}(y),\varepsilon}\big(\pi_{B,\mathrm{u}}(z)\big), \pi_{B,\mathrm{u}}(z)\right\rangle_{\omega} \\
&= \left\langle \mathscr{D}^{\theta}_{\phi(\pi_{A,\mathrm{u}}(x)),\psi(\pi_{A,\mathrm{u}}(y)),\varepsilon}\big(\pi_{B,\mathrm{u}}(z)\big), \pi_{B,\mathrm{u}}(z)\right\rangle_{\omega} \\
&= \left\langle \mathscr{D}^{\theta}_{\pi_{B,\mathrm{u}}(\phi(x)),\pi_{B,\mathrm{u}}(\psi(y)),\varepsilon}\big(\pi_{B,\mathrm{u}}(z)\big), \pi_{B,\mathrm{u}}(z)\right\rangle_{\omega}.
\end{aligned}$$

Proposition 2.1.34 shows

$$\pi_{B,\mathrm{u}}(v) = \int_{\mathscr{U}(N_B')} u v u^* \, d\nu_{N_B} \tag{2.75}$$

for all $v \in B$. Equation 2.75 expresses π_B as average of unitary conjugations. Note the application of perturbed noncommutative division operators is jointly convex (cf. proof of Lemma 2.2.17). We therefore apply the Jensen inequality [174] to estimate

$$\left\langle \mathscr{D}^{\theta}_{\pi_{B,\mathrm{u}}(\phi(x)),\pi_{B,\mathrm{u}}(\psi(y)),\varepsilon}\big(\pi_{B,\mathrm{u}}(z)\big), \pi_{B,\mathrm{u}}(z)\right\rangle_{\omega} \le \left\langle \mathscr{D}^{\theta}_{\phi(x),\psi(y),\varepsilon}(z), z\right\rangle_{\omega}. \tag{2.76}$$

Altogether, Equation 2.76 and 1) in Proposition 2.2.21 imply Equation 2.72. □

Remark 2.2.23. Equation 2.64 in Lemma 2.2.22 is the monotonicity of quasi-entropies under restriction maps, called monotonicity. We distinguish this from monotonicity of operators means implied by 2) in Proposition 2.2.10.

Lemma 2.2.24. *Assume f is symmetric. For all $x, y > 0$ in A and $u \in B$, we have*

1) $\|u\|_{\omega}^2 \le \mathscr{I}^{f,\theta}_{A,B}(x^\flat, y^\flat, u^\flat) \cdot 2^{-\theta}\big(\|x\|_\infty^\theta + \|y\|_\infty^\theta\big)$,

2) $\|u\|_1^2 \le \mathscr{I}^{f,\theta}_{A,B}(x^\flat, y^\flat, u^\flat) \cdot 2^{-\theta}\big(\|\phi\|_1^\theta \|x\|_1^\theta + \|\psi\|_1^\theta \|y\|_1^\theta\big) \cdot \omega(1_B)^{1-\theta}$.

Proof. The arithmetic operator mean is the maximal symmetric one (cf. Theorem 4.5 in [13]). Note $r \mapsto r^\theta$ on $[0,\infty)$ preserves order. For all $x, y > 0$ in A, get

$$\mathscr{M}^{\theta}_{x,y} = m_f^\theta\Big(L_x^\phi, R_y^\psi\Big) \le 2^{-\theta}\big(L_x^\phi + R_y^\psi\big)^\theta. \tag{2.77}$$

For all $x, y > 0$ in A and $u \in B$, apply $L_x^\phi + R_y^\psi \le \big(\|\phi\|_\infty \|x\|_\infty + \|\psi\|_\infty \|y\|_\infty\big) \cdot I$ to estimate

$$\|u\|_\omega^2 \le \big\|\mathscr{M}^{\theta}_{x,y}\big\| \cdot \big\|\mathscr{D}^{\frac{\theta}{2}}_{x,y}(u)\big\|_\omega^2 \le \mathscr{I}^{f,\theta}_{A,B}\Big(x^\flat, y^\flat, u^\flat\Big) \cdot 2^{-\theta}\big(\|\phi\|_\infty \|x\|_\infty + \|\psi\|_\infty \|y\|_\infty\big)^\theta \tag{2.78}$$

using Equation 2.77. Note $\|\phi\|_\infty = \|\psi\|_\infty = 1$ by 2) in Lemma 2.1.41. Further, $r \mapsto r^\theta$ is concave and therefore subadditive on $[0,\infty)$ since $\theta \in [0,1]$. Equation 2.78 shows 1).

We prove 2). For all $x,y>0$ in A and $u\in B$, we use the maximal symmetric operator mean property as above to estimate

$$\big|\omega\big(u^*z\big)\big|^2 \leq \big\|\mathscr{D}^{\frac{\theta}{2}}_{x,y}(u)\big\|^2_\omega \cdot \big\|\mathscr{M}^{\frac{\theta}{2}}_{x,y}(z)\big\|^2_\omega \leq \big\|\mathscr{D}^{\frac{\theta}{2}}_{x,y}(u)\big\|^2_\omega \cdot 2^{-\theta}\big\langle\big(L^\phi_x+R^\psi_y\big)^\theta(z),z\big\rangle_\omega. \tag{2.79}$$

Subadditivity of $r\mapsto r^\theta$ implies $(S+T)^\theta\leq S^\theta+T^\theta$ for commuting bounded operators $T,S\geq 0$ by functional calculus. Since L^ϕ and R^ψ are *-representations, we obtain

$$\big\langle\big(L^\phi_x+R^\psi_y\big)^\theta(z),z\big\rangle_\omega \leq \big\langle\phi(x)^\theta(z),z\big\rangle_\omega+\big\langle z\psi(y)^\theta,z\big\rangle_\omega \leq \big(\|\phi(x)^\theta\|_1+\|\psi(y)^\theta\|_1\big)\cdot\|z\|^2_B. \tag{2.80}$$

For all $v\in B_+$ and $\theta\in[0,1]$, $\|v^\theta\|_1\leq\omega(1_B)^{1-\theta}\|v\|^\theta_1$ by Jensen's inequality. Equation 2.79 and Equation 2.80 together show 2) as norm is obtained by testing on B. For this, note $\|\phi(x)\|_1\leq\|\phi\|_1\|x\|_1$ and $\|\psi(y)\|_1\leq\|\psi\|_1\|y\|_1$. □

Extending to AF-C^*-bimodules. Monotonicity extends quasi-entropies from the finite-dimensional setting to the AF-C^*-setting. Theorem 2.2.29 collects fundamental properties. We view each symmetric representing function f as determining a class of energetic structures with $\theta\in[0,1]$ as interpolation parameter. Proposition 3.1.53 shows $\theta=0$ gives quantum $(-1,2)$-Sobolev distance independent of f. In the logarithmic mean setting, i.e. f represents the logarithmic operator mean and $\theta=1$, we obtain quantum L^2-Wasserstein distances in direct analogy to the classical case [97].

Let (A,τ) and (B,ω) be tracial AF-C^*-algebras. Let (ϕ,ψ,γ) be an AF-A-bimodule structure on B. Let f be representing function of an operator mean and $\theta\in[0,1]$. For all $j\in\mathbb{N}$, we use induced AF-A_j-bimodule structure on B_j as per 4) in Definition 2.1.46.

Definition 2.2.25.

1) For all $j\in\mathbb{N}$, we call $\mathscr{I}^{f,\theta}_{A,B,j}:=\mathscr{I}^{f,\theta}_{A_j,B_j}$ the j-th restricted quasi-entropy.

2) We define quasi-entropy $\mathscr{I}^{f,\theta}_{A,B}:A^*_+\times A^*_+\times B^*\longrightarrow[0,\infty]$ by setting

$$\mathscr{I}^{f,\theta}_{A,B}(\mu,\eta,w):=\sup_{j\in\mathbb{N}}\mathscr{I}^{f,\theta}_{A,B,j}\big(\mu_j,\eta_j,w_j\big) \tag{2.81}$$

for all $\mu,\eta\in A^*_+$ and $w\in B^*$.

Notation 2.2.26. Unless stated otherwise, we suppress A and B in Definition 2.2.25.

Corollary 2.2.27. *For all $j\leq k$ in $\mathbb{N}$, we have*

1) $\mathscr{I}^{f,\theta}_j(\mu,\eta,w)=\mathscr{I}^{f,\theta}_k(\mu,\eta,w)$ for all $\mu,\eta\in A^*_{j,+}$ and $w\in B^*_j$,

2) $\mathscr{I}^{f,\theta}_j\big(\mu_j,\eta_j,w_j\big)\leq\mathscr{I}^{f,\theta}_k(\mu,\eta,w)$ for all $\mu,\eta\in A^*_{k,+}$ and $w\in B^*_k$.

Proof. For all $j\leq k$ in $\mathbb{N}$, apply Lemma 2.2.22 to $N_A=A_j$ and $N_B=B_j$ in the setting of the induced AF-A_k-bimodule B_k. This shows both claims at once. □

Definition 2.2.28. For all $j \in \mathbb{N}$, we define

1) $\mathrm{inc}_j : A^*_{j,+} \times A^*_{j,+} \times B^*_j \longrightarrow A^*_+ \times A^*_+ \times B^*$ by setting

$$\mathrm{inc}_j(\mu,\eta,w) := (\mu,\eta,w) \tag{2.82}$$

for all $\mu,\eta \in A^*_{j,+}$ and $w \in B^*_j$,

2) $\mathrm{res}_j : A^*_+ \times A^*_+ \times B^* \longrightarrow A^*_{j,+} \times A^*_{j,+} \times B^*_j$ by setting

$$\mathrm{res}_j(\mu,\eta,w) := \big(\mu_j,\eta_j,w_j\big) \tag{2.83}$$

for all $\mu,\eta \in A^*_+$ and $w \in B^*$.

Theorem 2.2.29. *Let (A,τ) and (B,ω) be tracial AF-C^*-algebras. Let (ϕ,ψ,γ) be an AF-A-bimodule structure on B. Let f be representing function of an operator mean and $\theta \in [0,1]$.*

1) $\mathscr{I}^{f,\theta}$ is jointly convex and l.s.c. in w^*-topology.

2) $\mathscr{I}^{f,\theta}_j = \mathscr{I}^{f,\theta}_k \circ \mathrm{inc}_{kj} = \mathscr{I}^{f,\theta} \circ \mathrm{inc}_j$ for all $j \le k$ in $\mathbb{N}$.

3) $\mathscr{I}^{f,\theta}_j \circ \mathrm{res}_j \le \mathscr{I}^{f,\theta}_k \circ \mathrm{res}_k$ for all $j \le k$ in $\mathbb{N}$.

4) Assume f is symmetric. For all $\mu,\eta \in A^*_+ \cap L^\infty(A,\tau)^\flat$ and $w \in B^* \cap L^2(B,\omega)^\flat$, we have

$$\|\sharp w\|^2_\omega \le \mathscr{I}^{f,\theta}(\mu,\eta,w) \cdot 2^{-\theta}\big(\|\sharp\mu\|^\theta_\infty + \|\sharp\eta\|^\theta_\infty\big). \tag{2.84}$$

5) Assume f is symmetric. For all $\mu,\eta \in A^*_+$ and $w \in B^*$, we have

$$\|w\|^2_{B^*} \le \mathscr{I}^{f,\theta}(\mu,\eta,w) \cdot 2^{-\theta}\big(\|\phi\|^\theta_1\|\mu\|^\theta_{A^*} + \|\psi\|^\theta_1\|\eta\|^\theta_{A^*}\big) \cdot \|\omega\|^{1-\theta} \tag{2.85}$$

with $\|\omega\| := \omega(1_B) = \sup_{j\in\mathbb{N}} \omega(1_{B_j})$ the volume and convention $\|\omega\|^0 := 1$.

Proof. Since restriction is w^*-continuous, Proposition 2.2.20 implies 1). Get 2) and 3) by Corollary 2.2.27. Following 2.1) in Proposition 2.1.31, all noncommutative L^1-, L^2- and L^∞-norms in use are the suprema over $j \in \mathbb{N}$ of their restrictions to A_j, resp. B_j. Writing norms as such, get 4) and 5) by Lemma 2.2.24. □

Remark 2.2.30. We know $\|\phi\|_1, \|\psi\|_1 < \infty$ by 2.1) in Lemma 2.1.41. If $(A,\tau) = (B,\omega)$ with self-adjoint local *-homomorphisms, then $\|\phi\|_1 = \|\psi\|_1 \le 2$. If $\theta = 1$, then the volume term in Equation 2.85 vanishes. This allows estimates for unbounded traces.

2.2.2 Noncommutative division operators from quasi-entropies

Quasi-entropies determine closed positive unbounded quadratic forms on symmetric W^*-bimodules given pairs of positive bounded functionals on tracial AF-C^*-algebras. Theorem 2.2.49 shows such quadratic forms have unique representing operators. These are, by definition, noncommutative division operators of positive bounded functionals on tracial AF-C^*-algebras. Under the modified standard pairing, normal positive bounded functionals on tracial AF-C^*-algebras are positive measurable operators. Using results in Theorem 2.2.53, Theorem 2.2.58 states necessary and sufficient conditions to recover noncommutative division operators of positive measurable operators.

We construct noncommutative division operators as follows using the Kato-Robinson theorem (cf. Theorem 10.4.2 in [88]). We define perturbed left- and right-division with positive bounded functionals on tracial AF-C^*-algebras. Inverses exist and are strongly commuting positive unbounded operators. Using their joint spectral calculus, we define perturbed noncommutative division operators in direct analogy to positive measurable operators. Theorem 2.2.49 shows strong resolvent limits exist as perturbation tends to zero. These limits are noncommutative division operators as above. Standard reference for unbounded quadratic forms and the Kato-Robinson theorem is [88].

Unbounded quadratic forms and the Kato-Robinson theorem. Let H be a Hilbert space. The Kato-Robinson theorem relates closed positive unbounded quadratic forms, their representing operators and strong resolvent limits as follows.

The full Kato-Robinson theorem, i.e. its general formulation, uses strong resolvent convergence of positive unbounded operators on Hilbert subspaces. Definition 2.2.31 generalises Definition A.2.1 accordingly. Proposition 2.2.34 shows uniform reducibility lets us restrict again to strong resolvent convergence on Hilbert spaces. For details on strong resolvent convergence on Hilbert spaces, we refer to Subsection A.2.1. For details on reducing subspaces, we refer to Subsection A.2.2.

Definition 2.2.31. Let $V \subset H$ be a Hilbert subspace. We call $\{T_n\}_{n\in\mathbb{N}} \subset \mathcal{UB}(H)_+$ strong resolvent convergent to $T \in \mathcal{UB}(V)_+$ on V in H if for all $a > 0$, we have

$$R_{-a}(T)(u) = \|.\|_V\text{-}\lim_{n\in\mathbb{N}} R_{-a}(T_n)\big(\pi_V(u)\big) \tag{2.86}$$

for all $u \in H$.

Notation 2.2.32. Let $T = \text{sr-}\lim_{n\in\mathbb{N}} T_n$ on V in H denote strong resolvent convergence. We drop *on* V if V is clear from context, resp. *in* H if H is.

Remark 2.2.33. The resolvents in Equation 2.86 are given by bounded measurable functional calculus in a priori different W^*-algebras. If $V = H$, then Equation 2.86 is strong convergence and we recover Definition A.2.1 for positive unbounded operators by Lemma A.2.5 and 1) in Proposition A.2.8.

Proposition 2.2.34. *If $T = \text{sr-lim}_{n\in\mathbb{N}} T_n$ on V in H and $\{T_n\}_{n\in\mathbb{N}} \subset \mathscr{UB}_V(H)$, then we have $T = \text{sr-lim}_{n\in\mathbb{N}} T_n|_V$ on V.*

Proof. Using 1) in Proposition A.2.24 and 2) in Lemma A.2.26, Equation 2.86 for fixed but arbitrary $a > 0$ restricts to

$$R_{-a}(T) = \text{s-}\lim_{n\in\mathbb{N}} R_{-a}(T_n|_V). \tag{2.87}$$

Using 1) in Proposition A.2.8, Equation 2.87 shows $T = \text{sr-lim}_{n\in\mathbb{N}} T_n|_V$ on V. □

Definition 2.2.35. Let H be a separable Hilbert space. For all positive unbounded quadratic forms $Q : H \longrightarrow [0,\infty]$, set

1) $\operatorname{dom} Q := \{u \in H \mid Q(u) < \infty\}$,
2) $H(Q) := \overline{\operatorname{dom} Q}^{\|\cdot\|_H}$.

Let H be a Hilbert space. Theorem 9.3.7 in [88] gives positivity-preserving bijection between closed positive unbounded quadratic forms and representing operators. If Q is a closed positive unbounded quadratic form on H, then it has representing operator $T \in \mathscr{UB}(H(Q))_+$ s.t. $\operatorname{dom} Q = \operatorname{dom} \sqrt{T}$ and

$$Q(u) = \langle \sqrt{T}(u), \sqrt{T}(u) \rangle_H \tag{2.88}$$

for all $u \in \operatorname{dom} Q$. For all monotonically increasing sequences $\{T_n\}_{n\in\mathbb{N}} \subset \mathscr{B}(H)_+$, we define closed positive unbounded quadratic form on H by setting

$$Q(u) := \sup_{n\in\mathbb{N}} \langle T_n(u), u \rangle_H \in [0,\infty] \tag{2.89}$$

for all $u \in H$. If T is its representing operator, then $T = \text{sr-lim}_{n\in\mathbb{N}} T_n$ on $H(Q)$ by the Kato-Robinson theorem. Remark 2.2.36 below fixes conventions for using uncountable monotonically decreasing sequences instead.

Remark 2.2.36. Consider monotonically increasing $\{T_\varepsilon\}_{\varepsilon>0} \subset \mathscr{B}(H)_+$ in dual order. All sequences $\{T_{\varepsilon_n}\}_{n\in\mathbb{N}}$ given fixed but arbitrary monotonically decreasing $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset (0,\infty)$ generate identical quadratic form as per Equation 2.89. Uniqueness ensures each has identical strong resolvent limit, denoted by $T = \text{sr-lim}_{\varepsilon\downarrow 0} T_\varepsilon$ in this case.

The unbounded operator representation of quasi-entropies. Let (A,τ) and (B,ω) be tracial AF-C^*-algebras. Let (ϕ,ψ,γ) be an AF-A-bimodule structure on B. Let f be representing function of an operator mean and $\theta \in [0,1]$. For all $\mu,\eta \in A^*_+$, we extend the map $u \mapsto \mathscr{I}^{f,\theta}(\mu,\eta,u)$ from B_0 to $L^2(B,\omega)$. Such extensions determine closed positive unbounded quadratic forms on $L^2(B,\omega)$. We equip $A^*_+ \times A^*_+ \times L^2(B,\omega)$ with the product topology of the given w^*-topologies. We use Notation 2.1.29.

Definition 2.2.37.

1) For all $\mu,\eta \in A_+^*$, set

   1.1) $Q^{f,\theta}_{\mu,\eta}(u) := \sup_{j\in\mathbb{N}} \mathscr{I}^{f,\theta}_j\big(\mu_j,\eta_j,u_j\big)$ for all $u \in L^2(B,\omega)$,

   1.2) $\operatorname{dom} Q^{f,\theta}_{\mu,\eta} := \big\{u \in L^2(B,\omega) \mid Q^{f,\theta}_{\mu,\eta}(u) < \infty\big\}$.

2) We define $Q^{f,\theta} : A_+^* \times A_+^* \times L^2(B,\omega) \longrightarrow [0,\infty]$ by setting $Q^{f,\theta}(\mu,\eta,u) := Q^{f,\theta}_{\mu,\eta}(u)$ for all $\mu,\eta \in A_+^*$ and $u \in L^2(B,\omega)$.

Remark 2.2.38. Note $L^{1,2}(B,\omega) \subset L^1(B,\omega) \subset B^*$. Further note $\mathscr{I}^{f,\theta}$ and $Q^{f,\theta}$ coincide on $A_+^* \times A_+^* \times L^{1,2}(B,\omega)$. If $\omega < \infty$, then $L^2(B,\omega) = L^{1,2}(B,\omega)$ and $Q^{f,\theta}$ is the restriction of $\mathscr{I}^{f,\theta}$ to $A_+^* \times A_+^* \times L^2(B,\omega)$.

Proposition 2.2.39. *We have*

1) $Q^{f,\theta}$ is jointly convex and l.s.c. in w^*-topology,

2) $Q^{f,\theta} \circ \mathrm{inc}_j = \mathscr{I}^{f,\theta}_j$ for all $j \in \mathbb{N}$.

Proof. Get 1) and 2) by arguing as for 1), resp. 2) in Theorem 2.2.29. □

We construct perturbed left- and right-division by positive bounded functionals. For all $\mu \in A_+^*$, $\varepsilon > 0$ and $j \in \mathbb{N}$, as well as fix but arbitrary $\eta \in A_+^*$, we have positive bounded quadratic form on $L^2(B,\omega)$ defined by

$$\mathbf{L}^{-\phi}_{\mu_j,\varepsilon}(u) := Q^{t,1}_{\mu_j+\varepsilon 1_{A_j},\eta_j+\varepsilon 1_{A_j}}(u_j) = \langle \pi^B_j\Big(\big(\phi(\mu_j)+\varepsilon I\big)^{-1}\pi^B_j(u)\Big), u\rangle_\omega \tag{2.90}$$

for all $u \in L^2(B,\omega)$ using $(t,s) \mapsto t$ as our representing function. The right-hand side of Equation 2.90 does not depend on $\eta \in A_+^*$. For all $j \le k$ in $\mathbb{N}$, get $\pi^B_{jk}(1_{B_k}) = 1_{B_j}$. Thus 2) in Proposition 2.2.39 and 3) in Theorem 2.2.29 yield monotonically increasing sequence of uniformly positive and bounded quadratic forms on $L^2(B,\omega)$ s.t.

$$0 \le \mathbf{L}^{-\phi}_{\mu_1,\varepsilon} \le \ldots \le \mathbf{L}^{-\phi}_{\mu_j,\varepsilon} \le \mathbf{L}^{-\phi}_{\mu_{j+1},\varepsilon} \le \ldots \le \varepsilon^{-1} I. \tag{2.91}$$

Note Equation 2.91 gives monotonically increasing sequence $\{\pi^B_j(L_{\phi(\mu_j)}+\varepsilon I)^{-1}\pi^B_j\}_{j\in\mathbb{N}}$ of uniformly positive and bounded operators as determined by Equation 2.90. Hence the Kato-Robinson theorem shows its strong limit is the unique positive bounded operator representing the positive bounded quadratic form defined by

$$\mathbf{L}^{-\phi}_{\mu,\varepsilon}(u) := \sup_{j\in\mathbb{N}} \mathbf{L}^{-\phi}_{\mu_j,\varepsilon}(u) = \sup_{j\in\mathbb{N}} \langle \pi^B_j\Big(\big(\phi(\mu_j)+\varepsilon I\big)^{-1}\pi^B_j(u)\Big), u\rangle_\omega \tag{2.92}$$

for all $\mu \in A_+^*$, $\varepsilon > 0$ and $u \in L^2(B,\omega)$.

We analogously construct perturbed right-division using $(t,s)\mapsto s$ as representing function. For all $\eta\in A_+^*$, $\varepsilon>0$ and $j\in\mathbb{N}$, we have positive bounded quadratic form on $L^2(B,\omega)$ defined by

$$\mathbf{R}^{-\psi}_{\eta_j,\varepsilon}(u):=\langle\pi_j^B\Big(\pi_j^B(u)\big(\psi(\eta_j)+\varepsilon I\big)^{-1}\Big),u\rangle_\omega \tag{2.93}$$

for all $u\in L^2(B,\omega)$. As above, we have monotonically increasing sequence of uniformly positive and bounded operators as determined by Equation 2.93. The Kato-Robinson theorem shows its strong limit is the unique positive bounded operator representing the positive bounded quadratic form defined by

$$\mathbf{R}^{-\psi}_{\eta,\varepsilon}(u):=\sup_{j\in\mathbb{N}}\mathbf{R}^{-\psi}_{\eta_j,\varepsilon}(u)=\sup_{j\in\mathbb{N}}\langle\pi_j^B\Big(\pi_j^B(u)\big(\psi(\eta_j)+\varepsilon I\big)^{-1}\Big),u\rangle_\omega \tag{2.94}$$

for all $\eta\in A_+^*$, $\varepsilon>0$ and $u\in L^2(B,\omega)$.

Remark 2.2.40. Note the Kato-Robinson theorem by itself only implies strong resolvent convergence. Using Proposition 10.1.13 in [88], we know uniform boundedness together with strong resolvent convergence implies strong convergence. If uniform boundedness is given when applying the Kato-Robinson theorem, then we have strong convergence.

Proposition 2.2.41. *For all $\mu,\eta\in A_+^*$ and $\varepsilon>0$, we have*

1) *positive bounded quadratic form $\mathbf{L}^{-\phi}_{\mu,\varepsilon}$ on $L^2(B,\omega)$ s.t.*
   1.1) *its representing operator $L^{-\phi}_{\mu,\varepsilon}\in\mathscr{B}\big(L^2(B,\omega)\big)_+$ is injective,*
   1.2) $0\le L^{-\phi}_{\mu,\varepsilon}=\text{s-lim}_{j\in\mathbb{N}}\pi_j^B\big(L_{\phi(\mu_j)}+\varepsilon I\big)^{-1}\pi_j^B\le\varepsilon^{-1}I$,
2) *positive bounded quadratic form $\mathbf{R}^{-\psi}_{\eta,\varepsilon}$ on $L^2(B,\omega)$ s.t.*
   2.1) *its representing operator $R^{-\psi}_{\eta,\varepsilon}\in\mathscr{B}\big(L^2(B,\omega)\big)_+$ is injective,*
   2.2) $0\le R^{-\psi}_{\eta,\varepsilon}=\text{s-lim}_{j\in\mathbb{N}}\pi_j^B\big(R_{\psi(\eta_j)}+\varepsilon I\big)^{-1}\pi_j^B\le\varepsilon^{-1}I$.

Proof. Let $\mu,\eta\in A_+^*$ and $\varepsilon>0$. Equation 2.92 and Equation 2.94 show $\mathbf{L}^{-\phi}_{\mu,\varepsilon}$ and $\mathbf{R}^{-\psi}_{\eta,\varepsilon}$ are positive bounded quadratic forms on $L^2(B,\omega)$. The Kato-Robinson theorem ensures the existence of the positive bounded representing operators.

For all $u\in L^2(B,\omega)$ and $j\in\mathbb{N}$, we have

$$\big(\|\mu_j\|_\infty+\varepsilon\big)^{-1}\|u_j\|_\omega^2\le\big\langle\big(\phi(\mu_j)+\varepsilon I\big)^{-1}u_j,u_j\big\rangle_\omega. \tag{2.95}$$

Since $\|u\|_\omega=\sup_{j\in\mathbb{N}}\|u_j\|_\omega$ in each case, Equation 2.95 implies injectivity of $L^{-\phi}_{\mu,\varepsilon}$. Get 1.1). We know 1.2) by Equation 2.91. Altogether, 1) holds. We show 2) analogously. □

Definition 2.2.42. For all $\mu,\eta \in A^*_+$ and $\varepsilon > 0$, we call

1) the representing operator $L^{-\phi}_{\mu,\varepsilon}$ of $\mathbf{L}^{-\phi}_{\mu,\varepsilon}$ left-division by μ perturbed with ε, and $L^{\phi}_{\mu,\varepsilon} := \big(L^{-\phi}_{\mu,\varepsilon}\big)^{-1}$ left-multiplication by μ perturbed with ε,

2) the representing operator $R^{-\psi}_{\eta,\varepsilon}$ of $\mathbf{R}^{-\psi}_{\eta,\varepsilon}$ right-division by η perturbed with ε, and $R^{\psi}_{\eta,\varepsilon} := \big(R^{-\psi}_{\eta,\varepsilon}\big)^{-1}$ right-multiplication by η perturbed with ε.

Notation 2.2.43. We suppress ϕ and ψ in Definition 2.2.42 if $\phi = \psi = \mathrm{id}_A$.

Remark 2.2.44. For all $\mu,\eta \in A^*_+$, $\varepsilon > 0$ and $j \in \mathbb{N}$, note $I = \text{s-lim}_{j\in\mathbb{N}} \pi^B_j$ implies

$$L^{-\phi}_{\mu_j,\varepsilon} = \big(L_{\phi(\mu_j)} + \varepsilon I\big)^{-1},\ R^{-\psi}_{\eta_j,\varepsilon} = \big(R_{\psi(\eta_j)} + \varepsilon I\big)^{-1} \tag{2.96}$$

and therefore $L^{\phi}_{\mu_j,\varepsilon} = L_{\phi(\mu_j)} + \varepsilon I$, $R^{\psi}_{\eta_j,\varepsilon} = R_{\psi(\eta_j)} + \varepsilon I$.

Proposition 2.2.45. *For all $\mu,\eta \in A^*_+$ and $\varepsilon > 0$, we have*

1) $L^{\phi}_{\mu,\varepsilon}, R^{\psi}_{\eta,\varepsilon} \in \mathscr{UB}\big(L^2(B,\omega)\big)_+$ commute strongly and $L^{\phi}_{\mu,\varepsilon}, R^{\psi}_{\eta,\varepsilon} \geq \varepsilon I$,

2) $L^{\phi}_{\mu,\varepsilon} = \text{sr-lim}_{j\in\mathbb{N}} L^{\phi}_{\mu_j,\varepsilon}$ and $R^{\psi}_{\eta,\varepsilon} = \text{sr-lim}_{j\in\mathbb{N}} R^{\psi}_{\eta_j,\varepsilon}$.

Proof. We know $L^{\phi}_{\mu,\varepsilon}, R^{\psi}_{\eta,\varepsilon} \in \mathscr{UB}(L^2(B,\omega))_+$ and $L^{\phi}_{\mu,\varepsilon}, R^{\psi}_{\eta,\varepsilon} \geq \varepsilon I$ by Proposition 2.2.41 as inversion reverts partial order (cf. Proposition A.2.30). We show strong commutativity. Since we have uniform lower bound $\varepsilon > 0$, resolvents in $a = 0$ are respective perturbed division operators. Using sequential strong continuity of multiplication and the inverse of Equation 2.96, we calculate

$$L^{-\phi}_{\mu,\varepsilon} R^{-\psi}_{\eta,\varepsilon} = \operatorname*{s-lim}_{j\in\mathbb{N}} L^{-\phi}_{\mu_j,\varepsilon} R^{-\psi}_{\eta_j,\varepsilon} = \operatorname*{s-lim}_{j\in\mathbb{N}} R^{-\psi}_{\eta_j,\varepsilon} L^{-\phi}_{\mu_j,\varepsilon} = R^{-\psi}_{\eta,\varepsilon} L^{-\phi}_{\mu,\varepsilon}. \tag{2.97}$$

Equation 2.97 is commutativity of resolvents in $a = 0$. Proposition 5.27 in [184] then implies strong commutativity. Get 1). We have 2) by 1) in Proposition A.2.8 for $a = 0$. □

Definition 2.2.46 uses bounded measurable joint functional calculus of strongly commuting self-adjoint unbounded operators (cf. Definition A.1.94). For details on spectral integration and the latter functional calculus, we refer to Subsection A.1.3.

Definition 2.2.46. For all $\mu,\eta \in A^*_+$ and $\varepsilon > 0$, we call $\mathscr{D}_{\mu,\eta,\varepsilon} := m_f^{-1}(L^{\phi}_{\mu,\varepsilon}, R^{\psi}_{\eta,\varepsilon})$ the noncommutative division operator of μ and η perturbed with ε.

Proposition 2.2.47. *Let $\mu,\eta \in A^*_+$.*

1) For all $\varepsilon > 0$, we have

1.1) $\mathscr{D}_{\mu,\eta,\varepsilon} = \text{s-lim}_{j\in\mathbb{N}} \mathscr{D}_{\mu_j,\eta_j,\varepsilon} \in \mathscr{B}\big(L^2(B,\omega)\big)_+$ and $\|\mathscr{D}_{\mu,\eta,\varepsilon}\|_{\mathscr{B}(L^2(B,\omega))} \leq \varepsilon^{-1}$,

1.2) $\mathscr{D}_{\mu_j,\eta_j,\varepsilon} = \mathscr{D}_{\mu_j+\varepsilon 1_A,\eta_j+\varepsilon 1_A}$ for all $j \in \mathbb{N}$.

2) We have monotonically increasing net $\{\mathscr{D}_{\mu,\eta,\varepsilon}\}_{\varepsilon>0} \subset \mathscr{B}\big(L^2(B,\omega)\big)_+$ in dual order.

Proof. Using Proposition 2.2.45 and Remark 2.2.4, get 1.1) by Lemma A.2.5 applied to m_f^{-1}. Using Proposition 2.1.54, get 1.2) by functional calculus upon taking inverses in Equation 2.96 since ϕ, ψ are unital. Altogether, get 1).

We show 2). For all $j \in \mathbb{N}$ and $\varepsilon_1 \geq \varepsilon_0 > 0$ in $\mathbb{R}$, we use 2) in Proposition 2.2.39 and 3) in Theorem 2.2.29 to estimate

$$\mathbf{L}_{\mu_j,\varepsilon_1}^{-\phi} \leq \mathbf{L}_{\mu_j,\varepsilon_0}^{-\phi},\ \mathbf{R}_{\eta_j,\varepsilon_1}^{-\psi} \leq \mathbf{R}_{\eta_j,\varepsilon_0}^{-\psi}. \tag{2.98}$$

Using positivity-preservation of representing operators, Equation 2.98 shows

$$L_{\mu_j,\varepsilon_1}^{-\phi} \leq L_{\mu_j,\varepsilon_0}^{-\phi},\ R_{\eta_j,\varepsilon_1}^{-\psi} \leq R_{\eta_j,\varepsilon_0}^{-\psi}. \tag{2.99}$$

Letting $j \uparrow \infty$ in Equation 2.99 yields

$$L_{\mu,\varepsilon_1}^{-\phi} \leq L_{\mu,\varepsilon_0}^{-\phi},\ R_{\eta,\varepsilon_1}^{-\psi} \leq R_{\eta,\varepsilon_0}^{-\psi} \tag{2.100}$$

in strong limit. Since inversion reverts partial order (cf. Proposition A.2.30), taking the inverses in Equation 2.100 shows

$$L_{\mu,\varepsilon_0}^{\phi} \leq L_{\mu,\varepsilon_1}^{\phi},\ R_{\eta,\varepsilon_0}^{\psi} \leq R_{\eta,\varepsilon_1}^{\psi}. \tag{2.101}$$

Using 1.2) and Proposition 2.2.3, Equation 2.101 implies 2) by functional calculus. □

Lemma 2.2.48. *For all $\mu, \eta \in A_+^*$ and $u \in L^2(B,\omega)$, we have*

1) $\mathscr{I}_j^{f,\theta}(\mu_j,\eta_j,u_j) = \sup_{\varepsilon>0} \langle \mathscr{D}_{\mu_j,\eta_j,\varepsilon}^{\theta}(u_j), u_j \rangle_\omega$ *for all $j \in \mathbb{N}$,*
2) $\sup_{j\in\mathbb{N}} \sup_{\varepsilon>0} \langle \mathscr{D}_{\mu_j,\eta_j,\varepsilon}^{\theta}(u_j), u_j \rangle_\omega = \sup_{\varepsilon>0} \sup_{j\in\mathbb{N}} \langle \mathscr{D}_{\mu_j,\eta_j,\varepsilon}^{\theta}(u_j), u_j \rangle_\omega$,
3) $\sup_{j\in\mathbb{N}} \langle \mathscr{D}_{\mu_j,\eta_j,\varepsilon}^{\theta}(u_j), u_j \rangle_\omega = \langle \mathscr{D}_{\mu,\eta,\varepsilon}^{\theta}(u), u \rangle_\omega$ *for all $\varepsilon > 0$.*

Proof. Let $\mu, \eta \in A_+^*$ and $u \in L^2(B,\omega)$. We use Corollary 2.1.63. For all $\varepsilon > 0$ and $j \in \mathbb{N}$, we see 1.2) in Proposition 2.2.47 and 2) in Lemma 2.2.13 show

$$\mathscr{D}_{\mu_j,\eta_j,\varepsilon}^{\theta}\Big|_{B_j} = \mathscr{D}_{\mu_j+\varepsilon 1_A,\eta_j+\varepsilon 1_A}^{\theta}\Big|_{B_j} = \mathscr{D}_{\mu_j,\eta_j,B_j,\varepsilon}^{\theta}. \tag{2.102}$$

Equation 2.102 shows 1) by construction of quasi-entropies. Note $\sup_{j\in J} \sup_{k\in K} a_{j,k} = \sup_{k\in K} \sup_{j\in J} a_{j,k}$ for all double-indexed real sequences. The latter shows 2) at once. For all $\varepsilon > 0$, monotonicity of quasi-entropies shows

$$\sup_{j\in\mathbb{N}} \langle \mathscr{D}_{\mu_j,\eta_j,\varepsilon}^{\theta}(u_j), u_j \rangle_\omega = \lim_{j\in\mathbb{N}} \langle \mathscr{D}_{\mu_j,\eta_j,\varepsilon}^{\theta}(u_j), u_j \rangle_\omega. \tag{2.103}$$

For all $j \in \mathbb{N}$, $u_j = \pi_j^B(u)$. Thus $u = \|.\|_\omega\text{-}\lim_{j\in\mathbb{N}} u_j$, hence 3) follows by Equation 2.103 and 1.1) in Proposition 2.2.47. We use uniform boundedness in our calculation. □

Theorem 2.2.49. *Let (A,τ) and (B,ω) be tracial AF-C^*-algebras. Let (ϕ,ψ,γ) be an AF-A-bimodule structure on B. Let f be representing function of an operator mean and $\theta\in[0,1]$. For all $\mu,\eta\in A^*_+$, we have*

1) *$Q^{f,\theta}_{\mu,\eta}: L^2(B,\omega)\longrightarrow[0,\infty]$ is closed positive unbounded quadratic form on $L^2(B,\omega)$ represented uniquely by the positive unbounded operator defined by*

$$\mathscr{D}^{\theta}_{\mu,\eta} := \text{sr-}\lim_{\varepsilon\downarrow 0}\mathscr{D}^{\theta}_{\mu,\eta,\varepsilon} \tag{2.104}$$

on $H\big(Q^{f,\theta}_{\mu,\eta}\big)$,

2) *$Q^{f,\theta}_{\mu,\eta}(u)=\big\|\mathscr{D}^{\frac{\theta}{2}}_{\mu,\eta}(u)\big\|^2_{\omega}=\sup_{\varepsilon>0}\big\langle\mathscr{D}^{\theta}_{\mu,\eta,\varepsilon}(u),u\big\rangle_{\omega}$ for all $u\in L^2(B,\omega)$.*

Proof. For all $u\in L^2(B,\omega)$, get $Q^{f,\theta}_{\mu,\eta}(u)=\sup_{j\in\mathbb{N}}\mathscr{I}^{f,\theta}_j(\mu_j,\eta_j,u_j)$ by definition. Consecutive application of 1) to 3) in Lemma 2.2.48 lets us calculate

$$Q^{f,\theta}_{\mu,\eta}(u)=\sup_{\varepsilon>0}\sup_{j\in\mathbb{N}}\big\langle\mathscr{D}^{\theta}_{\mu_j,\eta_j,\varepsilon}(u_j),u_j\big\rangle_{\omega}=\sup_{\varepsilon>0}\big\langle\mathscr{D}^{\theta}_{\mu,\eta,\varepsilon}(u),u\big\rangle_{\omega} \tag{2.105}$$

for all $u\in L^2(B,\omega)$. Equation 2.105 implies our claims by 2) in Proposition 2.2.47 and the Kato-Robinson theorem (cf. Theorem 10.4.2 in [88]). Note Remark 2.2.36 for uniqueness of strong resolvent limits for Equation 2.104. □

Definition 2.2.50. For all $\mu,\eta\in A^*_+$, we call $\mathscr{D}^{\theta}_{\mu,\eta}$ in Equation 2.104 the noncommutative division operator of μ and η.

Noncommutative division operators in the normal case. Definition 2.2.6 and Definition 2.2.50 are a priori different definitions of noncommutative division in the AF-C^*-setting. Using results in Theorem 2.2.53 and assuming the representing function induces operator mean vanishing on $[0,\infty)\times\{0\}\cup\{0\}\times[0,\infty)$, Theorem 2.2.58 implies Definition 2.2.50 reduces to Definition 2.2.6 if and only if operator means have finite inverses w.r.t. compressed joint spectral measures. Since we do not suppress the flat operator for positive integrable measurable operators, we distinguish inverses of canonical left- and right-actions from perturbed noncommutative left- and right-division.

Let (A,τ) and (B,ω) be tracial AF-C^*-algebras. Let (ϕ,ψ,γ) be an AF-A-bimodule structure on B. Let f be representing function of an operator mean and $\theta\in[0,1]$.

Proposition 2.2.51. *Let $p\in\{2,\infty\}$ and $x\in L^p(A,\tau)_h$.*

1) *For all $j\in\mathbb{N}$, we have $L_{x_j},R_{x_j}\in\mathscr{B}\big(L^2(A,\tau)\big)_h\cap\mathscr{U}\mathscr{B}_{A_j}\big(L^2(A,\tau)\big)$ and*

$$\pi^A_j L_{x_j}=\mathrm{com}_{A_j}L_x,\ \pi^A_j R_{x_j}=\mathrm{com}_{A_j}R_x. \tag{2.106}$$

2) *$L_x=\text{sr-lim}_{j\in\mathbb{N}}L_{x_j}$ and $R_x=\text{sr-lim}_{j\in\mathbb{N}}R_{x_j}$.*

Proof. For all $T \in \mathscr{B}(L^2(A,\tau))$, get $\mathrm{com}_{A_j} T = \pi_j^A T \pi_j^A$ (cf. Definition A.2.18). We prove all claims for canonical left-action. This readily transfers to canonical right-actions. For all $j \in \mathbb{N}$, we directly verify the identity of bounded operators

$$\pi_j^A L_{x_j} = \mathrm{com}_{A_j} L_x \tag{2.107}$$

on inner products. The above calculation uses A_j is a *-algebra. If x is self-adjoint, then Equation 2.107 implies A_j-reducibility. Get 1). We show 2). Assume x is self-adjoint. If $p = 2$, then 2) in Proposition A.2.8 for core $L^{2,\infty}(A,\tau)$ and Corollary B.1.68 show our claim. If $p = \infty$, then 2) in Proposition A.2.8 for core $L^2(A,\tau)$ and boundedness do. □

Lemma 2.2.52. *Let $x \in L^1(A,\tau)_+$.*

1) $L_x = \text{sr-lim}_{n\in\mathbb{N}} L_{\min\{x,n\}}$ *and* $R_x = \text{sr-lim}_{n\in\mathbb{N}} R_{\min\{x,n\}}$.
2) *For all $j \in \mathbb{N}$, we have*

$$\mathrm{com}_{A_j} L_{x_j} \le L^2_{\pi_j^A(\sqrt{x})},\ \mathrm{com}_{A_j} R_{x_j} \le R^2_{\pi_j^A(\sqrt{x})}. \tag{2.108}$$

3) *For all $\varepsilon > 0$, we have*

$$R_{-\varepsilon}(L_x) \le L^{-\mathrm{id}_A}_{x^\flat,\varepsilon},\ R_{-\varepsilon}(R_x) \le R^{-\mathrm{id}_A}_{x^\flat,\varepsilon}. \tag{2.109}$$

Proof. We prove all claims for canonical left-action. This readily transfers to canonical right-actions. By 2) in Lemma B.1.72 and functional calculus, we have monotonically increasing $\{L_{\min\{x,n\}}\}_{n\in\mathbb{N}} = \{\min\{L_x,n\}\}_{n\in\mathbb{N}} \subset \mathscr{B}(L^2(A,\tau))_+$. Applying the Kato-Robinson theorem, we directly verify 1) on closed positive unbounded quadratic forms.

We show 2). Let $j \in \mathbb{N}$. We know $\sqrt{x}_j = \pi_j^A(\sqrt{x})$. Using 1) in Proposition 2.2.51 and 1.3) in Proposition A.2.24, we have the identity of bounded operators

$$L_{\pi_j^A(\sqrt{x})} = \mathrm{com}_{A_j} L_{\sqrt{x}} + \left(I - \pi_j^A\right) L_{\pi_j^A(\sqrt{x})} \left(I - \pi_j^A\right). \tag{2.110}$$

Multiplying out terms as per Equation 2.110 lets us estimate

$$L^2_{\pi_j^A(\sqrt{x})} = \left(\mathrm{com}_{A_j} L_{\sqrt{x}}\right)^2 + \left(\left(I - \pi_j^A\right) L_{\pi_j^A(\sqrt{x})} \left(I - \pi_j^A\right)\right)^2 \ge \left(\mathrm{com}_{A_j} L_{\sqrt{x}}\right)^2. \tag{2.111}$$

Furthermore, we calculate

$$\begin{aligned}\left\langle \left(\mathrm{com}_{A_j} L_{\sqrt{x}}\right)^2(u), u\right\rangle_\tau &= \tau\left(x\pi_j^A(u)\pi_j^A(u)^*\right) + \left\langle \left(I - \pi_j^A\right)\left(L_{\sqrt{x}}\pi_j^A(u)\right), L_{\sqrt{x}}\pi_j^A(u)\right\rangle_\tau \\ &\ge \tau\left(x\pi_j^A(u)\pi_j^A(u)^*\right) = \left\langle \mathrm{com}_{A_j} L_{x_j}(u), u\right\rangle_\tau\end{aligned}$$

for all $u \in L^2(A,\tau)$.

The above calculation implies

$$\left(\mathrm{com}_{A_j} L_{\sqrt{x}}\right)^2 \geq \mathrm{com}_{A_j} L_{x_j}. \tag{2.112}$$

Equation 2.111 and Equation 2.112 show 2).

We show 3). Let $\varepsilon > 0$. For all $j \in \mathbb{N}$, Equation 2.108 yields

$$R_{-\varepsilon}\left(L^2_{\pi^A_j(\sqrt{x})}\right) \leq R_{-\varepsilon}\left(\pi^A_j L_{x_j} \pi^A_j\right) \tag{2.113}$$

since inversion reverts partial order (cf. Proposition A.2.30). Let $j \in \mathbb{N}$. Then using 2) in Lemma A.2.26, we directly verify

$$R_{-\varepsilon}\left(\mathrm{com}_{A_j} L_{x_j}\right) = \mathrm{com}_{A_j}\left(\mathrm{com}_{A_j} L_{x_j} + \varepsilon \pi^A_j\right)^{-1} + \varepsilon^{-1}\left(I - \pi^A_j\right), \tag{2.114}$$

and

$$\mathrm{com}_{A_j} L^{-\mathrm{id}_A}_{x^\flat_j,\varepsilon} = \mathrm{com}_{A_j}\left(\mathrm{com}_{A_j} L_{x_j} + \varepsilon \pi^A_j\right)^{-1}. \tag{2.115}$$

Equation 2.113, Equation 2.114 and Equation 2.115 let us estimate

$$R_{-\varepsilon}\left(L^2_{\pi^A_j(\sqrt{x})}\right) \leq R_{-\varepsilon}\left(\mathrm{com}_{A_j} L_{x_j}\right) = \mathrm{com}_{A_j} L^{-\mathrm{id}_A}_{x^\flat_j,\varepsilon} + \varepsilon^{-1}\left(I - \pi^A_j\right). \tag{2.116}$$

Note $I = \text{s-}\lim_{j\in\mathbb{N}} \pi^A_j$ is uniformly bounded in norm (cf. 3) in Proposition 2.1.26). By construction of perturbed left-division, sequential strong continuity of multiplication ensures

$$L^{-\mathrm{id}_A}_{x^\flat,\varepsilon} = \text{s-}\lim_{j\in\mathbb{N}} \mathrm{com}_{A_j} L^{-\mathrm{id}_A}_{x^\flat_j,\varepsilon} + \varepsilon^{-1}\left(I - \pi^A_j\right). \tag{2.117}$$

The map $t \mapsto R_{-\varepsilon}(t^2)$ lies in $C_b(\mathbb{R})$. Using Lemma A.2.5, we see 2) in Proposition 2.2.51 thus implies

$$R_{-\varepsilon}(L_x) = R_{-\varepsilon}\left(L^2_{\sqrt{x}}\right) = \text{s-}\lim_{j\in\mathbb{N}} R_{-\varepsilon}\left(L^2_{\pi^A_j(\sqrt{x})}\right) \tag{2.118}$$

by functional calculus.

Using Equation 2.116, Equation 2.117 and Equation 2.118, we calculate

$$\begin{aligned}R_{-\varepsilon}(L_x) &= \text{s-}\lim_{j\in\mathbb{N}} R_{-\varepsilon}\Big(L^2_{\pi^A_j(\sqrt{x})}\Big)\\ &\leq \text{s-}\lim_{j\in\mathbb{N}} \mathrm{com}_{A_j} L^{-\mathrm{id}_A}_{x^\flat_j,\varepsilon} + \varepsilon^{-1}\big(I-\pi^A_j\big)\\ &= L^{-\mathrm{id}_A}_{x^\flat,\varepsilon}.\end{aligned}$$

The above calculation shows 3) at once. □

Theorem 2.2.53. *Let* (A,τ) *and* (B,ω) *be tracial AF-*C^**-algebras. Let* (ϕ,ψ,γ) *be an AF-*A*-bimodule structure on* B*. Let* f *be representing function of an operator mean and* $\theta\in[0,1]$*. Let* $p\in\{1,2,\infty\}$*. For all* $x\in L^p(A,\tau)_+$*, we have*

$$L^\phi_x = \text{sr-}\lim_{j\in\mathbb{N}} L^\phi_{x_j},\ R^\psi_x = \text{sr-}\lim_{j\in\mathbb{N}} R^\psi_{x_j}. \tag{2.119}$$

Proof. We prove all claims for canonical left-action. This readily transfers to canonical right-actions. First, we show our claim for canonical AF-C^*-bimodules. Secondly, we extend to the general case. In this proof, γ is of no consequence. Let $x\in L^p(A,\tau)_+$.

Assume $(A,\tau)=(B,\omega)$ is equipped with its canonical AF-A-bimodule structure. By 1) in Proposition A.2.8, $R_{-\varepsilon}(L_x)=\text{s-}\lim_{j\in\mathbb{N}}R_{-\varepsilon}(L_{x_j})$ for fixed but arbitrary $\varepsilon>0$ implies $L_x=\text{sr-}\lim_{j\in\mathbb{N}}L_{x_j}$. Let $\varepsilon>0$. Using 1.2) in Proposition 2.2.41, uniform boundedness and sequential strong continuity of multiplication, we calculate

$$L^{-\mathrm{id}_A}_{x^\flat,\varepsilon} = \text{s-}\lim_{j\in\mathbb{N}} \mathrm{com}_{A_j}\big(L_{x_j}+\varepsilon I\big)^{-1} = \text{s-}\lim_{j\in\mathbb{N}}\big(L_{x_j}+\varepsilon I\big)^{-1} = \text{s-}\lim_{j\in\mathbb{N}} R_{-\varepsilon}\big(L_{x_j}\big). \tag{2.120}$$

It suffices to have $R_{-\varepsilon}(L_x)=L^{-\mathrm{id}_A}_{x^\flat,\varepsilon}$, ergo

$$R_{-\varepsilon}(L_x)\geq L^{-\mathrm{id}_A}_{x^\flat,\varepsilon} \tag{2.121}$$

by 3) in Lemma 2.2.52.

We use the following. For all $u\in L^2(A,\tau)$, get w^*-l.s.c. map

$$\mu\mapsto \mathbf{L}^{-\mathrm{id}_A}_{\mu,\varepsilon}(u) = \sup_{j\in\mathbb{N}} \mathbf{L}^{-\mathrm{id}_A}_{\mu_j,\varepsilon}(u) \tag{2.122}$$

defined on A^*_+ by 1) in Proposition 2.2.39. Let $y\in L^{1,\infty}(A,\tau)_+$. Using $\sup_{j\in\mathbb{N}}\|y_j\|_\infty = \|y\|_\infty<\infty$, we estimate

$$L^{-\mathrm{id}_A}_{y^\flat,\varepsilon}, L^{-\mathrm{id}_A}_{y^\flat_j,\varepsilon} \geq \big(\|y\|_\infty+\varepsilon\big)^{-1} I > 0 \tag{2.123}$$

in $\mathscr{B}(L^2(A,\tau))$ for all $j\in\mathbb{N}$.

Taking inverses in Equation 2.123 yields uniform bound s.t. 2) in Proposition 2.2.45 implies

$$L^{\mathrm{id}_A}_{y^\flat,\varepsilon} = \text{s-}\lim_{j\in\mathbb{N}} L^{\mathrm{id}_A}_{y^\flat_j,\varepsilon}. \tag{2.124}$$

Following Remark 2.96, Equation 2.124 shows

$$L^{\mathrm{id}_A}_{y^\flat,\varepsilon} = \text{s-}\lim_{j\in\mathbb{N}} L_{y_j} + \varepsilon I. \tag{2.125}$$

Normality implies $L_y = \text{w-}\lim_{j\in\mathbb{N}} L_{y_j}$ by 2) in Proposition 2.1.31 (cf. Remark A.1.10 and Remark A.1.30). Therefore, Equation 2.125 lets us calculate

$$L^{\mathrm{id}_A}_{y^\flat,\varepsilon} = \text{s-}\lim_{j\in\mathbb{N}} L_{y_j} + \varepsilon I = L_y + \varepsilon I. \tag{2.126}$$

Taking inverses in Equation 2.126, we have $L^{-\mathrm{id}_A}_{y^\flat,\varepsilon} = R_{-\varepsilon}(L_y)$ and therefore

$$\left\langle R_{-\varepsilon}(L_y)(u), u\right\rangle_\tau = \mathbf{L}^{-\mathrm{id}_A}_{y^\flat,\varepsilon}(u) \tag{2.127}$$

for all $u \in L^2(A,\tau)$.

For all $n \in \mathbb{N}$, get $x_n := \min\{x,n\} \in L^{1,\infty}(A,\tau)_+$ by positivity, as well as

$$\left\langle R_{-\varepsilon}\big(L_{x_n}\big)(u), u\right\rangle_\tau = \mathbf{L}^{-\mathrm{id}_A}_{x^\flat_n,\varepsilon}(u) \tag{2.128}$$

for all $u \in L^2(A,\tau)$ by Equation 2.127. Then 1) in Lemma 2.2.52 and Lemma A.2.5 show

$$R_{-\varepsilon}(L_x) = \text{s-}\lim_{n\in\mathbb{N}} R_{-\varepsilon}\big(L_{x_n}\big). \tag{2.129}$$

Equation 2.128 and Equation 2.129 let us calculate

$$\left\langle R_{-\varepsilon}(L_x)(u), u\right\rangle_\tau = \lim_{n\in\mathbb{N}} \left\langle R_{-\varepsilon}\big(L_{x_n}\big)(u), u\right\rangle_\tau = \lim_{n\in\mathbb{N}} \mathbf{L}^{-\mathrm{id}_A,\varepsilon}_{x^\flat_n}(u) \tag{2.130}$$

for all $u \in L^2(A,\tau)$. Finally, note $x^\flat = w^*\text{-}\lim_{n\in\mathbb{N}} x^\flat_n$ by 2.2) in Proposition 2.1.31. Since the map in Equation 2.122 is w^*-l.s.c. , Equation 2.130 shows

$$\left\langle R_{-\varepsilon}(L_x)(u), u\right\rangle_\tau = \liminf_{n\in\mathbb{N}} \mathbf{L}^{-\mathrm{id}_A,\varepsilon}_{x^\flat_n}(u) \geq \mathbf{L}^{-\mathrm{id}_A}_{x^\flat}(u) \tag{2.131}$$

for all $u \in L^2(A,\tau)$. Equation 2.131 implies Equation 2.121.

Thus our claim holds assuming canonical AF-C^*-bimodule structure. Assume the general case. The map $t \mapsto R_{\pm i}(t)$ lies in $C_b(\mathbb{R})$. Applying 2) in Lemma B.1.72 and using our above discussion, Lemma A.2.5 implies

$$\Gamma_{x,L^\infty(A,\tau)}(R_{\pm i}) = \operatorname*{s\text{-}lim}_{n\in\mathbb{N}} \Gamma_{x_j,L^\infty(A,\tau)}(R_{\pm i}). \tag{2.132}$$

For all $y \in L^0(A,\tau)_h$, Lemma 2.1.59 shows

$$L^\phi\big(\Gamma_{y,L^\infty(A,\tau)}(R_{\pm i})\big) = R_{\pm i}\big(L^\phi_y\big). \tag{2.133}$$

We know $L^\phi : L^\infty(A,\tau) \longrightarrow \mathscr{B}(L^2(B,\omega))$ is normal unital *-homomorphism. Thus L^ϕ is strongly continuous, hence Equation 2.132 and Equation 2.133 show

$$\begin{aligned} R_{\pm i}\big(L^\phi_x\big) &= L^\phi\big(\Gamma_{x,L^\infty(A,\tau)}(R_{\pm i})\big) \\ &= \operatorname*{s\text{-}lim}_{n\in\mathbb{N}} L^\phi\big(\Gamma_{x_j,L^\infty(A,\tau)}(R_{\pm i})\big) \\ &= \operatorname*{s\text{-}lim}_{n\in\mathbb{N}} R_{\pm i}\big(L^\phi_{x_j}\big). \end{aligned}$$

The above calculation shows our claim. □

Corollary 2.2.54. *For all $x, y \in L^1(A,\tau)_+$ and $\varepsilon > 0$, we have*

1) $L^\phi_{x^\flat,\varepsilon} = L^\phi_x + \varepsilon I$ *and* $R^\psi_{y^\flat,\varepsilon} = R^\psi_y + \varepsilon I$,

2) $\mathscr{D}^\theta_{x^\flat,y^\flat,\varepsilon} = \mathscr{D}^\theta_{x,y,\varepsilon}$.

Proof. Let $x, y \in L^1(A,\tau)_+$ and $\varepsilon > 0$. Equation 2.96 in Remark 2.2.44 rewrites as

$$L^{-\phi}_{x^\flat_j,\varepsilon} = \big(L^\phi_{x_j} + \varepsilon I\big)^{-1},\ R^{-\psi}_{y^\flat_j,\varepsilon} = \big(R^\psi_{y_j} + \varepsilon I\big)^{-1}. \tag{2.134}$$

Using Equation 2.49 to ensure perturbation tends to zero as required, get 1) by applying Theorem 2.2.53 to inverses in Equation 2.134. We then get 2) by 1) in Proposition 2.2.10 applied to the trivial compression. □

Definition 2.2.55. We say that f vanishes at the boundary if $m_f(\lambda, 0) = 0$ for all $\lambda \geq 0$.

Remark 2.2.56. If f vanishes at the boundary, then $m_f(\lambda,0) = m_f(0,\lambda) = 0$ for all $\lambda \geq 0$ by symmetry. Both the geometric and logarithmic operator means have representing function vanishing at the boundary. The arithmetic operator mean does not.

Lemma 2.2.57. *Let f vanish at the boundary.*

1) Let $p \in L^{\infty}(A,\tau)$ be a projection. For all $x,y \in L^1(A[p],\tau)_+$ and $\varepsilon > 0$, we have

$$\left[\mathscr{D}^{\theta}_{x,0,\varepsilon}, L^{\phi}_p\right] = \left[\mathscr{D}^{\theta}_{0,y,\varepsilon}, R^{\psi}_p\right] = 0. \tag{2.135}$$

2) For all $x,y \in L^1(A,\tau)_+$ and $u \in L^2(B,\omega)$, we have

$$\sup_{\varepsilon>0} \left\langle \mathscr{D}^{\theta}_{x,0,\varepsilon}(u), u\right\rangle_{\omega} = \sup_{\varepsilon>0} \left\langle \mathscr{D}^{\theta}_{0,y,\varepsilon}(u), u\right\rangle_{\omega} = \infty. \tag{2.136}$$

Proof. Let $x,y \in L^1(A,\tau)_+$. We prove all claims for x. Their proof readily transfers to the analogous one for y. We show 1). Let $p \in L^{\infty}(A,\tau)$ be a projection s.t. $x \in L^1(A[p],\tau)_+$ and $\varepsilon > 0$. By bounded measurable joint functional calculus (cf. Proposition A.1.100), we have

$$\mathscr{D}^{\theta}_{x,0,\varepsilon} = m^{-\theta}_{f,\varepsilon}\left(L^{\phi}_x, 0\right) \in W^*\left(L^{\phi}_x, I\right). \tag{2.137}$$

Note $W^*(L^{\phi}_x, I) = W^*(L^{\phi}_x) \otimes W^*(I) \cong W^*(L^{\phi}_x) \subset \mathscr{B}(L^2(B,\omega)$ following Equation A.33 in our construction of bounded measurable joint functional calculus. Since $x \in L^1(A[p],\tau)$, we know $\Gamma_{x,L^{\infty}(A,\tau)}(R_{\pm i}) \in W^*_{L^{\infty}(A,\tau)}(x)$.

Corollary B.2.36 therefore implies

$$\left[\Gamma_{x,L^{\infty}(A,\tau)}(R_{\pm i}), p\right] = \left[\Gamma_{x,L^{\infty}(A[p],\tau)}(R_{\pm i} - \mp i) + \mp i 1_A, p\right] = 0. \tag{2.138}$$

Using Lemma 2.1.59, 2) in Lemma B.1.72 and Equation 2.138, we calculate

$$\left[R_{\pm i}(L^{\phi}_x), L^{\phi}_p\right] = \left[L^{\phi}\left(\Gamma_{x,L^{\infty}(A,\tau)}(R_{\pm i})\right), L^{\phi}_p\right] = L^{\phi}\left(\left[\Gamma_{x,L^{\infty}(A,\tau)}(R_{\pm i}), p\right]\right) = 0. \tag{2.139}$$

Following Remark A.1.95, Equation 2.139 shows

$$\left[T, L^{\phi}_p\right] = 0 \tag{2.140}$$

for all $T \in W^*(L^{\phi}_x)$. Equation 2.137 and Equation 2.140 imply 1) at once.

We show 2). Let $u \in L^2(B,\omega)$. Note 2) in Proposition 2.2.10 shows

$$\sup_{\varepsilon>0} \langle \mathscr{D}^{\theta}_{x,0,\varepsilon}(u), u \rangle_{\omega} = \liminf_{\varepsilon \downarrow 0} \langle \mathscr{D}^{\theta}_{x,0,\varepsilon}(u), u \rangle_{\omega}. \tag{2.141}$$

Lemma B.1.72 and Lemma B.1.77 imply

$$\operatorname{spec}_{L^{\infty}(A,\tau)} x \times 0 = \operatorname{supp} E_{L^{\phi}_{x},0} \times \{0\} \subset \operatorname{supp} E_{L^{\phi}_{x}} \times \{0\} = \operatorname{spec}_{L^{\infty}(A,\tau)} x \times 0. \tag{2.142}$$

For all $\varepsilon > 0$, Equation 2.142 and Equation A.18 show

$$\langle \mathscr{D}^{\theta}_{x,0,\varepsilon}(u), u \rangle_{\omega} = \int_{\operatorname{spec}_{L^{\infty}(A,\tau)} x \times 0} m^{-\theta}_{f,\varepsilon}(t,0) dE^{u}_{L^{\phi}_{x},0}. \tag{2.143}$$

Let $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset (0,\infty)$ be a descending sequence converging to zero. Since f vanishes at the boundary, we have

$$\liminf_{n\in\mathbb{N}} m^{-\theta}_{f,\varepsilon_n}(t,0) = \infty \tag{2.144}$$

for all $t \in \operatorname{spec}_{L^{\infty}(A,\tau)} x$. Applying Fatou's Lemma to Equation 2.143, Equation 2.144 shows

$$\liminf_{n\in\mathbb{N}} \langle \mathscr{D}^{\theta}_{x,0,\varepsilon_n}(u), u \rangle_{\omega} \geq \int_{\operatorname{spec}_{L^{\infty}(A,\tau)} x \times 0} \liminf_{n\in\mathbb{N}} m^{-\theta}_{f,\varepsilon_n}(t,0) dE^{u}_{L^{\phi}_{x},0} = \infty. \tag{2.145}$$

The sequence used for Equation 2.144 is fixed but arbitrary. As such, Equation 2.141 and Equation 2.145 imply 2) since no descending sequence yields a finite value. $\square$

Theorem 2.2.58. *Let (A,τ) and (B,ω) be tracial AF-C^*-algebras. Let (ϕ,ψ,γ) be an AF-A-bimodule structure on B. Let f be representing function of an operator mean and $\theta \in [0,1]$. Let $p \in L^{\infty}(A,\tau)$ be a projection. For all $x, y \in L^1(A[p],\tau)_+$, we have*

1) $H\big(Q^{f,\theta}_{x^{\flat},y^{\flat}}\big) \subset L^2(B[p],\omega)$,

2) $\mathscr{D}^{\theta}_{x^{\flat},y^{\flat}} = \text{sr-lim}_{\varepsilon\downarrow 0} \mathscr{D}^{\theta}_{x,y,p,\varepsilon}$ *on* $H\big(Q^{f,\theta}_{x^{\flat},y^{\flat}}\big)$,

3) $Q^{f,\theta}_{x^{\flat},y^{\flat}}(u) = \sup_{\varepsilon>0} \langle \mathscr{D}^{\theta}_{x,y,p,\varepsilon}(u), u \rangle_{\omega}$ *for all* $u \in H\big(Q^{f,\theta}_{x^{\flat},y^{\flat}}\big)$.

Proof. Let $x, y \in L^1(A,\tau)_+$. We show 1). Theorem 2.2.49 and Corollary 2.2.54 imply

$$Q^{f,\theta}_{x^{\flat},y^{\flat}}(u) = \sup_{\varepsilon>0} \langle \mathscr{D}^{f,\theta}_{x,y,\varepsilon}(u), u \rangle_{\omega} \tag{2.146}$$

for all $u \in L^2(B,\omega)$.

Let $u \in L^2(B,\omega)$ and $\varepsilon > 0$. Equation 2.43 shows

$$L^2(B[p],\omega)^\perp = pL^2(B,\omega)p^\perp \oplus p^\perp L^2(B,\omega)p \oplus L^2\big(B\big(p^\perp\big),\omega\big). \tag{2.147}$$

Using 4) in Lemma 2.2.13 and 1) in Lemma 2.2.57, we have

$$\mathscr{D}^\theta_{x,y,\varepsilon} = \mathscr{D}^\theta_{x,y,p,\varepsilon} \oplus \left(\mathscr{D}^\theta_{x,0,\varepsilon} L^\phi_p R^\psi_{p^\perp} \oplus \mathscr{D}^\theta_{0,y,\varepsilon} L^\phi_{p^\perp} R^\psi_p \oplus \varepsilon^{-\theta}\pi_{p^\perp}\right) \tag{2.148}$$

w.r.t. $\mathscr{B}(L^2(B[p],\omega)) \oplus \mathscr{B}(pL^2(B,\omega)p^\perp) \oplus \mathscr{B}(p^\perp L^2(B,\omega)p) \oplus \mathscr{B}(L^2(B(p^\perp),\omega))$. Moreover, all bounded operators in Equation 2.148 are positive. Equation 2.148 lets us estimate

$$\begin{aligned}
\langle \mathscr{D}^\theta_{x,y,\varepsilon}(u), u\rangle_\omega &= \left\| \left(\mathscr{D}^\theta_{x,0,\varepsilon} L^\phi_p R^\psi_{p^\perp} \oplus \mathscr{D}^\theta_{0,y,\varepsilon} L^\phi_{p^\perp} R^\psi_p \oplus \varepsilon^{-\theta}\pi_{p^\perp}\right)^{\frac{1}{2}} \big(\pi_p^\perp(u)\big)\right\|^2_\omega + \left\|\mathscr{D}^{\frac{\theta}{2}}_{x,y,\varepsilon}\pi_p(u)\right\|^2_\omega \\
&\geq \left\| \left(\mathscr{D}^\theta_{x,0,\varepsilon} L^\phi_p R^\psi_{p^\perp} \oplus \mathscr{D}^\theta_{0,y,\varepsilon} L^\phi_{p^\perp} R^\psi_p \oplus \varepsilon^{-\theta}\pi_{p^\perp}\right)^{\frac{1}{2}} \big(\pi_p^\perp(u)\big)\right\|^2_\omega \\
&= \left\|\mathscr{D}^{\frac{\theta}{2}}_{x,0,\varepsilon}\big(pup^\perp\big)\right\|^2_\omega + \left\|\mathscr{D}^{\frac{\theta}{2}}_{0,y,\varepsilon}\big(p^\perp up\big)\right\|^2_\omega + \varepsilon^{-\theta}\left\|\pi_{p^\perp}(u)\right\|^2_\omega.
\end{aligned}$$

Using Equation 2.146 and 2) in Proposition 2.2.10, taking suprema in $\varepsilon > 0$ yields

$$Q^{f,\theta}_{x^\flat,y^\flat}(u) \geq \sup_{\varepsilon>0} \left\|\mathscr{D}^{\frac{\theta}{2}}_{x,0,\varepsilon}\big(pup^\perp\big)\right\|^2_\omega + \sup_{\varepsilon>0} \left\|\mathscr{D}^{\frac{\theta}{2}}_{0,y,\varepsilon}\big(p^\perp up\big)\right\|^2_\omega + \sup_{\varepsilon>0} \varepsilon^{-\theta}\left\|\pi_{p^\perp}(u)\right\|^2_\omega. \tag{2.149}$$

Using 2) in Lemma 2.2.57, Equation 2.149 implies 1) at once. We therefore get 2) and 3) by Equation 2.146, Equation 2.148 and Theorem 2.2.49. □

Corollary 2.2.59. *Let $p \in L^\infty(A,\tau)$ be a projection. If $x,y \in L^0(A[p],\tau)_+$ s.t. we have $m_f^{-1} \in \mathscr{S}_p(E_{x,y})$, then*

1) $H\big(Q^{f,\theta}_{x^\flat,y^\flat}\big) = L^2(B[p],\omega)$,

2) $\mathscr{D}^\theta_{x^\flat,y^\flat} = \mathscr{D}^\theta_{x,y,p} = \text{sr-lim}_{\varepsilon\downarrow 0}\mathscr{D}^\theta_{x,y,p,\varepsilon}$ *on* $L^2(B[p],\omega)$,

3) $Q^{f,\theta}_{x^\flat,y^\flat}(u) = \left\|\mathscr{D}^{\frac{\theta}{2}}_{x,y,p}(u)\right\|^2_\omega = \sup_{\varepsilon>0}\langle \mathscr{D}^\theta_{x,y,p,\varepsilon}(u),u\rangle_\omega$ *for all* $u \in L^2(B[p],\omega)$.

Proof. Note $\mathscr{D}^\theta_{x,y,p} = \text{sr-lim}_{\varepsilon\downarrow 0}\mathscr{D}^\theta_{x,y,p,\varepsilon}$ on $L^2(B[p],\omega)$ by 3) in Proposition 2.2.10. Thus 2) in Proposition 2.2.10 and the Kato-Robinson theorem imply

$$\sup_{\varepsilon>0}\langle \mathscr{D}^\theta_{x,y,p,\varepsilon}(u),u\rangle_\omega = \left\|\mathscr{D}^{\frac{\theta}{2}}_{x,y,p}(u)\right\|^2_\omega < \infty \tag{2.150}$$

for all $u \in \operatorname{dom}\mathscr{D}^{\frac{\theta}{2}}_{x,y,p}$. We know $\mathscr{D}^\theta_{x,y,p}$ is densely defined as per 1) in Proposition 2.2.5 by hypothesis and construction of compressed pulled-back joint functional calculus. Using Equation 2.150, Theorem 2.2.58 hence implies our claims. □

Remark 2.2.60. If $x, y \in L^1(A[p], \tau)_+$ s.t. $L_{x,p}$ and $R_{y,p}$ injective, then $m_f^{-1} \in \mathscr{S}_p(E_{x,y})$. Since $E_{x,L^\infty(A[p],\tau)}$ and $E_{y,L^\infty(A[p],\tau)}$ have no mass at zero if injectivity is given, we know $[0,\infty) \times \{0\} \cup \{0\} \times [0,\infty) \in \mathscr{N}(E_{x,y,L^\infty(A[p],\tau)})$. Zero may still lie in $\mathrm{spec}_{L^\infty(A[p],\tau)} x \times y$.

Corollary 2.2.61. *Let $p, q \in L^\infty(A, \tau)$ be projections. If $x, y \in L^0(A[p], \tau)_+ \cap L^0(A(q), \tau)_+$ s.t. $m_f^{-1} \in \mathscr{S}_p(E_{x,y}) \cap \mathscr{S}_q(E_{x,y})$, then $p = q$.*

Proof. We equip A with its canonical AF-A-bimodule structure. We have $\pi_p = \pi_q$ by 1) in Corollary 2.2.59. For all $j \in \mathbb{N}$, note $p1_{A_j}p = q1_{A_j}q$. Using sequential strong continuity of multiplication and Proposition 2.1.16, get $p = \text{s-lim}_{j\in\mathbb{N}} p1_{A_j}p = \text{s-lim}_{j\in\mathbb{N}} q1_{A_j}q = q$. □

2.3 Noncommutative gradients

Symmetric C^*-derivations are noncommutative gradients [63][65]. We introduce and consider the special case of symmetric W^*-derivations. Using symmetric W^*-bimodules induced by AF-C^*-bimodules as codomains, quantum gradients are, by construction, a class of symmetric W^*-derivations compatible with compression and finite-dimensional approximation. Their dualisation provides the weak formulation of continuity equations in the AF-C^*-setting. Thus Banach dual spaces of AF-C^*-bimodules serve as synthetic tangent spaces. Compatibility transfers to quantum Laplacians, their noncommutative heat semigroups, as well as continuity equations. Compatibility therefore transfers to quantum optimal transport.

Structure. In Subsection 2.3.1, we review symmetric C^*- and W^*-derivations. We study their compression. In Subsection 2.3.2, we define quantum gradients, collect properties and give standard constructions. We further construct dynamic quantum gradients from twisted conjugation groups. In Subsection 2.3.3, we define noncommutative differential structures, discuss compatibility and outline the coarse graining process.

2.3.1 Symmetric C*- and W*-derivations

Symmetric C^*-derivations are closable unbounded module derivations for symmetric C^*-bimodules intertwining adjoining and anti-linear involution. They determine noncommutative analogues of Dirichlet forms [117], called C^*-Dirichlet forms [1][63][65]. Following likewise generalised Beurling-Deny formula [26], representing operators of conservative C^*-Dirichlet forms are concatenations of symmetric C^*-derivations and their adjoints (cf. Theorem 8.3 in [65]). These in turn generate completely Markovian semigroups for tracial C^*-algebras (cf. Theorem 4.11 in [63]). Altogether, we say that symmetric C^*-derivations are noncommutative gradients which determine Laplacians and view completely Markovian semigroups generated by the latter as noncommutative heat semigroups. The relationship between gradients, heat semigroups and Dirichlet forms extends to the noncommutative setting [63][65].

We define symmetric W^*-derivations to be symmetric C^*-derivations for symmetric W^*-bimodules, moreover closable w.r.t. bounded strong convergence s.t. units are in the kernel upon closure. Using symmetric W^*-bimodules induced by AF-C^*-bimodules as codomains, we have compression based on compression of AF-C^*-bimodules. We then define quantum gradients to be symmetric W^*-derivations with sufficient compression to have finite-dimensional approximation. Standard references for C^*-bimodules and C^*-derivations are [63][65]. The latter are collected in [64] on p.161-276 in [27].

Unbounded module derivations. Definition 2.3.2 collects notions of unbounded module derivations we use, including symmetric C^*- and W^*-derivations. This yields a more general definition of symmetric C^*-derivations than in [65]. Remark 2.3.3 shows results for symmetric C^*-derivations in [63][65] apply regardless. Proposition 2.3.10 states the chain rule for symmetric W^*-derivations.

Let (M,τ) be a tracial W^*-algebra.

Notation 2.3.1. Unless stated otherwise, we use the identical symbols for unbounded operators and all of their closures. For all closable unbounded operators $T : H_0 \longrightarrow H_1$ of Hilbert spaces, let $\|.\|_T$ denote its graph norm.

Definition 2.3.2. Let $A \subset M$ be a σ-weakly dense C^*-subalgebra and H a symmetric W^*-bimodule over M. Let $\mathscr{A} \subset A$ be a *-subalgebra and $\nabla : \mathscr{A} \longrightarrow H$ a linear map.

1) We say that ∇ satisfies

   1.1) the Leibniz rule if $\nabla xy = \nabla x \cdot y + x\nabla y$ for all $x, y \in \mathscr{A}$,

   1.2) symmetry if $\nabla x^* = \gamma(\nabla x)$ for all $x \in \mathscr{A}$.

2) We say that ∇ is an $\mathscr{A}$-module derivation if it satisfies the Leibniz rule, and further call ∇ symmetric if it satisfies symmetry.

3) We say that ∇ is a symmetric C^*-derivation if

   3.1) ∇ is a symmetric $\mathscr{A}$-module derivation,

   3.2) $\mathscr{A} \subset A$ is $\|.\|_A$-dense and $\mathscr{A} \subset L^2(M,\tau)$ is $\|.\|_\tau$-dense,

   3.3) ∇ is $(\|.\|_A, \|.\|_H)$-closable and $\nabla|_{\mathscr{A}\cap L^2(M,\tau)}$ is $(\|.\|_\tau, \|.\|_H)$-closable.

   Then its $(\|.\|_\tau, \|.\|_H)$-closure defines the Laplacian $\Delta := \nabla^*\nabla$ of ∇.

4) We say that ∇ is a symmetric W^*-derivation if

   4.1) ∇ is a symmetric C^*-derivation,

   4.2) for all nets $\{x_k\}_{k\in K} \subset \mathscr{A}$ s.t. bds-$\lim_{k\in K} x_k =$ bds-$\lim_{k\in K} x_k^* = 0$, get existence of $\|.\|_H$-$\lim_{k\in K} \nabla x_k$ if and only if $\|.\|_H$-$\lim_{k\in K} \nabla x_k = 0$,

   4.3) there exists a net $\{x_k\}_{k\in K} \subset \mathscr{A}$ s.t. bds-$\lim_{k\in K} x_k =$ bds-$\lim_{k\in K} x_k^* = 1_M$ and $\|.\|_H$-$\lim_{k\in K} \nabla x_k = 0$.

Remark 2.3.3. Assume the setting of Definition 2.3.2. Let $\nabla : \mathscr{A} \longrightarrow H$ be a symmetric C^*-derivation. Restricting the bimodule action of M to A yields symmetric C^*-bimodule H over A as per Definition 2.1.48. If $\tau|_{A_+}$ is semi-finite, then (A,τ) is tracial C^*-algebra and ∇ is symmetric C^*-derivation used in [65]. Since semi-finiteness does not affect the chain rule, the relationship between gradients, heat semigroups and Dirichlet forms in the noncommutative setting uses the $(\|.\|_\tau, \|.\|_H)$-closure of ∇. We therefore know results for symmetric C^*-derivations in [63][65] apply to our general notion. However, we apply them only if A is unital and $\tau < \infty$. Note semi-finiteness of $\tau|_{A_+}$ is always given in this case (cf. 2) in Proposition B.1.12). If $\tau < \infty$, then replacing $(\|.\|_A, \|.\|_H)$-closable in 3.3) in Definition 2.3.2 by $(\|.\|_A, \|.\|_H)$-closed yields identical $(\|.\|_\tau, \|.\|_H)$-closures.

Definition 2.3.4. Let $A \subset M$ be a σ-weakly dense C^*-subalgebra and H a symmetric W^*-bimodule over M. Let $\nabla : \mathscr{A} \longrightarrow H$ be a symmetric W^*-derivation.

1) We call a net $\{x_k\}_{k\in K} \subset \mathscr{A}$ bounded strongly convergent to $x \in M$ for ∇ if

   1.1) $x = \text{bds-lim}_{k\in K} x_k$ and $x^* = \text{bds-lim}_{k\in K} x_k^*$,

   1.2) $\{\nabla x_k\}_{k\in K} \subset H$ is Cauchy net in norm.

   Let $x = \text{bds}^\nabla\text{-lim}_{k\in K} x_k$ denote bounded strong convergence for ∇.

2) Set $M_\nabla := \left\{ x \in M \mid \exists \{x_k\}_{k\in K} \subset \mathscr{A} : \ x = \text{bds}^\nabla\text{-lim}_{k\in K} x_k \right\}$.

Let $A \subset M$ be a σ-weakly dense C^*-subalgebra and H a symmetric W^*-bimodule over M. Let $\nabla : \mathscr{A} \longrightarrow H$ be a symmetric W^*-derivation. By 4.2) in Definition 2.3.2, we define bounded strong closure $\nabla : M_\nabla \longrightarrow H$ of ∇ by setting

$$\nabla x := \|.\|_H\text{-}\lim_{k\in K} \nabla x_k \tag{2.151}$$

for all $x \in M_\nabla$. In each case, we use fixed but arbitrary net $\{x_k\}_{k\in K} \subset \mathscr{A}$ bounded strongly convergent to $x \in M$ for ∇.

Definition 2.3.5. Let $A \subset M$ be a σ-weakly dense C^*-subalgebra and H a symmetric W^*-bimodule over M. For all symmetric W^*-derivations $\nabla : \mathscr{A} \longrightarrow H$, its bounded strong closure $\nabla : M_\nabla \longrightarrow H$ is defined by Equation 2.151.

Proposition 2.3.6. *Let $A \subset M$ be a σ-weakly dense C^*-subalgebra and H a symmetric W^*-bimodule over M. For all symmetric W^*-derivations $\nabla : \mathscr{A} \longrightarrow H$, we have*

1) $1_M \in M_\nabla$ and unital *-subalgebra $M_\nabla \subset M$,

2) symmetric M_∇-module derivation $\nabla : M_\nabla \longrightarrow H$ and $\nabla 1_M = 0$.

Proof. Multiplication in M is jointly continuous in strong operator topology. We thus know $M_\nabla \subset M$ is a *-subalgebra, further having unit $1_M \in M_\nabla$ with $\nabla 1_M = 0$ by 4.3) in Definition 2.3.2. Since we use normal unital *-homomorphisms to define the bimodule action of M on H as per Definition 2.1.48, the Leibniz rule extends from $\mathscr{A}$ to M_∇. Note symmetry follows by construction. Altogether, get 1) and 2). □

The Leibniz rule formulates a noncommutative chain rule using functional calculus of left- and right-bimodule actions of symmetric W^*-bimodules. Following notation in Definition 2.1.48, we use (ϕ,ψ)-action of M on H for normal unital *-homomorphisms $\phi,\psi : M \longrightarrow \mathscr{B}(H)$. For all $x,y \in M$, $\phi(x),\psi(y) \in \mathscr{B}(H)$ commute by definition.

Definition 2.3.7. Let $I \subset \mathbb{R}$ be a closed interval. For all $g \in C^1(I)$, we define functional derivative of g on $I \times I$ by setting

$$Dg(t,s) := \begin{cases} \frac{g(t)-g(s)}{t-s} & \text{if } t \neq s, \\ \frac{d}{dt} g(t) & \text{else}. \end{cases}$$

Remark 2.3.8. Note $Dg \in C(I \times I)$ s.t. $\|Dg\|_{C(I\times I)} \leq \|\frac{d}{dt} g\|_{C(I)}$ in each case.

Proposition 2.3.9. *Let $x \in M_h$. If $I \subset \mathbb{R}$ is a closed interval s.t. $\operatorname{spec}_M x \subset I$, then*

1) $C(I \times I) \subset C(\operatorname{spec}_M x \times \operatorname{spec}_M x) \subset L^\infty\big(\operatorname{spec} \phi(x) \times \psi(x), dE_{\phi(x),\psi(x)}\big)$,

2) $\|\Gamma_{\phi(x),\psi(x)}(h)\|_{\mathscr{B}(H)} \leq \|h\|_{C(I\times I)}$ *for all* $h \in C(I \times I)$.

Proof. Note $\operatorname{spec} \phi(x), \operatorname{spec} \psi(x) \subset \operatorname{spec}_M x$ as ϕ and ψ are unital *-homomorphisms. Get

$$\operatorname{spec} \phi(x) \times \psi(x) \subset \operatorname{spec}_M x \times \operatorname{spec}_M x \subset I \times I. \tag{2.152}$$

Equation 2.152 implies 1) by dualisation. Bounded measurable joint functional calculus $\Gamma_{\phi(x),\psi(x)} : L^\infty(\operatorname{spec} \phi(x) \times \psi(x), dE_{\phi(x),\psi(x)}) \longrightarrow \mathscr{B}(H)$ is a normal unital *-homomorphism (cf. 1) in Proposition A.1.100). Using $\|\Gamma_{\phi(x),\psi(x)}\| \leq 1$ and 1), we obtain 2) at once. □

Proposition 2.3.10. *Let $A \subset M$ be a σ-weakly dense C^*-subalgebra and H a symmetric W^*-bimodule over M. Let $\nabla : \mathscr{A} \longrightarrow H$ be a symmetric W^*-derivation, $x \in M_\nabla$ self-adjoint and $I \subset \mathbb{R}$ a closed interval s.t. $\operatorname{spec}_M x \subset I$. If $g \in C^1(I)$, then*

1) $g(x) \in M_\nabla$ *self-adjoint and* $\nabla g(x) = \Gamma_{\phi(x),\psi(x)}(Dg)(\nabla x)$,

2) $\|\nabla g(x)\|_H \leq \|\frac{d}{dt} g\|_{C(I)} \cdot \|\nabla x\|_H$.

Proof. Note $\nabla 1_M = 0$ by 2) in Proposition 2.3.6. If g is polynomial, then we directly verify 1) and 2) using the Leibniz rule, symmetry, and $\nabla 1_M = 0$. Let $I = [a,b]$ for $a \leq b$ in $\mathbb{R}$ and $g \in C^1(I)$. Since $\nabla 1_M = 0$, we assume $g(a) = 0$ without loss of generality.

We know $\frac{d}{dt} g \in C(I)$. Let $\{q_n\}_{n\in\mathbb{N}} \subset C(\mathbb{R})$ be polynomials s.t. $\frac{d}{dt} g = \|.\|_\infty\text{-}\lim_{n\in\mathbb{N}} q_n$. For all $n \in \mathbb{N}$, set $g_n(t) := \int_a^t q_n(s)ds$ for all $t \in I$. Get $g_n \in C^1(I)$ with derivative q_n in each case. Using standard arguments for integration [109][139][140], norm convergence of derivatives implies $g = \|.\|_\infty\text{-}\lim_{n\in\mathbb{N}} g_n$ since $g(a) = g_n(a) = 0$ for all $n \in \mathbb{N}$. Following our definition of bounded strong closure as per Equation 2.151, such approximation reduces our claims to the polynomial case by linearity of the functional derivative and Proposition 2.3.9. □

Compressing symmetric W*-derivations. Definition 2.3.13 gives compression of symmetric W^*-derivations. It is based on compression of AF-C^*-bimodules. The two classes of compression given in Subsection 2.1.2 each provide compression of symmetric W^*-derivations. First, we compress to induced AF-C^*-bimodules in Corollary 2.3.14. Secondly, we compress with projections in Corollary 2.3.15.

Let (A,τ) and (B,ω) be tracial AF-C^*-algebras. Let (ϕ,ψ,γ) be an AF-A-bimodule structure on B. Let $\nabla:\mathscr{A}\longrightarrow L^2(B,\omega)$ be a symmetric W^*-derivation.

Lemma 2.3.11. *Let H_0 and H_1 be Hilbert spaces, $V_0\subset H_0$ and $V_1\subset H_1$ Hilbert subspaces, and $T:H_0\longrightarrow H_1$ closed unbounded operator. If $\mathscr{C}$ is core of T s.t.*

1) $\pi^{H_0}_{V_0}(\mathscr{C})\subset \operatorname{dom} T$,

2) $\pi^{H_1}_{V_1}\big(T(x)\big)=T\big(\pi^{H_0}_{V_0}(x)\big)$ *for all* $x\in\mathscr{C}$,

then $\pi^{H_1}_{V_1}T\subset T\pi^{H_0}_{V_0}$.

Proof. Let $x\in\operatorname{dom}T$ and $x=\|.\|_T\text{-}\lim_{k\in K}x_k$ for a net $\{x_k\}_{k\in K}\subset\mathscr{C}$. Using 1) and 2), get

$$\pi^{H_1}_{V_1}\big(T(x)\big)=\|.\|_{H_1}\text{-}\lim_{k\in K}\pi^{H_1}_{V_1}\big(T(x_k)\big)=\|.\|_{H_1}\text{-}\lim_{k\in K}T\big(\pi^{H_0}_{V_0}(x_k)\big).\tag{2.153}$$

Equation 2.153 shows $\pi^{H_1}_{V_1}\big(T(x)\big)=T\big(\pi^{H_0}_{V_0}(x)\big)$ since T is closed. □

Remark 2.3.12. Assume $H:=H_0=H_1$ and $V:=V_0=V_1$ in the setting of Lemma 2.3.11. If $T\in\mathscr{UB}(H)_h$ has core as per Lemma 2.3.11, then T is V-reducible. If $T\in\mathscr{UB}_V(H)$ and $\mathscr{C}$ core of T, then $\mathscr{C}$ satisfies 1) and 2) in Lemma 2.3.11.

Definition 2.3.13. Let (ϕ,ψ,γ) be (N,V)-compressible. We say that $\nabla:\mathscr{A}\longrightarrow L^2(B,\omega)$ is (N,V)-compressible, and call (N,V) a compression of ∇, if

1) $\pi^A_{L^2(N,\tau)}(\mathscr{A})\subset N\cap L^\infty(A,\tau)_\nabla$,

2) $\pi^A_{L^2(N,\tau)}(\mathscr{A})\subset N$ is σ-weakly dense and $\pi^A_{L^2(N,\tau)}(\mathscr{A})\subset L^2(N,\tau)$ is $\|.\|_\tau$-dense,

3) $\pi^A_{L^2(N,\tau)}(\mathscr{A})\subset\operatorname{dom}\nabla$, $\pi^B_V(\operatorname{dom}\nabla^*)\subset\operatorname{dom}\nabla^*$, and

$$\pi^B_V(\nabla x)=\nabla\pi^A_{L^2(N,\tau)}(x),\ \pi^A_{L^2(N,\tau)}\big(\nabla^*u\big)=\nabla^*\pi^B_V(u)\tag{2.154}$$

for all $x\in\mathscr{A}$ and $u\in\operatorname{dom}\nabla^*$.

Corollary 2.3.14. *Let $\mathscr{A} = A_0$. If $j \in \mathbb{N}$ s.t.*

$$\nabla(A_j) \subset B_j,\ \nabla^*(B_j) \subset A_j, \tag{2.155}$$

then ∇ is (A_j, B_j)-compressible.

Proof. Let $j \in \mathbb{N}$ s.t. Equation 2.155 holds. We know (ϕ, ψ, γ) is (A_j, B_j)-compressible by Corollary 2.1.63. Using $\pi^A_{L^2(N,\tau)} = \pi^A_j$, $\pi^B_V = \pi^B_j$ and Equation 2.155, we directly verify 1) to 3) in Definition 2.3.13. □

Corollary 2.3.15. *If $p \in L^\infty(A,\tau)$ is a projection and $\{p_k\}_{k\in K} \subset \mathscr{A} \cap A_h$ a net s.t.*

1) $p = \text{bds-lim}_{k\in K}\, p_k$,

2) $p_k \in \ker\nabla$ *for all* $k \in K$,

then $p \in L^\infty(A,\tau)_\nabla$, $\nabla p = 0$, and ∇ is $(L^\infty(A[p],\tau), L^2(B[p],\omega))$-compressible.

Proof. We use the following results. Let $p \in L^\infty(A,\tau)$ be a projection. We know (ϕ, ψ, γ) is $(L^\infty(A[p],\tau), L^2(B[p],\omega))$-compressible by Corollary 2.1.65. The latter shows

$$\pi^A_{L^2(N,\tau)} = \pi^A_{L^2(A[p],\tau)} = L_p R_p,\ \pi^B_V = \pi^B_{L^2(B[p],\omega)} = L^\phi_p R^\psi_p. \tag{2.156}$$

Equation 2.156 in turn implies

$$\pi^A_{L^2(N,\tau)}(\mathscr{A}) = p\mathscr{A}p,\ \pi^B_V\big(\mathrm{dom}\,\nabla^*\big) = p\big(\mathrm{dom}\,\nabla^*\big)p. \tag{2.157}$$

Lemma 2.1.6 shows $p\mathscr{A}p \subset pL^\infty(A,\tau)p = L^\infty(A[p],\tau)$, as well as $p\mathscr{A}p \subset pL^2(A,\tau)p = L^2(A[p],\tau)$. We use these inclusions to show 1) to 3) in Definition 2.3.13.

Let $\{p_k\}_{k\in K} \subset \mathscr{A} \cap A_h$ be a net s.t. 1) and 2) holds. Get $p \in L^\infty(A,\tau)_\nabla$ and $\nabla p = 0$. Thus 1) in Proposition 2.3.6 yields $p\mathscr{A}p \subset L^\infty(A,\tau)_\nabla$, hence Equation 2.157 shows 1) in Definition 2.3.13. Using density of $\mathscr{A}$ as per 3.2) in Definition 2.3.2, Equation 2.157 further shows 2) in Definition 2.3.13. Using $p \in L^\infty(A,\tau)_\nabla$, Equation 2.157 and 2) in Proposition 2.3.6, we directly verify 3) in Definition 2.3.13 on inner products. □

Definition 2.3.16 gives *-subalgebras generated by compressions. Proposition 2.3.18 lifts properties in Definition 2.3.13 to such *-subalgebras. The latter therefore serve as domains of compressed symmetric W^*-derivations. Definition 2.3.16 gives compressed symmetric W^*-derivations. Proposition 2.3.19 collects their properties. Notation 2.3.21 fixes conventions.

Definition 2.3.16. For all compressions (N,V) of $\nabla : \mathscr{A} \longrightarrow L^2(B,\omega)$, let $\mathscr{A}_N \subset N$ be the *-subalgebra generated by $\pi^A_{L^2(N,\tau)}(\mathscr{A})$ in N.

Remark 2.3.17. We do not require *-subalgebras to be closed in any topology.

Proposition 2.3.18. *For all compressions (N,V) of $\nabla : \mathscr{A} \longrightarrow L^2(B,\omega)$, we have*

1) $\mathscr{A}_N \subset N \cap L^\infty(A,\tau)_\nabla$ *is a* **-subalgebra,*

2) $\mathscr{A}_N \subset N$ *is σ-weakly dense and* $\mathscr{A}_N \subset L^2(N,\tau)$ *is* $\|.\|_\tau$*-dense,*

3) $\mathscr{A}_N \subset \operatorname{dom}\nabla$, $\pi_V^B(\operatorname{dom}\nabla^*) \subset \operatorname{dom}\nabla^*$, *and*

$$\pi_V^B(\nabla x) = \nabla\pi_{L^2(N,\tau)}^A(x),\ \pi_{L^2(N,\tau)}^A\big(\nabla^* u\big) = \nabla^*\pi_V^B(u) \tag{2.158}$$

for all $x \in \mathscr{A}$ and $u \in \operatorname{dom}\nabla^*$.

Proof. Get 1) by 1) in Proposition 2.3.6. We have 2) as Hölder ensures $\mathscr{A}_N \subset L^2(N,\tau)$. Using 2) in Proposition 2.3.6, we obtain 3) by extending 3) in Definition 2.3.13. □

Proposition 2.3.19. *For all compressions (N,V) of $\nabla : \mathscr{A} \longrightarrow L^2(B,\omega)$, we have*

1) $\pi_V^B\nabla \subset \nabla\pi_{L^2(N,\tau)}^A$ *and* $\pi_{L^2(N,\tau)}^A\nabla^* \subset \nabla^*\pi_V^B$,

2) $\nabla|_{\mathscr{A}_N} : \mathscr{A}_N \longrightarrow V$ *is a symmetric W^*-derivation and*

   2.1) $\pi_{L^2(N,\tau)}^A(\operatorname{dom}\nabla) = L^2(N,\tau) \cap \operatorname{dom}\nabla$,

   2.2) $\nabla|_{L^2(N,\tau)} : L^2(N,\tau) \cap \operatorname{dom}\nabla \longrightarrow V$ *is* $(\|.\|_\tau, \|.\|_\omega)$*-closure of* $\nabla|_{\mathscr{A}_N}$,

3) $\big(\nabla|_V\big)^* : V \cap \operatorname{dom}\nabla^* \longrightarrow L^2(N,\tau)$ *is a closed unbounded operator and*

   3.1) $\pi_V^B(\operatorname{dom}\nabla^*) = V \cap \operatorname{dom}\nabla^*$,

   3.2) $\big(\nabla|_V\big)^* = \big(\nabla|_{L^2(N,\tau)}\big)^*$,

4) $\Delta \in \mathscr{UB}\big(L^2(A,\tau)\big)_+ \cap \mathscr{UB}_{L^2(N,\tau)}\big(L^2(A,\tau)\big)$ *and* $\Delta|_{L^2(N,\tau)} = \big(\nabla|_{L^2(N,\tau)}\big)^*\big(\nabla|_{L^2(N,\tau)}\big)$.

Proof. Proposition 2.3.18 ensures Lemma 2.3.11 applies to $\nabla : L^2(A,\tau) \longrightarrow L^2(B,\omega)$ for core $\mathscr{A}$ and $\nabla^* : L^2(B,\omega) \longrightarrow L^2(A,\tau)$ for $\operatorname{dom}\nabla^*$. Note (ϕ,ψ,γ) being (N,V)-compressible implies V is a symmetric W^*-bimodule over N as per 2) in Proposition 2.1.58. Set

$$A_N := \overline{\mathscr{A}_N}^{\|.\|_A} = C^*(\mathscr{A}_N). \tag{2.159}$$

The second identity in Equation 2.159 follows from 1) in Proposition 2.3.18. Using 2) in Proposition 2.3.18, note $A_N \subset N$ is a σ-weakly dense C^*-subalgebra. Using the Leibniz rule and symmetry, Equation 2.158 shows $\nabla(\mathscr{A}_N) \subset V$. Thus 2) in Proposition 2.3.6 and 1) in Proposition 2.3.18 show the restriction $\nabla : \mathscr{A}_N \longrightarrow V$ of $\nabla : L^\infty(A,\tau)_\nabla \longrightarrow L^2(B,\omega)$ to $\mathscr{A}_N$ is a symmetric $\mathscr{A}_N$-module derivation. It satisfies 3.2) in Definition 2.3.2 by 2) in Proposition 2.3.18. Since $\pi_{L^2(N,\tau)}^A(\operatorname{dom}\nabla)$ is $\|.\|_\nabla$-closure of $\mathscr{A}_N$, it satisfies 2.1) and in turn 2.2) by 3) in Proposition 2.3.18. Hence $\nabla : \mathscr{A}_N \longrightarrow V$ satisfies 3.3) in Definition 2.3.2 and is a symmetric C^*-derivation. We show it is a symmetric W^*-derivation.

We use the following. We already show and use 1) above. Being the restriction to $\mathscr{A}_N$ ensures 4.2) in Definition 2.3.2. Semi-finiteness of N moreover shows there exists noncommutative conditional expectation from $L^\infty(A,\tau)$ to N as per Remark 2.1.8, i.e. a normal unital bounded linear map

$$\pi_N^{L^\infty(A,\tau)} : L^\infty(A,\tau) \longrightarrow N \tag{2.160}$$

restricting to $\pi^A_{L^2(N,\tau)}$ on $L^{2,\infty}(A,\tau)$ and satisfying a trace identity (cf. Remark B.2.8).

Applying the noncommutative conditional expectation to an approximating net for $\nabla : \mathscr{A} \longrightarrow L^2(B,\omega)$ as per 4.3) in Definition 2.3.2 yields one for $\nabla : \mathscr{A}_N \longrightarrow V$. This uses 1) and restriction of the noncommutative conditional expectation to the Hilbert space projection. Thus $\nabla : \mathscr{A}_N \longrightarrow V$ is a symmetric W^*-derivation, hence 2) follows since we have 2.1) and 2.2). Using 1), we directly verify 3.1) and $\nabla^*(V \cap \operatorname{dom}\nabla^*) \subset L^2(N,\tau)$. The latter implies

$$\operatorname{dom}\big(\nabla|_{L^2(N,\tau)}\big)^* = \pi^B_V\big(\operatorname{dom}\nabla^*\big). \tag{2.161}$$

Equation 2.161 and 2.2) show 3.2). Altogether, get 1) to 3). Note 3) implies 4). □

Definition 2.3.20. For all compressions (N,V) of $\nabla : \mathscr{A} \longrightarrow L^2(B,\omega)$, set

1) $\nabla_N := \nabla|_{\mathscr{A}_N}$,

2) $\Delta_N := \nabla_N^* \nabla_N$.

Notation 2.3.21. Following Notation 2.3.1, we additionally use ∇_N to denote closures in Definition 2.3.20 and throughout our discussion. Proposition 2.3.19 therefore states $\nabla_N = \nabla|_{L^2(A,\tau)}$, $\nabla_N^* = \big(\nabla|_V\big)^*$ and $\Delta_N = \Delta|_{L^2(A,\tau)}$.

Proposition 2.3.22. *Let $\nabla : \mathscr{A} \longrightarrow L^2(B,\omega)$ be (N,V)-compressible.*

1) For all $g \in C_b([0,\infty))$, we have

$$g(\Delta) = g(\Delta_N) \oplus g\big(\Delta|_{L^2(N,\tau)^\perp}\big) \tag{2.162}$$

w.r.t. $\mathscr{B}\big(L^2(N,\tau)\big) \oplus \mathscr{B}\big(L^2(N,\tau)^\perp\big)$.

2) $g(\Delta_N) \in \mathscr{B}\big(L^2(N,\tau)\big) \subset \mathscr{B}_V\big(L^2(A,\tau)\big)$ and $g(\Delta_N) = \operatorname{com}_{L^2(A,\tau)} g(\Delta_N)$.

Proof. Note 4) in Proposition 2.3.19 shows $\Delta \geq 0$ on $L^2(A,\tau)$ is $L^2(N,\tau)$-reducible. Using the latter, get 1) and 2) by 2) in Corollary A.2.27 because $\Delta_N = \Delta|_{L^2(N,\tau)}$. □

2.3.2 Quantum gradients for AF-C*-bimodules

In the AF-C^*-setting, Equation 2.155 provides the sufficient condition for compressing symmetric W^*-derivations to induced AF-C^*-bimodules. We therefore define quantum gradients to be symmetric W^*-derivations s.t. Equation 2.155 holds for all $j \in \mathbb{N}$. Their compression is Definition 2.3.20. Compressing to induced AF-C^*-bimodules and taking limits is finite-dimensional approximation of quantum gradients. Proposition 2.3.19 and Proposition 2.3.22 imply such compatibility transfers as claimed.

Standard constructions of quantum gradients are direct sum, tensor product, as well as internal quantum gradients. We further construct dynamic quantum gradients by weak differentiation of twisted conjugation groups. We include a non-twisted case. In Subsection 3.1.3, standard constructions using dynamic quantum gradients provide fundamental example classes. Standard references for unbounded algebra derivations generating C^*-dynamical systems are [173] and [182]. We moreover refer to [35][36] as comprehensive treatment of C^*-dynamical systems in quantum statistical mechanics. Standard reference for the weak differentiation of, in general non-twisted, conjugation groups is [60]. Their weak derivatives generalise inner derivations [133].

Definition and properties. Definition 2.3.23 gives quantum gradients. They are symmetric W^*-derivations by Proposition 2.3.25. Proposition 2.3.25 collects properties. We compress quantum gradients. First, we compress to induced AF-C^*-bimodules as per Corollary 2.3.14. Secondly, we compress with projections as per Corollary 2.3.15 assuming additional properties. Using the first one, finite-dimensional approximation is 4) in Proposition 2.3.25. This is compatibility of quantum gradients with compression and finite-dimensional approximation.

Let (A,τ) and (B,ω) be tracial AF-C^*-algebras. Let (ϕ,ψ,γ) be an AF-A-bimodule structure on B. Note Remark 2.3.24 concerning closure of quantum gradients.

Definition 2.3.23. Let $\nabla : A_0 \longrightarrow L^2(B,\omega)$ be a symmetric A_0-module derivation.

1) We say that ∇ is a quantum gradient if $B_0 \subset \operatorname{dom}\nabla^*$ and

$$\nabla(A_j) \subset B_j,\ \nabla^*(B_j) \subset A_j \tag{2.163}$$

for all $j \in \mathbb{N}$. Equation 2.163 is called locality. We further call $\Delta := \nabla^*\nabla$ a quantum Laplacian.

2) Let ∇ be a quantum gradient. For all $j \in \mathbb{N}$, we call $\nabla_j := \nabla|_{A_j} : A_j \longrightarrow B_j$ the j-th restricted quantum gradient and $\Delta_j := \nabla_j^*\nabla_j$ the j-th restricted Laplacian.

Remark 2.3.24. Let $\nabla : A_0 \longrightarrow L^2(B,\omega)$ be a quantum gradient. Since $B_0 \subset L^2(B,\omega)$ is $\|.\|_\omega$-dense and $B_0 \subset \operatorname{dom}\nabla^*$, we see ∇ is $(\|.\|_\tau, \|.\|_\omega)$-closable (cf. Theorem 5.1.5 in [171]).

Proposition 2.3.25. *Let $\nabla : A_0 \longrightarrow L^2(B,\omega)$ be a quantum gradient.*

1) $\nabla : A_0 \longrightarrow L^2(B,\omega)$ *is a symmetric W^*-derivation.*

2) For all $j \leq k$ in $\mathbb{N}$, we have

2.1) $\pi_j^B \nabla \subset \nabla \pi_j^A$ *and* $\pi_j^A \nabla^* \subset \nabla^* \pi_j^B$,

2.2) $\pi_j^B \nabla \pi_k^A \subset \nabla \pi_j^A$ *and* $\pi_j^A \nabla^* \pi_k^B \subset \nabla^* \pi_j^B$.

3) For all $j \in \mathbb{N}$, we have

3.1) $\nabla : A_0 \longrightarrow L^2(B,\omega)$ *is (A_j, B_j)-compressible,*

3.2) $\nabla_j : A_j \longrightarrow B_j$ *is a quantum gradient,* $\nabla_j^* = \left(\nabla|_{B_j}\right)^*$ *and* $\Delta_j = \Delta_{A_j}$.

4) We have

4.1) A_0 *is core of* ∇ *and* $u = \|.\|_\nabla\text{-}\lim_{j\in\mathbb{N}} \pi_j^A(u)$ *for all* $u \in \operatorname{dom}\nabla$,

4.2) B_0 *is core of* ∇^* *and* $v = \|.\|_{\nabla^*}\text{-}\lim_{j\in\mathbb{N}} \pi_j^B(v)$ *for all* $v \in \operatorname{dom}\nabla^*$,

4.3) A_0 *is core of* Δ *and* $w = \|.\|_\Delta\text{-}\lim_{j\in\mathbb{N}} \pi_j^A(w)$ *for all* $w \in \operatorname{dom}\Delta$.

5) We have $\gamma(\operatorname{dom}\nabla^*) = \operatorname{dom}\nabla^*$. *For all* $u \in \operatorname{dom}\nabla^*$, *we have* $\nabla^*\gamma(u) = \left(\nabla^* u\right)^*$.

Proof. Using $(\|.\|_\tau, \|.\|_\omega)$-closure of $\nabla : A_0 \longrightarrow L^2(B,\omega)$, we have $\nabla : L^2(A,\tau) \longrightarrow L^2(B,\omega)$ with core $A_0 \subset \operatorname{dom}\nabla$. We further have $\nabla^* : L^2(B,\omega) \longrightarrow L^2(A,\tau)$ and $B_0 \subset \operatorname{dom}\nabla^*$. Note $A_0 \subset L^2(A,\tau)$ and $B_0 \subset L^2(B,\omega)$ are dense in respective Hilbert space norms.

We use 4.1) to show 1). We see 3) in Proposition 2.1.26 shows 4.1) if Equation 2.164 holds for all $j \in \mathbb{N}$. We use Lemma 2.3.11. Set $H_0 = A_j$ and $H_1 = B_j$ in each case. By testing on the inner product, density in Hilbert space norms and Equation 2.163 show Lemma 2.3.11 applies to $\nabla : \operatorname{dom}\nabla \longrightarrow L^2(B,\omega)$ using core A_0. For all $j \in \mathbb{N}$, we have

$$\pi_j^B \nabla \subset \nabla \pi_j^A. \tag{2.164}$$

Get 4.1). We show 1). Note $\nabla : A_0 \longrightarrow L^2(B,\omega)$ is a symmetric C^*-derivation if it is $(\|.\|_A, \|.\|_\omega)$-closable. Using $A_0 \subset L^\infty(A,\tau) = L^1(A,\tau)^*$ and $B_0 \subset L^2(B,\omega)$ $\|.\|_\omega$-dense, we directly verify closability. Using 4.1), we directly verify 4.2) and 4.3) in Definition 2.3.2. For 4.3) in Definition 2.3.2, we use $\{1_{A_j}\}_{j\in\mathbb{N}}$ as approximating sequence of 1_A. We see $\nabla : A_0 \longrightarrow L^2(B,\omega)$ is a symmetric W^*-derivation. Get 1).

We know 1). We therefore have 3) by Equation 2.163, Corollary 2.3.14, as well as 3) and 4) in Proposition 2.3.19. Equation 2.164 shows all claims for ∇ in 2). We directly verify all claims for ∇^* in 2) by restricting ∇^* as per 3.2). This shows 4.2) by 3) in Proposition 2.1.26. Then 4.3) follows by 1). Altogether, get 1) to 4).

We show 5). The anti-linear isometric property of γ is $\langle\gamma(v),\gamma(w)\rangle_\omega = \overline{\langle v,w\rangle}$ for all $v,w \in L^2(B,\omega)$. For all $x \in A_0$ and $u \in B_0$, we apply $\gamma(\nabla x) = \nabla x^*$ and the anti-linear isometry property to calculate

$$\left\langle \nabla^* \gamma(u), x\right\rangle_\tau = \left\langle \gamma(u), \nabla x\right\rangle_\omega = \overline{\left\langle u, \nabla x^*\right\rangle_\omega} = \left\langle \left(\nabla^* u\right)^*, x\right\rangle_\tau. \tag{2.165}$$

Using 4), Equation 2.165 shows 5) by closure. □

Definition 2.3.26. Let $\nabla : A_0 \longrightarrow L^2(B,\omega)$ be a quantum gradient.

1) Let $p \in L^\infty(A,\tau)$ be a projection. We say that ∇ is p-compressible if the conditions of Corollary 2.3.15 are satisfied for ∇ and p.

2) Let ∇ be p-compressible and $A_{0,L^\infty(A[p],\tau)}$ the *-subalgebra generated by pA_0p in $L^\infty(A[p],\tau)$. We call $\nabla_p := \nabla_{L^\infty(A[p],\tau)} : A_{0,L^\infty(A[p],\tau)} \longrightarrow L^2(B[p],\omega)$ a p-compressed quantum gradient and $\Delta_p := \Delta_{L^\infty(A[p],\tau)}$ its p-compressed Laplacian.

Proposition 2.3.27. *Let $\nabla : A_0 \longrightarrow L^2(B,\omega)$ be a quantum gradient.*

1) For all $j \in \mathbb{N}$, we have $\Delta_j = \Delta_{A_j} = \nabla_j^* \nabla_j$.

2) If ∇ is p-compressible, then $\Delta_p = \Delta|_{L^2(A[p],\tau)} = \nabla_p^* \nabla_p$.

Proof. Apply 4) in Proposition 2.3.19. □

Standard constructions. We use just three standard constructions for quantum gradients: direct sums, tensor products and internal products. We collect constructions and properties here for use throughout our discussion. For details on direct sums and tensor products of C^*- and W^*-algebras, we refer to Subsection A.1.2.

Notation 2.3.28. We use superscripts before subscripts to denote instances of objects whenever possible. If this does not yield suitable notation, in particular to prevent any overload of exponents, then we revert to subscripts even as it may introduce double subscripts. The latter must be clear from context, e.g. $A_{n,j}$ denotes the j-th generating C^*-subalgebra of an AF-C^*-algebra A_n for $n \in \mathbb{N}$. Let $m \in \mathbb{N}$. For all objects E with direct sums, set $E^m := \oplus_{n=1}^m E$. For all direct sums $\oplus_{n=1}^m H_n$ of Hilbert spaces, let

$$\pi_k : \oplus_{n=1}^m H_n \longrightarrow H_k \tag{2.166}$$

be the Hilbert space projection from $\oplus_{n=1}^m H_n$ to H_k for all $k \in \{1,\ldots,m\}$.

Definition 2.3.31 gives direct sum AF-C^*-bimodules and quantum gradients for the following data. Let $m \in \mathbb{N}$ and (A, τ) be a tracial AF-C^*-algebra. For all $n \in \{1, \ldots, m\}$, let (B_n, ω_n) be a tracial AF-C^*-algebra and $(\phi_n, \psi_n, \gamma_n)$ an AF-A-bimodule structure on B_n. We define f.s.n. trace $\oplus_{n=1}^m \omega_n$ on $\oplus_{n=1}^m L^\infty(B_n, \omega_n)$ by setting

$$\left(\oplus_{n=1}^m \omega_n\right)(x) := \sum_{n=1}^m \omega_n(x_n) \tag{2.167}$$

for all $x = (x_1, \ldots, x_n) \in \oplus_{n=1}^m L^\infty(B_n, \omega_n)_+$. Get tracial AF-$C^*$-algebra $(\oplus_{n=1}^m B_n, \oplus_{n=1}^m \omega_n)$ in $\oplus_{n=1}^m L^\infty(B_n, \omega_n)$ generated by $\{\oplus_{n=1}^m B_{n,j}\}_{j \in \mathbb{N}}$. Let $p \in \{1, 2, \infty\}$. Equation 2.167 shows $L^p(\oplus_{n=1}^m B_n, \oplus_{n=1}^m \omega_n) = \oplus_{n=1}^m L^p(B_n, \omega_n)$. We obtain local *-homomorphisms

$$\oplus_{n=1}^m \phi_n, \oplus_{n=1}^m \psi_n : A \longrightarrow \oplus_{n=1}^m B_n \tag{2.168}$$

by restricting direct sum *-homomorphisms to the diagonal $A \subset A^m$. We use direct sum anti-linear isometric involution

$$\oplus_{n=1}^m \gamma_n : L^2\left(\oplus_{n=1}^m B_n, \oplus_{n=1}^m \omega_n\right) \longrightarrow L^2\left(\oplus_{n=1}^m B_n, \oplus_{n=1}^m \omega_n\right). \tag{2.169}$$

Proposition 2.3.29. *Let $m \in \mathbb{N}$ and (A, τ) tracial AF-C^*-algebra. For all $n \in \{1, \ldots, m\}$, let (B_n, ω_n) be a tracial AF-C^*-algebra, $(\phi_n, \psi_n, \gamma_n)$ an AF-A-bimodule structure on B_n, and $\partial_n : A_0 \longrightarrow L^2(B_n, \omega_n)$ a quantum gradient. We have*

1) *AF-A-bimodule structure $\left(\oplus_{n=1}^m \phi_n, \oplus_{n=1}^m \psi_n, \oplus_{n=1}^m \gamma_n\right)$ on $\oplus_{n=1}^m B_n$,*

2) *quantum gradient $\nabla^\oplus := \oplus_{n=1}^m \partial_n : A_0 \longrightarrow L^2\left(\oplus_{n=1}^m B_n, \oplus_{n=1}^m \omega_n\right)$ defined by*

$$\nabla^\oplus x := (\partial_1 x, \ldots, \partial_n x) \tag{2.170}$$

for all $x \in A_0$,

3) *$\nabla^{\oplus,*} := \left(\oplus_{n=1}^m \partial_n\right)^* = \sum_{n=1}^m \partial_n^* \pi_n$ with core $\oplus_{n=1}^m B_{n,0}$ and given by*

$$\nabla^{\oplus,*} u = \sum_{n=1}^m \partial_n^* u_n \tag{2.171}$$

for all $u = (u_1, \ldots, u_m) \in \operatorname{dom} \nabla^{\oplus,*} = \oplus_{n=1}^m \operatorname{dom} \partial_n^*$,

4) *$\Delta^\oplus := \nabla^{\oplus,*} \nabla^\oplus = \sum_{n=1}^m \partial_n^* \partial_n$ with core A_0 and given by*

$$\Delta^\oplus u = \sum_{n=1}^m \partial_n^* \partial_n u \tag{2.172}$$

for all $u \in \operatorname{dom} \Delta^\oplus = \bigcap_{n=1}^m \operatorname{dom} \partial_n^* \partial_n$.

Proof. Get 1) by construction. Using direct sum construction, Equation 2.170 shows 2) and Equation 2.171 by reducing to summands. Equation 2.170 and Equation 2.171 thus imply 3) and 4) by Proposition 2.3.25. □

Proposition 2.3.30. *Assume the setting of Proposition 2.3.29. Let f be a representing function and $\theta \in [0,1]$. For all $\mu,\eta \in A_+^*$ and $w \in B^*$, we have*

$$\mathscr{I}^{f,\theta}(\mu,\eta,w) = \sum_{n=1}^{m} \mathscr{I}^{f,\theta}_{A,B_n}\big(\mu,\eta,w|_{B_n}\big). \tag{2.173}$$

Proof. We reduce to the finite-dimensional setting by 3) in Theorem 2.2.29. Note $\mathscr{I}^{f,\theta}$ and each $\mathscr{I}^{f,\theta}_{A,B_n}$ are l.s.c. in w^*-topology by 1) in Theorem 2.2.29. L.s.c. in w^*-topology shows Equation 2.173 if it holds for all $\mu,\eta \in A_+^*$ s.t. $\sharp\mu, \sharp\eta > 0$ in A. Equation 2.173 itself follows by construction of noncommutative division operators in this case. □

Definition 2.3.31. Assume the setting of Proposition 2.3.29. We call

1) $\big(\oplus_{n=1}^m \phi_n, \oplus_{n=1}^m \psi_n, \oplus_{n=1}^m \gamma_n\big)$ their direct sum AF-C^*-bimodule,
2) $\nabla^{\oplus} : A_0 \longrightarrow L^2\big(\oplus_{n=1}^m B_n, \oplus_{n=1}^m \omega_n\big)$ their direct sum quantum gradient,
3) ∂_n the n-th partial gradient, ∂_n^* the n-th partial adjoint, and $\Delta_n := \partial_n^*\partial_n$ the n-th Laplacian for all $n \in \{1,\ldots,m\}$.

Definition 2.3.33 gives tensor product AF-C^*-bimodules and quantum gradients for the following data. For all $n \in \{1,2\}$, let (A_n,τ_n) and (B_n,ω_n) be tracial AF-C^*-algebras with (ϕ_n,ψ_n,γ_n) an AF-A_n-bimodule structure on B_n. We determine f.s.n. trace $\tau_1 \otimes \tau_2$ on $L^\infty(A_1,\tau_1) \otimes L^\infty(A_2,\tau_2)$ by setting

$$\big(\tau_1 \otimes \tau_2\big)(x \otimes y) := \tau_1(x)\tau_2(y) \tag{2.174}$$

for all $x \in \mathfrak{m}_{\tau_1}$ and $y \in \mathfrak{m}_{\tau_2}$. Note both A_1 and A_2 are nuclear. Get tracial AF-C^*-algebra $(A_1 \otimes A_2, \tau_1 \otimes \tau_2)$ in $L^\infty(A_1,\tau_1) \otimes L^\infty(A_2,\tau_2)$ generated by $\{A_{1,j} \otimes A_{2,j}\}_{j\in\mathbb{N}}$.

Let $p \in \{2,\infty\}$. Equation 2.174 shows

$$L^p(A_1 \otimes A_2, \tau_1 \otimes \tau_2) = L^p(A_1,\tau_1) \otimes L^p(A_2,\tau_2). \tag{2.175}$$

Note the above construction likewise yields tracial AF-C^*-algebra $(B_1 \otimes B_2, \omega_1 \otimes \omega_2)$ in $L^\infty(B_1,\omega_1) \otimes L^\infty(B_2,\omega_2)$ generated by $\{B_{1,j} \otimes B_{2,j}\}_{j\in\mathbb{N}}$ s.t. Equation 2.175 is

$$L^p(B_1 \otimes B_2, \omega_1 \otimes \omega_2) = L^p(B_1,\omega_1) \otimes L^p(B_2,\omega_2) \tag{2.176}$$

in each case.

Note Corollary A.1.53 shows we obtain local *-homomorphisms

$$\phi_1 \otimes \phi_2, \psi_1 \otimes \psi_2 : A_1 \otimes A_2 \longrightarrow B_1 \otimes B_2 \tag{2.177}$$

by restricting tensored *-homomorphisms to $A_1 \otimes A_2 \subset L^\infty(A_1 \otimes A_2, \tau_1 \otimes \tau_2)$. We use tensor product anti-linear isometric involution

$$\gamma_1 \otimes \gamma_2 : L^2(B_1 \otimes B_2, \omega_1 \otimes \omega_2) \longrightarrow L^2(B_1 \otimes B_2, \omega_1 \otimes \omega_2). \tag{2.178}$$

Proposition 2.3.32. *For all $n \in \{1,2\}$, let (A_n, τ_n) and (B_n, ω_n) be tracial AF-C^*-algebras with $(\phi_n, \psi_n, \gamma_n)$ an AF-A_n-bimodule structure on B_n, as well as $\delta_n : A_{n,0} \longrightarrow L^2(B_n, \omega_n)$ a quantum gradient. We have*

1) *AF-$A_1 \otimes A_2$-bimodule structure $\left(\phi_1 \otimes \phi_2, \psi_1 \otimes \psi_2, \gamma_1 \otimes \gamma_2\right)$ on $B_1 \otimes B_2$,*
2) *quantum gradient $\nabla^\otimes : A_{1,0} \odot A_{2,0} \longrightarrow L^2(B_1 \otimes B_2, \omega_1 \otimes \omega_2)$ defined by*

$$\nabla^\otimes x \otimes y := \delta_1 x \otimes \psi_2(y) + \phi_1(x) \otimes \delta_2 y \tag{2.179}$$

for all $x \in A_{1,0}$ and $y \in A_{2,0}$,

3) *$\nabla^{\otimes,*} := \left(\nabla^\otimes\right)^*$ with core $\operatorname{dom} \delta_1^* \odot \operatorname{dom} \delta_2^*$ and determined by*

$$\nabla^{\otimes,*} u \otimes v = \delta_1^* u \otimes \psi_2^*(v) + \phi_1^*(u) \otimes \delta_2^* v \tag{2.180}$$

for all $u \in \operatorname{dom} \delta_1^*$ and $v \in \operatorname{dom} \delta_2^*$.

Proof. Get 1) by construction. Using tensor product construction, Equation 2.179 shows $\nabla^\otimes$ is a symmetric $A_{1,0} \odot A_{2,0}$-module derivation, implies Equation 2.180, and yields

$$B_{1,0} \otimes B_{2,0} \subset \operatorname{dom} \delta_1^* \odot \operatorname{dom} \delta_2^* \subset \operatorname{dom} \nabla^{\otimes,*} \tag{2.181}$$

by reducing to elementary tensors. Using inclusions in Equation 2.181, Equation 2.179 and Equation 2.180 imply locality. Proposition 2.3.25 implies all remaining claims. □

Definition 2.3.33. Assume the setting of Proposition 2.3.32. We call

1) $\left(\phi_1 \otimes \phi_2, \psi_1 \otimes \psi_2, \gamma_1 \otimes \gamma_2\right)$ their tensor product AF-C^*-bimodule,
2) $\nabla^\otimes : A_{1,0} \odot A_{2,0} \longrightarrow L^2(B_1 \otimes B_2, \omega_1 \otimes \omega_2)$ their tensor product quantum gradient.

Remark 2.3.34. Tensor product quantum gradients have Laplacians with mixed terms coupling their factors. This follows by construction. It differs from the decomposition given by Equation 2.172 for direct sum quantum gradients.

Definition 2.3.35 gives generalised discrete derivatives. Example 3.1.54 shows these specialise to discrete derivatives and internal quantum gradients. Let (A,τ) be unital tracial C^*-algebra in M s.t. $\tau<\infty$. We have *-homomorphisms

$$\phi^{\mathrm{Int}} := \mathrm{id}_{A\otimes A}\big|_{A\otimes\langle 1_A\rangle_{\mathbb{C}}}, \psi^{\mathrm{Int}} := \mathrm{id}_{A\otimes A}\big|_{\langle 1_A\rangle_{\mathbb{C}}\otimes A} : A \longrightarrow A\otimes A \tag{2.182}$$

given by restriction to unital C^*-subalgebras $A \cong A\otimes\langle 1_A\rangle_{\mathbb{C}} \cong \langle 1_A\rangle_{\mathbb{C}}\otimes A$ of $A\otimes A$. We have f.s.n. trace $\tau\otimes\tau$ on $M\otimes M$ and unital tracial C^*-algebra $A\otimes A$ in $M\otimes M$. Thus $L^2(A\otimes A,\tau\otimes\tau)$ is a symmetric C^*-bimodule over A equipped with $(L_M\circ\phi, R_M\circ\psi)$-action and $\gamma=\mathrm{Adj}$ as per Example 2.1.45 for $A\otimes A$ as anti-linear involution.

Definition 2.3.35. Let (A,τ) be unital tracial C^*-algebra in M s.t. $\tau<\infty$. We define symmetric C^*-bimodule $L^2(A\otimes A,\tau\otimes\tau)$ over A by Equation 2.182. We define generalised discrete derivative $\delta : A \longrightarrow L^2(A\otimes A,\tau\otimes\tau)$ on A by setting

$$\delta x := x\otimes 1_A - 1_A\otimes x \tag{2.183}$$

for all $x\in A$.

Proposition 2.3.36. *If (A,τ) is a unital tracial C^*-algebra in M s.t. $\tau<\infty$, then δ as per Definition 2.3.35 is a bounded symmetric A-module derivation.*

Proof. Note $A\otimes A\subset L^2(A\otimes A,\tau\otimes\tau)$ since $\tau\otimes\tau<\infty$. On $A\otimes A$, the symmetric C^*-bimodule action reduces to left- and right-algebra multiplication in $A\otimes A$ using *-homomorphisms given by Equation 2.182 as in Equation 2.20. For all $x,y\in A$, we use $\delta x,\delta y\in A\otimes A$ and Equation 2.183 to calculate $\delta xy = (x\otimes 1_A)(y\otimes 1_A - 1_A\otimes y) + (x\otimes 1_A - 1_A\otimes x)(1_A\otimes y) = x(\delta y)+(\delta x)y$. Thus δ satisfies the Leibniz rule. Symmetry follows at once. □

Let (A,τ) be a strongly unital tracial AF-C^*-algebra s.t. $\tau<\infty$. We equip A with its canonical AF-A-bimodule structure. Then $(A\otimes A,\tau\otimes\tau)$ has canonical AF-$A\otimes A$-bimodule structure and the *-homomorphisms given by Equation 2.182 are local.

Proposition 2.3.37. *Let (A,τ) be a strongly unital tracial AF-C^*-algebra s.t. $\tau<\infty$. For all $\lambda\geq 0$, we have*

1) *AF-A-bimodule structure $\left(\phi^{\mathrm{Int}},\psi^{\mathrm{Int}},\mathrm{Adj}\right)$ on $A\otimes A$,*
2) *bounded quantum gradient $\nabla : A_0 \longrightarrow L^2(A\otimes A,\tau\otimes\tau)$ defined by*

$$\nabla^\lambda x := \sqrt{\frac{\lambda}{2\tau(1_A)}}\cdot\left(x\otimes 1_A - 1_A\otimes x\right) \tag{2.184}$$

for all $x\in A_0$,

3) $\nabla^{\lambda,*} := \left(\nabla^{\lambda}\right)^*$ *bounded and determined by*

$$\nabla^{\lambda,*} y \otimes z = \sqrt{\frac{\lambda}{2\tau(1_A)}} \cdot \left(\left\langle 1_A, z\right\rangle_\tau y - \left\langle 1_A, y\right\rangle_\tau z\right) \tag{2.185}$$

for all $y, z \in A$,

4) $\Delta^\lambda := \nabla^{\lambda,*}\nabla^\lambda = \lambda \pi^A_{\ker\tau}$.

Proof. We have 1) by construction. Set $\lambda := 2\tau(1_A)$ without loss of generality. We then suppress λ in the superscript. We show 2). Note $\nabla = \delta|_{A_0}$. Proposition 2.3.36 shows it is a symmetric A_0-module derivation. Using Equation 2.184, we directly verify it is a quantum gradient. Equation 2.184 further shows boundedness upon closure. Get 2).

We show 3). Boundedness of ∇ implies ∇^* is bounded and determined on elementary tensors. For all $x \in A_0$ and $y, z \in A$, Equation 2.184 lets us calculate

$$\left\langle \nabla x, y \otimes z\right\rangle_{\tau\otimes\tau} = \left\langle x, \left\langle 1_A, z\right\rangle_\tau y\right\rangle_\tau - \left\langle x, \left\langle 1_A, y\right\rangle_\tau z\right\rangle_\tau = \left\langle x, \left\langle 1_A, z\right\rangle_\tau y - \left\langle 1_A, y\right\rangle_\tau z\right\rangle_\tau. \tag{2.186}$$

Equation 2.186 shows Equation 2.185. Get 3). We show 4). For all $x \in A_0$, Equation 2.184 and Equation 2.185 show $\Delta x = 2\tau(1_A)\pi^A_{\ker\tau}(x)$. Get 4) by boundedness. □

Definition 2.3.38. Let (A, τ) be a strongly unital tracial AF-C^*-algebra s.t. $\tau < \infty$. For all $\lambda \geq 0$, we call

1) $\left(\phi^{\mathrm{Int}}, \psi^{\mathrm{Int}}, \mathrm{Adj}\right)$ the internal AF-A-bimodule structure on $A \otimes A$,

2) $\nabla^\lambda : A_0 \longrightarrow L^2(A \otimes A, \tau \otimes \tau)$ the λ-internal quantum gradient on A.

Dynamic quantum gradients. Definition 2.3.42 gives sufficient conditions to construct quantum gradients by weak differentiation of conjugation groups twisted with self-adjoint involutive local *-homomorphisms. These dynamic quantum gradients are either twisted or non-twisted. Generators on Hilbert spaces control weak derivatives as per Equation 2.188 and Equation 2.189. We pull back along canonical left-actions upon weak differentiation in our construction, and twist as per Remark 2.3.41. We use one-parameter semigroups of bounded operators on Banach spaces [102].

Definition 2.3.45 gives dynamic quantum gradients. We give two classes of dynamic quantum gradient. First, we consider trace-preserving local C^*-dynamical systems in Corollary 2.3.49. Secondly, we consider intertwining self-adjoint unbounded operators generating suitable conjugation groups in Corollary 2.3.56. Whereas Corollary 2.3.49 yields only non-twisted examples, Corollary 2.3.56 yields both twisted and non-twisted ones. In Subsection 3.1.3, standard constructions using dynamic quantum gradients provide fundamental example classes. Those using tracial AF-C^*-algebras generating hyperfinite factors of type I and II by σ-weak closure are of particular interest.

Definition 2.3.39. Let H be a Hilbert space and $\mathscr{D} \in \mathscr{UB}(H)_h$. We define conjugation group $\mathrm{Ad}^{\mathscr{D}} : \mathbb{R} \longrightarrow \mathrm{Aut}(\mathscr{B}(H))$ of $\mathscr{D}$ by setting

$$\mathrm{Ad}_t^{\mathscr{D}}(S) := e^{it\mathscr{D}} S e^{-it\mathscr{D}} \tag{2.187}$$

for all $t \in \mathbb{R}$ and $S \in \mathscr{B}(H)$.

Let H be a Hilbert space and $\mathscr{D} \in \mathscr{UB}(H)_h$. For all $S \in \mathscr{B}(H)$, the weak derivative

$$\left.\frac{d}{dt}\right|_{t=0,\mathrm{w}} \mathrm{Ad}_t^{\mathscr{D}}(S) = \mathrm{w}\text{-}\lim_{t\to 0} t^{-1}\big(\mathrm{Ad}_t^{\mathscr{D}}(S) - S\big) \tag{2.188}$$

exists if and only if the following two conditions are satisfied (cf. Theorem 3.8 in [60]). First, $S(u) \in \mathrm{dom}\,\mathscr{D}$ for all $u \in \mathrm{dom}\,\mathscr{D}$. Secondly, that $\mathscr{D}S - S\mathscr{D} \in \mathscr{UB}(H)$ is bounded and closable. Then $\mathrm{dom}\,\mathscr{D}S - S\mathscr{D} = \mathrm{dom}\,\mathscr{D}$ and Equation 2.188 is

$$\left.\frac{d}{dt}\right|_{t=0,\mathrm{w}} \mathrm{Ad}_t^{\mathscr{D}}(S) = \overline{i(\mathscr{D}S - S\mathscr{D})} \in \mathscr{B}(H). \tag{2.189}$$

Note $\mathscr{D}S - S\mathscr{D}$ is bounded, but not a bounded operator in general. This necessitates closure. For conjugation groups, differentiation in weak and strong operator topologies is equivalent [60]. We use strong limits in Equation 2.188 without loss of generality. If $\mathscr{D} \in \mathscr{B}(H)$, then $\frac{d}{dt}\big|_{t=0,\mathrm{w}} \mathrm{Ad}_t^{\mathscr{D}}(S) = i[\mathscr{D}, S]$ for all $S \in \mathscr{B}(H)$. In fact, all bounded module derivations on W^*-algebras are inner (cf. Theorem XI.3.5 in [193]). This explains use of unbounded module derivations, resp. non-canonical AF-C^*-bimodule structures.

Definition 2.3.40. Let (A, τ) be a tracial AF-C^*-algebra. A local *-homomorphism $\phi : A \longrightarrow A$ is self-adjoint if $\phi \in \mathscr{B}(L^2(A,\tau))_h$ as per Definition 2.1.39.

Remark 2.3.41. Let (A, τ) be a tracial AF-C^*-algebra and $\phi : A \longrightarrow A$ a self-adjoint involutive local *-homomorphism. Thus its L^2-extension is self-adjoint by hypothesis and involutive since $A_0 \subset L^2(A,\tau)$ is $\|.\|_\tau$-dense, hence $\phi \in \mathscr{U}(\mathscr{B}(L^2(A,\tau)))$ and $\phi^\dagger$ as per Definition A.1.13. For all $T \in \mathscr{UB}(L^2(A,\tau))$, get $\phi^\dagger(T) = \phi T \phi$. For all $x \in L^0(A,\tau)$, we have $\phi^\dagger(L_x) = L_{\phi(x)}$ using canonical AF-C^*-bimodule action on A. We obtain

$$\phi^\dagger \circ L = L \circ \phi. \tag{2.190}$$

If $T \in L(A)$, then $\phi^\dagger(T) \in L(A)$. If $T \in \phi L(A)$, then $T\phi \in L(A)$.

Let $\mathscr{A}$ be a *-algebra. For all $x, y \in \mathscr{A}$, we use their commutator $[x,y] = xy - yx$ and anti-commutator $\{x,y\} = xy + yx$ throughout our discussion. Definition 2.3.42 gives twisted commutators and anti-commutators in an unbounded case.

Definition 2.3.42. Let (A,τ) be a tracial AF-C^*-algebra and $\phi : A \longrightarrow A$ a self-adjoint involutive local *-homomorphism. Note $\phi = \phi^2 \in \mathscr{B}(L^2(A,\tau))$. Let $\mathscr{D} \in \mathscr{UB}(L^2(A,\tau))_h$.

1) We call $(\mathscr{D},\phi)$ generator of a dynamic quantum gradient if

   1.1) for all $j \in \mathbb{N}$ and $x \in A_j$, we have

   1.1.1) $\frac{d}{dt}\big|_{t=0,\mathrm{w}} \mathrm{Ad}_t^{\mathscr{D}}(L_x) = \overline{i(\mathscr{D}L_x - L_x\mathscr{D})} \in \phi L(A_j)$,

   1.1.2) $\frac{d}{dt}\big|_{t=0,\mathrm{w}} \mathrm{Ad}_{-t}^{\mathscr{D}}(L_x\phi) = \overline{i(L_x\phi\mathscr{D} - \mathscr{D}L_x\phi)} \in L(A_j)$,

   1.2) for all $x, y \in A_0$, we have

$$\left\langle x, L^{-1}\left(\frac{d}{dt}\bigg|_{t=0,\mathrm{w}} \mathrm{Ad}_t^{\mathscr{D}}(L_y)\phi\right)\right\rangle_\tau = \left\langle L^{-1}\left(\frac{d}{dt}\bigg|_{t=0,\mathrm{w}} \mathrm{Ad}_{-t}^{\mathscr{D}}(L_x\phi)\right), y\right\rangle_\tau. \tag{2.191}$$

2) Let $(\mathscr{D},\phi)$ be generator of a dynamic quantum gradient. We define ϕ-twisted commutator $[\mathscr{D},-]_A^\phi : A_0 \longrightarrow L^2(A,\tau)$ and anti-commutator $\{\mathscr{D},-\}_A^\phi : A_0 \longrightarrow L^2(A,\tau)$ by setting

$$[\mathscr{D},x]_A^\phi := L^{-1}\left(\left(\overline{\mathscr{D}L_{\phi(x)} - L_{\phi(x)}\mathscr{D}}\right)\phi\right),\ \{\mathscr{D},x\}_A^\phi := L^{-1}\left(\phi\left(\overline{L_x\phi\mathscr{D} - \mathscr{D}L_x\phi}\right)\phi\right) \tag{2.192}$$

for all $x \in A_0$.

Remark 2.3.43. If $\phi = \mathrm{id}_A$, then Equation 2.192 reduces to commutators and their negatives. Using 2) in Lemma 2.3.55, we see non-trivial ϕ as per Example 3.1.59 yield anti-commutators up to twist generalising [48].

Let (A,τ) be a tracial AF-C^*-algebra. Let $\phi : A \longrightarrow A$ be a self-adjoint involutive local *-homomorphism. We define anti-linear isometric involution $\gamma^\phi : L^2(A,\tau) \longrightarrow L^2(A,\tau)$ by setting

$$\gamma^\phi(u) := \phi(u^*) \tag{2.193}$$

for all $u \in L^2(A,\tau)$.

Proposition 2.3.44. *Let (A,τ) be a tracial AF-C^*-algebra. For all generators $(\mathscr{D},\phi)$ of dynamic quantum gradients, we have*

1) *AF-A-bimodule structure $(\phi, \mathrm{id}_A, \gamma^\phi)$ on A,*

2) *quantum gradient $\nabla^{\mathscr{D},\phi} : A_0 \longrightarrow L^2(A,\tau)$ defined by*

$$\nabla^{\mathscr{D},\phi}x := L^{-1}\left(\frac{d}{dt}\bigg|_{t=0,\mathrm{w}} Ad_t^{\mathscr{D}}\left(L_{\phi(x)}\right)\phi\right) = i[\mathscr{D},x]_A^\phi \tag{2.194}$$

for all $x \in A_0$,

3) $\nabla^{\mathscr{D},\phi,*} := \left(\nabla^{\mathscr{D},\phi}\right)^*$ *with core* A_0 *and determined by*

$$\nabla^{\mathscr{D},\phi,*}x = L^{-1}\left(\phi\left(\frac{d}{dt}\bigg|_{t=0,\mathrm{w}} Ad^{\mathscr{D}}_{-t}(L_x\phi)\right)\phi\right) = i\{\mathscr{D},x\}^{\phi}_{A} \tag{2.195}$$

for all $x \in A_0$*,*

4) $\Delta^{\mathscr{D},\phi} := \nabla^{\mathscr{D},\phi,*}\nabla^{\mathscr{D},\phi}$ *with core* A_0 *and determined by*

$$\Delta^{\mathscr{D},\phi}x = -\left\{\mathscr{D},[\mathscr{D},x]^{\phi}_{A}\right\}^{\phi}_{A} \tag{2.196}$$

for all $x \in A_0$*.*

Proof. We have 1) by construction. For both Equation 2.194 and Equation 2.195, we see 1) in Definition 2.3.42 shows existence of weak derivatives. Equation 2.192 implies the second identity in Equation 2.194 and therefore Equation 2.194 itself.

Note weak differentiation in Equation 2.194 is strong differentiation [60]. We know all weak, resp. strong derivatives in use exist. Using the latter and sequential strong continuity of multiplication, we directly verify the Leibniz rule for $\nabla^{\mathscr{D},\phi}$. Equation 2.194 implies symmetry. Thus $\nabla^{\mathscr{D},\phi}$ is a symmetric A_0-derivation. Using self-adjointness of ϕ for the second identity below, Equation 2.190 and Equation 2.191 let us calculate

$$\begin{aligned}
\left\langle x, \nabla^{\mathscr{D},\phi} y\right\rangle_{\tau} &= \left\langle L^{-1}\left(\frac{d}{dt}\bigg|_{t=0,\mathrm{w}} \mathrm{Ad}^{\mathscr{D}}_{-t}(L_x\phi)\right), \phi(y)\right\rangle_{\tau} \\
&= \left\langle \phi\left(L^{-1}\left(\frac{d}{dt}\bigg|_{t=0,\mathrm{w}} \mathrm{Ad}^{\mathscr{D}}_{-t}(L_x\phi)\right)\right), y\right\rangle_{\tau} \\
&= \left\langle L^{-1}\left(\phi\left(\frac{d}{dt}\bigg|_{t=0,\mathrm{w}} \mathrm{Ad}^{\mathscr{D}}_{-t}(L_x\phi)\right)\phi\right), y\right\rangle_{\tau}
\end{aligned}$$

for all $t \in \mathbb{R}$ and $x, y \in A_0$. Since $A_0 \subset L^2(A,\tau)$ is $\|.\|_{\tau}$-dense, we see the above calculation implies Equation 2.195. Hence A_0 lies in domain of the adjoint. Locality follows by 1.1) in Definition 2.3.42. We have 2) and 3). They imply 4) and therefore Equation 2.196. □

Definition 2.3.45. Let (A,τ) be a tracial AF-C^*-algebra. For all generators $(\mathscr{D},\phi)$ of dynamic quantum gradients, we call

1) $\left(\phi, \mathrm{id}_A, \gamma^{\phi}\right)$ the ϕ-intertwined AF-A-bimodule structure on A,

2) $\nabla^{\mathscr{D},\phi}$ the dynamic quantum gradient generated by $(\mathscr{D},\phi)$,

3) $\nabla^{\mathscr{D},\phi}$ non-twisted if $\phi = \mathrm{id}_A$, else twisted.

Notation 2.3.46. We suppress ϕ in Definition 2.3.42 and Definition 2.3.45, as well as all objects in Proposition 2.3.44, if $\phi = \mathrm{id}_A$ or $\mathscr{D} = \mathscr{D}_{\phi}$ as per 2) in Definition 2.3.53.

Definition 2.3.47 gives trace-preserving local C^*-dynamical systems. Lemma 2.3.48 yields canonical extensions to conjugation groups. Corollary 2.3.49, by considering such extensions, gives non-twisted dynamic quantum gradients by norm differentiation of trace-preserving local C^*-dynamical systems. Note Remark 3.1.57.

Definition 2.3.47. Let $I \in \{\mathbb{R}, [0,\infty)\}$.

1) Let V be a Banach space. We say that a semigroup $G : I \longrightarrow \mathscr{B}(V)$ is strongly continuous if $x = \|.\|_V\text{-}\lim_{t\to 0} G_t(v)$ for all $v \in V$.

2) Let A be a C^*-algebra. Let $\mathrm{Aut}(A) \subset \mathscr{B}(A)$ be the automorphism group of A given by all *-isomorphisms. A C^*-dynamical system $(A,\mathbb{R},\alpha)$ is a strongly continuous group homomorphism $\alpha : \mathbb{R} \longrightarrow \mathrm{Aut}(A)$.

3) Let (A,τ) be a tracial AF-C^*-algebra. We call a C^*-dynamical system $(A,\mathbb{R},\alpha)$

   3.1) τ-preserving if α_t is τ-preserving for all $t \in \mathbb{R}$,

   3.2) local if $\alpha_t(A_j) \subset A_j$ for all $t \in \mathbb{R}$ and $j \in \mathbb{N}$.

Lemma 2.3.48. *For all τ-preserving local C^*-dynamical systems $(A,\mathbb{R},\alpha)$, there exists unique $\mathscr{D}_\alpha \in \mathscr{U}\mathscr{B}(L^2(A,\tau))_h$ s.t. $L \circ \alpha_t = \mathrm{Ad}_t^{\mathscr{D}_\alpha} \circ L$ for all $t \in \mathbb{R}$.*

Proof. Let $(A,\mathbb{R},\alpha)$ be a τ-preserving local C^*-dynamical system. We extend to strongly continuous unitary group on $L^2(A,\tau)$ s.t. Stone's theorem implies our claim. Let $t \in \mathbb{R}$. For all $u,v \in A_0$, get $\alpha_t(u), \alpha_t(v) \in L^2(A,\tau)$ by locality, as well as

$$\langle \alpha_t(u), \alpha_t(v) \rangle_\tau^2 = \tau\big(\alpha_t(u)^* \alpha_t(v)\big) = \tau\big(\alpha_t\big(u^* v\big)\big) = \langle u, v \rangle_\tau^2 \tag{2.197}$$

by the *-homomorphism property and τ-preservation. Using $A_0 \subset L^2(A,\tau)$ $\|.\|_\tau$-dense and Equation 2.197, we extend to $\alpha_t \in \mathscr{U}(\mathscr{B}(L^2(A,\tau)))$ here. We have unitary group $\alpha : \mathbb{R} \longrightarrow \mathscr{U}(\mathscr{B}(L^2(A,\tau)))$. We show its strong continuity. For all $j \in \mathbb{N}$, set $\alpha_t^j := \alpha_t|_{A_j}$ for all $t \in \mathbb{R}$. Locality shows we have strongly continuous group $\alpha^j : \mathbb{R} \longrightarrow \mathrm{Aut}(A_j)$ w.r.t. $\|.\|_A$ in each case. Finite-dimensionality further implies strong continuity w.r.t. $\|.\|_\tau$.

For all $t \in \mathbb{R}$, Equation 2.197 shows $\alpha_t \in \mathscr{U}(\mathscr{B}(L^2(A,\tau)))$ is an isometry. Using the latter get uniform bounds, note 3) in Proposition 2.1.26 implies $\alpha : \mathbb{R} \longrightarrow \mathscr{U}(\mathscr{B}(L^2(A,\tau)))$ is strongly continuous since α^j is for all $j \in \mathbb{N}$. Stone's theorem yields unique generator $\mathscr{D}_\alpha \in \mathscr{U}\mathscr{B}(L^2(A,\tau))_h$ s.t.

$$\alpha_t = e^{it\mathscr{D}_\alpha} \tag{2.198}$$

for all $t \in \mathbb{R}$ (cf. Theorem 5.6.36 in [134]). For all $t \in \mathbb{R}$ and $x \in A$, get $L_{\alpha_t(x)} = \alpha_t L_x \alpha_t^*$ by the *-homomorphism property. Using $A_0 \subset L^2(A,\tau)$ $\|.\|_\tau$-dense, Equation 2.198 then implies our claim. Altogether, our proof is extension of invariant C^*-dynamical systems in our special case (cf. Proposition 7.4.12 [173]). □

Corollary 2.3.49. *Let (A,τ) be a tracial AF-C^*-algebra. For all τ-preserving local C^*-dynamical systems $(A,\mathbb{R},\alpha)$, $(\mathscr{D}_\alpha,\mathrm{id}_A)$ is a generator of dynamic quantum gradient and we have*

1) *quantum gradient $\nabla^{\mathscr{D}_\alpha}: A_0 \longrightarrow L^2(A,\tau)$ given by*

$$\nabla^{\mathscr{D}_\alpha} x = i[\mathscr{D}_\alpha, x]_A = iL^{-1}\left(\overline{\mathscr{D}_\alpha L_x - L_x \mathscr{D}_\alpha}\right) \tag{2.199}$$

for all $x \in A_0$,

2) *$\nabla^{\mathscr{D}_\alpha,*} = \left(\nabla^{\mathscr{D}_\alpha}\right)^*$ with core A_0 and determined by*

$$\nabla^{\mathscr{D}_\alpha,*} x = -\nabla^{\mathscr{D}_\alpha} x = -i[\mathscr{D}_\alpha, x]_A \tag{2.200}$$

for all $x \in A_0$,

3) *$\Delta^{\mathscr{D}_\alpha} = \nabla^{\mathscr{D}_\alpha,*}\nabla^{\mathscr{D}_\alpha}$ with core A_0 and determined by*

$$\Delta^{\mathscr{D}_\alpha} x = -\left(\nabla^{\mathscr{D}_\alpha}\right)^2(x) = [\mathscr{D}_\alpha, [\mathscr{D}_\alpha, x]_A]_A \tag{2.201}$$

for all $x \in A_0$.

Proof. Let $j \in \mathbb{N}$ and $x \in A_j$. Note we use locality to define strongly continuous group $\alpha^j : \mathbb{R} \longrightarrow \mathrm{Aut}(A_j)$ in the proof of Lemma 2.3.48. It is local and τ-preserving since we have finite tracial AF-C^*-algebra (A_j,τ) as per Example 2.1.21. Applying Lemma 2.3.48 to (A_j,τ) and α^j yields $\mathscr{D}_j \in \mathscr{B}(A_j)_h$ s.t.

$$L_{\alpha_t^j(x),A_j} = \mathrm{Ad}_t^{\mathscr{D}_j}\left(L_{x,A_j}\right) \tag{2.202}$$

for all $t \in \mathbb{R}$. The conjugation group $\mathrm{Ad}^{\mathscr{D}_j} : \mathbb{R} \longrightarrow \mathrm{Aut}(\mathscr{B}(A_j))$ is norm differentiable at zero for all $S \in \mathscr{B}(A_j)$. Thus locality and Equation 2.202 imply

$$\left.\frac{d}{dt}\right|_{t=0,\|.\|_A} \alpha_t(x) = \left.\frac{d}{dt}\right|_{t=0,\|.\|_A} \alpha_t^j(x) = iL_{A_j}^{-1}\left(\mathscr{D}_j L_{x,A_j} - L_{x,A_j}\mathscr{D}_j\right) \in A_j. \tag{2.203}$$

Weak, strong and norm differentiation coincide in the finite-dimensional setting. Using normality of canonical left-actions, Lemma 2.3.48 and Equation 2.203 show

$$\left.\frac{d}{dt}\right|_{t=0,\mathrm{w}} \mathrm{Ad}_t^{\mathscr{D}_\alpha}(L_x) = L\left(\left.\frac{d}{dt}\right|_{t=0,\|.\|_A} \alpha_t(x)\right) \in L(A_j). \tag{2.204}$$

Since $\alpha_t^* = \alpha_{-t}$ for all $t \in \mathbb{R}$, Lemma 2.3.48 and Equation 2.204 imply

$$\frac{d}{dt}\bigg|_{t=0,\mathrm{w}} \mathrm{Ad}_{-t}^{\mathscr{D}_\alpha}(L_x) = -L\left(\frac{d}{dt}\bigg|_{t=0,\|.\|_A} \alpha_t(x)\right) \in L(A_j). \tag{2.205}$$

Equation 2.204 and Equation 2.205 imply 1.1) in Definition 2.3.42 at once. Using $\alpha_t^* = \alpha_{-t}$ for all $t \in \mathbb{R}$ and Lemma 2.3.48, note Equation 2.204 and Equation 2.205 show Equation 2.191 is given by

$$\big\langle x, \frac{d}{dt}\bigg|_{t=0,\|.\|_A} \alpha_{-t}(y)\big\rangle_\tau = \big\langle \frac{d}{dt}\bigg|_{t=0,\|.\|_A} \alpha_t(x), y\big\rangle_\tau \tag{2.206}$$

for all $x, y \in A_0$. Equation 2.206 shows 1.2) in Definition 2.3.42. We therefore have 1) in Definition 2.3.42, i.e. $(\mathscr{D}_\alpha, \mathrm{id}_A)$ is a generator of dynamic quantum gradient. Apply Proposition 2.3.44 to $(\mathscr{D}_\alpha, \mathrm{id}_A)$. Equation 2.194 shows Equation 2.199. Equation 2.199 and Equation 2.206 show Equation 2.200 and Equation 2.201. Get 1) to 3). □

Following identities in Lemma 2.3.55, Corollary 2.3.56 gives twisted and non-twisted dynamic quantum gradients by using intertwining self-adjoint unbounded operators as generators of twisted conjugation groups.

Definition 2.3.50 gives necessary local and strongly local properties underlying both twisted and non-twisted dynamic quantum gradients. Proposition 2.3.52 collects several implied properties, in particular splitting of induced semigroups as per Equation 2.209 applicable to heat semigroups of their quantum Laplacians, used in our discussion.

Definition 2.3.50. Let (A, τ) be a tracial AF-C^*-algebra and $T \in \mathscr{UB}(L^2(A,\tau))_h$.

1) We say that T is local if $A_0 \subset \operatorname{dom} T$ and $T(A_j) \subset A_j$ for all $j \in \mathbb{N}$. We say that T is strongly local if for all $j \in \mathbb{N}$ and $x \in A_j$, we have

$$T\pi_j^A L_x \in L(A_j),\ T\pi_j^\perp L_x = 0. \tag{2.207}$$

2) Let T be local. For all $j \in \mathbb{N}$, set $T_j := \mathrm{com}_{A_j} T$ and $T_j^\perp := \mathrm{com}_{A_j^\perp} T$.

Remark 2.3.51. Let (A, τ) be a tracial AF-C^*-algebra and $T \in \mathscr{UB}(L^2(A,\tau))_h$ strongly local. For all $j \in \mathbb{N}$ and $u \in A_j$, $T\pi_j^A L_u \in L(A_j)$ implies $u \in \operatorname{dom} T$ and

$$T(u) = T\big(\pi_j^A(u)\big) = T\big(\pi_j^A(u 1_{A_j})\big) = \big(T\pi_j^A L_u\big)(1_{A_j}) \in A_j. \tag{2.208}$$

Equation 2.208 shows T is local. Therefore, strongly local implies local.

Proposition 2.3.52. *Let (A,τ) be a tracial AF-C^*-algebra and $T \in \mathscr{UB}(L^2(A,\tau))_h$.*

1) *T is local if and only if $T : (\operatorname{dom} T, \|.\|_T) \longrightarrow L^2(A,\tau)$ has orthonormal eigenbasis $\{e_n\}_{n\in\mathbb{N}} \subset A_0$ s.t. it is furthermore orthonormal eigenbasis of π_j^A for all $j \in \mathbb{N}$.*

2) *Let $T \in \mathscr{UB}\big(L^2(A,\tau)\big)_h$ be local.*

   2.1) *T has core A_0. For all $j \in \mathbb{N}$, get $T \in \mathscr{UB}_{A_j}\big(L^2(A,\tau)\big)$.*

   2.2) *For all $t \in \mathbb{R}$ and $j \in \mathbb{N}$, we have*

$$e^{itT} = e^{itT_j} \oplus e^{itT_j^{\perp}} \tag{2.209}$$

   w.r.t. $\mathscr{B}(A_j) \oplus \mathscr{B}(A_j^{\perp})$.

3) *Let $T \in \mathscr{UB}\big(L^2(A,\tau)\big)_h$ be strongly local. For all $j \in \mathbb{N}$, $x \in A_j$ and $t \in \mathbb{R}$, we have*

   3.1) *$TL_x = T_jL_x \in L(A_j)$ and $L_xT = L_xT_j \in L(A_j)$,*

   3.2) *$e^{itT}L_x = e^{itT_j}L_x$ and $L_xe^{itT} = L_xe^{itT_j}$.*

Proof. We directly verify 1). Let T be local. Then 1) implies 2.1). Using reducibility as per 2.1), get 2.2) by 2) in Corollary A.2.27. Get 2). For all $j \in \mathbb{N}$ and $x \in A_j$, $[\pi_j^A, L_x] = 0$ by 1) in Proposition 2.2.51). Let T be strongly local. T is local by Remark 2.3.51. We see Equation 2.207 implies 3.1) by 1.3) in Proposition A.2.24 since we have reducibility. Moreover, Equation 2.207 and Equation 2.209 show 3.2). Get 3). □

Strong locality shows Equation 2.210 is well-defined by 3) in Proposition 2.3.52. We use Example 2.3.54 for sets of Clifford generators as per Definition 2.3.58.

Definition 2.3.53. Let (A,τ) be a tracial AF-C^*-algebra and $\phi : A \longrightarrow A$ a self-adjoint involutive local *-homomorphism. Let $\mathscr{D} \in \mathscr{UB}(L^2(A,\tau))_h$.

1) We say that $\mathscr{D}$ is ϕ-intertwining if

   1.1) $\mathscr{D}$ is strongly local,

   1.2) $\phi(\operatorname{dom} D) \subset \operatorname{dom}\mathscr{D}$, $\mathscr{D}\phi \neq 0$, and $\mathscr{D}\phi = \pm\phi\mathscr{D}$,

   1.3) for all $j \in \mathbb{N}$ and $x, y \in A_j$, we have

$$\big\langle x, L^{-1}\big(\mathscr{D}_j L_{\phi(y)} - L_y \mathscr{D}_j\big)\big\rangle_\tau = \big\langle L^{-1}\big(\operatorname{sgn}(\mathscr{D}) L_x \mathscr{D}_j - \mathscr{D}_j L_{\phi(x)}\big), y\big\rangle_\tau \ . \tag{2.210}$$

2) Let $\mathscr{D}$ be ϕ-intertwining. Let $\operatorname{sgn}(\mathscr{D}) \in \{\pm 1\}$ s.t. $\mathscr{D}\phi = \operatorname{sgn}(\mathscr{D})\phi\mathscr{D}$ be its sign. Its sign delta is $\delta(\mathscr{D}) := \delta_{-1}(\operatorname{sgn}(\mathscr{D})) \in \{0,1\}$. Set

$$\mathscr{D}_\phi := (-i)^{\delta(\mathscr{D})}\mathscr{D}\,\phi. \tag{2.211}$$

Example 2.3.54. Let (A,τ) be a tracial AF-C^*-algebra and $\phi : A \longrightarrow A$ a self-adjoint involutive local *-homomorphism. Let $d \in L^\infty(A,\tau)_h \setminus \{0\}$ s.t. L_d is strongly local and $\phi(d) = -d$. We show L_d is ϕ-intertwining s.t. $\operatorname{sgn}(L_d) = -1$.

We know 1.1) in Definition 2.3.53. Equation 2.190 implies $-L_d = L_{\phi(d)} = \phi L_d \phi$ and therefore 1.2) in Definition 2.3.53. We moreover have $\operatorname{sgn}(L_d) = -1$. Compressing as per Corollary 2.1.63 and using 1) in Proposition 2.2.51, we calculate

$$\begin{aligned}\left\langle x, L^{-1}\left(\pi_j^A L_d \pi_j^A L_{\phi(y)} - L_y \pi_j^A L_d \pi_j^A\right)\right\rangle_\tau &= \left\langle x, \left(L|_{A_j}\right)^{-1}\left(\pi_j^A (L_{d\phi(y)-yd})\pi_j^A\right)\right\rangle_\tau \\ &= \left\langle L^{-1}\left(-L_x \pi_j^A L_d \pi_j^A - \pi_j^A L_d \pi_j^A L_{\phi(x)}\right), y\right\rangle_\tau\end{aligned}$$

for all $j \in \mathbb{N}$ and $x, y \in A_j$. The above calculation shows Equation 2.210 in our case since $\operatorname{sgn}(L_d) = -1$. Thus 1.3) in Definition 2.3.53, hence our claim holds.

Lemma 2.3.55. *Let (A,τ) be a tracial AF-C^*-algebra and $\phi : A \longrightarrow A$ a self-adjoint involutive local *-homomorphism. For all ϕ-intertwining $\mathscr{D} \in \mathscr{UB}(L^2(A,\tau))_h$, $(\mathscr{D}_\phi, \phi)$ is a generator of dynamic quantum gradient and we have*

1) $[\mathscr{D}_\phi, x]_A^\phi = (-i)^{\delta(\mathscr{D})} L^{-1}\left(\overline{\mathscr{D} L_x - L_{\phi(x)} \mathscr{D}}\right)$,

2) $\{\mathscr{D}_\phi, x\}_A^\phi = -(-i)^{\delta(\mathscr{D})} L^{-1}\left(\overline{\operatorname{sgn}(\mathscr{D}) \mathscr{D} L_x - L_{\phi(x)} \mathscr{D}}\right)$,

3) $\left\{\mathscr{D}_\phi, [\mathscr{D}_\phi, x]_A^\phi\right\}_A^\phi = -L^{-1}\left(\overline{\mathscr{D}^2 L_x + L_x \mathscr{D}^2 - 2\mathscr{D} L_{\phi(x)} \mathscr{D}}\right)$,

for all $x \in A_0$.

Proof. Let $\mathscr{D} \in \mathscr{UB}(L^2(A,\tau))_h$ be ϕ-intertwining. Then 1.1) and 1.2) in Definition 2.3.53 imply $\mathscr{D}_\phi$ is strongly local since $[\pi_j^A, \phi] = 0$ for all $j \in \mathbb{N}$ by 3.1) in Proposition 2.1.40. In addition, $\phi(\operatorname{dom} \mathscr{D}_\phi) \subset \operatorname{dom} \mathscr{D}_\phi$ and $\mathscr{D}_\phi \phi = \operatorname{sgn}(\mathscr{D}) \phi \mathscr{D}_\phi$.

Let $j \in \mathbb{N}$ and $x \in A_j$. Note $[\pi_j^A, \phi] = 0$. Set

$$\mathscr{D}_{\phi,j} := \operatorname{com}_{A_j} \mathscr{D}_\phi = (-i)^{\delta(\mathscr{D})} \mathscr{D}_j \phi,\ \mathscr{D}_{\phi,j}^\perp := \operatorname{com}_{A_j^\perp} \mathscr{D}_\phi = (-i)^{\delta(\mathscr{D})} \mathscr{D}_j^\perp \phi. \tag{2.212}$$

The bounded operators in Equation 2.212 are those in Proposition 2.3.52 for $\mathscr{D}_\phi$. Using 3.2) in Proposition 2.3.52 and $L_{\phi(x)} = \phi L_x \phi$, the first identity in Equation 2.212 shows

$$\operatorname{Ad}_t^{\mathscr{D}_\phi}(L_x) = \operatorname{Ad}_t^{\mathscr{D}_{\phi,j}}(L_x) \tag{2.213}$$

and

$$\operatorname{Ad}_{-t}^{\mathscr{D}_\phi}\left(L_x \phi\right) = \operatorname{Ad}_{-t}^{\mathscr{D}_{\phi,j}}\left(L_x \phi\right) \tag{2.214}$$

for all $t \in \mathbb{R}$.

Equation 2.213 shows

$$\frac{d}{dt}\bigg|_{t=0,\mathrm{w}} \mathrm{Ad}_t^{\mathscr{D}_\phi}(L_x) = \overline{i\big(\mathscr{D}_\phi L_x - L_x \mathscr{D}_\phi\big)} = i\big(\mathscr{D}_{\phi,j} L_x - L_x \mathscr{D}_{\phi,j}\big), \tag{2.215}$$

whereas Equation 2.214 shows

$$\frac{d}{dt}\bigg|_{t=0,\mathrm{w}} \mathrm{Ad}_{-t}^{\mathscr{D}_\phi}\big(L_x\phi\big) = \overline{i\big(L_x\phi\mathscr{D}_\phi - \mathscr{D}_\phi L_x\phi\big)} = i\big(L_x\phi\mathscr{D}_{\phi,j} - \mathscr{D}_{\phi,j} L_x\phi\big). \tag{2.216}$$

Using 3.1) in Proposition 2.3.52 and $L_{\phi(x)} = \phi L_x \phi$, the first identity in Equation 2.212 further shows

$$\mathscr{D}_{\phi,j} L_x - L_x \mathscr{D}_{\phi,j} = (-i)^{\delta(\mathscr{D})} \operatorname{sgn}(\mathscr{D}) \phi\big(\mathscr{D}_j L_x - L_{\phi(x)} \mathscr{D}_j\big) \in \phi L(A_j) \tag{2.217}$$

and

$$L_x\phi\mathscr{D}_{\phi,j} - \mathscr{D}_{\phi,j} L_x\phi = (-i)^{\delta(\mathscr{D})}\big(\operatorname{sgn}(\mathscr{D}) L_x \mathscr{D}_j - \mathscr{D}_j L_{\phi(x)}\big) \in L(A_j). \tag{2.218}$$

Equation 2.215 and Equation 2.217 in turn show

$$\frac{d}{dt}\bigg|_{t=0,\mathrm{w}} \mathrm{Ad}_t^{\mathscr{D}_\phi}(L_x) = (-i)^{\delta(\mathscr{D})} \operatorname{sgn}(\mathscr{D}) \phi\big(\mathscr{D}_j L_x - L_{\phi(x)} \mathscr{D}_j\big) \in \phi L(A_j), \tag{2.219}$$

whereas Equation 2.216 and Equation 2.218 show

$$\frac{d}{dt}\bigg|_{t=0,\mathrm{w}} \mathrm{Ad}_{-t}^{\mathscr{D}_\phi}\big(L_x\phi\big) = (-i)^{\delta(\mathscr{D})}\big(\operatorname{sgn}(\mathscr{D}) L_x \mathscr{D}_j - \mathscr{D}_j L_{\phi(x)}\big) \in L(A_j). \tag{2.220}$$

Equation 2.219 and Equation 2.220 imply 1.1) in Definition 2.3.42 at once. Using Equation 2.219 for the first, Equation 2.210 for the second, and finally Equation 2.220 for the third identity below, we calculate

$$\begin{aligned}
\left\langle x, L^{-1}\left(\frac{d}{dt}\bigg|_{t=0,\mathrm{w}} \mathrm{Ad}_t^{\mathscr{D}_\phi}(L_y)\phi\right)\right\rangle_\tau &= (-i)^{\delta(\mathscr{D})}\big\langle x, L^{-1}\big(\operatorname{sgn}(\mathscr{D})\phi\big(\mathscr{D}_j L_y - L_{\phi(y)}\mathscr{D}_j\big)\phi\big)\big\rangle_\tau \\
&= (-i)^{\delta(\mathscr{D})}\big\langle L^{-1}\big(\operatorname{sgn}(\mathscr{D}) L_x \mathscr{D}_j - \mathscr{D}_j L_{\phi(x)}\big), y\big\rangle_\tau \\
&= \left\langle L^{-1}\left(\frac{d}{dt}\bigg|_{t=0,\mathrm{w}} \mathrm{Ad}_{-t}^{\mathscr{D}_\phi}\big(L_x\phi\big)\right), y\right\rangle_\tau
\end{aligned}$$

for all $j \in \mathbb{N}$ and $x, y \in A_j$. The above calculation shows 1.2) in Definition 2.3.42. We therefore have 1) in Definition 2.3.42, i.e. $(\mathscr{D}_\phi, \phi)$ is a generator of dynamic quantum gradient. Apply Proposition 2.3.44 to $(\mathscr{D}_\phi, \phi)$.

Using Equation 2.192, we directly verify 1) and 2). We show 3). Note $A_0 \subset \mathrm{dom}\mathscr{D}^2$ by locality. For all $x, u \in A_0$, we have $xu \in \mathrm{dom}\mathscr{D}^2$. Using Equation 2.192, strong locality and $(-i)^{2\delta(\mathscr{D}_\phi)} = \mathrm{sgn}(\mathscr{D})$, we apply 1) and 2) in each finite-dimensional case to get

$$L\Big(\{\mathscr{D}_\phi, [\mathscr{D}_\phi, x]^\phi_A\}^\phi_A\Big)(u) = -\Big(\mathscr{D}_j^2 L_x + L_x\mathscr{D}_j^2 - 2\mathscr{D}_j L_{\phi(x)}\mathscr{D}_j\Big)(u) \tag{2.221}$$

for all $j \in \mathbb{N}$, $x \in A_j$ and $u \in A_0$. Using strong locality, Equation 2.221 shows

$$L\Big(\{\mathscr{D}_\phi, [\mathscr{D}_\phi, x]^\phi_A\}^\phi_A\Big)(u) = -\Big(\pi^A_k\big(\mathscr{D}^2 L_x + L_x\mathscr{D}^2 - 2\mathscr{D} L_{\phi(x)}\mathscr{D}\big)\pi^A_k\Big)(u) \tag{2.222}$$

for all $j \le k$ in $\mathbb{N}$, $x \in A_j$ and $u \in A_0$. For fix but arbitrary $u \in A_0$, π^A_k on the right-hand side of the inner bracket in Equation 2.222 vanishes without loss of generality. Using 3) in Proposition 2.1.26, letting $k \uparrow \infty$ in Equation 2.222 yields

$$L\Big(\{\mathscr{D}_\phi, [\mathscr{D}_\phi, x]^\phi_A\}^\phi_A\Big)(u) = -\big(\mathscr{D}^2 L_x + L_x\mathscr{D}^2 - 2\mathscr{D} L_{\phi(x)}\mathscr{D}\big)(u) \tag{2.223}$$

for all $x \in A_j$ and $u \in A_0$. Note the left-hand side of Equation 2.223 evaluates a bounded operator. Since $A_0 \subset L^2(A,\tau)$ is $\|.\|_\tau$-dense, the right-hand side of Equation 2.222 in fact evaluates a bounded and closable operator defined on $\mathrm{dom}\mathscr{D}^2$. Get 3) by closure. □

Corollary 2.3.56. *Let (A,τ) be a tracial AF-C^*-algebra and $\phi : A \longrightarrow A$ a self-adjoint involutive local *-homomorphism. For all ϕ-intertwining $\mathscr{D} \in \mathscr{UB}(L^2(A,\tau))_h$, $(\mathscr{D}_\phi, \phi)$ is a generator of dynamic quantum gradient and we have*

1) *quantum gradient $\nabla^{\mathscr{D}_\phi} : A_0 \longrightarrow L^2(A,\tau)$ given by*

$$\nabla^{\mathscr{D}_\phi} x = i[\mathscr{D}_\phi, x]^\phi_A = i(-i)^{\delta(\mathscr{D})} L^{-1}\Big(\overline{\mathscr{D} L_x - L_{\phi(x)}\mathscr{D}}\Big) \tag{2.224}$$

for all $x \in A_0$,

2) *$\nabla^{\mathscr{D}_\phi,*} = \big(\nabla^{\mathscr{D}_\phi}\big)^*$ with core A_0 and determined by*

$$\nabla^{\mathscr{D}_\phi,*} x = i\{\mathscr{D}_\phi, x\}^\phi_A = -i(-i)^{\delta(\mathscr{D})} L^{-1}\Big(\overline{\mathrm{sgn}(\mathscr{D})\mathscr{D} L_x - L_{\phi(x)}\mathscr{D}}\Big) \tag{2.225}$$

for all $x \in A_0$,

3) *$\Delta^{\mathscr{D}_\phi} = \nabla^{\mathscr{D}_\phi,*}\nabla^{\mathscr{D}_\phi}$ with core A_0 and determined by*

$$\Delta^{\mathscr{D}_\phi} x = -\big\{\mathscr{D}_\phi, [\mathscr{D}_\phi, x]^\phi_A\big\}^\phi_A = L^{-1}\Big(\overline{\mathscr{D}^2 L_x + L_x\mathscr{D}^2 - 2\mathscr{D} L_{\phi(x)}\mathscr{D}}\Big) \tag{2.226}$$

for all $x \in A_0$.

Proof. Apply Proposition 2.3.44 and Lemma 2.3.55. □

Corollary 2.3.57. *Let (A,τ) be a tracial AF-C^*-algebra and $\phi : A \longrightarrow A$ a self-adjoint involutive local *-homomorphism. Let $\mathscr{D} \in \mathscr{UB}(L^2(A,\tau))_h$ be ϕ-intertwining. For all $j \in \mathbb{N}$ and $x \in A_j$, we have*

1) $\nabla^{\mathscr{D}_\phi} x = i(-i)^{\delta(\mathscr{D})} L^{-1}\big(\mathscr{D}_j L_x - L_{\phi(x)}\mathscr{D}_j\big)$,
2) $\nabla^{\mathscr{D}_\phi,*} x = -i(-i)^{\delta(\mathscr{D})} L^{-1}\big(\mathrm{sgn}(\mathscr{D})\mathscr{D}_j L_x - L_{\phi(x)}\mathscr{D}_j\big)$,
3) $\Delta^{\mathscr{D}_\phi} x = L^{-1}\Big(\mathscr{D}_j^2 L_x + L_x \mathscr{D}_j^2 - 2\mathscr{D}_j L_{\phi(x)}\mathscr{D}_j\Big)$.

Proof. Apply Corollary 2.3.56 in each finite-dimensional case. □

Definition 2.3.58 gives intertwining sets of Clifford generators. In the logarithmic mean setting, Example 4.3.20 shows their direct sum quantum gradients yield strictly positive lower Ricci bounds. This requires Lemma 2.3.59.

Definition 2.3.58. Let (A,τ) be a tracial AF-C^*-algebra and $\phi : A \longrightarrow A$ a self-adjoint involutive local *-homomorphism. Let $m \in \mathbb{N}$ and $\{d_n\}_{n=1}^m \subset L^\infty(A,\tau)_h$.

1) We say that $\{d_n\}_{n=1}^m$ is a ϕ-intertwining set of Clifford generators for $C > 0$ if for all $n,k \in \{1,\dots,m\}$, we have
   1.1) L_{d_n} strongly local and $\phi(d_n) = -d_n$,
   1.2) $d_n d_k + d_k d_n = 2C\delta_{nk} 1_A$.
2) Let $\{d_n\}_{n=1}^m$ be a ϕ-intertwining set of Clifford generators for $C > 0$ as above. For all $n \in \{1,\dots,m\}$, set $\partial_n := \nabla^{-iL_{d_n},\phi}$ and $\Delta_n := \Delta^{-iL_{d_n},\phi}$.

Lemma 2.3.59. *Let (A,τ) be a tracial AF-C^*-algebra, $\phi : A \longrightarrow A$ a self-adjoint involutive local *-homomorphism and $m \in \mathbb{N}$. If $\{d_n\}_{n=1}^m \subset L^\infty(A,\tau)_h$ is a ϕ-intertwining set of Clifford generators for $C > 0$, then*

$$\partial_n \Delta_k = \big(\Delta_k + \delta_{nk} 4C \cdot I\big)\partial_n \tag{2.227}$$

for all $n,k \in \{1,\dots,m\}$.

Proof. Let $\{d_n\}_{n=1}^m \subset L^\infty(A,\tau)_h$ be a ϕ-intertwining set of Clifford generators for $C > 0$. Lemma C.1.1 gives three identities we use in this proof. If $n,k \in \{1,\dots,m\}$ s.t. $n \neq k$, then we see Equation C.1 and Equation C.2 let us calculate

$$\partial_n \Delta_k = \partial_n \partial_k^* \partial_k = (-1)^2 \cdot \Delta_k \partial_n = \Delta_k \partial_n. \tag{2.228}$$

Equation C.1 implies $\partial_n^2 = 0$ in each case. If $n = k$, then we see Equation C.3 together with $\partial_n^2 = 0$ lets us calculate

$$\partial_n \Delta_n = 4C\partial_n = \big(\Delta_n + 4C \cdot I\big)\partial_n. \tag{2.229}$$

Equation 2.228 and Equation 2.229 show Equation 2.227. □

2.3.3 Noncommutative differential structures and compatibility

Noncommutative differential structures collect the data which define quantum optimal transport distances. Each consists of two components and one setting. The data collected is compatible with compression and finite-dimensional approximation. These are two general operations we formalise in a coarse graining process.

The notion of noncommutative differential structure. This chapter provides all necessary data. Definition 2.3.60 gives noncommutative differential structures. We explain our notions of compression and finite-dimensional approximation, as well as compatibility with either. To this end, we use the terms *noncommutative* and *quantum* in our discussion as means to distinguish classes of objects as per Figure 2.1.

We further explain the data for 1) in Definition 2.3.60 satisfies such compatibility by construction. In Subsection 3.1.1 and Subsection 3.1.2, we show compatibility transfers to quantum optimal transport. In Subsection 3.3.2, we then formalise compatibility in the coarse graining process as per Diagram 3.346. This completes our explanation.

Definition 2.3.60. Let (A,τ) and (B,ω) be tracial AF-C^*-algebras. Let (ϕ,ψ,γ) be an AF-A-bimodule structure on B. Let f be symmetric representing function of an operator mean and $\theta \in [0,1]$ s.t. $\|\omega\|^{1-\theta} < \infty$. Let $\nabla : A_0 \longrightarrow L^2(B,\omega)$ be a quantum gradient.

1) We call $(\phi,\psi,\gamma,\nabla)$ noncommutative differential structure for (A,τ) and (B,ω) in (f,θ)-setting.

2) For all $j \in \mathbb{N}$, we consider the induced AF-A_j-bimodule structure (ϕ_j,ψ_j,γ_j) on B_j as per 4) in Definition 2.1.46 together with the j-th restricted quantum gradient $\nabla_j : A_j \longrightarrow B_j$ as per 2) in Definition 2.3.23 and call $\big(\phi_j,\psi_j,\gamma_j,\nabla_j\big)$ the induced noncommutative differential structure for (A_j,τ) and (B_j,ω) in (f,θ)-setting.

Remark 2.3.61. Definition 2.3.60 is motivated by Definition 4.7 in [50]. The latter uses absolutely continuous finite weights [193] w.r.t. a given finite trace. Proposition 4.12 in [50] shows a detailed balance condition for Laplacians. We see [50] generalises [152]. Yet the detailed balance condition as per Proposition 4.12 in [50] implies ergodicity of the given noncommutative heat semigroup. As such, Definition 4.7 in [50] assumes the ergodic finite-dimensional setting but not traciality, whereas Definition 2.3.60 assumes it but allows for infinite dimensions, possibly non-finite traces, as well as non-ergodicity of noncommutative heat semigroups. We account for these differences.

	Commutative	Not Commutative
Quantum	$C_0(\mathbb{N})$	$\mathscr{K}(\ell^2(\mathbb{N}))$
Not Quantum	$C_0(\mathbb{R})$	$C_0(\mathbb{R}) \otimes \mathscr{K}(\ell^2(\mathbb{N}))$

Figure 2.1: Matrix for example C^*-algebras decomposing the noncommutative setting according to commutativity and inclusion in the AF-C^*-setting. The noncommutative setting subsumes the commutative and properly noncommutative one. Note all function spaces use elements evaluating in complex numbers and vanishing at infinity.

We use the two terms *noncommutative* and *quantum* in our discussion as means to distinguish classes of objects as per Figure 2.1. The former denotes objects in the full noncommutative setting, in particular the AF-C^*-setting. The latter denotes objects in the AF-C^*-setting compatible with compression and finite-dimensional approximation. Note tracial AF-C^*-algebras generating hyperfinite factors of type I and II by σ-weak closure, i.e. Example 3.1.55, Example 3.1.56 and Example 3.1.58, are common algebras of observables in quantum statistical mechanics [35][36][162].

We use the above to explain compression and finite-dimensional approximation, as well as compatibility with either. For compression, we apply compression maps to tracial AF-C^*-algebras as per Remark 2.1.8. It acts on and yields objects and properties in the noncommutative setting. For finite-dimensional approximation, we apply restriction maps, possibly up to rescaling as per 1) in Definition 3.1.12, to tracial AF-C^*-algebras as per Definition 2.1.30. It acts on objects and properties in the AF-C^*-setting and yields description of these as limits of restricted analogues in the finite-dimensional setting. If we have notions of compression and finite-dimensional approximation for a class of objects or properties, which we give explicitly for each use case in our discussion, then we say such a class is compatible with both. We use compression and finite-dimensional approximation for the coarse graining process as per Diagram 3.346. This demands data compatible with both. The data for 1) in Definition 2.3.60 satisfies such compatibility by their locality properties. The coarse graining process, hence compatibility, is essential for our discussion because it reduces the AF-C^*-setting to the finite-dimensional one s.t. ergodicity is recovered up to a controlled remainder. Note Remark 2.3.61.

3 | Quantum Optimal Transport

Quantum optimal transport is described using dynamic transport distances of states on tracial AF-C^*-algebras. Noncommutative differential structures collect the data which define such dynamic transport distances. First, quantum gradients define continuity equations for states on tracial AF-C^*-algebras. Continuity equations in turn define sets of admissible paths. Secondly, quasi-entropies define energy functionals by integrating their own evaluation on admissible paths. Minimising square roots of energy functionals over all admissible paths for fixed marginals defines dynamic transport distances, called quantum optimal transport distances. This follows the classical case [97]. We show our construction extends the discrete cases [152][159], as well as tracial finite-dimensional ones in [48][49][50]. We provide fundamental example classes. The latter themselves yield quantum optimal transport of normal states on hyperfinite factors of type I and II [173]. An application is given by first and second quantisation of spectral triples [54][55] [197][198]. This yields our ansatz to study noncommutative gauge theories based on a proposed internalised spectral action [51][52][53][197][198].

However, we defer a detailed discussion to future work as it requires generalisation to dynamic transport distances of states on continuous fields of AF-C^*-algebras. We still view quantum optimal transport as the pointwise case of a general parametrised one since this strongly motivates non-spatiality. First quantisation considers commutative spectral triples, i.e. first quantisation of compact spin manifolds [68]. We show quantum optimal transport is transversal to spatial optimal transport in this case. Second quantisation rectifies this by quantising all spatial coordinates. We apply a characterisation in [55] to obtain sufficient conditions s.t. the quantum gradients used are infinitesimal evolution of observables at thermal equilibrium determined by KMS-states [36]. Each assumes fixed gauge field [51][197][198]. Varying von Neumann entropy [163] of such KMS-states w.r.t. the canonical trace yields description of the spectral action on gauge fields [51][52][53] in terms of quantum statistical mechanics [35][36] using quantum relative entropy [55]. Upon passing to second quantisation, we introduce gauge fields as spatial coordinates. We consider it a model, and therefore expect several properties of quantum optimal transport: quantum gradients and thus continuity equations do not use spatial coordinates, we have a description of quantum Laplacians in terms of quantum statistical mechanics, and non-ergodic noncommutative heat semigroups are the rule. We avoid spatial interpretations of the classical case [97][151], e.g. as mass transport [8][199], but do require an alternative one for quantum optimal transport.

The coarse graining process provides such an alternative as it lets us view quantum optimal transport as transport of, suitably general, quantum information. We transport scaling limits of uniformly conditioned spin states encoding sequences of qubits. We avoid spatial interpretations because spin states have physical realisation [43][62][95] s.t. manipulation of encoded qubits does not consider spatial coordinates. We thereby have non-spatiality, as well as an immediate link to quantum statistical mechanics since information is physical [45][95][142][143]. This link ought to be noticeable if the given quantum system provides physical realisation of a quantum computer [18][62].

Non-ergodicity, defined as complex kernel dimension larger than one for quantum Laplacians, restricts information-bearing degrees of freedom. Since energy functionals are Γ-limits w.r.t. the coarse graining process, the latter reduces the AF-C^*-setting to the finite-dimensional one s.t. ergodicity is recovered up to a controlled remainder by reducing to accessibility components in the finite-dimensional setting. There may exist uncountable infinitely many since sets of states at finite distance have identical fixed parts under noncommutative heat semigroups of quantum Laplacians. Assuming spectral gaps of quantum Laplacians and fixed parts, we use such fixed parts to classify accessibility components of square integrable normal states. Altogether, we study a non-spatial transport of quantum information with restricted information-bearing degrees of freedom. In Chapter 4, we moreover obtain a description of quantum Laplacians in terms of both quantum statistical mechanics and quantum information theory.

Structure. In Section 3.1, we discuss quantum optimal transport distances given our noncommutative differential structures. We provide fundamental example classes. In Section 3.2, we review support projections of normal states, discuss our use of quantum Fokker-Planck equations, and subsequently study noncommutative heat semigroups of quantum Laplacians. Finally, we classify accessibility components of square integrable normal states. In Section 3.3, we explain the coarse graining process and use it to view quantum optimal transport as transport of quantum information.

3.1 Description using dynamic transport distances

Quantum optimal transport requires two notions. First, admissible paths determined by continuity equations. Secondly, energy functionals given by integrating quasi-entropies evaluated on admissible paths. Minimising square roots of energy functionals over all admissible paths for fixed marginals defines quantum optimal transport distances. We show existence of minimising geodesics. Energy functionals are Γ-limits if restricted to sets of admissible paths with identical interval and marginals, and therefore w.r.t. the coarse graining process. We formalise the latter as existence of sufficient minimising geodesics approximated in finite dimensions.

Structure. In Subsection 3.1.1, we use quasi-entropies to define energy functionals on admissible paths determined by continuity equations. In Subsection 3.1.2, we discuss quantum optimal transport distances, minimising geodesics and their approximation in finite dimensions. In Subsection 3.1.3, we provide all fundamental example classes. An application is given by first and second quantisation of spectral triples.

3.1.1 Energy functionals on admissible paths

Quantum gradients define continuity equations for states on tracial AF-C^*-algebras. Note each contains the codomain of the given quantum gradient. Continuity equations define sets of admissible paths. We formulate the latter using Banach dual spaces of Bochner L^2-spaces. Quasi-entropies define energy functionals by integrating their own evaluation on admissible paths. Altogether, we obtain energy functionals on admissible paths of states on tracial AF-C^*-algebras.

We use compression of quantum gradients and therefore continuity equations to show energy functionals are Γ-limits. Compressing to induced AF-C^*-bimodules yields energy functionals on admissible paths of states on generating C^*-subalgebras. We must initially extend inclusion and restriction maps for Banach dual spaces of tracial AF-C^*-algebras as per Definition 2.1.27 to sets of admissible paths. We then compress as above by restricting to induced AF-C^*-bimodules. Taking limits recovers the initial set of admissible paths. Using the latter, Theorem 3.1.31 shows energy functionals are Γ-limits if restricted to sets of admissible paths with identical interval and marginals. We thereby extend finite-dimensional approximation of quantum gradients to energy functionals. Standard reference for Bochner L^2-spaces and their Banach dual spaces is [129]. Standard reference for Γ-convergence of functionals is [74].

Banach dual spaces of Bochner $\mathbf{L^2}$-spaces. Bochner L^2-spaces have locally convex topological vector spaces as codomains of integration and are not reflexive in general [129]. We rectify this by considering w^*-topologies.

Let V be a separable Banach space.

Notation 3.1.1. Let $I \subset \mathbb{R}$ denote a closed interval. We commonly use $I = [a,b] \subset \mathbb{R}$.

We equip all closed intervals $I \subset \mathbb{R}$ with the Lebesgue measure. Radon measures are strictly localisable [170]. Theorem IV.5 in [129] therefore shows results in [129] used here apply. A map $h : I \longrightarrow V$ is Bochner measurable if and only if the map $t \mapsto \mu(h(t))$ is measurable for all $\mu \in V^*$. A map $g : I \longrightarrow V^*$ is w^*-measurable if and only if the map $t \mapsto g(t)(v)$ is measurable for all $v \in V$. Separability implies equivalence.

Definition 3.1.2. Let $I \subset \mathbb{R}$ be a closed interval.

1) Set $L^2(I,V) := \{h : I \longrightarrow V \mid \text{Bochner measurable, } \|h\|_V^2 \in L^1(I)\}$. We call $L^2(I,V)$ the Bochner L^2-space of functions from I to V. For all $h \in L^2(I,V)$, set

$$\|h\|_2 := \int_I \|h(t)\|_V^2 dt. \tag{3.1}$$

2) Set $L^2(I,V^*)_{\mathrm{w}} := \{g : I \longrightarrow V^* \mid w^*\text{-measurable, } \|g\|_{V^*}^2 \in L^1(I)\}$. We call $L^2(I,V^*)_{\mathrm{w}}$ the L^2-space of w^*-functions from I to V^*. For all $g \in L^2(I,V^*)_{\mathrm{w}}$, set

$$\|g\|_2 := \int_I \|g(t)\|_{V^*}^2 dt. \tag{3.2}$$

Proposition 3.1.3. *For all closed intervals $I \subset \mathbb{R}$, we have*

1) $\big(L^2(I,V), \|.\|_2\big)$ *and* $\big(L^2(I,V^*)_{\mathrm{w}}, \|.\|_2\big)$ *are Banach spaces,*
2) $L^2(I,V)^* = L^2(I,V^*)_{\mathrm{w}}$.

Proof. Let $I \subset \mathbb{R}$ be a closed interval. We use notation in [129]. Note $L^2(I,V) = L^2_{V'}$ and $L^2(I,V^*)_{\mathrm{w}} = L^2_{V'}[V]$ are Banach spaces. Get 1). For all $F \in L^2(I,V)^*$, Theorem VII.9 in [129] and its immediate corollary show there exists unique $g_F \in L^2(I,V^*)_{\mathrm{w}}$ s.t.

$$F(h) = \int_I g_F(t)(h(t))dt \tag{3.3}$$

for all $h \in L^2(I,V)$. Equation 3.1 and Equation 3.2 further imply

$$\|F\|_{L^2(I,V)^*} = \sup_{\substack{h \in L^2(I,V), \\ \|h\|_2 \le 1}} \left| \int_I g_F(t)(h(t))dt \right| = \int_I \|g_F(t)\|^2_{V^*} dt = \|g_F\|_2 \tag{3.4}$$

in each case. Therefore, $L^2(I,V)^* = L^2(I,V^*)_{\mathrm{w}}$. Get 2). □

Remark 3.1.4. Let V be a separable Banach space and $K \subset V^*$ norm bounded. Given $\{v_n\}_{n\in\mathbb{N}} \subset V\setminus\{0\}$ with $\|.\|_V$-dense linear span, set $d(\rho,\rho') := \sum_{n=1}^{\infty} 2^{-n}\|v_n\|_V^{-1}|\rho(v_n) - \rho'(v_n)|$ for all $\rho, \rho' \in K$. This defines a distance metricising the w^*-topology on K.

Admissible paths determined by continuity equations. Definition 3.1.5, in particular Equation 3.5, gives continuity equations. Definition 3.1.7 gives admissible paths determined by continuity equations. Admissible paths lie in state spaces of tracial AF-C^*-algebras as per Definition 2.1.11. Proposition 3.1.6 shows norm-preservation.

Let $(\phi,\psi,\gamma,\nabla)$ be noncommutative differential structure for tracial AF-C^*-algebras (A,τ) and (B,ω) in (f,θ)-setting.

Definition 3.1.5. Let $I \subset \mathbb{R}$ be a closed interval.

1) We say that $\mu : I \longrightarrow A_+^*$ is weakly absolutely continuous if $t \mapsto \mu(t)(x)$ is absolutely continuous for all $x \in A_0$.
2) Let $\mu : I \longrightarrow A_+^*$ be weakly absolutely continuous and $w \in L^2(I,B^*)_{\mathrm{w}}$. The pair (μ, w) satisfies the continuity equation for ∇ on I if

$$\frac{d}{dt}\mu(t)(x) = w(t)(\nabla x) \tag{3.5}$$

for all $x \in A_0$ and a.e. $t \in I$.

Proposition 3.1.6. *Let $\mu : I \longrightarrow A^*_+$ be weakly absolutely continuous and $w \in L^2(I, B^*)_{\mathrm{w}}$. If (μ, w) satisfies the continuity equation for ∇ on I, then*

$$\left\|\mu(t)\right\|_{A^*} = \left\|\mu(s)\right\|_{A^*} \tag{3.6}$$

for all $t, s \in I$.

Proof. For all $j \in \mathbb{N}$, note $\nabla 1_{A_j} = 0$ the Leibniz rule. Thus $\frac{d}{dr}\mu(r)(1_{A_j}) = 0$ for a.e. $r \in I$ for all $j \in \mathbb{N}$, hence

$$\mu(t)(1_{A_j}) = \mu(s)(1_{A_j}) \tag{3.7}$$

for all $t, s \in I$ and $j \in \mathbb{N}$. Set $\mu_j(t) := \mu(t)_j = \mu(t)|_{A_j}$ in each case. Positivity ensures

$$\left\|\mu_j(t)\right\|_{A^*} = \mu_j(t)(1_{A_j}) = \mu(t)(1_{A_j}) \tag{3.8}$$

in each case. Using 1.1) in Proposition 2.1.31, we see Equation 3.7 and Equation 3.8 imply Equation 3.6 at once. □

Definition 3.1.7. Let $\overline{\mathscr{S}(A)}$ denote the w^*-closure of $\mathscr{S}(A) \subset A^*_+$.

1) Let $I = [a, b] \subset \mathbb{R}$. Set

   1.1) $\mathrm{AC}(I, \overline{\mathscr{S}(A)}) := \{\mu : I \longrightarrow \overline{\mathscr{S}(A)} \mid \mu \text{ is weakly absolutely continuous}\}$,

   1.2) $\mathrm{AC}(I, \mathscr{S}(A)) := \{\mu \in \mathrm{AC}(I, \overline{\mathscr{S}(A)}) \mid \mathrm{im}\,\mu \subset \mathscr{S}(A)\}$.

2) We say that $(\mu, w) \in \mathrm{AC}([a,b], \mathscr{S}(A)) \times L^2([a,b], B^*)_{\mathrm{w}}$ is an admissible path if (μ, w) satisfies the continuity equation for ∇ on $[a, b]$. We further call $\mu(a), \mu(b) \in \mathscr{S}(A)$ the marginals of (μ, w), resp. μ.

3) For all $\mu^0, \mu^1 \in \mathscr{S}(A)$, let $\mathrm{Adm}^{[a,b]}(\mu^0, \mu^1)$ be the set of all admissible paths defined on $[a, b] \subset \mathbb{R}$ with marginals μ^0 and μ^1. Set

   3.1) $\mathrm{Adm}(\mu^0, \mu^1) := \bigcup_{[a,b] \subset \mathbb{R}} \mathrm{Adm}^{[a,b]}(\mu^0, \mu^1)$ for all $\mu^0, \mu^1 \in \mathscr{S}(A)$,

   3.2) $\mathrm{Adm}^{[a,b]} := \bigcup_{\mu^0, \mu^1 \in \mathscr{S}(A)} \mathrm{Adm}^{[a,b]}(\mu^0, \mu^1)$ for all $[a, b] \subset \mathbb{R}$,

   3.3) $\mathrm{Adm} := \bigcup_{[a,b] \subset \mathbb{R}} \bigcup_{\mu^0, \mu^1 \in \mathscr{S}(A)} \mathrm{Adm}^{[a,b]}(\mu^0, \mu^1)$.

Notation 3.1.8. For all $j \in \mathbb{N}$, we use Adm_j when denoting sets of admissible paths in Definition 3.1.7 for the induced noncommutative differential structure $(\phi_j, \psi_j, \gamma_j, \nabla_j)$.

Remark 3.1.9. Let $I \subset \mathbb{R}$ be a closed interval and $j \in \mathbb{N}$. We have $A_j \cong A_j^*$ and $B_j \cong B_j^*$ via musical isomorphisms and therefore

$$L^2(I, A_j^*)_{\mathrm{w}} \cong L^2(I, A_j),\ L^2(I, B_j^*)_{\mathrm{w}} \cong L^2(I, B_j). \tag{3.9}$$

Each Bochner L^2-space in Equation 3.9 is norm equivalent to the respective Hilbert space of square integrable functions. Up to musical isomorphisms applied to codomains of integration, each L^2-space of w^*-functions in Equation 3.9 is therefore likewise norm equivalent to such a Hilbert space of square integrable functions.

Definition 3.1.10 gives the canonical topology on sets of admissible paths alongside a related notion of convergence for the latter. Proposition 3.1.11 collects properties. Let $[a,b] \subset \mathbb{R}$. Since $\mathrm{AC}([a,b], \overline{\mathscr{S}(A)}) \subset L^2([a,b], A^*)_{\mathrm{w}}$ up to null sets, we obtain the canonical inclusion

$$\mathrm{Adm}^{[a,b]} \subset L^2([a,b], A^*)_{\mathrm{w}} \times L^2([a,b], B^*)_{\mathrm{w}}. \tag{3.10}$$

The relative topology on $\mathrm{Adm}^{[a,b]}$ w.r.t. the w^*-topology on $L^2([a,b], A^*)_{\mathrm{w}} \times L^2([a,b], B^*)_{\mathrm{w}}$ given by the above canonical inclusion is called the relative w^*-topology.

We define a second topology on $\mathrm{Adm}^{[a,b]}$ by equipping

$$\overline{\mathscr{S}(A)}^{[a,b]} := \prod_{t \in [a,b]} \overline{\mathscr{S}(A)} \tag{3.11}$$

with the product topology given by the w^*-topology on $\overline{\mathscr{S}(A)}$. Pointwise convergence in w^*-topology is convergence in the product topology. We further consider w^*-topology on $L^2([a,b], B)^* = L^2([a,b], B^*)_{\mathrm{w}}$ as per 2) in Proposition 3.1.3. The relative topology given by the canonical inclusion

$$\mathrm{Adm}^{[a,b]} \subset \overline{\mathscr{S}(A)}^{[a,b]} \times L^2([a,b], B^*)_{\mathrm{w}} \tag{3.12}$$

is called the canonical topology on $\mathrm{Adm}^{[a,b]}$.

Definition 3.1.10. For all $[a,b] \subset \mathbb{R}$, the relative topology as per Equation 3.12 is called the canonical topology on $\mathrm{Adm}^{[a,b]}$. We say that $(\mu^n, w^n)_{n \in \mathbb{N}} \subset \mathrm{Adm}^{[a,b]}$ converges to $(\mu, w) \in \mathrm{Adm}^{[a,b]}$ if

1.1) $\mu(t) = w^*\text{-}\lim_{n \in \mathbb{N}} \mu^n(t)$ in $\mathscr{S}(A)$ for all $t \in [a,b]$,

1.2) $w = w^*\text{-}\lim_{n \in \mathbb{N}} w^n$ in $L^2([a,b], B^*)_{\mathrm{w}}$.

We further write $(\mu, w) = \lim_{n \in \mathbb{N}} (\mu^n, w^n)$ in $\mathrm{Adm}^{[a,b]}$.

Proposition 3.1.11. *Let* $(\mu^n, w^n)_{n\in\mathbb{N}} \subset \mathrm{Adm}^{[a,b]}$.

1) *Let* $(\mu, w) \in \mathrm{AC}([a,b], \overline{\mathscr{S}(A)}) \times L^2([a,b], B^*)_{\mathrm{w}}$ *s.t.*

   1.1) $\mu(t) = w^*\text{-}\lim_{n\in\mathbb{N}} \mu^n(t)$ *for all* $t \in [a,b]$,

   1.2) $w = w^*\text{-}\lim_{n\in\mathbb{N}} w^n$.

   If there exists $t_0 \in [a,b]$ *s.t.* $\|\mu(t_0)\|_{A^*} = 1$, *then* $(\mu, w) \in \mathrm{Adm}^{[a,b]}$.

2) *If* $(\mu, w) = \lim_{n\in\mathbb{N}}(\mu^n, w^n)$ *in* $\mathrm{Adm}^{[a,b]}$, *then* $(\mu, w) = w^*\text{-}\lim_{n\in\mathbb{N}}(\mu^n, w^n)$.

Proof. We show 1). Assume its setting. For all $x \in A_0$, the map $t \mapsto g(t) := \nabla x$ defined on $[a,b]$ lies in $L^2([a,b], B)$ by locality if we identify as per Remark 3.1.9. For all $x \in A_0$ and $h \in (0,1)$, we apply the continuity equation in order to rewrite the difference quotient

$$\frac{1}{h}\big(\mu(t+h)(x) - \mu(0)(x)\big) = \lim_{n\in\mathbb{N}} \frac{1}{h}\int_0^{t+h} \frac{d}{ds} w^n(s)(\nabla x)ds = \frac{1}{h}\int_0^{t+h} w(s)(\nabla x)ds. \tag{3.13}$$

Letting $h \to 0$ in Equation 3.13 shows (μ, w) satisfies the continuity equation for ∇ on $[a,b]$. Proposition 3.1.6 shows norm-preservation. We see 1) at once. Moreover, standard arguments show 2) by dominated convergence. □

Definition 3.1.12 extends restriction maps in Definition 2.1.27 to all paths in Banach dual spaces of tracial AF-C^*-algebras. Proposition 3.1.14 further extends inclusion and restriction maps to sets of admissible paths. Restricting paths rescales norm.

Definition 3.1.12. Let $\mathscr{A}$ be a tracial AF-C^*-algebra, $I \subset \mathbb{R}$ a closed interval and $j \in \mathbb{N}$.

1) For all $\rho \in \mathscr{A}_+^*$, set

$$\bar{\rho}_j := \begin{cases} \rho(1_{\mathscr{A}_j})^{-1}\rho_j & \text{if } \rho(1_{\mathscr{A}_j}) \neq 0, \\ 0 & \text{else}. \end{cases}$$

2) Let $\rho : I \longrightarrow \mathscr{A}_+^*$ be defined for a.e. $t \in I$. We define $\rho_j : I \longrightarrow \mathscr{A}_j^*$ and $\bar{\rho}_j : I \longrightarrow \mathscr{A}_j^*$ by setting

$$\rho_j(t) := \rho(t)_j, \ \bar{\rho}_j(t) := \overline{\rho(t)}_j \tag{3.14}$$

   for a.e. $t \in I$.

Remark 3.1.13. For all $\rho \in \mathscr{A}_+^*$, we have $\|\rho_j\|_{A^*} = \rho(1_{A_j})$ for all $j \in \mathbb{N}$ by positivity. We obtain $\rho(1_{A_j}) \neq 0$ for a.e. $j \in \mathbb{N}$ by 1) in Proposition 2.1.31. For all $\mu \in \mathscr{S}(A)$, we have $\bar{\mu}_j \in \mathscr{S}(A_j)$ if and only if $\mu_j \neq 0$. We use this throughout our discussion.

Following Remark 3.1.15, we assume strictly positive norm for at least one marginal if we apply restriction maps as per Proposition 3.1.14. A more rigorous but cumbersome notation may further include marginals in sets of admissible paths.

Proposition 3.1.14. *For all* $[a,b] \subset \mathbb{R}$ *and* $j \le k$ *in* $\mathbb{N}$*, we define*

1) *the* j*-th inclusion and restriction*

$$\mathrm{inc}_j : \mathrm{Adm}_j^{[a,b]} \longrightarrow \mathrm{Adm}^{[a,b]},\ \mathrm{res}_j : \mathrm{Adm}^{[a,b]} \longrightarrow \mathrm{Adm}_j^{[a,b]} \tag{3.15}$$

by setting

$$\mathrm{inc}_j(\mu,w) := (\mu,w),\ \mathrm{res}_j(\mu,w) := \left(\bar{\mu}_j, \mu(a)(1_{A_j})^{-1} w_j\right) \tag{3.16}$$

for all $(\mu,w) \in \mathrm{Adm}_j^{[a,b]}$*, resp.* $(\mu,w) \in \mathrm{Adm}^{[a,b]}$.

2) *the* kj*-inclusion and* jk*-restriction*

$$\mathrm{inc}_{kj} : \mathrm{Adm}_j^{[a,b]} \longrightarrow \mathrm{Adm}_k^{[a,b]},\ \mathrm{res}_{jk} : \mathrm{Adm}_k^{[a,b]} \longrightarrow \mathrm{Adm}_j^{[a,b]} \tag{3.17}$$

by setting

$$\mathrm{inc}_{kj}(\mu,w) := (\mu,w),\ \mathrm{res}_{jk}(\mu,w) := \left(\bar{\mu}_j, \mu(a)(1_{A_j})^{-1} w_j\right) \tag{3.18}$$

for all $(\mu,w) \in \mathrm{Adm}_j^{[a,b]}$*, resp.* $(\mu,w) \in \mathrm{Adm}_k^{[a,b]}$.

Proof. We show 1), i.e. the case of $k = \infty$. We obtain 2) by analogous argument for $k < \infty$. We know w^*-continuity of inclusion and restriction maps by 1) in Proposition 2.1.28. Upon identifying as per Remark 3.1.9, inc_j maps to $\mathrm{AC}(I, \mathscr{S}(A)) \times L^2(I, B^*)_{\mathrm{w}}$ and res_j to $\mathrm{AC}(I, \mathscr{S}(A_j)) \times L^2(I, B_j)$. Using the latter and Proposition 2.3.25, we directly verify all claimed continuity equations. □

Remark 3.1.15. If (μ,w) satisfies the continuity equation for ∇ on I, then $\bar{\mu}(t)_j = \bar{\mu}(0)_j$ for all $t \in I$ and $j \in \mathbb{N}$ by Proposition 3.1.6. Non-trivial restriction requires $\mu(0)_j \neq 0$ in each case. If we apply restriction maps as per Proposition 3.1.14, then we either assume $\mu(0)_j \neq 0$ for all $j \in \mathbb{N}$ as part of a statement itself or we assume it implicitly without loss of generality since Remark 3.1.13 ensures $\mu(0)_j \neq 0$ for a.e. $j \in \mathbb{N}$.

Energy functionals from quasi-entropies. Definition 3.1.16 describes energy functionals given by integrating quasi-entropies evaluated on admissible paths. Note Remark 3.1.18. Definition 3.1.24 gives an a priori different description. They coincide on admissible paths. Proposition 3.1.19 extends results in Theorem 2.2.29 concerning inclusion and restriction maps to energy functionals.

Let $(\phi,\psi,\gamma,\nabla)$ be noncommutative differential structure for tracial AF-C^*-algebras (A,τ) and (B,ω) in (f,θ)-setting.

Definition 3.1.16. We define the energy functional $E^{f,\theta}:\mathrm{Adm}\longrightarrow[0,\infty]$ by setting

$$E^{f,\theta}(\mu,w):=\int_a^b \mathscr{I}^{f,\theta}\big(\mu(t),\mu(t),w(t)\big)dt \tag{3.19}$$

for all $[a,b]\subset\mathbb{R}$ and $(\mu,w)\in\mathrm{Adm}^{[a,b]}$.

Notation 3.1.17. For all $j\in\mathbb{N}$, let $E_j^{f,\theta}$ denote energy functional in Definition 3.1.16 for the induced noncommutative differential structure $(\phi_j,\psi_j,\gamma_j,\nabla_j)$.

Remark 3.1.18. For all $j\in\mathbb{N}$, Equation 3.19 is

$$E_j^{f,\theta}(\mu,w)=\int_a^b \mathscr{I}_j^{f,\theta}\big(\mu(t),\mu(t),w(t)\big)dt \tag{3.20}$$

for all $[a,b]\subset\mathbb{R}$ and $(\mu,w)\in\mathrm{Adm}_j^{[a,b]}$.

Proposition 3.1.19. *Let $[a,b]\subset\mathbb{R}$ and $j\le k$ in $\mathbb{N}$.*

1) For all $(\mu,w)\in\mathrm{Adm}_j^{[a,b]}\big(\mu^0,\mu^1\big)$, we have

$$E_j^{f,\theta}(\mu,w)=E_k^{f,\theta}\big(\mathrm{inc}_{kj}(\mu,w)\big)=E^{f,\theta}\big(\mathrm{inc}_j(\mu,w)\big). \tag{3.21}$$

2) Assume $\mu^0(1_{A_j})\neq 0$ in all statements below.

2.1) For all $(\mu,w)\in\mathrm{Adm}^{[a,b]}\big(\mu^0,\mu^1\big)$, we have

$$E_j^{f,\theta}\big(\mathrm{res}_j(\mu,w)\big)\le\mu^0(1_{A_j})^{-1}E^{f,\theta}(\mu,w). \tag{3.22}$$

2.2) For all $(\mu,w)\in\mathrm{Adm}_k^{[a,b]}\big(\mu^0,\mu^1\big)$, we have

$$E_j^{f,\theta}\big(\mathrm{res}_{jk}(\mu,w)\big)\le\mu^0(1_{A_j})^{-1}E_k^{f,\theta}(\mu,w). \tag{3.23}$$

Proof. Equation 3.20 shows 1) by 2) in Theorem 2.2.29. We show 2). Assume its setting. For all $\mu,\eta \in A_+^*$, $w \in B^*$ and $\lambda \geq 0$, get $\mathscr{I}^{f,\theta}(\lambda\mu,\lambda\eta,\lambda w) = \lambda\mathscr{I}^{f,\theta}(\mu,\eta,w)$ by construction of quasi-entropies. For all $t \in [0,1]$, Equation 3.7 shows $\mu(t)(1_{A_j}) = \mu^0(1_{A_j}) \neq 0$. Using the latter, we obtain 2) by 3) in Theorem 2.2.29. □

Proposition 3.1.21 gives a change of variables formula for energy functionals. For this, Remark 3.1.20 states a general one for reparametrisations of measurable functions (cf. Corollary 6 to Theorem 3 in [185]). We commonly use affine transformations as per Remark 3.1.22. Proposition 3.1.23 extends 5) in Theorem 2.2.29 to energy functionals and derives Lipschitz continuity.

Remark 3.1.20. Let $g : [a,b] \longrightarrow \mathbb{R}$ be Lebesgue integrable. If $\varphi : [c,d] \longrightarrow [a,b]$ is monotone and absolutely continuous, then $\dot{\varphi} \cdot (g \circ \varphi)$ is Lebesgue integrable and we have

$$\int_{\varphi(c)}^{\varphi(d)} g(t)dt = \int_a^b \dot{\varphi}(t)g\big(\varphi(t)\big)dt. \tag{3.24}$$

Proposition 3.1.21. *Let $\varphi : [c,d] \longrightarrow [a,b]$ be monotone and absolutely continuous with $\dot{\varphi}(t) \neq 0$ for a.e. $t \in [c,d]$. If $(\mu,w) \in \mathrm{Adm}^{[a,b]}(\mu^0,\mu^1)$, then $(\mu\circ\varphi, \dot{\varphi}\cdot(w\circ\varphi)) \in \mathrm{Adm}^{[c,d]}(\mu^0,\mu^1)$ and we have*

$$E^{f,\theta}(\mu,w) = \int_c^d \dot{\varphi}(t)^{-1}\mathscr{I}^{f,\theta}\Big(\mu\big(\varphi(t)\big),\mu\big(\varphi(t)\big),\dot{\varphi}(t)w\big(\varphi(t)\big)\Big)dt. \tag{3.25}$$

Proof. Since φ is monotone and $t \mapsto \mu(t)(x)$ is absolutely continuous for all $x \in A_0$, the chain rule holds for $\mu\circ\varphi$ upon evaluation by Theorem 2 and Corollary 4 in [185]. Thus $(\mu\circ\varphi, \dot{\varphi}\cdot(w\circ\varphi))$ satisfies the continuity equation for ∇ on $[c,d]$. All remaining properties of admissible paths are inherited. For all $\mu,\eta \in A_+^*$, $w \in B^*$ and $\lambda \geq 0$, get $\mathscr{I}^{f,\theta}(\mu,\eta,\lambda w) = \lambda^2\mathscr{I}^{f,\theta}(\mu,\eta,w)$ by construction of quasi-entropies. Using the latter, Equation 3.24 shows Equation 3.25 immediately. □

Remark 3.1.22. Let $[a,b],[c,d] \subset \mathbb{R}$ s.t. $a \neq b, c \neq d$. We define monotone and absolutely continuous homeomorphism $\varphi : [c,d] \longrightarrow [a,b]$ by setting

$$\varphi(t) := \frac{b-a}{d-c}(t-c)+a \tag{3.26}$$

for all $t \in [c,d]$. Using Proposition 3.1.21, Equation 3.25 shows

$$E^{f,\theta}(\mu,w) = \frac{d-c}{b-a}E^{f,\theta}\Big(\mu\circ\varphi, \frac{b-a}{d-c}\big(w\circ\varphi\big)\Big). \tag{3.27}$$

Proposition 3.1.23. *Let* $[a,b] \subset \mathbb{R}$.

1) For all $(\mu,w) \in \mathrm{Adm}^{[a,b]}$*, we have*

$$\|w\|^2_{L^2([a,b],B^*)_{\mathrm{w}}} \le E^{f,\theta}(\mu,w) \cdot 2^{-\theta}\big(\|\phi\|_1^\theta + \|\psi\|_1^\theta\big) \cdot \|\omega\|^{1-\theta}. \tag{3.28}$$

2) For all $(\mu,w) \in \mathrm{Adm}^{[a,b]}$*,* $x \in A_0$ *and* $t,s \in [a,b]$*, we have*

$$\big|\big(\mu(t)-\mu(s)\big)(x)\big|^2 \le |t-s| \cdot E^{f,\theta}(\mu,w) \cdot 2^{-\theta}\big(\|\phi\|_1^\theta + \|\psi\|_1^\theta\big) \cdot \|\omega\|^{1-\theta} \cdot \|\nabla x\|_B^2. \tag{3.29}$$

Proof. Note Equation 3.2 ensures 1) follows by 5) in Theorem 2.2.29. We show 2). For all $(\mu,w) \in \mathrm{Adm}^{[a,b]}$, $x \in A_0$ and $t,s \in [a,b]$, we use the continuity equation and apply Hölder in order to estimate

$$\begin{aligned}
\big|\big(\mu(t)-\mu(s)\big)(x)\big| &= \left|\int_s^t \frac{d}{dr}\mu(r)(x)dr\right| \\
&\le \left|\int_s^t \|w(r)\|_{B^*}\|\nabla x\|_B dr\right| \\
&\le \sqrt{|t-s|} \cdot \|w\|_{L^2([a,b],B^*)_{\mathrm{w}}} \cdot \|\nabla x\|_B.
\end{aligned}$$

We obtain 2) by applying Equation 3.28 to the above calculation. □

Definition 3.1.24 gives an a priori different, as well as more general, description of energy functionals than Definition 3.1.16 for a larger domain. Lemma 3.1.25 shows both descriptions coincide on admissible paths. Moreover, extensions of energy functionals as per Definition 3.1.24 are l.s.c in w^*-topology. Lemma 3.1.26 leverages the latter in order to show l.s.c. of energy functionals w.r.t. convergence in canonical topology, and further ensures the direct method in the calculus of variations [74][109] applies.

Definition 3.1.24. We define $\mathbf{E}^{f,\theta} : \bigcup_{[a,b]\subset\mathbb{R}} L^2([a,b],A^*)_{\mathrm{w}} \times L^2([a,b],B^*)_{\mathrm{w}} \longrightarrow [0,\infty]$ by setting

$$\mathbf{E}^{f,\theta}(\mu,w) := \sup_{j\in\mathbb{N}} \int_a^b \mathscr{I}_j^{f,\theta}\big(\mu_j(t),\mu_j(t),w_j(t)\big)dt \tag{3.30}$$

for all $[a,b] \subset \mathbb{R}$ and $(\mu,w) \in L^2([a,b],A^*)_{\mathrm{w}} \times L^2([a,b],B^*)_{\mathrm{w}}$.

For all $[a,b] \subset \mathbb{R}$, the inclusion in Equation 3.12 extends to

$$\mathrm{Adm}^{[a,b]} \subset L^2([a,b],A^*)_{\mathrm{w}} \times L^2([a,b],B^*)_{\mathrm{w}}. \tag{3.31}$$

Thus Equation 3.31 shows $\mathrm{Adm} \subset \bigcup_{[a,b]\subset\mathbb{R}} L^2([a,b],A^*)_{\mathrm{w}} \times L^2([a,b],B^*)_{\mathrm{w}}$, hence we have functional $\mathbf{E}^{f,\theta} : \mathrm{Adm} \longrightarrow [0,\infty]$ by restricting to Adm.

Lemma 3.1.25. *For all $[a,b] \subset \mathbb{R}$, $\mathbf{E}^{f,\theta}|_{\mathrm{Adm}^{[a,b]}}$ is l.s.c. in w^*-topology. We further have*

$$E^{f,\theta} = \mathbf{E}^{f,\theta}|_{\mathrm{Adm}}. \tag{3.32}$$

Proof. Let $[a,b] \subset \mathbb{R}$. We show $\mathbf{E}^{f,\theta}|_{\mathrm{Adm}^{[a,b]}}$ is l.s.c. in w^*-topology. For all $j \in \mathbb{N}$, we show

$$(\mu, w) \mapsto \int_a^b \mathscr{I}_j^{f,\theta}\big(\mu_j(t), \mu_j(t), w_j(t)\big)dt \tag{3.33}$$

is l.s.c. in w^*-topology. Compactness shows pointwise restriction yields w^*-continuous map from $L^2([a,b],A^*) \times L^2([a,b],B^*)$ to $L^2([a,b],A_j) \times L^2([a,b],B_j)$. We reduce l.s.c. in w^*-topology to the finite-dimensional setting. Assume A and B are finite-dimensional. Note $\mathscr{I}^{f,\theta}$ is jointly convex and l.s.c. in w^*-topology by 1) in Theorem 2.2.29. Further note joint convexity implies $\mathbf{E}^{f,\theta}$ is jointly convex. Following Remark 3.1.9, it suffices to show sequential l.s.c. in norm as the domain is norm equivalent to a product of Hilbert spaces. We extract pointwise a.e.-converging subsequences and conclude by l.s.c. of $\mathscr{I}^{f,\theta}$ in w^*-topology and Fatou's lemma. We obtain l.s.c. in w^*-topology as discussed above.

Return to the general setting. We show Equation 3.32. Let $(\mu, w) \in \mathrm{Adm}^{[a,b]}$. For all $k \in \mathbb{N}$, definition of quasi-entropy as suprema yields

$$\mathscr{I}_k^{f,\theta}\big(\mu_k(t), \mu_k(t), w_k(t)\big) \le \mathscr{I}^{f,\theta}\big(\mu(t), \mu(t), w(t)\big) \tag{3.34}$$

for a.e. $t \in [a,b]$. Note we restrict pointwise. Equation 3.34 shows $\mathbf{E}^{f,\theta}(\mu, w) \le E^{f,\theta}(\mu, w)$. Using 1.2) in Proposition 2.1.31, get $w^*\text{-}\lim_{j\in\mathbb{N}} \mu_j(t) = \mu(t)$ and $w^*\text{-}\lim_{j\in\mathbb{N}} w_j(t) = w(t)$ for a.e. $t \in [a,b]$. Then l.s.c. of $\mathscr{I}^{f,\theta}$ and Fatou's lemma imply

$$E^{f,\theta}(\mu, w) \le \liminf_{j\in\mathbb{N}} \int_a^b \mathscr{I}_j^{f,\theta}\big(\mu_j(t), \mu_j(t), w_j(t)\big)dt. \tag{3.35}$$

Yet the right-hand side of Equation 3.35 equals $\mathbf{E}^{f,\theta}(\mu, w)$ by 3) in Theorem 2.2.29. We altogether obtain our second claim. □

Lemma 3.1.26. *Let $(\mu^n, w^n)_{n\in\mathbb{N}} \subset \mathrm{Adm}^{[a,b]}$.*

1) *If $(\mu, w) = \lim_{n\in\mathbb{N}}(\mu^n, w^n)_{n\in\mathbb{N}}$ in $\mathrm{Adm}^{[a,b]}$, then $E^{f,\theta}(\mu, w) \le \liminf_{n\in\mathbb{N}} E^{f,\theta}(\mu^n, w^n)$.*

2) *If $\liminf_{n\in\mathbb{N}} E^{f,\theta}(\mu^n, w^n) < \infty$ and $t_0 \in [a,b]$ s.t. $w^*\text{-}\lim_{n\in\mathbb{N}} \mu^n(t_0) \in \mathscr{S}(A)$, then there exists a subsequence of $(\mu^n, w^n)_{n\in\mathbb{N}}$ converging in canonical topology.*

Proof. By 2) in Proposition 3.1.11, convergence in $\mathrm{Adm}^{[a,b]}$ implies w^*-convergence in $L^2([a,b],A^*)_{\mathrm{w}} \times L^2([a,b],B^*)_{\mathrm{w}}$. Thus 1) follows from Lemma 3.1.25. We show 2). Assume its setting. By passing to subsequences, we furthermore assume $\sup_{n\in\mathbb{N}} E^{f,\theta}(\mu^n, w^n) < \infty$ without loss of generality. This is necessary for uniform bounds.

Following Remark 3.1.4, we metricise the w^*-topology on $\overline{\mathscr{S}(A)}$ using $\{x_n\}_{n\in\mathbb{N}} \subset A_0$ for which the linear span lies $\|.\|_A$-dense in A and s.t. $\|\nabla x_n\|_B \leq 1$ for all $n \in \mathbb{N}$. Using bounded limit inferior and 2) in Proposition 3.1.23, we see $\{\mu^n\}_{n\in\mathbb{N}} \subset AC([a,b],\overline{\mathscr{S}(A)})$ is equicontinuous. Note the Arzelà-Ascoli theorem applies to paths in compact metric spaces [136]. We extract converging subsequence $\{\mu^n\}_{n\in\mathbb{N}}$. For all $t \in [a,b]$, we obtain $\mu(t) := w^*\text{-}\lim_{n\in\mathbb{N}} \mu^n(t) \in \overline{\mathscr{S}(A)}$. Using 1) in Proposition 3.1.23 instead, get uniform bound on $\{w^n\}_{n\in\mathbb{N}} \subset L^2([a,b],B^*)_{\mathrm{w}}$. We extract w^*-converging subsequence $\{w^n\}_{n\in\mathbb{N}}$. Finally, we conclude by applying 1) in Proposition 3.1.11 to $(\mu^n,w^n)_{n\in\mathbb{N}}$. □

Definition 3.1.27 gives suitable restriction of energy functionals. Let $[a,b] \subset \mathbb{R}$. For all $j \in \mathbb{N}$, we know $\mathrm{res}_j \circ \mathrm{inc}_j = \mathrm{inc}_j$ and $\mathrm{res}_{jk} \circ \mathrm{inc}_{kj} = \mathrm{inc}_{kj}$ by 2) in Proposition 2.1.28. We therefore identify

$$\mathrm{Adm}_j^{[a,b]} \cong \mathrm{inc}_j\left(\mathrm{Adm}_j^{[a,b]}\right) \subset \mathrm{Adm}^{[a,b]} \tag{3.36}$$

in each case. Notation 2.1.29 thereby likewise extends to admissible paths. For all $j \in \mathbb{N}$ and $(\mu,w) \in \mathrm{Adm}_j^{[a,b]}$, note 1) in Proposition 3.1.19 shows

$$E_j^{f,\theta}\big(\mathrm{res}_j(\mu,w)\big) = E_j^{f,\theta}(\mu,w) = E^{f,\theta}(\mu,w) \tag{3.37}$$

under identification as per Equation 3.36. Note Equation 3.37 shows Definition 3.1.27 extends Equation 3.20, i.e. Definition 3.1.16 for induced noncommutative differential structures. We account for rescaling of norm.

Definition 3.1.27. We define the j-th restricted energy functional $E^{f,\theta} : \mathrm{Adm} \longrightarrow [0,\infty]$ for $j \in \mathbb{N}$ by setting

$$E_j^{f,\theta}(\mu,w) := \begin{cases} E_j^{f,\theta}\big(\mathrm{res}_j(\mu,w)\big) & \text{if } \mu_j(a) \neq 0, \\ 0 & \text{else}. \end{cases}$$

Corollary 3.1.28. *Let $(\mu,w) \in \mathrm{Adm}^{[a,b]}$. If $E^{f,\theta}(\mu,w) < \infty$, then*

1) $(\mu,w) = \lim_{j\in\mathbb{N}} \mathrm{res}_j(\mu,w)$ in $\mathrm{Adm}^{[a,b]}$,

2) $E^{f,\theta}(\mu,w) = \lim_{j\in\mathbb{N}} E_j^{f,\theta}(\mu,w)$.

Proof. Let $(\mu,w) \in \mathrm{Adm}^{[a,b]}$ s.t. $E^{f,\theta}(\mu,w) < \infty$. Using 1) in Proposition 3.1.23 to have uniform bounds, get 1) by dominated convergence. The necessary pointwise convergence for μ and w holds by 1) in Proposition 2.1.31 as for Remark 3.1.13. Lemma 3.1.25 and 1) in Lemma 3.1.26 imply 2) since rescaling of norm vanishes in the limit. □

Upon restricting domains to sets of admissible paths with identical interval and marginals, Theorem 3.1.31 shows energy functionals are Γ-limits of restrictions as per Definition 3.1.27. Sequential descriptions of Γ-limits require first countability of the domain [74]. This does not hold for function spaces parametrised by intervals with more than one point (cf. Corollary 1.5 in [158]). We fix marginals up to restriction in order to get sequential descriptions as per Definition 3.1.29 for non-trivial intervals.

Definition 3.1.29. Let $[a,b] \subset \mathbb{R}$.

1) For all $(\mu,w) \in \mathrm{Adm}^{[a,b]}$, let $\mathscr{C}(\mu,w)$ be the set of all $\left(\mu^j,w^j\right)_{j\in\mathbb{N}} \subset \mathrm{Adm}^{[a,b]}$ s.t.

    1.1) $\left(\mu^j,w^j\right) \in \mathrm{Adm}_j^{[a,b]}\left(\bar{\mu}(a)_j,\bar{\mu}(b)_j\right)$ for a.e. $j \in \mathbb{N}$,

    1.2) $(\mu,w) = \lim_{j\in\mathbb{N}}\left(\mu^j,w^j\right)$ in $\mathrm{Adm}^{[a,b]}$.

2) We define the restricted lower Γ-limit, resp. the restricted upper Γ-limit of $E^{f,\theta}$ by setting

    2.1) $E_L^{f,\theta}(\mu,w) := \inf_{C(\mu,w)} \liminf_{j\in\mathbb{N}} E_j^{f,\theta}\left(\mu^j,w^j\right)$,

    2.2) $E_U^{f,\theta}(\mu,w) := \inf_{C(\mu,w)} \limsup_{j\in\mathbb{N}} E_j^{f,\theta}\left(\mu^j,w^j\right)$

    for all $(\mu,w) \in \mathrm{Adm}^{[a,b]}$.

Remark 3.1.30. Note 1) in Definition 3.1.29 equivalently uses truncation $j \geq m$ for fixed but arbitrary $m \in \mathbb{N}$ rather than a.e. $j \in \mathbb{N}$. The sets we obtain are identical. We use this in the proof of Theorem 3.1.31 if the norm vanishes for finitely many indices.

Theorem 3.1.31. *Let $(\phi,\psi,\gamma,\nabla)$ be noncommutative differential structure for tracial AF-C^*-algebras (A,τ) and (B,ω) in (f,θ)-setting. For all $[a,b] \subset \mathbb{R}$ and $\mu^0,\mu^1 \in \mathscr{S}(A)$, get*

$$E^{f,\theta}\Big|_{\mathrm{Adm}^{[a,b]}(\mu^0,\mu^1)} = \Gamma\text{-}\lim_{j\in\mathbb{N}} E_j^{f,\theta}\Big|_{\mathrm{Adm}^{[a,b]}(\mu^0,\mu^1)} \tag{3.38}$$

and

$$\Gamma\text{-}\lim_{j\in\mathbb{N}} E_j^{f,\theta}\Big|_{\mathrm{Adm}^{[a,b]}(\mu^0,\mu^1)} = E_L^{f,\theta}\Big|_{\mathrm{Adm}^{[a,b]}(\mu^0,\mu^1)} = E_U^{f,\theta}\Big|_{\mathrm{Adm}^{[a,b]}(\mu^0,\mu^1)}. \tag{3.39}$$

Proof. Let $[a,b] \subset \mathbb{R}$ and $\mu^0,\mu^1 \in \mathscr{S}(A)$. We use the canonical topology on $\mathrm{Adm}^{[a,b]}$. Set $X := \mathrm{Adm}^{[a,b]}(\mu^0,\mu^1)$. For all $(\mu,w) \in X$, let $\mathscr{N}(\mu,w) \subset X$ be its set of open neighbourhoods in the relative topology given by $X \subset \mathrm{Adm}^{[a,b]}$. We show Equation 3.38 in the first part of our proof using standard bounds for lower and upper Γ-limits. We show Equation 3.39 in the second part of our proof using Lemma 3.1.26 and Corollary 3.1.28. We truncate indices $j \geq m$ in $\mathbb{N}$ as per Remark 3.1.30 throughout this proof.

Let $(\mu,w)\in X$. We show the energy functional bounds its upper Γ-limit, i.e.

$$\sup_{U\in\mathcal{N}(\mu,w)}\limsup_{j\in\mathbb{N}}\inf_{(\eta,v)\in U}E_j^{f,\theta}(\eta,v)\le E^{f,\theta}(\mu,w). \tag{3.40}$$

Lemma 3.1.26 shows $E^{f,\theta}\big|_X=\mathbf{E}^{f,\theta}\big|_X$. We further know $\lim_{j\in\mathbb{N}}\mu^0(1_{A_j})=\lim_{j\in\mathbb{N}}\|\mu_j^0\|_{A^*}=1$ by 1.1) in Proposition 2.1.31. For all $U\in\mathcal{N}(\mu,w)$, we have $(\mu,w)\in U$ by definition of open neighbourhood and use 2.1) in Proposition 3.1.19 in order to estimate

$$\begin{aligned}\limsup_{j\in\mathbb{N}}\inf_{(\eta,v)\in U}E_j^{f,\theta}(\eta,v)&\le\limsup_{j\in\mathbb{N}}E_j^{f,\theta}(\mu,w)\\&\le\limsup_{j\in\mathbb{N}}\mu^0(1_{A_j})^{-1}\cdot E^{f,\theta}(\mu,w)\\&=E^{f,\theta}(\mu,w).\end{aligned}$$

Equation 3.40 follows by applying the supremum in $\mathcal{N}(\mu,w)$.

We show the energy functional is bounded by its lower Γ-limit, i.e.

$$E^{f,\theta}(\mu,w)\le\sup_{U\in\mathcal{N}(\mu,w)}\liminf_{j\in\mathbb{N}}\inf_{(\eta,v)\in U}E_j^{f,\theta}(\eta,v). \tag{3.41}$$

For all $U\in\mathcal{N}(\mu,w)$, the right-hand side of Equation 3.41 is either finite or our claim holds. We assume finiteness without loss of generality. Let $U\in\mathcal{N}(\mu,w)$. We construct a sequence associated to each such open set. For a.e. $j\in\mathbb{N}$, we have

$$E(U,j):=\inf_{(\eta,v)\in U}E_j^{f,\theta}(\eta,v)<\infty \tag{3.42}$$

by finiteness. We consider subsequence $\{E(U,j_n)\}_{n\in\mathbb{N}}\subset[0,\infty)$ s.t.

$$\lim_{n\in\mathbb{N}}E(U,j_n)=\liminf_{j\in\mathbb{N}}\inf_{(\eta,v)\in U}E_j^{f,\theta}(\eta,v)<\infty. \tag{3.43}$$

For a.e. $j\in\mathbb{N}$, select $(\mu^j,w^j)\in U$ s.t.

$$\inf_{(\eta,v)\in U}E_j^{f,\theta}(\eta,v)=E_j^{f,\theta}\big(\mu^j,w^j\big)+j^{-1}. \tag{3.44}$$

Since marginals are fixed, we use 2) in Lemma 3.1.26 for $t_0=0$ to get subsequence of $\mathrm{res}_{jn}(\mu^{j_n},w^{j_n})_{n\in\mathbb{N}}\subset X$ converging in $\mathrm{Adm}^{[a,b]}$. Note convergence in Equation 3.43 is invariant under passing to a subsequence. We relabel the subsequence obtained by Lemma 3.1.26 as $\mathrm{res}_{jn}(\mu^{j_n},w^{j_n})_{n\in\mathbb{N}}$. Let (μ^U,w^U) be its limit in canonical topology.

Using the respective sequence constructed as above in each case, i.e. s.t. we have $(\mu^U, w^U) = \lim_{n\in\mathbb{N}} \mathrm{res}_{jn}(\mu^{j_n}, w^{j_n})$ in $\mathrm{Adm}^{[a,b]}$, Equation 3.42 and Equation 3.44 show

$$\lim_{n\in\mathbb{N}} \left| E^{f,\theta}_{j_n}\big(\mu^{j_n}, w^{j_n}\big) - E(U, j_n) \right| = \lim_{n\in\mathbb{N}} j_n^{-1} = 0 \tag{3.45}$$

for all $U \in \mathcal{N}(\mu, w)$. Using 1) in Lemma 3.1.26, Equation 3.43 and Equation 3.45 in turn let us calculate

$$\begin{aligned} E^{f,\theta}\big(\mu^U, w^U\big) &\leq \liminf_{n\in\mathbb{N}} E^{f,\theta}_{j_n}\big(\mu^{j_n}, w^{j_n}\big) \\ &= \lim_{n\in\mathbb{N}} E(U, j_n) \\ &= \liminf_{j\in\mathbb{N}} \inf_{(\eta,v)\in U} E^{f,\theta}_{j}(\eta, v) \end{aligned}$$

in each case. Equation 3.41 therefore follows if

$$E^{f,\theta}(\mu, w) \leq \sup_{U\in\mathcal{N}(\mu,w)} E^{f,\theta}\big(\mu^U, w^U\big). \tag{3.46}$$

For all $U \in \mathcal{N}(\mu, w)$, we have $(\mu^U, w^U) \in \overline{U}$ by construction. We thus show Equation 3.46 by constructing a sequence of open sets s.t. Lemma 3.1.26 lets us extract subsequence converging to (μ, w) in $\mathrm{Adm}^{[a,b]}$ and apply l.s.c. of the energy functional.

Let $K \subset L^2([a,b], B^*)_{\mathrm{w}}$ be a norm bounded closed set s.t. $w \in K$. We consider $\mathcal{S}(A)$ and K as metric spaces using w^*-topology as per Remark 3.1.4. All open balls used in this proof are in one of these two metric spaces. Let $\{t_n\}_{n\in\mathbb{N}} \subset [a,b]$ be a dense and monotone increasing sequence. For all $n \in \mathbb{N}$ and $t \in [a,b]$, we define open $V_{n,t} \subset \mathcal{S}(A)$ by setting

$$V_{n,t} := \begin{cases} B_{n^{-1}}\big(\mu(t_l)\big) & \text{if } t = t_l \text{ for } l \leq n, \\ \mathcal{S}(A) & \text{else}. \end{cases}$$

For all $n \in \mathbb{N}$, set $V_n := \prod_{t\in[a,b]} V_{n,t}$. Each of the latter is an open set in

$$\mathcal{S}(A)^{[a,b]} := \prod_{t\in[a,b]} \mathcal{S}(A). \tag{3.47}$$

There exists open sets $\{W_n\}_{n\in\mathbb{N}} \subset \mathcal{P}(L^2([a,b], B^*)_{\mathrm{w}})$, the latter denoting the respective power set, s.t. for all $n \in \mathbb{N}$, we have $W_n \cap K = B_{n^{-1}}(w)$ and $W_{n+1} \subset W_n$. For all $n \in \mathbb{N}$, we obtain open set $V_n \times W_n \subset \mathcal{S}(A)^{[a,b]} \times L^2([a,b], B^*)_{\mathrm{w}}$ and therefore open set

$$U_n := \big(V_n \times W_n\big) \cap X \subset X. \tag{3.48}$$

The above construction further ensures

$$U_{n+1} \subset U_n \tag{3.49}$$

for all $n \in \mathbb{N}$, as well as

$$\{(\mu, w)\} = \bigcap_{n \in \mathbb{N}} \overline{U}_n. \tag{3.50}$$

For all $n \in \mathbb{N}$, let $(\mu^n, w^n) \in \overline{U}_n$. Using 2) in Lemma 3.1.26 for $t_0 = 0$, get subsequence converging to (μ, w) in $\mathrm{Adm}^{[a,b]}$ by Equation 3.49 and Equation 3.50. Equation 3.46 follows by applying 1) in Lemma 3.1.26 to such a subsequence. Equation 3.41 follows as discussed above. Altogether, we have Equation 3.40 and Equation 3.41. Using standard arguments for Γ-convergence from upper and lower Γ-limits [74], we see Equation 3.40 and Equation 3.41 show Equation 3.38 immediately.

We have $E_L^{f,\theta} \leq E_U^{f,\theta}$ by definition. We are left to show

$$E_U^{f,\theta}\Big|_X \leq E^{f,\theta}\big|_X \leq E_L^{f,\theta}\Big|_X. \tag{3.51}$$

Using Equation 3.37 and 1) in Lemma 3.1.26, we directly verify

$$E^{f,\theta}\big|_X \leq E_L^{f,\theta}\Big|_X. \tag{3.52}$$

Equation 3.52 reduces us to $(\mu, w) \in X$ s.t. $E^{f,\theta}(\mu, w) < \infty$. We assume the latter without loss of generality. Thus $\{\mathrm{res}_j(\mu, w)\}_{j \in \mathbb{N}} \in \mathscr{C}(\mu, w)$ by 1) in Corollary 3.1.28, hence

$$E_U^{f,\theta}\Big|_X \leq E^{f,\theta}\big|_X \tag{3.53}$$

by 2) in Corollary 3.1.28. Equation 3.52 and Equation 3.53 show Equation 3.51. □

3.1.2 Quantum optimal transport distances

We define quantum optimal transport distances. Theorem 3.1.47 collects properties of their metric geometries. Accessibility components are complete geodesic length-metric spaces. Theorem 3.1.52 gives existence of sufficient minimising geodesics approximated in finite dimensions. Standard references for metric geometry are [8] and [40].

Quantum optimal transport as dynamic transport distance. Let $(\phi,\psi,\gamma,\nabla)$ be noncommutative differential structure for tracial AF-C^*-algebras (A,τ) and (B,ω) in (f,θ)-setting. Definition 3.1.33 gives quantum optimal transport distances. It extends the tracial finite-dimensional cases in [48][49][50] by construction.

Notation 3.1.32. Let X be a set and $d : X \times X \longrightarrow [0,\infty]$ a metric, or distance function on X. We say that the metric space (X,d) is equipped with d-topology. For all subsets $Y \subset X$, we write $(Y,d) = (Y,d|_{Y\times Y})$ for its relative metric space.

Definition 3.1.33. We define the quantum optimal transport distance of $(\phi,\psi,\gamma,\nabla)$ on $\mathscr{S}(A)$ in (f,θ)-setting by setting

$$\mathscr{W}_{\nabla}^{f,\theta}\big(\mu^0,\mu^1\big) := \inf_{\mathrm{Adm}^{[0,1]}(\mu^0,\mu^1)} \sqrt{E^{f,\theta}(\mu,w)} \in [0,\infty] \tag{3.54}$$

for all $\mu^0,\mu^1 \in \mathscr{S}(A)$.

Remark 3.1.34. Neither symmetry of f nor $\|\omega\|^{1-\theta} < \infty$ is required to define admissible paths and energy functionals. They do ensure accessibility components are complete geodesic length-metric spaces. In the logarithmic mean setting, i.e. f represents the logarithmic operator mean and $\theta = 1$, we have symmetric f and $\|\omega\|^0 = 1$.

We require accessibility components of quantum optimal transport distances to be complete geodesic length-metric spaces. Definition 3.1.35 describes length functionals given by integrating square roots of quasi-entropies, i.e. speed, evaluated on admissible paths. Proposition 3.1.39 shows using square roots of quasi-entropies as speed defines length structures for state spaces in w^*-topology. Corollary 3.1.42, which uses constant speed parametrisations of admissible paths on the unit interval as per Lemma 3.1.40, in turn shows quantum optimal transport distances are intrinsic distances of such length structures by Proposition 2.4.1 in [40]. Equation 3.68 gives their necessary standard representation. Using our subsequent discussion, Corollary 3.1.50 shows accessibility components are complete geodesic length-metric spaces.

Definition 3.1.35.

1) For all $[a,b] \subset \mathbb{R}$ and $(\mu,w) \in \mathrm{Adm}^{[a,b]}$, set

$$\mathscr{N}^{f,\theta}\big(\mu(t),w(t)\big) := \sqrt{\mathscr{I}^{f,\theta}\big(\mu(t),\mu(t),w(t)\big)} \tag{3.55}$$

for a.e. $t \in [a,b]$.

2) We define the length functional $L^{f,\theta} : \mathrm{Adm} \longrightarrow [0,\infty]$ by setting

$$L^{f,\theta}(\mu,w) := \int_a^b \mathscr{N}^{f,\theta}\big(\mu(t),w(t)\big)dt \tag{3.56}$$

for all $[a,b] \subset \mathbb{R}$ and $(\mu,w) \in \mathrm{Adm}^{[a,b]}$.

We restrict admissible paths and therefore length functionals to subintervals as per Remark 3.1.36. Proposition 3.1.37 shows length functionals are invariant under change of variables. Proposition 3.1.38 derives Lipschitz continuity, as well as standard upper bounds involving energy functionals.

Remark 3.1.36. We restrict admissible paths to subintervals. Equation 3.57 restricts their length accordingly. Let $[a,b] \subset \mathbb{R}$. For all $[s,t] \subset [a,b]$ and $(\mu,w) \in \mathrm{Adm}^{[a,b]}$, we have $(\mu,w)|_{[s,t]} := (\mu|_{[s,t]}, w|_{[s,t]}) \in \mathrm{Adm}^{[s,t]}$ and set

$$L^{f,\theta}(\mu,w)\big|_{[s,t]} := L^{f,\theta}\big(\mu|_{[s,t]}, w|_{[s,t]}\big) = \int_s^t \mathcal{N}^{f,\theta}\big(\mu(r),w(r)\big)dr. \tag{3.57}$$

Proposition 3.1.37. *Let $\varphi : [c,d] \longrightarrow [a,b]$ be monotone and absolutely continuous. If $(\mu,w) \in \mathrm{Adm}^{[a,b]}(\mu^0,\mu^1)$, then $(\mu\circ\varphi, \dot\varphi\cdot(w\circ\varphi)) \in \mathrm{Adm}^{[c,d]}(\mu^0,\mu^1)$ and we have*

$$L^{f,\theta}(\mu,w) = L^{f,\theta}\big(\mu\circ\varphi, \dot\varphi\cdot\big(w\circ\varphi\big)\big). \tag{3.58}$$

Proof. We argue as in the proof of Proposition 3.1.21. However, we integrate over the evaluated square root $\mathcal{N}^{f,\theta} = \sqrt{\mathcal{I}^{f,\theta}}$. Thus we do not require $\dot\varphi$ to have t-a.e. defined inverse, hence Equation 3.24 shows Equation 3.58 immediately. □

Proposition 3.1.38. *Let $[a,b] \subset \mathbb{R}$.*

1) For all $(\mu,w) \in \mathrm{Adm}^{[a,b]}$, $x \in A_0$ and $[s,t] \subset [a,b]$, we have

$$\big|\big(\mu(t)-\mu(s)\big)(x)\big| \le L^{f,\theta}(\mu,w)\big|_{[s,t]} \cdot \sqrt{2^{-\theta}\big(\|\phi\|_1^\theta + \|\psi\|_1^\theta\big)\cdot\|\omega\|^{1-\theta}} \cdot \|\nabla x\|_B. \tag{3.59}$$

2) For all $(\mu,w) \in \mathrm{Adm}^{[a,b]}$, we have

$$L^{f,\theta}(\mu,w)^2 \le (b-a)\cdot E^{f,\theta}(\mu,w). \tag{3.60}$$

Furthermore, we have equality in Equation 3.59 if and only if $t \mapsto \mathcal{N}^{f,\theta}\big(\mu(t),w(t)\big)$ is t-a.e. constant on $[a,b]$.

Proof. We show 1). We argue as in the proof of 2) in Proposition 3.1.23, where we use the continuity equation to estimate. Rather than subsequent application of Hölder, we instead apply 5) in Theorem 2.2.29. Equation 3.59 holds. We show 2). We reduce to $[a,b] = [0,1]$ by applying Proposition 3.1.37 to the left- , resp. Proposition 3.1.21 to the right-hand side of Equation 3.60. We use affine transformations as per Remark 3.1.22 in both cases. Having reduced to $[0,1]$ as described, both Equation 3.60 and our claim concerning equality follow by Jensen's inequality. □

Proposition 3.1.39. $(\mathrm{Adm}, L^{f,\theta})$ *is a length structure for* $\mathscr{S}(A)$ *in* w^**-topology.*

Proof. Proposition 3.1.37 shows Adm is a class of admissible paths in the sense of metric geometry [40]. Our claim follows if $L^{f,\theta}$ satisfies conditions 1) to 4) on p.27 in [40]. Using Equation 3.56, we directly verify the first three conditions. The fourth one is equivalent to the following statement. If $\{\mu^n\}_{n\in\mathbb{N}} \subset \mathscr{S}(A)$ and $\mu^0 \in \mathscr{S}(A)$ s.t.

$$\lim_{n\in\mathbb{N}} \inf_{\mathrm{Adm}(\mu^0,\mu^n)} L^{f,\theta}(\mu,w) = 0, \tag{3.61}$$

then $\mu = w^*\text{-}\lim \mu^n$ in $\mathscr{S}(A)$. This is ensured by 1) in Proposition 3.1.38. □

Lemma 3.1.40. *Let* $\mu^0, \mu^1 \in \mathscr{S}(A)$ *and* $(\mu,w) \in \mathrm{Adm}^{[0,1]}(\mu^0,\mu^1)$*. If* $L^{f,\theta}(\mu,w) \in (0,\infty)$*, then there exists* $(\tilde{\mu},\tilde{w}) \in \mathrm{Adm}^{[0,1]}(\mu^0,\mu^1)$ *s.t.*

1) $L^{f,\theta}(\tilde{\mu},\tilde{w}) = L^{f,\theta}(\mu,w)$,

2) $t \mapsto \mathscr{N}^{f,\theta}\big(\tilde{\mu}(t),\tilde{w}(t)\big) \neq 0$ *is t-a.e. constant.*

Proof. Assume $L^{f,\theta}(\mu,w) \in (0,\infty)$. For all $t \in [0,1]$, set

$$\varphi(t) := \varphi^{\mu,w}(t) := L^{f,\theta}(\mu,w)^{-1} \cdot \int_0^t \mathscr{N}^{f,\theta}\big(\mu(s),w(s)\big)ds. \tag{3.62}$$

Since $L^{f,\theta}(\mu,w) > 0$, get $\mu^0 \neq \mu^1$ by Proposition 3.1.39. Since $L^{f,\theta}(\mu,w) < \infty$, we know φ is monotone and absolutely continuous. We reduce to φ strictly monotone.

Assume φ is not strictly monotone. There exists $[c,d] \subset [0,1]$ s.t. $t \mapsto \mathscr{N}^{f,\theta}(\mu(t),w(t))$ vanishes for a.e. $t \in [c,d]$. Thus $\mu|_{[c,d]}$ is constant by 1) in Proposition 3.1.38. We select $[c,d]$ maximal. If $[c,d] \subset I$ proper for a closed interval $I \subset \mathbb{R}$, then $\mu|_I$ is not constant on I. Since $\mu^0 \neq \mu^1$ and $[0,1]$ is compact, there exists $m \in \mathbb{N}$ and non-intersecting maximal intervals $\{[c_n,d_n]\}_{n=1}^m \subset \mathscr{P}(\mathbb{R})$ satisfying R.1) and R.2) below. For all $n \in \{1,\dots,m\}$, let

R.1) $0 < c_n < d_n < 1$ s.t. $\mu|_{[c_n,d_n]}$ is constant,

R.2) there exists no $(a,b) \subset [0,1] \setminus \big(\bigcup_{n=1}^m [c_n,d_n]\big)$ s.t. $\mu|_{[a,b]}$ is constant.

Set $d_0 := 0$ and $c_{m+1} := 1$. For all $n \in \{0,\dots,m\}$, get $(\mu,w)|_{[d_n,c_{n+1}]} \in \mathrm{Adm}$. Thus R.1) immediately yields

$$L^{f,\theta}(\mu,w) = \sum_{n=0}^m L^{f,\theta}(\mu,w)\big|_{[d_n,c_{n+1}]}. \tag{3.63}$$

For all $n \in \{0,\dots,m\}$, we reparametrise $(\mu,w)|_{[d_n,c_{n+1}]}$ to $[n(m+1)^{-1},(n+1)(m+1)^{-1}]$ using affine transformation as per Remark 3.1.22. We concatenate reparametrised paths via canonical topological path composition.

Altogether, we obtain a rectified path

$$(\tilde{\mu},\tilde{w}) := (\mu,w)|_{\left[0,\frac{1}{m+1}\right]} \circ \cdots \circ (\mu,w)|_{\left[\frac{m}{m+1},1\right]} \in \mathrm{Adm}^{[0,1]}. \tag{3.64}$$

Proposition 3.1.37 and Equation 3.63 show $L^{f,\theta}(\tilde{\mu},\tilde{w}) = L^{f,\theta}(\mu,w)$. Yet R.2) shows there exists no $(a,b) \subset [0,1]$ s.t. $\tilde{\mu}|_{[a,b]}$ is constant. Hence $\varphi^{\tilde{\mu},\tilde{w}}$ is strictly monotone by 1) in Proposition 3.1.38. We may reduce since its construction preserves length.

We assume $\varphi := \varphi^{\mu,w}$ is strictly monotone without loss of generality. Thus φ is a homeomorphism onto $[0,1]$, hence φ^{-1} exists and is monotone. Monotonicity ensures φ^{-1} has t-a.e. finite derivative $\frac{d}{dt}\varphi^{-1}$. The chain rule holds for $\mu \circ \varphi^{-1}$ upon testing with A_0 (cf. Corollary 4 in [185]). We therefore have

$$(\tilde{\mu},\tilde{w}) := \left(\mu \circ \varphi^{-1}, \frac{d}{dt}\varphi^{-1} \cdot \left(w \circ \varphi^{-1}\right)\right) \in \mathrm{Adm}^{[0,1]}\left(\mu^0,\mu^1\right). \tag{3.65}$$

Proposition 3.1.37 shows $L^{f,\theta}(\tilde{\mu},\tilde{w}) = L^{f,\theta}(\mu,w)$. Since we have t-a.e. finite derivatives for φ, φ^{-1} and $\mathrm{id}_{[0,1]}$, the chain rule holds for $t = \varphi(\varphi^{-1}(t))$ (cf. Theorem 2 in [185]). We use chain rule to derive the first, and Equation 3.62 for the second identity in

$$\left.\frac{d}{ds}\right|_{s=t} \varphi^{-1}(s) = \left(\left.\frac{d}{ds}\right|_{s=\varphi^{-1}(t)} \varphi(s)\right)^{-1} = L^{f,\theta}(\mu,w) \cdot \mathcal{N}^{f,\theta}\left(\mu\left(\varphi^{-1}(t)\right), w\left(\varphi^{-1}(t)\right)\right)^{-1} \tag{3.66}$$

for a.e. $t \in [0,1]$. Using Equation 3.66, we further calculate

$$\mathcal{N}^{f,\theta}\left(\tilde{\mu}(t),\tilde{w}(t)\right) = \left.\frac{d}{ds}\right|_{s=t} \varphi^{-1}(s) \cdot \mathcal{N}^{f,\theta}\left(\mu\left(\varphi^{-1}(t)\right), w\left(\varphi^{-1}(t)\right)\right) = L^{f,\theta}(\mu,w) \neq 0 \tag{3.67}$$

in each case. Equation 3.67 shows our claim. □

Remark 3.1.41. In the proof of Lemma 3.1.40, we alternatively show $\tilde{\mu}$ has constant and non-vanishing metric derivative. Minimality and finite length let us bound from below using the metric derivative in order to show $\mathcal{N}^{f,\theta}(\tilde{\mu}(t),\tilde{w}(t)) \neq 0$ for a.e. $t \in [0,1]$.

Corollary 3.1.42. *For all $\mu^0,\mu^1 \in \mathcal{S}(A)$, we have*

$$\mathcal{W}_{\nabla}^{f,\theta}\left(\mu^0,\mu^1\right) = \inf_{\mathrm{Adm}^{[0,1]}(\mu^0,\mu^1)} L^{f,\theta}(\mu,w) = \inf_{\mathrm{Adm}(\mu^0,\mu^1)} L^{f,\theta}(\mu,w). \tag{3.68}$$

Proof. Let $\mu^0,\mu^1 \in \mathcal{S}(A)$. Either $\mu^0 \neq \mu^1$ or all terms equal zero. We assume $\mu^0 \neq \mu^1$ without loss of generality. Proposition 3.1.37 shows

$$\inf_{\mathrm{Adm}^{[0,1]}(\mu^0,\mu^1)} L^{f,\theta}(\mu,w) = \inf_{\mathrm{Adm}(\mu^0,\mu^1)} L^{f,\theta}(\mu,w). \tag{3.69}$$

Moreover, 2) in Proposition 3.1.38 shows

$$\inf_{\mathrm{Adm}^{[0,1]}(\mu^0,\mu^1)} L^{f,\theta}(\mu,w) \leq \mathscr{W}_{\nabla}^{f,\theta}\big(\mu^0,\mu^1\big). \tag{3.70}$$

Let $\mathbf{S} := \{(\mu,w) \in \mathrm{Adm}^{[0,1]}(\mu^0,\mu^1) \mid t \mapsto \mathscr{N}^{f,\theta}(\mu(t),w(t))$ is t-a.e. constant$\}$. Lemma 3.1.40 implies $\mathbf{S} \neq \emptyset$ and further

$$\inf_{\mathbf{S}} L^{f,\theta}(\mu,w) \ = \inf_{\mathrm{Adm}^{[0,1]}(\mu^0,\mu^1)} L^{f,\theta}(\mu,w). \tag{3.71}$$

Using the statement on equality for Equation 3.60, we see Equation 3.71 shows

$$\mathscr{W}_{\nabla}^{f,\theta}\big(\mu^0,\mu^1\big) \leq \inf_{\mathbf{S}} L^{f,\theta}(\mu,w) \ = \inf_{\mathrm{Adm}^{[0,1]}(\mu^0,\mu^1)} L^{f,\theta}(\mu,w). \tag{3.72}$$

Equation 3.69, Equation 3.70 and Equation 3.72 imply Equation 3.68. □

Definition 3.1.43 gives minimising geodesics and distance minimisers [8][40]. The notions coincide by 4) in Theorem 3.1.47. In Section 4.3, we apply results in variational analysis for metric geometry using minimising geodesics [75][160].

Definition 3.1.43.

1) Let $\mu^0,\mu^1 \in \mathscr{S}(A)$ and $[a,b] \subset \mathbb{R}$. We call $(\mu,w) \in \mathrm{Adm}^{[a,b]}\big(\mu^0,\mu^1\big)$ a minimising geodesic from μ^0 to μ^1 if there exists $C \geq 0$ s.t.

$$\mathscr{W}_{\nabla}^{f,\theta}\big(\mu(t),\mu(s)\big) = C|t-s| \tag{3.73}$$

for all $t,s \in [a,b]$.

2) Let $\mu^0,\mu^1 \in \mathscr{S}(A)$. We call $(\mu,w) \in \mathrm{Adm}^{[0,1]}\big(\mu^0,\mu^1\big)$ a distance minimiser if

$$\mathscr{W}_{\nabla}^{f,\theta}\big(\mu^0,\mu^1\big) = \sqrt{E^{f,\theta}(\mu,w)} < \infty. \tag{3.74}$$

3) For all $\mu^0,\mu^1 \in \mathscr{S}(A)$, let $\mathrm{Geo}\big(\mu^0,\mu^1\big)$ be the set of all distance minimisers with marginals μ^0 and μ^1. Set $\mathrm{Geo} := \bigcup_{\mu^0,\mu^1 \in \mathscr{S}(A)} \mathrm{Geo}\big(\mu^0,\mu^1\big)$.

Notation 3.1.44. For all $j \in \mathbb{N}$, we use Geo_j when denoting sets of distance minimisers in Definition 3.1.43 for induced noncommutative differential structure $(\phi_j,\psi_j,\gamma_j,\nabla_j)$.

Proposition 3.1.45.

1) Let $\mu^0,\mu^1 \in \mathscr{S}(A)$. *If* $(\mu,w) \in \mathrm{Geo}\big(\mu^0,\mu^1\big)$, *then* $t \mapsto \mathscr{N}^{f,\theta}\big(\mu(t),w(t)\big)$ *is t-a.e. constant.*

2) For all $j \le k$ *in* $\mathbb{N}$ *and* $\mu^0,\mu^1 \in \mathscr{S}(A_j)$, *we have length- and energy-preserving maps*

2.1) $\mathrm{Geo}_j\big(\mu^0,\mu^1\big) \overset{\mathrm{inc}_{kj}}{\hookrightarrow} \mathrm{Geo}_k\big(\mu^0,\mu^1\big) \overset{\mathrm{inc}_j}{\hookrightarrow} \mathrm{Geo}\big(\mu^0,\mu^1\big)$,

2.2) $\mathrm{Geo}\big(\mu^0,\mu^1\big) \xrightarrow{\mathrm{res}_k} \mathrm{Geo}_k\big(\mu^0,\mu^1\big) \xrightarrow{\mathrm{res}_{jk}} \mathrm{Geo}_j\big(\mu^0,\mu^1\big)$.

Proof. We show 1). For all $(\mu,w) \in \mathrm{Geo}(\mu^0,\mu^1)$, 2) in Proposition 3.1.38, Corollary 3.1.42 and Equation 3.74 yield

$$\mathscr{W}_\nabla^{f,\theta}\big(\mu^0,\mu^1\big) \le L^{f,\theta} \le \sqrt{E^{f,\theta}(\mu,w)} = \mathscr{W}_\nabla^{f,\theta}\big(\mu^0,\mu^1\big). \tag{3.75}$$

Furthermore, we have equality in Equation 3.75 if and only if $t \mapsto \mathscr{N}^{f,\theta}(\mu(t),w(t))$ is t-a.e. constant on $[0,1]$. We have 1). We show 2). For all distance minimisers, equality in Equation 3.75 implies length and square root of energy coincide. It suffices to show energy is preserved. Inclusions in 2.1) preserve energy by 1) in Proposition 3.1.19, and restrictions in 2.2) do not increase energy by 2) in Proposition 3.1.19. We obtain 2) by Equation 3.74 since restriction maps are left-inverses of inclusion maps. □

Notation 3.1.46. For all $\mu^k \in \mathscr{S}(A)$ and $k \in \{0,1\}$, set $\bar{\mu}^k := \overline{\mu^k}$ as per Definition 3.1.12.

Theorem 3.1.47. *Let* $(\phi,\psi,\gamma,\nabla)$ *be noncommutative differential structure for tracial AF-*C^**-algebras* (A,τ) *and* (B,ω) *in* (f,θ)*-setting.*

1) $\big(\mathscr{S}(A),\mathscr{W}_\nabla^{f,\theta}\big)$ *is a length-metric space with topology stronger than* w^**-topology.*

2) For all $j \le k$ *in* $\mathbb{N}$, *we have isometric inclusions*

$$\big(\mathscr{S}(A_j),\mathscr{W}_{\nabla_j}^{f,\theta}\big) \overset{\mathrm{inc}_{kj}}{\hookrightarrow} \big(\mathscr{S}(A_k),\mathscr{W}_{\nabla_k}^{f,\theta}\big) \overset{\mathrm{inc}_k}{\hookrightarrow} \big(\mathscr{S}(A),\mathscr{W}_\nabla^{f,\theta}\big). \tag{3.76}$$

3) $\mathscr{W}_\nabla^{f,\theta}$ *is l.s.c. in* w^**-topology. For all* $\mu^0,\mu^1 \in \mathscr{S}(A)$, *we have*

$$\mathscr{W}_\nabla^{f,\theta}\big(\mu^0,\mu^1\big) = \lim_{j\in\mathbb{N}} \mathscr{W}_{\nabla_j}^{f,\theta}\big(\bar{\mu}_j^0,\bar{\mu}_j^1\big). \tag{3.77}$$

4) Let $\mu^0,\mu^1 \in \mathscr{S}(A)$.

4.1) If $\mathscr{W}_\nabla^{f,\theta}\big(\mu^0,\mu^1\big) < \infty$, *then* $\mathrm{Geo}\big(\mu^0,\mu^1\big) \neq \emptyset$.

4.2) For all $(\mu,w) \in \mathrm{Adm}^{[0,1]}\big(\mu^0,\mu^1\big)$, *we have* $(\mu,w) \in \mathrm{Geo}\big(\mu^0,\mu^1\big)$ *if and only if* μ *is a minimising geodesic from* μ^0 *to* μ^1.

Proof. We know 1) by Proposition 3.1.39 and Corollary 3.1.42. Then 2) follows from 2) in Proposition 3.1.45. We show 3). Let $\mu^0, \mu^1 \in \mathscr{S}(A)$. For all $k \in \{0,1\}$, let $\{\mu^{n,k}\}_{n\in\mathbb{N}} \subset \mathscr{S}(A)$ s.t. $\mu^k = w^*\text{-}\lim_{n\in\mathbb{N}} \mu^{n,k}$, $\mathscr{W}_\nabla^{f,\theta}(\mu^{n,0},\mu^{n,1}) < \infty$ for all $n \in \mathbb{N}$, as well as

$$\liminf_{n\in\mathbb{N}} \mathscr{W}_\nabla^{f,\theta}\left(\mu^{n,0},\mu^{n,1}\right) < \infty. \tag{3.78}$$

In order to show l.s.c. in w^*-topology, it suffices to consider such subsequences. For all $n \in \mathbb{N}$, let $(\mu^n, w^n) \in \mathrm{Adm}^{[0,1]}(\mu^{n,0},\mu^{n,1})$ s.t.

$$E^{f,\theta}(\mu^n, w^n) = \mathscr{W}_\nabla^{f,\theta}\left(\mu^{n,0},\mu^{n,1}\right)^2 + n^{-1}. \tag{3.79}$$

Using w^*-convergence of marginals, Equation 3.79 shows Lemma 3.1.26 for $t_0 = 0$ yields $(\mu, w) \in \mathrm{Adm}^{[0,1]}(\mu^0,\mu^1)$ s.t. we have estimate

$$\mathscr{W}_\nabla^{f,\theta}\left(\mu^0,\mu^1\right) \leq \sqrt{E^{f,\theta}(\mu,w)} \leq \liminf_{n\in\mathbb{N}} \mathscr{W}_\nabla^{f,\theta}\left(\mu^{n,0},\mu^{n,1}\right). \tag{3.80}$$

Equation 3.80 shows l.s.c. in w^*-topology. In particular, we see Equation 3.77 follows at once if

$$\limsup_{j\in\mathbb{N}} \mathscr{W}_{\nabla_j}^{f,\theta}\left(\bar{\mu}_j^0,\bar{\mu}_j^1\right) \leq \mathscr{W}_\nabla^{f,\theta}\left(\mu^0,\mu^1\right). \tag{3.81}$$

Equation 3.81 holds by 2.1) in Proposition 3.1.19. Get 3).

We show 4). Lemma 3.1.26 for $t_0 = 0$ implies 4.1). Let $(\mu, w) \in \mathrm{Geo}(\mu^0,\mu^1)$. Using Corollary 3.1.42 and 1) in Proposition 3.1.45, get $C := \mathscr{N}^{f,\theta}(\mu(t), w(t))$ for a.e. $t \in [0,1]$ and estimate

$$\mathscr{W}_\nabla^{f,\theta}\left(\mu(s),\mu(t)\right) \leq \int_s^t \mathscr{N}^{f,\theta}\left(\mu(r),w(r)\right)dr = C|t-s| \tag{3.82}$$

for all $t, s \in [0,1]$. Let $[s,t] \subset [0,1]$ proper. If equality in Equation 3.81 does not hold for $[s,t] \subset [0,1]$, then there exists a distance minimiser from $\mu(s)$ to $\mu(t)$ with strictly less length than $(\mu,w)|_{[s,t]}$. Note Remark 3.1.36. This contradicts minimality on $[0,1]$. Thus equality holds in each case, hence μ is a minimising geodesic. The converse then follows by Equation 3.73 and Theorem 2.7.6 in [40]. Get 4.2). Altogether, get 4). □

Accessibility components and minimising geodesics. Definition 3.1.48 gives accessibility components of quantum optimal transport distances. They are maximal sets of states at finite distance. Corollary 3.1.50 shows accessibility components are complete geodesic length-metric spaces s.t. intrinsic distances of their length structures are quantum optimal transport distances. Thus accessibility components are maximal sets of points connected by minimising geodesics, hence metric geometry reduces to the latter. We use this throughout our discussion.

Let $(\phi,\psi,\gamma,\nabla)$ be noncommutative differential structure for tracial AF-C^*-algebras (A,τ) and (B,ω) in (f,θ)-setting.

Definition 3.1.48.

1) We call $\mathscr{C}\subset\mathscr{S}(A)$ accessible if $\mathscr{W}_{\nabla}^{f,\theta}(\mu^0,\mu^1)<\infty$ for all $\mu^0,\mu^1\in\mathscr{C}$.

2) We say that $\mathscr{C}\subset\mathscr{S}(A)$ is an accessibility component if there exists no accessible $\mathscr{C}'\subset\mathscr{S}(A)$ s.t. $\mathscr{C}'\subset\mathscr{C}$ proper. If $\mathscr{C}\subset\mathscr{S}(A)$ is an accessibility component, then we write

$$\mathscr{C}\subset\big(\mathscr{S}(A),\mathscr{W}_{\nabla}^{f,\theta}\big). \tag{3.83}$$

3) For all $\mathscr{C}\subset\big(\mathscr{S}(A),\mathscr{W}_{\nabla}^{f,\theta}\big)$, set $\mathrm{Adm}_{\mathscr{C}}:=\bigcup_{\mu,\eta\in\mathscr{C}}\mathrm{Adm}(\mu,\eta)$.

Corollary 3.1.49.

1) An equivalence relation on $\mathscr{S}(A)$ is given by

$$\mu\sim\eta\Leftrightarrow\mu,\eta\in\mathscr{C}\subset\big(\mathscr{S}(A),\mathscr{W}_{\nabla}^{f,\theta}\big) \tag{3.84}$$

for all $\mu,\eta\in\mathscr{S}(A)$.

2) For all $\mu,\eta\in\mathscr{S}(A)$, we have $\mu\sim\eta$ if and only if

2.1) $\bar{\mu}_j\sim\bar{\eta}_j$ for a.e. $j\in\mathbb{N}$,

2.2) $\limsup_{j\in\mathbb{N}}\mathscr{W}_{\nabla_j}^{f,\theta}\big(\bar{\mu}_j,\bar{\eta}_j\big)<\infty$.

Proof. Let $\mathscr{C}\subset(\mathscr{S}(A),\mathscr{W}_{\nabla}^{f,\theta})$. If $\mu^0\in\mathscr{C}$, then 1) in Theorem 3.1.47 and maximality of $\mathscr{C}$ as set of finite-length admissible paths shows

$$\mathscr{C}=\Big\{\mu^1\in\mathscr{S}(A)\ \Big|\ \exists(\mu,w)\in\mathrm{Adm}^{[0,1]}\big(\mu^0,\mu^1\big):\ L^{f,\theta}(\mu,w)<\infty\Big\}. \tag{3.85}$$

Using Equation 3.85, we directly verify Equation 3.84 defines an equivalence relation on $\mathscr{S}(A)$. Thus 1) holds, hence 2) follows from 3) in Theorem 3.1.47. □

Corollary 3.1.50. *For all $\mathscr{C} \subset (\mathscr{S}(A), \mathscr{W}_{\nabla}^{f,\theta})$, we have*

1) $\left(L^{f,\theta}, \mathrm{Adm}_{\mathscr{C}}\right)$ *is a length structure for* $\mathscr{C}$ *in* w^**-topology,*

2) $\mathscr{W}_{\nabla|\mathscr{C}\times\mathscr{C}}^{f,\theta}$ *is the unique intrinsic distance of* $\left(L^{f,\theta}, \mathrm{Adm}_{\mathscr{C}}\right)$ *on* $\mathscr{C}$*,*

3) $\left(\mathscr{C}, \mathscr{W}_{\nabla}^{f,\theta}\right)$ *is a complete geodesic length-metric space.*

Proof. Let $\mathscr{C} \subset (\mathscr{S}(A), \mathscr{W}_{\nabla}^{f,\theta})$. Equation 3.85 shows 1) and 2) alike. We see $(\mathscr{C}, \mathscr{W}_{\nabla}^{f,\theta})$ is a length-metric space. Furthermore, maximality of $\mathscr{C}$ and 4) in Theorem 3.1.47 imply it is geodesic. Thus 3) follows if we show its completeness.

Let $\{\mu^n\}_{n\in\mathbb{N}} \subset \mathscr{C}$ be a Cauchy sequence. Using 1) in Theorem 3.1.47, get $\mu \in \overline{\mathscr{S}(A)}$ s.t. $\mu = w^*\text{-}\lim_{n\in\mathbb{N}} \mu^n$. Since $\mu^n \sim \mu^m$ for all $n, m \in \mathbb{N}$, Equation 3.7 further implies

$$\mu^n(1_{A_j}) = \mu^m(1_{A_j}) \tag{3.86}$$

for all $j, n, m \in \mathbb{N}$. Using 1.1) in Proposition 2.1.31, Equation 3.86 lets us calculate

$$\|\mu\|_{A^*} = \lim_{j\in\mathbb{N}} \mu(1_{A_j}) = \lim_{j\in\mathbb{N}} \lim_{n\in\mathbb{N}} \mu^n(1_{A_j}) = \lim_{j\in\mathbb{N}} \mu^1(1_{A_j}) = \|\mu^1\|_{A^*} = 1. \tag{3.87}$$

Equation 3.87 shows $\mu \in \mathscr{S}(A)$. For all $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ s.t. $\mathscr{W}_{\nabla}^{f,\theta}(\mu^n, \mu^m) < \varepsilon$ for all $n, m \geq n_\varepsilon$. For all $\varepsilon > 0$ and $m \geq n_\varepsilon$, l.s.c. in w^*-topology as per 3) in Theorem 3.1.47 implies

$$\mathscr{W}_{\nabla}^{f,\theta}\left(\mu, \mu^m\right) \leq \liminf_{n\in\mathbb{N}} \mathscr{W}_{\nabla}^{f,\theta}\left(\mu^n, \mu^m\right) < \varepsilon. \tag{3.88}$$

Equation 3.88 shows $\lim_{m\in\mathbb{N}} \mathscr{W}_{\nabla}^{f,\theta}(\mu, \mu^m) = 0$. We obtain $\mu \in \mathscr{C}$. $\square$

We formalise here, to the extend necessary for the study of metric geometry, energy functionals being Γ-limits w.r.t. the coarse graining process as existence of sufficient minimising geodesics approximated in finite dimensions. Motivated by the sequential descriptions as per Definition 3.1.29 and used in Theorem 3.1.31, Definition 3.1.51 gives finite-dimensional approximation of minimising geodesics. We consider closure of

$$\mathrm{Geo}_0 := \bigcup_{j\in\mathbb{N}} \mathrm{Geo}_j \subset \mathrm{Geo} \tag{3.89}$$

w.r.t. suitable notion of convergence. Note 2.1) in Proposition 3.1.45 shows inclusion used in Equation 3.89. Theorem 3.1.52 gives existence of sufficient minimising geodesics approximated in finite dimensions. For details on the coarse graining process, we refer to Subsection 3.3.2.

Definition 3.1.51. Let $\mu^0,\mu^1\in\mathscr{S}(A)$ s.t. $\mathscr{W}_{\nabla}^{f,\theta}(\mu^0,\mu^1)<\infty$. We call $(\mu,w)\in\mathrm{Geo}(\mu^0,\mu^1)$ approximated in finite dimensions if there exists $m\in\mathbb{N}$ and $(\mu^j,w^j)_{j\geq m}\subset\mathrm{Geo}_0$ s.t.

1) $\left(\mu^j,w^j\right)\in\mathrm{Geo}_j\left(\bar{\mu}_j^0,\bar{\mu}_j^1\right)$ for all $j\geq m$,

2) $\left(\mu^j,w^j\right)_{j\geq m}$ has subsequence converging to (μ,w) in $\mathrm{Adm}^{[0,1]}$.

Theorem 3.1.52. *Let $(\phi,\psi,\gamma,\nabla)$ be noncommutative differential structure for tracial AF-C^*-algebras (A,τ) and (B,ω) in (f,θ)-setting. If $\mu^0,\mu^1\in\mathscr{S}(A)$ and $\mathscr{W}_{\nabla}^{f,\theta}(\mu^0,\mu^1)<\infty$, then there exists $(\mu,w)\in\mathrm{Geo}(\mu^0,\mu^1)$ approximated in finite dimensions.*

Proof. Let $\mu^0,\mu^1\in\mathscr{S}(A)$ s.t. $\mathscr{W}_{\nabla}^{f,\theta}(\mu^0,\mu^1)<\infty$. Apply 3) and 4) in Theorem 3.1.47 to get $m\in\mathbb{N}$ s.t. for all $j\geq m$, we have

$$\mathrm{Geo}_j\left(\bar{\mu}_j^0,\bar{\mu}_j^1\right)\neq\emptyset. \tag{3.90}$$

For all $j\geq m$, let $(\mu^j,w^j)\in\mathrm{Geo}_j(\bar{\mu}_j^0,\bar{\mu}_j^1)$. Using 1) in Proposition 3.1.19 and further 3) in Theorem 3.1.47, i.e. Equation 3.77, we calculate

$$\liminf_{j\in\mathbb{N}}E^{f,\theta}\left(\mu^j,w^j\right)=\liminf_{j\in\mathbb{N}}\mathscr{W}_{\nabla_j}^{f,\theta}\left(\bar{\mu}_j^0,\bar{\mu}_j^1\right)^2=\mathscr{W}_{\nabla}^{f,\theta}\left(\mu^0,\mu^1\right)^2<\infty. \tag{3.91}$$

Equation 3.91 ensures we may extract suitable subsequence. Using 2) in Lemma 3.1.26 for $t_0=0$, get subsequence of $(\mu^j,w^j)_{j\geq m}$ converging to a $(\mu,w)\in\mathrm{Adm}^{[0,1]}$. Using 1) in Lemma 3.1.26, we obtain $(\mu,w)\in\mathrm{Geo}(\mu^0,\mu^1)$ as claimed. □

The interpolation parameter. We view each symmetric representing function f as determining a class of energetic structures with $\theta\in[0,1]$ as interpolation parameter. Proposition 3.1.53 shows $\theta=0$ gives quantum $(-1,2)$-Sobolev distance independent of f. In the logarithmic mean setting, variation of $\theta\in[0,1]$ interpolates between, due to independence from f, non-geometric quantum $(-1,2)$-Sobolev distances and quantum L^2-Wasserstein distances. This follows the classical case [97].

Proposition 3.1.53. *Let $(\phi,\psi,\gamma,\nabla)$ be noncommutative differential structure for tracial AF-C^*-algebras (A,τ) and (B,ω) in $(f,0)$-setting. For all $\mu^0,\mu^1\in\mathscr{S}(A)$, we have*

$$\mathscr{W}_{\nabla}^{f,0}\left(\mu^0,\mu^1\right)=\sup\left\{\left|(\mu^1-\mu^0)(x)\right|\;\middle|\;x\in A_0,\ \|\nabla x\|_{\omega}\leq 1\right\}. \tag{3.92}$$

Proof. We reduce to the finite-dimensional setting. Assume Equation 3.92 holds in the latter. For all $\mu^0, \mu^1 \in \mathscr{S}(A)$, we use 3) in Theorem 3.1.47 to calculate

$$\begin{aligned}\mathscr{W}^{f,0}\big(\mu^0,\mu^1\big) &= \limsup_{j\in\mathbb{N}} \mu^0(1_{A_j})^{-1} \sup\left\{\left|\big(\mu^1-\mu^0\big)(x)\right| \;\middle|\; x \in A_j,\ \|\nabla x\|_\omega \le 1\right\} \\ &= \sup_{j\in\mathbb{N}} \sup\left\{\left|\big(\mu^1-\mu^0\big)(x)\right| \;\middle|\; x \in A_j,\ \|\nabla x\|_\omega \le 1\right\} \\ &= \sup\left\{\left|\big(\mu^1-\mu^0\big)(x)\right| \;\middle|\; x \in A_0,\ \|\nabla x\|_\omega \le 1\right\}.\end{aligned}$$

It suffices to show Equation 3.92 in the finite-dimensional setting.

Assume A and B are finite-dimensional. Equation 3.93 below states μ, η and f are irrelevant if $\theta = 0$. This follows since their contributions are perturbed noncommutative division operators to the power of zero, i.e. the identity operator. For all $\mu, \eta \in \mathscr{S}(A)$ and $w \in B^*$, we have

$$\mathscr{I}^{f,0}(\mu,\eta,w) = \sup_{j\in\mathbb{N}} \|w\|_\omega^2 \in [0,\infty]. \tag{3.93}$$

For all $\mu^0, \mu^1 \in \mathscr{S}(A)$, set

$$d\big(\mu^0,\mu^1\big) := \sup\left\{\left|\big(\mu^1-\mu^0\big)(x)\right| \;\middle|\; x \in A,\ \|\nabla x\|_\omega \le 1\right\}. \tag{3.94}$$

Let $\mu^0, \mu^1 \in \mathscr{S}(A)$ s.t. $d(\mu^0,\mu^1) < \infty$. Finiteness implies $\mu^1(x) = \mu^0(x)$ for all $x \in \ker\nabla$ by scaling with strictly positive constants. We therefore define bounded linear functional $F_{\mu^0,\mu^1} : \operatorname{im}\nabla \cong \ker\nabla^\perp \longrightarrow \mathbb{C}$ by setting

$$F_{\mu^0,\mu^1}(x) := \big(\mu^1-\mu^0\big)(x) \tag{3.95}$$

for all $\nabla x \in \operatorname{im}\nabla$. Equation 3.95 determines unique $w \in \operatorname{im}\nabla$ s.t. $\|w\|_\omega = d(\mu^0,\mu^1)$ and $(\mu^1-\mu^0)(x) = \langle w, \nabla x\rangle_\omega$ for all $x \in A$. We define $(\mu, w) \in \mathrm{Adm}^{[0,1]}(\mu^0,\mu^1)$ by setting

$$\mu(t) := (1-t)\mu^0 + t\mu^1,\ w(t) := w \tag{3.96}$$

for all $t \in [0,1]$. Equation 3.93 and Equation 3.96 imply

$$\mathscr{W}_\nabla^{f,0}\big(\mu^0,\mu^1\big) \le L^{f,\theta}(\mu,w) = \|w\|_\omega = d\big(\mu^0,\mu^1\big). \tag{3.97}$$

We show the converse. If $(\mu,w)\in \mathrm{Adm}^{[0,1]}(\mu^0,\mu^1)$, then Equation 3.93 shows

$$\int_0^1 \|w(t)\|_\omega dt = L^{f,0}(\mu,w). \tag{3.98}$$

Equation 3.98 in turn shows

$$\left|(\mu^1-\mu^0)(x)\right| \le \int_0^1 \|w(t)\|_\omega dt = L^{f,0}(\mu,w) \tag{3.99}$$

for all $x\in A$ s.t. $\|\nabla x\|_\omega \le 1$. Take the infimum over all admissible paths with marginals μ^0 and μ^1 in Equation 3.99, followed by the supremum over all $x\in A$ s.t. $\|\nabla x\|_\omega \le 1$. This yields the converse to Equation 3.97. Note our use of Corollary 3.1.42. Equation 3.92 holds in the finite-dimensional setting. The general case follows as discussed above. □

3.1.3 Fundamental example classes

We provide fundamental example classes. We specify neither symmetric representing function nor interpolation parameter. First, we use generalised discrete derivatives to construct quantum optimal transport distances for tracial AF-C^*-algebras parametrised over finite sets. This generalises the discrete cases [152][159] and those using internal quantum gradients. Secondly, we use dynamic quantum gradients to construct quantum optimal transport distances for tracial AF-C^*-algebras generating hyperfinite factors of type I and II by σ-weak closure. These are common algebras of observables in quantum statistical mechanics [35][36][162].

In the non-twisted case, we have an iterative construction. Self-adjoint unbounded operator with compact resolvent induce examples for type I-factors. We extend to the type II_1-factor using natural extensions of bounded operators on separable Hilbert space to elements in CAR-algebras [162] under Clifford representations [114][177]. We tensor both to the type II_∞-factor. In the twisted case, we show intertwining sets of Clifford generators yield direct sums of dynamic quantum gradients for tracial AF-C^*-algebras closing to the type II_∞-factor. In the logarithmic mean setting, the non-twisted and twisted case have non-negative, resp. strictly positive lower Ricci bounds. Thirdly, examples using non-twisted dynamic quantum gradients are given by first and second quantisation of spectral triples [54][55][197][198]. First quantisation of spectral triples gives examples for type I-factors induced by noncommutative Dirac operators. Second quantisation of spectral triples is extension to the type II_1-factor. Finally, we outline how second quantisation of spectral triples yields our ansatz to study noncommutative gauge theories [51][52][53][197][198] if we generalise to quantum optimal transport parametrised by gauge fields. We view quantum optimal transport as the pointwise case. We therefore see our discussion lies in the intersection of noncommutative gauge theory, quantum statistical mechanics and quantum information theory [62].

Standard references for factor W^*-algebras, in particular hyperfinite ones, are [173] and [192][193][194]. We refer to [162] for details on CAR-algebras, as well as [114] and [177] for Clifford representations over anti-symmetric Fock space. Standard references for noncommutative geometry are [114][198] and [197]. Whereas [114] provides a rather comprehensive treatment, note [198] gives a condensed version of the former. Standard references for quantum statistical mechanics are [35][36], [82], [121], [163] and [188].

Generalised discrete derivatives over finite sets. We use generalised discrete derivatives to construct quantum optimal transport distances in Example 3.1.54 for tracial AF-C^*-algebras parametrised over finite sets. This generalises the discrete cases [152][159] and those using internal quantum gradients.

Let X be a finite set and $u \in C(X)_+$. We define f.s.n. trace ν_u on $C(X)$ by setting

$$\nu_u(F) := \sum_{x\in X} F(x)u(x) \tag{3.100}$$

for all $F \in C(X)$. We have finite tracial AF-C^*-algebra $(C(X),\nu_u)$ as per Example 2.1.21. Let (A,τ) be a tracial AF-C^*-algebra. Since $|X| < \infty$, note $C(X,A) \cong C(X)\otimes A \cong A^{|X|}$ as AF-C^*-algebras. Proposition 2.3.32 yields tracial AF-C^*-algebra $(C(X,A),\nu_u\otimes\tau)$ in $C(X,L^\infty(A,\tau))$ generated by $\{C(X,A_j)\}_{j\in\mathbb{N}}$. We have f.s.n. trace $\nu_u\otimes\tau$ on $C(X,L^\infty(A,\tau))$ given by

$$\big(\nu_u\otimes\tau\big)(F) = \sum_{x\in X} \tau\big(F(x)\big)u(x) \tag{3.101}$$

for all $F \in C(X,L^\infty(A,\tau))_+$.

Example 3.1.54. Let X be a finite set and $K \in C(X\times X)_+$ an irreducible Markov kernel with steady state $u_K \in C(X)_+$ having full support. Let (A,τ) be a strongly unital tracial AF-C^*-algebra s.t. $\tau < \infty$. We tensor $(C(X),\nu_{u_K})$ and (A,τ) as per Equation 3.101. We likewise tensor $(C(X\times X),\nu_K)$ and $(A\otimes A,\tau\otimes\tau)$.

We have f.s.n. trace τ_K on $C(X,L^\infty(A,\tau))$ given by

$$\tau_K(F) := \big(\nu_{u_K}\otimes\tau\big)(F) = \sum_{x\in X} \tau\big(F(x)\big)u_K(x) \tag{3.102}$$

for all $F \in C(X,L^\infty(A,\tau))_+$, as well as f.s.n. trace ω_K on $C(X\times X,L^\infty(A,\tau)\otimes L^\infty(A,\tau))_+$ given by

$$\omega_K(G) := \big(\nu_K\otimes(\tau\otimes\tau)\big)(G) = \sum_{x,y\in X} (\tau\otimes\tau)\big(G(x,y)\big)K(x,y) \tag{3.103}$$

for all $G \in C(X\times X,L^\infty(A,\tau)\otimes L^\infty(A,\tau))_+$. Altogether, we obtain strongly unital tracial AF-C^*-algebra $(C(X,A),\tau_K)$ generated by $\{C(X,A_j)\}_{j\in\mathbb{N}}$, as well as $(C(X\times X,A\otimes A),\omega_K)$ generated by $\{C(X\times X,A_j\odot A_j)\}_{j\in\mathbb{N}}$.

We know Equation 3.102 shows $L^2(C(X,A),\tau_K) = C(X,L^2(A,\tau))$ and Equation 3.103 shows $L^2(C(X\times X,A\otimes A),\omega_K) = C(X\times X,L^2(A\otimes A,\tau\otimes\tau))$. Equation 2.182 further shows we define local *-homomorphisms $\phi,\psi : C(X,A) \longrightarrow C(X\times X,A\otimes A)$ by setting

$$\phi(F)(x,y) := \phi^{\mathrm{Int}}\big(F(x)\big),\ \psi(F)(x,y) := \psi^{\mathrm{Int}}\big(F(y)\big) \tag{3.104}$$

for all $F \in C(X,A)$ and $x,y \in X$. Finally, pointwise algebra involution defines anti-linear isometric involution $\gamma : C(X\times X,L^2(A\otimes A,\tau\otimes\tau)) \longrightarrow C(X\times X,L^2(A\otimes A,\tau\otimes\tau))$. We have AF-$C(X,A)$-bimodule structure (ϕ,ψ,γ) on $C(X\times X,A\otimes A)$.

Let $\lambda \geq 0$. Following Equation 3.104, we define the (K,λ)-parametrised quantum gradient $\nabla^\lambda_K : C(X,A_0) \longrightarrow C(X\times X,L^2(A\otimes A,\tau\otimes\tau))$ by setting

$$\big(\nabla^\lambda_K F\big)(x,y) := \sqrt{\frac{\lambda}{2\tau(1_A)}}\cdot\big(F(x)\otimes 1_A - 1_A\otimes F(y)\big) \tag{3.105}$$

for all $F \in C(X,A_0)$ and $x,y \in X$. We have $C(X\times X,A\otimes A) \cong C(X,A)\otimes C(X,A)$. Using the latter and up to positive constant, ∇^λ_K is the generalised discrete derivative on $C(X,A)$ as per Definition 2.3.35 restricted to $C(X,A_0)$. Proposition 2.3.36 shows said generalised discrete derivative is a bounded symmetric $C(X,A)$-module derivation. Equation 3.105 shows ∇^λ_K commutes with Hilbert space projections to generating C^*-subalgebras. Thus ∇^λ_K is a quantum gradient. If $(A,\tau) = (\mathbb{C},1)$, then Equation 3.105 specialises to the discrete derivative. If $|X| = 1$, then Equation 3.105 instead specialises to the λ-internal quantum gradient on A as per Definition 2.3.38. If $|X| > 1$, then $\nu_K \neq \nu_{u_K}\otimes\nu_{u_K}$ since K and u_K are stochastic. Hence ∇^λ_K is internal quantum gradient if and only if $|X| = 1$. Parametrised quantum gradients as per Equation 3.105 therefore generalise discrete derivatives and internal quantum gradients by using $(A,\tau) = (\mathbb{C},1)$, resp. $|X| = 1$.

We obtain noncommutative differential structures which define quantum optimal transport distances of discrete densities evaluating in tracial AF-C^*-algebras. If we use $(A,\tau) = (\mathbb{C},1)$ here with K as in [152], then we recover discrete Wasserstein distances associated to Markov chains with detailed balance condition [152]. We likewise recover [159]. In summary, we generalise the discrete cases [152][159] and any using internal quantum gradients. We recover these by using trivial codomain, resp. domain.

Dynamic quantum gradients for hyperfinite factors of type I and II. We use dynamic quantum gradients to construct quantum optimal transport distances for tracial AF-C^*-algebras generating hyperfinite factors of type I and II by σ-weak closure [35][36][162]. The iterative construction of non-twisted dynamic quantum gradients is given by following Example 3.1.55, Example 3.1.56 and Example 3.1.58 in order. Note Example 3.1.64 clarifies their importance. We construct direct sums of twisted dynamic quantum gradients, each induced by a Clifford generator, in Example 3.1.59. In the logarithmic mean setting, Example 4.3.19 and Example 4.3.20 in Subsection 4.3.2, both of which use Theorem 4.3.18, imply all examples constructed here have non-negative lower Ricci bounds. Example 4.3.20 shows strict positivity in the twisted case.

We give the iterative construction of non-twisted dynamic quantum gradients. Each step in the construction is induced by trace-preserving local C^*-dynamical systems as per Corollary 2.3.49. We cover type I-factors in Example 3.1.55, the type II_1-factor in Example 3.1.56, and the type II_∞-factor Example 3.1.58. We apply Example 3.1.55 to get first, and Example 3.1.56 to get second quantisation of spectral triples.

Example 3.1.55. Hyperfinite factors of type I are of form $\mathscr{B}(H)$ for a separable Hilbert space H. Let H be a separable Hilbert space, $D \in \mathscr{U}\mathscr{B}(H)_h$ with compact resolvent and $\{e_j\}_{j\in\mathbb{N}}$ an orthonormal eigenbasis of the latter. For all $j \in \mathbb{N}$, let $P_j : H \longrightarrow \langle e_1, \ldots, e_j\rangle_{\mathbb{C}}$ be the Hilbert space projection. The orthonormal eigenbasis determines unique unitary $U : H \longrightarrow \ell^2(\mathbb{N})$. For all $j \in \mathbb{N}$, set $H_j := P_j H$, $U_j := \mathrm{com}_{H_j} U = P_j U P_j$ and

$$A_j := \mathscr{B}(H_j) = U_j^* M_j(\mathbb{C}) U_j. \tag{3.106}$$

We have tracial AF-C^*-algebra $(\mathscr{K}(H), \mathrm{tr})$ in $\mathscr{B}(H)$ generated by $\{A_j\}_{j\in\mathbb{N}}$. We equip $\mathscr{K}(H)$ with its canonical AF-$\mathscr{K}(H)$-bimodule structure.

For all $j \in \mathbb{N}$, Equation 3.106 shows

$$\mathrm{Ad}_t^D(A_j) \subset A_j \tag{3.107}$$

for all $t \in \mathbb{R}$. Equation 3.107 shows we have tr-preserving local C^*-dynamical system $(\mathscr{K}(H), \mathbb{R}, \mathrm{Ad}^D|_{\mathscr{K}(H)})$. Note $L^2(\mathscr{K}(H), \mathrm{tr}) = S^2(H)$ for the Hilbert-Schmidt operators on H [29]. We apply Lemma 2.3.48 to get unique $\mathscr{D}_{\mathrm{Ad}} \in \mathscr{U}\mathscr{B}(S^2(H))$ s.t.

$$L_{\mathrm{Ad}_t^D(x)} = \mathrm{Ad}_t^{\mathscr{D}_{\mathrm{Ad}}}(L_x) \tag{3.108}$$

for all $t \in \mathbb{R}$ and $x \in \mathscr{K}(H)$. By norm differentiation of Equation 3.108, Corollary 2.3.49 yields non-twisted dynamic quantum gradient given by

$$\nabla^{\mathscr{D}_{\mathrm{Ad}}} x = i[\mathscr{D}_{\mathrm{Ad}}, x]_A = iL^{-1}\left(\overline{\mathscr{D}_{\mathrm{Ad}} L_x - L_x \mathscr{D}_{\mathrm{Ad}}}\right) \tag{3.109}$$

for all $x \in \mathscr{K}(H)_0$. Equation 3.108 and Equation 3.109 show $\mathscr{D}_{\mathrm{Ad}}$ is $\mathrm{id}_{\mathscr{K}(H)}$-intertwining. We know the identities for dynamic quantum gradient, its adjoint and finally Laplacian given in Corollary 2.3.56 and Corollary 2.3.57 apply.

We pull back along L to $\mathscr{K}(H)$ as follows. For all $j \in \mathbb{N}$, note Equation 3.106 shows D is H_j-reducible and set $D_j := \mathrm{com}_{H_j} D = P_j D P_j$. For all $t \in \mathbb{R}$ and $j \in \mathbb{N}$, arguing as for Equation 2.209 in Proposition 2.3.52 shows

$$e^{itD} = e^{itD_j} \oplus e^{itD_j^\perp} \tag{3.110}$$

w.r.t. $\mathscr{B}(H_j) \oplus \mathscr{B}(H_j^\perp)$.

Note $\mathscr{K}(H)_0 \subset \mathscr{B}(H)$ is strongly dense and $\|e^{itD}\|_{\mathscr{B}(H)} = 1$ in each case. Using the latter, Equation 3.110 and sequential strong continuity of multiplication show

$$t \mapsto L_{e^{itD}} \in \mathscr{U}\big(\mathscr{B}\big(S^2(H)\big)\big) \tag{3.111}$$

is a strongly continuous unitary group. Equation 3.108 additionally shows

$$L_{e^{itD}} L_x L_{e^{-itD}} = \mathrm{Ad}_t^{\mathscr{D}_{\mathrm{Ad}}}(L_x) \tag{3.112}$$

for all $t \in \mathbb{R}$, $j \in \mathbb{N}$ and $x \in A_j$. Equation 3.112 extends to $\mathscr{K}(H)$ by norm density. Then uniqueness, Corollary 2.3.57, and Equation 3.112 imply

$$\nabla^D x := \nabla^{\mathscr{D}_{\mathrm{Ad}}} x = i\big[D_j, x\big] = i\big(D_j x - x D_j\big) = \overline{i(Dx - xD)} \tag{3.113}$$

for all $j \in \mathbb{N}$ and $x \in A_j$. The identities for dynamic quantum gradient, its adjoint and Laplacian given in Corollary 2.3.56 and Corollary 2.3.57 pull back accordingly.

We obtain noncommutative differential structures which define quantum optimal transport distances of density operators. Note all constructions of non-twisted dynamic quantum gradients reduce to Equation 3.113 in this example. In the logarithmic mean setting, which uses $\theta = 1$, accessibility components are complete geodesic length-metric spaces even for $\dim_{\mathbb{C}} H = \infty$. We use this for first quantisation of spectral triples.

Equation 3.114 are abstract canonical anti-commutation relations of CAR-algebras [162]. Clifford representations determined by Equation 3.120 provide natural concrete realisations [114][177]. Let H be a separable Hilbert space. The CAR-algebra $\mathscr{A}(H)$ over H is defined as the unique unital C^*-algebra, up to isometric *-isomorphism, equipped with a bounded anti-linear map $a : H \longrightarrow \mathscr{A}(H)$ s.t. $C^*(\operatorname{im} a, 1_{\mathscr{A}(H)}) = \mathscr{A}(H)$ and

$$a(u)^* a(v) + a(v) a(u)^* = \langle u, v\rangle_H \cdot 1_{\mathscr{A}(H)},\ a(u)a(v) + a(v)a(u) = 0 \tag{3.114}$$

for all $u, v \in H$. We consider CAR-algebras as Clifford algebras here. For all $u \in H$, set $b(u) := a(u) + a(u)^*$. Then Equation 3.114 are equivalent to the Clifford relations

$$b(u)b(v) + b(v)b(u) = 2\,\mathrm{Re}\,\langle u, v\rangle_H \cdot 1_{\mathscr{A}(H)} \tag{3.115}$$

for all $u, v \in H$. Thus the universal property of Clifford algebras applies and lets us extend bounded linear maps preserving Equation 3.114, hence Equation 3.115, from H to $\mathscr{A}(H)$ (cf. Proposition 5.1 in [114]). For all $\varphi \in \mathscr{U}(\mathscr{B}(H))$, said universal property determines the Bogoliubov automorphism $\mathrm{Cliff}(\varphi) \in \mathrm{Aut}(\mathscr{A}(H))$ of φ by setting

$$\mathrm{Cliff}(\varphi)\big(a(u)\big) := a\big(\varphi(u)\big) \tag{3.116}$$

for all $u \in H$ (cf. Example 5.2 in [114]).

We determine l.s.c. faithful semi-finite trace τ on $\mathscr{A}(H)$ by setting

$$\tau\big(a(u_1)^* \dots a(u_n)^* a(v_m) \dots a(v_1)\big) := \delta_{nm} \det\left(\frac{1}{2}\big(\langle u_k, v_l\rangle_H\big)_{k,l=1}^n\right) \tag{3.117}$$

for all $n, m \in \mathbb{N}$ and $\{u_k\}_{k=1}^n, \{v_l\}_{l=1}^m \subset H$ [162]. Note τ is the unique normalised trace on $\mathscr{A}(H)$. If $\dim_{\mathbb{C}} H = n$, then $(\mathscr{A}(H), \tau) \cong (\otimes_{k=1}^n M_2(\mathbb{C}), 2^{-n} \otimes_{k=1}^n \mathrm{tr}_2) \cong (M_{2^n}(\mathbb{C}), 2^{-n}\,\mathrm{tr}_{2^n})$ as tracial C^*-algebras [162]. If $\dim_{\mathbb{C}} H = \infty$, then $\mathscr{A}(H)''$ is the hyperfinite factor of type II_1 up to choice of faithful unital *-representation [173].

We associate faithful unital *-representations of CAR-algebras over anti-symmetric Fock space to orthogonal complex structures. Such representations are called Clifford representations [114][177]. We equip H with Euclidean structure of $\|.\|_H$. Let J be an orthogonal complex structure on H. Using J as imaginary unital left-multiplication, we complexify to $H[J] = H \oplus iH$. We define inner product of $H[J]$ by setting

$$\langle u, v\rangle_{H[J]} := \mathrm{Re}\,\langle u, v\rangle_H + i\,\mathrm{Re}\,\langle u, J(v)\rangle_H \tag{3.118}$$

for all $u, v \in H[J]$. Thus $(H[J], \|.\|_J)$ is a Hilbert space. Equation 3.118 induces inner product $\bigwedge \|.\|_J$ of anti-symmetric Fock space $\mathscr{F}(H[J]) := \bigwedge H[J]$ by universal property of the exterior algebra [114]. Hence $(\mathscr{F}(H[J]), \bigwedge \|.\|_J)$ is a Hilbert space. We define bounded anti-linear map $a_J : H \longrightarrow \mathscr{B}(\mathscr{F}(H[J]))$ by setting

$$\big(a_J(u)\big)^*(v) := u \wedge v \tag{3.119}$$

for all $u \in H$ and $v \in \mathscr{F}(H[J])$. Using $\bigwedge \|.\|_J$ and Equation 3.119 to obtain adjoints in $\mathscr{B}(\mathscr{F}(H[J]))$, we directly verify Equation 3.114 for a_J (cf. pp.186-187 in [114]).

Finally, we determine the Clifford representation $\rho_J : \mathscr{A}(H) \longrightarrow \mathscr{B}(\mathscr{F}(H[J]))$ for J by setting

$$\rho_J(u) := a_J(u) + a_J(u)^* \tag{3.120}$$

for all $u \in H$. Note $\rho_J(u) = \rho_J(b(u))$ in each case since we consider $H \cong b(H) \subset \mathscr{A}(H)$ as set of generators. Thus $2a_J(u) = \rho_J(u) - i\rho_J(J(u))$ for all $u \in H$ by (anti-)linearity, hence ρ_J is a faithful unital *-representation s.t.

$$\mathscr{A}(H[J]) := \rho_J(\mathscr{A}(H)) \cong \mathscr{A}(H) \tag{3.121}$$

is CAR-algebra over H and Clifford algebra of $\|.\|_H^2$. Note the unique normalised and l.s.c. faithful semi-finite trace τ on $\mathscr{A}(H[J])$ is determined by Equation 3.117 for a_J. We have tracial C^*-algebra $(\mathscr{A}(H[J]), \tau)$ in $\mathscr{A}(H[J])'' \subset \mathscr{B}(\mathscr{F}(H[J]))$.

Example 3.1.56. The hyperfinite factor of type II_1 is $\mathscr{A}(H)''$ for a separable Hilbert space H. Let H be a separable Hilbert space, J orthogonal complex structure on H and $H_1 \subset H_2 \subset \ldots \subset H$ Hilbert subspaces with

$$H = \overline{\bigcup_{j\in\mathbb{N}} H_j}^{\|.\|_H} \tag{3.122}$$

and s.t.

$$J(H_j) \subset H_j \tag{3.123}$$

for all $j \in \mathbb{N}$. Equation 3.123 shows J is orthogonal complex structure on H_j and

$$H_j[J] \subset H_{j+1}[J] \subset H[J] \tag{3.124}$$

in each case. Equation 3.122 and Equation 3.124 show $H[J] = \overline{\bigcup_{j\in\mathbb{N}} H_j[J]}^{\|.\|_{H[J]}}$. They moreover show analogous restriction properties for ρ_J and $\mathscr{A}(H[J])$. We have tracial AF-C^*-algebra $(\mathscr{A}(H[J]),\tau)$ in $\mathscr{A}(H[J])'' \subset \mathscr{B}(\mathscr{F}(H[J]))$ generated by $\{\mathscr{A}(H_j[J])\}_{j\in\mathbb{N}}$. We equip $\mathscr{A}(H[J])$ with its canonical AF-$\mathscr{A}(H[J])$-bimodule structure.

We construct τ-preserving local C^*-dynamical system. Let $\varphi \in \mathscr{U}(\mathscr{B}(H))$ s.t. we have $\varphi(H_j), \varphi^{-1}(H_j) \subset H_j$ for all $j \in \mathbb{N}$. Using $\varphi^{-1} = \varphi^*$, we directly verify $\mathrm{com}_{H_j} \varphi \in \mathscr{U}(\mathscr{B}(H_j))$ for all $j \in \mathbb{N}$. We obtain the J-twisted Bogoliubov automorphism

$$\mathrm{Cliff}_J(\varphi) := \rho_J \circ \mathrm{Cliff}(\varphi) \circ \rho_J^{-1} \in \mathrm{Aut}(\mathscr{A}(H[J])) \tag{3.125}$$

s.t. $\mathrm{Cliff}_J(\varphi)|_{\mathscr{A}(H_j[J])} = \rho_J \circ \mathrm{Cliff}(\mathrm{com}_{H_j} \varphi) \circ \rho_J^{-1} \in \mathrm{Aut}(\mathscr{A}(H_j[J]))$ for all $j \in \mathbb{N}$. We select compatible Dirac operator. Let $D \in \mathscr{U}\mathscr{B}(H)_h$ with compact resolvent and orthonormal eigenbasis $\{e_j\}_{j\in\mathbb{N}}$. For all $j \in \mathbb{N}$, let $H_j = \langle e_1, \ldots, e_j\rangle_{\mathbb{C}}$. For all $t \in \mathbb{R}$, Equation 3.110 shows $e^{itD} \in \mathscr{U}(\mathscr{B}(H))$ satisfies our assumptions on φ in Equation 3.125. For all $t \in \mathbb{R}$, set

$$\alpha_t := \mathrm{Cliff}_J\left(e^{itD}\right) \in \mathrm{Aut}(\mathscr{A}(H[J])). \tag{3.126}$$

For all $t \in \mathbb{R}$ and $j \in \mathbb{N}$, Equation 3.117 shows α_t is τ-preserving and Equation 3.125 shows $\alpha_t(\mathscr{A}(H_j[J])) \subset \mathscr{A}(H_j[J])$. We show strong continuity of $\alpha : \mathbb{R} \longrightarrow \mathrm{Aut}(\mathscr{A}(H[J]))$ to conclude. Since $\|e^{itD}\|_{\mathscr{B}(H)} = 1$ for all $t \in \mathbb{R}$ by functional calculus, we see locality of α as above in fact reduces to $\|.\|_{\mathscr{A}(H[J])}$-continuity upon evaluation on $\mathscr{A}(H[J])_0$. Taken together with the *-homomorphism property, we further reduce to $\|.\|_{\mathscr{A}(H[J])}$-continuity upon evaluation on $a(\bigcup_{j\in\mathbb{N}} H_j)$.

For all $u \in \bigcup_{j\in\mathbb{N}} H_j$, Equation 3.116 shows the map

$$t \mapsto \alpha_t\big(a_J(u)\big) = \rho_J\big(\mathrm{Cliff}\big(e^{itD}\big)\big(a(u)\big)\big) = a_J\big(e^{itD}u\big) \tag{3.127}$$

is $\|.\|_{\mathscr{A}(H[J])}$-continuous. Equation 3.127 implies strong continuity. We have τ-preserving local C^*-dynamical system $(\mathscr{A}(H[J]),\mathbb{R},\alpha)$. Corollary 2.3.49 yields non-twisted dynamic quantum gradient. In Example 3.1.62, note Equation 3.146 gives an explicit formula for Equation 3.126 using wedged conjugation group of $|D|$ and for suitable J depending on eigenvalues of D. The formula is taken from Proposition 2.6 in [55]. However, we explicitly solve the associated implementation problem [177] in Lemma C.2.1.

We obtain noncommutative differential structures which define quantum optimal transport distances of states on CAR-algebras. Note the non-twisted dynamic quantum gradients used are induced by trace-preserving local C^*-dynamical systems lifted from Example 3.1.55 via Clifford representations to Equation 3.126. We use this to get second quantisation of spectral triples as extension of their first quantisation.

Remark 3.1.57. In contrast to Example 3.1.55, the construction of dynamic quantum gradients in Example 3.1.56 does not pull back along canonical left-actions. We derive explicit formula for Equation 3.126 in Example 3.1.62 using wedged conjugation groups analogous to the construction in Example 3.1.55. Choice of orthogonal complex structure is necessary to solve the associated implementation problem.

Example 3.1.58. The hyperfinite factor of type II_∞ is W^*-tensor product $\mathscr{B}(H)\otimes\mathscr{A}(\mathscr{H})''$ for infinite-dimensional separable Hilbert spaces H and $\mathscr{H}$. We do not identify $H \cong \mathscr{H}$ since their finite-dimensional approximation differs in general. Let H be a separable Hilbert space and assume the setting of Example 3.1.55 for $D \in \mathscr{U}\mathscr{B}(H)_h$. Let $\mathscr{H}$ be a separable Hilbert space and assume the setting of Example 3.1.56 for $\mathscr{D} \in \mathscr{U}\mathscr{B}(\mathscr{H})_h$. By 1) in Proposition 2.3.32, the tensor product construction yields tracial AF-C^*-algebra $(\mathscr{K}(H)\otimes\mathscr{A}(\mathscr{H}[J]),\mathrm{tr}\otimes\tau)$ in $\mathscr{B}(H)\otimes\mathscr{A}(\mathscr{H}[J])''$ generated by $\{\mathscr{K}(H_j)\odot\mathscr{A}(\mathscr{H}_j[J])\}_{j\in\mathbb{N}}$. We equip $\mathscr{K}(H)\otimes\mathscr{A}(\mathscr{H}[J])$ with its canonical AF-$\mathscr{K}(H)\otimes\mathscr{A}(\mathscr{H}[J])$-bimodule structure.

We require τ-preserving local C^*-dynamical system. For all $t \in \mathbb{R}$, set

$$\alpha_t := \mathrm{Ad}_t^D \otimes \mathrm{Cliff}_J\big(e^{it\mathscr{D}}\big) \in \mathrm{Aut}(\mathscr{K}(H)\otimes\mathscr{A}(\mathscr{H}[J])). \tag{3.128}$$

Example 3.1.55 and Example 3.1.56 show we have $\mathrm{tr}\otimes\tau$-preserving local C^*-dynamical system $(\mathscr{K}(H)\otimes\mathscr{A}(\mathscr{H}[J]),\mathbb{R},\alpha)$ by reducing to elementary tensors. Corollary 2.3.49 yields non-twisted dynamic quantum gradient. Since the latter are furthermore defined by norm differentiation, it is the tensor product quantum gradient of the dynamic ones as per Example 3.1.55 and Example 3.1.56 given by Proposition 2.3.32.

We obtain noncommutative differential structures which define quantum optimal transport distances of density operators evaluating in CAR-algebras. The non-twisted dynamic quantum gradients used are induced by trace-preserving local C^*-dynamical systems which are tensor products of those in Example 3.1.55 and Example 3.1.56. This finalises our three-step iterative construction.

We construct direct sums of twisted dynamic quantum gradients induced by Clifford generators. For this, we use tracial AF-C^*-algebras in the setting of Example 3.1.58.

Example 3.1.59. Let H and $\mathscr{H}$ be separable Hilbert spaces. Let $T \in \mathscr{B}(H)_h$ s.t. we have $\operatorname{spec} T \subset \{\pm 1\}$ and with orthonormal eigenbasis $\{u_j\}_{j\in\mathbb{N}}$. Let $\{v_j\}_{j\in\mathbb{N}}$ be orthonormal basis of $\mathscr{H}$. Let $m \in \mathbb{N}$. For all $j \in \mathbb{N}$, set

$$H_j := \langle u_1, \dots, u_j \rangle_{\mathbb{C}}, \ \mathscr{H}_j := \langle v_1, \dots, v_{m-1}, \dots v_{m-1+j} \rangle_{\mathbb{C}}. \tag{3.129}$$

We use trivial orthogonal complex structure $J := iI_{\mathscr{H}}$ on $\mathscr{H}$ and suppress it. Using finite-dimensional approximation given by Equation 3.129 and following construction in Example 3.1.58, we have tracial AF-C^*-algebra $(\mathscr{K}(H) \otimes \mathscr{A}(\mathscr{H}), \operatorname{tr} \otimes \tau)$ generated by $\{\mathscr{K}(H_j) \odot \mathscr{A}(\mathscr{H}_j)\}_{j\in\mathbb{N}}$. We determine the principle automorphism $\varphi \in \operatorname{Aut}(\mathscr{A}(H))$ of $\mathscr{A}(H)$ by setting

$$\varphi\big(\rho(u)\big) = -\rho(u) \tag{3.130}$$

for all $u \in \mathscr{H}$. Since φ is a self-adjoint involutive local *-homomorphism, we know $\phi := \mathrm{id}_{\mathscr{B}(H)} \otimes \varphi \in \operatorname{Aut}(\mathscr{K}(H) \otimes \mathscr{A}(\mathscr{H}))$ is one. We have AF-$\mathscr{K}(H) \otimes \mathscr{A}(\mathscr{H})$-bimodule structure $(\phi, \mathrm{id}_{\mathscr{K}(H)\otimes\mathscr{A}(\mathscr{H})}, \gamma^{\phi})$ on $\mathscr{K}(H) \otimes \mathscr{A}(\mathscr{H})$ as per 1) in Proposition 2.3.44.

Let $C > 0$. For all $n \in \{1, \dots, m\}$, set $d_n := T \otimes C^{\frac{1}{2}} \rho(v_n)$. Get $T^2 = I_H$ by $\operatorname{spec} T \subset \{\pm 1\}$. Equation 3.115 and Equation 3.129 show $\{d_n\}_{n=1}^m \subset \mathscr{B}(H) \otimes \mathscr{A}(\mathscr{H})''$ is a ϕ-intertwining set of Clifford generators for $C > 0$ as per 1) in Definition 2.3.58. For all $n \in \{1, \dots, m\}$, we know Corollary 2.3.56 yields twisted dynamic quantum gradient $\partial_n = \nabla^{-iL_{d_n}, \phi}$ and its Laplacian $\Delta_n = \partial_n^* \partial_n$ as per 2) in Definition 2.3.58. Proposition 2.3.29 yields direct sum quantum gradient $\nabla^{\oplus} = \oplus_{n=1}^m \partial_n : \mathscr{K}(H)_0 \odot \mathscr{A}(\mathscr{H})_0 \longrightarrow L^2(\oplus_{n=1}^m \mathscr{K}(H) \otimes \mathscr{A}(\mathscr{H}), \oplus_{n=1}^m \operatorname{tr} \otimes \tau)$ given by

$$\nabla^{\oplus} x = (\partial_1 x, \dots, \partial_m x) = \big(\nabla^{-iL_{d_1}, \phi} x, \dots, \nabla^{-iL_{d_m}, \phi} x\big) \tag{3.131}$$

for all $x \in \mathscr{K}(H)_0 \odot \mathscr{A}(\mathscr{H})_0$. Since $\Delta^{\oplus} = \sum_{n=1}^m \Delta_n$ by 4) in Proposition 2.3.29, Lemma 2.3.59 implies

$$\partial_n \Delta^{\oplus} = \big(\Delta^{\oplus} + 4C \cdot I\big) \partial_n \tag{3.132}$$

for all $n \in \{1, \dots, m\}$. Note Equation 3.132 lets us apply Theorem 4.3.18 to show strictly positive lower Ricci bounds in Example 4.3.20. If we rescale each partial gradient of $\nabla^{\oplus}$ as $\partial_n \mapsto \lambda \partial_n$ for $\lambda > 0$, then $4C$ in Equation 3.132 is $\lambda \cdot 4C$ instead.

We obtain noncommutative differential structures which define quantum optimal transport distances of density operators evaluating in CAR-algebras. Yet in contrast to Example 3.1.58, we use direct sum quantum gradients of twisted dynamic quantum gradients induced by intertwining sets of Clifford generators. Equation 3.132 holds and implies strictly positive lower Ricci bounds. We therefore obtain arbitrary lower Ricci bounds by rescaling this example.

Remark 3.1.60. Example 3.1.59 for $H = \mathbb{C}$, $\dim_{\mathbb{C}} \mathscr{H} < \infty$, and fixed $C = \frac{1}{4}$ is introduced in [48]. For all $j \in \mathbb{N}$, note $\rho(v_{m+j+1})\mathscr{A}(\mathscr{H}_j) \subset \mathscr{A}(\mathscr{H}_j)^{\perp} \subset L^2(\mathscr{A}(\mathscr{H}),\tau)$. If $m = \infty$, then we cannot use $\{\partial_n\}_{n\in\mathbb{N}}$ as noncommutative directional derivatives since locality is violated if we do not fix $m < \infty$ for Equation 3.129 in Example 3.1.59.

First and second quantisation of spectral triples. Connes' program of noncommutative geometry [67][69][137][138] unifies continuous and discrete geometries [114][197][198] using operator theory [29][192][193][194]. His spectral reconstruction theorem shows commutative real spectral triples are operator algebraic formulation of compact spin geometry [68]. All real spectral triples define noncommutative gauge theories [51][197][198]. Inner fluctuations of noncommutative Dirac operators [51][54][55][197][198], the latter being given as part of real spectral triples, determine a spectral action on gauge fields [51][52][53]. Following the spectral action principle of Connes and Chamseddine [52], it is used as action functional driving the dynamics of bosonic gauge fields [51][197][198]. This spectral paradigm derives the Standard Model of particle physics [118] from almost commutative geometries [53], i.e. mixed continuous-discrete noncommutative geometries. We review noncommutative gauge theory, give first and second quantisation of spectral triples, and outline how the latter yields our ansatz to study noncommutative gauge theories based on a proposed internalised spectral action if we generalise to quantum optimal transport parametrised by gauge fields.

We review noncommutative gauge theory. All spectral triples $(\mathfrak{A},H,D)$ consist of a unital pre-C^*-algebra $\mathfrak{A}$, faithful unital *-representation $\pi : \mathfrak{A} \longrightarrow \mathscr{B}(H)$ over separable Hilbert space H, as well as $D \in \mathscr{U}\mathscr{B}(H)_h$ with compact resolvent (cf. Definition 4.30 in [197]). Moreover, note D satisfies properties showing it is a noncommutative analogue of an Atiyah–Singer–Dirac operator. We say that $(\mathfrak{A},H,D)$ is a real spectral triple if it is further equipped with real structure J on H intertwining with D s.t. the commutant property and first-order condition

$$\left[\pi(x), J\pi(y)^* J^{-1}\right] = 0,\ \left[\overline{D\pi(x) - \pi(x)D}, J\pi(y)^* J^{-1}\right] = 0 \tag{3.133}$$

are satisfied for all $x,y \in \mathfrak{A}$ (cf. Equation 4.3.1 in [197]). The first-order condition as per Equation 3.133 is an operator algebraic characterisation of D as differential operator of order one. We ignore gradient operators here as they only signify even or odd dimension. We may disregard the first-order condition [54] but do not do so here. First quantisation of compact spin manifolds as per Example 3.1.61 clarifies the above analogies as it gives all commutative real spectral triples [68]. Equation 3.138 implies triviality of gauge groups in this case. We see general real spectral triples are necessary to describe abelian and non-abelian gauge theories [123]. Second quantisation of spectral triples as per Example 3.1.62 yields description of the spectral action as per Equation 3.137 in terms of quantum statistical mechanics [35][36] as per Equation 3.150 using quantum relative entropy as per Definition 4.1.12. Two essential results in Example 3.1.62 are taken from [55]. It leads us to formulate the internalised spectral action as per Equation 3.160.

To this end, we summarise relevant parts of noncommutative gauge theories defined by real spectral triples satisfying the first-order condition. Let $(\mathfrak{A},H,D,J)$ be such a real spectral triple. Norm closure of $\mathfrak{A}$ generates unital C^*-algebra A s.t. $\pi : A \longrightarrow \mathscr{B}(H)$ is faithful unital *-representation. Its $\mathfrak{A}$-bimodule of differential one-forms is defined by

$$\Omega^1_D(\mathfrak{A}) := \left\{ T \in \mathscr{B}(H) \;\middle|\; \exists (x_k,y_k)_{k=1}^n \subset \mathfrak{A}\times\mathfrak{A} :\; T = \sum_{k=1}^n \pi(x_k)\cdot \overline{D\pi(y_k) - \pi(y_k)D} \right\} \tag{3.134}$$

(cf. Definition 4.36 in [197]). Closure of unbounded commutators in Equation 3.134 is ensured by the axioms of spectral triples. Moreover, get $\epsilon \in \{\pm 1\}$ s.t. $JDJ^{-1} = \epsilon D$. For all hermitian connections $\nabla : \mathfrak{A} \longrightarrow \Omega^1_D(\mathfrak{A})$, Theorem 6.15 and Theorem 6.16 in [197] imply the inner fluctuation of D defined by

$$D_T := D + T + \epsilon J T J^{-1} \tag{3.135}$$

with $T := \nabla 1_A \in \Omega^1_D(\mathfrak{A}) \cap \mathscr{B}(H)_h$ yields real spectral triple $(\mathfrak{A},H,D_T,J)$ (cf. pp.112-114 in [197]). We say that $T \in \Omega^1_D(\mathfrak{A}) \cap \mathscr{B}(H)_h$ is a gauge field in this case. Proposition 6 in [54] shows we have gauge semigroup

$$\mathrm{Inn}(\mathfrak{A},H,D) := \left\{ T \in \Omega^1_D(\mathfrak{A}) \cap \mathscr{B}(H)_h \;\middle|\; T \text{ is a gauge field} \right\} \tag{3.136}$$

of $(\mathfrak{A},H,D,J)$. Its semigroup structure is not relevant to us. The map $T \mapsto D_T$ defined on $\mathrm{Inn}(\mathfrak{A},H,D)$ is a deformation of noncommutative Dirac operators parametrised by gauge fields. Assuming even $h : \mathbb{R} \longrightarrow [0,\infty)$ s.t. finite trace is ensured in Equation 3.137 below, the spectral action $S_b : \mathrm{Inn}(\mathfrak{A},H,D) \longrightarrow \mathbb{R}$ is defined by

$$S_b(T) := \mathrm{tr}\left(h(D_T)\right) \tag{3.137}$$

for all $T \in \mathrm{Inn}(\mathfrak{A},H,D)$ (cf. Definition 7.1 in [197]). We give suitable h for our purposes in Example 3.1.62. The subscript of S_b denotes its use as action functional driving the dynamics of bosonic gauge fields (cf. Theorem 11.10 in [197]). There exist alternatives for other gauge fields, e.g. the fermionic action (cf. Definition 7.1 in [197]).

The spectral action is a spectral invariant of $(\mathfrak{A},H,D,J)$. Proposition 6.17 in [197] shows each unitary Morita self-equivalences of $(\mathfrak{A},H,D,J)$ is implemented by a unique $U = \pi(u)J\pi(u)J^{-1} \in \mathscr{U}(H)$ for $u \in \mathscr{U}(\mathfrak{A})$ s.t. $T_U = \pi(u)\overline{D\pi(u) - \pi(u)D} \in \mathrm{Inn}(\mathfrak{A},H,D)$ is a gauge field. Proposition 6.5 in [197] shows we have gauge group

$$\mathfrak{G}(\mathfrak{A},H,D) := \left\{ U \in \mathscr{U}(H) \;\middle|\; \exists u \in \mathscr{U}(\mathfrak{A}) :\; U = \pi(u)J\pi(u)J^{-1} \right\} \cong \mathscr{U}(\mathfrak{A})/\mathscr{U}(\mathfrak{A}_J) \tag{3.138}$$

of $(\mathfrak{A},H,D,J)$ (cf. Definition 6.4 in [197]). Note $\mathscr{U}(A_j) \lhd \mathscr{U}(A)$ as for Equation 3.138 since $A_J = \{x \in A \mid \pi(x)J = J^*\pi(x)\} \subset Z(A)$. For all $U \in \mathfrak{G}(\mathfrak{A},H,D)$, we have $D_{T_U} = UDU^*$ by the first-order condition. We therefore see Equation 3.137 is invariant under gauge transformations (cf. Lemma A.1.89 and Lemma A.1.92).

We give first and second quantisation of spectral triples. Example 3.1.61 gives first quantisation of compact spin manifolds [68]. We further include general spectral triples as their own first quantisation by convention. Example 3.1.62 gives second quantisation of spectral triples [55]. Remark 3.1.63 briefly reviews the terminology of first and second quantisation as used in our discussion. Both underlying fundamental example classes use non-twisted dynamic quantum gradients arising from weak, equivalently norm, differentiation of trace-preserving C^*-dynamical systems determined by noncommutative Dirac operators, i.e. assumes fixed gauge field. Example 3.1.61 and Example 3.1.62 give quantum optimal transport without considering spatial coordinates. Upon passing to second quantisation, we introduce gauge fields as spatial coordinates. Example 3.1.64 generalises to quantum optimal transport parametrised by gauge fields via deforming noncommutative Dirac operators as per Equation 3.135.

We assume fixed gauge field and summarise results. First, Example 3.1.61 arises from a conjugation group which splits into a spatial and quantum component as per Equation 3.142. We see quantum optimal transport is transversal to spatial optimal transport in this case. Equation 3.140 shows the spatial component is generated by a quantisation of the gradient w.r.t. the given Riemannian metric using the Clifford action [197][198]. Secondly, Example 3.1.62 rectifies transversality by quantising all spatial coordinates as per Equation 3.146. We instead have a description in terms of quantum statistical mechanics without considering spatial coordinates. Equation 3.146 gives an explicit formula for Equation 3.126. The formula is taken from Proposition 2.6 in [55]. However, note we explicitly solve the associated implementation problem [177] in Lemma C.2.1. Assuming trace-class, Equation 3.146 shows the given non-twisted dynamic quantum gradient is infinitesimal evolution of observables in wedged Heisenberg representation at thermal equilibrium determined by a KMS-state [36] of the given trace-preserving local C^*-dynamical system. Up to sign, Corollary 2.3.49 shows such description transfers to quantum Laplacians by twice application. We therefore expect properties of quantum optimal transport as stated in the introduction of this chapter.

Example 3.1.61. We assume commutative real spectral triples, i.e. first quantisation of compact spin manifolds [68]. Let (X,g) be a compact spin manifold, $S \longrightarrow X$ its spinor bundle and D its Atiyah–Singer–Dirac operator [68][195][197][198].

We have unital pre-C^*-algebra $C^\infty(X)$ and C^*-algebra $C(X)$ (cf. Example A.1.18). For all $x \in X$, the finite-dimensional Clifford algebra $S_x = \mathscr{A}(T_x^*X)$ has inner product induced by the cotangent Riemannian metric. We extend pointwise left-multiplication of scalars from $C^\infty(X,T^*X)$ to $L^2(X,S)$. This defines faithful unital *-representation $L : C(X) \longrightarrow \mathscr{B}(L^2(X,S))$. Fibrewise right-multiplication of elements in Clifford algebras defines Clifford action $c : C^\infty(X,T^*X) \longrightarrow \mathscr{B}(L^2(X,S))$. Up to L_{-i}, D is the concatenation of c and the spin connection of (X,g), i.e. the unique lift of the Levi-Civita connection associated to (X,g) from T^*X to S. The charge conjugation J_X of S is a suitable real structure on $L^2(X,S)$. Altogether, we construct the canonical commutative real spectral triple $(\mathfrak{A},H,D,J) = (C^\infty(X),L^2(X,S),D,J_X)$ [68]. We assume the latter without loss of generality. For details on the construction and our application of its properties, we refer to Chapter 4 in [197] and Chapter 3 in [198].

We know spectra of elements in $\mathcal{K}(L^2(X,S))_h$ are discrete by the spectral theorem for self-adjoint unbounded operators (cf. Theorem 5.7 in [184]). Continuity of elements in $C(X)$ implies $L(C(X)) \cap \mathcal{K}(L^2(X,S)) = 0$ by the intermediate value theorem as spectra of continuous functions on X are subsets of their images by compactness.

We claim the conjugation group $\mathrm{Ad}^D : \mathbb{R} \longrightarrow \mathrm{Aut}(\mathcal{B}(L^2(X,S)))$ of D given by

$$\mathrm{Ad}^D_t(S) = e^{itD} S e^{-itD} \tag{3.139}$$

for all $t \in \mathbb{R}$ and $S \in \mathcal{B}(L^2(X,S))$ splits into a spatial and quantum component as per Equation 3.142 upon restriction to $L(C(X)) \oplus \mathcal{K}(L^2(X,S)) \subset \mathcal{B}(L^2(X,S))$. We consider their generators. If $h \in C^\infty(X)$, then the two conditions for Equation 2.189 are met for $H = L^2(X,S)$, $\mathcal{D} = D$ and $S = L_h$. As per Theorem 4.20 in [197] and explained on pp.8-10 in [198], we obtain

$$\nabla^{\mathrm{spt}} h := \frac{d}{dt}\bigg|_{t=0,\mathrm{w}} \mathrm{Ad}^D_t(L_h) = \overline{i(DL_h - L_h D)} = -ic(dh) \tag{3.140}$$

for all $h \in C^\infty(X)$. Smoothness and Equation 3.140 imply $-ic(dh) \in L(C^\infty(X))$ in each case by the first-order property (cf. Equation 4.3.1 in [197]). As such, Equation 3.140 integrates to $\mathrm{Ad}^D|_{L(C(X))} : \mathbb{R} \longrightarrow \mathrm{Aut}(L(X))$. Applying constructions in Example 3.1.55 to $H = L^2(X,S)$ and $D \in \mathcal{U}\mathcal{B}(L^2(X,S))$, we see Equation 3.113 yields non-twisted dynamic quantum gradient $\nabla^{\mathrm{qtm}} : \mathcal{K}(L^2(X,S))_0 \longrightarrow S^2(L^2(X,S))$ given by

$$\nabla^{\mathrm{qtm}} x := \nabla^D x = \frac{d}{dt}\bigg|_{t=0,\mathrm{w}} \mathrm{Ad}^D_t(T) = \overline{i(Dx - xD)} \tag{3.141}$$

for all $x \in \mathcal{K}(L^2(X,S))_0$. Note $\overline{i(Dx - xD)} \in \mathcal{K}(L^2(X,S))_0$ in each case by construction. As such, Equation 3.141 integrates to $\mathrm{Ad}^D|_{\mathcal{K}(L^2(X,S))} : \mathbb{R} \longrightarrow \mathrm{Aut}(\mathcal{K}(L^2(X,S)))$.

Using $L(C(X)) \cap \mathcal{K}(L^2(X,S)) = 0$, note Equation 3.140 and Equation 3.141 together integrate to $\mathrm{Ad}^D|_{L(C(X)) \oplus \mathcal{K}(L^2(X,S))} : \mathbb{R} \longrightarrow \mathrm{Aut}(L(C(X)) \oplus \mathcal{K}(L^2(X,S)))$ given by

$$\mathrm{Ad}^D_t|_{L(C(X)) \oplus \mathcal{K}(L^2(X,S))} = \mathrm{Ad}^D_t|_{L(C(X))} \oplus \mathrm{Ad}^D_t|_{\mathcal{K}(L^2(X,S))} \tag{3.142}$$

for all $t \in \mathbb{R}$. Note Equation 3.140 and Equation 3.141 are infinitesimal evolution of observables in Heisenberg representation (cf. pp.3-15 in [35]). They are first quantisations in the sense of Remark 3.1.63. Equation 3.140 in fact quantises the gradient on X w.r.t. g using the Clifford action. We say that $\mathrm{Ad}^D|_{L(C(X))}$ is the spatial, and $\mathrm{Ad}^D|_{\mathcal{K}(L^2(X,S))}$ the quantum component of the time-evolution Ad^D of observables.

Example 3.1.55 shows the quantum gradient $\nabla^{\mathrm{qtm}} = \nabla^D$ lets us define quantum optimal transport for the tracial AF-C^*-algebra $(\mathcal{K}(L^2(X,S)), \mathrm{tr})$. Assuming the spatial gradient ∇^{spt} yields a notion of spatial optimal transport for the tracial C^*-algebra $(C(X), \int d|\mathrm{vol}|)$, e.g. as per [200], we see quantum optimal transport is transversal to spatial optimal transport by the direct sum decomposition in Equation 3.142.

Spatiality of $C(X)$, resp. $L(C(X))$, is obvious as spatial coordinates parametrise the Riemannian manifold X and therefore observables given by elements in $L(C(X))$. We consider $L(C(X)) \subset \mathscr{B}(L^2(X,S))$ to formulate a necessary condition for spatiality here. We use quantum relative entropy as per Definition 4.1.12. We apply said condition in Example 3.1.62 to argue second quantisation quantises all, ergo considers no, spatial coordinates. For all $\mu \in \mathscr{S}(\mathscr{K}(L^2(X,S))$, $\mathrm{Ent}(\mu,\mathrm{tr}) \in [-\infty,\infty]$ is the relative entropy of μ w.r.t. tr as per Equation 4.12. Theorem 4.1.25 ensures it measures information required to discriminate μ and tr through observation by extending its use from the strongly unital finite-trace case (cf. pp.1-11 in [163]). If $h \in C(X)_+$, then Lemma 4.1.17 shows there exists no weakly dense subset $K \subset \mathscr{K}(L^2(X,S)) \cap S^1(L^2(X,S)) = S^1(L^2(X,S))$ and $C > 0$ s.t. the map $\tilde{\mu}_h : K \longrightarrow \mathbb{C}$ defined by

$$\tilde{\mu}_h(T) := C^{-1}\,\mathrm{tr}(L_x T) \tag{3.143}$$

for all $T \in K$ extends to a $\mu_h \in \mathscr{S}(\mathscr{K}(L^2(X,S)))$ with

$$\left|\mathrm{Ent}(\mu_h,\mathrm{tr})\right| < \infty. \tag{3.144}$$

Indeed, 1) in Lemma 4.1.17 shows Equation 3.144 implies $h \in L(C(X)) \cap S^1(L^2(X,S)) = 0$ since $S^1(L^2(X,S)) \subset \mathscr{K}(L^2(X,S)$ (cf. Example A.1.33). Assuming hyperfinite factor, our necessary condition for spatiality is non-extension w.r.t. the canonical trace.

We motivate our condition. The volume form $d|\mathrm{vol}|$ is a non-atomic Radon measure on X (cf. pp.299-306 in [144]). Non-extension implies our measuring process of quantum information as difference of observables from quantum white noise, up to musical isomorphisms, fails for all positive continuous ones parametrised by spatial coordinates. Corollary 4.1.27 shows said process, in contrast to any associated to relative entropy w.r.t. $\int d|\mathrm{vol}|$, only considers differences on discrete spectra. Naturally, such a countable process cannot discern observables as above by the intermediate value theorem. We see our measuring process fails since it requires us to measure with absolute precision [84] and this is prevented by infinitesimal length elements [67][144].

Example 3.1.62. Let H be a separable Hilbert space and $D \in \mathscr{U}\mathscr{B}(H)_h$ with compact resolvent, e.g. given by a real spectral triple. We use finite-dimensional approximation $\{H_j\}_{j\in\mathbb{N}}$ of H via eigenvectors as per Example 3.1.55. The setting of Example 3.1.56 requires orthogonal complex structure J on H s.t. it is H_j-reducible for all $j \in \mathbb{N}$. We use the one in [55]. Let $P_\pm : H \longrightarrow E_\pm$ be Hilbert space projections onto the eigenvectors of D with non-negative, resp. non-positive eigenvalues. Set $J := i(P_+ - P_-)$. We directly verify J is orthogonal complex structure on H s.t.

$$DJ = JD. \tag{3.145}$$

Equation 3.145 shows we are in the setting of Example 3.1.56. The second quantisation map $T \mapsto \bigwedge T$ from $\mathscr{B}(H)$ to $\mathscr{B}(\mathscr{F}(H[J]))$ exists (cf. pp.6-10 in [36]). Example 3.1.56 also gives τ-preserving local C^*-dynamical system $(\mathscr{A}(H[J]),\mathbb{R},\mathrm{Cliff}_J(e^{itD}))$.

For all $t \in \mathbb{R}$ and $x \in \mathscr{A}(H[J])$, Lemma C.2.1 shows

$$\mathrm{Cliff}_J\big(e^{itD}\big)(x) = \bigwedge e^{it|D|} x \bigwedge e^{-it|D|} \in \mathscr{A}(H[J]). \tag{3.146}$$

Equation 3.146 is the claimed explicit formula for Equation 3.126. Passing from D to $|D|$ in the second quantisation map avoids negative eigenvalues, i.e. the Dirac sea [55][195]. We prove Lemma C.2.1 by solving the associated implementation problem [177]. We say that $\mathrm{Cliff}_J(e^{itD})$ is implemented on $\mathscr{F}(H[J])$ by $\bigwedge e^{it|D|}$ in each case.

Following Equation 3.146, Equation 3.147 links quantum optimal transport and quantum statistical mechanics [35][36]. We use KMS-states below (cf. Definition 5.3.1 in [36]). If $e^{-\beta|D|} \in S^1(H)$ for given inverse temperature $\beta \in \mathbb{R}$, then Proposition 2.6 in [55] specifies Example 5.3.2 in [36] by showing the unique KMS_β-state of the τ-preserving local C^*-dynamical system $(\mathscr{A}(H[J]), \mathbb{R}, \mathrm{Cliff}_J(e^{itD}))$ has density operator

$$P_D := \mathrm{tr}\Big(\bigwedge e^{-\beta|D|}\Big)^{-1} \cdot \bigwedge e^{-\beta|D|} \in S^1(\mathscr{F}(H[J]))_+. \tag{3.147}$$

Applying constructions in Example 3.1.56 to $\alpha_t = \mathrm{Cliff}_J(e^{itD})$, Corollary 2.3.49 yields non-twisted dynamic quantum gradient $\nabla^{\mathrm{qtm}} : \mathscr{A}(H[J])_0 \longrightarrow L^2(\mathscr{A}(H[J]), \tau)$ given by

$$\nabla^{\mathrm{qtm}} x := \nabla^{\mathscr{D}_\alpha}(x) = \frac{d}{dt}\bigg|_{t=0,\mathrm{w}} \bigwedge e^{it|D|} x \bigwedge e^{-it|D|} \tag{3.148}$$

for all $x \in \mathscr{A}(H[J])_0$. Note Equation 3.147 then implies Equation 3.148 is infinitesimal evolution of observables in wedged Heisenberg representation at thermal equilibrium determined by P_D. This is a second quantisation in the sense of Remark 3.1.63. Whereas Equation 3.141 has closed form as unbounded commutator, use of the infinite exterior algebra on the right-hand side of Equation 3.148 introduces converging double sums with varying left-and right-multiplication of $\pm i|D|$ preventing a ready closed form.

Example 3.1.56 shows the quantum gradient $\nabla^{\mathrm{qtm}} = \nabla^{\mathscr{D}_\alpha}$ lets us define quantum optimal transport for the tracial AF-C^*-algebra $(\mathscr{A}(H[J]), \tau)$. Following our discussion at the end of Example 3.1.61, $\tau < \infty$ implies our necessary condition for spatiality is not satisfied. For all $\mu \in \mathscr{S}(\mathscr{A}(H[J]))$, $\mathrm{Ent}(\mu, \tau) \in [-\infty, \infty]$ is the relative entropy of μ w.r.t. τ as per Equation 4.12. We know $\mathscr{A}(H[J]) \subset L^1(\mathscr{A}(H[J]), \tau)$ is weakly dense since $\tau < \infty$ (cf. Proposition B.1.54). For all $x \in \mathscr{A}(H[J])_+$, Corollary 4.1.27 for $p = 1_A$ shows

$$\mu_x := \tau(x)^{-1} x^\flat \in \mathscr{S}^{\mathrm{N}}(\mathscr{A}(H[J])) \tag{3.149}$$

as per Equation 3.143 for $K = \mathscr{A}(H[J])$ has $|\mathrm{Ent}(\mu_x, \tau)| < \infty$. Our necessary condition is not satisfied. We see ∇^{qtm} quantises all, ergo considers no, spatial coordinates. Assuming commutative real spectral triple, ∇^{qtm} subsumes the generators of both components on the right-hand side of Equation 3.142 because Equation 3.146 is a second quantisation of the unrestricted time-evolution as per Equation 3.139.

If H and $D \in \mathscr{U}\mathscr{B}(H)_h$ are given by a real spectral triple, commutative or not, then we describe the spectral action as per Equation 3.137 using the negative of quantum relative entropy w.r.t. tr, i.e. von Neumann entropy (cf. p.17 in [163]). Let T be the fixed gauge field, $D_T := D$ and $P_T := P_D$. For all $\lambda \in \mathbb{R}$, set $h(\lambda) := \log(1+e^{-\lambda}) + \lambda e^{-\lambda}(1+e^{-\lambda})^{-1}$. Corollary 3.2 in [55] implies $h : \mathbb{R} \longrightarrow [0,\infty)$ is even. Theorem 3.4 in [55] shows

$$S_b(T) = \operatorname{tr}\big(h(D_T)\big) = -\operatorname{tr}\big(P_T \log P_T\big) = -\operatorname{Ent}\big(P_T^\flat, \operatorname{tr}\big) < \infty. \tag{3.150}$$

Unfortunately, Equation 3.150 uses quantum relative entropy w.r.t. tr and not τ. We want the latter for an ansatz to study the dynamics of gauge fields driven by varying Equation 3.150 via deforming Equation 3.135. We propose to internalise Equation 3.150 as per Equation 3.151 and generalise to Equation 3.160 in Example 3.1.64. Note [55] is based on [52][54]. We moreover refer to [197] as comprehensive treatment of the latter. The general noncommutative geometric approach to quantum thermodynamics used in [55] is introduced and explained as part of [71].

Remark 3.1.63. First and second quantisation denotes, to our knowledge, Hamiltonian formalism for a single quantum system, resp. multiple, often countable infinitely many interacting ones (cf. pp.1-38 in [188]). The latter arises as infinitely many copies of the former by applying to it the second quantisation map. If we consider time-evolution of fermions in Heisenberg representation (cf. pp.3-15 in [35] and pp.6-10 in [36]), then Example 3.1.62 indeed lifts time-evolution in Example 3.1.61 as per Equation 3.146 by mapping both given constituent semigroups to their wedged form.

Example 3.1.64 outlines how Example 3.1.62, specifically Equation 3.150, yields an ansatz to study noncommutative gauge theories based on the internalised spectral action as per Equation 3.160 if we generalise to quantum optimal transport parametrised by gauge fields. Let $(\mathfrak{A}, H, D, J)$ be a real spectral triple. We suppress J below as we use its symbol for orthogonal complex structures as per Example 3.1.62. For all gauge fields $T \in \operatorname{Inn}(\mathfrak{A}, H, D)$, we have J_T as per Example 3.1.62 for D_T as per Equation 3.135. If we further have a map $\operatorname{Inn} : \operatorname{Inn}(\mathfrak{A}, H, D) \longrightarrow \mathscr{S}(\mathscr{A}(H))$, then its associated internalisation of Equation 3.150 using quantum relative entropy w.r.t. τ is given by

$$S_b^{\operatorname{Inn}}(T) = -\operatorname{Ent}\Big(\big(\rho_{J_T}^{-1}\big)^*\big(\operatorname{Inn}(T)\big), \tau\Big) \tag{3.151}$$

for all $T \in \operatorname{Inn}(\mathfrak{A}, H, D)$. Note Equation 3.151 uses ρ_{J_T} as per Equation 3.120 in each case. We generalise Equation 3.151 to Equation 3.160 in Example 3.1.64 by considering all normalised Radon measures on finite-dimensional spaces of admissible gauge fields evaluating in $\mathscr{A}(H)$ up to varying ρ_{J_T} as per Equation 3.152, i.e. states on continuous fields of AF-C^*-algebras. If key technical challenges are solved in future work, then we hope to study the dynamics of such generalised gauge fields described as gradient flows driven by the internalised spectral action for the given parametrised quantum optimal transport. We are motivated by the classical approach of Jordan, Kinderlehrer and Otto for Fokker-Planck equations [131][167][169].

Example 3.1.64. Let $(\mathfrak{A},H,D,J)$ be a real spectral triple. We suppress J. For all gauge fields $T\in\mathrm{Inn}(\mathfrak{A},H,D)$, we have J_T as per Example 3.1.62 for D_T as per Equation 3.135. We do not know of a locally compact topology on $\mathrm{Inn}(\mathfrak{A},H,D)$ allowing for constructions as below. We instead consider $X\subset\mathrm{Inn}(\mathfrak{A},H,D)$ s.t. four conditions are satisfied.

First, let (X,g) be a smooth Riemannian manifold. We equip $T^*X\cong TX$ with its canonical dual Riemannian metric. Secondly, let $d|\mathrm{vol}|$ be a finite unoriented volume form, also called volume element, on X (cf. pp.299-306 in [144]). Thirdly, let $\beta:X\longrightarrow\mathbb{R}$ be smooth s.t. $e^{-\beta(T)|D_T|}\in S^1(H)$ for all $T\in X$. Fourthly, let

$$A_X:=\coprod_{T\in X}\mathscr{A}(H[J_T])=\coprod_{T\in X}\rho_{J_T}(\mathscr{A}(H)) \tag{3.152}$$

determine both a smooth vector bundle and u.s.c. C^*-bundle over X (cf. Definition 6.18 in [197]). Its space of continuous sections $\Gamma(A_X)$ is a C^*-algebra with norm given by $\|F\|_{\Gamma(A_X)}:=\sup_{T\in X}\|F(T)\|_{\mathscr{A}(H[J_T])}$ for all $F\in\Gamma(A_X)$ (cf. Proposition 6.19 in [197]). We define l.s.c. faithful semi-finite trace $\int_X\tau d|\mathrm{vol}|$ on $\Gamma(A_X)$ by setting

$$\left(\int_X\tau d|\mathrm{vol}|\right)(F):=\int_X\tau\big(F(T)\big)d|\mathrm{vol}| \tag{3.153}$$

for all $F\in\Gamma(A_X)_+$. We have tracial C^*-algebra $(\Gamma(A_X),\int_X\tau d|\mathrm{vol}|)$ in the space of bounded measurable sections $L^\infty(\Gamma(A_X),\int_X\tau d|\mathrm{vol}|)$ (cf. Proposition B.1.7 and Proposition B.1.10). Note $L^2(\Gamma(A_X),\int_X\tau d|\mathrm{vol}|)$ equipped with canonical left- and right-actions and pointwise algebra involution is a symmetric W^*-bimodule over $L^\infty(\Gamma(A_X),\int_X\tau d|\mathrm{vol}|)$.

We define noncommutative gradient as per Equation 3.157 with domain

$$\Gamma_0^\infty(A_X):=\Big\{F\in\Gamma^\infty(A_X)\ \Big|\ \forall T\in X:\ F(T)\in\mathscr{A}(H[J_T])_0\Big\}. \tag{3.154}$$

Equation 3.155 gives the fundamental compatibility condition for spatial and quantum components. The latter assumes continuous action of $\Gamma^\infty(T^*X)$ on $\Gamma^\infty(A_X)$ motivated by tensor contraction. We allow for loss of regularity. Let $\nabla^X:\Gamma^\infty(A_X)\longrightarrow\Gamma^\infty(T^*X\otimes A_X)$ be a covariant derivative and $\mathfrak{C}:\Gamma^\infty(T^*X\otimes A_X)\longrightarrow\Gamma(A_X)$ a bounded linear map. Assume we define symmetric W^*-derivation $\nabla^h:\Gamma_0^\infty(A_X)\longrightarrow L^2(\Gamma(A_X),\int_X\tau d|\mathrm{vol}|)$ by setting

$$\big(\nabla^hF\big)(T):=\mathfrak{C}\big(\nabla^XF\big)(T) \tag{3.155}$$

for all $F\in\Gamma_0^\infty(A_X)$ and $T\in X$. We call it a spatial, or horizontal gradient. Applying constructions in Example 3.1.62 to J_T and D_T in each case, we see Equation 3.146 lets us define symmetric W^*-derivation $\nabla^v:\Gamma_0^\infty(A_X)\longrightarrow\Gamma(A_X)$ by setting

$$(\nabla^vF)(T):=\frac{d}{dt}\bigg|_{t=0,\mathrm{w}}\bigwedge e^{it|D_T|}F(T)\bigwedge e^{-it|D_T|} \tag{3.156}$$

for all $F\in\Gamma_0^\infty(A_X)$ and $T\in X$. We call it a total quantum, or vertical gradient.

We define symmetric W^*-derivation $\nabla : \Gamma_0^\infty(A_X) \longrightarrow L^2(\Gamma(A_X), \int_X \tau d|\mathrm{vol}|)$ by setting

$$\big(\nabla F\big)(T) := \big(\nabla^h F\big)(T) + \big(\nabla^v F\big)(T) \tag{3.157}$$

for all $F \in \Gamma_0^\infty(A_X)$ and $T \in X$. Equation 3.157 yields noncommutative gradient for a mixed continuous-discrete noncommutative geometry. We define continuity equations as per 2) in Definition 3.1.5 by testing on $\Gamma_0^\infty(A_X)$ and therefore admissible paths as per Definition 3.1.7. Let f be symmetric representing function of an operator mean and $\theta \in [0,1]$. For all $F, G \in L^1(\Gamma(A_X), \int_X \tau d|\mathrm{vol}|)$, we define closed positive unbounded quadratic form on $L^2(\Gamma(A_X), \int_X \tau d|\mathrm{vol}|)$ as per Theorem 2.2.58 by setting

$$Q^{f,\theta}_{F^\flat, G^\flat}(W) := \int_X \mathscr{I}^{f,\theta}_{\mathscr{A}(H[J_T]),\mathscr{A}(H[J_T])}\big(F(U)^\flat, G(U)^\flat, W(U)^\flat\big) d|\mathrm{vol}| \tag{3.158}$$

for all $W \in L^2(\Gamma(A_X), \int_X \tau d|\mathrm{vol}|)$. If we show Equation 3.158 extends to a quasi-entropy for A_X, then it defines energy functionals as per Definition 3.1.16. Altogether, we define dynamic transport distances as per Definition 3.1.33.

We call these generalised quantum optimal transport distances parametrised by gauge fields, or parametrised quantum optimal transport distances. If $\nabla^h = 0$, then $\nabla = \nabla^v$ determines a mean quantum optimal transport for normalised averages of positive bounded functionals on CAR-algebras as per Example 3.1.62. The latter is recovered as the singular case of dimension zero given by $X = \{\mathrm{pt}\}$ and $d|\mathrm{vol}| = \delta_{\mathrm{pt}}$. We therefore know it is indeed ∇^v allowing for cross-fibre transport. How much non-ergodicity in the AF-C^*-setting is in fact due to a lack of such cross-fibre transport is unknown to us.

We generalise Equation 3.151 and define the internalised spectral action. For all $F \in \mathscr{S}(\Gamma(A_X))$, $\mathrm{Ent}^{\int_X \tau d|\mathrm{vol}|}(F) := \mathrm{Ent}(F, \int_X \tau d|\mathrm{vol}|) \in [-\infty, \infty]$ is the relative entropy of F w.r.t. $\int_X \tau d|\mathrm{vol}|$ as per Equation 4.7. Assume smooth map $\mathrm{Inn} : X \longrightarrow \mathscr{S}(\mathscr{A}(H))$ using the w^*-topology on $\mathscr{S}(\mathscr{A}(H))$. For all $T \in X$, we rewrite Equation 3.151 as

$$S_b^{\mathrm{Inn}}\big(\mathrm{id}_{\mathscr{A}(H[J_T])}\delta_T\big) = -\mathrm{Ent}\Big(\big(\rho_{J_T}^{-1}\big)^*\big(\mathrm{Inn}(T)\big)\delta_T, \int_X \tau d|\mathrm{vol}|\Big). \tag{3.159}$$

Note the right-hand side of Equation 3.159 is infinite in general since it evaluates Dirac measures. We consider a more direct definition by further subsuming precomposition in Equation 3.159 using more general internalisation maps. If we have weakly smooth map $\mathrm{Inn} : \mathscr{S}(\Gamma(A_X)) \longrightarrow \mathscr{S}(\Gamma(A_X))$ w.r.t. the w^*-topology on $\mathscr{S}(\Gamma(A_X))$, then we define its associated internalised spectral action by setting

$$S_b^{\mathrm{Inn}}(\mu) := -\mathrm{Ent}\Big(\mathrm{Inn}(\mu), \int_X \tau d|\mathrm{vol}|\Big) \tag{3.160}$$

for all $\mu \in \mathscr{S}(\Gamma(A_X))$. Note Equation 3.160 transforms the spectral action into an action functional on generalised gauge fields rather than mere points. An obvious but trivial choice for the internalisation map $\mathrm{Inn} : \mathscr{S}(\Gamma(A_X)) \longrightarrow \mathscr{S}(\Gamma(A_X))$ is the identity map.

We see our choice of internalisation map is essential. Specific forms, e.g. all of those utilising $\beta : X \longrightarrow \mathbb{R}$ due to its use for density operators as per Equation 3.147, are of interest. If we have a regularisation property for internalisation maps w.r.t. a weak Riemannian geometry in the logarithmic mean setting, then Equation 3.162 suggests a gradient flow description of the dynamics of generalised gauge fields driven by the internalised spectral action. For details on relative entropy for W^*-algebras, the logarithmic mean setting and non-spatial lower Ricci bounds, we refer to Chapter 4.

Let $\mathscr{S}^{\mathrm{N},\infty}_{-1}(\Gamma(A_X))$ as per 2) in Definition 2.1.11 equipped with $\|.\|_\infty$-topology. We may have to weaken it. Assume it has a weak Riemannian metric induced by the given quasi-entropy as per Equation 3.158 in the logarithmic mean setting analogous to the finite-dimensional case as per Definition 3.2.52. We use identical notation. Assume $\Delta := \nabla^*\nabla$ has $\ker\Delta = \langle 1_{A_X}\rangle_{\mathbb{C}}$ to avoid non-ergodicity. Moreover, we demand stronger smooth regularisation $\mathrm{Inn} : \mathscr{S}(\Gamma(A_X)) \longrightarrow \mathscr{S}^{\mathrm{N},\infty}_{-1}(\Gamma(A_X))$ from $\|.\|_{\Gamma(A_X)^*}$- to $\|.\|_\infty$-topology. We see a weaker topology weakens our regularity assumption. Let $F : (-\varepsilon,\varepsilon) \longrightarrow \mathscr{S}(\Gamma(A_X))$ with $F(0) = \mu$ be smooth. Equation 3.160 implies

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} S^{\mathrm{Inn}}_b(F(\varepsilon)) = -d_{\mathrm{Inn}(\mu)}\mathrm{Ent}^{\int_X \tau d|\mathrm{vol}|}\Big(d_\mu \mathrm{Inn}\left(\dot{F}(0)\right)\Big). \tag{3.161}$$

We want $\mathrm{grad}_\eta \mathrm{Ent}^{\int_X \tau d|\mathrm{vol}|} = (\sharp\Delta\eta)^\flat$ for all $\eta \in \mathscr{S}^{\mathrm{N},\infty}_{-1}(\Gamma(A_X)) \cap (\mathrm{dom}\,\Delta)^\flat$ in direct analogy to Equation 4.145 in the proof of Proposition 4.2.24. If we do lift said finite-dimensional pointwise case, then, for $\xi := (\int \tau(1_{A_X}) d|\mathrm{vol}|)^{-1} 1_{A_X} \in \mathscr{S}^{\mathrm{N}}(\Gamma(A_X))$, Equation 3.161 is

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} S^{\mathrm{Inn}}_b(F(\varepsilon)) = -g^{\xi}_{\mathrm{Inn}(\mu)}\Big(d_\mu \mathrm{Inn}\left(\dot{F}(0)\right), \left(\Delta\sharp\mathrm{Inn}(\mu)\right)^\flat\Big). \tag{3.162}$$

If regularisation allows pointwise adjoining of the derivatives in Equation 3.162 s.t. we adjoin to the given quasi-entropy precomposed with a well-behaved map, then we may use it to express metric slopes as per Equation 4.196 and control any EVI_λ-gradient flow of S^{Inn}_b [8][160]. If we show lower Ricci bounds are Hessian lower bounds as per H) in Definition 4.3.6 for our choice of weak Riemannian geometry, then a given one may use the above adjoining relation to impact the dynamics of $\mathscr{S}(\Gamma(A_X))$ driven by S^{Inn}_b.

We must solve key technical challenges, ranging from our initial construction to choice of internalisation map, its interplay with a suitable weak Riemannian structure and the EVI_λ-gradient flow property for $\mathrm{Ent}^{\int_X \tau d|\mathrm{vol}|}$. We may therefore seek to relax the problem as follows. We use, as in the AF-C^*-setting, canonical C^*-bimodule structures. If we instead consider general u.s.c. C^*-bundles and C^*-bimodule actions s.t. each fibre in Equation 3.152 is a tracial AF-C^*-algebra, then we also consider Equation 3.157 for more general noncommutative gradients. Such disintegration of tracial W^*-algebras into direct integrals of factors follows from the von Neumann disintegration theorem in operator theory (cf. Theorem IV.8.21 in [192]). We see fundamental example classes using tracial AF-C^*-algebras generating hyperfinite factors of type I and II by σ-weak closure are of particular interest. We thereby define general parametrised quantum optimal transport. We view quantum optimal transport as its pointwise case since the latter is the singular case of dimension zero.

3.2 Accessibility components

Accessibility components of quantum optimal transport distances are complete geodesic length-metric spaces. Metric geometry reduces to accessibility components. There may exist uncountable infinitely many since sets of states at finite distance have identical fixed parts under noncommutative heat semigroups of quantum Laplacians. Assuming spectral gaps of quantum Laplacians and fixed parts, we use such fixed parts to classify accessibility components of square integrable normal states. We in turn use the latter classification for the coarse graining process since its assumptions are satisfied for all accessibility components in the finite-dimensional setting.

Classification uses regularisation of normal states under heat flow. Assuming fixed parts with integrable support, we show heat flow instantaneously regularises normal states to be, possibly unboundedly, invertible up to fixed part. This uses compatibility with compression and finite-dimensional approximation. Note we avoid any regularity assumptions for noncommutative heat semigroups. Under assumptions as above, we use such regularisation for classification since spectral gaps of square integrable normal fixed parts imply integrable support. In the logarithmic mean setting and assuming finitely supported fixed parts, we further use it to show heat flow induces finite-energy admissible paths for all states with finite quantum relative entropy.

We show classification and regularisation by passing through the finite-dimensional setting. In the latter setting, accessibility components are norm closed convex subsets of states having identical fixed part. States at finite distance have support projections in the unique compressed C^*-subalgebra which is given by compressing with the support projection of their common fixed part. Relative interiors consist of all invertible states on, resp. densities in, such a given compressed C^*-subalgebra. They are also connected Riemannian manifolds with Riemannian metric induced by the given quasi-entropy. Using finite-dimensionality, we directly verify heat flow yields finite energy paths from relative boundaries to relative interiors. We thereby connect all states with identical fixed part. This yields classification and regularisation in the finite-dimensional case. Under assumptions as above, we extend regularisation and classification to the general case. We require the notion of reducible support as finite-dimensional approximation of support projections. We show it is implied by integrable support. We are therefore able to pass through the finite-dimensional setting.

Structure. In Subsection 3.2.1, we review support projections of normal states, as well as spectral gaps. We introduce the notion of reducible support. In Subsection 3.2.2, we discuss both completely Markovian semigroups and Lindblad master equations, our use of quantum Fokker-Planck equations, and subsequently study noncommutative heat semigroups of quantum Laplacians. In Subsection 3.2.3, we apply the latter to classify accessibility components of square integrable normal states as discussed above.

3.2.1 Support projections of normal states

We review canonical order-preserving bijections from projections of W^*-algebras to faces of normal state spaces. They are determined by support projections of normal states.

In the AF-C^*-setting, reducible support is finite-dimensional approximation of such support projections. Theorem 3.2.18 shows integrable support implies reducible support as required. Standard references for convex geometry of norm closed convex subsets in pre-duals of W^*-algebras are [2][3]. Standard reference for differential and Riemannian geometry is [144].

Faces of normal state spaces. Lemma 3.2.5 represents faces of normal state spaces of abstract tracial C^*-algebras as per Definition 2.1.11 and Remark 2.1.12. This uses normal state spaces of compressed C^*-subalgebras and their canonical inclusions as per 1) in Proposition 2.1.13. We use the modified standard pairing, in particular their flat and sharp operators as per Definition 2.1.1 and Remark 2.1.2.

Let (M,τ) be a tracial W^*-algebra and $A \subset M$ a σ-weakly dense C^*-subalgebra. Ergo $M = L^\infty(A,\tau)$ and $M_* = L^1(A,\tau)$. For all $x \in L^1(A,\tau)_+$, we have unique carrier projection $\operatorname{supp} x \in L^\infty(A,\tau)$ of $\{x^\flat\}$ (cf. Definition 3.20 and Lemma 3.21 in [2]). Each $\operatorname{supp} x$ is the minimal projection in $L^\infty(A,\tau)$ s.t. $x = x \cdot \operatorname{supp} x$ holds. If we have $x = xp$ for a projection $p \in L^\infty(A,\tau)$, then $\operatorname{supp} x \leq p$. Note $x = xp$, $x = px$ and $x = pxp$ are equivalent.

Definition 3.2.1. Let $x \in L^1(A,\tau)_+$.

1) The carrier projection $\operatorname{supp} x \in L^\infty(A,\tau)$ of $\{x^\flat\}$ is the support projection of x.

2) If $x \in L^0(N,\tau)$ for $N \subset (L^\infty(A,\tau),\tau)$, then we say that $\operatorname{supp}^{\mathrm{c}}_N x := 1_N - \operatorname{supp} x$ is the kernel projection of x in N.

Notation 3.2.2. We suppress N in Definition 3.2.1 if $N = L^\infty(A,\tau)$.

Proposition 3.2.3. *Let $N \subset (L^\infty(A,\tau),\tau)$.*

1) *Let $x \in L^1(N,\tau)_+$. We have $\operatorname{supp} x \in N$. Furthermore, we have $\operatorname{supp}^{\mathrm{c}} x \in N[1_A]$ and $\operatorname{supp}^{\mathrm{c}}_N x = \operatorname{com}_{1_N} \operatorname{supp}^{\mathrm{c}} x \in N$.*

2) *Let $x, y \in L^1(A,\tau)_+$. If $\tau(yp) = 0$ for all projections $p \in L^\infty(A,\tau)$ s.t. $\tau(xp) = 0$, then $\operatorname{supp} y \leq \operatorname{supp} x$.*

Proof. Since $\operatorname{com}_{1_N} 1_A = 1_N$, we know 1) by definition. Get 2) by Lemma 3.25 in [2]. □

Proposition 3.2.4 shows support projections are invariant under change of canonical left- and right-actions. Let $N \subset (L^\infty(A,\tau),\tau)$ and $x \in L^0(N,\tau)_+$. Using the latter and following Remark A.1.87, note 2) in Lemma B.1.72 shows

$$\Gamma_{x,N}\big(\chi_{(0,\infty]}\big) = L_N^{-1}\Big(\pi^A_{\operatorname{im} L_{x,N}}\Big) = R_N^{-1}\Big(\pi^A_{\operatorname{im} R_{x,N}}\Big) \tag{3.163}$$

and

$$\Gamma_{x,N}(\delta_0) = L_N^{-1}\Big(\pi^A_{\ker L_{x,N}}\Big) = R_N^{-1}\Big(\pi^A_{\ker R_{x,N}}\Big). \tag{3.164}$$

Proposition 3.2.4. *Let $N \subset (L^\infty(A,\tau),\tau)$. For all $x \in L^1(N,\tau)_+$, we have*

1) $\operatorname{supp} x = \Gamma_{x,N}\big(\chi_{(0,\infty]}\big) = L_N^{-1}\Big(\pi^A_{\operatorname{im} L_{x,N}}\Big) = R_N^{-1}\Big(\pi^A_{\operatorname{im} R_{x,N}}\Big)$,

2) $\operatorname{supp}^{\mathrm{c}}_N x = \Gamma_{x,N}(\delta_0) = L_N^{-1}\Big(\pi^A_{\ker L_{x,N}}\Big) = R_N^{-1}\Big(\pi^A_{\ker R_{x,N}}\Big)$,

3) $L_{\operatorname{supp} x,N} = \operatorname{com}_{L^2(N,\tau)} L_{\operatorname{supp} x, L^\infty(A,\tau)}$ *and* $R_{\operatorname{supp} x,N} = \operatorname{com}_{L^2(N,\tau)} R_{\operatorname{supp} x, L^\infty(A,\tau)}$.

Proof. Let $x \in L^1(N,\tau)_+$. Note we have $x = x\Gamma_{x,N}(\chi_{(0,\infty]})$ and $0 = x\Gamma_{x,N}(\delta_0)$ by functional calculus. Thus minimality of support projections implies $\operatorname{supp} x \leq \Gamma_{x,N}(\chi_{(0,\infty]})$, hence $\operatorname{supp} x \cdot \Gamma_{x,N}(\chi_{(0,\infty]}) = \operatorname{supp} x$ since both are projections. We prove the converse. For all $u \in L^2(A,\tau)$, let $\{u_n\}_{n\in\mathbb{N}} \subset \operatorname{dom} L_{x,N}$ s.t. $\pi^A_{\operatorname{im} L_x}(u) = \|.\|_\tau\text{-}\lim_{n\in\mathbb{N}} x u_n$. Using the latter in each case and further $\operatorname{supp} x \cdot \Gamma_{x,N}(\chi_{(0,\infty]}) = \operatorname{supp} x$, Equation 3.163 lets us calculate

$$\begin{aligned}
\big\| \operatorname{supp} x \cdot u \big\|_\tau &= \Big\| L_{\operatorname{supp} x \cdot \Gamma_{x,N}(\chi_{(0,\infty]}),N}\Big(\pi^A_{L^2(N,\tau)}(u)\Big)\Big\|_\tau \\
&= \Big\| L_{\operatorname{supp} x,N}\Big(\pi^A_{\operatorname{im} L_{x,N}}(u)\Big)\Big\|_\tau \\
&= \lim_{n\in\mathbb{N}} \big\| L_{\operatorname{supp} x,N}(x u_n)\big\|_\tau \\
&= \big\| \pi^A_{\operatorname{im} L_x}(u)\big\|_\tau = \big\| \Gamma_{x,N}\big(\chi_{(0,\infty]}\big) \cdot u \big\|_\tau
\end{aligned}$$

for all $u \in L^2(A,\tau)$. Since $\operatorname{supp} x \leq \Gamma_{x,N}(\chi_{(0,\infty]})$, get $\operatorname{supp} x = \Gamma_{x,N}(\chi_{(0,\infty]})$ and therefore 1) by Equation 3.163. Then $\operatorname{supp}^{\mathrm{c}}_N x = 1_N - \Gamma_{x,N}(\chi_{(0,\infty]}) = \Gamma_{x,N}(\delta_0)$ by functional calculus and we have 2) by Equation 3.164. Using 1), get 3) by Corollary B.2.35. □

Theorem 3.35 in [2] classifies norm closed convex subsets of normal state space using support projections. We review this below for abstract tracial C^*-algebras. Let V be a normed vector space and $K \subset V$ a norm closed convex subset. Its relative interior

$$\operatorname{relint} K = \Big\{ \mu \in K \;\Big|\; \forall \eta \in K\ \exists t > 1 :\ t\mu + (1-t)\eta \in K \Big\} \tag{3.165}$$

is open, and its relative boundary $\partial K = K \setminus \operatorname{relint} K$ is closed in relative topology. A norm closed convex subset $\mathscr{F} \subset K$ is a face of K if for all $x, y \in K$, we know $(1-t)x + ty \in \mathscr{F}$ for any $t \in (0,1)$ implies $x, y \in \mathscr{F}$.

Lemma 3.2.5. *For all projections $p \in L^\infty(A,\tau)$, we know*

$$\mathscr{F}_A(p) := \Big\{ x \in L^1(A,\tau)_+ \;\Big|\; \|x\|_1 = 1,\ \operatorname{supp} x \leq p \Big\}^\flat = \mathscr{S}^{\mathrm{N}}(A[p]) \tag{3.166}$$

is a face of $\mathscr{S}^{\mathrm{N}}(A)$. Furthermore, the map $p \mapsto \mathscr{F}_A(p)$ from projections in $L^\infty(A,\tau)$ to faces of $\mathscr{S}^{\mathrm{N}}(A)$ is an order-preserving bijection.

Proof. We use 1) in Proposition 2.1.13 here and throughout our discussion. Using 1) in Theorem 3.35 and Lemma 3.21 in [2], we obtain a face

$$\mathscr{F}_A(p) = \left\{\mu \in \mathscr{S}^{\mathrm{N}}(A) \mid \mu(p) = 1\right\} = \left\{x \in L^1(A,\tau)_+ \mid \|x\|_1 = 1,\ \operatorname{supp} x \leq p\right\}^{\flat} \tag{3.167}$$

of $\mathscr{S}^{\mathrm{N}}(A)$ in each case. Theorem 3.35 in [2] states the map $p \mapsto \mathscr{F}_A(p)$ from projections in $L^\infty(A,\tau)$ to faces of $\mathscr{S}^{\mathrm{N}}(A)$ is an order-preserving bijection. Let $p \in L^\infty(A,\tau)$ be a projection. If $x \in \mathscr{F}_A(p)$, then $\operatorname{supp} x \leq p$ implies $\operatorname{supp} x \cdot p = \operatorname{supp} x$ and therefore $x = xp$ by minimality. Thus $x \in L^0(A[p],\tau)_+$ by Lemma 2.1.6, hence $x^\flat \in \mathscr{S}^{\mathrm{N}}(A[p])$. The converse follows because $x^\flat \in \mathscr{S}^{\mathrm{N}}(A[p])$ likewise implies $x = xp$, which in turn implies $\operatorname{supp} x \leq p$ by Lemma 3.21 in [2]. ☐

Definition 3.2.6. For all $\mu \in L^1(A,\tau)^\flat_+$, set

1) $\operatorname{supp}\mu := \operatorname{supp}\sharp\mu$ and call $\operatorname{supp}\mu$ the support projection of μ,

2) $\mathscr{F}_A(\mu) := \mathscr{F}_A\big(\operatorname{supp}\mu\big)$ and call $\mathscr{F}_A(\mu)$ the face of μ on A.

Remark 3.2.7. Let $\mu \in L^1(A,\tau)^\flat$. For all projections $p \in L^\infty(A,\tau)$ s.t. $\operatorname{supp}\mu \leq p$, we have $\mu \in L^1(A[p],\tau)^\flat$ and therefore $\mathscr{F}_A(\mu) = \mathscr{F}_{A[p]}(\mu)$ by Lemma 3.2.5.

Corollary 3.2.8. *Let $p,q \in L^\infty(A,\tau)$ be projections.*

1) We have $p \leq q$ if and only if $\mathscr{S}^{\mathrm{N}}(A[p]) \subset \mathscr{S}^{\mathrm{N}}(A[q])$.

2) Assume $p \leq q$. If $K \subset \mathscr{S}^{\mathrm{N}}(A[p])$ is a face, then $K \subset \mathscr{S}^{\mathrm{N}}(A[q])$ is a face.

Proof. Apply Lemma 3.2.5. ☐

Remark 3.2.9. Let $p \in L^\infty(A,\tau)$ be a projection. If $\mu \in \mathscr{S}^{\mathrm{N}}(A[p])$, then $\operatorname{supp}\mu \leq p$ by Lemma 3.2.5 and therefore $\mathscr{F}_A(\mu) \subset \mathscr{S}^{\mathrm{N}}(A[p])$ by 1) in Corollary 3.2.8. If $K \subset \mathscr{S}^{\mathrm{N}}(A[p])$ is a face, then $K \subset \mathscr{S}^{\mathrm{N}}(A)$ is a face by 2) in Corollary 3.2.8.

Corollary 3.2.10. *For all $\mu \in \mathscr{S}^{\mathrm{N}}(A)$, we have $\mathscr{F}_A(\mu) = \{\mu\}$ if and only if μ is pure.*

Proof. Let $\mu \in \mathscr{S}^{\mathrm{N}}(A)$. If $\mathscr{F}_A(\mu) = \{\mu\}$, then purity of μ follows by the face property. Assume μ is pure. We know $K := \{\mu\} \subset \mathscr{S}^{\mathrm{N}}(A[p])$ is a face. Using 2) in Corollary 3.2.8 and following Remark 3.2.9, Lemma 3.2.5 yields unique projection $q \in L^\infty(A[p],\tau)$ s.t. we have $K = \mathscr{S}(A[q])$. Since $\mu \in K$, the lemma further shows $\operatorname{supp}\mu \leq q$ and therefore $\mathscr{F}_A(\mu) \subset K$. We obtain $\mathscr{F}_A(\mu) = \{\mu\}$ as claimed. ☐

Let $p \in L^\infty(A,\tau)$ be a projection. Note $A[p]^*_+ \cap \mathrm{GL}(L^\infty(A[p],\tau))^\flat \subset A[p]^*_h$ open in norm topology. Using real vector space structure, we see $A[p]^*_+ \cap \mathrm{GL}(L^\infty(A[p],\tau))^\flat$ is a Banach manifold. We have $\mathscr{S}^{\mathrm{N},\infty}_{-1}(A[p]) = \mathscr{S}^{\mathrm{N},\infty}(A[p]) \cap \mathrm{GL}(L^\infty(A[p],\tau))^\flat$ by boundedness.

Corollary 3.2.11. *Let $p \in L^\infty(A,\tau)$ be a projection s.t. $\tau(p) < \infty$.*

1) We have embedded Banach submanifold

$$\mathscr{S}^{\mathrm{N},\infty}_{-1}(A[p]) = \operatorname{relint} \mathscr{S}^{\mathrm{N},\infty}(A[p]) \subset A[p]^*_+ \cap \mathrm{GL}(L^\infty(A[p],\tau))^\flat. \tag{3.168}$$

2) For all $\mu \in \mathscr{S}^{\mathrm{N}}(A[p])$, we have

2.1) $\mathscr{F}_A(\mu) = \mathscr{S}^{\mathrm{N}}(A[p])$ if and only if $\mu \in \mathscr{S}^{\mathrm{N}}_{>0}(A[p])$,

2.2) $\mathscr{F}_A(\mu) \subset \partial \mathscr{S}^{\mathrm{N}}(A[p])$ if and only if $\mu \notin \mathscr{S}^{\mathrm{N}}_{>0}(A[p])$.

Proof. Set $\xi_p := \tau(p)^{-1} p^\flat$. We show 1). For all $\mu \in \mathscr{S}^{\mathrm{N},\infty}(A[p])$, we have $\mu \in \mathscr{S}^{\mathrm{N},\infty}_{-1}(A[p])$ if and only if $\sharp\mu \in \mathrm{GL}(L^\infty(A[p],\tau))$ by 2) in Definition 2.1.11. In particular, the equivalence ensures $\xi_p \in \mathscr{S}^{\mathrm{N},\infty}_{-1}(A[p])$. For all $\mu \in \operatorname{relint} \mathscr{S}^{\mathrm{N},\infty}(A[p])$, there exists $t > 1$ s.t.

$$\mu \geq \frac{t-1}{t} \xi_p \geq 0. \tag{3.169}$$

Since $\xi_p \in \mathscr{S}^{\mathrm{N},\infty}_{-1}(A[p])$, note Equation 3.169 shows $\operatorname{relint} \mathscr{S}^{\mathrm{N},\infty}(A[p]) \subset \mathscr{S}^{\mathrm{N},\infty}_{-1}(A[p])$. We directly verify the converse. Thus Equation 3.168 holds, hence

$$\operatorname{relint} \mathscr{S}^{\mathrm{N},\infty}(A[p]) = \left(\tau|_{A[p]^*_+ \cap \mathrm{GL}(L^\infty(A[p],\tau))^\flat}\right)^{-1}(1). \tag{3.170}$$

Equation 3.170 implies 1) by the submersion theorem [144].

We show 2). For all $\mu \in \mathscr{S}^{\mathrm{N}}(A[p])$, $\mu \in \mathscr{S}^{\mathrm{N}}_{>0}(A[p])$ if and only if $\Gamma_{\sharp\mu, L^\infty(A[p],\tau)}(\delta_0) = 0$ by 1) in Definition 2.1.11. Proposition 3.2.4 further shows the latter is equivalent to $\operatorname{supp}\mu = p$. Then 1) in Corollary 3.2.8 yields 2.1). Lemma 3.2.5 shows $\operatorname{supp}\mu \leq p$ in each case, i.e. $\mathscr{F}_A(\mu) \subset \mathscr{S}^{\mathrm{N}}(A[p])$ by 1) in Corollary 3.2.8. Using the latter and 2.1), note Equation 3.169 for $\operatorname{relint} \mathscr{S}^{\mathrm{N}}(A[p])$ derives 2.2) by contradiction. □

Support projections in the AF-C*-setting. Definition 3.2.17 gives reducible support. Theorem 3.2.18 shows integrable support implies reducible support. Spectral gaps imply integrable support. Lemma 3.2.16 shows spectral gaps of square integrable positive elements are limits of spectral gaps of their restrictions. This shows the utility of assuming spectral gaps in order to use finite-dimensional approximation.

Let H be a Hilbert space. Let (A,τ) be a tracial AF-C^*-algebra.

Lemma 3.2.12. *Let $T = \text{sr-lim}_{n\in\mathbb{N}} T_n$ on H. If $T_n \geq 0$ for all $n \in \mathbb{N}$, then*

$$0 \leq \limsup_{j\in\mathbb{N}} \left\| \left(\chi_{(0,\infty]}(T) - \chi_{(0,\infty]}(T_n)\right)(u) \right\|_H \leq 2 \left\| \delta_0(T)(u) \right\|_H \tag{3.171}$$

for all $u \in H$.

Proof. For all $\varepsilon > 0$, we define $g_\varepsilon \in C_b([0,\infty))$ by setting

$$g_\varepsilon(t) := \begin{cases} \varepsilon^{-1} t & \text{if } t \leq \varepsilon, \\ 1 & \text{else}. \end{cases}$$

By construction, get $g_\varepsilon(0) = 0$, $\|g_\varepsilon\|_\infty = 1$ and $0 \leq \chi_{(0,\infty]} - g_\varepsilon \leq I - g_\varepsilon$ in each case. We moreover have pointwise convergence $\chi_{(0,\infty]} = \lim_{\varepsilon \downarrow 0} g_\varepsilon$ on $[0,\infty)$. Let $S \in \mathscr{UB}(H)_+$. Note $\chi_{(0,\infty]}(S) = \pi_{\operatorname{im} S}$ and $\delta_0(S) = \pi_{\ker S}$ (cf. Remark A.1.87). Thus $\chi_{(0,\infty]}(S) = \text{s-}\lim_{\varepsilon \downarrow 0} g_\varepsilon(S)$ by uniform boundedness, hence $\delta_0(S) = \text{s-}\lim_{\varepsilon \downarrow 0} 1 - g_\varepsilon(S)$.

Let $u \in H$ and $\varepsilon > 0$. Then $\|(\chi_{(0,\infty]}(T) - g_\varepsilon(T))(u)\|_H \leq \|(I - g_\varepsilon(T))(u)\|_H$, as well as $\|(\chi_{(0,\infty]}(T_n) - g_\varepsilon(T))(u)\|_H \leq \|(I - g_\varepsilon(T_n))(u)\|_H$ for all $n \in \mathbb{N}$, by functional calculus. For all $n \in \mathbb{N}$, we therefore bound $\|(\chi_{(0,\infty]}(T) - \chi_{(0,\infty]}(T_n))(u)\|_H$ from above by

$$\big\|\big(I - g_\varepsilon(T)\big)(u)\big\|_H + \big\|\big(I - g_\varepsilon(T_n)\big)(u)\big\|_H + \big\|\big(g_\varepsilon(T) - g_\varepsilon(T_n)\big)(u)\big\|_H. \tag{3.172}$$

Since $g_\varepsilon \in C_b([0,\infty)$, we know $g_\varepsilon(T) = \text{s-}\lim_{n \in \mathbb{N}} g_\varepsilon(T_n)$ by Lemma A.2.5. Applying the latter to Equation 3.172 shows

$$0 \leq \limsup_{n \in \mathbb{N}} \big\|\big(\chi_{(0,\infty]}(T) - \chi_{(0,\infty]}(T_n)\big)(u)\big\|_H \leq 2 \big\|\big(I - g_\varepsilon(T)\big)(u)\big\|_H. \tag{3.173}$$

Using $\pi_{\ker T} = \text{s-}\lim_{\varepsilon \downarrow 0} I - g_\varepsilon(T)$, letting $\varepsilon \downarrow 0$ in Equation 3.173 yields Equation 3.171. $\square$

For all $T \in \mathscr{UB}(H)_+$, we have spectral gap $\sigma(T) = \inf\{\lambda > 0 \mid \lambda \in \operatorname{spec} T\}$ and say that T has spectral gap if $\sigma(T) > 0$ (cf. Definition A.2.31). Definition 3.2.13 gives spectral gaps of positive measurable operators and normal positive bounded functionals. Using canonical left- and right-actions, Proposition 3.2.14 recovers spectral gaps of positive unbounded operators. Spectral gaps are invariant under compression. For details on spectral gaps of positive unbounded operators, we refer to Subsection A.2.2.

Definition 3.2.13. Let $N \subset (L^\infty(A,\tau),\tau)$.

1) For all $x \in L^0(N,\tau)_+$, we call $\sigma(x) := \inf\{\lambda > 0 \mid \lambda \in \operatorname{spec}_{L^\infty(A,\tau)} x\}$ the spectral gap of x. We further say that x has spectral gap if $\sigma(x) > 0$.

2) For all $\mu \in L^1(A,\tau)^\flat$, set $\sigma(\mu) := \sigma(\sharp\mu)$ and call $\sigma(\mu)$ the spectral gap of μ. We further say that μ has spectral gap if $\sigma(\mu) > 0$.

Proposition 3.2.14. *Let $N \subset (L^\infty(A,\tau),\tau)$. For all $x \in L^0(N,\tau)_+$, we have*

$$\sigma(x) = \inf\Big\{\lambda > 0 \;\Big|\; \lambda \in \operatorname{spec}_N x\Big\} = \sigma\big(L_{x,N}\big) = \sigma\big(R_{x,N}\big). \tag{3.174}$$

Proof. Let $x \in L^0(N,\tau)$. Thus $\operatorname{spec}_{L^\infty(A,\tau)} x = \operatorname{spec}_N x \cup \{0\}$ by 1) in Corollary B.2.35, hence we obtain the first identity in Equation 3.174 by positivity. The second and third one follow at once from 2) in Proposition B.1.70 and 2) in Lemma B.1.72. $\square$

Remark 3.2.15. Let $N \subset (L^{\infty}(A,\tau),\tau)$ and $\mu \in L^1(N,\tau)^{\flat}$. Note Proposition B.1.51 shows we have $\mu \in L^1(N,\tau)^{\flat}$ if and only if $\sharp\mu \in L^1(N,\tau)_+$. Proposition 3.2.14 further implies

$$\sigma(\mu) = \inf\left\{\lambda > 0 \mid \lambda \in \mathrm{spec}_N \sharp\mu\right\} = \sigma\big(L_{\sharp\mu,N}\big) = \sigma\big(R_{\sharp\mu,N}\big). \tag{3.175}$$

Equation 3.175 holds if $N \subset (L^{\infty}(A,\tau),\tau)$ lies in one of the two classes of compression given in Subsection 2.1.2, i.e. either if we compress to induced AF-C^*-bimodules or if we compress with projections. We use this throughout our discussion.

We use the following estimate. For all $z \in L^1(A,\tau)_+$ and $j \in \mathbb{N}$, 1) in Proposition 2.2.51 and 2) in Lemma 2.2.52 show

$$0 \leq L_{z_j,A_j} = \pi_j^A L_{z_j} \pi_j^A \leq L^2_{\pi_j^A(\sqrt{z})}. \tag{3.176}$$

Lemma 3.2.16. *For all $x \in L^2(A,\tau)_+$, we have*

1) $\sigma(x) = \lim_{j\in\mathbb{N}} \sigma(x_j)$*,*

2) $\chi_{(0,\infty]}(x) = \text{s-}\lim_{j\in\mathbb{N}} \chi_{(0,\infty]}(x_j)$ *if* $\tau\big(\chi_{(0,\infty]}(x)\big) < \infty$*.*

Proof. Following Remark 3.2.15, Proposition 3.2.14 ensures results for spectral gaps of positive unbounded operators likewise apply to spectral gaps of positive measurable operators, resp. normal positive bounded functionals. Let $x \in L^2(A,\tau)_+$. For all $j \in \mathbb{N}$, get $\sigma(x_j) = \sigma(L_{x,A_j})$ by Proposition 3.2.14. Theorem 2.2.53 states $L_x = \text{sr-}\lim_{j\in\mathbb{N}} L_{x_j}$.

We show 1). Strong resolvent convergence as above implies $\limsup_{j\in\mathbb{N}} \sigma(x_j) \leq \sigma(x)$ by Lemma A.2.35. We show the converse. If $\sigma(x) = 0$, then our claim follows by positivity of spectral gaps. We assume $\sigma(x) > 0$ without loss of generality. Thus $x \neq 0$, hence $x_j \neq 0$ and therefore $\sigma(x_j) > 0$ for a.e. $j \in \mathbb{N}$ by finite-dimensionality. We assume $x_j \neq 0$ and thereby $\sigma(x_j) > 0$ for all $j \in \mathbb{N}$ without loss of generality. For all $j \in \mathbb{N}$, we have $u^j \in A_j$ s.t. $x_j u_j = \sigma(x_j) u_j$ and $\|u_j\|_\tau = 1$ by finite-dimensionality.

Let $j \in \mathbb{N}$. Let $v \in \mathrm{im} L_{x_j,A_j}$ and $w \in A_j$ s.t. $v = x_j w$. Note $x_j = \pi_j^A(x)$ by construction. Get $A_0 \subset \mathrm{dom} L_x \cap \mathrm{dom} I - L_{\pi_j^A(x)}$ by square integrability. We have

$$v = x_j w = xw - \big(I - \pi_j^A\big)(x)w. \tag{3.177}$$

We know $A_j^{\perp} A_j \subset A_j^{\perp}$. Moreover, we have $\chi_{(0,\infty]}(x)x = x$ by functional calculus. Using each of the latter, Equation 3.177 implies

$$v = \pi_j^A(v) = \pi_j^A(xw) \tag{3.178}$$

and

$$\chi_{(0,\infty]}(x)v = xw - \chi_{(0,\infty]}(x)\big(I - \pi_j^A\big)(x)w. \tag{3.179}$$

Equation 3.178 and Equation 3.179 show

$$\pi_j^A\big(\chi_{(0,\infty]}(x)v\big) = v - \pi_j^A\Big(\chi_{(0,\infty]}(x)\big(I-\pi_j^A\big)(x)w\Big). \tag{3.180}$$

Expanding $xw = x_j w + (I-\pi_j^A)(x)w$ and using $A_j^{\perp}A_j \subset A_j^{\perp}$ for the second summand in the final term below, we apply Equation 3.180 in order to estimate

$$\begin{aligned}
\langle \pi_j^A\big(\chi_{(0,\infty]}(x)v\big), v\rangle_\tau &= \|v\|_\tau^2 - \langle \big(I-\pi_j^A\big)(x)w, \chi_{(0,\infty]}(x)v\rangle_\tau \\
&= \|v\|_\tau^2 - \langle \big(I-\pi_j^A\big)(x)w, xw - \chi_{(0,\infty]}(x)\big(I-\pi_j^A\big)(x)w\rangle_\tau \\
&= \|v\|_\tau^2 - \big\|\big(I-\pi_j^A\big)(x)w\big\|_\tau^2 + \big\|\chi_{(0,\infty]}(x)\big(I-\pi_j^A\big)(x)w\big\|_\tau^2 \\
&\geq \|v\|_\tau^2 - \big\|\big(I-\pi_j^A\big)(x)w\big\|_\tau^2.
\end{aligned}$$

Assume $v = u^j$ and $w = \sigma(x_j)^{-1}u^j$. The above estimate yields

$$\big\|\chi_{(0,\infty]}(x)u^j\big\|_\tau^2 = \langle \pi_j^A\big(\chi_{(0,\infty]}(x)u^j\big), u^j\rangle_\tau \geq 1 - \sigma(x_j)^{-2}\big\|\big(I-\pi_j^A\big)(x)u^j\big\|_\tau^2. \tag{3.181}$$

We show $\|(I-\pi_j^A)(x)u^j\|_\tau^2 = 0$. We calculate

$$\begin{aligned}
\big\|\big(I-\pi_j^A\big)(x)u^j\big\|_\tau^2 &= \big\|xu^j - x_j u^j\big\|_\tau^2 \\
&= \big\|xu^j\big\|_\tau^2 - 2\,\mathrm{Re}\,\langle xu^j, x_j u^j\rangle_\tau + \big\|x_j u^j\big\|_\tau^2 \\
&= \big\|xu^j\big\|_\tau^2 - 2\sigma(x_j)\,\mathrm{Re}\,\langle x_j u^j, u^j\rangle_\tau + \sigma(x_j)^2 \\
&= \big\|xu^j\big\|_\tau^2 - \sigma(x_j)^2.
\end{aligned}$$

Set $y := x^2 \in L^1(A,\tau)_+$. Then $\sqrt{y} = x$ by definition. Equation 3.176 for $z = y$ shows

$$\pi_j^A L_{y_j} \pi_j^A \leq L^2_{\pi_j^A(\sqrt{y})} = L_{x_j^2}. \tag{3.182}$$

Self-adjointness and Equation 3.182 show

$$\big\|xu^j\big\|_\tau^2 = \langle yu^j, u^j\rangle_\tau = \langle y_j u^j, u^j\rangle_\tau \leq \langle x_j^2 u^j, u^j\rangle_\tau = \sigma(x_j)^2. \tag{3.183}$$

Thus $0 \leq \|(I-\pi_j^A)(x)u^j\|_\tau^2 = \|xu^j\|_\tau^2 - \sigma(x_j)^2 \leq 0$, hence $\|(I-\pi_j^A)(x)u^j\|_\tau^2 = 0$. Applying the latter to Equation 3.181 yields

$$\big\|\chi_{(0,\infty]}(x)u^j\big\|_\tau^2 = \langle \pi_j^A\big(\chi_{(0,\infty]}(x)u^j\big), u^j\rangle_\tau \geq 1. \tag{3.184}$$

For all $j \in \mathbb{N}$, choice of u^j and Equation 3.184 show

$$\sigma(x_j) = \langle x_j u^j, u^j \rangle_\tau = \langle x u^j, u^j \rangle_\tau \geq \sigma(x) \| \chi_{(0,\infty]}(x) u^j \|_\tau^2 \geq \sigma(x). \tag{3.185}$$

Equation 3.185 implies $\limsup_{j\in\mathbb{N}} \sigma(x_j) \geq \sigma(x)$. Then 1) follows as discussed above.

We show 2). Using 2) in Lemma B.1.72, note Equation 3.163 and Equation 3.164 ensure Lemma 3.2.12 implies 2) if

$$\limsup_{j\in\mathbb{N}} \| \chi_{(0,\infty]}(L_{x_j}) u \|_\tau = 0 \tag{3.186}$$

for all $u \in \operatorname{im} \ker L_x$. We reduce to $u \in \ker L_x \cap L^\infty(A,\tau)$. For all $v \in L^{2,\infty}(A,\tau)$, we see Equation 3.164 shows

$$\pi^A_{\ker L_x}(v) = L_{\delta_0(x)}(v) = \delta_0(x) v \in \ker L_x \cap L^\infty(A,\tau). \tag{3.187}$$

Note 2) in Proposition 2.1.20 shows $A_0 \subset L^{2,\infty}(A,\tau) \subset L^2(A,\tau)$ is $\|.\|_\tau$-dense. Let $u \in \ker L_x$ and fix arbitrary $\{u_n\}_{n\in\mathbb{N}} \subset L^{2,\infty}(A,\tau)$ s.t. $u = \|.\|_\tau\text{-}\lim_{n\in\mathbb{N}} u_n$. For all $j \in \mathbb{N}$ and $n \in \mathbb{N}$, we have $\pi^A_{\ker x}(u_n) \in \ker L_x \cap L^\infty(A,\tau)$ by Equation 3.187 and estimate

$$\| \chi_{(0,\infty]}(x_j) u \|_\tau = \| \chi_{(0,\infty]}(x_j) \pi^A_{\ker L_x}(u) \|_\tau \leq \| u - u_n \|_\tau + \| \chi_{(0,\infty]}(x_j) \pi^A_{\ker L_x}(u_n) \|_\tau \tag{3.188}$$

as non-trivial projections have norm one. Equation 3.188 implies Equation 3.186 if

$$\limsup_{j\in\mathbb{N}} \| \chi_{(0,\infty]}(x_j) u \|_\tau = 0 \tag{3.189}$$

for all $u \in \ker L_x \cap L^\infty(A,\tau)$.

We show Equation 3.189. Assume $\tau(\chi_{(0,\infty]}(x)) < \infty$. Set $y := x + \chi_{(0,\infty]}(x) \in L^2(A,\tau)_+$. Note $\chi_{(0,\infty]}(y) = \chi_{(0,\infty]}(x)$ and $\sigma(y) \geq 1$ by functional calculus. We know restriction maps are positivity-preserving by Proposition 2.1.28. For all $j \in \mathbb{N}$, get $x_j \leq y_j = x_j + \chi_{(0,\infty]}(x)_j$ and therefore

$$\chi_{(0,\infty]}(x_j) = \operatorname{supp} x_j \leq \operatorname{supp} y_j = \chi_{(0,\infty]}(y_j) \tag{3.190}$$

by 2) in Proposition 3.2.3 and 1) in Proposition 3.2.4. Note 2) in the latter proposition shows $\ker L_x = \ker L_y$ since we have $\chi_{(0,\infty]}(y) = \chi_{(0,\infty]}(x)$. For all $u \in \ker L_x \cap L^\infty(A,\tau) = \ker L_y \cap L^\infty(A,\tau)$ and $j \in \mathbb{N}$, we calculate

$$\begin{aligned} 0 = \langle yu, u \rangle_\tau &= \langle y_j u, u \rangle + \langle (I - \pi^A_j)(y) u, u \rangle_\tau \\ &\geq \sigma(y_j) \| \chi_{(0,\infty]}(y_j) u \|_\tau^2 + \langle (I - \pi^A_j)(y) u, u \rangle_\tau. \end{aligned}$$

For all $j \in \mathbb{N}$, we have $\|\chi_{(0,\infty]}(x_j)u\|_\tau^2 \le \|\chi_{(0,\infty]}(y_j)u\|_\tau^2$ by Equation 3.190 and further $|\langle (I-\pi_j^A)(y)u, u\rangle_\tau| \le \|(I-\pi_j^A)(x)\|_\tau \|u\|_\infty \|u\|_\tau < \infty$ by reducing to $\ker L_x \cap L^\infty(A,\tau)$. We also use 1) for $\lim_{j\in\mathbb{N}} \sigma(y_j) = 1$, and note 3) in Proposition 2.1.26. Altogether, we have

$$0 = \langle yu, u\rangle = \limsup_{j\in\mathbb{N}} \langle y_j u, u\rangle_\tau \ge \limsup_{j\in\mathbb{N}} \left\| \chi_{(0,\infty]}(x_j)u \right\|_\tau^2 \ge 0 \tag{3.191}$$

for all $u \in \ker x \cap L^\infty(A,\tau)$. Equation 3.191 shows Equation 3.189. We see Equation 3.186 and therefore 2) follows as discussed above. □

Definition 3.2.17. Let $x \in L^1(A,\tau)_+$. We say that x has

1) integrable support if $\tau(\operatorname{supp} x) < \infty$,

2) reducible support if $\operatorname{supp} x = \text{s-}\lim_{j\in\mathbb{N}} \operatorname{supp} x_j$.

Theorem 3.2.18. *Let (A,τ) be a tracial AF-C^*-algebra. Let $x \in L^1(A,\tau)_+$. If we have $\tau(\operatorname{supp} x) < \infty$, then $\operatorname{supp} x = \text{s-}\lim_{j\in\mathbb{N}} \operatorname{supp} x_j$.*

Proof. Theorem 2.2.53 states $L_x = \text{sr-}\lim_{j\in\mathbb{N}} L_{x_j}$. Using 2) in Lemma B.1.72, as well as 1) and 2) in Proposition 3.2.4, Equation 3.163 and Equation 3.164 show Lemma 3.2.12 implies our claim if

$$\limsup_{j\in\mathbb{N}} \left\| \operatorname{supp} x_j \cdot u \right\|_\tau = 0 \tag{3.192}$$

for all $u \in \ker L_x$. Note 1) in Proposition 3.2.4 shows $\operatorname{supp} x = \chi_{(0,\infty]}(\sqrt{x})$ by positivity and functional calculus. For all $j \in \mathbb{N}$, get $\operatorname{supp} x_j = \chi_{(0,\infty]}(\sqrt{x_j})$. Equation 3.176 for $z = x$ further shows $\tau(\sqrt{x_j}p) = 0$ for all projections $p \in A_j$ s.t. $\tau(\pi_j^A(\sqrt{x})p) = 0$.

For all $j \in \mathbb{N}$, 2) in Proposition 3.2.3 and 1) in Proposition 3.2.4 therefore show

$$\chi_{(0,\infty]}\left(\sqrt{x_j}\right) \le \chi_{(0,\infty]}\left(\pi_j^A(\sqrt{x})\right). \tag{3.193}$$

Thus 1) in Proposition 3.2.4 and 2) in Lemma 3.2.16 show

$$\operatorname{supp} x = \text{s-}\lim_{j\in\mathbb{N}} \operatorname{supp} \pi_j^A(\sqrt{x}) = \text{s-}\lim_{j\in\mathbb{N}} \chi_{(0,\infty]}\left(\pi_j^A(\sqrt{x})\right), \tag{3.194}$$

hence Equation 3.193 and Equation 3.194 let us estimate

$$0 = \left\| \operatorname{supp} x \cdot u \right\|_\tau^2 = \lim_{j\in\mathbb{N}} \left\| \chi_{(0,\infty]}\left(\pi_j^A(\sqrt{x})\right) \cdot u \right\|_\tau^2 \ge \limsup_{j\in\mathbb{N}} \left\| \operatorname{supp} x_j \cdot u \right\|_\tau^2 \ge 0 \tag{3.195}$$

for all $u \in \ker L_x$. Equation 3.195 immediately shows Equation 3.192. We obtain our claim as described above. □

Corollary 3.2.19. *If $\tau < \infty$, then all $x \in L^1(A,\tau)_+$ have reducible support.*

Proof. Apply Theorem 3.2.18. □

Corollary 3.2.20. *If $x \in L^1(A,\tau)_+$ has spectral gap, then x has reducible support.*

Proof. Note 1) in Proposition 3.2.4 shows $x \geq \sigma(x) \cdot \operatorname{supp} x$ by functional calculus. Thus $\tau(\operatorname{supp} x) \leq \sigma(x)^{-1}\tau(x) < \infty$ since $\sigma(x) > 0$, hence Theorem 3.2.18 applies. □

Theorem 3.2.18 gives sufficient conditions for reducible support. Non-integrability does not exclude reducible support in general. All injective $x \in L^1(A,\tau)_+$ have reducible support by Lemma 3.2.12 (cf. Proposition A.1.88). If $(A,\tau) = (\mathscr{K}(H),\operatorname{tr})$ for a separable Hilbert space H, then Example 3.2.21 shows integrable support is equivalent to being a finite-dimensional matrix.

Example 3.2.21. Let H be a separable Hilbert space. Assume $(A,\tau) = (\mathscr{K}(H),\operatorname{tr})$. Let $x \in S_1(H)$. There exists $U \in \mathscr{U}(\mathscr{B}(H))$ s.t. UxU^* has diagonal form. We know the latter is determined by $\{\lambda_n\}_{n\in\mathbb{N}} \subset [0,\infty)$ up to reordering. Applications of unitary conjugations are normal unital *-homomorphisms. Thus $\operatorname{supp} x = U^*(\operatorname{supp} UxU^*)U$ by Lemma A.1.92 and Corollary A.1.93, hence $\operatorname{tr}(\operatorname{supp} x) = \operatorname{tr}(\operatorname{supp} UxU^*)$. Ergo $\tau(\operatorname{supp} x) < \infty$ if and only if $\lambda_n = 0$ for a.e. $n \in \mathbb{N}$, i.e. $\operatorname{tr}(\operatorname{supp} x) < \infty$ if and only if $\operatorname{supp} x \in \mathscr{K}(H)_0 = \bigcup_{n\in\mathbb{N}} M_n(\mathbb{C})$.

3.2.2 Noncommutative heat semigroups of quantum Laplacians

Noncommutative heat semigroups of quantum Laplacians are trace-preserving, as well as completely Markovian. In the finite-dimensional setting, self-adjointness implies quantum Laplacians satisfy, up to sign, a quantum Fokker-Planck equation with vanishing drift term [121], i.e. only diffusion term. The latter solve special cases of general Lindblad master equations [82][121][187] describing purely irreversible time-evolution of dissipative quantum systems [35][36][82][121][163][188]. Of course, the sign occurs since negatives of quantum Laplacians generate noncommutative heat semigroups.

We view such diffusion terms of quantum Fokker-Planck equations as infinitesimal applications of quantum channels [28][73] transmitting change of states of the given quantum system determined by irreversible interaction with its environment [62][141]. The extension [45][95] of Landauer's principle [142][143] gives strictly positive lower bounds on production of quantum entropy upon application of quantum channels due to minimal heat dissipation [15][44][181]. Following a maximum entropy production principle [91][92][155], we select noise diffusion terms in the finite-dimensional setting by maximising production of quantum entropy under constraints on energy spent and assume stability under scaling limits. Following our discussion of the coarse graining process in Subsection 3.3.2, we show quantum Laplacians satisfy, up to sign, a quantum Fokker-Planck equation with vanishing drift term in scaling limit, i.e. only noise diffusion term. Altogether, we therefore view quantum Laplacians as generators of quantum noise evolution in Subsection 4.2.3, and obtain a description of quantum Laplacians in terms of both quantum statistical mechanics [35][36] and quantum information theory [62] as claimed in the introduction of this chapter.

We require regularisation of normal states under heat flow. Theorem 3.2.40 shows such regularisation by combining compressing with support projections of normal fixed states and finite-dimensional approximation. This uses compatibility with compression and finite-dimensional approximation. As such, each step of the coarse graining process terminates at accessibility components in the finite-dimensional setting s.t. heat flow maps to their relative interiors for all non-zero times. Standard references for quantum statistical mechanics are [35][36], [82], [121], [163] and [188]. Standard reference for quantum information theory is [62]. We further use and refer to [45] as comprehensive treatment of the quantum statistical mechanics of quantum information.

Completely Markovian semigroups. We discuss both completely Markovian semigroups and Lindblad master equations, as well as their special case of quantum Fokker-Planck equations. Generalising the uniformly continuous case [61][150][149] applied to open quantum systems [79][80][81][115], completely Markovian semigroups [83][85][86] describe time-evolution of dissipative quantum systems weakly coupled to a heat bath [82][121][187]. Symmetric C^*-derivations are noncommutative gradients and define Laplacians generating completely Markovian noncommutative heat semigroups [63][65]. Following Remark 2.3.3, we specialise to the AF-C^*-setting in order to study noncommutative heat semigroups of quantum Laplacians.

Definition 3.2.22 gives completely Markovian semigroups for tracial C^*-algebras. We use completely positive and completely Markovian maps (cf. Definition A.1.45 and Definition A.1.54). Lemma 3.2.23 gives sufficient conditions for satisfying Equation 3.196 as special case of general Lindblad master equations [82][121][187]. This yields Lindblad decompositions as per Definition 3.2.24. Following Remark 3.2.26, Equation 3.196 is a quantum Fokker-Planck equation with drift and diffusion terms as per Equation 3.209. We view such diffusion terms as infinitesimal applications of quantum channels [28][73] transmitting change of states of the given quantum system determined by irreversible interaction with its environment [62][141]. If self-adjointness in the finite-dimensional setting is given, then Corollary 3.2.25 shows we may assume vanishing drift term.

Let (A,τ) be a tracial C^*-algebra.

Definition 3.2.22. A semigroup $G:[0,\infty)\longrightarrow\mathscr{B}(L^\infty(A,\tau))$ is completely Markovian if $G_t:L^\infty(A,\tau)\longrightarrow L^\infty(A,\tau)$ is a completely Markovian normal map for all $t\geq 0$.

Lemma 3.2.23. *Assume $\tau<\infty$. Let $S\in\mathscr{B}(L^2(A,\tau))_h$ s.t. $S\neq 0$, $S(L^\infty(A,\tau))\subset L^\infty(A,\tau)$ and $S(1_A)=0$. We have semigroup $G^S:[0,\infty)\longrightarrow\mathscr{B}(L^\infty(A,\tau))$ by setting $G^S_t:=e^{tS}$ for all $t\geq 0$. If $S:L^\infty(A,\tau)\longrightarrow L^\infty(A,\tau)$ is normal and $G^S:[0,\infty)\longrightarrow\mathscr{B}(L^\infty(A,\tau))$ is a completely Markovian semigroup, then there exists $H\in L^\infty(A,\tau)_h$, completely positive normal $\varphi:L^\infty(A,\tau)\longrightarrow L^\infty(A,\tau)$ with $\|\varphi(1_A)\|_\infty=1$, and $C>0$ satisfying the Lindblad master equation*

$$S(x)=i[H,x]+\frac{C}{2}\Big(2\varphi(x)-\big\{\varphi(1_A),x\big\}\Big) \tag{3.196}$$

for all $x\in L^\infty(A,\tau)$.

Proof. Note $G^S : [0,\infty) \longrightarrow \mathscr{B}(L^\infty(A,\tau))$ is a semigroup by boundedness and functional calculus. Assume $S : L^\infty(A,\tau) \longrightarrow L^\infty(A,\tau)$ is normal and $G^S : [0,\infty) \longrightarrow \mathscr{B}(L^\infty(A,\tau))$ is a completely Markovian semigroup. Set $\mathscr{A} := L_{L^\infty(A,\tau)}(A)$. Then $\mathscr{A}'' = L_{L^\infty(A,\tau)}(L^\infty(A,\tau))$ is the σ-weak closure (cf. Proposition A.1.34 and Proposition B.1.9). Theorem 3.1 in [61] applies to the canonical lift of G^S to $\mathscr{A}''$. For all $t \geq 0$, set

$$S^\dagger := L_{L^\infty(A,\tau)} \circ S \circ L^{-1}_{L^\infty(A,\tau)},\ G^{S,\dagger}_t := L_{L^\infty(A,\tau)} \circ G^S_t \circ L^{-1}_{L^\infty(A,\tau)}. \tag{3.197}$$

We have $G^{S,\dagger}_t = e^{tS^\dagger}$ in each case by norm differentiation. Since *-homomorphisms are completely positive (cf. Example A.1.47), conjugation with canonical left-actions as per Equation 3.197 preserves complete positivity. Moreover, normality is preserved by the GNS-construction (cf. Proposition B.1.9). Thus $G^{S,\dagger} : [0,\infty) \longrightarrow \mathscr{B}(\mathscr{A}'')$ is a uniformly $\|.\|_{\mathscr{A}''}$-continuous semigroup s.t. $G^{S,\dagger}_t : \mathscr{A}'' \longrightarrow \mathscr{A}''$ is a completely Markovian normal map for all $t \geq 0$, hence Theorem 3.1 in [61] applies.

We apply Theorem 3.1 in [61]. The theorem yields $H^\dagger \in \mathscr{A}''_h$ and completely positive $\varphi^\dagger : \mathscr{A}'' \longrightarrow \mathscr{A}''$ s.t.

$$S^\dagger\big(L_{x,L^\infty(A,\tau)}\big) = i\big[H^\dagger, L_{x,L^\infty(A,\tau)}\big] + \varphi^\dagger\big(L_{x,L^\infty(A,\tau)}\big) - \frac{1}{2}\big\{\varphi^\dagger(I), L_{x,L^\infty(A,\tau)}\big\} \tag{3.198}$$

for all $x \in L^\infty(A,\tau)$. Using $S^\dagger : \mathscr{A}'' \longrightarrow \mathscr{A}''$ normal, Equation 3.198 implies $\varphi^\dagger : \mathscr{A}'' \longrightarrow \mathscr{A}''$ is normal by rearranging terms accordingly. Set

$$H := L^{-1}_{L^\infty(A,\tau)} \circ H^\dagger \circ L_{L^\infty(A,\tau)},\ \varphi^S := L^{-1}_{L^\infty(A,\tau)} \circ \varphi^\dagger \circ L_{L^\infty(A,\tau)}. \tag{3.199}$$

Since we conjugate with normal *-homomorphisms, get $H \in L^\infty(A,\tau)_h$ and completely positive normal $\varphi^S : L^\infty(A,\tau) \longrightarrow L^\infty(A,\tau)$. Using the latter, applying Equation 3.197 and Equation 3.199 to Equation 3.198 shows

$$S(x) = i[H,x] + \frac{1}{2}\Big(2\varphi^S(x) - \big\{\varphi^S(1_A), x\big\}\Big) \tag{3.200}$$

for all $x \in L^\infty(A,\tau)$. Note $\varphi^S(1_A) = 0$ implies $\varphi^S = 0$ by positivity-preservation. Since $H \in L^\infty(A,\tau)_h$, as well as $S \in \mathscr{B}(L^2(A,\tau))_h$ and $S \neq 0$, Equation 3.200 shows $\varphi^S(1_A) \neq 0$ by self-adjointness. Equation 3.200 therefore shows

$$H,\ \varphi := \big\|\varphi^S(1_A)\big\|^{-1}_\infty \varphi^S,\ C := \big\|\varphi^S(1_A)\big\|_\infty \tag{3.201}$$

satisfy Equation 3.196 for S as claimed. $\square$

Definition 3.2.24. Assume the setting of Lemma 3.2.23.

1) We call $G^S : [0,\infty) \longrightarrow \mathscr{B}(L^\infty(A,\tau))$ the induced semigroup of S.
2) If $S|_{L^\infty(A,\tau)} : L^\infty(A,\tau) \longrightarrow L^\infty(A,\tau)$ is normal and G^S completely Markovian, then we call (H,φ,C) as per Equation 3.196 a Lindblad decomposition of S.

Corollary 3.2.25. *Assume A is finite-dimensional. Let $S \in \mathscr{B}(A)_h$ s.t. $S \neq 0$ and $S(1_A) = 0$. If S has completely Markovian induced semigroup, then there exists completely positive self-adjoint normal $\varphi : A \longrightarrow A$ with $\|\varphi(1_A)\|_\infty = 1$ and $C > 0$ s.t. $(0,\varphi,C)$ is a Lindblad decomposition of S.*

Proof. We have finite-dimensional tracial W^*-algebra $(\mathscr{B}(A),\mathrm{tr})$. For all $T \in \mathscr{B}(A)$, we decompose $T = \mathrm{Re}(T) + i\,\mathrm{Im}(T)$ into real and imaginary parts

$$\mathrm{Re}(T) = \frac{T+T^*}{2},\ \mathrm{Im}(T) = -i\frac{T-T^*}{2} \tag{3.202}$$

as per 1) in Proposition B.1.47. Equation 3.202 yields $\mathscr{B}(A) = \mathscr{B}(A)_h \oplus \mathscr{B}(A)_h$ using direct sum of real vector spaces. Let $T \in \mathscr{B}(A)$. We have $T \in \mathscr{B}(A)_h$ if and only if $\mathrm{Im}(T) = 0$. For all $u,v \in A$, set $x := v^*v, y := uu^* \in A_+$ and calculate

$$\langle L_{T(x)}u,u\rangle_\tau = \langle v^*v, T^*(y)\rangle_\tau = \langle v, vT^*(y)\rangle_\tau = \langle v, R_{T^*(y)}(v)\rangle_\tau = \langle R_{(T^*(y))^*}(v), v\rangle_\tau. \tag{3.203}$$

For all $y \in A_+$, we have $T^*(y) \geq 0$ if and only if $(T^*(y))^* \geq 0$ since $A_+ \subset A_h$. Using the latter and 3) in Proposition B.1.70, Equation 3.203 implies T is positivity-preserving if and only if T^* is. For all $n \in \mathbb{N}$, we argue analogously upon replacing (A,τ) with the finite-dimensional tracial C^*-algebra $(A \otimes M_n(\mathbb{C}), \tau \otimes \mathrm{tr}_n)$. Altogether, we know T is completely positive if and only if T^* is. We may also use Proposition 2.1.24 and reduce to Choi's theorem [82] for pairs of summands in $A \cong \oplus_{l=1}^n M_{n_l}(\mathbb{C})$, i.e. representations as per Equation 3.205 up to conjugation with projections, for alternative proof. If $T \in \mathscr{B}(A)_h$ is completely positive, then the first identity in Equation 3.202 shows $\mathrm{Re}(T)$ is completely positive, and the second one $\mathrm{Re}(T)(1_A) = T(1_A)$ since $T(1_A) = T^*(1_A)$ by $\mathrm{Im}(T) = 0$.

Normality is equivalent to boundedness in the finite-dimensional setting. Assume S has completely Markovian induced semigroup. We are in the setting of Lemma 3.2.23. Let (H,φ,C) be a Lindblad decomposition of S. Note $[H,-] \in \mathscr{B}(A)_h$. Using the latter and $S \in \mathscr{B}(A)_h$, Equation 3.196 and Equation 3.202 show

$$S = \mathrm{Re}(S) = \frac{C}{2}\Big(2\,\mathrm{Re}(\varphi) - \{\varphi(1_A), -\}\Big),\ 0 = \mathrm{Im}(S) = i[H,-] + iC\,\mathrm{Im}(\varphi). \tag{3.204}$$

Using $[H,1_A] = 0$, the second identity in Equation 3.204 shows $\mathrm{Im}(S)(1_A) = 0$ at once and therefore $\mathrm{Re}(\varphi)(1_A) = \varphi(1_A)$. Thus $\|\mathrm{Re}(\varphi)(1_A)\|_\infty = \|\varphi(1_A)\|_\infty = 1$, hence we have completely positive self-adjoint normal $\mathrm{Re}(\varphi) : A \longrightarrow A$ with $\|\mathrm{Re}(\varphi)(1_A)\|_\infty = 1$ since $\varphi : A \longrightarrow A$ is completely positive normal with $\|\varphi(1_A)\|_\infty = 1$ by hypothesis. The first identity in Equation 3.204 shows $(0,\mathrm{Re}(\varphi),C)$ is Lindblad decomposition of S. □

We show Equation 3.196 is a special case of a general Lindblad master equation (cf. Equation 5.2.29 in [121]). Assume the setting of Lemma 3.2.23. We use notation from its proof. Assume A is separable. Note $\tau < \infty$ ensures $L^2(A,\tau)$ is separable.

Let (H,φ,C) be a Lindblad decomposition of S. Upon conjugation with canonical left-actions as per Equation 3.197, Theorem 3.1 in [61] yields Lindblad decomposition $(H^\dagger,\varphi^\dagger,C)$ of $S^\dagger$. Using separability of $L^2(A,\tau)$ in order to have a sequence, Theorem 2.3 in Chapter 9 in [82] shows there exist $\{W_n\}_{n\in\mathbb{N}} \subset \mathscr{B}(L^2(A,\tau))$ s.t. we have $\sum_{n\in\mathbb{N}} W_n^* T W_n = \text{w-lim}_{m\in\mathbb{N}} \sum_{n=1}^m W_n^* T W_n$ and further

$$\varphi^\dagger(T) = \sum_{n\in\mathbb{N}} W_n^* T W_n \tag{3.205}$$

for all $T \in \mathscr{A}''$. Using unitality of canonical left-actions of tracial W^*-algebras, we have $\sum_{n\in\mathbb{N}} W_n^* W_n \leq I$ since $\varphi^\dagger : \mathscr{A}'' \longrightarrow \mathscr{A}''$ is positivity-preserving with $\|\varphi^\dagger(I)\|_\infty = 1$. This lets us relax unitality $\sum_{n\in\mathbb{N}} W_n^* W_n = I$ in the definition of quantum channels [62][141].

Equation 3.205 is a Kraus operator representation of $\varphi^\dagger$ with $\{W_n\}_{n\in\mathbb{N}} \subset \mathscr{B}(L^2(A,\tau))$ its Kraus operators [141]. Applying Equation 3.205 to Equation 3.196 for $S^\dagger$ yields

$$S^\dagger(T) = i\left[H^\dagger, T\right] + \sum_{n\in\mathbb{N}} W_n^* T W_n - \frac{1}{2}\left\{W_n^* W_n, T\right\} \tag{3.206}$$

for all $T \in \mathscr{A}''$. Pulled-back along the canonical left-action, we have

$$\varphi(x) = L^{-1}_{L^\infty(A,\tau)}\left(\sum_{n\in\mathbb{N}} W_n^* L_{x,L^\infty(A,\tau)} W_n\right) \tag{3.207}$$

for all $x \in L^\infty(A,\tau)$. Equation 3.206 and Equation 3.207 show

$$S(x) = i[H,x] + \frac{C}{2} L^{-1}_{L^\infty(A,\tau)}\left(\sum_{n\in\mathbb{N}} W_n^* L_{x,L^\infty(A,\tau)} W_n - \frac{1}{2}\left\{W_n^* W_n, L_{x,L^\infty(A,\tau)}\right\}\right) \tag{3.208}$$

for all $x \in L^\infty(A,\tau)$. Equation 3.208 is called a Kraus operator representation of S and Equation 3.196. Up to strictly positive constants, Equation 3.206, i.e. Equation 3.196 via Kraus operator representation as per Equation 3.208, is a general Lindblad master equation. Following Remark 3.2.26, we additionally know Equation 3.196 is a quantum Fokker-Planck equation with drift and diffusion terms as per Equation 3.209 s.t. their diffusion terms are infinitesimal applications of quantum channels.

Remark 3.2.26. Note general Lindblad master equations (cf. Equation 5.2.29 in [121]) specialise to quantum Fokker-Planck equations as follows. If quantum white noise is the input for a given quantum system, then its associated quantum Langevin equation (cf. Equation 5.3.15 in [121]) determines a quantum stochastic differential equation in Itô form (cf. Equation 5.3.50 in [121]) based on a quantum Wiener process.

Using reduced trace obtained by the weak coupling assumption, dualisation yields a linear differential equation of density operators (cf. Equation 5.4.12 in [121]). It is a quantum Fokker-Planck equation describing time-evolution of the given quantum system under quantum white noise similar to the classical case [180]. Indeed, it is a general Lindblad master equation s.t. commutators are taken w.r.t. the Hamiltonian of the given quantum system, and separates into distinct drift and diffusion terms arising from corresponding terms with physical meaning in the quantum Langevin equation. The former arise from all reversible interactions within quantum systems, whereas the latter do from all irreversible ones with their given environments. For details on general Lindblad master equations and the above derivation, we refer to Chapter 5 in [121]. For details on their many applications, we refer to [82], [121] and [187].

Assume the setting of Lemma 3.2.23. Let (H,φ,C) be a Lindblad decomposition of S. We consider H as Hamiltonian of a quantum system. Using the latter and following our above discussion, note Equation 3.196 is a quantum Fokker-Planck equation s.t. its commutator is taken w.r.t. H. We have drift term $S^{\mathrm{Drift}} \in i\mathscr{B}(L^2(A,\tau))_h$ and diffusion term $S^{\mathrm{Diff}} \in \mathscr{B}(L^2(A,\tau))_h$ given by

$$S^{\mathrm{Drift}}(x) = i[H,x],\ S^{\mathrm{Diff}}(x) = \frac{C}{2}\cdot\left(2\varphi(x) - \{\varphi(1_A),x\}\right) \tag{3.209}$$

for all $x \in L^\infty(A,\tau)$. Following our above discussion, S^{Drift} is the reversible part, and S^{Diff} the irreversible part of Equation 3.196. Altogether, Equation 3.196 is described in terms of quantum statistical mechanics [35][36]. We view S^{Diff} as infinitesimal application of the quantum channel $\varphi : L^\infty(A,\tau) \longrightarrow L^\infty(A,\tau)$ below. If $H = 0$, then we thereby describe Equation 3.196 in terms of quantum information theory [62].

Completely positive normal unital maps are quantum channels (cf. pp.353-373 in [62]). We may relax unitality in Kraus operator representations (cf. p.360 in [62]). Each quantum channel describes a state change due to measurement (cf. pp.360-364 in [62] or [84][141][163]), i.e. each transmits a corresponding change of information encoded in states of the given quantum system (cf. 365-373 in [62]) providing physical realisation of a quantum computer (cf. Chapter 7 in [62] or [18][43]). We therefore have quantum channel $\varphi : L^\infty(A,\tau) \longrightarrow L^\infty(A,\tau)$. The second identity in Equation 3.209 shows

$$S^{\mathrm{Diff}}(x) = C\cdot\left(\left(\varphi(x) - x\right) - \left[\frac{1}{2}\{\varphi(1_A),x\} - x\right]\right) \tag{3.210}$$

for all $x \in L^\infty(A,\tau)$. If φ is unital, then the second term in Equation 3.210 vanishes. Up to strictly positive constant, Equation 3.210 shows S^{Diff} is the difference operator given by φ minus a correction term controlling for non-unitality. The latter uses anti-commutator given by the arithmetic operator mean for two variables evaluated on $\varphi(1_A)$ [13]. It is a quantum channel and the correction terms its difference operator. Up to energy scale but accounting for non-unitality, Equation 3.210 shows φ transmits change of states of the given quantum system arising from irreversible interactions with its environment as per S^{Diff} for a discrete time-step, resp. applying S^{Diff} yields such change as per φ but infinitesimally [28][73]. We therefore view S^{Diff} as infinitesimal application of φ.

Definition and properties. Definition 3.2.31 gives noncommutative heat semigroups of quantum Laplacians by extending Definition 3.2.27 via the modified standard pairing. Following Remark 2.3.3, this is based on the extension of completely Markovian semigroups in [63] and uses results in [63][65]. Proposition 3.2.32 and Proposition 3.2.34 collect properties. In particular, note 3) in Proposition 3.2.34 shows sets of states at finite distance have identical fixed parts.

Let $(\phi,\psi,\gamma,\nabla)$ be noncommutative differential structure for tracial AF-C^*-algebras (A,τ) and (B,ω) in (f,θ)-setting.

Definition 3.2.27. We define heat semigroup $h:[0,\infty)\longrightarrow\mathscr{B}(L^2(A,\tau))$ of Δ by setting

$$h_t(u) := e^{-t\Delta}(u) \tag{3.211}$$

for all $t\geq 0$ and $u\in L^2(A,\tau)$.

Notation 3.2.28. For all $j\in\mathbb{N}$, let $h^j:[0,\infty)\longrightarrow\mathscr{B}(A_j)$ denote heat semigroup of Δ_j in Definition 3.2.27 for the induced noncommutative differential structure $(\phi_j,\psi_j,\gamma_j,\nabla_j)$.

Remark 3.2.29. Note $\Delta\in\mathscr{U}\mathscr{B}(L^2(A,\tau))$ is local by 4) in Proposition 2.3.19 and 3.1) in Proposition 2.3.25. Thus Proposition 2.3.52 applies, hence 1) therein yields orthonormal eigenbasis $\{e_n\}_{n\in\mathbb{N}}\subset A_0$ of Δ s.t. it is furthermore orthonormal eigenbasis of π^A_j for all $j\in\mathbb{N}$. By testing on A_0 using 4) in Proposition 2.3.19, 3.1) in Proposition 2.3.25 shows

$$\left[\pi^A_j,\pi^A_{\ker\Delta}\right]=0 \tag{3.212}$$

for all $j\in\mathbb{N}$ since $A_0\subset L^2(A,\tau)$ is $\|.\|_\tau$-dense. Alternatively, we derive Equation 3.212 by calculating on an orthonormal basis as above. Equation 3.212 thereby generalises to 2.2) in Proposition 3.2.32.

The heat semigroup of Δ extends as follows. For all $j\in\mathbb{N}$, following Remark 3.2.29 note 3.1) in Proposition 2.3.25 lets us apply 1) in Proposition 2.3.22 in order to get

$$h_t(x)=\left(e^{-t\Delta_j}\oplus e^{-t\Delta_j^\perp}\right)(x)=e^{-t\Delta_j}(x)=h^j_t(x)\in A_j \tag{3.213}$$

for all $t\geq 0$ and $x\in A_j$. For all $j\in\mathbb{N}$, we have symmetric C^*-derivation $\nabla_j:A_j\longrightarrow B_j$ by 1) in Proposition 2.3.25. Theorem 8.3 in [65] shows we have C^*-Dirichlet form $u\mapsto\|\nabla_j u\|^2_\tau$ on A_j in each case. Using the latter, Theorem 4.11 in [63] shows we have completely Markovian semigroup $h^j:[0,\infty)\longrightarrow\mathscr{B}(A_j)$ as well. Note our argument here initially yields Markovianity. Completeness follows by likewise application of both theorems to extensions of symmetric C^*-derivations to full matrix algebras over finite-dimensional tracial C^*-algebras. Theorem 2.12 in [63] shows completely Markovian semigroups and their extensions to Banach dual spaces are given by completely positive dilations.

For all $j \in \mathbb{N}$, we therefore have

$$\left\| h_t^j(x) \right\|_\infty \leq \|x\|_\infty \tag{3.214}$$

for all $x \in A_j$. Using $A_0 \subset A$ $\|.\|_\infty$-dense, Equation 3.212 and Equation 3.214 then yield extension $h_t \in \mathscr{B}(A)$ of Equation 3.211 for all $t \geq 0$. Dualisation of such an extended Equation 3.211 defines semigroup $h : [0,\infty) \longrightarrow \mathscr{B}(A^*)$ by setting

$$h_t(\mu)(x) := e^{-t\Delta}(\mu)(x) := \mu(h_t(x)) \tag{3.215}$$

for all $t \geq 0$, $\mu \in A^*$ and $x \in A$. Following Remark 2.1.2, normality moreover restricts Equation 3.215 to

$$\sharp\left(h_t|_{L^1(A,\tau)^\flat}\right) \circ \flat \in \mathscr{B}\left(L^1(A,\tau)\right) \tag{3.216}$$

for all $t \geq 0$. Equation 3.216 defines semigroup $h : [0,\infty) \longrightarrow \mathscr{B}(L^1(A,\tau))$ by setting

$$h_t(x) := e^{-t\Delta}(x) := \sharp\left(h_t(x^\flat)\right) \tag{3.217}$$

for all $t \geq 0$ and $x \in L^1(A,\tau)$. Finally, dualisation of Equation 3.217 and accounting for using the modified standard pairing $L^\infty(A,\tau) = L^1(A,\tau)^*$ as per Equation 3.216 defines semigroup $h : [0,\infty) \longrightarrow \mathscr{B}(L^\infty(A,\tau))$ by setting

$$h_t(x)(y) := e^{-t\Delta}(x)(y) := x^\flat(h_t(y)) \tag{3.218}$$

for all $t \geq 0$, $x \in L^\infty(A,\tau)$ and $y \in L^1(A,\tau)$. Note Equation 3.218 restricts to extension of Equation 3.211 to A for all $x \in A$. Up to musical isomorphisms, all extensions coincide on intersections of domains. Altogether, we have noncommutative heat semigroup of Δ mapping to $\mathscr{B}(V)$ if $V = A^*$ or $V = L^p(A,\tau)$ for $p \in \{1,2,\infty\}$.

Proposition 3.2.30. *Let $V = A^*$ or $V = L^p(A,\tau)$ for $p \in \{1,2,\infty\}$.*

1) For all $v \in V$, $h_\infty(v) := w^*\text{-}\lim_{t\to\infty} h_t(v)$ exists.

2) For all $t \geq 0$ and $u \in L^2(A,\tau)$, we have

2.1) $h_\infty(u) = \pi^A_{\ker\Delta}(u)$,

2.2) $h_t(u) \neq 0$ if $u \neq 0$.

Proof. Following Remark 2.1.2, density of A_0 and normality imply $\|.\|_V$ is determined by testing on A_0. Let $v \in V$. Equation 3.214 shows $\sup_{t\geq 0} \|h_t(v)\|_V \leq 4\|v\|_V$. Thus 1) follows if $\lim_{t\to\infty} h_t(v)(x)$ exists for all $x \in A_0$. We require 2.1). Following Remark 3.2.29 and using Equation 3.213, we calculate $\pi^A_{\ker\Delta}(x) = \|.\|_\tau\text{-}\lim_{t\to\infty} h_t(v)(x)$ for all $x \in A_0$ on an orthonormal eigenbasis $\{e_n\}_{n\in\mathbb{N}} \subset A_0$ of Δ as per the remark. We obtain 2.1) by density. Then 2.1) implies 1). We directly verify 2.2) by likewise calculation. Get 2). □

Definition 3.2.31. Let $V = A^*$ or $V = L^p(A,\tau)$ for $p \in \{1,2,\infty\}$. We define heat semigroup $h : [0,\infty] \longrightarrow \mathscr{B}(L^2(A,\tau))$ of Δ by setting

$$h_t(v) := e^{-t\Delta}(v) \tag{3.219}$$

for all $t \geq 0$ and $v \in V$.

Proposition 3.2.32. *Let $V = A^*$ or $V = L^p(A,\tau)$ for $p \in \{1,2,\infty\}$.*

1) We have strongly continuous semigroup $h : [0,\infty) \longrightarrow \mathscr{B}(V)$. In particular, we have trace-preserving and completely Markovian semigroup $h : [0,\infty) \longrightarrow \mathscr{B}(L^\infty(A,\tau))$.

2) For all $t \in [0,\infty]$, we have

2.1) h_t is positivity-preserving and w^*-continuous on norm bounded sets,

2.2) $h_t^j \circ \mathrm{res}_j = h_t \circ \mathrm{res}_j = \mathrm{res}_j \circ h_t$ for all $j \in \mathbb{N}$,

2.3) $\|h_t\|_{\mathscr{B}(V)} \leq 1$ and $h_t(1_A) = 1_A$,

2.4) $h_t \in \mathscr{B}\big(L^2(A,\tau)\big)_h$ is local.

Proof. By construction, $h : [0,\infty) \longrightarrow \mathscr{B}(V)$ is a semigroup s.t. h_t is w^*-continuous on norm bounded sets for all $t \geq 0$. We show 1). For all $t \geq 0$, testing for $\|.\|_V$ on A_0 lets us apply Equation 3.214 in order to calculate

$$\|h_t\|_{\mathscr{B}(V)} \leq 1 \tag{3.220}$$

for all $v \in V$ and $t \geq 0$. Equation 3.220 implies strong continuity. We extend to $t = \infty$ by letting $t \uparrow \infty$ in the latter equation. Assume $V = L^\infty(A,\tau)$. For all $j \in \mathbb{N}$, note $\Delta_j 1_{A_j} = 0$ by the Leibniz rule. Using the latter and 2) in Proposition 2.1.16, Equation 3.213 lets us calculate $h_t(1_A) = \text{s-lim}_{j\in\mathbb{N}} h_t(1_{A_j}) = \text{s-lim}_{j\in\mathbb{N}} 1_{A_j} = 1_A$ for all $t \geq 0$. We extend to $t = \infty$ by letting $t \uparrow \infty$ in our calculation. Moreover, we see $h_t \in \mathscr{B}(L^\infty(A,\tau))$ is trace-preserving for all $t \geq 0$ by testing all $x \in L^{1,\infty}(A,\tau)$ with $y = 1_A$ as per Equation 3.218.

For all $j \in \mathbb{N}$, our construction ensures $h^j : [0,\infty) \longrightarrow \mathscr{B}(A_j)$ is completely Markovian. Using 2.2) in Proposition 2.1.31, resp. 2) in Proposition 2.1.16, we calculate

$$h_t(x) \otimes I_n = w^*\text{-}\lim_{j\in\mathbb{N}} h_t(x_j) \otimes I_n \geq 0 \tag{3.221}$$

and

$$h_t(1_A) \otimes I_n = w^*\text{-}\lim_{j\in\mathbb{N}} h_t(1_{A_j}) \otimes I_n \leq w^*\text{-}\lim_{j\in\mathbb{N}} 1_{A_j} \otimes I_n = 1_A \otimes I_n \tag{3.222}$$

for all $n \in \mathbb{N}$ and $x \in L^\infty(A,\tau)_+$. Equation 3.221 uses restrictions are positivity-preserving by Proposition 2.1.28. For all $t \geq 0$, Equation 3.221 shows h_t is completely positive and Equation 3.222 shows h_t is completely Markovian. We are left to show normality in each case. Complete positivity and Proposition A.1.49 reduce to σ-weak continuity. Note the latter is equivalent to w^*-continuity on norm bounded sets (cf. Lemma II.2.5 in [192] and Proposition A.1.34). Get 1).

Assume the general case. We show 2). Since we have w^*-continuity on norm bounded sets, positivity-preservation and therefore 2.1) follows by arguing as for Equation 3.221 in the general case without tensoring. We know all extensions coincide on intersections of domains. Equation 3.213 shows 2.2) and Equation 3.214 shows 2.3). Then 2.2) implies 2.4) at once. Altogether, get 2). □

Definition 3.2.33 gives fixed parts of positive bounded functionals, and thereby fixed states, under noncommutative heat semigroups of quantum Laplacians. Note states are preserved by 1) in Proposition 3.2.34, and have identical fixed parts if at finite distance by 3) in Proposition 3.2.34. Following this, Definition 3.2.35 gives sets of states which are determined by fixed parts. These help to classify accessibility components.

Definition 3.2.33. For all $\mu \in A^*$, $h(\mu) := h_\infty(\mu)$ is its fixed part and $h^\perp(\mu) := \mu - h(\mu)$ its image part. We call $\xi \in \mathscr{S}(A)$ a fixed state, or fixed if $h(\xi) = \xi$.

Proposition 3.2.34.

1) *For all* $\mu \in A^*_+$, $t \in [0,\infty]$ *and* $j \in \mathbb{N}$, *we have*

   1.1) $\|h_t(\mu)\|_{A^*} = \|\mu\|_{A^*}$,

   1.2) $\mu = 0$ *if* $h(\mu) = 0$,

   1.3) $\overline{h_t(\mu)}_j = h_t(\bar{\mu}_j)$.

2) *For all* $t \in [0,\infty]$, *we have*

   2.1) $h_t(\mathscr{S}(A)) \subset \mathscr{S}(A)$,

   2.2) $h_t(\mathscr{S}^{\mathrm{N}}(A)) \subset \mathscr{S}^{\mathrm{N}}(A)$.

3) *For all* $(\mu, w) \in \mathrm{Adm}^{[0,1]}$, *we have* $h(\mu(0)) = h(\mu(1))$. *In particular, states at finite distance have identical fixed part.*

Proof. Note 1.1) and 1.2) follows from 1) in Proposition 2.1.31 and trace-preservation as per 1) in Proposition 3.2.32. Using 1.1) for rescaling as per 1) in Definition 3.1.12, get 1.3) by 2.2) in Proposition 3.2.32. Note Remark 3.1.15. Equation 3.216 shows normality is preserved under $h_t \in \mathscr{B}(A^*)$ for all $t \in [0,\infty]$. Then 1) implies 2). For 3), we reduce to the finite-dimensional setting by 2) in Corollary 3.1.49 and 1.3).

Assume A and B are finite-dimensional. Let $(\mu, w) \in \mathrm{Adm}^{[0,1]}$. Thus the continuity equation and finite-dimensionality imply

$$\sharp\dot{\mu}(t) = \nabla^* \pi_{\mathrm{im}\nabla}(w(t)) \in \mathrm{im}\,\Delta \tag{3.223}$$

for a.e $t \in [0,1]$. We moreover have $h(\mu(t)) = \pi^A_{\ker\Delta}(\mu(t)) \in \ker\Delta$ for all $t \in [0,1]$ by 2.1) in Proposition 3.2.30. Using the latter, Equation 3.223 implies 3) in the finite-dimensional setting. The general case follows as discussed above. □

Definition 3.2.35.

1) For all norm closed convex $K \subset \mathscr{S}(A)$, set $\mathrm{Fix}_A(K) := \{\mu \in \mathscr{S}(A) \mid h(\mu) \in K\}$.

2) For all fixed states $\xi \in \mathscr{S}(A)$, set

   2.1) $\mathrm{Fix}_A(\xi) := \mathrm{Fix}(\{\xi\}, A)$ and $\mathrm{Fix}_A^{\mathrm{N}}(\xi) := \mathrm{Fix}_A(\xi) \cap \mathscr{S}^{\mathrm{N}}(A)$,

   2.2) $\mathscr{C}_A(\xi) := \{\mu \in \mathscr{S}(A) \mid \mu \sim \xi\}$ and $\mathscr{C}_A^{\mathrm{N}}(\xi) := \mathscr{C}_A(\xi) \cap \mathscr{S}^{\mathrm{N}}(A)$.

Proposition 3.2.36. *Let $K \subset \mathscr{S}(A)$ be a norm closed convex subset. If $K \subset \mathscr{S}(A)$ is a face, then $\mathrm{Fix}_A(K)$ is a face.*

Proof. Let $\mu \in \mathrm{Fix}_A(K)$, $\eta_0, \eta_0 \in \mathscr{S}(A)$ and $t \in (0,1)$ s.t. $\mu = t\eta_0 + (1-t)\eta_1 \in \mathrm{Fix}(K, A)$. We have $h(\mathrm{Fix}_A(K)) \subset K$ and therefore $h(\mu) = th(\eta_0) + (1-t)h(\eta_1) \in K$. Assume K is a face. Thus $h(\eta_0), h(\eta_0) \in K$, hence $\eta_0, \eta_1 \in \mathrm{Fix}_A(K)$. Norm closedness of $\mathrm{Fix}_A(K)$ follows by 2.1) in Proposition 3.2.32. Altogether, our claim follows. □

Regularisation of normal states under heat flow. Assuming fixed parts with integrable support, Theorem 3.2.40 shows heat flow instantaneously regularises normal states to be, possibly unboundedly, invertible up to fixed part. The latter is equivalent to injectivity up to fixed part. Following Remark 2.2.60, we know Theorem 2.2.58 applies to noncommutative densities in form of Corollary 2.2.59 given injectivity up to fixed part. Note Remark 3.2.41. Theorem 3.2.40 uses Lemma 3.2.39. In the finite-dimensional setting, Lemma 3.2.38 shows Lemma 3.2.39, itself obtained from Lemma 3.2.37. We show the latter two lemmas by adapting [186] to the AF-C^*-setting.

Let $(\phi, \psi, \gamma, \nabla)$ be noncommutative differential structure for tracial AF-C^*-algebras (A, τ) and (B, ω) in (f, θ)-setting.

Lemma 3.2.37. *Let $T \in \mathscr{B}(L^2(A,\tau))_h$ be positivity-preserving. If $T(u) \neq 0$ for all non-zero $u \in L^2(A,\tau)_+$, then $\langle u, v\rangle_\tau > 0$ implies $\langle T(u), T(v)\rangle_\tau > 0$ for all $u, v \in L^2(A,\tau)_+$.*

Proof. We adapt Lemma 1 in [186]. For this, we require infima in $L^2(A,\tau)_+$ using partial order generated by positive elements. Definition 4.3 in [63] gives a wedge operation on $L^2(A,\tau)_h$ using projections onto closed convex sets of Hilbert spaces. These describe the infima we use as follows. For all $x \in L^2(A,\tau)_h$, Proposition B.1.47 yields $x_+, x_- \in L^2(A,\tau)_+$ s.t. $x = x_+ - x_-$, $-x = x_- - x_+$ and $x_+x_- = x_-x_+ = 0$. Lemma 4.4 in [63] states

$$\inf\{u, v\} = v - (u-v)_- = u \wedge v = v \wedge u = u - (v-u)_- = \inf\{v, u\} \tag{3.224}$$

for all $u, v \in L^2(A,\tau)_+$. If $u, v \in L^2(A,\tau)_+$ s.t. $u \wedge v = 0$, then Equation 3.224 shows we have $u = (v-u)_-$ and $v = (u-v)_- = (v-u)_+$. For all $u, v \in L^2(A,\tau)_+$, we use decomposition as per Proposition B.1.47 and thereby see $u \wedge v = 0$ implies $uv = vu = 0$.

We show our claim using the above. Assume $T(u) \neq 0$ for all non-zero $u \in L^2(A,\tau)_+$. Let $u,v \in L^2(A,\tau)_+$ s.t. $\langle u,v\rangle_\tau > 0$. Thus traciality and faithfulness imply $uv \neq 0$, hence Equation 3.224 shows $u \wedge v \neq 0$ as discussed above. Note $u,v \geq u \wedge v \geq 0$ by the infimum property. In particular, $u \wedge v \in L^2(A,\tau)_+$. Ergo $T(u \wedge v) \neq 0$ by hypothesis. We have

$$\left\langle T(u),T(v)\right\rangle_\tau = \left\|T(u \wedge v)\right\|_\tau^2 + \left\langle T(u-u\wedge v),T(u\wedge v)\right\rangle_\tau + \left\langle T(u),T(v-u\wedge v)\right\rangle_\tau. \qquad (3.225)$$

For all $x,y \in L^2(A,\tau)_+$, we know $\langle x,y\rangle_\tau \geq 0$ by traciality. Positivity-preservation implies the second and third summand in Equation 3.225 are non-negative. Since $T(u \wedge v) \neq 0$ implies $\|T(u\wedge v)\|_\tau^2 > 0$, Equation 3.225 shows our claim. □

Lemma 3.2.38. *For all $x \in L^{1,\infty}(A,\tau)_+$ and $u \in L^2(A,\tau)$, we have*

1) $\left\langle xu,u\right\rangle_\tau > 0$ implies $\left\langle h_t(x)u,u\right\rangle_\tau > 0$ for all $t \geq 0$,

2) the map $t \mapsto h_t(x,u) := \left\langle h_t(x)u,u\right\rangle_\tau$ defined on $(0,\infty)$ is either identically zero or has at most finitely many zeros in each open interval $I \subset (0,\infty)$.

Proof. For all $t \geq 0$, note 2.2) in Proposition 3.2.30 and 2.1) in Proposition 3.2.32 imply Lemma 3.2.37 applies to $T = h_t \in \mathscr{B}(L^2(A,\tau))_h$. We show 1). Let $x \in L^{1,\infty}(A,\tau)_+$ and $u \in L^2(A,\tau)$ s.t. $\langle xu,u\rangle_\tau > 0$. Corollary B.1.67 reduces the general case to $u \in L^{2,\infty}(A,\tau)$ and Lemma 3.2.37 shows our claim in this special case.

We reduce to $u \in L^{2,\infty}(A,\tau)$. For all $y \in L^\infty(A,\tau)_+$ and $w \in L^2(A,\tau)$, traciality yields

$$y^\flat\left(ww^*\right) = \tau\left(yww^*\right) = \left\langle yw,w\right\rangle_\tau = \left(w^*w\right)^\flat(y). \qquad (3.226)$$

Set $v := u^*u \in L^1(A,\tau)_+$. For all $n \in \mathbb{N}$, set $v_n := \min\{v,n\} \in L^{1,\infty}(A,\tau) \subset L^{2,\infty}(A,\tau)$. We have $0 \leq v_n \leq v$ in each case. Using Equation 3.226, we therefore estimate

$$\left\langle h_t(x)\sqrt{v_n},\sqrt{v_n}\right\rangle_\tau = v_n^\flat(h_t(x)) \leq v_{n+1}^\flat(h_t(x)) \leq v^\flat(h_t(x)) = \left\langle h_t(x)u,u\right\rangle_\tau \qquad (3.227)$$

for all $t \geq 0$ and $n \in \mathbb{N}$. We have $v = \|.\|_1\text{-}\lim_{n\in\mathbb{N}} v_n$ (cf. 2) in Corollary B.1.67). Using the latter, Equation 3.227 shows

$$\left\langle h_t(x)u,u\right\rangle_\tau = \sup_{n\in\mathbb{N}} \left\langle h_t(x)\sqrt{v_n},\sqrt{v_n}\right\rangle_\tau = \lim_{n\in\mathbb{N}} \left\langle h_t(x)\sqrt{v_n},\sqrt{v_n}\right\rangle_\tau \qquad (3.228)$$

for all $t \geq 0$. Equation 3.228 shows it suffices to consider $u \in L^{2,\infty}(A,\tau)$.

We know $x \in L^{2,\infty}(A,\tau)$. Let $u \in L^{2,\infty}(A,\tau)$. We obtain $uu^* \in L^2(A,\tau)$. Thus 2.1) in Proposition 3.2.32 implies there exists maximal $\varepsilon \in (0,\infty]$ s.t.

$$\left\langle h_{\frac{t}{2}}(x),h_{\frac{t}{2}}\left(uu^*\right)\right\rangle_\tau = \tau\left(h_t(x)uu^*\right) = \left\langle h_t(x)u,u\right\rangle_\tau > 0 \qquad (3.229)$$

for all $t \in [0,\varepsilon)$. If $\varepsilon = \infty$, then our claim follows. If $\varepsilon < \infty$, then Lemma 3.2.37 shows $\langle h_{\frac{\varepsilon}{2}}(x),h_{\frac{\varepsilon}{2}}(uu^*)\rangle_\tau > 0$ contradicting maximality. Hence 1) holds. The general case follows as discussed above.

We show 2). We adapt Lemma 2 in [186]. Let $x \in L^{1,\infty}(A,\tau)_+$ and $u \in L^2(A,\tau)$. Note $u \in L^{2,\infty}(A,\tau)$ is not required. Following Remark 3.2.29, we have orthonormal eigenbasis $\{e_n\}_{n\in\mathbb{N}} \subset A_0$ of Δ. For all $n \in \mathbb{N}$, let λ_n be the eigenvalue of e_n. Expressing $x = \sum_{n\in\mathbb{N}} \alpha_n e_n$ and using uniform convergence shows the non-negative map

$$t \mapsto h_t(x,u) = \sum_{m\in\mathbb{N}} \left(\sum_{n\in\mathbb{N}} \frac{(-1)^m \cdot \alpha_n \lambda_n^m}{m!} \langle e_n u, u \rangle_\tau \right) \cdot (t-0)^m \tag{3.230}$$

is analytic in the right half plane. Using standard arguments for analytic maps [145], we see 1) implies we either have $h_t(x,u) = 0$ for all $t \geq 0$ or for at most finitely many $t \in I$ in each open interval $I \subset (0,\infty)$. Get 2). □

Lemma 3.2.39. *Let $\xi \in \mathscr{S}(A)$ be a fixed state and $j \in \mathbb{N}$ s.t. $\xi_j \neq 0$.*

1) We have

1.1) $\mathrm{Fix}^{\mathrm{N}}_{A_j}(\mathscr{F}_{A_j}(\bar{\xi}_j)) = \mathscr{F}_{A_j}(\bar{\xi}_j)$,

1.2) $\bar{\xi}_j \in \mathscr{S}^{\mathrm{N},\infty}_{-1}(A_j[\operatorname{supp}\bar{\xi}_j])$.

2) For all $\mu \in \mathrm{Fix}^{\mathrm{N}}_{A_j}(\bar{\xi}_j)$, we have

$$h_t(\mu) \in \mathscr{S}^{\mathrm{N},\infty}_{-1}(A_j[\operatorname{supp}\bar{\xi}_j]) \tag{3.231}$$

for all $t \in (0,\infty]$.

Proof. Note 1.3) in Proposition 3.2.34 shows $\bar{\xi}_j \in \mathscr{S}(A_j)$ is a fixed state. Lemma 3.2.38 and Proposition 3.2.36 in particular apply to the induced noncommutative differential structure $(\phi_j, \psi_j, \gamma_j, \nabla_j)$ using fixed state $\bar{\xi}_j \in \mathscr{S}(A_j)$. We reduce to the finite-dimensional setting by 1.3) in Proposition 3.2.34.

Assume A and B are finite-dimensional. All states are normal. Lemma 3.2.5 shows $\mathscr{F}_A(\xi) \subset \mathscr{S}(A)$ is a face. Thus Proposition 3.2.36 shows $\mathrm{Fix}_A(\mathscr{F}_A(\xi)) \subset \mathscr{S}(A)$ is one, hence Lemma 3.2.5 yields projection $p \in A$ s.t.

$$\mathrm{Fix}_A(\mathscr{F}_A(\xi)) = \mathscr{S}(A[p]). \tag{3.232}$$

We have $\tau(p) < \infty$ as $A_0 = A \subset \mathfrak{m}_\tau$. The semigroup property and Equation 3.232 imply

$$h_t(\mathscr{S}(A[p])) \subset \mathscr{S}(A[p]) \tag{3.233}$$

for all $t \in [0,\infty]$. Finite-dimensionality ensures injectivity and invertibility coincide. In particular, get $\mathscr{S}^{\mathrm{N},\infty}_{>0}(A[p]) = \mathscr{S}^{\mathrm{N},\infty}_{-1}(A[p])$. We apply Corollary 3.2.11 accordingly.

Note 1) in Corollary 3.2.11 states

$$\mathscr{S}_{-1}^{\mathrm{N},\infty}(A[p]) = \operatorname{relint}\mathscr{S}(A[p]) \subset A[p]_+^* \cap \mathrm{GL}(A)^\flat \tag{3.234}$$

open in norm topology. Equation 3.234 ensures the following equivalence holds. For all $\eta \in \mathscr{S}(A[p])$, we have $\eta \in \mathscr{S}_{-1}^{\mathrm{N},\infty}(A[p])$ if and only if

$$\langle \eta u, u \rangle_\tau \geq \sigma(\eta) \cdot \|u\|_\tau^2 \tag{3.235}$$

for all $u \in A[p]$. Note the below estimate uses strong continuity and trace-preservation as per 1) in, as well as positivity-preservation as per 2.1) in Proposition 3.2.32. For all $\eta \in \mathscr{S}_{-1}^{\mathrm{N},\infty}(A[p])$, Proposition 3.2.32, Equation 3.235 and traciality let us estimate

$$\begin{aligned}\langle h(\eta)u, u \rangle_\tau = \lim_{t\to\infty} \tau\big(\eta h_t(uu^*)\big) &\geq \sigma(\eta) \cdot \lim_{t\to\infty} \tau\big(h_t(uu^*)\big) \\ &= \sigma(\eta) \cdot \lim_{t\to\infty} \tau\big(uu^*\big) \\ &= \sigma(\eta) \cdot \|u\|_\tau^2\end{aligned}$$

for all $u \in A[p]$. Equation 3.233 and the above estimate, either as stated for $t = \infty$ or without taking limits for $t < \infty$, show

$$h_t\big(\mathscr{S}_{-1}^{\mathrm{N},\infty}(A[p])\big) \subset \mathscr{S}_{-1}^{\mathrm{N},\infty}(A[p]) \tag{3.236}$$

for all $t \in [0, \infty]$. Note 2) in Corollary 3.2.11 states we have $\mathscr{F}_A(\xi) = \mathscr{S}(A[p])$ if and only if $\xi \in \mathscr{S}_{-1}^{\mathrm{N},\infty}(A[p])$, resp. $\mathscr{F}_A(\xi) \subset \partial\mathscr{S}(A[p])$ if and only if $\xi \notin \mathscr{S}_{-1}^{\mathrm{N},\infty}(A[p])$. If $\xi \in \mathscr{S}_{-1}^{\mathrm{N},\infty}(A[p])$ holds, then $\mathscr{F}_A(\xi) = \mathscr{S}(A[p])$ shows $\operatorname{supp}\xi = p$ by 1) in Corollary 3.2.8. Equation 3.232 and Equation 3.234 therefore imply 1) in this case.

We show 1). Assume $\xi \notin \mathscr{S}_{-1}^{\mathrm{N},\infty}(A[p])$. Since $\tau(p) < \infty$, $\tau(p)^{-1}p^\flat \in \mathscr{S}_{-1}^{\mathrm{N},\infty}(A[p])$. Note Equation 3.234. Thus $\partial\mathscr{S}(A[p]) \subset \mathscr{S}(A[p])$ proper, hence

$$\mathscr{F}_A(\xi) \subset \partial\mathscr{S}(A[p]) \subset \mathscr{S}(A[p]) \tag{3.237}$$

proper as well. For all $\eta \in \mathscr{S}_{-1}^{\mathrm{N},\infty}(A[p]) \neq \emptyset$, Equation 3.232 and Equation 3.237 imply $h(\eta) \in \partial\mathscr{S}(A[p])$. This contradicts Equation 3.236 for $t = \infty$. Ergo $\xi \in \mathscr{S}_{-1}^{\mathrm{N},\infty}(A[p])$. Get 1) as discussed above. We show 2). Let $\mu \in \mathrm{Fix}_A(\xi)$. Using 1.2), openness in norm topology as per Equation 3.234 shows there exists $t_0 \geq 0$ s.t.

$$h_t(\mu) \in \mathscr{S}_{-1}^{\mathrm{N},\infty}\big(A[\operatorname{supp}\xi]\big) \tag{3.238}$$

for all $t \in (t_0, \infty]$.

Equation 3.239 shows there exists minimal $t_\mu \geq 0$ s.t. Equation 3.238 is satisfied for all $t \in (t_\mu, \infty]$. Minimality and Equation 3.236 moreover imply

$$h_t(\mu) \notin \mathscr{S}^{\mathrm{N},\infty}_{-1}\big(A[\operatorname{supp}\xi]\big) \tag{3.239}$$

for all $t \in [0, t_\mu]$. If $t_\mu > 0$, then finite-dimensionality ensures Equation 3.239 derives a contradiction to 2) in Lemma 3.2.38. Thus $t_\mu = 0$ in each case. Get 2). The general case follows as discussed above. □

Theorem 3.2.40. *Let $(\phi, \psi, \gamma, \nabla)$ be noncommutative differential structure for tracial AF-C^*-algebras (A, τ) and (B, ω) in (f, θ)-setting. Let $\xi \in \mathscr{S}(A)$ be a fixed state.*

1) Assume $\xi \in \mathscr{S}^{\mathrm{N}}(A)$ has reducible support.

1.1) We have

1.1.1) $\operatorname{Fix}^{\mathrm{N}}_A(\mathscr{F}_A(\xi)) = \mathscr{F}_A(\xi)$,

1.1.2) $\operatorname{supp}\xi \in L^\infty(A, \tau)_\nabla$, $\operatorname{supp}\xi \in \ker\nabla$, *and* ∇ *is* $\operatorname{supp}\xi$*-compressible.*

1.2) For all $\mu \in \mathscr{F}_A(\xi)$, we have

$$h_t(\mu) \in \mathscr{F}_A(\xi) \tag{3.240}$$

for all $t \in [0, \infty]$.

1.3) For all $\mu \in \operatorname{Fix}^{\mathrm{N}}_A(\xi)$ and $j \in \mathbb{N}$ s.t. $\xi_j \neq 0$, we have

$$h_t(\bar{\mu}_j) \in \mathscr{S}^{\mathrm{N},\infty}_{-1}\big(A_j[\operatorname{supp}\bar{\xi}_j]\big) \tag{3.241}$$

for all $t \in (0, \infty]$.

2) Assume $\xi \in \mathscr{S}^{\mathrm{N}}(A)$ has integrable support.

2.1) We have

2.1.1) $\operatorname{Fix}^{\mathrm{N}}_A(\mathscr{F}_A(\xi)) = \mathscr{F}_A(\xi)$,

2.1.2) $\xi \in \mathscr{S}^{\mathrm{N}}_{>0}\big(A[\operatorname{supp}\xi]\big)$.

2.2) For all $\mu \in \operatorname{Fix}^{\mathrm{N}}_A(\xi)$, we have

$$h_t(\mu) \in \mathscr{S}^{\mathrm{N}}_{>0}\big(A[\operatorname{supp}\xi]\big) \tag{3.242}$$

for all $t \in (0, \infty]$.

Proof. We use 1.2) in Proposition 2.1.31 for weak continuity. Furthermore, we use 1.3) in Proposition 3.2.34 to commute restriction and rescaling with application of heat flow. Assume $\xi \in \mathscr{S}^{\mathrm{N}}(A)$ has reducible support, i.e. $\operatorname{supp}\xi = \text{s-lim}_{j\in\mathbb{N}} \operatorname{supp}\xi_j$.

For all $j \in \mathbb{N}$, we see $\bar{\xi}_j \in \mathscr{S}(A_j)$ is a fixed state if and only if $\xi_j \neq 0$. Thus 1.1) in Lemma 3.2.39 implies

$$\mathrm{Fix}_A^{\mathrm{N}}(\mathscr{F}_A(\xi)) = \left\{\mu \in \mathscr{S}^{\mathrm{N}}(A) \;\middle|\; \bar{\mu}_j \in \mathscr{F}_{A_j}\big(\bar{\xi}_j\big) \text{ for a.e. } j \in \mathbb{N}\right\} \tag{3.243}$$

by restricting elements on the left-hand side for all $j \in \mathbb{N}$ s.t. $\xi_j \neq 0$, resp. taking limits of elements on the right-hand side in w^*-topology. For all $j \in \mathbb{N}$ s.t. $\xi_j \neq 0$, Lemma 3.2.5 and 2) in Lemma 3.2.39 show $h_t(\bar{\mu}_j) \in \mathscr{F}_{A_j}(\bar{\xi}_j)$ and therefore

$$\sharp h_t\big(\bar{\mu}_j\big) = \operatorname{supp}\bar{\xi}_j \cdot \sharp h_t\big(\bar{\mu}_j\big) \cdot \operatorname{supp}\bar{\xi}_j. \tag{3.244}$$

Equation 3.243 and Equation 3.244 let us calculate

$$\sharp h_t(\mu) = w^*\text{-}\lim_{j\in\mathbb{N}} \sharp h_t\big(\bar{\mu}_j\big) = w^*\text{-}\lim_{j\in\mathbb{N}} \operatorname{supp}\bar{\xi}_j \cdot \sharp h_t\big(\bar{\mu}_j\big) \cdot \operatorname{supp}\bar{\xi}_j \tag{3.245}$$

for all $\mu \in \mathrm{Fix}_A^{\mathrm{N}}(\mathscr{F}_A(\xi))$ and $t \in [0,\infty]$. We show the right-hand side of Equation 3.245 is $\operatorname{supp}\xi \cdot \sharp h_t(\mu) \cdot \operatorname{supp}\xi$ in each case. For all $x \in L^\infty(A,\tau)$, we know $x = \text{bds-lim}_{j\in\mathbb{N}} x_j$ by 3) in Proposition 2.1.31. Using weak continuity as for Equation 3.245 and sequential strong continuity of multiplication, Equation 3.244 together with traciality and normality lets us calculate

$$\begin{aligned}
\tau\big(\sharp h_t(\mu)x\big) &= \lim_{j\in\mathbb{N}} \tau\big(\sharp h_t\big(\bar{\mu}_j\big)x_j\big) = \lim_{j\in\mathbb{N}} \tau\big(\sharp h_t\big(\bar{\mu}_j\big) \cdot \big(\operatorname{supp}\bar{\xi}_j \cdot x_j \cdot \operatorname{supp}\bar{\xi}_j\big)\big) \\
&= \lim_{j\in\mathbb{N}} \tau\big(\sharp h_t(\mu) \cdot \big(\operatorname{supp}\bar{\xi}_j \cdot x_j \cdot \operatorname{supp}\bar{\xi}_j\big)\big) \\
&= \tau\big(\big(\operatorname{supp}\xi \cdot \sharp h_t(\mu) \cdot \operatorname{supp}\xi\big) \cdot x\big)
\end{aligned}$$

for all $\mu \in \mathrm{Fix}_A^{\mathrm{N}}(\mathscr{F}_A(\xi))$, $t \in [0,\infty]$ and $x \in L^\infty(A,\tau)$. The above calculation at once shows the right-hand side of Equation 3.245 is of claimed form. We therefore have

$$\sharp h_t(\mu) = w^*\text{-}\lim_{j\in\mathbb{N}} \operatorname{supp}\bar{\xi}_j \cdot \sharp h_t\big(\bar{\mu}_j\big) \cdot \operatorname{supp}\bar{\xi}_j = \operatorname{supp}\xi \cdot \sharp h_t(\mu) \cdot \operatorname{supp}\xi \tag{3.246}$$

for all $\mu \in \mathrm{Fix}_A^{\mathrm{N}}(\mathscr{F}_A(\xi))$ and $t \in [0,\infty]$.

We show 1). Equation 3.246 shows $\mathrm{Fix}_A^{\mathrm{N}}(\mathscr{F}_A(\xi)) \subset \mathscr{F}_A(\xi)$ by Lemma 3.2.5. We obtain the converse as follows. Using strong continuity as per 1) in Proposition 3.2.32 to have norm closure, Lemma 3.2.5 and Proposition 3.2.36 yield inclusion of faces and therefore projection $p \in L^\infty(A,\tau)$ s.t.

$$\mathrm{Fix}_A^{\mathrm{N}}(\mathscr{F}_A(\xi)) = \mathscr{S}^{\mathrm{N}}(A[p]) \subset \mathscr{F}_A(\xi) = \mathscr{S}^{\mathrm{N}}\big(A[\operatorname{supp}\xi]\big) \subset \mathscr{S}^{\mathrm{N}}(A). \tag{3.247}$$

We have $\xi \in \mathrm{Fix}_A^{\mathrm{N}}(\mathscr{F}_A(\xi))$. Thus $\operatorname{supp}\xi \leq p$ by Lemma 3.2.5, hence Equation 3.247 shows $\mathscr{F}_A(\xi) \subset \mathrm{Fix}_A^{\mathrm{N}}(\mathscr{F}_A(\xi))$ by 1) in Corollary 3.2.8. Get 1.1.1). For all $j \in \mathbb{N}$, note $\Delta 1_{A_j} = 0$ by the Leibniz rule and $\xi_j \in \ker\Delta$ by 2.1) in Proposition 3.2.30. Thus 1) in Proposition 3.2.4 implies

$$\operatorname{supp}\xi_j \in C^*\big(\xi_j, 1_{A_j}\big) \subset A_j \cap \ker\Delta \tag{3.248}$$

in each case. Using Corollary 2.3.15, Equation 3.248 and reducible support of ξ shows 1.1.2) since $\ker\nabla = \ker\Delta \subset L^2(A,\tau)$. Get 1.1). Note 1.1.1) shows 1.2) and 1.3) are claims concerning states on A with fixed part ξ. Equation 3.243 and Equation 3.246 further reduce to the finite-dimensional setting as per Lemma 3.2.39. The latter lemma shows 1.2) and 1.3) at once. Altogether, get 1).

We show 2). Assume $\xi \in \mathscr{S}^{\mathrm{N},\infty}(A)$ has integrable support, i.e. $\tau(\operatorname{supp}\xi) < \infty$. Ergo Theorem 3.2.18 shows ξ has reducible support. Thus 1) holds, hence 1.1.1) implies 2.1.1) at once. We further have 2.1.2) by 2.1) in Corollary 3.2.11 since $\mathscr{F}_A(\xi) = \mathscr{S}(A[\operatorname{supp}\xi])$ by definition. Get 2.1). We reformulate 1.2) to

$$h_t\Big(\mathscr{S}^{\mathrm{N}}\big(A[\operatorname{supp}\xi]\big)\Big) \subset \mathscr{S}^{\mathrm{N}}\big(A[\operatorname{supp}\xi]\big) \tag{3.249}$$

for all $t \in [0,\infty]$. Let $\mu \in \mathscr{F}_A(\xi)$. For all $t \in [0,\infty]$, Equation 3.249 and Lemma 3.2.5 imply $\operatorname{supp} h_t(\mu) \leq \operatorname{supp}\xi$. Ergo Theorem 3.2.18 shows each $h_t(\mu)$ has reducible support. Using the latter, 2) in Lemma 3.2.39 shows

$$\operatorname{supp} h_t(\mu) = \operatorname*{s\text{-}lim}_{j\in\mathbb{N}} \operatorname{supp} h_t\big(\bar{\mu}_j\big) = \operatorname*{s\text{-}lim}_{j\in\mathbb{N}} \operatorname{supp}\bar{\xi}_j = \operatorname{supp}\xi \tag{3.250}$$

for all $t \in (0,\infty]$. Finally, Equation 3.250 shows 2.2) by 1) in Corollary 3.2.8 and 2.1) in Corollary 3.2.11. Altogether, get 2). □

Remark 3.2.41. We have injectivity of noncommutative densities in general, but do not get smoothing under heat flow as per Equation 3.231. Injectivity suffices to apply Theorem 2.2.58 as per Corollary 2.2.59. Coarse graining recovers smoothing under heat flow as per Equation 3.241. This depends on fixed parts. Such dependence is a uniform condition on accessibility components by 3) in Proposition 3.2.34.

We assume integrable support. Theorem 3.2.18 ensures reducible support. As per Corollary 2.3.15 and following Definition 2.3.26, note it is 1.1.2) in Theorem 3.2.40 which lets us compress quantum gradients with support projections of normal fixed states. We use this throughout our discussion. As per 3) in Corollary 3.2.43, we moreover combine compressing with such support projections and finite-dimensional approximation. This gives rise to our coarse graining process. Notation 3.2.42 fixes conventions. For details on compressing quantum gradients, we refer to Subsection 2.3.1.

Notation 3.2.42. Let $\xi \in \mathscr{S}^{\mathrm{N}}(A)$ be a fixed state with integrable support.

1) We write $A_\xi := A[\operatorname{supp}\xi]$, $\mathscr{A}_\xi := \mathscr{A}_{L^\infty(A_\xi,\tau)}$ and $L^\infty(A_\xi,\tau)_\nabla := L^\infty(A_\xi,\tau)_{\nabla_{\operatorname{supp}\xi}}$, as well as $L^2(B_\xi,\omega) := \pi_{\operatorname{supp}\xi}\big(L^2(B,\omega)\big)$. If A and B are finite-dimensional, then we have $A_\xi = L^2(A_\xi,\tau)$ and write $B_\xi := L^2(B_\xi,\omega)$.
2) For all $x \in L^0(A_\xi,\tau)_+$, we write $\mathscr{M}_{x,\xi} := \mathscr{M}_{x,\operatorname{supp}\xi}$ and further $\mathscr{D}_{x,\xi} := \mathscr{D}_{x^\flat,x^\flat} = \mathscr{D}_{x,\operatorname{supp}\xi}$ if $m_f^{-1} \in \mathscr{S}_{\operatorname{supp}\xi}\big(E_{x,x}\big)$.
3) We write $\nabla_\xi := \nabla_{\operatorname{supp}\xi} = \nabla_{L^\infty(A_\xi,\tau)}$ and $\Delta_\xi := \Delta_{\operatorname{supp}\xi} = \Delta_{L^\infty(A_\xi,\tau)}$.

Corollary 3.2.43. *Let $\xi \in \mathscr{S}^{\mathrm{N}}(A)$ be a fixed state with integrable support.*

1) We have

1.1) $\operatorname{supp}\xi$*-compressed symmetric W^*-derivation* $\nabla_\xi : \mathscr{A}_\xi \longrightarrow L^2(B_\xi,\omega)$*,*

1.2) $\operatorname{supp}\xi$*-compressed Laplacian* $\Delta_\xi = \nabla_\xi^*\nabla_\xi$*.*

2) For all $t \geq 0$ and $h_t \in \mathscr{B}\big(L^2(A,\tau)\big)$, we have

2.1) $\big[h_t, \pi_{\operatorname{supp}\xi}\big] = 0$*,*

2.2) $\operatorname{com}_{L^2(A_\xi,\tau)} h_t = e^{-t\Delta_\xi}$*.*

3) We have $L^\infty(A_\xi,\tau)_\nabla \subset \operatorname{dom}\nabla$ and

3.1) $\pi_{\operatorname{supp}\xi}(u) = \|.\|_\nabla\text{-}\lim_{j\in\mathbb{N}} \pi_{\operatorname{supp}\xi_j}(u_j)$ *for all* $u \in \operatorname{dom}\nabla$*,*

3.2) $x = \operatorname{bds}^\nabla\text{-}\lim_{j\in\mathbb{N}} \pi_{\operatorname{supp}\xi_j}(x_j)$ *for all* $x \in L^\infty(A_\xi,\tau)_\nabla$*.*

4) We have $\operatorname{dom}\nabla \cap L^2(A_\xi,\tau) \subset \operatorname{dom}\nabla_\xi$ *and*

4.1) $\operatorname{dom}\nabla_\xi = \big\{u \in L^2(A_\xi,\tau) \mid u = \|.\|_\nabla\text{-}\lim_{j\in\mathbb{N}} \pi_{\operatorname{supp}\xi_j}(u_j)\big\}$*,*

4.2) $L^\infty(A_\xi,\tau)_\nabla = \big\{x \in L^\infty(A_\xi,\tau) \mid x = \operatorname{bds}^\nabla\text{-}\lim_{j\in\mathbb{N}} \pi_{\operatorname{supp}\xi_j}(x_j)\big\}$*.*

Proof. We see 1) in Corollary 2.1.65 implies $\mathscr{A}_\xi = \operatorname{supp}\xi \cdot A_0 \cdot \operatorname{supp}\xi \subset A$ using algebra multiplication as per Definition 2.3.16, resp. $L^2(B_\xi,\omega) = \operatorname{supp}\xi \cdot L^2(B,\omega) \cdot \operatorname{supp}\xi$ using AF-C^*-bimodule action. We know ∇ is $\operatorname{supp}\xi$-compressible by 1.1.2) in Theorem 3.2.40. We have 1) by Corollary 2.3.15 and 2) in Proposition 2.3.27. By testing on A_0 using 4) in Proposition 2.3.19, 1) implies 2.1) since $A_0 \subset L^2(A,\tau)$ is $\|.\|_\tau$-dense. Moreover, 1) implies 2.2) by Corollary 2.3.22. Get 2).

We show 3). The latter implies 4) immediately. Using sequential strong continuity of multiplication, we readily see reducible support implies 3.1) since $u = \|.\|_{\nabla}\text{-}\lim_{j\in\mathbb{N}} u_j$ for all $u \in \operatorname{dom}\nabla$ by 4.1) in Proposition 2.3.25. We likewise obtain 3.2) if $x = \mathrm{bds}^{\nabla}\text{-}\lim_{j\in\mathbb{N}} x_j$ for all $x \in L^{\infty}(A_{\xi},\tau)_{\nabla}$. Arguing as for Equation 3.246, we use 3) in Proposition 2.1.31 and reducible support to calculate

$$x = \mathrm{s}\text{-}\lim_{j\in\mathbb{N}} \operatorname{supp}\xi_j \cdot x_j \cdot \operatorname{supp}\xi_j = \operatorname{supp}\xi \cdot x \cdot \operatorname{supp}\xi \tag{3.251}$$

for all $x \in L^{\infty}(A_{\xi},\tau)_{\nabla}$. We further have $\operatorname{supp}\xi \in L^2(A,\tau)$ and therefore

$$L^{\infty}(A_{\xi},\tau) = \pi_{\operatorname{supp}\xi}(L^{\infty}(A,\tau)) \subset L^2(A_{\xi},\tau) \tag{3.252}$$

by integrable support (cf. Proposition B.2.30). Equation 3.251 and Equation 3.252 show $L^{\infty}(A_{\xi},\tau)_{\nabla} \subset \operatorname{dom}\nabla$. Using sequential strong continuity of multiplication and 3.1), note reducible support implies 3.2) by 3) in Proposition 2.1.31. Thus 3), hence 4) holds. □

Remark 3.2.44. Following 2) in Corollary 3.2.43, the noncommutative heat semigroup of ∇_{ξ} considered as symmetric C^*-derivation is given by

$$t \longmapsto \mathrm{com}_{L^2(A_{\xi},\tau)} h_t = e^{-t\Delta_{\xi}} \in \mathscr{B}\big(L^2(A_{\xi},\tau)\big). \tag{3.253}$$

Since we only consider semigroups as above if we compress with support projections of normal fixed states, we do not distinguish any from $h : [0,\infty) \longrightarrow \mathscr{B}(L^2(A,\tau))$.

3.2.3 Classifying normal accessibility components

Assuming spectral gaps of quantum Laplacians and fixed parts, Theorem 3.2.65 classifies accessibility components of square integrable normal states using fixed parts by showing each one is a norm closed convex subsets of all such states with identical fixed part. Spectral gaps ensure such fixed parts themselves are square integrable normal states with integrable support. In the finite-dimensional setting, assumptions as above are satisfied and we classify all accessibility components using fixed parts.

In the finite-dimensional setting, relative interiors are embedded submanifolds, as well as connected Riemannian manifolds with Riemannian metric induced by the given quasi-entropy. Theorem 3.2.62 shows each in turn induces the given quantum optimal transport distance, and Theorem 3.2.65 ensures their norm closures are accessibility components. Theorem 3.2.40 therefore links the finite-dimensional Riemannian case to the general one by compression, finite-dimensional approximation and heat flow. In Chapter 4, we commonly reduce to the finite-dimensional Riemannian setting. This is a fundamental reason to require, from a purely technical point of view, compatibility with compression and finite-dimensional approximation. Standard reference for differential and Riemannian geometry is [144].

Embedded submanifolds of states in the finite-dimensional setting. We prepare our discussion further below. Let $(\phi,\psi,\gamma,\nabla)$ be noncommutative differential structure for tracial AF-C^*-algebras (A,τ) and (B,ω) in (f,θ)-setting. Assume A and B are finite-dimensional.

Proposition 3.2.45. *Let $p\in A$ be a projection. For all $x,y>0$ in $A[p]$ and $u\in B[p]$, we have*

1) $\mathscr{I}^{f,\theta}\big(x^\flat,y^\flat,u^\flat\big)=\big\langle \mathscr{D}^\theta_{x,y,p}(u),u\big\rangle_\omega$,

2) $0<\sigma(x)^{\frac{\theta}{2}}\sigma(y)^{\frac{\theta}{2}}\cdot\|u\|^2_\omega\leq\big\langle \mathscr{M}^\theta_{x,y,p}(u),u\big\rangle_\omega$.

Proof. Following Remark 2.2.38, we have 1) by 3) in Corollary 2.2.59. We show 2). The geometric operator mean is the minimal symmetric one (cf. Theorem 4.5 in [13]). Since $x,y>0$ in $A[p]$, evaluating the geometric operator mean in $L_{x,p}$ and $R_{y,p}$ yields

$$0<\sigma\big(L_{x,p}\big)^{\frac{\theta}{2}}\sigma\big(R_{y,p}\big)^{\frac{\theta}{2}}\cdot\|u\|^2_\omega\leq\big\langle \mathscr{M}^\theta_{x,y,p}(u),u\big\rangle_\omega \tag{3.254}$$

for all $u\in B[p]$. Equation 3.254 shows 2) by Proposition 3.2.14. □

Let $\xi\in\mathscr{S}(A_\xi)$ be a fixed state. We use Notation A.1.2. Restricting the GNS-inner product of τ yields real Hilbert space inner product of $A_{\xi,h}=A_\xi\cap A_h$.

Proposition 3.2.46. *Let $\xi\in\mathscr{S}(A)$ be a fixed state.*

1) *We have $\Delta_\xi\in\mathscr{B}(A_\xi)_h$, $\operatorname{supp}\xi\in\ker\Delta_\xi$ and $\operatorname{im}\Delta_\xi=\operatorname{im}\Delta\cap A_\xi$.*

2) *Setting $I(\Delta_\xi):=\operatorname{im}\Delta_\xi\cap A_{\xi,h}$ and $K(\Delta_\xi):=\langle\operatorname{supp}\xi\rangle^\perp_{\mathbb{R}}\subset\ker\Delta_\xi\cap A_{\xi,h}$ yields orthogonal decomposition*

$$A_{\xi,h}=I(\Delta_\xi)\oplus\langle\operatorname{supp}\xi\rangle_{\mathbb{R}}\oplus K(\Delta_\xi). \tag{3.255}$$

Proof. We known 1) by 1.1.2) in Theorem 3.2.40 and 1.2) in Corollary 3.2.43. We have $\Delta(A_h)\subset A_h$ by symmetry of ∇. Thus 1) implies 2) at once. □

We have real Hilbert space projections

$$\pi^A_{I(\Delta_\xi)}:A_{\xi,h}\longrightarrow I(\Delta_\xi),\ \pi^A_{K(\Delta_\xi)}:A_{\xi,h}\longrightarrow K(\Delta_\xi). \tag{3.256}$$

We know $I(\Delta_\xi),K(\Delta_\xi)\subset\ker\tau$. Furthermore, we know $\tau(\operatorname{supp}\xi)>0$ by faithfulness and have $\dim_{\mathbb{R}}\operatorname{im}_{\mathbb{R}}\tau|_{A^*_{\xi,h}}=1$. For all $\mu\in A^*_{\xi,h}$, Equation 3.255 yields decomposition

$$\mu=\pi^A_{I(\Delta_\xi)}\big(\sharp\mu\big)^\flat+\|\mu\|_{A^*}\cdot\tau(\operatorname{supp}\xi)^{-1}\operatorname{supp}\xi^\flat+\pi^A_{K(\Delta_\xi)}\big(\sharp\mu\big)^\flat. \tag{3.257}$$

Definition 3.2.47. Let $\xi \in \mathscr{S}(A)$ be a fixed state.

1) We define $\mathfrak{P}_\xi : A_\xi^* \longrightarrow K(\Delta_\xi)^\flat$ by setting

$$\mathfrak{P}_\xi(\mu) := \pi^A_{K(\Delta_\xi)}\big(\sharp\mu\big)^\flat \tag{3.258}$$

for all $\mu \in A_\xi^*$.

2) Set $\vartheta(\xi) := \mathfrak{P}^{-1}_{\xi|\mathscr{S}^{\mathrm{N},\infty}_{-1}(A_\xi)}\Big(\pi^A_{K(\Delta_\xi)}\big(\sharp\xi\big)^\flat\Big)$.

Notation 3.2.48. Let X be a smooth manifold. We write TX for its tangent bundle. We further write $T_\mu X$ for the tangent space upon evaluation at $\mu \in X$.

Proposition 3.2.49. *Let $\xi \in \mathscr{S}(A)$ be a fixed state. We have*

1) embedded submanifold

$$\vartheta(\xi) = \operatorname{relint}\operatorname{Fix}^{\mathrm{N}}_A(\xi) \subset \mathscr{S}^{\mathrm{N},\infty}_{-1}(A_\xi), \tag{3.259}$$

2) trivial tangent bundle $T\vartheta(\xi) = \vartheta(\xi) \times I(\Delta_\xi)^\flat$.

Proof. Using 2.1) in Proposition 3.2.30, Equation 3.257 shows

$$h(\mu)^\perp = \pi^A_{I(\Delta_\xi)}\big(\sharp h^\perp(\mu)\big)^\flat,\ h(\mu) = \tau(\operatorname{supp}\xi)^{-1}\operatorname{supp}\xi^\flat + \pi^A_{K(\Delta_\xi)}\big(\sharp h(\mu)\big)^\flat \tag{3.260}$$

for all $\mu \in \mathscr{S}(A_\xi)$. Equation 3.260 implies

$$\operatorname{Fix}_A(\xi) = \mathfrak{P}^{-1}_{\xi|\mathscr{S}(A_\xi)}\Big(\pi^A_{K(\Delta_\xi)}\big(\sharp\xi\big)^\flat\Big). \tag{3.261}$$

Arguing as for 1) in Corollary 3.2.11 but using $\xi \in \mathscr{S}^{\mathrm{N},\infty}_{-1}(A_\xi)$ in Equation 3.169 rather than rescaled $\operatorname{supp}\xi$ under the flat operator, we directly verify

$$\operatorname{relint}\operatorname{Fix}_A(\xi) = \operatorname{Fix}_A(\xi) \cap \mathscr{S}^{\mathrm{N},\infty}_{-1}(A_\xi). \tag{3.262}$$

Equation 3.261 and Equation 3.262 show Equation 3.259. Thus Equation 3.260 shows smooth paths in $\mathscr{S}^{\mathrm{N},\infty}_{-1}(A_\xi)$ with image in $\vartheta(\xi)$ vary in $I(\Delta_\xi)^\flat$ only, hence Equation 3.259 implies 1) and therefore 2) by the submersion theorem [144]. $\square$

Riemannian metrics induced by quasi-entropies. Using bounded operators in Definition 3.2.50 determined by quasi-entropies, Definition 3.2.52 gives Riemannian metrics on embedded submanifolds as per Proposition 3.2.49. Restricted to each such embedded submanifold, Theorem 3.2.62 shows the Riemannian distance is the quantum optimal transport distance given by the quasi-entropy inducing Riemannian metric.

Let $(\phi,\psi,\gamma,\nabla)$ be noncommutative differential structure for tracial AF-C^*-algebras (A,τ) and (B,ω) in (f,θ)-setting. Assume A and B are finite-dimensional. Let $\xi \in \mathscr{S}(A)$ be a fixed state. This is the finite-dimensional Riemannian setting. We use the following throughout our discussion. Get $\Delta|_{\operatorname{im}\Delta} > 0$ in $\mathscr{B}(\operatorname{im}\Delta)$ by finite-dimensionality. Note 1) in Corollary 3.2.43 shows $\nabla(A_\xi) \subset B_\xi$ and $\nabla^*(B_\xi) \subset A_\xi$ by $\operatorname{supp}\xi$-compressibility. For all $x \in A_{\xi,+}$, we see 1) in Lemma 2.2.13 implies

$$\mathscr{M}^\theta_{x,\xi} = \mathscr{M}^\theta_x\big|_{B_\xi}. \tag{3.263}$$

Equation 3.263 lets us suppress, upon restriction to B_ξ, compressing with $\operatorname{supp}\xi$. We suppress accordingly in Definition 3.2.50.

Definition 3.2.50. For all $\mu \in \vartheta(\xi)$, set

1) $\mathfrak{F}_\mu := \nabla^* \mathscr{M}^\theta_{\sharp\mu} \nabla \in \mathscr{B}(\operatorname{im}\Delta_\xi, A_\xi)$,

2) $\mathfrak{G}_\mu := \mathscr{M}^\theta_{\sharp\mu} \nabla \in \mathscr{B}(\operatorname{im}\Delta_\xi, B_\xi)$.

For all $\mu \in \vartheta(\xi)$, we have $\sharp\mu > 0$ in A_ξ and therefore $\mathfrak{F}_\mu, \mathfrak{F}_\mu^{-1} > 0$ in $\mathscr{B}(\operatorname{im}\Delta_\xi)$ by 1) in Proposition 3.2.49. Note $\nabla^*\mathfrak{G}_\mu = \mathfrak{F}_\mu$ in each case by definition.

Proposition 3.2.51. *For all $\mu \in \vartheta(\xi)$, we have*

1) $\mathfrak{F}_\mu, \mathfrak{F}_\mu^{-1} > 0$ *in* $\mathscr{B}(\operatorname{im}\Delta_\xi)$ *and* $\|\mathfrak{F}_\mu^{-1}\|_{\mathscr{B}(\operatorname{im}\Delta_\xi)} \le \sigma(\Delta)^{-1}\sigma(\mu)^{-\theta}$,

2) $\mathfrak{F}_\mu\big(I(\Delta_\xi)\big) \subset I(\Delta_\xi)$ *and* $\mathfrak{F}_\mu^{-1}\big(I(\Delta_\xi)\big) \subset I(\Delta_\xi)$,

3) $\nabla^*\mathfrak{G}_\mu\mathfrak{F}_\mu^{-1} = \operatorname{id}_{\operatorname{im}\Delta_\xi}$.

Proof. Let $\mu \in \vartheta(\xi)$. Get 1) by 1) in Proposition 3.2.49, resp. 2) in Proposition 3.2.45. If $\mathfrak{F}_\mu(I(\Delta_\xi)) \subset I(\Delta_\xi)$, then $\mathfrak{F}_\mu^{-1}(I(\Delta_\xi)) \subset I(\Delta_\xi)$. Note ∇ and ∇^* intertwine adjoining and γ by symmetry, resp. 5) in Proposition 2.3.25. Symmetry of f implies $\mathscr{M}^\theta_{\sharp\mu} \circ \gamma = \gamma \circ \mathscr{M}^\theta_{\sharp\mu}$ by 1) Corollary 2.2.12. Get 2). We have 3) by definition. □

Definition 3.2.52. For all $\mu \in \vartheta(\xi)$, set

$$g^\xi_\mu(u,v) := \left\langle \mathfrak{F}_\mu^{-1}(\sharp u), \sharp v \right\rangle_\tau \tag{3.264}$$

for all $u,v \in I(\Delta_\xi)^\flat$.

Proposition 3.2.53.

1) *We have connected Riemannian manifold $(\vartheta(\xi), g^{\xi})$.*

2) *For all $\mu \in \vartheta(\xi)$, we have*

$$\mathscr{I}^{f,\theta}\Big(\mu,\mu,\big(\mathfrak{G}_\mu\mathfrak{F}_\mu^{-1}(\sharp u)\big)^\flat\Big) = g_\mu^\xi(u,u) \le \sigma(\Delta)^{-1}\sigma(\mu)^{-\theta}\cdot \|\sharp u\|_\tau^2 \tag{3.265}$$

for all $u \in I(\Delta_\xi)^\flat$.

3) *Let $\mu \in \vartheta(\xi)$, $u \in I(\Delta_\xi)^\flat$ and $w \in B^*$. If $\sharp u = \nabla^*\sharp w$, then*

$$\mathscr{I}^{f,\theta}\Big(\mu,\mu,\big(\mathfrak{G}_\mu\mathfrak{F}_\mu^{-1}(\sharp u)\big)^\flat\Big) \le \mathscr{I}^{f,\theta}(\mu,\mu,w). \tag{3.266}$$

Furthermore, we have equality in Equation 3.266 if and only if $\sharp w = \mathfrak{G}_\mu\mathfrak{F}_\mu^{-1}(\sharp u)$.

Proof. The map $\mu \mapsto \mathfrak{F}_\mu$ from $\vartheta(\xi)$ to $\mathscr{B}(\operatorname{im}\Delta_\xi)_{>0} \subset \mathrm{GL}(\mathscr{B}(\operatorname{im}\Delta))$ is smooth and invertible by 1) and 2) in Proposition 3.2.51. Get 1). The identity in Equation 3.265 follows by 1) in Proposition 3.2.45, its subsequent estimate by 1) in Proposition 3.2.51. Get 2).

We show 3). Let $\mu \in \vartheta(\xi)$, $u = x^\flat \in I(\Delta_\xi)^\flat$ and $w \in B^*$. Assume $x = \nabla^*\sharp w$. Then $\nabla^*\sharp w = \nabla^*\mathfrak{G}_\mu\mathfrak{F}_\mu^{-1}(x)$ by 3) in Proposition 3.2.51. Set $y := \sharp w - \mathfrak{G}_\mu\mathfrak{F}_\mu^{-1}(x) \in \ker\nabla^*$. Using 2), get

$$\mathscr{I}^{f,\theta}\Big(\mu,\mu,\big(\mathfrak{G}_\mu\mathfrak{F}_\mu^{-1}(x)\big)^\flat + y^\flat\Big) = g_\mu^\xi(u,u) + 2\operatorname{Re}\big\langle y, \mathscr{D}^\theta_{\sharp\mu,\xi}\mathfrak{G}_\mu\mathfrak{F}_\mu^{-1}(x)\big\rangle_\omega + \mathscr{I}^{f,\theta}\Big(\mu,\mu,y^\flat\Big). \tag{3.267}$$

Note $\mathscr{D}^\theta_{\sharp\mu,\xi}\mathfrak{G}_\mu\mathfrak{F}_\mu^{-1}(x) = \nabla\mathfrak{F}_\mu^{-1}(x)$. Using $y \in \ker\nabla^*$, the latter implies

$$\operatorname{Re}\big\langle y, \mathscr{D}^\theta_{\sharp\mu,\xi}\mathfrak{G}_\mu\mathfrak{F}_\mu^{-1}(x)\big\rangle_\omega = 0. \tag{3.268}$$

Equation 3.267 and Equation 3.268 show Equation 3.266. Since $\sharp\mu > 0$ in A_ξ, we further have $\mathscr{I}^{f,\theta}(\mu,\mu,y^\flat) = 0$ if and only if $y = 0$. This shows equivalence. Get 3). ☐

We know $T\vartheta(\xi) = \vartheta(\xi) \times I(\Delta_\xi)^\flat$ by 2) in Proposition 3.2.49. Definition 3.2.54 gives smooth map $\Theta : T\vartheta(\xi) \longrightarrow B^*_\xi$. Proposition 3.2.56 shows evaluating the latter on square integrable absolutely continuous paths to $\vartheta(\xi)$ induces admissible paths. Their vector fields minimise energy along a given absolutely continuous path.

Definition 3.2.54.

1) We define $\Theta : T\vartheta(\xi) \longrightarrow B^*_\xi$ by setting

$$\Theta(\mu,u) := \big(\mathfrak{G}_\mu\mathfrak{F}_\mu^{-1}(\sharp u)\big)^\flat \tag{3.269}$$

for all $\mu \in \vartheta(\xi)$ and $u \in I(\Delta_\xi)^\flat$.

2) For all absolutely continuous $\mu : [a,b] \longrightarrow \vartheta(\xi)$, set

$$\Theta(\mu,\dot{\mu})(t) := \Theta\big(\mu(t),\dot{\mu}(t)\big) \tag{3.270}$$

for a.e. $t \in [0,1]$.

Remark 3.2.55. Following 1) in Definition 3.2.54, note Equation 3.265 yields

$$\mathscr{I}^{f,\theta}\big(\mu,\mu,\Theta(\mu,u)\big) = g^{\xi}_{\mu}(u,u) \tag{3.271}$$

for all $\mu \in \vartheta(\xi)$ and $u \in I(\Delta_\xi)^\flat$. We use this throughout our discussion.

Proposition 3.2.56. *We consider Riemannian manifold $(\vartheta(\xi), g^\xi)$. Let $\mu : [a,b] \longrightarrow \vartheta(\xi)$ be absolutely continuous. If $\int_a^b \|\dot{\mu}(t)\|^2_{A^*} dt < \infty$, then*

1) $\big(\mu,\Theta(\mu,\dot{\mu})\big) \in \mathrm{Adm}^{[a,b]}\big(\mu(a),\mu(b)\big)$,

2) $E^{f,\theta}\big(\mu,\Theta(\mu,\dot{\mu})\big) = \int_a^b g^{\xi}_{\mu(t)}\big(\dot{\mu}(t),\dot{\mu}(t)\big)dt < \infty$,

3) $E^{f,\theta}\big(\mu,\Theta(\mu,\dot{\mu})\big) \le E(\mu,w)$ *for all* $(\mu,w) \in \mathrm{Adm}^{[a,b]}\big(\mu(a),\mu(b)\big)$.

Furthermore, we have equality in 3) *if and only if $w(t) = \Theta(\mu,\dot{\mu})(t)$ for a.e. $t \in [a,b]$.*

Proof. Assume $\int_a^b \|\dot{\mu}(t)\|^2_{A^*} dt < \infty$. Note continuity by itself implies

$$\sup_{t\in[0,1]} \sigma\big(\mu(t)\big)^{-1} = \sup_{t\in[0,1]} \big\|L^{-1}_{\sharp\mu(t),\operatorname{supp}\xi}\big\| < \infty. \tag{3.272}$$

All Banach space norms we consider here are equivalent by finite-dimensionality. Using 2) in Proposition 3.2.53, Equation 3.272 yields $C > 0$ s.t.

$$\mathscr{I}^{f,\theta}\big(\mu(t),\mu(t),\Theta(\mu,\dot{\mu})(t)\big) = g^{\xi}_{\mu(t)}\big(\dot{\mu}(t),\dot{\mu}(t)\big) \le C \cdot \big\|\dot{\mu}(t)\big\|^2_{A^*} \tag{3.273}$$

for a.e. $t \in [a,b]$. Using 5) in Theorem 2.2.29, Equation 3.273 yields $C', C'' > 0$ s.t.

$$\int_a^b \big\|\Theta(\mu,\dot{\mu})(t)\big\|^2_{B^*} dt \le C' \cdot \int_a^b g^{\xi}_{\mu(t)}\big(\dot{\mu}(t),\dot{\mu}(t)\big)dt \le C'' \cdot \int_a^b \big\|\dot{\mu}(t)\big\|^2_{A^*} dt < \infty. \tag{3.274}$$

Equation 3.274 shows $\Theta(\mu,\dot{\mu})$ is square integrable. We calculate

$$\dot{\mu}(t) = \nabla^* \mathfrak{G}_\mu \mathfrak{F}_\mu^{-1}\big(\dot{\mu}(t)\big) = \nabla^* \Theta(\mu,\dot{\mu})(t) \tag{3.275}$$

for a.e. $t \in [a,b]$. Equation 3.273 and Equation 3.275 show 1) and 2). We show 3). For all $(\mu,w) \in \mathrm{Adm}^{[a,b]}(\mu(a),\mu(b))$, note $\sharp\dot{\mu}(t) = \nabla^* \sharp w(t)$ for a.e. $t \in [a,b]$ by the continuity equation. Using 3) in Proposition 3.2.53, the latter implies 3) at once. □

Theorem 3.2.62 uses Lemma 3.2.61. The latter shows minimising geodesics with marginals in $\vartheta(\xi)$ are suitably approximated by minimising geodesics in $\vartheta(\xi)$ without change of marginals. Corollary 3.2.63 implies $\vartheta(\xi) \subset \mathscr{C}_A(\xi)$ is a geodesic subspace as per 2) in Definition 4.3.1. The statement of Lemma 3.2.61 is more general. We show 1) in the lemma by extending convolution with Dirac sequences [109] to the AF-C^*-setting. We show 2) in the lemma by adapting the proof of Lemma 3.30 in [152].

Lemma 3.2.61 uses the convolution of bounded Bochner measurable maps to A^*_ξ with smooth maps on $\mathbb{R}$ having integrable first derivative. Definition 3.2.57 gives such Bochner convolutions. Note Remark 3.2.58 and Remark 3.2.59.

Definition 3.2.57.

1) Set $C^{\infty,1}(\mathbb{R}) := \{\varphi \in C^\infty(\mathbb{R}) \mid \forall k \in \mathbb{N} : \frac{d^k}{dt^k}\varphi \in L^1(\mathbb{R})\}$. For all closed intervals $I \subset \mathbb{R}$, we say that a Bochner measurable map $\eta : I \longrightarrow A^*$ [129] is bounded measurable if $\|\eta\|_\infty := \operatorname{ess\,sup}_{t\in I} \|\eta(t)\|_{A^*} < \infty$.

2) Let $\eta : \mathbb{R} \longrightarrow A^*_\xi$ be bounded measurable. For all $\varphi \in C^{\infty,1}(\mathbb{R})$, we define the Bochner convolution map $\eta * \varphi : \mathbb{R} \longrightarrow A^*_\xi$ by setting

$$\big(\eta * \varphi\big)(t) := \int_{-\infty}^{\infty} \eta(s)\varphi(t-s)ds \tag{3.276}$$

for all $t \in \mathbb{R}$.

Remark 3.2.58. In the finite-dimensional setting, Bochner integration specialises to one-dimensional analogues in components. Let $\eta : \mathbb{R} \longrightarrow A^*$ be bounded measurable. For all $\varphi \in C^{\infty,1}(\mathbb{R})$, the map $s \mapsto \eta(s)\varphi(t-s)$ is indeed integrable for all $t \in \mathbb{R}$.

Let $\eta : \mathbb{R} \longrightarrow A^*_\xi$ be bounded measurable and $\varphi \in C^{\infty,1}(\mathbb{R})$. For all $x \in A$, we consider the map $s \mapsto \eta_x(s) := \eta(s)(x)$ and have

$$\big(\eta * \varphi\big)(t)^\flat(x) = \int_{-\infty}^{\infty} \eta(s)^\flat(x)\varphi(t-s)ds = \big(\eta_x * \varphi\big)(t) \tag{3.277}$$

for all $t \in \mathbb{R}$. Equation 3.277 shows standard results for convolutions apply [109]. We have $\|\eta * \varphi\|_\infty \le \|\eta\|_\infty \|\varphi\|_1$ by Hölder. For all $k \in \mathbb{N}$, we moreover have

$$\frac{d^k}{dt^k}\big(\eta * \varphi\big)(t) = \Big(\eta * \frac{d^k}{dt^k}\varphi\Big)(t) \tag{3.278}$$

for all $t \in \mathbb{R}$. If η is t-a.e. differentiable and $\dot\eta$ bounded measurable, then

$$\frac{d}{dt}\big(\eta * \varphi\big)(t) = \big(\dot\eta * \varphi\big)(t) \tag{3.279}$$

for a.e. $t \in \mathbb{R}$.

Remark 3.2.59. For all bounded measurable $\eta : \mathbb{R} \longrightarrow A^*_\xi$, we have bounded measurable $h^\perp(\eta) : \mathbb{R} \longrightarrow A^*_\xi$ by setting $h^\perp(\eta)(t) := h^\perp(\eta(t))$ for all $t \in \mathbb{R}$.

Let $\mu : [0,1] \longrightarrow \vartheta(\xi)$ be absolutely continuous. We extend to bounded measurable $\mu : \mathbb{R} \longrightarrow \vartheta(\xi)$ by setting $\mu(t) := \xi$ if $t \notin [0,1]$. Thus $h^{\perp}(\mu)(t) = 0$ if $t \notin [0,1]$, hence $h^{\perp}(\mu)$ is bounded measurable with compact support in $[0,1]$. Assume $\|\dot{\mu}\|_{\infty} < \infty$ and $\varphi \in C^{\infty,1}(\mathbb{R})$ s.t. $\varphi \geq 0$ and $\|\varphi\|_1 = 1$. For all $\eta \in \vartheta(\xi)$, get $\sharp\eta > 0$ in A_{ξ} by 1) in Proposition 3.2.49. Since further $\xi \in \vartheta(\xi)$ by 2) in Theorem 3.2.40, continuity implies

$$\inf_{t\in\mathbb{R}} \sigma\big(\mu(t)\big) > 0. \tag{3.280}$$

Using $\varphi \geq 0$ and $\|\varphi\|_1 = 1$, Equation 3.277 and Equation 3.280 show

$$\sharp\big(\mu * \varphi\big)(t) \geq \inf_{t\in\mathbb{R}} \sigma\big(\mu(t)\big) \cdot \operatorname{supp} \xi > 0 \tag{3.281}$$

in A_{ξ} for all $t \in \mathbb{R}$. Equation 3.281 shows $(\mu * \varphi)(t) \in \vartheta(\xi)$ for all $t \in \mathbb{R}$. Taken together with Equation 3.278, we have smooth $\mu * \varphi : \mathbb{R} \longrightarrow \vartheta(\xi)$. Equation 3.279 shows

$$\frac{d}{dt}\big(\mu * \varphi\big)(t) = \frac{d}{dt}\big(h^{\perp}(\mu) * \varphi\big)(t) = \big(\dot{\mu} * \varphi\big)(t) \in I(\Delta_{\xi})^{\flat} \tag{3.282}$$

for a.e. $t \in \mathbb{R}$.

Remark 3.2.60. For all $n \in \mathbb{N}$, we consider normal distribution for $\sigma^2 = n^{-1}$ given by

$$\varphi_n(t) := \sqrt{\frac{n}{2\pi}} \exp\Big(-\frac{t^2 n}{2}\Big) \tag{3.283}$$

for all $t \in \mathbb{R}$ [170]. We have $\varphi_n \in C^{\infty,1}(\mathbb{R})$, $\varphi_n \geq 0$ and $\|\varphi_n\|_1 = 1$ in each case. We use such Dirac sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset C^{\infty,1}(\mathbb{R})$ [109] for Bochner convolutions in Lemma 3.2.61.

Lemma 3.2.61. *Let $\mu^0, \mu^1 \in \vartheta(\xi)$ and $(\mu, w) \in \mathrm{Adm}^{[0,1]}(\mu^0, \mu^1)$ s.t. $E^{f,\theta}(\mu, w) < \infty$.*

1) *If $\mu : [0,1] \longrightarrow \vartheta(\xi)$ and $\|\dot{\mu}\|_{\infty} < \infty$, then there exists family $\{\mu^n : [0,1] \longrightarrow \vartheta(\xi)\}_{n\in\mathbb{N}}$ of smooth paths s.t.*

   1.1) $\big(\mu^n, \Theta\big(\mu^n, \dot{\mu}^n\big)\big) \in \mathrm{Adm}^{[0,1]}$ *for all $n \in \mathbb{N}$,*

   1.2) $\lim_{n\in\mathbb{N}} \big(\mu^n, \Theta\big(\mu^n, \dot{\mu}^n\big)\big) = \big(\mu, \Theta(\mu, \dot{\mu})\big)$ *in* $\mathrm{Adm}^{[0,1]}$,

   1.3) $\lim_{n\in\mathbb{N}} E^{f,\theta}\big(\mu^n, \Theta\big(\mu, \dot{\mu}^n\big)\big) = E^{f,\theta}\big(\mu, \Theta(\mu, \dot{\mu})\big) \leq E^{f,\theta}(\mu, w)$.

2) *If $\|\dot{\mu}\|_{\infty} < \infty$, then there exists $(\mu^n, w^n)_{n\in\mathbb{N}} \subset \mathrm{Adm}\big(\mu^0, \mu^1\big)$ and $C > 0$ s.t.*

   2.1) $\mu^n : [0,1] \longrightarrow \vartheta(\xi)$ *and* $\|\dot{\mu}^n\|_{\infty} \leq C\|\dot{\mu}\|_{\infty}$ *for all $n \in \mathbb{N}$,*

   2.2) $\liminf_{n\in\mathbb{N}} E^{f,\theta}(\mu^n, w^n) \leq E^{f,\theta}(\mu, w)$.

3) *If $(\mu, w) \in \mathrm{Geo}\big(\mu^0, \mu^1\big)$, then $\|\dot{\mu}\|_{\infty} < \infty$.*

Proof. We show 1). Assume its setting. In particular, we have $(\mu, \Theta(\mu, \dot{\mu})) \in \mathrm{Adm}^{[0,1]}$ and $E^{f,\theta}(\mu, \Theta(\mu, \dot{\mu})) \leq E^{f,\theta}(\mu, w)$ by 1), resp. 3) in Proposition 3.2.56. Continuity implies $\|\mu\|_\infty = \operatorname{ess\,sup}_{t\in[0,1]} \|\mu\|_{A^*} < \infty$.

For all $n \in \mathbb{N}$, let $\varphi_n \in C^{1,\infty}(\mathbb{R})$ be the normal distribution as per Equation 3.283 and set $\mu^n := \mu * \varphi_n$. Then $\|\mu\|_\infty, \|\dot{\mu}\|_\infty < \infty$ implies $\mu^n : \mathbb{R} \longrightarrow \vartheta(\xi)$ is smooth s.t. $\dot{\mu}^n = \dot{\mu} * \varphi_n$ in each case. For all $n \in \mathbb{N}$, we directly verify

$$\|\mu^n\|_\infty \leq \|\mu\|_\infty < \infty, \; \|\dot{\mu}^n\|_\infty \leq \|\dot{\mu}\|_\infty < \infty \tag{3.284}$$

using $\|\varphi\|_1 = 1$. We show $\{\mu^n|_{[0,1]} : [0,1] \longrightarrow \vartheta(\xi)\}_{n\in\mathbb{N}}$ is a sequence as claimed. Get 1.1) by 1) in Proposition 3.2.56. Testing on A, standard properties of Dirac sequences imply $\mu(t) = w^*\text{-}\lim_{n\in\mathbb{N}} \mu^n(t)$ for all $t \in [0,1]$ and $\dot{\mu}(t) = w^*\text{-}\lim_{n\in\mathbb{N}} \dot{\mu}^n(t)$ for a.e. $t \in [0,1]$ [109]. All norms and operator topologies here are equivalent by finite-dimensionality. Thus 1.2) and 1.3) follow by dominated convergence if there exists $C > 0$ s.t.

$$\left\|\sharp\Theta\big(\mu^n(t), \dot{\mu}^n(t)\big)\right\|_\omega, g^\xi_{\mu^n(t)}\big(\dot{\mu}(t), \dot{\mu}(t)\big) \leq C \tag{3.285}$$

for a.e. $t \in [0,1]$ and all $n \in \mathbb{N}$.

We show there exists $C > 0$ as for Equation 3.285. Using Equation 3.284, applying $\|\dot{\mu}^n\|_\infty \leq \|\dot{\mu}\|_\infty$ in each case lets us estimate

$$\left\|\sharp\Theta\big(\mu^n(t), \dot{\mu}^n(t)\big)\right\|_\omega \leq \left\|\mathfrak{G}_{\mu^n(t)}\right\| \cdot \left\|\mathfrak{F}^{-1}_{\mu^n(t)}\right\| \cdot \|\dot{\mu}\|_\infty \tag{3.286}$$

and

$$g^\xi_{\mu^n(t)}\big(\dot{\mu}^n(t), \dot{\mu}^n(t)\big) \leq \left\|\mathfrak{F}^{-\frac{1}{2}}_{\mu^n(t)}\right\| \cdot \|\dot{\mu}\|_\infty \tag{3.287}$$

for a.e. $t \in [0,1]$ and all $n \in \mathbb{N}$. Since moreover $\|\mu^n\|_\infty \leq \|\mu\|_\infty$ in each case, get uniform bound for $\{\|\mathfrak{G}_{\mu^n(t)}\|\}_{t\in\mathbb{R}, n\in\mathbb{N}}$ by continuity. Uniform bound for $\{\|\mathfrak{F}^{-1}_{\mu^n(t)}\|\}_{t\in\mathbb{R}, n\in\mathbb{N}}$ follows by 1) in Proposition 3.2.51 if

$$\inf_{t\in\mathbb{R}} \sigma\big(\mu^n(t)\big) \geq \inf_{t\in\mathbb{R}} \sigma\big(\mu(t)\big) > 0 \tag{3.288}$$

for all $n \in \mathbb{N}$. Using Lemma A.2.33, Equation 3.281 shows Equation 3.288 by maximality of spectral gaps. Applying uniform bounds to Equation 3.286 and Equation 3.287 yields $C > 0$ as required. Get 1.2) and 1.3) by dominated convergence. Altogether, get 1).

We show 2). Assume $\|\dot{\mu}\|_\infty < \infty$. We adapt the proof of Lemma 3.30 in [152]. We construct two types of perturbed paths and concatenate them. For all $\varepsilon \in (0,1)$, set

$$\mu^\varepsilon(t) := (1-\varepsilon)\mu(t) + \varepsilon\xi,\ v^\varepsilon(t) := (1-\varepsilon)w(t) \tag{3.289}$$

for all $t \in [0,1]$. Since $\sharp\mu^0, \sharp\mu^1, \sharp\xi > 0$ in A_ξ, we see $\mu^\varepsilon(t) \in \vartheta(\xi)$ in each case. Moreover, we directly verify $(\mu^\varepsilon, v^\varepsilon) \in \mathrm{Adm}^{[0,1]}$. This is the first type of perturbed path.

For all $\varepsilon \in (0,1)$ and $k \in \{0,1\}$, set

$$\mu^{k,\varepsilon}(t) := (1-t)\mu(k) + t\mu^\varepsilon(k) \tag{3.290}$$

for all $t \in [0,1]$. Since $\sharp\mu^0, \sharp\mu^1, \sharp\xi > 0$ in A_ξ, we see $\mu^{k,\varepsilon}(t) \in \vartheta(\xi)$ in each case. There further exists $C > 0$ s.t.

$$\sharp\mu^{k,\varepsilon}(t) \geq C \cdot \operatorname{supp}\xi \tag{3.291}$$

for all $\varepsilon \in (0,1)$, $k \in \{0,1\}$ and $t \in [0,1]$. For all $\varepsilon \in (0,1)$ and $k \in \{0,1\}$, set

$$v^{k,\varepsilon}(t) := \varepsilon \cdot \Theta\big(\mu^{k,\varepsilon}(t), \xi - \mu(k)\big) \tag{3.292}$$

for all $t \in [0,1]$. Since $\frac{d}{dt}\mu^{k,\varepsilon}(t) = \varepsilon(\xi - \mu(k))$ in each case, get $(\mu^{k,\varepsilon}, v^{k,\varepsilon}) \in \mathrm{Adm}(\mu^0, \mu^1)$ at once by 1) in Proposition 3.2.56. This is the second type of perturbed path.

We concatenate these two types of paths. For all $\varepsilon \in (0,1)$, we define concatenated path on $[0,1]$ by setting

$$\big(\mu^\varepsilon, w^\varepsilon\big)(t) := \begin{cases} \big(\mu^{0,\varepsilon}, \varepsilon^{-1}v^{0,\varepsilon}\big)\big(\varepsilon^{-1}t\big) & \text{if } t \leq \varepsilon, \\ \big(\mu^\varepsilon, (1-2\varepsilon)^{-1}v^\varepsilon\big)\big((1-2\varepsilon)^{-1}(t-\varepsilon)\big) & \text{if } \varepsilon < t < 1-\varepsilon, \\ \big(\mu^{1,\varepsilon}, \varepsilon^{-1}v^{1,\varepsilon}\big)\big(\varepsilon^{-1}(1-t)\big) & \text{if } t \geq 1-\varepsilon. \end{cases}$$

We have $\mu^\varepsilon : [0,1] \longrightarrow \vartheta(\xi)$ and $(\mu^\varepsilon, w^\varepsilon) \in \mathrm{Adm}(\mu^0, \mu^1)$ in each case. Moreover, we directly verify there exists $C > 0$ s.t. $\sup_{\varepsilon\in(0,1)} \|\dot{\mu}^\varepsilon\|_\infty \leq C\|\dot{\mu}\|_\infty$. We readily see 2.1) is satisfied for all countable subsequences of $(\mu^\varepsilon, w^\varepsilon)_{\varepsilon>0}$. We claim 2.2) is likewise satisfied.

We show 2.2). For all $\varepsilon \in (0,1)$ and $k \in \{0,1\}$, joint convexity of quasi-entropies as per 1) in Theorem 2.2.29 shows

$$E^{f,\theta}\big(\mu^\varepsilon, v^\varepsilon\big) \leq (1-\varepsilon) \cdot E^{f,\theta}(\mu, w). \tag{3.293}$$

Using Lemma A.2.33, note Equation 3.291 shows $\inf_{t\in[0,1]} \sigma(\mu^{k,\varepsilon}(t)) \geq C$ in each case by maximality of spectral gaps. We invert the latter to get $\sup_{t\in[0,1]} \sigma(\mu^{k,\varepsilon}(t))^{-1} \leq C^{-1}$.

Using 2) in Proposition 3.2.53, we therefore have

$$E^{f,\theta}\left(\mu^{k,\varepsilon},v^{k,\varepsilon}\right)\leq\varepsilon^2\sigma(\Delta)^{-1}C^{-\theta}\cdot\left\|\sharp\xi-\sharp\mu(k)\right\|_\tau^2 \tag{3.294}$$

for all $\varepsilon\in(0,1)$ and $k\in\{0,1\}$. Rescaling as per Remark 3.1.22 shows

$$E^{f,\theta}\left(\mu^\varepsilon,w^\varepsilon\right)=\varepsilon^{-1}E^{f,\theta}\left(\mu^{0,\varepsilon},v^{0,\varepsilon}\right)+(1-2\varepsilon)^{-1}E^{f,\theta}\left(\mu^\varepsilon,v^\varepsilon\right)+\varepsilon^{-1}E^{f,\theta}\left(\mu^{1,\varepsilon},v^{1,\varepsilon}\right) \tag{3.295}$$

for all $\varepsilon\in(0,1)$. Applying Equation 3.293 and Equation 3.294 to Equation 3.295 yields

$$E^{f,\theta}\left(\mu^\varepsilon,w^\varepsilon\right)\leq 2\varepsilon\cdot\sigma(\Delta)^{-1}C^{-\theta}\cdot\left\|\sharp\xi-\sharp\mu(k)\right\|_\tau^2+\frac{1-\varepsilon}{1-2\varepsilon}E^{f,\theta}(\mu,w) \tag{3.296}$$

for all $\varepsilon\in(0,1)$. Letting $\varepsilon\downarrow 0$ in Equation 3.296 yields $E^{f,\theta}(\mu,w)$ on its right-hand side. We therefore have 2.2) as claimed. Altogether, get 2).

We show 3). Assume $\mu\in\mathrm{Geo}(\mu^0,\mu^1)$. Minimising geodesics have t-a.e. constant speed by 1) in Proposition 3.1.45. By definition, the quasi-entropy evaluated on (μ,w) is thus t-a.e. constant. There exists $C>0$ s.t. $\sup_{t\in[0,1]}\|\sharp w(t)\|_\omega\leq C$ by 4) in Theorem 2.2.29. We have $\|\nabla^*\|<\infty$ by finite-dimensionality. The continuity equation lets us calculate

$$\left\|\dot\mu(t)\right\|_\tau=\left\|\nabla^*\sharp w(t)\right\|_\tau\leq\left\|\nabla^*\right\|\cdot\left\|\sharp w(t)\right\|_\omega\leq\left\|\nabla^*\right\|\cdot C<\infty \tag{3.297}$$

for a.e. $t\in[0,1]$. Equation 3.297 implies 3) at once. □

Theorem 3.2.62. *Let $(\phi,\psi,\gamma,\nabla)$ be noncommutative differential structure for tracial AF-C^*-algebras (A,τ) and (B,ω) in (f,θ)-setting. Assume A and B are finite-dimensional. If $\xi\in\mathscr{S}(A)$ is a fixed state, then $\mathscr{W}^{f,\theta}_{\nabla|\vartheta(\xi)\times\vartheta(\xi)}$ is the distance induced by g^ξ.*

Proof. Let $\xi\in\mathscr{S}(A)$ be a fixed state. Proposition 3.2.56 shows the induced distance d^ξ of g^ξ is given by minimising

$$\sqrt{E^{f,\theta}\left(\mu,\Theta(\mu,\dot\mu)\right)}=\sqrt{\int_0^1 g^\xi_{\mu(t)}\left(\dot\mu(t),\dot\mu(t)\right)dt} \tag{3.298}$$

over smooth paths $\mu:[0,1]\longrightarrow\vartheta(\xi)$. Thus 1) and 2) in Lemma 3.2.61 show d^ξ is given by minimising over absolutely continuous path with marginals in $\vartheta(\xi)$ and bounded measurable derivative, hence we conclude by 3) in Lemma 3.2.61. □

Corollary 3.2.63. *For all $\mu^0,\mu^1\in\vartheta(\xi)$, there exists $(\mu,w)\in\mathrm{Geo}(\mu^0,\mu^1)$ s.t. $\mu(t)\in\vartheta(\xi)$ for all $t\in[0,1]$ and $\mu:[0,1]\longrightarrow\vartheta(\xi)$ is a minimising geodesic in distance induced by g^ξ.*

Proof. Let $\mu^0,\mu^1\in\vartheta(\xi)$. Get $(\mu,w)\in\mathrm{Geo}(\mu^0,\mu^1)$ by 3) in Corollary 3.1.50. Lemma 3.2.61 implies $\mu(t)\in\vartheta(\xi)$ for all $t\in[0,1]$ by minimality. We conclude by Theorem 3.2.62. □

Accessibility components of square integrable normal states. Assuming spectral gaps of quantum Laplacians and fixed parts, Theorem 3.2.65 classifies accessibility components of square integrable normal states by showing each one is a norm closed convex subsets of all such states with identical fixed part. Theorem 3.2.65 uses Lemma 3.2.64. We show the lemma by twice reduction. This lets us adapt the proof of Proposition 9.2 in [50]. In the finite-dimensional setting, assumptions as above are satisfied and Corollary 3.2.66 classifies all accessibility components using fixed parts.

Moreover, the coarse graining process reveals more general classification schemes by intersecting with convex subsets of states other than square integrable normal ones. In the logarithmic mean setting and assuming strictly positive lower Ricci bounds, as well as finitely supported fixed part but not spectral gaps, Theorem 4.3.12 classifies accessibility components of normal states with finite quantum relative entropy using fixed parts. Here, Example 3.2.67 constructs quantum Laplacians having spectral gaps for the unique hyperfinite type II_1-factor.

Let $(\phi,\psi,\gamma,\nabla)$ be noncommutative differential structure for tracial AF-C^*-algebras (A,τ) and (B,ω) in (f,θ)-setting. Assume $\sigma(\Delta)>0$.

Lemma 3.2.64. *Let $\xi\in\mathscr{S}^{\mathrm{N},2}_{-1}(A_\xi)$ be a fixed state. For all $\mu,\eta\in\mathrm{Fix}_A(\xi)\cap\mathscr{S}^{\mathrm{N},2}(A)$ and $\varepsilon\in(0,1]$, we have*

$$\mathscr{W}^{f,\theta}_{\nabla}(\mu,\eta)\le 2\sigma(\Delta)^{-\frac{1}{2}}\sigma(\xi)^{-\frac{\theta}{2}}\varepsilon^{-\frac{\theta}{2}}\cdot\left(\left\|\varepsilon h^{\perp}(\sharp\mu)\right\|_{\tau}+\left\|(1-\varepsilon)h^{\perp}(\sharp\mu)-h^{\perp}(\sharp\eta)\right\|_{\tau}\right)<\infty. \quad (3.299)$$

Proof. We reduce twice in order to adapt the proof of Proposition 9.2 in [50]. First, we reduce to $\mu,\eta\in\mathrm{Fix}_A(\xi)\cap\mathscr{S}^{\mathrm{N},2}(A)$ s.t. $\bar{\mu}_j,\bar{\eta}_j\in\vartheta(\bar{\xi}_j)$ for a.e. $j\in\mathbb{N}$. Secondly, we reduce to the finite-dimensional setting. Let $\varepsilon\in(0,1]$. Set

$$C_\varepsilon := 2\sigma(\Delta)^{-\frac{1}{2}}\sigma(\xi)^{-\frac{\theta}{2}}\varepsilon^{-\frac{\theta}{2}}. \quad (3.300)$$

We engage in the first reduction. Let $\mu,\eta\in\mathrm{Fix}_A(\xi)\cap\mathscr{S}^{\mathrm{N},2}(A)$. Note $\mathscr{W}^{f,\theta}_{\nabla}$ is l.s.c. in w^*-topology by 3) in Theorem 3.1.47. In addition, we know 2.1) in Proposition 3.2.32 ensures $h:[0,\infty]\longrightarrow\mathscr{B}(A^*)$ is w^*-continuous on $\mathscr{S}(A)$. We obtain

$$\mathscr{W}^{f,\theta}_{\nabla}(\mu,\eta)\le\liminf_{t\downarrow 0}\mathscr{W}^{f,\theta}_{\nabla}\big(h_t(\mu),h_t(\eta)\big). \quad (3.301)$$

Strong continuity of $h:[0,\infty)\longrightarrow\mathscr{B}(L^2(A,\tau))$ as per 1) in Proposition 3.2.32 together with $[h_t,h^{\perp}]=0$ for all $t\ge 0$ further yields

$$\left\|\varepsilon h^{\perp}(\sharp\mu)\right\|_{\tau}=\lim_{t\downarrow 0}\left\|\varepsilon h^{\perp}(\sharp h_t(\mu))\right\|_{\tau} \quad (3.302)$$

and

$$\left\|(1-\varepsilon)h^{\perp}(\sharp\mu)-h^{\perp}(\sharp\eta)\right\|_{\tau}=\lim_{t\downarrow 0}\left\|(1-\varepsilon)h^{\perp}(\sharp h_t(\mu))-h^{\perp}(\sharp h_t(\eta))\right\|_{\tau}. \quad (3.303)$$

Equation 3.301, Equation 3.302 and Equation 3.303 imply Equation 3.299 if

$$\mathscr{W}_{\nabla}^{f,\theta}\left(h_t(\mu),h_t(\eta)\right) \le C_{\varepsilon}\cdot\left(\left\|\varepsilon h^{\perp}\left(\sharp h_t(\mu)\right)\right\|_{\tau}+\left\|(1-\varepsilon)h^{\perp}\left(\sharp h_t(\mu)\right)-h^{\perp}\left(\sharp h_t(\eta)\right)\right\|_{\tau}\right) \tag{3.304}$$

for all $t>0$. If $\xi_j \neq 0$ for $j \in \mathbb{N}$, then $\overline{h_t(\mu)}_j = h_t(\bar{\mu}_j), \overline{h_t(\eta)}_j = h_t(\bar{\eta}_j) \in \vartheta(\bar{\xi}_j)$ for all $t>0$ by 1.3) in Proposition 3.2.34 and 1.3) in Theorem 3.2.40. Since $\xi_j \neq 0$ for a.e. $j \in \mathbb{N}$, we see Equation 3.304 lets us apply the first reduction by 3) in Theorem 3.1.47.

We engage in the second reduction. Let $\mu,\eta \in \mathrm{Fix}_A(\xi)\cap\mathscr{S}^{\mathrm{N},2}(A)$. Assume there exists $k\in\mathbb{N}$ s.t. $\bar{\mu}_j,\bar{\eta}_j \in \vartheta(\bar{\xi}_j)$ for all $j \ge k$ in $\mathbb{N}$. Let $0<\delta<\sigma(\xi)$. Set

$$C_{\delta} := \sigma(\xi)-\delta. \tag{3.305}$$

Following Remark 3.2.15, 1) in Lemma 3.2.16 implies there exists $l \in \mathbb{N}$ s.t.

$$0 < C_{\delta} \le \sigma\left(\bar{\xi}_j\right) \tag{3.306}$$

for all $j \ge l$ in $\mathbb{N}$. Set $m := \max\{k,l\}$. Further set

$$\mu^{\varepsilon} := (1-\varepsilon)\mu+\varepsilon\xi \tag{3.307}$$

for all $t\in[0,1]$. For all $j\in\mathbb{N}$, set $\bar{\mu}^{\varepsilon}_j := \mu^{\varepsilon}(1_{A_j})^{-1}\mu^{\varepsilon}_j$ as per 1) in Definition 3.1.12.

Assume

$$\mathscr{W}_{\nabla_j}^{f,\theta}\left(\bar{\mu}_j,\bar{\mu}^{\varepsilon}_j\right) \le 2\sigma\left(\Delta_j\right)^{-\frac{1}{2}} C_{\delta}^{-\frac{\theta}{2}}{}^{2}\varepsilon^{-\frac{\theta}{2}}\left\|\varepsilon h^{\perp}\left(\sharp\bar{\mu}_j\right)\right\|_{\tau} \tag{3.308}$$

and

$$\mathscr{W}_{\nabla_j}^{f,\theta}\left(\bar{\eta}_j,\bar{\mu}^{\varepsilon}_j\right) \le 2\sigma\left(\Delta_j\right)^{-\frac{1}{2}} C_{\delta}^{-\frac{\theta}{2}}{}^{2}\varepsilon^{-\frac{\theta}{2}}\left\|\sharp\bar{\mu}^{\varepsilon}_j-\sharp\bar{\eta}_j\right\|_{\tau} \tag{3.309}$$

for all $j \ge m$ in $\mathbb{N}$. Using triangle inequality, Equation 3.308 and Equation 3.309 show

$$\mathscr{W}_{\nabla_j}^{f,\theta}\left(\bar{\mu}_j,\bar{\eta}_j\right) \le 2\sigma\left(\Delta_j\right)^{-\frac{1}{2}} C_{\delta}^{-\frac{\theta}{2}}\varepsilon^{-\frac{\theta}{2}}\cdot\left(\left\|\varepsilon h^{\perp}\left(\sharp\bar{\mu}_j\right)\right\|_{\tau}+\left\|\sharp\bar{\mu}^{\varepsilon}_j-\sharp\bar{\eta}_j\right\|_{\tau}\right) \tag{3.310}$$

for all $j \ge m$ in $\mathbb{N}$. Note 2) in Theorem 3.1.47 shows

$$\mathscr{W}_{\nabla}^{f,\theta}\left(\bar{\mu}_j,\bar{\eta}_j\right) = \mathscr{W}_{\nabla_j}^{f,\theta}\left(\bar{\mu}_j,\bar{\eta}_j\right) \tag{3.311}$$

in each case as well. Applying Equation 3.311 to Equation 3.310 yields

$$\mathscr{W}_{\nabla}^{f,\theta}\left(\bar{\mu}_j,\bar{\eta}_j\right) \le 2\sigma\left(\Delta_j\right)^{-\frac{1}{2}} C_{\delta}^{-\frac{\theta}{2}}\varepsilon^{-\frac{\theta}{2}}\cdot\left(\left\|\varepsilon h^{\perp}\left(\sharp\bar{\mu}_j\right)\right\|_{\tau}+\left\|\sharp\bar{\mu}^{\varepsilon}_j-\sharp\bar{\eta}_j\right\|_{\tau}\right) \tag{3.312}$$

for all $j \ge m$ in $\mathbb{N}$.

Finally, 4) in Proposition 2.3.19 implies

$$0 < \sigma(\Delta) \leq \inf_{j \in \mathbb{N}} \sigma\big(\Delta_j\big) \tag{3.313}$$

by Proposition A.2.32. Applying Equation 3.313 to Equation 3.312 yields

$$\mathscr{W}_{\nabla}^{f,\theta}\big(\bar{\mu}_j, \bar{\eta}_j\big) \leq 2\sigma(\Delta)^{-\frac{1}{2}} C_{\delta}^{-\frac{\theta}{2}} \varepsilon^{-\frac{\theta}{2}} \cdot \Big(\big\| \varepsilon h^{\perp}\big(\sharp \bar{\mu}_j\big) \big\|_{\tau} + \big\| \sharp \bar{\mu}_j^{\varepsilon} - \sharp \bar{\eta}_j \big\|_{\tau} \Big) \tag{3.314}$$

for all $j \geq m$ in $\mathbb{N}$.

We apply limit inferior to both sides in Equation 3.314. In addition, we use l.s.c. in w^*-topology for the left-hand side and $I = \text{s-lim}_{j \in \mathbb{N}} \pi_j^A$ for the right-hand side to get its $\|.\|_{\tau}$-limit. Altogether, applying limit inferior to Equation 3.314 lets us estimate

$$\mathscr{W}_{\nabla}^{f,\theta}(\mu, \eta) \leq 2\sigma(\Delta)^{-\frac{1}{2}} C_{\delta}^{-\frac{\theta}{2}} \varepsilon^{-\frac{\theta}{2}} \cdot \Big(\big\| \varepsilon h^{\perp}\big(\sharp \mu\big) \big\|_{\tau} + \big\| \sharp \mu^{\varepsilon} - \sharp \eta \big\|_{\tau} \Big). \tag{3.315}$$

Note $\|\mu^{\varepsilon} - \eta\|_{\tau} = \|(1-\varepsilon) h^{\perp}(\mu) - h^{\perp}(\eta)\|_{\tau}$ since $\mu, \eta \in \mathrm{Fix}_A(\xi)$. Equation 3.315 shows

$$\mathscr{W}_{\nabla}^{f,\theta}(\mu, \eta) \leq 2\sigma(\Delta)^{-\frac{1}{2}} C_{\delta}^{-\frac{\theta}{2}} \varepsilon^{-\frac{\theta}{2}} \cdot \Big(\big\| h^{\perp}\big(\sharp \mu\big) \big\|_{\tau} + \big\| (1-\varepsilon) h^{\perp}\big(\sharp \mu\big) - h^{\perp}\big(\sharp \eta\big) \big\|_{\tau} \Big). \tag{3.316}$$

If Equation 3.308 and Equation 3.309 hold for all $j \geq m$ in $\mathbb{N}$, then Equation 3.316 in turn holds for $0 < \delta < \sigma(\xi)$ fixed but arbitrary. Letting $\delta \downarrow 0$ in Equation 3.316 therefore yields Equation 3.304. The latter lets us apply the second reduction.

Assume A and B are finite-dimensional. Let $\mu, \eta \in \mathrm{Fix}_A(\xi)$. We show Equation 3.308 and Equation 3.309. We suppress subscript $j \in \mathbb{N}$ without loss of generality. Set

$$\mu^{\varepsilon}(s) := (1-s)\mu + s\xi, \ \eta^{\varepsilon}(t) := (1-t)\eta + t\mu^{\varepsilon} \tag{3.317}$$

for all $s \in [0, \varepsilon]$ and $t \in [0,1]$. Note $\mu^{\varepsilon} : [0, \varepsilon] \longrightarrow \vartheta(\xi)$ and $\eta^{\varepsilon} : [0,1] \longrightarrow \vartheta(\xi)$ are absolutely continuous. Using the map $t \mapsto \varphi(t) := \varepsilon t$, rescaling $\mu := \mu^{\varepsilon} \circ \varphi$ as per Remark 3.1.22 yields absolutely continuous $\mu : [0, \varepsilon] \longrightarrow \vartheta(\xi)$. We may use double notation for μ and μ^{ε}, each denoting state and path, since their meaning is clear from context. We have $\dot{\mu}(t) = -\varepsilon h^{\perp}(\mu)$ and $\dot{\eta}^{\varepsilon} = \mu^{\varepsilon} - \eta$ in each case. Proposition 3.2.56 shows $\mu, \eta^{\varepsilon} : [0,1] \longrightarrow \vartheta(\xi)$ induce admissible paths

$$\big(\mu, \Theta\big(\mu, -\varepsilon h^{\perp}(\mu)\big)\big) \in \mathrm{Adm}^{[0,1]}\big(\mu, \mu^{\varepsilon}\big), \ \big(\eta^{\varepsilon}, \Theta\big(\eta^{\varepsilon}, \mu^{\varepsilon} - \eta\big)\big) \in \mathrm{Adm}^{[0,1]}\big(\eta, \mu^{\varepsilon}\big). \tag{3.318}$$

Since $\sharp\xi > 0$ in A_ξ, we have

$$\sharp\mu(t), \sharp\eta^\varepsilon(t) \geq t\varepsilon \cdot \sharp\xi \geq t\varepsilon \cdot \sigma(\xi) \cdot \operatorname{supp}\xi \tag{3.319}$$

in A_ξ for all $t \in (0,1]$. Using Lemma A.2.33, Equation 3.319 shows

$$\sigma\big(\mu(t)\big), \sigma\big(\eta^\varepsilon(t)\big) \geq t\varepsilon \cdot \sigma(\xi) \tag{3.320}$$

for all $t \in (0,1]$ by maximality of spectral gaps. Using 1) in Proposition 3.2.51, then note Equation 3.320 in turn shows

$$\mathfrak{F}^{-1}_{\mu(t)}, \mathfrak{F}^{-1}_{\eta^\varepsilon(t)} \leq \sigma(\Delta)^{-1} t^{-\theta} \varepsilon^{-\theta} \sigma(\xi)^{-\theta} \cdot I \tag{3.321}$$

on $\operatorname{im}\Delta_\xi$. We evaluate paths on lengths functionals. Using 2) in Proposition 3.2.53 in order to evaluate on the Riemannian metric, Equation 3.321 lets us estimate

$$L^{f,\theta}\big(\mu, \Theta\big(\mu, -\varepsilon h^\perp(\mu)\big)\big) \leq \sigma(\Delta)^{-\frac{1}{2}} \sigma(\xi)^{-\frac{\theta}{2}} \varepsilon^{-\frac{\theta}{2}} \cdot \big\|\varepsilon h^\perp\big(\sharp\mu\big)\big\|_\tau \cdot \int_0^1 t^{-\frac{\theta}{2}} dt \tag{3.322}$$

and

$$L^{f,\theta}\big(\eta^\varepsilon, \Theta\big(\eta^\varepsilon, \mu^\varepsilon - \eta\big)\big) \leq \sigma(\Delta)^{-\frac{1}{2}} \sigma(\xi)^{-\frac{\theta}{2}} \varepsilon^{-\frac{\theta}{2}} \cdot \big\|\sharp\mu^\varepsilon - \sharp\eta\big\|_\tau \cdot \int_0^1 t^{-\frac{\theta}{2}} dt. \tag{3.323}$$

Note $\int_0^1 t^{-\frac{\theta}{2}} dt = \big(1 - \frac{\theta}{2}\big)^{-1} \leq 2 < \infty$ since $\theta \in [0,1]$. Following Corollary 3.1.42, we obtain the required estimates at once by minimising the left-hand sides in Equation 3.322 and Equation 3.323 over admissible paths with marginals chosen accordingly. $\square$

The statement of Theorem 3.2.65, resp. Corollary 3.2.66, refers to continuity defined on accessibility components of square integrable normal states by norm topology under the standard modified pairing.

Theorem 3.2.65. *Let $(\phi, \psi, \gamma, \nabla)$ be noncommutative differential structure for tracial AF-C^*-algebras (A, τ) and (B, ω) in (f, θ)-setting. Assume $\sigma(\Delta) > 0$. If $\xi \in \mathscr{S}^{\mathrm{N},2}_{-1}(A_\xi)$ is a fixed state, then*

1) $\mathscr{C}^{\mathrm{N},2}_A(\xi) := \mathscr{C}_A(\xi) \cap \mathscr{S}^{\mathrm{N},2}(A) = \operatorname{Fix}_A(\xi) \cap \mathscr{S}^{\mathrm{N},2}(A)$,

2) $\mathscr{W}^{f,\theta}_{\nabla|\mathscr{C}^2_A(\xi)\times\mathscr{C}^2_A(\xi)}$ is finite and $\|.\|_\tau$-continuous.

Proof. Let $\xi \in \mathscr{S}^{\mathrm{N},2}_{-1}(A_\xi)$ be a fixed state. Using 1.2) in Proposition 2.1.31 and 1.3) in Proposition 3.2.34, 2) in Corollary 3.1.49 implies

$$\mathscr{C}_A(\xi) \cap \mathscr{S}^{\mathrm{N},2}(A) \subset \operatorname{Fix}_A(\xi) \cap \mathscr{S}^{\mathrm{N},2}(A). \tag{3.324}$$

Lemma 3.2.64 shows $\mathscr{W}^{f,\theta}_\nabla$ is finite on $\operatorname{Fix}_A(\xi) \cap \mathscr{S}^{\mathrm{N},2}(A)$, i.e. converse to Equation 3.324. Get 1). Note $h^\perp \in \mathscr{B}(L^2(A, \tau))$ is a projection by 2.1) in Proposition 3.2.30.

We show 2). Let $\mu \in \mathscr{C}_A^{\mathrm{N},2}(\xi)$ and $\{\mu^n\}_{n\in\mathbb{N}} \subset \mathscr{C}_A^{\mathrm{N},2}(\xi)$ s.t. $\sharp\mu = \|.\|_\tau\text{-}\lim_{n\in\mathbb{N}} \sharp\mu^n$. Using Lemma 3.2.64, there exists $C > 0$ s.t.

$$0 \leq \limsup_{n\in\mathbb{N}} \mathscr{W}_\nabla^{f,\theta}\big(\mu,\mu^n\big) \leq C\varepsilon^{1-\frac{\theta}{2}} \cdot \|h^\perp\|_{\mathscr{B}(L^2(A,\tau))} \cdot \|\sharp\mu\|_\tau = C\varepsilon^{1-\frac{\theta}{2}} \cdot \|\sharp\mu\|_\tau \tag{3.325}$$

for all $\varepsilon \in (0,1]$. Letting $\varepsilon \downarrow 0$ in Equation 3.325 shows $\|.\|_\tau$-continuity by l.s.c. as per 3) in Theorem 3.1.47. Get 2). ☐

Corollary 3.2.66. *Assume A and B are finite-dimensional. If $\xi \in \mathscr{S}(A) = \mathscr{S}^{\mathrm{N}}(A)$ is a fixed state, then*

1) $\mathscr{C}_A^{\mathrm{N},2}(\xi) = \mathscr{C}_A(\xi) = \overline{\vartheta(\xi)}^{\|.\|_A} = \mathrm{Fix}_A(\xi) = \mathrm{Fix}_A^{\mathrm{N}}(\xi)$,

2) $\mathscr{W}^{f,\theta}_{\nabla|\mathscr{C}_A(\xi)\times\mathscr{C}_A(\xi)}$ *is finite and $\|.\|_A$-continuous.*

Proof. Get $\overline{\vartheta(\xi)}^{\|.\|_A} = \mathrm{Fix}_A(\xi)$ since $\vartheta(\xi) = \mathrm{relint}\,\mathrm{Fix}_A(\xi)$ by 1) in Proposition 3.2.49. We see Theorem 3.2.65 implies 1) and 2) as $\xi \in \mathscr{S}_{-1}^{\mathrm{N},2}(A_\xi)$ by finite-dimensionality. ☐

Example 3.2.67. Assume the setting of Example 3.1.56. Let $\{e_j\}_{j\in\mathbb{N}}$ be orthonormal eigenbasis of $D \in \mathscr{UB}(H)_h$. For all $j \in \mathbb{N}$, get $H_j = \langle e_1,\dots,e_j\rangle_{\mathbb{C}}$ and $A_j = \mathscr{A}(H_j[J])$. Note $(A_j,\tau) \cong (\otimes_{k=1}^j M_2(\mathbb{C}), 2^{-j}\otimes_{k=1}^j \mathrm{tr}_2) \cong (M_{2^j}(\mathbb{C}), 2^{-j}\mathrm{tr}_{2^j})$ in each case (cf. p.288 in [162]).

We give the C^*-isomorphism. Let $j \in \mathbb{N}$. Set

$$V_J(e_j) := a_J(e_j)a_J(e_j)^* - a_J(e_j)^* a_J(e_j). \tag{3.326}$$

For all $k \in \{1,\dots,j\}$ and $n,m \in \{1,2\}$, let E_{nm}^k be the (n,m)-unit matrix of $M_2(\mathbb{C})$ in the k-th factor of $\otimes_{k=1}^j M_2(\mathbb{C})$. We define C^*-isomorphism from $\otimes_{k=1}^j M_2(\mathbb{C})$ to A_j by setting

$$E_{nm}^k \cong \begin{cases} a_J(e_k)a_J(e_k)^* & \text{if } n = m = 1, \\ a_J(e_k)^* a_J(e_k) & \text{if } n = m = 2, \\ a_J(e_k)V_J(e_1)\cdots V_J(e_{k-1}) & \text{if } n = 1,\ m = 2, \\ a_J(e_k)^* V_J(e_1)\cdots V_J(e_{k-1}) & \text{if } n = 2,\ m = 1. \end{cases}$$

Letting $j \uparrow \infty$ provides orthonormal basis of $L^2(\mathscr{A}(H),\tau)$ as follows, moreover suited to calculate sufficient conditions for quantum Laplacian with spectral gaps. Indexed over $k \in \mathbb{N}$ and $n,m \in \{1,2\}$, set

$$\mathbf{E}_{nm}^k := \begin{cases} E_{11}^k - E_{22}^k & \text{if } n = m, \\ 2E_{nm}^k & \text{if } n \neq m. \end{cases}$$

The set of all finite products $\mathbf{E}$ using factors in $\{I, \mathbf{E}^k_{nm}\}_{k\in\mathbb{N}, n,m\in\{1,2\}}$ is orthonormal basis of $L^2(\mathscr{A}(H),\tau)$. For all $j\in\mathbb{N}$, let v_j be the eigenvalue of e_j and use Equation 3.127 in order to calculate

$$\nabla a_J(e_j) = \left.\frac{d}{dt}\right|_{t=0,\mathrm{w}} \alpha_t\big(a_J(e_j)\big) = \left.\frac{d}{dt}\right|_{t=0,\mathrm{w}} e^{-itv_j}\cdot a_J(e_j) = -iv_j\cdot a_J(e_j) \tag{3.327}$$

and therefore

$$\nabla a_J(e_j)^* = \big(\nabla a_J(e_j)\big)^* = iv_j\cdot a_J(e_j). \tag{3.328}$$

Equation 3.327 and Equation 3.328 show $\nabla V_J(e_j) = 0$ in each case. If $n = m$, then we further have $V_J(e_j) = \mathbf{E}^j_{nm}$. For all $\mathbf{E}^k_{nm}\in\mathbf{E}$, the Leibniz rule therefore implies

$$\nabla\mathbf{E}^k_{nm} = \sum_{j\in I} v_j\cdot\mathbf{E}^k_{nm} \tag{3.329}$$

for finite $I\subset\mathbb{N}$ depending on $\mathbf{E}^k_{nm}$. Since $\nabla^* = -\nabla$, we see Equation 3.329 consequently shows all eigenvalues λ of Δ have form

$$\lambda = \left|\sum_{j\in I} v_j\right|^2. \tag{3.330}$$

Assume there exists $C>0$ s.t. $v_j\in C\mathbb{Z}$ for all $j\in\mathbb{N}$. Then Equation 3.330 shows $\lambda = C^2q^2$ for $q\in\mathbb{Z}$ in each case. We have $\lambda = 0$ if and only if $q = 0$. Thus $\lambda\neq 0$ implies $|q|\geq 1$ and therefore $\lambda\geq C^2>0$, hence Δ either vanishes or has spectral gap.

3.3 Coarse graining and transport of quantum information

We consider states on tracial AF-C^*-algebras as scaling limits of uniformly conditioned spin states encoding sequences of qubits. Scaling limits arise from a coarse graining process associated to noncommutative differential structures. We view quantum optimal transport as transport of quantum information. Since energy functionals are Γ-limits w.r.t. the coarse graining process, resp. using our formalisation of the latter notion in Subsection 3.1.2, we view minimising geodesics approximated in finite dimensions as optimal transport of information encoded in scaling limits as above.

Structure. In Subsection 3.3.1, we discuss coarse graining and scaling limits. We consider states on tracial AF-C^*-algebras as scaling limits of uniformly conditioned spin states encoding sequences of qubits. In Subsection 3.3.2, we give the coarse graining process and view quantum optimal transport as transport of quantum information.

3.3.1 Information encoded in states on tracial AF-C*-algebras

The fundamental unit of quantum information is the quantum bit, or qubit [62][95]. We consider spin states encoding qubits [42] since spin qubit quantum computers [43][62] operationalise [18] spin states according to DiVicenzo's criteria [93][95]. We generalise to scaling limits of uniformly conditioned spin states encoding sequences of qubits. We show states on tracial AF-C^*-algebras encode information in such form.

We do not claim they have physical realisation in general. However, we show such states are noncommutative analogues of scaling limits arising from projective limits of Banach dual spaces. These are themselves dualisations of direct limits in the category of commutative C^*-algebras obtained by means of a coarse graining process for locally compact Hausdorff spaces. Spin states are a special case and have well-known physical realisations as spin qubits [42][43][62][93][95]. Standard reference for approaches and methods of coarse graining in the commutative setting is [122]. Standard reference for category theory is [153]. Standard reference for quantum information theory is [62].

Coarse graining and scaling limits. Following a general description of coarse graining via renormalisation group transformations (cf. pp.180-182 in [122]), we obtain scaling limits from direct limits in the category of commutative C^*-algebras by means of a coarse graining process for locally compact Hausdorff spaces. Dualisation furthermore yields projective limits of their Banach dual spaces. Examples arise from Ehrenfest coarse graining processes for continuity equations (cf. pp.117-140 in [122]). We show the AF-C^*-setting yields noncommutative analogues of scaling limits.

We review the classical case. We use Gelfand-Naimark functor defined by Gelfand duality (cf. Theorem I.3.11 in [192]). It yields natural transformation for the categories of locally compact Hausdorff spaces and commutative C^*-algebras (cf. Theorem I.4.4 in [192]). The classical case is in said commutative setting (cf. Example A.1.18). We use direct and projective limits (cf. pp.62-72 in [153]). Let X be a locally compact Hausdorff space. We view X as phase space of a physical system [122][188]. Let $\{X_j\}_{j\in\mathbb{N}}$ be locally compact Hausdorff spaces s.t. we have diagram of continuous surjective maps

$$X \twoheadrightarrow \cdots \twoheadrightarrow X_j \twoheadrightarrow \cdots \twoheadrightarrow X_1 \tag{3.331}$$

in the category of locally compact Hausdorff spaces. Assume Diagram 3.331 maps, under the Gelfand-Naimark functor, to the direct limit diagram

$$C_0(X_1) \hookrightarrow \cdots \hookrightarrow C_0(X_j) \hookrightarrow \cdots \hookrightarrow C_0(X) = \varinjlim C_0(X_j) \tag{3.332}$$

in the category of commutative C^*-algebras. Dualisation reverts arrows and therefore maps Diagram 3.332 to the projective limit diagram

$$C_0(X)^* = \varprojlim C_0(X_j)^* \twoheadrightarrow \cdots \twoheadrightarrow C_0(X_j)^* \twoheadrightarrow \cdots \twoheadrightarrow C_0(X_1)^* \tag{3.333}$$

in the category of Banach spaces.

The set of pure states on $C_0(X)$ is the set of Dirac measures on X (cf. Theorem I.3.11 and Definition I.3.12 in [192]). We view the latter as pointwise measurement of phase space X. If each $X_j = X/\sim_j$ is a quotient space for a directed set $\{\sim_j\}_{j\in\mathbb{N}}$ of equivalence relations on X in dual order, then each step in Diagram 3.331 identifies certain sets of pointwise measurements. We thereby define renormalisation group transformations and obtain a coarse graining process (cf. p.181 in [122]). Examples arise from identifying interiors of certain cells in Ehrenfest coarse graining (cf. pp.117-123 in [122]).

We see injections in Diagram 3.332 are inclusions of observables on phase space X invariant under certain pointwise measurements. Each step in Diagram 3.332 increases the set of observables s.t. more pointwise measurements are separated. Diagram 3.333 further extends Diagram 3.331 by extending it to all totally finite signed outer regular Radon measures on X (cf. Theorem 6.3.4 in [171]) with separability increasing in each step. Sets of identified, i.e. non-separated, pointwise measurements have characteristic scale, e.g. the volume of cell interiors. Letting $j\uparrow\infty$ implies these tend to zero since

$$C_0(X) = \overline{\bigcup_{j\in\mathbb{N}} C_0(X_j)}^{\|.\|_\infty}. \tag{3.334}$$

We say that elements in $C_0(X)$ and $C_0(X)^*$, as well as compatible objects or properties using the latter, are scaling limits.

We show the AF-C^*-setting yields noncommutative analogues of scaling limits. Let (A,τ) be a tracial AF-C^*-algebra. Definition 2.1.14 shows direct limit diagram

$$A_1 \hookrightarrow \cdots \hookrightarrow A_j \hookrightarrow \cdots \hookrightarrow A = \overline{A_0}^{\|.\|_A} \tag{3.335}$$

in the category of C^*-algebras. For all $j\in\mathbb{N}$, 2) in Proposition 2.1.28 shows $A_j^* \subset A^*$. We use the modified standard pairing. Dualisation maps Diagram 3.335 to the projective limit diagram

$$A^* = \overline{A_0^*}^{\|.\|_{A^*}} \twoheadrightarrow \cdots \twoheadrightarrow A_j^* \twoheadrightarrow \cdots \twoheadrightarrow A_1^* \tag{3.336}$$

in the category of Banach spaces. Following our discussion immediately above, we see elements in A and A^*, as well as compatible objects or properties using the latter, are noncommutative analogues of scaling limits. Diagram 3.336 gives the coarse graining process without rescaling or consideration for the metric geometry of quantum optimal transport distances. In Subsection 3.3.2, Diagram 3.346 extends Diagram 3.336.

We are motivated by Ehrenfest coarse graining since it provides a coarse graining process lifting kinetic equations on phase spaces to continuity equations on state spaces by cell averaging (cf. pp.123-129 in [122]). However, we neither coarse grain time nor use entropy production to control scaling limits. As such, we do not see Diagram 3.346 to be a noncommutative analogue of Ehrenfest coarse graining. The maximum entropy production principle given in Subsection 4.2.3 is, to our knowledge, unrelated.

Spin states. We view tracial C^*-algebras as algebras of observables [82][84][121][163][188][192] used in Hamiltonian formalism [35][36][82][121][163][188] for a given quantum system. The set of all propositions P on a given quantum system is a lattice of projections (cf. pp.1-11 in [163]). If P is equipped with f.s.n. weight $\omega : P \longrightarrow [0,\infty]$ [193], then the GNS-construction for weights defines a faithful unital *-representation. This yields generated W^*-algebra $W^*(P)$ (cf. Proposition A.1.34 and Definition A.1.36). All tracial W^*-algebras arise in this manner (cf. Proposition A.1.37). Let (A,τ) be a tracial C^*-algebra. We have f.s.n. trace $\tau := \omega : P(L^\infty(A,\tau)) \longrightarrow [0,\infty]$ and $L^\infty(A,\tau) = W^*(P(L^\infty(A,\tau)))$ (cf. Proposition A.1.37). It suffices to consider $A \subset L^\infty(A,\tau)$ as algebra of observables since it is a σ-weakly dense C^*-subalgebra. Altogether, we view A as algebra of observables for the quantum system described by the set of all propositions $P(L^\infty(A,\tau))$. We view $\mathscr{S}(A)$ as its set of states [163][192]. Following Remark 3.2.26, we know precomposition with quantum channels transmits change of such states.

We consider spin states encoding qubits under quantum noise. We do not specify the latter here. However, Example 4.2.37 gives the depolarising channel as canonical choice of quantum noise operator (cf. pp.378-379 in [62]). Let $n \in \mathbb{N}$. Up to scaling of density operators (cf. pp.98-105 in [62]), pure states of n qubits are given by all Hilbert space projections onto one-dimensional subspaces of $H := \otimes_{k=1}^n \mathbb{C}^2$ (cf. pp.13-17 in [62]). They generate, by construction as a subset of all propositions on a given quantum system with state vectors in H, the lattice P of all Hilbert space projections onto any subspace of H. Assume $(A,\tau) = (\otimes_{k=1}^n M_2(\mathbb{C}), 2^{-n} \otimes_{k=1}^n \mathrm{tr}_2)$. Thus $A = W^*(P)$, hence A is an algebra of observables as above. Corollary 3.2.10 implies pure states on A, i.e. the extreme points of $\mathscr{S}(A)$, are pure states of n qubits. Superposition shows $\mathscr{S}(A)$ are states of n qubits. Spin qubit quantum computer [39][42][43][94] use spin-entangled electrons [41][43] as physical realisation of $\mathscr{S}(A)$ in order to achieve scalable quantum computing according to DiVicenzo's criteria [93][95]. If initialisation prepares pure states and quantum gates are unitary operations, then 1) in Corollary 3.2.11 implies quantum computations are restricted to $\partial\mathscr{S}(A)$. This is a desired feature but does require challenging control of quantum noise in form of sufficient quantum error correction [43][62]. The latter may be relaxed to initialisation preparing mixed states while retaining an edge over classical computing [116]. We consider each $\mu \in \mathscr{S}(A)$ as spin state of n qubits under quantum noise and say that it encodes the latter. We ignore the rôle of quantum noise here.

Spin is an intrinsic property of elementary particles, e.g. electrons, in the Standard Model of particle physics [53][118][197]. Its independence from mass, in contrast to angular momentum, necessitates use of spinors [177][197][198] in the Dirac equation [195]. Together with non-spatiality as per Example 3.1.61 and Example 3.1.62, this motivates our view of quantum optimal transport as non-spatial transport of quantum information. If we obtain the latter as analogue quantum simulation [18] for sufficiently small $n \in \mathbb{N}$, then we have physical realisation of our interpretation. Noncommutative analogues of push-forward measure representations [72][156] given by precomposition with quantum channels as per Remark 4.3.11 provide an ansatz but are not known to exist. If we further obtain the classical case as analogue simulation [32][154], e.g. for fluid dynamics [24][97] but without any spatial discretisation of observables, then we suspect similarities and differences of either arise from distinct physical realisations.

Scaling limits of uniformly conditioned spin states. Note all formulations of the classical case implicitly assume pure states have vanishing support, i.e. are Dirac measures. Assuming non-atomic Radon measure, Dirac delta sequences [109][139][140] show infinitesimal length elements [67][144] imply all pure states have infinite relative entropy w.r.t. the given Radon measure. We consider a different but equally well-known idealisation by letting $n \in \mathbb{N}$ tend to infinity. We thereby allow countable infinitely many interacting quantum systems, e.g. second quantisation as per Example 3.1.62, as initial approximation for a finite but large number of interacting ones (cf. pp.3-5 in [36]). In Chapter 4, we rectify the latter for our main contributions by restricting to the domain of quantum relative entropy. We therefore generalise spin states encoding qubits to scaling limits of uniformly conditioned spin states encoding sequences of qubits.

We show states on tracial AF-C^*-algebras are of such form, i.e. we consider scaling limits of uniformly conditioned spin states encoding sequences of qubits. Let (A,τ) be a tracial AF-C^*-algebra. Remark 3.1.15 explains use of restrictions in Equation 3.337 below. For all $\mu \in \mathscr{S}(A)$, we have

$$\mu = w^*\text{-}\lim_{j\in\mathbb{N}} \mu_j = w^*\text{-}\lim_{j\in\mathbb{N}} \bar{\mu}_j. \tag{3.337}$$

Following Diagram 3.336, note Equation 3.337 lets us consider each $\mu \in \mathscr{S}(A)$ as scaling limit. We rescale in each step for a given state but not uniformly on sets of states. We do so for Diagram 3.346. Here, we show how to consider a.e. $\bar{\mu}_j \in A^*_{j,+}$ in Equation 3.337 as uniformly conditioned spin state encoding qubits. We therefore consider each $\mu \in \mathscr{S}(A)$ as scaling limit of uniformly conditioned spin states encoding a sequence of qubits.

We consider uniformly conditioned spin states encoding qubits. For all $n \in \mathbb{N}$, note Example 3.2.67 gives an isomorphism $(\otimes_{k=1}^n M_2(\mathbb{C}), 2^{-n} \otimes_{k=1}^n \mathrm{tr}_2) \cong (M_{2^n}(\mathbb{C}), 2^{-n}\,\mathrm{tr}_{2^n})$ of tracial C^*-algebras [162]. Let $j \in \mathbb{N}$. There exists minimal $q_j \in \mathbb{N}$ s.t.

$$A_j \overset{r_{A_j}}{\cong} \oplus_{l=1}^{n_j} M_{n_{j,l}}(\mathbb{C}) \subset M_{2^{q_j}}(\mathbb{C}) \cong \otimes_{k=1}^{q_j} M_2(\mathbb{C}) \tag{3.338}$$

using inclusion $\oplus_{l=1}^{n_j} M_{n_{j,l}}(\mathbb{C}) \subset M_{2^{q_j}}(\mathbb{C})$ into the upper left corner. Equation 3.338 uses Notation 2.1.15. Set

$$N := \oplus_{l=1}^{n_j} M_{n_{j,l}}(\mathbb{C}),\ M := \otimes_{k=1}^{q_j} M_2(\mathbb{C}). \tag{3.339}$$

We suppress the second C^*-isomorphism in Equation 3.338 and consider C^*-subalgebra $N \subset M$. Using the latter, 1) in Proposition B.2.13 yields noncommutative conditional expectation

$$\pi_j^{\mathrm{sp}} := \pi_N^M : M \longrightarrow N \tag{3.340}$$

from M to N. Note π_j^{sp} is unital, surjective and positivity-preserving. Moreover, we know it conditions the set of all propositions $P(M)$ on the given quantum system to a subset of propositions $P(N)$ [192].

We obtain positivity-preserving injective Banach dual

$$\pi_j^{\mathrm{sp},*} := \left(\pi_N^M\right)^* : N^* \longrightarrow M^* \tag{3.341}$$

s.t. $\pi_j^{\mathrm{sp},*}(\mathscr{S}(N)) \subset \mathscr{S}(M)$. Precomposing with π_j^{sp} restricts each $\mu \in \mathscr{S}(N)$ from M to N by conditioning $P(M)$ to $P(N)$. We consider each $\mu \in \mathscr{S}(N)$ as uniformly conditioned spin state of q_j qubits and say that it encodes the latter. The first identity in Equation 3.338 and Equation 3.341 show we have positivity-preserving injective Banach dual

$$\iota_j^{\mathrm{sp}} := \left(r_{A_j} \circ \pi_j^{\mathrm{sp}}\right)^* : A_j^* \longrightarrow M^* \tag{3.342}$$

s.t. $\iota_j^{\mathrm{sp}}(\mathscr{S}(A_j)) = \pi_j^{\mathrm{sp},*}(\mathscr{S}(N)) \subset \mathscr{S}(M)$. Precomposing with r_{A_j} transforms the set of all propositions from $P(N)$ to $P(A_j)$ by equivalent formulation of observables. We consider each $\mu \in \mathscr{S}(A_j)$ as uniformly conditioned spin state of q_j qubits and say that it encodes the latter. We furthermore consider scaling limits of uniformly conditioned spin states encoding a sequence of qubits as discussed above.

3.3.2 Transport of quantum information

We give the coarse graining process and view quantum optimal transport as transport of quantum information. The coarse graining process involves rescaling and considers the metric geometry of quantum optimal transport distances. We use compression and finite-dimensional approximation as used for classification of accessibility components in Subsection 3.2.3 for its construction. We thereby formalise compatibility with both in the coarse graining process as claimed in Subsection 2.3.3.

The coarse graining process applies to accessibility components. These have unique common fixed parts ensuring existence of scaled restriction maps. In order to respect scaling limit description of marginals and fixed parts as per Subsection 3.3.1, we only consider minimising geodesics approximated in finite dimensions as optimal transport of scaling limits of of quantum information, i.e. of uniformly conditioned spin states encoding sequences of qubits. Non-ergodicity restricts information-bearing degrees of freedom by the continuity equation. Moreover, the coarse graining process reduces the AF-C^*-setting to the finite-dimensional one s.t. ergodicity is recovered up to fixed parts by reducing to those accessibility components in the finite-dimensional setting arising from scaled restriction of the given fixed part. For this, we use classification to determine accessibility components in the finite-dimensional setting.

The coarse graining process. Diagram 3.346 extends Diagram 3.336 and gives the coarse graining process. We use compression for all its vertical chains of arrows and finite-dimensional approximation for its horizontal ones. The coarse graining process decomposes global pictures, objects and properties into sequences of local ones together with a uniformity condition ensuring convergence of limits. For details on the notions of compression and finite-dimensional approximation, we refer to Subsection 2.3.3.

Let $(\phi,\psi,\gamma,\nabla)$ be noncommutative differential structure for tracial AF-C^*-algebras (A,τ) and (B,ω) in (f,θ)-setting. The coarse graining process applies to accessibility components. These may differ yet have states with identical fixed part. The latter are unique only in that each accessibility component has exactly one. For all fixed states $\xi\in\mathscr{S}(A)$, note 3) in Proposition 3.2.34 implies $\mathscr{C}_A(\xi)\subset\mathrm{Fix}_A(\xi)$ and decomposition

$$\mathrm{Fix}_A(\xi)=\coprod_{\mathscr{C}\subset\mathrm{Fix}_A(\xi)}\mathscr{C}. \tag{3.343}$$

Definition 3.3.1. Let $\xi\in\mathscr{S}(A)$ be a fixed state. We say that $\mathscr{C}\subset(\mathscr{S}(A),\mathscr{W}^{f,\theta}_{\nabla})$ has fixed part ξ if $\mathscr{C}\subset\mathrm{Fix}_A(\xi)$.

Remark 3.3.2. If $\mathscr{C}\subset(\mathscr{S}(A),\mathscr{W}^{f,\theta}_{\nabla})$, then the above shows $\mathscr{C}$ has a unique fixed part ξ as per Definition 3.3.1. Yet ξ is only unique among all $\mu\in\mathscr{C}$. As such, we cannot exclude $\mathscr{C}\neq\mathscr{C}_A(\xi)$ unless we intersect with a suitable convex subset of states, e.g. $\mathscr{S}^{\mathrm{N},2}(A)$ as per 1) in Theorem 3.2.65. This is classification and reason for K in Diagram 3.346.

The lowest horizontal chain of arrows in Diagram 3.346 gives the coarse graining process for the following data. Let $\xi\in\mathscr{S}(A)$ be a fixed state. For all $j\in\mathbb{N}$, we know $\bar{\xi}_j\in\mathscr{S}(A_j)$ is a fixed state if and only if $\xi_j\neq 0$. If $\xi_j\neq 0$ for $j\in\mathbb{N}$, then $\xi_k\neq 0$ for all $j\leq k$ in $\mathbb{N}$. Let $j_{\min}\in\mathbb{N}$ minimal among all $j\in\mathbb{N}$ s.t. $\xi_j\neq 0$. For all $j\geq j_{\min}$ in $\mathbb{N}$ and up to rescaling as per 1) in Definition 3.1.12, note 1.3) and 3) in Proposition 3.2.34 imply

$$\mathrm{res}_j(\mathscr{F}_A(\xi))=\mathscr{F}_{A_j}\big(\bar{\xi}_j\big). \tag{3.344}$$

We rescale subsets of $\mathscr{F}_A(\xi)$ as per Equation 3.344. Let $K\subset\mathscr{S}(A)$ be a convex subset s.t. for all $j\geq j_{\min}$ in $\mathbb{N}$ and up to rescaling as per 1) in Definition 3.1.12, we have

$$\mathrm{res}_j(\mathscr{C}\cap K)=\mathscr{C}_{A_j}\big(\bar{\xi}_j\big). \tag{3.345}$$

Corollary 3.2.66, which uses Theorem 3.2.65, shows Equation 3.345 is satisfied if K equals $\mathscr{S}(A)$, $\mathscr{S}^{\mathrm{N}}(A)$, or $\mathscr{S}^{\mathrm{N},2}(A)$. Corollary 4.1.27 shows Equation 3.345 is satisfied if K is the domain of quantum relative entropy as per Definition 4.1.12. This lets us apply the coarse graining process in Chapter 4. Theorem 3.2.65 and Theorem 4.3.12 yield classification if K equals $\mathscr{S}^{\mathrm{N},2}(A)$, resp. the domain of quantum relative entropy.

However, each choice implies restriction of the coarse graining process to suitable fixed states. If K is the domain of quantum relative entropy, then our discussion in Section 4.1 yields natural interpretation. For all $\mu\in\mathscr{S}(A)$, $\mathrm{Ent}(\mu,\tau)\in[-\infty,\infty]$ is the relative entropy of μ w.r.t. τ as per Equation 4.12. Theorem 4.1.25 ensures it measures information required to discriminate μ and τ through observation by extending its use from the strongly unital finite-trace case (cf. pp.1-11 in [163]). Restriction implies we only consider normal states, fixed or not, encoding a finite amount of information.

We assume data $\xi \in \mathscr{S}(A)$ and K as above. Using canonical inclusion maps for all vertical arrows, restriction maps for all uppermost horizontal arrows, as well as scaled restriction maps as per Equation 3.344, resp. Equation 3.345 for all lower horizontal arrows, we have diagram

$$\begin{tikzcd}
A^* \arrow[r, two heads] & \cdots \arrow[r, two heads] & A_j^* \arrow[r, two heads] & \cdots \arrow[r, two heads] & A_{j_{\min}}^* \\
\mathscr{F}_A(\xi) \arrow[u, hook] \arrow[r, two heads] & \cdots \arrow[r, two heads] & \mathscr{F}_{A_j}\left(\bar{\xi}_j\right) \arrow[u, hook] \arrow[r, two heads] & \cdots \arrow[r, two heads] & \mathscr{F}_{A_{j_{\min}}}\left(\bar{\xi}_{j_{\min}}\right) \arrow[u, hook] \\
\mathscr{C} \cap K \arrow[u, hook] \arrow[r, two heads] & \cdots \arrow[r, two heads] & \mathscr{C}_{A_j}\left(\bar{\xi}_j\right) \arrow[u, hook] \arrow[r, two heads] & \cdots \arrow[r, two heads] & \mathscr{C}_{A_{j_{\min}}}\left(\bar{\xi}_{j_{\min}}\right) \arrow[u, hook]
\end{tikzcd} \tag{3.346}$$

Diagram 3.346 extends Diagram 3.336. Assuming a fixed state is necessary for having scaled restriction maps in Diagram 3.346. We use compression for each vertical chain of arrows in Diagram 3.346 and finite-dimensional approximation for each horizontal one. This demands data compatible with both. Diagram 3.346 relates a global picture given by the leftmost vertical chain of arrows to a sequence of local pictures given by vertical chains of arrows obtained as images of scaled restriction maps.

We explain our notion of *locality*. For all $j \geq j_{\min}$ in $\mathbb{N}$, note $A_j^* \subset A^*$ restricts as per Equation 3.342 to an equivalent formulation represented on a finite-dimensional model algebra of observables. We thereby restrict $\mathscr{S}(A)$ to a standard representation of $\mathscr{S}(A_j)$ by conditioned testing on direct sums of full matrix algebras. We view the latter as local pictures in direct analogy to notions of locality for pure state spaces in the commutative setting, i.e. locally compact Hausdorff spaces. Altogether, Diagram 3.346 decomposes global pictures, objects and properties into sequences of local ones together with a uniformity condition ensuring convergence of limits.

Transport of information encoded in states on tracial AF-C*-algebras. Let $(\phi, \psi, \gamma, \nabla)$ be noncommutative differential structure for tracial AF-C^*-algebras (A, τ) and (B, ω) in (f, θ)-setting. Following our discussion in Subsection 3.3.1, we consider each $\mu \in \mathscr{S}(A)$ as scaling limit of uniformly conditioned spin states encoding sequences of qubits. Note minimising geodesics do not restrict to other minimising geodesics in general. However, we expect a form of finite-dimensional approximation related to and well-behaved w.r.t. the coarse graining process, at least for marginals, if transport of quantum information arises from quantum optimal transport. We therefore consider minimising geodesics approximated in finite dimensions in order to respect scaling limit description of marginals and fixed parts as above while retaining geodicity.

For all $\mu^0, \mu^1 \in \mathscr{S}(A)$, Theorem 3.1.52 shows we have $\mathscr{W}_{\nabla}^{\log}(\mu^0, \mu^1) < \infty$ if and only if there exists $(\mu, w) \in \mathrm{Geo}(\mu^0, \mu^1)$ approximated in finite dimensions by a sequence $(\mu^j, w^j)_{j \geq m} \subset \mathrm{Geo}_0$. We moreover have

$$\left(\mu^j, w^j\right) \in \mathrm{Geo}_j\left(\bar{\mu}_j^0, \bar{\mu}_j^1\right) \tag{3.347}$$

for all $j \geq m$ and may pass to a subsequence converging to (μ, w) in $\mathrm{Adm}^{[0,1]}$ in this case. We consider each $\mu^j : [0,1] \longrightarrow \mathscr{S}(A_j)$ as optimal transport of uniformly conditioned spin states encoding qubits and therefore transport of quantum information. Corollary 3.1.49 shows convergence to (μ, w) in $\mathrm{Adm}^{[0,1]}$ yields the global picture, here itself scaling limit w.r.t. the coarse graining process, using a sequence of local pictures for transport of quantum information. Equation 3.347 shows marginals are elements in the scaling limit sequence of marginals as per Equation 3.337.

We consider each $(\mu, w) \in \mathrm{Geo}$ approximated in finite-dimensions as optimal transport of scaling limits of uniformly conditioned spin states encoding sequences of qubits. We therefore view quantum optimal transport as transport of quantum information and say that it is compatible with the coarse graining process. Thus non-ergodicity restricts information-bearing degrees of freedom by the continuity equation, as visible from 3) in Proposition 3.2.34 in general, resp. 2) in Proposition 3.2.46 and Proposition 3.2.49 upon coarse graining. Moreover, our description of transport of quantum information extends suitably to Example 3.1.64 and its generalisations.

We restrict to Example 3.1.64 and use its notation. The given state space $\mathscr{S}(A_X)$ consists of normalised Radon measures on X evaluating in $\mathscr{A}(H)$ up to C^*-isometry as per Equation 3.121. Dualising the minimal C^*-tensor product [135][192] yields

$$\mathscr{S}(A_X) \cong \left\{ \mu \in C_0(X)^* \otimes \mathscr{A}(H)^* \;\middle|\; \mu \geq 0,\ \|\mu\|_{C_0(X)^* \otimes \mathscr{A}(H)^*} = 1 \right\}, \tag{3.348}$$

where $C_0(X)^* \cong C_c(X)^*$ is the Banach space of totally finite signed Radon measures on X by σ-compactness (cf. Proposition 6.3.6 in [171]). Each gauge field $T \in X$ determines an encoding scheme of $\mathscr{A}(H)_+^*$ as per Diagram 3.346. These vary since Example 3.1.62 applied to obtain each fibre depends entirely on the given inner fluctuation D_T of D as per Equation 3.135. If $\mu : [a,b] \longrightarrow \mathscr{S}(A_X)$ is given by an admissible path s.t.

$$\mu(t) = \delta_{\rho(t)} \otimes \nu(t) \in \left(C_0(X)^* \otimes \mathscr{A}(H)^*\right)_+ \tag{3.349}$$

under the isomorphism in Equation 3.348 for a.e. $t \in [a,b]$, then $\nu(t) \in \mathscr{S}(\mathscr{A}(H))$ for a.e. $t \in [a,b]$ as well. This suppresses encoding schemes. Upon considering said path in $\mathscr{S}(A_X)$, i.e. we know $t \mapsto \nu(t) \in \mathscr{S}(\mathscr{A}(H[J_{\rho(t)}]))$ are states on varying CAR-algebras $t \mapsto \mathscr{A}(H[J_{\rho(t)}])$ as per Equation 3.152, we see minimising geodesics transporting Dirac measures are transport of quantum information under varying encoding schemes. We are therefore motivated, in direct analogy to the classical case [8][97][199] generalising from transport of point mass to transport of mass distributions, to view parametrised quantum optimal transport as transport of densities of quantum information over those encoding schemes of $\mathscr{A}(H)_+^*$ as per Diagram 3.346 parametrised by X.

4 Metric Geometry of Quantum L^2-Wasserstein Distances

The logarithmic mean setting uses the logarithmic operator mean for interpolation parameter one. This defines quantum L^2-Wasserstein distances in direct analogy to the classical case [97]. The logarithmic operator mean is characterised as the one inducing the Kubo-Mori-Bogoliubov inner product [176]. Up to coarse graining in the logarithmic mean setting, the given noncommutative chain rule ensures heat flow is gradient flow of quantum relative entropy. In our logarithmic mean setting, which does assume the AF-C^*-setting, yet neither ergodicity nor finite trace, we extend results in [48][49][50] and [106] to the general case and view lower Ricci bounds as measurement convexity of quantum information. Non-ergodicity and non-finite trace ensure fundamental example classes in Subsection 3.1.3 are covered. We summarise our contributions below.

We extend quantum relative entropy in the sense of Araki [16][17] and Umegaki [196] to the AF-C^*-setting. Note our construction ensures it measures information required to discriminate a given state and, possibly non-finite, trace through observation by extending its use from the strongly unital finite-trace case [163]. If EVI_λ-gradient flow of quantum relative entropy exist, then it is heat flow. We show claimed equivalence of EVI_λ-gradient flow, λ-convexity, Bakry-Émery and Hessian lower bound conditions by means of the coarse graining process. We then define lower Ricci bounds of quantum gradients using any one of said equivalent conditions, give sufficient conditions for lower Ricci bounds of direct sum quantum gradients and, assuming lower Ricci bounds, derive functional inequalities HWI_λ, MLSI_λ and TW_λ in the AF-C^*-setting.

We view quantum Laplacians as generators of quantum noise evolution in order to have non-spatiality of lower Ricci bounds and associated energy-information trade-offs. Following Landauer's principle [142][143] and its extension to quantum information theory [45][95], erasure of quantum information implies strictly positive production of quantum entropy. Yet it is unclear how the EVI_λ-gradient flow property selects noise diffusion terms, i.e. generators of quantum noise evolution, in our case. To this end, we formulate a maximum entropy production principle [91][92][155]. We show quantum Laplacians satisfy, up to sign, a quantum Fokker-Planck equation with vanishing drift term in scaling limit, i.e. only noise diffusion term. Altogether, we obtain a description of quantum Laplacians in terms of both quantum statistical mechanics [35][36] and quantum information theory [62] as claimed in the introduction of Chapter 3.

Structure. In Section 4.1, we discuss quantum relative entropy. We extend to, possibly non-finite, traces in the second variable. In Section 4.2, we discuss the logarithmic mean setting and quantum L^2-Wasserstein distances. Moreover, we formulate our maximum entropy production principle. In Section 4.3, we consider heat flow as EVI_λ-gradient flow of quantum relative entropy and show our equivalence theorem. We discuss non-spatial lower Ricci bounds and energy-information trade-offs parametrised by lower bounds on quantum noise, give sufficient conditions and derive functional inequalities.

4.1 Quantum relative entropy

Quantum relative entropy is an extension of relative entropy for tracial C^*-algebras to the AF-C^*-setting. We construct it by extending Kosaki's formula [163] to traces in the second variable. Relative entropy for tracial C^*-algebras is the fundamental example of quasi-entropies and therefore quantum f-divergences [125][126]. We also know it measures information required to discriminate two given states through observation [163]. Since it is given by extension of Kosaki's formula, our construction ensures quantum relative entropy likewise measures information required to discriminate a given state and, possibly non-finite, trace through observation.

In Subsection 4.3.1, we consider heat flow as EVI_λ-gradient flow of quantum relative entropy. This uses two most essential properties of quantum relative entropy. First, we show the latter is compatible with compression and finite-dimensional approximation. Secondly, we show it satisfies a suitable notion of l.s.c. in topology of the given quantum optimal transport distance. However, finite-dimensional approximation and l.s.c. do not hold for all states in general. The latter requires strong unitality and finite trace. Upon restriction to finitely supported accessibility components, i.e. having finitely supported fixed state, we satisfyingly recover the strongly unital finite-trace case depending on the given finitely supported fixed state by compressing with uniform majorants of their local support. Examples of finitely supported fixed states arise from fixed states on tracial AF-C^*-algebras generating hyperfinite factors of type I and II by σ-weak closure.

Structure. In Subsection 4.1.1, we review relative entropy for C^*-algebras expressed using Kosaki's formula. We construct quantum relative entropy by extending to traces in the second variable. In Subsection 4.1.2, we discuss uniform majorisation, finitely supported fixed states and show all properties required of quantum relative entropy.

4.1.1 Quantum relative entropy for tracial AF-C*-algebras

Theorem 5.11 in [163] states Kosaki's formula. It is a variational expression of relative entropy for normal positive bounded functionals on W^*-algebras w.r.t. each other. This determines relative entropy for W^*-algebras. We construct quantum relative entropy by two consecutive extensions of Kosaki's formula. First, we extend to positive bounded functionals on C^*-algebras by evaluating their canonical normal extensions to universal enveloping W^*-algebras. This determines relative entropy for C^*-algebras. Secondly, we extend to positive bounded functionals on tracial AF-C^*-algebras w.r.t. the given trace.

This determines quantum relative entropy. Lemma 4.1.17 recovers Kosaki's formula as per Theorem 5.11 in [163] for normal positive bounded functionals with integrable support. Standard reference for Kosaki's formula, as well as relative entropy for C^*- and W^*-algebras alike, is [163]. We refer to pp.35-36 and pp.98-99 in [163] for a review.

Relative entropy for tracial C^*-algebras. Umegaki defined relative entropy for semi-finite W^*-algebras [196]. Using relative modular operators, Araki generalised to all W^*-algebras [16][17]. Equation 4.1 is Kosaki's formula as per Theorem 5.11 in [163]. Using universal enveloping W^*-algebras in Kosaki's formula, we engage in our first extension by adapting constructions in [163] but with additional detail required for our second one. Assuming tracial C^*-algebra, Lemma 4.1.8 shows Kosaki's formula uses spaces of bounded measurable operators. Proposition 4.1.6 and Proposition 4.1.9 collect properties. We consider two instructive examples here. Example 4.1.10 gives the finite-dimensional setting. Example 4.1.11 shows necessity of strong unitality.

Let (M,τ) be a tracial W^*-algebra and $A\subset M$ a σ-weakly dense C^*-subalgebra. Ergo $M=L^\infty(A,\tau)$ and $M_*=L^1(A,\tau)$. Following Remark 2.1.2, we have $L^1(A,\tau)^\flat\subset A^*$ as partially ordered Banach spaces.

Definition 4.1.1. Let $V\subset L^\infty(A,\tau)$ be a linear subspace s.t. $1_A\in V$. Let $n\in\mathbb{N}$.

1) Let $\mathscr{T}_n(V)$ be the set of all step functions $F:(n^{-1},\infty)\longrightarrow V$ s.t. $|\mathrm{im}F|<\infty$. Using the constant map $t\mapsto 1_M=1_A$ on (n^{-1},∞), set $F^\perp:=1_A-F$ for all $F\in\mathscr{T}_n(V)$.

2) $\mathscr{T}_n^u(V):=\{F\in\mathscr{T}_n(V)\mid \exists t\in(n^{-1},\infty)\ \forall s\geq t:\ F(s)=1_A\}$.

Definition 4.1.2 gives the relative entropy $\mathrm{Ent}:L^1(A,\tau)^\flat_+\times L^1(A,\tau)^\flat_+\longrightarrow(-\infty,\infty]$. Equation 4.1 is Kosaki's formula which we extend to variational expressions using positive bounded functionals on C^*-algebras, and w.r.t. traces in the second variable. We call extensions relative entropy, resp. Kosaki's formula as well. All extensions coincide on intersections of domains. For all $\mu,\eta\in L^1(A,\tau)^\flat_+$, note $\mathrm{Ent}(\mu,\eta)$ measures information required to discriminate μ and η through observation (cf. pp.1-11 in [163]). As expected in the commutative setting, Umegaki's definition shows Kosaki's formula yields relative entropy of probability densities, i.e. Kullback-Leibler divergence (cf. pp.35-36 in [163]). Theorem 4.1.25 extends the above notion of discriminating information.

Definition 4.1.2. For all $\mu\in L^1(A,\tau)^\flat_+$, set $\|x\|_\mu:=\sqrt{\mu(x^*x)}$ for all $x\in L^\infty(A,\tau)$. For all $\mu,\eta\in L^1(A,\tau)^\flat_+$, the relative entropy of μ w.r.t. η is defined by

$$\mathrm{Ent}(\mu,\eta):=\sup_{\substack{n\in\mathbb{N},\\ F\in\mathscr{T}_n(L^\infty(A,\tau))}}\left\{\|\mu\|_{A^*}\log n-\int_{n^{-1}}^{\infty}t^{-1}\left\|F^\perp(t)\right\|_\mu^2+t^{-2}\left\|F(t)^*\right\|_\eta^2\,dt\right\}. \tag{4.1}$$

Remark 4.1.3. Let $V\subset L^\infty(A,\tau)$ be a strong*-dense linear subspace s.t. $1_A\in V$. Then Theorem 5.11 in [163] shows we may replace the supremum over all $\mathscr{T}_n(L^\infty(A,\tau))$ with the one over all $\mathscr{T}_n(V)$ in Kosaki's formula, hence the one over all $\mathscr{T}_n^u(V)$. We use this throughout our discussion.

We review properties of relative entropy for W^*-algebras. We take the supremum over all $\mathscr{T}^u_n(L^\infty(A,\tau))$ in Kosaki's formula and apply Fatou's lemma. Kosaki's formula therefore shows the relative entropy is jointly convex and l.s.c. in w^*-topology given by $L^\infty(A,\tau) = L^1(A,\tau)^*$. Let $\mu,\eta \in L^1(A,\tau)^\flat_+$. If $\mu,\eta \neq 0$, then Proposition 5.1 in [163] shows

$$\operatorname{Ent}(\mu,\eta) \geq \left(\log\|\mu\|_{A^*} - \log\|\eta\|_{A^*}\right) \cdot \|\mu\|_{A^*} > -\infty \tag{4.2}$$

as $\|\mu\|_{A^*}, \|\eta\|_{A^*} \in (0,\infty)$. Kosaki's formula further implies $\operatorname{Ent}(0,\eta) = 0$ and $\operatorname{Ent}(\mu,0) = \infty$ in general (cf. proof of Proposition 4.1.6). If $N \subset L^\infty(A,\tau)$ is a unital W^*-subalgebra, then Corollary 5.12 in [163] shows we have restriction

$$\operatorname{Ent}(\mu,\eta) \geq \operatorname{Ent}\left(\mu|_N, \eta|_N\right) \tag{4.3}$$

since unital W^*-algebra inclusions are normal unital Schwarz maps. Altogether, we know $\operatorname{Ent} : L^1(A,\tau)^\flat_+ \times L^1(A,\tau)^\flat_+ \longrightarrow (-\infty,\infty]$ is jointly convex, l.s.c. in w^*-topology of $L^\infty(A,\tau)$ and has restriction property as per Equation 4.3. Moreover, we may replace suprema in Kosaki's formula as per Remark 4.1.3.

If (A,τ) is a strongly unital AF-C^*-algebra with finite trace, then the relative entropy satisfies the following consequence of the martingale property (cf. iv) in Corollary 5.12 in [163]). For all $\mu \in L^1(A,\tau)^\flat_+$, we have finite-dimensional approximation

$$\operatorname{Ent}(\mu,\tau) = \lim_{j \in \mathbb{N}} \operatorname{Ent}\left(\mu_j, \tau_j\right). \tag{4.4}$$

The martingale property requires l.s.c. in w^*-topology of $L^\infty(A,\tau)$ and Equation 4.3 for each generating C^*-subalgebra. If we extend to, possibly non-finite, traces in the second variable, then either may fail. Following Remark 4.1.14, l.s.c. in w^*-topology of $L^\infty(A,\tau)$ fails in general if the trace is non-finite and the relative entropy takes negative infinity as value. Example 4.1.11 shows Equation 4.3 may fail if (A,τ) is not strongly unital. Uniform majorisation of local support suffices to prevent failure and recover finite-dimensional approximation property as per Equation 4.4. Theorem 4.1.29 shows the latter on finitely supported accessibility components.

Definition 4.1.4 extends Definition 4.1.2 to $\operatorname{Ent} : A^*_+ \times A^*_+ \longrightarrow (-\infty,\infty]$. We require the following. We have separable Hilbert space H_U, universal faithful *-representation $\pi_U : A \longrightarrow \mathscr{B}(H_U)$, and universal enveloping W^*-algebra $U(A) := \pi_U(A)''$ of A [192]. For all $\mu \in A^*_+$, get unique $U(\mu) \in U(A)_{*,+} \subset U(A)^*_+$ s.t. $U(\mu)|_A = \mu$. These are called canonical normal extensions. Note $\|U(\mu)\|_{U(A)^*} = \|\mu\|_{A^*}$ in each case by construction.

Definition 4.1.4. For all $\mu,\eta \in A^*_+$, the relative entropy of μ w.r.t. η is defined by

$$\operatorname{Ent}(\mu,\eta) := \sup_{\substack{n \in \mathbb{N}, \\ F \in \mathscr{T}_n(U(A))}} \left\{ \|\mu\|_{A^*} \log n - \int_{n^{-1}}^{\infty} t^{-1} \left\|F^\perp(t)\right\|^2_{U(\mu)} + t^{-2} \left\|F(t)^*\right\|^2_{U(\eta)} \, dt \right\}. \tag{4.5}$$

Remark 4.1.5. Note Definition 4.1.2, as well as those properties of relative entropy for W^*-algebras given above, do not require traciality. Definition 4.1.4 therefore gives relative entropy for $U(A)$. We use this throughout our discussion.

Proposition 4.1.6. *For all $\mu,\eta \in A^*_+$ and $a,b>0$ in $\mathbb{R}$, we have*

1) $\mathrm{Ent}(a\mu,b\eta) = a\,\mathrm{Ent}(\mu,\eta) + a\big(\log a - \log b\big)\cdot \|\mu\|_{A^*}$,

2) $\mathrm{Ent}(\mu,\eta) \geq \big(\log\|\mu\|_{A^*} - \log\|\eta\|_{A^*}\big)\cdot\|\mu\|_{A^*}$ *if* $\eta \neq 0$,

3) $\mathrm{Ent}(0,\eta) = 0$ *and* $\mathrm{Ent}(\mu,0)=\infty$ *if* $\mu\neq 0$,

4) $\mathrm{Ent}(\mu,\eta) > -\infty$.

Proof. Let $\mu,\eta \in A^*_+$ and $a,b>0$ in $\mathbb{R}$. Proposition 5.1 in [163] is 1) and 2). Kosaki's formula implies $\mathrm{Ent}(0,\eta)=0$ by selecting $F=0$ for all $n\in\mathbb{N}$ in order to estimate the supremum. If $\mu\neq 0$, then Kosaki's formula likewise implies $\mathrm{Ent}(\mu,0)=\infty$ by selecting $F=1_{U(A)}$ in each case. Get 3). We see 2) and 3) imply 4) at once. □

We have σ-weakly dense C^*-subalgebras $A\subset U(A)$ and $A\subset L^\infty(A,\tau)$. Universal property implies there exists unique normal *-homomorphism $\varphi : U(A) \longrightarrow L^\infty(A,\tau)$ s.t. $\varphi\circ\pi_U = \mathrm{id}_A$. It is unital and surjective, further mapping the unit ball in $U(A)$ to the one in $L^\infty(A,\tau)$ as per Remark 4.1.7. We define normal trace $U(\tau)$ on $U(A)$ by setting

$$U(\tau)(x) := \tau\big(\varphi(x)\big) \tag{4.6}$$

for all $x\in U(A)_+$. We neither claim nor use semi-finiteness.

Remark 4.1.7. Since $\varphi|_A = \mathrm{id}_A$, the Kaplansky density theorem shows φ maps the unit ball in $U(A)$ to the one in $L^\infty(A,\tau)$ (cf. Theorem 5.3.5 in [134]). Thus φ is surjective. It is unital by normality and Proposition 2.1.16.

Lemma 4.1.8 ensures Definition 4.1.4 is well-behaved w.r.t. normality. We use the following. For all *-subalgebras of W^*-algebras, closure in strong and weak topology are equivalent. Such closures are equivalent to closure w.r.t. bounded strong, as well as bounded weak convergence (cf. Proposition A.1.38). Note (σ-)weak- and w^*-convergence coincide on bounded sets (cf. Lemma II.2.5 in [192] and Proposition A.1.34). Bounded sets in tracial W^*-algebras are compact in w^*-topology, ergo weakly compact.

Lemma 4.1.8. *For all $\mu\in A^*_+$, the following are equivalent:*

1) *There exists unique normal extension of μ to $L^\infty(A,\tau)$ s.t. $U(\mu)=\mu\circ\varphi$.*

2) *For all projections $p\in U(A)$, $U(\mu)(p)=0$ if $U(\tau)(p)=0$.*

3) $\ker\varphi \subset \ker U(\mu)$.

4) $\mu\in L^1(A,\tau)^\flat_+$.

Proof. Note Remark 4.1.7. Let $\mu \in A_+^*$. For all projections $p \in U(A)$, faithfulness of τ implies $U(\tau)(p) = 0$ if and only if $\varphi(p) = 0$. Thus 3) implies 2). We know $U(\mu)$ and φ are completely positive normal maps (cf. Example A.1.46 and Example A.1.47), and therefore bounded weakly continuous by normality (cf. Proposition A.1.49). Ergo $\ker\varphi$ is a W^*-subalgebra. Note W^*-algebras are bounded weakly generated by their projections (cf. Proposition A.1.37). Hence 2) implies 3). Altogether, get equivalence of 2) and 3).

Clearly, 1) implies 2). Assume $\ker\varphi \subset \ker U(\mu)$. For all $x \in L^\infty(A,\tau)$, get $\varphi^{-1}(x) \neq \emptyset$ by surjectivity and set $\mu(x) := U(\mu)(y)$ for fixed but arbitrary $y \in \varphi^{-1}(x)$. This is independent of our choice as 3) ensures $\ker\varphi \subset \ker U(\mu)$. We thereby define a positivity-preserving linear map $\mu : L^\infty(A,\tau) \longrightarrow \mathbb{C}$ s.t. $U(\mu) = \mu \circ \varphi$. Thus $\|\mu\|_{L^\infty(A,\tau)^*} = \|U(\mu)\|_{U(A)^*} = \|\mu\|_{A^*}$ since φ is unital, hence we have extension $\mu \in L^\infty(A,\tau)_+^*$. If $x = \text{bdw-lim}_{k\in K} x_k$ implies $\lim_{k\in K} |\mu(x - x_k)| = 0$ for all nets $\{x_k\}_{k\in K} \subset L^\infty(A,\tau)$, then complete positivity of μ shows its normality (cf. Example A.1.46 and Proposition A.1.49). Let $x = \text{bdw-lim}_{k\in K} x_k$. By considering all accumulation points of $\{\mu(x_k)\}_{k\in K} \subset \mathbb{R}$ and showing they are in fact zero as claimed above, we assume $\lim_{k\in K} |\mu(x - x_k)|$ exists without loss of generality.

Since φ is surjective on unit balls, we have both weakly convergent bounded subnet $\{x_k\}_{k\in K} \subset L^\infty(A,\tau)$ and weakly convergent bounded net $\{y_k\}_{k\in K} \subset U(A)$ s.t. $x_k = \varphi(y_k)$ for all $k \in K$. Set $y := \text{bdw-lim}_{k\in K} y_k$. Get $x = \varphi(y)$ by normality of φ. Thus $\lim_{k\in K} \mu(x - x_k) = \lim_{k\in K} U(\mu)(y - y_k) = 0$ by normality of $U(\mu)$, hence $\mu \in L^\infty(A,\tau)_+^*$ is normal as discussed above and therefore a unique extension as required. Ergo 1) implies 2). Altogether, get equivalence of 1) and 2). Note Remark 2.1.2. In particular, $L^1(A,\tau)_+^\flat \subset A_+^*$ is determined by normality. Thus 1) implies 4). Assume $\mu \in L^1(A,\tau)_+^\flat$. We obtain $U(\mu)|_A = \mu \circ \varphi|_A$ by construction. Normality of μ and φ extends the latter identity to $U(A)$. Hence 4) implies 1). Altogether, get equivalence of 1) and 4). All statements are equivalent. □

Proposition 4.1.9 collects further properties. Lemma 4.1.8 implies Equation 4.9 and therefore Equation 4.7. Example 4.1.10 gives the finite-dimensional setting. Quantum entropy is negative quantum relative entropy. Example 4.1.11 shows Equation 4.3 may fail in the finite-dimensional setting if (A,τ) is not strongly unital.

Proposition 4.1.9. $\mathrm{Ent} : L^1(A,\tau)_+^\flat \times L^1(A,\tau)_+^\flat \longrightarrow (-\infty,\infty]$ *is jointly convex and l.s.c. in w^*-topology of $A[1_A]^*$. Furthermore,* Ent *satisfies the following.*

1) For all $\mu,\eta \in L^1(A,\tau)_+^\flat$, we have

$$\mathrm{Ent}(\mu,\eta) = \sup_{\substack{n\in\mathbb{N},\\ F\in\mathcal{T}_n^u(A[1_A])}} \left\{ \|\mu\|_{A^*} \log n - \int_{n^{-1}}^{\infty} t^{-1} \left\|F^\perp(t)\right\|_\mu^2 + t^{-2} \left\|F(t)^*\right\|_\eta^2 \, dt \right\}. \tag{4.7}$$

2) Let $N \subset L^\infty(A,\tau)$ be a unital W^*-subalgebra. For all $\mu,\eta \in L^1(A,\tau)_+^\flat$, we have

$$\mathrm{Ent}(\mu,\eta) \geq \mathrm{Ent}\big(\mu|_N, \eta|_N\big). \tag{4.8}$$

Proof. Note 4) in Proposition 4.1.6 shows Ent $> -\infty$ on norm bounded sets. Kosaki's formula implies Ent is jointly convex. Let $\mu,\eta \in L^1(A,\tau)^\flat_+$. We know $U(\mu) = \mu \circ \varphi$ and $U(\eta) = \eta \circ \varphi$ by Lemma 4.1.8. Since φ is a unital surjective *-homomorphism, mapping $\mathscr{T}_n^u(U(A))$ to $\mathscr{T}_n^u(L^\infty(A,\tau))$ via $F \mapsto G := \varphi \circ F$ for all $n \in \mathbb{N}$ shows

$$\mathrm{Ent}(\mu,\eta) = \sup_{\substack{n\in\mathbb{N},\\ F\in\mathscr{T}_n^u(L^\infty(A,\tau))}} \left\{ \|\mu\|_{A^*} \log n - \int_{n^{-1}}^{\infty} t^{-1}\left\|G^\perp(t)\right\|_\mu^2 + t^{-2}\left\|G(t)^*\right\|_\eta^2 \, dt \right\}. \tag{4.9}$$

Since $\mu,\eta \in L^\infty(A,\tau)^*$, Equation 4.9 shows $\mathrm{Ent}(\mu,\eta)$ is the relative entropy of μ w.r.t. η as per Definition 4.1.2. Get 1) by replacing $L^\infty(A,\tau)$ with $A[1_A]$ in the second suprema of the equation. Applying Fatou's lemma to Equation 4.7 then shows l.s.c. in w^*-topology of $A[1_A]^*$. Equation 4.3 and Equation 4.9 show 2) immediately. □

Example 4.1.10. Assume (A,τ) is finite-dimensional. Following Proposition 2.1.24, we moreover assume $(A,\tau) = (M_n(\mathbb{C}), \mathrm{tr}_n)$ for $n \in \mathbb{N}$ without loss of generality. The general finite-dimensional case is therefore given by a weighted sum of terms having following form up to pull-back along C^*-isometries as per Equation 2.7. For all $\mu,\eta \in M_n(\mathbb{C})^*_+$, the relative entropy of μ w.r.t. η is given by

$$\mathrm{Ent}(\mu,\eta) = \begin{cases} 0 & \text{if } \mu = 0, \\ \mathrm{tr}_n\left(\sharp\mu \cdot \left(\log\sharp\mu - \log\sharp\eta\right)\right) & \text{if } \mu \neq 0 \text{ and } \operatorname{supp}\mu \leq \operatorname{supp}\eta, \\ \infty & \text{else}. \end{cases}$$

The above characterisation is Umegaki's definition, except we make vanishing for $\mu = 0$ explicit. It generalises to Araki's definition (cf. p.77 in [163]), which in turn coincides with Kosaki's formula by Theorem 5.11 in [163]. The negative of Umegaki's definition is quantum entropy, i.e. von Neumann entropy (cf. p.17 in [163]). Corollary 4.1.27 extends such description to the general case.

Example 4.1.11. Assume $(A,\tau) = (M_n(\mathbb{C}), \mathrm{tr})$ for $n \geq 2$ in $\mathbb{N}$. Note $M_{n-1}(\mathbb{C}) \subset M_n(\mathbb{C})$ is non-unital. For all $k \in \{1,\dots,n\}$, let $\lambda_k \in (0,1)$. Following Example 4.1.10, the diagonal matrix $D := (\lambda_1,\dots,\lambda_n) \in M_n(\mathbb{C})_+$ yields quantum relative entropy

$$\mathrm{Ent}\left(D^\flat, I_n^\flat\right) = \sum_{k=1}^{n} \lambda_k \log \lambda_k. \tag{4.10}$$

We know $\left(D|_{M_{n-1}(\mathbb{C})}\right)^\flat = (\lambda_1,\dots,\lambda_{n-1})^\flat \in M_{n-1}(\mathbb{C})_+$ and $\left(I|_{M_{n-1}(\mathbb{C})}\right)^\flat = I_{n-1}^\flat$. Moreover, we have $\lambda_n \log \lambda_n < 0$ by hypothesis. Equation 4.10 lets us estimate

$$\mathrm{Ent}\left(D^\flat, I_n^\flat\right) = \mathrm{Ent}\left(\left(D|_{M_{n-1}(\mathbb{C})}\right)^\flat, I_{n-1}^\flat\right) + \lambda_n \log \lambda_n < \mathrm{Ent}\left(\left(D|_{M_{n-1}(\mathbb{C})}\right)^\flat, I_{n-1}^\flat\right) < \infty. \tag{4.11}$$

Equation 4.11 shows Equation 4.3 fails since $M_{n-1}(\mathbb{C}) \subset M_n(\mathbb{C})$ is non-unital.

Extending to traces in the second variable. Note Equation 4.5 does not let us take relative entropy w.r.t. non-finite traces. We extend accordingly. Let (A,τ) be a tracial AF-C^*-algebra. Definition 4.1.12 gives the relative entropy $\mathrm{Ent}^\tau : A^*_+ \longrightarrow [-\infty,\infty]$ w.r.t. τ, i.e. quantum relative entropy. Proposition 4.1.9 shows Lemma 4.1.17 recovers Equation 4.1 for normal positive bounded functionals with integrable support.

Definition 4.1.12. Set extended trace norm $\|x\|_{U(\tau)} := \sqrt{U(\tau)(x^*x)}$ for all $x \in U(A)$. For all $\mu \in A^*_+$, the relative entropy of μ w.r.t. τ is defined by

$$\mathrm{Ent}(\mu,\tau) := \sup_{\substack{n\in\mathbb{N},\\ F\in\mathscr{T}_n(U(A))}} \left\{ \|\mu\|_{A^*}\log n - \int_{n^{-1}}^{\infty} t^{-1}\left\|F^\perp(t)\right\|^2_{U(\mu)} + t^{-2}\left\|F(t)\right\|^2_{U(\tau)} dt \right\}. \tag{4.12}$$

Set $\mathrm{Ent}^\tau := \mathrm{Ent}(.,\tau) : A^*_+ \longrightarrow [-\infty,\infty]$ and $\mathrm{dom}\,\mathrm{Ent}^\tau := \{\mu \in A^*_+ \mid |\mathrm{Ent}(\mu,\tau)| < \infty\}$. We call Ent^τ quantum relative entropy w.r.t. τ, or quantum relative entropy.

Notation 4.1.13. For all $j \in \mathbb{N}$ and $\mu \in A^*_{j,+}$, let $\mathrm{Ent}(\mu,\tau_j)$ denote the relative entropy of μ w.r.t. $\tau_j = \tau|_{A_j}$ for the tracial AF-C^*-algebra $(A_j,\tau) = (A_j,\tau_j)$ as per Definition 2.1.22.

Remark 4.1.14. For all $n \in \mathbb{N}$ and $F \in \mathscr{T}_n(U(A))$, traciality implies

$$\left\|F(t)\right\|^2_{U(\tau)} = U(\tau)\big(F(t)^*F(t)\big) = U(\tau)\big(F(t)F(t)^*\big). \tag{4.13}$$

Note $\|F(t)\|^2_{U(\eta)} = U(\eta)(F(t)F(t)^*)$ in Equation 4.5. Compare to Equation 4.13, i.e. use of extended trace norm, in Equation 4.12. If $\tau < \infty$, then Equation 4.6 shows Equation 4.12 is Equation 4.5 using $\eta = \tau$. If τ is non-finite, then its joint convexity implies Ent^τ is not l.s.c. in w^*-topology of A^* on weakly closed convex $K \subset A^*_+$ for which there exists $\mu \in K$ s.t. $\mathrm{Ent}(\mu,\tau) = -\infty$. We argue as for Example 4.4 in [189].

For all $j \in \mathbb{N}$ and $\mu \in A^*_{j,+}$, using quantum relative entropy for A_j yields

$$\mathrm{Ent}\big(\mu,\tau_j\big) = \sup_{\substack{n\in\mathbb{N},\\ F\in\mathscr{T}^u_n(A_j)}} \left\{ \|\mu\|_{A^*_j}\log n - \int_{n^{-1}}^{\infty} t^{-1}\left\|F^\perp(t)\right\|^2_{\mu} + t^{-2}\left\|F(t)\right\|^2_{\tau} dt \right\}. \tag{4.14}$$

For all $\mu \in A^*_+$, we expect $\mathrm{Ent}(\mu,\tau) = \lim_{j\in\mathbb{N}} \mathrm{Ent}(\mu_j,\tau_j)$ if we indeed measure information required to discriminate μ and τ through observation. Theorem 4.1.25 shows this given uniform majorant of local support. The latter uses Lemma 4.1.16 and Lemma 4.1.17.

Proposition 4.1.15. *If $\mu \in L^1(A,\tau)^\flat_+$ s.t. $\mathrm{Ent}(\mu,\tau) > -\infty$, then we have*

$$\mathrm{Ent}(\mu,\tau) = \sup_{\substack{n\in\mathbb{N},\\ F\in\mathscr{T}^u_n(L^{2,\infty}(A,\tau))}} \left\{ \|\mu\|_{A^*}\log n - \int_{n^{-1}}^{\infty} t^{-1}\left\|F^\perp(t)\right\|^2_{\mu} + t^{-2}\left\|F(t)\right\|^2_{\tau} dt \right\}. \tag{4.15}$$

Proof. Let $\mu \in L^1(A,\tau)^\flat_+$ s.t. $\mathrm{Ent}(\mu,\tau) > -\infty$. As for 1) in Proposition 4.1.9, normality lets us drop φ in Kosaki's formula while taking the supremum over all $\mathscr{T}^u_n(L^\infty(A,\tau))$.

Let $n \in \mathbb{N}$ and $F \in \mathcal{T}_n^u(L^\infty(A,\tau))$. If there exists $t_0 \in (n^{-1},\infty)$ s.t. $F(t_0) \notin L^2(A,\tau)$, then F being a step function implies

$$\int_{n^{-1}}^{\infty} t^{-1}\|F^\perp(t)\|_\mu^2 + t^{-2}\|F(t)\|_\tau^2 dt \geq \int_{n^{-1}}^{\infty} t^{-2}\|F(t)\|_\tau^2 dt = \infty. \tag{4.16}$$

Thus Equation 4.16 and $\operatorname{Ent}(\mu,\tau) > -\infty$ imply we may restrict to $L^{2,\infty}(A,\tau)$, hence we see Equation 4.6 shows Equation 4.15. □

Lemma 4.1.16. *For all $j \in \mathbb{N}$ and $\mu \in A_{j,+}^*$, we have* $\operatorname{Ent}(\mu,\tau) = \operatorname{Ent}(\mu,\tau_j)$.

Proof. Let $j \in \mathbb{N}$ and $\mu \in A_{j,+}^*$. Note $\|\mu\|_{A_j^*} = \|\mu\|_{A^*}$. For all $x \in A_j$, we further know

$$\|1_{A_j} - x\|_\mu = \|1_A - x\|_\mu. \tag{4.17}$$

Lemma 4.1.8 shows $U(\mu) = \mu \circ \varphi$. Since φ is a unital surjective *-homomorphism, we map $\mathcal{T}_n^u(A_j)$ to $\mathcal{T}_n^u(U(A))$ via $F \mapsto G := \varphi^{-1} \circ F$ for all $n \in \mathbb{N}$ by choosing pre-images in each case. Equation 4.17 and unitality show

$$\|1_{A_j} - F(t)\|_\mu = \|1_A - F(t)\|_\mu = \|G^\perp(t)\|_{U(\mu)} \tag{4.18}$$

in each case. Equation 4.18 implies $\operatorname{Ent}(\mu,\tau_j) \leq \operatorname{Ent}(\mu,\tau)$ by Kosaki's formula. We show the converse. Since $\operatorname{Ent}(\mu,\tau_j) > -\infty$ by 4) in Proposition 4.1.6, Proposition 4.1.15 ensures we may use Equation 4.15 as Kosaki's formula. For all $n \in \mathbb{N}$ and $F \in \mathcal{T}_n^u(L^{2,\infty}(A,\tau))$, set $F_j(t) := F(t)_j$ for all $t \in (n^{-1},\infty)$. Note $F_j \in \mathcal{T}_n^u(A_j)$ in each case.

Let $n \in \mathbb{N}$, $F \in \mathcal{T}_n^u(L^{2,\infty}(A,\tau))$ and $t \in (n^{-1},\infty)$. Then $F_j(t) = \pi_j^A(F(t))$ and $I - F_j(t) \in A_j^\perp$ by square integrability. We therefore have

$$\|F(t)\|_\tau^2 \geq \|\pi_j^A(F(t))\|_\tau^2 = \|F_j(t)\|_\tau^2. \tag{4.19}$$

We use $A_jA_j^\perp = A_j^\perp A_j = 0$. Proposition 2.1.28 implies restriction maps commute with adjoining as they are positivity-preserving (cf. Proposition A.1.6). We calculate

$$\|F(t)\|_\mu^2 = \|F_j(t)\|_\mu^2 + \|(I - \pi_j^A)(F_j(t))\|_\mu^2 \geq \|F_j(t)\|_\mu^2. \tag{4.20}$$

Since $\mu \in A_j$ implies $\mu(u) = \mu(u_j)$ for all $u \in L^2(A,\tau)$, multiplying out terms yields

$$\|1_A - F(t)\|_\mu^2 = \mu(1_{A_j}) - \mu(F_j(t)^*) - \mu(F_j(t)) + \|F(t)\|_\mu^2. \tag{4.21}$$

Note Equation 4.20 lets us estimate the final summand in Equation 4.21. We moreover collect terms on the right-hand side of the resulting estimate. In summary, we obtain

$$\|F^\perp(t)\|_\mu^2 = \|1_A - F(t)\|_\mu^2 \geq \|1_{A_j} - F_j(t)\|_\mu^2 = \|F_j^\perp(t)\|_\mu^2. \tag{4.22}$$

For all $n \in \mathbb{N}$ and $F \in \mathscr{T}_n^u(L^{2,\infty}(A,\tau))$, applying Equation 4.19 and Equation 4.22 to integrands on the left-hand side below lets us estimate

$$\int_{n^{-1}}^{\infty} t^{-1}\left\|F^{\perp}(t)\right\|_{\mu}^{2} + t^{-2}\left\|F(t)\right\|_{\tau}^{2} dt \geq \int_{n^{-1}}^{\infty} t^{-1}\left\|F_j^{\perp}(t)\right\|_{\mu}^{2} + t^{-2}\left\|F_j(t)\right\|_{\tau}^{2} dt. \tag{4.23}$$

Using Equation 4.14, resp. Equation 4.15 as Kosaki's formula, Equation 4.23 lets us estimate $\operatorname{Ent}(\mu,\tau_j) \geq \operatorname{Ent}(\mu,\tau)$. Altogether, get $\operatorname{Ent}(\mu,\tau) = \operatorname{Ent}(\mu,\tau_j)$. □

Lemma 4.1.17. *Let $\mu \in A_+^*$.*

1) *If $\mu \notin L^1(A,\tau)_+^\flat$, then $\mu \notin \operatorname{dom}\operatorname{Ent}^\tau$.*
2) *If $\mu \in L^1(A,\tau)_+^\flat$ and $p \in L^{1,\infty}(A,\tau)$ is a projection s.t.* $\operatorname{supp}\mu \leq p$*, then* $\operatorname{Ent}(\mu,\tau) > -\infty$ *and we have*

$$\operatorname{Ent}(\mu,\tau) = \sup_{\substack{n\in\mathbb{N},\\ F\in\mathscr{T}_n^u(A[p])}} \left\{\|\mu\|_{A[p]^*}\log n - \int_{n^{-1}}^{\infty} t^{-1}\left\|F^{\perp}(t)\right\|_{\mu}^{2} + t^{-2}\left\|F(t)\right\|_{\tau}^{2} dt\right\}. \tag{4.24}$$

Proof. We show 1). Assume $\mu \notin L^1(A,\tau)_+^\flat$. If $\operatorname{Ent}(\mu,\tau) = -\infty$, then our claim follows at once. We assume $\operatorname{Ent}(\mu,\tau) > -\infty$ without loss of generality. Proposition 4.1.15 ensures we may use Equation 4.15 as Kosaki's formula. Using Equation 4.15, each step function is constant for sufficiently large $t > 0$ and maps to $L^{2,\infty}(A,\tau)$. Since $\operatorname{Ent}(\mu,\tau) > -\infty$, there exist $x \in U(A)$ s.t. $\|1_A - x\|_{U(\mu)} = 0$ and $\varphi(x) \in L^{2,\infty}(A,\tau)$. Since $\mu \notin L^1(A,\tau)_+^\flat$, Lemma 4.1.8 yields projection $p \in U(A)$ s.t. $U(\tau)(p) = 0$ and $U(\mu)(p) > 0$ holds, and Lemma 2.1.6 shows $\|\mu\|_{A[p]^*} = \|\mu\|_{A^*}$. Let $C > 0$ s.t. $2CU(\mu)(p) > \|\mu\|_{A^*}$.

We require suitable sequence to estimate. For all $n \in \mathbb{N}$, set

$$F_n(t) := \begin{cases} Cp & \text{if } t \in (n^{-1}, n), \\ x & t \geq n. \end{cases}$$

Note $F_n \in \mathscr{T}_n^u(U(A))$ in each case. Selecting F_n for all $n \in \mathbb{N}$, we estimate

$$\operatorname{Ent}(\mu,\tau) \geq \sup_{n\in\mathbb{N}}\left\{\|\mu\|_{A^*}\log n - U(\mu)\big(1_{U(A)} - Cp\big)\int_{n^{-1}}^{n} t^{-1}dt - \left\|\varphi(x)\right\|_{\tau}^{2}\int_{n}^{\infty} t^{-2}dt\right\}. \tag{4.25}$$

Since $\int_{n^{-1}}^{n} t^{-1}dt = 2\log n$ and $\int_{n}^{\infty} t^{-2}dt = n^{-1}$ for all $n \in \mathbb{N}$, Equation 4.25 implies

$$\operatorname{Ent}(\mu,\tau) \geq \sup_{n\in\mathbb{N}}\big(2C\bar{\mu}(p) - \|\mu\|_{A^*}\big)\cdot\log n - \left\|\varphi(x)\right\|_{\tau}^{2}\cdot n^{-1} = \infty. \tag{4.26}$$

Equation 4.26 shows $\operatorname{Ent}(\mu,\tau) = \infty$ if $\operatorname{Ent}(\mu,\tau) > -\infty$. Altogether, get 1).

We show 2). Assume $\mu \in L^1(A,\tau)^\flat_+$. Let $p \in L^{1,\infty}(A,\tau)$ be a projection s.t. $\operatorname{supp}\mu \leq p$. Lemma 3.2.5 therefore implies $\mu \in L^1(A[p],\tau)^\flat_+$ and $\sharp\mu = \sharp\mu \cdot p = p \cdot \sharp\mu$. Surjectivity of φ yields element $x \in \varphi^{-1}(p) \in U(A)$. For all $n \in \mathbb{N}$, set $F_n(t) := x$ for all $t \in (n^{-1},\infty)$. Note $F_n \in \mathcal{T}^u_n(U(A))$ in each case. Selecting F_n for all $n \in \mathbb{N}$, we estimate

$$\operatorname{Ent}(\mu,\tau) \geq \sup_{n\in\mathbb{N}} \|\mu\|_{A^*} \log n - \int_{n^{-1}}^{\infty} t^{-1}\mu(1_A - p) + t^{-2}\tau(p)dt \geq -\tau(p) > -\infty. \tag{4.27}$$

Equation 4.27 shows Proposition 4.1.15 ensures we may use Equation 4.15 as Kosaki's formula. We may furthermore take the supremum over all $\mathcal{T}^u_n(L^\infty(A,\tau))$ (cf. proof of Proposition 4.1.15). Since $A[p] \subset L^\infty(A,\tau)$ is a C^*-subalgebra, we bound the variational expression on the right-hand side of Equation 4.24 from above by $\operatorname{Ent}(\mu,\tau)$. We show the converse. For all $n \in \mathbb{N}$ and $F \in \mathcal{T}^u_n(L^{2,\infty}(A,\tau))$, set $F_p(t) := \operatorname{com}_p F(t) = pF(t)p$ for all $t \in (n^{-1},\infty)$. Since $pAp \subset A[p]$ by definition, note $F_p \in \mathcal{T}^u_n(A[p])$.

Let $n \in \mathbb{N}$, $F \in \mathcal{T}^u_n(L^{2,\infty}(A,\tau))$ and $t \in (n^{-1},\infty)$. Using $\sharp\mu = \sharp\mu \cdot p = p \cdot \sharp\mu$, $p^2 = p$ and traciality, we calculate

$$\left\|F^\perp(t)\right\|^2_\mu = \left\|p - F_p(t)\right\|^2_\mu + \mu\big(pF(t)^*(1_A - p)F(t)p\big) \tag{4.28}$$

and

$$\left\|F(t)\right\|^2_\tau = \left\|F_p(t)\right\|^2_\tau + \tau\big(pF(t)(1_A - p)F(t)^*\big) + \tau\big((1_A - p)F(t)F(t)^*\big). \tag{4.29}$$

We know $p, 1_A - p \in L^\infty(A,\tau)_+$ by hypothesis. For all $y \in L^\infty(A,\tau)$, we therefore have $py^*(1-p)yp, y(1_A - p)y^*, yy^* \in L^\infty(A,\tau)_+$. Using such positivity, Equation 4.28 lets us estimate

$$\left\|F^\perp(t)\right\|^2_\mu \geq \left\|p - F_p(t)\right\|^2_\mu, \tag{4.30}$$

and Equation 4.29 lets us estimate

$$\left\|F(t)\right\|^2_\tau \geq \left\|F_p(t)\right\|^2_\tau. \tag{4.31}$$

We conclude by estimating integral terms in Kosaki's formula as follows. For all $n \in \mathbb{N}$ and $F \in \mathcal{T}^u_n(L^{2,\infty}(A,\tau))$, applying Equation 4.30 and Equation 4.31 to integrands on the left-hand side below lets us estimate

$$\int_{n^{-1}}^{\infty} t^{-1}\left\|F^\perp(t)\right\|^2_\mu + t^{-2}\left\|F(t)\right\|^2_\tau dt \geq \int_{n^{-1}}^{\infty} t^{-1}\left\|F^\perp_p(t)\right\|^2_\mu + t^{-2}\left\|F_p(t)\right\|^2_\tau dt. \tag{4.32}$$

Using Equation 4.15 as Kosaki's formula, Equation 4.32 shows we bound the variational expression on the right-hand side of Equation 4.24 from below by $\operatorname{Ent}(\mu,\tau)$. Thus get Equation 4.24, hence using relative entropy for $A[p]$ yields $\operatorname{Ent}(\mu,\tau) = \operatorname{Ent}(\mu,p^\flat) > -\infty$ by 4) in Proposition 4.1.6 and 1) in Proposition 4.1.9. Altogether, get 2). □

4.1.2 Restriction to finitely supported accessibility components

Theorem 4.1.25 shows compatibility of quantum relative entropy with compression and finite-dimensional approximation, as well as suitable l.s.c. used in Theorem 4.1.29 to show l.s.c. in topology of the given quantum optimal transport distance. We compress quantum relative entropy with uniform majorants of local support. Finite-dimensional approximation and l.s.c. are given for states with uniform majorant of local support. As such, Theorem 4.1.25 shows we recover the strongly unital finite-trace case, and thereby the notion of discriminating information for quantum relative entropy as claimed in the introduction of this section, by compressing with uniform majorants of local support.

We further show all states in finitely supported accessibility components have such uniform majorants. Theorem 4.1.29 lets us restrict quantum relative entropy to each one s.t. compatibility and l.s.c. as above are satisfied. Assuming finitely supported fixed states, we are therefore able to apply the coarse graining process using Diagram 3.346 for K the domain of quantum relative entropy. In Section 4.3, we use the latter for our discussion, in particular our equivalence Theorem 4.3.8. Examples of finitely supported fixed states arise from fixed states on tracial AF-C^*-algebras generating hyperfinite factors of type I and II by σ-weak closure.

Uniform majorants of local support. Definition 4.1.18 gives local support and uniform majorants of local support. Using the latter, we introduce finitely supported accessibility components. Strongly unital tracial AF-C^*-algebras with finite trace have units as uniform majorants of local support. We give examples for the non-unital and non-finite-trace case. Following Corollary 4.1.27, Example 4.1.22 and Example 4.1.23 give examples of finitely supported fixed states.

Let (A,τ) be a tracial AF-C^*-algebra.

Definition 4.1.18. Let $p \in L^{1,\infty}(A,\tau)$ a projection.

1) For all $\mu \in A^*_+$, we say that p majorises the local support of μ if

$$\operatorname{supp}\mu_j \leq p \tag{4.33}$$

in $L^\infty(A,\tau)$ for a.e. $j \in \mathbb{N}$. We further call p a majorant of the local support of μ and write $\operatorname{supp}\mu \subset p$.

2) The set of local supports in $L^\infty(A,\tau)$ uniformly majorised by p is defined by

$$\mathrm{C}[p] := \left\{\mu \in A^*_+ \;\middle|\; \operatorname{supp}\mu \subset p\right\}. \tag{4.34}$$

We further call p a uniform majorant of local support of all $\mu \in \mathrm{C}[p]$.

Remark 4.1.19. If $\tau < \infty$, then 1_A majorises the local support of all $\mu \in A^*_+$.

Lemma 4.1.20. *Let $p \in L^{1,\infty}(A,\tau)$ be a projection. If $\mu \in L^1(A,\tau)^\flat_+ \cap \mathrm{C}[p]$, then* $\operatorname{supp}\mu \leq p$ *and μ has integrable support.*

Proof. Let $\mu \in L^1(A,\tau)^\flat_+ \cap \mathrm{C}[p]$. Let $j \in \mathbb{N}$. Using 2) in Lemma 2.2.52, applying com_{A_j} to Equation 2.108 yields

$$S_j := \mathrm{com}_{A_j} L_{\sharp\mu_j} = L_{\sharp\mu_j,A_j} \leq T_j := \mathrm{com}_{A_j} L^2_{\pi^A_j\left(\sqrt{\sharp\mu}\right)} = L^2_{\pi^A_j\left(\sqrt{\sharp\mu}\right),A_j}. \tag{4.35}$$

Equation 4.35 shows $\ker T_j \subset \ker S_j$ and therefore

$$\pi^A_{\ker T_j} \leq \pi^A_{\ker S_j}. \tag{4.36}$$

Using 2) in Proposition 3.2.4, Equation 4.36 implies

$$\operatorname{supp}^c_{A_j}\mu_j \leq \operatorname{supp}^c_{A_j}\pi^A_j\left(\sqrt{\sharp\mu}\right). \tag{4.37}$$

Applying 1) in Proposition 3.2.4 to Equation 4.37 yields

$$\operatorname{supp}\pi^A_j\left(\sqrt{\sharp\mu}\right) \leq \operatorname{supp}\mu_j \leq p. \tag{4.38}$$

Note 1) in Proposition 3.2.4 shows $\operatorname{supp}\mu = \operatorname{supp}\sharp\mu = \chi_{(0,\infty]}\left(\sqrt{\sharp\mu}\right)$ by positivity and functional calculus. Thus 1) in Proposition 3.2.4 and 2) in Lemma 3.2.16 show

$$\chi_{(0,\infty]}\left(\sqrt{\sharp\mu}\right) = \operatorname{s-lim}_{j\in\mathbb{N}} \chi_{(0,\infty]}\left(\pi^A_j\left(\sqrt{\sharp\mu}\right)\right) = \operatorname{s-lim}_{j\in\mathbb{N}} \operatorname{supp}\pi^A_j\left(\sqrt{\sharp\mu}\right), \tag{4.39}$$

hence Equation 4.38 and Equation 4.39 lets us estimate

$$\operatorname{supp}\mu = \chi_{(0,\infty]}\left(\sqrt{\sharp\mu}\right) = \operatorname{s-lim}_{j\in\mathbb{N}} \operatorname{supp}\pi^A_j\left(\sqrt{\sharp\mu}\right) \leq p. \tag{4.40}$$

Equation 4.40 shows $\operatorname{supp}\mu \leq p$. In particular, $\tau(\operatorname{supp}\mu) \leq \tau(p) < \infty$ as required. □

Corollary 4.1.21. *Let $p \in L^{1,\infty}(A,\tau)$ be a projection.*

1) $\mathrm{C}[p] \subset A[p]^*_+$ *and* $\mathscr{S}(A) \cap \mathrm{C}[p] \subset \mathscr{S}(A[p])$.
2) $L^1(A,\tau)^\flat_+ \cap \mathrm{C}[p] \subset L^1(A[p],\tau)^\flat_+$ *and* $\mathscr{S}^{\mathrm{N}}(A) \cap \mathrm{C}[p] \subset \mathscr{S}^{\mathrm{N}}(A[p])$.
3) *If $\mu \in \mathrm{C}[p]$ and $\{\mu^k\}_{k\in K} \subset \mathrm{C}[p]$ is a net s.t. we have both $\mu = w^*\text{-}\lim_{k\in K}\mu^k$ in A^* and $\|\mu\|_{A^*} = \lim_{k\in K}\|\mu^k\|_{A^*}$, then $\mu = w^*\text{-}\lim_{k\in K}\mu^k$ in $A[p]^*$.*

Proof. Lemma 2.1.6 yields $A[p]^*_+ \cap L^1(A[p],\tau)^\flat \subset A^*_+ \cap L^1(A,\tau)^\flat$. Using normality, 2) in Proposition 2.1.16 shows $\mathrm{inc}_p = \mathrm{com}^*_p : A[p]^*_+ \cap L^1(A[p],\tau)^\flat \longrightarrow A^*_+ \cap L^1(A,\tau)^\flat$ is injective. Thus Lemma 4.1.20 implies 2), hence Proposition 2.1.9 shows 3) at once. We show 1). For all $\mu \in \mathrm{C}[p]$, 1) in Proposition 2.1.31 implies $\mu = w^*\text{-}\lim_{j\in\mathbb{N}}\mu_j$ in A^* and $\|\mu\|_{A^*} = \lim_{j\in\mathbb{N}}\|\mu_j\|_{A^*}$, and Lemma 3.2.5 shows $\{\mu_j\}_{j\in\mathbb{N}} \subset \mathrm{C}[p]$ by scaling. Then 3) implies 1). □

Example 4.1.22. Let H be a separable Hilbert space. Assume $(A,\tau)=(\mathscr{K}(H),\mathrm{tr})$. Let $\mu\in S^1(H)^\flat_+$. Following Example 3.2.21, we know μ has integrable support if and only if $\operatorname{supp}\mu\in\mathscr{K}(H)_0=\bigcup_{n\in\mathbb{N}}M_n(\mathbb{C})$. Lemma 4.1.20 therefore implies μ has integrable support if and only if there exists a majorant of its local support.

Example 4.1.23. Let H and $\mathscr{H}$ be infinite-dimensional separable Hilbert spaces. We assume the setting of Example 3.1.58, i.e. assume $(A,\tau)=(\mathscr{K}(H)\otimes\mathscr{A}(\mathscr{H}[J]),\mathrm{tr}\otimes\tau)$. Set $M:=L^\infty(\mathscr{K}(H)\otimes\mathscr{A}(\mathscr{H}[J]),\mathrm{tr}\otimes\tau)$. Let $n\in\mathbb{N}$. We consider σ-weak closure

$$N:=\overline{M_n(\mathbb{C})\odot\mathscr{A}(H[J])}\subset M. \tag{4.41}$$

Note $L^1(N,\mathrm{tr}\otimes\tau)=\overline{N}^{\|.\|_1}=\overline{M_n(\mathbb{C})\odot\mathscr{A}(H[J])}^{\|.\|_1}$. Since $I_n\otimes 1_{\mathscr{A}(H[J])}\in M_n(\mathbb{C})\odot\mathscr{A}(H[J])$ is the unit, we have $1_N=I_n\otimes 1_{\mathscr{A}(H[J])}$ by density in σ-weak topology. Thus $(\mathrm{tr}\otimes\tau)(1_N)=n<\infty$, hence $\mathrm{tr}\otimes\tau<\infty$ and therefore $N\subset(M,\mathrm{tr}\otimes\tau)$ (cf. 1) in Proposition B.1.12 and 1) in Proposition B.2.13). We show 1_N majorises local support of all $\mu\in L^1(N,\mathrm{tr}\otimes\tau)^\flat_+$.

Let $\mu\in L^1(N,\mathrm{tr}\otimes\tau)^\flat_+$. Using separability to obtain sequences, Equation 4.41 yields $\{m_k\}_{k\in\mathbb{N}}\subset\mathbb{N}$ and $\{x^k_l\otimes y^k_l\}_{k,l\in\mathbb{N}}\subset M_n(\mathbb{C})\odot\mathscr{A}(H[J])$ s.t. $\sharp\mu=\|.\|_1\text{-}\lim_{k\in\mathbb{N}}\sum_{l=1}^{m_k}x^k_l\otimes y^k_l$. If $j\geq n$ in $\mathbb{N}$, then construction of $\mathrm{tr}\otimes\tau$ shows

$$\pi_j^{\mathscr{K}(H)\otimes\mathscr{A}(\mathscr{H}[J])}\big(M_n(\mathbb{C})\odot\mathscr{A}(H[J])\big)\subset M_n(\mathbb{C})\odot\mathscr{A}(H_j[J])\subset N. \tag{4.42}$$

Using 2.1) in Proposition 2.1.28, applying the flat operator to Equation 4.42 yields dual equivalent. For all $j\geq n$ in $\mathbb{N}$, w^*-continuity and linearity therefore imply

$$\sharp\mu_j=\|.\|_\infty\text{-}\lim_{k\in\mathbb{N}}\sum_{l=1}^{m_k}x^k_l\otimes y^k_{l,j}\in M_n(\mathbb{C})\odot\mathscr{A}(H_j[J])\subset N. \tag{4.43}$$

Finite-dimensionality ensures a priori $\|.\|_1$-convergence for Equation 4.43 is equivalent to $\|.\|_\infty$-convergence as used. If $j\leq n$, then $\sharp\mu_j\in M_j(\mathbb{C})\odot\mathscr{A}(H_j[J])\subset N$ shows $\operatorname{supp}\mu_j\in N$ by 1) in Proposition 3.2.4. If instead $j\geq n$, then Equation 4.43 shows $\operatorname{supp}\mu_j\in N$ by said proposition. For all $j\in\mathbb{N}$, we therefore have $\operatorname{supp}\mu_j\leq 1_N$ since each is a projection.

Quantum relative entropy given uniform majorant of local support. Using Lemma 4.1.17, Lemma 4.1.20 and Lemma 4.1.24, Theorem 4.1.25 shows all properties we require of quantum relative entropy. We compress quantum relative entropy as per 1) in Theorem 4.1.25. Finite-dimensional approximation is 3) in Theorem 4.1.25. This is compatibility of quantum relative entropy with compression and finite-dimensional approximation. Its suitable l.s.c. in topology of the given quantum optimal transport distance is 2) in Theorem 4.1.25. Assuming boundedness, Corollary 4.1.27 gives closed term trace description of quantum relative entropy. Example 4.1.10 shows its negative is quantum entropy, i.e. von Neumann entropy (cf. p.17 in [163]).

Let (A,τ) be a tracial AF-C^*-algebra.

Lemma 4.1.24. *Let $p \in L^{1,\infty}(A,\tau)$ be a projection.*

1) If $\mu \in \mathrm{C}[p]$, then $\mathrm{Ent}(\mu,\tau) = \mathrm{Ent}(\mu,p^\flat) > -\infty$ and we have

$$\mathrm{Ent}(\mu,p^\flat) = \sup_{\substack{n\in\mathbb{N},\\ F\in\mathcal{T}_n^u(A[p])}} \left\{ \|\mu\|_{A[p]^*}\log n - \int_{n^{-1}}^{\infty} t^{-1}\left\|F^\perp(t)\right\|_\mu^2 + t^{-2}\left\|F(t)\right\|_{p^\flat}^2 dt \right\}. \quad (4.44)$$

2) If $\mu \in \mathrm{C}[p]$ and $\{\mu^n\}_{n\in\mathbb{N}} \subset \mathrm{C}[p]$ s.t. we have both $\mu = w^*\text{-}\lim_{n\in\mathbb{N}}\mu^n$ in A^* and $\|\mu\|_{A^*} = \lim_{n\in\mathbb{N}}\|\mu^n\|_{A^*}$, then

$$\mathrm{Ent}(\mu,\tau) \le \liminf_{n\in\mathbb{N}} \mathrm{Ent}(\mu^n,\tau). \quad (4.45)$$

Proof. Let $\mu \in \mathrm{C}[p]$. We have tracial AF-C^*-algebra $(A[p],\tau)$ in $L^\infty(A[p],\tau)$ generated by $\{A_j[p]\}_{j\in\mathbb{N}}$. As such, we may apply our results concerning quantum relative entropy for tracial AF-C^*-algebras given in Subsection 4.1.1. We use unit p in $A[p]$.

We show 1). Compression uses general W^*-algebras (cf. Definition A.2.15). Since $A[p] \subset A$, construction of universal enveloping W^*-algebras using σ-weak closure yields W^*-subalgebra $U(A[p]) = \overline{\pi_U(A[p])} \subset U(A)$. Note $1_{U(A[p])} = \pi_U(p)$. Since $\varphi \circ \pi_U = \mathrm{id}_A$ extends to $\mathrm{id}_{L^\infty(A,\tau)}$ by normality, get $\varphi(1_{U(A[p])}) = p$. We therefore have

$$U(A[p]) = U(A)\left[1_{U(A[p])}\right] \subset U(A). \quad (4.46)$$

Get $\mu \in A[p]^*_+$ by 1) in Corollary 4.1.21. Equation 4.46 shows $U(\mu)|_{U(A[p])}$ is canonical normal extension of μ to $U(A[p])$. We know $p^\flat$ and τ coincide on $A[p]_+$. Equation 4.46 shows $U(p^\flat) = U(\tau)|_{U(A[p])}$ as per Equation 4.6. We use relative entropy for A, resp. $A[p]$ in Equation 4.47 below. Equation 4.46 lets us estimate

$$\mathrm{Ent}(\mu,\tau) \ge \mathrm{Ent}(\mu,p^\flat). \quad (4.47)$$

If $\mu \notin \mathrm{dom}\,\mathrm{Ent}^\tau$, then 4) in Proposition 4.1.6 and 1) in Lemma 4.1.17 imply $\mu \notin L^1(A,\tau)^\flat_+$ and $\mathrm{Ent}(\mu,\tau) = \mathrm{Ent}(\mu,p^\flat) = \infty$. If $\mu \in \mathrm{dom}\,\mathrm{Ent}^\tau$, then 2) in Lemma 4.1.17 further implies $\mu \in L^1(A,\tau)^\flat_+$ and $\mathrm{Ent}(\mu,\tau) = \mathrm{Ent}(\mu,p^\flat) > -\infty$. Get 1).

We show 2). Assume its setting. Note 3) in Corollary 4.1.21 shows $\mu = w^*\text{-}\lim_{n\in\mathbb{N}}\mu^n$ in $A[p]^*$. Thus l.s.c. in Proposition 4.1.9 for $A[p]$ implies

$$\mathrm{Ent}(\mu,p^\flat) \le \liminf_{n\in\mathbb{N}} \mathrm{Ent}(\mu^n,p^\flat), \quad (4.48)$$

hence 1) and Equation 4.48 imply Equation 4.45. □

Theorem 4.1.25. *Let (A,τ) be a tracial AF-C^*-algebra. Let $p \in L^{1,\infty}(A,\tau)$ be a projection.*

1) If $\mu \in \mathscr{S}(A) \cap \mathrm{C}[p]$, then $\mathrm{Ent}(\mu,\tau) = \mathrm{Ent}(\mu, p^\flat) > -\infty$ and we have

$$\mathrm{Ent}(\mu, p^\flat) = \sup_{\substack{n\in\mathbb{N},\\ F\in\mathscr{T}_n^u(A[p])}} \left\{ \|\mu\|_{A[p]^*} \log n - \int_{n^{-1}}^{\infty} t^{-1} \left\|F^\perp(t)\right\|_\mu^2 + t^{-2} \left\|F(t)\right\|_{p^\flat}^2 dt \right\}. \quad (4.49)$$

2) If $\mu \in \mathscr{S}(A) \cap \mathrm{C}[p]$ and $\{\mu^n\}_{n\in\mathbb{N}} \subset \mathscr{S}(A) \cap \mathrm{C}[p]$ s.t. $\mu = w^*\text{-}\lim_{n\in\mathbb{N}} \mu^n$, then

$$\mathrm{Ent}(\mu,\tau) \le \liminf_{n\in\mathbb{N}} \mathrm{Ent}(\mu^n, \tau). \quad (4.50)$$

3) If $\mu \in \mathscr{S}(A) \cap \mathrm{C}[p]$, then

$$\mathrm{Ent}(\mu,\tau) = \lim_{j\in\mathbb{N}} \mathrm{Ent}(\mu_j, \tau) = \lim_{j\in\mathbb{N}} \mathrm{Ent}(\bar{\mu}_j, \tau). \quad (4.51)$$

Proof. Let $\mu \in \mathscr{S}(A) \cap \mathrm{C}[p]$. Lemma 4.1.24 shows 1) and 2) at once. Since $\mu = w^*\text{-}\lim_{j\in\mathbb{N}} \mu_j$ by 1.2) in Proposition 2.1.31, Equation 4.51 follows from 2) if

$$\mathrm{Ent}(\mu,\tau) \ge \liminf_{j\in\mathbb{N}} \mathrm{Ent}(\mu_j, \tau). \quad (4.52)$$

Note 1.1) in Proposition 2.1.31 and Proposition 4.1.6 ensure scaling upon restriction is of no consequence in Equation 4.51. We show Equation 4.52. For this, we engage in several reduction steps. If $\mu \notin L^1(A,\tau)_+^\flat$, then $\mu \notin \mathrm{dom}\,\mathrm{Ent}^\tau$ by 1) in Lemma 4.1.17. Thus $\mathrm{Ent}(\mu,\tau) > -\infty$ further implies $\mathrm{Ent}(\mu,\tau) = \infty$, hence Equation 4.52 as well. We assume $\mu \in L^1(A,\tau)_+^\flat$ without loss of generality. We use the following. Lemma 4.1.20 shows μ has integrable support. Theorem 3.2.18 ensures μ has reducible support.

We engage in the first reduction. We define remainder terms in Equation 4.55 below and show the latter implies Equation 4.52. For all $j \in \mathbb{N}$, we consider the C^*-subalgebra $\mathscr{A}_j := \langle 1_{A_j}^\perp \rangle_{\mathbb{C}} \subset L^\infty(A,\tau)$ and set

$$\mu_j^\perp := \mu|_{\mathscr{A}_j}, \nu_j := \left(\mathrm{supp}\,\mu\right)^\flat\Big|_{\mathscr{A}_j} \in \mathscr{A}_{j,+}^*. \quad (4.53)$$

We further define the j-th remainder term $\mathrm{R}_j \in \mathbb{R}$ by setting

$$\mathrm{R}_j(\mu) := \begin{cases} \mathrm{Ent}\left(\mu_j^\perp, \nu_j\right) & \text{if } 1_{A_j}^\perp \ne 0, \\ 0 & \text{else}. \end{cases}$$

Using $1_A = \text{s-lim}_{j\in\mathbb{N}} 1_{A_j}$ as per 2) in Proposition 2.1.16, Example 4.1.10 for $n = 1$ shows we either have $\mathrm{R}_j(\mu) = 0$ for a.e. $j \in \mathbb{N}$ or normality and positivity let us estimate

$$\liminf_{j\in\mathbb{N}} \mathrm{R}_j(\mu) = \liminf_{j\in\mathbb{N}} \mu\big(1_{A_j}^{\perp}\big)\log\Big(\mu\big(1_{A_j}^{\perp}\big)\Big) - \mu\big(1_{A_j}^{\perp}\big)\log\Big(\tau\big(\operatorname{supp}\mu\cdot 1_{A_j}^{\perp}\big)\Big) \geq 0. \tag{4.54}$$

For details on our estimate of Equation 4.54, we refer to Remark 4.1.26. Equation 4.54 shows $\liminf_{j\in\mathbb{N}} \mathrm{R}_j(\mu) \geq 0$. We claim Equation 4.52 follows if

$$\operatorname{Ent}(\mu,\tau) \geq \operatorname{Ent}\big(\mu_j,\tau\big) + \mathrm{R}_j(\mu) \tag{4.55}$$

for all $j \in \mathbb{N}$. If we do have Equation 4.55, then we apply $\liminf_{j\in\mathbb{N}} \mathrm{R}_j(\mu) \geq 0$ to estimate $\operatorname{Ent}(\mu,\tau) \geq \liminf_{j\in\mathbb{N}} \operatorname{Ent}(\mu_j,\tau) + \mathrm{R}_j(\mu) \geq \liminf_{j\in\mathbb{N}} \operatorname{Ent}(\mu_j,\tau)$. Thus Equation 4.55 implies Equation 4.52, hence it suffices to show the former.

We engage in the second reduction. Let $j \in \mathbb{N}$. Note Equation 4.55 follows if

$$\operatorname{Ent}(\mu,\tau) \geq \operatorname{Ent}\Big(\mu_j, (\operatorname{supp}\mu)_j^{\flat}\Big) + \mathrm{R}_j(\mu),\ \operatorname{Ent}\Big(\mu_j, (\operatorname{supp}\mu)_j^{\flat}\Big) \geq \operatorname{Ent}\big(\mu_j,\tau\big). \tag{4.56}$$

Set $\eta_j := (\operatorname{supp}\mu)_j^{\flat}$ and $\nu_j := (\operatorname{supp}\mu_j)^{\flat}$. We show Equation 4.56. For this, we show

$$\operatorname{Ent}\big(\mu_j,\eta_j\big) \geq \operatorname{Ent}\big(\mu_j,\tau\big). \tag{4.57}$$

Applying Lemma 4.1.16 and 1) to μ_j for $p = \operatorname{supp}\mu_j$ shows

$$\operatorname{Ent}\big(\mu_j,\tau\big) = \operatorname{Ent}\big(\mu_j,\tau_j\big) = \operatorname{Ent}\big(\mu_j,\nu_j\big). \tag{4.58}$$

Using Equation 4.14 as Kosaki's formula, Equation 4.58 implies Equation 4.57 if $\nu_j \geq \eta_j$ in A_j^*. Following 2) in Proposition 3.2.4, we equivalently estimate

$$x_j := 1_{A_j} - \sharp\eta_j \geq 1_{A_j} - \operatorname{supp}\mu_j = L_{A_j}^{-1}\left(\pi^{A}_{\ker L_{\sharp\mu_j,A_j}}\right) \tag{4.59}$$

in A_j. Note $x_j = \pi_j^A(1_A - \operatorname{supp}\mu) \geq 0$ by Proposition 2.1.28. Since $A_0 \subset \operatorname{dom} L_{\sqrt{\sharp\mu}}$, get

$$\big\|\sqrt{\sharp\mu}\cdot u\big\|_{\tau}^2 = \big\langle \sharp\mu_j u, u\big\rangle_{\tau} = 0 \tag{4.60}$$

for all $u \in \ker L_{\sharp\mu_j,A_j}$. Equation 4.60 shows

$$\ker L_{\sharp\mu_j,A_j} \subset \ker L_{\sqrt{\sharp\mu}}. \tag{4.61}$$

Note 1) in Proposition 3.2.4 shows $\operatorname{supp}\mu = \operatorname{supp}\sharp\mu = \chi_{(0,\infty]}\left(\sqrt{\sharp\mu}\right)$ by positivity and functional calculus. Thus 1) and 2) in Proposition 3.2.4 imply

$$\operatorname{supp}\mu = \chi_{(0,\infty]}\left(\sqrt{\sharp\mu}\right) = L^{-1}\left(I - \pi^{A}_{\ker\sqrt{\sharp\mu}}\right), \tag{4.62}$$

hence Equation 4.61 shows

$$\left\|\sqrt{\eta_j}\cdot u\right\|_{\tau}^{2} = \left\langle \operatorname{supp}\mu\cdot u, u\right\rangle_{\tau} = 0 \tag{4.63}$$

for all $u \in \ker L_{\sharp\mu_j, A_j}$. Equation 4.63 shows

$$\ker L_{\sharp\mu_j, A_j} \subset \ker L_{\sharp\eta_j, A_j}. \tag{4.64}$$

For all $u \in A_j$, we decompose as per Equation 4.65 below using the following. For the left-hand side of Equation 4.65, apply 1) and 2) in Proposition 3.2.4. For the right-hand side of Equation 4.65, we use Equation 4.64. Altogether, we have

$$u = \operatorname{supp}\mu_j\cdot u + \pi^{A}_{\ker L_{\sharp\mu_j,A_j}}(u),\ x_j\pi^{A}_{\ker L_{\sharp\mu_j,A_j}}(u) = \pi^{A}_{\ker L_{\sharp\mu_j,A_j}}(u) \tag{4.65}$$

for all $u \in A_j$. Equation 4.65 lets us calculate

$$\left\langle x_j u, u\right\rangle_{\tau} = \left\langle x_j \operatorname{supp}\mu_j\cdot u, \operatorname{supp}\mu_j\cdot u\right\rangle_{\tau} + \left\langle \pi^{A}_{\ker L_{\sharp\mu_j,A_j}}(u), u\right\rangle_{\tau} \tag{4.66}$$

for all $u \in A_j$. As $x_j \geq 0$ yields $\langle x_j \operatorname{supp}\mu_j\cdot u, \operatorname{supp}\mu_j\cdot u\rangle_{\tau} \geq 0$ in each case, Equation 4.66 lets us estimate

$$\left\langle x_j u, u\right\rangle_{\tau} \geq \left\langle \pi^{A}_{\ker L_{\sharp\mu_j,A_j}}(u), u\right\rangle_{\tau} \tag{4.67}$$

for all $u \in A_j$. Equation 4.67 shows Equation 4.59. Using the latter, Equation 4.58 then implies Equation 4.57 as discussed above.

We engage in the third reduction. Following Equation 4.57, we show

$$\operatorname{Ent}(\mu,\tau) \geq \operatorname{Ent}\left(\mu_j, \eta_j\right) + \mathrm{R}_j(\mu) \tag{4.68}$$

in order to have Equation 4.56 and therefore 3) as discussed above. Set $\eta := (\operatorname{supp}\mu)^{\flat}$. Note 2) in Proposition 4.1.9 for $N = A_j[1_A] \subset L^{\infty}(A,\tau)$ and 1) applied to $\mu \in L^{1}(A,\tau)^{\flat}_{+}$ for $p = \operatorname{supp}\mu$ lets us estimate

$$\operatorname{Ent}(\mu,\tau) = \operatorname{Ent}(\mu,\eta) \geq \operatorname{Ent}\left(\mu|_{A_j[1_A]}, \eta|_{A_j[1_A]}\right). \tag{4.69}$$

Note $\mathscr{A}_j = \langle 1^{\perp}_{A_j}\rangle_{\mathbb{C}} \subset L^{\infty}(A,\tau)$. Equation 4.69 implies Equation 4.68 if

$$\mathrm{Ent}\big(\mu|_{A_j[1_A]}, \eta|_{A_j[1_A]}\big) = \mathrm{Ent}\big(\mu_j, \eta_j\big) + \mathrm{R}_j(\mu). \tag{4.70}$$

If $1^{\perp}_{A_j} = 0$, then Equation 4.70 holds since $\mathscr{A}_j = 0$ and $\mathrm{R}_j(\mu) = 0$. Assume $1^{\perp}_{A_j} \neq 0$. Get

$$A_j[1_A] = A_j \oplus \mathscr{A}_j \tag{4.71}$$

by hypothesis (cf. Proposition A.1.65). For all $\nu \in A_j[1_A]^*_+$, we decompose its norm

$$\|\nu\|_{A_j[1_A]^*} = \|\nu\|_{A_j^*} + \|\nu\|_{\mathscr{A}_j^*} \tag{4.72}$$

over the direct sum of C^*-algebras as per Equation 4.71. For all $n \in \mathbb{N}$, decomposing as per Equation 4.72 at each time yields further product decomposition

$$\mathscr{T}^u_n\big(A_j[1_A]\big) = \mathscr{T}^u_n(A_j) \times \mathscr{T}^u_n(\mathscr{A}_j). \tag{4.73}$$

Using Equation 4.49 as Kosaki's formula, Equation 4.73 implies

$$\mathrm{Ent}\big(\mu|_{A_j[1_A]}, \eta|_{A_j[1_A]}\big) = \mathrm{Ent}\big(\mu_j, \eta_j\big) + \mathrm{Ent}\Big(\mu|_{\mathscr{A}_j}, (\operatorname{supp}\mu)^{\flat}\Big|_{\mathscr{A}_j}\Big). \tag{4.74}$$

The second summand on the right-hand side of Equation 4.74 is $\mathrm{R}_j(\mu)$. Equation 4.70 holds. Equation 4.68 and therefore 3) follows as discussed above. □

Remark 4.1.26. We elaborate on our estimate of Equation 4.54. We have

$$\lim_{j\in\mathbb{N}} \mu\big(1^{\perp}_{A_j}\big) = \lim_{j\in\mathbb{N}} \tau\big(\operatorname{supp}\mu \cdot 1^{\perp}_{A_j}\big) = 0,\ \mu\big(1^{\perp}_{A_j}\big), \tau\big(\operatorname{supp}\mu \cdot 1^{\perp}_{A_j}\big) \geq 0 \tag{4.75}$$

by normality, resp. for all $j \in \mathbb{N}$ by positivity. Using $\lim_{\lambda\to 0} \lambda \log \lambda = 0$, Equation 4.75 lets us estimate

$$\begin{aligned}
\liminf_{j\in\mathbb{N}} \mathrm{R}_j(\mu) &= \liminf_{j\in\mathbb{N}} \mu\big(1^{\perp}_{A_j}\big)\log\Big(\mu\big(1^{\perp}_{A_j}\big)\Big) - \mu\big(1^{\perp}_{A_j}\big)\log\Big(\tau\big(\operatorname{supp}\mu \cdot 1^{\perp}_{A_j}\big)\Big)\\
&\geq \liminf_{j\in\mathbb{N}} -\mu\big(1^{\perp}_{A_j}\big)\log\Big(\tau\big(\operatorname{supp}\mu \cdot 1^{\perp}_{A_j}\big)\Big)\\
&\geq 0
\end{aligned}$$

since $\lim_{j\in\mathbb{N}} \log\Big(\tau\big(\operatorname{supp}\mu \cdot 1^{\perp}_{A_j}\big)\Big) = -\infty$ by normality.

Corollary 4.1.27. *Let $p \in L^{1,\infty}(A,\tau)$ be a projection. If $\mu \in L^{1,\infty}(A,\tau)_+^\flat \cap \mathrm{C}[p]$, then* $\mathrm{Ent}(\mu,\tau) \in (0,\infty)$ *and we have*

$$\mathrm{Ent}(\mu,\tau) = \tau\big(\sharp\mu \log \sharp\mu\big) = \tau\big(\mathrm{com}_p \sharp\mu \log \mathrm{com}_p \sharp\mu\big). \tag{4.76}$$

Proof. Let $\mu \in L^{1,\infty}(A,\tau)_+^\flat \cap \mathrm{C}[p]$. We have $\mathrm{com}_p \sharp\mu = p \cdot \sharp\mu \cdot p \in L^\infty(A[p],\tau)$. We use the following. Lemma 4.1.20 shows μ has integrable support. Theorem 3.2.18 ensures μ has reducible support. Using Lemma 4.1.16 and 3) in Theorem 4.1.25, we calculate

$$\mathrm{Ent}(\mu,\tau) = \lim_{j\in\mathbb{N}} \mathrm{Ent}\big(\mu_j,\tau\big) = \lim_{j\in\mathbb{N}} \mathrm{Ent}\big(\mu_j,\tau_j\big). \tag{4.77}$$

Reduction using Proposition 2.1.24 in Example 4.1.10 yields

$$\mathrm{Ent}\big(\mu_j,\tau_j\big) = \tau\big(\sharp\mu_j \log \sharp\mu_j\big) \tag{4.78}$$

for all $j \in \mathbb{N}$. Equation 4.77 and Equation 4.78 imply

$$\mathrm{Ent}(\mu,\tau) = \lim_{j\in\mathbb{N}} \tau\big(\sharp\mu_j \log \sharp\mu_j\big). \tag{4.79}$$

Note Equation 4.79 shows Equation 4.76 if

$$\tau\big(\sharp\mu \log \sharp\mu\big) = \lim_{j\in\mathbb{N}} \tau\big(\sharp\mu_j \log \sharp\mu_j\big) \tag{4.80}$$

and further

$$\tau\big(\mathrm{com}_p \sharp\mu \log \mathrm{com}_p \sharp\mu\big) = \lim_{j\in\mathbb{N}} \tau\big(\mathrm{com}_p \sharp\mu_j \log \mathrm{com}_p \sharp\mu_j\big) = \lim_{j\in\mathbb{N}} \tau\big(\sharp\mu_j \log \sharp\mu_j\big). \tag{4.81}$$

Moreover, 1) in Theorem 4.1.25 and Equation 4.76 show $\mathrm{Ent}(\mu,\tau) \in (-\infty,\infty)$. It suffices to show the two equations above.

We show Equation 4.80 and Equation 4.81. Compressing with projections decreases norm. Thus 3) in Proposition 2.1.31 shows

$$\sharp\mu = \text{bds-}\lim_{j\in\mathbb{N}} \sharp\mu_j,\ \mathrm{com}_p \sharp\mu = \text{bds-}\lim_{j\in\mathbb{N}} \mathrm{com}_p \sharp\mu_j \tag{4.82}$$

by sequential strong continuity of multiplication, hence we additionally have uniform boundedness for all operators used in Equation 4.82.

Note Lemma A.2.5 requires such uniform boundedness. Using Lemma A.2.5, we see Equation 4.82 shows

$$\sharp\mu\log\sharp\mu = \text{s-}\lim_{j\in\mathbb{N}}\sharp\mu_j\log\sharp\mu_j,\ \operatorname{com}_p\sharp\mu\log\operatorname{com}_p\sharp\mu = \text{bds-}\lim_{j\in\mathbb{N}}\operatorname{com}_p\sharp\mu_j\log\operatorname{com}_p\sharp\mu_j \tag{4.83}$$

since $\lambda \mapsto \lambda\log\lambda$ is continuous on $[0,\infty)$ (cf. Remark A.2.3 and Remark A.2.4). Using $\mu \in L^{1,\infty}(A,\tau)^{\flat}_{+} \cap \mathrm{C}[p]$ and $\lim_{\lambda\to 0}\lambda\log\lambda = 0$, 3) in Corollary B.2.35 and Corollary B.2.36 therefore imply

$$\sharp\mu\log\sharp\mu = \operatorname{com}_p\sharp\mu\log\operatorname{com}_p\sharp\mu,\ \sharp\mu_j\log\sharp\mu_j = \operatorname{com}_p\sharp\mu_j\log\operatorname{com}_p\sharp\mu_j \tag{4.84}$$

for all $j \in \mathbb{N}$. Equation 4.83 shows $\tau(\operatorname{com}_p\sharp\mu\log\operatorname{com}_p\sharp\mu) = \lim_{j\in\mathbb{N}}\tau(\operatorname{com}_p\sharp\mu_j\log\operatorname{com}_p\sharp\mu_j)$ by strong convergence since $\tau(p) < \infty$. Using the latter, Equation 4.83 and Equation 4.84 let us calculate

$$\begin{aligned}\tau\big(\sharp\mu\log\sharp\mu\big) &= \tau\big(\operatorname{com}_p\sharp\mu\log\operatorname{com}_p\sharp\mu\big)\\ &= \lim_{j\in\mathbb{N}}\tau\big(\operatorname{com}_p\sharp\mu_j\log\operatorname{com}_p\sharp\mu_j\big)\\ &= \lim_{j\in\mathbb{N}}\tau\big(\sharp\mu_j\log\sharp\mu_j\big).\end{aligned}$$

The above calculation shows Equation 4.80 and Equation 4.81 at once. Equation 4.76 and therefore $\operatorname{Ent}(\mu,\tau) \in (0,\infty)$ follows as discussed above. □

Finitely supported accessibility components. Definition 4.1.28 gives finitely supported fixed states and finitely supported accessibility components. The latter are defined by having finitely supported fixed state. Upon restriction, Theorem 4.1.29 shows we recover the strongly unital finite-trace case as per Theorem 4.1.25 depending on the given finitely supported fixed state. Theorem 4.3.8 uses Corollary 4.1.30.

Let $(\phi,\psi,\gamma,\nabla)$ be noncommutative differential structure for tracial AF-C^*-algebras (A,τ) and (B,ω) in (f,θ)-setting.

Definition 4.1.28. Let $\xi \in \mathscr{S}(A)$ be a fixed state.

1) We say that ξ is finitely supported if $\xi \in \operatorname{dom}\operatorname{Ent}^{\tau}$ and there exists a majorant of its local support.

2) We say that $\mathscr{C} \subset \big(\mathscr{S}(A),\mathscr{W}^{\log}_{\nabla}\big)$ is finitely supported with fixed part ξ if $\mathscr{C}$ has fixed part ξ and the latter is finitely supported.

Theorem 4.1.29. *Let $(\phi,\psi,\gamma,\nabla)$ be noncommutative differential structure for tracial AF-C^*-algebras (A,τ) and (B,ω) in (f,θ)-setting. Let $\xi\in\mathscr{S}(A)$ be a finitely supported fixed state. Let $\mathscr{C}\subset(\mathscr{S}(A),\mathscr{W}_{\nabla}^{f,\theta})$ be finitely supported with fixed part ξ. Let $p\in L^{1,\infty}(A,\tau)$ be a projection s.t. $\xi\in\mathrm{C}[p]$.*

1) We have $\mathscr{C}\subset\mathrm{Fix}_A(\xi)\subset\mathscr{S}(A)\cap\mathrm{C}[p]$ and $\mathscr{C}\cap\mathrm{dom}\,\mathrm{Ent}^{\tau}\subset\mathrm{Fix}_A^{\mathrm{N}}(\xi)\subset\mathscr{S}^{\mathrm{N}}(A)\cap\mathrm{C}[p]$.

2) For a.e. $j\in\mathbb{N}$, we have $\mathscr{C}_{A_j}\big(\bar{\xi}_j\big)\subset\mathrm{Fix}_{A_j}\big(\bar{\xi}_j\big)\subset\mathscr{S}^{\mathrm{N}}(A)\cap\mathrm{C}[p]$.

3) For all $\mu\in\mathrm{Fix}_A^{\mathrm{N}}(\xi)$, we have

3.1) $\mathrm{supp}\,\mu\le p$ and μ has integrable support,

3.2) $\mathrm{Ent}(\mu,\tau)=\lim_{j\in\mathbb{N}}\mathrm{Ent}\big(\mu_j,\tau\big)=\lim_{j\in\mathbb{N}}\mathrm{Ent}\big(\bar{\mu}_j,\tau\big)$.

4) $\mathrm{Ent}^{\tau}:\mathrm{Fix}_A^{\mathrm{N}}(\xi)\longrightarrow(-\infty,\infty]$ is l.s.c. in $\mathscr{W}_{\nabla}^{f,\theta}$-topology.

Proof. If $\mathrm{supp}\,\bar{\xi}_j=\mathrm{supp}\,\xi_j\le p$ for $j\in\mathbb{N}$, then note 1) in Proposition 2.1.33 for the tracial AF-C^*-algebra $(A[p],\tau)$ and 1) in Corollary 3.2.8 for the tracial AF-C^*-algebra (A_j,τ) yield inclusions

$$\mathscr{S}\left(A_{j,\bar{\xi}_j}\right)\subset\mathscr{S}\big(A_j[p]\big)\subset\mathscr{S}^{\mathrm{N}}(A[p]).\tag{4.85}$$

Using 1.3) in Theorem 3.2.40 and for a.e. $j\in\mathbb{N}$, Equation 4.85 shows

$$h_t\big(\mathscr{C}_{A_j}\big(\bar{\xi}_j\big)\big)\subset h_t\big(\mathrm{Fix}_{A_j}\big(\bar{\xi}_j\big)\big)\subset\mathscr{S}_{-1}^{\mathrm{N},\infty}\big(A_{j,\bar{\xi}_j}\big)\subset\mathscr{S}^{\mathrm{N}}(A[p])\tag{4.86}$$

for all $t>0$. Note 3) in Proposition 3.2.34 ensures the first inclusion in Equation 4.86. Letting $t\downarrow 0$ in Equation 4.86 implies 2) by 1) in Proposition 3.2.32. Using 2), we readily see 2) in Corollary 3.1.49 yields

$$\mathscr{C}\subset\mathrm{Fix}_A(\xi)\subset\mathscr{S}(A)\cap\mathrm{C}[p]\tag{4.87}$$

as per Diagram 3.346 for $K=\mathrm{dom}\,\mathrm{Ent}^{\tau}$. Using Lemma 4.1.17, Equation 4.87 shows

$$\mathscr{C}\cap\mathrm{dom}\,\mathrm{Ent}^{\tau}\subset\mathrm{Fix}_A^{\mathrm{N}}(\xi)\subset\mathscr{S}^{\mathrm{N}}(A)\cap\mathrm{C}[p].\tag{4.88}$$

Equation 4.87 and Equation 4.88 show 1) at once. Using the latter, Lemma 4.1.20 in turn implies 3.1), whereas 3) in Theorem 4.1.25 implies 3.2). Altogether, get 3). Using 1) in Theorem 3.1.47, we readily see 2) in Theorem 4.1.25 shows 4). □

Corollary 4.1.30. *Let $\xi \in \mathscr{S}(A)$ be a finitely supported fixed state. Let $\mathscr{C} \subset (\mathscr{S}(A), \mathscr{W}_{\nabla}^{f,\theta})$ be finitely supported with fixed part ξ. We consider marginals $\mu^0, \mu^1 \in \mathscr{C} \cap \operatorname{dom}\operatorname{Ent}^{\tau}$ and $(\mu, w) \in \operatorname{Geo}(\mu^0, \mu^1)$ approximated by $(\mu^j, w^j)_{j\in\mathbb{N}} \subset \operatorname{Geo}_0$ in finite dimensions. If there exists $C > 0$ s.t. for all $t \in (0,1)$, we have*

$$\operatorname{Ent}\big(\mu^j(t), \tau\big) \le C \cdot \max\Big\{\operatorname{Ent}\big(\bar{\mu}_j^0, \tau\big), \operatorname{Ent}\big(\bar{\mu}_j^1, \tau\big)\Big\} \tag{4.89}$$

for a.e. $j \in \mathbb{N}$, then $\mu(t) \in \operatorname{dom}\operatorname{Ent}^{\tau}$ for all $t \in [0,1]$.

Proof. Let $p \in L^{1,\infty}(A, \tau)$ be a projection s.t. $\xi \in \mathrm{C}[p]$. Let $m \in \mathbb{N}$ s.t. $(\mu^j, w^j)_{j\ge m} \subset \operatorname{Geo}_0$ as per Definition 3.1.51 for (μ, w). Using 2) in Theorem 4.1.29, we assume $m \in \mathbb{N}$ s.t.

$$\mathscr{C}_{A_j}\big(\bar{\xi}_j\big) \subset \mathscr{S}^{\mathrm{N}}(A) \cap \mathrm{C}[p] \tag{4.90}$$

for all $j \ge m$ without loss of generality. Following 1) in Definition 3.1.51 and further 2) in Corollary 3.1.49, Equation 4.90 ensures we may in fact assume

$$\mu^j(t) \in \mathscr{C}_{A_j}\big(\bar{\xi}_j\big) \subset \mathscr{S}^{\mathrm{N}}(A) \cap \mathrm{C}[p] \tag{4.91}$$

for all $t \in [0,1]$ and $j \ge m$ without loss of generality. Note $\mu^j(0) = \mu_j^0$ and $\mu^j(1) = \mu_j^1$ in each case by hypothesis.

Following 2) in Definition 3.1.51, we select a subsequence $(\mu^j, w^j)_{j\ge m}$ converging to (μ, w) in $\operatorname{Adm}^{[0,1]}$. If there exists $C > 0$ as per Equation 4.89, then Equation 4.91 lets us apply 2) and 3) in Theorem 4.1.25 to Equation 4.89. We calculate

$$\begin{aligned}
\operatorname{Ent}\big(\mu(t), \tau\big) &\le \liminf_{j\in\mathbb{N}} \operatorname{Ent}\big(\mu^j(t), \tau\big) \\
&\le C \cdot \max\Big\{\liminf_{j\in\mathbb{N}} \operatorname{Ent}\big(\bar{\mu}_j^0, \tau\big), \liminf_{j\in\mathbb{N}} \operatorname{Ent}\big(\bar{\mu}_j^1, \tau\big)\Big\} \\
&= C \cdot \max\Big\{\lim_{j\in\mathbb{N}} \operatorname{Ent}\big(\bar{\mu}_j^0, \tau\big), \lim_{j\in\mathbb{N}} \operatorname{Ent}\big(\bar{\mu}_j^1, \tau\big)\Big\} \\
&= C \cdot \max\Big\{\operatorname{Ent}\big(\mu^0, \tau\big), \operatorname{Ent}\big(\mu^1, \tau\big)\Big\} < \infty
\end{aligned}$$

for all $t \in [0,1]$. Moreover, get $\operatorname{Ent}^{\tau} > -\infty$ on $\mathscr{S}(A) \cap \mathrm{C}[p]$ by 1) in Theorem 4.1.25. $\square$

4.2 The logarithmic mean setting

We use quantum relative entropy as measure of quantum information. Assume the logarithmic mean setting. Assuming finitely supported fixed parts, heat flow induces finite-energy admissible paths for all states with finite quantum relative entropy. Up to coarse graining, heat flow is gradient flow of quantum relative entropy. Heat flow further satisfies a steepest entropy ascent property [25] by considering the steepest descent property of gradient flows in smooth Riemannian manifolds [144] and taking limits. We seek conditions s.t. steepest entropy ascent implies quantum noise evolution. If we are able to do so, then we obtain slopes of maximal entropy production, i.e. erasure of quantum information, for sufficiently regular subsets of all bounded normal states. We accomplish this with our maximum entropy production principle [91][92][155].

In Subsection 4.3.1, we consider heat flow as EVI_λ-gradient flow of quantum relative entropy. We use Euler-Lagrange equations of energy functionals and results concerning Hessians of quantum relative entropy in the finite-dimensional setting. If heat flow is EVI_λ-gradient flow of quantum relative entropy, then we have metric slopes as per Equation 4.196 [8][160]. These generalise slopes of maximal entropy production, even absent differential structures, to all normal states with finitely supported fixed part and finite quantum relative entropy. By locality, we restrict our maximum entropy production principle to selection of noise diffusion terms in the finite-dimensional setting and assume such selection is stable under scaling limits. We therefore view quantum Laplacians as generators of quantum noise evolution. In Subsection 4.3.2, we use such description to show strictly positive lower Ricci bounds determine energy-information trade-offs parametrised by lower bounds on quantum noise.

Structure. In Subsection 4.2.1, we discuss fundamental properties of the logarithmic mean setting, define quantum L^2-Wasserstein distances and show heat flow induces finite-energy admissible paths. In Subsection 4.2.2, we show Euler-Lagrange equations and give, to us, necessary results concerning Hessians of quantum relative entropy. In Subsection 4.2.3, we formulate our maximum entropy production principle.

4.2.1 Quantum $\mathbf{L^2}$-Wasserstein distances

Quantum L^2-Wasserstein distances are quantum optimal transport distances in the logarithmic mean setting. Assuming the latter and finitely supported fixed parts, note Theorem 4.2.10 shows heat flow induces finite-energy admissible paths for all states with finite quantum relative entropy. Energy is controlled by time and relative entropy of marginals. Moreover, quantum relative entropy decreases along heat flow.

The logarithmic operator mean and representing function. Definition 4.2.1 gives the logarithmic operator mean. Equation 4.92 induces the Kubo-Mori-Bogoliubov inner product [176]. Note Remark 4.2.2 for its functional derivative. Proposition 4.2.3 gives its symmetric representing function. For details on such representing functions of operator means, we refer to Subsection 2.2.1.

Definition 4.2.1. We define logarithmic operator mean $m_{\log} : (0,\infty) \times (0,\infty) \longrightarrow (0,\infty)$ by setting

$$m_{\log}(t,s) := \frac{t-s}{\log t - \log s} = \int_0^1 t^{\alpha} s^{1-\alpha} d\alpha \tag{4.92}$$

for all $t,s > 0$.

Remark 4.2.2. Note $m_{\log}$ extends to $t,s \geq 0$ since $t \longmapsto t^{\alpha}$ is monotone on $[0,\infty)$ for all $\alpha \in [0,1]$. We have $m_{\log}(0,0) = 0$. Using functional derivative as per Definition 2.3.7 and in the noncommutative chain rule given by Proposition 2.3.10, we have

$$m_{\log}^{-1}(t,s) = (D\log)(t,s) = \int_0^{\infty} (t+\alpha 1)^{-1}(s+\alpha 1)^{-1} d\alpha \tag{4.93}$$

for all $t,s > 0$. Integral characterisations of $m_{\log}$ and $m_{\log}^{-1}$ are well-known [172].

Proposition 4.2.3. *We define* $f_{\log} : (0,\infty) \longrightarrow (0,\infty)$ *by setting*

$$f_{\log}(t) := \frac{t-1}{\log t} \tag{4.94}$$

for all $t > 0$. *Then* $f_{\log}$ *is the unique symmetric representing function s.t.* $m_{f_{\log}} = m_{\log}$.

Proof. If $f_{\log}$ is a symmetric representing function s.t. we have $m_{f_{\log}}(t,s) = f_{\log}(ts^{-1})s = m_{\log}(t,s)$ for all $t,s > 0$, then Definition 2.2.1 yields our claim at once. We directly verify symmetry, as well as $f_{\log}(1) = 1$ and $f_{\log}(ts^{-1})s = m_{\log}(t,s)$ for all $t,s > 0$. The map $t \longmapsto t^{\alpha}$ is operator monotone for all $\alpha \in [0,1]$. Since Equation 4.94 is Equation 4.92 for $s = 1$ in each case, we know operator monotonicity of $f_{\log}$ by its integral characterisation. □

Definition and relation to quantum relative entropy. Using symmetric representing function as per Proposition 4.2.3, Definition 4.2.4 gives the logarithmic mean setting. Proposition 4.2.6 shows the noncommutative chain rule intertwines logarithmic operator mean and noncommutative division operators. Equation 4.95 links quantum optimal transport and noncommutative heat semigroups of quantum Laplacians. The latter uses both Notation 3.2.42 and 1.1) in Corollary 3.2.43. For details on compressing quantum gradients, we refer to Subsection 2.3.1.

Definition 4.2.4. Let $(\phi,\psi,\gamma,\nabla)$ be noncommutative differential structure for tracial AF-C^*-algebras (A,τ) and (B,ω) in (f,θ)-setting. We are in the logarithmic mean setting if $f = f_{\log}$ represents $m_{\log}$ and $\theta = 1$. We further say that it is finite-dimensional if A and B are finite-dimensional.

Notation 4.2.5. Assume the logarithmic mean setting. We write $\mathscr{I}^{\log} := \mathscr{I}^{f,1}$, as well as $E^{\log} := E^{f,1}$ and $\mathscr{W}_{\nabla}^{\log} := \mathscr{W}_{\nabla}^{f,1}$.

Proposition 4.2.6. *Assume the logarithmic mean setting. Let $\xi \in \mathscr{S}^{\mathrm{N}}(A)$ be a fixed state with integrable support. If $x \in L^\infty(A_\xi,\tau)_\nabla$ s.t. $x > 0$ in $L^\infty(A_\xi,\tau)$, then $\log x \in L^\infty(A_\xi,\tau)_\nabla$ and we have*

$$\nabla_\xi \log x = \mathscr{D}_{x,\xi} \nabla_\xi x. \tag{4.95}$$

Proof. Let $x \in L^\infty(A_\xi,\tau)_\nabla$ s.t. $x > 0$ in $L^\infty(A_\xi,\tau)$. The latter implies $0 \notin \mathrm{spec}_{L^\infty(A_\xi,\tau)} x$. Note $\mathrm{spec}_{L^\infty(A,\tau)} x = \mathrm{spec}_{L^\infty(A_\xi,\tau)} x \cup \{0\}$ (cf. 1) in Corollary B.2.35). Let $I \subset [0,\infty)$ be a closed interval s.t. $\mathrm{spec}_{L^\infty(A,\tau)} x \subset I$. Zero is isolated as both spectra are compact.

Let $g \in C^1(I)$ s.t. $g(0) = 0$ and $g(t) = \log t$ for all $t \in \mathrm{spec}_{L^\infty(A_\xi,\tau)} x$. Such g exists by the above discussion. We have $\log x = g(x) \in L^\infty(A_\xi,\tau)$ (cf. Corollary B.2.36). Using $m_{\log}^{-1}(t,s) = (D\log)(t,s)$ for all $t,s > 0$ as per Equation 4.93, we calculate

$$\begin{aligned}\mathscr{D}_{x,\xi} &= m_{\log}^{-1}\big(L_{x,\mathrm{supp}\,\xi}, R_{x,\mathrm{supp}\,\xi}\big)\\ &= D\log\big(L_{x,\mathrm{supp}\,\xi}, R_{x,\mathrm{supp}\,\xi}\big)\\ &= Dg\big(L_{x,\mathrm{supp}\,\xi}, R_{x,\mathrm{supp}\,\xi}\big).\end{aligned}$$

Following 1.1) in Corollary 3.2.43, compressing $\nabla : A_0 \longrightarrow L^2(B,\omega)$ with $\mathrm{supp}\,\xi$ yields symmetric W^*-derivation $\nabla_\xi : \mathscr{A}_\xi \longrightarrow L^2(B_\xi,\omega)$. Using the above calculation in order to account for $\mathscr{D}_{x,\xi}$, applying 1) in Proposition 2.3.10 for ∇_ξ to g selected as above shows $\log x = g(x) \in L^\infty(A_\xi,\tau)_\nabla$ and furthermore Equation 4.95. □

Remark 4.2.7. Note $\nabla = \nabla_\xi$ on $L^\infty(A_\xi,\tau)_\nabla$ by 4.2) in Corollary 3.2.43. We may therefore suppress ξ in the subscript of ∇_ξ in Equation 4.95.

Theorem 4.2.10 uses Lemma 4.2.8, resp. its immediate Corollary 4.2.9. Up to coarse graining, Lemma 4.2.8 implies heat flow is gradient flow of quantum relative entropy. Moreover, Theorem 4.2.35 generalises arguments in their proof without assuming the finite-dimensional setting. We use operator differentiable functions [172]. We review its general case. Let H be a separable Hilbert space and $T > 0$ in $\mathscr{B}(H)$. Equation 4.96 uses Fréchet derivatives in $\mathscr{B}(H)$. For all $S \in \mathscr{B}(H)$, we define $d\log_T(S) \in \mathscr{B}(H)$ by setting

$$d\log_T(S) := \frac{d}{dt}\bigg|_{t=0,\|.\|_{\mathscr{B}(H)}} \log\varphi(t) \tag{4.96}$$

for all Fréchet differentiable maps $t \mapsto \varphi(t) \in \mathscr{B}(H)_{>0}$ s.t. $\varphi(0) = T$ and $\dot\varphi(0) = S$. We obtain $d\log_T : \mathscr{B}(H) \longrightarrow \mathscr{B}(H)$. For all $S \in \mathscr{B}(H)$, we have

$$d\log_T(S) = \int_0^\infty (\alpha I + T)^{-1} S (\alpha I + T)^{-1} d\alpha, \quad \int_0^1 T^\alpha d\log_T(S) T^{1-\alpha} d\alpha = S \tag{4.97}$$

by Subsection 4.3 in [172]. Identities in Equation 4.97 determine $d\log_T$. Equation 4.96 and Equation 4.97 pull back along compressed canonical left- and right-action as given below in the proof of Lemma 4.2.8.

Lemma 4.2.8. *Assume the finite-dimensional logarithmic mean setting. Let $\xi \in \mathscr{S}(A)$ be a fixed state and $[a,b] \subset \mathbb{R}$.*

1) If $\mu : [a,b] \longrightarrow \vartheta(\xi)$ is differentiable for a.e. $t \in [a,b]$, then

$$\frac{d}{dt}\mathrm{Ent}^{\tau}\big(\mu(t)\big) = \tau\big(\sharp\dot{\mu}(t)\log\sharp\mu(t)\big) \tag{4.98}$$

for a.e. $t \in [a,b]$.

2) If $(\mu,w) \in \mathrm{Adm}^{[a,b]}$ s.t. $\mu(t) \in \vartheta(\xi)$ for all $t \in [a,b]$ and furthermore $\sharp w(t) \in B_{\xi}$ for a.e. $t \in [a,b]$, then

$$\frac{d}{dt}\mathrm{Ent}^{\tau}\big(\mu(t)\big) = \tau\big(\sharp\dot{\mu}(t)\log\sharp\mu(t)\big) = \big\langle \mathscr{D}_{\sharp\mu(t),\xi}\sharp w(t), \nabla\sharp\mu(t)\big\rangle_{\omega} \tag{4.99}$$

for a.e. $t \in [a,b]$.

Proof. We use Hilbert space $(A_{\xi}, \|.\|_{\tau})$. Pull-back along compressed canonical left- and right-action preserves Fréchet derivatives and therefore identities as above. These use $A_{\xi,>0}$ and A_{ξ}, rather than $\mathscr{B}(A_{\xi})_{>0}$ and $\mathscr{B}(A_{\xi})$. If $\mu : [a,b] \longrightarrow \vartheta(\xi)$ is differentiable for a.e. $t \in [a,b]$, then Proposition 3.2.49 ensures $\sharp\mu(t) > 0$ in A_{ξ} for all $t \in [a,b]$ and $\sharp\dot{\mu}(t) \in I(\Delta_{\xi})$ for a.e. $t \in [a,b]$. Thus $\log\sharp\mu(t) \in A_{\xi}$ in each case (cf. Corollary B.2.36), hence the map $s \mapsto \log\sharp\mu(s) \in A_{\xi}$ is Fréchet differentiable for all $t \in (a,b)$. Corollary 4.1.27 shows $\mathrm{Ent}(\mu(t),\tau) = \tau(\sharp\mu(t)\log\sharp\mu(t))$ for all $t \in [a,b]$. Note 2) in Proposition 3.2.46 implies $I(\Delta_{\xi}) \subset \ker\tau$. Using the latter, we argue as follows.

We show 1). Assume its setting. If $\dot{\mu}(t)$ exists for $t \in [a,b]$, then traciality, the second identity in Equation 4.97, and $\sharp\dot{\mu}(t) \in \ker\tau$ imply

$$\tau\big(\sharp\mu(t)d\log_{\sharp\mu(t)}\big(\sharp\dot{\mu}(t)\big)\big) = \int_0^1 \tau\big(\sharp\mu(t)^{\alpha}d\log_{\sharp\mu(t)}\big(\sharp\dot{\mu}(t)\big)\sharp\mu(t)^{1-\alpha}\big)d\alpha = \tau\big(\sharp\dot{\mu}(t)\big) = 0. \tag{4.100}$$

We know $\mathrm{Ent}(\mu(t),\tau) = \tau(\sharp\mu(t)\log\sharp\mu(t))$ for all $t \in [a,b]$. Using the latter and the Leibniz rule for Fréchet derivatives, Equation 4.100 lets us calculate

$$\frac{d}{dt}\mathrm{Ent}^{\tau}\big(\mu(t)\big) = \tau\big(\sharp\dot{\mu}(t)\log\sharp\mu(t)\big) + \tau\big(\sharp\mu(t)d\log_{\sharp\mu(t)}\big(\sharp\dot{\mu}(t)\big)\big) = \tau\big(\sharp\dot{\mu}(t)\log\sharp\mu(t)\big) \tag{4.101}$$

for a.e. $t \in [a,b]$. Equation 4.101 shows Equation 4.98. We obtain 1). In particular, we see 1) applies to all elements in $\mathrm{Adm}^{[a,b]}$.

We show 2). Assume its setting. Note finite-dimensionality ensures ξ has integrable support. Using Proposition 4.2.6, Equation 4.98 lets us calculate

$$\frac{d}{dt}\operatorname{Ent}^{\tau}\big(\mu(t)\big) = \tau\big(\sharp\dot{\mu}(t)\log\sharp\mu(t)\big) = \big\langle \sharp w(t), \mathscr{D}_{\sharp\mu(t),\xi}\nabla\sharp\mu(t)\big\rangle_{\omega} \tag{4.102}$$

for a.e. $t \in [a,b]$. We suppress ξ in the subscript in Equation 4.102 as per Remark 4.2.7. We swap $\mathscr{D}_{\sharp\mu(t),\xi} \in \mathscr{B}(B_{\xi})_h$ to the left-hand side of the inner product. Note this requires $\sharp w(t) \in B_{\xi}$. Equation 4.102 shows Equation 4.99. We have 2). $\square$

Corollary 4.2.9. *Assume the finite-dimensional logarithmic mean setting. Let $\xi \in \mathscr{S}(A)$ be a fixed state. For all $\mu \in \operatorname{Fix}_A(\xi)$ and $t > 0$, we have*

1) $-\frac{d}{dt}\operatorname{Ent}^{\tau}\big(h_t(\mu)\big) = \tau\big(\Delta h_t(\sharp\mu)\log h_t(\sharp\mu)\big)$,

2) $\tau\big(\Delta h_t(\sharp\mu)\log h_t(\sharp\mu)\big) = \big\|\mathscr{D}^{\frac{1}{2}}_{h_t(\sharp\mu),\xi}\nabla h_t(\sharp\mu)\big\|^2_{\omega} = \mathscr{I}^{\log}\Big(h_t(\mu), h_t(\mu), \big(\nabla h_t(\sharp\mu)\big)^{\flat}\Big)$.

Proof. For all $t \in \mathbb{R}$, set $\mu(t) := h_t(\mu)$ and $w(t) := -(\nabla\sharp\mu(t))^{\flat}$. Thus $\sharp\dot{\mu}(t) = -\Delta\sharp\mu(t) = \nabla^*\sharp w(t)$ for all $t > 0$ by definition of $h : [0,\infty) \longrightarrow \mathscr{B}(A)$ and construction of extension $h : [0,\infty) \longrightarrow \mathscr{B}(A^*)$, hence $(\mu, w) \in \operatorname{Adm}^{[a,b]}$ for all $[a,b] \subset \mathbb{R}$ by finite-dimensionality.

If $t > 0$, then $\sharp\mu(t) > 0$ in A_{ξ} by 2.2) in Theorem 3.2.40 and furthermore $\sharp w(t) \in B_{\xi}$ by 1.1) in Corollary 3.2.43. Using the latter and 1.2) in Corollary 3.2.43, applying 2) in Lemma 4.2.8 for all $[a,b] \subset [0,\infty)$ yields both 1) and the first identity in 2) at once. Note 1) in Proposition 3.2.45 shows the second one immediately. $\square$

Theorem 4.2.10. *Let $(\phi, \psi, \gamma, \nabla)$ be noncommutative differential structure for tracial AF-C^*-algebras (A, τ) and (B, ω) in the logarithmic mean setting. Let $\xi \in \mathscr{S}(A)$ be a finitely supported fixed state. For all $\mu \in \operatorname{Fix}^{\mathrm{N}}_A(\xi) \cap \operatorname{dom}\operatorname{Ent}^{\tau}$ and $t \geq 0$, we have*

1) $h_t(\mu) \in \operatorname{dom}\operatorname{Ent}^{\tau}$,

2) $\operatorname{Ent}(\xi, \tau) \leq \operatorname{Ent}\big(h_t(\mu), \tau\big) \leq \operatorname{Ent}(\mu, \tau)$,

3) $\mathscr{W}^{\log}_{\nabla}\big(\mu, h_t(\mu)\big)^2 \leq t \cdot \Big(\operatorname{Ent}(\mu, \tau) - \operatorname{Ent}\big(h_t(\mu), \tau\big)\Big) < \infty$.

Proof. Note 2) implies 1). We show 2) and 3). Let $\mu \in \operatorname{Fix}^{\mathrm{N}}_A(\xi) \cap \operatorname{dom}\operatorname{Ent}^{\tau}$. If $\xi_j \neq 0$ for $j \in \mathbb{N}$, then $\overline{h_t(\mu)}_j = h_t(\bar{\mu}_j) \in \mathscr{S}\left(A_{j,\bar{\xi}_j}\right)$ for all $t \in [0,\infty]$ by 1.3) in Proposition 3.2.34 and 1.3) in Theorem 3.2.40. Using the latter and 1.2) in Proposition 2.1.31, we reduce to the finite-dimensional setting. For all $t \in [0,\infty]$, we have

$$\operatorname{Ent}\big(h_t(\mu), \tau\big) = \lim_{j \in \mathbb{N}} \operatorname{Ent}\big(h_t(\bar{\mu}_j), \tau\big) \tag{4.103}$$

by 3) in Theorem 4.1.29, as well as

$$\mathscr{W}_{\nabla}^{\log}\big(\mu, h_t(\mu)\big) = \lim_{j\in\mathbb{N}} \mathscr{W}_{\nabla_j}^{\log}\big(\bar{\mu}_j, h_t(\bar{\mu}_j)\big) \tag{4.104}$$

by 1) and 3) in Theorem 3.1.47. We use both equations above to reduce as follows.

Equation 4.103 by itself shows 2) if for a.e. $j \in \mathbb{N}$, we have

$$\mathrm{Ent}\big(\bar{\xi}_j, \tau\big) \le \mathrm{Ent}\big(h_t(\bar{\mu}_j), \tau\big) \le \mathrm{Ent}\big(\bar{\mu}_j, \tau\big) \tag{4.105}$$

for all $t \ge 0$. Equation 4.103 and Equation 4.104 show 3) if for a.e. $j \in \mathbb{N}$, we have

$$\mathscr{W}_{\nabla_j}^{\log}\big(\bar{\mu}_j, h_t(\bar{\mu}_j)\big)^2 \le t \cdot \Big(\mathrm{Ent}\big(\bar{\mu}_j, \tau\big) - \mathrm{Ent}\big(h_t(\bar{\mu}_j), \tau\big)\Big) \tag{4.106}$$

for all $t \ge 0$. Since $\xi_j \neq 0$ for a.e. $j \in \mathbb{N}$, Equation 4.105 reduces 2) and Equation 4.106 reduces 3) to the finite-dimensional setting.

Assume A and B are finite-dimensional. In particular, $\xi \neq 0$. We show 2). Note 2.3) in Proposition 3.2.32 ensures $\sup_{t\in[0,\infty]} \|h_t(\mu)\|_\infty \le \|\mu\|_\infty$. Get compact $K \subset [0,\infty)$ s.t. $\mathrm{spec}_A\, h_t(\mu) \subset K$ for all $t \in [0,\infty]$. Let $g \in C_b(\mathbb{R})$ s.t. $g(\lambda) = \lambda \log \lambda$ for all $\lambda \in K$. Using Lemma A.2.5 and Corollary 4.1.27, we see $g \in C_b(\mathbb{R})$ ensures we define continuous map $F : [0,\infty) \longrightarrow \mathbb{R}$ by setting

$$t \mapsto F(t) := \mathrm{Ent}\big(h_t(\mu), \tau\big) = \tau\big(\sharp h_t(\mu) \log \sharp h_t(\mu)\big) = \tau\big(g\big(\sharp h_t(\mu)\big)\big) \tag{4.107}$$

for all $t \ge 0$ (cf. Remark A.2.3 and Remark A.2.4). This requires strong continuity as per 1) in Proposition 3.2.32. We obtain $\mathrm{Ent}(\xi, \tau) = \lim_{t\to\infty} F(t)$. Corollary 4.2.9 shows

$$-\frac{d}{dt}F(t) = \big\|\mathscr{D}^{\frac{1}{2}}_{h_t(\sharp\mu),\xi} \nabla h_t\big(\sharp\mu\big)\big\|_\omega^2 = \mathscr{I}^{\log}\Big(h_t(\mu), h_t(\mu), \big(\nabla h_t\big(\sharp\mu\big)\big)^\flat\Big) \ge 0 \tag{4.108}$$

for all $t > 0$. Equation 4.107 and Equation 4.108 show 2).

We show 3). If $t = 0$, then our claim follows since $\mathscr{W}_{\nabla}^{\log}$ is a metric. Assume $t > 0$. For all $s \in [0,t]$, set $\mu(s) := h_s(\mu)$ and $w(s) := -(\nabla \sharp \mu(s))^\flat$. We show $(\mu, w) \in \mathrm{Adm}^{[0,t]}(\mu, h_t(\mu))$ in the proof of Corollary 4.2.9. Using the map $s \mapsto \varphi(s) := ts$, we rescale $(\mu, w) \in \mathrm{Adm}^{[0,t]}$ to $(\mu', w') \in \mathrm{Adm}^{[0,1]}$ as per Remark 3.1.22. Using the map $s \mapsto \varphi^{-1}(s) := t^{-1}s$, we likewise rescale $(\mu', w') \in \mathrm{Adm}^{[0,1]}$ to $(\mu, w) \in \mathrm{Adm}^{[0,t]}$. We further apply Proposition 3.1.21 to the latter below. Equation 4.108 lets us calculate

$$\begin{aligned}
\mathscr{W}_{\nabla}^{\log}\big(\mu, h_t(\mu)\big)^2 &\le E^{\log}\big(\mu', w'\big) \\
&= t \cdot \int_0^t \mathscr{I}^{\log}\big(\mu(s), w(s), w(s)\big) ds \\
&= t \cdot \Big(\mathrm{Ent}(\mu, \tau) - \mathrm{Ent}\big(h_t(\mu), \tau\big)\Big).
\end{aligned}$$

Note 2) ensures the right-hand side is finite. As such, the above calculation shows 3) at once. The general case follows as discussed above. □

4.2.2 The finite-dimensional setting

We discuss the finite-dimensional logarithmic mean setting. We introduce Λ-operations to simplify those of our calculations involving derivatives and noncommutative division operators. Theorem 4.2.19 formulates Euler-Lagrange equations. Theorem 4.2.22 gives two differential equations for Hessians of quantum relative entropy.

Euler-Lagrange equations. Theorem 4.2.22 requires Euler-Lagrange equations as per Theorem 4.2.19. We introduce Λ-operations in order to simplify calculations. Let $(\phi,\psi,\gamma,\nabla)$ be noncommutative differential structure for tracial AF-C^*-algebras (A,τ) and (B,ω) in the finite-dimensional logarithmic mean setting. Let $\xi \in \mathscr{S}(A)$ be a fixed state. Finite-dimensionality ensures finite support.

Following Definition 4.2.1 and Remark 4.2.2, note Proposition 4.2.11 gives integral characterisation of multiplication and division operators in our setting. These allow for direct calculations. We moreover obtain smooth maps in Definition 4.2.13.

Proposition 4.2.11. *For all $x \in A_{\xi,+}$ and $u \in B_\xi$, we have*

1) $\mathscr{M}_{x,\xi}(u) = \int_0^1 \phi(x)^\alpha u \psi(x)^{1-\alpha} d\alpha$,

2) $\mathscr{D}_{x,\xi}(u) = \int_0^\infty \big(\phi(x)+\alpha 1_B\big)^{-1} u \big(\psi(x)+\alpha 1_B\big)^{-1} d\alpha$ *if* $x>0$ *in* A_ξ.

Proof. We show 1). Let $x \in A_{\xi,+}$ and $u \in B_\xi$. Using 1) in Lemma 2.2.13, Equation 4.92 lets us calculate

$$\begin{aligned}\mathscr{M}_{x,\xi}(u) = \mathscr{M}_x(u) &= m_{\log}\Big(L_x^\phi, R_x^\psi\Big)(u)\\ &= \int_0^1 \big(L_x^\phi\big)^\alpha \Big(\big(R_x^\psi\big)^{1-\alpha}(u)\Big) d\alpha\\ &= \int_0^1 \phi(x)^\alpha u \psi(x)^{1-\alpha} d\alpha.\end{aligned}$$

The above calculation shows 1). We show 2). Assume $x>0$ in A_ξ. For all $\alpha>0$, we define $g^\alpha \in C_b([0,\infty)\times[0,\infty))$ by setting

$$g^\alpha(t,s) := \big(t+\alpha\big)^{-1}\big(s+\alpha\big)^{-1} \tag{4.109}$$

for all $t,s \geq 0$. Since $u \in B_\xi$, 2.3) in Lemma 2.1.67 implies

$$g^\alpha\Big(L_x^\phi, R_x^\psi\Big)(u) = g^\alpha\Big(L^\phi_{x,\operatorname{supp}\xi}, R^\psi_{x,\operatorname{supp}\xi}\Big)(u) \tag{4.110}$$

for all $\alpha>0$. Using 2) in Lemma 2.2.13, Equation 4.93 and Equation 4.110 show

$$\begin{aligned}\mathscr{D}_{x,\xi}(u) = \left(\mathscr{M}_{x,\xi}\right)^{-1}(u) = m_{\log}^{-1}\left(L^{\phi}_{x,\operatorname{supp}\xi}, R^{\psi}_{x,\operatorname{supp}\xi}\right)(u)\\ &= \int_0^\infty \left(L^{\phi}_{x,\operatorname{supp}\xi} + \alpha I_{B_\xi}\right)^{-1}\left(\left(R^{\psi}_{x,\operatorname{supp}\xi} + \alpha I_{B_\xi}\right)^{-1}(u)\right) d\alpha\\ &= \int_0^\infty g^\alpha\left(L^{\phi}_{x,\operatorname{supp}\xi}, R^{\psi}_{x,\operatorname{supp}\xi}\right)(u)\, d\alpha\\ &= \int_0^\infty g^\alpha\left(L^{\phi}_{x}, R^{\psi}_{x}\right)(u)\, d\alpha\\ &= \int_0^\infty \left(L^{\phi}_{x} + \alpha I_{B}\right)^{-1}\left(\left(R^{\psi}_{x} + \alpha I_{B}\right)^{-1}(u)\right) d\alpha\\ &= \int_0^\infty \left(\phi(x) + \alpha 1_B\right)^{-1} u \left(\psi(x) + \alpha 1_B\right)^{-1} d\alpha.\end{aligned}$$

Get 2). Note $x > 0$ in A_ξ is required for $\mathscr{D}_{x,\xi} = (\mathscr{M}_{x,\xi})^{-1}$ to be defined. □

We have Riemannian manifold $(\vartheta(\xi), g^\xi)$ as per Definition 3.2.52 embedded in $\mathscr{S}(A_\xi)$ as per Proposition 3.2.49 s.t. its tangent bundle is indeed trivial with fibre $I(\Delta_\xi)$. For all $\mu \in \vartheta(\xi)$, we introduce operators

$$\mathfrak{F}_\mu = \nabla^* \mathscr{M}_{\sharp\mu,\xi} \nabla = \nabla^* \mathscr{M}_{\sharp\mu} \nabla,\ \mathfrak{G}_\mu = \mathscr{M}_{\sharp\mu,\xi}\nabla = \mathscr{M}_{\sharp\mu}\nabla \tag{4.111}$$

with domain $\operatorname{im}\Delta_\xi$ as per Definition 3.2.50 by Proposition 3.2.51. In each case, get $\mathfrak{F}_\mu > 0$ in $\mathscr{B}(\operatorname{im}\Delta_\xi)$ s.t. $[\mathfrak{F}_\mu, \mathrm{Adj}] = 0$. Moreover, note $\mathfrak{G}_\mu \in \mathscr{B}(\operatorname{im}\Delta_\xi, B_\xi)$ intertwines adjoining. We therefore restrict to $I(\Delta_\xi) = \operatorname{im}\Delta_\xi \cap A_{\xi,h}$.

Notation 4.2.12. Let X and Y be smooth manifolds. We write $dg : TX \longrightarrow TY$ for the first differential form of a smooth map $g : X \longrightarrow Y$ [144], i.e. its total derivative. We further write $d_\mu g \in \mathrm{Hom}(T_\mu X, T_{g(\mu)}Y)$ upon evaluation at $\mu \in X$.

Definition 4.2.13. We consider Riemannian manifold $(\vartheta(\xi), g^\xi)$.

1) We define $\mathscr{M}_\xi : \vartheta(\xi) \longrightarrow \mathrm{GL}(\mathscr{B}(B_\xi))$ by setting $\mathscr{M}_\xi(\mu) := \mathscr{M}_{\sharp\mu,\xi}$ for all $\mu \in \vartheta(\xi)$. We define $\mathscr{D}_\xi : \vartheta(\xi) \longrightarrow \mathrm{GL}(\mathscr{B}(B_\xi))$ by setting $\mathscr{D}_\xi(\mu) := \mathscr{D}_{\sharp\mu,\xi}$ for all $\mu \in \vartheta(\xi)$.
2) We define $\mathfrak{F} : \vartheta(\xi) \longrightarrow \mathrm{GL}(\mathscr{B}(\operatorname{im}\Delta_\xi))$ by setting $\mathfrak{F}(\mu) := \mathfrak{F}_\mu$ for all $\mu \in \vartheta(\xi)$. We define $\mathfrak{G} : \vartheta(\xi) \longrightarrow \mathscr{B}(\operatorname{im}\Delta_\xi, B_\xi)$ by setting $\mathfrak{G}(\mu) := \mathfrak{G}_\mu$ for all $\mu \in \vartheta(\xi)$.

Remark 4.2.14. Proposition 4.2.11 shows all maps in Definition 4.2.13 are smooth. We use this throughout our discussion.

Definition 4.2.15 gives Λ-operations. Proposition 4.2.17 and Lemma 4.2.18 simplify calculations involving derivatives and noncommutative division operators. Using their results, Theorem 4.2.19 formulates Euler-Lagrange equations.

Definition 4.2.15. Let $\mu \in \vartheta(\xi)$.

1) For all $x \in A_\xi$ and $u \in B_\xi$, set

$$\Lambda_\mu(x,u) := \Lambda_\mu^\phi(x,u) + \Lambda_\mu^\psi(x,u) \in B_\xi \tag{4.112}$$

using

$$\Lambda_\mu^\phi(x,u) := \int_0^\infty \big(\phi(\sharp\mu) + \alpha 1_B\big)^{-1} \phi(x) \big(\phi(\sharp\mu) + \alpha 1_B\big)^{-1} u \big(\psi(\sharp\mu) + \alpha 1_B\big)^{-1} d\alpha,$$

$$\Lambda_\mu^\psi(x,u) := \int_0^\infty \big(\phi(\sharp\mu) + \alpha 1_B\big)^{-1} u \big(\psi(\sharp\mu) + \alpha 1_B\big)^{-1} \psi(x) \big(\psi(\sharp\mu) + \alpha 1_B\big)^{-1} d\alpha.$$

2) For all $u, v \in B_\xi$, set

$$\Lambda_\mu^*(u,v) := \Lambda_\mu^{\phi,*}(u,v) + \Lambda_\mu^{\psi,*}(u,v) \in A_\xi \tag{4.113}$$

using

$$\Lambda_\mu^{\phi,*}(u,v) := \phi^*\left(\int_0^\infty \big(\phi(\sharp\mu) + \alpha 1_B\big)^{-1} v \big(\psi(\sharp\mu) + \alpha 1_B\big)^{-1} u^* \big(\phi(\sharp\mu) + \alpha 1_B\big)^{-1} d\alpha \right),$$

$$\Lambda_\mu^{\psi,*}(u,v) := \psi^*\left(\int_0^\infty \big(\psi(\sharp\mu) + \alpha 1_B\big)^{-1} u^* \big(\phi(\sharp\mu) + \alpha 1_B\big)^{-1} v \big(\psi(\sharp\mu) + \alpha 1_B\big)^{-1} d\alpha \right).$$

Proposition 4.2.16. *Let V be a unital Banach *-algebra. If a map $F : [a,b] \longrightarrow \mathrm{GL}(V)$ is Fréchet differentiable in an open neighbourhood of $t_0 \in (a,b)$, then*

$$\left.\frac{d}{dt}\right|_{t=t_0, \|\cdot\|_V} F(t)^{-1} = -F(t_0)^{-1} \cdot \left.\frac{d}{dt}\right|_{t=t_0, \|\cdot\|_V} F(t) \cdot F(t_0)^{-1}. \tag{4.114}$$

Proof. Let $F : [a,b] \longrightarrow \mathrm{GL}(V)$ be Fréchet differentiable in an open neighbourhood of $t_0 \in (a,b)$. The Leibniz rule lets us calculate

$$0 = \left.\frac{d}{dt}\right|_{t=t_0, \|\cdot\|_V} F(t)F(t)^{-1} = \left.\frac{d}{dt}\right|_{t=t_0, \|\cdot\|_V} F(t) \cdot F(t_0)^{-1} + F(t_0) \cdot \left.\frac{d}{dt}\right|_{t=t_0, \|\cdot\|_V} F(t)^{-1}. \tag{4.115}$$

Equation 4.114 follows by solving Equation 4.115 for $\left.\frac{d}{dt}\right|_{t=t_0, \|\cdot\|_V} F(t)^{-1}$. $\square$

Proposition 4.2.17. *For all $\mu \in \vartheta(\xi)$, $x \in I(\Delta_\xi)$ and $u,v \in B_\xi$, we have*

1) $\langle \Lambda_\mu(x,u), v\rangle_\omega = \langle x, \Lambda_\mu^*(u,v)\rangle_\tau$,

2) $d_\mu \mathscr{D}_\xi(x^\flat)(u) = -\Lambda_\mu(x,u)$.

Proof. For all $\mu \in \vartheta(\xi)$, $x \in I(\Delta_\xi)$ and $u,v \in B_\xi$, we directly verify

$$\langle \Lambda_\mu^\phi(x,u), v\rangle_\omega = \langle x, \Lambda_\mu^{\phi,*}(u,v)\rangle_\tau,\ \langle \Lambda_\mu^\psi(x,u), v\rangle_\omega = \langle x, \Lambda_\mu^{\psi,*}(u,v)\rangle_\tau. \tag{4.116}$$

Using Equation 4.112 and Equation 4.113, note Equation 4.116 implies 1) by definition.

We show 2). Let $\mu \in \vartheta(\xi)$, $x \in I(\Delta_\xi)$ and $u \in B_\xi$. Let $\varepsilon > 0$ and $\mu : (-\varepsilon,\varepsilon) \longrightarrow \vartheta(\xi)$ smooth map s.t. $\mu(0) = \mu$ and $\dot\mu(0) = x^\flat$. Then 2) in Proposition 4.2.11 shows

$$\mathscr{D}_{\sharp\mu(t),\xi}(u) = \int_0^\infty \left(\phi\left(\sharp\mu(t)\right) + \alpha 1_B\right)^{-1} u \left(\psi\left(\sharp\mu(t)\right) + \alpha 1_B\right)^{-1} d\alpha \tag{4.117}$$

for all $t \in (-\varepsilon,\varepsilon)$. Using $d_\mu \mathscr{D}_\xi(x^\flat)(u) = \frac{d}{dt}\big|_{t=0,\|.\|_B} \mathscr{D}_{\sharp\mu(t),\xi}(u)$ and further applying Fréchet derivative to the integrand in Equation 4.117, the Leibniz rule lets us calculate

$$\begin{aligned}\frac{d}{dt}\bigg|_{t=0,\|.\|_B} \mathscr{D}_{\sharp\mu(t),\xi}(u) &= \int_0^\infty \frac{d}{dt}\bigg|_{t=0,\|.\|_B} \left(\phi\left(\sharp\mu(t)\right) + \alpha 1_B\right)^{-1} u \left(\psi\left(\sharp\mu\right) + \alpha 1_B\right)^{-1} d\alpha \\ &+ \int_0^\infty \left(\phi\left(\sharp\mu\right) + \alpha 1_B\right)^{-1} u \frac{d}{dt}\bigg|_{t=0,\|.\|_B} \left(\psi\left(\sharp\mu(t)\right) + \alpha 1_B\right)^{-1} d\alpha.\end{aligned}$$

The above is the integral characterisation of $\frac{d}{dt}\big|_{t=0,\|.\|_B} \mathscr{D}_{\sharp\mu(t),\xi}(u)$. Proposition 4.2.16 further implies

$$\frac{d}{dt}\bigg|_{t=0,\|.\|_B} \left(\phi\left(\sharp\mu(t)\right) + \alpha 1_B\right)^{-1} = -\left(\phi\left(\sharp\mu\right) + \alpha 1_B\right)^{-1} \phi(x) \left(\phi\left(\sharp\mu\right) + \alpha 1_B\right)^{-1} \tag{4.118}$$

and

$$\frac{d}{dt}\bigg|_{t=0,\|.\|_B} \left(\psi\left(\sharp\mu(t)\right) + \alpha 1_B\right)^{-1} = -\left(\psi\left(\sharp\mu\right) + \alpha 1_B\right)^{-1} \psi(x) \left(\psi\left(\sharp\mu\right) + \alpha 1_B\right)^{-1} \tag{4.119}$$

for all $\alpha > 0$. Equation 4.118 lets us calculate

$$\int_0^\infty \frac{d}{dt}\bigg|_{t=0,\|.\|_B} \left(\phi\left(\sharp\mu(t)\right) + \alpha 1_B\right)^{-1} u \left(\psi\left(\sharp\mu\right) + \alpha 1_B\right)^{-1} d\alpha = -\Lambda_\mu^\phi(x,u), \tag{4.120}$$

whereas Equation 4.119 lets us calculate

$$\int_0^\infty \left(\phi\left(\sharp\mu\right) + \alpha 1_B\right)^{-1} u \frac{d}{dt}\bigg|_{t=0,\|.\|_B} \left(\psi\left(\sharp\mu(t)\right) + \alpha 1_B\right)^{-1} d\alpha = -\Lambda_\mu^\psi(x,u). \tag{4.121}$$

Using Equation 4.112, applying Equation 4.120 and Equation 4.121 to the integral characterisation of $\frac{d}{dt}\big|_{t=0,\|.\|_B}\mathscr{D}_{\sharp\mu(t),\xi}(u)$ yields

$$\frac{d}{dt}\bigg|_{t=0,\|.\|_B}\mathscr{D}_{\sharp\mu(t),\xi}(u) = -\Big(\Lambda_\mu^\phi(x,u) + \Lambda_\mu^\psi(x,u)\Big) = -\Lambda_\mu(x,u). \tag{4.122}$$

Equation 4.122 shows 2) at once. □

Lemma 4.2.18. *For all $\mu \in \vartheta(\xi)$ and $x, y, z \in I(\Delta_\xi)$, we have*

$$\langle d_\mu \mathfrak{F}^{-1}(x^\flat)(y), z\rangle_\tau = -\langle x, \Lambda_\mu^*\big(\Theta(\mu, y^\flat), \Theta(\mu, z^\flat)\big)\rangle_\tau. \tag{4.123}$$

Proof. Let $\mu \in \vartheta(\xi)$ and $x, y, z \in I(\Delta_\xi)$. We calculate $d_\mu\mathfrak{F}^{-1}(x^\flat)(y)$. Note

$$\mathfrak{F}_\mu^{-1} = \left(\nabla^* \mathscr{M}_{\sharp\mu,\xi}\nabla\right)^{-1},\ \mathscr{M}_{\sharp\mu,\xi} = \mathscr{D}_{\sharp\mu,\xi}^{-1} \in \mathrm{GL}(\mathscr{B}(B_\xi)). \tag{4.124}$$

Using the first identity in Equation 4.124, Proposition 4.2.16 implies

$$d_\mu\mathfrak{F}^{-1}(x^\flat)(y) = -\mathfrak{F}_\mu^{-1}\Big(d_\mu\big(\nabla^*\mathscr{M}_\xi\nabla\big)(x^\flat)\big(\mathfrak{F}_\mu^{-1}(y)\big)\Big). \tag{4.125}$$

Since ∇ and ∇^* are bounded linear, get $d_\mu(\nabla^*\mathscr{M}_\xi\nabla)(x^\flat) = \nabla^* d_\mu\mathscr{M}_\xi(x^\flat)\nabla$. Applying the latter to Equation 4.125 yields

$$d_\mu\mathfrak{F}^{-1}(x^\flat)(y) = -\mathfrak{F}_\mu^{-1}\Big(\nabla^* d_\mu\mathscr{M}_\xi(x^\flat)\big(\nabla\mathfrak{F}_\mu^{-1}(y)\big)\Big). \tag{4.126}$$

We therefore calculate $d_\mu\mathscr{M}_\xi(x^\flat)(\nabla\mathfrak{F}_\mu^{-1}(y))$ in order to calculate $d_\mu\mathfrak{F}^{-1}(x^\flat)(y)$. Using the second identity in Equation 4.124, Proposition 4.2.16 implies

$$d_\mu\mathscr{M}_\xi(x^\flat) = -\mathscr{M}_{\sharp\mu,\xi}d_\mu\mathscr{D}_\xi(x^\flat)\mathscr{M}_{\sharp\mu,\xi}. \tag{4.127}$$

Applying 2) in Proposition 4.2.17 for $u = \mathscr{M}_{\sharp\mu,\xi}(\nabla\mathfrak{F}_\mu^{-1}(y))$ to Equation 4.127 evaluated on $\nabla\mathfrak{F}_\mu^{-1}(y)$ yields

$$\begin{aligned} d_\mu\mathscr{M}_\xi(x^\flat)\big(\nabla\mathfrak{F}_\mu^{-1}(y)\big) &= -\mathscr{M}_{\sharp\mu,\xi}\Big(d_\mu\mathscr{D}_\xi(x^\flat)\big(\mathscr{M}_{\sharp\mu,\xi}\big(\mathfrak{F}_\mu^{-1}(y)\big)\big)\Big) \\ &= \mathscr{M}_{\sharp\mu,\xi}\big(\Lambda_\mu\big(x, \mathscr{M}_{\sharp\mu,\xi}\big(\nabla\mathfrak{F}_\mu^{-1}(y)\big)\big)\big). \end{aligned}$$

Using $\mathscr{M}_{\sharp\mu,\xi}(\nabla\mathfrak{F}_\mu^{-1}(y)) = \sharp\Theta\big(\mu, y^\flat\big)$, we therefore obtain

$$d_\mu\mathscr{M}_\xi(x^\flat)\big(\nabla\mathfrak{F}_\mu^{-1}(y)\big) = \mathscr{M}_{\sharp\mu,\xi}\Big(\Lambda_\mu\big(x, \sharp\Theta\big(\mu, y^\flat\big)\big)\Big). \tag{4.128}$$

Equation 4.126 and Equation 4.128 show

$$d_\mu \mathfrak{F}^{-1}(x^\flat)(y) = -\mathfrak{F}_\mu^{-1}\Big(\nabla^* \mathscr{M}_{\sharp\mu,\xi}\Big(\Lambda_\mu\big(x, \sharp\Theta(\mu, y^\flat)\big)\Big)\Big). \tag{4.129}$$

We show Equation 4.123. Equation 4.129, together with 1) in Proposition 4.2.17 applied to the third identity in our below calculation, lets us calculate

$$\begin{aligned}\langle d_\mu \mathfrak{F}^{-1}(x^\flat)(y), z\rangle_\tau &= -\langle -\mathfrak{F}_\mu^{-1}\Big(\nabla^* \mathscr{M}_{\sharp\mu,\xi}\Big(\Lambda_\mu\big(x, \sharp\Theta(\mu, y^\flat)\big)\Big)\Big), z\rangle_\omega \\ &= -\langle \Lambda_\mu\big(x, \sharp\Theta(\mu, y^\flat)\big), \sharp\Theta(\mu, z^\flat)\rangle_\omega \\ &= -\langle x, \Lambda_\mu^*\big(\sharp\Theta(\mu, y^\flat), \sharp\Theta(\mu, z^\flat)\big)\rangle_\tau.\end{aligned}$$

The above calculation shows Equation 4.123. □

Theorem 4.2.19. *Let $(\phi, \psi, \gamma, \nabla)$ be noncommutative differential structure for tracial AF-C^*-algebras (A, τ) and (B, ω) in the finite-dimensional logarithmic mean setting. Let $\xi \in \mathscr{S}(A)$ be a fixed state. A smooth path $\mu : [0,1] \longrightarrow \vartheta(\xi)$ satisfies the Euler-Lagrange equations of the energy functional induced by g^ξ if and only if*

$$\frac{d}{ds}\Big|_{s=t, \|\cdot\|_{A^*}} \mathfrak{F}_{\mu(s)}^{-1}\big(\dot\mu(s)\big) = -\frac{1}{2}\Lambda_{\mu(t)}^*\big(\sharp\Theta\big(\mu(t), \dot\mu(t)\big), \sharp\Theta\big(\mu(t), \dot\mu(t)\big)\big) \tag{4.130}$$

for all $t \in (0,1)$.

Proof. We consider first variation of energy [144]. Note $T\vartheta(\xi) = \vartheta(\xi) \times I(\Delta_\xi)^\flat$ is trivial by 2) in Proposition 3.2.49. It suffices to solve for critical points of the energy functional induced by g^ξ on variations of form $\mu(t,\varepsilon) = \mu(t) + \varepsilon\eta(t)$ using $\eta \in C_0^\infty([0,1], I(\Delta_\xi)^\flat)$ and $\varepsilon \in (-\delta, \delta)$ for $\delta > 0$ sufficiently small. The latter is chosen s.t. $\mu(t,\varepsilon) \in \vartheta(\xi)$ in each case.

Let $\mu(t,\varepsilon) := \mu(t) + \varepsilon\eta(t)$ be such a variation. Lemma 4.2.18 shows

$$\langle d_{\mu(t)}\mathfrak{F}^{-1}\big(\eta(t)\big)\big(\sharp\dot\mu(t)\big), \sharp\dot\mu(t)\rangle_\tau = -\langle \sharp\eta(t), \Lambda_{\mu(t)}^*\big(\sharp\Theta\big(\mu(t), \dot\mu(t)\big), \sharp\Theta\big(\mu(t), \dot\mu(t)\big)\big)\rangle_\tau \tag{4.131}$$

for all $t \in [0,1]$. Note $g^\xi \cong \mathfrak{F}^{-1}$ via GNS-inner product of τ restricted to A_ξ. Using the latter, we calculate

$$\begin{aligned}&\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\int_0^1 g_{\mu(t,\varepsilon)}^\xi\big(\dot\mu(t) + \varepsilon\dot\eta(t), \dot\mu(t) + \varepsilon\dot\eta(t)\big)dt \\ &= \int_0^1 2g_{\mu(t)}^\xi\big(\dot\mu(t), \dot\eta(t)\big) + \langle \frac{d}{d\varepsilon}\Big|_{\varepsilon=0, \|\cdot\|_{A^*}} \mathfrak{F}^{-1}\big(\mu(t,\varepsilon)\big)\big(\sharp\dot\mu(t)\big), \sharp\dot\mu(t)\rangle_\tau dt \\ &= \int_0^1 2g_{\mu(t)}^\xi\big(\dot\mu(t), \dot\eta(t)\big) + \langle d_{\mu(t)}\mathfrak{F}^{-1}\big(\eta(t)\big)\big(\sharp\dot\mu(t)\big), \sharp\dot\mu(t)\rangle_\tau dt.\end{aligned}$$

We thus apply Equation 4.131, symmetry of the real inner product and integration by parts in order to calculate

$$
\begin{aligned}
&\frac{d}{d\varepsilon}\Big|_{\varepsilon=0,\|.\|_{A^*}} \int_0^1 g^{\xi}_{\mu(t,\varepsilon)}\big(\dot\mu(t)+\varepsilon\dot\eta(t),\dot\mu(t)+\varepsilon\dot\eta(t)\big)dt\\
&=\int_0^1 2g^{\xi}_{\mu(t)}\big(\dot\mu(t),\dot\eta(t)\big)-\big\langle\Lambda^*_{\mu(t)}\big(\sharp\Theta\big(\mu(t),\dot\mu(t)\big),\sharp\Theta\big(\mu(t),\dot\mu(t)\big)\big),\sharp\eta(t)\big\rangle_\tau dt\\
&=-\Bigg(\int_0^1 2\Big\langle\frac{d}{ds}\Big|_{s=t,\|.\|_{A^*}}\mathfrak{F}^{-1}_{\mu(s)}\big(\sharp\dot\mu(s)\big),\sharp\eta(t)\Big\rangle_\tau+\big\langle\Lambda^*_{\mu(t)}\big(\sharp\Theta\big(\mu(t),\dot\mu(t)\big),\sharp\Theta\big(\mu(t),\dot\mu(t)\big)\big),\sharp\eta(t)\big\rangle_\tau dt\Bigg).
\end{aligned}
$$

We solve for critical points of the energy functional induced by g^{ξ}. Using the formula for first variation of energy (cf. proof of Theorem IX.4.3 in [144]), the above calculation hence shows Equation 4.130 gives Euler-Lagrange equations. □

Hessians of quantum relative entropy. Theorem 4.2.22 gives two differential equations for Hessians of quantum relative entropy used in Lemma 4.3.7, i.e. required for our equivalence theorem. Let $(\phi,\psi,\gamma,\nabla)$ be noncommutative differential structure for tracial AF-C^*-algebras (A,τ) and (B,ω) in the finite-dimensional logarithmic mean setting. Let $\xi\in\mathscr{S}(A)$ be a fixed state. Finite-dimensionality ensures finite support.

Note 2.2) in Theorem 3.2.40 and 1) in Corollary 4.2.9 imply smoothness of quantum relative entropy restricted to relative interiors. Proposition 4.2.21 expresses Hessians of quantum relative entropy in terms of Λ-operations.

Notation 4.2.20. Let $\operatorname{Hess}\operatorname{Ent}^\tau$ denote the Hessian of Ent^τ restricted to $\vartheta(\xi)$. We write $\operatorname{Hess}_\mu\operatorname{Ent}^\tau(\eta):=\operatorname{Hess}_\mu\operatorname{Ent}^\tau(\eta,\eta)$ upon evaluation at $\mu\in\vartheta(\xi)$ and $(\eta,\eta)\in T_\mu\vartheta(\xi)^2$.

Proposition 4.2.21. *For all $\mu\in\vartheta(\xi)$ and $\eta\in I(\Delta_\xi)^\flat$, we have*

$$
\operatorname{Hess}_\mu\operatorname{Ent}^\tau(\eta)=-\frac{1}{2}\big\langle\Lambda^*_\mu\big(\sharp\Theta(\mu,\eta),\sharp\Theta(\mu,\eta)\big),\Delta\sharp\mu\big\rangle_\tau+g^{\xi}_\mu\big(\eta,(\Delta\sharp\eta)^\flat\big). \tag{4.132}
$$

Proof. Let $\mu\in\vartheta(\xi)$ and $\eta\in I(\Delta_\xi)^\flat$. Note $\mathfrak{F}_\mu=\nabla^*\mathscr{M}_{\sharp\mu,\xi}\nabla$ commutes with adjoining. Using $\log\sharp\mu\in A_\xi$, Proposition 4.2.6 lets us calculate

$$
\tau\big(\sharp\eta\log\sharp\mu\big)=\big\langle\mathfrak{F}^{-1}_\mu\big(\sharp\eta\big),\nabla^*\mathscr{M}_{\sharp\mu,\xi}\nabla\log\sharp\mu\big\rangle_\tau=\big\langle\mathfrak{F}^{-1}_\mu\big(\sharp\eta\big),\Delta\sharp\mu\big\rangle_\tau=\tau\big(\mathfrak{F}^{-1}_\mu\big(\sharp\eta\big)\Delta\sharp\mu\big). \tag{4.133}
$$

Using 1) in Lemma 4.2.8, Equation 4.133 implies

$$
\frac{d}{dt}\operatorname{Ent}^\tau\big(\mu(t)\big)=\tau\big(\sharp\dot\mu(t)\log\sharp\mu(t)\big)=\tau\Big(\mathfrak{F}^{-1}_{\mu(t)}\big(\sharp\dot\mu(t)\big)\Delta\sharp\mu(t)\Big) \tag{4.134}
$$

for all smooth paths $\mu:[a,b]\longrightarrow\vartheta(\xi)$.

Let $\mu:[0,1]\longrightarrow \vartheta(\xi)$ be a geodesic s.t. $\mu=\mu(0)$ and $\dot{\mu}(0)=\eta$. Using the chain rule of Riemannian metrics involving covariant derivatives [144], we argue as [169] to get

$$\operatorname{Hess}_{\mu}\operatorname{Ent}^{\tau}(\eta)=\frac{d^2}{dt^2}\bigg|_{t=0}\operatorname{Ent}^{\tau}\big(\mu(t)\big). \tag{4.135}$$

Equation 4.134 and Equation 4.135 let us calculate

$$\operatorname{Hess}_{\mu}\operatorname{Ent}^{\tau}(\eta)=\frac{d^2}{dt^2}\bigg|_{t=0}\tau\big(\sharp\mu(t)\log\sharp\mu(t)\big)=\frac{d}{dt}\bigg|_{t=0}\tau\Big(\mathfrak{F}^{-1}_{\mu(t)}\big(\sharp\dot{\mu}(t)\big)\Delta\sharp\mu(t)\Big). \tag{4.136}$$

We show Equation 4.134. All geodesics are critical points of the energy functional induced by g^{ξ} [144]. Using Equation 4.136 for the first and Theorem 4.2.19 for the third identity in our below calculation, we therefore calculate

$$\begin{aligned}\operatorname{Hess}_{\mu}\operatorname{Ent}^{\tau}(\eta)&=\frac{d}{dt}\bigg|_{t=0}\tau\Big(\mathfrak{F}^{-1}_{\mu(t)}\big(\sharp\dot{\mu}(t)\big)\Delta\sharp\mu(t)\Big)\\&=\langle\frac{d}{dt}\bigg|_{t=0}\mathfrak{F}^{-1}_{\mu(t)}\big(\sharp\dot{\mu}(t)\big),\Delta\sharp\mu\rangle_{\tau}+g^{\xi}_{\mu}\big(\eta,(\Delta\sharp\eta)^{\flat}\big)\\&=-\frac{1}{2}\langle\Lambda^{*}_{\mu}\big(\Theta(\mu,x^{\flat}),\Theta(\mu,x^{\flat})\big),\Delta\mu\rangle_{\tau}+g^{\xi}_{\mu}\big(\eta,(\Delta\sharp\eta)^{\flat}\big).\end{aligned}$$

The above calculation shows Equation 4.134. □

Theorem 4.2.22. *Let $(\phi,\psi,\gamma,\nabla)$ be noncommutative differential structure for tracial AF-C^*-algebras (A,τ) and (B,ω) in the finite-dimensional logarithmic mean setting. Let $\xi\in\mathscr{S}(A)$ be a fixed state. Let $\varphi_0,\varphi_1:U\longrightarrow(0,\infty)$ be smooth maps for $U\subset(0,\infty)\times(0,\infty)$ open.*

1) *Assume $U=(0,\infty)\times(0,\infty)$ and $\varphi:=\varphi_0=\varphi_1$. Let $\mu:[0,1]\longrightarrow\vartheta(\xi)$ be smooth. Using the latter, we define smooth map $\eta:U\longrightarrow\vartheta(\xi)$ by setting*

$$\eta(t,s):=h_{\varphi(t,s)}\big(\mu(t)\big) \tag{4.137}$$

for all $t,s>0$. We have

$$\frac{1}{2}\frac{\partial}{\partial s}g^{\xi}_{\eta}\left(\frac{\partial}{\partial t}\eta,\frac{\partial}{\partial t}\eta\right)+\frac{\partial^2}{\partial s\partial t}\varphi\cdot\frac{\partial}{\partial t}\operatorname{Ent}^{\tau}(\eta)=-\frac{\partial}{\partial s}\varphi\cdot\operatorname{Hess}_{\eta}\operatorname{Ent}^{\tau}\left(\frac{\partial}{\partial t}\eta\right) \tag{4.138}$$

on $(0,\infty)\times(0,\infty)$.

2) *Assume* $\frac{\partial}{\partial s}\varphi_0 = -\frac{\partial}{\partial s}\varphi_1$. *Let* $\mu \in \vartheta(\xi)$ *and* $x \in I(\Delta_\xi)$. *Using the latter, we define smooth maps* $\eta : U \longrightarrow \vartheta(\xi)$ *and* $X : U \longrightarrow I(\Delta_\xi)$ *by setting*

$$\eta(t,s) := h_{\varphi_0(t,s)}(\mu),\ X(t,s) := h_{\varphi_1(t,s)}(x) \tag{4.139}$$

for all $(t,s) \in U$. *We have*

$$\frac{1}{2}\frac{\partial}{\partial s}\left\|\mathscr{M}_\eta^{\frac{1}{2}}\nabla X\right\|_\omega^2 = -\frac{\partial}{\partial s}\varphi_1 \cdot \operatorname{Hess}_\eta \operatorname{Ent}^\tau\left(\mathfrak{F}_\eta(X)^\flat\right) \tag{4.140}$$

on U.

Proof. We show 1). Assume its setting. Note $\frac{\partial}{\partial s}\sharp\eta(t,s) = -\frac{\partial}{\partial s}\varphi(t,s)\cdot\Delta\sharp\eta(t,s)$ and further $\frac{\partial^2}{\partial s\partial t}\sharp\eta(t,s) = -\frac{\partial^2}{\partial s\partial t}\varphi(t,s)\cdot\Delta\sharp\eta(t,s) - \frac{\partial}{\partial s}\varphi(t,s)\cdot\Delta\frac{\partial}{\partial t}\sharp\eta(t,s)$. We calculate

$$\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial s}g_\eta^\xi\left(\frac{\partial}{\partial t}\eta,\frac{\partial}{\partial t}\eta\right)(t,s) &= -\frac{\partial}{\partial s}\varphi(t,s)\cdot\frac{1}{2}\left\langle d_{\eta(t,s)}\mathfrak{F}^{-1}\left(\left(\Delta\sharp\eta(t,s)\right)^\flat\right)\left(\frac{\partial}{\partial t}\sharp\eta(t,s)\right),\frac{\partial}{\partial t}\sharp\eta(t,s)\right\rangle_\tau \\
&\quad + g_{\eta(t,s)}^\xi\left(\frac{\partial^2}{\partial s\partial t}\eta(t,s),\frac{\partial}{\partial t}\eta(t,s)\right) \\
&= -\frac{\partial}{\partial s}\varphi(t,s)\cdot\frac{1}{2}\left\langle d_{\eta(t,s)}\mathfrak{F}^{-1}\left(\left(\Delta\sharp\eta(t,s)\right)^\flat\right)\left(\frac{\partial}{\partial t}\sharp\eta(t,s)\right),\frac{\partial}{\partial t}\sharp\eta(t,s)\right\rangle_\tau \\
&\quad - \frac{\partial^2}{\partial s\partial t}\varphi(t,s)\cdot g_{\eta(t,s)}^\xi\left(\left(\Delta\sharp\eta(t,s)\right)^\flat,\frac{\partial}{\partial t}\eta(t,s)\right) \\
&\quad - \frac{\partial}{\partial s}\varphi(t,s)\cdot g_{\eta(t,s)}^\xi\left(\left(\Delta\frac{\partial}{\partial t}\sharp\eta(t,s)\right)^\flat,\frac{\partial}{\partial t}\eta(t,s)\right).
\end{aligned}$$

Using 1) in Lemma 4.2.8 and symmetry of the real inner product, we calculate

$$\begin{aligned}
\frac{\partial}{\partial t}\operatorname{Ent}^\tau\left(\eta(t,s)\right) &= \left\langle\frac{\partial}{\partial t}\sharp\eta(t,s),\log\sharp\eta(t,s)\right\rangle_\tau \\
&= g_{\eta(t,s)}\left(\frac{\partial}{\partial t}\eta(t,s),\mathfrak{F}_{\eta(t,s)}\left(\log\sharp\eta(t,s)\right)^\flat\right) \\
&= g_{\eta(t,s)}^\xi\left(\left(\Delta\sharp\eta(t,s)\right)^\flat,\frac{\partial}{\partial t}\eta(t,s)\right).
\end{aligned}$$

We combine the two calculations above. We obtain

$$\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial s}g_\eta^\xi\left(\frac{\partial}{\partial t}\eta,\frac{\partial}{\partial t}\eta\right)(t,s) &= -\frac{\partial}{\partial s}\varphi(t,s)\cdot\frac{1}{2}\left\langle d_{\eta(t,s)}\mathfrak{F}^{-1}\left(\left(\Delta\sharp\eta(t,s)\right)^\flat\right)\left(\frac{\partial}{\partial t}\sharp\eta(t,s)\right),\frac{\partial}{\partial t}\sharp\eta(t,s)\right\rangle_\tau \\
&\quad - \frac{\partial^2}{\partial s\partial t}\varphi(t,s)\cdot\frac{\partial}{\partial t}\operatorname{Ent}^\tau\left(\eta(t,s)\right) \\
&\quad - \frac{\partial}{\partial s}\varphi(t,s)\cdot g_{\eta(t,s)}^\xi\left(\left(\Delta\frac{\partial}{\partial t}\sharp\eta(t,s)\right)^\flat,\frac{\partial}{\partial t}\eta(t,s)\right).
\end{aligned}$$

We readily see adding $\frac{\partial^2}{\partial s \partial t}\varphi(t,s) \cdot \frac{\partial}{\partial t}\operatorname{Ent}^\tau(\eta(t,s))$ to both sides of the above identity shows Equation 4.138 if

$$\begin{aligned}\operatorname{Hess}_\eta \operatorname{Ent}^\tau\left(\frac{\partial}{\partial t}\eta\right)(t.s) =& \frac{1}{2}\left\langle d_{\eta(t,s)}\mathfrak{F}^{-1}\left(\left(\Delta\sharp\eta(t,s)\right)^\flat\right)\left(\frac{\partial}{\partial t}\sharp\eta(t,s)\right), \frac{\partial}{\partial t}\sharp\eta(t,s)\right\rangle_\tau \\ &+ g^\xi_{\eta(t,s)}\left(\left(\Delta\frac{\partial}{\partial t}\sharp\eta(t,s)\right)^\flat, \frac{\partial}{\partial t}\eta(t,s)\right)\end{aligned}$$

for all $t,s>0$. We show the above identity. Let $t,s>0$. Proposition 4.2.21 implies

$$\begin{aligned}\operatorname{Hess}_\eta \operatorname{Ent}^\tau\left(\frac{\partial}{\partial t}\eta\right)(t.s) =& -\frac{1}{2}\left\langle \Lambda^*_{\eta(t,s)}\left(\sharp\Theta\left(\eta(t,s), \frac{\partial}{\partial t}\eta(t,s)\right), \sharp\Theta\left(\eta(t,s), \frac{\partial}{\partial t}\eta(t,s)\right)\right), \Delta\sharp\eta(t,s)\right\rangle_\tau \\ &+ g^\xi_{\eta(t,s)}\left(\frac{\partial}{\partial t}\eta(t,s), \left(\Delta\frac{\partial}{\partial t}\sharp\eta(t,s)\right)^\flat\right).\end{aligned}$$

Applying Lemma 4.2.18 to the first term and symmetry of the real inner product to the second one above, we obtain the claimed identity. Thus $\operatorname{Hess}_\eta \operatorname{Ent}^\tau\left(\frac{\partial}{\partial t}\eta\right)$ is of required form, hence 1) holds.

We show 2). Assume its setting. For all $(t,s)\in U$, set $\mathfrak{F}_{\eta,X}(t,s) := \mathfrak{F}_{\eta(t,s)}(X(t,s))$. Let $(t,s)\in U$. We have

$$\left\|\mathcal{M}^{\frac{1}{2}}_{\eta(t,s)}\nabla X(t,s)\right\|^2_\omega = \left\langle \mathfrak{F}_{\eta,X}(t,s), X(t,s)\right\rangle_\tau. \tag{4.141}$$

Using Equation 4.141, the Leibniz rule lets us calculate

$$\frac{\partial}{\partial s}\left\|\mathcal{M}^{\frac{1}{2}}_{\eta(t,s)}\nabla X(t,s)\right\|^2_\omega = \left\langle \frac{\partial}{\partial s}\mathfrak{F}_{\eta,X}(t,s), X(t,s)\right\rangle_\tau + 2\left\langle \mathfrak{F}_{\eta(t,s)}\left(\frac{\partial}{\partial s}X(t,s)\right), X(t,s)\right\rangle_\tau. \tag{4.142}$$

We therefore calculate the two summands on the right-hand side of Equation 4.142 in order. Note $\frac{\partial}{\partial s}\eta(t,s) = -\frac{\partial}{\partial s}\varphi_0(t,s)\cdot\Delta\sharp\eta(t,s)$. Applying Proposition 4.2.16 to $\mathfrak{F} = (\mathfrak{F}^{-1})^{-1}$ and further using $\frac{\partial}{\partial s}\varphi_0 = -\frac{\partial}{\partial s}\varphi_1$, we calculate

$$\begin{aligned}\left\langle \frac{\partial}{\partial s}\mathfrak{F}_{\eta,X}(t,s), X(t,s)\right\rangle_\tau &= -\left\langle \frac{\partial}{\partial s}\mathfrak{F}^{-1}_{\eta(t,s)}\left(\mathfrak{F}_{\eta,X}(t,s)\right), \mathfrak{F}_{\eta,X}(t,s)\right\rangle_\tau \\ &= \frac{\partial}{\partial s}\varphi_0(t,s)\cdot\left\langle d_{\eta(t,s)}\mathfrak{F}^{-1}\left(\left(\Delta\sharp\eta(t,s)\right)^\flat\right)\left(\mathfrak{F}_{\eta,X}(t,s)\right), \mathfrak{F}_{\eta,X}(t,s)\right\rangle_\tau \\ &= -\frac{\partial}{\partial s}\varphi_1(t,s)\cdot\left\langle d_{\eta(t,s)}\mathfrak{F}^{-1}\left(\left(\Delta\sharp\eta(t,s)\right)^\flat\right)\left(\mathfrak{F}_{\eta,X}(t,s)\right), \mathfrak{F}_{\eta,X}(t,s)\right\rangle_\tau.\end{aligned}$$

Using $\frac{\partial}{\partial s}X(t,s) = -\frac{\partial}{\partial s}\varphi_1(t,s)\cdot\Delta X(t,s)$, we moreover calculate

$$\begin{aligned}\left\langle \mathfrak{F}_{\eta(t,s)}\left(\frac{\partial}{\partial s}X(t,s)\right), X(t,s)\right\rangle_\tau &= -\frac{\partial}{\partial s}\varphi_1(t,s)\cdot\left\langle \mathfrak{F}_{\eta(t,s)}(\Delta X(t,s)), X(t,s)\right\rangle_\tau \\ &= -\frac{\partial}{\partial s}\varphi_1(t,s)\cdot\left\langle X(t,s), \Delta\mathfrak{F}_{\eta,X}(t,s)\right\rangle_\tau \\ &= -\frac{\partial}{\partial s}\varphi_1(t,s)\cdot g^\xi_{\eta(t,s)}\left(\left(\mathfrak{F}_{\eta,X}(t,s)\right)^\flat, \left(\Delta\mathfrak{F}_{\eta,X}(t,s)\right)^\flat\right).\end{aligned}$$

We combine the two calculations above with Equation 4.142. We obtain

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial s}\left\|\mathscr{M}^{\frac{1}{2}}_{\eta(t,s)}\nabla X(t,s)\right\|^2_\omega = -\frac{\partial}{\partial s}\varphi_1(t,s)\Bigg(&\frac{1}{2}\left\langle d_{\eta(t,s)}\mathfrak{F}^{-1}\left(\left(\Delta\sharp\eta(t,s)\right)^\flat\right)\left(\mathfrak{F}_{\eta,X}(t,s)\right), \mathfrak{F}_{\eta,X}(t,s)\right\rangle_\tau \\ &+ g^\xi_{\eta(t,s)}\left(\left(\mathfrak{F}_{\eta,X}(t,s)\right)^\flat, \left(\Delta\mathfrak{F}_{\eta,X}(t,s)\right)^\flat\right)\Bigg).\end{aligned}$$

We see the above identity implies Equation 4.140 if

$$\begin{aligned}\operatorname{Hess}_{\eta(t,s)}\operatorname{Ent}^\tau\left(\left(\mathfrak{F}_{\eta,X}(t,s)\right)^\flat\right) = &\frac{1}{2}\left\langle d_{\eta(t,s)}\mathfrak{F}^{-1}\left(\left(\Delta\sharp\eta(t,s)\right)^\flat\right)\left(\mathfrak{F}_{\eta,X}(t,s)\right), \mathfrak{F}_{\eta,X}(t,s)\right\rangle_\tau \\ &+ g^\xi_{\eta(t,s)}\left(\left(\mathfrak{F}_{\eta,X}(t,s)\right)^\flat, \left(\Delta\mathfrak{F}_{\eta,X}(t,s)\right)^\flat\right)\end{aligned}$$

for all $(t,s)\in U$. We show the above identity. We argue as for 1) using Proposition 4.2.21 and Lemma 4.2.18. We likewise use symmetry of the real inner product. Thus

$$\operatorname{Hess}_\eta\operatorname{Ent}^\tau\left(\mathfrak{F}_\eta(X)^\flat\right) = \operatorname{Hess}_\eta\operatorname{Ent}^\tau\left(\mathfrak{F}^\flat_{\eta,X}\right) \tag{4.143}$$

is of required form, hence 2) holds. □

4.2.3 Quantum noise evolution

We view quantum Laplacians as generators of quantum noise evolution in order to have non-spatiality of lower Ricci bounds and associated energy-information trade-offs. If EVI_λ-gradient flow of quantum relative entropy exist, then our Corollary 4.3.9 shows it is heat flow. Its curves of maximal slope [160] determine slopes of maximal entropy production, i.e. erasure of quantum information. A priori, it is nevertheless unclear how the EVI_λ-gradient flow property selects noise diffusion terms, i.e. generators of quantum noise evolution, without their selection being an isolated assumption unrelated to the underlying metric geometry. We require finer model assumptions for a selection process to justify viewing quantum Laplacians as above. To this end, we formulate a maximum entropy production principle as the latter may determine erasure of information [91][92] [124] motivated by fluctuation-dissipation principles [4][5][31][155] in non-equilibrium classical [23][178] and quantum statistical mechanics [188].

Up to coarse graining, Lemma 4.2.8 implies heat flow is gradient flow of quantum relative entropy. Theorem 4.2.35 shows heat flow further satisfies a steepest entropy ascent property [25] by considering the steepest descent property of gradient flows in smooth Riemannian manifolds [144] and taking limits. Note Corollary 4.1.27 shows production of quantum entropy is erasure of quantum information. We seek conditions s.t. steepest entropy ascent implies quantum noise evolution. If we are able to do so, then Theorem 4.2.35 obtains slopes of maximal entropy production, i.e. erasure of quantum information, for sufficiently regular subsets of all bounded normal states. Metric slopes as per Equation 4.196 generalise to larger sets of unbounded normal states. We restrict our maximum entropy production principle to selection of noise diffusion terms in the finite-dimensional setting and assume such selection is stable under scaling limits.

Accordingly, our maximum entropy production principle selects from candidates for noise diffusion terms in the finite-dimensional setting. Each candidate is determined by a quantum Fokker-Planck equation with vanishing drift term s.t. the kernel of the given quantum Laplacian is the solution set for zero. Following Remark 3.2.26, generators of induced semigroups as per Lemma 3.2.23 satisfying a quantum Fokker-Planck equation with vanishing drift term are diffusion terms. These describe purely irreversible time-evolution of dissipative quantum systems weakly coupled to a heat bath [35][36][82][121][163][188]. Following Landauer's principle [142][143] and its extension to quantum information theory [45][95], we expect they produce quantum entropy at each state. We show this is the case for candidates but with arbitrary energy scales. If we fix these, then we may formulate our selection rule. Note Corollary 3.2.25 shows the given quantum Laplacian has vanishing drift term, i.e. is itself a candidate for noise diffusion terms.

We consider four model assumptions. The first three assume the finite-dimensional setting, and the latter is stability under scaling limits. We summarise the first three. First, we assume production of quantum entropy, i.e. erasure of quantum information, is transport of quantum information along information-bearing degrees of freedom. This amounts to assuming the logarithmic mean setting and our above notion of candidate. Secondly, we select noise diffusion terms from all candidates for arbitrary energy scales by maximising production of quantum entropy under constraints on energy spent. Maximisation constraints are given by suitable evaluation of quantum Fisher information at each state. The latter links the information structure of quantum relative entropy to the energy structure of the given quasi-entropy, i.e. the underlying metric geometry. Thirdly, we use fixed energy scales normalised relative to the given quantum Laplacian. We obtain normalisation from an equivalent but expected least dissipation of energy principle [31]. This ensures unique solutions and avoids implausible ones.

Under assumptions as above, our maximum entropy production principle then states self-adjoint local unbounded operators are generators of quantum noise evolution if they restrict to unique solutions in each case. Corollary 4.2.33 implies these are indeed negatives of quantum Laplacians. Following our discussion of the coarse graining process in Subsection 3.3.2, Theorem 4.2.35 shows quantum Laplacians satisfy, up to sign, a quantum Fokker-Planck equation with vanishing drift term in scaling limit, i.e. only noise diffusion term. Of course, the sign occurs since negatives of quantum Laplacians generate noncommutative heat semigroups as per Lemma 3.2.23.

The maximum entropy production principle. We motivate our formulation in the finite-dimensional setting by fluctuation-dissipation principles [4][5][31][155] in non-equilibrium classical [23][178] and quantum statistical mechanics [188]. The latter exist in form of both minimum and maximum entropy production principles depending on constraints imposed on the given time-evolution [4][5][31]. The variational approach in [179] derives L^2-Wasserstein gradient flows by considering infinitesimal constraints on energy spent. This extends Onsager's least dissipation of energy principle [165][166]. In the setting of linear non-equilibrium thermodynamics [4][5][23][178], Onsager's least dissipation of energy principle is equivalent to a maximum entropy production principle [31]. There exist efforts to give a sensible description of the latter exclusively in terms of information theory [91][92]. However, such a description is contested [124]. We still arrive at three formal conditions for a suitable maximum entropy production principle. First, it must consider exclusively infinitesimal data for its maximisation constraints on energy spent. Secondly, it must be equivalent to a least dissipation of energy principle for the given thermodynamics by choice of such constraints. Thirdly, these constraints must be described only in terms of quantum information theory [62]. We show all three formal conditions are satisfied by our maximum entropy production principle.

We in fact derive it from an equivalent least dissipation of energy principle. As part of our discussion, we make explicit the first three model assumptions. Equation 4.159 gives maximal production of quantum entropy for candidates of noise diffusion terms as per the first model assumptions. This lets us select noise diffusion terms for arbitrary energy scales as per the second model assumption. Unless we fix energy scales, Proposition 4.2.27 implies we do not have unique solutions. Lemma 4.2.30, which assumes Equation 4.159, leads us to normalised energy scales as per the third model assumption and thereby our least dissipation of energy principle s.t. heat flow serves as fluctuated gradient flow. Equation 4.183 gives the latter. Example 4.2.37 shows our choice kills implausible solutions in the essential case of depolarising channels [62]. Lemma 4.2.32 shows Equation 4.185, i.e. Equation 4.159 for normalised energy scales, is derived from Equation 4.183 in Corollary 4.2.33. Equation 4.185 selects noise diffusion terms in the finite-dimensional setting as per our maximum entropy production principle.

Let $(\phi,\psi,\gamma,\nabla)$ be noncommutative differential structure for tracial AF-C^*-algebras (A,τ) and (B,ω) in the finite-dimensional logarithmic mean setting. Proposition 4.2.24 shows heat flow is gradient flow of quantum relative entropy. Remark 4.2.26 explains Proposition 4.2.27 gives maximisation constraints on energy spent for Equation 4.159.

Notation 4.2.23. Let $\xi\in\mathscr{S}(A)$ be a fixed state. Let $\operatorname{grad}\operatorname{Ent}^\tau$ denote the gradient of Ent^τ restricted to $\vartheta(\xi)$. We write $\operatorname{grad}_\mu\operatorname{Ent}^\tau$ upon evaluation at $\mu\in\vartheta(\xi)$.

Proposition 4.2.24. *Let $\xi\in\mathscr{S}(A)$ be a fixed state. For all $\mu\in\vartheta(\xi)$, we have*

$$-\operatorname{grad}_{h_t(\mu)}\operatorname{Ent}^\tau = -\big(\Delta\sharp h_t(\mu)\big)^\flat = \frac{d}{dt}h_t(\mu) \tag{4.144}$$

for all $t\geq 0$.

Proof. Let $\mu \in \vartheta(\xi)$. Since $\frac{d}{dt}h_t(\mu) = -(\Delta\sharp h_t(\mu))^\flat$ for all $t \geq 0$ by construction, we know Equation 4.144 follows if

$$\operatorname{grad}_\eta \operatorname{Ent}^\tau = \big(\Delta\sharp\eta\big)^\flat \tag{4.145}$$

for all $\eta \in \vartheta(\xi)$. We show Equation 4.145. Let $\eta \in \vartheta(\xi)$ and $u \in T_\eta\vartheta(\xi)$. Let $\varepsilon > 0$ and $\rho : [-\varepsilon, \varepsilon] \longrightarrow \vartheta(\xi)$ smooth s.t. $\rho(0) = \eta$ and $\dot\rho(0) = u$. We directly verify having admissible path $(\rho, \Theta(\rho, \dot\rho)) \in \operatorname{Adm}^{[-\varepsilon,\varepsilon]}$ satisfying the conditions of 2) in Lemma 4.2.8. Using the latter and $\mathfrak{G}_\eta = \mathscr{M}_{\sharp\eta,\xi}\nabla$ ensured by Equation 3.263, we calculate

$$\begin{aligned}
\frac{d}{dt}\bigg|_{t=0} \operatorname{Ent}^\tau\big(\rho(t)\big) &= \big\langle \mathscr{D}_{\sharp\eta,\xi}\sharp\Theta\big(\rho,\dot\rho\big)(0), \nabla\sharp\eta\big\rangle_\omega \\
&= \big\langle \mathscr{D}_{\sharp\eta,\xi}\mathscr{M}_{\sharp\eta,\xi}\nabla\mathfrak{F}_\eta^{-1}(\sharp u), \nabla\sharp\eta\big\rangle_\omega \\
&= \big\langle \nabla\mathfrak{F}_\eta^{-1}(\sharp u), \nabla\sharp\eta\big\rangle_\omega \\
&= \big\langle \mathfrak{F}_\eta^{-1}(\sharp u), \Delta\sharp\eta\big\rangle_\tau \\
&= g_\eta^\xi\big(u, \big(\Delta\sharp\eta\big)^\flat\big).
\end{aligned}$$

The above calculation implies Equation 4.145 and therefore Equation 4.144. □

Definition 4.2.25. Let $\xi \in \mathscr{S}(A)$ be a fixed state and $\mu \in \vartheta(\xi)$.

1) We define $\mathfrak{H}_{\xi,\mu} : A_{\xi,h} \longrightarrow \mathbb{R}$ by setting

$$\mathfrak{H}_{\xi,\mu}(x) := g_\mu^\xi\Big(\big(\Delta x\big)^\flat, \big(\Delta\sharp\mu\big)^\flat\Big) \tag{4.146}$$

for all $x \in A_{\xi,h}$.

2) For all $C \geq 0$, set $\mathfrak{S}_{\xi,\mu}(C) := \Big\{x \in A_{\xi,h} \mid g_\mu^\xi\Big(\big(\Delta x\big)^\flat, \big(\Delta x\big)^\flat\Big) = C\Big\}$.

Remark 4.2.26. Let $\xi \in \mathscr{S}(A)$ be a fixed state and $\mu \in \vartheta(\xi)$. Using definition of gradient flows [144] and following 1) in Definition 4.2.25, Proposition 4.2.24 shows

$$-\frac{d}{dt}\operatorname{Ent}^\tau\big(h_t(\mu)\big) = g_{h_t(\mu)}^\xi\Big(\big(\Delta\sharp h_t(\mu)\big)^\flat, \big(\Delta\sharp h_t(\mu)\big)^\flat\Big) = \mathfrak{H}_{\xi,h_t(\mu)}\big(\sharp h_t(\mu)\big) \tag{4.147}$$

for all $t \geq 0$. Equation 4.147 for $t = 0$ yields $-\frac{d}{dt}\big|_{t=0}\operatorname{Ent}^\tau(h_t(\mu)) = \mathfrak{H}_{\xi,\mu}(\sharp\mu)$ at once. We use this throughout our discussion.

Proposition 4.2.27. *Let $\xi \in \mathscr{S}(A)$ be a fixed state and $\mu \in \vartheta(\xi)\setminus\{\xi\}$. Let $C > 0$. For all $x \in \mathfrak{S}_{\xi,\mu}(C)$, we have $\mathfrak{H}_{\xi,\mu}(x) = \sup_{y\in\mathfrak{S}_{\xi,\mu}(C)} \mathfrak{H}_{\xi,\mu}(y)$ if and only if*

$$\Delta x = C^{\frac{1}{2}} \cdot \mathfrak{H}_{\xi,\mu}\big(\sharp\mu\big)^{-\frac{1}{2}} \cdot \Delta\sharp\mu. \tag{4.148}$$

Proof. We consider Riemannian manifold $(\vartheta(\xi), g^{\xi})$ as per 1) in Proposition 3.2.53. We know $T_\mu\vartheta(\xi) = I(\Delta_\xi)^\flat$ by 2) in Proposition 3.2.49. Pull-back of g^{ξ}_{μ} along the flat operator yields real Hilbert space $(I(\Delta_\xi), g^{\xi}_{\mu})$.

We have orthogonal decomposition $I(\Delta_\xi) = \langle\Delta\sharp\mu\rangle_{\mathbb{R}} \oplus \langle\Delta\sharp\mu\rangle_{\mathbb{R}}^{\perp}$. For all $x \in \mathfrak{S}_{\xi,\mu}(C)$, get unique $C_x \in \mathbb{R}$ and $r_x \in \langle\Delta\sharp\mu\rangle_{\mathbb{R}}^{\perp}$ s.t.

$$\Delta x = C_x \cdot \Delta\sharp\mu + r_x. \tag{4.149}$$

Using 2) in Definition 4.2.25, Equation 4.149 shows

$$C = C_x^2 \cdot \mathfrak{H}_{\xi,\mu}\big(\sharp\mu\big) + g^{\xi}_{\mu}(r_x, r_x) \tag{4.150}$$

for all $x \in \mathfrak{S}_{\xi,\mu}(C)$. Since moreover $r_x \in \langle\Delta\sharp\mu\rangle_{\mathbb{R}}^{\perp}$, Equation 4.149 further shows

$$\mathfrak{H}_{\xi,\mu}(x) = C_x \cdot \mathfrak{H}_{\xi,\mu}\big(\sharp\mu\big) \tag{4.151}$$

in each case. In addition, note Corollary 3.2.66 states $\xi \in \vartheta(\xi)$ is the only fixed state in $\vartheta(\xi)$. Yet $\mu \neq \xi$. Thus Proposition 4.2.24 implies $\mathfrak{H}_{\xi,\mu}(\sharp\mu) = -\frac{d}{dt}\big|_{t=0}\mathrm{Ent}^{\tau}(h_t(\mu)) > 0$, hence we see Equation 4.150 shows

$$|C_x| = \sqrt{C - g^{\xi}_{\mu}(r_x, r_x)} \cdot \mathfrak{H}_{\xi,\mu}\big(\sharp\mu\big)^{-\frac{1}{2}} \tag{4.152}$$

for all $x \in \mathfrak{S}_{\xi,\mu}(C)$ by rearranging terms accordingly. Let $x \in \mathfrak{S}_{\xi,\mu}(C)$. Equation 4.151 shows we have $\mathfrak{H}_{\xi,\mu}(x) = \sup_{y\in\mathfrak{S}_{\xi,\mu}(C)} \mathfrak{H}_{\xi,\mu}(y)$ if and only if $C_x = \sup_{y\in\mathfrak{S}_{\xi,\mu}(C)} C_y$ holds. Up to positive constant, note $\sharp\mu \in \mathfrak{S}_{\xi,\mu}(C)$. We assume $C_x \geq 0$ without loss of generality since we are concerned with the supremum. Equation 4.152 therefore implies we have $|C_x| = C_x = \sup_{y\in\mathfrak{S}_{\xi,\mu}(C)} C_y$ if and only if

$$g^{\xi}_{\mu}(r_x, r_x) = 0. \tag{4.153}$$

Equation 4.153 states $r_x = 0$ by positive definiteness of Riemannian metrics. Using the latter, Equation 4.149 and Equation 4.152 show the claimed equivalence. □

We give explicit formulation of the first and second model assumption. For this, we must describe candidates for noise diffusion terms in the finite-dimensional setting and use the extension [45][95] of Landauer's principle [142][143] to justify strictly positive production of quantum entropy. Equation 4.154 shows such candidates are diffusion terms. Using $\ker\Delta$ as the solution set for zero in each case, Equation 4.158 gives upper bounds on production of quantum entropy under constraints on energy spent. The latter ensure finiteness. Definition 4.2.28 gives maximal production of quantum entropy as per Equation 4.159 for arbitrary energy scales by maximising Equation 4.158.

We describe our notion of candidate. We use completely Markovian semigroups on $L^\infty(A,\tau) = A$ as per Definition 3.2.22. Let $S \in \mathscr{B}(A)_h$ s.t. $S \neq 0$ and $\ker S = \ker\Delta$. Assume S has completely Markovian induced semigroup $G^S : [0,\infty) \longrightarrow \mathscr{B}(A)$ given by $G^S_t = e^{tS}$ for all $t \geq 0$ as per 1) in Definition 3.2.24. We extend to positivity-preserving semigroup $G^S : [0,\infty) \longrightarrow \mathscr{B}(A^*)$ s.t. $G^S_t(\mathscr{S}(A)) \subset \mathscr{S}(A)$ for all $t \geq 0$ by dualisation. Self-adjointness implies S is a diffusion term as follows. Corollary 3.2.25, which uses Lemma 3.2.23 in the finite-dimensional setting, shows there exists Lindblad decomposition $(0,\varphi,C)$ of S as per 2) in Definition 3.2.24. Following Remark 3.2.26, we therefore have a quantum Fokker-Planck equation given by

$$S(x) = \frac{C}{2}\big(2\varphi(x) - \{\varphi(1_A), x\}\big) \tag{4.154}$$

for all $x \in A$. Equation 3.209 shows Equation 4.154 has vanishing drift term. We say that S is a candidate for noise diffusion terms. As we show below, this notion of candidate is part of the first model assumption and leads us to the second one.

The first model assumption states production of quantum entropy, i.e. erasure of quantum information, is transport of quantum information along information-bearing degrees of freedom. This description requires choice of quasi-entropy and measure of quantum information. We use $\mathscr{I}^{\log}$ and Ent^τ in our formulation here. Remark 4.2.29 explains our choice of the logarithmic mean setting. For all fixed states $\xi \in \mathscr{S}(A)$, we replace $\mathscr{I}^{\log}$ with g^ξ on $\vartheta(\xi)$ as per Remark 3.2.55. In Subsection 3.3.2, we explain non-ergodicity restricts information-bearing degrees of freedom by the continuity equation. Thus $\ker S = \ker\Delta$ restricts, hence $G^S : [0,\infty) \times \mathscr{S}(A) \longrightarrow \mathscr{S}(A)$ induces finite-energy admissible paths as follows. For all fixed states $\xi \in \mathscr{S}(A)$, note $T\vartheta(\xi) = \vartheta(\xi) \times I(\Delta_\xi)^\flat$ by 2) in Proposition 3.2.49 and $\operatorname{im} S \cap A_{\xi,h} = \operatorname{im}\Delta \cap A_{\xi,h} = I(\Delta_\xi)$ since $\ker S = \ker\Delta$. Using the latter, Corollary 3.2.66 then implies $\operatorname{im} S \cap A_{\xi,h} = I(\Delta_\xi)$ is equivalent to the following statement in the finite-dimensional setting. For all fixed states $\xi \in \mathscr{S}(A)$, we have

$$G^S_t(\vartheta(\xi)) \subset \vartheta(\xi) \tag{4.155}$$

for all $t \geq 0$. We have $\Delta|_{\operatorname{im}\Delta} > 0$ in $\mathscr{B}(\operatorname{im}\Delta)$ by finite-dimensionality. Equation 4.155 yields finite-energy admissible paths in relative interiors. The first model assumption is use of noncommutative differential structure and notion of candidate as above.

The second model assumption states we select noise diffusion terms from all candidates for arbitrary energy scales by maximising production of quantum entropy under constraints on energy spent. This requires candidates produce quantum entropy at each state. Following Remark 3.2.26, we view diffusion terms as infinitesimal applications of quantum channels [28][73] transmitting change of states of the given quantum system determined by irreversible interaction with its environment [62][141]. The extension [45][95] of Landauer's principle [142][143] gives strictly positive lower bounds on production of quantum entropy upon application of quantum channels due to minimal heat dissipation [15][44][181]. Under assumptions identical to those for general Lindblad master equations (cf. Equation 5.2.29 in [121]), Equation 3.8 in [45] shows erasure of quantum information implies strictly positive production of quantum entropy.

We expect $G^S : [0,\infty) \times \mathscr{S}(A) \longrightarrow \mathscr{S}(A)$ produces quantum entropy at each state since S is infinitesimal application of φ. For all fixed states $\xi \in \mathscr{S}(A)$, Equation 4.155 and differentiation at $t = 0$ yield unique $x_\mu \in I(\Delta_\xi)$ s.t.

$$S\left(\sharp\mu\right) = -\Delta x_\mu \tag{4.156}$$

for all $\mu \in \vartheta(\xi)$. Following Example 4.1.10, Corollary 4.1.27 shows quantum entropy is negative quantum relative entropy. We give production of quantum entropy, i.e. erasure of quantum information, at each state. For all fixed states $\xi \in \vartheta(\xi)$, 2) in Lemma 4.2.8 as in the proof of Proposition 4.2.24 and Equation 4.156 let us calculate

$$-\frac{d}{dt}\bigg|_{t=0} \operatorname{Ent}^\tau\left(G_t^S(\mu)\right) = \tau\left(\Delta x_\mu \log \sharp\mu\right) = g_\mu^\xi\Big(\left(\Delta x_\mu\right)^\flat, \left(\Delta\sharp\mu\right)^\flat\Big) = \mathfrak{H}_{\xi,\mu}\left(x_\mu\right) \tag{4.157}$$

for all $\mu \in \vartheta(\xi)$. Set $C_{\xi,\mu} := g_\mu^\xi((\Delta x_\mu)^\flat, (\Delta x_\mu)^\flat)$ in each case. Proposition 4.2.27 shows these are energy scales, varying in each tangent space and which determine strictly positive constants in Equation 4.148 for the following maximisation problem. For all fixed states $\xi \in \vartheta(\xi)$, we have $x_\mu \in \mathfrak{S}_{\xi,\mu}(C_{\xi,\mu})$ and Equation 4.157 shows

$$-\frac{d}{dt}\bigg|_{t=0} \operatorname{Ent}^\tau\left(G_t^S(\mu)\right) \le \sup_{y \in \mathfrak{S}_{\xi,\mu}(C_{\xi,\mu})} \mathfrak{H}_{\xi,\mu}(y) \tag{4.158}$$

for all $\mu \in \vartheta(\xi)$. Maximising Equation 4.158 gives rise to Definition 4.2.28, in particular to Equation 4.159. The second model assumption is selection of noise diffusion terms for arbitrary energy scales from all candidates through maximal production of quantum entropy as per Equation 4.159 by maximising Equation 4.158.

Definition 4.2.28. Let $S \in \mathscr{B}(A)_h$ s.t. $S \neq 0$ and $\ker S = \ker \Delta$. Assume S has completely Markovian induced semigroup $G^S : [0,\infty) \longrightarrow \mathscr{B}(A)$. We say that S produces maximal quantum entropy for ∇ if for all fixed states $\xi \in \mathscr{S}(A)$ and $\mu \in \vartheta(\xi)$, we have $C \ge 0$ s.t.

$$-\frac{d}{dt}\bigg|_{t=0} \operatorname{Ent}^\tau\left(G_t^S(\mu)\right) = \sup_{y \in \mathfrak{S}_{\xi,\mu}(C)} \mathfrak{H}_{\xi,\mu}(y). \tag{4.159}$$

Remark 4.2.29. Note 1) in Proposition 3.2.32 ensures $-\Delta$ is a candidate. Following Remark 4.2.26 for $t = 0$, Proposition 4.2.24 and Proposition 4.2.27 show $-\Delta$ produces maximal quantum entropy using energy scale

$$C_{\xi,\mu} = \mathfrak{H}_{\xi,\mu}(\sharp\mu)^{\frac{1}{2}} \tag{4.160}$$

for all fixed states $\xi \in \mathscr{S}(A)$ and $\mu \in \vartheta(\xi)$. If we do use both the first and second model assumptions, then $-\Delta$ is a noise diffusion term for energy scale as per Equation 4.160. We expect this but require Proposition 4.2.24 and Proposition 4.2.27. Moreover, the two propositions are necessary to derive Equation 4.158 and therefore Equation 4.159. This in turn requires us to assume the logarithmic mean setting.

Example 4.2.37 shows selection of noise diffusion terms as per the second model assumption must discern multiples of $-\Delta$. Unless we fix energy scales, Proposition 4.2.27 shows we do not. Lemma 4.2.30 shows candidates producing maximal quantum entropy are determined by energy maps varying $-\Delta$. This leads us to normalised energy scales as per the third model assumption and thereby our least dissipation of energy principle s.t. heat flow serves as fluctuated gradient flow.

Lemma 4.2.30. *Let $S \in \mathscr{B}(A)_h$ s.t. $S \neq 0$ and $\ker S = \ker \Delta$. Assume S has completely Markovian induced semigroup $G^S : [0,\infty) \longrightarrow \mathscr{B}(A)$. If S produces maximal quantum entropy for ∇, then we know there exist two unique maps $E_S : \partial\mathscr{S}(A) \times [0,\infty) \longrightarrow [0,\infty)$ and $\lambda_S : \partial\mathscr{S}(A) \times (0,\infty) \longrightarrow (0,\infty)$ satisfying the following.*

1) *The map $E_S|_{\partial\mathscr{S}(A)} : \partial\mathscr{S}(A) \times \{0\} \longrightarrow (0,\infty)$ is norm continuous.*
2) *For all $\mu \in \partial\mathscr{S}(A)$, the map $E_S(\mu,-) : (0,\infty) \longrightarrow (0,\infty)$ is continuously differentiable and the map $\lambda_S : (\mu,-) : (0,\infty) \longrightarrow (0,\infty)$ is continuous.*
3) *For all $\mu \in \partial\mathscr{S}(A)$, we have*
   3.1) $E_S(\mu,t) = \left\|S\left(G_t^S(\sharp\mu)\right)\right\|_\tau \left\|\Delta G_t^S(\sharp\mu)\right\|_\tau^{-1}$ *for all $t > 0$,*
   3.2) $E_S|_{\partial\mathscr{S}(A)}(\mu) = E_S(\mu,0) = \lim_{t\downarrow 0} \left\|S\left(G_t^S(\sharp\mu)\right)\right\|_\tau \left\|\Delta G_t^S(\sharp\mu)\right\|_\tau^{-1}$.
4) *For all $\mu \in \partial\mathscr{S}(A)$, we have*

$$S\left(G_t^S(\sharp\mu)\right) = -E_S(\mu,t)\cdot\Delta G_t^S(\sharp\mu) \tag{4.161}$$

   for all $t \geq 0$.

5) *For all $\mu \in \partial\mathscr{S}(A)$, we have*

$$\frac{d}{dt}E_S(\mu,t) = \lambda_S(\mu,t)\cdot E_S(\mu,t) \tag{4.162}$$

   for all $t > 0$.

Proof. Let $\xi \in \mathscr{S}(A)$ be a fixed state and $\mu \in \vartheta(\xi) \setminus \{\xi\}$. Note $\Delta x_\mu \neq 0$ and $\mathfrak{H}_{\xi,\mu}(\sharp\mu) > 0$ since we have $\sharp\mu \notin \ker S = \ker \Delta$. Equation 4.158 shows

$$C_{\xi,\mu} := g^{\xi}_{\mu}\Big(\big(\Delta x_\mu\big)^\flat, \big(\Delta x_\mu\big)^\flat\Big) > 0 \tag{4.163}$$

is the unique constant in Equation 4.159. Then Equation 4.157 and Equation 4.159 let us calculate

$$\mathfrak{H}_{\xi,\mu}\big(x_\mu\big) = -\frac{d}{dt}\bigg|_{t=0} \mathrm{Ent}^\tau\left(G^S_t(\mu)\right) = \sup_{y \in \mathfrak{S}_{\xi,\mu}(C_{\xi,\mu})} \mathfrak{H}_{\xi,\mu}(y). \tag{4.164}$$

Using Proposition 4.2.27 for the second identity in Equation 4.165 below, Equation 4.156 and Equation 4.164 show

$$S\big(\sharp\mu\big) = -\Delta x_\mu = -C^{\frac{1}{2}}_{\xi,\mu} \cdot \mathfrak{H}_{\xi,\mu}\big(\sharp\mu\big)^{-\frac{1}{2}} \cdot \Delta\sharp\mu. \tag{4.165}$$

Equation 4.167 uses constants on the right-hand side of Equation 4.165 in order to define the claimed energy map on $\partial\mathscr{S}(A) \times (0,\infty)$. Equation 4.169 extends to $t = 0$ in the second variable. For all fixed states $\xi \in \mathscr{S}(A)$, $\mu \in \mathscr{C}_A(\xi) \setminus \{\xi\}$ and $t \geq 0$, we calculate $G^S_t(\mu) \neq \xi$ and therefore

$$\Delta G^S_t\big(\sharp\mu\big) \neq 0 \tag{4.166}$$

on an orthonormal eigenbasis of S. We define $E_S : \partial\mathscr{S}(A) \times (0,\infty) \longrightarrow (0,\infty)$ by setting

$$E_S(\mu, t) := C^{\frac{1}{2}}_{\xi,\mu} \cdot \mathfrak{H}_{\xi,\mu}\big(G^S_t\big(\sharp\mu\big)\big)^{-\frac{1}{2}} \tag{4.167}$$

for all $\mu \in \partial\mathscr{S}(A)$ and $t > 0$. Equation 4.166 and Equation 4.167 show

$$E_S(\mu, t) = \big\|S\big(G^S_t\big(\sharp\mu\big)\big)\big\|_\tau \cdot \big\|\Delta G^S_t\big(\sharp\mu\big)\big\|^{-1}_\tau \tag{4.168}$$

in each case by taking Hilbert space norms and then the inverses in Equation 4.165. Using boundedness of S and Δ, Equation 4.166 and Equation 4.168 show we extend to $E_S : \partial\mathscr{S}(A) \times [0,\infty) \longrightarrow (0,\infty)$ by setting

$$E_S(\mu, 0) := \lim_{t \downarrow 0} \big\|S\big(G^S_t\big(\sharp\mu\big)\big)\big\|_\tau \big\|\Delta G^S_t\big(\sharp\mu\big)\big\|^{-1}_\tau \tag{4.169}$$

for all $\mu \in \partial\mathscr{S}(A)$. With the exception of 5), Equation 4.168 and Equation 4.169 show all claims involving E_S here.

We show 5). Equation 4.173 uses Equation 4.172 in order to define the claimed map. Let $\mu \in \operatorname{relint}\mathscr{S}(A)$. Using boundedness of S and Δ, as well as norm differentiability as per 1) and the Leibniz rule, Equation 4.165 and Equation 4.167 let us calculate

$$\begin{aligned}
\left.\frac{d}{dr}\right|_{r=0} S\big(G_r^S(\sharp\mu)\big) &= \left.\frac{d}{dr}\right|_{r=0} G_r^S\big(S(\sharp\mu)\big) \\
&= -E_S(\mu,0)\cdot S\big(\Delta\sharp\mu\big) \\
&= -E_S(\mu,0)\cdot\big(\Delta S(\sharp\mu) + [S,\Delta](\sharp\mu)\big) \\
&= -E_S(\mu,0)\cdot\big(-E_S(\mu,0)\Delta^2\cdot\sharp\mu + [S,\Delta](\sharp\mu)\big) \\
&= E_S(\mu,0)^2\cdot\Delta^2\sharp\mu - E_S(\mu,0)\cdot[S,\Delta](\sharp\mu),
\end{aligned}$$

$$\begin{aligned}
\left.\frac{d}{dr}\right|_{r=0} S\big(G_r^S(\sharp\mu)\big) &= -\left.\frac{d}{dr}\right|_{r=0} E_S(\mu,r)\cdot\Delta G_r^S(\sharp\mu) \\
&= -\left.\frac{d}{dr}\right|_{r=0} E_S(\mu,r)\cdot\Delta\sharp\mu - E_S(\mu,0)\cdot\Delta\left.\frac{d}{dr}\right|_{r=0} G_r^S(\sharp\mu) \\
&= -\left.\frac{d}{dr}\right|_{r=0} E_S(\mu,r)\cdot\Delta\sharp\mu - E_S(\mu,0)\cdot\Delta S(\sharp\mu) \\
&= -\left.\frac{d}{dr}\right|_{r=0} E_S(\mu,r)\cdot\Delta\sharp\mu + E_S(\mu,0)^2\Delta^2\cdot\sharp\mu.
\end{aligned}$$

We combine the two calculations above. We obtain

$$E_S(\mu,0)\cdot[S,\Delta](\sharp\mu) \;=\left.\frac{d}{dr}\right|_{r=0} E_S(\mu,r)\cdot\Delta\sharp\mu. \tag{4.170}$$

For all $t \geq 0$, Equation 4.170 shows $E_S(\mu,t) = E_S(G_t^S(\mu),0)$. Using the latter together with the semigroup property of $G^S : [0,\infty) \longrightarrow \mathscr{B}(A)$, Equation 4.170 generalises to

$$E_S(\mu,t)\cdot[S,\Delta]\big(G_t^S(\sharp\mu)\big) \;=\left.\frac{d}{dr}\right|_{r=t} E_S(\mu,r)\cdot\Delta G_t^S(\sharp\mu) \tag{4.171}$$

in each case. Equation 4.168 therefore shows we have

$$[S,\Delta]\big(G_t^S(\sharp\mu)\big) \;= E_S(\mu,t)^{-1}\cdot\left.\frac{d}{dr}\right|_{r=t} E_S(\mu,r)\cdot\Delta G_t^S(\sharp\mu) \tag{4.172}$$

for all $t > 0$ by taking the inverses in Equation 4.171.

Equation 4.172 shows we define $\lambda_S : \partial\mathscr{S}(A) \times (0,\infty) \longrightarrow (0,\infty)$ by setting

$$\lambda_S(\mu,t) := E_S(\mu,t)^{-1} \cdot \frac{d}{dr}\bigg|_{r=t} E_S(\mu,r) \tag{4.173}$$

for all $\mu \in \partial\mathscr{S}(A)$ and $t > 0$. Equation 4.166 and Equation 4.172 show continuity in the second variable. Altogether, get 1) to 4). Equation 4.172 and Equation 4.173 yield

$$\left(\lambda_S(\mu,t) \cdot E_S(\mu,t) - \frac{d}{dr}\bigg|_{r=t} E_S(\mu,r)\right) \cdot \Delta G_t^S(\sharp\mu) = 0 \tag{4.174}$$

in each case. Equation 4.166 and Equation 4.174 imply Equation 4.162. Get 5). □

We give explicit formulation of the third model assumption. For this, we use our least dissipation of energy principle. Lemma 4.2.30 lets us construct infinitesimal energy dissipation maps as per Equation 4.179, resp. its reformulation as Equation 4.182. We use Equation 4.182 as measure of energy dissipation when evolving induced semigroups to heat flow through dissipating fluctuations of its integral curves. Definition 4.2.31 gives least dissipation of energy as per Equation 4.183 s.t. heat flow serves as fluctuated gradient flow by minimising Equation 4.182. Accordingly, 3) in Definition 4.2.31 gives candidates for noise diffusion terms with normal energy scale, i.e. candidates satisfying Equation 4.180. The latter equation normalises energy scales relative to $-\Delta$.

We derive Equation 4.179 and Equation 4.180. Let $S \in \mathscr{B}(A)_h$ as per Lemma 4.2.30. For all $\mu \in \partial\mathscr{S}(A)$, Equation 4.162 readily shows $E_S(\mu,-) : [0,\infty) \longrightarrow [0,\infty)$ satisfies a homogeneous linear differential equation with 3.2) in Lemma 4.2.30 as its initial value at $t = 0$ using standard arguments for extension [102][139][140]. We therefore obtain

$$E_S(\mu,t) = \exp\left(\int_0^t \lambda_S(\mu,r)dr\right) \cdot E_S(\mu,0) \tag{4.175}$$

for all $\mu \in \partial\mathscr{S}(A)$ and $t \geq 0$. Lemma 4.2.30 ensures

$$\exp\left(\int_0^t \lambda_S(\mu,r)dr\right) \geq 1 \tag{4.176}$$

since $\int_0^t \lambda_S(\mu,r)dr \geq 0$ in each case. Note 2.2) in Theorem 3.2.40 and Corollary 4.2.9 show $h : [0,\infty) \times \partial\mathscr{S}(A) \longrightarrow \mathscr{S}(A)$ is a norm continuous injective map s.t. fixed states are the only elements not in its image. Moreover, Corollary 3.2.66 ensures all fixed states are limits in time of initial states in $\partial\mathscr{S}(A)$. Thus $\partial\mathscr{S}(A) \times [0,\infty)$ is a complete product space description of heat flow, hence we adopt it to measure infinitesimal energy dissipation when evolving the Hamiltonian of a given quantum system with initial state $\mu \in \partial\mathscr{S}(A)$ from S to $-\Delta$ at time $t \geq 0$. We formally view such evolutions as arising from dissipating small time-varying out-of-equilibrium perturbations of $-\Delta$, i.e. fluctuations of integral curves $t \mapsto h_t(\mu)$ describing evolution of temperature over time [23][178][188].

We construct a suitable pointwise direct sum norm. Equation 4.175 itself leads us to consider an energy gradient of S given at $(\mu,t)\in\partial\mathscr{S}(A)\times[0,\infty)$ by

$$\exp\left(\int_0^t \lambda_S(\mu,r)dr\right)\cdot\left(\inf_{\mu\in\partial\mathscr{S}(A)} E_S(\mu,0) - \sup_{\mu\in\partial\mathscr{S}(A)} E_S(\mu,0)\right) \tag{4.177}$$

for all $\mu\in\partial\mathscr{S}(A)$ and $t\geq 0$. Equation 4.177 is composed into two factors. The right-hand factor is the energy gradient of S at $t=0$, or initial energy gradient of S as per 2.1) in Definition 4.2.31. The left-hand factor is an exponential fluctuation term. If the initial energy gradient of S is zero, then Equation 4.175 and Equation 4.176 imply variation of $G^S:[0,\infty)\times\mathscr{S}(A)\longrightarrow\mathscr{S}(A)$ away from heat flow is determined by the exponential fluctuation term up to homogeneous initial energy

$$e_S := \inf_{\mu\in\partial\mathscr{S}(A)} E_S(\mu,0) = \sup_{\mu\in\partial\mathscr{S}(A)} E_S(\mu,0) \tag{4.178}$$

relative to $-\Delta$. If the exponential fluctuation term equals one, then Equation 4.175 shows such variation is instead determined by initial states. We consequently measure infinitesimal energy dissipation when evolving S to $-\Delta$ at initial state $\mu\in\partial\mathscr{S}(A)$ and time $t\geq 0$ using the pointwise direct sum norm

$$\sqrt{\left|\exp\left(\int_0^t \lambda_S(\mu,r)dr\right)-1\right|^2+\left|\inf_{\mu\in\partial\mathscr{S}(A)} E_S(\mu,0) - \sup_{\mu\in\partial\mathscr{S}(A)} E_S(\mu,0)\right|^2}. \tag{4.179}$$

Equation 4.176 shows Equation 4.179 has zero as its minimum. Unless we restrict values of homogeneous initial energy as per Equation 4.178, Equation 4.175 implies minimisers are given by $-C\Delta$ for all $C>0$. We expect this from Proposition 4.2.27 but instead due to energy scales varying away from Equation 4.160, i.e. the energy scale of $-\Delta$, rather than from $e_{-\Delta}=1$. Note 3) in Lemma 4.2.30 shows the latter. We therefore normalise energy scales relative to $-\Delta$ by letting

$$\inf_{\mu\in\partial\mathscr{S}(A)} E_S(\mu,0)\leq 1\leq \sup_{\mu\in\partial\mathscr{S}(A)} E_S(\mu,0). \tag{4.180}$$

Equation 4.180 shows $-\Delta$ is the unique minimiser of Equation 4.179, i.e. we have zero variation if and only if $E_S(\mu,t)=E_{-\Delta}(\mu,t)=1$ for all $\mu\in\partial\mathscr{S}(A)$ and $t\geq 0$. The third model assumption is use of fixed energy scales normalised as per Equation 4.180.

Definition 4.2.31. Let $S \in \mathscr{B}(A)_h$ s.t. $S \neq 0$ and $\ker S = \ker \Delta$. Assume S has completely Markovian induced semigroup $G^S : [0,\infty) \longrightarrow \mathscr{B}(A)$ and produces maximal quantum entropy for ∇.

1) We call $E_S : \partial\mathscr{S}(A) \times [0,\infty) \longrightarrow (0,\infty)$ the energy map of S. We further say that $\lambda_S : \partial\mathscr{S}(A) \times (0,\infty) \longrightarrow (0,\infty)$ is its fluctuation.

2) Set $E_S^{\min} := \inf_{\mu \in \partial\mathscr{S}(A)} E_S(\mu, 0)$ and $E_S^{\max} := \sup_{\mu \in \partial\mathscr{S}(A)} E_S(\mu, 0)$.

   2.1) We define the initial energy gradient $\operatorname{grad}_S := E_S^{\min} - E_S^{\max}$ of S. We define its variance $\operatorname{var}_S : \partial\mathscr{S}(A) \times [0,\infty) \longrightarrow [0,\infty)$ by setting

$$\operatorname{var}_S(\mu, t) := \exp\left(\int_0^t \lambda_S(\mu, r)dr\right) - 1 \tag{4.181}$$

   for all $\mu \in \partial\mathscr{S}(A)$ and $t \geq 0$.

   2.2) We define infinitesimal energy dissipation map $E_S^{\mathrm{dis}} : \partial\mathscr{S}(A) \times [0,\infty) \longrightarrow [0,\infty)$ of S by setting

$$E_S^{\mathrm{dis}}(\mu, t) := \sqrt{\left|\operatorname{var}_S(\mu, t)\right|^2 + \left|\operatorname{grad}_S\right|^2} \tag{4.182}$$

   for all $\mu \in \partial\mathscr{S}(A)$ and $t \geq 0$.

3) We say that S is a candidate for generating quantum noise evolution for ∇ with normal energy scale if $E_S^{\min} \leq 1 \leq E_S^{\max}$. We further say that S is the generator of quantum noise evolution for ∇ if

$$E_S^{\mathrm{dis}}(\mu, t) = 0 \tag{4.183}$$

for all $\mu \in \partial\mathscr{S}(A)$ and $t \geq 0$.

Lemma 4.2.32 gives equivalent conditions for minimising Equation 4.182. Note 4) in the lemma shows $-\Delta$ is the unique minimiser. Moreover, Corollary 4.2.33 gives maximal production of quantum entropy as per Equation 4.185 for normalised energy scales by maximising Equation 4.158. This gives our maximum entropy production principle in the finite-dimensional setting. Lemma 4.2.32 ensures we do select $-\Delta$ as claimed. Maximisation constraints on energy spent in Equation 4.185 are indeed given by suitable evaluation of quantum Fisher information as per Definition 4.3.21 at each state.

Lemma 4.2.32. *For all $S \in \mathscr{B}(A)_h$ which are candidates for generating quantum noise evolution for ∇, the following are equivalent:*

1) *S is the generator of quantum noise evolution for ∇,*
2) *$\mathrm{grad}_S = 0$ and $\mathrm{var}_S(\mu,t) = 0$ for all $\mu \in \partial\mathscr{S}(A)$ and $t \geq 0$,*
3) *$E_S(\mu,t) = 1$ for all $\mu \in \partial\mathscr{S}(A)$ and $t \geq 0$,*
4) *$S = -\Delta$.*

Proof. Let $\mathbf{S}$ be the set of all candidates for generating quantum noise evolution for ∇. For all $S \in \mathbf{S}$, Equation 4.175 and Equation 4.183 show S is the generator of quantum noise evolution for ∇ if and only if

$$E_S^{\mathrm{dis}}(\mu,t) = \inf_{S' \in \mathbf{S}} E_{S'}^{\mathrm{dis}}(\mu,t) = 0 = E_{-\Delta}^{\mathrm{dis}}(\mu,t) \tag{4.184}$$

for all $\mu \in \partial\mathscr{S}(A)$ and $t \geq 0$. Following our discussion of Equation 4.179, we know $-\Delta$ is the unique minimiser in our case. Using the latter, get 1) to 4). □

Corollary 4.2.33. *Let $S \in \mathscr{B}(A)_h$ be a candidate for generating quantum noise evolution for ∇. Then S is the generator of quantum noise evolution for ∇ if and only if for all fixed states $\xi \in \mathscr{S}(A)$, we have*

$$-\frac{d}{dt}\bigg|_{t=0} \mathrm{Ent}^{\tau}\left(G_t^S(\mu)\right) = \sup_{y \in \mathfrak{S}_{\xi,\mu}(\mathfrak{H}_{\xi,\mu}(\sharp\mu))} \mathfrak{H}_{\xi,\mu}(y) = \sup_{y \in \mathfrak{S}_{\xi,\mu}(\mathscr{I}^{\log}(\mu,\mu,(\nabla\sharp\mu)^\flat))} \mathfrak{H}_{\xi,\mu}(y) \tag{4.185}$$

for all $\mu \in \vartheta(\xi)$.

Proof. Let $\xi \in \mathscr{S}(A)$ be a fixed state and $\mu \in \vartheta(\xi)$. Proposition 4.2.6 shows $\sharp\Theta\big(\mu,(\Delta\sharp\mu)^\flat\big) = \mathfrak{G}_\mu\mathfrak{F}_\mu^{-1}(\mathfrak{F}_\mu(\log\sharp\mu)) = \nabla\sharp\mu$ by twice application. Using the latter, 2) in Proposition 3.2.53 lets us calculate

$$\mathfrak{H}_{\xi,\mu}(\sharp\mu) = g_\mu^\xi\Big(\big(\Delta\sharp\mu\big)^\flat,\big(\Delta\sharp\mu\big)^\flat\Big) = \mathscr{I}^{\log}\Big(\mu,\mu,\Theta\big(\mu,\big(\Delta\sharp\mu\big)^\flat\big)\Big) = \mathscr{I}^{\log}\Big(\mu,\mu,\big(\nabla\sharp\mu\big)^\flat\Big). \tag{4.186}$$

Equation 4.186 shows the second identity in Equation 4.185. Lemma 4.2.32 shows S is the generator of quantum noise evolution for ∇ if and only if $S = -\Delta$. Proposition 4.2.27 thus implies the first identity in Equation 4.185. □

Generators of quantum noise evolution. Definition 4.2.34 gives our maximum entropy production principle. The fourth model assumption is locality therein. Following our discussion of the coarse graining process in Subsection 3.3.2, we justify locality as a natural complement to the first model assumption since Theorem 3.1.52 lets us describe quantum optimal transport itself as scaling limit w.r.t. the coarse graining process.

We therefore view quantum Laplacians as generators of quantum noise evolution as per 1) in Theorem 4.2.35. Fittingly, 2) in Theorem 4.2.35 shows quantum Laplacians satisfy, up to sign, a quantum Fokker-Planck equation with vanishing drift term in scaling limit, i.e. only noise diffusion term. Thus 3) in Theorem 4.2.35 shows heat flow satisfies a steepest entropy ascent property [25] by considering the steepest descent property of gradient flows in smooth Riemannian manifolds [144] as per Equation 4.147 and taking limits. We thereby obtain slopes of maximal entropy production, i.e. erasure of quantum information, as per Equation 4.188 for the given subsets of all bounded normal states. If heat flow is EVI_λ-gradient flow of quantum relative entropy, then Equation 4.188 generalises to metric slopes as per Equation 4.196 for all normal states with finitely supported fixed part and finite quantum relative entropy. Note Remark 4.2.36.

Let $(\phi,\psi,\gamma,\nabla)$ be noncommutative differential structure for tracial AF-C^*-algebras (A,τ) and (B,ω) in the logarithmic mean setting.

Definition 4.2.34. Let $S \in \mathscr{U}\mathscr{B}(L^2(A,\tau))_h$ be local. We say that S is the generator of quantum noise evolution for ∇ if for all $j \in \mathbb{N}$, $S_j \in \mathscr{B}(A_j)_h$ is the generator of quantum noise evolution for ∇_j.

Theorem 4.2.35. *Let $(\phi,\psi,\gamma,\nabla)$ be noncommutative differential structure for tracial AF-C^*-algebras (A,τ) and (B,ω) in the logarithmic mean setting.*

1) $-\Delta$ is the generator of quantum noise evolution for ∇.

2) For all $j \in \mathbb{N}$, let $\big(0,\varphi_j,C_j\big)$ be a Lindblad decomposition of $-\Delta_j$. We have

$$-\Delta u = \|.\|_\tau\text{-}\lim_{j\in\mathbb{N}} -\Delta_j u_j = \|.\|_\tau\text{-}\lim_{j\in\mathbb{N}} \frac{C_j}{2}\big(2\varphi_j(u_j) - \{\varphi_j(1_{A_j}),u_j\}\big) \tag{4.187}$$

for all $u \in \operatorname{dom}\Delta$.

3) Let $\xi \in \mathscr{S}(A)$ be a finitely supported fixed state. Let $p \in L^{1,\infty}(A,\tau)$ be a projection s.t. we have $\xi \in \mathrm{C}[p]$. For all $\mu \in \mathrm{Fix}_A^{\mathrm{N}}(\xi) \cap \mathscr{S}_{-1}^{\mathrm{N},\infty}(A_\xi) \cap (\operatorname{dom}\Delta)^\flat$, there exists maximal $\varepsilon \in (0,\infty]$ s.t.

$$-\frac{d}{dt}\mathrm{Ent}^\tau\big(h_t(\mu)\big) = \tau\big(\Delta\sharp h_t(\mu)\log\sharp h_t(\mu)\big) = \mathscr{I}^{\log}\Big(h_t(\mu),h_t(\mu),\big(\nabla\sharp h_t(\mu)\big)^\flat\Big) \tag{4.188}$$

for all $t \in [0,\varepsilon)$.

Proof. We show 1) and 2). Note 4.3) in Proposition 2.3.25 shows A_0 is core of Δ. For all $j \in \mathbb{N}$, 1) in Proposition 2.3.27 shows $\Delta_j = \mathrm{com}_{A_j}\Delta \in \mathscr{B}(A_j)_h$. Thus $-\Delta$ is local, hence Lemma 4.2.32 shows it is the generator of quantum noise evolution for ∇. If we use $\|.\|_\tau$-limits as per 4.3) in Proposition 2.3.25, then Equation 4.187 is given by considering Equation 4.154 for each $-\Delta_j$ and letting $j \uparrow \infty$. Altogether, get 1) and 2).

We show 3). Assume its setting. Let $\mu \in \mathrm{Fix}_A^{\mathrm{N}}(\xi) \cap \mathscr{S}_{-1}^{\mathrm{N},\infty}(A_\xi) \cap (\mathrm{dom}\,\Delta)^\flat$. Following 1) in Proposition 3.2.32 for $p = 2$, the Hille-Yosida theorem applies to heat flow (cf. p.79 in [102]). Using the latter, 3) in Proposition 3.2.34 and 2.2) in Theorem 3.2.40, we show

$$h_t(\mu) \in \mathrm{Fix}_A^{\mathrm{N}}(A) \cap \mathscr{S}^{\mathrm{N},\infty}(A_\xi) \cap (\mathrm{dom}\,\Delta)^\flat \tag{4.189}$$

for all $t \geq 0$. Note $\mathrm{GL}(L^\infty(A_\xi, \tau)) \subset L^\infty(A_\xi, \tau)$ open in norm topology. Using the latter and strong continuity as per 1) in Proposition 3.2.32, we obtain $\varepsilon > 0$ s.t.

$$h_t(\mu) \in \mathscr{S}_{-1}^{\mathrm{N},\infty}(A_\xi) \tag{4.190}$$

for all $t \in [0, \varepsilon]$. Equation 4.189 and Equation 4.190 show

$$h_t(\mu) \in \mathrm{Fix}_A^{\mathrm{N}}(A) \cap \mathscr{S}_{-1}^{\mathrm{N},\infty}(A_\xi) \cap (\mathrm{dom}\,\Delta)^\flat \tag{4.191}$$

for all $t \in [0, \varepsilon]$. Note $\xi \in \mathrm{C}[p]$. We have $\mathrm{Fix}_A^{\mathrm{N}}(\xi) \subset \mathrm{C}[p]$ by 1) in Theorem 4.1.29. Using the latter and Corollary 4.1.27, Equation 4.189 implies

$$\mathrm{Ent}\big(h_t(\mu), \tau\big) = \tau\big(\sharp h_t(\mu) \log \sharp h_t(\mu)\big) = \tau\big(\mathrm{com}_p \sharp h_t(\mu) \log \mathrm{com}_p \sharp h_t(\mu)\big) \tag{4.192}$$

for all $t \in [0, \varepsilon]$. Since $\xi \in \mathrm{dom}\,\mathrm{Ent}^\tau$, note $\xi \in \mathscr{S}^{\mathrm{N}}(A) \cap \mathrm{C}[p]$ by Lemma 4.1.17. We have $\mathrm{supp}\,\xi \leq p$ by Lemma 4.1.20. Equation 4.191 shows we may replace p with $\mathrm{supp}\,\xi$ in Equation 4.192. Using the latter and 4.2) in Corollary 3.2.43, Equation 4.191 implies we have Fréchet differentiable map $t \mapsto \sharp h_t(\mu) \in L^\infty(A_\xi, \tau)_{>0} \cap L^\infty(A_\xi, \tau)_\nabla$ defined on $[0, \varepsilon)$.

We thereby extend calculations in Lemma 4.2.8 and Corollary 4.2.9, in particular those involving operator differentiable functions [172], to our setting. Lemma 4.1.20 shows ξ has integrable support. Using Proposition 4.2.6, we calculate

$$\begin{aligned}
-\frac{d}{dt}\mathrm{Ent}^\tau\big(h_t(\mu)\big) &= \tau\big(\Delta \sharp h_t(\mu) \log \sharp h_t(\mu)\big) + \tau\big(\sharp h_t(\mu) d\log_{\sharp h_t(\mu)}\big(\Delta \sharp h_t(\mu)\big)\big) \\
&= \tau\big(\Delta \sharp h_t(\mu) \log \sharp h_t(\mu)\big) \\
&= \big\langle \mathscr{D}_{\sharp h_t(\mu),\xi} \nabla \sharp h_t(\mu), \nabla \sharp h_t(\mu)\big\rangle_\omega \\
&= \mathscr{I}^{\log}\Big(h_t(\mu), h_t(\mu), \big(\nabla \sharp h_t(\mu)\big)^\flat\Big)
\end{aligned}$$

for all $[0, \varepsilon)$. Note Remark 4.2.7. The above calculation shows Equation 4.188. Since $\varepsilon > 0$ by construction, there exists maximal choice of $\varepsilon \in (0, \infty]$ as claimed. □

Remark 4.2.36. Note 3) in Theorem 4.2.35 generalises Corollary 4.2.9 by the semigroup property. Using standard arguments for interchanging derivatives and limits [109][139][140], we see 3.2) in Theorem 4.1.29 and 3) in Theorem 4.2.35 let us calculate

$$
\begin{aligned}
-\frac{d}{dt}\operatorname{Ent}^{\tau}\big(h_t(\mu)\big) &= \lim_{j\in\mathbb{N}} -\frac{d}{dt}\operatorname{Ent}^{\tau}\big(h_t(\bar{\mu}_j)\big) \\
&= \lim_{j\in\mathbb{N}} \mathscr{I}^{\log}\Big(h_t(\bar{\mu}_j), h_t(\bar{\mu}_j), \big(\nabla\sharp h_t(\bar{\mu}_j)\big)^{\flat}\Big) \\
&= \mathscr{I}^{\log}\Big(h_t(\mu), h_t(\mu), \big(\nabla\sharp h_t(\mu)\big)^{\flat}\Big)
\end{aligned}
$$

for all $\mu \in \mathscr{S}^{\mathrm{N},2}(A) \cap (\operatorname{dom}\Delta)^{\flat}$ and $t > 0$ if uniform convergence in the second identity is given in each case and to finite terms. We might thereby extend Equation 4.188 to a maximal definition domain by means of coarse graining. Here, we do not consider such assumptions on uniform convergence since we do not know of any examples.

Example 4.2.37 gives the depolarising channel as canonical choice of quantum noise operator (cf. pp.378-379 in [62]). We see internal quantum gradients induce quantum Laplacians which are, up to sign, infinitesimal applications of depolarising channels. This shows quantum Fokker-Planck equations with vanishing drift term in scaling limit as per Equation 4.187 may have closed form description.

Example 4.2.37. Assume (A,τ) is a strongly unital tracial AF-C^*-algebra s.t. $\tau < \infty$, as well as $(B,\omega) = (A \otimes A, \tau \otimes \tau)$ equipped with the internal AF-A-bimodule structure on $A \otimes A$ as per 1) in Definition 2.3.38. Let $\lambda \in [0,1]$. We consider the λ-internal quantum gradient $\nabla^{\lambda} : A_0 \longrightarrow L^2(A \otimes A, \tau \otimes \tau)$ on A as per 2) in Definition 2.3.38. We therefore have quantum Laplacian $\Delta^{\lambda} = \lambda \pi^{A}_{\ker\tau} \in \mathscr{B}(L^2(A,\tau))_h$ by 4) in Proposition 2.3.37.

We obtain $-\Delta^{\lambda} \neq 0$, $-\Delta^{\lambda}(L^{\infty}(A,\tau)) \subset L^{\infty}(A,\tau)$ and $-\Delta^{\lambda}1_A = 0$. We have completely Markovian semigroup $h : [0,\infty) \longrightarrow \mathscr{B}(L^{\infty}(A,\tau))$ by 1) in Proposition 3.2.32. Using the latter, Lemma 3.2.23 shows $-\Delta^{\lambda}$ has Lindblad decomposition. We show $-\Delta^{\lambda}$ satisfies a quantum Fokker-Planck equation with vanishing drift term. We define the depolarising channel $\varphi^{\lambda} : L^{\infty}(A,\tau) \longrightarrow L^{\infty}(A,\tau)$ with depolarisation probability λ by setting

$$
\varphi^{\lambda}(x) := (1-\lambda)x + \lambda\Big(I - \pi^{A}_{\ker\tau}\Big)(x) = \Big(I - \lambda\pi^{A}_{\ker\tau}\Big)(x) = \big(I - \Delta^{\lambda}\big)(x) \tag{4.193}
$$

for all $x \in L^{\infty}(A,\tau)$ (cf. pp.378-379 in [62]). Moreover, we directly verify all maps of form $x \mapsto C\tau(x)1_A$ defined on $L^{\infty}(A,\tau)$ for $C > 0$ are completely positive. Yet

$$
\Big(I - \pi^{A}_{\ker\tau}\Big)(x) = \frac{\tau(x)}{\tau(1_A)}1_A \tag{4.194}
$$

for all $x \in L^{\infty}(A,\tau)$.

Equation 4.194 shows $\varphi^\lambda : A \longrightarrow L^\infty(A,\tau)$ is a completely positive trace-preserving unital map. Following Remark 3.2.26, Equation 4.193 shows φ^λ is the quantum channel transmitting change of states given by complete mixing with uniform probability λ for all states. Using $\varphi^\lambda = I - \Delta^\lambda$, we calculate

$$-\Delta^\lambda x = -\big(I - \varphi^\lambda\big)(x) = \frac{1}{2}\big(2\varphi^\lambda(x) - \{\varphi^\lambda(1_A), x\}\big) \tag{4.195}$$

for all $x \in L^\infty(A,\tau)$. Equation 4.195 yields Lindblad decomposition $(0, \varphi^\lambda, 1)$ of $-\Delta^\lambda$ and closed form of Equation 4.187. Following Remark 3.2.26, Equation 4.195 is a quantum Fokker-Planck equation with vanishing drift term. If we do not use fixed energy scales normalised as per Equation 4.180, then Equation 4.195 does not suffice to determine unique quantum noise evolution even as the depolarising probability itself is fixed.

4.3 EVI$_\lambda$-gradient flow of quantum relative entropy

We emulate the classical case in the infinitesimally Hilbertian setting [105]. Following work of Jordan, Kinderlehrer and Otto for Fokker-Planck equations [131], resp. Otto for porous medium equations [167][169], Ambrosio, Gigli and Savaré give EVI$_\lambda$-gradient flows of proper l.s.c. functionals defined on metric spaces [8] to study evolution partial differential equations using gradient flows absent differential structures [75][160]. Note EVI$_\lambda$-gradient flows generalise gradient flows in smooth Riemannian manifolds driven by smooth functionals with Hessians bounded from below [8][103][160]. We therefore apply results in variational analysis for metric geometry using minimising geodesics [75][160] to study quantum relative entropy in the logarithmic mean setting.

Analogous L^2-Wasserstein distances in the classical case [97] are those determined by weak upper gradients [7][56] inducing Dirichlet forms [117]. If EVI$_\lambda$-gradient flow of relative entropy exists in this case, then it is heat flow [9][10]. Existence is equivalent to λ-convexity of relative entropy [9][10] and Bakry-Émery conditions [19][20] linking heat flow to a weak Riemannian structure [8][103] for the given classical L^2-Wasserstein distance [11][12][105]. Sturm [189][190], as well as Lott and Villani [151], each established λ-convexity of relative entropy [72][156] as synthetic lower Ricci bounds [191]. They imply following chain of functional inequalities [151][168] probing the underlying metric geometry. As for Riemannian manifolds [113][191], there exists a HWI$_\lambda$-interpolation inequality and Talagrand inequality TW$_\lambda$ for $\lambda \geq 0$, and a modified logarithmic Sobolev inequality MLSI$_\lambda$ for $\lambda > 0$. If we do have lower Ricci bound $\lambda > 0$, then λ-convexity of relative entropy implies HWI$_\lambda$, in turn implying MLSI$_\lambda$, finally implying TW$_\lambda$ [151]. If we want lower Ricci bounds and functional inequalities for quantum L^2-Wasserstein distances in direct analogy to the classical case, then we initially require equivalent characterisations for EVI$_\lambda$-gradient flow of quantum relative entropy in the logarithmic mean setting. Since we cover all fundamental example classes in Subsection 3.1.3, we also face complications arising from non-ergodicity commonly avoided in the classical case by assumption. We cannot expect ergodicity in the AF-C^*-setting because dynamic quantum gradients generalise the ubiquitous case of inner derivations [133].

In the ergodic finite-dimensional logarithmic mean setting, Carlen and Maas extend equivalent characterisations and functional inequalities [48][49][50]. Equivalence in [50] uses arguments fully given by Erbar and Maas in [106] alone. We use [50] as foundation and apply the coarse graining process to reduce to the finite-dimensional Riemannian setting s.t. ergodicity is recovered. We extend results upon replacing some essential arguments used in [50] and [106]. Ours and independent work of Wirth [200] together with Zhang [202] are the first infinite-dimensional extensions of the results in [48][49][50]. Wirth [200] gives noncommutative optimal transport distances determined by suitable symmetric C^*-derivations inducing C^*-Dirichlet forms on noncommutative L^2-spaces of tracial W^*-algebras [63][65]. Assuming tracial state and ergodicity, Wirth shows a, possibly infinite-dimensional, Bakry-Émery condition [200] as per [50] implies heat flow is EVI_λ-gradient flow of relative entropy for W^*-algebras [163]. However, [200] does not show its equivalence. Assuming tracial state, Wirth and Zhang give sufficient conditions for satisfying Bakry-Émery conditions [202] as per [50] and obtain functional inequalities HWI_λ, MLSI_λ [202] and TW_λ [200] as per [50] using relative entropy for W^*-algebras conditioned to fixed-point subalgebras. Such a priori conditioning handles non-ergodicity but does not emerge from an underlying metric geometry. As part of the overall introduction, we show their results are insufficient for our purposes.

We contribute the following. In our logarithmic mean setting, which does assume the AF-C^*-setting, yet neither ergodicity nor finite trace, we extend results in [48][49][50] and [106] to the general case and view lower Ricci bounds as measurement convexity of quantum information. Non-ergodicity and non-finite trace ensure fundamental example classes in Subsection 3.1.3 are covered. We extend results in four parts by means of the coarse graining process. This lets us modify arguments in [50] and [106] for the known ergodic finite-dimensional case. First, we show claimed equivalence of EVI_λ-gradient flow, λ-convexity, Bakry-Émery and Hessian lower bound conditions. Secondly, we then define lower Ricci bounds of quantum gradients. Thirdly, we give sufficient conditions for lower Ricci bounds of direct sum quantum gradients. Fourthly, we derive functional inequalities in the AF-C^*-setting. This requires quantum Fisher information. Apart from extension and following our view of quantum Laplacians as generators of quantum noise evolution in Subsection 4.2.3, lower Ricci bounds are given by λ-convexity of quantum information along minimising geodesics measured by quantum relative entropy. If we have noncommutative analogues of displacement interpolations [72][156], then such measurement convexity in the Schrödinger picture is convexity under measurement of observables in the Heisenberg picture. Unfortunately, existence results are unknown to us. We instead show strictly positive lower Ricci bounds determine energy-information trade-offs parametrised by lower bounds on quantum noise. Lower resolution implies lower energy paths. We avoid spatial interpretations of the classical case [97][151].

Structure. In Subsection 4.3.1, we discuss EVI_λ-gradient flows in metric spaces and λ-convexity of proper l.s.c. functionals. We consider heat flow as EVI_λ-gradient flow of quantum relative entropy and show our equivalence theorem. In Subsection 4.3.2, we discuss lower Ricci bounds, energy-information trade-offs parametrised by lower bounds on quantum noise and derive functional inequalities.

4.3.1 The equivalence theorem

Following our discussion of the coarse graining process in Subsection 3.3.2, we define the EVI$_\lambda$-gradient flow, λ-convexity and Bakry-Émery conditions in global and local form. We furthermore consider a Hessian lower bound condition. Such conditions are, in their global form, properties on all finitely supported accessibility components having non-trivial intersection with the domain of quantum relative entropy. Non-ergodicity requires us to consider accessibility components. Compatibility with compression and finite-dimensional approximation requires finitely supported ones. Theorem 4.3.8 shows equivalence of all conditions by means of the coarse graining process.

EVI$_\lambda$-gradient flows and λ-convexity. Metric-functional systems provide the general setting [8][160]. Definition 4.3.1 gives EVI$_\lambda$-gradient flows and λ-convexity as per [8][160]. We use continuous semigroups on metric spaces [8] generalising those on Banach spaces [102]. Note 2) in Definition 4.3.1 is called strong geodesic λ-convexity if it is to be distinguished from weaker formulations. We only use the former and therefore call it λ-convexity throughout our discussion. We use minimising geodesics defined on the unit interval [8][40]. Proposition 4.3.3 collects properties of EVI$_\lambda$-gradient flows.

We review gradient flows in metric spaces determined by evolution variational inequalities, or EVI$_\lambda$-gradient flows. Let (X,d) be a complete geodesic length-metric space and $F : X \longrightarrow (-\infty,\infty]$ a proper functional l.s.c. in d-topology. Let $Y \subset \overline{\mathrm{dom}F}$ s.t. for all $\mu^0,\mu^1 \in Y \cap \mathrm{dom}F$, there exists minimising geodesic $\mu : [0,1] \longrightarrow Y \cap \mathrm{dom}F$ from μ^0 to μ^1. Let $\lambda \in \mathbb{R}$. If $S : [0,\infty) \times Y \longrightarrow Y$ is EVI$_\lambda$-gradient flow as per 1) in Definition 4.3.1, then it is λ-contracting as per 1) in Proposition 4.3.3 and its curves are of maximal slope by Theorem 3.5 in [160], i.e. each $t \mapsto S_t(\mu)$ is absolutely continuous and satisfies

$$\frac{d^+}{dt}F\big(S_t(\mu)\big) = -|\partial F|^2\big(S_t(\mu)\big) \tag{4.196}$$

for a.e. $t \geq 0$. We use metric slope $\mu \mapsto |\partial F|(\mu)$ of F [8][160]. Equation 4.196 recovers the steepest descent property of gradient flows in smooth Riemannian manifolds [144]. Note existence of EVI$_\lambda$-gradient flows implies λ-convexity of F as per 2) in Definition 4.3.1 by 4) in Theorem 3.10 in [160]. The chain rule of Riemannian metrics involving covariant derivatives [144] implies λ-convexity generalises lower bounds for Hessians of smooth functionals [169]. Theorem 4.2 in [160] conversely shows λ-convexity of F implies S is the unique EVI$_\lambda$-gradient flow given by the generalised minimising movements scheme [87]. Altogether, EVI$_\lambda$-gradient flows generalise gradient flows in smooth Riemannian manifolds driven by smooth functionals with Hessians bounded from below.

If EVI$_\lambda$-gradient flow of quantum relative entropy exist, then Corollary 4.3.9 shows it is heat flow. We further generalise slopes of maximal entropy production, i.e. erasure of quantum information, as per Equation 4.188 to Equation 4.196 for all normal states with finitely supported fixed part and finite quantum relative entropy as claimed in the introduction of Section 4.2. We avoid the extension problems in Remark 4.2.36.

Definition 4.3.1. Let (X,d) be a metric space s.t. $d < \infty$ and $F : X \longrightarrow (-\infty,\infty]$ a proper functional l.s.c. in d-topology. We call (X,d,F) a metric-functional system. Let $Y \subset \overline{\mathrm{dom} F}$ and $\lambda \in \mathbb{R}$.

1) We say that a continuous semigroup $S : [0,\infty) \times Y \longrightarrow Y$ is EVI_λ-gradient flow of F in Y if for all $\mu \in Y$ and $\eta \in \mathrm{dom} F$, we have

$$\frac{1}{2}\frac{d^+}{dt} d\big(S_t(\mu),\eta\big)^2 + \frac{\lambda}{2} d\big(S_t(\mu),\eta\big)^2 \le F(\eta) - F\big(S_t(\mu)\big) \tag{EVI_λ}$$

for all $t \ge 0$.

2) Assume (X,d) is a complete geodesic length-metric space. We call $Y \cap \mathrm{dom} F \subset X$ a geodesic subspace if for all $\mu^0, \mu^1 \in Y \cap \mathrm{dom} F$, there exists a minimising geodesic $\mu : [0,1] \longrightarrow X$ from μ^0 to μ^1 s.t. we have $\mu(t) \in Y \cap \mathrm{dom} F$ for all $t \in [0,1]$. Assume $Y \cap \mathrm{dom} F \subset X$ is a geodesic subspace. We say that F is λ-convex in Y if for all minimising geodesics $\mu : [0,1] \longrightarrow Y \cap \mathrm{dom} F$, we have

$$F\big(\mu(t)\big) \le (1-t)F\big(\mu(0)\big) + tF\big(\mu(1)\big) - \frac{\lambda}{2} t(1-t) d\big(\mu(0),\mu(1)\big)^2 \tag{CNV_λ}$$

for all $t \in [0,1]$.

Remark 4.3.2. We have following integral characterisation of EVI_λ-gradient flows. Let (X,d,F) be a metric-functional system, $Y \subset \overline{\mathrm{dom} F}$ and $\lambda \in \mathbb{R}$. Theorem 3.3 in [160] shows a continuous semigroup $S : [0,\infty) \times Y \longrightarrow Y$ is EVI_λ-gradient flow of F in Y if and only if for all $\mu \in Y$ and $\eta \in \mathrm{dom} F$, the map $t \mapsto F(S_t(\mu))$ is strictly decreasing and we have

$$\frac{e^{\lambda(t-s)}}{2} d\big(S_t(\mu),\eta\big)^2 - \frac{1}{2} d\big(S_s(\mu),\eta\big)^2 \le \int_0^{t-s} e^{\lambda r} dr \cdot \Big(F(\eta) - F\big(S_t(\mu)\big)\Big) \tag{$\mathrm{EVI}_\lambda^\int$}$$

for all $0 < s < t < \infty$.

Proposition 4.3.3. *Let (X,d,F) be a metric-functional system, $Y \subset \overline{\mathrm{dom} F}$ and $\lambda \in \mathbb{R}$. Let $S : [0,\infty) \times Y \longrightarrow Y$ be an EVI_λ-gradient flow of F in Y.*

1) For all $\mu,\eta \in Y$, we have

$$d\big(S_t(\mu),S_t(\eta)\big) \le e^{-\lambda t} d(\mu,\eta) \tag{4.197}$$

for all $t \ge 0$.

2) Assume $F : X \longrightarrow (-\infty,\infty)$ has complete sublevels in d-topology. If $\lambda > 0$, then F has a unique minimum $\mu_{\min} \in Y \cap \mathrm{dom} F$.

3) Assume (X,d) is a complete length-metric space. If $Y \cap \mathrm{dom} F \subset X$ is a geodesic subspace, then F is λ-convex in Y.

Proof. The statement on λ-contraction as per Theorem 3.5 in [160] for $s = 0$ shows 1) at once. Moreover, the statements on asymptotic behaviour as $t \to \infty$ as per Theorem 3.5 in [160] for $\lambda > 0$ imply 2) by rearranging terms. Finally, note 4) in Theorem 3.10 in [160] implies 3) if $Y \cap \operatorname{dom} F \subset X$ is a geodesic subspace since the latter ensures existence of minimising geodesics. □

Equivalence in the logarithmic mean setting. Following our discussion of the coarse graining process in Subsection 3.3.2, Definition 4.3.6 gives the EVI_λ-gradient flow, λ-convexity and Bakry-Émery conditions in global and local form. We furthermore consider a Hessian lower bound condition. In the finite-dimensional logarithmic mean setting, Lemma 4.3.7 shows all conditions are equivalent. We are motivated in our proof by analogous arguments in [50] and [106]. However, Theorem 4.2.22 replaces essential steps therein letting us argue using Riemannian metrics on relative interiors induced by the given quasi-entropy. Theorem 4.3.8 uses Lemma 4.3.7 to show equivalence of all conditions by means of the coarse graining process. Corollary 4.3.9 shows restriction to finitely supported accessibility components suffices to determine EVI_λ-gradient flows.

Let $(\phi, \psi, \gamma, \nabla)$ be noncommutative differential structure for tracial AF-C^*-algebras (A, τ) and (B, ω) in the logarithmic mean setting. Notation 4.3.4 summarises use of 2) in Theorem 3.1.47 and Lemma 4.1.16. Proposition 4.3.5 gives metric-functional systems equipped with restricted $h : [0, \infty) \times \mathscr{S}(A) \longrightarrow \mathscr{S}(A)$ as continuous semigroup. We use these in Definition 4.3.6.

Notation 4.3.4. Let $\xi \in \mathscr{S}(A)$ be a finitely supported fixed state. Let $\mathscr{C} \subset (\mathscr{S}(A), \mathscr{W}_\nabla^{\log})$ be finitely supported with fixed part ξ. For all $j \in \mathbb{N}$ s.t. $\xi_j \neq 0$, we have

$$\mathscr{W}_\nabla^{\log}\Big|_{\mathscr{C}_{A_j}(\bar{\xi}_j) \times \mathscr{C}_{A_j}(\bar{\xi}_j)} = \mathscr{W}_{\nabla_j}^{\log},\ h|_{\mathscr{C}_{A_j}(\bar{\xi}_j)} = h^j,\ \mathrm{Ent}^\tau|_{\mathscr{C}_{A_j}(\bar{\xi}_j)} = \mathrm{Ent}^{\tau_j}. \tag{4.198}$$

We suppress $j \in \mathbb{N}$ upon restriction as per Equation 4.198.

Proposition 4.3.5. *Let $\xi \in \mathscr{S}(A)$ be a finitely supported fixed state. Let $\mathscr{C} \subset (\mathscr{S}(A), \mathscr{W}_\nabla^{\log})$ be finitely supported with fixed part ξ s.t. $\mathscr{C} \cap \operatorname{dom} \mathrm{Ent}^\tau \neq \emptyset$. We have*

1) *the metric-functional system $\left(\mathscr{C} \cap \operatorname{dom} \mathrm{Ent}^\tau, \mathscr{W}_\nabla^{\log}, \mathrm{Ent}^\tau\right)$ equipped with continuous semigroup $h : [0, \infty) \times \mathscr{C} \cap \operatorname{dom} \mathrm{Ent}^\tau \longrightarrow \mathscr{C} \cap \operatorname{dom} \mathrm{Ent}^\tau$,*

2) *the metric-functional system $\left(\mathscr{C}_{A_j}\left(\bar{\xi}_j\right), \mathscr{W}_\nabla^{\log}, \mathrm{Ent}^\tau\right)$ equipped with continuous semigroup $h : [0, \infty) \times \mathscr{C}_{A_j}\left(\bar{\xi}_j\right) \longrightarrow \mathscr{C}_{A_j}\left(\bar{\xi}_j\right)$ for a.e. $j \in \mathbb{N}$,*

3) *complete sublevels of $\mathrm{Ent}^\tau : \mathscr{C} \cap \operatorname{dom} \mathrm{Ent}^\tau \longrightarrow (-\infty, \infty)$ in $\mathscr{W}_\nabla^{\log}$-topology.*

Proof. Using $\mathscr{C} \cap \operatorname{dom} \mathrm{Ent}^\tau \neq \emptyset$, 3) in Corollary 3.1.50 and 4) in Theorem 4.1.29 show we have metric-functional system $(\mathscr{C} \cap \operatorname{dom} \mathrm{Ent}^\tau, \mathscr{W}_\nabla^{\log}, \mathrm{Ent}^\tau)$. Then 3) in Theorem 4.2.10 implies we obtain continuous semigroup $h : [0, \infty) \times \mathscr{C} \cap \operatorname{dom} \mathrm{Ent}^\tau \longrightarrow \mathscr{C} \cap \operatorname{dom} \mathrm{Ent}^\tau$ by considering $h_t|_{\mathscr{C} \cap \operatorname{dom} \mathrm{Ent}^\tau}$ for all $t \geq 0$. Get 1). We see 2) follows since $\mathscr{C}_{A_j}(\bar{\xi}_j) \subset \operatorname{dom} \mathrm{Ent}^\tau$ by Corollary 4.1.27 if $\xi_j \neq 0$. Using 3) in Corollary 3.1.50, l.s.c. of Ent^τ implies 3). □

The conditions in Definition 4.3.6 are subdivided in order as follows. First, three global conditions. Secondly, three local conditions with each one being the analogue of the respective global condition sharing its numbering. Thirdly, our Hessian lower bound condition. We ensure all such conditions are well-defined. For this, we collect results in our case concerning minimising geodesics, the coarse graining process and EVI_λ-gradient flows of quantum relative entropy. We use Notation 4.3.4.

We collect results. Let $\xi \in \mathscr{S}(A)$ be a finitely supported fixed state. Lemma 4.1.20 shows ξ has integrable support. Let $\mathscr{C} \subset (\mathscr{S}(A), \mathscr{W}_\nabla^{\log})$ be finitely supported with fixed part ξ s.t. $\mathscr{C} \cap \operatorname{dom}\operatorname{Ent}^\tau \neq \emptyset$. We have metric-functional systems equipped with heat flow as continuous semigroups as per 1) and 2) in Proposition 4.3.5. They coincide if A and B are finite-dimensional. Diagram 3.346 for $K = \operatorname{dom}\operatorname{Ent}^\tau$ shows we restrict, up to rescaling as per 1) in Definition 3.1.12, to

$$\operatorname{res}_j\big(\mathscr{C} \cap \operatorname{dom}\operatorname{Ent}^\tau\big) = \mathscr{C}_{A_j}\big(\bar{\xi}_j\big),\ \operatorname{res}_j \circ h = h|_{A_j^*} = h^j \tag{4.199}$$

for a.e. $j \in \mathbb{N}$. Remark 3.1.15 ensures we assume such rescaling here without loss of generality. Following 1) in Definition 4.3.1, $h : [0,\infty) \times \mathscr{C} \cap \operatorname{dom}\operatorname{Ent}^\tau \longrightarrow \mathscr{C} \cap \operatorname{dom}\operatorname{Ent}^\tau$ is EVI_λ-gradient flow of Ent^τ in $\mathscr{C} \cap \operatorname{dom}\operatorname{Ent}^\tau$ if for all $\mu, \eta \in \mathscr{C} \cap \operatorname{dom}\operatorname{Ent}^\tau$, we have

$$\frac{1}{2}\frac{d^+}{dt}\mathscr{W}_\nabla^{\log}\big(h_t(\mu),\eta\big)^2 + \frac{\lambda}{2}\mathscr{W}_\nabla^{\log}\big(h_t(\mu),\eta\big)^2 \leq \operatorname{Ent}(\eta,\tau) - \operatorname{Ent}\big(h_t(\mu),\tau\big) \tag{4.200}$$

for all $t \geq 0$. Remark 4.3.2 gives the following equivalent integral characterisation. For all $\mu, \eta \in \mathscr{C} \cap \operatorname{dom}\operatorname{Ent}^\tau$, we have

$$\frac{e^{\lambda(t-s)}}{2}\mathscr{W}_\nabla^{\log}\big(h_t(\mu),\eta\big)^2 - \frac{1}{2}\mathscr{W}_\nabla^{\log}\big(h_s(\mu),\eta\big)^2 \leq \int_0^{t-s} e^{\lambda r}dr \cdot \Big(\operatorname{Ent}(\eta,\tau) - \operatorname{Ent}\big(h_t(\mu),\tau\big)\Big) \tag{4.201}$$

for all $0 < s < t < \infty$. Note 4.2) in Theorem 3.1.47 ensures minimising geodesics and distance minimisers are identical. We assume 2.1) in Definition 4.3.6. Following 2) in Definition 4.3.1, Ent^τ is λ-convex if for all $\mu^0, \mu^1 \in \mathscr{C} \cap \operatorname{dom}\operatorname{Ent}^\tau$ and $(\mu, w) \in \operatorname{Geo}(\mu^0, \mu^1)$ s.t. $\mu(t) \in \operatorname{dom}\operatorname{Ent}^\tau$ for all $t \geq 0$, we have

$$\operatorname{Ent}\big(\mu(t),\tau\big) \leq (1-t)\operatorname{Ent}\big(\mu^0,\tau\big) + t\operatorname{Ent}\big(\mu^1,\tau\big) - \frac{\lambda}{2}t(1-t)\mathscr{W}_\nabla^{\log}\big(\mu^0,\mu^1\big)^2 \tag{4.202}$$

for all $t \in [0,1]$. Following Definition 4.3.1 and Remark 4.3.2, let EVI_λ, $\mathrm{EVI}_\lambda^\int$, resp. CNV_λ reference the above equations accordingly. Note G.3) in Definition 4.3.6, i.e. BE_λ, uses both Notation 3.2.42 and 1.1) in Corollary 3.2.43. Equation 4.199 shows we restrict in each step of the coarse graining process by replacing $\mathscr{C} \cap \operatorname{dom}\operatorname{Ent}^\tau$ with $\mathscr{C}_{A_j}(\bar{\xi}_j)$. We thereby obtain local forms from global ones. For H) in Definition 4.3.6, i.e. H_λ, there exists no local form. Referenced equations do not use subscripts upon restriction.

Definition 4.3.6. Let $\lambda \in \mathbb{R}$.

G.1) We say that Ent^τ satisfies EVI_λ if for all finitely supported $\mathscr{C} \subset \left(\mathscr{S}(A), \mathscr{W}_\nabla^{\log}\right)$ s.t. $\mathscr{C} \cap \operatorname{dom}\operatorname{Ent}^\tau \neq \emptyset$, $h : [0,\infty) \times \mathscr{C} \cap \operatorname{dom}\operatorname{Ent}^\tau \longrightarrow \mathscr{C} \cap \operatorname{dom}\operatorname{Ent}^\tau$ is EVI_λ-gradient flow of Ent^τ in $\mathscr{C} \cap \operatorname{dom}\operatorname{Ent}^\tau$.

G.2) We say that Ent^τ satisfies λ-convexity if for all finitely supported $\mathscr{C} \subset \left(\mathscr{S}(A), \mathscr{W}_\nabla^{\log}\right)$ s.t. $\mathscr{C} \cap \operatorname{dom}\operatorname{Ent}^\tau \neq \emptyset$, we have

   2.1) $\left(\mathscr{C} \cap \operatorname{dom}\operatorname{Ent}^\tau, \mathscr{W}_\nabla^{\log}\right)$ is a complete geodesic length-metric space,

   2.2) Ent^τ is λ-convex in $\mathscr{C} \cap \operatorname{dom}\operatorname{Ent}^\tau$.

G.3) We say that h satisfies BE_λ if for all finitely supported fixed states $\xi \in \mathscr{S}(A)$ and $\mathscr{C} \subset \left(\mathscr{S}(A), \mathscr{W}_\nabla^{\log}\right)$ with fixed part ξ, we have

$$\left\|\mathscr{M}_{\sharp\mu}^{\frac{1}{2}} \nabla h_t(u)\right\|_\omega^2 \leq e^{-2\lambda t} \left\|\mathscr{M}_{h_t(\sharp\mu)}^{\frac{1}{2}} \nabla u\right\|_\omega^2 \tag{BE_λ}$$

for all $\mu \in \mathscr{C} \cap L^{2,\infty}(A_\xi, \tau)^\flat$, $u \in \operatorname{dom}\nabla_\xi$ and $t \geq 0$.

L.1) We say that Ent^τ satisfies EVI_λ locally if for all finitely supported fixed states $\xi \in \mathscr{S}(A)$, $h : [0,\infty) \times \mathscr{C}_{A_j}(\bar{\xi}_j) \longrightarrow \mathscr{C}_{A_j}(\bar{\xi}_j)$ is EVI_λ-gradient flow of Ent^τ in $\mathscr{C}_{A_j}(\bar{\xi}_j)$ for a.e. $j \in \mathbb{N}$.

L.2) We say that Ent^τ satisfies λ-convexity locally if for all finitely supported fixed states $\xi \in \mathscr{S}(A)$, Ent^τ is λ-convex in $\mathscr{C}_{A_j}(\bar{\xi}_j)$ for a.e. $j \in \mathbb{N}$.

L.3) We say that h satisfies BE_λ locally if for all finitely supported fixed states $\xi \in \mathscr{S}(A)$ and a.e. $j \in \mathbb{N}$ in each case, we have

$$\left\|\mathscr{M}_{\sharp\mu}^{\frac{1}{2}} \nabla h_t(u)\right\|_\omega^2 \leq e^{-2\lambda t} \left\|\mathscr{M}_{h_t(\sharp\mu)}^{\frac{1}{2}} \nabla u\right\|_\omega^2 \tag{$\mathrm{BE}_\lambda^{\mathrm{loc}}$}$$

for all $\mu \in \mathscr{C}_{A_j}(\bar{\xi}_j)$, $u \in A_{j,\bar{\xi}_j}$ and $t \geq 0$.

H) We say that $\operatorname{Hess}\operatorname{Ent}^\tau$ has lower bound λ if for all for all finitely supported fixed states $\xi \in \mathscr{S}(A)$ and a.e. $j \in \mathbb{N}$ in each case, we have

$$\operatorname{Hess}_\mu \operatorname{Ent}^\tau(\eta) \geq \lambda g_\mu^{\bar{\xi}_j}(\eta, \eta) \tag{H_λ}$$

for all $\mu \in \vartheta(\bar{\xi}_j)$ and $\eta \in I\left(\Delta_{\bar{\xi}_j}\right)^\flat$. We further call λ a lower bound of the Hessian of quantum relative entropy and write $\operatorname{Hess}\operatorname{Ent}^\tau \geq \lambda$.

Following results in Section 3.1, Section 3.2 and Section 4.1, our discussion here and in Subsection 4.3.2 additionally uses below equations arising from applying the coarse graining process to quantum objects in the AF-C^*-setting. As such, they use compatibility with compression and finite-dimensional approximation. Let $\xi \in \mathscr{S}(A)$ and $\mathscr{C} \subset (\mathscr{S}(A), \mathscr{W}_{\nabla}^{\log})$ as above. We use 2.2) in Proposition 3.2.32. The latter immediately reduces to Equation 3.212 in the square integrable case. For all $\mu^0, \mu^1 \in \mathscr{S}(A)$, note 3) in Theorem 3.1.47 shows we have

$$\mathscr{W}_{\nabla}^{\log}\big(h_t(\mu^0), h_s(\mu^1)\big) = \lim_{j \in \mathbb{N}} \mathscr{W}_{\nabla}^{\log}\big(h_t(\bar{\mu}_j^0), h_s(\bar{\mu}_j^1)\big) \tag{4.203}$$

for all $t, s \geq 0$. For all $u \in L^2(A_\xi, \tau)$, 2) and 4.1) in Corollary 3.2.43 together show we have $u \in \operatorname{dom} \nabla_\xi$ if and only if limits

$$h_t(u) = \|.\|_{\nabla}\text{-}\lim_{j \in \mathbb{N}} \pi_{\operatorname{supp} \xi_j}\big(h_t(u_j)\big) = \|.\|_{\nabla}\text{-}\lim_{j \in \mathbb{N}} \operatorname{supp} \xi_j h_t(u_j) \operatorname{supp} \xi_j \tag{4.204}$$

exists for all $t \geq 0$. For all $\mu \in \mathscr{C}$, we see 3.2) in Theorem 4.1.29 shows we have

$$\operatorname{Ent}\big(h_t(\mu), \tau\big) = \lim_{j \in \mathbb{N}} \operatorname{Ent}\big(h_t(\mu_j), \tau\big) = \lim_{j \in \mathbb{N}} \operatorname{Ent}\big(h_t(\bar{\mu}_j), \tau\big) \tag{4.205}$$

for all $t \in [0, \infty]$. Compare Equation 4.205 to Equation 4.4.

Lemma 4.3.7. *Assume A and B are finite-dimensional. Let $\xi \in \mathscr{S}(A)$ be a fixed state. For all $\lambda \in \mathbb{R}$, the following are equivalent:*

1) $h : [0,\infty) \times \mathscr{C}_A(\xi) \longrightarrow \mathscr{C}_A(\xi)$ is EVI_λ-gradient flow of Ent^τ in $\mathscr{C}_A(\xi)$,

2) Ent^τ is λ-convex in $\mathscr{C}_A(\xi)$,

3) Ent^τ satisfies BE_λ for all $\mu \in \mathscr{C}_A(\xi)$, $u \in A_\xi$ and $t \geq 0$,

4) Ent^τ satisfies H_λ for all $\mu \in \vartheta(\xi)$ and $\eta \in I(\Delta_\xi)^\flat$.

Proof. Let $\lambda \in \mathbb{R}$. We show 4) implies 1), then 1) implies 2), and finally 2) implies 4). We further show equivalence of 3) and 4). We thereby show our claim. Theorem 4.2.22 lets us apply Theorem 2.2 in [75] to show 4) implies 1). We are motivated by analogous arguments in the proof of Theorem 4.5 in [106]. However, Theorem 4.2.22 replaces the essential steps in [106] necessary to apply Theorem 2.2 in [75] here. Finally, further note Theorem 4.2.22 lets us apply a standard semigroup interpolation argument as in the proof of Theorem 10.4 in [50] to show 3) implies 4).

We show 4) implies 1). Using Corollary 3.2.66, we know Theorem 3.3 in [75] implies 1) at once if $h:[0,\infty)\times\vartheta(\xi)\longrightarrow\vartheta(\xi)$ is EVI_λ-gradient flow of Ent^τ in $\vartheta(\xi)$. For all smooth $\mu:[0,1]\longrightarrow\vartheta(\xi)$, set $\eta(t,s):=h_{ts}(\mu(t))$ for all $t,s\geq 0$. Using the latter, Theorem 2.2 in [75] further shows $h:[0,\infty)\times\vartheta(\xi)\longrightarrow\vartheta(\xi)$ is EVI_λ-gradient flow of Ent^τ in $\vartheta(\xi)$ if for all smooth $\mu:[0,1]\longrightarrow\vartheta(\xi)$, we have

$$\frac{1}{2}\frac{\partial}{\partial s}g^{\xi}_{\eta(t,s)}\left(\frac{\partial}{\partial t}\eta(t,s),\frac{\partial}{\partial t}\eta(t,s)\right)+\frac{\partial}{\partial t}\mathrm{Ent}^\tau\big(\eta(t,s)\big)\leq -t\lambda g^{\xi}_{\eta(t,s)}\left(\frac{\partial}{\partial t}\eta(t,s),\frac{\partial}{\partial t}\eta(t,s)\right) \tag{4.206}$$

for all $t,s\geq 0$. Assume 4). Set $\varphi(t,s):=ts$ for all $t,s\geq 0$. Thus $\eta(t,s)=h_{\varphi(t,s)}(\mu(t))$ in each case, hence 1) in Theorem 4.2.22 yields

$$\frac{1}{2}\frac{\partial}{\partial s}g^{\xi}_{\eta(t,s)}\left(\frac{\partial}{\partial t}\eta(t,s),\frac{\partial}{\partial t}\eta(t,s)\right)+\frac{\partial}{\partial t}\mathrm{Ent}^\tau\big(\eta(t,s)\big)=-t\,\mathrm{Hess}_{\eta(t,s)}\mathrm{Ent}^\tau\left(\frac{\partial}{\partial t}\eta(t,s)\right) \tag{4.207}$$

for all $t,s>0$. We extend to $t,s\geq 0$ by continuity. We apply H_λ to the right-hand side of Equation 4.207 and obtain Equation 4.206. Altogether, 4) implies 1).

Note 3) in Proposition 4.3.3 shows 1) implies 2) since we know $\mathscr{S}(A)\subset\mathrm{dom}\,\mathrm{Ent}^\tau$ by finite-dimensionality. We show 2) implies 4). Assume 2). Let $\mu:[0,1]\longrightarrow\vartheta(\xi)$ be a minimising geodesic. Set $\mu:=\mu(0)$ and $\eta:=\dot\mu(0)$. Equation 4.135 states

$$\mathrm{Hess}_\mu\mathrm{Ent}^\tau(\eta)=\left.\frac{d^2}{dr^2}\right|_{r=0}\mathrm{Ent}^\tau\big(\mu(r)\big). \tag{4.208}$$

We write out both differential quotients on the right-hand side of Equation 4.208. The latter equation therefore shows $\mathrm{Hess}_\mu\mathrm{Ent}^\tau(\eta)$ equals

$$\lim_{t\downarrow 0}\lim_{s\downarrow 0}t^{-1}s^{-1}\Big(\mathrm{Ent}\big(\mu(t+s),\tau\big)-\mathrm{Ent}\big(\mu(t),\tau\big)-\mathrm{Ent}\big(\mu(s),\tau\big)+\mathrm{Ent}(\mu,\tau)\Big). \tag{4.209}$$

Equation 4.214 below lets us estimate parenthesis terms in Equation 4.209. Using the latter, we directly calculate lower Hessian bounds. For all $t,s\in(0,1)$ s.t. $t+s<1$, set $\rho(r):=\mu((t+s)r)$ and $\nu(r):=\mu((t+s)(1-r))$ for all $r\in[0,1]$. Segments of minimising geodesics reparametrised to constant speed on the unit interval as per Remark 3.1.22 are minimising geodesics. We obtain $\rho\in\mathrm{Geo}(\mu,\mu(t+s))$ and $\nu\in\mathrm{Geo}(\mu(t+s),\mu)$ in each case, where we suppress canonical vector fields along minimising geodesics as per 1) in Proposition 3.2.56 in our notation here. We estimate by applying CNV_λ to the latter. We require additional considerations.

Set $h(t,s) := t(t+s)^{-1}$ for all $t,s > 0$. Using $h(t,s) = 1 - h(s,t)$, get $\mu(t) = \rho(h(t,s))$ and $\mu(s) = \nu(1-h(s,t)) = \nu(h(t,s))$ in each case. Let $t,s \in (0,1)$ s.t. $t+s < 1$. Minimising geodesics have t-a.e. constant speed by 1) in Proposition 3.1.45. Since we have $(t+s)\eta = \dot{\rho}(0)$ by construction, symmetry of distances and constant speed of minimising geodesics let us calculate

$$\mathscr{W}_{\nabla}^{\log}\big(\mu(t+s),\mu\big)^2 = \mathscr{W}_{\nabla}^{\log}\big(\mu,\mu(t+s)\big)^2 = (t+s)^2 \cdot g_{\mu}^{\xi}(\eta,\eta). \tag{4.210}$$

For all $t,s > 0$ s.t. $t+s < 1$, CNV_λ and Equation 4.210 let us calculate the following two estimates. First, CNV_λ to $\mu(t) = \rho(h(t,s))$ in order to estimate

$$\mathrm{Ent}\big(\mu(t),\tau\big) \le \big(1-h(t,s)\big)\cdot \mathrm{Ent}(\mu,\tau) + h(t,s)\cdot \mathrm{Ent}\big(\mu(t+s),\tau\big) - \frac{\lambda}{2} ts \cdot g_{\mu}^{\xi}(\eta,\eta). \tag{4.211}$$

Secondly, we apply CNV_λ to $\mu(s) = \nu(h(t,s))$ in order to estimate

$$\mathrm{Ent}\big(\mu(s),\tau\big) \le \big(1-h(t,s)\big)\cdot \mathrm{Ent}\big(\mu(t+s),\tau\big) + h(t,s)\cdot \mathrm{Ent}(\mu,\tau) - \frac{\lambda}{2} ts \cdot g_{\mu}^{\xi}(\eta,\eta). \tag{4.212}$$

We moreover add Equation 4.211 and Equation 4.212 to obtain

$$\mathrm{Ent}\big(\mu(t),\tau\big) + \mathrm{Ent}\big(\mu(s),\tau\big) \le \mathrm{Ent}(\mu,\tau) + \mathrm{Ent}\big(\mu(t+s),\tau\big) - \lambda ts \cdot g_{\mu}^{\xi}(\eta,\eta) \tag{4.213}$$

in each case.

For all $t,s \in (0,1)$ s.t. $t+s < 1$, Equation 4.213 implies

$$\lambda ts \cdot g_{\mu}^{\xi}(\eta,\eta) \le \mathrm{Ent}\big(\mu(t+s),\tau\big) - \mathrm{Ent}\big(\mu(t),\tau\big) - \mathrm{Ent}\big(\mu(s),\tau\big) + \mathrm{Ent}(\mu,\tau) \tag{4.214}$$

by rearranging terms accordingly. Assuming $t+s < 1$ in Equation 4.209, which we may do since we consider a double limit, Equation 4.214 lets us estimate parenthesis terms in Equation 4.209. We therefore calculate

$$\begin{aligned}
\mathrm{Hess}_{\mu}\,\mathrm{Ent}^{\tau}(\eta) &= \frac{d^2}{dr^2}\bigg|_{r=0} \mathrm{Ent}^{\tau}\big(\mu(r)\big) \\
&= \lim_{t\downarrow 0}\lim_{s\downarrow 0} t^{-1}s^{-1}\Big(\mathrm{Ent}\big(\mu(t+s),\tau\big) - \mathrm{Ent}\big(\mu(t),\tau\big) - \mathrm{Ent}\big(\mu(s),\tau\big) + \mathrm{Ent}(\mu,\tau)\Big) \\
&\ge \lambda g_{\mu}^{\xi}(\eta,\eta).
\end{aligned}$$

The above calculation shows 2) implies 4). Corollary 3.2.63 ensures we have sufficient minimising geodesics in $\vartheta(\xi)$. Altogether, we obtain a chain of implications as claimed.

We show equivalence of 3) and 4). Assume 3). Let $\mu \in \vartheta(\xi)$ and $x \in I(\Delta_\xi)$. Set

$$l(t) := e^{2\lambda t}\left\|\mathcal{M}_{\sharp\mu}^{\frac{1}{2}}\nabla h_t(x)\right\|_\omega^2,\ r(t) := \left\|\mathcal{M}_{h_t(\sharp\mu)}^{\frac{1}{2}}\nabla x\right\|_\omega^2 \tag{4.215}$$

for all $t \geq 0$. Applying BE_λ to Equation 4.215 shows $r(t) \geq l(t)$ for all $t \geq 0$ and therefore $r'(0) \geq l'(0)$ as well. We directly verify

$$l'(0) = -2g_\mu^\xi\Big(\mathfrak{F}_\mu(\Delta x)^\flat, \mathfrak{F}_\mu(x)^\flat\Big) + 2\lambda g_\mu^\xi\Big(\mathfrak{F}_\mu(x)^\flat, \mathfrak{F}_\mu(x)^\flat\Big). \tag{4.216}$$

Applying Proposition 4.2.16 to $\mathfrak{F} = (\mathfrak{F}^{-1})^{-1}$ and further using Lemma 4.2.18, we thus argue as in the proof of 2) in Theorem 4.2.22 in order to calculate

$$r'(0) = -\Big\langle \Lambda_\mu^*\Big(\sharp\Theta\big(\mu, \mathfrak{F}_\mu(x)^\flat\big), \sharp\Theta\big(\mu, \mathfrak{F}_\mu(x)^\flat\big)\Big), \Delta\mu\Big\rangle_\tau. \tag{4.217}$$

Proposition 4.2.21 shows

$$\begin{aligned}\mathrm{Hess}_\mu \mathrm{Ent}^\tau\Big(\mathfrak{F}_\mu(x)^\flat\Big) = &-\Big\langle \frac{1}{2}\Lambda_\mu^*\Big(\sharp\Theta\big(\mu, \mathfrak{F}_\mu(x)^\flat\big), \sharp\Theta\big(\mu, \mathfrak{F}_\mu(x)^\flat\big)\Big), \Delta\mu\Big\rangle_\tau \\ &+ g_\mu^\xi\Big(\mathfrak{F}_\mu(\Delta x)^\flat, \mathfrak{F}_\mu(x)^\flat\Big).\end{aligned}$$

Using $r'(0) \geq l'(0)$ and the above identity, Equation 4.216 and Equation 4.217 imply

$$\mathrm{Hess}_\mu \mathrm{Ent}^\tau\Big(\mathfrak{F}_\mu(x)^\flat\Big) \geq \lambda g_\mu^\xi\Big(\mathfrak{F}_\mu(x)^\flat, \mathfrak{F}_\mu(x)^\flat\Big) \tag{4.218}$$

by rearranging terms accordingly. Note $\mathfrak{F}_\mu$ in Equation 4.218 is of no consequence by 1) in Proposition 3.2.51. Thus Equation 4.218 shows Ent^τ satisfies H_λ. Get 4).

Assume 4). It suffices to consider $\mu \in \vartheta(\xi)$ by Corollary 3.2.66, as well as $x \in I(\Delta_\xi)$ by 1) in Corollary 2.2.12 and symmetry of ∇. Let $U := \{(t,s) \in (0,\infty) \times (0,\infty) \mid t > s\}$. Set $\varphi_0(t,s) := s$ and $\varphi_1(t,s) := t - s$, as well as

$$\eta(t,s) := h_{\varphi_0(t,s)}(\mu) = h_s(\mu),\ X(t,s) := h_{\varphi_1(t,s)}(x) = h_{t-s}(x) \tag{4.219}$$

for all $(t,s) \in U$. Thus $\frac{\partial}{\partial s}\varphi_0 = -\frac{\partial}{\partial s}\varphi_1$, hence 2) in Theorem 4.2.22 yields

$$\frac{1}{2}\frac{\partial}{\partial s}\left\|\mathcal{M}_{h_s(\sharp\mu)}^{\frac{1}{2}}\nabla h_{t-s}(u)\right\|_\omega^2 = \mathrm{Hess}_{h_s(\mu)} \mathrm{Ent}^\tau\Big(\mathfrak{F}_{h_s(\mu)}\big(h_{t-s}(x)\big)^\flat\Big) \tag{4.220}$$

for all $(t,s) \in U$. If $t > 0$, then we extend to all $s \in [0,t]$ by continuity.

For all $t > 0$, set

$$l(s) := e^{-2\lambda s} \cdot \left\| \mathscr{M}_{h_s(\sharp\mu)}^{\frac{1}{2}} \nabla h_{t-s}(x) \right\|_\omega^2 \tag{4.221}$$

for all $s \in [0,t]$. Applying Equation 4.220 to derivatives of terms in Equation 4.221 lets us calculate

$$\begin{aligned} l'(s) &= 2e^{-2\lambda s} \cdot \left(\frac{1}{2} \frac{\partial}{\partial s} \left\| \mathscr{M}_{h_s(\sharp\mu)}^{\frac{1}{2}} \nabla h_{t-s}(x) \right\|_\omega^2 - \lambda \left\| \mathscr{M}_{h_s(\sharp\mu)}^{\frac{1}{2}} \nabla h_{t-s}(x) \right\|_\omega^2 \right) \\ &= 2e^{-2\lambda s} \cdot \left(\mathrm{Hess}_{h_s(\mu)} \mathrm{Ent}^\tau \left(\mathfrak{F}_{h_s(\sharp\mu)} \left(h_{t-s}(x) \right)^\flat \right) - \lambda \left\| \mathscr{M}_{h_s(\sharp\mu)}^{\frac{1}{2}} \nabla h_{t-s}(x) \right\|_\omega^2 \right) \end{aligned}$$

in each case. Further note $\mathfrak{F}_{h_s(\mu)} = \nabla^* \mathscr{M}_{h_s(\mu)} \nabla$ on $I(\Delta_\xi)$. We obtain

$$\left\| \mathscr{M}_{h_s(\sharp\mu)}^{\frac{1}{2}} \nabla h_{t-s}(x) \right\|_\omega^2 = g_{h_s(\mu)}^\xi \left(\mathfrak{F}_{h_s(\mu)} \left(h_{t-s}(x) \right)^\flat, \mathfrak{F}_{h_s(\mu)} \left(h_{t-s}(x) \right)^\flat \right) \tag{4.222}$$

for all $s \in [0,t]$. Then applying Equation 4.222 to its preceding calculation yields

$$\begin{aligned} l'(s) = 2e^{-2\lambda s} \cdot \Big(& \mathrm{Hess}_{h_s(\mu)} \mathrm{Ent}^\tau \left(\mathfrak{F}_{h_s(\mu)} h_{t-s}(x) \right) \\ & - \lambda g_{h_s(\mu)}^\xi \left(\mathfrak{F}_{h_s(\mu)} \left(h_{t-s}(x) \right)^\flat, \mathfrak{F}_{h_s(\mu)} \left(h_{t-s}(x) \right)^\flat \right) \Big). \end{aligned}$$

If $t > 0$, then the above calculation shows H_λ implies $l'(s) \geq 0$ for all $s \in [0,t]$. For all $t \geq 0$, we therefore have $l(t) \geq l(0)$. Using the latter, Equation 4.221 implies

$$\left\| \mathscr{M}_{\sharp\mu}^{\frac{1}{2}} \nabla h_t(x) \right\|_\omega^2 \leq e^{-2\lambda t} \left\| \mathscr{M}_{h_t(\sharp\mu)}^{\frac{1}{2}} \nabla x \right\|_\omega^2 \tag{4.223}$$

for all $t \geq 0$. Equation 4.223 shows Ent^τ satisfies BE_λ at once. Get 3). Altogether, get equivalence of 3) and 4). □

Theorem 4.3.8. *Let $(\phi, \psi, \gamma, \nabla)$ be noncommutative differential structure for tracial AF-C^*-algebras (A,τ) and (B,ω) in the logarithmic mean setting. For all $\lambda \in \mathbb{R}$, the conditions in Definition 4.3.6 are equivalent.*

Proof. Let $\lambda \in \mathbb{R}$. Note Equation 4.199 at once shows Lemma 4.3.7 implies equivalence of L.1), L.2), L.3) and H). It suffices to show equivalence of G.1) and L.1), of G.2) and L.2), as well as of G.3) and L.3) each. We do so by passing from global to local properties and vice versa by means of the coarse graining process. We consider the following fixed but arbitrary. Let $\xi \in \mathscr{S}(A)$ be a finitely supported fixed state. Let $\mathscr{C} \subset (\mathscr{S}(A), \mathscr{W}_\nabla^{\log})$ be finitely supported with fixed part ξ s.t. $\mathscr{C} \cap \operatorname{dom} \mathrm{Ent}^\tau \neq \emptyset$. We test all statements on the latter without loss of generality.

We show equivalence of G.1) and L.1). Assume G.1). For a.e. $j \in \mathbb{N}$, note $\mathscr{C}_A(\bar{\xi}_j)$ is finitely supported s.t. $\bar{\xi}_j \in \mathscr{C}_A(\bar{\xi}_j) \cap \operatorname{dom}\operatorname{Ent}^\tau \neq \emptyset$. If the latter is satisfied, then G.1) implies $h : [0,\infty) \times \mathscr{C}_A(\bar{\xi}_j) \cap \operatorname{dom}\operatorname{Ent}^\tau \longrightarrow \mathscr{C}_A(\bar{\xi}_j) \cap \operatorname{dom}\operatorname{Ent}^\tau$ is EVI_λ-gradient flow of Ent^τ in $\mathscr{C}_A(\bar{\xi}_j)$. Moreover, 2) in Theorem 3.1.47 yields isometric inclusion

$$\left(\mathscr{C}_{A_j}\left(\bar{\xi}_j\right), \mathscr{W}_\nabla^{\log}\right) \subset \left(\mathscr{C}_A\left(\bar{\xi}_j\right) \cap \operatorname{dom}\operatorname{Ent}^\tau, \mathscr{W}_\nabla^{\log}\right) \tag{4.224}$$

in each case. For a.e. $j \in \mathbb{N}$, Equation 4.224 reduces EVI_λ as per Equation 4.200 from $\mathscr{C}_A(\bar{\xi}_j)$ to $\mathscr{C}_{A_j}(\bar{\xi}_j)$, i.e. we see $h : [0,\infty) \times \mathscr{C}_{A_j}(\bar{\xi}_j) \longrightarrow \mathscr{C}_{A_j}(\bar{\xi}_j)$ is EVI_λ-gradient flow of Ent^τ in $\mathscr{C}_{A_j}(\bar{\xi}_j)$ in each case. Get L.1).

Assume L.1). Let $\mu, \eta \in \mathscr{C} \cap \operatorname{dom}\operatorname{Ent}^\tau$. For a.e. $j \in \mathbb{N}$, note L.1) shows $\mathrm{EVI}_\lambda^\int$ as per Equation 4.201 for $\bar{\mu}_j, \bar{\eta}_j \in \mathscr{C}_{A_j}(\bar{\xi}_j)$. Using the latter, Equation 4.203 and Equation 4.205 let us estimate

$$\begin{aligned}
&\frac{e^{\lambda(t-s)}}{2}\mathscr{W}_\nabla^{\log}\big(h_t(\mu),\eta\big)^2 - \frac{1}{2}\mathscr{W}_\nabla^{\log}\big(h_s(\mu),\eta\big)^2\\
&= \lim_{j\in\mathbb{N}} \frac{e^{\lambda(t-s)}}{2}\mathscr{W}_\nabla^{\log}\big(h_t(\bar{\mu}_j),\bar{\eta}_j\big)^2 - \frac{1}{2}\mathscr{W}_\nabla^{\log}\big(h_s(\bar{\mu}_j),\bar{\eta}_j\big)^2\\
&\leq \lim_{j\in\mathbb{N}} \int_0^{t-s} e^{\lambda r}\,dr \cdot \Big(\operatorname{Ent}(\bar{\eta}_j,\tau) - \operatorname{Ent}\big(h_t(\bar{\mu}_j),\tau\big)\Big)\\
&= \int_0^{t-s} e^{\lambda r}\,dr \cdot \Big(\operatorname{Ent}(\eta,\tau) - \operatorname{Ent}\big(h_t(\mu),\tau\big)\Big)
\end{aligned}$$

for all $0 < s < t < \infty$. The above calculation readily lifts $\mathrm{EVI}_\lambda^\int$ as per Equation 4.201 from $\{\mathscr{C}_{A_j}(\bar{\xi}_j)\}_{j\in\mathbb{N}}$ to $\mathscr{C} \cap \operatorname{dom}\operatorname{Ent}^\tau$, i.e. we see $h : [0,\infty) \times \mathscr{C} \cap \operatorname{dom}\operatorname{Ent}^\tau \longrightarrow \mathscr{C} \cap \operatorname{dom}\operatorname{Ent}^\tau$ is EVI_λ-gradient flow of Ent^τ in $\mathscr{C} \cap \operatorname{dom}\operatorname{Ent}^\tau$ in each case. Get G.1). Altogether, get equivalence of G.1) and L.1).

We show equivalence of G.2) and L.2). Assume G.2). We then reduce from global to local property as above. For a.e. $j \in \mathbb{N}$, we see 2.1) in Proposition 3.1.45 shows

$$\operatorname{Geo}_j\left(\bar{\mu}_j^0, \bar{\mu}_j^1\right) \subset \operatorname{Geo}\left(\bar{\mu}_j^0, \bar{\mu}_j^1\right) \tag{4.225}$$

for all $\mu^0, \mu^1 \in \mathscr{C}_{A_j}(\bar{\xi}_j)$. For a.e. $j \in \mathbb{N}$, Equation 4.224 and Equation 4.225 reduce CNV_λ as per Equation 4.202 from $\mathscr{C}_A(\bar{\xi}_j)$ to $\mathscr{C}_{A_j}(\bar{\xi}_j)$, i.e. we see Ent^τ is λ-convex in $\mathscr{C}_{A_j}(\bar{\xi}_j)$ in each case. Get L.2). Assume L.2). We show G.2) by using equivalence of G.1) and L.1) to apply 3) in Proposition 4.3.3. We show 2.1) in Definition 4.3.6. Let $\mu^0, \mu^1 \in \mathscr{C} \cap \operatorname{dom}\operatorname{Ent}^\tau$. Since they are at finite distance, Theorem 3.1.52 shows there exists $(\mu, w) \in \operatorname{Geo}(\mu^0, \mu^1)$ approximated in finite dimensions by a sequence $(\mu^j, w^j)_{j \geq m} \subset \operatorname{Geo}_0$. For a.e. $j \in \mathbb{N}$, L.2) shows CNV_λ as per Equation 4.202 for the minimising geodesic $\mu^j : [0,1] \longrightarrow \mathscr{C}_{A_j}(\bar{\xi}_j)$.

Upon passing to a subsequence converging to (μ, w) in $\mathrm{Adm}^{[0,1]}$, we consider CNV_λ in each case and take limits in $j \in \mathbb{N}$ on both sides. Equation 4.203 and Equation 4.205 show they exist. We therefore have $C > 0$ s.t.

$$\mathrm{Ent}\big(\mu^j(t), \tau\big) \leq C \cdot \max\Big\{\mathrm{Ent}\big(\bar{\mu}_j^0, \tau\big), \mathrm{Ent}\big(\bar{\mu}_j^1, \tau\big)\Big\} \tag{4.226}$$

for a.e. $j \in \mathbb{N}$. Equation 4.226 shows Corollary 4.1.30 applies. The latter in turn shows $\mu(t) \in \mathscr{C} \cap \operatorname{dom} \mathrm{Ent}^\tau$ for all $t \in [0,1]$. Ergo 2.1) as claimed. If G.1) holds, then G.2) follows by 3) in Proposition 4.3.3. Lemma 4.3.7 shows L.2) implies L.1). The latter is equivalent to G.1). Get G.2). Altogether, get equivalence of G.2) and L.2).

We show equivalence of G.3) and L.3). Assume G.3). We reduce from global to local property as above. For a.e. $j \in \mathbb{N}$, Equation 4.224 reduces BE_λ from $\mathscr{C}_A(\bar{\xi}_j)$ to $\mathscr{C}_{A_j}(\bar{\xi}_j)$. Get L.3). Assume L.3). Using 3) in Proposition 2.1.31, 2.2) in Proposition 3.2.32, which reduces to Equation 3.212 here, and Equation 4.204 show

$$h_t\big(\sharp\mu\big) = \operatorname*{s\text{-}lim}_{j \in \mathbb{N}} h_t\big(\sharp\bar{\mu}_j\big),\ h_t(u) = \|.\|_\nabla\text{-}\lim_{j \in \mathbb{N}} \pi_{\operatorname{supp}\xi_j}\big(h_t(u_j)\big) \tag{4.227}$$

for all $\mu \in \mathscr{C} \cap L^{2,\infty}(A_\xi, \tau)^\flat$, $u \in \operatorname{dom} \nabla_\xi$ and $t \geq 0$. Using Lemma A.2.5, for which we ensure necessary and suitable uniform boundedness by 2.1) in Proposition 2.1.31, the left-hand side of Equation 4.227 implies

$$\mathscr{M}^{\frac{1}{2}}_{h_t(\sharp\mu)} = \operatorname*{s\text{-}lim}_{j \in \mathbb{N}} \mathscr{M}^{\frac{1}{2}}_{h_t(\sharp\bar{\mu}_j)} \tag{4.228}$$

for all $\mu \in \mathscr{C} \cap L^{2,\infty}(A_\xi, \tau)^\flat$ and $t \geq 0$ (cf. Remark A.2.3 and Remark A.2.4).

Finally, we estimate. For a.e. $j \in \mathbb{N}$, note L.3) shows BE_λ for all $\mu \in \mathscr{C}_{A_j}(\bar{\xi}_j)$, $u_j \in A_{j,\bar{\xi}_j}$ and $t \geq 0$. Using the latter, the right-hand side of Equation 4.227 and Equation 4.228 let us estimate

$$\begin{aligned}
\big\|\mathscr{M}^{\frac{1}{2}}_{\sharp\mu} \nabla h_t(u)\big\|_\omega^2 &= \lim_{j \in \mathbb{N}} \big\|\mathscr{M}^{\frac{1}{2}}_{\sharp\bar{\mu}_j} \nabla h_t(u_j)\big\|_\omega^2 \\
&\leq \lim_{j \in \mathbb{N}} e^{-2\lambda t} \big\|\mathscr{M}^{\frac{1}{2}}_{h_t(\sharp\bar{\mu}_j)} \nabla u_j\big\|_\omega^2 \\
&= e^{-2\lambda t} \big\|\mathscr{M}^{\frac{1}{2}}_{h_t(\sharp\mu)} \nabla u\big\|_\omega^2
\end{aligned}$$

for all $\mu \in \mathscr{C} \cap L^{2,\infty}(A_\xi, \tau)^\flat$, $u \in \operatorname{dom} \nabla_\xi$ and $t \geq 0$. The above calculation lifts BE_λ from $\{\mathscr{C}_{A_j}(\bar{\xi}_j)\}_{j \in \mathbb{N}}$ to $\mathscr{C} \cap \operatorname{dom} \mathrm{Ent}^\tau$. Get G.3). Altogether, get equivalence of G.3) and L.3). □

Corollary 4.3.9. *Let* Ent^τ *satisfy* EVI_λ *for* $\lambda \in \mathbb{R}$. *Let* $S : [0,\infty) \times \mathscr{S}(A) \longrightarrow \mathscr{S}(A)$ *be a continuous semigroup s.t.* $S_t : \mathscr{S}(A) \longrightarrow \mathscr{S}(A)$ *is* w^**-continuous for all* $t \geq 0$. *If we know* $S : [0,\infty) \times \mathscr{C} \cap \mathrm{dom}\,\mathrm{Ent}^\tau \longrightarrow \mathscr{C} \cap \mathrm{dom}\,\mathrm{Ent}^\tau$ *is* EVI_λ*-gradient flow of* Ent^τ *in* $\mathscr{C} \cap \mathrm{dom}\,\mathrm{Ent}^\tau$ *for all finitely supported* $\mathscr{C} \subset (\mathscr{S}(A), \mathscr{W}_\nabla^{\log})$ *s.t.* $\mathscr{C} \cap \mathrm{dom}\,\mathrm{Ent}^\tau \neq \emptyset$, *then* $S = h$.

Proof. Let $\lambda \in \mathbb{R}$ as per hypothesis. For all finitely supported fixed states $\xi \in \mathscr{S}(A)$, we know $S : [0,\infty) \times \mathscr{C}_A(\bar{\xi}_j) \cap \mathrm{dom}\,\mathrm{Ent}^\tau \longrightarrow \mathscr{C}_A(\bar{\xi}_j) \cap \mathrm{dom}\,\mathrm{Ent}^\tau$ is EVI_λ-gradient flow of Ent^τ in $\mathscr{C}_A(\bar{\xi}_j) \cap \mathrm{dom}\,\mathrm{Ent}^\tau \neq \emptyset$ for a.e. $j \in \mathbb{N}$. Uniqueness of EVI_λ-gradient flows [160] implies $S_t(\mu) = h_t(\mu)$ for all $\mu \in \mathscr{C}_A(\bar{\xi}_j)$ and $t \geq 0$ in each case. Diagram 3.346 for $K = \mathrm{dom}\,\mathrm{Ent}^\tau$ furthermore shows

$$\mathscr{S}(A) = \overline{\bigcup_{\xi \in \mathscr{S}(A)} \bigcup_{j \in \mathbb{N}} \mathscr{C}_A(\bar{\xi}_j)} \tag{4.229}$$

in w^*-topology. However, each non-vanishing $\bar{\xi}_j \in \mathscr{S}(A)$ is a finitely supported fixed state itself. Equation 4.199 therefore implies we may reduce to finitely supported $\xi \in \mathscr{S}(A)$ in the first product on the right-hand side of Equation 4.229. The latter therefore implies $S_t = h_t$ for all $t \geq 0$ by w^*-continuity. □

4.3.2 Lower Ricci bounds

We define lower Ricci bounds of quantum gradients using conditions in Definition 4.3.6. Theorem 4.3.8 ensures all such conditions are indeed equivalent. Lower Ricci bounds are given by λ-convexity of quantum information along minimising geodesics measured by quantum relative entropy. Their non-spatiality is further visible beyond the given description in terms of quantum information theory [62] as follows. Assuming strictly positive lower Ricci bounds and finitely supported fixed part, Theorem 4.3.12 classifies accessibility components of normal states with finite quantum relative entropy using fixed parts. Using the latter, we show strictly positive lower Ricci bounds determine energy-information trade-offs parametrised by lower bounds on quantum noise.

Moreover, we extend remaining results in [48][49][50] as claimed. Theorem 4.3.18 gives sufficient conditions for lower Ricci bounds of direct sum quantum gradients. Apart from generalised discrete derivatives over finite sets, Theorem 4.3.18 applies to all fundamental example classes in Subsection 3.1.3. Theorem 4.3.25 derives functional inequalities and their chain of implications. Note all terms correcting for non-ergodicity are given by quantum relative entropy evaluated on finitely supported fixed parts since conditioning is determined by the underlying metric geometry as restriction to finitely supported accessibility components.

Definition and energy-information trade-offs from quantum noise. We use quantum relative entropy as measure of quantum information. Assume the logarithmic mean setting. Note our discussion concerning quantum optimal transport as transport of quantum information in Subsection 3.3.2.

Theorem 4.3.8 ensures we may use any condition in Definition 4.3.6 as equivalent characterisation. Definition 4.3.10 gives lower Ricci bounds of quantum gradients. We view them as measurement convexity of quantum information. Specifically, note CNV_λ as per Equation 4.202 shows lower Ricci bounds are given by λ-convexity of quantum information along minimising geodesics measured by quantum relative entropy. In light of our discussion in Subsection 3.3.2, this is a non-spatial description of λ-convexity but not one we have related to computation. If we do have noncommutative analogues of displacement interpolations [72][156], then precomposition with quantum channels as per Remark 4.3.11 transforms such measurement convexity in the Schrödinger picture into convexity under measurement of observables in the Heisenberg picture. We may view such channels as computations of a quantum computer [18][62] to get a computational interpretation of lower Ricci bounds. Unfortunately, existence results are unknown to us. We instead show strictly positive lower Ricci bounds determine energy-information trade-offs parametrised by lower bounds on quantum noise. Lower resolution implies lower energy paths. We avoid spatial interpretations of the classical case [97][151].

Let $(\phi,\psi,\gamma,\nabla)$ be noncommutative differential structure for tracial AF-C^*-algebras (A,τ) and (B,ω) in the logarithmic mean setting.

Definition 4.3.10. We say that $\lambda \in \mathbb{R}$ is a lower Ricci bound on $\mathscr{S}(A)$ given $(\phi,\psi,\gamma,\nabla)$ if any condition in Definition 4.3.6 is satisfied for λ. We further write $\operatorname{Ric}\nabla \geq \lambda$ and say that λ is a lower Ricci bound of ∇.

Remark 4.3.11. We know lower Ricci bounds [151][189][190] for optimal transport on continuous geometries [8][97][199] are displacement convexity of relative entropy in the sense of McCann [72][156]. Let (X,g) be a complete connected smooth Riemannian manifold and $d|\mathrm{vol}|$ the Riemannian density on X (cf. pp.299-306 in [144]). Get metric measure space $(X,d^g,d|\mathrm{vol}|)$ with d^g given by g and exponential map $\exp : TX \longrightarrow X$ on TX by the Hopf-Rinow theorem (cf. pp.216-224 in [144]). If $\mu : [0,1] \longrightarrow \mathscr{S}^{\mathrm{N}}(C_0(X))$ is a minimising geodesic for the classical L^2-Wasserstein distance [97], then Theorem 3.2 and Corollary 5.2 in [72] imply there exists a $d|\mathrm{vol}|$-a.e. differentiable map $u : X \longrightarrow \mathbb{R}$ and homotopy $F : [0,1] \times X \longrightarrow X$ defined by

$$F(t)(x) := \exp_x\big(-t \cdot \mathrm{grad}_x u\big) \tag{4.230}$$

for all $x \in X$ and $t \in [0,1]$ s.t. its dualisation $F^* : [0,1] \times C_0(X) \longrightarrow C_0(X)$ in the second variable satisfies

$$\mu(t)(h) = \int_X h(x)d\mu(t) = \int_X h(x)dF(t)_\sharp\big(\mu(0)\big) = \int_X h(F(t)(x))d\mu = \mu\big(F(t)^*(h)\big) \tag{4.231}$$

for all $h \in C_0(X)$ and $t \in [0,1]$. Homotopies as per Equation 4.230 extend the pointwise case in [157] and are called displacement interpolations generalising terminology in the Euclidian case [156]. Functionals satisfying strong convexity, resp. a weaker form as per 2) in Definition 4.3.1, along interpolation lines determined by Equation 4.230 are called displacement convex. Equation 4.231 is a push-forward measure representation transforming the Eulerian picture into the Lagrangian one (cf. pp.224-225 in [72]).

Noncommutative analogues of Equation 4.230 are given by deforming the identity operator using quantum channels. Indeed, precomposition with any continuous function is unital and positive, ergo completely positive by commutativity (cf. Corollary IV.3.5 in [192]). Following Remark 3.2.26, we see analogues of homotopies as per Equation 4.230 in the AF-C^*-setting are given by $\varphi : [0,1] \times \mathscr{B}(A) \longrightarrow \mathscr{B}(A)$ s.t. $\varphi(t) \in \mathscr{B}(A)$ is a quantum channel for all $t \geq 0$ and $\varphi(0) = I$. If $\mu : [0,1] \longrightarrow \mathscr{S}^{\mathrm{N}}(A)$ is a minimising geodesic, then we want such deformation $\varphi : [0,1] \times \mathscr{B}(A) \longrightarrow \mathscr{B}(A)$ of the identity operator s.t.

$$\mu(t)(x) = \varphi(t)^*(\mu)(x) = \mu\big(\varphi(t)(x)\big) \tag{4.232}$$

for all $x \in A$ and $t \in [0,1]$. Passing from points $x \in X$ to observables formally replaces the Lagrangian with the Heisenberg picture as we replace vectors of real numbers with bounded operators (cf. pp.xix-xx in [193]). Since each $\varphi(t)$ in Equation 4.232 moreover describes a state change due to measurement [62][84][141][163], i.e. each transmits a corresponding change of information encoded in states of the given quantum system [62] providing physical realisation of a quantum computer [18][43], Equation 4.232 shows measurement convexity in the Schrödinger picture as per Definition 4.3.10 is convexity under measurement of observables in the Heisenberg picture.

We show conditioning in Definition 4.3.23 is determined by the underlying metric geometry as restriction to finitely supported accessibility components. Assuming strictly positive lower Ricci bounds and finitely supported fixed part, Theorem 4.3.12 classifies accessibility components of normal states with finite quantum relative entropy using fixed parts. Strictly lower Ricci bounds avoid assumptions on spectral gaps required by Theorem 3.2.65. We use Corollary 4.3.13 to formulate energy-information trade-offs.

Theorem 4.3.12. *Let $(\phi, \psi, \gamma, \nabla)$ be noncommutative differential structure for tracial AF-C^*-algebras (A, τ) and (B, ω) in the logarithmic mean setting. Assume $\operatorname{Ric}\nabla \geq \lambda > 0$. If $\xi \in \mathscr{S}(A)$ is a finitely supported fixed state, then*

1) $\mathscr{C}_A^{\mathrm{Ent}}(\xi) := \mathscr{C}_A(\xi) \cap \operatorname{dom}\operatorname{Ent}^\tau = \operatorname{Fix}_A(\xi) \cap \operatorname{dom}\operatorname{Ent}^\tau \neq \emptyset$,

2) $\mathscr{W}^{\log}_{\nabla|\mathscr{C}_A^{\mathrm{Ent}}(\xi)\times\mathscr{C}_A^{\mathrm{Ent}}(\xi)}$ *is finite and* $\mathscr{C}_A^{\mathrm{Ent}}(\xi) \subset \mathscr{C}_A(\xi)$ *is a geodesic subspace.*

Proof. Let $\xi \in \mathscr{S}(A)$ be a finitely supported fixed state. Let $\mathscr{C} \subset (\mathscr{S}(A), \mathscr{W}_\nabla^{\log})$ be finitely supported with fixed part ξ s.t. $\mathscr{C} \cap \operatorname{dom}\operatorname{Ent}^\tau \neq \emptyset$. Note $\operatorname{Ent}^\tau : \mathscr{C} \cap \operatorname{dom}\operatorname{Ent}^\tau \longrightarrow (-\infty, \infty)$ has complete sublevels in $\mathscr{W}_\nabla^{\log}$-topology by 3) in Proposition 4.3.5. We see Ent^τ has a unique minimum $\mu_{\min} \in \mathscr{C} \cap \operatorname{dom}\operatorname{Ent}^\tau$ by 2) in Proposition 4.3.3. Theorem 4.2.10 yields $\mu_{\min} = \xi$ by minimality. Ergo $\mathscr{C} = \mathscr{C}_A(\xi)$ by uniqueness of fixed states. Using the latter in each case, we have 1) by decomposing $\operatorname{Fix}_A(\xi)$ as per Equation 3.343. Theorem 4.3.8 shows 2) by 2.1) in Definition 4.3.6. ☐

Corollary 4.3.13. *Assume* $\operatorname{Ric}\nabla \geq \lambda > 0$. *If* $\xi \in \mathscr{S}(A)$ *is a finitely supported fixed state and* $\mathscr{C} \subset (\mathscr{S}(A), \mathscr{W}_{\nabla}^{\log})$ *is finitely supported with fixed part* ξ*, then either* $\mathscr{C} = \mathscr{C}_A(\xi)$ *or* $\mathscr{C} \cap \operatorname{dom}\operatorname{Ent}^{\tau} = \emptyset$.

Proof. If $\mathscr{C} \cap \operatorname{dom}\operatorname{Ent}^{\tau} \neq \emptyset$, then our proof of 1) in Theorem 4.3.12 shows $\mathscr{C} = \mathscr{C}_A(\xi)$. If $\mathscr{C} \cap \operatorname{dom}\operatorname{Ent}^{\tau} = \emptyset$, then $\xi \notin \mathscr{C}$ since $\xi \in \operatorname{dom}\operatorname{Ent}^{\tau}$. As such, $\mathscr{C} \neq \mathscr{C}_A(\xi)$ in this case. □

We use strictly positive lower Ricci bounds in order to determine energy-information trade-offs parametrised by lower bounds on quantum noise. Lower resolution, i.e. higher lower bounds on quantum noise, implies lower energy paths. We give one trade-off for each finitely supported accessibility component having non-trivial intersection with the domain of quantum relative entropy. Assume $\operatorname{Ric}\nabla \geq \lambda > 0$. Let $\xi \in \mathscr{S}(A)$ be a finitely supported fixed state. Let $\mathscr{C} \subset (\mathscr{S}(A), \mathscr{W}_{\nabla}^{\log})$ be finitely supported with fixed part ξ s.t. $\mathscr{C} \cap \operatorname{dom}\operatorname{Ent}^{\tau} \neq \emptyset$. Corollary 4.3.13 shows $\mathscr{C} = \mathscr{C}_A(\xi)$.

Following our maximum entropy production principle in Subsection 4.2.3, we view quantum Laplacians as generators of quantum noise evolution. Thus applying heat flow to a state for $t > 0$ introduces quantum noise. We use resolutions to define lower bounds on quantum noise. We define minimal and maximal resolution on $\mathscr{C}_A^{\operatorname{Ent}}(\xi)$ by setting

$$-\infty < \rho_A^{\min}(\xi) := \inf_{\mu \in \mathscr{C}_A^{\operatorname{Ent}}(\xi)} \operatorname{Ent}(\mu, \tau) < \rho_A^{\max}(\xi) := \sup_{\mu \in \mathscr{C}_A^{\operatorname{Ent}}(\xi)} \operatorname{Ent}(\mu, \tau) \leq \infty. \tag{4.233}$$

Get $\rho_A^{\min}(\xi) = \operatorname{Ent}(\xi, \tau)$ by 2) in Theorem 4.2.10. We say that $\rho \in (\rho_A^{\min}(\xi), \rho_A^{\max}(\xi))$ is a resolution. For all $\rho \in (\rho_A^{\min}(\xi), \rho_A^{\max}(\xi))$, we define the resolution surface and resolution sublevel of ρ by setting

$$\mathrm{R}_A^{\operatorname{Ent}}(\xi, \rho) := \left(\operatorname{Ent}^{\tau}|_{\mathscr{C}_A(\xi)}\right)^{-1}(\rho),\ \mathrm{S}_A^{\operatorname{Ent}}(\xi, \rho) := \bigcup_{\rho' \leq \rho} \left(\operatorname{Ent}^{\tau}|_{\mathscr{C}_A(\xi)}\right)^{-1}(\rho'). \tag{4.234}$$

Each $\mathrm{R}_A^{\operatorname{Ent}}(\xi, \rho)$ is determined by all states for which 2) in Theorem 4.2.10 prohibits gain in quantum information above ρ by reducing quantum noise. We thereby use resolutions to define lower bounds on quantum noise. Of course, each $\mathrm{S}_A^{\operatorname{Ent}}(\xi, \rho)$ is a sublevel of $\operatorname{Ent}^{\tau} : \mathscr{C}_A(\xi) \longrightarrow (-\infty, \infty]$. For all $\rho \in (\rho_A^{\min}(\xi), \rho_A^{\max}(\xi))$, 3) in Proposition 4.3.5 and CNV_{λ} as per Equation 4.202 show $\mathrm{S}_A^{\operatorname{Ent}}(\xi, \rho) \subset \mathscr{C}_A^{\operatorname{Ent}}(\xi)$ is a geodesic subspace and therefore a complete geodesic length-metric space.

Let $\rho \in (\rho_A^{\min}(\xi), \rho_A^{\max}(\xi))$. We obtain metric-functional system $(\mathrm{S}_A^{\operatorname{Ent}}(\xi, \rho), \mathscr{W}_{\nabla}^{\log}, \operatorname{Ent}^{\tau})$ equipped with continuous semigroup $h : [0, \infty) \times \mathrm{S}_A^{\operatorname{Ent}}(\xi, \rho) \longrightarrow \mathrm{S}_A^{\operatorname{Ent}}(\xi, \rho)$. We define the maximal lower Ricci bound of ∇ given ρ by setting

$$\lambda_A^{\max}(\xi, \rho) := \sup_{\lambda' \geq \lambda} \lambda', \tag{4.235}$$

where the supremum on the right-hand side of Equation 4.235 is taken over all $\lambda' \geq \lambda$ s.t. $h : [0, \infty) \times \mathrm{S}_A^{\operatorname{Ent}}(\xi, \rho) \longrightarrow \mathrm{S}_A^{\operatorname{Ent}}(\xi, \rho)$ is $\mathrm{EVI}_{\lambda'}$-gradient flow of $\operatorname{Ent}^{\tau}$ in $\mathrm{S}_A^{\operatorname{Ent}}(\xi, \rho)$.

For all $\mu,\eta \in \mathrm{S}_A^{\mathrm{Ent}}(\xi,\rho)$, 1) in Proposition 4.3.3 shows

$$\mathscr{W}_{\nabla}^{\log}\big(h_t(\mu),h_t(\eta)\big) \le e^{-t\lambda_A^{\max}(\xi,\rho)}\mathscr{W}_{\nabla}^{\log}(\mu,\eta) \tag{4.236}$$

for all $t \ge 0$. Moreover, 3) in Proposition 4.3.3 shows Ent^{τ} is $\lambda_A^{\max}(\xi,\rho)$-convex in $\mathscr{C}_A^{\mathrm{Ent}}(\xi)$. Note EVI_{λ} as per Equation 4.200 shows $\lambda_A^{\max}(\xi,\rho) \ge \lambda > 0$. Equation 4.236 further shows introducing quantum noise relative to ρ, i.e. $t > 0$, implies lower energy paths.

We obtain monotonically decreasing map $\lambda_A^{\max}(\xi,-) : (\rho_A^{\min}(\xi),\rho_A^{\max}(\xi)) \longrightarrow [\lambda,\infty)$. As such, Equation 4.236 shows a decrease in resolution, i.e. from ρ to $\rho' < \rho$, implies lower energy paths if $\lambda_A^{\max}(\xi,\rho') > \lambda_A^{\max}(\xi,\rho)$. For all $\rho \in (\rho_A^{\min}(\xi),\rho_A^{\max}(\xi))$, $\mu^0,\mu^1 \in \mathrm{S}_A^{\mathrm{Ent}}(\xi,\rho)$ and $(\mu,w) \in \mathrm{Geo}(\mu^0,\mu^1)$ s.t. $\mu(t) \in \operatorname{dom}\mathrm{Ent}^{\tau}$ for all $t \ge 0$, we have

$$\mathrm{Ent}\big(\mu(t),\tau\big) \le \rho - \frac{\lambda_A^{\max}(\xi,\rho)}{2}t(1-t)\cdot\mathscr{W}_{\nabla}^{\log}\big(\mu^0,\mu^1\big)^2 \tag{4.237}$$

for all $t \in [0,1]$. Equation 4.237 shows we obtain lower energy paths since energy costs of introducing and reducing quantum noise along minimising geodesics are lowered if resolutions are lowered.

Equation 4.240 gives the energy-information trade-off for $\mathscr{C}_A^{\mathrm{Ent}}(\xi)$ parametrised by lower bounds on quantum noise, i.e. by resolutions. We define strictly monotonically increasing map $\mathrm{diam}_A^{\xi} : (\rho_A^{\min}(\xi),\rho_A^{\max}(\xi)) \longrightarrow (0,\infty)$ by setting

$$\mathrm{diam}_A^{\xi}(\rho) := \sqrt{\frac{8}{\lambda_A^{\max}(\xi,\rho)}\big(\rho-\rho_{\min}^A(\xi)\big)} \tag{4.238}$$

for all $\rho \in (\rho_A^{\min}(\xi),\rho_A^{\max}(\xi))$. For all $(\rho_A^{\min}(\xi),\rho_A^{\max}(\xi))$, Equation 3.18a in the statements on asymptotic behaviour as $t \to \infty$ as per Theorem 3.5 in [160] for $\lambda > 0$ shows

$$\mathscr{W}_{\nabla}^{\log}(\mu,\xi) \le \sqrt{\frac{2}{\lambda_A^{\max}(\xi,\rho)}\big(\rho-\rho_{\min}^A(\xi)\big)} \tag{4.239}$$

for all $\mu \in \mathrm{S}_A^{\mathrm{Ent}}(\xi,\rho)$. Equation 4.239 is the Talagrand inequality TW_{λ} for $\lambda \ge 0$ as per 3) in Definition 4.3.23. Using triangle inequality, Equation 4.238 and Equation 4.239 let us calculate

$$\mathrm{diam}\ \mathrm{S}_A^{\mathrm{Ent}}(\xi,\rho) \le \mathrm{diam}_A^{\xi}(\rho) \tag{4.240}$$

for all $\rho \in (\rho_A^{\min}(\xi),\rho_A^{\max}(\xi))$. Equation 4.240 gives, on $\mathscr{C}_A^{\mathrm{Ent}}(\xi)$, a global description of our above discussion. Lower resolution implies lower energy paths since energy costs of introducing and reducing quantum noise along minimising geodesics are lowered if resolutions are lowered. Equation 4.240 formulates an energy-information trade-off since lower energy paths are obtained by introducing quantum noise.

Sufficient conditions. Theorem 4.3.18 gives sufficient conditions for lower Ricci bounds of direct sum quantum gradients. We adapt the proof of Theorem 10.9 in [50] to the AF-C^*-setting by means of the coarse graining process. Corollary 4.3.16, which uses Lemma 4.3.15, is essential for this. Lemma 4.3.15 provides detailed proof of a necessary extension of Theorem 5 in [127] to all finite-dimensional C^*-algebras. Example 4.3.19 and Example 4.3.20 derive non-negative, resp. strictly positive lower Ricci bounds.

We consider the following direct sum noncommutative differential structures. Let $m \in \mathbb{N}$. Let (A,τ) be a tracial AF-C^*-algebra and (ϕ,ψ,γ) an AF-A-bimodule structure on A. For all $n \in \{1,\dots,m\}$, let $\partial_n : A_0 \longrightarrow L^2(A,\tau)$ be a quantum gradient. We view each as noncommutative directional derivative. Proposition 2.3.29 yields their direct sum quantum gradient $\nabla^{\oplus} = \oplus_{n=1}^m \partial_n : A_0 \longrightarrow L^2(\oplus_{n=1}^m A, \oplus_{n=1}^m \tau)$. Set

$$\left(\phi^m,\psi^m,\gamma^m,\nabla^{\oplus}\right) := \left(\oplus_{n=1}^m \phi, \oplus_{n=1}^m \psi, \oplus_{n=1}^m \gamma, \oplus_{n=1}^m \partial_n\right) \tag{4.241}$$

for tracial AF-C^*-algebras (A,τ) and $(B,\omega) := (\oplus_{n=1}^m A, \oplus_{n=1}^m \tau)$ in the logarithmic mean setting. We use Notation 2.3.28. For details on direct sum quantum gradients, we refer to Subsection 2.3.2.

Notation 4.3.14. We write $\mathscr{I}_{A,A}^{\log}$ for the quasi-entropy of the canonical AF-A-bimodule structure on A in the logarithmic mean setting. Compare to Notation 2.2.26. For all $n \in \mathbb{N}$ and tracial AF-C^*-algebra $(M_n(\mathbb{C}),\mathrm{tr}_n)$ using non-normalised canonical trace, we further write $\mathscr{I}_{n,\mathrm{tr}}^{\log}$ for the quasi-entropy of the canonical AF-$M_n(\mathbb{C})$-bimodule structure on $M_n(\mathbb{C})$ in the logarithmic mean setting

Lemma 4.3.15. *Assume A is finite-dimensional. If $\varphi : A \longrightarrow A$ is a completely positive trace-preserving map, then we have*

$$\mathscr{I}_{A,A}^{\log}\left(\varphi\left(\sharp\mu\right)^{\flat}, \varphi\left(\sharp\eta\right)^{\flat}, \varphi(\sharp w)^{\flat}\right) \leq \mathscr{I}_{A,A}^{\log}(\mu,\eta,w) \tag{4.242}$$

for all $\mu,\eta \in A_+^*$ and $w \in A^*$.

Proof. Let $n,q \in \mathbb{N}$. We consider $(M_n(\mathbb{C}),\mathrm{tr}_n)$ and $(M_q(\mathbb{C}),\mathrm{tr}_q)$ both as finite-dimensional tracial AF-C^*-algebras, as well as Hilbert spaces using GNS-inner product of their respective non-normalised canonical traces. Let $\beta : M_n(\mathbb{C}) \longrightarrow M_q(\mathbb{C})$ be completely positive trace-preserving. Theorem 5 in [127] shows we have

$$\beta^* \circ \mathscr{D}_{\beta(X),\beta(Y)} \circ \beta \leq \mathscr{D}_{X,Y} \tag{4.243}$$

in $\mathscr{B}(M_n(\mathbb{C}))$ for all $X,Y > 0$ in $M_n(\mathbb{C})$. Equation 4.243 shows

$$\mathscr{I}_{q,\mathrm{tr}}^{\log}\left(\beta(X)^{\flat}, \beta(Y)^{\flat}, \beta(U)^{\flat}\right) \leq \mathscr{I}_{n,\mathrm{tr}}^{\log}\left(X^{\flat}, Y^{\flat}, U^{\flat}\right) \tag{4.244}$$

for all $X,Y > 0$ in $M_n(\mathbb{C})$ and $U \in M_q(\mathbb{C})$. We suppress sharp operators in all equations here. We show our claim by reducing Equation 4.242 to Equation 4.244.

Note $\mathscr{I}^{\log}_{A,A}$ is jointly convex and l.s.c. in w^*-topology by 1) in Theorem 2.2.29. We scale with strictly positive constants as in the proof of Proposition 3.1.19 by construction of quasi-entropies. Let $\varphi : A \longrightarrow A$ be completely positive trace-preserving. Since we know φ is w^*-continuous by finite-dimensionality, l.s.c. in w^*-topology implies Equation 4.242 if it holds for all $\mu, \eta \in \mathscr{S}(A)$ s.t. $\sharp\mu, \sharp\eta > 0$ in A. Let

$$(A,\tau) \overset{r_A}{\cong} (A',\tau') := \left(\oplus_{l=1}^m M_{n_l}(\mathbb{C}), \oplus_{l=1}^m C_l \mathrm{tr}_{n_l}\right). \tag{4.245}$$

Equation 4.245 uses Notation 2.1.15. We know such r_A is completely positive since it is a *-homomorphism (cf. Example A.1.47). It is furthermore trace-preserving by 2) in Proposition 2.1.24. We see $\varphi' := r_A \circ \varphi \circ r_A^{-1}$ is completely positive trace-preserving.

Proposition 2.1.24 and 2) in Proposition 2.2.14 imply

$$\mathscr{I}^{\log}_{A,A}(x,y,u) = \mathscr{I}^{\log}_{A',A'}\left(r_A(x)^\flat, r_A(y)^\flat, r_A(u)^\flat\right) \tag{4.246}$$

for all $x, y > 0$ in A and $u \in A$. Equation 4.246 implies Equation 4.242 if and only if

$$\mathscr{I}^{\log}_{A',A'}\left(\varphi'(X)^\flat, \varphi'(Y)^\flat, \varphi'(U)^\flat\right) \leq \mathscr{I}^{\log}_{A',A'}\left(X^\flat, Y^\flat, U^\flat\right) \tag{4.247}$$

for all $X, Y > 0$ in A' and $U \in A'$. We reduce Equation 4.247 to Equation 4.244.

We assume $(A,\tau) = (A',\tau')$ without loss of generality. Thus $r_A = \mathrm{id}_A$, hence $\varphi = \varphi'$. We require several identities and completely positive trace-preserving maps in order to apply Equation 4.244. For all $l \in \{1,\dots,m\}$, set $X_l := \pi_l(X)$ for all $X \in A$. The latter uses Notation 2.3.28. Proposition 2.1.24 and 2) in Proposition 2.2.14 imply

$$\mathscr{I}^{\log}_{A,A}\left(X^\flat, Y^\flat, U^\flat\right) = \sum_{l=1}^m C_l \mathscr{I}^{\log}_{n_l,\mathrm{tr}}\left(X_l^\flat, Y_l^\flat, U_l^\flat\right) \tag{4.248}$$

for all $X, Y > 0$ in A and $U \in A$. Set $q := \sum_{l=1}^m n_l$. We consider the diagonal $A \subset M_q(\mathbb{C})$. If we moreover consider $C_l = 1$ for all $l \in \{1,\dots,m\}$, then Equation 4.248 yields

$$\mathscr{I}^{\log}_{q,\mathrm{tr}}\left(X^\flat, Y^\flat, U^\flat\right) = \sum_{l=1}^m \mathscr{I}^{\log}_{n_l,\mathrm{tr}}\left(X_l^\flat, Y_l^\flat, U_l^\flat\right) \tag{4.249}$$

for all $X, Y > 0$ in A, ergo $M_q(\mathbb{C})$, and $U \in A$. Set $M_C(X) := \sum_{l=1}^m C_l X_l$ for all $X \in A$. The direct sum construction implies $M_C(X) > 0$ in $M_q(\mathbb{C})$ for all $X > 0$ in A as $C_l > 0$ in each case by assumption.

By scaling with strictly positive constants, Equation 4.248 and Equation 4.249 let us calculate

$$\begin{aligned}
\mathscr{I}^{\log}_{A,A}\left(X^\flat, Y^\flat, U^\flat\right) &= \sum_{l=1}^{m} C_l \mathscr{I}^{\log}_{n_l,\mathrm{tr}}\left(X_l^\flat, Y_l^\flat, U_l^\flat\right) \\
&= \sum_{l=1}^{m} \mathscr{I}^{\log}_{n_l,\mathrm{tr}}\left(C_l X_l^\flat, C_l Y_l^\flat, C_l U_l^\flat\right) \\
&= \mathscr{I}^{\log}_{q,\mathrm{tr}}\left(M_C(X)^\flat, M_C(Y)^\flat, M_C(U)^\flat\right)
\end{aligned}$$

in each case. Precomposing with φ in the above calculation shows

$$\mathscr{I}^{\log}_{A,A}\left(\varphi(X)^\flat, \varphi(Y)^\flat, \varphi(U)^\flat\right) = \mathscr{I}^{\log}_{q,\mathrm{tr}}\left(M_C\big(\varphi(X)\big)^\flat, M_C\big(\varphi(Y)\big)^\flat, M_C\big(\varphi(U)\big)^\flat\right) \tag{4.250}$$

for all $X, Y > 0$ in A and $U \in A$. Altogether, we have the required identities.

For all $l \in \{1, \ldots, m\}$, we define $\varphi_l : M_{n_l}(\mathbb{C}) \longrightarrow M_q(\mathbb{C})$ by setting

$$\varphi_l(X) := C_l^{-1} M_C\big(\varphi(X)\big) \tag{4.251}$$

for all $X \in M_{n_l}(\mathbb{C})$. We know the diagonal $A \subset M_q(\mathbb{C})$ is completely positive because it is a *-homomorphism (cf. Example A.1.47). Since φ is as well, Equation 4.251 readily shows each φ_l is completely positive. For all $l \in \{1, \ldots, m\}$, trace-preservation of φ implies $\mathrm{tr}_q(M_C\varphi(X)) = \sum_{l=1}^{m} C_l \,\mathrm{tr}_{n_l}(X_l) = \tau(\varphi(X)) = C_l \,\mathrm{tr}_{n_l}(X)$ and therefore

$$\mathrm{tr}_q\big(\varphi_l(X)\big) = C_l^{-1}\mathrm{tr}_q\big(M_C\varphi(X)\big) = C_l^{-1}\tau\big(\varphi(X)\big) = \mathrm{tr}_{n_l}(X_l) \tag{4.252}$$

for all $X \in M_{n_l}(\mathbb{C})$. Equation 4.252 shows each φ_l is trace-preserving. The latter holds for non-normalised canonical traces on full matrix algebras. Altogether, we have completely positive trace-preserving map $\varphi_l : M_{n_l}(\mathbb{C}) \longrightarrow M_q(\mathbb{C})$ for all $l \in \{1, \ldots, m\}$.

We consider our final reduction and apply Equation 4.244. Let $X, Y > 0$ in A, $U \in A$ and $\{\lambda_l\}_{l=1}^{m} \subset (0,1]$ s.t. $\sum_{l=1}^{m} \lambda_l = 1$. Then joint convexity and scaling with strictly positive constants followed by Equation 4.250 lets us calculate

$$\begin{aligned}
\mathscr{I}^{\log}_{A,A}\left(\varphi(X)^\flat, \varphi(Y)^\flat, \varphi(U)^\flat\right) &= \mathscr{I}^{\log}_{A,A}\left(\sum_{l=1}^{m} \lambda_l \varphi\big(\lambda_l^{-1} X_l\big)^\flat, \sum_{l=1}^{m} \lambda_l \varphi\big(\lambda_l^{-1} Y_l\big)^\flat, \sum_{l=1}^{m} \lambda_l \varphi\big(\lambda_l^{-1} U_l\big)^\flat\right) \\
&\le \sum_{l=1}^{m} \lambda_l \mathscr{I}^{\log}_{A,A}\left(\varphi\big(\lambda_l^{-1} X_l\big)^\flat, \varphi\big(\lambda_l^{-1} Y_l\big)^\flat, \varphi\big(\lambda_l^{-1} U_l\big)^\flat\right) \\
&= \sum_{l=1}^{m} \mathscr{I}^{\log}_{A,A}\left(\varphi(X_l)^\flat, \varphi(Y_l)^\flat, \varphi(U_l)^\flat\right) \\
&= \sum_{l=1}^{m} \mathscr{I}^{\log}_{q,\mathrm{tr}}\left(M_C\big(\varphi(X_l)\big)^\flat, M_C\big(\varphi(Y_l)\big)^\flat, M_C\big(\varphi(U_l)\big)^\flat\right).
\end{aligned}$$

For all $l \in \{1,\dots,m\}$, scaling with strictly positive constants implies

$$\mathscr{I}^{\log}_{q,\mathrm{tr}}\Big(M_C\big(\varphi(X_l)\big)^\flat, M_C\big(\varphi(Y_l)\big)^\flat, M_C\big(\varphi(U_l)\big)^\flat\Big) = \mathscr{I}^{\log}_{q,\mathrm{tr}}\Big(\varphi_l(X_l)^\flat, \varphi_l(Y_l)^\flat, \varphi_l(U_l)^\flat\Big). \tag{4.253}$$

Taken together, the above calculation and Equation 4.253 show

$$\mathscr{I}^{\log}_{A,A}\Big(\varphi(X)^\flat, \varphi(Y)^\flat, \varphi(U)^\flat\Big) \leq \sum_{l=1}^{m} C_l \mathscr{I}^{\log}_{q,\mathrm{tr}}\Big(\varphi_l(X_l)^\flat, \varphi_l(Y_l)^\flat, \varphi_l(U_l)^\flat\Big). \tag{4.254}$$

Equation 4.244 applies to each summand on the right-hand side of Equation 4.254 since each $\varphi_l : M_{n_l}(\mathbb{C}) \longrightarrow M_q(\mathbb{C})$ is completely positive trace-preserving. Using Equation 4.244 accordingly, applying Equation 4.254 and Equation 4.248 in order lets us calculate

$$\begin{aligned}
\mathscr{I}^{\log}_{A,A}\Big(\varphi(X)^\flat, \varphi(Y)^\flat, \varphi(U)^\flat\Big) &\leq \sum_{l=1}^{m} C_l \mathscr{I}^{\log}_{q,\mathrm{tr}}\Big(\varphi_l(X_l)^\flat, \varphi_l(Y_l)^\flat, \varphi_l(U_l)^\flat\Big) \\
&\leq \sum_{l=1}^{m} C_l \mathscr{I}^{\log}_{q,\mathrm{tr}}\Big(X_l^\flat, Y_l^\flat, U_l^\flat\Big) \\
&= \mathscr{I}^{\log}_{A,A}\Big(X^\flat, Y^\flat, U^\flat\Big).
\end{aligned}$$

This yields Equation 4.247. The general case follows as discussed above. □

Corollary 4.3.16. *Assume A is finite-dimensional. Let $\lambda \in \mathbb{R}$ and set $h_t^\dagger := \oplus_{n=1}^m e^{-\lambda t} h_t$ in $\mathscr{B}(B)$ for all $t \geq 0$. If $[\phi, \Delta_n] = [\psi, \Delta_n] = 0$ for all $n \in \{1,\dots,m\}$, then we have*

$$\mathscr{I}^{\log}\Big(h_t(\mu), h_t(\eta), h_t^\dagger(\sharp w)^\flat\Big) \leq e^{-2\lambda t} \mathscr{I}^{\log}(\mu, \eta, w) \tag{4.255}$$

for all $\mu, \eta \in A_+^*$, $w \in B^*$ and $t \geq 0$.

Proof. We suppress sharp operators in all equations here. We show our claim by reducing Equation 4.255 to Lemma 4.3.15. Let $x, y \in A_+$, $u \in B$ and $t \geq 0$. Since $\Delta^\oplus = \sum_{n=1}^m \Delta_n$ by 4) in Proposition 2.3.29, we see $[\phi, \Delta_n] = [\psi, \Delta_n] = 0$ for all $n \in \{1,\dots,m\}$ implies

$$[\phi, h_t] = [\psi, h_t] = 0 \tag{4.256}$$

for all $t \geq 0$. Equation 4.256 in turn shows

$$\mathscr{I}^{\log}\Big(h_t(x)^\flat, h_t(y)^\flat, h_t^\dagger(u)^\flat\Big) = \mathscr{I}^{\log}_{A,B}\Big(h_t\big(\phi(x)\big)^\flat, h_t\big(\psi(y)\big)^\flat, h_t^\dagger\big(\phi(u)\big)^\flat\Big) \tag{4.257}$$

by construction of quasi-entropies.

Moreover, Proposition 2.3.30 shows

$$\mathscr{I}^{\log}_{A,B}\Big(h_t\big(\phi(x)\big)^\flat, h_t\big(\psi(y)\big)^\flat, h_t^\dagger\big(\phi(u)\big)^\flat\Big) = \sum_{n=1}^{m} \mathscr{I}^{\log}_{A,A}\Big(h_t\big(\phi(x)\big)^\flat, h_t\big(\psi(y)\big)^\flat, \pi_n\big(h_t^\dagger(u)\big)^\flat\Big). \quad (4.258)$$

We combine Equation 4.257 and Equation 4.258. We obtain

$$\mathscr{I}^{\log}\Big(h_t(x)^\flat, h_t(y)^\flat, h_t^\dagger(u)^\flat\Big) = \sum_{n=1}^{m} \mathscr{I}^{\log}_{A,A}\Big(h_t\big(\phi(x)\big)^\flat, h_t\big(\psi(y)\big)^\flat, \pi_n\big(h_t^\dagger(u)\big)^\flat\Big). \quad (4.259)$$

Note 1) in Proposition 3.2.32 shows $h_t : A \longrightarrow A$ is completely positive trace-preserving. Applying Equation 4.259, Lemma 4.3.15 and finally Proposition 2.3.30 in order lets us calculate

$$\begin{aligned}
\mathscr{I}^{\log}\Big(h_t(x)^\flat, h_t(y)^\flat, h_t^\dagger(u)^\flat\Big) &= \sum_{n=1}^{m} \mathscr{I}^{\log}_{A,A}\Big(h_t\big(\phi(x)\big)^\flat, h_t\big(\psi(y)\big)^\flat, \pi_n\big(h_t^\dagger(u)\big)^\flat\Big) \\
&= e^{-2\lambda t} \cdot \sum_{n=1}^{m} \mathscr{I}^{f,\theta}_{A,A}\Big(h_t\big(\phi(x)\big)^\flat, h_t\big(\psi(y)\big)^\flat, h_t\big(\pi_n(u)\big)^\flat\Big) \\
&\leq e^{-2\lambda t} \cdot \sum_{n=1}^{m} \mathscr{I}^{\log}\Big(x^\flat, y^\flat, \pi_n(u)^\flat\Big) \\
&= e^{-2\lambda t} \cdot \mathscr{I}^{\log}\Big(x^\flat, y^\flat, u^\flat\Big).
\end{aligned}$$

The above calculation shows Equation 4.255. □

Definition 4.3.17. We call $(\phi^m, \psi^m, \gamma^m, \nabla^\oplus)$ as per Equation 4.241 their direct sum noncommutative differential structure. Let $\lambda \in \mathbb{R}$. If

1) $[\phi, \Delta_n] = [\psi, \Delta_n] = 0$,

2) $\partial_n \Delta^\oplus = (\Delta^\oplus + \lambda \cdot I)\partial_n$,

on A_0 for all $n \in \{1, \ldots, m\}$, then we say that $\nabla^\oplus$ is λ-intertwining.

Theorem 4.3.18. *Let $m \in \mathbb{N}$. Let (A, τ) be a tracial AF-C^*-algebra and (ϕ, ψ, γ) an AF-A-bimodule structure on A. For all $n \in \{1, \ldots, m\}$, let $\partial_n : A_0 \longrightarrow L^2(A, \tau)$ be a quantum gradient. We consider their direct sum noncommutative differential structure. If $\nabla^\oplus$ is λ-intertwining, then* $\operatorname{Ric} \nabla^\oplus \geq \lambda$.

Proof. We adapt the proof of Theorem 10.9 in [50] to the AF-C^*-setting by means of the coarse graining process. We reduce to the finite-dimensional setting. This is necessary to apply Corollary 4.3.16. Theorem 4.3.8 ensures H) in Definition 4.3.6 is a condition for lower Ricci bounds. For a.e. $j \in \mathbb{N}$, note H) and Definition 4.3.17 restrict to the induced noncommutative differential structure $(\phi_j^m, \psi_j^m, \gamma_j^m, \oplus_{n=1}^m \partial_{n,j})$ without changing λ.

Assume A is finite-dimensional. Then B is finite-dimensional. It suffices to show H_λ for all fixed states $\xi \in \mathscr{S}(A)$, $\mu \in \vartheta(\xi)$ and $\eta \in I(\Delta_\xi^\oplus)^\flat$. Using Corollary 3.2.66, we readily see Theorem 3.3 in [75] and Lemma 4.3.7 show the latter if $h : [0,\infty) \times \vartheta(\xi) \longrightarrow \vartheta(\xi)$ is EVI_λ-gradient flow of Ent^τ in $\vartheta(\xi)$ for all fixed states $\xi \in \mathscr{S}(A)$. We further reduce as follows. For all fixed states $\xi \in \mathscr{S}(A)$, we claim

$$\frac{1}{2}\frac{d^+}{ds}\bigg|_{s=0} \mathscr{W}_{\nabla^\oplus}^{\log}\big(\mu, h_s(\eta)\big)^2 + \frac{\lambda}{2}\mathscr{W}_{\nabla^\oplus}^{\log}(\mu,\eta)^2 \leq \mathrm{Ent}(\mu,\tau) - \mathrm{Ent}(\eta,\tau) \tag{4.260}$$

for all $\mu, \eta \in \vartheta(\xi)$. If Equation 4.260 holds, then symmetry of distances, the semigroup property and Equation 4.260 itself let us calculate

$$\begin{aligned}\frac{1}{2}\frac{d^+}{dt}\mathscr{W}_{\nabla^\oplus}^{\log}\big(h_t(\mu),\eta\big)^2 &= \frac{1}{2}\frac{d^+}{ds}\bigg|_{s=0}\mathscr{W}_{\nabla^\oplus}^{\log}\big(\eta, h_s\big(h_t(\mu)\big)\big)^2 \\ &\leq \mathrm{Ent}(\eta,\tau) - \mathrm{Ent}\big(h_t(\mu),\tau\big) - \frac{\lambda}{2}\mathscr{W}_{\nabla^\oplus}^{\log}\big(h_t(\mu),\eta\big)\end{aligned}$$

for all $t \geq 0$ in each case. The above calculation in turn shows $h : [0,\infty) \times \vartheta(\xi) \longrightarrow \vartheta(\xi)$ is EVI_λ-gradient flow of Ent^τ in $\vartheta(\xi)$ for all fixed states $\xi \in \mathscr{S}(A)$. In summary, it suffices to show Equation 4.260.

We require several identities in order to show Equation 4.260. Set $h_t^\dagger := \oplus_{n=1}^m e^{-\lambda t} h_t$ in $\mathscr{B}(B)$ for all $t \geq 0$. The latter extends to $B^* = \oplus_{n=1}^m A^*$ by dualisation in each summand as per construction of noncommutative heat semigroups. Note 1) in Definition 4.3.17 ensures Corollary 4.3.16 applies. Using the latter, we have

$$\mathscr{I}^{\log}\Big(h_t(\mu), h_t(\eta), h_t^\dagger(w)\Big) \leq e^{-2\lambda t}\mathscr{I}^{\log}(\mu,\eta,w) \tag{4.261}$$

for all $\mu, \eta \in \mathscr{S}(A)$, $w \in B^*$ and $t \geq 0$. We dualise 2) in Definition 4.3.17 by taking adjoints. Using the latter, 3) in Proposition 2.3.29 implies

$$h_t \nabla^{\oplus,*} = \nabla^{\oplus,*} h_t^\dagger \tag{4.262}$$

for all $t \geq 0$. Altogether, we have the required identities.

We show Equation 4.260. Let $\xi \in \mathscr{S}(A)$ be a fixed state. Let $\mu^0, \mu^1 \in \vartheta(\xi)$. Heat flow varies minimising geodesics as follows. Let $\mu : [0,1] \longrightarrow \vartheta(\xi)$ be a minimising geodesic from μ^0 to μ^1. Proposition 3.2.56 shows the canonical vector field along μ is given by $w_t := \Theta(\mu(t), \dot{\mu}(t))$ for all $t \geq 0$. We have $(\mu, w) \in \mathrm{Adm}^{[0,1]}(\mu^0, \mu^1)$. Minimising geodesics have t-a.e. constant speed by 1) in Proposition 3.1.45. The latter lets us calculate

$$E^{\log}(\mu, w) = \mathscr{I}^{\log}\big(\mu(t), \mu(t), w(t)\big) \tag{4.263}$$

for all $t \in [0,1]$.

For all $s \in [0,1]$, set

$$\mu_s(t) := h_{ts}\big(\mu(t)\big),\ w_s(t) := h_{ts}^{\dagger}(w(t)) - s\big(\nabla^{\oplus}\sharp\mu_s(t)\big)^{\flat} \tag{4.264}$$

for all $t \in [0,1]$. For all $s \in [0,1]$, Equation 4.262 and Equation 4.264 let us calculate

$$\begin{aligned}\frac{d}{dt}\mu_s(t) &= h_{ts}\big(\nabla^{\oplus,*}\sharp w(t)\big)^{\flat} - s\big(\Delta^{\oplus}\sharp\mu_s(t)\big)^{\flat} \\ &= \Big(\nabla^{\oplus,*}\Big(\sharp h_{ts}^{\dagger}(w(t)) - s\nabla^{\oplus}\sharp\mu_s(t)\Big)\Big)^{\flat} = \big(\nabla^{\oplus,*}\sharp w_s(t)\big)^{\flat}\end{aligned}$$

for all $t \in (0,1)$. The above calculation shows $(\mu_s, w_s) \in \mathrm{Adm}^{[0,1]}(\mu^0, h_s(\mu^1))$ for all $s \in [0,1]$.
We estimate $E^{\log}(\mu^s, w^s)$ in each case. Let $s \in (0,1]$. Set

$$F_s(t) := -2s\big\langle \mathscr{D}_{\sharp\mu_s(t),\xi}\sharp h_{ts}^{\dagger}(w(t)), \nabla^{\oplus}\sharp\mu_s(t)\big\rangle_{\omega} + s^2\big\|\mathscr{D}_{\sharp\mu_s(t),\xi}^{\frac{1}{2}}\nabla^{\oplus}\sharp\mu_s(t)\big\|_{\omega}^{2} \tag{4.265}$$

for all $t \in (0,1]$. Equation 4.261, Equation 4.263 and Equation 4.264 let us calculate

$$E^{\log}\big(\mu_s, w_s\big) \le \int_0^1 e^{-2\lambda ts}dt \cdot E^{\log}(\mu, w) + \int_0^1 F_s(t)dt. \tag{4.266}$$

We therefore define the integrand F_s precisely as per Equation 4.265 in order to have Equation 4.266. We directly verify

$$\int_0^1 e^{-2\lambda ts}dt = \frac{1 - e^{-2\lambda s}}{2\lambda s}. \tag{4.267}$$

Taken together, Equation 4.266 and Equation 4.267 show

$$E^{\log}\big(\mu_s, w_s\big) \le \frac{1 - e^{-2\lambda s}}{2\lambda s} \cdot E^{\log}(\mu, w) + \int_0^1 F_s(t)dt. \tag{4.268}$$

Equation 4.268 clearly shows we must estimate the integrand F_s. Using $\sharp h_{ts}^{\dagger}(w(t)) = \sharp w_s(t) + s\nabla^{\oplus}\sharp\mu_s(t)$ in each case, 2) in Lemma 4.2.8 lets us calculate

$$\begin{aligned}\big\langle \mathscr{D}_{\sharp\mu_s(t),\xi}\sharp h_{ts}^{\dagger}(w(t)), \nabla^{\oplus}\sharp\mu_s(t)\big\rangle_{\omega} &= \big\langle \mathscr{D}_{\sharp\mu_s(t),\xi}\sharp w_s(t), \nabla^{\oplus}\sharp\mu_s(t)\big\rangle_{\omega} + s\big\|\mathscr{D}_{\sharp\mu_s(t),\xi}^{\frac{1}{2}}\nabla^{\oplus}\sharp\mu_s(t)\big\|_{\omega}^{2} \\ &= \frac{d}{dt}\mathrm{Ent}^{\tau}\big(\mu_s(t)\big) + s\big\|\mathscr{D}_{\sharp\mu_s(t),\xi}^{\frac{1}{2}}\nabla^{\oplus}\sharp\mu_s(t)\big\|_{\omega}^{2}\end{aligned}$$

for all $t \in (0,1)$.

The above calculation shows

$$-2s\frac{d}{dt}\operatorname{Ent}^{\tau}\big(\mu_s(t)\big) = -2\big\langle \mathscr{D}_{\sharp\mu_s(t),\xi}\sharp h_{ts}^{\dagger}(w(t)), \nabla^{\oplus}\sharp\mu_s(t)\big\rangle_{\omega} + 2s^2\big\|\mathscr{D}_{\sharp\mu_s(t),\xi}^{\frac{1}{2}}\nabla^{\oplus}\sharp\mu_s(t)\big\|_{\omega}^2 \quad (4.269)$$

in each case by rearranging terms accordingly. Finally, we readily see Equation 4.265 and Equation 4.269 let us calculate

$$F_s(t) = -2s\frac{d}{dt}\operatorname{Ent}^{\tau}\big(\mu_s(t)\big) - s^2\big\|\mathscr{D}_{\sharp\mu_s(t),\xi}^{\frac{1}{2}}\nabla^{\oplus}\sharp\mu_s(t)\big\|_{\omega}^2 \leq -2s\frac{d}{dt}\operatorname{Ent}^{\tau}\big(\mu_s(t)\big) \quad (4.270)$$

for all $t \in (0,1)$. We combine Equation 4.268 and Equation 4.270. We obtain

$$E^{\log}\big(\mu_s, w_s\big) \leq \frac{1-e^{-2\lambda s}}{2\lambda s}E^{\log}(\mu, w) + s\cdot\Big(\operatorname{Ent}\big(\mu^0,\tau\big) - \operatorname{Ent}\big(\mu^1,\tau\big)\Big). \quad (4.271)$$

Equation 4.271 yields our required estimate of $E^{\log}(\mu^s, w^s)$ for all $s \in (0,1]$. We engage in our final estimate. Equation 4.271 implies

$$\frac{1}{2}\mathscr{W}_{\nabla^{\oplus}}^{\log}\big(\mu^0, h_s\big(\mu^1\big)\big)^2 \leq \frac{1-e^{-2\lambda s}}{4\lambda s}\cdot E^{\log}(\mu, w) + s\cdot\Big(\operatorname{Ent}\big(\mu^0,\tau\big) - \operatorname{Ent}\big(\mu^1,\tau\big)\Big) \quad (4.272)$$

for all $s \in (0,1]$. We use the energy $E^{\log}(\mu, w) = \mathscr{W}_{\nabla^{\oplus}}^{\log}(\mu^0,\mu^1)^2$ of the minimising geodesic $\mu : [0,1] \longrightarrow \vartheta(\xi)$ from μ^0 to μ^1. Corollary 3.2.63 ensures we have sufficient minimising geodesics in $\vartheta(\xi)$. Equation 4.272 therefore equals

$$\frac{1}{2}\mathscr{W}_{\nabla^{\oplus}}^{\log}\big(\mu^0, h_s\big(\mu^1\big)\big)^2 \leq \frac{1-e^{-2\lambda s}}{4\lambda s}\cdot \mathscr{W}_{\nabla^{\oplus}}^{\log}\big(\mu^0,\mu^1\big)^2 + s\cdot\Big(\operatorname{Ent}\big(\mu^0,\tau\big) - \operatorname{Ent}\big(\mu^1,\tau\big)\Big) \quad (4.273)$$

for all $s \in (0,1]$. We see multiplying with s^{-1} on and subtracting $\frac{1}{2s}\mathscr{W}_{\nabla^{\oplus}}^{\log}(\mu^0,\mu^1)^2$ from both sides of Equation 4.273 yields

$$\begin{aligned}&\frac{1}{2}s^{-1}\Big(\mathscr{W}_{\nabla^{\oplus}}^{\log}\big(\mu^0, h_s\big(\mu^1\big)\big)^2 - \mathscr{W}_{\nabla^{\oplus}}^{\log}\big(\mu^0,\mu^1\big)^2\Big)\\ &\leq \frac{1-e^{-2\lambda s}-2\lambda s}{4\lambda s^2}\cdot \mathscr{W}_{\nabla^{\oplus}}^{\log}\big(\mu^0,\mu^1\big)^2 + \operatorname{Ent}\big(\mu^0,\tau\big) - \operatorname{Ent}\big(h_s\big(\mu^1\big),\tau\big)\end{aligned}$$

for all $s \in (0,1]$. We directly verify

$$\lim_{s\downarrow 0}\frac{1-e^{-2\lambda s}-2\lambda s}{4\lambda s^2} = -\frac{\lambda}{2}. \quad (4.274)$$

Note Equation 4.274 shows letting $s \downarrow 0$ in the final estimate yields Equation 4.260. The general case follows as discussed above. $\square$

Example 4.3.19 and Example 4.3.20 derive non-negative, resp. strictly positive lower Ricci bounds. Whereas Example 4.3.19 covers Example 3.1.55, Example 3.1.56 and Example 3.1.58 in Subsection 3.1.3, Example 4.3.20 covers Example 3.1.59 therein.

Example 4.3.19. Assume the following setting. Let (A,τ) be a tracial AF-C^*-algebra and $(A,\mathbb{R},\alpha)$ a τ-preserving local C^*-dynamical system. We equip A with its canonical AF-A-bimodule structure. We use $m=1$. Corollary 2.3.49 yields non-twisted dynamic quantum gradient $\nabla^{\mathscr{D}_\alpha,\mathrm{id}_A}$ and shows

$$\Delta^{\mathscr{D}_\alpha}x = -\left(\nabla^{\mathscr{D}_\alpha}\right)^2(x) \tag{4.275}$$

for all $x\in A_0$. Equation 4.275 shows $\nabla^{\mathscr{D}_\alpha,\mathrm{id}_A}$ is λ-intertwining for $\lambda=0$. Theorem 4.3.18 implies $\mathrm{Ric}\,\nabla^{\mathscr{D}_\alpha,\mathrm{id}_A}\geq 0$ as claimed.

Example 4.3.20. Assume the following setting. Let (A,τ) be a tracial AF-C^*-algebra and $\phi: A\longrightarrow A$ a self-adjoint involutive local *-homomorphism. Let $m\in\mathbb{N}$ and further $\{d_n\}_{n=1}^m\subset L^\infty(A,\tau)_h$ be a ϕ-intertwining set of Clifford generators for $C>0$ as per 1) in Definition 2.3.58. For all $n\in\{1,\dots,m\}$, Corollary 2.3.56 yields twisted dynamic quantum gradient $\partial_n=\nabla^{-iL_{d_n},\phi}$ and its Laplacian $\Delta_n=\partial_n^*\partial_n$ as per 2) in Definition 2.3.58.

Note Equation C.5 lets us calculate 1) in Definition 4.3.17. Since $\Delta^\oplus=\sum_{n=1}^m\Delta_n$ by 4) in Proposition 2.3.29, Lemma 2.3.59 implies

$$\partial_n\Delta^\oplus = \left(\Delta^\oplus + 4C\cdot I\right)\partial_n \tag{4.276}$$

for all $n\in\{1,\dots,m\}$. Equation 4.276 shows 2) in Definition 4.3.17. Altogether, we see ∇ is λ-intertwining for $\lambda=4C$. Theorem 4.3.18 implies $\mathrm{Ric}\,\nabla\geq 4C>0$ as claimed.

Functional inequalities. Assuming lower Ricci bounds, Theorem 4.3.25 derives functional inequalities HWI_λ, MLSI_λ and TW_λ as per Definition 4.3.23. Non-ergodicity requires relative entropy of finitely supported fixed states in their formulation. We introduce quantum Fisher information in the AF-C^*-setting. Its rôle mirrors the classical case [151][168]. We adapt the proofs of Theorem 11.3, Theorem 11.4 and Theorem 11.5 in [50] to the AF-C^*-setting by means of the coarse graining process. Lemma 4.3.24 provides sufficient control of metric derivatives using quantum Fisher information.

Let $(\phi,\psi,\gamma,\nabla)$ be noncommutative differential structure for tracial AF-C^*-algebras (A,τ) and (B,ω) in the logarithmic mean setting. Definition 4.3.21 gives quantum Fisher information. Indeed, note 3) in Theorem 4.2.35 and 3) in Proposition 4.3.22 imply it is a noncommutative analogue for parametrisations $\{h_t(\mu)\}_{t\geq 0}$ given $\mu\in\mathscr{S}^{\mathrm{N}}(A)$.

Definition 4.3.21. We define quantum Fisher information $\mathrm{I}^{\log}: A_+^*\longrightarrow[0,\infty]$ by setting

$$\mathrm{I}^{\log}(\mu) := \sup_{j\in\mathbb{N}}\mathscr{I}_j^{\log}\Big(\mu_j,\mu_j,\left(\nabla\sharp\mu_j\right)^\flat\Big) \tag{4.277}$$

for all $\mu\in A_+^*$.

Proposition 4.3.22.

1) $\mathrm{I}^{\log}$ *is convex and l.s.c. in w^*-topology.*

2) *For all $\mu \in A_+^*$, we have*

   2.1) $\mathrm{I}^{\log}(\bar{\mu}_j) = \mathscr{I}_j^{\log}\Big(\bar{\mu}_j, \bar{\mu}_j, \big(\nabla\sharp\bar{\mu}_j\big)^\flat\Big)$ *for all $j \in \mathbb{N}$,*

   2.2) $\mathrm{I}^{\log}(\mu) = \lim_{j\in\mathbb{N}} \mathrm{I}^{\log}(\bar{\mu}_j)$.

3) *For all finitely supported fixed states $\xi \in \mathscr{S}(A)$, we have*

$$\mathrm{I}^{\log}(\mu) = -\frac{d}{dt}\bigg|_{t=0} \mathrm{Ent}^\tau\big(h_t(\mu)\big) \tag{4.278}$$

for all $\mu \in \mathrm{Fix}_A^{\mathrm{N}}(\xi) \cap \mathscr{S}_{-1}^{\mathrm{N},\infty}(A_\xi) \cap (\mathrm{dom}\,\Delta)^\flat$.

Proof. We have 1) and 2.1) by 1), resp. 2) in Theorem 2.2.29. Using 2.1), Equation 4.277 shows 3) in Theorem 2.2.29 implies 2.2). We show 3). Let $\xi \in \mathscr{S}(A)$ be a finitely supported fixed state. Using 2.1) and 4.1) in Proposition 2.3.25, note 2) lets us calculate

$$\mathrm{I}^{\log}(\mu) = \lim_{j\in\mathbb{N}} \mathrm{I}^{\log}(\bar{\mu}_j) = \mathscr{I}^{\log}\Big(\mu, \mu, \big(\nabla\sharp\mu\big)^\flat\Big) \tag{4.279}$$

for all $\mu \in \mathrm{Fix}_A^{\mathrm{N}}(\xi) \cap \mathscr{S}_{-1}^{\mathrm{N},\infty}(A_\xi) \cap (\mathrm{dom}\,\Delta)^\flat$. The second identity in Equation 4.279 uses $\sharp\mu \in \mathrm{dom}\,\Delta \subset \mathrm{dom}\,\nabla$ and therefore $\nabla\sharp\mu = \|.\|_\omega\text{-}\lim_{j\in\mathbb{N}} \nabla\sharp\bar{\mu}_j$ in each case. Equation 4.279 shows 3) in Theorem 4.2.35 implies 3) by differentiation at $t = 0$. □

Definition 4.3.23. Let $\lambda \in \mathbb{R}$

1) We say that Ent^τ satisfies HWI_λ if for all finitely supported fixed states $\xi \in \mathscr{S}(A)$ and $\mathscr{C} \subset \big(\mathscr{S}(A), \mathscr{W}_\nabla^{\log}\big)$ with fixed part ξ s.t. $\mathscr{C} \cap \mathrm{dom}\,\mathrm{Ent}^\tau \neq \emptyset$, we have

$$\mathrm{Ent}(\mu,\tau) \leq \mathscr{W}_\nabla^{\log}(\mu,\xi)\sqrt{\mathrm{I}^{\log}(\mu)} - \frac{\lambda}{2}\mathscr{W}_\nabla^{\log}(\mu,\xi)^2 + \mathrm{Ent}(\xi,\tau) \tag{HWI_λ}$$

   for all $\mu \in \mathscr{C}$.

2) Assume $\lambda > 0$. We say that Ent^τ satisfies MLSI_λ if for all finitely supported fixed states $\xi \in \mathscr{S}(A)$ and $\mathscr{C} \subset \big(\mathscr{S}(A), \mathscr{W}_\nabla^{\log}\big)$ with fixed part ξ s.t. $\mathscr{C} \cap \mathrm{dom}\,\mathrm{Ent}^\tau \neq \emptyset$, we have

$$\mathrm{Ent}(\mu,\tau) \leq \frac{1}{2\lambda}\mathrm{I}^{\log}(\mu) + \mathrm{Ent}(\xi,\tau) \tag{MLSI_λ}$$

   for all $\mu \in \mathscr{C}$.

3) Assume $\lambda > 0$. We say that Ent^τ satisfies TW_λ if for all finitely supported fixed states $\xi \in \mathscr{S}(A)$ and $\mathscr{C} \subset \left(\mathscr{S}(A), \mathscr{W}_\nabla^{\log}\right)$ with fixed part ξ s.t. $\mathscr{C} \cap \operatorname{dom}\operatorname{Ent}^\tau \neq \emptyset$, we have

$$\mathscr{W}_\nabla^{\log}(\mu,\xi) \leq \sqrt{\frac{2}{\lambda}\left(\operatorname{Ent}(\mu,\tau) - \operatorname{Ent}(\xi,\tau)\right)} \tag{TW_λ}$$

for all $\mu \in \mathscr{C}$.

Lemma 4.3.24. *For all $\mu, \eta \in \mathscr{S}(A)$, we have*

$$\limsup_{j\in\mathbb{N}} \frac{d^+}{dt} \mathscr{W}_\nabla^{\log}\left(h_t(\bar{\mu}_j), \bar{\eta}_j\right) \leq \sqrt{\mathrm{I}^{\log}\left(h_t(\mu)\right)} \tag{4.280}$$

for all $t \geq 0$.

Proof. We adapt the proof of Proposition 11.2 in [50] to the AF-C^*-setting by means of the coarse graining process. We reduce to the finite-dimensional setting. Note 2.2) in Proposition 3.2.32 reduces to Equation 3.212 in the square integrable case. For all $\mu \in \mathscr{S}(A)$, 2.2) in Proposition 4.3.22 therefore shows

$$\mathrm{I}^{\log}\left(h_t(\mu)\right) = \lim_{j\in\mathbb{N}} \mathrm{I}^{\log}\left(h_t(\bar{\mu}_j)\right) = \limsup_{j\in\mathbb{N}} \mathrm{I}^{\log}\left(h_t(\bar{\mu}_j)\right) \tag{4.281}$$

for all $t \geq 0$. Equation 4.281 implies Equation 4.280 if for all $\mu, \eta \in \mathscr{S}(A)$, we have

$$\frac{d^+}{dt} \mathscr{W}_\nabla^{\log}\left(h_t(\bar{\mu}_j), \bar{\eta}_j\right) \leq \sqrt{\mathrm{I}^{\log}\left(h_t(\bar{\mu}_j)\right)} \tag{4.282}$$

for all $t \geq 0$ and a.e. $j \in \mathbb{N}$. Taken together, Equation 4.281 and Equation 4.282 reduce our claim to the finite-dimensional setting.

Assume A and B are finite-dimensional. We show Equation 4.280. Let $\mu, \eta \in \mathscr{S}(A)$. Using the semigroup property, 1) in Corollary 4.2.9 and 3) in Proposition 4.3.22 let us calculate

$$\mathrm{I}^{\log}\left(h_t(\mu)\right) = -\frac{d}{dt}\operatorname{Ent}^\tau\left(h_t(\mu)\right) = \tau\left(\Delta h_t(\sharp\mu) \log h_t(\sharp\mu)\right) \tag{4.283}$$

for all $t > 0$. We extend to $t \geq 0$ by continuity. Equation 4.283 shows $t \mapsto \sqrt{\mathrm{I}^{\log}(h_t(\mu))}$ is continuous on $[0,\infty)$. Using triangle inequality, we calculate

$$\begin{aligned}
\frac{d^+}{dt} \mathscr{W}_\nabla^{\log}\left(h_t(\mu), \eta\right) &= \limsup_{s\downarrow 0} s^{-1}\left(\mathscr{W}_\nabla^{\log}\left(h_{t+s}(\mu), \eta\right) - \mathscr{W}_\nabla^{\log}\left(h_t(\mu), \eta\right)\right) \\
&\leq \limsup_{s\downarrow 0} s^{-1} \mathscr{W}_\nabla^{\log}\left(h_t(\mu), h_{t+s}(\mu)\right)
\end{aligned}$$

for all $t \geq 0$.

For all $s > 0$, we claim

$$s^{-1}\mathscr{W}_{\nabla}^{\log}\big(h_t(\mu), h_{t+s}(\mu)\big) \le s^{-1} \cdot \int_t^{t+s} \sqrt{\mathrm{I}^{\log}\big(h_r(\mu)\big)}dr \tag{4.284}$$

for all $t \ge 0$. If Equation 4.284 holds, then continuity of $t \mapsto \sqrt{\mathrm{I}^{\log}(h_t(\mu))}$ on $[0,\infty)$ and Equation 4.284 itself let us calculate

$$\frac{d^+}{dt}\mathscr{W}_{\nabla}^{\log}\big(h_t(\mu), \eta\big) \le \limsup_{s \downarrow 0} s^{-1} \int_t^{t+s} \sqrt{\mathrm{I}^{\log}\big(h_t(\mu)\big)}dr = \sqrt{\mathrm{I}^{\log}\big(h_t(\mu)\big)} \tag{4.285}$$

for all $t \ge 0$. Equation 4.285 shows Equation 4.280. We therefore show Equation 4.284. Let $t \ge 0$. For all $s > 0$, set $\mu(r) := h_r(\mu)$ and $w(r) := -(\nabla \sharp \mu(r))^\flat$ for all $r \in [t, t+s]$. We show $(\mu, w) \in \mathrm{Adm}^{[t,t+s]}(h_t(\mu), h_{t+s}(\mu))$ in the proof of Corollary 4.2.9. Let $L^{\log}$ denote the length functional in our case. Using scale invariance of length functionals as per Proposition 3.1.37, we directly verify

$$L^{\log}(\mu, w) = \int_t^{t+s} \sqrt{\mathrm{I}^{\log}\big(h_t(\mu)\big)}dr. \tag{4.286}$$

Equation 4.286 shows Equation 4.284. The general case follows as discussed above. □

Theorem 4.3.25. *Let $(\phi, \psi, \gamma, \nabla)$ be noncommutative differential structure for tracial AF-C^*-algebras (A, τ) and (B, ω) in the logarithmic mean setting.*

1) If $\mathrm{Ric}\,\nabla \ge \lambda$, then Ent^τ satisfies HWI_λ.

2) If Ent^τ satisfies HWI_λ for $\lambda > 0$, then Ent^τ satisfies MLSI_λ.

3) If Ent^τ satisfies MLSI_λ, then Ent^τ satisfies TW_λ.

Proof. We adapt the proofs of Theorem 11.3, Theorem 11.4 and Theorem 11.5 in [50] to the AF-C^*-setting by means of the coarse graining process. Non-ergodicity requires us to consider relative entropy of finitely supported fixed states.

We reduce to the finite-dimensional setting. Let $\xi \in \mathscr{S}(A)$ be a finitely supported fixed state. Let $\mathscr{C} \subset (\mathscr{S}(A), \mathscr{W}_{\nabla}^{\log})$ be finitely supported with fixed part ξ s.t. we have $\mathscr{C} \cap \mathrm{dom}\,\mathrm{Ent}^\tau \neq \emptyset$. For all $j \in \mathbb{N}$ s.t. $\xi_j \neq 0$, Equation 4.199 and Equation 4.224 together with 2.1) in Proposition 4.3.22 show HWI_λ restricts to

$$\mathrm{Ent}\big(\bar{\mu}_j, \tau\big) \le \mathscr{W}_{\nabla}^{\log}\big(\bar{\mu}_j, \bar{\xi}_j\big)\sqrt{\mathrm{I}^{\log}\big(\bar{\mu}_j\big)} - \frac{\lambda}{2}\mathscr{W}_{\nabla}^{\log}\big(\bar{\mu}_j, \bar{\xi}_j\big)^2 + \mathrm{Ent}\big(\bar{\xi}_j, \tau\big) \tag{EVI_λ^j}$$

on $\mathscr{C}_{A_j}(\bar{\xi}_j)$ for all $\lambda \in \mathbb{R}$, resp. MLSI_λ and TW_λ restrict to

$$\mathrm{Ent}\big(\bar{\mu}_j,\tau\big) \le \frac{1}{2\lambda}\,\mathrm{I}^{\log}\big(\bar{\mu}_j\big) + \mathrm{Ent}\big(\bar{\xi}_j,\tau\big) \tag{MLSI^j_λ}$$

and

$$\mathscr{W}^{\log}_{\nabla}\big(\bar{\mu}_j,\bar{\xi}_j\big) \le \sqrt{\frac{2}{\lambda}\Big(\mathrm{Ent}\big(\bar{\mu}_j,\tau\big) - \mathrm{Ent}\big(\bar{\xi}_j,\tau\big)\Big)} \tag{TW^j_λ}$$

on $\mathscr{C}_{A_j}(\bar{\xi}_j)$ for all $\lambda > 0$. If conversely HWI^j_λ, MLSI^j_λ, resp. TW^j_λ holds for a.e. $j \in \mathbb{N}$, then note Equation 4.203 and Equation 4.205 together with 2.2) in Proposition 4.3.22 show letting $j \uparrow \infty$ on both sides of the given inequality recovers HWI_λ, MLSI_λ, resp. TW_λ on $\mathscr{C} \cap \operatorname{dom}\mathrm{Ent}^\tau$. We therefore reduce to the finite-dimensional setting.

Assume A and B are finite-dimensional. Let $\mu \in \mathscr{C}$. We show 1). Assume $\operatorname{Ric}\nabla \ge \lambda$. If $\mathrm{I}^{\log}(\mu) = \infty$, then there is nothing to show. Assume $\mathrm{I}^{\log}(\mu) < \infty$. Theorem 4.3.8 shows EVI_λ as per Equation 4.200 applies. Corollary 4.3.13 shows $\mu, \xi \in \mathscr{C}_A(\xi)$. We obtain

$$\mathrm{Ent}(\mu,\tau) \le -\frac{1}{2}\frac{d^+}{dt}\bigg|_{t=0} \mathscr{W}^{\log}_{\nabla}\big(h_t(\mu),\xi\big)^2 - \frac{\lambda}{2}\mathscr{W}^{\log}_{\nabla}(\mu,\xi)^2 + \mathrm{Ent}(\xi,\tau) \tag{4.287}$$

for $t = 0$ by rearranging terms accordingly. Equation 4.287 shows Ent^τ satisfies HWI_λ if

$$-\frac{1}{2}\frac{d^+}{dt}\bigg|_{t=0} \mathscr{W}^{\log}_{\nabla}\big(h_t(\mu),\xi\big)^2 \le \mathscr{W}^{\log}_{\nabla}(\mu,\xi)\sqrt{\mathrm{I}^{\log}(\mu)}. \tag{4.288}$$

We show Equation 4.288. Note 2) in Corollary 3.2.66 shows $\mathscr{W}^{\log}_{\nabla|\mathscr{C}_A(\xi)\times\mathscr{C}_A(\xi)}$ is finite and $\|.\|_A$-continuous. Using the latter, we have

$$\limsup_{t\downarrow 0} \mathscr{W}^{\log}_{\nabla}\big(h_t(\mu),\mu\big) = 0,\ \limsup_{t\downarrow 0} \mathscr{W}^{\log}_{\nabla}\big(h_t(\mu),\xi\big) = \mathscr{W}^{\log}_{\nabla}(\mu,\xi). \tag{4.289}$$

Using triangle inequality, symmetry of distances and Equation 4.289 let us calculate

$$\begin{aligned}
&-\frac{1}{2}\frac{d^+}{dt}\bigg|_{t=0} \mathscr{W}^{\log}_{\nabla}\big(h_t(\mu),\xi\big)^2\\
&= \limsup_{t\downarrow 0} \frac{1}{2}t^{-1}\Big(\mathscr{W}^{\log}_{\nabla}(\mu,\xi)^2 - \mathscr{W}^{\log}_{\nabla}\big(h_t(\mu),\xi\big)^2\Big)\\
&\le \limsup_{t\downarrow 0} \frac{1}{2}t^{-1}\Big(\Big(\mathscr{W}^{\log}_{\nabla}\big(h_t(\mu),\mu\big) + \mathscr{W}^{\log}_{\nabla}\big(h_t(\mu),\xi\big)\Big)^2 - \mathscr{W}^{\log}_{\nabla}\big(h_t(\mu),\xi\big)^2\Big)\\
&= \limsup_{t\downarrow 0} \frac{1}{2}t^{-1}\Big(\mathscr{W}^{\log}_{\nabla}\big(h_t(\mu),\mu\big)^2 + 2\mathscr{W}^{\log}_{\nabla}\big(h_t(\mu),\mu\big)\mathscr{W}^{\log}_{\nabla}\big(h_t(\mu),\xi\big)\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \limsup_{t\downarrow 0} \frac{1}{2} t^{-1} \mathscr{W}_{\nabla}^{\log}\big(h_t(\mu),\mu\big)^2 + \left(\limsup_{t\downarrow 0} t^{-1} \mathscr{W}_{\nabla}^{\log}\big(h_t(\mu),\mu\big)\right) \cdot \mathscr{W}_{\nabla}^{\log}(\mu,\xi) \\
&= 0 + \left(\limsup_{t\downarrow 0} t^{-1} \mathscr{W}_{\nabla}^{\log}\big(h_t(\mu),\mu\big)\right) \cdot \mathscr{W}_{\nabla}^{\log}(\mu,\xi) \\
&= \mathscr{W}_{\nabla}^{\log}(\mu,\xi) \cdot \left(\limsup_{t\downarrow 0} t^{-1}\Big(\mathscr{W}_{\nabla}^{\log}\big(h_t(\mu),\mu\big) - \mathscr{W}_{\nabla}^{\log}(\mu,\mu)\Big)\right) \\
&= \mathscr{W}_{\nabla}^{\log}(\mu,\xi) \cdot \frac{d^+}{dt}\bigg|_{t=0} \mathscr{W}_{\nabla}^{\log}\big(h_t(\mu),\mu\big).
\end{aligned}
$$

Applying Lemma 4.3.24 to the final term in the above calculation yields Equation 4.288 as required. As such, we know Ent^{τ} satisfies HWI_{λ}. Get 1).

We show 2). Assume Ent^{τ} satisfies HWI_{λ} for $\lambda > 0$. Note Young's inequality implies $xy \le Cx^2 + (4C)^{-1}y^2$ for all $x, y \in \mathbb{R}$ and $C > 0$ [106]. Using $C = 2^{-1}\lambda$, we obtain

$$
\mathscr{W}_{\nabla}^{\log}(\mu,\xi)\sqrt{\mathrm{I}^{\log}(\mu)} - \frac{\lambda}{2}\mathscr{W}_{\nabla}^{\log}(\mu,\xi)^2 \le \frac{1}{2\lambda}\,\mathrm{I}^{\log}(\mu) \tag{4.290}
$$

by rearranging terms accordingly. HWI_{λ} and Equation 4.290 let us calculate

$$
\mathrm{Ent}(\mu,\tau) \le \mathscr{W}_{\nabla}^{\log}(\mu,\xi)\sqrt{\mathrm{I}^{\log}(\mu)} - \frac{\lambda}{2}\mathscr{W}_{\nabla}^{\log}(\mu,\xi)^2 + \mathrm{Ent}(\xi,\tau) \le \frac{1}{2\lambda}\,\mathrm{I}^{\log}(\mu) + \mathrm{Ent}(\xi,\tau). \tag{4.291}
$$

Equation 4.291 shows Ent^{τ} satisfies MLSI_{λ}. Get 2).

We show 3). Assume Ent^{τ} satisfies MLSI_{λ}. We know $\lambda > 0$ by hypothesis. Set

$$
F(t) := \mathscr{W}_{\nabla}^{\log}\big(\mu, h_t(\mu)\big) + \sqrt{\frac{2}{\lambda}\Big(\mathrm{Ent}\big(h_t(\mu),\tau\big) - \mathrm{Ent}(\xi,\tau)\Big)} \tag{4.292}
$$

for all $t \ge 0$. Using 1) in Theorem 3.2.40 and 3) in Proposition 4.3.22, we directly verify Equation 4.292 defines continuous map $F : (0,\infty) \longrightarrow \mathbb{R}$ s.t. $\frac{d^+}{dt}F$ exists for all $t \ge 0$. Norm continuity and Theorem 4.2.10 imply

$$
F(0) := \lim_{t\downarrow 0} F(t) = \sqrt{\frac{2}{\lambda}\big(\mathrm{Ent}(\mu,\tau) - \mathrm{Ent}(\xi,\tau)\big)},\ F(\infty) := \lim_{t\uparrow\infty} F(t) = \mathscr{W}_{\nabla}^{\log}(\mu,\xi). \tag{4.293}
$$

Integrating over $[0,\infty)$, Equation 4.293 implies Ent^{τ} satisfies TW_{λ} if $\frac{d^+}{dt}F(t) \le 0$ for all $t > 0$. We show the latter.

Using the semigroup property, 3) in Proposition 4.3.22 and MLSI_λ let us calculate

$$\frac{d}{dt}\sqrt{\frac{2}{\lambda}\cdot\Big(\mathrm{Ent}\big(h_t(\mu),\tau\big)-\mathrm{Ent}(\xi,\tau)\Big)} = -\frac{\mathrm{I}^{\log}\big(h_t(\mu)\big)}{\sqrt{2\lambda\cdot\Big(\mathrm{Ent}\big(h_t(\mu),\tau\big)-\mathrm{Ent}(\xi,\tau)\Big)}}$$

$$\leq -\sqrt{\mathrm{I}^{\log}\big(h_t(\mu)\big)}$$

in each case. Note we use MLSI_λ in the denominator. Applying Lemma 4.3.24 and the above calculation to Equation 4.292 shows

$$\frac{d^+}{dt}F(t) \leq \sqrt{\mathrm{I}^{\log}\big(h_t(\mu)\big)} - \frac{d}{dt}\sqrt{\frac{2}{\lambda}\Big(\mathrm{Ent}\big(h_t(\mu),\tau\big)-\mathrm{Ent}(\xi,\tau)\Big)} \leq 0 \tag{4.294}$$

for all $t > 0$. Equation 4.294 shows $\frac{d^+}{dt}F(t) \leq 0$ for all $t > 0$ as required. As such, we know Ent^τ satisfies TW_λ. Get 3). □

Corollary 4.3.26. *If* $\mathrm{Ric}\,\nabla \geq \lambda > 0$*, then* Ent^τ *satisfies* HWI_λ*,* MLSI_λ *and* TW_λ*.*

Proof. Apply Theorem 4.3.25. □

A | Operator Theory

We review operator theory. In Section A.1, we cover fundamental results for unbounded operators, C^*- and W^*-algebras, as well as functional calculus used in our discussion. In Section A.2, we discuss strong resolvent convergence, resolvent-preserving maps of unbounded operators, and introduce compression maps.

A.1 Fundamental operator theory

In Subsection A.1.1, we review partial orders generated by positive elements, as well as spaces of bounded and unbounded operators on Hilbert spaces. We further discuss twisting maps on spaces of unbounded operators induced by Hilbert space isometries. In Subsection A.1.2, C^*- and W^*-algebras are covered. We give direct sums and tensor products. We discuss normal, completely positive and completely Markovian maps.

In Subsection A.1.3, we review functional calculus. Spectral measures of self-adjoint unbounded operators are given by the well-established bounded measurable functional calculus for W^*-algebras. Joint spectral measures are given by tensoring such spectral measures of strongly commuting self-adjoint unbounded operators. Functional calculus is integration w.r.t. spectral measures. Joint functional calculus is integration w.r.t. joint spectral measures. We introduce two related standard operations for further use in our discussion. In Lemma A.1.101, we establish pull-back along tensor products of normal unital *-homomorphisms. In Subsection A.2.2, we study compression.

A.1.1 Unbounded operators

Standard references for unbounded operators are [171], [184] and [192].

Partial orders generated by positive elements. We use $\mathbb{K} \in \{\mathbb{C}, \mathbb{R}\}$ as field.

Definition A.1.1. Let V be a complex vector space. A convex cone $C \subset V$ is proper if $0 \in C$ and $C \cap -C = \{0\}$. Let $\gamma : V \longrightarrow V$ be anti-linear involution. Its set of hermitian elements is $V_h := \{v \in V \mid \gamma(v) = v\}$.

1) For all $v \in V$, set $\mathrm{Re}(v) := \frac{1}{2}\big(v + \gamma(v)\big)$ and $\mathrm{Im}(v) := \frac{1}{2i}\big(v - \gamma(v)\big)$.

2) If V_h has partial order, then we call it generated by its set $V_+ := \{v \in V_h \mid v \geq 0\}$ of positive elements if V_+ is a proper cone generating the partial order.

Notation A.1.2. We use vector spaces V_s with subscript $s \in S$ in an index set. If V_s has partial order generated by positive elements, then we write $V_{s,h}$ and $V_{s,+}$ to denote its set of hermitian, resp. positive elements.

Remark A.1.3. If V_h has partial order generated by its set V_+ of positive elements, then $V_h = \langle V_+ \rangle_{\mathbb{R}} \oplus \langle V_+ \rangle_{\mathbb{R}}$ using direct sum of real vector spaces. Since further $V = \langle V_h \rangle_{\mathbb{R}} \oplus \langle V_h \rangle_{\mathbb{R}}$ using decomposition as per Equation A.1, we say that V has partial order generated by its set V_+ of positive elements in this case.

Proposition A.1.4. *Let V be a complex vector space. We consider anti-linear involution $\gamma : V \longrightarrow V$. For all $v \in V$, we have* $\mathrm{Re}(v), \mathrm{Im}(v) \in V_h$ *and*

$$v = \mathrm{Re}(v) + i\,\mathrm{Im}(v), \ \gamma(v) = \mathrm{Re}(v) - i\,\mathrm{Im}(v). \tag{A.1}$$

Proof. Apply anti-linearity of γ. □

Definition A.1.5. Let V and W be complex vector spaces. We consider anti-linear involutions $\gamma^V : V \longrightarrow V$ and $\gamma^W : W \longrightarrow W$. Let $\phi : V \longrightarrow W$ be a linear map.

1) We call ϕ order-preserving if

   1.1) $\phi(V_h) \subset W_h$,

   1.2) $v_1 \leq v_2$ in V_h implies $\phi(v_1) \leq \phi(v_2)$ in W_h.

2) Assume V_h and W_h have partial orders generated by positive elements. We call ϕ positivity-preserving if $\phi(V_+) \subset W_+$.

Proposition A.1.6. *Let V and W be complex vector spaces. We consider anti-linear involutions $\gamma^V : V \longrightarrow V$ and $\gamma^W : W \longrightarrow W$. Let $\phi : V \longrightarrow W$ be a linear map.*

1) ϕ is order-preserving if and only if $\phi \circ \gamma^V = \gamma^W \circ \phi$.

2) If V_h and W_h have partial orders generated by positive elements, then ϕ is order-preserving if and only if ϕ is positivity-preserving.

Proof. We have $\phi(V_h) \subset W_h$ if and only if $\phi(\mathrm{Re}(v)) = \mathrm{Re}(\phi(v))$ and $\phi(\mathrm{Im}(v)) = \mathrm{Im}(\phi(v))$ for all $v \in V$. This implies 1). Get 2) since $V_h = \langle V_+ \rangle_{\mathbb{R}} \oplus \langle V_+ \rangle_{\mathbb{R}}$ and $W_h = \langle W_+ \rangle_{\mathbb{R}} \oplus \langle W_+ \rangle_{\mathbb{R}}$ generate the respective partial orders. □

Bounded and unbounded operators. For spaces of bounded operators, we fix notation. This includes operator topologies used throughout our discussion. For spaces of unbounded operators, we fix notation, set partial order in Definition A.1.11, and give twisting maps of Hilbert space isometries in Definition A.1.13. We collect properties of such twisting maps in Proposition A.1.14.

Definition A.1.7. Let $(V, \|.\|_V)$ and $(W, \|.\|_W)$ be Banach spaces.

1) Let $\mathscr{B}(V,W)$ be the set of all $(\|.\|_V, \|.\|_W)$-bounded operators and let $\|.\|_{\mathscr{B}(V,W)}$ be its operator norm.

2) Set $\mathscr{B}(V) := \mathscr{B}(V,V)$ and $I_V := \mathrm{id}_{\mathscr{B}(V)}$. We call $V^* := \mathscr{B}(V,\mathbb{C})$ Banach dual of V.

Notation A.1.8. Unless stated otherwise, we suppress Banach space norms.

Operator norms determine uniform operator topology, also called norm topology. Let H be a Hilbert space. We equip $\mathscr{B}(H)$ with several other operator topologies aside from the uniform one: the σ-strong and σ-weak, as well as the strong and weak operator topology. For details on these operator topologies, we refer to Chapter II.2 in [192].

Notation A.1.9. For a normed vector space $(V, \|.\|_V)$, let $v = \|.\|_V\text{-}\lim_{k\in K} v_k$ denote norm convergence of nets in V and $u = w^*\text{-}\lim_{k\in K} u_k$ denote w^*-convergence of nets in V^*. For a Hilbert space $(H, \|.\|_H)$, let $x = \text{s-}\lim_{k\in K} x_k$ denote strong and $x = \text{w-}\lim_{k\in K} x_k$ denote weak convergence of nets in $\mathscr{B}(H)$.

Remark A.1.10. The σ-strong and strong topologies are equivalent on norm bounded sets (cf. Lemma II.2.5 in [192]). Equally, the σ-weak and weak topologies are.

For details on elementary unbounded operator theory, we refer to [171].

Definition A.1.11. Let H be a Hilbert space, $\mathscr{UB}(H)$ the set of all unbounded operators on H, and $\mathscr{UB}(H)_h$ the set of all self-adjoint unbounded operators on H.

1) For all $T,S \in \mathscr{UB}(H)_h$, set $T \geq S$ if and only if

   1.1) $\operatorname{dom} T \subset \operatorname{dom} S$,

   1.2) $\langle T(u), u\rangle_H \geq \langle S(u), u\rangle_H$ for all $u \in \operatorname{dom} T$.

2) We call $T \in \mathscr{UB}(H)_h$ positive if $\langle T(u), u\rangle_H \geq 0$ for all $u \in \operatorname{dom} T$. Let $\mathscr{UB}(H)_+$ be the set of all positive unbounded operators on H.

Remark A.1.12. We equip $\mathscr{UB}(H)$ with canonical addition and scalar multiplication (cf. Chapter 5 in [171]). We obtain complex unital semi-module $\mathscr{UB}(H)$ satisfying all vector space axioms except additive inverses. Linear maps, inclusions and proper cones are defined as for complex vector spaces. Functional calculus shows $\mathscr{UB}(H)_+ \subset \mathscr{UB}(H)_h$ is a proper cone generating partial order defined as per 1) in Definition A.1.11.

Definition A.1.13. Let H_0 and H_1 be Hilbert spaces. Let $\phi : H_0 \longrightarrow H_1$ be a linear or anti-linear isometric isomorphism. For all $T \in \mathscr{UB}(H_0)$, we define $\phi^\dagger(T) \in \mathscr{UB}(H_1)$ as

1) $\operatorname{dom} \phi^\dagger(T) := \{u \in H_1 \mid \phi^{-1}(u) \in \operatorname{dom} T\}$,

2) $\phi^\dagger(T)(u) := \phi\big(T\big(\phi^{-1}(u)\big)\big)$ for all $u \in \operatorname{dom} \phi^\dagger(T)$.

This defines map $\phi^\dagger : \mathscr{UB}(H_0) \longrightarrow \mathscr{UB}(H_1)$ by $T \mapsto \phi^\dagger(T)$. Using $\phi^{-1} : H_1 \longrightarrow H_0$ defines map $\phi^{-\dagger} : \mathscr{UB}(H_1) \longrightarrow \mathscr{UB}(H_0)$ by $T \mapsto \phi^{-\dagger}(T) := (\phi^{-1})^\dagger(T)$.

Proposition A.1.14 shows twisting preserves standard operations for densely defined closable unbounded operators if they are defined in the domain.

Proposition A.1.14. *Let $\phi : H_0 \longrightarrow H_1$ be a linear or anti-linear isometric isomorphism of Hilbert spaces.*

1) *We have bijective linear maps $\phi^\dagger$ and $\phi^{-\dagger} = \left(\phi^{-1}\right)^\dagger = \left(\phi^\dagger\right)^{-1}$.*
2) *If $T \in \mathscr{UB}(H_0)$ is densely defined closable, then $\phi^\dagger(T) \in \mathscr{UB}(H_1)$ is.*
3) *Let $T, S \in \mathscr{UB}(H_0)$ be densely defined closable s.t. $T+S$ and TS are densely defined closable. For all $\lambda_0, \lambda_1 \in \mathbb{C}$, we have*
   - *3.1)* $\phi^\dagger(T^*) = \phi^\dagger(T)^*$,
   - *3.2)* $\phi^\dagger\left(\overline{\lambda_0 T + \lambda_1 S}\right) = \overline{\lambda_0 \phi^\dagger(T) + \lambda_1 \phi^\dagger(S)}$,
   - *3.3)* $\phi^\dagger\left(\overline{TS}\right) = \overline{\phi^\dagger(T)\phi^\dagger(S)}$.

Proof. Since ϕ is anti-linear if and only if ϕ^{-1} is, we assume ϕ is linear without loss of generality. Get 1) by construction. Let $T \in \mathscr{UB}(H_0)$. Written as graph, $\phi^\dagger$ maps T to

$$\phi^\dagger(T) = \left\{ \left(\phi(u), \phi(T(u))\right) \in \phi(\operatorname{dom} T) \times H_1 \;\middle|\; u \in \operatorname{dom} T \right\}. \tag{A.2}$$

Equation A.2 shows 2) and 3) because ϕ is isometric isomorphism of Hilbert spaces. □

A.1.2 C*- and W*-algebras

Standard references for the theory of C^*- and W^*-algebras are [29] and [192][193][194]. We use [134][135] and [173] as supplement. Moreover, [38] focuses on the approximately finite-dimensional, or AF-C^*-setting, and [78] is a source of examples.

C*-algebras. The C^*-identity, i.e. Equation A.5, defines C^*-algebras. It imposes a rigid structure on such Banach *-algebras. All *-homomorphisms of C^*-algebras are bounded of norm at most one, and isometries if injective. Thus all *-isomorphisms of C^*-algebras are isometries, hence a *-algebra has at most one C^*-norm.

Definition A.1.15. Let $(A, \|.\|_A)$ be a Banach *-algebra. It is unital if it has unit $1_A \in A$.

1) The hermitian, resp. positive elements in A are

$$A_h := \left\{ x \in A \;\middle|\; x = x^* \right\}, \; A_+ := \left\{ x \in A_h \;\middle|\; \exists y \in A : \; x = y^* y \right\}. \tag{A.3}$$

2) The hermitian, resp. positive bounded functionals on A are

$$A_h^* := \left\{ \mu \in A^* \;\middle|\; \forall x \in A : \; \mu(x^* x) \in \mathbb{R} \right\}, \; A_+^* := \left\{ \mu \in A_h^* \;\middle|\; \forall x \in A : \; \mu(x^* x) \geq 0 \right\}. \tag{A.4}$$

3) A is a C^*-algebra if all $x \in A$ satisfy the C^*-identity

$$\|x^*x\|_A = \|x\|_A^2. \tag{A.5}$$

Notation A.1.16. Let A be a C^*-algebra. Rather than hermitian, we say that $x \in A_h$ is self-adjoint and call $\mu \in A_h^*$ real.

Example A.1.17. For all Hilbert spaces H, its space $(\mathscr{B}(H), \|.\|_{\mathscr{B}(H)})$ of bounded and its space $(\mathscr{K}(H), \|.\|_{\mathscr{B}(H)})$ of compact operators are C^*-algebras.

Example A.1.18. Let X be a locally compact Hausdorff space. Let $C_0(X)$ be the set of all continuous $g : X \longrightarrow \mathbb{C}$ vanishing at infinity. Pointwise operations equip it with Banach *-algebra structure. For all $g \in C_0(X)$, set $\|g\|_\infty := \sup_{x \in X} |g(x)|$. We see $C_0(X)$ equipped with $\|.\|_\infty$ is a C^*-algebra. Up to natural isomorphisms [153], all commutative C^*-algebras are of such form by Gelfand duality (cf. Theorem I.3.11 in [192]). If X is compact, then $C_0(X) = C(X)$. Note $C_b(X)$ equipped with $\|.\|_\infty$ is a C^*-algebra.

We use standard definitions for *-algebras. Homomorphisms of *-algebras are called *-homomorphisms. For C^*-algebras, Proposition A.1.21 shows boundedness follows by the C^*-identity if *-algebra structures are preserved. This leads to Definition A.1.19.

Definition A.1.19. Let A and B be C^*-algebras.

1) A *-homomorphism $\phi : A \longrightarrow B$ of C^*-algebras is a *-homomorphism. If A and B are unital, then $\phi : A \longrightarrow B$ is unital if $\phi(1_A) = 1_B$.
2) If $A \subset B$, then A is a C^*-subalgebra of B if $A \subset B$ is a *-homomorphism. If A and B are furthermore unital, then A is a unital C^*-subalgebra of B if $A \subset B$ is a unital *-homomorphism.

Example A.1.20. Let A be a C^*-algebra and H a Hilbert space. We call $\pi : A \longrightarrow \mathscr{B}(H)$ a *-representation of A over H if it is a *-homomorphism. It is faithful if injective. It is unital if A is unital and $\pi(1_A) = I_H$, i.e. if it is a unital *-homomorphism.

Proposition A.1.21. *Let $\phi : A \longrightarrow B$ be a *-homomorphism of C^*-algebras.*

1) $\phi \in \mathscr{B}(A,B)$ and $\|\phi\|_{\mathscr{B}(A,B)} \leq 1$.

2) If ϕ is injective, then it is an isometry.

3) If ϕ is a *-isomorphism, then ϕ^{-1} is a *-isomorphism.

Proof. Proposition I.5.2 and Proposition I.5.3 in [192] show 1), resp. 2) at once. Using 2), we directly verify 3). □

Proposition A.1.22. *Let A be a C^*-algebra. There exists Hilbert space H and faithful *-representation $\pi : A \longrightarrow \mathscr{B}(H)$. If A is unital, then we may ask π to be unital.*

Proof. Apply Theorem I.9.18 in [192]. □

Faithful *-representations of C^*-algebras are direct sums of cyclic *-representations. The latter arise as GNS-constructions, which are standard constructions associated to positive functionals on C^*-algebras (cf. Theorem I.9.14 in [192]). If we further demand normality given W^*-algebras, i.e. σ-weak closed C^*-algebras, then we obtain semi-cyclic *-representations (cf. Definition VII.1.5 in [193]). Relevant to us are canonical left- and right-actions associated to f.s.n. traces constructed in Subsection B.1.1.

Proposition A.1.23. *Let A be a C^*-algebra.*

1) *The partial order generated on A_h by the proper cone A_+ is given by*

$$x \geq y \Leftrightarrow x - y \in A_+ \tag{A.6}$$

for all $x, y \in A_h$. Using algebra involution as anti-linear involution on A, the set A_+ of positive elements generates the partial order.

2) *The partial order generated on A_h^* by the proper cone A_+^* is given by*

$$\mu \geq \eta \Leftrightarrow \mu - \eta \in A_+ \tag{A.7}$$

for all $\mu, \eta \in A_h^*$. Using pointwise conjugation as anti-linear involution on A^*, the set A_+ of positive elements generates the partial order.

Proof. Apply Theorem I.6.1 and Proposition III.2.1 in [192]. □

Definition A.1.24. Let A be a C^*-algebra. We equip A_h with the partial order defined by Equation A.6, resp. A_h^* with the partial order defined by Equation A.7.

Remark A.1.25. If H is a Hilbert space, then partial order on $\mathscr{B}(H)_h$ given by 1) in Definition A.1.11 is the one fixed here by Definition A.1.24. Note Example A.1.47 shows all *-homomorphisms are positivity-preserving. Altogether, we know partial orders on C^*-algebras reduce to Definition A.1.11 by Proposition A.1.6 and Proposition A.1.22.

We use three standard constructions for C^*-algebras: generation, direct sums and tensor products. Definition A.1.26 gives generated C^*-algebras. Let A be a C^*-algebra and $S \subset A$. Let $\mathrm{Poly}(S)$ be the set of all finite polynomials with elements in S or S^*. We know $\mathrm{Poly}(S)^* = \mathrm{Poly}(S) \subset A$ by construction. For all $n \in \mathbb{N}$, set

$$\mathrm{Poly}(S)^n := \left\{ x \in A \;\middle|\; \exists \{y_k\}_{k=1}^n \subset \mathrm{Poly}(S) : \; x = \prod_{k=1}^n y_k \right\}. \tag{A.8}$$

Note Equation A.8 implies the complex linear span $C_0^*(S) := \langle \bigcup_{n \in \mathbb{N}} \mathrm{Poly}(S)^n \rangle_{\mathbb{C}} \subset A$ is in fact a *-subalgebra. The C^*-identity is therefore inherited from A.

Definition A.1.26. Let A be a C^*-algebra. For all $S \subset A$, we call $C^*(S) := \overline{C_0^*(S)}^{\|\cdot\|_A}$ the C^*-algebra generated by S. If $\{S_k\}_{k=1}^n \subset \mathscr{P}(A)$, then set $C^*(S_1, \ldots, S_n) := C^*(\bigcup_{k \in K} S_k)$.

Definition A.1.27 gives direct sum C^*-algebras. Let $m \in \mathbb{N}$. For all $n \in \{1,\dots,m\}$, let A_n be a C^*-algebra. Let $\oplus_{n=1}^m A_n$ be the direct sum of Banach spaces. Thus

$$\|x\|_{\oplus_{n=1}^m A_n} = \max_{1 \le n \le m} \|x_n\|_{A_n} \tag{A.9}$$

for all $x = (x_1,\dots,x_n) \in \oplus_{n=1}^m A_n$. Multiplication and adjoining is defined on summands. Equation A.9 ensures the C^*-identity.

Definition A.1.27. Let $m \in \mathbb{N}$. For all $n \in \{1,\dots,m\}$, let A_n be a C^*-algebra. We call $\oplus_{n=1}^m A_n$ the direct sum C^*-algebra of $\{A_n\}_{n=1}^m$.

Definition A.1.28 gives tensor product C^*-algebras. Note we assume nuclearity of at least one factor (cf. Definition XV.1.4 in [194]). This ensures unique cross norms up to *-isomorphism. For details on C^*-tensor products, we refer to Section IV.4 in [192] and Chapter 11 in [135]. The latter discusses infinite tensor products.

Let A be a C^*-algebra and B a nuclear C^*-algebra. We construct minimal C^*-tensor product $A \otimes B := A \otimes_{\min} B$ via norm closure of algebraic tensor product $A \odot B$ under the unique norm satisfying the cross norm identity

$$\|x \otimes y\|_{A \otimes B} = \|x\|_A \|y\|_B \tag{A.10}$$

for all $x \in A$ and $y \in B$. Multiplication and adjoining are defined on factors and therefore elementary tensors. Equation A.10 ensures the C^*-identity.

Definition A.1.28. Let A be a C^*-algebra A and B a nuclear C^*-algebra. We call $A \otimes B$ the C^*-tensor product of A and B.

Remark A.1.29. We are able to tensor suitable bounded linear maps via the algebraic tensor product, in particular bounded linear functionals and *-homomorphisms. Both sets of linear maps are stable under C^*-tensoring.

W*-algebras. Upon faithful *-representation, closures of C^*-algebras in σ-weak operator topology are unital C^*-algebras. This defines W^*-algebras concretely but it is their *-algebra structures which determine σ-weak operator topology. Definition A.1.31 gives an equivalent abstract definition as C^*-algebras which are Banach duals. Their pre-duals are unique up to isometric isomorphism, including noncommutative L^1-spaces of tracial W^*-algebras as per Definition B.1.41. Upon faithful normal *-representation as per Proposition A.1.34, the induced w^*-topology is σ-weak operator topology.

Remark A.1.30. Proposition A.1.49 for weakly continuous faithful *-representations as per Proposition A.1.34 shows normality of *-representations is continuity w.r.t. w^*- and σ-weak operator topology. Unitality is not necessary.

Definition A.1.31. Let M be a C^*-algebra. We say that M is a W^*-algebra if there exists a Banach space M_* s.t. $M = (M_*)^*$. In this case, we call M_* the pre-dual of M.

Remark A.1.32. If M is a W^*-algebra, then M_* is unique up to isometric isomorphism of Banach spaces (cf. Corollary III.3.9 in [192]).

Example A.1.33. In Subsection B.1.1, we cover tracial W^*-algebras. Their pre-duals are noncommutative L^1-spaces. Fundamental example is $\mathscr{B}(H) = S^1(H)^*$ for a Hilbert space H and $S^1(H)$ its trace-class operators. The σ-weak operator topology is defined as the w^*-topology on $\mathscr{B}(H) = S^1(H)^*$ in this case. This mirrors the commutative case of X a locally compact Hausdorff space and $\mathscr{N}$ a σ-ideal of null sets of the Borel σ-algebra $\mathfrak{B}(X)$. We define W^*-algebra $L^\infty(X,\mathscr{N})$ using $\|.\|_\infty$ modulo $\mathscr{N}$, i.e. essential supremum. If $\mathscr{N} = \mathscr{N}_\mu$ for $\mu \in C_c(X)^*$ as per Riesz–Markov–Kakutani theorem (cf. Theorem 6.3.4 in [171]), then we know $L^\infty(X,\mu) := L^\infty(X,\mathscr{N}_\mu) = L^1(X,\mu)^*$ depending only on $\mathscr{N}_\mu$.

Proposition A.1.34. *Let M be a C^*-algebra. M is a W^*-algebra if and only if M is unital and there exists a faithful unital *-representation $\pi : M \longrightarrow \mathscr{B}(H)$ satisfying one of the following:*

1) $\pi(M) = \pi(M)''$,

2) $\pi(M)$ is (σ-)strongly closed,

3) $\pi(M)$ is (σ-)weakly closed.

If M is a W^*-algebra, then there exists a faithful unital *-representation $\pi : M \longrightarrow \mathscr{B}(H)$ s.t. the w^*-topology on $M = (M_*)^*$ is the σ-weak operator topology on $\pi(M) \subset \mathscr{B}(H)$.

Proof. For all Hilbert spaces H and $S \subset \mathscr{B}(H)$, let $S' \subset \mathscr{B}(H)$ be the commutant of S. Theorem II.3.9 and Theorem III.3.5 in [192] show all claims, with exception of the weak topologies on M and $\pi(M)$ coinciding. Theorem 7.4.2 in [135] shows the latter. □

Proposition A.1.49 states σ-weak continuity is normality as per Definition A.1.44 for completely positive maps as per Definition A.1.45. Note Example A.1.47 shows all *-homomorphisms are completely positive. Thus we see all σ-weakly *-homomorphisms are normal, hence all those faithful unital *-representations weakly continuous as per Proposition A.1.34 are also normal, i.e. Remark A.1.30. Altogether, we know normality as per 1) in Definition A.1.35 is a special case of Definition A.1.44. If we consider any W^*-subalgebras as per 2) in Definition A.1.35, then we do not assume unitality unless stated otherwise. For details on the choice of unit, we refer to Subsection B.2.2.

Definition A.1.35. Let M and N be W^*-algebras.

1) A normal *-homomorphism $\phi : M \longrightarrow N$ of W^*-algebras is σ-weakly continuous *-homomorphism.

2) If $N \subset M$, then N is W^*-subalgebra of M if $N \subset M$ is normal *-homomorphism. If it is also unital, then N is a unital W^*-subalgebra of M.

Standard constructions for C^*-algebras specialise to W^*-algebras. The direct sum construction is unchanged. Definition A.1.36 gives generated W^*-algebras by σ-weak closure of generated C^*-algebras. Definition A.1.51 gives tensor product W^*-algebras.

Definition A.1.36. Let M be a W^*-algebra. For all $S \subset M$, we call $W^*(S) := \overline{C_0^*(S)}^{\mathrm{w}}$ the W^*-algebra generated by S. If $\{S_k\}_{k=1}^n \subset \mathscr{P}(M)$, then set $W^*(S_1,\ldots,S_n) := W^*(\bigcup_{k\in K} S_k)$.

Proposition A.1.37. *For all unital C^*-algebras A, we have $A = C^*(\mathscr{U}(A))$ for the set $\mathscr{U}(A) := \{x \in A \mid x^* = x^{-1}\}$ of unitary operators in A. For all W^*-algebras M, we have $M = W^*(P(M))$ for the set $P(M) := \{p \in M_h \mid p^2 = p\}$ of projections in M.*

Proof. Let A be a unital C^*-algebra. For all $x \in A_h$, $C(\mathrm{spec}_A x) = C^*(\mathscr{U}(C(\mathrm{spec}_A x)))$ by Stone-Weierstrass [171]. Since $\mathscr{U}(C(\mathrm{spec}_A x)) \subset \mathscr{U}(A)$ in each case, decomposing into real and imaginary parts shows $A = C^*(\mathscr{U}(A))$. Let M be a W^*-algebra. We readily see $M = C^*(\mathscr{U}(M))$ by unitality. Theorem 5.2.5 in [134] implies $\mathscr{U}(M) \subset C^*(P(M))$. Thus we combine both to $M \subset C^*(P(M))$, hence $M = W^*(P(M))$ as claimed. □

For our discussion, it commonly suffices to have bounded linear maps preserving strong or weak convergence of uniformly bounded nets. Proposition A.1.49 shows such bounded convergence is equivalent to normality if we assume complete positivity.

Proposition A.1.38. *Let M be a W^*-algebra, $S \subset M$ a *-subalgebra and $\overline{S}$ its strong closure. For all $x \in \overline{S}$, there exists net $\{x_k\}_{k\in K} \subset S$ s.t.*

$$x = \operatorname*{s-lim}_{k\in K} x_k, \ \sup_{k\in K} \|x_k\|_M \leq \|x\|_M. \tag{A.11}$$

Proof. If $x = 0$, then $x \in S$. If $x \neq 0$, then the Kaplansky density theorem yields a net as claimed up to rescaling by a positive constant (cf. Theorem 5.3.5 in [134]). □

Definition A.1.39. Let M be a W^*-algebra. We call a net $\{x_k\}_{k\in K} \subset M$ bounded strongly convergent if it is bounded and converges strongly, resp. bounded weakly convergent if it is bounded and converges weakly.

Notation A.1.40. Let $x = \text{bds-lim}_{k\in K} x_k$ denote bounded strong and $x = \text{bdw-lim}_{k\in K} x_k$ bounded weak convergence of nets.

Remark A.1.41. The uniform boundedness principle shows bounded strong and strong convergence coincide on sequences (cf. Theorem 2.2.9 in [171]). Equally, bounded weak and weak convergence coincide on sequences.

Definition A.1.42. Let $\phi : M \longrightarrow N$ be a bounded linear map of W^*-algebras.

1) We call ϕ bounded strongly continuous if for all nets $\{x_k\}_{k\in K} \subset M$, $x = \text{bds-lim}_{k\in K} x_k$ implies $\phi(x) = \text{bds-lim}_{k\in K} \phi(x_k)$,

2) We call ϕ bounded weakly continuous if for all nets $\{x_k\}_{k\in K} \subset M$, $x = \text{bdw-lim}_{k\in K} x_k$ implies $\phi(x) = \text{bdw-lim}_{k\in K} \phi(x_k)$.

Remark A.1.43. Definition A.1.42 extends to bounded multi-linear maps. We commonly use multiplication in W^*-algebras is bounded strongly continuous and therefore further sequentially strongly continuous by Remark A.1.41.

Normal, completely positive and completely Markovian maps. We consider properties of bounded linear maps of C^*- and W^*-algebras. Completely positive normal bounded linear maps of W^*-algebras are continuous in all operator topologies we use and stable under tensoring. Examples are positive bounded normal functionals on and normal *-homomorphisms of W^*-algebras, as well as compression maps. The notions of completely positive map and completely Markovian map are used to define completely Markovian semigroups [83][85][86] describing irreversible time-evolution of dissipative quantum systems weakly coupled to a heat bath [35][36][82][121][163][188]. For details on the latter, we refer to Subsection 3.2.2.

Normality is preservation of suprema under a given map. Unique suprema exist for W^*-algebras. For all bounded increasing nets $\{x_k\}_{k\in K} \subset M_h$ in a given W^*-algebra M, we have unique supremum $\sup_{k\in K} x_k \in M_h$ in partial order. We furthermore have $\sup_{k\in K} x_k = \text{s-lim}_{k\in K} x_k$ in M. Lemma 5.1.4 in [134] shows both statements.

Definition A.1.44. Let $\phi : M \longrightarrow N$ be a positivity-preserving bounded linear map of W^*-algebras. We call ϕ normal if for all bounded increasing nets $\{x_k\}_{k\in K} \subset M_h$, get

$$\phi\left(\sup_{k\in K} x_k\right) = \sup_{k\in K} \phi(x_k). \tag{A.12}$$

In the general noncommutative setting, positivity-preservation is not stable under tensoring. The latter requires complete positivity. For all completely positive maps of W^*-algebras, Proposition A.1.49 shows normality is equivalent to σ-weak continuity. Example A.1.47 therefore leads to Definition A.1.35. Full matrix algebras are nuclear C^*-algebras. For all $n \in \mathbb{N}$, let $I_n \in M_n(\mathbb{C})$ be the identity. For all bounded linear maps $\phi : A \longrightarrow B$ of C^*-algebras, the bounded linear maps $\phi \otimes \mathrm{id}_{M_n(\mathbb{C})} : A \otimes M_n(\mathbb{C}) \longrightarrow B \otimes M_n(\mathbb{C})$ of C^*-algebras obtained for all $n \in \mathbb{N}$ are determined on algebraic tensor products.

Definition A.1.45. We call a bounded linear map $\phi : A \longrightarrow B$ of C^*-algebras completely positive if $\phi \otimes \mathrm{id}_{M_n(\mathbb{C})} : A \otimes M_n(\mathbb{C}) \longrightarrow B \otimes M_n(\mathbb{C})$ is positivity-preserving for all $n \in \mathbb{N}$.

Example A.1.46. All positivity-preserving bounded linear functionals $\mu : A \longrightarrow \mathbb{C}$ of C^*-algebras are completely positive (cf. Corollary IV.3.5 in [192]).

Example A.1.47. If $\phi : A \longrightarrow B$ is a *-homomorphisms of C^*-algebras, then $\phi(x^*x) = \phi(x)^*\phi(x)$ for all $x \in A$ ensures ϕ is positivity-preserving. Since each $\phi \otimes \mathrm{id}_{M_n(\mathbb{C})}$ itself is a *-homomorphism if ϕ is, *-homomorphisms are completely positive. Proposition A.1.49 shows σ-weak continuity is normality, i.e. Equation A.12, for *-homomorphisms.

Example A.1.48. Let M be a W^*-algebra and $p \in M$ a projection. We obtain W^*-algebra $M[p] := pMp$ and define positivity-preserving compression map $\mathrm{com}_p : M \longrightarrow M[p]$ by setting $\mathrm{com}_p\, x := pxp$ for all $x \in M$. For all $n \in \mathbb{N}$, $p \otimes I_n \in M \otimes M_n(\mathbb{C})$ is a projection and $\mathrm{com}_p \otimes \mathrm{id}_{M_n(\mathbb{C})} = \mathrm{com}_{p\otimes I_n}$ upon repeat construction. Thus com_p is completely positive. We define compression maps in Definition A.2.15. Note 2) in Proposition B.2.13 shows com_p is the unique noncommutative conditional expectation from M to $M[p]$.

Proposition A.1.49. *For all completely positive maps $\phi : M \longrightarrow N$ of W^*-algebras, the following are equivalent:*

1) *ϕ is normal,*

2) *ϕ is σ-weakly continuous,*

3) *ϕ is σ-strongly continuous,*

4) *ϕ is bounded weakly continuous,*

5) *ϕ is bounded strongly continuous.*

Proof. Proposition III.2.2.2 in [29] shows 1) to 3). As ϕ is bounded, the unit ball in M is mapped to a bounded ball in N. Note σ-strong and strong, as well as σ-weak and weak topologies are equivalent on norm bounded sets of W^*-algebras (cf. Lemma II.2.5 in [192]). For all bounded increasing nets $\{x_k\}_{k\in K} \subset M_h$, $\sup_{k\in K} x_k$ is the σ-strong and therefore σ-weak limit of $\{x_k\}_{k\in K}$. Equivalence of 1) and 4), as well as 1) and 5), thus hold by equivalence of the operator topologies on norm bounded sets. □

Remark A.1.50. By Remark A.1.41 and Proposition A.1.49, completely positive normal bounded linear maps of W^*-algebras are sequentially strongly and sequentially weakly continuous. We use this throughout our discussion.

We refer to Section IV.4 in [192] for details on W^*-tensor products. We do not assume nuclearity for tensor products of W^*-algebras as their construction uses unique minimal C^*-tensor products. Let M and N be W^*-algebras. Their minimal C^*-tensor product is $M \otimes_{\min} N$. Let M_* and N_* denote their respective pre-dual. Get $M_* \odot N_* \subset (M \otimes_{\min} N)^*$ for the algebraic tensor product of pre-duals by letting

$$\big(\mu \otimes \eta\big)(x \otimes y) := \mu(x)\eta(y) \tag{A.13}$$

for all $\mu \otimes \eta \in M_* \otimes N_*$ and $x \otimes y \in M \otimes_{\min} N$.

Definition A.1.51. Let M and N be W^*-algebras. Set

$$M_* \otimes N_* := \overline{M_* \odot N_*} \subset (M \otimes_{\min} N)^* \tag{A.14}$$

using norm closure. We call $M \otimes N := (M_* \otimes N_*)^*$ the W^*-tensor product of M and N.

Lemma A.1.52. *Let $\phi : M_0 \longrightarrow M_1$ and $\psi : N_0 \longrightarrow N_1$ be completely positive normal maps of W^*-algebras. We define completely positive normal map $\phi \otimes \psi : M_0 \otimes N_0 \longrightarrow M_1 \otimes N_1$ by setting $(\phi \otimes \psi)(x \otimes y) := \phi(x) \otimes \psi(y)$ for all $x \in M_0$ and $y \in N_0$.*

Proof. By Proposition A.1.49, this is Proposition IV.5.13 in [192]. □

Corollary A.1.53. *Let $\phi : M_0 \longrightarrow M_1$ and $\psi : N_0 \longrightarrow N_1$ be normal *-homomorphisms of W^*-algebras. We define normal *-homomorphism $\phi \otimes \psi : M_0 \otimes N_0 \longrightarrow M_1 \otimes N_1$ by setting $(\phi \otimes \psi)(x \otimes y) := \phi(x) \otimes \psi(y)$ for all $x \in M_0$ and $y \in N_0$. If ϕ and ψ are unital, then $\phi \otimes \psi$ is.*

Proof. Example A.1.47 and Lemma A.1.52 yield completely positive normal $\phi \otimes \psi$. By Proposition A.1.6 and Proposition A.1.49, $\phi \otimes \psi$ intertwines adjoining and is σ-strongly continuous. This is equivalent to bounded strong convergence for bounded nets. Let S be the linear span of all elementary tensors. By strong density, Proposition A.1.38 shows $M_0 \otimes N_0$ is the bounded strong closure of S. As multiplication is bounded strongly continuous and $\phi \otimes \psi$ is bounded, we directly verify our claim on elementary tensors. □

Definition A.1.54. We call a completely positive map $\phi : A \longrightarrow A$ of unital C^*-algebras Markovian if $\phi(x) \leq \|x\|_A 1_A$ for all $x \in A_+$. We call such maps completely Markovian if $\phi \otimes \mathrm{id}_{M_n(\mathbb{C})} : A \otimes M_n(\mathbb{C}) \longrightarrow A \otimes M_n(\mathbb{C})$ is Markovian for all $n \in \mathbb{N}$.

A.1.3 Functional calculus

Standard references for continuous and bounded measurable functional calculus for C^*-, resp. W^*-algebras are [134] and [192]. Standard references for spectral integration and functional calculus of self-adjoint unbounded operators are [171] and [184].

Integration of spectral measures. Let H be a Hilbert space. Spectral measures of self-adjoint unbounded operators on H are projection-valued measures taking values in $\mathscr{B}(H)$. Image lattices of projections are noncommutative Borel σ-algebras.

Notation A.1.55. For all $n \in \mathbb{N}$ and Borel measurable $X \subset \mathbb{C}^n$, let $\mathfrak{B}(X)$ denote the Borel σ-algebra of X. Let χ_Z denote the characteristic function of a set $Z \subset \mathbb{C}^n$.

Definition A.1.56. Let $X \in \mathfrak{B}(\mathbb{C}^n)$. A map $E : \mathfrak{B}(X) \longrightarrow \mathscr{B}(H)$ is a spectral measure on X with values in $\mathscr{B}(H)$ if

1) $E(X) = I$ and $E(Z)$ is a projection for all $Z \in \mathfrak{B}(X)$,

2) $Z \mapsto E^u(Z) := \langle E(Z)(u), u \rangle_H$ is a measure on X for all $u \in H$.

Let E be a spectral measure on X. Its support $\operatorname{supp} E$ is the set of all $x \in X$ s.t. $E(N_x) \neq 0$ for all open neighbourhoods N_x of x. Its null ideal is $\mathscr{N}(E) := \{Z \in \mathfrak{B}(X) \mid E(Z) = 0\}$.

Spectral measures $E : \mathfrak{B}(\mathbb{R}) \longrightarrow \mathscr{B}(H)$ map bijectively to resolutions of the identity $\{E((-\infty, \lambda])\}_{\lambda \in \mathbb{R}}$ (cf. Theorem 4.6 in [184]). A spectral measure $E : \mathfrak{B}(\mathbb{R}) \longrightarrow \mathscr{B}(H)$ is thus determined by its resolution of the identity $\{E((-\infty, \lambda])\}_{\lambda \in \mathbb{R}}$.

Proposition A.1.57. *For all spectral measures E on $X \in \mathfrak{B}(\mathbb{C}^n)$, we have*

1) E^u is a finite measure for all $u \in H$,

2) $\operatorname{supp} E$ is minimal among closed $Z \in \mathfrak{B}(X)$ s.t. $E(Z) = I_H$.

Proof. By definition of spectral measures. □

The null ideal $\mathcal{N}(E)$ of a given spectral measure E yields notions of E-a.e. defined and E-a.e. finite map. The set of all E-a.e. finite maps $g : X \longrightarrow \mathbb{C}$ is the domain of spectral integration w.r.t. E.

Definition A.1.58. Let E be spectral measure on $X \in \mathfrak{B}(\mathbb{C}^n)$. Let $\mathcal{S}(E)$ denote the set of all E-a.e. defined Borel measurable $g : X \longrightarrow \mathbb{C}$ s.t. $|g|$ is E-a.e. finite. We say that $\{Z_k\}_{k\in\mathbb{N}} \subset \mathfrak{B}(X)$ is a bounding sequence for $\mathcal{G} \subset \mathcal{S}(E)$ if

1) $Z_k \subset Z_{k+1}$ for all $k \in \mathbb{N}$ and $E(\bigcup_{k\in\mathbb{N}} Z_k) = I$,
2) $|g|_{Z_k}|$ is bounded for all $k \in \mathbb{N}$.

Remark A.1.59. For all spectral measures E on $X \in \mathfrak{B}(\mathbb{C}^n)$ and finite $\mathcal{G} \subset \mathcal{S}(E)$, there exists a bounding sequence (cf. Subsection 4.3.2 in [184]).

Let E be a spectral measure on $X \in \mathfrak{B}(\mathbb{C}^n)$. We define spectral integration as per Lemma 4.11 and Theorem 4.13 in [184]. Theorem 4.16 and Subsection 4.3.3 in [184] show fundamental properties. For all simple functions $g = \sum_{l=1}^{n} c_l \chi_{Z_l}$ on X, the spectral integral of g w.r.t. E is defined by

$$\mathrm{I}_E(g) := \sum_{l=1}^{n} c_l E(Z_l). \tag{A.15}$$

Lemma 4.11 in [184] states $\|\mathrm{I}_E(g)\|_{\mathcal{B}(H)} \le \sup_{x\in X} |g(x)|$ in each case. Density of simple functions in uniform norm extends spectral integration w.r.t. E to all bounded Borel functions on X. For all bounded Borel functions $g : X \longrightarrow \mathbb{C}$ and simple functions $\{g_n\}_{n\in\mathbb{N}}$ on X $\|.\|_\infty$-converging to g, get $\mathrm{I}_E(g) = \|.\|_{\mathcal{B}(H)}\text{-}\lim_{n\in\mathbb{N}} \mathrm{I}_E(g_n)$.

Let $g \in \mathcal{S}(E)$. The domain of I_E is defined by

$$\operatorname{dom} \mathrm{I}_E(g) := \left\{ u \in H \;\middle|\; \int_X |g(x)|^2 dE^u < \infty \right\}. \tag{A.16}$$

For all $u \in H$, we have $u \in \operatorname{dom} \mathrm{I}_E(g)$ if and only if

$$\mathrm{I}_E(g)(u) := \|.\|_H\text{-}\lim_{k\in\mathbb{N}} \mathrm{I}_E\big(g\chi_{Z_k}\big)(u) \tag{A.17}$$

exists for a bounding sequence $\{Z_k\}_{k\in\mathbb{N}}$ of g. If $u \in \operatorname{dom} \mathrm{I}_E(g)$, then $\mathrm{I}_E(g)(u)$ exists and is independent of choice of bounding sequence of g by Theorem 4.13 in [184].

Definition A.1.60. Let E be a spectral measure on $X \in \mathfrak{B}(\mathbb{C}^n)$. For all $g \in \mathcal{S}(E)$, we call $\int g dE := \mathrm{I}_E(g)$ the spectral integral of g w.r.t. E.

Remark A.1.61. Let $g \in \mathcal{S}(E)$. Its domain as per Equation A.16 and the identity

$$\big\langle \mathrm{I}_E(g)(u), u \big\rangle_H = \int_X g(x) dE^u \tag{A.18}$$

for all $u \in \operatorname{dom} \mathrm{I}_E(g)$, i.e. $\|\mathrm{I}_E(g)(u)\|_H^2 = \int_X |g(x)|^2 dE^u < \infty$, determine $\mathrm{I}_E(g)$.

Proposition A.1.62. *Let E be a spectral measure on $X \in \mathfrak{B}(\mathbb{C}^n)$. For all $g \in \mathscr{S}(E)$, $\mathrm{I}_E(g)$ is a closed normal operator s.t. $\mathrm{I}_E(g)^* = \mathrm{I}_E(\overline{g})$ and $\mathrm{I}_E(g)^*\mathrm{I}_E(g) = \mathrm{I}_E(g)\mathrm{I}_E(g)^* = \mathrm{I}_E(g\overline{g})$.*

Proof. Apply Theorem 4.16 in [184]. □

Remark A.1.63. Proposition A.1.62 defines invertible map $E \mapsto \mathrm{I}_E(\mathrm{id}_{\mathbb{R}})$ from all spectral measures $E : \mathfrak{B}(\mathbb{R}) \longrightarrow \mathscr{B}(H)$ to $\mathscr{U}\mathscr{B}(H)_h$. Note invertibility is the spectral theorem for self-adjoint unbounded operators (cf. Theorem 5.7 in [184]).

Bounded measurable functional calculus. Functional calculus of self-adjoint unbounded operators is based on the use of spectral measures. We construct these using bounded measurable functional calculus for W^*-algebras. The latter in turn extends continuous functional calculus for unital C^*-algebras.

The choice of unit matters. If a Banach *-algebra is unital, then the unit is unique. If however $N \subset M$ is a W^*-subalgebra s.t. $1_N \neq 1_M$, then all normal elements in N have two a priori distinct bounded measurable functional calculi. Equation A.19 shows how they may differ. If they differ, then they differ only at zero but generate distinct spectral measures. This impacts spectral integration, in particular taking inverses.

Definition A.1.64. Let B be a unital C^*-algebra. For all C^*-subalgebras $A \subset B$, we call $A[1_B] = C^*(A, 1_B)$ the unitalisation of A in B.

Proposition A.1.65. *Let A and B be unital C^*-algebras and $A \subset B$ a C^*-subalgebra. If $A \subset B$ is not a unital C^*-subalgebra, then $A[1_B] = A \oplus \langle 1_B - 1_A \rangle_{\mathbb{C}}$.*

Proof. Get $1_B - 1_A \in A[1_B]$ and $(1_B - 1_A)A = A(1_B - 1_A) = 0$. Thus $A \oplus \langle 1_B - 1_A \rangle_{\mathbb{C}}$. □

Definition A.1.66. Let A be a C^*-algebra.

1) We call $x \in A$ normal if $x^*x = xx^*$.

2) Let A be unital. Set $\mathrm{GL}(A) := \{x \in A \mid x^{-1} \in A\}$. For all normal $x \in A$, its spectrum in A is $\mathrm{spec}_A x := \{\lambda \in \mathbb{C} \mid x - \lambda 1_A \notin \mathrm{GL}(A)\}$.

Lemma A.1.67 states continuous functional calculus for unital C^*-algebras. For all normal $x \in A$ in a unital C^*-algebra A, Example A.1.18 explains how $C(\mathrm{spec}_A x)$ is a C^*-algebra using uniform norm.

Lemma A.1.67. *Let A be a unital C^*-algebra. If $x \in A$ is normal, then*

1) $\mathrm{spec}_A x \subset \mathbb{C}$ *is non-empty and compact,*

2) there exists unital *-isomorphism $\Gamma_{x,A} : C(\mathrm{spec}_A x) \longrightarrow C^*(x, x^*, 1_A)$,

3) $\Gamma_{x,A}$ is determined by unitality and $\Gamma_{x,A}\big(\mathrm{id}_{\mathrm{spec}_A x}\big) = x$.

Proof. Get 1) by Proposition I.4.2, resp. 2) and 3) by Proposition I.4.6 in [192]. □

Remark A.1.68. Let $x \in A$ be normal. We call $\Gamma_{x,A}$ the continuous functional calculus of x in A. For all $x \in C(\operatorname{spec}_A x)$, set $g(x) := \Gamma_{x,A}(g)$. We adopt analogues convention for all functional calculus. If $\operatorname{spec}_A x \subset X \subset \mathbb{C}$ for locally compact Hausdorff X, then $g(x) = h(x)$ for all $g, h \in C_0(X)$ s.t. $g|_{\operatorname{spec}_A x} = h|_{\operatorname{spec}_A x}$.

Corollary A.1.69 shows continuous functional calculus extends uniquely to normal elements in non-unital C^*-algebras if we restrict to functions vanishing at zero. Note it further shows choice of unit only involves values at zero.

Corollary A.1.69. *Let B be a unital C^*-algebra and $A \subset B$ a C^*-subalgebra. If $x \in A$ is normal and $\pi : A \longrightarrow \mathscr{B}(H)$ a faithful *-representation, then*

1) $\operatorname{spec}_B x \setminus \{0\} = \operatorname{spec}_{\mathscr{B}(H)} \pi(x) \setminus \{0\}$,

2) $\Gamma_{x,B}(g) \in A$ *and* $\pi\big(\Gamma_{x,B}(g)\big) = \Gamma_{\pi(x),\mathscr{B}(H)}(g)$ *for all* $g \in C_0\big(\operatorname{spec}_{\mathscr{B}(H)} \pi(x) \setminus \{0\}\big)$.

Proof. Get 1) by Proposition A.1.65. Instead of $\operatorname{spec}_{\mathscr{B}(H)} \pi(x) \setminus \{0\}$, we consider compact $K \subset \mathbb{C}$ s.t. $\{0\} \cup \operatorname{spec}_B x \cup \operatorname{spec}_{\mathscr{B}(H)} \pi(x) \subset K$ as per Remark A.1.68. If g is a polynomial on K vanishing at zero, then it is expressed without the constant function. Thus $\Gamma_{x,B}(g) \in A$ and $\pi(\Gamma_{x,B}(g)) = \Gamma_{\pi(x),\mathscr{B}(H)}(g)$ by Lemma A.1.67. If $g \in C(K)$ vanishes at zero, then we approximate g uniformly in norm by polynomial on K vanishing at zero. We conclude by boundedness of *-homomorphisms. □

Corollary A.1.70. *Let A be a C^*-algebra. For all $x \in A$, we have*

1) $x \in A_h$ *if and only if* $\operatorname{spec}_A x \subset \mathbb{R}$,

2) $x \in A_+$ *if and only if* $\operatorname{spec}_A x \subset [0, \infty)$,

3) $x = x_+ - x_-$ *for* $x_+ := \max\{x, 0\}, x_- := -\min\{x, 0\} \in A_+$ *if* $x \in A_h$.

Proof. By Corollary A.1.69, we assume A is unital without loss of generality. Thus 1) and 2) are Proposition I.4.3 and Theorem I.6.1 in [192]. Writing $g(x) := \Gamma_{x,A}(g)$ in each case, we see 3) is decomposition in Proposition A.1.23 to have proper cone. □

Lemma A.1.72 extends to bounded measurable functional calculus. Corollary A.1.93 shows bounded measurable calculus of self-adjoint elements is preserved under normal unital *-homomorphisms. In the proof of Lemma A.1.72, abstract spectral measures yield bounded measurable functional calculus. Note functional calculus of self-adjoint unbounded operators instead uses concrete ones as it assumes faithful normal unital *-representations as per Remark A.1.86 in general. In Subsection B.1.3, we unify these approaches for spaces of measurable operators.

Proposition A.1.71. *If $N \subset M$ is a unital W^*-subalgebra, then $N[1_M] = N$. If $N \subset M$ is a non-unital W^*-subalgebra, then $N[1_M] = N \oplus \langle 1_M - 1_N \rangle_{\mathbb{C}}$.*

Proof. Proposition A.1.65. Note C^*-direct sums of W^*-algebras are W^*-algebras. □

Lemma A.1.72. *Let M be a W^*-algebra. If $x \in M$ is normal, then there exists unique σ-ideal $\mathcal{N}_{x,M}$ of null sets of the Borel σ-algebra $\mathfrak{B}(\mathrm{spec}_M x)$ s.t.*

1) *$\big(L^\infty(\mathrm{spec}_M x, \mathcal{N}_{x,M}), \|.\|_\infty\big)$ is a W^*-algebra s.t. $C(\mathrm{spec}_M x)$ is σ-weakly dense,*

2) *$\Gamma_{x,M}$ extends to a normal unital *-isomorphism*

$$\Gamma_{x,M} : L^\infty\big(\mathrm{spec}_M x, \mathcal{N}_{x,M}\big) \longrightarrow W^*_M(x) := W^*(x, x^*, 1_M), \tag{A.19}$$

3) *$\Gamma_{x,M}$ is determined by unitality and $\Gamma_{x,M}\big(\mathrm{id}_{\mathrm{spec}_M x}\big) = x$.*

Proof. Let $x \in M_h$. For details and the normal case, we refer to Section 5.2 in [134]. Let $\pi : M \longrightarrow \mathscr{B}(H)$ be faithful normal unital *-representation. Following Theorem 5.2.2 in [134], get unique resolution of the identity in $\mathscr{B}(H)$ associated to x. It determines unique spectral measure $E_{x,M} : \mathfrak{B}(\mathbb{R}) \longrightarrow \mathscr{B}(H)$. Pulled-back along π, uniqueness implies $E_{x,M}$ is independent of our choice of faithful normal *-representation.

Let $\mathcal{N}_{x,M} := \{Z \in \mathfrak{B}(\mathbb{R}) \mid E_{x,M}(Z) = 0\}$. Intersecting with $\mathrm{spec}_M x$ shows $\mathcal{N}_{x,M}$ is a σ-ideal of null sets of the Borel σ-algebra $\mathfrak{B}(\mathrm{spec}_M x)$. Following the construction in Example A.1.33, get W^*-algebra $L^\infty(\mathrm{spec}_M x, \mathcal{N}_{x,M})$ s.t. $C(\mathrm{spec}_M x) \subset L^\infty(\mathrm{spec}_M x, \mathcal{N}_{x,M})$ is σ-weakly dense. This shows 1). For 2), see [134]. Get 3) by Lemma A.1.67. $\square$

Definition A.1.73. Let M be a W^*-algebra. For all normal $x \in M$, we call

1) $\Gamma_{x,M}$ as in Equation A.19 the bounded measurable functional calculus of x in M,

2) $W^*_M(x)$ as in Equation A.19 the W^*-algebra generated by x in M.

Notation A.1.74. Unless stated otherwise, we suppress W^*-algebras in subscripts of spectral measures, spectra, bounded measurable functional calculus and generated W^*-algebras. We extend to measurable operators in Notation B.1.79.

Functional calculus of self-adjoint unbounded operators. Let H be Hilbert space. For all normal $T \in \mathscr{B}(H)$, the map $Z \mapsto E_T(Z) := \chi_Z(T)$ defined on $\mathfrak{B}(\mathbb{C})$ is spectral measure on $\mathbb{C}$ with values in $\mathscr{B}(H)$. We extend to self-adjoint unbounded operators.

Let $T \in \mathscr{U}\mathscr{B}(H)_h$. We call $B(T) := T(1+T^2)^{-\frac{1}{2}} \in \mathscr{B}(H)$ its bounded transform [184]. We have $\mathrm{spec}\, B_T \subset [-1,1]$. For all $t \in [-1,1]$, set

$$\varphi(t) := t\big(1-t^2\big)^{-\frac{1}{2}}. \tag{A.20}$$

Note φ is $E_{B(T)}$-a.e. finite measurable and invertible on $[-1,1]$. Formally, $B(T) = \varphi^{-1}(T)$ by change of variable $t \mapsto T$ in Equation A.20. For all $Z \in \mathfrak{B}(\mathbb{R})$, set

$$E_T(Z) := E_{B(T)}\big(\varphi^{-1}(Z)\big). \tag{A.21}$$

Equation A.21 defines spectral measure $E_T : \mathfrak{B}(\mathbb{R}) \longrightarrow \mathscr{B}(H)$ [184].

Definition A.1.75. Let $T \in \mathscr{UB}(H)_h$.

1) We call E_T the spectral measure of T.

2) For all $g \in \mathscr{S}(E_T)$, set

$$\Gamma_T(g) := g(T) := \mathrm{I}_{E_T}(g) = \int g dE_T. \tag{A.22}$$

3) We call $\Gamma_T : \mathscr{S}(E_T) \longrightarrow \mathscr{UB}(H)$ the functional calculus of T.

Remark A.1.76. Note $B(T)$ is denoted by Z_T in [184]. Instead of $B(T)$, [171] uses the Cayley transform $C(T) := \frac{T-i}{T+i} \in \mathscr{B}(H)$. The induced spectral measure is E_T. Thus $B(T)$ and $C(T)$ define identical spectral measure, hence functional calculus.

Theorem 5.9 and Proposition 5.10 in [184] collect elementary properties of functional calculus. The spectral theorem for self-adjoint unbounded operators further shows each $E_T : \mathfrak{B}(\mathbb{R}) \longrightarrow \mathscr{B}(H)$ is the unique spectral measure s.t. $T = \mathrm{I}_{E_T}(\mathrm{id}_\mathbb{R}) = \int t dE_T$.

Functional calculus restricts to bounded measurable functional calculus. This uses spectra of densely defined operators. For self-adjoint unbounded operators, spectra are the support of spectral measures. Definition A.1.66 is subsumed if we are given faithful normal unital *-representation. Unitality is necessary.

Definition A.1.77. Let T be a densely defined closable operator on H. Its resolvent set is $\operatorname{rsl} T := \{\lambda \in \mathbb{C} \mid (T - \lambda I)^{-1} \in \mathscr{B}(H)\}$ and its spectrum is $\operatorname{spec} T := \mathbb{C} \setminus \operatorname{rsl} T$.

Remark A.1.78. For all faithful normal unital *-representation $\pi : M \longrightarrow \mathscr{B}(H)$ of a W^*-algebra M, get $\operatorname{spec}_M x = \operatorname{spec} \pi(x)$ for all normal $x \in M$.

If $T \in \mathscr{UB}(H)_h$, then $\operatorname{spec} T \subset \mathbb{R}$ and $\pm i \in \operatorname{rsl} T$. For all $g \in C(\operatorname{spec} T)$, get $\operatorname{spec} g(T) = \overline{g(\operatorname{spec} T)} \subset \mathbb{R}$ by the spectral mapping theorem (cf. Proposition 5.25 in [184]). If moreover $g, g^{-1} \in C(\operatorname{spec} T)$, then $\operatorname{spec} g(T) = g(\operatorname{spec} T)$.

Proposition A.1.79. *If $T \in \mathscr{UB}(H)_h$, then* $\operatorname{supp} E_T = \operatorname{spec} T \subset \mathbb{R}$.

Proof. Proposition 5.10 in [184]. □

Definition A.1.80. Let $T \in \mathscr{UB}(H)_h$.

1) Let $a \in \mathbb{C}$. For all $z \in \mathbb{C} \setminus \{a\}$, set

$$R_a(z) := \left(z - a\right)^{-1}. \tag{A.23}$$

If $a \in \operatorname{rsl} T$, then $R_a(T) \in \mathscr{B}(H)$ is the resolvent of T in a.

2) Set $L^\infty(\operatorname{spec} T, dE_T) := L^\infty(\operatorname{spec} T, \mathscr{N}(E_T))$.

Notation A.1.81. For all $T \in \mathscr{UB}(H)_h$, let $R_{\pm i}(T)$ denote both $R_i(T)$ or $R_{-i}(T)$. Note $\pm i \in \mathbb{C} \setminus \mathbb{R}$ lies in the resolvent set of all self-adjoint unbounded operators.

Let $T \in \mathscr{UB}(H)_h$ and $g \in \mathscr{S}(E_T)$. Bounding sequences let us write $g(T)$ as pointwise $\|.\|_H$-limit. For all $Z \in \mathfrak{B}(\mathbb{R})$, note Equation A.15 ensures $g(T)E_T(Z) = (g\chi_Z)(T)$. For all $u \in H$, we have $u \in \operatorname{dom} g(T)$ if and only if

$$g(T)(u) = \|.\|_H\text{-}\lim_{k\in\mathbb{N}} g(T)E_T(Z_k)(u) \tag{A.24}$$

exists for a bounding sequence $\{Z_k\}_{k\in\mathbb{N}}$ of g. In fact, Equation A.24 is Equation A.17 for spectral integration w.r.t. E_T. If $u \in \operatorname{dom} g(T)$, then $g(T)(u)$ exists and is independent of choice of bounding sequence of g by Theorem 4.13 in [184].

Proposition A.1.82. *For all $T \in \mathscr{UB}(H)_h$, $W^*(B(T)) = W^*(C(T)) = W^*(R_{\pm i}(T))$.*

Proof. Following Remark A.1.76, we know $W^*(B(T)) = W^*(C(T))$. Since we further have $R_{\pm i} \in L^\infty(\operatorname{spec} T, dE_T)$, get $R_{\pm i}(T) \in W^*(C(T))$ by Theorem 5.3.8 in [171]. We directly verify $C(T) = R_i(T)R_{-i}(T)$ and get $W^*(C(T)) = W^*(R_i(T), R_{-i}(T)) =: W^*(R_{\pm i}(T))$. □

Definition A.1.83. For all $T \in \mathscr{UB}(H)_h$, we call $W^*(T) := W^*(B(T))$ the W^*-algebra generated by T.

Remark A.1.84. If $T \in \mathscr{B}(H)_h$, then $W^*(T) = W^*_{\mathscr{B}(H)}(T) = W^*(T, I_H)$.

If $T \in \mathscr{UB}(H)_h$, then Γ_T restricts to $L^\infty(\operatorname{spec} T, dE_T)$. Proposition A.1.85 uses the latter to formulate bounded measurable functional calculus.

Proposition A.1.85. *Let $T \in \mathscr{UB}(H)_h$.*

1) We have normal unital *-isomorphism $\Gamma_T : L^\infty(\operatorname{spec} T, dE_T) \longrightarrow W^*(T)$.

2) If $M \subset \mathscr{B}(H)$ is a W^*-subalgebra s.t. $E_T(Z) \in M$ for all $Z \in \mathfrak{B}(\mathbb{R})$, then $W^*(T) \subset M$.

Proof. Since $W^*(T) = W^*(C(T))$ by Proposition A.1.82, we use the functional calculus in [171]. $W^*(T) = W^*(T)''$ by Proposition A.1.34. Since T is self-adjoint, Lemma 5.2.8 and Theorem 5.3.8 in [171] therefore show 1). In the setting of 2), $P(W^*(T)) \subset P(M)$ and Proposition A.1.37 imply $W^*(T) = W^*(P(W^*(T)) \subset W^*(P(M)) = M$. □

Remark A.1.86. If $\pi : M \longrightarrow \mathscr{B}(H)$ is a faithful normal unital *-representation of a W^*-algebra M, then $\pi \circ \Gamma_{x,M} = \Gamma_{\pi(x)}$ for all $x \in M_h$. Unitality is necessary.

Bounded measurable functional calculus lets us test for injectivity by considering the mass of $\{0\}$ under E_T as per Remark A.1.87.

Remark A.1.87. Following Remark A.1.61, note $E_T(\{0\}) = \chi_{\{0\}}(T) = \delta_0(T)$ is the Hilbert space projection onto $\ker T$ since $u \in \ker T$ if and only if $\operatorname{supp} E_T^u = \{0\}$ for all $u \in H$.

Proposition A.1.88. *If $T \in \mathscr{UB}(H)_h$, then T is injective if and only if $E_T(\{0\}) = 0$.*

Proof. If T is injective, then get $E_T(\{0\}) = 0$ as per Remark A.1.87. If $E_T(\{0\}) = 0$, then $t \mapsto t^{-1}$ is E_T-a.e. finite. Thus T^{-1} is densely defined closed by functional calculus, hence T is injective if $E_T(\{0\}) = 0$. □

We use Lemma A.1.89 to directly verify affiliation with W^*-algebras. Lemma A.1.91 and Lemma A.1.92 provide necessary and sufficient conditions for preserving bounded measurable functional calculus.

Lemma A.1.89. *If $T \in \mathscr{UB}(H)_h$ and $U \in \mathscr{U}(\mathscr{B}(H))$, then we have $TU = UT$ if and only if $[E_T(Z), U] = 0$ for all $Z \in \mathfrak{B}(\mathbb{R})$.*

Proof. Assume $TU = UT$. Proposition A.1.82 shows $[R_{\pm i}(T), U] = 0$ yields $[E_T(Z), U] = 0$ in each case. Note $T = U^*TU$ and $U \operatorname{dom} T \subset \operatorname{dom} T$ imply $R_{\pm i}(T) = U^* R_{\pm i}(T) U$. Thus $TU = UT$ implies $[E_T(Z), U] = 0$ for all $Z \in \mathfrak{B}(\mathbb{R})$.

Assume $[E_T(Z), U] = 0$, i.e. $E_T(Z) = UE_T U^*$ for all $Z \in \mathfrak{B}(\mathbb{R})$. Thus $E_T^v = E_T^{U^*v}$ for all $v \in H$, hence $v \in \operatorname{dom} T$ if and only if $U^*v \in \operatorname{dom} T$. Get $\operatorname{dom} T = \operatorname{dom} TU^*$. We also know $[g(T), U] = 0$ for all $g(T) \in W^*(T)$ since $W^*(T)$ is generated by all $E_T(Z)$ for all $Z \in \mathfrak{B}(\mathbb{R})$ by Proposition A.1.37 and Proposition A.1.85. The spectral theorem and Equation A.24 imply $w \in \operatorname{dom} T$ if and only if there exists bounding sequence $\{Z_k\}_{k \in \mathbb{N}}$ of $\mathrm{id}_{\mathbb{R}}$ s.t.

$$T(w) = \int t dE_T^w = \|.\|_H\text{-}\lim_{n \in \mathbb{N}} g_n(T)(w). \tag{A.25}$$

If $w \in \operatorname{dom} T$, then the limit in Equation A.1.89 exists and is independent of choice of bounding sequence of $\mathrm{id}_{\mathbb{R}}$. For all $v \in \operatorname{dom} T = \operatorname{dom} TU^*$, we see Equation A.25 implies $T(v) = U(\|.\|_H\text{-}\lim_{n \in \mathbb{N}} g_n(T) U^*(v)) = UTU^*(v)$. Thus $T = UTU^*$, hence $TU = UT$. □

Definition A.1.90. Let H_0 and H_1 be Hilbert spaces. Let $M \subset \mathscr{B}(H_0)$ be W^*-algebra and $\phi : M \longrightarrow \mathscr{B}(H_1)$ normal unital *-homomorphism. If $T \in \mathscr{UB}(H_0)_h$ s.t. $\operatorname{im} E_T \subset M$, then we define the push-forward spectral measure $\phi(E_T)$ of T under ϕ by setting

$$\phi(E_T)(Z) := \phi(E_T(Z)) \tag{A.26}$$

for all $Z \in \mathfrak{B}(\mathbb{R})$.

Equation A.26 defines spectral measure $\phi(E_T) : \mathfrak{B}(\mathbb{R}) \longrightarrow \mathscr{B}(H_1)$ if we are in the setting of Definition A.1.90. Lemma A.1.92 shows push-forward spectral measures link bounded measurable functional calculus across Hilbert spaces.

Lemma A.1.91. *Let H_0 and H_1 be Hilbert spaces. Let $M \subset \mathscr{B}(H_0)$ be W^*-algebra and $\phi : M \longrightarrow \mathscr{B}(H_1)$ normal unital *-homomorphism. If $T_0 \in \mathscr{UB}(H_0)_h$ and $T_1 \in \mathscr{UB}(H_1)_h$ s.t. $\operatorname{im} E_T \subset M$ and $\phi(g(T_0)) = g(T_1)$ for all $g \in C_c(\mathbb{R})$, then $\phi(E_{T_0}) = E_{T_1}$.*

Proof. Let $\lambda \in \mathbb{R}$, and $\{g_n^\lambda\}_{n \in \mathbb{N}} \subset C_b(\mathbb{R})$ s.t. $\sup_{n \in \mathbb{N}} \|g_n^\lambda\| < \infty$ and $\chi_{(-\infty,\lambda]}(t) = \lim_{n \in \mathbb{N}} g_n^\lambda(t)$ for all $t \in \mathbb{R}$. For all self-adjoint S on arbitrary Hilbert space H and $u \in H$, E_S^u is finite and we have

$$\left\| \left(E_S((-\infty,\lambda]) - g_n^\lambda(S)\right)(u) \right\|_H^2 = \int_{\mathbb{R}} \left(\chi_{(-\infty,\lambda]}(t) - g_n^\lambda(t)\right)^2 dE_S^u \tag{A.27}$$

by functional calculus. Thus $E_S((-\infty,\lambda]) = \text{s-}\lim_{n \in \mathbb{N}} g_n^\lambda(S)$ by dominated convergence.

Let $\{\varphi_n\}_{n\in\mathbb{N}} \subset C_c(\mathbb{R})$ s.t. $0 \le \varphi_n \le \varphi_{n+1} \le 1$ for all $n \in \mathbb{N}$ and $\lim_{n\in\mathbb{N}} \varphi_n(t) = 1$ for all $t \in \mathbb{R}$. Arguing as for Equation A.27 shows s-$\lim_{n\in\mathbb{N}} g_n^\lambda(S)$ = s-$\lim_{n\in\mathbb{N}}(g_n^\lambda \varphi_n)(S)$. We therefore assume $\{g_n\}_{n\in\mathbb{N}} \subset C_c(\mathbb{R})$ in Equation A.27 without loss of generality.

If $\phi(E_{T_0}((-\infty,\lambda])) = E_{T_1}((-\infty,\lambda])$ for all $\lambda \in \mathbb{R}$, then $\phi(E_{T_0}) = E_{T_1}$ as resolutions of the identity are unique. We show the former by approximating in strong operator topology as above. For fixed but arbitrary $\{g_n^\lambda\}_{n\in\mathbb{N}} \subset C_c(\mathbb{R})$ and for all $n \in \mathbb{N}$, note Equation A.27 holds uniformly for all self-adjoint unbounded operators. Sequential strong continuity of ϕ therefore implies

$$\begin{aligned}\phi\big(E_{T_0}((-\infty,\lambda])\big) &= \operatorname*{s-lim}_{n\in\mathbb{N}} \phi(g_n(T_0)) \\ &= \operatorname*{s-lim}_{n\in\mathbb{N}} g_n(T_1) \\ &= E_{T_1}((-\infty,\lambda])\end{aligned}$$

for all $\lambda \in \mathbb{R}$. The above calculation uses Remark A.1.10. □

Lemma A.1.92. *Let H_0 and H_1 be Hilbert spaces. Let $T_0 \in \mathscr{UB}(H_0)_h$ and $T_1 \in \mathscr{UB}(H_1)_h$. If $\phi : W^*(T_0) \longrightarrow W^*(T_1)$ is a normal unital *-homomorphism s.t. $\phi(E_{T_0}) = E_{T_1}$, then*

1) $\operatorname{spec} T_1 \subset \operatorname{spec} T_0$ *and* $\mathscr{N}\big(E_{T_0}\big) \subset \mathscr{N}\big(E_{T_1}\big)$,
2) $\phi(g(T_0)) = g(T_1)$ *for all* $g \in L^\infty\big(\operatorname{spec} T_0, dE_{T_0}\big)$,
3) *we have commutative diagram of normal unital surjective *-homomorphisms*

$$\begin{tikzcd} L^\infty\big(\operatorname{spec} T_0, dE_{T_0}\big) \arrow[r, "\Gamma_{T_0}"] \arrow[d, "\mathrm{res}"] & W^*(T_0) \arrow[d, "\phi"] \\ L^\infty\big(\operatorname{spec} T_1, dE_{T_1}\big) \arrow[r, "\Gamma_{T_1}"] & W^*(T_1) \end{tikzcd} \tag{A.28}$$

with res *the restriction map given by* $\operatorname{spec} T_1 \subset \operatorname{spec} T_0$.

Proof. We directly verify $\mathscr{N}(E_{T_0}) \subset \mathscr{N}(E_{T_1})$. Since we have $\lambda \in \operatorname{spec} T_1 = \operatorname{supp} E_{T_1}$ if and only if $\phi(E_{T_0}(N_\lambda)) = E_{T_1}(N_\lambda) \neq 0$ for all open neighbourhoods N_λ of λ, get 1) at once. If $g \in \mathscr{S}(E_{T_0})$ is E_{T_0}-a.e. bounded, then $g \in \mathscr{S}(E_{T_1})$ is E_{T_1}-a.e. bounded.

For all $n \in \mathbb{N}$, let $\{Z_{k,m}\}_{k,m\in\mathbb{N}} \subset \mathfrak{B}(\mathbb{R})$ s.t. following pointwise E_{T_0}-a.e. approximation of $\mathrm{id}_{\mathbb{R}}\, \chi_{[-n,n]}$ on $\operatorname{spec} T_0$ holds. For E_{T_0}-a.e. $t \in \operatorname{spec} T$, get $\{a_{k,m}\}_{k,m\in\mathbb{N}} \subset \mathbb{R}$ and

$$t\chi_{[-n,n]}(t) = \lim_{m\in\mathbb{N}} \sum_{k=1}^{m} a_{k,m}\chi_{Z_{k,m}}(t). \tag{A.29}$$

Using 1), Equation A.29 further yields E_{T_1}-a.e. approximation of $\mathrm{id}_{\mathbb{R}}\, \chi_{[-n,n]}$ on $\operatorname{spec} T_1$. The approximations we use here are uniformly bounded, hence yield bounded strong limits upon evaluation using T_0, resp. T_1.

Normality and $\phi(E_{T_0}) = E_{T_1}$ yield $\phi(T_0E_{T_0}([-n,n])) = T_1E_{T_1}([-n,n])$ for all $n \in \mathbb{N}$. For all $g \in L^\infty(\operatorname{spec} T_0, dE_{T_0})$, we see 1) implies $g \in L^\infty(\operatorname{spec} T_1, dE_{T_1})$ upon restriction. Strong convergence of sequences further implies

$$g(T_0) = \operatorname*{s\text{-}lim}_{n\in\mathbb{N}} g\big(T_0E_{T_0}([-n,n])\big),\ g(T_1) = \operatorname*{s\text{-}lim}_{n\in\mathbb{N}} g\big(T_1E_{T_1}([-n,n])\big). \tag{A.30}$$

If $\phi(g(T_0E_{T_0}([-n,n])) = g(T_1E_{T_1}([-n,n]))$ for all $g(T_0) \in W^*(T_0)$ and $n \in \mathbb{N}$, then 2) holds by Equation A.30. Ergo 2), and thereby 3), reduces to the bounded case.

Assume T_0 and T_1 are bounded. Thus $\phi(T_0) = T_1$, hence $g(\phi(T_0)) = g(T_1)$ for all $g \in C(\operatorname{spec} T_k)$. For all $k \in \{0,1\}$, Proposition A.1.82 shows $R_{\pm i} \in C(\operatorname{spec} T_k)$ and $W^*(T_k) = W^*(R_{\pm i}(T_k))$. Normality implies $g(\phi(T_0)) = g(T_1)$ for all $g \in L^\infty(\operatorname{spec} T_0, dE_{T_0})$. Get 2). Using the latter, we directly verify 3). The general case follows as discussed above. □

Corollary A.1.93. *Let H_0 and H_1 be Hilbert spaces. Let $M \subset \mathscr{B}(H_0)$ and $N \subset \mathscr{B}(H_1)$ be W^*-algebras. We consider $x \in M_h$ and $y \in N_h$. If $\phi : W^*_M(x) \longrightarrow W^*_N(y)$ is a normal unital *-homomorphism s.t. $\phi(x) = y$, then $\phi(E_{x,M}) = E_{y,N}$ and Lemma A.1.92 applies to ϕ.*

Proof. Note $\operatorname{supp} E_{x,M} = \operatorname{spec}_M x$ and $\operatorname{supp} E_{y,N} = \operatorname{spec}_N y$. Let $\operatorname{spec}_M x, \operatorname{spec}_N y \subset K$ for compact $K \subset \mathbb{R}$. Lemma A.1.91 shows $\phi(g(x)) = g(y)$ for all $g \in C(K)$ suffices. We reduce to polynomials by approximating in norms. The *-homomorphism property concludes. □

Joint functional calculus of strongly commuting self-adjoint unbounded operators. Let H be a Hilbert space. Let $T, S \in \mathscr{UB}(H)_h$. If $[E_T(Z_0), E_S(Z_1)] = 0$ for all $Z_0, Z_1 \in \mathfrak{B}(\mathbb{R})$, then we determine joint spectral measure by setting

$$E_{T,S}(Z_0 \times Z_1) := E_T(Z_0)E_S(Z_1) \tag{A.31}$$

for all $Z_0, Z_1 \in \mathfrak{B}(\mathbb{R})$. Equation A.31 defines spectral measure $E_{T,S} : \mathfrak{B}(\mathbb{R}\times\mathbb{R}) \longrightarrow \mathscr{B}(H)$ by Theorem 4.10 in [184].

Definition A.1.94. Let $T, S \in \mathscr{UB}(H)_h$. We say that T and S commute strongly if $[E_T(Z_0), E_S(Z_1)] = 0$ for all $Z_0, Z_1 \in \mathfrak{B}(\mathbb{R})$. Assume T and S commute strongly.

1) We call $E_{T,S}$ the joint spectral measure of T and S.

2) For all $g \in \mathscr{S}\big(E_{T,S}\big)$, set

$$\Gamma_{T,S}(g) := g(T,S) := \mathrm{I}_{E_{T,S}}(g) = \int g dE_{T,S}. \tag{A.32}$$

3) We call $\Gamma_{T,S} : \mathscr{S}\big(E_{T,S}\big) \longrightarrow \mathscr{UB}(H)$ the joint functional calculus of T and S.

Remark A.1.95. The commutator $[-,-] : \mathscr{B}(H) \times \mathscr{B}(H) \longrightarrow \mathscr{B}(H)$ in $\mathscr{B}(H)$ is given by $[T,S] := TS - ST$ for all $T,S \in \mathscr{B}(H)$. It is separately continuous in strong operator topology. If $M,N \subset \mathscr{B}(H)$ are W^*-algebras with strongly dense subsets $M_0 \subset M$ and $N_0 \subset N$ s.t. $[M_0,N_0] = 0$, then $[M,N] = 0$ by separate strong continuity.

Proposition A.1.96. *For all $T,S \in \mathscr{U}\mathscr{B}(H)_h$, the following are equivalent:*

1) *T and S commute strongly,*
2) $[R_a(T),R_b(S)] = 0$ *for* $a \in \operatorname{rsl} T$ *and* $b \in \operatorname{rsl} S$,
3) $[g(T),h(S)] = 0$ *for all* $g \in L^\infty(\operatorname{spec} T, dE_T)$ *and* $h \in L^\infty(\operatorname{spec} S, dE_S)$,
4) $[B(T),B(S)] = 0$,
5) $[C(T),C(S)] = 0$.

Proof. Equivalence of 1) and 2) is Proposition 5.27 in [184]. Continuity of commutators as per Remark A.1.95 ensures Proposition A.1.82 shows equivalence of 2) to 5). □

Proposition A.1.97. *If $T,S \in \mathscr{U}\mathscr{B}(H)_h$ commute strongly, then*

1) *$E_{T,S}$ is the unique spectral measure s.t. $T = \int t dE_{T,S}$ and $S = \int s dE_{T,S}$,*
2) $\operatorname{supp} E_{T,S} \subset \operatorname{spec} T \times \operatorname{spec} S$.

Proof. Get 1) by Lemma 5.22 in [184]. Thus $E_{T,S}$ is joint spectral measure given in the proof of Theorem 5.23 in [184], hence 2) follows by Proposition 5.24 in [184]. □

Remark A.1.98. Note $\operatorname{supp} E_{T,S} \neq \operatorname{spec} T \times \operatorname{spec} S$ in general as $\mathscr{B}(H)$ has zero divisors. Inequality therefore occurs if $E_S(N_s)H \subset (E_T(N_t)H)^\perp$ for a product open neighbourhood $N_t \times N_s$ of $(t,s) \in \operatorname{spec} T \times \operatorname{spec} S$.

Let $M,N \subset \mathscr{B}(H)$ be commutative W^*-subalgebras s.t. $W^*(M,N) \subset \mathscr{B}(H)$ is also commutative W^*-subalgebra. We determine normal unital injective *-homomorphism $M \otimes N \longrightarrow \mathscr{B}(H)$ by mapping

$$x \otimes y \mapsto xy = yx \tag{A.33}$$

for all $x \in M$ and $y \in N$. Get W^*-subalgebra $M \otimes N \subset \mathscr{B}(H)$. Proposition A.1.100 thereby extends Proposition A.1.85 using jointly generated W^*-algebras.

Definition A.1.99. Let $T,S \in \mathscr{U}\mathscr{B}(H)_h$ commute strongly.

1) The joint spectrum of T and S is $\operatorname{spec} T \times S := \operatorname{supp} E_{T,S}$.
2) Set $L^\infty\big(\operatorname{spec} T \times S, dE_{T,S}\big) := L^\infty\big(\operatorname{spec} T \times S, \mathscr{N}\big(E_{T,S}\big)\big)$.
3) We call $W^*(T,S) := W^*(T) \otimes W^*(S)$ the W^*-algebra generated by T and S.

If $T,S \in \mathcal{UB}(H)_h$ commute strongly, then $\Gamma_{T,S}$ restricts to $L^\infty(\operatorname{spec} T \times S, dE_{T,S})$ as in the case of one self-adjoint unbounded operator. Proposition A.1.100 uses the latter to formulate bounded measurable joint functional calculus.

Proposition A.1.100. *Let $T,S \in \mathcal{UB}(H)_h$ commute strongly.*

1) *We have normal unital *-isomorphism $\Gamma_{T,S} : L^\infty\big(\operatorname{spec} T \times S, dE_{T,S}\big) \longrightarrow W^*(T,S)$.*

2) *If $M \subset \mathcal{B}(H)$ is a W^*-subalgebra s.t. $E_{T,S}(Z_0 \times Z_1) \in M$ for all $Z_0, Z_1 \in \mathfrak{B}(\mathbb{R})$, then $W^*(T,S) \subset M$.*

Proof. Note $L^\infty(\operatorname{spec} T \times S, dE_{T,S}) = L^\infty(\operatorname{spec} T \times \operatorname{spec} S, dE_{T,S})$ by construction of joint spectral measures, and $L^\infty(\operatorname{spec} T \times \operatorname{spec} S, dE_{T,S}) \cong L^\infty(\operatorname{spec} T, dE_T) \otimes L^\infty(\operatorname{spec} S, dE_S)$ naturally. All claims reduce to elementary tensors. Apply Proposition A.1.85. □

Lemma A.1.101. *Let H_0 and H_1 be Hilbert spaces. Let $T_0, S_0 \in \mathcal{UB}(H_0)_h$, as well as $T_1, S_1 \in \mathcal{UB}(H_1)_h$ commute strongly. If $\phi : W^*(T_0) \longrightarrow W^*(T_1)$ and $\psi : W^*(S_0) \longrightarrow W^*(S_1)$ are normal unital *-homomorphisms s.t. $\phi(E_{T_0}) = E_{T_1}$ and $\psi(E_{S_0}) = E_{S_1}$, then*

1) $\operatorname{spec} T_1 \times S_1 \subset \operatorname{spec} T_0 \times S_0$ *and* $\mathcal{N}\big(E_{T_0,S_0}\big) \subset \mathcal{N}\big(E_{T_1,S_1}\big)$,

2) $\big(\phi \otimes \psi\big)(g(T_0,S_0)) = g(T_1,S_1)$ *for all* $g \in L^\infty\big(\operatorname{spec} T_0 \times S_0, dE_{T_0,S_0}\big)$,

3) *we have commutative diagram of normal unital surjective *-homomorphisms*

$$\begin{tikzcd}
L^\infty\big(\operatorname{spec} T_0 \times S_0, dE_{T_0,S_0}\big) \arrow[r, "\Gamma_{T_0,S_0}"] \arrow[d, "\mathrm{res}"] & W^*(T_0,S_0) \arrow[d, "\phi\otimes\psi"] \\
L^\infty\big(\operatorname{spec} T_1 \times S_1, dE_{T_1,S_1}\big) \arrow[r, "\Gamma_{T_1,S_1}"] & W^*(T_1,S_1)
\end{tikzcd} \tag{A.34}$$

with res *the restriction map given by* $\operatorname{spec} T_1 \times S_1 \subset \operatorname{spec} T_0 \times S_0$.

Proof. We apply Lemma A.1.92 and Corollary A.1.53 to ϕ and ψ. This constructs normal unital surjective *-homomorphism $\phi\otimes\psi : W^*(T_0,S_0) \longrightarrow W^*(T_1,S_1)$. Note Equation A.13 and Equation A.33 show $\phi \otimes \psi$ is determined by $(\phi \otimes \psi)(g(T_0)h(S_0)) = g(T_1)h(S_1)$ for all $g \in L^\infty(\operatorname{spec} T_0, dE_{T_0})$ and $h \in L^\infty(\operatorname{spec} S_0, dE_{S_0})$. For all $Z, Z' \in \mathfrak{B}(\mathbb{R})$, construction of joint spectral measures shows

$$\begin{aligned}
\big(\phi \otimes \psi\big)\big(E_{T_0,S_0}(Z \times Z')\big) &= \big(\phi \otimes \psi\big)\big(E_{T_0}(Z)E_{S_0}(Z')\big) \\
&= E_{T_1}(Z)E_{S_1}(Z') = E_{T_1,S_1}(Z \times Z').
\end{aligned}$$

Arguing as in the proof of Lemma A.1.92, the above calculation implies 1). We see the restriction map res is well-defined. Using Proposition A.1.49, we directly verify 2) and 3) on elementary tensors. Using σ-strong continuity, we conclude by strong density. □

A.2 Maps of unbounded operators

In Subsection A.2.1, we discuss strong resolvent convergence and resolvent-preserving maps of unbounded operators. Strong resolvent convergence provides suitable notion of continuity. Given uniform neighbourhood of supports, evaluation of functional calculus on fixed bounded continuous functions is strong resolvent continuous. We extend to joint functional calculus. Resolvent-preserving maps are strong resolvent continuous and preserve functional calculus. Examples are twisting and compression maps.

In Subsection A.2.2, we introduce abstract and concrete compression maps. They are given by left- and right-multiplication with projections. In the abstract case, we compress W^*-algebras. In the concrete case, we thus compress self-adjoint unbounded operators on a Hilbert space by reducing subspaces. We extend abstract compression maps to spaces of measurable operators in Subsection B.2.1.

A.2.1 Strong resolvent continuity and resolvent-preservation

We define strong resolvent convergence, prove strong resolvent continuity of functional calculus in Lemma A.2.5 and review sufficient conditions. We then give two standard approximations and discuss resolvent-preserving maps. Standard reference for strong resolvent convergence is [88].

Strong resolvent convergence of self-adjoint unbounded operators. Note Definition A.2.1 gives strong resolvent convergence on Hilbert spaces. We use the latter to extend 2) in Lemma A.1.101 to suitable unbounded functions in Corollary A.2.6.

In Subsection 2.2.2, Definition 2.2.31 generalises to strong resolvent convergence on Hilbert subspaces for use in the Kato-Robinson theorem (cf. Theorem 10.4.2 in [88]). In the appendix, we only use strong resolvent convergence on Hilbert spaces.

Definition A.2.1. Let H be a Hilbert space. We call $\{T_n\}_{n\in\mathbb{N}} \subset \mathscr{UB}(H)_h$ strong resolvent convergent to $T \in \mathscr{UB}(H)_h$ on H if $R_i(T) = \text{s-lim}_{n\in\mathbb{N}} R_i(T_n)$.

Notation A.2.2. Let $T = \text{sr-lim}_{n\in\mathbb{N}} T_n$ on H denote strong resolvent convergence. We drop *on H* if H is clear from context. We extend to strong resolvent convergence on Hilbert subspaces in Notation 2.2.32.

Remark A.2.3. We equivalently use R_{-i} in Definition A.2.1 (cf. Lemma 10.1.5 in [88]). Moreover, note uniform boundedness and strong resolvent convergence together are equivalent to strong convergence (cf. Proposition 10.1.13 in [88]).

Note Lemma A.2.5 is based on the case of one self-adjoint unbounded operator as per Remark A.2.4. We recover this one-variable case using the identity as second one.

Remark A.2.4. Proposition 10.1.9 in [88] shows we have $T = \text{sr-lim}_{n\in\mathbb{N}} T_n$ if and only if $g(T) = \text{s-lim}_{n\in\mathbb{N}} g(T_n)$ for all $g \in C_b(\mathbb{R})$. Lemma A.2.5 yields the first direction given two strongly commuting self-adjoint unbounded operators. We recover the one-variable case by setting $S = S_n = I$ for all $n \in \mathbb{N}$ and $g = g \cdot 1 \in C_b(\mathbb{R})$ in Lemma A.2.5.

Lemma A.2.5. *Let $T = \text{sr-lim}_{n\in\mathbb{N}} T_n$ and $S = \text{sr-lim}_{n\in\mathbb{N}} S_n$ on H. Let T and S commute strongly. For all $n \in \mathbb{N}$, let T_n and S_n commute strongly. Set*

$$X_{T,S} := \overline{\bigcup_{n\in\mathbb{N}} \operatorname{spec} T_n \times \operatorname{spec} S_n} \subset \mathbb{R} \times \mathbb{R}. \tag{A.35}$$

If $g \in C_b(X_{T,S})$, then $g(T,S) = \text{s-lim}_{n\in\mathbb{N}} g(T_n, S_n)$.

Proof. As $X_{T,S}$ is closed by hypothesis and contains all spectral measure supports in use (cf. Corollary 10.2.2 in [88]), we assume $g \in C_b(\mathbb{R}\times\mathbb{R})$ without loss of generality. For all $g_0, g_1 \in C_0(\mathbb{R})$, sequential strong continuity of multiplication yields

$$\begin{aligned}
\big(g_0 \otimes g_1\big)(T,S) &= g_0(T) g_1(S) \\
&= \text{s-}\lim_{n\in\mathbb{N}} g_0(T_n) g_1(S_n) \\
&= \text{s-}\lim_{n\in\mathbb{N}} \big(g_0 \otimes g_1\big)(T_n, S_n)
\end{aligned}$$

using the one-variable case as per Remark A.2.4. Thus approximating uniformly in norm shows our claim for all $g \in C_0(\mathbb{R}\times\mathbb{R})$. If $g \in C_b(\mathbb{R}\times\mathbb{R})$, then fix a monotone sequence of mollifying functions and argue as in the proof of Proposition 10.1.9 in [88]. □

Corollary A.2.6. *Assume the setting of Lemma A.1.101. For all real $g \in \mathscr{S}(E_{T_0,S_0})$ s.t*

1) *$(t,s) \mapsto g_\varepsilon(t,s) := g(t+\varepsilon, s+\varepsilon)$ lies in $C_b(\operatorname{spec} T_0 \times S_0)$ for all $\varepsilon > 0$,*
2) *$g(T_1,S_1) = \text{sr-lim}_{\varepsilon\downarrow 0} g_\varepsilon(T_1,S_1)$ on H_1,*

we have $g \in \mathscr{S}(E_{T_1,S_1})$ with $g(T_1,S_1) = \text{sr-lim}_{\varepsilon\downarrow 0}(\phi\otimes\psi)(g_\varepsilon(T_0,S_0))$ on H_1.

Proof. We know $g \in \mathscr{S}(E_{T_1,S_1})$ by 1) in Lemma A.1.101. For all $\varepsilon > 0$, we apply 2) in Lemma A.1.101 to $g_\varepsilon \in C_b(\operatorname{spec} T_0 \times S_0)$. Thus $(\phi\otimes\psi)(g_\varepsilon(T_0,S_0)) = g_\varepsilon(T_1,S_1)$, hence we conclude by 2) and $\varepsilon \downarrow 0$. □

Remark A.2.7. In the sense of Corollary A.2.6, Lemma A.1.101 gives conditions to pull back unbounded joint functional calculus. Lemma A.1.101 and Lemma A.2.5 further show Corollary A.2.6 applies to all $g \in C_b(X)$ for compact $X \subset \mathbb{R}\times\mathbb{R}$ with $\delta > 0$ s.t.

$$\bigcup_{0<\varepsilon<\delta} \operatorname{spec} T_1 + \varepsilon I \times \operatorname{spec} S_1 + \varepsilon I \subset X. \tag{A.36}$$

Note $X_{T,S} \subset X$ as per Equation A.35 in this case.

Proposition A.2.8. *We have $T = \text{sr-lim}_{n\in\mathbb{N}} T_n$ on H if there exists*

1) *$a \in \operatorname{rsl} T$ s.t. $a \in \bigcap_{n\in\mathbb{N}} \operatorname{rsl} T_n$ and $R_a(T) = \text{s-lim}_{n\in\mathbb{N}} R_a(T_n)$,*
2) *or core $\mathscr{H}$ of T s.t. $\mathscr{H} \subset \bigcap_{n\in\mathbb{N}} \operatorname{dom} T_n$ and $T(u) = \|.\|_H\text{-lim}_{n\in\mathbb{N}} T_n(u)$ for all $u \in \mathscr{H}$.*

Proof. Get 1) by Proposition 10.1.23, resp. 2) by Proposition 10.1.18 in [88]. □

Two standard approximations. Using cut-off sequences, functional calculus lets us approximate self-adjoint and positive unbounded operators on Hilbert spaces in strong resolvent convergence. Let H be a Hilbert space.

Lemma A.2.9. *For all $T \in \mathscr{UB}(H)_+$, we have*

1) $T = \text{sr-lim}_{n\in\mathbb{N}} \min\{T,n\}$,

2) $\lambda \notin \operatorname{spec} T$ *if and only if* $\lambda \notin \operatorname{spec}\min\{T,n\}$ *for a.e.* $n \in \mathbb{N}$.

Proof. For all $n \in \mathbb{N}$, set $T_n := \min\{T,n\}$ and note $T_{n+1} \geq T_n$. For all $u \in H$, we have

$$\sup_{n\in\mathbb{N}} \langle T_n(u),u\rangle_H = \sup_{n\in\mathbb{N}} \int_{\operatorname{spec} T} \min\{t,n\} dE_T^u = \begin{cases} \langle T(u),u\rangle_H & \text{if } u \in \operatorname{dom}\sqrt{T}, \\ \infty & \text{else}. \end{cases}$$

Thus $T_n \uparrow T$ monotonically in the sense of closed positive unbounded quadratic forms on H, hence get 1) by the Kato-Robinson theorem (cf. Theorem 10.4.2 in [88]).

We show 2). We know $\operatorname{spec} T = \bigcup_{n\in\mathbb{N}} \{\lambda \leq n \mid \lambda \in \operatorname{spec} T\}$. For all $n \in \mathbb{N}$, the spectral mapping theorem (cf. Proposition 5.25 in [184]) implies

$$\operatorname{spec} T_n = \begin{cases} \operatorname{spec} T & \text{if } \|T\|_{\mathscr{B}(H)} \leq n, \\ \{\lambda \leq n \mid \lambda \in \operatorname{spec} T\} \cup \{n\} & \text{else}. \end{cases}$$

Let $\lambda \geq 0$. If $\lambda \notin \operatorname{spec} T$, then $\lambda \notin \{\lambda \leq n \mid \lambda \in \operatorname{spec} T\} = \operatorname{spec} T_n \setminus \{n\}$ for all $n \in \mathbb{N}$. If further $\lambda \in \operatorname{spec} T_{n_0}$ for $n_0 \in \mathbb{N}$, then $\lambda = n_0$ and $n_0 \notin \operatorname{spec} T$. We see $\lambda \notin \operatorname{spec} T$ implies $\lambda \notin \operatorname{spec} T_n$ for a.e. $n \in \mathbb{N}$. We know $\{\lambda \leq n \mid \lambda \in \operatorname{spec} T\} \subset \mathscr{P}(\operatorname{spec} T)$ is a monotonically increasing sequence. Thus $\lambda \notin \operatorname{spec} T_n$ for a.e. $n \in \mathbb{N}$ implies $\lambda \notin \operatorname{spec} T$, hence 2) follows. □

Corollary A.2.10. *For all $T \in \mathscr{UB}(H)_h$, $T = \text{sr-lim}_{n\in\mathbb{N}} E_T([-n,n])T$.*

Proof. Set $T_+ := E_T([0,\infty))T$ and $T_- := -E_T((-\infty,0])T$. Get $T = T_+ - T_-$ by functional calculus. Using Proposition A.1.96, we know $T_+, T_- \in \mathscr{UB}(H)_+$ commute strongly since $R_i(T_+), R_i(T_-) \in W^*(T)$ commute. For all $n \in \mathbb{N}$, functional calculus implies

$$E_T([-n,n])T = E_{T_+}([0,n])T_+ - E_{T_-}([0,n])T_-. \tag{A.37}$$

Summands in Equation A.37 commute strongly. Note $(t,s) \mapsto g(t,s) := R_i(t-s)$ lies in $C_b(\mathbb{R}\times\mathbb{R})$. If $S = \text{sr-lim}_{n\in\mathbb{N}} E_S([0,n])S$ for all $S \geq 0$ on H, then Lemma A.2.5 shows

$$\begin{aligned} R_i(T) = R_i(T_+ - T_-) = g(T_+,T_-) \\ &= \operatorname*{s\text{-}lim}_{n\in\mathbb{N}} g\big(E_{T_+}([0,n])T_+, E_{T_-}([0,n])T_-\big) \\ &= \operatorname*{s\text{-}lim}_{n\in\mathbb{N}} R_i\big(E_{T_+}([0,n])T_+ - E_{T_-}([0,n])T_-\big) \\ &= \operatorname*{s\text{-}lim}_{n\in\mathbb{N}} R_i(E_T([-n,n])T). \end{aligned}$$

If $S = \text{sr-lim}_{n\in\mathbb{N}} E_S([0,n])S$ for all $S \geq 0$ on H, then the above calculation shows our claim follows. Let $S \in \mathscr{U}\mathscr{B}(H)_+$. For all $n \in \mathbb{N}$, functional calculus implies

$$E_S([0,n])S = \min\{S,n\} - n \cdot \big(I - E_S([0,n])\big). \tag{A.38}$$

Summands in Equation A.38 commute strongly. Note Lemma A.2.9 ensures we have $S = \text{sr-lim}_{n\in\mathbb{N}} \min\{S,n\}$. We moreover have pointwise convergence $\lim_{n\in\mathbb{N}} n(1-\chi_{[0,n]}) = 0$ on $[0,\infty)$, i.e. uniformly bounded pointwise limit $\lim_{n\in\mathbb{N}} R_i(n(1-\chi_{[0,n]})) = R_i(0)$ in $C_0([0,\infty))$. Thus $\text{sr-lim}_{n\in\mathbb{N}} n(I - E_S([0,n])) = 0$. Using Lemma A.2.5 as above for each summand on the right-hand side of Equation A.38 a separate variable, we obtain our claim. □

Resolvent-preserving maps. Lemma A.2.12 shows resolvent-preserving maps are strong resolvent continuous and preserve functional calculus. Both twisting and compression maps are resolvent-preserving. Let H_0 and H_1 be Hilbert spaces.

Definition A.2.11. Let $\phi : \mathscr{U}\mathscr{B}(H_0) \longrightarrow \mathscr{U}\mathscr{B}(H_1)$ be a linear map s.t. $\phi(\mathscr{B}(H_0)) \subset \mathscr{B}(H_1)$ and $\mathscr{D} \subset \mathscr{U}\mathscr{B}(H_0)_h$. We say that ϕ is resolvent-preserving using $\mathscr{D}(\phi) := \mathscr{D}$ if

1) $\phi : \mathscr{B}(H_0) \longrightarrow \mathscr{B}(H_1)$ is bounded and normal,

2) $\phi\big(R_{\pm i}(T)\big) = R_{\pm i}\big(\phi(T)\big)$ for all $T \in \mathscr{D}(\phi)$.

Lemma A.2.12. *Let $\phi : \mathscr{U}\mathscr{B}(H_0) \longrightarrow \mathscr{U}\mathscr{B}(H_1)$ be a resolvent-preserving map.*

1) Let $T \in \mathscr{D}(\phi)$. If $T = \text{sr-lim}_{n\in\mathbb{N}} T_n$ on H_0 and $\{T_n\}_{n\in\mathbb{N}} \subset \mathscr{D}(\phi)$, then

$$\phi(T) = \text{sr-}\lim_{n\in\mathbb{N}} \phi(T_n) \text{ on } H_1. \tag{A.39}$$

2) For all $T \in \mathscr{D}(\phi)$, we have

2.1) $\phi : W^*(T) \longrightarrow W^*\big(\phi(T)\big)$ is a normal unital *-isomorphism,

2.2) $\phi(E_T) = E_{\phi(T)}$ and Lemma A.1.92 applies to $\phi : W^*(T) \longrightarrow W^*\big(\phi(T)\big)$.

Proof. Let $T \in \mathscr{D}(\phi)$. If $T = \text{sr-lim}_{n\in\mathbb{N}} T_n$ on H_0 and $\{T_n\}_{n\in\mathbb{N}} \subset \mathscr{D}(\phi)$, then we calculate

$$\begin{aligned} R_i\big(\phi(T)\big) = \phi\big(R_i(T)\big) &= \phi\big(\text{s-}\lim_{n\in\mathbb{N}} R_i(T_n)\big) \\ &= \text{s-}\lim_{n\in\mathbb{N}} \phi\big(R_i(T_n)\big) = \text{s-}\lim_{n\in\mathbb{N}} R_i\big(\phi(T_n)\big). \end{aligned}$$

This shows 1). We show 2). Note $\phi : C^*(R_{\pm i}(T)) \longrightarrow C^*(R_{\pm i}(\phi(T)))$ maps C^*-generators onto by hypothesis. Ergo $\phi|_{C^*(R_{\pm i}(T))}$ is unital *-isomorphism s.t. $\phi(g(T)) = g(\phi(T))$ for all $g \in C_c(\mathbb{R})$. We know $\phi|_{\mathscr{B}(H_0)}$ is normal. Therefore, σ-weak closure of $\phi|_{C^*(R_{\pm i}(T))}$ exists and is normal unital *-isomorphism $\phi : W^*(T) \longrightarrow W^*(\phi(T))$. Thus Lemma A.1.91, hence Lemma A.1.92 applies as claimed. Get 2). □

Corollary A.2.13. *Let $\phi : H_0 \longrightarrow H_1$ be a linear or anti-linear isometric isomorphism.*

1) *$\phi^\dagger$ is resolvent-preserving using $\mathscr{D}(\phi^\dagger) = \mathscr{U}\mathscr{B}(H_0)_h$.*

2) *If $T_0 \in \mathscr{U}\mathscr{B}(H_0)_h$ and $T_1 \in \mathscr{U}\mathscr{B}(H_1)_h$ s.t. $\phi^\dagger(E_{T_0}) = E_{T_1}$, then $\phi^\dagger(T_0) = T_1$.*

Proof. Using Proposition A.1.14, we directly verify 1). Thus $E_{\phi^\dagger(T_0)} = \phi^\dagger(E_{T_0}) = E_{T_1}$ by 2) in Lemma A.2.12 and hypothesis, hence 2) follows by the spectral theorem. □

A.2.2 Compression maps, reducing subspaces and spectral gaps

We introduce abstract and concrete compression maps. Reducing subspaces are used to define subsets for which compression maps are resolvent-preserving. We then apply compression to get useful standard results concerning spectral gaps. Standard reference for reducing subspaces is [88].

Compression maps. Definition A.2.15 gives abstract compression maps as per Remark A.2.14, and Definition A.2.18 gives concrete ones. Following our discussion in Subsection B.2.1, Definition B.2.31 extends Definition A.2.15 to spaces of measurable operators. We unify in Corollary B.2.32 for spaces of measurable operators.

Remark A.2.14. Following [192][193][194], *abstract* signifies that an object or property is independent of representation whereas *concrete* assumes representation.

Let M be a W^*-algebra and $p \in M$ a projection. If $A \subset M$ is a C^*-subalgebra, then $pC^*(A,p)p \subset M$ is one. If $N \subset M$ is a W^*-subalgebra, then $pC^*(N,p)p \subset M$ is one.

Definition A.2.15. Let M be a W^*-algebra. We consider C^*-subalgebra $A \subset M$. For all projections $p \in M$, we define

1) orthogonal projection $p^\perp := 1_M - p \in M$,

2) compressed C^*-subalgebra $A[p] := pC^*(A,p)p \subset M$,

3) the compression map $\mathrm{com}_p : A[1_M] \longrightarrow A[p]$ by setting

$$\mathrm{com}_p\, x := pxp \tag{A.40}$$

for all $x \in A[1_M]$.

Remark A.2.16. If $p \in A$, then $A[p] = pAp$. If $p = 1_M$, then we recover unitalisation.

Proposition A.2.17. *Let M be a W^*-algebra and $N \subset M$ a W^*-subalgebra. If $p \in M$ is a projection, then $\mathrm{com}_p : N[1_M] \longrightarrow N[p]$ is a completely positive, normal, unital and surjective bounded linear map.*

Proof. Complete positivity is given in Example A.1.48. Bounded weak continuity and Proposition A.1.49 imply normality. All remaining claims follow by construction. □

Let $N \subset M$ be a W^*-subalgebra. Proposition A.1.71 states $N[1_M] = N \oplus \langle 1_M^\perp \rangle_{\mathbb{C}}$. We directly verify $N 1_N^\perp = 1_N^\perp N = 0$. We have $N[1_M][1_N] = N$ and commutative diagram

$$\begin{tikzcd} N \arrow[r, hook] \arrow[rr, bend right, "\mathrm{id}_N"'] & N \oplus \langle 1_M^\perp \rangle_{\mathbb{C}} \arrow[r, "\mathrm{com}_{1_N}"] & N \end{tikzcd} \tag{A.41}$$

of normal *-homomorphisms. We extend Diagram A.41 in Subsection B.2.1. Following this, Corollary B.2.36 shows choice of unit only involves values at zero.

Let H be a Hilbert space. If $V \subset H$ is a Hilbert subspace and $\pi_V : H \longrightarrow V$ its Hilbert space projection, we use positivity-preserving canonical inclusion $\mathscr{UB}(V) \subset \mathscr{UB}(H)$ by setting

$$T = \pi_V T \pi_V \tag{A.42}$$

for all $T \in \mathscr{UB}(V)$. For details on inclusions and partial order for spaces of unbounded operators, we refer to Subsection A.1.1, in particular Remark A.1.12.

Definition A.2.18. Let H be a Hilbert space. For all Hilbert subspaces $V \subset H$, i.e. for $\|.\|_H$-closed ones, let $\pi_V : H \longrightarrow V$ denote its Hilbert space projection and we define

1) orthogonal projection $\pi_V^\perp := I_H - \pi_V \in \mathscr{B}(H)$,

2) inclusion $\mathscr{UB}(V) \subset \mathscr{UB}(H)$ as per Equation A.42,

3) the compression map $\mathrm{com}_V : \mathscr{UB}(H) \longrightarrow \mathscr{UB}(V)$ by setting

$$\mathrm{com}_V T := \pi_V T \pi_V \tag{A.43}$$

for all $T \in \mathscr{UB}(H)$.

Proposition A.2.19. *Let H be a Hilbert space. If $V \subset H$ is a Hilbert subspace, then $\mathscr{B}(H)[\pi_V] = \mathscr{B}(V)$ and $\mathrm{com}_V : \mathscr{B}(H) \longrightarrow \mathscr{B}(V)$ is a completely positive, normal, unital and surjective bounded linear map.*

Proof. Apply Proposition A.2.17 for $M = N = \mathscr{B}(H)$ and $p = \pi_V$. □

Reducing subspaces. Proposition A.2.19 shows compression maps satisfy 1) in Definition A.2.11. Reducing subspaces, resp. reducible operators, yield the definition domains for concrete compression maps. Note Equation A.44 below reduces to the obvious commutation relation in the bounded case.

Let H be a Hilbert space.

Definition A.2.20. Let $V \subset H$ be a Hilbert subspace.

1) We say that $T \in \mathscr{UB}(H)_h$ is V-reducible and call V a reducing subspace of T if

$$\pi_V T \subset T\pi_V. \tag{A.44}$$

2) Let $\mathscr{UB}_V(H)$ be the set of all V-reducible $T \in \mathscr{UB}(H)_h$. For all $T \in \mathscr{UB}(H)$, set

$$T|_V := \mathrm{com}_V T. \tag{A.45}$$

Remark A.2.21. Note $\mathscr{UB}(V) \subset \mathscr{UB}_V(H)$. For all $T \in \mathscr{UB}(V)$, get $T|_V = \mathrm{com}_V T = T$.

Notation A.2.22. Let $V \subset H$ be a Hilbert subspace. For all $T \in \mathscr{UB}(H)$, we write $T|_V$ if we consider $\mathrm{com}_V T$ as operator on V.

Let $V \subset H$ be a Hilbert subspace. Lemma 9.8.4 in [88] shows we have $T \in \mathscr{UB}_V(H)$ if and only if $\pi_V(\mathrm{dom}\, T) \subset \mathrm{dom}\, T$ and $T\pi_V(\mathrm{dom}\, T) \subset V$. Since $\mathscr{UB}_V(H) = \mathscr{UB}_{V^\perp}(H)$, we may replace V with $V^\perp$ in all statements concerning reducing subspaces.

Example A.2.23. If $T \in \mathscr{UB}(H)_h$, then T is reduced by $\overline{\mathrm{im}\, T} := \overline{\mathrm{im}\, T}^{\|.\|_H}$ and $\ker T$.

Proposition A.2.24. *Let $V \subset H$ be a Hilbert subspace.*

1) For all $T \in \mathscr{UB}_V(H)$, we have

1.1) $T|_V \in \mathscr{UB}(V)_h$ and $T\pi_V = \mathrm{com}_V T$,

1.2) $T|_V \in \mathscr{UB}(V)_+$ if $T \in \mathscr{UB}(H)_+$,

1.3) $T = \mathrm{com}_V T + \mathrm{com}_{V^\perp} T$,

2) For all Hilbert subspaces $W \subset V$, we have

2.2) $\mathrm{com}_W = \mathrm{com}_W \circ \mathrm{com}_V$,

2.3) $\mathscr{UB}_W(H) \subset \mathscr{UB}_V(H)$.

Proof. Theorem 9.8.3 in [88] implies 1) at once. We directly verify 2). □

Proposition A.2.25. *If $V \subset H$ is a Hilbert subspace, then com_V is resolvent-preserving using $\mathscr{D}(\mathrm{com}_V) = \mathscr{UB}_V(H)$.*

Proof. We directly verify $\mathrm{com}_V : \mathscr{UB}(H) \longrightarrow \mathscr{UB}(V)$ is linear. Proposition A.2.19 and 1.1) in Proposition A.2.24 immediately imply all conditions except 2) in Definition A.2.11 are satisfied. We show the latter.

Let $T \in \mathscr{UB}_V(H)$. Since $T|_V \in \mathscr{UB}(V)_h$ and thereby $\pm i \in \operatorname{rsl}(T|_V)$, get

$$\operatorname{im} T|_V \mp iI_V = \operatorname{dom} R_{\pm i}(T|_V) = V. \tag{A.46}$$

For all $v \in V$, let $u_v \in \operatorname{im} T|_V \mp iI_V \subset V$ s.t. $v = (T|_V \mp iI_V)(u_v)$. We calculate

$$\operatorname{com}_V R_{\pm i}(T)(v) = \pi_V\Big(R_{\pm i}(T)\big(\pi_V\big(T(u_v) \mp iu_v\big)\big)\Big) = \pi_V\Big(R_{\pm i}(T)\big(T(u_v) \mp iu_v\big)\Big) = u_v. \tag{A.47}$$

Injectivity of $T|_V \mp iI_V$ ensures Equation A.47 implies

$$\operatorname{com}_V R_{\pm i}(T) = R_{\pm i}(T|_V). \tag{A.48}$$

Under canonical inclusion $\mathscr{UB}(V) \subset \mathscr{UB}(H)$ mapping $S \mapsto \pi_V S \pi_V$, note $T|_V = \operatorname{com}_V T$ by definition. Equation A.48 therefore shows

$$\operatorname{com}_V R_{\pm i}(T) = \operatorname{com}_V R_{\pm i}(\operatorname{com}_V T) = \pi_V R_{\pm i}(\operatorname{com}_V T)\pi_V. \tag{A.49}$$

This is 2) in Definition A.2.11. □

Lemma A.2.26. *If $T \in \mathscr{UB}_V(H)$, then $T|_V \in \mathscr{UB}(V)_h$ and we have*

1) $\operatorname{spec} T|_V \subset \operatorname{spec} T$ *and* $\mathscr{N}\big(E_{T|_V}\big) \subset \mathscr{N}(E_T)$,

2) *normal unital surjective* **-homomorphism* $\operatorname{com}_V : W^*(T) \longrightarrow W^*(T|_V)$ *s.t.*

$$\operatorname{com}_V g(T) = g(T)|_V = g(T|_V) = \operatorname{com}_V g(\operatorname{com}_V T) \tag{A.50}$$

for all $g \in L^\infty(\operatorname{spec} T, dE_T)$,

3) *commutative diagram of normal unital surjective* **-homomorphisms*

$$\begin{tikzcd} L^\infty(\operatorname{spec} T, dE_T) \arrow[r, "\Gamma_T"] \arrow[d, "\operatorname{res}"] & W^*(T_0) \arrow[d, "\operatorname{com}_V"] \\ L^\infty\big(\operatorname{spec} T|_V, dE_{T|_V}\big) \arrow[r, "\Gamma_{T|_V}"] & W^*(T|_V) \end{tikzcd} \tag{A.51}$$

with res *the restriction map given by* $\operatorname{spec} T|_V \subset \operatorname{spec} T$.

Proof. Proposition A.2.25 shows Lemma A.2.12 applies. Thus Lemma A.1.92 applies to $\phi = \operatorname{com}_V$ since $\mathscr{D}(\operatorname{com}_V) = \mathscr{UB}_V(H)$ by hypothesis, hence our claims follow. □

Corollary A.2.27. *Let $T \in \mathscr{UB}_V(H)$.*

1) $W^*(T) \subset \mathscr{UB}_V(H)$.

2) *For all* $g \in L^\infty(\operatorname{spec} T, dE_T)$, $g(T) = g(T|_V) \oplus g(T|_{V^\perp}) \in \mathscr{B}(V) \oplus \mathscr{B}(V^\perp) \subset \mathscr{B}(H)$.

3) *If* $T = \text{sr-lim}_{n\in\mathbb{N}} T_n$ *on* H *and* $\{T_n\}_{n\in\mathbb{N}} \subset \mathscr{UB}_V(H)$, *then* $T|_V = \text{sr-lim}_{n\in\mathbb{N}} T_n|_V$ *on* V.

Proof. Get 1) and 2) by Proposition A.2.24 and Lemma A.2.26. Proposition A.2.25 shows 3) is 1) in Lemma A.2.12 applied to $\phi = \mathrm{com}_V$ using $\mathscr{D}(\mathrm{com}_V) = \mathscr{UB}_V(H)$. □

Corollary A.2.28. *Let $T \in \mathscr{UB}(H)_h$. We have $T \in \mathscr{UB}_V(H)$ if and only if $[E_T(Z), \pi_V] = 0$ for all $Z \in \mathfrak{B}(\mathbb{R})$. If $T \in \mathscr{UB}_V(H)$, then $[g(T), \pi_V] = 0$ for all $g \in L^\infty(\operatorname{spec} T, dE_T)$.*

Proof. If $T \in \mathscr{UB}_V(H)$, then $[E_T(Z), \pi_V]$ for all $Z \in \mathfrak{B}(\mathbb{R})$ by 2) in Corollary A.2.27. The converse is Lemma 9.8.6 in [88]. Apply 2) in Corollary A.2.27 for our final claim. □

Lemma A.2.29. *If $T, S \in \mathscr{UB}_V(H)$ commute strongly, then $T|_V, S|_V \in \mathscr{UB}(V)_h$ commute strongly and we have*

1) $\operatorname{spec} T|_V \times S|_V \subset \operatorname{spec} T \times S$ *and* $\mathscr{N}\big(E_{T,S}\big) \subset \mathscr{N}\big(E_{T|_V, S|_V}\big)$,

2) *normal unital surjective* **-homomorphism* $\mathrm{com}_V : W^*(T,S) \longrightarrow W^*(T|_V, S|_V)$ *s.t.*

$$\mathrm{com}_V\, g(T,S) = g(T,S)|_V = g(T|_V, S|_V) = \mathrm{com}_V\, g(\mathrm{com}_V T, \mathrm{com}_V S) \tag{A.52}$$

for all $g \in L^\infty\big(\operatorname{spec} T \times S, dE_{T,S}\big)$,

3) *commutative diagram of normal unital surjective* **-homomorphisms*

$$\begin{tikzcd} L^\infty\big(\operatorname{spec} T \times S, dE_{T,S}\big) \arrow[r, "\Gamma_{T,S}"] \arrow[d, "\mathrm{res}"] & W^*(T,S) \arrow[d, "\mathrm{com}_V"] \\ L^\infty\big(\operatorname{spec} T|_V \times S|_V, dE_{T|_V, S|_V}\big) \arrow[r, "\Gamma_{T|_V, S|_V}"] & W^*(T|_V, S|_V) \end{tikzcd} \tag{A.53}$$

with res *the restriction map given by* $\operatorname{spec} T_1 \times S_1 \subset \operatorname{spec} T_0 \times S_0$.

Proof. Let $T, S \in \mathscr{UB}_V(H)$ commute strongly. Apply 2) in Corollary A.2.27, which uses C^*-algebra direct sum, to show $T|_V, S|_V \in \mathscr{UB}(V)_h$ commute strongly. Lemma A.2.26 shows our claims follow from Lemma A.1.101 if

$$\mathrm{com}_V = \mathrm{com}_V \otimes \mathrm{com}_V \tag{A.54}$$

on $W^*(T,S) = W^*(T) \otimes W^*(S)$ as per Equation A.33. Since Proposition A.2.19 ensures com_V has normal extension from $W^*(T) \odot W^*(S)$, we directly verify Equation A.54 on elementary tensors using 2) in Corollary A.2.27. □

Spectral gaps. Lemma A.2.35 states spectral gaps are upper semi-continuous in strong resolvent convergence. Corollary A.2.36 shows spectral gaps of local positive unbounded operators are limits of spectral gaps of compressions.

Let H be a Hilbert space.

Proposition A.2.30. *Let $T \geq S \geq 0$ in $\mathcal{UB}(H)$. If S is injective, then T is injective and $S^{-1} \geq T^{-1} \geq 0$ in $\mathcal{UB}(H)$.*

Proof. Let $L \in \mathcal{UB}(H)_+$ be injective. For all $\varepsilon_1 \geq \varepsilon_0 > 0$ in $\mathbb{R}$, functional calculus yields

$$0 < R_{-\varepsilon_1}(L) \leq R_{-\varepsilon_0}(L) \leq L^{-1}. \tag{A.55}$$

Note Equation A.55 gives monotonically increasing sequence $\{R_{-n^{-1}}(L)\}_{n\in\mathbb{N}} \subset \mathcal{B}(H)$ of uniformly positive and bounded operators.

The Kato-Robinson theorem (cf. Theorem 10.4.2 in [88]) shows

$$L^{-1} = \operatorname{sr-lim}_{n\in\mathbb{N}} R_{-n^{-1}}(L), \tag{A.56}$$

and we obtain unique closed positive unbounded quadratic form

$$u \mapsto \|\sqrt{L^{-1}}(u)\|_H^2 = \sup_{n\in\mathbb{N}} \langle R_{-n^{-1}}(L)(u), u\rangle_H \in [0,\infty] \tag{A.57}$$

on H represented by L^{-1}.

Let S be injective. Then T is injective by partial order. Applying Equation A.56 and Equation A.57 to T, resp. S, we calculate

$$\begin{aligned}\|\sqrt{T^{-1}}(u)\|_H^2 &= \sup_{n\in\mathbb{N}} \langle R_{-n^{-1}}(T)(u), u\rangle_H \\ &\leq \sup_{n\in\mathbb{N}} \langle R_{-n^{-1}}(S)(u), u\rangle_H \\ &= \|\sqrt{S^{-1}}(u)\|_H^2\end{aligned}$$

for all $u \in H$. The above calculation implies Theorem 9.3.7 in [88] yields $S^{-1} \geq T^{-1}$. □

For all $T \in \mathcal{UB}(H)_+$, we know $\operatorname{spec} T \subset [0,\infty)$ by definition of partial order.

Definition A.2.31. Let $T \in \mathcal{UB}(H)_+$.

1) The spectral gap of T is $\sigma(T) := \inf\{\lambda > 0 \mid \lambda \in \operatorname{spec} T\}$.

2) We say that T has spectral gap if $\sigma(T) > 0$.

Proposition A.2.32. *If $T \in \mathcal{UB}_V(H)$, then $\sigma(T) \leq \sigma(T|_V)$.*

Proof. Apply 1) in Lemma A.2.26. □

Lemma A.2.33. *For all $T \in \mathscr{UB}(H)_+$, we have*

$$\sigma(T) = \sup\left\{\lambda \geq 0 \;\middle|\; T|_{\overline{\operatorname{im} T}} \geq \lambda \pi_{\overline{\operatorname{im} T}}\right\}. \tag{A.58}$$

Proof. We write $\overline{\operatorname{im} T} = \overline{\operatorname{im} T}^{\|\cdot\|_H}$ as per Example A.2.23. Note $\pi_{\overline{\operatorname{im} T}} = E_T((0,\infty))$ as $T|_{\overline{\operatorname{im} T}}$ is injective by Proposition A.1.88. For all $Z \in \mathfrak{B}(\mathbb{R})$, Lemma A.2.26 implies

$$E_{T|_{\overline{\operatorname{im} T}}}(Z) = E_T((0,\infty)) \cdot E_T(Z) \cdot E_T((0,\infty)) = E_T(Z \cap (0,\infty)). \tag{A.59}$$

Positivity ensures $\operatorname{supp} E_T = \operatorname{supp} T \subset [0,\infty)$ and $\operatorname{supp} E_{T|_{\overline{\operatorname{im} T}}} = \operatorname{spec} T|_{\overline{\operatorname{im} T}} \subset [0,\infty)$. Then Equation A.59 implies we have $\operatorname{spec} T|_{\overline{\operatorname{im} T}} = \operatorname{spec} T$ if and only if $\sigma(T) = 0$, as well as $\operatorname{spec} T|_{\overline{\operatorname{im} T}} = \operatorname{spec} T \setminus \{0\}$ if and only if $\sigma(T) > 0$. Set

$$\zeta(T) := \sup\left\{\lambda \geq 0 \;\middle|\; T|_{\overline{\operatorname{im} T}} \geq \lambda \pi_{\overline{\operatorname{im} T}}\right\}. \tag{A.60}$$

Assume $\sigma(T) = 0$. Thus $0 \in \operatorname{spec} T = \operatorname{spec} T|_{\overline{\operatorname{im} T}}$ by $\operatorname{spec} T$ closed. If $\zeta(T) > 0$, then there exists $\lambda > 0$ s.t.

$$T|_{\overline{\operatorname{im} T}} \geq \lambda \pi_{\overline{\operatorname{im} T}} > 0 \tag{A.61}$$

in $\mathscr{UB}(\overline{\operatorname{im} T})$. Get $0 \notin \operatorname{spec} T|_{\overline{\operatorname{im} T}}$ by Proposition A.2.30 and Equation A.61. We obtain $0 = \sigma(T) = \zeta(T)$ as claimed.

Assume $\sigma(T) > 0$. Since their spectra are closed, positive unbounded operators are injective with closed image if and only if they are bounded from below. Thus $\operatorname{im} T$ is closed as $T|_{\overline{\operatorname{im} T}}$ is positive and injective. Closedness further shows $\sigma(T) \in \operatorname{spec} T$.

Get $\operatorname{rsl} T|_{\operatorname{im} T} = \operatorname{rsl} T \cup \{0\}$ by $\operatorname{spec} T|_{\operatorname{im} T} = \operatorname{spec} T \setminus \{0\}$. Thus $[0,\sigma(T)) \subset \operatorname{rsl} T|_{\operatorname{im} T}$, hence $\operatorname{supp} E_{T|_{\operatorname{im} T}} = \operatorname{spec} T|_{\operatorname{im} T}$ shows we have $\operatorname{id}_{\mathbb{R}} - \lambda \geq 0$ $E_{T|_{\operatorname{im} T}}$-a.e. for all $\lambda \in [0,\sigma(T))$. We see $T|_{\operatorname{im} T} \geq \lambda \pi_{\operatorname{im} T}$ for all $\lambda \in [0,\sigma(T))$ by functional calculus and therefore $\sigma(T) \leq \zeta(T)$ by continuity. We show the converse. For all $\lambda \in [0,\zeta(T))$, Equation A.60 implies

$$T|_{\operatorname{im} T} \geq \lambda \pi_{\operatorname{im} T}. \tag{A.62}$$

We claim $[0,\zeta(T)) \subset \operatorname{rsl} T$. If this holds, then $\sigma(T) \geq \zeta(T)$. Let $\lambda \in [0,\zeta(T))$. Since we have $0 < \sigma(T) \leq \zeta(T)$, there exists $\delta > 0$ s.t. $\lambda + \delta < \zeta(T)$. Equation A.62 shows

$$T|_{\operatorname{im} T} \geq (\lambda + \delta) \cdot \pi_{\operatorname{im} T}. \tag{A.63}$$

Subtracting $\lambda \pi_{\operatorname{im} T}$ in Equation A.63 shows $T|_{\operatorname{im} T} - \lambda \pi_{\operatorname{im} T} \geq \delta \pi_{\operatorname{im} T}$ for $\delta > 0$ and $\pi_{\operatorname{im} T} = I_{\operatorname{im} T}$. Thus $T|_{\operatorname{im} T} - \lambda \pi_{\operatorname{im} T} > 0$ in $\mathscr{UB}(\operatorname{im} T)$, hence $R_\lambda(T|_{\operatorname{im} T}) \in \mathscr{B}(\operatorname{im} T)$ as well. Using $\operatorname{rsl} T|_{\operatorname{im} T} = \operatorname{rsl} T \cup \{0\}$, we see $\lambda \in \operatorname{rsl} T$ since $\lambda > 0$. □

Corollary A.2.34. *For all $T \in \mathscr{UB}(H)_+$, we either have*

1) $\sigma(T) = 0$ *and* $\overline{\operatorname{im} T} \neq \operatorname{im} T$*,*

2) *or* $\sigma(T) > 0$ *and* $\overline{\operatorname{im} T} = \operatorname{im} T$*.*

Proof. Since $\operatorname{spec} T|_{\overline{\operatorname{im} T}}$ is closed, the injective positive unbounded operator $T|_{\overline{\operatorname{im} T}}$ has closed image if and only if it is bounded from below. Apply Lemma A.2.33. □

Lemma A.2.35. *Let $T = \text{sr-lim}_{n\in\mathbb{N}} T_n$ on H. If $\{T_n\}_{n\in\mathbb{N}} \subset \mathscr{UB}(H)_+$, then $T \in \mathscr{UB}(H)_+$ and we have*

$$\limsup_{n\in\mathbb{N}} \sigma(T_n) \leq \sigma(T). \tag{A.64}$$

Proof. Corollary 10.2.2 in [88] implies $T \in \mathscr{UB}(H)_+$. If $\limsup_{n\in\mathbb{N}} \sigma(T_n) = 0$, then our claim follows. We assume $\limsup_{n\in\mathbb{N}} \sigma(T_n) > 0$ without loss of generality. Let $\{\sigma(T_{n_k})\}_{k\in\mathbb{N}}$ be a converging subsequence s.t. $\lambda := \lim_{k\in\mathbb{N}} \sigma(T_{n_k}) > 0$. For all $\varepsilon \in (0,\lambda)$, let $k_\varepsilon \in \mathbb{N}$ s.t. $\{\sigma(T_{n_k})\}_{k\geq k_\varepsilon} \subset (\lambda-\varepsilon, \lambda+\varepsilon)$. Get $(0, \lambda-\varepsilon) \subset \bigcap_{k\geq k_\varepsilon} \operatorname{rsl} T_{n_k}$. Theorem 10.2.1 in [88] applies to this inclusion as $T = \text{sr-lim}_{k\in\mathbb{N}} T_{n_k}$, implying $(0, \lambda-\varepsilon) \subset \operatorname{rsl} T$ for all $\varepsilon \in (0,\lambda)$. Letting $\varepsilon \downarrow 0$ shows $(0,\lambda) \subset \operatorname{rsl} T$. Altogether, we estimate $\lambda \leq \sigma(T)$ for all non-zero accumulation points λ of $\{\sigma(T_n)\}_{n\in\mathbb{N}} \subset [0,\infty)$. This is Equation A.64. □

Corollary A.2.36. *Let $H_1 \subset H_2 \subset \ldots \subset H$ be Hilbert subspaces s.t. $H = \overline{\bigcup_{n\in\mathbb{N}} H_n}^{\|.\|_H}$. If $T \in \mathscr{UB}(H)_+$ is H_n-reducible for all $n \in \mathbb{N}$, then $T = \text{sr-lim}_{n\in\mathbb{N}} \operatorname{com}_{H_n} T$ and we have*

1) $\operatorname{com}_{H_n} T \in \mathscr{B}(H)_+$, $T|_{H_n} \in \mathscr{B}(H_n)_+$ *and* $\sigma\big(\operatorname{com}_{H_n} T\big) = \sigma\big(T|_{H_n}\big)$ *for all* $n \in \mathbb{N}$*,*

2) *monotonically decreasing sequence* $\big\{\sigma\big(T|_{H_n}\big)\big\}_{n\in\mathbb{N}} \subset [0,\infty)$*,*

3) $\sigma(T) = \lim_{n\in\mathbb{N}} \sigma\big(T|_{H_n}\big)$*.*

Proof. Let $T \in \mathscr{UB}(H)_+$ be H_n-reducible for all $n \in \mathbb{N}$. Using 2) in Corollary A.2.27 and $I_H = \text{s-lim}_{n\in\mathbb{N}} \pi_{H_n}$, get $T = \text{sr-lim}_{n\in\mathbb{N}} \operatorname{com}_{H_n} T$. Moreover, we see $T \in \mathscr{UB}(H)_+$ and 1.1) in Proposition A.2.24 show $\operatorname{com}_{H_n} T \in \mathscr{B}(H)_+$ and $T|_{H_n} \in \mathscr{B}(H_n)_+$ as per Notation A.2.22 for all $n \in \mathbb{N}$. Proposition A.2.32 and Lemma A.2.35 thus imply our claims if

$$\sigma\big(\operatorname{com}_{H_n} T\big) = \sigma\big(T|_{H_n}\big) \tag{A.65}$$

for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$. For all $\lambda \in \mathbb{R}$, we decompose

$$\operatorname{com}_{H_n} T - \lambda I_H = \big(T|_{H_n} - \lambda I_{H_n}\big) \oplus -\lambda I_{H_n^\perp} \tag{A.66}$$

w.r.t. $\mathscr{B}(H_n) \oplus \mathscr{B}(H_n^\perp)$. If $H_n = H$, then there is nothing to show. We assume $H_n \neq 0$ without loss of generality. Using decomposition as per Equation A.66, we directly verify Equation A.65 by definition of spectra. □

B Noncommutative Measure and Integration Theory

Theorem B.2.44 states sufficient conditions for compressing joint functional calculus pulled-back to joint functional calculus of self-adjoint measurable operators. The latter are noncommutative measurable functions. Tracial W^*-algebras define such spaces of measurable operators. For all $p \in [1,\infty]$, we define noncommutative L^p-spaces of measurable operators equipped with L^p-norm [130][161]. They fulfil Hölder inequalities. We have a modified standard pairing encoding duality [193].

In Section B.1, we discuss tracial W^*-algebras, spaces of measurable operators and noncommutative integration theory. We study canonical left- and right-actions of spaces of measurable operators. In Section B.2, we prove Theorem B.2.44 using compression maps given by change of canonical left- and right-actions. We formulate compressed pulled-backed joint functional calculus of self-adjoint measurable operators.

B.1 Spaces of measurable operators

In Subsection B.1.1, we discuss tracial C^*- and W^*-algebras. The GNS-construction for traces defines canonical left-actions. Each is a faithful normal unital *-representation over noncommutative L^2-space, i.e. Hilbert space given by GNS-construction. Tracial C^*-algebras are a preliminary step useful for the AF-C^*-setting. Canonical right-actions are canonical left-actions of opposite tracial W^*-algebras.

In Subsection B.1.2, we discuss spaces of measurable operators and noncommutative integration theory. Spaces of measurable operators are uniformly completed *-algebras in measure topologies of tracial W^*-algebras. Traces extend. For all $p \in [1,\infty]$, we define noncommutative L^p-spaces via L^p-norms using traces of measurable operators. Hölder inequalities apply and we have a modified standard pairing.

In Subsection B.1.3, we further extend canonical left- and right-actions to spaces of measurable operators using *-algebra multiplication. We account for noncommutative L^2-spaces different from Hilbert spaces given by GNS-construction. Whereas canonical left-actions represent *-algebras of measurable operators, canonical right-actions are twisted canonical left-actions defined on opposite *-algebras. Using canonical left- and right-actions, we define spectral and joint spectral measures of self-adjoint measurable operators. This lets us formulate their bounded measurable joint functional calculus.

B.1.1 Tracial C*- and W*-algebras

Tracial W^*-algebras have f.s.n. traces. Applying GNS-construction, each is represented over noncommutative L^2-space via canonical left-actions. Canonical right-actions arise using opposite tracial W^*-algebras. Remark B.1.65 explains there is no twisting in the bounded case. Standard references for tracial C^*- and W^*-algebras are [96] and [192] [193]. Note [193] discusses general weights as an extension of the tracial case.

Tracial W*-algebras and canonical left-actions. In Subsection A.1.2, we cover C^*- and W^*-algebras. Definition A.1.24 fixes partial orders.

Definition B.1.1. Let A be a C^*-algebra. Set $\infty \cdot 0 = 0 \cdot \infty = 0$ as convention.

1) A map $\tau : A_+ \longrightarrow [0,\infty]$ is a trace on A if

   1.1) $\tau(x+y) = \tau(x) + \tau(y)$ for all $x, y \in A_+$, (Linearity)

   1.2) $\tau(\lambda x) = \lambda\tau(x)$ for all $x \in A_+$ and $\lambda \geq 0$, (Homogeneity)

   1.3) $\tau(x^*x) = \tau(xx^*)$ for all $x \in A$. (Traciality)

2) Let τ be a trace on A. We say that τ is

   2.1) l.s.c. if it is l.s.c. in $\|.\|_A$,

   2.2) faithful if $\tau(x) = 0$ implies $x = 0$ for all $x \in A_+$,

   2.3) semi-finite if $\tau(x) = \sup\left\{\tau(y) \mid y \in A_+ :\ y \leq x,\ \tau(y) < \infty\right\}$ for all $x \in A_+$.

3) Let τ be a faithful trace on A. Set $\mathfrak{n}_\tau := \left\{x \in A \mid \tau(x^*x) < \infty\right\}$. We call

$$\mathfrak{m}_\tau := \left\{x \in A \;\middle|\; \exists \{y_k\}_{k=1}^n, \{z_k\}_{k=1}^n \subset \mathfrak{n}_\tau :\ x = \sum_{k=1}^n y_k^* z_k\right\} \tag{B.1}$$

   the definition domain of τ.

4) We call (A,τ) a tracial C^*-algebra if τ is a l.s.c. faithful semi-finite trace on A.

5) Let (A,τ) be a tracial C^*-algebra and $\phi : \mathfrak{m}_\tau \longrightarrow A$. We say that ϕ is a dilation if $0 \leq \tau(\phi(x)) \leq \tau(x)$ for all $x \in \mathfrak{m}_\tau \cap A_+$. We call ϕ trace-, or τ-preserving if $\tau(\phi(x)) = \tau(x)$ for all $x \in \mathfrak{m}_\tau$.

Let A be a C^*-algebra and τ a faithful trace on A. Note $\mathfrak{n}_\tau, \mathfrak{m}_\tau = \mathfrak{n}_\tau^2 = \langle \mathfrak{m}_\tau \cap A_+ \rangle_{\mathbb{C}} \subset A$ are self-adjoint two-sided ideals (cf. Lemma 4.5.1 and Proposition 6.1.2 in [96]). There exists unique linear extension of τ to $\mathfrak{m}_\tau$ since $\tau(\mathfrak{m}_\tau \cap A_+) < \infty$ (cf. Proposition 6.1.2 in [96]). We denote extension by τ. For all $x, y \in \mathfrak{m}_\tau$, $|\tau(x)| < \infty$ and $\tau(xy) = \tau(yx)$. The notion of τ-preserving map as per 5) in Definition B.1.1 is well-defined.

Remark B.1.2. For all $x, y \in \mathfrak{m}_\tau$ self-adjoint, $x \geq y$ implies $\tau(x) \geq \tau(y)$. In this sense, τ is positivity-preserving. This corresponds to Definition A.1.24.

Let (A,τ) be a tracial C^*-algebra. The GNS-inner product of τ on $\mathfrak{n}_\tau$ is given by

$$\langle x,y\rangle_\tau := \tau(x^* y) \tag{B.2}$$

for all $x,y\in\mathfrak{n}_\tau$. For all $x\in\mathfrak{n}_\tau$, faithfulness shows $\|x\|_\tau = 0$ if and only if $x=0$. Its Hilbert space completion is noncommutative L^2-space $\mathscr{H}(A,\tau)$. For all $x\in A$, set $\mathscr{L}_x(y) := xy\in\mathfrak{n}_\tau$ for all $y\in\mathfrak{n}_\tau$ and extend to $\mathscr{L}_x\in\mathscr{B}(\mathscr{H}(A,\tau))$. This is the GNS-construction for τ. Thus $\mathscr{L}$ is a semi-cyclic *-representation (cf. Theorem I.9.14 in [192] and Definition VII.1.5 in [193]), hence a faithful *-representation of A over $\mathscr{H}(A,\tau)$. It is non-degenerate by l.s.c. (cf. Lemma VII.4.1 in [193]). We see unitality of A implies that of $\mathscr{L}$.

Definition B.1.3. For all tracial C^*-algebras (A,τ), we call $\mathscr{H}(A,\tau) := \overline{\mathfrak{n}_\tau}^{\|\cdot\|_\tau}$ the concrete noncommutative L^2-space and $\mathscr{L}$ the canonical left-action of A on $\mathscr{H}(A,\tau)$.

Remark B.1.4. Canonical right-actions are given in Definition B.1.14. For this, we use the opposite *-algebra construction given in Definition B.1.15. Note Definition B.1.55 subsumes canonical left- and right-actions in this subsection.

Traces on W^*-algebras must have canonical normal left-action in order to preserve bounded measurable functional calculus. Faithful, semi-finite and normal traces, or f.s.n. traces on W^*-algebras have canonical normal left-action. Tracial W^*-algebras are all W^*-algebras equipped with an f.s.n. trace.

Proposition A.1.34, fundamentally a useful reformulation of the double commutant theorem [135][192], states double commutants of concrete C^*-algebra are, up to normal faithful unital *-representations, all W^*-algebras. Proposition B.1.7 implies each tracial C^*-algebra induces unique f.s.n. extension of their trace to the double commutant of their image C^*-algebra. Finally, Proposition B.1.9 ensures each tracial W^*-algebra is a tracial C^*-algebra with image C^*-algebra being its own double commutant. We thereby reduce from tracial C^*- to tracial W^*-algebras.

Definition B.1.5. Let M be a W^*-algebra.

1) A trace τ on M is normal if for all bounded increasing nets $\{x_k\}_{k\in K}\subset M_+$, get

$$\tau\left(\sup_{k\in K} x_k\right) = \sup_{k\in K}\tau(x_k). \tag{B.3}$$

2) A trace τ on M is f.s.n. if it is faithful, semi-finite and normal.

3) We call (M,τ) a tracial W^*-algebra if τ is a f.s.n. trace on M.

Remark B.1.6. Equation B.3 corresponds to Equation A.12, i.e. normality for bounded linear maps of W^*-algebras. The two notions coincide assuming boundedness.

Proposition B.1.7. *Let (A,τ) be a tracial C^*-algebra.*

1) *There exists unique f.s.n. trace τ_∞ on $\mathscr{L}(A)''$ extending τ from A_+ to $\mathscr{L}(A)''_+$.*

2) *$\left(\mathscr{L}(A)'', \tau_\infty\right)$ is a tracial W^*-algebra.*

Proof. Get τ_∞ by applying Lemma 6.1.5 in [96] to $\mathscr{L}$ (cf. A.60 in [96]). We have $\tau_\infty = \tau$ on $A_+ \subset \mathscr{L}(A)''_+$ by Proposition 6.6.5 in [96]. Thus $(\mathscr{L}(A)'', \tau_\infty)$ is tracial W^*-algebra. $\square$

Notation B.1.8. Let (A,τ) be a tracial C^*-algebra. We write $\tau = \tau_\infty$ on $\mathscr{L}(A)''$.

Proposition B.1.9. *Let (M,τ) be a tracial W^*-algebra.*

1) *(M,τ) is a tracial C^*-algebra and $\mathscr{L}$ is faithful normal unital *-representation s.t. w^*-topology on M maps to σ-weak topology on $\mathscr{L}(M)$,*

2) *$\left(\mathscr{L}(M)'', \tau\right) = (\mathscr{L}(M), \tau)$.*

Proof. Normality of τ shows l.s.c. in σ-weak topology by Theorem VII.1.11 in [193]. Thus τ is l.s.c. in norm, hence (M,τ) is tracial C^*-algebra and we know $\mathscr{L}$ is faithful unital *-representation. Equation B.3 shows normality of $\mathscr{L}$. Its construction and normality then show $\mathscr{L}$ maps w^*-topology on M to σ-weak topology on $\mathscr{L}(M)$. Proposition A.1.34 implies $\mathscr{L}(M) = \mathscr{L}(M)''$ at once. $\square$

Proposition B.1.10. *If (A,τ) is a tracial C^*-algebra, then $\mathscr{H}(\mathscr{L}(A)'', \tau) = \mathscr{H}(\mathscr{L}(A), \tau)$.*

Proof. Apply Proposition B.1.7 and Proposition B.1.9. $\square$

Finite faithful traces on unital C^*-algebras are well-behaved.

Definition B.1.11. Let A be a C^*-algebra and τ a trace on A. We call τ finite if $\tau(x) < \infty$ for all $x \in A_+$ and further write $\tau < \infty$.

Proposition B.1.12. *Let A be a unital C^*-algebra and τ a faithful trace on A.*

1) *$\tau < \infty$ if and only if $\tau(1_A) < \infty$.*

2) *If $\tau < \infty$, then τ is semi-finite.*

3) *If $\tau < \infty$, then (A,τ) is a tracial C^*-algebra, $\tau \in A^*_+$ and $\ker\tau^\perp = \langle 1_A \rangle_{\mathbb{C}}$.*

Proof. If $\tau < \infty$, then $\tau(1_A) < \infty$. Assume $\tau(1_A) < \infty$. For all $x \in A_+$, get $x \leq \|x\|_A 1_A$ by functional calculus and therefore $|\tau(x)| \leq \|x\|_A \tau(1_A) < \infty$ by positivity-preservation on $\mathfrak{m}_\tau$ as per Remark B.1.2. Get 1). Assume τ is finite. Thus $A = \mathfrak{m}_\tau$, hence $\tau \in A^*_+$. We see τ is semi-finite and l.s.c. in norm. In particular, (A,τ) is a tracial C^*-algebra and we have $1_A \in \mathscr{H}(A,\tau)$. Since $\dim_{\mathbb{C}} \ker\tau^\perp = 1$ by $\tau \in A^*$ and $1_A \in \ker\tau^\perp$ by faithfulness, get $\ker\tau^\perp = \langle 1_A \rangle_{\mathbb{C}} \subset \mathscr{H}(A,\tau)$. Altogether, get 2) and 3). $\square$

Proposition B.1.13. *Let M be a W^*-algebra and τ a faithful normal trace on M. If $\tau < \infty$, then τ is f.s.n. trace on M.*

Proof. Apply 2) in Proposition B.1.12. $\square$

Opposite tracial W^*-algebras and canonical right-actions. Proposition B.1.7 and Proposition B.1.10 show canonical right-actions for tracial C^*-algebras reduce to tracial W^*-algebras. Let (M,τ) be a tracial W^*-algebra. For all $x \in M$, set $\mathscr{R}_x(y) := yx \in \mathfrak{n}_\tau$ for all $y \in \mathfrak{n}_\tau$ and extend to $\mathscr{R}_x \in \mathscr{B}(\mathscr{H}(M,\tau))$.

Definition B.1.14. Let (M,τ) be a tracial W^*-algebra. Following Definition B.1.3, we call $\mathscr{R}$ the canonical right-action of M on $\mathscr{H}(M,\tau)$.

Definition B.1.15 gives an opposite *-algebra construction. Proposition B.1.17 shows (M,τ) yields opposite tracial W^*-algebra (M^{op},τ) s.t. $\mathscr{H}(M^{\mathrm{op}},\tau) = \mathscr{H}(M,\tau)$. Using the latter, Proposition B.1.19 shows $\mathscr{R}$ is canonical left-action $\mathscr{L}^{\mathrm{op}}$ of M^{op} on $\mathscr{H}(M,\tau)$ and Proposition B.1.21 implies our discussion concerning canonical left-actions translates to canonical right-actions as per Diagram B.4.

Definition B.1.15. Let $\mathscr{A}$ be a *-algebra and $\mathrm{Adj} : \mathscr{A} \longrightarrow \mathscr{A}$ its algebra involution. Its opposite *-algebra $\mathscr{A}^{\mathrm{op}}$ has $\mathscr{A}$ as complex vector space and is equipped with

1) opposite algebra action given by $x \cdot^{\mathrm{op}} y := yx$ for all $x, y \in \mathscr{A}$,

2) $\mathrm{Adj} : \mathscr{A}^{\mathrm{op}} \longrightarrow \mathscr{A}^{\mathrm{op}}$ as algebra involution.

Remark B.1.16. If $\mathscr{A}$ is a topological vector space, then $\mathscr{A}^{\mathrm{op}}$ is one using the identical topology. For all W^*-algebras, we use identical norm and w^*-topology on opposites.

Proposition B.1.17. *For all tracial W^*-algebras (M,τ), we have*

1) τ is f.s.n. trace on M^{op} and (M^{op},τ) is a tracial W^*-algebra,

2) $\left(\mathscr{H}(M^{\mathrm{op}},\tau), \|.\|_{\tau,\mathrm{op}}\right) = (\mathscr{H}(M,\tau), \|.\|_\tau)$.

Proof. Note M^{op} is a C^*-algebra with norm and algebra involution of M. This implies $M^{\mathrm{op}} = M = (M_*)^*$ as Banach spaces. Thus M is a W^*-algebra s.t. $M_+^{\mathrm{op}} = M_+$, hence τ is f.s.n. trace on M^{op}. We obtain 1). Traciality moreover ensures $\mathfrak{n}_\tau$ defined by τ on M and M^{op} are identical. Get 2) by construction. □

Notation B.1.18. We write $\mathscr{L}^{\mathrm{op}}$ for the canonical left-action $\mathscr{L}^{\mathrm{op}}$ of M^{op} on $\mathscr{H}(M,\tau)$. We write $\mathfrak{n}_{\tau,\mathrm{op}}$ as per 3) in Definition B.1.1 for f.s.n. trace τ on M^{op}.

Proposition B.1.19 shows $\mathscr{R} = \mathscr{L}^{\mathrm{op}}$ on M^{op}. Note $\mathscr{R} \neq \mathscr{L}^{\mathrm{op}}$ in general as extensions of M and M^{op} are different spaces of measurable operators. We show $\mathscr{R} \cong \mathscr{L}^{\mathrm{op}}$ naturally extends the bounded case. For details on the latter, we refer to Subsection B.1.2.

Proposition B.1.19. *Let (M,τ) be a tracial W^*-algebra.*

1) $\mathscr{R} = \mathscr{L}^{\mathrm{op}}$ is faithful normal unital *-representation s.t. w^*-topology on M^{op} maps to σ-weak topology on $\mathscr{R}(M)$.

2) $\left(\mathscr{R}(M^{\mathrm{op}})'', \tau\right) = (\mathscr{L}(M)^{\mathrm{op}}, \tau)$.

Proof. For τ on M, resp. M^{op} traciality ensures $\mathfrak{n}_\tau = \mathfrak{n}_{\tau,\mathrm{op}}$. For all $x \in M$, we calculate $\mathscr{R}_x(y) = xy = y \cdot^{\mathrm{op}} x = \mathscr{L}_x^{\mathrm{op}}$ for all $y \in \mathfrak{n}_\tau$. Get 1) and 2) by Proposition B.1.9. □

We have anti-linear isometric involution $\mathrm{Adj} : \mathscr{H}(M,\tau) \longrightarrow \mathscr{H}(M,\tau)$ by closing $\mathrm{Adj}|_{\mathfrak{n}_\tau}$ w.r.t $\|.\|_\tau$. Get $\mathrm{Adj}^\dagger : \mathscr{B}(\mathscr{H}(M,\tau)) \longrightarrow \mathscr{B}(\mathscr{H}(M,\tau))$ as per Definition A.1.13.

Definition B.1.20. Let (M,τ) be a tracial W^*-algebra. $\mathrm{Adj} : \mathscr{H}(M,\tau) \longrightarrow \mathscr{H}(M,\tau)$ is called adjoining on $\mathscr{H}(M,\tau)$.

Proposition B.1.21. *Let (M,τ) be a tracial W^*-algebra. We have commutative diagram*

$$\begin{tikzcd} M \arrow[r, "\mathscr{L}"] \arrow[d, "\mathrm{Adj}"] & \mathscr{B}(\mathscr{H}(M,\tau)) \arrow[d, "\mathrm{Adj}^\dagger"] \\ M^{\mathrm{op}} \arrow[r, "\mathscr{R}"] & \mathscr{B}(\mathscr{H}(M,\tau)) \end{tikzcd} \tag{B.4}$$

s.t. horizontal maps are normal unital injective *-homomorphisms and vertical ones are isometric involutions of Banach spaces.

Proof. We directly verify Diagram B.4 and all claims. □

B.1.2 Noncommutative integration for tracial W*-algebras

We discuss spaces of measurable operators and noncommutative integration theory. Traces extend. For all $p \in [1,\infty]$, we define noncommutative L^p-spaces of measurable operators equipped with L^p-norm [130][161]. They fulfil Hölder inequalities. We have a modified standard pairing encoding duality [193]. In particular, tracial W^*-algebras are noncommutative L^∞-spaces and have noncommutative L^1-spaces as pre-duals. We see their f.s.n. traces are, possibly unbounded [170][171], noncommutative Radon measures. Standard references for their spaces of measurable operators and resulting notion of noncommutative integration are pp.1461-1470 in [130], [161] and [192][193].

Spaces of measurable operators. Let (M,τ) be a tracial W^*-algebra. Its space $L^0(M,\tau)$ of measurable operators is uniform completion in measure topology and serves as setting for noncommutative integration theory. For $p=\infty$, get $M \subset L^0(M,\tau)$. For all $p \in [1,\infty]$, get $L^p(M,\tau) \subset L^0(M,\tau)$ as per Definition B.1.41. Uniform completion extends the *-algebra structure and trace from M to $L^0(M,\tau)$ as per Remark B.1.24.

We thereby extend canonical left-action $\mathscr{L} : M \longrightarrow \mathscr{B}(\mathscr{H}(M,\tau))$ to an unbounded faithful unital *-representation $\mathscr{L} : L^0(M,\tau) \longrightarrow \mathscr{UB}(\mathscr{H}(M,\tau))$. Remark B.1.22 explains $\mathscr{L}$ does not equal canonical left-action of measurable operators in general.

Remark B.1.22. If we twist $\mathscr{L}$ as per Definition A.1.13 using the natural isometric isomorphism $\mathscr{H}(M,\tau) \cong L^2(M,\tau)$ implied by Proposition B.1.42, then we obtain canonical left-action L using left-multiplication in $L^0(M,\tau)$ as per Definition B.1.56 and based on Definition B.1.55. This subsumes canonical left-action in the bounded case.

Note $P(M)$ is the set of all projections in M. The measure topology of (M,τ) is defined by the following fundamental system of neighbourhoods of zero. For all $\varepsilon,\delta>0$, set

$$N(\varepsilon,\delta) := \Big\{x\in M \;\Big|\; \exists p\in P(M):\; \|xp\|_M<\varepsilon,\; \tau\big(p^\perp\big)<\delta\Big\}. \tag{B.5}$$

The fundamental system of entourages given by $U(\varepsilon,\delta) := \{(x,y)\in M\times M \mid x-y\in N(\varepsilon,\delta)\}$ for all $\varepsilon,\delta>0$ defines uniform structure of measure topology on M. Convergence in measure topology is called convergence in measure.

Definition B.1.23. Let $L^0(M,\tau)$ be the uniform closure of M in measure topology. We call it the space of measurable operators for (M,τ), or τ-measurable operator algebra.

Remark B.1.24. Theorem IX.2.2 in [193] shows the *-algebra structure of M extends to $L^0(M,\tau)$. Lemma IX.2.3 in [193] shows $L^0(M,\tau)$ is Hausdorff and $M\subset L^0(M,\tau)$.

We additionally have measure topology on $\mathscr{H}(M,\tau)$, as well as subsequent notion of convergence in measure. The measure topology of $\mathscr{H}(M,\tau)$ is defined by the following fundamental system of neighbourhoods of zero. For all $\varepsilon,\delta>0$, set

$$O(\varepsilon,\delta) := \Big\{u\in \mathscr{H}(M,\tau) \;\Big|\; \exists p\in P(M):\; \|p(u)\|_\tau<\varepsilon,\; \tau\big(p^\perp\big)<\delta\Big\}. \tag{B.6}$$

Uniform structure of measure topology on $\mathscr{H}(M,\tau)$ follows as for $L^0(N,\tau)$. Convergence in measure topology is called convergence in measure. $\mathscr{H}(M,\tau)$ is not complete.

Let $x\in L^0(M,\tau)$. We construct densely defined closed operator $\mathscr{L}_x$ on $\mathscr{H}(M,\tau)$. Let $\operatorname{dom}\mathscr{L}_x$ be the set of all $u\in\mathscr{H}(M,\tau)$ s.t. there exists a net $\{x_k\}_{k\in K}\subset M$ converging to x in measure and for which $\{x_k u\}_{k\in K}\subset\mathscr{H}(M,\tau)$ converges in measure to an element in $\mathscr{H}(M,\tau)$. For all $u\in\operatorname{dom}\mathscr{L}_x$, set

$$\mathscr{L}_x(u) := \lim_{k\in K}\mathscr{L}_{x_k}(u)\in\mathscr{H}(M,\tau) \tag{B.7}$$

using limit in measure topology on $\mathscr{H}(M,\tau)$. Equation B.7 defines $\mathscr{L}_x(u)$ independent of converging net $\{x_k\}_{k\in K}\subset M$. Theorem IX.2.5 in [193] shows $\mathscr{L}_x$ is a densely defined closed operator on $\mathscr{H}(M,\tau)$. Moreover, Proposition B.1.31 implies those operators as per Equation B.7 for all $x\in L^0(M,\tau)$ define unbounded faithful unital *-representation

$$\mathscr{L}: L^0(M,\tau)\longrightarrow \mathscr{U}\mathscr{B}(\mathscr{H}(M,\tau)). \tag{B.8}$$

Restricting to $M\subset L^0(M,\tau)$, we recover canonical left-action of M on $\mathscr{H}(M,\tau)$ as per Definition B.1.3. We understand *-algebra structure using $\mathscr{L}$. It has image the set of all τ-measurable operators on $\mathscr{H}(M,\tau)$. Their definition requires the notion of M-affiliated operator. The commutant $\mathscr{L}(M)'\subset\mathscr{B}(\mathscr{H}(M,\tau))$ is a W^*-algebra.

Definition B.1.25. A densely defined closed unbounded operator T on $\mathscr{H}(M,\tau)$ is called M-affiliated if $TU=UT$ for all unitaries $U\in\mathscr{U}(\mathscr{L}(M)')\subset\mathscr{B}(\mathscr{H}(M,\tau))$.

Proposition B.1.26. *If $T \in \mathscr{U}\mathscr{B}(\mathscr{H}(M,\tau))_h$, then we know T is M-affiliated if and only if $W^*(T) \subset \mathscr{L}(M)$.*

Proof. Note $W^*(T) = W^*(T, \mathscr{L}_{1_M})$ since $\mathscr{L}$ is unital. Proposition A.1.37 and 1) in Proposition A.1.85 imply spectral projections in T generate $W^*(T)$. Apply Lemma A.1.89. □

Remark B.1.27. For all densely defined closed operators T on a Hilbert space H, get T^*T self-adjoint and set absolute value $|T| := \sqrt{T^*T}$ (cf. Theorem 5.1.9 in [171]). If T is M-affiliated, then $E_{|T|}(Z) = \chi_Z(|T|) \in \mathscr{L}(M)$ for all $Z \in \mathfrak{B}(\mathbb{R})$ by Proposition B.1.26.

Following Notation B.1.8, let τ further denote the push-forward of $\tau : M_+ \longrightarrow [0,\infty]$ along $\mathscr{L}$ to $\mathscr{L}(M)$. The f.s.n. trace $\tau : \mathscr{L}(M) \longrightarrow [0,\infty]$ has definition domain $\mathscr{L}(\mathfrak{m}_\tau)$.

Definition B.1.28. We call an M-affiliated operator T on $\mathscr{H}(M,\tau)$ τ-measurable, or just measurable if there exists $\lambda > 0$ s.t.

$$\tau\big(E_{|T|}([\lambda,\infty))\big) < \infty. \tag{B.9}$$

Remark B.1.29. Corollary IX.2.9 in [193] ensures Definition B.1.28 is τ-measurability as used in [193]. Proposition B.1.30 further breaks down τ-measurability for self-adjoint unbounded operators on $\mathscr{H}(M,\tau)$.

Proposition B.1.30. *If $T \in \mathscr{U}\mathscr{B}(\mathscr{H}(M,\tau))_h$, then T is τ-measurable if and only if*

1) *$E_T(Z) \in \mathscr{L}(M)$ for all $Z \in \mathfrak{B}(\mathbb{R})$,*
2) *there exists $\lambda > 0$ s.t. $E_T((-\infty,-\lambda]), E_T([\lambda,\infty)) \in \mathscr{L}(\mathfrak{m}_\tau)$.*

Proof. Proposition B.1.26 at once implies 1) is equivalent to T being M-affiliated. For all $\lambda > 0$, get $\chi_{[\lambda,\infty)}(|t|) = \chi_{(-\infty,-\lambda]}(t) + \chi_{[\lambda,\infty)}(t)$ for all $t \in \mathbb{R}$. Equation B.9 therefore shows 2) is equivalent to τ-measurability for all self-adjoint M-affiliated operators. □

Proposition B.1.31 collects properties of $\mathscr{L}$. In particular, 3) states the maximality property. Using maximality and Remark B.1.32, we readily see extending the *-algebra structure of M to $L^0(M,\tau)$ yields a *-algebra. Note closure is necessary for this.

Proposition B.1.31. *We know each $\mathscr{L}_x$ is a τ-measurable operator on $\mathscr{H}(M,\tau)$ for all $x \in L^0(M,\tau)$ and furthermore have the following.*

1) *For all $x, y \in L^0(M,\tau)$ and $\lambda \in \mathbb{C}$, we have*
   1.1) $\mathscr{L}_{\lambda_1 x + \lambda_2 y} = \overline{\lambda_1 \mathscr{L}_x + \lambda_2 \mathscr{L}_y}$,
   1.2) $\mathscr{L}_{xy} = \overline{\mathscr{L}_x \mathscr{L}_y}$,
   1.3) $\mathscr{L}_{x^*} = \mathscr{L}_x^*$.
2) *If T is τ-measurable, then there exists unique $x \in L^0(M,\tau)$ s.t. $T = \mathscr{L}_x$.*
3) *If $x, y \in L^0(M,\tau)$ s.t. $\mathscr{L}_y \subset \mathscr{L}_x$, then $\mathscr{L}_x = \mathscr{L}_y$.*

Proof. Apply Theorem IX.2.5 in [193]. □

Remark B.1.32. For all densely defined closable operators T on a Hilbert space H, get T^* densely defined closed and $T^* = (T^*)^{**} = (\overline{T})^*$ by $\overline{T} = T^{**}$ (cf. Theorem 5.15 in [171]).

Partial order on $\mathscr{UB}(\mathscr{H}(M,\tau))_h$ is fixed by Definition A.1.11. We pull back partial order to $L^0(M,\tau)_h$ along $\mathscr{L}$. The set $L^0(M,\tau)_h$ of hermitian elements in Definition B.1.33 below is given using algebra involution.

Definition B.1.33. The hermitian, resp. positive elements in $L^0(M,\tau)$ are

$$L^0(M,\tau)_h := \left\{x \in L^0(M,\tau) \mid \mathscr{L}_x = \mathscr{L}_x^*\right\},\ L^0(M,\tau)_+ := \left\{x \in L^0(M,\tau)_h \mid \mathscr{L}_x \geq 0\right\}. \tag{B.10}$$

Notation B.1.34. Rather than hermitian, we say that $x \in L^0(M,\tau)_h$ is self-adjoint.

Remark B.1.35. Corollary IX.2.10 in [193] shows $L^0(M,\tau)_+$ is positive cone generating the partial order on $L^0(M,\tau)_h$. Proposition B.1.49 implies the set $L^0(M,\tau)_+$ of positive elements generates the partial order on $L^0(M,\tau)$ as per Definition A.1.1.

If an application of functional calculus preserves τ-measurability, then we obtain a unique element in $L^0(M,\tau)$ by 2) in Proposition B.1.31. Taking absolute values preserves τ-measurability. This lets us define generalised singular numbers by Equation B.11.

Definition B.1.36. Let $x \in L^0(M,\tau)_h$. If $g \in \mathscr{S}(E_{\mathscr{L}_x})$ s.t. $g(\mathscr{L}_x)$ is τ-measurable, then let $g(x) \in L^0(M,\tau)$ be the unique element s.t. $\mathscr{L}_{g(x)} = g(\mathscr{L}_x)$.

Proposition B.1.37. *Let $x \in L^0(M,\tau)$.*

1) If $x \in L^0(M,\tau)_h$ and $g \in L^\infty\big(\operatorname{spec}\mathscr{L}_x, dE_{\mathscr{L}_x}\big)$, then $g(\mathscr{L}_x)$ is τ-measurable.

2) If $x \in L^0(M,\tau)_+$ and $p \in [1,\infty)$, then $\mathscr{L}_x^p$ is τ-measurable.

3) $|\mathscr{L}_x|$ is τ-measurable and $|\mathscr{L}_x| = \mathscr{L}_{\sqrt{x^*x}}$.

Proof. If $x \in L^0(M,\tau)_h$ and $g \in L^\infty(\operatorname{spec}\mathscr{L}_x, dE_{\mathscr{L}_x})$, then $g(\mathscr{L}_x) \in \mathscr{L}(M)$ is τ-measurable by Proposition B.1.31. The latter further implies 3) if 2) holds. For this, merely apply 2) using $p = 2$. Get 2) since Equation B.9 demands fix but arbitrarily large $\lambda \in (0,\infty)$ while $\lambda^{-p} \uparrow \infty$ as $\lambda \uparrow \infty$ for all $p \in [1,\infty)$. □

Definition B.1.38. For all $x \in L^0(M,\tau)$, set $|x| := \sqrt{x^*x}$.

We extend τ to $L^0(M,\tau)_+$ (cf. pp.1461-1470 in [130]). The extension is linear. For all $x \in L^0(M,\tau)$, we have $E_{\mathscr{L}_{|x|}}([\lambda,\infty)) \in \mathscr{L}(M)$ as per Remark B.1.27. For all $x \in L^0(M,\tau)$, we define the generalised singular number $\mu(x) : (0,\infty) \longrightarrow [0,\infty)$ of x by setting

$$\mu_t(x) := \inf\left\{\lambda > 0 \mid \tau\big(E_{\mathscr{L}_{|x|}}([\lambda,\infty))\big) \leq t\right\} \tag{B.11}$$

for all $t > 0$.

Definition B.1.39. For all $x \in L^0(M,\tau)_+$, the trace of x is defined by

$$\tau(x) := \int_0^\infty \mu_t(x)dt. \tag{B.12}$$

Remark B.1.40. For all $x \in L^0(M,\tau)$ and $t > 0$, note Equation B.11 immediately shows we have $\mu_t(x) = \mu_t(x^*) = \mu_t(|x|)$ by definition.

Noncommutative L^p-spaces and integration. Extension of integration theory to the noncommutative setting is fundamental to our discussion. Proposition B.1.7 and Proposition B.1.10 reduce the case of tracial C^*- to tracial W^*-algebras.

Let (M,τ) be a tracial W^*-algebra. For all $p \in [1,\infty)$, 2) in Proposition B.1.37 and Equation B.12 let us define noncommutative L^p-spaces. For $p = \infty$, we use M.

Definition B.1.41. Let $p \in [1,\infty]$.

1) Assume $p < \infty$. The noncommutative L^p-space of (M,τ) is

$$L^p(M,\tau) := \left\{ x \in L^0(M,\tau) \mid \tau\left(|x|^p\right)^{\frac{1}{p}} < \infty \right\}. \tag{B.13}$$

For all $x \in L^p(M,\tau)$, set $\|x\|_p := \tau(|x|^p)^{\frac{1}{p}}$. We further call $\|.\|_p$ the noncommutative L^p-norm. The self-adjoint, resp. positive elements in $L^p(M,\tau)$ are

$$L^p(M,\tau)_h := L^p(M,\tau) \cap L^0(M,\tau)_h,\ L^0(M,\tau)_+ := L^p(M,\tau) \cap L^0(M,\tau)_+. \tag{B.14}$$

2) The noncommutative L^∞-space of (M,τ) is $L^\infty(M,\tau) := M$.

Proposition B.1.42. *For all $p \in [1,\infty]$, we have*

1) $\left(L^p(M,\tau), \|.\|_p\right)$ is a Banach space s.t. $M \cap L^p(M,\tau) \subset L^p(M,\tau)$ is $\|.\|_p$-dense,

2) $\mathrm{Adj} : L^p(M,\tau) \longrightarrow L^p(M,\tau)$ is anti-linear isometric involution,

3) $\tau \in L^1(M,\tau)^*$ s.t. $\tau(x) \geq 0$ for all $x \in L^1(M,\tau)_+$.

Proof. All claims are given by (i) and (ii) in Theorem IX.2.13 in [193]. Of course, its proof shows τ has linear extension to $L^1(M,\tau)$. We therefore have $\tau \in L^1(M,\tau)^*$ as claimed. □

As $\mathfrak{n}_\tau = M \cap L^2(M,\tau) \subset L^2(M,\tau)$ is $\|.\|_2$-dense by 1) in Proposition B.1.42, the identity $\mathrm{id} : (\mathfrak{n}_\tau, \|.\|_\tau) \longrightarrow (\mathfrak{n}_\tau, \|.\|_2)$ closes to an isometric isomorphism $\mathrm{id}_\tau : \mathscr{H}(M,\tau) \longrightarrow L^2(M,\tau)$. Equivalence classes w.r.t. $\|.\|_\tau$ are mapped to equivalence classes in uniform closure which are represented by square integrable τ-measurable operators.

Definition B.1.43. We call $\mathrm{id}_\tau : \mathscr{H}(M,\tau) \longrightarrow L^2(M,\tau)$ identity in measure topology.

Notation B.1.44. Let $\mathrm{id}_{\tau,\mathrm{op}} : (\mathscr{H}(M,\tau), \|.\|_\tau) \longrightarrow L^2(M^{\mathrm{op}},\tau)$ denote identity in measure topology using (M^{op},τ) instead. We may consider it by 2) in Proposition B.1.17.

We have $\mathrm{id}_\tau^{-\dagger} = (\mathrm{id}_\tau^{-1})^\dagger$ as per Definition A.1.13.

Proposition B.1.45. *The maps* id_τ *and* id_τ^{-1} *are continuous w.r.t. measure topology on* $L^0(M,\tau)$ *and* $\mathscr{H}(M,\tau)$*. For all* $x \in L^0(M,\tau)$*, we have*

1) $\mathrm{dom}\,\mathrm{id}_\tau^{-\dagger}(\mathscr{L}_x) = \{u \in L^2(M,\tau) \mid xu \in L^2(M,\tau)\}$,

2) $\mathrm{id}_\tau^{-\dagger}(\mathscr{L}_x)(u) = xu$ *in* $L^0(M,\tau)$ *for all* $u \in \mathrm{dom}\,\mathrm{id}_\tau^{-\dagger}(\mathscr{L}_x)$.

Proof. We directly verify continuity of id_τ and id_τ^{-1} in measure topologies on uniform structures given by Equation B.5 and Equation B.6. For all $x \in M$, our claims follow since they reduce to canonical left-action of M on $\mathscr{H}(M,\tau)$. Construction of $\mathscr{L}$ therefore implies the general case since id_τ and id_τ^{-1} are continuous in measure topologies. □

Proposition B.1.46. *We have* $x \in L^1(M,\tau)_+$ *if and only if* $\sqrt{x} \in L^2(M,\tau)_+$.

Proof. By definition of noncommutative L^1-, resp. L^2-spaces. □

Proposition B.1.47. *For all* $x \in L^0(M,\tau)$ *and* $p \in [1,\infty]$*, we have*

1) $x = \mathrm{Re}(x) + i\,\mathrm{Im}(x)$ *and* $\mathrm{Re}(x) := \frac{x+x^*}{2}, \mathrm{Im}(x) := -i\frac{x-x^*}{2} \in L^0(M,\tau)_h$,

2) $x = x_+ - x_-$ *for* $x_+ := \max\{x,0\}, x_- := -\min\{x,0\} \in L^0(M,\tau)_+$ *if* $x \in L^0(M,\tau)_h$,

3) $x \in L^p(M,\tau)$ *if and only if* $\mathrm{Re}(x)_+, \mathrm{Re}(x)_-, \mathrm{Im}(x)_+, \mathrm{Im}(x)_- \in L^p(M,\tau)$.

Proof. Get 1) by 1) in Proposition B.1.31. Get 2) by Proposition B.1.30 together with Proposition B.1.31. We see 3) follows from 2) since $|x|^p = (x_+)^p + (x_-)^p$ in each case. □

Remark B.1.48. Re and Im are $\mathbb{R}$-linear maps on $L^0(M,\tau)$. For all $x \in L^0(M,\tau)$, we have $x^* = \mathrm{Re}(x) - i\,\mathrm{Im}(x)$ by anti-linearity of taking adjoints.

Proposition B.1.49. $L^0(M,\tau)_+$ *generates the partial order on* $L^0(M,\tau)$.

Proof. We use $L^0(M,\tau)_h$ as hermitian elements. Definition A.1.11 fixes partial order. Corollary IX.2.10 in [193] and 2) in Proposition B.1.47 show $L^0(M,\tau)_+$ is a proper cone generating the partial order on $L^0(M,\tau)_h$. □

The modified standard pairing. Let (M,τ) be a tracial W^*-algebra. We know $\tau \in L^1(M,\tau)_+^*$ by 3) in Proposition B.1.42. Equation B.15 are Hölder inequalities. These in turn yield a modified standard pairing defined by Equation B.16.

Let $p,q \in [1,\infty]$ s.t. $1 = p^{-1} + q^{-1}$. For all $x \in L^p(M,\tau)$ and $y \in L^q(M,\tau)$, we apply (iv) in Theorem IX.2.13 in [193] to get $xy \in L^1(M,\tau)$ and

$$|\tau(xy)| \leq \|x\|_p \|y\|_q. \tag{B.15}$$

For $p = \infty$, we use $\|.\|_M$. By (iv) in Theorem IX.2.13 in [193], note Equation B.15 defines bounded non-degenerate pairing $S : L^p(M,\tau) \times L^q(M,\tau) \longrightarrow \mathbb{C}$ by setting $S(x,y) := \tau(xy)$ for all $x \in L^p(M,\tau)$ and $y \in L^q(M,\tau)$. We call S the standard pairing.

In order to recover the GNS-inner product of τ for $p = q = 2$ as per Equation B.2, we modify the fist variable by taking adjoints. We therefore define the modified standard pairing by setting

$$x^\flat(y) := \tau(x^* y) \tag{B.16}$$

for all $x \in L^p(M,\tau)$ and $y \in L^q(M,\tau)$. We have $\tau(xy) = \tau(yx)$ and $\overline{\tau(xy)} = \tau(x^* y^*)$ in each case. For $p = q = 2$, get $\tau(xy) = \langle x^*, y\rangle_\tau$ for all $x, y \in L^2(M,\tau)$. The modified standard pairing is bounded and non-degenerate. The standard and modified standard pairing are identical upon restriction to self-adjoint elements in the first variable.

Definition B.1.50. For all $p, q \in [1,\infty]$ s.t. $1 = p^{-1} + q^{-1}$, the modified standard pairing on $L^p(M,\tau) \times L^q(M,\tau)$ is defined by Equation B.16.

Proposition B.1.51. *Let M_* be the pre-dual of M.*

1) $\flat : L^1(M,\tau) \longrightarrow M^*$ is an anti-linear isometry onto $M_* \subset M^*$.

2) $L^1(M,\tau)^\flat$ is the set of all normal bounded functionals on M.

3) For all $x \in L^1(M,\tau)$, we have

3.1) x is self-adjoint if and only if $x^\flat$ is real,

3.2) x is positive if and only if $x^\flat$ is positive.

Proof. Get 1) by (iv) in Theorem IX.2.13 in [193]. Using 1), get 2) by Corollary III.3.11 in [192]. We directly verify 3) using Proposition B.1.46 and Proposition B.1.47. □

Remark B.1.52. Following Proposition B.1.51, set $M_* := L^1(M,\tau)$. We readily see 3) in the proposition shows the partial order induced by $L^1(M,\tau) \subset L^0(M,\tau)$ equals the dual space partial order induced by $M_* \subset M^*$ as per 2) in Proposition A.1.23.

Definition B.1.53. We define the state space $\mathscr{S}(M) := \{\mu \in M^*_+ \mid \|\mu\|_M = 1\}$ and the normal state space $\mathscr{S}^{\mathrm{N}}(M) := \mathscr{S}(M) \cap L^1(M,\tau)^\flat$ of M.

Proposition B.1.54. *Let $A \subset M$ be a strongly dense C^*-subalgebra. If $\mathscr{A} \subset A \cap L^2(M,\tau)$ is $\|.\|_A$-dense in A and $\|.\|_2$-dense in $L^2(M,\tau)$, then*

1) $\mathscr{A} \subset M$ strongly dense,

2) finite convex combinations of $\mathscr{A}^* \cdot \mathscr{A} \subset L^1(M,\tau)$ are $\|.\|_1$-dense in $L^1(M,\tau)$.

Proof. Get 1) as $A \subset M$ is strongly dense and $\mathscr{A} \subset A$ is $\|.\|_A$-dense. Mazur's lemma [192] implies w^*-density of $\mathscr{A}^* \cdot \mathscr{A} \subset L^1(M,\tau)_+$ suffices for $\|.\|_1$-density. Let $x \in L^1(M,\tau)_+$ and $\{y_n\}_{n\in\mathbb{N}} \subset \mathscr{A}$ s.t. $\sqrt{x} = \|.\|_2\text{-}\lim_{n\in\mathbb{N}} y_n$. For all $n \in \mathbb{N}$, get $y_n^* y_n \in L^1(M,\tau)_+$. For all $z \in M$, we see $\tau(xz) = \langle \sqrt{x}, \sqrt{x}z\rangle_2 = \lim_{n\in\mathbb{N}} \langle y_n, y_n z\rangle_2 = \lim_{n\in\mathbb{N}} \tau(y_n^* y_n z)$. This is w^*-density. □

B.1.3 Canonical left- and right-actions of measurable operators

We extend canonical left- and right-actions to spaces of measurable operators. We keep natural identifications as per Remark B.1.22 and Remark B.1.65 explicit to ensure their consistent use. This subsumes the bounded case. Using canonical left- and right-actions accordingly, we define spectral and joint spectral measures of self-adjoint measurable operators. This lets us formulate their bounded measurable joint functional calculus.

Definition using *-algebra multiplication. Let (M,τ) be a tracial W^*-algebra. Following Proposition B.1.45, note Definition B.1.55 gives canonical left- and canonical right-action of $L^0(M,\tau)$ on $L^2(M,\tau)$ using multiplication in $L^0(M,\tau)$.

Definition B.1.55. Let $x \in L^0(M,\tau)$. Set

$$\operatorname{dom} L_{x,M} := \left\{ u \in L^2(M,\tau) \;\middle|\; xu \in L^2(M,\tau) \right\}, \tag{B.17}$$

$$\operatorname{dom} R_{x,M} := \left\{ u \in L^2(M,\tau) \;\middle|\; ux \in L^2(M,\tau) \right\}. \tag{B.18}$$

We define canonical left-action $L_{x,M}$ of x on M, resp. canonical right-action $R_{x,M}$ of x on M by setting

$$L_{x,M}(u) := xu,\ R_{x,M}(u) := ux \tag{B.19}$$

for all $u \in \operatorname{dom} L_{x,M}$, resp. for all $u \in \operatorname{dom} R_{x,M}$.

We equip the opposite algebra $L^0(M,\tau)^{\mathrm{op}}$ of $L^0(M,\tau)$ as per Definition B.1.15 with the measure topology of $L^0(M,\tau)$. Using Corollary B.1.64, we readily see Definition B.1.55 determines, by Equation B.19, two unbounded faithful unital *-representations

$$L_M : L^0(M,\tau) \longrightarrow \mathscr{UB}\big(L^2(M,\tau)\big),\ R_M : L^0(M,\tau)^{\mathrm{op}} \longrightarrow \mathscr{UB}\big(L^2(M,\tau)\big). \tag{B.20}$$

Definition B.1.56. We call L_M and R_M in Equation B.20 canonical left- , resp. canonical right-action of $L^0(M,\tau)$ on $L^2(M,\tau)$.

Notation B.1.57. Unless stated otherwise, we suppress W^*-algebras in subscripts of canonical left- and right-actions. We require subscripts in Section B.2. We further write L^{op}, suppressing subscripts, for the canonical left-action of $L^0(M^{\mathrm{op}},\tau)$ on $L^2(M,\tau)$.

Proposition B.1.58 states L is $\mathscr{L}$ up to twisting with id_τ^{-1}. We see L subsumes our discussion for $\mathscr{L}$ in Subsection B.1.2, in particular the bounded case. Concerning results for R, get $R \cong L^{\mathrm{op}}$ naturally but $R \neq L^{\mathrm{op}}$ in general. Following Remark B.1.65, we know results for L^{op} apply to R if and only if they are preserved under $R \cong L^{\mathrm{op}}$.

Proposition B.1.58. *For all $x \in L^0(M,\tau)$, we have $L_x = \mathrm{id}_\tau^{-\dagger}(\mathscr{L}_x)$.*

Proof. Apply Proposition B.1.45. □

Lemma B.1.62 shows we obtain $R \cong L^{\mathrm{op}}$ by twisting with natural *-isomorphism $L^0(M,\tau)^{\mathrm{op}} \cong L^0(M^{\mathrm{op}},\tau)$ extending $\mathrm{id}_{M^{\mathrm{op}}}$. This natural *-isomorphism is called opposite algebra map and constructed in Lemma B.1.59.

Lemma B.1.59. *There exists unique *-isomorphism* $\mathrm{op}: L^0(M,\tau)^{\mathrm{op}} \longrightarrow L^0(M^{\mathrm{op}},\tau)$ *s.t.*

1) op *is a homeomorphism w.r.t. measure topologies on* $L^0(M,\tau)^{\mathrm{op}}$ *and* $L^0(M^{\mathrm{op}},\tau)$*,*
2) $\mathrm{op}^{-1}: L^0(M^{\mathrm{op}},\tau) \longrightarrow L^0(M,\tau)^{\mathrm{op}}$ *is a *-isomorphism,*
3) *we have commutative diagram*

$$\begin{tikzcd} M^{\mathrm{op}} \arrow[r, hook] \arrow[d, "\mathrm{id}_M"] & L^0(M,\tau)^{\mathrm{op}} \arrow[d, "\mathrm{op}"] \\ M^{\mathrm{op}} \arrow[r, hook] & L^0(M^{\mathrm{op}},\tau) \end{tikzcd} \tag{B.21}$$

of injective horizontal and bijective vertical maps.

Proof. If $\{x_k\}_{k\in N} \subset M$ is a net, then Equation B.5 and continuity of Adj in measure topology shows $\{x_k\}_{k\in K} \subset M$ is Cauchy in measure if and only if $\{x_k\}_{k\in K} \subset M^{\mathrm{op}}$ is. For all $x = [\{x_k\}_{k\in N}] \in L^0(M,\tau)$, let $x^{\mathrm{op}} := [\{x_k\}_{k\in N}] \in L^0(M^{\mathrm{op}},\tau)$ and set

$$\mathrm{op}(x) := x^{\mathrm{op}}. \tag{B.22}$$

By construction, $\mathrm{op}: L^0(M,\tau)^{\mathrm{op}} \longrightarrow L^0(M^{\mathrm{op}},\tau)$ satisfies Diagram B.21 and is continuous in measure topology. Equation B.22 in turn is fully determined by Diagram B.21 and continuity in measure topology. We see op is unique. It is a homeomorphism since op^{-1} is determined by mapping Cauchy nets in M^{op} to Cauchy nets in M. We are left to show op, ergo op^{-1}, is a *-homomorphism. The *-algebras M and M^{op} extend suitably. □

Definition B.1.60. We call $\mathrm{op}: L^0(M,\tau)^{\mathrm{op}} \longrightarrow L^0(M^{\mathrm{op}},\tau)$ defined by Equation B.22 the opposite algebra map.

Corollary B.1.61. *We have commutative diagram*

$$\begin{tikzcd} (\mathscr{H}(M,\tau), \|.\|_\tau) \arrow[r, "\mathrm{id}_\tau"] \arrow[d, "\mathrm{id}_{\mathscr{H}(M,\tau)}"] & \big(L^2(M,\tau), \|.\|_2\big) \arrow[d, "\mathrm{op}"] \\ (\mathscr{H}(M,\tau), \|.\|_\tau) \arrow[r, "\mathrm{id}_{\tau,\mathrm{op}}"] & \big(L^2(M^{\mathrm{op}},\tau), \|.\|_2\big) \end{tikzcd} \tag{B.23}$$

of isometric isomorphisms of Hilbert spaces.

Proof. Apply the construction of op in the proof of Lemma B.1.59. □

Note 2) in Proposition B.1.42 shows $\mathrm{Adj} : L^2(M,\tau) \longrightarrow L^2(M,\tau)$ defines $\mathrm{Adj}^\dagger$ as per Definition A.1.13. Corollary B.1.61 shows $\mathrm{op} : L^2(M,\tau) \longrightarrow L^2(M^{\mathrm{op}},\tau)$ defines $\mathrm{op}^\dagger$. If we restrict to the bounded case, then we have commutative diagram

$$\begin{tikzcd}
M \arrow[r, "L"] \arrow[d, "\mathrm{Adj}"] & \mathscr{B}\big(L^2(M,\tau)\big) \arrow[d, "\mathrm{Adj}^\dagger"] \\
M^{\mathrm{op}} \arrow[r, "R"] \arrow[d, "\mathrm{id}_M"] & \mathscr{B}\big(L^2(M,\tau)\big) \arrow[d, "\mathrm{op}^\dagger"] \\
M^{\mathrm{op}} \arrow[r, "L^{\mathrm{op}}"] & \mathscr{B}\big(L^2(M^{\mathrm{op}},\tau)\big)
\end{tikzcd} \tag{B.24}$$

by Diagram B.4 and Diagram B.21. We thus recover $R = L^{\mathrm{op}}$, given as $\mathscr{R} = \mathscr{L}^{\mathrm{op}}$ in 1) in Proposition B.1.19, if we collapse the lower part of Diagram B.24 by pull-back along Diagram B.23. We further account for twisting of $\mathscr{L}^{\mathrm{op}}$ with $\mathrm{id}_{\tau,\mathrm{op}}$.

Lemma B.1.62. *We have commutative diagram*

$$\begin{tikzcd}
L^0(M,\tau) \arrow[r, "L"] \arrow[d, "\mathrm{Adj}"] & \mathscr{U}\mathscr{B}\big(L^2(M,\tau)\big) \arrow[d, "\mathrm{Adj}^\dagger"] \\
L^0(M,\tau)^{\mathrm{op}} \arrow[r, "R"] \arrow[d, "\mathrm{op}"] & \mathscr{U}\mathscr{B}\big(L^2(M,\tau)\big) \arrow[d, "\mathrm{op}^\dagger"] \\
L^0(M^{\mathrm{op}},\tau) \arrow[r, "L^{\mathrm{op}}"] & \mathscr{U}\mathscr{B}\big(L^2(M^{\mathrm{op}},\tau)\big)
\end{tikzcd} \tag{B.25}$$

of injective horizontal and bijective vertical maps.

Proof. For all $x \in L^0(M,\tau)$, we directly verify $\mathrm{dom}\, R_x = \mathrm{dom}\,\mathrm{Adj}^\dagger(L_{x^*})$ and $R = \mathrm{Adj}^\dagger \circ L$. This is the upper diagram. Lemma B.1.59 and 1) in Proposition A.1.14 ensure vertical maps are bijective. In particular, taking adjoints is. Thus L, $R = \mathrm{Adj}^\dagger \circ L$ and L^{op} are injective, hence we are left to show the lower diagram.

Let $x \in L^0(M,\tau)$. Note we have $\mathrm{dom}\, R_x = \{u \in L^2(M,\tau) \mid ux \in L^2(M,\tau)\}$ and $\mathrm{dom}\, L^{\mathrm{op}}_{x^{\mathrm{op}}} = \{v \in L^2(M^{\mathrm{op}},\tau) \mid x^{\mathrm{op}} v \in L^2(M^{\mathrm{op}},\tau)\}$ by definition. For all $u \in L^2(M,\tau)$, we calculate

$$R_x(u) = x \cdot^{\mathrm{op}} u = \mathrm{op}^{-1}\big(x^{\mathrm{op}}\big) \cdot^{\mathrm{op}} \mathrm{op}^{-1}\big(u^{\mathrm{op}}\big) = \mathrm{op}^{-1}\big(L^{\mathrm{op}}_{x^{\mathrm{op}}}(\mathrm{op}(u))\big). \tag{B.26}$$

Equation B.26 implies $\mathrm{op}(\mathrm{dom}\, R_x) = \mathrm{dom}\, L^{\mathrm{op}}_{x^{\mathrm{op}}}$. Thus $R_x = \mathrm{op}^{-\dagger}(L^{\mathrm{op}}_{x^{\mathrm{op}}})$, hence $\mathrm{op}^\dagger(R_x) = L^{\mathrm{op}}_{x^{\mathrm{op}}}$ upon applying the given dagger map. This is the lower diagram. □

Corollary B.1.63. *For all $x \in L^0(M,\tau)$, we have*

1) $\mathrm{Adj}^\dagger L_x = R_{x^*}$

2) $R_x = \mathrm{op}^{-\dagger}(L^{\mathrm{op}}_{x^{\mathrm{op}}})$.

Proof. This reformulates the upper, resp. lower part of Diagram B.25. □

Corollary B.1.64. *For all $x \in L^0(M,\tau)$, L_x and R_x are densely defined closed operators on $L^2(M,\tau)$. For all $x, y \in L^0(M,\tau)$ and $\lambda \in \mathbb{C}$, we have*

1) $L_{\lambda_1 x + \lambda_2 y} = \overline{\lambda_1 L_x + \lambda_2 L_y}$ *and* $R_{\lambda_1 x + \lambda_2 y} = \overline{\lambda_1 R_x + \lambda_2 R_y}$,

2) $L_{xy} = \overline{L_x L_y}$ *and* $R_{xy} = \overline{R_y R_x}$,

3) $L_{x^*} = L_x^*$ *and* $R_{x^*} = R_x^*$.

Proof. Proposition B.1.31 shows analogous claims for $\mathscr{L}$, and Proposition A.1.14 implies they are preserved under twisting with id_τ. As such, Proposition B.1.58 shows all claims for L at once. Proposition A.1.14 implies they are preserved under twisting with op. We therefore obtain all claims for R by reducing to L^{op} using 2) in Corollary B.1.63. □

Remark B.1.65. All canonical left- and right-actions are multiplication by measurable operators. Corollary B.1.64 shows they are unbounded faithful unital *-representations extending the bounded case. This requires Proposition B.1.58, i.e. Proposition B.1.45. We twist with id_τ in case of L, as well as with $\mathrm{id}_{\tau,\mathrm{op}}$ in case of L^{op}. These identities in measure topology induce distinct measure topologies on $\mathscr{H}(M,\tau) = \mathscr{H}(M^{\mathrm{op}},\tau)$.

We obtain $L^2(M,\tau)$ and $L^2(M^{\mathrm{op}},\tau)$ accordingly. Thus $R \neq L^{\mathrm{op}}$ by Lemma B.1.62, even as $R \cong L^{\mathrm{op}}$ is $R = L^{\mathrm{op}}$ upon pull-back along Diagram B.23. Note R is not defined on an algebra of measurable operators, but on an opposite algebra of one. Since $R \cong L^{\mathrm{op}}$ up to twisting with opposite algebra maps, results for canonical left-actions apply if and only if they are compatible with such twisting. We use this in Corollary B.1.64, as well as in Section B.2. Altogether, we have consistent use of canonical left- and right-actions for joint functional calculus of self-adjoint measurable operators.

Lemma B.1.66. *For all $x \in L^0(M,\tau)_h$, the following are equivalent:*

1) $u \in \mathrm{dom} L_x$,

2) $\mathrm{Re}(u)_+, \mathrm{Re}(u)_-, \mathrm{Im}(u)_+, \mathrm{Im}(u)_- \in \mathrm{dom} L_x$.

Proof. Let $x \in L^0(M,\tau)_h$. For all $n \in \mathbb{N}$, set $x_n := \chi_{[-n,n]}(x)x$. We know $\{x_n\}_{n\in\mathbb{N}} \in M_h$ by 1) in Proposition B.1.37. Moreover, $|x_n| \leq |x_{n+1}| \leq |x|$ for all $n \in \mathbb{N}$. We know $u \in \mathrm{dom} L_x$ if and only if $\|xu\|_2 = \int_{\mathrm{spec} L_x} \lambda^2 dE^u_{L_x} < \infty$. Fatou's lemma implies

$$\int_{\mathrm{spec} L_x} \lambda^2 dE^u_{L_x} \leq \liminf_{n\in\mathbb{N}} \int_{\mathrm{spec} L_x} \left(\lambda \cdot \chi_{[-n,n]}(\lambda)\right)^2 dE^u_{L_x} = \|x_n u\|_2^2. \tag{B.27}$$

Since $|x_n| \leq |x|$ for all $n \in \mathbb{N}$, Equation B.27 implies

$$\|xu\|_2 = \sup_{n\in\mathbb{N}} \|x_n u\|_2 = \lim_{n\in\mathbb{N}} \|x_n u\|_2 \in [0,\infty] \tag{B.28}$$

for all $u \in L^2(M,\tau)$. We use decomposition as per Proposition B.1.47 and apply limits in $n \in \mathbb{N}$ as per Equation B.28 to show equivalence as claimed.

Let $u \in L^2(M,\tau)$. Proposition B.1.47 implies

$$\|zu\|_2^2 = \left\|z\operatorname{Re}(u)\right\|_2^2 + \left\|z\operatorname{Im}(u)\right\|_2^2 \tag{B.29}$$

for all $z \in M_h$. Note mixed terms $i2\operatorname{Re}\langle z\operatorname{Re}(u), z\operatorname{Im}(u)\rangle_2$ do not appear in Equation B.29 as $\|zu\|_2 \in \mathbb{R}$ ensures they vanish in each case. Since all positive and negative parts involved have disjoint support, multiplying out terms yields

$$\|zu\|_2^2 = \left\|z\operatorname{Re}(u)_+\right\|_2^2 + \left\|z\operatorname{Re}(u)_-\right\|_2^2 + \left\|z\operatorname{Im}(u)_+\right\|_2^2 + \left\|z\operatorname{Im}(u)_-\right\|_2^2 \tag{B.30}$$

for all $z \in M_h$. In particular, Equation B.30 is satisfied using $z = x_n$ for all $n \in \mathbb{N}$. Thus applying the limit in $n \in \mathbb{N}$ as per Equation B.28 for given u extends Equation B.30 to $z = x$ s.t. the resulting limit is finite if and only if u satisfies 1) and 2). □

Corollary B.1.67. *Let $x \in L^1(M,\tau)_h$.*

1) *For all $n \in \mathbb{N}$, set $x_n := \chi_{[-n,n]}(x)x$. Then $\{x_n\}_{n\in\mathbb{N}} \subset L^1(M,\tau)_h$ and $x = \|.\|_1\text{-}\lim_{n\in\mathbb{N}} x_n$.*
2) *Assume $x \in L^1(M,\tau)_+$. For all $n \in \mathbb{N}$, let $x_n := \min\{x,n\}$. Then $\{x_n\}_{n\in\mathbb{N}} \subset L^1(M,\tau)_+$ and $x = \|.\|_1\text{-}\lim_{n\in\mathbb{N}} x_n$. We have $u \in \operatorname{dom} L_x$ if and only if $\sup_{n\in\mathbb{N}} \|x_n u\|_2 < \infty$ or $\sup_{n\in\mathbb{N}} \|u x_n\|_2 < \infty$.*

Proof. Arguing as for Equation B.28 in the proof of Lemma B.1.66, we have 1) and our first claim in 2). Let $x \in L^1(M,\tau)_+$. For all $u \in L^2(M,\tau)$, $\|xu\|_2^2 = \sup_{n\in\mathbb{N}} \|x_n u\|_2^2 \in [0,\infty]$ by monotone convergence. Thus $u \in \operatorname{dom} L_x$ if and only if $\sup_{n\in\mathbb{N}} \|x_n u\|_\tau^2 < \infty$, hence if and only if $\sup_{n\in\mathbb{N}} \|u x_n\|_\tau^2 < \infty$ by 2) in Proposition B.1.42. □

Corollary B.1.68. *For all $x \in L^2(M,\tau)_h$, $M \cap L^2(M,\tau)$ is core of L_x and R_x.*

Proof. Since $x \in L^0(M,\tau)_h$, note 1) in Corollary B.1.63 ensures it suffices to show our claim for L_x. Lemma B.1.66 lets us reduce further to showing $\operatorname{dom} L_x \cap L^2(M,\tau)_+$ lies in the closure of $M \cap L^2(A,\tau)$ w.r.t. the graph norm of L_x.

Let $u \in \operatorname{dom} L_x \cap L^2(M,\tau)_+$ and set $u_n := \min\{u,n\} \in L^2(M,\tau)$ for all $n \in \mathbb{N}$. For all $\lambda \geq 0$ and $n \in \mathbb{N}$, get $\min\{\lambda,n\}^2 = \min\{\lambda^2,n^2\} \leq \lambda \cdot \min\{\lambda,n\}$. Multiplying out terms of the inner product lets us estimate

$$\left\|u - u_n\right\|_2^2 = \left\|u^2\right\|_1 + \left\|\min\{u^2,n^2\}\right\|_1 - 2\tau(uu_n) \leq \left\|u^2\right\|_1 - \left\|\min\{u^2,n^2\}\right\|_1 \tag{B.31}$$

for all $n \in \mathbb{N}$. Equation B.31 and 2) in Corollary B.1.67 imply $\lim_{n\in\mathbb{N}} \|u - u_n\|_2^2 = 0$.

Note $x \in \operatorname{dom} R_u$ since $u \in \operatorname{dom} L_x$. For all $n \in \mathbb{N}$, get

$$\|x(u-u_n)\|_2^2 = \int_{\operatorname{spec} R_u} \big(\lambda - \min\{\lambda, n\}\big)^2 dE^x_{R_u} < \infty. \tag{B.32}$$

Since $(\lambda - \min\{\lambda, n\})^2 \leq \lambda^2$ on $[0,\infty)$ for all $n \in \mathbb{N}$ by definition, applying Fatou's lemma to Equation B.32 shows $\lim_{n\in\mathbb{N}} \|x(u-u_n)\|_2 = 0$. □

Spectral measures of self-adjoint measurable operators. Using inverses of canonical left- and right-actions, we extend Subsection A.1.3 to self-adjoint measurable operators. This yields abstract notion of spectral and joint spectral measure, as well as bounded measurable functional and joint functional calculus of self-adjoint measurable operators. This subsumes Definition A.1.73. Notation B.1.79 fixes conventions.

Let (M,τ) be a tracial W^*-algebra.

Definition B.1.69. Let $x \in L^0(M,\tau)_h$.

1) For all $Z \in \mathfrak{B}(\mathbb{R})$, set

$$E_{x,M}(Z) := L_M^{-1}\big(E_{L_{x,M}}(Z)\big). \tag{B.33}$$

We call $E_{x,M}$ the spectral measure of x in M.

2) The spectrum of x in M is $\operatorname{spec}_M x := \operatorname{spec} L_{x,M}$. We call

$$W^*_M(x) := L_M^{-1}\big(W^*\big(L_{x,M}\big)\big) \tag{B.34}$$

the W^*-algebra generated by x in M.

Proposition B.1.70. *If $x \in L^0(M,\tau)_h$, then*

1) $L_{E_{x,M}(Z),M} = E_{L_{x,M}}(Z)$ *and* $R_{E_{x,M}(Z),M} = E_{R_{x,M}}(Z)$ *for all* $Z \in \mathfrak{B}(\mathbb{R})$*,*

2) $\operatorname{spec}_M x = \operatorname{spec} L_{x,M} = \operatorname{spec} R_{x,M}$*,*

3) $W^*_M(x) = L_M^{-1}\big(W^*_M\big(L_{x,M}\big)\big) = R_M^{-1}\big(W^*_M\big(R_{x,M}\big)\big)$*.*

Proof. Let $x \in L^0(M,\tau)_h$. For all $Z \in \mathfrak{B}(\mathbb{R})$, we know $E_{x,M}(Z) \in M$. Thus our claim in 1) concerning L_M holds. For R_M, we instead use $R_{x,M} = \operatorname{Adj}^\dagger L_{x,M}$ by 1) in Corollary B.1.63 and reduce to L_M. We directly verify it suffices to show $\operatorname{Adj}^\dagger(E_{L_{x,M}}(Z)) = E_{\operatorname{Adj}^\dagger L_{x,M}}(Z)$ for all $Z \in \mathfrak{B}(\mathbb{R})$ to obtain our claim in 1) concerning R_M.

Since $\operatorname{Adj}^\dagger(R_{\pm i}(T)) = R_{\pm i}(\operatorname{Adj}^\dagger(T))$, Lemma A.1.91 shows Lemma A.1.92 applies to $T = L_{x,M}$, $S = \operatorname{Adj}^\dagger(T)$ and $\phi = \operatorname{Adj}^\dagger$. Thus the required identity, hence 1) holds. Get 2) since the spectrum of a self-adjoint unbounded operator is the support of its spectral measure. Get 3) because all W^*-algebras involved are commutative. □

If $x \in L^0(M,\tau)_h$, then $\operatorname{spec}_M x$ is a locally compact Hausdorff space and with σ-ideal $\mathcal{N}(E_{x,M}) \subset \mathfrak{B}(\operatorname{spec}_M x)$ of null sets as per 1) in Definition B.1.71.

Definition B.1.71. Let $x \in L^0(M,\tau)_h$. Set

1) $\mathcal{N}\big(E_{x,M}\big) := \big\{Z \in \mathfrak{B}(\mathbb{R}) \mid E_{x,M}(Z) = 0\big\}$,
2) $L^\infty\big(\operatorname{spec}_M x, dE_{x,M}\big) := L^\infty\big(\operatorname{spec}_M x, \mathcal{N}\big(E_{x,M}\big)\big)$.

Lemma B.1.72. *If $x \in L^0(M,\tau)_h$, then*

1) $\big(L^\infty\big(\operatorname{spec}_M x, dE_{x,M}\big), \|.\|_\infty\big)$ *is a W^*-algebra s.t. $C_0(\operatorname{spec}_M x)$ is σ-weakly dense,*

2) there exists normal unital *-isomorphism

$$\Gamma_{x,M} : L^\infty\big(\operatorname{spec}_M x, dE_{x,M}\big) \longrightarrow W^*_M(x) \tag{B.35}$$

s.t. $\Gamma_{L_{x,M}} = L_M \circ \Gamma_{x,M}$ *and* $\Gamma_{R_{x,M}} = R_M \circ \Gamma_{x,M}$,

3) $\Gamma_{x,M}$ *is determined by unitality and*

$$\Gamma_{x,M}\big(\chi_Z\big) = E_{x,M}(Z) \tag{B.36}$$

for all $Z \in \mathfrak{B}(\mathbb{R})$.

Proof. Note $L^\infty(\operatorname{spec}_M x, dE_{x,M}) = L^\infty(\operatorname{spec} L_{x,M}, dE_{L_{x,M}})$. Get 1). Proposition B.1.70 implies $\Gamma_{L_{x,M}} = L_M \circ \Gamma_{x,M}$ and $\Gamma_{R_{x,M}} = R_M \circ \Gamma_{x,M}$. Proposition A.1.37 and Proposition A.1.85 thus show reducing to $\Gamma_{L_{x,M}} = L_M \circ \Gamma_{x,M}$ yields 2) and 3) in full. □

Definition B.1.73. Let $x \in L^0(M,\tau)_h$. We call $\Gamma_{x,M}$ the bounded measurable functional calculus of x in M. For all $g \in L^\infty(\operatorname{spec}_M x, dE_{x,M})$, set

$$g(x) := \Gamma_{x,M}(g). \tag{B.37}$$

Remark B.1.74. Let $x \in L^0(M,\tau)_h$. For all $g \in L^\infty(\operatorname{spec}_M x, dE_x)$, get $g(x) \in M \subset L^0(M,\tau)$ consistent with Definition B.1.36. If $x \in M_h$, then we recover Definition A.1.73 since 3) in Lemma B.1.72 reduces to 3) in Lemma A.1.72.

Let $x, y \in L^0(M,\tau)_h$. Using Proposition A.1.96, we know 2) in Lemma B.1.72 at once implies $L_{x,M}, R_{y,M} \in \mathscr{UB}(L^2(M,\tau))_h$ commute strongly. Equation A.33 shows

$$W^*\big(L_{x,M}\big) \otimes W^*\big(R_{y,M}\big) = W^*\big(L_{x,M}, R_{y,M}\big) \subset \mathscr{B}\big(L^2(M,\tau)\big). \tag{B.38}$$

Note Equation B.38 ensures Corollary A.1.53 lets us tensor $L_M : W^*_M(x) \longrightarrow W^*(L_{x,M})$ and $R_M : W^*_M(y) \longrightarrow W^*(R_{y,M})$ to a normal unital *-isomorphism

$$L_M \otimes R_M : W^*_M(x) \otimes W^*_M(y) \longrightarrow W^*\big(L_{x,M}, R_{y,M}\big). \tag{B.39}$$

Definition B.1.75. Let $x, y \in L^0(M,\tau)_h$.

1) For all $Z \in \mathfrak{B}(\mathbb{R}\times\mathbb{R})$, set

$$E_{x,y,M}(Z) := \left(L_M \otimes R_M\right)^{-1}\left(E_{L_{x,M},R_{y,M}}(Z)\right). \tag{B.40}$$

We call $E_{x,y,M}$ the joint spectral measure of $x \otimes y$ in $M \otimes M^{\mathrm{op}}$.

2) The joint spectrum of $x \otimes y$ in $M \otimes M^{\mathrm{op}}$ is $\mathrm{spec}_M\, x \times y := \mathrm{spec}\, L_{x,M} \times R_{y,M}$. We call

$$W^*_M(x,y) := W^*_M(x) \otimes W^*_M(y) = \left(L_M \otimes R_M\right)^{-1}\left(W^*\left(L_{x,M}, R_{y,M}\right)\right) \tag{B.41}$$

the W^*-algebra generated by $x \otimes y$ in $M \otimes M^{\mathrm{op}}$.

If $x, y \in L^0(M,\tau)_h$, then $\mathrm{spec}_M\, x \times y$ is a locally compact Hausdorff space and with σ-ideal $\mathcal{N}(E_{x,y,M}) \subset \mathfrak{B}(\mathrm{spec}_M\, x \times y)$ of null sets as per 1) in Definition B.1.76.

Definition B.1.76. Let $x, y \in L^0(M,\tau)_h$. Set

1) $\mathcal{N}\left(E_{x,y,M}\right) := \left\{Z \in \mathfrak{B}(\mathbb{R}) \mid E_{x,y,M}(Z) = 0\right\}$,

2) $L^\infty\left(\mathrm{spec}_M\, x \times y, dE_{x,y,M}\right) := L^\infty\left(\mathrm{spec}_M\, x \times y, \mathcal{N}\left(E_{x,y,M}\right)\right)$.

Lemma B.1.77. *If $x, y \in L^0(M,\tau)_h$, then*

1) $\left(L^\infty\left(\mathrm{spec}_M\, x \times y, dE_{x,y,M}\right), \|.\|_\infty\right)$ is a W^*-algebra s.t. $C_0(\mathrm{spec}_M\, x \times y)$ is σ-weakly dense,

2) there exists normal unital *-isomorphism

$$\Gamma_{x,y,M} : L^\infty\left(\mathrm{spec}_M\, x \times y, dE_{x,y,M}\right) \longrightarrow W^*_M(x,y) \tag{B.42}$$

s.t. $\Gamma_{L_{x,M},R_{y,M}} = \left(L_M \otimes R_M\right) \circ \Gamma_{x,M}$,

3) $\Gamma_{x,M}$ is determined by unitality and

$$\Gamma_{x,y,M}\left(\chi_{Z_0} \otimes \chi_{Z_1}\right) = E_{x,M}(Z_0) E_{y,M}(Z_1) \tag{B.43}$$

for all $Z_0, Z_1 \in \mathfrak{B}(\mathbb{R})$.

Proof. We know 1) by definition. For 2), note it reduces to factors by construction of joint spectral measures and apply Lemma B.1.72. Likewise get 3) by Proposition A.1.100. □

Definition B.1.78. Let $x, y \in L^0(M,\tau)_h$. We call $\Gamma_{x,y,M}$ the bounded measurable joint functional calculus of $x \otimes y$ in $M \otimes M^{\mathrm{op}}$. For all $g \in L^\infty(\mathrm{spec}_M x \times y, dE_{x,y,M})$, set

$$g(x,y) := \Gamma_{x,y,M}(g). \tag{B.44}$$

Notation B.1.79. Unless stated otherwise, we suppress W^*-algebras in subscripts of spectral measures, spectra, bounded measurable functional calculus and generated W^*-algebras. In general, we use subscripts to keep track of W^*-(sub-)algebras apart from the algebra of bounded operators on a Hilbert space.

Lemma B.1.80. *Let $x, y \in L^0(M,\tau)_+$. If $g \in C_b([0,\infty) \times [0,\infty))$, then*

$$\Gamma_{x,y,M}(g) = \mathrm{s}\text{-}\lim_{\varepsilon \downarrow 0} \Gamma_{x+\varepsilon 1_M, y+\varepsilon 1_M, M}(g). \tag{B.45}$$

Proof. Note 2) in Lemma B.1.77 implies Equation B.45 is equivalent to

$$\Gamma_{L_x,R_y}(g) = \mathrm{s}\text{-}\lim_{\varepsilon \downarrow 0} \Gamma_{L_x+\varepsilon I, R_y+\varepsilon I}(g). \tag{B.46}$$

Let $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset (0,\infty)$ be a descending sequence converging to zero. Proposition 10.1.8 in [88] implies $L_x = \mathrm{sr}\text{-}\lim_{n\in\mathbb{N}} L_x + \varepsilon_n I$ and $R_y = \mathrm{sr}\text{-}\lim_{n\in\mathbb{N}} R_y + \varepsilon_n I$. All unbounded operators used here are positive, and each limit is clearly independent of the given descending sequence. Using standard arguments, we see Lemma A.2.5 implies Equation B.46. □

B.2 Compressed pull-back of joint functional calculus

In Subsection B.2.1, we discuss semi-finite W^*-subalgebras of tracial W^*-algebras and associated L^2-reducible measurable operators. Assuming such semi-finiteness upon inclusion, f.s.n. traces restrict to f.s.n. traces. Theorem B.2.28 gives structure-preserving canonical inclusion for spaces of measurable operators. These let us extend abstract compression maps from W^*-algebras to spaces of measurable operators.

In Subsection B.2.2, we formulate compressed pulled-back joint functional calculus of self-adjoint measurable operators. Theorem B.2.44 states sufficient conditions. For its proof, we express change of canonical left- and right-actions as abstract compression maps. We use Theorem B.2.44 to define compressed pulled-back bounded measurable joint functional calculus of self-adjoint measurable operators and extend to suitable unbounded functions following its Corollary B.2.45.

B.2.1 L^2-reducible measurable operators

Semi-finite W^*-subalgebras are tracial W^*-algebras. We construct structure-preserving canonical inclusions of the resulting spaces of measurable operators in Theorem B.2.28 by mapping to L^2-reducible measurable operators.

Inclusion of pre-duals ensure semi-finite W^*-algebras have unique noncommutative conditional expectations by dualisation [192]. Abstract compression maps are one of two example classes. While we do not use them in the appendix, we do use noncommutative conditional expectations to show monotonicity of quasi-entropies in Subsection 2.2.1.

Semi-finite W*-subalgebras. Let (M,τ) be a tracial W^*-algebra. If $N \subset M$ is a W^*-subalgebra, then $\tau|_{N_+}$ is faithful normal trace on N since $N_+ \subset M_+$.

Definition B.2.1. If $N \subset M$ is a W^*-subalgebra s.t. $\tau|_{N_+}$ is semi-finite, then we call N a semi-finite W^*-subalgebra. We write $N \subset (M,\tau)$ in this case.

Notation B.2.2. Let $N \subset (M,\tau)$. We write $\tau = \tau|_{N_+}$ on N.

Proposition B.2.3. *Let $N \subset M$ be a W^*-subalgebra.*

1) If $N \subset (M,\tau)$, then τ is f.s.n. trace on N and (N,τ) is a tracial W^*-algebra.

2) $N \subset (M,\tau)$ if and only if $N[1_M] \subset (M,\tau)$.

3) $N \subset (M,\tau)$ if and only if $N^{\mathrm{op}} \subset (M^{\mathrm{op}},\tau)$.

Proof. We have 1) by definition. If $N \subset (M,\tau)$, then we know $N[1_M] = N \oplus \langle 1_N^\perp \rangle_{\mathbb{C}} \subset (M,\tau)$ by Proposition A.1.71. If $(N[1_M],\tau) \subset (M,\tau)$, then $N \subset (N[1_M],\tau)$ shows $N \subset (M,\tau)$ at once. Get 2). We obtain 3) since partial orders on N and N^{op} are identical. □

Let $N \subset (M,\tau)$. By construction, $\mathscr{H}(N,\tau) \subset \mathscr{H}(M,\tau)$. We define isometric inclusion $\mathrm{inc}_2 : L^2(N,\tau) \longrightarrow L^2(M,\tau)$ of Hilbert spaces as the unique bounded linear map s.t. we have commutative diagram

$$\begin{tikzcd} L^2(N,\tau) \arrow[r, "\mathrm{inc}_2"] & L^2(M,\tau) \\ \mathscr{H}(N,\tau) \arrow[u, "\mathrm{id}_\tau"] \arrow[r, hook] & \mathscr{H}(M,\tau) \arrow[u, "\mathrm{id}_\tau"'] \end{tikzcd} \tag{B.47}$$

of Hilbert space isometries.

Definition B.2.4. For all $N \subset (M,\tau)$, set

$$\mathbf{L}^2(N,\tau) := \overline{\mathrm{inc}_2\big(L^2(N,\tau)\big)}^{\|.\|_2} = \mathrm{inc}_2\big(L^2(N,\tau)\big) \tag{B.48}$$

for inc_2 as per Diagram B.47.

Remark B.2.5. Note $\mathrm{inc}_2 = \mathrm{id}_{L^2(N,\tau)}$ upon identifying as per Remark B.2.29 following Theorem B.2.28, i.e. a natural identification by extending Diagram B.49 to $L^0(M,\tau)$.

The isometric isomorphism $\mathrm{inc}_2 : L^2(N,\tau) \longrightarrow \mathbf{L}^2(N,\tau)$ of Hilbert spaces defines $\mathrm{inc}_2^\dagger$ as per Definition A.1.13. We introduce compression maps in Subsection A.2.2.

Proposition B.2.6. *Let $N \subset (M,\tau)$. We have commutative diagram*

$$\begin{tikzcd} N \arrow[r, hook] \arrow[d, "\mathrm{id}_N"] & N[1_M] \arrow[r, "L_M"] \arrow[d, "\mathrm{com}_{1_N}"] & \mathscr{B}\big(L^2(M,\tau)\big) \arrow[d, "\mathrm{inc}_2^{-\dagger} \circ \mathrm{com}_{\mathbf{L}^2(N,\tau)}"] \\ N \arrow[r, "\mathrm{id}_N"] & N \arrow[r, "L_N"] & \mathscr{B}\big(L^2(N,\tau)\big) \end{tikzcd} \tag{B.49}$$

s.t. horizontal maps are normal unital injective *-homomorphisms and vertical ones are positivity-preserving surjections of Banach spaces.

Proof. Diagram A.41 is the left diagram in Diagram B.49. Get unital W^*-subalgebra $L_M(N)'' = L_M(N[1_M]) \subset \mathscr{B}(L^2(M,\tau))$. Proposition A.1.34 and Proposition B.1.9 show

$$N[1_M] = \Big\{x \in M \;\Big|\; L_{x,M} \text{ is } L_M(N)''\text{-affiliated}\Big\}. \tag{B.50}$$

For all $x \in N[1_M]_h$, Proposition B.1.26 shows the affiliation property in Equation B.50 lets us apply Corollary A.2.28 to get

$$\big[L_{x,M}, \pi_{\mathbf{L}^2(N[1_M],\tau)}\big] = 0. \tag{B.51}$$

Equation B.51 holds for all $x \in N[1_M]$ by decomposing into real and imaginary parts.

Note $N[1_M] = N \oplus \langle 1_N^\perp \rangle_{\mathbb{C}}$ using direct sum of C^*-algebras as per Proposition A.1.71 since $1_N^\perp = 1_M - 1_N$ by definition. Using $N1_N^\perp = 1_N^\perp N = 0$, Equation B.50 shows

$$N = \Big\{x \in M \;\Big|\; L_{x,M} \text{ is } L_M(N)''\text{-affiliated, } x = 1_N x\Big\}. \tag{B.52}$$

Using $\mathbf{L}^2(N,\tau) \subset \mathbf{L}^2(N[1_M],\tau)$, we directly verify

$$\pi_{\mathbf{L}^2(N,\tau)} = L_{1_N,M}\pi_{\mathbf{L}^2(N[1_M],\tau)} = \pi_{\mathbf{L}^2(N[1_M],\tau)}L_{1_N,M} \tag{B.53}$$

by testing on the inner product. For all $x \in N[1_M]_h$, we apply $[x, 1_N] = 0$, Equation B.51 and Equation B.53 to calculate

$$\big[L_{x,M}, \pi_{\mathbf{L}^2(N,\tau)}\big] = 0. \tag{B.54}$$

Equation B.54 holds for all $x \in N[1_M]$ by decomposing into real and imaginary parts.

We show the right diagram in Diagram B.49. We directly verify $\mathrm{com}_{1_N}(1_N^{\perp}) = 0$ and $\mathrm{com}_{\mathbf{L}^2(N,\tau)}(L_{1_N^{\perp},M}) = 0$. We are left to consider $x \in N$. Let $x \in N$ and $u \in L^2(N,\tau)$. Thus

$$L_{x,M}(\mathrm{inc}_2(u)) = \mathrm{inc}_2\big(L_{x,N}(u)\big) \tag{B.55}$$

by $\|.\|_2$-density. Equation B.54 and Equation B.55 let us calculate

$$\begin{aligned}
\mathrm{com}_{\mathbf{L}^2(N,\tau)} L_{x,M}(\mathrm{inc}_2(u)) &= \pi_{\mathbf{L}^2(N,\tau)}\big(L_{x,M}(\mathrm{inc}_2(u))\big) \\
&= \pi_{\mathbf{L}^2(N,\tau)}\big(\mathrm{inc}_2\big(L_{x,N}(u)\big)\big) \\
&= \mathrm{inc}_2\big(L_{x,N}(u)\big).
\end{aligned}$$

Apply inc_2^{-1} to get the right diagram in Diagram B.49. Altogether, get Diagram B.49. We are left to show positivity-preservation. This follows from Proposition A.2.17, as well as Proposition A.2.19 based on the former. □

Arguing as in the proof of Proposition V.2.36 in [192], we construct noncommutative conditional expectations of semi-finite W^*-algebras. We may use Theorem B.2.28 below for this since noncommutative conditional expectations are not used in its proof.

Let $N \subset (M,\tau)$. We identify as per Remark B.2.29 following Theorem B.2.28. Thus $N_* = L^1(N,\tau) \subset L^1(M,\tau) = M_*$ and $L^2(N,\tau) \subset L^2(M,\tau)$ by $L^0(N,\tau) \subset L^0(M,\tau)$. Dualising this inclusion map $\iota : N_* \longrightarrow M_*$ yields unique noncommutative conditional expectation from M to N. Definition B.2.7 gives its defining properties.

Definition B.2.7. Let $N \subset (M,\tau)$. We say that a normal unital map $P : M \longrightarrow N$ is a noncommutative conditional expectation from M to N if

1) $P(x) = x$ for all $x \in N$, (Projection)

2) $P(x) = 0$ implies $x = 0$ for all $x \in M_+$, (Faithfulness)

3) $P(x)(y) = x(y)$ for all $x \in M$ and $y \in N_*$. (Trace identity)

Remark B.2.8. Following Remark B.1.52, we use the modified standard pairing as per Definition B.1.50 to have noncommutative L^1-spaces as pre-duals. The trace identity is equivalent to the following. For all $x \in M$ and $y \in L^1(N,\tau)$, we have

$$\tau\big(P(x)^* y\big) = \tau\big(P(x^*)y\big) = \tau(x^* y). \tag{B.56}$$

Equation B.56 shows P is unique if it exists. If $\tau < \infty$, then $M \subset L^2(M,\tau) \subset L^1(M,\tau)$ by Hölder and we have analogous chain of subspaces for N. We therefore see $\tau < \infty$ ensures P extends to the Hilbert space projection $\pi_N^M : L^2(M,\tau) \longrightarrow L^2(N,\tau)$.

Definition B.2.9. Let $N \subset (M,\tau)$ and let $\iota : N_* \longrightarrow M_*$ denote the canonical inclusion given by the modified standard pairing. We call $\pi_N^M := \iota^* : M \longrightarrow N$ the noncommutative conditional expectation from M to N.

Proposition B.2.10. *If $N \subset (M,\tau)$, then $\pi_N^M : M \longrightarrow N$ is noncommutative conditional expectation from M to N. If $N \subset M$ is furthermore a unital W^*-subalgebra, then π_N^M is trace-preserving.*

Proof. We may argue here as in the proof of Proposition V.2.36 in [192] to show π_N^M is a noncommutative conditional expectation. This assumes unitality. However, the latter is only used to show $\tau \circ \pi_N^M = \tau$ on $\mathfrak{m}_\tau$. Equation B.56 implies uniqueness. □

We give two classes of noncommutative conditional expectations used throughout our discussion in Proposition B.2.13. First, we decompose Hilbert space projections. We use such decomposition in order to reduce non-unital to unital cases if the given trace is finite. Secondly, we compress with projections using abstract compression maps as per Definition A.2.15 for compressed W^*-subalgebras as per Example A.1.48.

Remark B.2.11. Assume $\tau < \infty$. If $N \subset M$ is a W^*-subalgebra, then we know $N \subset (M,\tau)$ by Proposition B.1.13. The latter shows semi-finiteness is satisfied for finite faithful normal traces. We use this for Definition B.2.12 and 1) in Proposition B.2.13.

Definition B.2.12. Assume $\tau < \infty$. Let $N \subset M$ be a W^*-subalgebra. For all $x \in M$, set

$$\kappa_N^M(x) := \begin{cases} \tau(1_N^\perp)^{-1} \cdot \tau\Big(\pi^M_{\langle 1_N^\perp\rangle_{\mathbb{C}}}(x)\Big) & \text{if } 1_M \neq 1_N, \\ 0 & \text{else}. \end{cases}$$

Proposition B.2.13. *Let $N \subset M$ be a W^*-subalgebra.*

1) *If $\tau < \infty$, then $N \subset (M,\tau)$ and $\pi_N^M = \pi^M_{N[1_M]} - \kappa_N^M 1_N^\perp$.*
2) *If $p \in M$ is a projection, then $M[p] \subset (M,\tau)$ and $\pi^M_{M[p]} = \mathrm{com}_p$.*

Proof. Assume $\tau < \infty$. Proposition B.1.13 shows $N \subset (M,\tau)$. By our construction of noncommutative L^2-spaces, we have orthogonal decomposition

$$L^2(N[1_M],\tau) = L^2(N,\tau) \oplus \langle 1_N^\perp\rangle_{\mathbb{C}} \subset L^2(M,\tau) \tag{B.57}$$

since $N[1_M] = N \oplus \langle 1_N^\perp\rangle_{\mathbb{C}}$ by Proposition A.1.71. Extending to Hilbert space projections as per Remark B.2.8, Equation B.57 shows

$$\pi^M_{N[1_M]} = \pi_N^M \oplus \pi^M_{\langle 1_N^\perp\rangle_{\mathbb{C}}} \tag{B.58}$$

w.r.t. $\mathscr{B}(L^2(N,\tau)) \oplus \mathscr{B}(\langle 1_N^\perp\rangle_{\mathbb{C}})$. Equation B.58 implies 1) at once.

We show 2). Let $p \in M$ be a projection. Let $x \in N_+$ be non-zero. Semi-finiteness of τ yields $y \in M_+$ s.t. $y \leq x$ and $\tau(y) < \infty$. Note $pyp \in M[p]_+$. Get $pyp \leq x$ in $M[p]$ since $x = pxp$. In addition, traciality implies

$$0 \leq \tau(pyp) + \tau\big((1_M - p)y(1_M - p)\big) = \tau(y) < \infty. \tag{B.59}$$

Equation B.59 implies $\tau(pyp) < \infty$. We obtain $M[p] \subset (M,\tau)$. Equation B.56 determines noncommutative conditional expectations. Moreover, $M[p] \cap L^1(M[p],\tau) \subset L^1(M[p],\tau)$ is $\|.\|_1$-dense by construction as per Definition B.1.41. It suffices to show

$$\tau\Big(\pi^M_{M[p]}(x)^* y\Big) = \tau\big((\mathrm{com}_p x)^* y\big) \tag{B.60}$$

for all $x \in M$ and $y \in M[p] \cap L^1(M[p],\tau)$. Applying Equation B.56 and using $y = pyp$ in each case, we directly verify Equation B.60. Get 2). □

In Subsection 2.1.1, we write noncommutative conditional expectations in the unital finite-dimensional case as as averages of unitary conjugations. We ultimately obtain the general non-unital finite-dimensional one by 1) in Proposition B.2.13.

$\mathbf{L}^2$-reducible measurable operators. Let (M,τ) be a tracial W^*-algebra. For all $N \subset (M,\tau)$, Theorem B.2.28 yields canonical inclusion $L^0(N,\tau) \subset L^0(M,\tau)$ preserving noncommutative L^p-norms. Equation B.52 leads to Definition B.2.14.

Definition B.2.14. For all $N \subset (M,\tau)$, we call

$$\mathbf{L}^0(N,\tau) := \Big\{x \in L^0(M,\tau) \;\Big|\; L_{x,M} \text{ is } L_M(N)''\text{-affiliated, } x = 1_N x\Big\} \tag{B.61}$$

the space of $L^2(N,\tau)$-reducible measurable operators in $L^0(M,\tau)$.

Proposition B.2.15. *Let $N \subset (M,\tau)$.*

1) $\mathbf{L}^0(N,\tau) \subset \mathbf{L}^0(N[1_M],\tau) \subset L^0(M,\tau)$ are *-subalgebras.

2) $\mathbf{L}^0(N,\tau) = \overline{N}$ for uniform closure in measure topology of (M,τ).

Proof. Note $L_M(N)'' = L_M(N[1_M])$. The construction of spaces of measurable operators reviewed in Subsection B.1.2 taken from [193] is in fact independent of choice of normal faithful unital *-representation. Using L_M and f.s.n. trace $\tau : L_M(N[1_M]) \longrightarrow [0,\infty]$, we see L_M maps $\mathbf{L}^0(N[1_M],\tau)$ onto $L^0(L_M(N[1_M]),\tau)$. This implies 1) and 2) for $N[1_M]$ since uniform structure is determined by the measure topology on M, resp. $\mathscr{L}(M)$.

By definition, $\mathbf{L}^0(N,\tau) \subset L^0(N[1_M],\tau)$ is a *-subalgebra. Get 1). Equation B.52 shows $N \subset \mathbf{L}^0(N,\tau)$. We have $\mathbf{L}^0(N,\tau) \subset \overline{N}$ by 2) for $N[1_M]$ and continuity of multiplication on bounded subsets of $L^0(M,\tau)$ (cf. Theorem IX.2.2 in [193] and [161]). We therefore get 2) by taking uniform closure. □

Let $N \subset (M,\tau)$. Let $x \in \mathbf{L}^0(N,\tau)$ be self-adjoint. For all $Z \in \mathfrak{B}(\mathbb{R})$, Proposition B.1.26 ensures the affiliation property in Equation B.61 implies $E_{L_{x,M}}(Z) \in L_M(N[1_M])$. For all $Z \in \mathfrak{B}(\mathbb{R})$, get $E_{x,M}(Z) \in N[1_M]$ by 1) in Proposition B.1.70 and we have decomposition

$$E_{x,M}(Z) = 1_N E_{x,M}(Z) 1_N \oplus 1_N^{\perp} E_{x,M}(Z) 1_N^{\perp} = \mathrm{com}_{1_N} E_{x,M}(Z) \oplus \nu_{x,N}(Z) 1_N^{\perp} \tag{B.62}$$

w.r.t. $N[1_M] = N \oplus \langle 1_N^{\perp} \rangle_{\mathbb{C}}$. Note $\nu_{x,N}(Z) \in \{0,1\}$ in each case. Equation B.62 in turn yields two compressed spectral measures. For all $Z \in \mathfrak{B}(\mathbb{R})$, set

$$E_{x,N}(Z) := \mathrm{com}_{1_N} E_{x,M}(Z). \tag{B.63}$$

The map $Z \mapsto \nu_{x,N}(Z) \in \{0,1\}$ defined on $\mathfrak{B}(\mathbb{R})$ is determined by $Z \mapsto 1_N^{\perp} E_{x,M}(Z) 1_N^{\perp}$. If $N \subset M$ is a unital W^*-subalgebra, then set $\nu_x := 0$. If not, then $E_{x,M}$ spectral measure of x in M and $\nu_{x,N}(Z) \in \{0,1\}$ in each case implies there exists unique $\nu_x \in \mathbb{R}$ s.t.

$$\nu_{x,N}(Z) = \chi_Z(\nu_x) \tag{B.64}$$

for all $Z \in \mathfrak{B}(\mathbb{R})$. Equation B.64 shows ν_x determines $\nu_{x,N}$.

Definition B.2.16. Let $N \subset (M,\tau)$. For all self-adjoint $x \in \mathbf{L}^0(N,\tau)$, we define

1) the map $Z \mapsto E_{x,N}(Z)$ on $\mathfrak{B}(\mathbb{R})$ as per Equation B.63,

2) $\nu_x = 0$ if $N \subset M$ is a unital W^*-subalgebra, and $\nu_x \in \mathbb{R}$ as per Equation B.64 if not.

Remark B.2.17. Upon identifying $\mathbf{L}^0(N,\tau) = L^0(N,\tau)$ as per Remark B.2.29, we readily see $E_{x,N}$ as per 1) in Definition B.2.16 is in fact the spectral measure of x in N as per 1) in Definition B.1.69 for all $x \in L^0(N,\tau)$.

Proposition B.2.18. *Let $N \subset (M,\tau)$. If $x \in \mathbf{L}^0(N[1_M],\tau)$ is self-adjoint, then*

1) we define spectral measure $L_N\big(E_{x,N}\big)$ on $\mathbb{R}$ with values in $\mathscr{B}\big(L^2(N,\tau)\big)$ by setting

$$L_N\big(E_{x,N}\big)(Z) := L_N\big(E_{x,N}(Z)\big) \tag{B.65}$$

for all $Z \in \mathfrak{B}(\mathbb{R})$,

2) $L_{x,M}$ is $\mathbf{L}^2(N[1_M],\tau)$-, $\mathbf{L}^2(N,\tau)$- and $\mathbf{L}^2(\langle 1_N^{\perp}\rangle_{\mathbb{C}},\tau)$-reducible,

3) $\mathrm{com}_{\mathbf{L}^2(N[1_M],\tau)} L_{x,M} = \mathrm{com}_{\mathbf{L}^2(N,\tau)} L_{x,M} + \mathrm{com}_{\mathbf{L}^2(\langle 1_N^{\perp}\rangle_{\mathbb{C}},\tau)} L_{x,M}$.

Proof. Let $x \in \mathbf{L}^0(N[1_M],\tau)$ be self-adjoint. Get 1) by Proposition B.2.6. We show 2). For all $Z \in \mathfrak{B}(\mathbb{R})$, we use 1) in Proposition B.1.70 and Equation B.51 to calculate

$$\left[E_{L_{x,M}}(Z), \pi_{\mathbf{L}^2(N[1_M],\tau)}\right] = \left[L_{E_{x,M}(Z),M}, \pi_{\mathbf{L}^2(N[1_M],\tau)}\right] = 0. \tag{B.66}$$

Equation B.66 shows $L_{x,M}$ is $\mathbf{L}^2(N[1_M],\tau)$-reducible by Corollary A.2.28. If we instead use Equation B.52, then we calculate

$$\left[E_{L_{x,M}}(Z), \pi_{\mathbf{L}^2(N,\tau)}\right] = \left[L_{E_{x,M}(Z),M}, \pi_{\mathbf{L}^2(N,\tau)}\right] = 0 \tag{B.67}$$

in each case. Equation B.67 implies $L_{x,M}$ is $\mathbf{L}^2(N,\tau)$-reducible by Corollary A.2.28. If $\tau(1_N^\perp) = \infty$, then $\mathbf{L}^2(\langle 1_N^\perp\rangle_\mathbb{C},\tau) = 0$ by construction. If not, then $\mathbf{L}^2(\langle 1_N^\perp\rangle_\mathbb{C},\tau) = \langle 1_N^\perp\rangle_\mathbb{C}$. For all $Z \in \mathfrak{B}(\mathbb{R})$, we obtain

$$E_{L_{x,M}}(Z)\pi_{\mathbf{L}^2(\langle 1_N^\perp\rangle_\mathbb{C},\tau)} = \pi_{\mathbf{L}^2(\langle 1_N^\perp\rangle_\mathbb{C},\tau)}E_{L_{x,M}}(Z) = \nu_{x,N}(Z)\cdot \pi_{\mathbf{L}^2(\langle 1_N^\perp\rangle_\mathbb{C},\tau)}. \tag{B.68}$$

Equation B.68 shows $L_{x,M}$ is $\mathbf{L}^2(\langle 1_N^\perp\rangle_\mathbb{C},\tau)$-reducible by Corollary A.2.28. Get 2).

We show 3). Using 2), 1.3) in Proposition A.2.24 shows

$$L_{x,M} = \mathrm{com}_{\mathbf{L}^2(N,\tau)} L_{x,M} + \mathrm{com}_{\mathbf{L}^2(N,\tau)^\perp} L_{x,M}. \tag{B.69}$$

Applying $\mathrm{com}_{\mathbf{L}^2(N[1_M],\tau)}$ to Equation B.69 yields

$$\mathrm{com}_{\mathbf{L}^2(N[1_M],\tau)} L_{x,M} = \mathrm{com}_{\mathbf{L}^2(N,\tau)} L_{x,M} + \mathrm{com}_{\mathbf{L}^2(N[1_M],\tau)}\left(\mathrm{com}_{\mathbf{L}^2(N,\tau)^\perp} L_{x,M}\right) \tag{B.70}$$

since $\mathbf{L}^2(N,\tau) \subset \mathbf{L}^2(N[1_M],\tau)$. We directly verify

$$\pi_{\mathbf{L}^2(\langle 1_N^\perp\rangle_\mathbb{C},\tau)} = \pi_{\mathbf{L}^2(N,\tau)^\perp}\pi_{\mathbf{L}^2(N[1_M],\tau)} = \pi_{\mathbf{L}^2(N[1_M],\tau)}\pi_{\mathbf{L}^2(N,\tau)^\perp} \tag{B.71}$$

by testing on the inner product. Equation B.71 implies

$$\mathrm{com}_{\mathbf{L}^2(N[1_M],\tau)}\left(\mathrm{com}_{\mathbf{L}^2(N,\tau)^\perp} L_{x,M}\right) = \mathrm{com}_{\mathbf{L}^2(\langle 1_N^\perp\rangle_\mathbb{C},\tau)} L_{x,M}. \tag{B.72}$$

Applying Equation B.72 to the right-hand side of Equation B.70 shows 4). $\square$

Lemma B.2.19. *Let $N \subset (M,\tau)$. If $x \in \mathbf{L}^0(N[1_M],\tau)$ is self-adjoint, then $\int \lambda dL_N(E_{x,N})$ is a τ-measurable self-adjoint unbounded operator on $L^2(N,\tau)$.*

Proof. Let $x \in \mathbf{L}^0(N[1_M],\tau)$ be self-adjoint. Set $T_x := \int \lambda dL_N(E_{x,N})$. Proposition B.1.26 shows the affiliation property in Equation B.61 ensures T_x is N-affiliated. We are left to show τ-measurability as claimed. We use Notation B.1.8.

Let $Z \in \mathfrak{B}(\mathbb{R})$. Proposition B.2.6 shows

$$L_N\big(E_{x,N}(Z)\big) = \mathrm{inc}_2^{-\dagger}\big(\mathrm{com}_{\mathbf{L}^2(N,\tau)} L_M\big(E_{x,M}(Z)\big)\big). \tag{B.73}$$

Note 2) in Proposition B.2.18 implies $L_{x,M}$ is $\mathbf{L}^2(N,\tau)$-reducible. Lemma A.2.26 shows

$$\mathrm{com}_{\mathbf{L}^2(N,\tau)} L_M\big(E_{x,M}(Z)\big) = \mathrm{com}_{\mathbf{L}^2(N,\tau)} L_{E_{x,M}(Z),M} = E_{\mathrm{com}_{\mathbf{L}^2(N,\tau)} L_{x,M}}(Z). \tag{B.74}$$

We combine Equation B.73 and Equation B.74 to

$$L_N\big(E_{x,N}(Z)\big) = \mathrm{inc}_2^{-\dagger}\Big(E_{\mathrm{com}_{\mathbf{L}^2(N,\tau)} L_{x,M}}(Z)\Big). \tag{B.75}$$

Upon inversion of $\mathrm{inc}_2^{\dagger}$, we see Equation B.75 ensures 2) in Corollary A.2.13 applies here. Applying said corollary accordingly, get

$$T_x = \mathrm{inc}_2^{-\dagger}\big(\mathrm{com}_{\mathbf{L}^2(N,\tau)} L_{x,M}\big). \tag{B.76}$$

Equation B.76 implies $T_x^2 = T_{x^2}$ and therefore

$$|T_x| = T_{|x|}. \tag{B.77}$$

Equation B.62 and Equation B.77 let us calculate

$$\tau\big(E_{|x|,M}(Z)\big) = \tau\big(L_M\big(E_{|x|,M}(Z)\big)\big) = \tau\big(E_{|T_x|,M}(Z)\big) + \nu_{|x|,N}(Z)\cdot\tau(1_N^{\perp}) \tag{B.78}$$

for all $Z \in \mathfrak{B}(\mathbb{R})$. Equation B.78 shows τ-measurability of $L_{x,M}$ implies τ-measurability of T_x. This is our claim by construction. □

Definition B.2.20. Let $N \subset (M,\tau)$. For all $x \in \mathbf{L}^0(N[1_M],\tau)$, set

$$\mathrm{com}_N x := L_N^{-1}\Big(\int \lambda dL_N\big(E_{\mathrm{Re}(x),N}\big)\Big) + iL_N^{-1}\Big(\int \lambda dL_N\big(E_{\mathrm{Im}(x),N}\big)\Big). \tag{B.79}$$

Lemma B.2.21. *Let $N \subset (M,\tau)$. If $x \in \mathbf{L}^0(N[1_M],\tau)$ is self-adjoint, then*

1) $\mathrm{com}_{\mathbf{L}^2(N[1_M],\tau)} L_{x,M} = \mathrm{com}_{\mathbf{L}^2(N,\tau)} \mathrm{inc}_2^\dagger\big(L_{\mathrm{com}_N x,N}\big) + \nu_x \cdot \pi_{\mathbf{L}^2(\langle 1_N^\perp\rangle_{\mathbb{C}},\tau)}$.

2) $\mathrm{com}_{\mathbf{L}^2(N,\tau)} L_{x,M} = \mathrm{com}_{\mathbf{L}^2(N,\tau)} \mathrm{inc}_2^\dagger\big(L_{\mathrm{com}_N x,N}\big)$,

3) $\mathrm{com}_{\mathbf{L}^2(\langle 1_N^\perp\rangle_{\mathbb{C}},\tau)} L_{x,M} = \nu_x \cdot \pi_{\mathbf{L}^2(\langle 1_N^\perp\rangle_{\mathbb{C}},\tau)}$.

Proof. If we have 2) and 3), then 3) in Proposition B.2.18 implies 1). Get 2) by applying $\mathrm{inc}_2^\dagger$ to Equation B.76. Using Lemma A.2.26, Equation B.68 shows the spectral theorem implies 3) since the given spectral measures coincide. □

Corollary B.2.22. *Let $N \subset (M,\tau)$. For all $x \in \mathbf{L}^0(N[1_M],\tau)$, we have*

1) $1_N x 1_N \in \mathbf{L}^0(N,\tau)$,

2) $\mathrm{com}_N x = \mathrm{com}_N 1_N x 1_N$.

Proof. We know 1) since the affiliation property in Equation B.61 is identical. Note 1) in Proposition B.2.15 shows all claims reduce to self-adjoint elements. Let $x \in \mathbf{L}^0(N[1_M],\tau)$ be self-adjoint. Using Corollary B.1.64, Equation B.53 lets us calculate

$$\begin{aligned}
\mathrm{com}_{\mathbf{L}^2(N,\tau)} L_{x,M} &= \mathrm{com}_{\mathbf{L}^2(N[1_M],\tau)} \overline{L_{1_N,M} L_{x,M} L_{1_N,M}} \\
&= \mathrm{com}_{\mathbf{L}^2(N[1_M],\tau)} L_{1_N,M} \cdot \overline{L_{x,M} L_{1_N,M}} \\
&= \mathrm{com}_{\mathbf{L}^2(N[1_M],\tau)} L_{1_N,M} \cdot \overline{L_{x,M}} \cdot L_{1_N,M} \\
&= \mathrm{com}_{\mathbf{L}^2(N[1_M],\tau)} L_{1_N,M} L_{x,M} L_{1_N,M} \\
&= \mathrm{com}_{\mathbf{L}^2(N[1_M],\tau)} L_{1_N x 1_N,M}
\end{aligned}$$

by boundedness of left- and right-multiplication with 1_N. Equation B.76 shows applying $\mathrm{com}_{\mathbf{L}^2(N,\tau)}$ to both sides of the above calculation yields 2). □

Upon identifying $\mathbf{L}^0(N,\tau) = L^0(N,\tau)$ as per Remark B.2.29, note Lemma B.2.24 lets us extend $N[1_M] = N \oplus \langle 1_N^\perp\rangle_{\mathbb{C}}$ to $L^0(N[1_M],\tau) = L^0(N,\tau) \oplus \langle 1_N^\perp\rangle_{\mathbb{C}}$ as per Equation B.100 using direct sum of *-algebras s.t. integrability is preserved. Theorem B.2.28 and its Corollary B.2.32 ensure this extends to L^p-norms for all $p \in [1,\infty]$. We may therefore forget all a priori complications underlying Definition B.2.20, treating $\mathrm{com}_N = \mathrm{com}_{1_N}$ and $\mathrm{com}_{1_N^\perp}$ as in the bounded case. We make this explicit in Diagram B.101.

Definition B.2.23. For all $x \in \mathbf{L}^0(N[1_M],\tau)$, set $\nu_x := \nu_{\mathrm{Re}(x)} + i\nu_{\mathrm{Im}(x)}$.

Lemma B.2.24. *Let $N \subset (M,\tau)$. For all $x \in \mathbf{L}^0(N[1_M],\tau)$, we have*

1) $x = 1_N x 1_N + \nu_x 1_N^\perp$,

2) $x \in \mathbf{L}^0(N,\tau)$ *if and only if* $\nu_{\mathrm{Re}(x)} = \nu_{\mathrm{Im}(x)} = 0$,

3) $\tau(|x|) = \tau(|\mathrm{com}_N x|) \in [0,\infty]$ *if* $x \in \mathbf{L}^0(N,\tau)$.

Proof. If $N \subset M$ is a unital W^*-subalgebra, then we reduce to the non-unital case for $\nu_x = 0$. We assume $N \subset M$ is a non-unital W^*-subalgebra without loss of generality. Let $x \in \mathbf{L}^0(N[1_M],\tau)$. We require 1) to show 2) and 3).

We show 1). As $\mathbf{L}^0(N[1_M],\tau)$ is a *-subalgebra by 1) in Proposition B.2.15, we assume x is self-adjoint without loss of generality. We show Equation B.86 to get decomposition as per 1) by the spectral theorem. Using 2) in Lemma B.2.21 for the first and third identity, as well as 2) in Corollary B.2.22 for the second one, we calculate

$$\begin{aligned}
\mathrm{com}_{\mathbf{L}^2(N,\tau)} L_{x,M} &= \mathrm{com}_{\mathbf{L}^2(N,\tau)} \mathrm{inc}_2^\dagger\big(L_{\mathrm{com}_N x, N}\big) \\
&= \mathrm{com}_{\mathbf{L}^2(N,\tau)} \mathrm{inc}_2^\dagger\big(L_{\mathrm{com}_N 1_N x 1_N, N}\big) \\
&= \mathrm{com}_{\mathbf{L}^2(N,\tau)} L_{1_N x 1_N, M}.
\end{aligned}$$

We show Equation B.86. Let $Z \in \mathfrak{B}(\mathbb{R})$. Using the above calculation, Equation B.75 immediately implies

$$L_N\big(E_{x,N}(Z)\big) = L_N\big(E_{1_N x 1_N, N}(Z)\big). \tag{B.80}$$

Note Equation B.63 ensures $E_{x,N}(Z), E_{1_N x 1_N, N}(Z) \in N$ by definition. Moreover, we know $L_N : N \longrightarrow \mathscr{B}(L^2(N,\tau))$ is faithful by 1) in Proposition B.2.3. Applying L_N^{-1} to both sides of Equation B.80 yields

$$E_{x,N}(Z) = E_{1_N x 1_N, N}(Z). \tag{B.81}$$

Equation B.62 and Equation B.81 show we have decomposition

$$E_{x,M}(Z) = E_{1_N x 1_N, N}(Z) \oplus \nu_{x,N} 1_N^\perp(Z) \tag{B.82}$$

w.r.t. $N[1_M] = N \oplus \langle 1_N^\perp \rangle_{\mathbb{C}}$.

Using $N1_N^\perp = 1_N^\perp N = 0$, we directly verify

$$R_{\pm i}\big(1_N x 1_N + \nu_x 1_N^\perp\big) = 1_N(1_N x 1_N \mp i 1_M)^{-1} 1_N + 1_N^\perp\big(\nu_x 1_N^\perp \mp i 1_M\big)^{-1} 1_N^\perp \tag{B.83}$$

in $L^0(M,\tau)$. Using 3) in Proposition B.1.70 and bounded measurable functional calculus as per Definition B.1.73 inside compression terms, Equation B.83 implies

$$E_{1_N x 1_N + \nu_x 1_N^\perp, M}(Z) = E_{1_N x 1_N, N}(Z) \oplus 1_N^\perp E_{\nu_x 1_N^\perp, M}(Z) 1_N^\perp \tag{B.84}$$

w.r.t. $N[1_M] = N \oplus \langle 1_N^\perp \rangle_{\mathbb{C}}$. Note Equation B.63 ensures $E_{1_N x 1_N, N}(Z) = 1_N E_{1_N x 1_N, M}(Z) 1_N$ by definition. Lemma B.1.72 shows elements in $W_M^*(\nu_x 1_N)$ as per 3) in Proposition B.1.70 are strong limits of finite polynomials with elements in $\{\nu_x 1_N^\perp, 1_M\}$. Using the latter, get $\nu_{x,N} 1_N^\perp(Z) + 1_N = E_{\nu_x 1_N^\perp, M}(Z)$ since $E_{\nu_x 1_N^\perp, M}(\mathbb{R}) = 1_M$. This implies

$$\nu_{x,N} 1_N^\perp(Z) = 1_N^\perp E_{\nu_x 1_N^\perp, M} 1_N^\perp(Z). \tag{B.85}$$

Equation B.85 shows the right-hand sides of Equation B.84 and Equation B.82 are identical in each case. For all $Z \in \mathfrak{B}(\mathbb{R})$, we therefore have

$$E_{x,M}(Z) = E_{1_N x 1_N + \nu_x 1_N^\perp, M}(Z). \tag{B.86}$$

Using 2) in Lemma B.1.72, Equation B.86 shows the spectral theorem implies 1) since the given spectral measures coincide. Note 1) shows 2) at once.

We show 3). Equation B.77 shows $|\mathrm{com}_N x| = \mathrm{com}_N |x|$. We directly verify $|1_N x 1_N| = 1_N |x| 1_N$. Thus $|x| \in \mathbf{L}^0(N,\tau)$ if $x \in \mathbf{L}^0(N,\tau)$, hence we assume $x \in \mathbf{L}^0(N,\tau)$ is positive without loss of generality. Then 2) implies $\nu_x = 0$. Note the infimum in Equation B.11 runs over all $\lambda > 0$. Equation B.62 and $\nu_x = 0$ therefore imply the generalised singular number of x is given by

$$\mu_t(x) = \begin{cases} \mu_{t-\tau(1_N^\perp)}(\mathrm{com}_N x) & \text{if } t \geq \tau(1_N^\perp), \\ 0 & \text{else}. \end{cases}$$

We use the explicit expression above to calculate

$$\tau(x) = \int_0^\infty \mu_t(x) dt = \int_{\tau(1_N^\perp)}^\infty \mu_{t-\tau(1_N^\perp)}(\mathrm{com}_N x) dt = \int_0^\infty \mu_t(\mathrm{com}_N x) dt = \tau(\mathrm{com}_N x). \tag{B.87}$$

Following Remark B.1.40, Equation B.87 shows $\tau(|x|) = \tau(|\mathrm{com}_N x|)$. Get 3). □

Corollary B.2.25. *Let $N \subset (M,\tau)$. For all $x \in \mathbf{L}^0(N,\tau)$, we have*

1) $x \in L^1(M,\tau)$ *if and only if* $\mathrm{com}_N x \in L^1(N,\tau)$,

2) $\|x\|_1 = \|\mathrm{com}_N x\|_1$ *and* $\tau(x) = \tau(\mathrm{com}_N x)$ *if* $x \in L^1(M,\tau)$.

Proof. Note 3) in Lemma B.2.24 shows 1). If we furthermore extend Equation B.87 to all $x \in \mathbf{L}^0(N,\tau) \cap L^1(M,\tau)$, then 2) follows. Equation B.76 shows $\mathrm{com}_N : \mathbf{L}^0(N,\tau) \longrightarrow L^0(N,\tau)$ is linear and positivity-preserving by 1) in Corollary A.2.13 and Proposition A.2.25. For all $\lambda \in \mathbb{R}$, $\max\{\lambda,0\} = \frac{1}{2}(\lambda + |\lambda|)$ and $\min\{\lambda,0\} = \frac{1}{2}(\lambda - |\lambda|)$. We use decomposition as per Proposition B.1.47 and thereby see linearity, positivity-preservation and Equation B.77 extend Equation B.87 to all $x \in \mathbf{L}^0(N,\tau) \cap L^1(M,\tau)$. □

Lemma B.2.26. *If $N \subset (M,\tau)$, then*

1) $\mathrm{com}_N : \mathbf{L}^0(N[1_M],\tau) \longrightarrow L^0(N,\tau)$ *is a surjective* **-homomorphism,*

2) $\mathrm{com}_N : \mathbf{L}^0(N,\tau) \longrightarrow L^0(N,\tau)$ *is a* **-isomorphism,*

3) *we have commutative diagram*

$$\begin{tikzcd}
\mathbf{L}^0(N,\tau) \arrow[r, hook] \arrow[d, "\mathrm{com}_N"] & \mathbf{L}^0(N[1_M],\tau) \arrow[r, "L_M"] \arrow[d, "\mathrm{com}_N"] & \mathscr{UB}\big(L^2(M,\tau)\big) \arrow[d, "\mathrm{inc}_2^{-\dagger} \circ \mathrm{com}_{\mathbf{L}^2(N,\tau)}"] \\
L^0(N,\tau) \arrow[r, "\mathrm{id}_{L^0(N,\tau)}"] & L^0(N,\tau) \arrow[r, "L_N"] & \mathscr{UB}\big(L^2(N,\tau)\big)
\end{tikzcd} \tag{B.88}$$

s.t. horizontal maps are normal unital injective **-homomorphisms and vertical ones are positivity-preserving linear surjections,*

3) $\mathbf{L}^2(N,\tau) = \mathrm{com}_N{}^{-1}\big(L^2(N,\tau)\big)$ *and* $\mathrm{com}_N{}^{-1} : L^2(N,\tau) \longrightarrow \mathbf{L}^2(N,\tau)$ *is an isometric isomorphism of Hilbert spaces restricting to the identity on N.*

Proof. Equation B.76 shows Diagram B.88 for self-adjoint elements. Said equation also shows $\mathrm{com}_N : \mathbf{L}^0(N[1_M],\tau) \longrightarrow L^0(N,\tau)$ is linear. Twisting and concrete compression maps are linear and positivity-, hence order-preserving by 1) in Corollary A.2.13 and Proposition A.2.25. Get Diagram B.88 from the case of self-adjoint elements.

Thus $\mathrm{com}_N : \mathbf{L}^0(N[1_M],\tau) \longrightarrow L^0(N,\tau)$ is a positivity-, hence order-preserving linear map. In particular, com_N commutes with algebra involution. If we have

$$\mathrm{com}_{\mathbf{L}^2(N,\tau)} L_{xy,M} = \overline{\mathrm{com}_{\mathbf{L}^2(N,\tau)} L_{x,M} \cdot \mathrm{com}_{\mathbf{L}^2(N,\tau)} L_{y,M}} \tag{B.89}$$

for all $x, y \in \mathbf{L}^0(N[1_M],\tau)$, then we see 3.3) in Proposition A.1.14 and Diagram B.89 imply com_N is a *-homomorphism.

For all $y \in \mathbf{L}^0(N[1_M],\tau)$, we know $\mathbf{L}^2(N,\tau)$-reducibility by 2) in Proposition B.2.18 and use $\pi_{\mathbf{L}^2(N,\tau)} L_{y,M} \subset L_{y,M}\pi_{\mathbf{L}^2(N,\tau)}$ as per Equation A.44 to get

$$\mathrm{com}_{\mathbf{L}^2(N,\tau)} L_{y,M} = \pi_{\mathbf{L}^2(N,\tau)} \cdot L_{y,M} \cdot \pi_{\mathbf{L}^2(N,\tau)} = L_{y,M} \cdot \pi_{\mathbf{L}^2(N,\tau)}. \tag{B.90}$$

Note left- and right-multiplication with $\pi_{\mathbf{L}^2(N,\tau)}$ commute with taking closure. Using Corollary B.1.64 and Equation B.90, we directly verify Equation B.89. Get 1) and 3).

We show 2). We must show injectivity. It suffices to consider self-adjoint elements. By Lemma B.2.21 and 2) in Lemma B.2.24, we assume $N \subset M$ is a unital W^*-subalgebra without loss of generality. For all $x \in \mathbf{L}^0(N,\tau)$, get $E_{x,M} = E_{x,N}$. The spectral theorem then shows $L_{\mathrm{com}_N x,N}$ determines $L_{x,M}$ uniquely in each case. Injectivity as required for 2) holds. We show 4). For all $x \in \mathbf{L}^0(N,\tau)$, $\mathrm{com}_N |x|^2 = |\mathrm{com}_N x|^2$ because com_N is a positivity-preserving *-homomorphism. Get 4) by 2) and Corollary B.2.25. □

Remark B.2.27. Note $\mathrm{inc}_2 = {\mathrm{com}_N}^{-1}$ on $L^2(N,\tau)$ since both are isometries restricting to the identical map on a dense subset.

We have $\mathbf{L}^2(N,\tau) = {\mathrm{com}_N}^{-1}(L^2(N,\tau))$ and identify along inc_2. This is subsumed by the general convention fixed in Remark B.2.29.

Theorem B.2.28. *Let (M,τ) be a tracial W^*-algebra. If $N \subset (M,\tau)$, then*

1) ${\mathrm{com}_N}^{-1} : L^0(N,\tau) \longrightarrow \mathbf{L}^0(N,\tau)$ *is a *-isomorphism,*

2) we have commutative diagram

$$\begin{tikzcd}
\mathbf{L}^0(N,\tau) \arrow[r, hook] \arrow[d, "\mathrm{com}_N"] & \mathbf{L}^0(N[1_M],\tau) \arrow[r, "L_M"] \arrow[d, "\mathrm{com}_N"] & \mathscr{U}\mathscr{B}(L^2(M,\tau)) \arrow[d, "\mathrm{com}_{L^2(N,\tau)}"] \\
L^0(N,\tau) \arrow[r, "\mathrm{id}_{L^0(N,\tau)}"] & L^0(N,\tau) \arrow[r, "L_N"] & \mathscr{U}\mathscr{B}(L^2(N,\tau))
\end{tikzcd} \tag{B.91}$$

s.t. horizontal maps are normal unital injective *-homomorphisms and vertical ones are positivity-preserving linear surjections,

3) for all $p \in [1,\infty]$, $\mathbf{L}^p(N,\tau) = {\mathrm{com}_N}^{-1}(L^p(N,\tau))$ and ${\mathrm{com}_N}^{-1} : L^p(N,\tau) \longrightarrow \mathbf{L}^p(N,\tau)$ is an isometric isomorphism of Banach spaces restricting to the identity on N.

Proof. We identify along inc_2. Lemma B.2.26 implies 1) and 2) at once. We show 3). For this, we instead require the following. Let $x \in \mathbf{L}^0(N[1_M],\tau)$. If $\alpha \in \mathbb{Q}$, then we know

$$\mathrm{com}_N |x|^\alpha = |\mathrm{com}_N x|^\alpha \tag{B.92}$$

because com_N is a positivity-preserving *-homomorphism by 1) in Lemma B.2.26. We show Equation B.92 holds for all $\alpha \geq 0$.

Let $\beta \geq 0$. For all $\lambda \geq 0$, set $g_\beta(\lambda) := \lambda^\beta$. Get $R_i(g_\beta) \in C_0([0,\infty))$ and $R_i(|x|^\beta) \in \mathbf{L}^0(N,\tau)$. Since com_N is a positivity-preserving *-homomorphism, get

$$\mathrm{com}_N R_i\big(|x|^\beta\big) = R_i\big(\mathrm{com}_N |x|^\beta\big). \tag{B.93}$$

Let $\alpha \geq 0$ and $\{\alpha_n\}_{n\in\mathbb{N}} \subset \mathbb{Q} \cap (0,\infty)$ s.t. $\alpha = \lim_{n\in\mathbb{N}} \alpha_n$. We thereby have uniformly bounded pointwise limit $R_i(g_\alpha) = \lim_{n\in\mathbb{N}} R_i(g_{\alpha_n})$ in $C_0([0,\infty))$ and calculate

$$R_i\big(|x|^\alpha\big) = \text{s-}\lim_{n\in\mathbb{N}} R_i\big(|x|^{\alpha_n}\big),\ R_i\big(|\,\mathrm{com}_N x|^\alpha\big) = \text{s-}\lim_{n\in\mathbb{N}} R_i\big(|\,\mathrm{com}_N x|^{\alpha_n}\big). \tag{B.94}$$

Note 2) in Corollary B.2.22 shows $\mathrm{com}_N = \mathrm{com}_{1_N}$ on $N[1_M]$. Thus com_N restricted to $N[1_M]$ is a completely positive normal bounded linear map by Proposition A.2.17, hence bounded strongly continuous by Proposition A.1.49. Using bounded strong continuity in each case, Equation B.92, Equation B.93 and Equation B.94 let us calculate

$$\begin{aligned}\mathrm{com}_N R_i\big(|x|^\alpha\big) &= \text{s-}\lim_{n\in\mathbb{N}} \mathrm{com}_N R_i\big(|x|^{\alpha_n}\big)\\ &= \text{s-}\lim_{n\in\mathbb{N}} R_i\big(\mathrm{com}_N |x|^{\alpha_n}\big)\\ &= \text{s-}\lim_{n\in\mathbb{N}} R_i\big(|\,\mathrm{com}_N x|^{\alpha_n}\big)\\ &= R_i\big(|\,\mathrm{com}_N x|^\alpha\big).\end{aligned}$$

The above calculation shows

$$\mathrm{com}_N R_i\big(|x|^\alpha\big) = R_i\big(|\,\mathrm{com}_N x|^\alpha\big). \tag{B.95}$$

Since $\mathrm{com}_N 1_M = \mathrm{com}_{1_N} 1_M = 1_N$, get $\mathrm{com}_N\,(|x|^\alpha - i1_M) = \mathrm{com}_N |x|^\alpha - i1_N$. Using the latter as first and Equation B.95 for the second identity below, we calculate

$$\mathrm{com}_N |x|^\alpha - i1_N = \big(\mathrm{com}_N R_i\big(|x|^\alpha\big)\big)^{-1} = |\,\mathrm{com}_N x|^\alpha - i1_N. \tag{B.96}$$

Equation B.96 implies Equation B.92 for all $\alpha \geq 0$.

Let $p \in [1,\infty)$. For all $x \in \mathbf{L}^0(N,\tau)$, Equation B.92 shows $\mathrm{com}_N |x|^p = |\,\mathrm{com}_N x|^p$. If $p < \infty$, then we know the latter identity implies 3) by 1) and Corollary B.2.25. The case of $p = \infty$ is clear. Get 3). □

Remark B.2.29. Let (M,τ) be a tracial W^*-algebra. If $N \subset (M,\tau)$, then Theorem B.2.28 ensures we identify along com_N without loss of generality. Note identification preserves *-algebra structure, positivity and noncommutative L^p-norms. Corollary B.2.25 ensures identification further preserves trace.

Compression maps on spaces of measurable operators. If (M,τ) is a tracial W^*-algebra and $p \in M$ is a projection, then $M[p] \subset (M,\tau)$ by 2) in Proposition B.2.13 and $L^0(M[p],\tau) \subset L^0(M,\tau)$ by Theorem B.2.28.

Proposition B.2.30. *Let (M,τ) be a tracial W^*-algebra. For all projections $p \in M$ and $q \in [1,\infty]$, we have $L^0(M[p],\tau) = pL^0(M,\tau)p$ and $L^q(M[p],\tau) = pL^q(M,\tau)p$.*

Proof. Multiplication in $L^0(M,\tau)$ is continuous in measure topology on bounded subsets (cf. Theorem IX.2.2 in [193] and [161]). Using the latter, 2) in Proposition B.2.15 and $M[p] = pMp$, we have $L^0(M[p],\tau) = \overline{M[p]} = pL^0(M,\tau)p$ by approximating in measure topology. For all $q \in [1,\infty]$, get $L^q(M[p],\tau) = pL^q(M,\tau)p$ by 3) in Theorem B.2.28. □

Definition B.2.31. Let (M,τ) be a tracial W^*-algebra. For all projections $p \in M$, we define the compression map $\mathrm{com}_p : L^0(M,\tau) \longrightarrow L^0(M[p],\tau)$ by setting

$$\mathrm{com}_p\, x := pxp \tag{B.97}$$

for all $x \in L^0(M,\tau)$.

Corollary B.2.32. *If $N \subset (M,\tau)$, then*

1) $\mathrm{com}_N = \mathrm{com}_{1_N}$ *on* $L^0(N[1_M],\tau)$,

2) *we have commutative diagrams*

$$\begin{tikzcd} L^0(N,\tau) \arrow[r, hook] \arrow[d, "\mathrm{id}_{L^0(N,\tau)}"] & L^0(N[1_M],\tau) \arrow[r, "L_M"] \arrow[d, "\mathrm{com}_{1_N}"] & \mathscr{U}\mathscr{B}\big(L^2(M,\tau)\big) \arrow[d, "\mathrm{com}_{L^2(N,\tau)}"] \\ L^0(N,\tau) \arrow[r, "\mathrm{id}_{L^0(N,\tau)}"] & L^0(N,\tau) \arrow[r, "L_N"] & \mathscr{U}\mathscr{B}\big(L^2(N,\tau)\big) \end{tikzcd} \tag{B.98}$$

$$\begin{tikzcd} L^0(N,\tau)^{\mathrm{op}} \arrow[r, hook] \arrow[d, "\mathrm{id}_{L^0(N,\tau)}"] & L^0(N[1_M],\tau)^{\mathrm{op}} \arrow[r, "R_M"] \arrow[d, "\mathrm{com}_{1_N}"] & \mathscr{U}\mathscr{B}\big(L^2(M,\tau)\big) \arrow[d, "\mathrm{com}_{L^2(N,\tau)}"] \\ L^0(N,\tau)^{\mathrm{op}} \arrow[r, "\mathrm{id}_{L^0(N,\tau)}"] & L^0(N,\tau)^{\mathrm{op}} \arrow[r, "R_N"] & \mathscr{U}\mathscr{B}\big(L^2(N,\tau)\big) \end{tikzcd} \tag{B.99}$$

s.t. horizontal maps are normal unital injective *-homomorphisms and vertical ones are positivity-preserving linear surjections.

Proof. Get 1) by Corollary B.2.22. Thus Diagram B.98 is 2) in Theorem B.2.28, hence Diagram B.99 follows from Diagram B.98 by 2) in Corollary B.1.63. □

B.2.2 Compressed pulled-back joint functional calculus

We prove Theorem B.2.44 and give compressed pulled-back joint functional calculus of self-adjoint measurable operators in Definition B.2.46. In Subsection 2.1.2, we apply Theorem B.2.44 and its corollaries.

Change of canonical left- and right-actions. We express change of canonical left- and right-actions as abstract compression maps. Let (M,τ) be a tracial W^*-algebra and $N \subset (M,\tau)$. We identify as per Remark B.2.29. For all $x \in L^0(N[1_M],\tau)$, note 1) in Lemma B.2.24 shows we have decomposition

$$x = \mathrm{com}_{1_N} x \oplus \nu_x 1_N^{\perp} \tag{B.100}$$

extending $N[1_M] = N \oplus \langle 1_N^{\perp}\rangle_{\mathbb{C}}$ to $L^0(N[1_M],\tau) = L^0(N,\tau) \oplus \langle 1_N^{\perp}\rangle_{\mathbb{C}}$ using the direct sum of *-algebras induced by canonical inclusion in $L^0(M,\tau)$. Mapping as per Equation B.100 using left diagram in Diagram B.98, we indeed have $L^0(N[1_M],\tau) = L^0(N,\tau) \oplus \langle 1_N^{\perp}\rangle_{\mathbb{C}}$ and commutative diagram

$$\begin{tikzcd} L^0(N,\tau) \arrow[r, hook] \arrow[rr, bend right, "\mathrm{id}_{L^0(N,\tau)}"'] & L^0(N,\tau) \oplus \langle 1_N^{\perp}\rangle_{\mathbb{C}} \arrow[r, "\mathrm{com}_{1_N}"] & L^0(N,\tau) \end{tikzcd} \tag{B.101}$$

of *-homomorphisms. Diagram B.101 extends Diagram A.41.

Lemma B.2.33. *If $x \in L^0(N[1_M],\tau)_h$, then we have*

1) $\mathrm{spec}_M x = \mathrm{spec}_N \mathrm{com}_{1_N} x \cup \{\nu_x\}$ *and* $\mathcal{N}\big(E_{x,M}\big) = \mathcal{N}\big(E_{\mathrm{com}_{1_N} x,N}\big) \cap \mathcal{N}(\nu_x)$,
2) *normal unital surjective *-homomorphism* $\mathrm{com}_{1_N} : W_M^*(x) \longrightarrow W_N^*\big(\mathrm{com}_{1_N} x\big)$ *s.t.*

$$\mathrm{com}_{1_N} \Gamma_{x,M}(g) = \Gamma_{\mathrm{com}_{1_N} x,N}(g) \tag{B.102}$$

for all $g \in L^\infty\big(\mathrm{spec}_M x, dE_{x,M}\big)$,

3) *commutative diagram of normal unital surjective *-homomorphisms*

$$\begin{tikzcd} L^\infty\big(\mathrm{spec}_M x, dE_{x,M}\big) \arrow[r, "\Gamma_{x,M}"] \arrow[d, "\mathrm{res}"] & W_M^*(x) \arrow[d, "\mathrm{com}_{1_N}"] \\ L^\infty\Big(\mathrm{spec}_N \mathrm{com}_{1_N} x, dE_{\mathrm{com}_{1_N} x,N}\Big) \arrow[r, "\Gamma_{\mathrm{com}_{1_N} x,N}"] & W_N^*\big(\mathrm{com}_{1_N} x\big) \end{tikzcd} \tag{B.103}$$

with res *the restriction map given by* $\mathrm{spec}_N \mathrm{com}_{1_N} x \subset \mathrm{spec}_M x$.

Proof. Let $x \in L^0(N[1_M],\tau)_h$. Decomposing $L^0(N[1_M],\tau) = L^0(N,\tau) \oplus \langle 1_N^\perp \rangle_{\mathbb{C}}$, we directly verify 1). We show 2) and 3) using Lemma B.1.72. Note 1) in Corollary B.2.32 shows $\mathrm{com}_N = \mathrm{com}_{1_N}$ on $L^0(N[1_M],\tau)$. Since we identify as per Remark B.2.29, we know 2) in Lemma B.2.21 therefore implies

$$L_{x,M}\big|_{L^2(N,\tau)} = \mathrm{com}_{L^2(N,\tau)} L_{x,M} = L_{\mathrm{com}_{1_N} x,N}. \tag{B.104}$$

Up to representation under canonical left-actions, Equation B.104 shows 2) and 3) in Lemma B.1.72 are 2) and 3) as claimed. We therefore invert canonical left-actions and conclude by 2) in Lemma B.1.72. □

Remark B.2.34. Theorem B.2.28 and Corollary B.2.32 show Lemma B.2.33 subsumes Corollary A.1.69 and Lemma A.2.26 for canonical left- and right-actions of L^2-reducible measurable operators. Choice of unit only involves values at zero.

Corollary B.2.35. *If $x \in L^0(N,\tau)_h$, then we have*

1) $\mathrm{spec}_M x = \mathrm{spec}_N x \cup \{0\}$ *and* $\mathcal{N}(E_{x,M}) \subset \mathcal{N}(E_{x,N})$,
2) $g(L_{x,M})\big|_{L^2(N,\tau)} = g(L_{x,N})$, $g(R_{x,M})\big|_{L^2(N,\tau)} = g(R_{x,N})$ *for all* $g \in L^\infty(\mathrm{spec}_M x, dE_{x,M})$,
3) *commutative diagrams of normal unital surjective* **-homomorphisms*

$$\begin{tikzcd} L^\infty(\mathrm{spec}_M x, dE_{x,M}) \arrow[r, "\Gamma_{x,M}"] \arrow[d, "\mathrm{res}"] & W^*_M(x) \arrow[r, hook] \arrow[d, "\mathrm{com}_{1_N}"] & M \arrow[r, "L_M"] & \mathscr{B}(L^2(M,\tau)) \arrow[d, "\mathrm{com}_{L^2(N,\tau)}"] \\ L^\infty(\mathrm{spec}_N x, dE_{x,N}) \arrow[r, "\Gamma_{x,N}"] & W^*_N(x) \arrow[r, hook] & N \arrow[r, "L_N"] & \mathscr{B}(L^2(N,\tau)) \end{tikzcd} \tag{B.105}$$

$$\begin{tikzcd} L^\infty(\mathrm{spec}_M x, dE_{x,M}) \arrow[r, "\Gamma_{x,M}"] \arrow[d, "\mathrm{res}"] & W^*_M(x) \arrow[r, hook] \arrow[d, "\mathrm{com}_{1_N}"] & M^{\mathrm{op}} \arrow[r, "R_M"] & \mathscr{B}(L^2(M,\tau)) \arrow[d, "\mathrm{com}_{L^2(N,\tau)}"] \\ L^\infty(\mathrm{spec}_N x, dE_{x,N}) \arrow[r, "\Gamma_{x,N}"] & W^*_N(x) \arrow[r, hook] & N^{\mathrm{op}} \arrow[r, "R_N"] & \mathscr{B}(L^2(N,\tau)) \end{tikzcd} \tag{B.106}$$

with res *the restriction map given by* $\mathrm{spec}_N x \subset \mathrm{spec}_M x$.

Proof. Get 1) by 2) in Lemma B.2.24 and 1) in Lemma B.2.33. Diagram B.103 is the left diagram in both Diagram B.105 and Diagram B.106. Diagram B.98 and Diagram B.99 yield right diagrams by restriction to bounded operators. This shows both 2) and 3). □

Corollary B.2.36. *If $x \in L^0(N,\tau)_h$ and $g \in L^\infty(\mathrm{spec}_M x, dE_{x,M})$ s.t. $g(0) = 0$, then*

$$\Gamma_{x,M}(g) = \Gamma_{x,N}(g). \tag{B.107}$$

Proof. Note $N \subset N[1_M] \subset (M,\tau)$ by 2) in Proposition B.2.13. By Theorem B.2.28, we assume $M = N[1_M]$ without loss of generality. Let $x \in L^0(N,\tau)_h$. We know $\nu_x = 0$ by 2) in Lemma B.2.24. Let $g \in L^\infty(\mathrm{spec}_M x, dE_{x,M})$ s.t. $g(0) = 0$. Equation B.102 for $N \subset (N[1_M],\tau)$ and $\langle 1_N^\perp\rangle_{\mathbb{C}} \subset (N[1_M],\tau)$ each lets us calculate

$$\Gamma_{x,M}(g) = \mathrm{com}_{1_N}\Gamma_{x,M}(g) + \mathrm{com}_{1_N^\perp}\Gamma_{x,M}(g) = \Gamma_{x,N}(g) + \Gamma_{0,\langle 1_N^\perp\rangle_{\mathbb{C}}}(g). \tag{B.108}$$

Get $\Gamma_{0,\langle 1_N^\perp\rangle_{\mathbb{C}}}(g) = g(0) = 0$ by hypothesis. Equation B.108 shows Equation B.107. □

Corollary B.2.37. *If $x, y \in L^0(N,\tau)_h$, then we have*

1) $\mathrm{spec}_M x \times y = \big(\mathrm{spec}_N x \cup \{0\}\big) \times \big(\mathrm{spec}_N x \cup \{0\}\big)$ *and* $\mathcal{N}\big(E_{x,y,M}\big) \subset \mathcal{N}\big(E_{x,y,N}\big)$,
2) $g\big(L_{x,M}, R_{y,M}\big)\big|_{L^2(N,\tau)} = g\big(L_{x,N}, R_{y,N}\big)$ *for all* $g \in L^\infty\big(\mathrm{spec}_M x \times y, dE_{x,y,M}\big)$,
3) *commutative diagram of normal unital surjective *-homomorphisms*

$$\begin{tikzcd} L^\infty\big(\mathrm{spec}_M x \times y, dE_{x,y,M}\big) \arrow[r, "\Gamma_{x,y,M}"] \arrow[d, "\mathrm{res}"] & W_M^*(x,y) \arrow[r, "L_M \otimes R_M"] \arrow[d, "\mathrm{com}_{1_N \otimes 1_N}"] & \mathscr{B}\big(L^2(M,\tau)\big) \arrow[d, "\mathrm{com}_{L^2(N,\tau)}"'] \\ L^\infty\big(\mathrm{spec}_N x \times y, dE_{x,y,N}\big) \arrow[r, "\Gamma_{x,y,N}"] & W_N^*(x,y) \arrow[r, "L_N \otimes R_N"] & \mathscr{B}\big(L^2(N,\tau)\big) \end{tikzcd} \tag{B.109}$$

with res *the restriction map given by* $\mathrm{spec}_N x \times y \subset \mathrm{spec}_M x \times y$.

Proof. Note 2) Lemma B.1.77 and 1) in Corollary B.2.35 show Lemma A.2.29 applies to the outer diagram in Diagram B.109. Said lemma therefore shows 1), 2) and the outer diagram. Normality of all maps involved reduces the left diagram in Diagram B.109 to elementary tensors. Apply 3) in Corollary B.2.37. □

The compression theorem. Let (M,τ) be a tracial W^*-algebra and H a Hilbert space. Let $N \subset (M,\tau)$ and $V \subset H$ be a Hilbert subspace.

Definition B.2.38. We say that a normal unital *-homomorphism $\phi : M \longrightarrow \mathscr{B}(H)$ is (N,V)-compressible if $\phi(N) \subset \mathscr{B}(V)$ and $\pi_V = \phi(1_N)\pi_V$.

Proposition B.2.39. *If $\phi : M \longrightarrow \mathscr{B}(H)$ is (N,V)-compressible, then $\phi|_N : N \longrightarrow \mathscr{B}(V)$ is a normal unital *-homomorphism.*

Proof. Note $\phi(1_N) \in \mathscr{B}(V)$ and $\pi_V = \phi(1_N)\pi_V$ shows $\phi(1_N) = \mathrm{com}_V \phi(1_N) = \pi_V$. Since $\phi(N) \subset \mathscr{B}(V)$, we see $\phi|_N : N \longrightarrow \mathscr{B}(V)$ is a normal unital *-homomorphism. □

Remark B.2.40. Let $\phi : M \longrightarrow \mathscr{B}(H)$ be (N,V)-compressible. For all $x \in L^0(N,\tau)_h$, we have $\operatorname{im} E_{x,N} \subset N$ as per 1) in Definition B.1.69 and therefore $\phi(\operatorname{im} E_{x,N}) \subset \mathscr{B}(V)$.

Definition B.2.41. Let $\phi : M \longrightarrow \mathscr{B}(H)$ be (N,V)-compressible. For all $x \in L^0(N,\tau)_h$, we define the push-forward spectral measure $\phi(E_{x,N})$ of x in N under ϕ to V by setting

$$\phi\big(E_{x,N}\big)(Z) := \phi\big(E_{x,N}(Z)\big) \tag{B.110}$$

for all $Z \in \mathfrak{B}(\mathbb{R})$.

Lemma B.2.42. *Let $\phi : M \longrightarrow \mathscr{B}(H)$ be (N,V)-compressible. If $x \in L^0(N,\tau)_h$ and further $T \in \mathscr{UB}_V(H)$ s.t. $\phi(\Gamma_{x,M}(R_{\pm i})) = R_{\pm i}(T)$, then we have $\phi(\Gamma_{x,N}(R_{\pm i})) = R_{\pm i}(T|_V)$ and $\phi(E_{x,N}) = E_{T|_V}$.*

Proof. For all $y \in N$, we have $\phi(1_N) \in \mathscr{B}(V)$, $[\phi(1_N), \pi_V] = 0$ and

$$\operatorname{com}_V \phi(y) = \operatorname{com}_V\big(\phi(1_N)\phi(y)\phi(1_N)\big) = \operatorname{com}_V \phi\big(\operatorname{com}_{1_N} y\big) \tag{B.111}$$

since ϕ is (N,V)-compressible. If $y = \Gamma_{z,M}(g)$ for $z \in N$ and $g \in L^\infty(\operatorname{spec}_M z, dE_{z,M})$, then Equation B.111 and 2) in Lemma B.2.33 show

$$\operatorname{com}_V \phi\big(\Gamma_{z,M}(g)\big) = \operatorname{com}_V \phi\big(\operatorname{com}_{1_N} \Gamma_{z,M}(g)\big) = \phi\big(\Gamma_{z,N}(g)\big). \tag{B.112}$$

Let $x \in L^0(N,\tau)_h$ and $T \in \mathscr{UB}_V(H)$ s.t. $\phi(\Gamma_{x,M}(R_{\pm i})) = R_{\pm i}(T)$. Then using 2) in Lemma A.2.26, Equation B.112 lets us calculate

$$R_{\pm i}(T|_V) = \operatorname{com}_V R_{\pm i}(T) = \operatorname{com}_V \phi\big(\Gamma_{x,M}(R_{\pm i})\big) = \phi\big(\Gamma_{x,N}(R_{\pm i})\big). \tag{B.113}$$

Proposition B.2.39 shows $\phi \circ L_N^{-1} : L^\infty(N,\tau) \longrightarrow \mathscr{B}(V)$ is normal unital *-homomorphism. Set $\phi_{L,N} := \phi \circ L_N^{-1}$. Using 2) in Lemma B.1.72, Equation B.113 implies $\phi_{L,N}(R_{\pm i}(L_{x,N})) = \phi(\Gamma_{x,N}(R_{\pm i})) = R_{\pm i}(T|_V)$. We see approximating in norm shows

$$\phi\big(\Gamma_{x,N}(g)\big) = \phi_{L,N}\big(g\big(L_{x,N}\big)\big) = g(T|_V) \tag{B.114}$$

for all $g \in C_0(\mathbb{R})$. We have push-forward measure $\phi_{L,N}(E_{L_{x,N}})$ as per Definition A.1.90. Precomposing with L_N^{-1} maps Equation B.110 to Equation A.26 for $\phi_{L,N}(E_{L_{x,N}})$, i.e.

$$\phi\big(E_{x,N}\big) = \phi_{L,N}\big(E_{L_{x,N}}\big). \tag{B.115}$$

Equation B.114 shows Lemma A.1.91 applies to $\phi_{L,N}(E_{L_{x,N}})$. As such, Equation B.115 shows $\phi(E_{x,N}) = \phi_{L,N}(E_{L_{x,N}}) = E_{T|_V}$ by Lemma A.1.91. We therefore know $\phi(E_{x,N})$ is a spectral measure on $\mathbb{R}$ with values in $\mathscr{B}(V)$. $\square$

Definition B.2.43. Let $\phi : M \longrightarrow \mathscr{B}(H)$ be (N,V)-compressible and $\psi : M^{\mathrm{op}} \longrightarrow \mathscr{B}(H)$ (N^{op},V)-compressible. The pair (ϕ,ψ) is (N,V)-compressible. Set $\phi \otimes_V \psi := \phi|_N \otimes \psi|_{N^{\mathrm{op}}}$.

Theorem B.2.44. *Let $N \subset (M,\tau)$ and $V \subset H$ be a Hilbert subspace. Let (ϕ,ψ) be an (N,V)-compressible pair. If $x,y \in L^0(N,\tau)_h$ and further $T,S \in \mathscr{U}\mathscr{B}_V(H)$ commute strongly s.t. $\phi(\Gamma_{x,M}(R_{\pm i})) = R_{\pm i}(T)$ and $\psi(\Gamma_{y,M}(R_{\pm i})) = R_{\pm i}(S)$, then we have*

1) $\operatorname{spec}_N x \times y = \operatorname{spec} T|_V \times S|_V$ *and* $\mathscr{N}\big(E_{x,y,N}\big) = \mathscr{N}\big(E_{T|_V,S|_V}\big)$,

2) $\big(\phi \otimes_V \psi\big)(g(x,y)) = g(T|_V,S|_V)$ *for all* $g \in L^\infty\big(\operatorname{spec}_N x \times y, dE_{x,y,N}\big)$,

3) *commutative diagram of normal unital surjective* **-homomorphisms*

$$\begin{tikzcd}
L^\infty\big(\operatorname{spec} T \times S, dE_{T,S}\big) \arrow[rr, "\Gamma_{T,S}"] \arrow[ddd, bend right, "\mathrm{res}"'] & & W^*(T,S) \arrow[ddd, "\mathrm{com}_V"] \\
L^\infty\big(\operatorname{spec}_M x \times y, dE_{x,y,M}\big) \arrow[u, "\mathrm{id}"] \arrow[r, "\Gamma_{x,y,M}"] \arrow[d, "\mathrm{res}"] & W^*_M(x,y) \arrow[ur, "\phi\otimes\psi"] \arrow[d, "\mathrm{com}_{1_N\otimes 1_N}"] & \\
L^\infty\big(\operatorname{spec}_N x \times y, dE_{x,y,N}\big) \arrow[r, "\Gamma_{x,y,N}"] \arrow[d, "\mathrm{id}"] & W^*_N(x,y) \arrow[dr, "\phi\otimes_V\psi"] & \\
L^\infty\big(\operatorname{spec} T|_V \times S|_V, dE_{T|_V,S|_V}\big) \arrow[rr, "\Gamma_{T|_V,S|_V}"] & & W^*(T|_V,S|_V)
\end{tikzcd} \tag{B.116}$$

with restriction maps given by $\operatorname{spec} T|_V \times S|_V \subset \operatorname{spec} T \times S$, $\operatorname{spec}_N x \times y \subset \operatorname{spec}_M x \times y$.

Proof. Let $x,y \in L^0(N,\tau)_h$ and $T,S \in \mathscr{U}\mathscr{B}_V(H)$ commute strongly s.t. $\phi(\Gamma_{x,M}(R_{\pm i})) = R_{\pm i}(T)$ and $\psi(\Gamma_{y,M}(R_{\pm i})) = R_{\pm i}(S)$. By construction of W^*-tensor products, the normal unital *-isomorphism $L_N \otimes R_N : W^*_M(x,y) \longrightarrow W^*(L_{x,M}, R_{y,M})$ has inverse $(L_N \otimes R_N)^{-1} = L_N^{-1} \otimes R_N^{-1}$ b . Thus 2) in Lemma B.1.77 shows we have commutative diagram

$$\begin{tikzcd}
L^\infty\big(\operatorname{spec} L_{x,N} \times R_{y,N}, dE_{L_{x,N},R_{y,N}}\big) \arrow[r, "\Gamma_{L_{x,N},R_{y,N}}"] \arrow[d, "\mathrm{id}"] & W^*\big(L_{x,N}, R_{y,N}\big) \arrow[d, "L_N^{-1}\otimes R_N^{-1}"] \\
L^\infty\big(\operatorname{spec}_N x \times y, dE_{x,y,N}\big) \arrow[r, "\Gamma_{x,y,N}"] & W^*_N(x,y)
\end{tikzcd} \tag{B.117}$$

of normal unital *-isomorphisms.

Lemma B.2.42 implies $\phi(\Gamma_{x,N}(R_{\pm i})) = R_{\pm i}(T|_V)$ and $\psi(\Gamma_{y,N}(R_{\pm i})) = R_{\pm i}(S|_V)$. Arguing as in the proof of Lemma A.2.12, mapping C^*-generators onto and closing in σ-weak operator topology provides normal unital *-isomorphisms $\phi : W^*_N(x) \longrightarrow W^*(T|_V)$ and $\phi : W^*_N(y) \longrightarrow W^*(S|_V)$. Corollary A.1.53 lets us tensor these two *-isomorphisms to the normal unital *-isomorphism $\phi \otimes_V \psi : W^*_N(x,y) \longrightarrow W^*(T|_V, S|_V)$.

Arguing as in the proof of Lemma B.2.42, set $\phi_{L,N} := \phi \circ L_N^{-1}$ and $\psi_{R,N} := \psi \circ R_N^{-1}$. Lemma B.1.62 shows $R_N = L_N^{\mathrm{op}}$ on N. Moreover, 2) in Lemma B.1.72 and Lemma B.2.42 show we in fact have normal unital *-isomorphisms $\phi_{L,N} : W^*(L_{x,N}) \longrightarrow W^*(T|_V)$ and $\psi_{R,N} : W^*(R_{y,N}) \longrightarrow W^*(S|_V)$ s.t. $\phi_{L,N}(E_{L_{x,N}}) = E_{T|_V}$ and $\psi_{R,N}(E_{R_{x,N}}) = E_{S|_V}$.

Thus Lemma A.1.101 applies using $\phi_{L,N}$ and $\psi_{R,N}$, resp. their inverses. The concrete analogue of 1) hence follows by 1) in Lemma A.1.101. We pull back along Diagram B.117 to the abstract case. This is 1). In our setting, Diagram A.34 in Lemma A.1.101 is the commutative diagram

$$\begin{tikzcd}
L^\infty\big(\operatorname{spec} L_{x,N} \times R_{y,N}, dE_{L_{x,N},R_{y,N}}\big) \arrow[r, "\Gamma_{L_{x,N},R_{y,N}}"] \arrow[d, "\mathrm{id}"] & W^*\big(L_{x,N}, R_{y,N}\big) \arrow[d, "\phi_{L,N}\otimes\psi_{R,N}"] \\
L^\infty\big(\operatorname{spec} T|_V \times S|_V, dE_{T|_V,S|_V}\big) \arrow[r, "\Gamma_{T|_V,S|_V}"] & W^*(T|_V, S|_V)
\end{tikzcd} \tag{B.118}$$

of normal unital *-isomorphisms. Using σ-weak closure, we directly verify

$$\phi_{L,N} \otimes \psi_{R,N} = \big(\phi \otimes_V \psi\big) \circ \big(L_N^{-1} \otimes R_N^{-1}\big) \tag{B.119}$$

on elementary tensors. Equation B.119 shows Diagram B.118 factors into the upper and lower diagrams

$$\begin{tikzcd}
L^\infty\big(\operatorname{spec} L_{x,N} \times R_{y,N}, dE_{L_{x,N},R_{y,N}}\big) \arrow[r, "\Gamma_{L_{x,N},R_{y,N}}"] \arrow[d, "\mathrm{id}"] & W^*\big(L_{x,N}, R_{y,N}\big) \arrow[d, "L_N^{-1}\otimes R_N^{-1}"] \\
L^\infty\big(\operatorname{spec}_N x \times y, dE_{x,y,N}\big) \arrow[r, "\Gamma_{x,y,N}"] \arrow[d, "\mathrm{id}"] & W^*_N(x,y) \arrow[d, "\phi\otimes_V\psi"] \\
L^\infty\big(\operatorname{spec} T|_V \times S|_V, dE_{T|_V,S|_V}\big) \arrow[r, "\Gamma_{T|_V,S|_V}"] & W^*(T|_V, S|_V)
\end{tikzcd} \tag{B.120}$$

of normal unital *-isomorphisms. The outer diagram in Diagram B.120 is therefore given by Diagram B.118, whereas the upper one is Diagram B.117. Thus both outer and upper diagrams commute, hence the lower diagram in Diagram B.120 commutes.

We apply the above discussion to (N,V) and its special case $(N,V)=(M,H)$. We thus combine both to have commutative diagram

$$
\begin{tikzcd}
L^\infty\big(\operatorname{spec} T\times S, dE_{T,S}\big) \arrow[rr, "\Gamma_{T,S}"] & & W^*(T,S) \\
L^\infty\big(\operatorname{spec}_M x\times y, dE_{x,y,M}\big) \arrow[u, "\mathrm{id}"] \arrow[r, "\Gamma_{x,y,M}"] & W^*_M(x,y) \arrow[ur, "\phi\otimes\psi"] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
L^\infty\big(\operatorname{spec}_N x\times y, dE_{x,y,N}\big) \arrow[d, "\mathrm{id}"] \arrow[r, "\Gamma_{x,y,N}"] & W^*_N(x,y) \arrow[dr, "\phi\otimes_V\psi"] & \\
L^\infty\big(\operatorname{spec} T|_V\times S|_V, dE_{T|_V,S|_V}\big) \arrow[rr, "\Gamma_{T|_V,S|_V}"] & & W^*(T|_V,S|_V)
\end{tikzcd}
\tag{B.121}
$$

of normal unital *-isomorphisms. We further have commutative diagram

$$
\begin{tikzcd}
L^\infty\big(\operatorname{spec} T\times S, dE_{T,S}\big) \arrow[rrr, "\Gamma_{T,S}"] \arrow[ddd, bend right, "\mathrm{res}"'] & & & W^*(T,S) \arrow[ddd, "\mathrm{com}_V"] \\
& L^\infty\big(\operatorname{spec}_M x\times y, dE_{x,y,M}\big) \arrow[r, "\Gamma_{x,y,M}"] \arrow[d, "\mathrm{res}"] & W^*_M(x,y) \arrow[d, "\mathrm{com}_{1_N\otimes 1_N}"] & \\
& L^\infty\big(\operatorname{spec}_N x\times y, dE_{x,y,N}\big) \arrow[r, "\Gamma_{x,y,N}"] & W^*_N(x,y) & \\
L^\infty\big(\operatorname{spec} T|_V\times S|_V, dE_{T|_V,S|_V}\big) \arrow[rrr, "\Gamma_{T|_V,S|_V}"] & & & W^*(T|_V,S|_V)
\end{tikzcd}
\tag{B.122}
$$

of normal unital *-homomorphisms. Indeed, note Diagram A.53 as per Lemma A.2.29 is the outer diagram in Diagram B.122, whereas the left diagram in Diagram B.109 as per Corollary B.2.37 is the inner one.

We combine Diagram B.121 and Diagram B.122 to Diagram B.116. Commutativity of the latter therefore implies both 2) and 3) follow if the diagram

$$\begin{tikzcd}
& W^*(T,S) \arrow[ddd, "\mathrm{com}_V"] \\
W_M^*(x,y) \arrow[ur, "\phi\otimes\psi"] \arrow[d, "\mathrm{com}_{1_N\otimes 1_N}"] & \\
W_N^*(x,y) \arrow[dr, "\phi\otimes_V\psi"] & \\
& W^*(T|_V,S|_V)
\end{tikzcd} \tag{B.123}$$

of normal unital *-homomorphisms commutes. Normality reduces to commutativity on elementary tensors. Note $\mathrm{com}_{1_N\otimes 1_N} = \mathrm{com}_{1_N}\otimes\mathrm{com}_{1_N}$. Equation B.112 for $\phi_{L,N}$ and $\psi_{R,N}$ implies commutativity on elementary tensors. □

Corollary B.2.45. *Assume the setting of Theorem B.2.44. For all real $g\in\mathscr{S}(E_{x,y,N})$ s.t*

1) *$(t,s)\mapsto g_\varepsilon(t,s) := g(t+\varepsilon, s+\varepsilon)$ lies in $C_b(\operatorname{spec} T_0\times S_0)$ for all $\varepsilon>0$,*

2) *$g(T|_V,S|_V) = \text{sr-lim}_{\varepsilon\downarrow 0}\, g_\varepsilon(T|_V,S|_V)$ on V,*

we have $g\in\mathscr{S}(E_{T|_V,S|_V})$ with $g(T|_V,S|_V) = \text{sr-lim}_{\varepsilon\downarrow 0}(\phi\otimes_V\psi)(g_\varepsilon(x,y))$ on V.

Proof. Apply Lemma B.2.42 and Theorem B.2.44 to reduce to Corollary A.2.6. □

Definition B.2.46. Assume the setting of Theorem B.2.44.

1) We call $\Gamma^{\phi,\psi}_{x,y,N} := \left(\phi\otimes_V\psi\right)\circ\Gamma_{x,y,N}$ the bounded measurable joint functional calculus of $x\otimes y$ in $N\otimes N^{\mathrm{op}}$ under $\phi\otimes_V\psi$.

2) Let $\mathscr{S}_V\left(E_{x,y,N}\right)$ be the set of all real $g\in\mathscr{S}\left(E_{x,y,N}\right)$ s.t. 1) and 2) in Corollary B.2.45 are satisfied. For all $g\in\mathscr{S}_V\left(E_{x,y,N}\right)$, set

$$\Gamma^{\phi,\psi}_{x,y,N}(g) := g(T|_V,S|_V). \tag{B.124}$$

3) We call $\Gamma^{\phi,\psi}_{x,y,N} : \mathscr{S}_V\left(E_{x,y,N}\right) \longrightarrow \mathscr{UB}(V)_h$ the joint functional calculus of $x\otimes y$ in $N\otimes N^{\mathrm{op}}$ under $\phi\otimes_V\psi$.

Corollary B.2.47. *Let $N_0 \subset N_1 \subset (M,\tau)$ and $V_0 \subset V_1 \subset H$ be Hilbert subspaces. Let (ϕ,ψ) be an (N_0,V_0)- and (N_1,V_1)-compressible pair. If $x,y \in L^0(N_0,\tau)_h$ and $T,S \in \mathscr{UB}_V(H)$ commute strongly s.t. $\phi(\Gamma_{x,M}(R_{\pm i})) = R_{\pm i}(T)$ and $\psi(\Gamma_{y,M}(R_{\pm i})) = R_{\pm i}(S)$, then we have*

1) $\operatorname{spec}_{N_1} x\times y \subset \operatorname{spec}_M x\times y$ *and* $\mathscr{N}\big(E_{x,y,M}\big) \subset \mathscr{N}\big(E_{x,y,N_1}\big)$,
2) $\operatorname{spec}_{N_0} x\times y \subset \operatorname{spec}_{N_1} x\times y$ *and* $\mathscr{N}\big(E_{x,y,N_1}\big) \subset \mathscr{N}\big(E_{x,y,N_0}\big)$,
3) *commutative diagram of normal unital surjective *-homomorphisms*

$$\begin{tikzcd}
L^\infty\big(\operatorname{spec}_M x\times y, dE_{x,y,M}\big) \arrow[r, "\Gamma^{\phi,\psi}_{x,y}"] \arrow[d, "\mathrm{res}"] & \mathscr{B}(H) \arrow[d, "\mathrm{com}_{V_1}"] \\
L^\infty\big(\operatorname{spec}_{N_1} x\times y, dE_{x,y,N_1}\big) \arrow[r, "\Gamma^{\phi,\psi}_{x,y,N_1}"] \arrow[d, "\mathrm{res}"] & \mathscr{B}(V_1) \arrow[d, "\mathrm{com}_{V_0}"] \\
L^\infty\big(\operatorname{spec}_{N_0} x\times y, dE_{x,y,N_0}\big) \arrow[r, "\Gamma^{\phi,\psi}_{x,y,N_0}"] & \mathscr{B}(V_0)
\end{tikzcd} \tag{B.125}$$

with restriction maps given by $\operatorname{spec}_{N_1} x\times y \subset \operatorname{spec}_M x\times y$, $\operatorname{spec}_{N_0} x\times y \subset \operatorname{spec}_{N_1} x\times y$.

Proof. Get 1) and 2) by Corollary B.2.37. Lemma B.2.42 shows $\phi(\Gamma_{x,N_1}(R_{\pm i})) = R_{\pm i}(T|_{V_1})$ and $\psi(\Gamma_{y,N_1}(R_{\pm i})) = R_{\pm i}(S|_{V_1})$. Theorem B.2.44 applies using $N_1 \subset (M,\tau)$ and V_1, as well as $N_0 \subset (N_1,\tau)$ and V_0. We use ϕ and ψ in both cases. Using 3) in Corollary B.2.37 and 3) in Theorem B.2.44 applied twice accordingly, we directly verify Diagram B.125. □

Corollary B.2.48. *Assume the setting of Theorem B.2.44. If $x,y \in L^0(N,\tau)_+$, $\alpha,\beta \geq 0$ and $g \in C_b([0,\infty)\times[0,\infty))$, then*

$$\operatorname{com}_{1_N\otimes 1_N}\Big(\Gamma_{x+\alpha 1_N^\perp,\, y+\beta 1_N^\perp,\, M}(g)\Big) = \Gamma_{x,y,N}(g). \tag{B.126}$$

Proof. Let $x,y \in L^0(N,\tau)_+$, $\alpha,\beta \geq 0$ and $g \in C_b([0,\infty)\times[0,\infty))$. Proposition A.2.17 shows abstract compression maps are normal. By normality and Lemma B.1.80, we assume $\{0\} \in \mathscr{N}(E_{x,N}) \cap \mathscr{N}(E_{y,N})$ without loss of generality.

Set $Z_x := \operatorname{spec}_M x \setminus \{0\}$ and $Z_y := \operatorname{spec}_M y \setminus \{0\}$. Note 1) in Corollary B.2.35 shows

$$\Gamma_{x,N}\big(\chi_{Z_x}\big) = 1_N,\ \Gamma_{y,N}\big(\chi_{Z_y}\big) = 1_N. \tag{B.127}$$

Using Corollary B.2.36, Equation B.127 implies

$$\Gamma_{x,M}\big(\chi_{Z_x}\big) = \Gamma_{x,N}\big(\chi_{Z_x}\big) = 1_N = \Gamma_{y,N}\big(\chi_{Z_y}\big) = \Gamma_{y,M}\big(\chi_{Z_y}\big). \tag{B.128}$$

Equation B.128 yields $1_N^{\perp} \in W_M^*(x) \cap W_M^*(y)$ and

$$\Gamma_{x,M}(\delta_0) = 1_N^{\perp} = \Gamma_{y,M}(\delta_0). \tag{B.129}$$

For all $t, s \in \mathbb{R}$, let $g^{\alpha,\beta}(t,s) := g(t + \alpha\delta_0(t), s + \beta\delta_0(s))$. We obtain $g^{\alpha,\beta} \in C_b(\mathbb{R} \times \mathbb{R})$ and

$$\Gamma_{x,y,N}\left(g^{\alpha,\beta}\right) = \Gamma_{x,y,N}(g). \tag{B.130}$$

Using Theorem B.2.44, we calculate

$$\mathrm{com}_{1_N \otimes 1_N}\left(\Gamma_{x+\alpha 1_N^{\perp}, y+\beta 1_N^{\perp}, M}(g)\right) = \mathrm{com}_{1_N \otimes 1_N}\left(\Gamma_{x,y,M}\left(g^{\alpha,\beta}\right)\right) = \Gamma_{x,y,N}\left(g^{\alpha,\beta}\right). \tag{B.131}$$

Equation B.130 and Equation B.131 show Equation B.126. □

C | Clifford Calculations

We give calculations in Clifford algebras for our discussion. In Section C.1, Lemma C.1.1 gives three identities for twisted dynamic quantum gradients induced by intertwining sets of Clifford generators. In Section C.2, Lemma C.2.1 gives, in detail, implementation of Bogoliubov automorphisms as per Equation 3.126 on anti-symmetric Fock space.

C.1 Identities for intertwining sets of Clifford generators

We use the three identities in Lemma C.1.1 to prove Lemma 2.3.59.

Lemma C.1.1. *Assume the setting of Lemma 2.3.59.*

1) For all $n,k \in \{1,\dots,m\}$, we have

$$\partial_n \partial_k = -\partial_k \partial_n. \tag{C.1}$$

2) For all $n,k \in \{1,\dots,m\}$ s.t. $n \neq k$, we have

$$\partial_n \partial_k^* = -\partial_k^* \partial_n. \tag{C.2}$$

3) For all $n \in \{1,\dots,m\}$, we have

$$\partial_n \Delta_n = 4C\partial_n. \tag{C.3}$$

Proof. For all $n \in \{1,\dots,m\}$, Example 2.3.54 shows $L_{d_n} \in \mathscr{B}(L^2(A,\tau))_h$ is ϕ-intertwining with $\operatorname{sgn}(L_{d_n}) = -1$. We pull back along L^{-1}. Note ϕ is involutive by hypothesis. Let $n,k \in \{1,\dots,m\}$ and $x \in A_0$. Using $\{d_n\}_{n=1}^m \in L^\infty(A,\tau)_h$, Corollary 2.3.56 shows

$$\partial_n x = d_n x - \phi(x) d_n,\ \partial_n^* x = d_n x + \phi(x) d_n \tag{C.4}$$

and

$$\Delta_n = d_n^2 x + x d_n^2 - 2 d_n \phi(x) d_n. \tag{C.5}$$

Note 1.2) in Definition 2.3.58 gives the Clifford relation $d_n d_k + d_k d_n = 2C\delta_{nk} 1_A$. We use the first identity in Equation C.4 to calculate

$$\partial_n \partial_k x = d_n d_k x - d_n \phi(x) d_k + d_k \phi(x) d_n - x d_k d_n. \tag{C.6}$$

Interchanging n and k in Equation C.6 yields $\partial_k \partial_n x$. We apply the Clifford relation to the first and fourth summand on the right-hand side of Equation C.6. We obtain $-\partial_k \partial_n x$. Equation C.6 therefore implies Equation C.1 by Clifford relations. Get 1). We likewise use both identities in Equation C.4 to calculate

$$\partial_n \partial_k^* x = d_n d_k x + d_n \phi(x) d_k + d_k \phi(x) d_n + x d_k d_n \tag{C.7}$$

and

$$\partial_k^* \partial_n x = d_k d_n x - d_k \phi(x) d_n - d_n \phi(x) d_k + x d_n d_k. \tag{C.8}$$

If $n \neq k$, then $d_n d_k = -d_k d_n$. Using the latter, we see Equation C.7 and Equation C.8 imply Equation C.2 by Clifford relations. Get 2).

If $n = k$, then $d_n^2 = C1_A$. Equation C.5 is $\Delta_n x = 2Cx - 2d_n \phi(x) d_n$ in this case. We use the latter and the first identity in Equation C.4 to calculate

$$\partial_n \Delta_n x = 2C d_n x - 2C\phi(x) d_n - 2C\phi(x) d_n + 2C d_n x = 4C\partial_n x. \tag{C.9}$$

Equation C.9 shows Equation C.3. Get 3). □

C.2 Implementation on anti-symmetric Fock space

We use Lemma C.2.1 to derive explicit formula for Equation 3.126 in Example 3.1.62.

Lemma C.2.1. *Assume the setting of Example 3.1.62. For all $t \in \mathbb{R}$ and $x \in \mathscr{A}(H)$, get*

$$\rho_J\big(\mathrm{Cliff}\big(e^{itD}\big)(x)\big) = \bigwedge e^{it|D|} \rho_J(x) \bigwedge e^{-it|D|} \in \rho_J(\mathscr{A}(H)). \tag{C.10}$$

Proof. We solve the associated implementation problem at each time, i.e. we show each $\mathrm{Cliff}(e^{itD})$ to be the Bogoliubov automorphism implemented on $\mathscr{F}(H[J])$ using $\bigwedge e^{it|D|}$ as its unique unitary operator (cf. Section 3.2 and Section 3.3 in [177]). For this, we construct suitable unitary equivalences of faithful unital *-representations of $\mathscr{A}(H)$ over $\mathscr{F}(H[J])$ using cyclic vectors.

For all $t \in \mathbb{R}$, set $\rho_J^t := \rho_J \circ \mathrm{Cliff}(e^{itD})$ and note a cyclic unit vector $\Omega \in H$ of ρ_J^t satisfies the J-vacuum condition if

$$\rho_J^t\big(u + iJ(u)\big)(\Omega) = 0 \tag{C.11}$$

for all $u \in H$. Let Ω_J be the Fock state of $\mathscr{F}(H[J])$. It is the unique cyclic unit vector of $\rho_J^0 := \rho_J$ satisfying the J-vacuum condition. For all $t \in \mathbb{R}$, set $\Omega_J^t := \bigwedge e^{it|D|}(\Omega_J)$.

Let $t \in \mathbb{R}$. Theorem 3.2.5 in [177] states $e^{it|D|} \in \mathscr{U}(\mathscr{B}(H))$ is implemented on $\mathscr{F}(H[J])$ by $\bigwedge e^{it|D|} \in \mathscr{U}(\mathscr{B}(\mathscr{F}(H[J])))$. Using $H \cong b(H) \subset \mathscr{A}(H)$ as set of generators and by norm continuity, Equation C.10 reduces to $\rho_J^t = \rho_J \circ e^{itD}$ and

$$\rho_J^t(u) = \bigwedge e^{it|D|} \rho_J(u) \bigwedge e^{-it|D|} \in \rho_J(\mathscr{A}(H)) \tag{C.12}$$

for all $u \in H$. Note $[e^{itD}, J] = 0$ ensures Theorem 3.3.3 in [177] yields implementation as per Equation C.12 for unique but unspecified unitary operators. We require $\bigwedge e^{it|D|}$ to be the unique unitary operator used. Since Ω is a cyclic unit vector of ρ_J, we know Ω_J^t is a cyclic unit vector of ρ_J^t s.t. $\bigwedge e^{it|D|}(\Omega_J) = \Omega_J^t$ by construction. If Ω_J^t satisfies the J-vacuum condition, then Theorem 2.4.7 in [177] shows Equation C.12.

We show the J-vacuum condition for Ω_J^t. Since $[e^{itD}, J] = 0$, we calculate

$$\rho_J\big(e^{itD} u\big)(\Omega_J) = -i\rho_J\big(J\big(e^{itD} u\big)\big)(\Omega_J) = \rho_J\big(e^{itD}(P_+ - P_-)(u)\big)(\Omega_J) \tag{C.13}$$

for all $u \in H$. We have $e^{itD}(P_+ - P_-) = e^{it|D|}$ on non-negative, and $e^{itD}(P_+ - P_-) = e^{it|D|} - 2I$ on negative eigenvalues. For all $v \in H$, note $P_-(v) = v$ implies $J(v) = -iv$ and therefore $2v = v + iJ(v)$. Using the J-vacuum condition for Ω_J, Equation C.13 implies

$$\rho_J\big(e^{itD} u\big)(\Omega_J) = \rho_J\big(e^{it|D|} u\big)(\Omega_J) \tag{C.14}$$

for all $u \in H$. Using implementation of $e^{it|D|}$ on $\mathscr{F}(H[J])$ by $\bigwedge e^{it|D|}$, Equation C.14 lets us calculate

$$\rho_J\big(e^{itD} u\big)(\Omega_J) = \rho_J\big(e^{it|D|} u\big)(\Omega_J) = \Big(\bigwedge e^{it|D|} \rho_J(u) \bigwedge e^{-it|D|}\Big)(\Omega_J) \tag{C.15}$$

for all $u \in H$. Equation C.15 shows

$$\bigwedge e^{it|D|}\big(\rho_J(u)(\Omega_J)\big) = \bigwedge e^{it|D|}\big(\rho_J\big(e^{-itD} e^{itD} u\big)(\Omega_J)\big) = \rho_J^t(u)(\Omega_J^t) \tag{C.16}$$

for all $u \in H$. Using $u := v + iJ(v)$ for all $v \in H$, the J-vacuum condition for Ω_J and Equation C.16 imply Ω_J^t satisfies the J-vacuum condition. Thus Theorem 2.4.7 in [177] shows Equation C.12, hence Equation C.10. □

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