Singular Brascamp-Lieb Forms and Multilinear Fourier Multipliers with Rough or Oscillatory Multipliers

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Summary

In this thesis, we study the boundedness of several multilinear operators including singular Brascamp-Lieb forms and certain multilinear Fourier multipliers with rough or oscillatory multipliers.

Several important multilinear singular integral operators in harmonic analysis such as the Coifman-Meyer multipliers, the bilinear Hilbert transform, and twisted paraproducts all fall within the class of singular Brascamp-Lieb forms. The boundedness of a singular Brascamp-Lieb form is invariant under certain linear changes of variables. Given specific dimension data, we classify singular Brascamp-Lieb forms up to equivalence and characterize their boundedness in this setting.

Typically, for a given singular Brascamp-Lieb form, one imposes the Mihlin's condition on its multiplier, which is the Fourier transform of the singular kernel. This Mihlin's condition can be generalized to Hörmander's condition, which allows for fractional regularity. Naturally, this raises the question: what is the minimal regularity required of the multiplier to ensure boundedness? We address this question for multipliers that may exhibit Lipschitz-type singularities.

Furthermore, the linear projections appearing in singular Brascamp-Lieb forms can be replaced by nonlinear maps. This line of research traces back to the 1970s, when singular Radon transforms were first studied. We also explore multilinear generalizations of the singular Radon transform, where the associated multipliers may exhibit oscillatory behavior. A crucial step in establishing the boundedness of such multilinear oscillatory multipliers is to prove a suitable smoothing inequality. We provide partial answers regarding which classes of multilinear oscillatory multipliers admit such smoothing effects.

This thesis consists of four chapters.

In Chapter 0, we introduce the historical background and explain how our work fits into this broader framework.

In Chapter 1, we classify singular Brascamp-Lieb forms in ambient space of dimension 3, involving functions of dimensions $1, 2, 2$, and a kernel of dimension 1. We also characterize the boundedness of singular Brascamp-Lieb forms within this family. This chapter is based on a single-author paper [71].

In Chapter 2, we establish a local L^2 bound for multilinear Fourier multipliers with Lipschitz singularities under the sharp Hörmander's condition. This chapter is based on a joint work with Jiao Chen and Martin Hsu [16].

In Chapter 3, we present a blueprint for Lean, a computer-assisted verification pro-

gram. This blueprint concerns a smoothing inequality for multipliers associated with the triangular Hilbert transform along curves. It is a single-author work based on the paper [53], jointly written with Martin Hsu.

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Contents

Introduction

The introduction is organized as follows. It consists of three sections, each corresponding to one of the main topics of this dissertation. In each section, we first present the historical background and highlight key developments in that direction. This is followed by a subsection that explains how our research fits within this framework, along with the main ideas and techniques employed in our work.

0.1 Singular Brascamp-Lieb forms

Given linear maps $B = \{B_j\}_{j=1}^n$, with $B_j : \mathbb{R}^{d_0} \to \mathbb{R}^{d_j}$, one may ask: for which exponents $p = (p_1, \cdots, p_n)$ does the following inequality hold?

$$\left| \int_{\mathbb{R}^{d_0}} \prod_{j=1}^n f_j(B_j x) dx \right| \leq C_{B,p} \prod_{j=1}^n \|f_j\|_{L^{p_j}} . \tag{0.1.1}$$

Inequalities of this type are known as Brascamp-Lieb inequalities, a broad family that encompasses several fundamental inequalities, including Hölder's inequality, Young's convolution inequality, and the Loomis-Whitney inequality. In 2008, Bennet, Carbery, Christ, and Tao [4] fully resolved this question, providing a sharp criterion to determine the range of exponents p, based on the linear maps B.

Beyond the classical Brascamp-Lieb setting, there are other important multilinear forms arising in harmonic analysis, especially those connected to singular integrals. Let $B = \{B_j\}_{j=1}^{n+1}$, with $B_j : \mathbb{R}^{d_0} \to \mathbb{R}^{d_j}$, be linear maps and let K be a singular kernel whose Fourier transform satisfies the Mihlin's condition:

$$\left| (\partial^\alpha \widehat{K})(\xi) \right| \leq C_\alpha |\xi|^{-|\alpha|} \tag{0.1.2}$$

for all $\xi \neq 0$ and for all multi-indices α up to some large enough integer $|\alpha| \leq N_0$. Analogous to the Brascamp-Lieb setting, one may ask: for which exponents $p = (p_1, \cdots, p_n)$, does the following singular Brascamp-Lieb inequality hold?

$$\left| \int_{\mathbb{R}^{d_0}} \prod_{j=1}^n f_j(B_j x) K(B_{j+1}) dx \right| \leq C_{K,B,p} \prod_{j=1}^n \|f_j\|_{L^{p_j}} . \tag{0.1.3}$$

Note that, since K is a singular kernel, the integral should be considered in the principle value sense though we omit writing the *p.v.* in front of the integral. The tuple

$d = (d_0, d_1, \cdots, d_{n+1})$ will be referred to as the dimension data associated with the singular Brascamp-Lieb form. Many objects within this class of multilinear forms have their own distinct background and motivation. Establishing their boundedness has applications across various areas of mathematics. In this work, we will not delve into the detailed historical background of each individual object but will instead focus on how they fit into the broader general framework.

This general form includes many well-known linear and multilinear singular integral operators. For instance, when $n = 2$, $d = (2, 1, 1, 1)$, $K(y) = \frac{1}{y}$, the singular Brascamp-Lieb form corresponds to the dual form of the Hilbert transform, defined by

$$Hf(x) := \int_{\mathbb{R}} f(x - y)\frac{1}{y}dt\,, \tag{0.1.4}$$

which serves as a prototype of the Calderon-Zygmund singular integral operator. The boundedness of the Hilbert transform in $L^p(\mathbb{R})$ for $1 < p < \infty$ was first established in 1928 by Riesz [86] using methods from complex analysis. Later, in 1952, Calderón and Zygmund [12] gave a real-variable proof via their Calderón-Zygmund decomposition which becomes an essential tool in modern singular integral theory.

Later in 1978, Coifman and Meyer [24, 26] initiated the study of multilinear singular integrals, establishing boundedness results in the open Banach range for the following bilinear operator:

$$T(f, g)(x) := \int_{\mathbb{R}^{2d_0}} f(x - y_1)g(x - y_2)K(y_1, y_2)dy_1dy_2 \tag{0.1.5}$$

where the kernel K satisfies the Mihlin's condition (0.1.2). The dual form of (0.2.3) corresponds to the case $n = 3$ and dimension data $d = (3d_0, d_0, d_0, d_0, 2d_0)$ in (0.1.3). If we require that all functions are defined on spaces of the same dimension, and that the bounds obtained are of Hölder-type exponents, then by scaling, the dimension data must take the form $(a + b, a, \cdots, a, b)$. Once the dimension a of the function is fixed and the kernel dimension b becomes lower, the corresponding operator becomes more singular, making boundedness results harder to prove. For the Coifman-Meyer multiplier, if we set $d_0 = 1$, the dimension data is $(3, 1, 1, 1, 2)$. There is a more singular case with dimension data $(2, 1, 1, 1, 1)$ corresponding to a kernel of lower dimension. In this context, Lacey and Thiele [63] first established a local L^2 bound in 1997 and later in 1999 [64], they extended the range for the bilinear Hilbert transform:

$$T_\alpha(f, g)(x) := \int_{\mathbb{R}} f(x - y)g(x + \alpha y)\frac{1}{y}dy\,, \tag{0.1.6}$$

whose dual form fits into the dimension data $(2, 1, 1, 1, 1)$. The method they introduced, now known as time-frequency analysis, has become a fundamental tool in dealing with operators exhibiting modulation symmetry, including the Carleson operator [65] which is related to the almost everywhere convergence of Fourier series. The study of the bilinear Hilbert transform was originally motivated by the work of Calderón, Coifman, McIntosh, and Meyer [11, 25], which established the L^p boundedness of the Calderón commutator and, consequently, the Cauchy integral on Lipschitz curves.

Once uniform estimates in α for the bilinear Hilbert transform are obtained, we obtain an alternative proof of the boundedness of the Calderón commutator. The first uniform bounds were proven by Thiele [92] in 2002 for a certain range, and later, this range was extended by Grafakos and Li [44], Li [67], and Uraltsev and Warchalski [96]. In 2014, Muscalu [74, 75] provided a new proof of the boundedness of the Calderón commutator. Instead of proving uniform bounds for the bilinear Hilbert transform, his approach directly tackled Coifman-Meyer type multiplier operators, where the multipliers satisfy a weaker regularity condition.

If we maintain the function and kernel in one dimension but increase the linearity to $n = 4$, we encounter the trilinear Hilbert transform for which boundedness remains a major open problem:

Conjecture 0.1.1 (Trilinear Hilbert transform)**.** *For $f_j \in \mathcal{S}(\mathbb{R})$, $j = 1, 2, 3, 4$, the following inequality holds:*

$$\left| \int_{\mathbb{R}^2} f_1(x+y) f_2(x+2y) f_3(x+3y) f_4(x) \frac{1}{y} dy dx \right| \leq C \prod_{j=1}^{4} \|f_j\|_{L^4} . \tag{0.1.7}$$

Alternatively, we may keep $n = 3$, but raise the dimension of the function to 2. In 2010, Thiele and Demeter [28] studied the following two-dimensional bilinear Hilbert transform whose dual form are of dimension data $(4, 2, 2, 2, 2)$:

$$T(f,g)(x_1,x_2) = \int_{\mathbb{R}^2} f((x_1,x_2) + A_1(y_1,y_2)) g((x_1,x_2) + A_2(y_1,y_2)) K(y_1,y_2) dy_1 dy_2 , \tag{0.1.8}$$

where A_1, A_2 are 2×2 matrices. Bounds in certain range were obtained for most cases except for one particularly case, which is now called the twisted paraproduct:

$$T(f,g)(x_1,x_2) := \int_{\mathbb{R}^2} f(x_1+y_1, x_2) g(x_1, x_2+y_2) K(y_1,y_2) dy_1 dy_2 . \tag{0.1.9}$$

In 2012, Kovač [59] proved boundedness for the twisted paraproduct in the loca L^2 using a novel technique called twisted technology. In the same year, Bernicot [5] extended the range of boundedness using a fiberwise Calderón-Zygmund decomposition. Through a change of variables, the twisted paraproduct can be reformulated as a symmetric quadralinear form:

$$\Lambda(f_1,f_2,f_3,f_4) = \int_{\mathbb{R}^4} f_1(x_1,x_2) f_2(x_2,x_3) f_3(x_3,x_4) f_4(x_4,x_1) K(x_1-x_3, x_2-x_4) dx_1 dx_2 dx_3 dx_4 \tag{0.1.10}$$

with $f_4 = 1$. In [59], Kovač first established boundedness for Λ with $f_4 = 1$ in the Walsh model, and then transferred the result to Euclidean space using the square function estimate of Jones, Seeger, and Wright [55]. Later, in 2015 and 2017, Durcik directly prove bounds for the entangled quadrilinear form Λ by further introducing a Gaussian domination techniques.

By keeping the functions two-dimensional but lowering the dimension of the kernel, we arrive at a highly singular object with dimension data $(3, 2, 2, 2, 1)$ known as the triangular Hilbert transform, whose boundedness remains as another major open problem:

Conjecture 0.1.2 (Triangular Hilbert transform)**.** *For $f_j \in \mathcal{S}(\mathbb{R})$, $j = 1, 2, 3$, the following inequality holds:*

$$\left| \int_{\mathbb{R}^3} f_1(x,y) f_2(y,z) f_3(z,x) \frac{1}{x+y+z} dx dy dz \right| \leq C \prod_{j=1}^{3} \|f_j\|_{L^3} . \tag{0.1.11}$$

This operator is of particular importance because establishing its boundedness would imply, as a consequence, the boundedness of both the bilinear Hilbert transform and the Carleson operator within their respective ranges.

As we move to higher dimensions, we encounter increasingly diverse cases. Therefore, it becomes essential to establish a complete classification and a clear hierarchy among singular Brascamp-Lieb forms before addressing the estimates for each specific case. Our primary focus lies in understanding the boundedness properties of these multilinear forms. It is important to note that boundedness is invariant under changes of variables in the ambient space and under shearing each functions by invertible linear transformations with non-zero Jacobian. This observation naturally leads to the following notion of equivalence for singular Brascamp-Lieb data. Specifically, we say that two singular Brascamp-Lieb data $B = \{B_j\}_{j=1}^n$, $B_j : \mathbb{R}^{d_0} \to \mathbb{R}^{d_j}$ and $B' = \{B'_j\}_{j=1}^n$, $B_j : \mathbb{R}^{d_0} \to \mathbb{R}^{d_j}$ are equivalent if and only if there exist invertible linear maps $A : \mathbb{R}^{d_0} \to \mathbb{R}^{d_0}$, $C_j : \mathbb{R}^{d_j} \to \mathbb{R}^{d_j}$, $1 \leq j \leq n$ such that for all $1 \leq j \leq n$,

$$B'_j = C_j B_j A\,. \tag{0.1.12}$$

Singular Brascamp-Lieb forms that are equivalent in this sense share the same range of boundedness exponents p. In Chapter 1, we first classify singular Brascamp-Lieb forms with dimension data $(3, 1, 2, 2, 1)$ under this equivalence and characterize the boundedness properties of this class. Later, in collaboration with Lars Becker and Polona Durcik, and using tools from quiver representation theory, we extend this classification to trilinear singular Brascamp-Lieb forms with arbitrary dimension data [1], and establish boundedness results for certain subclasses.

0.1.1 Our results

In Chapter 1, we study a special case of the singular Brascamp-Lieb form with dimension data $(3, 1, 2, 2, 1)$. Our interest in this particular dimension data arises from the question of what kind of bounds one might obtain if, in the triangular Hilbert transform, one of the two-dimensional functions is replaced by a one-dimensional function. Notably, due to scaling considerations, the exponents p in this case do not correspond to Hölder-type exponents.

This object serves as a testing ground in several respects. First, we attempt to classify this family of forms through the notion of equivalence introduced earlier. To our knowledge, this is the first complete classification of a singular Brascamp-Lieb form with a specific dimension data. Previous works have addressed partial classifications—for instance, in [28], the authors divides the two-dimensional bilinear Hilbert transforms into different classes according to the spectrum of the two matrices, but they did not establish that objects in different classes are inequivalent. Later, we realized that establishing the inequivalence and classification of trilinear singular Brascamp-Lieb forms is directly related to a well-known problem in quiver representation theory, known as the four subspace problem. It is my advisor Prof. Christoph Thiele who suggested us to take the approach through quiver representation. By employing tools from quiver representation, and in collaboration with Lars Becker and Polona Durcik, we were able to classify trilinear singular Brascamp-Lieb forms for general dimension data [1].

Second, we develop an approach to argue the unboundedness of singular Brascamp-Lieb forms when the kernel is a genuinely singular kernel, such as $\frac{1}{t}$. Typically, singular

Brascamp-Lieb inequalities are stated for all kernels K satisfying Mihlin's condition, and one seeks to establish bounds within some range. To disprove such bounds, a common strategy is to let K be a Dirac delta, reducing the singular Brascamp-Lieb form to a classical Brascamp-Lieb form, and then verify whether the corresponding Brascamp-Lieb criterion fails. However, our method tackles the unboundedness directly when K is a genuinely singular kernel.

Our result shows that no bound exists in the open range for this particular form — bounds hold only at certain endpoints. We conjecture this phenomenon reflects a general principle for most non-Hölder-type singular Brascamp-Lieb forms.

0.2 Multilinear Fourier multipliers with rough multipliers

If a multiplier m satisfies the Mihlin's condition (0.1.2) for all multi-indices α with $|\alpha| \leq s$ for some integer s, then we say m satisfies the s-Mihlin's condition. A natural question arises: Given a multiplier m, how rough can m be? More precisely, what is the minimal integer s for which the associated singular Brascamp–Lieb form remains bounded in a certain range? Moreover, we are interested in exploring the relationship between the regularity exponent s and the range of exponents p for which boundedness holds.

Linear and multilinear singular integrals can often be reformulated in terms of multipliers. In the linear setting, Mihlin's classical theorem [73] from 1956 shows that if m satisfies the $(\lfloor \frac{d}{2} \rfloor + 1)$-Mihlin's condition then the associate multiplier operator T_m defined below

$$T_m f(x) := \int_{\mathbb{R}^d} m(\xi) \widehat{f}(\xi) e^{2\pi i x \xi} d\xi \tag{0.2.1}$$

is bounded in the open Banach range. Furthermore, the regularity requirement $\lfloor \frac{d}{2} \rfloor + 1$ is sharp. There is also a weaker, yet closely related, condition known as the Hörmander's condition. We say a multiplier m satisfies the s-Hörmander's condition if

$$\sup_{j \in \mathbb{Z}} \| m(2^j \cdot) \psi \|_{H^s(\mathbb{R}^d)} < +\infty \tag{0.2.2}$$

where ψ is a smooth bump function compactly supported away from origin and H^s denotes the L^2-based inhomogeneous Sobolev norm. By direct calculation, if m satisfies th s-Mihlin's condition, then m also satisfies the s-Hörmander's condition. Note that Hörmander's condition allows for fractional regularity exponents s. In 1960, Hörmander [52] proved that if m satisfies the s-Hörmander's condition for $s > \frac{d}{2}$ then the multiplier operator T_m defined in (0.2.1) is also bounded in the open Banach range. Moreover, the number $\frac{d}{2}$ is also sharp.

In the multilinear setting, the Coifman–Meyer operator serves as a fundamental example of a singular Brascamp–Lieb form. This operator can also be expressed via a multiplier form

$$T_m(f_1, \cdots, f_n)(x) = \int_{\mathbb{R}^{nd}} m(\xi) \prod_{j=1}^{n} \widehat{f_j}(\xi_j) e^{2\pi i x(\xi_1 + \cdots + \xi_n)} d\xi_1 \cdots d\xi_n \,. \tag{0.2.3}$$

In a pair of seminal papers from 1978, Coifman and Meyer [24, 26] established that if m satisfies s-Mihlin's condition for sufficient large s, then T_m defined in (0.2.3) is bounded in the open Banach range. Later in 2010, Tomita [94] identified the sharp Hörmander-type condition for this class of operators, proving that if m satisfies the s-Hörmander's condition with $s > \frac{nd}{2}$, then T_m is bounded in the open Banach range. For results concerning boundedness beyond the Banach range, we refer to [56], [47], [46], and [66].

Historically, another important class of kernels or multipliers has drawn significant interest. These are homogeneous kernels of the form

$$K(x) = \frac{\Omega(x/|x|)}{|x|^d} \tag{0.2.4}$$

where Ω is a function defined on the unit sphere unit sphere $\mathbb{S}^{d-1}$. Remarkably, the associated singular integral operator can sometimes be bounded over a certain range without any regularity assumption on Ω. In the linear setting, Calderón and Zygmund [13] showed in 1956 that if Ω has mean value zero on the sphere and belongs to the Orlicz space $L \operatorname{Log} L(\mathbb{S}^{d-1})$, then the singular integral operator

$$Tf(x) = \int_{\mathbb{R}^d} f(x-y)K(y)dy \tag{0.2.5}$$

is bounded in the open Banach range. Moreover, there are notable endpoint results established in 1988 by Christ, Rubio de Francia, and Hofmann [19, 22, 51], as well as further results by Seeger in 1996 [89].

In the multilinear setting, let us again consider a Coifman–Meyer-type operator, but now associated with a rough kernel. Define

$$T_\Omega^n(f_1, \cdots, f_n)(x) := \int_{\mathbb{R}^{nd}} K(y_1, \cdots, y_n) \prod_{j=1}^n f_j(x-y_j)dy_1 \cdots dy_n\,. \tag{0.2.6}$$

Starting around 2010, there has been a series of contributions by Grafakos, He, Honzík, Lenka, and Dosidis [29, 42, 43, 49, 32, 31], investigating the boundedness of T_Ω^n under minimal integrability assumptions on Ω. In particular, the recent work [31] in 2024 establishes an estimate that can be applied to the boundedness of Coifman–Meyer type operators involving both Hörmander multipliers and rough kernels.

It is natural to ask whether, for other singular Brascamp–Lieb forms, boundedness can still be obtained when the multiplier is of Hörmander type, or when the kernel is a rough kernel. To explore this, we first reformulate some important examples of singular Brascamp–Lieb forms in the multiplier setting. For instance, the dual form of the bilinear Hilbert transform can be written as

$$\int_V \operatorname{sgn}(\xi_1 - \xi_2) \prod_{j=1}^3 \widehat{f}_j(\xi_j) d\mathcal{H}^2(\xi) \tag{0.2.7}$$

where V is the hyperplane $\xi_1+\xi_2+\xi_3 = 0$ in $\mathbb{R}^3$ and $\mathcal{H}^2$ denotes the 2-dimensional Hausdorff measure on V. Unlike the Coifman–Meyer operator, whose multiplier singularity is a single point, here the singularity of the multiplier is one-dimensional. This higher-dimensional singularity makes the associated multilinear form significantly harder to control and bound.

In general, let $f_j \in \mathcal{S}(\mathbb{R})$ and let V be the hyperplane $\sum_{j=1}^n \xi_j = 0$ in $\mathbb{R}^n$. Suppose Γ is a linear subspace of V and let m be a L^∞ function defined on $V \setminus \Gamma$. We can then associate to m the multilinear form

$$\Lambda_m(f_1, \cdots, f_n) := \int_V m(\xi) \prod_{j=1}^{n} \widehat{f}_j(\xi_j) d\mathcal{H}^{n-1}(\xi)\,. \tag{0.2.8}$$

In 2002, Muscalu, Tao, Thiele [77] showed that if m satisfies the Mihlin-type condition

$$|\partial^\alpha m(\xi)| \lesssim \operatorname{dist}(\xi, \Gamma)^{-|\alpha|} \tag{0.2.9}$$

for all multi-indices α with $|\alpha| \leq N_0$, for some sufficiently large integer N_0, and if the singularity Γ is nondegenerate and satisfies $\dim \Gamma < \frac{n}{2}$, then Λ_m is bounded in a suitable range of exponents. The singularity set can also be generalized. Curved singularites were first studied by Muscalu [78] in 2000. Later, the bound of bilinear disk multiplier was obtained by Grafakos and Li [45] in 2006.

In Chapter 2, we extend the notion of s-Hörmander condition to settings where the singularity is no longer a point but a higher-dimensional set

$$\sup_{\boldsymbol{\beta} \in V \setminus \Gamma} \left\| \left(\operatorname{Dil}^{\infty}_{d_\Gamma(\boldsymbol{\beta})^{-1}} \operatorname{Tr}_{-\boldsymbol{\beta}} m \right) \cdot \Phi \right\|_{H^s(V)} < +\infty. \tag{0.2.10}$$

where $d_\Gamma(\boldsymbol{\beta})$ is the distance between Γ and $\boldsymbol{\beta}$. We then show that, when $n = 3$, the multilinear form Λ_m is bounded in the local L^2 range with the sharp regularity exponent $s > 1$. Furthermore, the singularity set Γ can be generalized beyond linear subspaces to Lipschitz curves in our work.

0.2.1 Our results

In Chapter 2, we study a trilinear Fourier multiplier form where the multiplier m satisfies the generalized Hörmander condition (0.2.10) for a Lipschitz singularity Γ. The study of such Lipschitz singularity was suggested by Prof. Thiele. Classically, to handle Coifman–Meyer operators with Hörmander multipliers, one typically performs a global Hölder inequality, encodes the relevant information into a shifted square function, and then seeks better bounds on that square function. However, in our setting, such a global Hölder inequality is no longer viable. Instead, we apply a local Hölder inequality on each tent object which is analogous to a "tree" in some classical literatures.

Usually, when a multiplier satisfies Mihlin's condition, the standard approach is to first perform a Whitney decomposition around the singular set. Then, within each Whitney cube, one applies a Fourier series decomposition, which expresses the multiplier as a tensor product of exponential functions. The s-Mihlin condition ensures that the Fourier series coefficients are summable when s is sufficiently large. Usually, through this approach, we are not able to obtain the sharp exponent s. After this decomposition, one passes back to the spatial side, where the projections of the center of the Whitney cube in three different directions determine the frequency positions of the associated tiles. Since the Fourier transform of an exponential is a Dirac delta, this spatial side is concentrated in the diagonal, meaning that the three tiles share the same spatial interval.

However, when the multiplier satisfies only a Hörmander condition rather than a Mihlin condition, the associated tiles may correspond to three different spatial intervals, instead of overlapping perfectly. This creates substantial new difficulties. Unlike the classical single-tree estimate for the bilinear Hilbert transform, where a local Hölder inequality on each tree suffices, our situation requires a more delicate analysis.

In particular, we distinguish between two scenarios based on the relation between the length of the spatial intervals and the distance separating them:

1. The number of scales for which the intervals are smaller than their mutual distance is only logarithmic, we can apply a trivial estimate.

2. When the distance between intervals is less than their length, they are essentially overlapped. By a fixed dilation factor, we can embed all three intervals into a common one. In this case, similar to the classical single tree estimate, we perform a Hölder inequality to decouple the three functions and bound them by the product some suitable sizes.

This is the key how we handle the difficulties arising from the Hörmander's condition.

On the other hand, the Lipschitz nature of Γ is addressed separately at the phase of selection algorithm, and where we build strong disjointness among tents. To this end, we develop geometric lemmas that ensure the projection of the frequency part of each tent remains well-controlled, preventing certain "leakage". The key idea is that when the curve does not oscillate excessively, the frequency components of the tents remain well-ordered in a desirable way. However, we believe that the upper bound on the Lipschitz constant may not be necessary, provided there is a method to effectively manage the leakage.

One feature of our approach is that we work entirely in the continuous setting, rather than performing an initial discretization — marking a difference from some classical treatments in time-frequency analysis. This approach is inspired by the work of Do and Thiele [30], who introduced L^p outer measure theory in 2015. Their framework reinterprets certain arguments in time-frequency analysis as a form of generalized Carleson embedding. For further developments in L^p outer measure theory, see the work of Uraltsev [95]. However, when the singularity is not a hyperplane, we currently have no clear way to reduce the problem to a form where the generalized Carleson embedding from [30] can be applied. This limitation is why we did not work entirely within the L^p outer measure theory framework.

Recently, Fraccaroli, Saari, and Thiele [36] introduced a new approach, constructing phase space localization operators with useful properties. This method provides an alternative to classical time-frequency analysis and L^p outer measure theory for handling multipliers with nontrivial singularities. An interesting question is whether these phase space localization operators can also be applied to multipliers with curved singularities.

0.3 Multilinear Fourier multipliers with oscillatory multipliers

In the singular Brascamp–Lieb form (0.1.3), it is natural to ask what happens if we replace the linear projection maps B_j with nonlinear ones. Can certain boundedness properties still hold in such a nonlinear setting? In the linear operator case, beginning in the 1970s, there has been a series of fundamental results by Stein, Wainger, Nagel, Christ, Seeger, and Wright [91, 79, 17, 18, 14] on the Hilbert transform along various classes of curves, culminating in the work [21] in 1999.

More recently, in the multilinear operator setting, consider the bilinear Hilbert transform along a curve γ:

$$T_\gamma(f,g)(x) := \int_{\mathbb{R}} f(x+t)g(x+\gamma(t))\frac{dt}{t} \tag{0.3.1}$$

In 2013, Li [68] established the $L^2 \times L^2 \to L^1$ boundedness for the parabola $\gamma(t) = t^2$. Later, in 2015, Lie [69] extended this result to certain non-flat curves γ, and in 2018, Lie [70] further generalized it to a Hölder-type bound in the full Banach range.

Pushing beyond bilinear settings, in 2023, Lie and Hu proved a Hölder-type bound in the Banach range for the trilinear Hilbert transform along the moment curve:

$$T(f,g,h)(x) := \int_{\mathbb{R}} f(x-t)g(x+t^2)h(x+t^3)\frac{dt}{t}\,. \tag{0.3.2}$$

Furthermore, moving to higher dimensions, in 2021, Christ, Durcik, and Roos [20] proved a Hölder-type $L^p \times L^q \to L^r$ bound for exponents $p,q \in (1,\infty)$, $r \in [1,2)$ for the triangular Hilbert transform along parabola $\gamma(t) = t^2$:

$$T(f,g)(x,y) := \int_{\mathbb{R}} f(x+t,y)g(x,y+\gamma(t))\frac{dt}{t} \tag{0.3.3}$$

An important step in their proof involves a powerful sublevel set estimate for certain measurable functions. Most recently, in 2024, Gaitán and Lie [38] provided an alternative proof of the boundedness of the triangular Hilbert transform along parabola, employing a novel approach via the LGC method combined with the sparse-uniform dichotomy.

These multilinear operators can also be expressed in multiplier form. For example, consider the bilinear Hilbert transform along a curve (0.3.1), which can be written as

$$T_\gamma(f,g)(x) = \int_{\mathbb{R}} m_\gamma(\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i(\xi+\eta)x}d\xi d\eta\,, \tag{0.3.4}$$

where the multiplier

$$m_\gamma(\xi,\eta) := \int_{\mathbb{R}} e^{2\pi i(\xi t+\eta\gamma(t))}\frac{1}{t}dt \tag{0.3.5}$$

is an example of an oscillatory multiplier.

Similarly, the triangular Hilbert transform along a curve (0.3.3) can also be written in multiplier form:

$$T_\gamma(f,g)(x,y) = \int_{\mathbb{R}} m_\gamma(\xi,\eta)(\mathcal{F}_{(1)}f)(\xi,y)(\mathcal{F}_{(2)}g)(x,\eta)e^{2\pi i(x\xi+y\eta)}d\xi d\eta\,, \tag{0.3.6}$$

where $\mathcal{F}_{(i)}$ denotes the partial Fourier transform in the i-th variable, and m_γ is is the same multiplier as in (0.3.5). One difference between the bilinear and triangular Hilbert transforms along curves lies in how the parameter t interacts with the functions: for the triangular Hilbert transform, t appears only in the first variable of f and the second variable of g. As a result, partial Fourier transforms are applied to the relevant components of each function, rather than full Fourier transforms.

How to utilize the oscillaotry nature of these multiplier in the high frequency regimes becomes a main task when trying to bound these multilinear oscillatory Fourier multipliers.

Establishing suitable bounds for these operators often relies on a smoothing inequality, which plays a crucial role in proving the boundedness of both the bilinear and triangular Hilbert transforms along curves.

To illustrate this, let us focus on the case of the triangular Hilbert transform along curves. By decomposing the kernel $\frac{1}{t}$ into dyadic scales and localizing the operator in space, one arrives at a localized single-scale multiplier operator:

$$T_{\gamma,0}(f,g)(x,y) := \varphi(x,y) \int_{\mathbb{R}} m_{\gamma,0}(\xi,\eta)(\mathcal{F}_{(1)}f)(\xi,y)(\mathcal{F}_{(2)}g)(x,\eta)e^{2\pi i(x\xi+y\eta)} d\xi d\eta\,, \quad (0.3.7)$$

where φ is a bump function supported in the unit ball and

$$m_{\gamma,0} := \int_{\mathbb{R}} e^{2\pi i(\xi t+\eta\gamma(t))}\psi(t)dt \quad (0.3.8)$$

with ψ a bump function supported away from 0. Through a Littlewood-Paley decomposition applied to both f and g on their respective components, the smoothing inequality asserts that there exist constants $C > 0$, $\sigma > 0$, such that for all $\lambda > 1$ and for all functions f, g satisfying the frequency localization conditions

$$\operatorname{supp} \mathcal{F}_{(1)}f \subseteq [\lambda, 2\lambda] \times \mathbb{R}\,, \quad \operatorname{supp} \mathcal{F}_{(2)}g \subseteq \mathbb{R} \times [\lambda, 2\lambda]\,, \quad (0.3.9)$$

the bilinear operator T satisfies the bound

$$\|T_{\gamma,0}(f,g)\|_{L^1} \leq C\lambda^{-\sigma}\|f\|_{L^2}\|g\|_{L^2}\,. \quad (0.3.10)$$

This smoothing inequality has several other important applications. One notable consequence is a Roth-type theorem in the Euclidean setting. Specifically, consider a measurable subset $E \subseteq [0,1]^2$ with Lebesgue measure ε where $0 < \varepsilon < \frac{1}{2}$. Then, there exists a triple of points

$$(x,y), \quad (x+t,y), \quad (x,y+t^2) \in E \quad (0.3.11)$$

with $t > \exp(-\exp(\varepsilon^{-C}))$ for some constant $C > 0$ not depending on E or ε. This means that, we can find a "corner-type" nonlinear configuration (0.3.11) within E with a large gap t depending only on the measure ε.

There is a rich and extensive history surrounding Roth's theorem, which itself has inspired a substantial body of literature. Here, we only briefly mention results related to nonlinear Roth-type theorems, with a particular focus on the Euclidean setting.

The study of nonlinear Roth theorems in Euclidean space originates from the work of Bourgain [9] in 1998, where he first established the existence of three-term nonlinear patterns of the form

$$x,\ x+t,\ x+\gamma(t) \in E \subseteq [0,N] \quad (0.3.12)$$

for $\gamma(t) = t^2$ with some large gap t depending on the density of the set $\frac{|E|}{N}$ and the scale N. Later in 2019, Durcik, Guo, and Roos [33] extended Bourgain's result the result to γ a polynomial. More recently, in 2024, Krause, Mirek, Peluse, and Wright [61] further generalized this framework to m-term nonlinear patterns of the form

$$x,\ x+\gamma_1(t),\ \cdots, x+\gamma_m(t) \in E \subseteq [0,N] \quad (0.3.13)$$

for $\gamma_1, \cdots, \gamma_m$ some general polynomials with distinct degree and some large scale N with some large gap t depending on the density of the set $\frac{|E|}{N}$ and the scale N. The work [61] builds on techniques developed over a series of works by Peluse and Prendiville [85, 81, 82, 84, 83], beginning in 2014 where quantitative bounds for polynomial Roth theorems in finite fields and integers were established.

Another line of research focuses on polynomial progressions that can be found within fractal sets of large Hausdorff dimension, under additional constraints on the Fourier dimension. Notable contributions in this direction include works by Laba, Malabika, Henriot, Guo, and Fraser [62, 50, 37].

These smoothing inequalities can also be applied to the linear and multilinear spherical maximal operator. There is a long history of research in this direction, leading to several related questions. For example, one may ask about L^p improving estimates, generalize the set to which the dilation parameter belongs, or consider surfaces of different codimensions. The study of these questions has made this branch a vibrant and dynamic area in harmonic analysis. In this last part on the introduction, In this final part of the introduction, I will highlight several results that are important in this direction and relevant to the main theme of this thesis. The linear spherical maximal operator is defined as follows:

$$S(f)(x) := \sup_{t>0} \left| \int_{S^{d-1}} f(x - ty) d\sigma(y) \right| . \tag{0.3.14}$$

Given a dimension d, one may ask for which exponents p the operator $S(f)$ is bounded on $L^p(\mathbb{R}^d)$? This question was resolved by Stein by Stein [90] for $d \geq 3$ in 1976 and later by Bourgain [8] for $d = 2$ in 1986.

f we replace the sphere with a curve of codimension $d - 1$ such as the moment curve, several breakthroughs have been made in the past five years. The sharp result for $d = 3$ was obtained by Ko, Lee, and Oh [57], as well as by Beltran, Guo, Hickman, and Seeger [3]. Partial results for $d \geq 4$ was obtained by Ko, Lee, and Oh [58].

Turning to the bilinear case, the natural analogue is given by the operator

$$S(f, g)(x) := \sup_{t>0} \left| \int_{S^{2d-1}} f(x - ty) g(x - tz) d\sigma(y, z) \right| . \tag{0.3.15}$$

Several recent works [54, 6, 23, 7] have investigated the boundedness of this bilinear operator, leading to important new developments in this area.

The final section of this dissertation is presented in the form of a "blueprint" written for Lean. To set the stage, we begin with a brief introduction to proof assistants, the Lean programming language, and the "blueprint" format. A proof assistant is a software tool designed to verify the correctness of formal proofs written by humans. Typically, it involves a specialized programming language based on logic and type theory. Since the 1960s, several proof assistants have been developed, including Mizar [72], Isabelle [80], and Lean [27].

Lean, a programming language and proof assistant developed by Leonardo de Moura at Microsoft Research in 2013. Over the past decade, Lean has gone through several updates. Lean features a robust mathematical library known as mathlib including many formalized definitions and theorems from modern mathematics.

The collaboration model behind Lean-based formalization works as follows. Starting from an original mathematics paper, a group of contributors writes a detailed "blueprint".

This blueprint expands on the original work, spelling out every new definition not already found in mathlib and elaborating on steps that may have been omitted in the original paper. Collaborating with members of the Lean community, the team then works together to formalize the content into Lean code.

This model has led to several successful projects in recent years. One notable example is the formalization of Perfectoid spaces, sophisticated objects in arithmetic geometry introduced by Scholze in 2012 [88]. This project was led by Buzzard, Commelin, and Massot [10]. Another major achievement is the formalization of the Polynomial Freiman-Ruzsa (PFR) conjecture [40], originally proved by Gowers, Green, Manners, and Tao in 2023. Remarkably, the formalization was completed in just three weeks thanks to an extraordinary collaborative effort within the Lean community.

This model demonstrates another significant advantage. By breaking down a complex proof into smaller, manageable parts, it enables contributions from individuals who may not be experts in the specific area of mathematics involved.

An ongoing example of this approach is the formalization of Carleson's theorem in harmonic analysis. Originally proved by Carleson in 1966 [15], with alternative proofs later provided by Fefferman in 1973 [35] and by Lacey and Thiele in 2000 [65], this project is currently being led by Becker, van Doorn, Jamneshan, Srivastava, and Thiele [2].

0.3.1 Our results

In Chapter 3, we study the class of multipliers m for which the associated localized single-scale multiplier operator (0.3.7) satisfies the smoothing inequality (0.3.10). To structure the argument clearly and for future reusability, we modularize the proof into several independent parts, each of which can potentially be applied in other contexts.

The first part of the proof consists several inequalities of bounding the L^1 norm of the operator by products of certain mixed L^p norm of $\mathcal{F}_{(1)}f$ and $\mathcal{F}_{(2)}g$, together with two special quantities $\|m\|_u$ and $\|m\|_U$, which capture the oscillatory nature of the multipliers. These quantities are defined as follows:

$$\|m\|_u := \left\|(\mathcal{F}_{(1)}\mathcal{D}_{(0,s)}m)(x,\eta)\right\|_{L^\infty_{s\eta}L^2_x([-1,1])}^{\frac{1}{2}}, \tag{0.3.16}$$

$$\|m\|_U := \left\|\int_{\mathbb{R}}\mathcal{D}_{(0,s)}\mathcal{D}_{(u,v)}m(\xi,\eta)d\xi\right\|_{L^\infty_{u\eta}L^1_sL^2_v}^{\frac{1}{4}}, \tag{0.3.17}$$

where $\mathcal{D}_{(u,v)}$ denotes the two-dimensional multiplicative derivative:

$$\mathcal{D}_{(u,v)}f(x,y) := f(x+u,y+v)\overline{f(x,y)}\,. \tag{0.3.18}$$

These quantities can be viewed as variants of the Gowers uniformity norm on a finite abelian group G, defined by

$$\|f\|_{U^d(G)}^{2^d} := \mathop{\mathbb{E}}_{x,h_1,\cdots,h_k\in G}\mathcal{D}_{h_1}\cdots\mathcal{D}_{h_k}f(x)\,. \tag{0.3.19}$$

The key technical challenge lies in estimating $\|m_{\gamma,\lambda}\|_u$ and $\|m_{\gamma,\lambda}\|_U$ for multipliers of the form

$$m_{\gamma,\lambda}(\xi,\eta) := \psi\left(\frac{\xi}{\lambda}\right)\psi\left(\frac{\eta\gamma'(1)}{\lambda}\right)\int_{\mathbb{R}}e^{2\pi i(\xi t+\eta\gamma(t))}\psi(t)dt\,. \tag{0.3.20}$$

Our main estimate shows that there exist constants $c_1, c_2 > 0$ such that

$$\|m_{\gamma,\lambda}\|_u \lesssim \lambda^{-c_1}, \quad \|m_{\gamma,\lambda}\|_U \lesssim (\log \lambda)^{c_2}\,, \tag{0.3.21}$$

which, after interpolation, still yields polynomial decay in λ.

An interesting question is whether one can find explicit examples of multipliers, not necessarily arising from singular Brascamp-Lieb forms along curves, that nevertheless satisfy good bounds in terms $\|\cdot\|_u$, $\|\cdot\|_U$ and hence the associate operator will admit a smoothing inequality. Such results may have further applications in other areas.

Following a suggestion by Prof. Thiele, Chapter 3 is written in the form of a blueprint for Lean, a computer-assisted formal verification system. In particular, we calculate explicit dependencies on higher-order derivatives of the curve γ. A recent project on generalized Carleson operators [2, 54] serves as a reference for this type of formalization.

Chapter 1

On the family of singular Brascamp-Lieb inequalities with dimension datum $(1, 2, 2, 1)$

1.1 Introduction

For $F, G, H \in \mathcal{S}(\mathbb{R}^2)$, the triangular Hilbert form Λ is defined by

$$\Lambda(F, G, H) := \mathrm{p.\,v.} \int_{\mathbb{R}^3} F(x, y)G(y, z)H(z, x) \frac{dxdydz}{x + y + z}, \tag{1.1.1}$$

which has been introduced by Demeter and Thiele [28] motivated by an open probem in ergodic theory, on the pointwise convergence of bilinear averages with respect to two commuting transformations. A celebrated open problem in harmonic analysis is whether there exists a constant C such that for all Schwartz functions F, G, H, the a priori inequality

$$|\Lambda(F, G, H)| \leq C\|F\|_{L^{p_1}(\mathbb{R}^2)}\|G\|_{L^{p_2}(\mathbb{R}^2)}\|H\|_{L^{p_3}(\mathbb{R}^2)} \tag{1.1.2}$$

holds for any exponents $p_1, p_2, p_3 > 1$ with $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$, in particular for the case $p_1 = p_2 = p_3 = 3$. Such bounds are stronger than several important results in harmonic analysis related to Carleson's operator [15], [35], [65] and the bilinear Hilbert transform [63], [64] and also the latter's uniform estimates [92], [44], [67], [96]. Inequality (1.1.2) falls in the realm of singular Brascamp-Lieb inequalities as in the survey [34].

To make small progress towards the boundedness of the triangular Hilbert form, we discuss a simpler related family of singular Brascamp-Lieb forms. If we consider a function F of the form

$$F(x, y) := f(x + \alpha y)$$

for a one dimensional Schwartz function f, then the integral (1.1.1) is still well defined and becomes

$$\widetilde{\Lambda}_\alpha(f, G, H) := \mathrm{p.\,v.} \int_{\mathbb{R}^3} f(x + \alpha y)G(y, z)H(z, x) \frac{dxdydz}{x + y + z}. \tag{1.1.3}$$

This is a singular Brascamp-Lieb form with dimensions $1, 2, 2$ of the functions f, G, H and dimension one of the singular kernel. If f is in $L^\infty(\mathbb{R})$, then F is in $L^\infty(\mathbb{R}^2)$ and singular

Brascamp-Lieb bounds for (1.1.3) can be viewed as special cases of (1.1.2) with $p_1 = \infty$. For f in $L^p(\mathbb{R})$ with $p < \infty$, inequality (1.1.3) is not a special case of inequality (1.1.3).

More generally, our family of interest of singular Brascamp-Lieb forms with dimension datum $(1, 2, 2, 1)$ is a generalization of (1.1.3) in the spirit of [34] towards general projections with dimensions $(1, 2, 2, 1)$ in the arguments, that is

$$\Lambda_\Pi(f, G, H) := \text{p. v.} \int_{\mathbb{R}^3} f(\Pi_1 x) G(\Pi_2 x) H(\Pi_3 x) \frac{1}{\Pi_4 x} dx, \tag{1.1.4}$$

with the projection datum

$$\Pi = (\Pi_1, \Pi_2, \Pi_3, \Pi_4), \tag{1.1.5}$$

where

$$\Pi_1, \Pi_4 : \mathbb{R}^3 \to \mathbb{R}, \quad \Pi_2, \Pi_3 : \mathbb{R}^3 \to \mathbb{R}^2 \tag{1.1.6}$$

are surjective linear maps. To avoid some trivial cases, we assume that

$$\text{Im}(\Pi_4^*) \not\subset \text{Im}(\Pi_j^*) \tag{1.1.7}$$

for $j = 1, 2, 3$, where the star denotes the adjoint.

Our first theorem classifies the forms (1.1.4) up to equivalence, where Λ_Π is equivalent to $\Lambda_{\Pi'}$, if there exist

$$B \in GL(\mathbb{R}^3),\ A_1, A_4 \in GL(\mathbb{R}),\ A_2, A_3 \in GL(\mathbb{R}^2),$$

such that for each $1 \le j \le 4$

$$\Pi'_j = A_j \Pi_j B.$$

Note that then we have

$$\sup_{f,G,H} \frac{|\Lambda_{\Pi'}(f, G, H)|}{\|f\|_{p_1} \|G\|_{p_2} \|H\|_{p_3}} = \frac{A_4^{\frac{1}{p_1} + \frac{2}{p_2} + \frac{2}{p_3} - 3}}{\text{detB} \cdot \prod_{j=1}^3 \det(A_j)^{\frac{1}{p_j}}} \sup_{f,G,H} \frac{|\Lambda_\Pi(f, G, H)|}{\|f\|_{p_1} \|G\|_{p_2} \|H\|_{p_3}}.$$

Hence, it suffices to consider the boundedness of standard forms listed in Theorem 1.1.1.

Theorem 1.1.1. *Let Π be a datum as in* (1.1.5)*,* (1.1.6)*, and* (1.1.7)*. If Λ_Π is nonzero, it is equivalent to one of*

$$\Lambda_{(1)}(f, G, H) := \text{p. v.} \int_{\mathbb{R}^3} f(x) G(x, y) H(x, y + t) \frac{1}{t} dt dx dy, \tag{1.1.8}$$

$$\Lambda_{(2)}(f, G, H) := \text{p. v.} \int_{\mathbb{R}^3} f(x) G(x, y) H(x + t, y) \frac{1}{t} dt dx dy, \tag{1.1.9}$$

$$\Lambda_{(3)}(f, G, H) := \text{p. v.} \int_{\mathbb{R}^3} f(x + t) G(x, y) H(x, y + t) \frac{1}{t} dt dx dy, \tag{1.1.10}$$

or it is equivalent to

$$\Lambda_{(4,\beta)}(f, G, H) := \text{p. v.} \int_{\mathbb{R}^3} f(x + t) G(x, y) H(x + \beta t, y) \frac{1}{t} dt dx dy \tag{1.1.11}$$

for some $\beta \in \mathbb{R}$. Furthermore, any two forms in the above three discrete cases and in the one parameter family are mutually not equivalent to each other.

In particular, form (1.1.3) is equivalent to (1.1.10) if $\alpha = 1$ and to (1.1.11) with $\beta = 1-\alpha$ otherwise.

Our second goal is to discuss L^p bounds of the forms in Theorem 1.1.1. Although there's a singular kernel p. v. $1/t$ in our case, we may still able to perform a similar scaling argument as in [4] and obtain the region of exponents where one may have such L^p bounds:

$$p_1 = \infty, \ \frac{1}{p_2} + \frac{1}{p_3} = 1, \ 1 < p_2, p_3 < \infty. \tag{1.1.12}$$

Theorem 1.1.2 describes the most interesting case (1.1.11), while Theorem 1.1.3 considers the remaining cases.

Theorem 1.1.2. *Let $\beta \neq 0, 1$, assume (p_1, p_2, p_3) is in the range* (1.1.12)*. There exists a constant C such that for $f \in \mathcal{S}(\mathbb{R})$, $G, H \in \mathcal{S}(\mathbb{R}^2)$, the following holds:*

$$|\Lambda_{(4,\beta)}(f, G, H)| \leq \ C\|f\|_{L^{p_1}(\mathbb{R})}\|G\|_{L^{p_2}(\mathbb{R}^2)}\|H\|_{L^{p_3}(\mathbb{R}^2)}. \tag{1.1.13}$$

If (p_1, p_2, p_3) is not in the range (1.1.12)*, then there is no constant C such that the a priori inequality (1.1.13) holds.*

The proof of the positive result in this theorem by freezing a variable can easily be adapted to the L^∞ case of an estimate in [60] in the dyadic setting. Somehow this easy case of the family of estimates in [60] does not appear there. Note that the trick of freezing a variable then apply bilinear Hilbert transform to bound certain trilinear singular integral form has already appeared in [48].

For the remaining cases of the classification in Theorem 1.1.1, we summarize the much easier L^p theory in next theorem.

Theorem 1.1.3. *The three variants of Theorem 1.1.2 with $\Lambda_{(4,\beta)}$ replaced by $\Lambda_{(1)}$ or $\Lambda_{(2)}$ or $\Lambda_{(4,1)}$ remain true. On the other hand, for any $1 \leq p_1, p_2, p_3 \leq \infty$, there is no constant C such that the two variants of* (1.1.13) *with $\Lambda_{(4,\beta)}$ replaced by Λ_3 or $\Lambda_{4,0}$ hold.*

After discussing the boundedness of the family of singular Brascamp-Lieb form with dimension datum $(1, 2, 2, 1)$, we end up with showing the unboundedness of triangular Hilbert transform in the endpoint region (∞, p_2, p_3).

Theorem 1.1.4. *Let Λ be the triangular Hilbert transform (1.1.1). Given exponents p_1, p_2, p_3 satisfying (1.1.12), and C be any constant, there exists $F, G, H \in \mathcal{S}(\mathbb{R}^2)$ such that*

$$|\Lambda(F, G, H)| \geq C\|F\|_{L^\infty}\|G\|_{L^{p_2}}\|H\|_{L^{p_3}}. \tag{1.1.14}$$

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1.2 Proof of Theorem 1.1.1

Let Π be a datum as in (1.1.5),(1.1.6), and (1.1.7). For $1 \leq j \leq 4$, let V_j be the image of Π_j^*. As Π_j is surjective, the dimension of V_j is one if $j = 1, 4$, and is two if $j = 2, 3$. We do a case distinction according to the relative positions of these subspaces. The following diagram shows the case distinction.

- $V_1 \nsubseteq V_2$
  - $V_2 \neq V_3$
    - $V_1 \nsubseteq \mathrm{span}(V_2 \cap V_3, V_4)$ $\rightsquigarrow \Lambda_{(4,\beta)}, \beta \neq 0$
    - $V_1 \subseteq \mathrm{span}(V_2 \cap V_3, V_4)$ $\rightsquigarrow \Lambda_{(3)}$
  - $V_2 = V_3 \rightsquigarrow \Lambda_{(4,\beta)}, \beta = 0$
- $V_1 \subseteq V_2$
  - $V_1 \nsubseteq V_3$
    - $V_2 \neq V_3 \rightsquigarrow \Lambda_{(1)}$
    - $V_2 = V_3 \rightsquigarrow 0$
  - $V_1 \nsubseteq V_3 \rightsquigarrow \Lambda_{(2)}$

We start with the most interesting case.

Case 1: On the one hand $V_1 \nsubseteq V_2$ and on the other hand $V_2 = V_3$ or $V_1 \nsubseteq \mathrm{span}(V_2 \cap V_3, V_4)$.
As V_4 is not contained in V_2 by (1.1.7), we have that $V_2 + V_4$ is the full space $\mathbb{R}^3$. Hence we may pick $v_1 \in V_2$ and $v_3 \in V_4$ such that $v_1 + v_3$ spans the one dimensional space V_1. The vectors v_1 and v_3 are linearly independent, because V_1 is not in V_2 by the first assumption in Case 1, and not in V_4 by (1.1.7). Now choose a vector v_2 in $V_2 \cap V_3$ which is linearly independent of v_1. This is possible if $V_2 = V_3$, because then $V_2 \cap V_3$ is two dimensional. It is also possible if $V_1 \nsubseteq \mathrm{span}(V_2 \cap V_3, V_4)$. Namely, let v_2 be any nonzero vector in $V_2 \cap V_3$ and assume to get a contradiction that v_1 is a multiple of v_2. Then $v_1 + v_3$ is in $\mathrm{span}(V_2 \cap V_3, V_4)$. This contradicts that $v_1 + v_3$ spans V_1. Hence we have seen that under the assumption of Case 1, we can choose v_2 as above.

As v_1 and v_3 are linearly independent and v_3 is not in V_3, there is a $\beta \in \mathbb{R}$ such that $v_1 + \beta v_3 \in V_3$. Since $V_2 + V_4 = \mathbb{R}^3$, $\mathrm{span}\{v_1, v_2\} = V_2$, and $\mathrm{span}\{v_3\} = V_4$, we have found a basis $\{v_1, v_2, v_3\}$ for $\mathbb{R}^3$.

We choose B so that B^* maps v_1, v_2, v_3 to the standard unit vectors e_1, e_2, e_3. We choose A_j so that A_j^* maps the standard unit vectors of $\mathbb{R}^1$ or $\mathbb{R}^2$ to preimages under Π_j^* of the spanning vectors of V_j expressed as above in terms of v_1, v_2, or v_3. This allows to directly write down the matrix for $B^* \Pi_j^* A_j^*$ in the standard basis. The transposes of these matrices are as follows.

$$A_1 \Pi_1 B = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}, \quad A_2 \Pi_2 B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$A_3 \Pi_3 B = \begin{pmatrix} 1 & 0 & \beta \\ 0 & 1 & 0 \end{pmatrix}, \quad A_4 \Pi_4 B = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}.$$

Hence

$$\mathrm{p.\,v.} \int_{\mathbb{R}^3} f(A_1 \Pi_1 B X) G(A_2 \Pi_2 B X) H(A_3 \Pi_3 B X) \frac{1}{A_4 \Pi_4 B X} dX \tag{1.2.1}$$

$$= \text{p. v.} \int_{\mathbb{R}^3} f(x+t)G(x,y)H(x+\beta t, y)\frac{1}{t}dtdxdy,$$

which is $\Lambda_{(4,\beta)}$.

Case 2: $V_1 \nsubseteq V_2$, $V_2 \neq V_3$, and $V_1 \subseteq \operatorname{span}(V_2 \cap V_3, V_4)$.

By the last assumption of Case 2, pick $v_1 \in V_2 \cap V_3$ and $v_3 \in V_4$ such that $v_1 + v_3$ spans V_1. The vectors v_1, v_3 are linearly independent, otherwise one of them is a nonzero vector in V_1 which is impossible by (1.1.7) and the first assumption of Case 2. Choose v_2 to be a vector in V_2 which is linearly independent of v_1 such that $v_2 + v_3$ is a nonzero vector in V_3. This is possible because $V_2 \neq V_3$ by the second assumption of Case 2 and $V_4 \cap V_3 = \{0\}$ by (1.1.7). Since $V_1 \nsubseteq V_2$, we have $v_2 \notin \operatorname{span}\{v_1, v_3\}$. Hence $\{v_1, v_2, v_3\}$ forms a basis of $\mathbb{R}^3$.

Choosing A_j and B similarly as above, we have

$$A_1\Pi_1 B = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}, \quad A_2\Pi_2 B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$A_3\Pi_3 B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad A_4\Pi_4 B = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}.$$

Hence (1.2.1) is equal to

$$\text{p. v.} \int_{\mathbb{R}^3} f(x+t)G(x,y)H(x, y+t)\frac{1}{t}dtdxdy,$$

which is $\Lambda_{(3)}$.

Case 3: $V_1 \subseteq V_2$ and $V_1 \nsubseteq V_3$.

Let v_2 be a vector in $V_2 \cap V_3$. Choose $v_1 \in V_1$ and $v_3 \in V_4$ such that $v_1 + v_3 \in V_3$. This is possible because $V_1 \nsubseteq V_3$ and $V_4 \nsubseteq V_3$. Since $V_4 \nsubseteq V_2$ and $V_1 \subseteq V_2$, we have $V_2 \cap V_3 \nsubseteq V_1 + V_4$. This shows that $\{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$. Choosing A_j and B similarly as above, we have

$$A_1\Pi_1 B = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \quad A_2\Pi_2 B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$A_3\Pi_3 B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_4\Pi_4 B = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}.$$

Hence (1.2.1) is equal to

$$\text{p. v.} \int_{\mathbb{R}^3} f(x)G(x,y)H(x+t, y)\frac{1}{t}dtdxdy,$$

which is $\Lambda_{(2)}$.

Case 4: $V_1 \subseteq V_2$, $V_1 \subseteq V_3$, and $V_2 \neq V_3$.

Let v_1 be a vector in V_1 and v_3 be a vector in V_4. Choose another vector $v_2 \in V_2$ which is linearly independent of v_1 such that $v_2 + v_3 \in V_3$. This is possible because $V_2 \neq V_3$ and $V_4 \nsubseteq V_3$. Since $V_2 + V_4 = \mathbb{R}^3$, we have that $\{v_1, v_2, v_3\}$ forms a basis of $\mathbb{R}^3$.

Choosing A_j and B suitably as above, we have

$$A_1\Pi_1 B = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \quad A_2\Pi_2 B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$A_3\Pi_3 B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad A_4\Pi_4 B = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}.$$

Similarly as above, we obtain for (1.2.1)

$$\text{p. v.} \int_{\mathbb{R}^3} f(x)G(x,y)H(x,y+t)\frac{1}{t}dtdxdy,$$

which is $\Lambda_{(1)}$.

Notice that beside the above cases, there's a trivial case, $V_1 \subseteq V_2$ and $V_2 = V_3$. We then have the following data.

$$A_1\Pi_1 B = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \quad A_2\Pi_2 B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$A_3\Pi_3 B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_4\Pi_4 B = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}.$$

and the corresponding trilinear form

$$\text{p. v.} \int_{\mathbb{R}^3} f(x)G(x,y)H(x,y)\frac{1}{t}dtdxdy.$$

This integral is zero since the principal value is interpreted as limit as $\epsilon \to 0$ of the truncation of the integral to $t \in [-\epsilon^{-1}, \epsilon^{-1}] \setminus [-\epsilon, \epsilon]$.

Since the inclusion relation of subspaces maintain the same after basis change, $\Lambda_{(1)}$, $\Lambda_{(2)}$, $\Lambda_{(3)}$, $\Lambda_{(4,0)}$, and $\Lambda_{(4,1)}$ are mutually not equivalent to each other and all not equivalent to $\Lambda_{(4,\beta)}$ for $\beta \neq 0, 1$. In the following, we define a quantity in projective geometry to distinguish $\Lambda_{(4,\beta)}$ for different β. Let V_1' be the space spanned by V_1 and $V_2 \cap V_3$. Let V_4' be the space spanned by V_4 and $V_2 \cap V_3$. We define the cross ratio of these four planes as follows. Take an arbitrary line L in $\mathbb{R}^3$ not intersecting $V_2 \cap V_3$. Let x_1, x_2, x_3, and x_4 be the intersection of L with these four planes then project to $\mathbb{R}$ respectively. Then the cross ratio

$$\frac{(x_1 - x_4)(x_3 - x_2)}{(x_1 - x_2)(x_3 - x_4)}$$

is independent of the choice of L and is an invariant under basis change. We calculate the cross ratio for $\Lambda_{4,\beta}$ with $\beta \neq 0, 1$. Consider the line $L : x + z = 1$, $y = 0$. The intersection of L with these four planes are $(\frac{1}{2}, 0, \frac{1}{2})$, $(1, 0, 0)$, $(\frac{1}{1+\beta}, 0, \frac{\beta}{1+\beta})$, $(0, 0, 1)$ respectively. To calculate the cross ratio of these four points, it suffices to focus on its x variable.

$$\frac{(\frac{1}{1+\beta} - 1) \cdot \frac{1}{2}}{\frac{1}{1+\beta} \cdot (-\frac{1}{2})} = \beta.$$

With this invariant, we then finish all the classification in the last case.

Example 1.2.1. We will write (1.1.3) in the standard form.

For $\alpha \neq 1$, (1.1.3) is equivalent to $\Pi_{(4,1-\alpha)}$. For $\alpha = 1$ (1.1.3) is equivalent to $\Pi_{(3)}$. Moreover, we can explicitly show A_1, A_2, A_3, A_4, and B. Let Π be the datum of (1.1.3).

For $\alpha \neq 1$, take

$$A_1 = 1, \quad A_2 = \begin{pmatrix} \alpha - 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} -\alpha & -\alpha + 1 \\ 1 & 0 \end{pmatrix}, \quad A_4 = 1, \tag{1.2.2}$$

and

$$B = \begin{pmatrix} \frac{-1}{\alpha-1} & \frac{-\alpha}{\alpha-1} & 1 \\ \frac{1}{\alpha-1} & \frac{1}{\alpha-1} & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{1.2.3}$$

Then

$$A_1\Pi_1 B = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}, \quad A_2\Pi_2 B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$A_3\Pi_3 B = \begin{pmatrix} 1 & 0 & 1-\alpha \\ 0 & 1 & 0 \end{pmatrix}, \quad A_4\Pi_4 B = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}.$$

For $\alpha = 1$, take

$$A_1 = 1, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \quad A_4 = 1, \tag{1.2.4}$$

and

$$B = \begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{1.2.5}$$

Then

$$A_1\Pi_1 B = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}, \quad A_2\Pi_2 B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$A_3\Pi_3 B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad A_4\Pi_4 B = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}.$$

1.3 Proof of Theorem 1.1.2

Let $\beta \in \mathbb{R}$, assume (p_1, p_2, p_3) is in the range (1.1.12). The letter C will denote a sufficiently large positive number that may be implicitly re-adjusted from inequality to inequality and that may depend on β and p_1, p_2, p_3. We write $A \lesssim B$ if $A \leq CB$ for such number C. We write $A \sim B$ if both $A \lesssim B$ and $B \lesssim A$. We will adopt this convention in the rest of this paper.

First consider exponents (p_1, p_2, p_3) in the range (1.1.12). Assuming momentarily that we can pass the p. v inside, then freezing the y variable, we may identify the form as the trilinear form associated with the bilinear Hilbert transform and obtain the desired estimate (1.1.13) as follows:

$$
\begin{aligned}
&\left|\mathrm{p.\,v.}\int_{\mathbb{R}^3} f(x+t)G(x,y)H(x+\beta t,y)\frac{1}{t}dtdxdy\right| \\
=&\left|\int_{\mathbb{R}}\ \mathrm{p.\,v.}\int_{\mathbb{R}^2} f(x+t)G(x,y)H(x+\beta t,y)\frac{1}{t}dtdxdy\right| \\
\lesssim&\int_{\mathbb{R}}\|f\|_{L^\infty}\|G_y\|_{L^{p_2}}\|H_y\|_{L^{p_3}}dy \\
\lesssim&\|f\|_{L^\infty}\|G\|_{L^{p_2}}\|H\|_{L^{p_3}}.
\end{aligned}
$$

To show that we can pass the p. v. inside, by the dominated convergence theorem, it suffices to show that $M \in L^1(\mathbb{R})$, where

$$
M(y) := \sup_{\varepsilon>0}\left|\int_{|t|>\varepsilon} f(x+t)G(x,y)H(x+\beta t,y)\frac{1}{t}dtdx\right|. \tag{1.3.1}
$$

We set $G_y(x) := G(x,y)$, $H_y(x) := H(x,y)$. We split the integrand into $|t|>1$ and $|t|<1$. By the triangle inequality we have

$$
\begin{aligned}
|M(y)| \le &\int_{|t|\le 1}\left|\frac{f(x+t)-f(x)}{t}G_y(x)H_y(x+\beta t)\right|dtdx \\
&+\int_{|t|\le 1}\left|f(x)G_y(x)\frac{H_y(x+\beta t)-H_y(x)}{t}\right|dtdx \\
&+\int_{|t|\le 1}|f(x)G_y(x)H_y(x)|\,dtdx \\
&+\int_{1<|t|}\left|f(x+t)G_y(x)H_y(x+\beta t)\frac{1}{t}\right|dtdx.
\end{aligned}
$$

Then by mean value theorem,

$$
\begin{aligned}
\le &\int_{|t|\le 1}\big|\|f'\|_{L^\infty}G_y(x)H_y(x+\beta t)\big|\,dtdx \\
&+\int_{|t|\le 1}\big|f(x)G_y(x)\|H_y'\|_{L^\infty}\big|\,dtdx \\
&+\int_{|t|\le 1}|f(x)G_y(x)H_y(x)|\,dtdx \\
&+\int_{1<|t|}|f(x+t)G_y(x)H_y(x+\beta t)|\,dtdx.
\end{aligned}
$$

Since f, G, H are all Schwartz functions, these four terms are all L^1 integrable. This completes the proof of estimate (1.1.13) for (p_1,p_2,p_3) in the range (1.1.12).

Now assume (p_1,p_2,p_3) do not satisfy (1.1.12), we will show that the a priori inequality (1.1.13) does not hold. In this proof, we will use H to denote the Hilbert transform. To distinguish from the Hilbert transform, we take E to denote the third function in our trilinear form. We will prove by contradiction. Expanding the trilinear form, we obtain

$$\Lambda_\Pi(T_k g, G, T^{(1)}_{\beta k}E) = \mathrm{p.\,v.}\int_{\mathbb{R}^3} f(x-k+t)G(x,y)E(x-\beta k+\beta t, y)\frac{1}{t}dtdxdy$$
$$= \mathrm{p.\,v.}\int_{\mathbb{R}^3} f(x+t)G(x,y)E(x+\beta t, y)\frac{1}{t+k}dtdxdy.$$

Now fix a $m \in L^1$ with $\operatorname{supp}\widehat{m} \subseteq (0,\infty)$. Pairing the above form with $Hm(-k)$, we have the following estimate

$$\begin{aligned}
&\left|\int_{\mathbb{R}}(Hm)(-k)\Lambda_\Pi(T_k g, G, T^{(1)}_{\beta k}E)dk\right| \\
\lesssim &\int_{\mathbb{R}} |Hm(-k)|\cdot|\Lambda_\Pi(T_k g, G, T^{(1)}_{\beta k}E)|dk \\
\lesssim &\int_{\mathbb{R}} |Hm(-k)|\cdot\|T_k f\|_{L^{p_1}}\|G\|_{L^{p_2}}\|T^{(1)}_{\beta k}E\|_{L^{p_3}}dk \\
\lesssim &\int_{\mathbb{R}} |Hm(-k)|\cdot\|f\|_{L^{p_1}}\|G\|_{L^{p_2}}\|E\|_{L^{p_3}}dk.
\end{aligned}$$

By $Hm = -im$, where H is the Hilbert transform,

$$\left|\int_{\mathbb{R}}(Hm)(-k)\Lambda_\Pi(T_k f, G, T^{(1)}_{\beta k}E)dk\right| \lesssim \|f\|_{L^{p_1}}\|G\|_{L^{p_2}}\|E\|_{L^{p_3}}\|m\|_{L^1}. \tag{1.3.2}$$

On the other hand, by $H^2 m = -m$, i.e.

$$m(x) = -\,\mathrm{p.\,v.}\int_{\mathbb{R}}(Hm)(-y)\frac{1}{x+y}dy,$$

we obtain

$$\begin{aligned}
&\left|\int_{\mathbb{R}}(Hm)(-k)\Lambda_\Pi(T_k g, G, T^{(1)}_{\beta k}E)dk\right| \\
= &\left|\int_{\mathbb{R}^3} f(x+t)G(x,y)E(x+\beta t, y)m(t)dtdxdy\right|.
\end{aligned} \tag{1.3.3}$$

It's tempting to check the Brascamp-Lieb conditions in [4] at this point.
For scaling condition,

$$\frac{1}{p_1}+\frac{2}{p_2}+\frac{2}{p_3}+1 = 3. \tag{1.3.4}$$

As for dimension condition, we may take the subspace
$\operatorname{span}\{(0,1,0)\}$,

$$1 \le \frac{0}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}+0. \tag{1.3.5}$$

However, in this case, m is not an arbitrary function. We cannot take m as Gaussian, hence we need to mimic the proof in [4] and modify it slightly. Take $f(x) = e^{-\pi x^2}$, $G(x,y) = e^{-\pi(x^2+\varepsilon y^2)}$, $E(x,y) = e^{-\pi(x^2+\varepsilon y^2)}$, $\widehat{m}$ is a nonnegative bump function support in $(0,\infty)$. Then

$$\|f\|_{L^{p_1}}\|G\|_{L^{p_2}}\|E\|_{L^{p_3}}\|m\|_{L^1} \sim \varepsilon^{-\frac{1}{2}(\frac{1}{p_2}+\frac{1}{p_3})}. \tag{1.3.6}$$

On the other hand

$$\int_{\mathbb{R}^3} f(x+t)G(x,y)E(x+\beta t,y)m(t)dtdxdy$$

$$= \int_{\mathbb{R}^3} e^{-\pi(2\varepsilon y^2)} \cdot e^{-\pi(3[x+\frac{1}{3}(1+\beta t)]^2)} \cdot e^{-\pi(\frac{1}{3}(2\beta^2+\beta+2)t^2)} m(t)dydxdt. \tag{1.3.7}$$

First integrate in y, we may obtain a factor $\varepsilon^{-\frac{1}{2}}$. Second, integrate in x, we may get a constant. Hence, we may simplify (1.3.7) into

$$\sim \varepsilon^{-\frac{1}{2}} \int_{\mathbb{R}^3} e^{-\pi(\frac{1}{3}(2\beta^2+\beta+2)t^2)} m(t)dt. \tag{1.3.8}$$

By Plancherel identity, we may further reduce (1.3.8) to

$$\varepsilon^{-\frac{1}{2}}(\frac{1}{3}(2\beta^2+\beta+2))^{-\frac{1}{2}} \int_{\mathbb{R}^3} e^{-\pi((3\beta^2+\beta+2)^{-1}t^2)} \widehat{m}(t)dt. \tag{1.3.9}$$

The integral in (1.3.9) is a positive constant away from 0 and $2\beta^2+\beta+2$ is positive for all $\beta \in \mathbb{R}$. Taking $\varepsilon \to 0$ and compare (1.3.6) and (1.3.9), we may have the exponents p_1, p_2, p_3 in the quadrilinear form (1.3.2) is impossible to hold for the range other than (1.1.12), which is a contradiction. Hence (1.1.13) cannot hold for exponents (p_1, p_2, p_3) other than (1.1.12).
This completes the proof of Theorem 1.1.2.

1.4 Proof of Theorem 1.1.3

Now we discuss the L^p bounded for the form (1.1.4). Since equivalent forms share the same L^p boundedness property, suffice to consider the boundedness of standard forms listed in Theorem 1.1.1.

First, we deal with the endpoint range (1.1.12). The cases (1.1.10) and (1.1.11) with $\beta = 0$ are not bounded. On the other hand, the cases (1.1.8), (1.1.9), and (1.1.11) with $\beta = 1$ are bounded.

For (1.1.10) and (1.1.11) with $\beta = 0$, we prove the unboundedness for the case (1.1.10), the others are similar. Suppose we have the bound, take

$$G(x,y) = \operatorname{sgn} g(x)|g(x)|^{\frac{1}{2}}(D_N^\infty\varphi)(y),\ H(x,y) = |g(x)|^{\frac{1}{2}}(D_N^\infty\varphi)(y).$$

Then

$$\begin{aligned} &\left|\text{p. v.} \int_{\mathbb{R}^3} f(x+t)g(x)(D_N^\infty\varphi)(y)(D_N^\infty\varphi)(y+t)\frac{1}{t}dxdy\right| \\ \lesssim &\|f\|_{L^\infty} \cdot (\|g\|_{L^1}^{\frac{1}{2}} N^{\frac{1}{2}}) \cdot (\|g\|_{L^1}^{\frac{1}{2}} N^{\frac{1}{2}}) \\ = &N\|f\|_{L^\infty} \cdot \|g\|_{L^1}. \end{aligned} \tag{1.4.1}$$

Notice that taking $N \to \infty$ and integrate over y, the quantity

$$\frac{1}{N}(D_N^\infty\varphi)(y)(D_N^\infty\varphi)(y+t)$$

will tend to a constant. This implies Hilbert transform is bounded at L^∞, a contradiction.

For the cases (1.1.8), (1.1.9), and (1.1.11) with $\beta = 1$, we prove the boundedness for the case (1.1.11) with $\beta = 1$, the proof for other cases are similar.

$$
\begin{aligned}
&\left| \mathrm{p.v.} \int_{\mathbb{R}^3} f(x+t)G(x,y)H(x+t,y)\frac{1}{t}dtdxdy \right| \\
\lesssim &\int_{\mathbb{R}} \|fH_y\|_{L^{p_2}} \|G_y\|_{L^{p_3}} dy \\
\lesssim &\|f\|_{L^\infty} \int_{\mathbb{R}} \|H_y\|_{L^{p_2}} \|G_y\|_{L^{p_3}} dy \\
\lesssim &\|f\|_{L^\infty} \|G\|_{L^{p_2}} \|H\|_{L^{p_3}}.
\end{aligned} \tag{1.4.2}
$$

Freeze y, pair f and G_y together and use the estimate of the Hilbert transform. Then use the Hölder inequality twice to get the desired estimate. For the impossibility of L^p bound of all other cases for exponents (p_1, p_2, p_3) other than the range (1.1.12), the proof is similar to the case in Theorem 1.1.2.

1.5 Proof of Theorem 1.1.4

Let exponents p_1, p_2, p_3 be given satisfying (1.1.12), and let C be any constant. Shearing some functions, we rewrite the triangular Hilbert transform

$$
\begin{aligned}
\Lambda(F,G,H) &= \mathrm{p.v.} \int_{\mathbb{R}^3} F(t-y-z,y)G(y,z)H(z,t-y-z)\frac{1}{t}dtdydz \\
&= \mathrm{p.v.} \int_{\mathbb{R}^3} \widetilde{F}(z-t,y)G(y,z)\widetilde{H}(z,y-t)\frac{1}{t}dtdydz,
\end{aligned} \tag{1.5.1}
$$

where $\widetilde{F}(x,y) := F(-x-y,y)$, $\widetilde{H}(x,y) := H(x,-y-x)$.
For a parameter $N > 2e^{2C}$, let

$$
\begin{aligned}
G_N(y,z) &= 1_{[0,N]}(y)1_{[0,1]}(z), \\
\widetilde{H}_N(z,y) &:= 1_{[0,N]}(y)1_{[0,1]}(z), \\
\widetilde{F}(x,y) &:= 1_{[-\infty,-1)}(x).
\end{aligned}
$$

Then the integrand of (1.5.1) is non-negative, and we may estimate (1.5.1) from below as

$$
\Lambda(F,G,H) \geq \int_0^1 \int_{\mathbb{R}} \int_{1\leq t} G(y,z)\widetilde{H}(z,y-t)\frac{1}{t}dtdydz. \tag{1.5.2}
$$

$$
\geq \int_0^{\frac{N}{2}} \int_1^{\frac{N}{2}} \frac{1}{t}dtdy = \frac{N}{2}\log(\frac{N}{2}) > NC. \tag{1.5.3}
$$

From penultimate to ultimate line, the we integrated z from 0 to 1 and used that if $0 < y, t < \frac{N}{2}$, then both y and $y+t$ are in $[0,N]$. On the other hand, as shearing leaves the L^p norm invariant,

$$
\|F\|_\infty \|G\|_{p_2} \|H\|_{p_3} = \|\tilde{F}\|_\infty \|G\|_{p_2} \|\tilde{H}\|_{p_3} = N^{\frac{1}{p_2}} N^{\frac{1}{p_3}} = N. \tag{1.5.4}
$$

This together with (1.5.3) shows (1.1.14) and completes the proof of Theorem (1.1.4).

Chapter 2

A sharp Hörmander condition for bilinear Fourier multipliers with Lipschitz singularities

2.1 Introduction

An n-linear Fourier multiplier m is a function on the space V of all points $\xi = (\xi_1, \cdots, \xi_{n+1}) \in \mathbb{R}^{n+1}$ such that

$$\sum_{j=1}^{n+1} \xi_j = 0.$$

It is associated with an $(n+1)$-linear form acting on functions on the real line defined by

$$\Lambda_m(f_1, \cdots, f_{n+1}) := \int_V m(\xi) \prod_{j=1}^{n+1} \widehat{f_j}(\xi_j) d\mathcal{H}^n(\xi). \tag{2.1.1}$$

Here $\mathcal{H}^n$ denotes the n-dimension Hausdorff measure on V.

We call such a multiplier n-linear as classically one associates to it an n-linear operator dual to this $(n+1)$-linear form.

For a multiplier m and a tuple $p = (p_1, \cdots, p_{n+1})$ of Lebesgue norm exponents in $(1, \infty)^{n+1}$ with

$$\sum_{j=1}^{n+1} \frac{1}{p_j} = 1, \tag{2.1.2}$$

we define the constant $\mathcal{C}(m, p)$ to be the infimum of all constants $C > 0$ satisfying

$$|\Lambda_m(f_1, \cdots, f_{n+1})| \leq C \prod_{j=1}^{n+1} \|f_j\|_{L^{p_j}} \tag{2.1.3}$$

for all tuples of Schwartz functions f_j.

We say the form Λ_m is bounded in the open Banach range if $\mathcal{C}(m, p)$ is finite on all tuples p with (2.1.2) in the range $(1, \infty)^{n+1}$. We say it is bounded in the local L^2 range if it is bounded for all tuples p with (2.1.2) in $(2, \infty)^{n+1}$.

Classical works concern classes of multipliers singular at one point, typically the origin. These include the Mikhlin class M_s, which is all multipliers satisfying away from the origin the symbol bounds

$$|(\partial^\alpha m)(\xi)| \leq C_\alpha |\xi|^{-\alpha} \tag{2.1.4}$$

for all multi-indices α up to order $|\alpha| \leq s$. Another slightly larger class is Hörmander class H_s, which is the set of all multipliers satisfying

$$\sup_{j\in\mathbb{Z}} \|m(2^j \cdot)\Psi\|_{H^s(V)} < \infty \tag{2.1.5}$$

where Ψ is a smooth bump function compactly supported away from 0. Here H^s is the L^2-based inhomogeneous Sobolev norm defined by

$$\|f\|_{H^s(V)} := \left\|(1+|x|^2)^{\frac{s}{2}} \widehat{f \cdot d\mathcal{H}^n}(x)\right\|_{L^2_x(V)}, \tag{2.1.6}$$

where

$$\widehat{f \cdot d\mathcal{H}^n}(x) := \int_V f(y) e^{-2\pi i x\cdot y} d\mathcal{H}^n(y). \tag{2.1.7}$$

The classical Mikhlin multiplier theorem [73] gives boundedness in the open Banach range for linear multipliers $m \in M_1$. In [52], Hörmander proved boundedness in the open Banach range for linear multipliers m in H_s with $s > \frac{1}{2}$. Boundedness for the general n-linear case in the open Banach range was shown by Coifman and Meyer [24], [26] for $m \in M_s$ with s sufficiently large and by Tomita [94] for $m \in H_s$ with the sharp condition $s > \frac{n}{2}$. For results concerning exponents outside the Banach range, see [56],[47],[46],[66].

More recently, people studied multilinear multipliers with higher dimensional singularities. Lacey and Thiele [63] proved bounds in the local L^2 range for $n = 2$ and $m = \mathrm{sgn}(\alpha_1\xi_1 + \alpha_2\xi_2)$, the so-called bilinear Hilbert transform, for all vectors $\alpha = (\alpha_1, \alpha_2)$. The bound is non-trivial only for α outside the three so-called degenerate one-dimensional subspaces. This result was extended to the open Banach range and beyond in [64]. That this m is a particular instance of more general multipliers singular along a line was noted by Gilbert and Nahmod [39], who extended the result accordingly. Muscalu, Tao, and Thiele [77] proved bounds in the open Banach range for n-linear multipliers satisfying

$$|\partial^\alpha m(\xi)| \lesssim \mathrm{dist}(\xi, \Gamma)^{-|\alpha|} \tag{2.1.8}$$

for singularity Γ a non-degenerate subspace with $\dim \Gamma < \frac{n+1}{2}$ and for α up to some large degree that has not been specified in [77].

The bounds in [63] are not uniform in α. Uniform bounds were proven in [44], [67] by Grafakos and Li, and later the range was extended by Uraltsev and Warchalski in [96].

Curved singularites were first studied by Muscalu [78]. Later, the bound of bilinear disk multiplier was obtained by Grafakos and Li [45].

The main theorem of this paper establishes the sharp Sobolev exponent for the Hörmander condition associated with bilinear multipliers whose singularities are unions of Lipschitz curves away from degenerate directions. This is the first work that provides this sharp Hörmander condition for multilinear multipliers with singularities of dimension larger than zero. Moreover, we work in a continuous model without discretization in the vein of [30] and develop a suitable setting to analyze the geometry arising from the presence of Lipschitz singularity.

Define dilation, translation, and modulation operators

$$(D_a^p f)(x) := a^{-\frac{n}{p}} f\left(\frac{x}{a}\right)$$

$$(T_a f)(x) := f(x-a)$$

$$(M_a f)(x) := e^{2\pi i a x} f(x).$$

For $\boldsymbol{\beta} \in V$, define the distance function $d_\Gamma(\boldsymbol{\beta}) := \inf_{\boldsymbol{\xi}\in\Gamma} |\boldsymbol{\beta} - \boldsymbol{\xi}|$. Let $B_r(x)$ denote the open ball with radius r centered at x. Let η be a L^1 normalized function supported on $[-1,1]$ defined by

$$\eta(x) = \left(\int_{-1}^{1} e^{\frac{-1}{1-t^2}}\,dt\right)^{-1} \cdot e^{\frac{-1}{1-x^2}} \cdot 1_{[-1,1]}(x).$$

Define

$$\widetilde{\eta} := 1_{B_{\frac{3}{20}}(0)} * D^1_{\frac{1}{100}}\eta$$

which is constant one in $B_{\frac{1}{10}}(0)$ and supported on $B_{\frac{2}{10}}(0)$. Define a smooth function Φ on V

$$\Phi(x) := \widetilde{\eta}(|x|).$$

For a subspace $A \subseteq \mathbb{R}^n$ and a vector $v \in \mathbb{R}^n$, we denote the orthogonal projection of v onto A as $P_A v$. Let $0 \le \theta_0 < \frac{\pi}{6}$. For $j \in \{1,2,3\}$, let $\mathcal{K}_j(\theta_0)$ be the open double cone of all vectors $\boldsymbol{\beta}$ in V which have angle less than θ_0 to the line spanned by by $P_V e_j$, i.e., as the length of $P_V e_j$ is $\frac{\sqrt{6}}{3}$, $\mathcal{K}_j(\theta_0)$ contains points $\boldsymbol{\beta} \in V$ satisfying

$$|\langle \boldsymbol{\beta}, e_j\rangle| = |\langle \boldsymbol{\beta}, P_V e_j\rangle| > \frac{\sqrt{6}}{3}|\boldsymbol{\beta}|\cos\theta_0. \tag{2.1.9}$$

Theorem 2.1.1. *Let $n = 2$. Let $2 < p_1, p_2, p_3 < \infty$ with $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$. Let $0 \le \theta_0 < \frac{\pi}{6}$. Let $s > 1$. There is a constant $C(p_1, p_2, p_3, \theta_0, s, N)$ such that the following holds.*

For every $1 \le \iota \le N$, let $\Gamma_\iota \subset V$ be a closed set such that there exists an index $j \in \{1,2,3\}$ such that for every distinct $\boldsymbol{\gamma}, \boldsymbol{\gamma}' \in \Gamma_\iota$, we have $\boldsymbol{\gamma} - \boldsymbol{\gamma}' \in \mathcal{K}_j(\theta_0)$. Let Γ be the union of the sets Γ_ι for $1 \le \iota \le N$. Let m be a function on V satisfying

$$\sup_{\boldsymbol{\beta}\in V\setminus\Gamma} \left\| \left(D^\infty_{d_\Gamma(\boldsymbol{\beta})^{-1}} T_{-\boldsymbol{\beta}} m\right) \cdot \Phi \right\|_{H^s(V)} \le 1. \tag{2.1.10}$$

Then we have for the form bound (2.1.3) *the inequality*

$$\mathcal{C}(m, p_1, p_2, p_3) \le C(p_1, p_2, p_3, \theta_0, s, N).$$

Note that the theorem applies in particular to the case

$$m(\xi_1, \xi_2, \xi_3) = \widetilde{m}(\xi_1 - \xi_2)$$

with $\widetilde{m}$ satisfying the Hörmander condition on real line for $s > 1$.

The exponent s in this theorem is sharp. It suffices to establish the sharpness of the exponent s in the case where Γ is a point. This sharpness has been discussed in [41].

The assumption (2.1.9) says that the tangencies of each Lipschitz curve stay away from a fixed angle from the degenerate direction. While bounds for specific examples such as

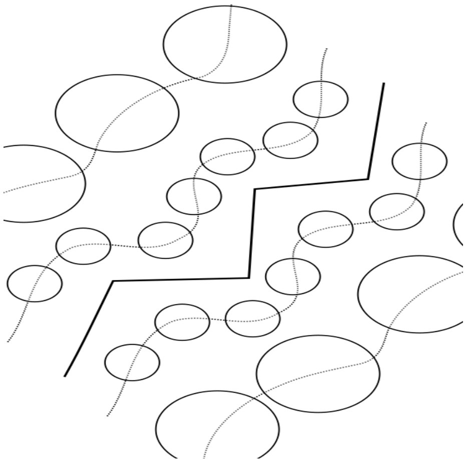

Figure 2.1: We may view (2.1.10) as testing Sobolev norm of m on scaled Whitney bumps.

circular arcs with degenerate tangencies have been established in the literature [45], even the question for convex arcs in general with degenerate tangencies appears to be very difficult, as discussed in [87].

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2.2 Overview of the proof

For the proof of Theorem 2.1.1, fix $2 < p_1, p_2, p_3 < \infty$ with $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$. Fix also $0 \leq \theta_0 < \frac{\pi}{6}$. Fix $s > 1$. Let $N \in \mathbb{N}$. For any quantities A, B depending on these and possibly further parameters, which most prominently will be Γ, m, we will write $A \lesssim B$ whenever $A \leq CB$ for some number C depending on $p_1, p_2, p_3, \theta_0, s, N$ only but not on the parameters. Analogously, we write $A \gtrsim B$ whenever $B \lesssim A$. If in particular, $A \lesssim B$ and $A \gtrsim B$ simultaneously we write $A \sim B$.

For $(S, \mathcal{A}, \mu)$ a measure space and f a measurable function on this space, the L^p norm of f will be expressed as

$$\|f\|_{L^p_\mu(S)} = \|f(x)\|_{L^p_{\mu(x)}(S)} := \left(\int_S |f|^p(x) d\mu(x)\right)^{\frac{1}{p}}.$$

Furthermore, if $\mu(S) < \infty$, we define the average L^p norm of f as

$$\|f\|_{\text{Ł}^p_\mu(S)} = \|f(x)\|_{\text{Ł}^p_{\mu(x)}(S)} := \left(\frac{1}{\mu(S)} \int_S |f|^p(x) d\mu(x)\right)^{\frac{1}{p}} = \left(⨍_S |f|^p(x) d\mu(x)\right)^{\frac{1}{p}}.$$

If it's clear from the context that the integration is over a space V isomorphic to an n-dimensional Euclidean space with the usual n-dimensional Hausdorff measure, we simply write $\|f(x)\|_{L^p_x(V)}$ instead of $\|f(x)\|_{L^p_{\mathcal{H}^n(x)}(V)}$.

Theorem 2.1.1 will be proven in Section 2.3 by reducing to Proposition 2.2.1 which states a bound of a model form.

We write $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ for a typical element in $\mathbb{R}^3$ and $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3)$ for a typical element on V. Define μ a measure on V which assigns zero measure to Γ and has density

$$d\mu(\boldsymbol{\beta}) := \frac{d\mathcal{H}^2(\boldsymbol{\beta})}{d_\Gamma(\boldsymbol{\beta})^2}$$

on $V \setminus \Gamma$. Define a measure on $\mathbb{R} \times V$ by

$$d\nu(\alpha, \boldsymbol{\beta}) := d\alpha \otimes d\mu(\boldsymbol{\beta}).$$

Let P_V be the orthogonal projection from $\mathbb{R}^3$ onto V. Define a smooth function on $\mathbb{R}$

$$\widehat{\varphi} := D^\infty_{2\varepsilon} \widetilde{\eta},$$

which is constant one in $B_{\frac{2\varepsilon}{10}}(0)$ and supported on $B_{\frac{4\varepsilon}{10}}(0)$. The number ε is a small constant which only depends on θ_0. The specific value of ε will be determined in Section 2.9.

Proposition 2.2.1 (Bound of the model form)**.** *Let $K : \mathbb{R}^3 \times V \to \mathbb{C}$ be a continuous function satisfying* (2.2.1) *and* (2.2.2) *below. For all $\boldsymbol{\alpha} \in \mathbb{R}^3$ and $\boldsymbol{\beta} \in V$,*

$$K(\boldsymbol{\alpha}, \boldsymbol{\beta}) = K(P_V \boldsymbol{\alpha}, \boldsymbol{\beta}). \tag{2.2.1}$$

For all $\boldsymbol{\beta} \in V$, $s > 1$,

$$\left\| \left(1 + |d_\Gamma(\boldsymbol{\beta})\boldsymbol{\alpha}|^2\right)^{\frac{s}{2}} \cdot K(\boldsymbol{\alpha}, \boldsymbol{\beta}) \right\|_{L^2_{\boldsymbol{\alpha}}(V)} \lesssim d_\Gamma(\boldsymbol{\beta}). \tag{2.2.2}$$

Then for all Schwartz functions f_1, f_2, f_3 on $\mathbb{R}$, we have the bound

$$\left| \int_V \int_{\mathbb{R}^3} K(\boldsymbol{\alpha}, \boldsymbol{\beta}) \cdot \prod_{j=1}^3 \left((\mathrm{Mod}_{\beta_j} D^1_{d_\Gamma(\boldsymbol{\beta})^{-1}} \varphi) * f_j \right) (\alpha_j) d\boldsymbol{\alpha} d\mu(\boldsymbol{\beta}) \right| \lesssim \prod_{j=1}^3 \|f_j\|_{p_j}. \tag{2.2.3}$$

Proposition 2.2.1 is proven in Section 2.10 by first reducing to the special case $N = 1$. Fix from now on K as in Proposition 2.2.1. Define $\delta_0 := \frac{\sqrt{6}}{3} \cos(\theta_0 + \frac{\pi}{3})$.

Lemma 2.2.2. *Assume $N = 1$, for all $j = 1, 2, 3$, we have for any $\boldsymbol{\gamma}, \boldsymbol{\gamma}' \in \Gamma$,*

$$|\langle \boldsymbol{\gamma} - \boldsymbol{\gamma}', e_j \rangle| = |\langle \boldsymbol{\gamma} - \boldsymbol{\gamma}', P_V e_j \rangle| \geq \delta_0 |\boldsymbol{\gamma} - \boldsymbol{\gamma}'|. \tag{2.2.4}$$

Proof. The equality on the left of (2.2.4) is true because $\boldsymbol{\gamma} - \boldsymbol{\gamma}'$ belongs to V. Notice from (2.1.9) that the angle between $\boldsymbol{\gamma} - \boldsymbol{\gamma}'$ and $P_V e_{j_0}$ is $\theta_0 < \frac{\pi}{6}$. For $j \neq j_0$, the angle between the orthogonal complement of $P_V e_j$ and $P_V e_{j_0}$ is $\frac{\pi}{6}$. Hence the angle between $\boldsymbol{\gamma} - \boldsymbol{\gamma}'$ and the orthogonal complement of $P_V e_j$ is a least $\frac{\pi}{6} - \theta_0$, and the angle between $\boldsymbol{\gamma} - \boldsymbol{\gamma}'$ and $P_V e_j$ is at most $\theta_0 + \frac{\pi}{3}$. Then since $|P_V e_{j_0}| = \frac{\sqrt{6}}{3}$, we have (2.2.4). $\square$

Let $\theta_1 = \frac{\pi}{18} - \frac{\theta_0}{3}$ be a fixed constant such that $\frac{\pi}{3} + \theta_0 + \theta_1 < \frac{\pi}{2}$ and $\sin \theta_1 < \frac{\sqrt{6}}{3} \cos(\frac{\pi}{3} + \theta_0 + \theta_1)$. Let $\delta_1 = \sin \theta_1$ and $\delta_2 = \frac{\sqrt{6}}{3} \cos(\frac{\pi}{3} + \theta_0 + \theta_1)$.

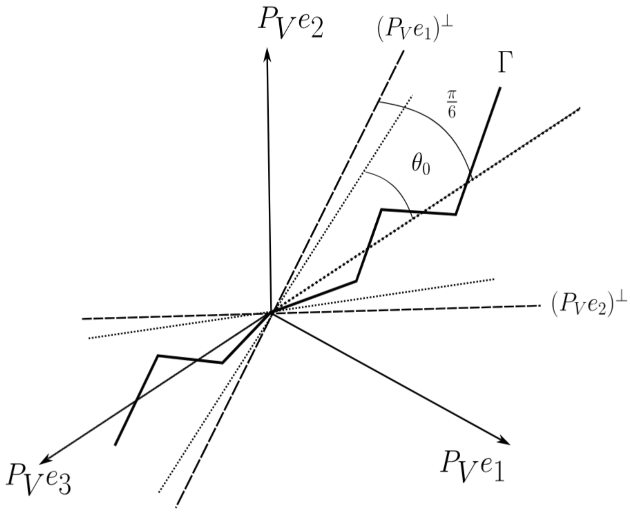

Figure 2.2: Lemma 2.2.2 explains that Γ is also away from all the three degenerate directions.

For $\boldsymbol{\gamma} \in \Gamma$, $t \geq 0$, and $j \in \{1,2,3\}$ define the sets $W_{\boldsymbol{\gamma},t}$ and $U^j_{\boldsymbol{\gamma}}$ as follow

$$W_{\boldsymbol{\gamma},t} := \{\boldsymbol{\beta} \in V : t \leq |\boldsymbol{\beta} - \boldsymbol{\gamma}| \leq \frac{1}{\delta_1} d_\Gamma(\boldsymbol{\beta})\}. \tag{2.2.5}$$

$$U^j_{\boldsymbol{\gamma}} := \{\boldsymbol{\beta} \in V : |\langle \boldsymbol{\beta} - \boldsymbol{\gamma}, e_j \rangle| \leq \delta_2 |\boldsymbol{\beta} - \boldsymbol{\gamma}|\}. \tag{2.2.6}$$

Let $\mathcal{I}$ be the collection of all intervals in $\mathbb{R}$. Let $\mathbf{T}$ be the set of all pairs $T = (I, \boldsymbol{\gamma})$ with $I \in \mathcal{I}$ and $\boldsymbol{\gamma} \in \Gamma$. For such T, we associate a region $D_T := I \times W_{\boldsymbol{\gamma},1/|I|}$. We choose the letter T here because parts of the literature [30] refer to closely related objects as tents. From here on, the term "tent" refers to either the pairing $T = (I, \boldsymbol{\gamma})$ or the region D_T, which will be clear from the context. We define for $j \in \{1,2,3\}$ and a function f on $\mathbb{R}$ the function $F_j f$ on $\mathbb{R} \times V$ by

$$(F_j f)(\alpha_j, \boldsymbol{\beta}) := \left((\mathrm{Mod}_{\beta_j} D^1_{d_\Gamma(\boldsymbol{\beta})^{-1}} \varphi) * f \right)(\alpha_j). \tag{2.2.7}$$

For a set $I \times W_{\gamma,t} \subseteq \mathbb{R} \times V$, an index $j \in \{1,2,3\}$, and a function F on $\mathbb{R} \times V$, we define a local size S^j of F associated with $I \times W_{\gamma,t}$

$$\|F\|_{S^j(I,\boldsymbol{\gamma},t)} := |I|^{-\frac{1}{2}} \|F\|_{L^2_\nu(I \times (W_{\boldsymbol{\gamma},t} \setminus U^j_{\boldsymbol{\gamma}}))} \vee \|F\|_{L^\infty(I \times W_{\gamma,t})} \tag{2.2.8}$$

$$= \left(\frac{1}{|I|} \int_{I \times \left(W_{\boldsymbol{\gamma},t} \setminus U^j_{\boldsymbol{\gamma}}\right)} |F(\alpha, \boldsymbol{\beta})|^2 d\alpha d\mu(\boldsymbol{\beta}) \right)^{\frac{1}{2}} \vee \|F\|_{L^\infty(I \times W_{\gamma,t})}.$$

If in particular $t = \frac{1}{|I|}$, we write

$$\|F\|_{S^j(I,\boldsymbol{\gamma})} := \|F\|_{S^j(I,\boldsymbol{\gamma},t)}.$$

We also define a global size

$$\|F\|_{S^j} := \sup_{I\in\mathcal{I},\boldsymbol{\gamma}\in\Gamma} \|F\|_{S^j(I,\boldsymbol{\gamma})}.$$

The model form is estimated first on certain regions associated to tents in Proposition 2.2.3. To obtain the sharp regularity s in the form of condition (2.1.10), we prove Proposition 2.2.3 in Section 2.4 by splitting the frequency region into small and large scale, then performing different estimates on the respective pieces.

Proposition 2.2.3 (Tent Estimate)**.** *Assume $N = 1$. Let $i \in \{1, 2, 3\}$. Let $I \in \mathcal{I}$ and $\boldsymbol{\gamma} \in \Gamma$. Then we have the inequality*

$$\left\| K(\boldsymbol{\alpha},\boldsymbol{\beta}) \cdot \prod_{j=1}^{3} (F_j f_j)(\alpha_j,\boldsymbol{\beta}) \right\|_{L^1_{\boldsymbol{\alpha},\mu(\boldsymbol{\beta})}((I\boldsymbol{e}_i \oplus \boldsymbol{e}_i^{\perp})\times W_{\boldsymbol{\gamma},1/|I|})} \lesssim |I| \cdot \prod_{j=1}^{3} \|F_j f_j\|_{S^j}. \tag{2.2.9}$$

Naturally, we aim to control the right-hand side of (2.2.9). On the one hand, a simple L^∞ bound can be obtained as follows.

Proposition 2.2.4 (Global Estimate)**.** *Assume $N = 1$. Given $f \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, we have*

$$\|F_j f\|_{S^j} \lesssim \|f\|_{L^\infty}. \tag{2.2.10}$$

On the other hand, we aim to obtain a certain L^2 estimate to serve as the other endpoint and perform an interpolation argument. At this stage, the main difficulty of proving Proposition 2.2.1 is to be efficient in summing all the pieces on the left-hand side of (2.2.9) in Proposition 2.2.3. Therefore, we must derive certain orthogonality among objects associated with tents. To address the orthogonality issue, we build up Proposition 2.2.5 to treat the distribution and the interaction among Whitney balls associated with a Lipschitz singularity.

Proposition 2.2.5 (Geometry of Tents)**.** *Assume $N = 1$. Let $1 \le j \le 3$. We define*

$$\rho := \frac{\delta_2 - \delta_1}{1 + \delta_1}. \tag{2.2.11}$$

Let $\boldsymbol{\gamma}, \boldsymbol{\gamma}'$ be two distinct points on Γ and $t > 0$.
(1) Let $\boldsymbol{\gamma}''$ be another point on Γ satisfying

$$\boldsymbol{\gamma}_j \le \boldsymbol{\gamma}''_j \le \boldsymbol{\gamma}'_j \le \boldsymbol{\gamma}_j + \delta_0(1-\delta_1)t. \tag{2.2.12}$$

Then

$$W_{\boldsymbol{\gamma}'',t} \subseteq W_{\boldsymbol{\gamma},\delta_1 t} \cup W_{\boldsymbol{\gamma}',\delta_1 t}. \tag{2.2.13}$$

(2) Given two points

$$\boldsymbol{\beta} \in W_{\boldsymbol{\gamma},t} \setminus U^j_{\boldsymbol{\gamma}}, \quad \boldsymbol{\beta}' \in W_{\boldsymbol{\gamma}',0} \setminus W_{\boldsymbol{\gamma},\delta_1 t} \tag{2.2.14}$$

with $\beta_j < \boldsymbol{\gamma}_j < \boldsymbol{\gamma}'_j$, then

$$B_{\rho d_\Gamma(\boldsymbol{\beta})}(\beta_j) \cap B_{\rho d_\Gamma(\boldsymbol{\beta}')}(\beta'_j) = \varnothing. \tag{2.2.15}$$

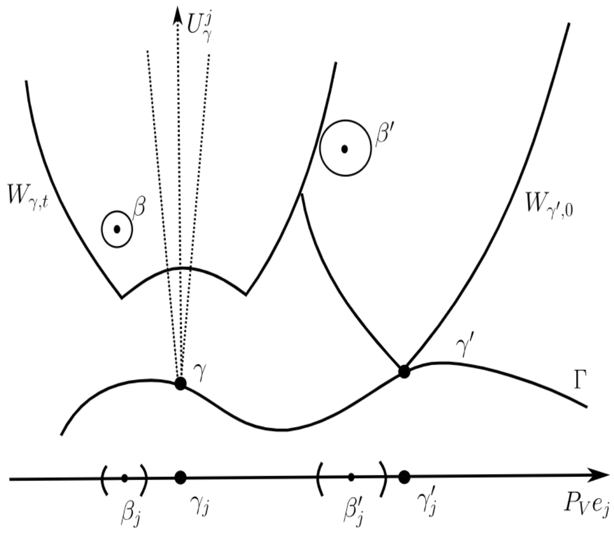

Figure 2.3: In (2.2.14) and (2.2.15), we describe certain orthogonality.

The geometry in Proposition 2.2.5 serves as a base for the two algorithms introduced in Proposition 2.2.6 and 2.2.7. Essentially, both algorithms extract collections of countable tents with desired geometric properties from collections of uncountable tents.

Proposition 2.2.6 (Selection Algorithm, L^∞ Component)**.** *Let $\Omega \subseteq \mathbb{R}\times(V\setminus\Gamma)$ be compact and $\lambda>0$ be a threshold. For $j=1,2,3$ and $f\in L^2(\mathbb{R})\cap L^\infty(\mathbb{R})$, we have the following: There is a countable collection of tents $\mathbf{T}$ and a countable collection of points $\mathbf{P}$ in Ω that satisfy the following properties:*

- *Covering:*
$$\mathbf{P}\subseteq\Omega\cap|F_j f|^{-1}(\lambda,2\lambda]\subseteq\bigcup_{T=(I,\boldsymbol{\gamma})\in\mathbf{T}} D_T. \tag{2.2.16}$$

- *Estimate:*
$$\sum_{T=(I,\boldsymbol{\gamma})\in\mathbf{T}}|I|\sim\sum_{(\alpha,\boldsymbol{\beta})\in\mathbf{P}} d_\Gamma(\boldsymbol{\beta})^{-1}\lesssim\sum_{(\alpha,\boldsymbol{\beta})\in\mathbf{P}}\frac{|F_j f(\alpha,\boldsymbol{\beta})|^2}{d_\Gamma(\boldsymbol{\beta})\lambda^2}. \tag{2.2.17}$$

- *Orthogonality: For distinct $(\alpha,\boldsymbol{\beta}),(\alpha',\boldsymbol{\beta}')\in\mathbf{P}$, at least one of the following statements holds.*
$$|\alpha-\alpha'|\geq 2\left(d_\Gamma(\boldsymbol{\beta})^{-1}+d_\Gamma(\boldsymbol{\beta}')^{-1}\right). \tag{2.2.18}$$
$$|\beta_j-\beta'_j|\geq\rho\left(d_\Gamma(\boldsymbol{\beta})+d_\Gamma(\boldsymbol{\beta}')\right). \tag{2.2.19}$$

To state the next proposition, we introduce two auxiliary sets. Given $\boldsymbol{\gamma}\in\Gamma$ and $t\geq 0$, we define
$$\left(W_{\boldsymbol{\gamma},t}\setminus U^j_{\boldsymbol{\gamma}}\right)^{<j}:=\{\boldsymbol{\beta}\in V:\beta_j<\gamma_j\}\cap\left(W_{\boldsymbol{\gamma},t}\setminus U^j_{\boldsymbol{\gamma}}\right) \tag{2.2.20}$$
and
$$\left(W_{\boldsymbol{\gamma},t}\setminus U^j_{\boldsymbol{\gamma}}\right)^{>j}:=\{\boldsymbol{\beta}\in V:\beta_j>\gamma_j\}\cap\left(W_{\boldsymbol{\gamma},t}\setminus U^j_{\boldsymbol{\gamma}}\right). \tag{2.2.21}$$

Proposition 2.2.7 (Selection Algorithm, L^2 Component)**.** *Let $\Omega \subseteq \mathbb{R} \times (V \setminus \Gamma)$ be compact and $\lambda > 0$ a threshold. For $j = 1, 2, 3$ and $f \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, we have the following: There is a countable collection $\mathbf{T}$ of tents and a countable collection $\mathbf{S}$ of the form $(I, \boldsymbol{\gamma}, S)$ with $(I, \boldsymbol{\gamma})$ a tent and S a measurable subset of $\Omega \cap I \times \left(W_{\boldsymbol{\gamma},1/|I|} \setminus U^j_{\boldsymbol{\gamma}}\right)^{<j}$ that satisfy the following properties:*

- *Covering: For any tent $(I, \boldsymbol{\gamma})$,*

$$|I|^{-\frac{1}{2}} \left\| 1_{\Omega \setminus \bigcup_{T \in \mathcal{T}} D_T} F_j f \right\|_{L^2_\nu\left(I \times (W_{\boldsymbol{\gamma},1/|I|} \setminus U^j_{\boldsymbol{\gamma}})^{<j}\right)} \leq \frac{\lambda}{\sqrt{2}} \tag{2.2.22}$$

- *Estimate:*

$$\sum_{(I,\boldsymbol{\gamma}) \in \mathcal{T}} |I| \sim \sum_{(I,\boldsymbol{\gamma},S) \in \mathbf{S}} |I| \lesssim \sum_{(I,\boldsymbol{\gamma},S) \in \mathbf{S}} \frac{\|F_j f\|^2_{L^2_\nu(S)}}{\lambda^2} \tag{2.2.23}$$

  In particular, for $(I, \boldsymbol{\gamma}, S) \in \mathbf{S}$, we have $|I|^{-\frac{1}{2}} \|F_j f\|_{L^2_\nu(S)} \gtrsim \lambda$.

- *Orthogonality: For distinct $(I, \boldsymbol{\gamma}, S), (I', \boldsymbol{\gamma}', S') \in \mathbf{S}$ with $\gamma_j \leq \gamma'_j$, any pair of $(\alpha, \boldsymbol{\beta}) \in S$ and $(\alpha', \boldsymbol{\beta}') \in S'$ satisfies at least one of the following.*

$$\left|\alpha - \alpha'\right| \geq 2|I|. \tag{2.2.24}$$

$$\left|\beta_j - \beta'_j\right| \geq \rho\left(d_\Gamma(\boldsymbol{\beta}) + d_\Gamma(\boldsymbol{\beta}')\right). \tag{2.2.25}$$

Symmetrically, the proposition holds for $\left(W_{\boldsymbol{\gamma},\frac{1}{|I|}} \setminus U^j_{\boldsymbol{\gamma}}\right)^{<j}$ replaced by $\left(W_{\boldsymbol{\gamma},\frac{1}{|I|}} \setminus U^j_{\boldsymbol{\gamma}}\right)^{>j}$.

Once Propositions 2.2.6 and 2.2.7 guarantee the existence of well-behaved configurations of tents, the proof of the next proposition mainly follows the same line of argument as in [30] and [76]. For completeness, we include the proof in section 2.9.

Proposition 2.2.8 (Bessel Type Estimate)**.** *Assume $N = 1$. Given a compact set Ω in $\mathbb{R} \times (V \setminus \Gamma)$, a function $f \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, and $\lambda > 0$, there is a countable collection of tents $\mathcal{T}$ such that*

$$\sum_{(I,\boldsymbol{\gamma}) \in \mathcal{T}} |I| \lesssim \frac{\|f\|^2_{L^2}}{\lambda^2} \tag{2.2.26}$$

and

$$\left\| 1_{\Omega \setminus \bigcup_{T \in \mathcal{T}} D_T} \cdot F_j f \right\|_{S^j} \leq \lambda. \tag{2.2.27}$$

2.3 Proof of Theorem 2.1.1

Define a smooth function on V

$$\chi(\boldsymbol{\beta}) := D^\infty_\varepsilon \widetilde{\eta}(|\boldsymbol{\beta}|).$$

For $\boldsymbol{\beta} \in V \setminus \Gamma$, define the bump function adapted to position $\boldsymbol{\beta}$ by

$$\chi_{\boldsymbol{\beta}} := T_{\boldsymbol{\beta}} D^\infty_{d_\Gamma(\boldsymbol{\beta})} \chi\,.$$

Define the normalized bump function as

$$\widetilde{\chi}_{\boldsymbol{\beta}} := X_\Gamma^{-1} \cdot \chi_{\boldsymbol{\beta}},$$

where

$$X_\Gamma := \int_V \chi_{\boldsymbol{\beta}} d\mu(\boldsymbol{\beta}).$$

Then we have the identity

$$1_{V\setminus\Gamma} = \int_V \widetilde{\chi}_{\boldsymbol{\beta}} d\mu(\boldsymbol{\beta}).$$

As χ is supported in $B_{\frac{2\varepsilon}{10}}(0)$ and $\widehat{\varphi}$ is constant one on $B_{\frac{2\varepsilon}{10}}(0)$, we have

$$1_{V\setminus\Gamma}(\boldsymbol{\xi}) = \int_V \widetilde{\chi}_{\boldsymbol{\beta}}(\boldsymbol{\xi}) \prod_{j=1}^{3} \left(T_{\beta_j} D^\infty_{d_\Gamma(\boldsymbol{\beta})}\widehat{\varphi}\right)(\xi_j) d\mu(\boldsymbol{\beta}).$$

Inserting this into (2.1.1), in our case $n=2$, we obtain for $\Lambda(f_1, f_2, f_3)$ the expression

$$\int_V \int_V \left(m(\boldsymbol{\xi})\widetilde{\chi}_{\boldsymbol{\beta}}(\boldsymbol{\xi})\right) \cdot \prod_{j=1}^{3} \left(\left(T_{\beta_j} D^\infty_{d_\Gamma(\boldsymbol{\beta})}\widehat{\varphi}\right)(\xi_j)\widehat{f}_j(\xi_j)\right) d\mathcal{H}^2(\boldsymbol{\xi}) d\mu(\boldsymbol{\beta}). \tag{2.3.1}$$

Let $m_{\boldsymbol{\beta}} = m \cdot \widetilde{\chi}_{\boldsymbol{\beta}}$ and $\mathcal{F}f = \widehat{f}$ be the Fourier transform of f. We define

$$K(\boldsymbol{\alpha}, \boldsymbol{\beta}) := \mathcal{F}(m_{\boldsymbol{\beta}} \cdot d\mathcal{H}^2_V)(\boldsymbol{\alpha}) = \int_V m_{\boldsymbol{\beta}}(\boldsymbol{\xi}) e^{-2\pi i \boldsymbol{\alpha}\cdot\boldsymbol{\xi}} d\mathcal{H}^2(\boldsymbol{\xi}). \tag{2.3.2}$$

Note that $K(\boldsymbol{\alpha}, \boldsymbol{\beta})$ satisfies the invariance property (2.2.1). Applying Plancherel to the inner integral in (2.3.1), we obtain

$$\begin{aligned}\Lambda(f_1, f_2, f_3) &= \int_V \int_{\mathbb{R}^3} K(\boldsymbol{\alpha}, \boldsymbol{\beta}) \cdot \prod_{j=1}^{3} \left((\mathrm{Mod}_{\beta_j} D^1_{d_\Gamma(\boldsymbol{\beta})^{-1}}\varphi) * f_j\right)(\alpha_j) d\boldsymbol{\alpha} d\mu(\boldsymbol{\beta}) \\ &= \int_V \int_{\mathbb{R}^3} K(\boldsymbol{\alpha}, \boldsymbol{\beta}) \cdot \prod_{j=1}^{3} (F_j f_j)(\alpha_j, \boldsymbol{\beta}) d\boldsymbol{\alpha} d\mu(\boldsymbol{\beta}), \end{aligned} \tag{2.3.3}$$

where we use the notation F_j as defined in (2.2.7).

We verify the kernel condition (2.2.2) for K. Let $\boldsymbol{\beta} \in V \setminus \Gamma$ and $s > 1$. Expanding the kernel, we observe

$$\begin{aligned}&\left\| d_\Gamma(\boldsymbol{\beta})^{-1} \left(1 + |d_\Gamma(\boldsymbol{\beta})\boldsymbol{\alpha}|^2\right)^{\frac{s}{2}} \cdot K(\boldsymbol{\alpha}, \boldsymbol{\beta}) \right\|_{L^2_{\boldsymbol{\alpha}}(V)} \qquad (2.3.4)\\ &= \left\| d_\Gamma(\boldsymbol{\beta})^{-1} \left(1 + |d_\Gamma(\boldsymbol{\beta})\boldsymbol{\alpha}|^2\right)^{\frac{s}{2}} \cdot \mathcal{F}\left(m \cdot \left(T_{\boldsymbol{\beta}} D^\infty_{d_\Gamma(\boldsymbol{\beta})}\chi\right) \cdot X_\Gamma^{-1} \cdot d\mathcal{H}^2\right)(\boldsymbol{\alpha}) \right\|_{L^2_{\boldsymbol{\alpha}}(V)}.\end{aligned}$$

Applying the L^2 isometry $D^2_{d_\Gamma(\boldsymbol{\beta})}\mathrm{Mod}_{-\boldsymbol{\beta}}$ on the function inside the L^2 norm and distributing powers of $d_\Gamma(\boldsymbol{\beta})$ equates (2.3.4) with

$$\left\| \left(1 + |\boldsymbol{\alpha}|^2\right)^{\frac{s}{2}} \cdot \mathcal{F}\left(\left(D^\infty_{d_\Gamma(\boldsymbol{\beta})^{-1}} T_{-\boldsymbol{\beta}} m\right) \cdot \chi \cdot \left(D^\infty_{d_\Gamma(\boldsymbol{\beta})^{-1}} T_{-\boldsymbol{\beta}} X_\Gamma^{-1}\right) \cdot d\mathcal{H}^2\right)(\boldsymbol{\alpha}) \right\|_{L^2_{\boldsymbol{\alpha}}(V)}.$$

Applying the definition of $H^s(V)$ and the fact that $H^s(V)$ is a Banach algebra when $s > 1$, we then estimate (2.3.4) by

$$\left\| \left(D^{\infty}_{d_\Gamma(\boldsymbol{\beta})^{-1}} T_{-\boldsymbol{\beta}} m \right) \cdot \Phi \cdot \left(D^{\infty}_{d_\Gamma(\boldsymbol{\beta})^{-1}} T_{-\boldsymbol{\beta}} X_\Gamma^{-1} \right) \cdot \chi \right\|_{H^s(V)}$$

$$\lesssim \left\| \left(\mathrm{Dil}^{\infty}_{d_\Gamma(\boldsymbol{\beta})^{-1}} \mathrm{Tr}_{-\boldsymbol{\beta}} m \right) \cdot \Phi \right\|_{H^s(V)} \cdot \left\| \left(\mathrm{Dil}^{\infty}_{d_\Gamma(\boldsymbol{\beta})^{-1}} \mathrm{Tr}_{-\boldsymbol{\beta}} X_\Gamma^{-1} \right) \cdot \chi \right\|_{H^s(V)} \tag{2.3.5}$$

The first factor in (2.3.5) is bounded by 1 by (2.1.10). We introduce three lemmas to prove the bound for the second factor.

Lemma 2.3.1. *For $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2 \in V \setminus \Gamma$, we have*

$$d_\Gamma(\boldsymbol{\beta}_1) \leq d_\Gamma(\boldsymbol{\beta}_2) + |\boldsymbol{\beta}_1 - \boldsymbol{\beta}_2| \tag{2.3.6}$$

Proof. For all $\epsilon > 0$, there exists $\boldsymbol{\gamma}_2 \in \Gamma$ such that

$$|\boldsymbol{\gamma}_2 - \boldsymbol{\beta}_2| < d_\Gamma(\boldsymbol{\beta}) + \epsilon. \tag{2.3.7}$$

By the triangle inequality,

$$d_\Gamma(\boldsymbol{\beta}_1) \leq |\boldsymbol{\gamma}_2 - \boldsymbol{\beta}_1| \leq |\boldsymbol{\gamma}_2 - \boldsymbol{\beta}_2| + |\boldsymbol{\beta}_1 - \boldsymbol{\beta}_2| < d_\Gamma(\boldsymbol{\beta}_2) + |\boldsymbol{\beta}_1 - \boldsymbol{\beta}_2| + \epsilon$$

Since (2.3.7) holds for all $\varepsilon > 0$, we obtain (2.3.6). □

Lemma 2.3.2. *For $x \in V \setminus \Gamma$, then*

$$|X_\Gamma(x)| \gtrsim 1. \tag{2.3.8}$$

Proof. For $\boldsymbol{\beta} \in B_r(x)$, where $r = \frac{\varepsilon}{20} d_\Gamma(x)$, by Lemma 2.3.1, we have

$$d_\Gamma(x) \leq d_\Gamma(\boldsymbol{\beta}) + |\boldsymbol{\beta} - x| \leq d_\Gamma(\boldsymbol{\beta}) + \frac{\varepsilon}{20} d_\Gamma(x).$$

Hence $d_\Gamma(x)$ can be dominated by a constant times $d_\Gamma(\boldsymbol{\beta})$.

$$d_\Gamma(x) < \frac{1}{1 - \varepsilon/20} d_\Gamma(\boldsymbol{\beta}).$$

Then

$$|\boldsymbol{\beta} - x| \leq \frac{\varepsilon}{20} d_\Gamma(x) \leq \frac{\varepsilon}{10} d_\Gamma(\boldsymbol{\beta}).$$

That is $\chi_{\boldsymbol{\beta}}(x) = 1$. On the other hand, for $\boldsymbol{\beta} \in B_r(x)$,

$$d_\Gamma(\boldsymbol{\beta}) \leq d_\Gamma(x) + |\boldsymbol{\beta} - x| < (1 + \frac{\varepsilon}{20}) d_\Gamma(x).$$

Therefore, we obtain a lower bound of $X_\Gamma(x)$.

$$X_\Gamma(x) = \int_V \chi_{\boldsymbol{\beta}} \frac{d\mathcal{H}^2(\boldsymbol{\beta})}{d_\Gamma(\boldsymbol{\beta})^2} \geq \int_{B_r(x)} (1 + \frac{\varepsilon}{20})^{-2} d_\Gamma(x)^{-2} d\mathcal{H}^2(\boldsymbol{\beta}) \gtrsim 1.$$

□

Lemma 2.3.3. *For $x \in V \setminus \Gamma$, and multi-index α,*

$$|(\partial^\alpha X_\Gamma)(x)| \lesssim d_\Gamma(x)^{-|\alpha|}. \tag{2.3.9}$$

Proof. Assume $\boldsymbol{\beta} \in V \setminus \Gamma$ satisfies

$$|\boldsymbol{\beta} - x| < \frac{2\varepsilon}{10} d_\Gamma(\boldsymbol{\beta}). \tag{2.3.10}$$

Then

$$d_\Gamma(\boldsymbol{\beta}) \leq d_\Gamma(x) + |\boldsymbol{\beta} - x| \leq d_\Gamma(x) + \frac{2\varepsilon}{10} d_\Gamma(\boldsymbol{\beta}).$$

Hence $d_\Gamma(\boldsymbol{\beta})$ can be dominated by a constant times $d_\Gamma(x)$.

$$d_\Gamma(\boldsymbol{\beta}) \leq \frac{1}{1 - 2\varepsilon/10} d_\Gamma(x).$$

Then

$$|\boldsymbol{\beta} - x| \leq \frac{2\varepsilon}{10 - 2\varepsilon} d_\Gamma(x) \leq \frac{2\varepsilon}{5} d_\Gamma(x).$$

Also note that for $\boldsymbol{\beta}$ satisfies (2.3.10),

$$d_\Gamma(x) \leq d_\Gamma(\boldsymbol{\beta}) + |\boldsymbol{\beta} - x| \leq (1 + \frac{2\varepsilon}{10}) d_\Gamma(\boldsymbol{\beta}).$$

Let $r = \frac{2\varepsilon}{5} d_\Gamma(x)$. We have an upper bound of derivatives of $X_\Gamma(x)$.

$$|(\partial^\alpha X_\Gamma)(x)| \leq \int_{B_r(x)} \|\partial^\alpha \chi\|_{L^\infty} \cdot d_\Gamma(\boldsymbol{\beta})^{-|\alpha|} \frac{d\mathcal{H}^2(\boldsymbol{\beta})}{d_\Gamma(\boldsymbol{\beta})^2} \lesssim d_\Gamma(x)^{-|\alpha|}.$$

$\square$

Back to the estimate of the second factor of (2.3.5). Let A be the least integer larger than s. Denote $\mathrm{Dil}^\infty_{d_\Gamma(\boldsymbol{\beta})^{-1}} \mathrm{Tr}_{-\boldsymbol{\beta}} X_\Gamma$ as $X_{\Gamma,\boldsymbol{\beta}}$ and note that by Lemma 2.3.2 and Lemma 2.3.3, we have $|X^{-1}_{\Gamma,\boldsymbol{\beta}}| \lesssim 1$ and $|\partial^\alpha X_{\Gamma,\boldsymbol{\beta}}| \lesssim 1$ on the support of χ. By chain rule and Leibniz rule, this implies

$$\begin{aligned}\left\| X^{-1}_{\Gamma,\boldsymbol{\beta}} \cdot \chi \right\|_{H^s} &\lesssim \left\| X^{-1}_{\Gamma,\boldsymbol{\beta}} \cdot \chi \right\|_{L^2} + \sum_{|\alpha|=A} \left\| \partial^\alpha \left(X^{-1}_{\Gamma,\boldsymbol{\beta}} \cdot \chi \right) \right\|_{L^2} \\ &\lesssim 1 + \sum_{|\alpha| \leq A} \left\| \partial^\alpha \left(X^{-1}_{\Gamma,\boldsymbol{\beta}} \right) \right\|_{L^\infty(B_{\frac{2\varepsilon}{10}}(0))} \lesssim 1. \end{aligned} \tag{2.3.11}$$

This completes the estimate of (2.3.5).

2.4 Proof of Proposition 2.2.3: Tent Estimate

Let i be given, and let i' and i'' be the indices in $\{1,2,3\}$ different from i. We decompose the kernel K

$$K = \sum_{k\geq 0} K 1_{E_k} \tag{2.4.1}$$

where

$$E_k := \begin{cases} \left\{(\boldsymbol{\alpha},\boldsymbol{\beta}) \in \mathbb{R}^3 \times V : |d_\Gamma(\boldsymbol{\beta}) P_V \boldsymbol{\alpha}| \in \left(2^{k-1}, 2^k\right]\right\}, & k \in \mathbb{N}. \\ \left\{(\boldsymbol{\alpha},\boldsymbol{\beta}) \in \mathbb{R}^3 \times V : |d_\Gamma(\boldsymbol{\beta}) P_V \boldsymbol{\alpha}| \leq 1\right\}, & k = 0. \end{cases} \tag{2.4.2}$$

Identity (2.4.1) holds because $\{E_k\}_{k=0}^\infty$ partitions $\mathbb{R}^3 \times V$. To show (2.2.3), it suffices to show for each $k \geq 0$

$$\left\| K(\boldsymbol{\alpha},\boldsymbol{\beta}) \cdot \prod_{j=1}^3 (F_j f_j)(\alpha_i, \boldsymbol{\beta}) \right\|_{L^1_{\boldsymbol{\alpha},\mu(\boldsymbol{\beta})}\left(E_k \cap (I\boldsymbol{e}_i \oplus \boldsymbol{e}_i^\perp) \times W_{\boldsymbol{\gamma},1/|I|}\right)} \lesssim (1+k)\, 2^{k(1-s)} |I| \prod_{j=1}^3 \|F_j f_j\|_{S^j} \tag{2.4.3}$$

because summing the right-hand side over k gives the desired result since $s > 1$.

Lemma 2.4.1. *For all $k \geq 0$, $(\boldsymbol{\alpha},\boldsymbol{\beta}) \in E_k$, and $j, j' \in \{1,2,3\}$*

$$\left|\alpha_j - \alpha_{j'}\right| \leq \frac{2^{k+1}}{d_\Gamma(\boldsymbol{\beta})}. \tag{2.4.4}$$

Proof. Define c implicitly by the condition

$$(\alpha_1, \alpha_2, \alpha_3) + c(1,1,1) \in V.$$

By construction, (2.4.2) implies

$$|(\alpha_1 + c, \alpha_2 + c, \alpha_3 + c)| \leq \frac{2^k}{d_\Gamma(\boldsymbol{\beta})}.$$

The triangle inequality yields

$$|\alpha_{j'} - \alpha_j| = |(\alpha_{j'} + c) + (-\alpha_j - c)| \leq |\alpha_{j'} + c| + |\alpha_j + c| \leq \frac{2^{k+1}}{d_\Gamma(\boldsymbol{\beta})}.$$

$\square$

In the rest of this section, to simplify the notation, we denote $W_{\boldsymbol{\gamma}\cdot\frac{\tau}{|I|}}$ as W_τ. We apply Cauchy Schwarz in the integration over $\alpha_{i'}$ and $\alpha_{i''}$ to estimate the left-hand side of (2.4.3) by

$$\int_{W_1} \int_I \|\left(1_{E_k} K\right)(\boldsymbol{\alpha},\boldsymbol{\beta})\|_{L^2_{\alpha_{i'},\alpha_{i''}}} \cdot \left\| \prod_{j=1}^3 (F_j f_j)(\alpha_j, \boldsymbol{\beta}) \right\|_{L^2_{\alpha_{i'},\alpha_{i''}}(Q^2_{\alpha_i})} d\alpha_i d\mu(\boldsymbol{\beta}), \tag{2.4.5}$$

where we use Lemma 2.4.1 to restrict the domain of the last L^2 norm to $Q^2_{a_i}$ with

$$Q_{\alpha_i} = \left[\alpha_i - \frac{2^{k+1}}{d_\Gamma(\boldsymbol{\beta})},\ \alpha_i + \frac{2^{k+1}}{d_\Gamma(\boldsymbol{\beta})}\right].$$

As K is constant on any fiber parallel to $(1,1,1)$, the first factor in (2.4.5) is a universal constant times

$$\| (1_{E_k} K)(\boldsymbol{\alpha}, \boldsymbol{\beta})\|_{L^2_{\boldsymbol{\alpha}}(V)}. \tag{2.4.6}$$

We estimate (2.4.6) via the kernel condition (2.2.2) and obtain for (2.4.5) the bound

$$\lesssim \int_{W_1}\int_I 2^{-ks}\cdot d_\Gamma(\boldsymbol{\beta})\cdot \left\| \prod_{j=1}^{3}(F_j f_j)(\alpha_j,\boldsymbol{\beta})\right\|_{L^2_{\alpha_{i'},\alpha_{i''}}(Q^2_{\alpha_i})} d\alpha_i d\mu(\boldsymbol{\beta}). \tag{2.4.7}$$

Now we split W_1 as

$$(W_1 \setminus W_{2^k}) \cup W_{2^k}.$$

We split (2.4.7) accordingly and estimate the pieces separately.

Starting with the first piece, estimating the triple product by its sup norm, and using that length of Q_{α_i} is $\frac{2^{k+2}}{d_\Gamma(\boldsymbol{\beta})}$, we estimate this piece by

$$|I|\, 2^{-k(1-s)} \left\|\prod_{j=1}^{3}(F_j f_j)(\alpha_j,\boldsymbol{\beta})\right\|_{L^\infty_{\alpha,\mu(\boldsymbol{\beta})}} \int_{W_1\setminus W_{2^k}} d\mu(\boldsymbol{\beta}). \tag{2.4.8}$$

By definition (2.2.5) and direct calculation via polar coordinates $r = |\boldsymbol{\beta} - \boldsymbol{\gamma}|$, we obtain

$$\int_{W_1\setminus W_{2^k}} \frac{d\mathcal{H}^2(\boldsymbol{\beta})}{d_\Gamma(\boldsymbol{\beta})^2} \lesssim \int_{W_1\setminus W_{2^k}} \frac{d\mathcal{H}^2(\boldsymbol{\beta})}{|\boldsymbol{\beta}-\boldsymbol{\gamma}|^2} \lesssim \int_{1/|I|}^{2^k/|I|} \frac{1}{r^2}\cdot r dr \lesssim k. \tag{2.4.9}$$

This together with the definition of S_j for (2.4.8) gives the bound

$$\le |I|\, k\cdot 2^{k(1-s)}\cdot \prod_{j=1}^{3}\|F_j\|_{S^j}, \tag{2.4.10}$$

which completes our estimation of the first piece of (2.4.7).

Next, to calculate the second piece, we rewrite the remaining piece of (2.4.7) into L^2 average

$$2^{k(1-s)}\int_{W_{2^k}}\int_I (F_i f_i)(\alpha_i,\boldsymbol{\beta})\cdot\prod_{j\neq i}\|(F_j f_j)(\alpha_j,\boldsymbol{\beta})\|_{\text{Ł}^2_{\alpha_j}(Q_{\alpha_i})}\, d\alpha_i d\mu(\boldsymbol{\beta}). \tag{2.4.11}$$

We decompose further the domain of integration into

$$W_{2^k} \subseteq V_i \cup V_{i'} \cup V_{i''} \tag{2.4.12}$$

where

$$V_a := \left(W_{2^k} \setminus \bigsqcup_{b\neq a} U^b_{\boldsymbol{\gamma}} \right). \tag{2.4.13}$$

The inclusion (2.4.12) holds because $U^a_{\boldsymbol{\gamma}} \cap U^b_{\boldsymbol{\gamma}} = \emptyset$ for $a \neq b$. By the triangle inequality, it suffices to estimate each piece independently. First, we deal with the piece $V_{i'}$. The the treatment of $V_{i''}$ is similar.

$$|I| \cdot 2^{k(1-s)} \int_{V_{i'}} \frac{1}{|I|} \int_I (F_i f_i)(\alpha_i, \boldsymbol{\beta}) \cdot \prod_{j \neq i} \|(F_j f_j)(\alpha_j, \boldsymbol{\beta})\|_{\text{Ł}^2_{\alpha_j}(Q_{\alpha_i})} \, d\alpha_i d\mu(\boldsymbol{\beta}) \tag{2.4.14}$$

We perform a Hölder's inequality over the measure $d\alpha_i d\mu(\boldsymbol{\beta})$ where we place a L^∞ norm on i'-th coordinate and L^2 norm on the rest. Then (2.4.14) is dominated by

$$\begin{aligned} &|I| \cdot 2^{k(1-s)} \left(\frac{1}{|I|} \int_{V_{i'}} \int_I |(F_i f_i)(\alpha_i, \boldsymbol{\beta})|^2 d\alpha_i d\mu(\boldsymbol{\beta}) \right)^{\frac{1}{2}} \\ &\cdot \left\| \|(F_{i'} f_{i'})(\alpha_{i'}, \boldsymbol{\beta})\|_{\text{Ł}^2_{\alpha_{i'}}(Q_{\alpha_i})} \right\|_{L^\infty_{\alpha_i, \mu(\boldsymbol{\beta})}} \\ &\cdot \left(\frac{1}{|I|} \int_{V_{i'}} \int_I \|(F_{i''} f_{i''})(\alpha_{i''}, \boldsymbol{\beta})\|^2_{\text{Ł}^2_{\alpha_{i''}}(Q_{\alpha_i})} \, d\alpha_i d\mu(\boldsymbol{\beta}) \right)^{\frac{1}{2}}. \end{aligned}$$

Then by definition of size, we can further estimate

$$\lesssim 2^{k(1-s)} |I| \|F_i f_i\|_{S^i} \cdot \prod_{j \neq i} \left\| \|(F_j f_j)(\alpha_j, \boldsymbol{\beta})\|_{\text{Ł}^2_{\alpha_j}(Q_{\alpha_i})} \right\|_{S^j(I, \boldsymbol{\gamma}, \frac{2^k}{|I|})}. \tag{2.4.15}$$

Next, we deal with the piece V_i. Again, We perform a Hölder's inequality over the measure $d\alpha_i d\mu(\boldsymbol{\beta})$ where we place a L^∞ norm on i-th coordinate and L^2 norm on the rest. Then (2.4.14) with $V_{i'}$ replaced by V_i is dominated by

$$\begin{aligned} &|I| \cdot 2^{k(1-s)} \|(F_i f_i)(\alpha_i, \boldsymbol{\beta})\|_{L^\infty_{\alpha_i, \mu(\boldsymbol{\beta})}} \\ &\cdot \left(\frac{1}{|I|} \int_{V_i} \int_I \|(F_{i'} f_{i'})(\alpha_{i'}, \boldsymbol{\beta})\|^2_{\text{Ł}^2_{\alpha_{i'}}(Q_{\alpha_i})} \, d\alpha_i d\mu(\boldsymbol{\beta}) \right)^{\frac{1}{2}} \cdot \\ &\cdot \left(\frac{1}{|I|} \int_{V_i} \int_I \|(F_{i''} f_{i''})(\alpha_{i''}, \boldsymbol{\beta})\|^2_{\text{Ł}^2_{\alpha_{i''}}(Q_{\alpha_i})} \, d\alpha_i d\mu(\boldsymbol{\beta}) \right)^{\frac{1}{2}}. \end{aligned}$$

Again, by the definition of size, the above term can be dominated by (2.4.15). We may rewrite each factor in the product of (2.4.15) into

$$\left\| \left\| (F_j f_j) \left(\alpha + x \frac{2^{k+1}}{d_\Gamma(\boldsymbol{\beta})}, \, \boldsymbol{\beta} \right) \right\|_{\text{Ł}^2_x([-1,1])} \right\|_{S^j(I, \boldsymbol{\gamma}, \frac{2^k}{|I|})}. \tag{2.4.16}$$

Since size is the maximum of a L^2 quantity and a L^∞ quantity, by Minkowski inequality, we can estimate (2.4.16) by

$$\left\| \left\| (F_j f_j) \left(\alpha + x \frac{2^{k+1}}{d_\Gamma(\boldsymbol{\beta})}, \, \boldsymbol{\beta} \right) \right\|_{S^j(I, \boldsymbol{\gamma}, \frac{2^k}{|I|})} \right\|_{\text{Ł}^2_x([-1,1])} \tag{2.4.17}$$

Fix x and consider the inner norm in (2.4.17). Observe that

$$T : (\alpha, \boldsymbol{\beta}) \mapsto \left(\alpha + x\frac{2^{k+1}}{d_\Gamma(\boldsymbol{\beta})}, \boldsymbol{\beta}\right)$$

is a map that preserves measure and for $\boldsymbol{\beta} \in W_{2^k}$

$$\left|x\frac{2^{k+1}}{d_\Gamma(\boldsymbol{\beta})}\right| \leq \frac{2^{k+1}}{\delta_1|\boldsymbol{\beta}-\boldsymbol{\gamma}|} \leq \frac{2^{k+1}}{\delta_1 2^k/|I|} \leq 2\delta_1^{-1}|I|.$$

Thus we have for $C = \left(1 + 4\delta_1^{-1}\right)$,

$$T\left(I \times W_{2^k}\right) \subseteq CI \times W_{2^k} \subseteq CI \times W_{\frac{1}{C}}.$$

Then (2.4.15) can be estimated by

$$\lesssim |I| \cdot 2^{k(1-s)} \|(F_i f_i)\|_{S^i} \cdot \prod_{j\neq i} \left\| \|(F_j f_j)(\alpha, \boldsymbol{\beta})\|_{S^j\left(CI\times W_{\boldsymbol{\gamma},2^k/|CI|}\right)} \right\|_{L_x^2([-1,1])}. \tag{2.4.18}$$

Then by the definition of size (2.2.8),

$$(2.4.18) \lesssim |I| \cdot 2^{k(1-s)} \prod_{j=1}^{3} \|F_j f_j\|_{S^j}.$$

Together with (2.4.10), we complete the proof of Proposition 2.2.3.

2.5 Proof of Proposition 2.2.4: Global Estimate

For $(\alpha, \boldsymbol{\beta}) \in \mathbb{R} \times V$, we introduce the L^1 normalized wave-packet

$$\varphi^j_{\alpha,\boldsymbol{\beta}} := T_\alpha \operatorname{Mod}_{-\beta_j} D^1_{d_\Gamma(\boldsymbol{\beta})^{-1}} \overline{\varphi}. \tag{2.5.1}$$

Since φ is even, direct calculation gives

$$(F_j f)(\alpha, \boldsymbol{\beta}) = \left((\operatorname{Mod}_{\beta_j} D^1_{d_\Gamma(\boldsymbol{\beta})^{-1}} \varphi) * f_j\right)(\alpha) = \left\langle f_j, \varphi^j_{\alpha,\boldsymbol{\beta}} \right\rangle.$$

By the definition of the global size, it suffices to show

$$\|F_j f\|_{S^j(I,\boldsymbol{\gamma})} \lesssim \|f\|_{L^\infty}, \quad \forall I \in \mathcal{I}, \boldsymbol{\gamma} \in \Gamma.$$

Fix a pair $(I, \boldsymbol{\gamma}) \in \mathcal{I} \times \Gamma$ and recall

$$\|F_j f\|_{S^j(I,\boldsymbol{\gamma})} := \|F_j f\|_{L^\infty\left(I\times W_{\boldsymbol{\gamma},1/|I|}\right)} \vee |I|^{-\frac{1}{2}} \|F_j f\|_{L^2_\nu\left(I\times\left(W_{\boldsymbol{\gamma},1/|I|}\setminus U^j_{\boldsymbol{\gamma}}\right)\right)}. \tag{2.5.2}$$

Trivially, we have

$$|F_j f(\alpha, \boldsymbol{\beta})| = \left|\left\langle f, \varphi^j_{\alpha,\boldsymbol{\beta}} \right\rangle\right| \leq \|f\|_{L^\infty}. \tag{2.5.3}$$

Hence, the L^∞ component in (2.5.2) is dominated by $\|f\|_{L^\infty}$.

To control the remaining L^2 component, we split the function f into the local part $1_{3I}f$ and the tail part $1_{3I^c}f$. By linearity of F_j and the triangle inequality, it suffices to obtain the following estimate for the local part

$$|I|^{-\frac{1}{2}} \left\| F_j(1_{3I}f) \right\|_{L^2_\nu\left(I\times\left(W_{\boldsymbol{\gamma},1/|I|}\setminus U^j_{\boldsymbol{\gamma}}\right)\right)} \lesssim \|f\|_{L^\infty} \tag{2.5.4}$$

and the analogous estimate for the tail part

$$|I|^{-\frac{1}{2}} \left\| F_j(1_{3I^c}f) \right\|_{L^2_\nu\left(I\times\left(W_{\boldsymbol{\gamma},1|I|}\setminus U^j_{\boldsymbol{\gamma}}\right)\right)} \lesssim \|f\|_{L^\infty}. \tag{2.5.5}$$

Starting with the treatment of the local part (2.5.4), we observe that it's enough to show the following L^2 estimate

$$\|F_j g\|_{L^2_\nu\left(\mathbb{R}\times\left(W_{\boldsymbol{\gamma},0}\setminus U^j_{\boldsymbol{\gamma}}\right)\right)} \lesssim \|g\|_{L^2}. \tag{2.5.6}$$

This can be seen by taking $g = 1_{3I}f$,

$$|I|^{-\frac{1}{2}} \left\| F_j(1_{3I}f) \right\|_{L^2_\nu\left(I\times\left(W_{\boldsymbol{\gamma},1/|I|}\setminus U^j_{\boldsymbol{\gamma}}\right)\right)} \le |I|^{-\frac{1}{2}} \|F_j g\|_{L^2_\nu\left(\mathbb{R}\times\left(W_{\boldsymbol{\gamma},0}\setminus U^j_{\boldsymbol{\gamma}}\right)\right)} \lesssim |I|^{-\frac{1}{2}} \|g\|_{L^2}.$$

The localization gives the trivial bound $|I|^{-\frac{1}{2}} \|g\|_{L^2} \lesssim \|f\|_{L^\infty}$, which finishes (2.5.4). Return to the proof of (2.5.6). Recall that

$$F_j g\,(\alpha, \boldsymbol{\beta}) := \left(M_{\beta_j} D^1_{d_\Gamma(\boldsymbol{\beta})^{-1}}\varphi\right) * g\,(\alpha).$$

Applying Plancherel on the spatial variable α to the left-hand side of (2.5.6) yields

$$\left\| \left\| \left(T_{\beta_j} D^\infty_{d_\Gamma(\boldsymbol{\beta})}\widehat{\varphi}\right)(\xi) \cdot \widehat{g}\,(\xi) \right\|_{L^2_\xi} \right\|_{L^2_{\mu(\boldsymbol{\beta})}\left(W_{\boldsymbol{\gamma},0}\setminus U^j_{\boldsymbol{\gamma}}\right)} \tag{2.5.7}$$

then interchange the order of the L^2 norms equates (2.5.7) to

$$\left\| \left\| \left(T_{\beta_j} D^\infty_{d_\Gamma(\boldsymbol{\beta})}\widehat{\varphi}\right)(\xi) \right\|_{L^2_{\mu(\boldsymbol{\beta})}\left(W_{\boldsymbol{\gamma},0}\setminus U^j_{\boldsymbol{\gamma}}\right)} \cdot \widehat{g}\,(\xi) \right\|_{L^2_\xi}. \tag{2.5.8}$$

It remains to show

$$\left\| T_{\beta_j} D^\infty_{d_\Gamma(\boldsymbol{\beta})}\widehat{\varphi} \right\|_{L^2_{\mu(\boldsymbol{\beta})}\left(W_{\boldsymbol{\gamma},0}\setminus U^j_{\boldsymbol{\gamma}}\right)} \lesssim 1. \tag{2.5.9}$$

By developing the L^2 norm, the left-hand side of (2.5.9) equals to

$$\int_{W_{\boldsymbol{\gamma},0}\setminus U^j_{\boldsymbol{\gamma}}} |\widehat{\varphi}|^2 \left(\frac{\xi - \beta_j}{d_\Gamma(\boldsymbol{\beta})}\right) \frac{d\mathcal{H}^2(\boldsymbol{\beta})}{d_\Gamma(\boldsymbol{\beta})^2}. \tag{2.5.10}$$

Recall that for $\boldsymbol{\beta} \in W_{\boldsymbol{\gamma},0} \setminus U^j_{\boldsymbol{\gamma}}$,

$$\delta_1 |\beta_j - \gamma_j| \le \delta_1 |\boldsymbol{\beta} - \boldsymbol{\gamma}| \le d_\Gamma(\boldsymbol{\beta}) \le |\boldsymbol{\beta} - \boldsymbol{\gamma}| \le \frac{1}{\delta_2} |\beta_j - \gamma_j|.$$

On the other hand, since $\operatorname{supp}\widehat{\varphi} \subset B_{\frac{4\epsilon}{10}}(0)$,

$$|\widehat{\varphi}|^2\left(\frac{\xi-\beta_j}{d_\Gamma(\boldsymbol{\beta})}\right) \neq 0 \implies |\xi-\beta_j| \leq \frac{4\epsilon}{10} d_\Gamma(\boldsymbol{\beta}) \leq \frac{2\epsilon}{5\delta_2}|\beta_j-\gamma_j|.$$

By triangle inequality,

$$|\beta_j-\gamma_j| - |\xi-\beta_j| \leq |\xi-\gamma_j| \leq |\beta_j-\gamma_j| + |\xi-\beta_j|.$$

Hence,

$$\left(1-\frac{2\epsilon}{5\delta_2}\right)|\beta_j-\gamma_j| \leq |\xi-\gamma_j| \leq \left(1+\frac{2\epsilon}{5\delta_2}\right)|\beta_j-\gamma_j|.$$

This shows that $|\boldsymbol{\beta}-\boldsymbol{\gamma}| \sim d_\Gamma(\boldsymbol{\beta}) \sim |\xi-\gamma_j|$. As a direct consequence, we can dominate (2.5.10) by

$$\|\widehat{\varphi}\|_{L^\infty}^2 \cdot ⨍_{\{\boldsymbol{\beta}\in V \mid |\boldsymbol{\beta}-\boldsymbol{\gamma}|\sim|\xi-\gamma_j|\}} d\mathcal{H}^2(\boldsymbol{\beta}) \sim 1$$

and thus, verify (2.5.9) and complete the proof of (2.5.6).

As for the tail part (2.5.5), we have a trivial bound

$$\begin{aligned} &|I|^{-\frac{1}{2}}\|F_j(1_{3I^c}f)\|_{L^2_\nu\left(I\times\left(W_{\boldsymbol{\gamma},1/|I|}\setminus U^j_{\boldsymbol{\gamma}}\right)\right)} \\ \leq\ &|I|^{-\frac{1}{2}}\|F_j(1_{3I^c}f)\|^{\frac{1}{2}}_{L^1_\nu\left(I\times\left(W_{\boldsymbol{\gamma},1/|I|}\setminus U^j_{\boldsymbol{\gamma}}\right)\right)} \cdot \|F_j(1_{3I^c}f)\|^{\frac{1}{2}}_{L^\infty}. \end{aligned} \tag{2.5.11}$$

By (2.5.3), we dominate (2.5.11) by

$$|I|^{-\frac{1}{2}}\|F_j(1_{3I^c}f)\|^{\frac{1}{2}}_{L^1_\nu\left(I\times\left(W_{\boldsymbol{\gamma},1/|I|}\setminus U^j_{\boldsymbol{\gamma}}\right)\right)} \cdot \|f\|^{\frac{1}{2}}_{L^\infty}.$$

Therefore, to show (2.5.5), it suffices to show the following inequality

$$|I|^{-1}\|F_j(1_{3I^c}f)\|_{L^1_\nu\left(I\times W_{\boldsymbol{\gamma},1/|I|}\right)} \lesssim \|f\|_{L^\infty}. \tag{2.5.12}$$

We may dominate the left-hand side of (2.5.12) by

$$\|f\|_{L^\infty}\int_{3I^c} ⨍_I \int_{W_{\boldsymbol{\gamma},1/|I|}} \left|\varphi^j_{\alpha,\boldsymbol{\beta}}\right|(x)\, d\mu(\boldsymbol{\beta})\, d\alpha dx.$$

Then, it suffices to show

$$\int_{3I^c} ⨍_I \int_{W_{\boldsymbol{\gamma},1/|I|}} \left|\varphi^j_{\alpha,\boldsymbol{\beta}}\right|(x)\, d\mu(\boldsymbol{\beta})\, d\alpha dx \lesssim 1. \tag{2.5.13}$$

Recall again that $\boldsymbol{\beta} \in W_{\boldsymbol{\gamma},1/|I|}$ implies that $d_\Gamma(\boldsymbol{\beta}) \sim |\boldsymbol{\beta}-\boldsymbol{\gamma}|$ and thus,

$$\left|\varphi^j_{\alpha,\boldsymbol{\beta}}\right|(x) \lesssim |\boldsymbol{\beta}-\boldsymbol{\gamma}|\,(1+|\boldsymbol{\beta}-\boldsymbol{\gamma}|\cdot|x-\alpha|)^{-N}.$$

By polar coordinates with center at $\boldsymbol{\gamma}$, we have

$$\int_{W_{\boldsymbol{\gamma},1/|I|}} \left|\varphi^j_{\alpha,\boldsymbol{\beta}}\right|(x)\, d\mu(\boldsymbol{\beta}) \lesssim \int_{1/|I|}^\infty t\,(1+t|x-\alpha|)^{-N}\frac{dt}{t}.$$

As a result, the left-hand side of (2.5.13) is dominated by

$$\lesssim \int_{3I^c} \fint_I \int_{1/|I|}^{\infty} t\,(1+t\,|x-\alpha|)^{-N}\,\frac{dt}{t}d\alpha dx. \tag{2.5.14}$$

After a change of variable,

$$(2.5.14) = \int_{[-\frac{3}{2},\frac{3}{2}]} \int_{[-\frac{1}{2},\frac{1}{2}]} \int_{1}^{\infty} t\,(1+t\,|x-\alpha|)^{-N}\,\frac{dt}{t}d\alpha dx.$$

It becomes apparent that the above quantity can be estimated by

$$\sim \int_{[-\frac{3}{2},\frac{3}{2}]} \int_{[-\frac{1}{2},\frac{1}{2}]} \int_{1}^{\infty} t^{1-N}\frac{dt}{t} \cdot |x-\alpha|^{-N}\,d\alpha dx \sim \int_{[-\frac{3}{2},\frac{3}{2}]} \int_{[-\frac{1}{2},\frac{1}{2}]} |x-\alpha|^{-N}\,d\alpha dx \sim 1.$$

This verifies (2.5.13) and completes the proof of (2.5.5).

2.6 Proof of Proposition 2.2.5: Geometry of Tents

For $x_1, x_2 \in \mathbb{R}^n$, $0 < r < 1$, define the Apollonian circle

$$B_r(x_1,x_2) := \left\{ y \in \mathbb{R}^n : \frac{|y-x_1|}{r} < \frac{|y-x_2|}{1} \right\}. \tag{2.6.1}$$

We begin with a geometric lemma concerning the relation between two Apollonian circles.

Lemma 2.6.1. *Let $x_0, x_1, x_2 \in \mathbb{R}^n$ and $0 < r < 1$. Suppose that*

$$r|x_2 - x_1| \le |x_2 - x_0| - |x_1 - x_0|, \tag{2.6.2}$$

then $B_r(x_0,x_1) \subseteq B_r(x_0,x_2)$. This inclusion relation is equivalent to the fact that if $y \in \mathbb{R}^n$ satisfies

$$r|y-x_2| \le |y-x_0|,$$

it must also satisfy

$$r|y-x_1| \le |y-x_0|.$$

Proof. By direct calculation, we have the center $C(B_r(x_0,x_i))$ and radius $R(B_r(x_0,x_i))$ of these two Apollonian circles

$$C(B_r(x_0,x_i)) = \frac{x_0 - r^2 x_i}{1-r^2}, \quad R(B_r(x_0,x_i)) = \frac{r|x_0 - x_i|}{1-r^2}. \tag{2.6.3}$$

Then by assumption (2.6.2),

$$\begin{aligned} R(B_r(x_0,x_2)) - R(B_r(x_0,x_1)) &= \frac{r(|x_2-x_0| - |x_1-x_0|)}{1-r^2} \\ &\ge \frac{r^2|x_2-x_1|}{1-r^2} = |C(B_r(x_0,x_2)) - C(B_r(x_0,x_1))|. \end{aligned}$$

Hence $B_r(x_0,x_1) \subseteq B_r(x_0,x_2)$. $\square$

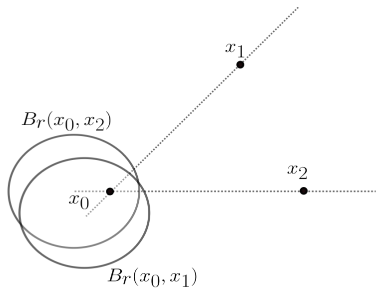

Figure 2.4: The inclusion relation between two Apollonian circles.

Lemma 2.6.2. *For $\boldsymbol{\gamma} \in \Gamma$ and $j = 1, 2, 3$, we have the inclusion relation*

$$U^j_{\boldsymbol{\gamma}} \subseteq W_{\boldsymbol{\gamma},0}. \tag{2.6.4}$$

Proof. Notice that

$$W_{\boldsymbol{\gamma},0} = V \setminus \bigcup_{\boldsymbol{\gamma}' \in \Gamma} B_{\delta_1}(\boldsymbol{\gamma}', \boldsymbol{\gamma}). \tag{2.6.5}$$

For $\boldsymbol{\beta} \in \bigcup\limits_{\boldsymbol{\gamma}' \in \Gamma} B_{\delta_1}(\boldsymbol{\gamma}', \boldsymbol{\gamma})$, there is $\boldsymbol{\gamma}' \in \Gamma$ such that $\boldsymbol{\beta} \in B_{\delta_1}(\boldsymbol{\gamma}', \boldsymbol{\gamma})$. Recall that $B_{\delta_1}(\boldsymbol{\gamma}', \boldsymbol{\gamma})$ is a ball with center

$$C(B_{\delta_1}(\boldsymbol{\gamma}', \boldsymbol{\gamma})) = \frac{\boldsymbol{\gamma}' - \delta_1^2 \boldsymbol{\gamma}}{1 - \delta_1^2} = \boldsymbol{\gamma} + \frac{\boldsymbol{\gamma}' - \boldsymbol{\gamma}}{1 - \delta_1^2}$$

and radius

$$R(B_{\delta_1}(\boldsymbol{\gamma}', \boldsymbol{\gamma})) = \delta_1 \cdot \frac{\boldsymbol{\gamma}' - \boldsymbol{\gamma}}{1 - \delta_1^2}.$$

The angle between $\overline{\boldsymbol{\gamma}\,\boldsymbol{\beta}}$ and $\overline{\boldsymbol{\gamma}\, C(B_{\delta_1}(\boldsymbol{\gamma}', \boldsymbol{\gamma}))}$ is at most

$$\arcsin\left(\frac{C(B_{\delta_1}(\boldsymbol{\gamma}', \boldsymbol{\gamma})) - \boldsymbol{\gamma}}{R(B_{\delta_1}(\boldsymbol{\gamma}', \boldsymbol{\gamma}))}\right) = \arcsin \delta_1 = \theta_1.$$

Since $\boldsymbol{\gamma}, \boldsymbol{\gamma}' \in \Gamma$, by Lemma 2.2.2, the angle between $\overline{\boldsymbol{\gamma}\,\boldsymbol{\gamma}'}$ and $P_V e_j$ is at most $\frac{\pi}{3} + \theta_0$. Using the fact that $\boldsymbol{\gamma}, \boldsymbol{\gamma}', C(B_{\delta_1}(\boldsymbol{\gamma}', \boldsymbol{\gamma}))$ are on the same line, we conclude that the angle between $\overline{\boldsymbol{\gamma}\boldsymbol{\beta}}$ and $P_V e_j$ is at most $\frac{\pi}{3} + \theta_0 + \theta_1$. Finally, we recall the definition (2.2.6) of $U^j_{\boldsymbol{\gamma}}$ and obtain $U^j_{\boldsymbol{\gamma}} \subseteq W_{\boldsymbol{\gamma},0}$. □

Lemma 2.6.3. *For $\boldsymbol{\gamma}, \boldsymbol{\gamma}', \boldsymbol{\gamma}'' \in \Gamma$. Suppose we have an order relation $\boldsymbol{\gamma}_j \leq \boldsymbol{\gamma}'_j \leq \boldsymbol{\gamma}''_j$ in some direction $j \in \{1, 2, 3\}$, then the order is either preserved $\boldsymbol{\gamma}_i \leq \boldsymbol{\gamma}'_i \leq \boldsymbol{\gamma}''_i$ or reversed $\boldsymbol{\gamma}''_i \leq \boldsymbol{\gamma}'_i \leq \boldsymbol{\gamma}_i$ in other directions $i \neq j$, $i \in \{1, 2, 3\}$. Furthermore, $\boldsymbol{\gamma}, \boldsymbol{\gamma}', \boldsymbol{\gamma}''$ satisfy the condition* (2.6.2),

$$\delta_1 |\boldsymbol{\gamma}'' - \boldsymbol{\gamma}'| \leq |\boldsymbol{\gamma}'' - \boldsymbol{\gamma}| - |\boldsymbol{\gamma}' - \boldsymbol{\gamma}|. \tag{2.6.6}$$

Proof. Translate $\boldsymbol{\gamma}'$ to origin. By condition (2.1.9), we have $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}''$ in two octants diagonal to each other. This implies that in i-th direction, the order is either preserved or reversed. Notice that by (2.1.9), we have

$$\langle \boldsymbol{\gamma} - \boldsymbol{\gamma}', \boldsymbol{\gamma}'' - \boldsymbol{\gamma}' \rangle \leq -\frac{1}{2} |\boldsymbol{\gamma} - \boldsymbol{\gamma}'| \cdot |\boldsymbol{\gamma}'' - \boldsymbol{\gamma}'|.$$

Then

$$\begin{aligned}|\boldsymbol{\gamma}'' - \boldsymbol{\gamma}|^2 &= |\boldsymbol{\gamma}' - \boldsymbol{\gamma}|^2 + |\boldsymbol{\gamma}'' - \boldsymbol{\gamma}'|^2 - 2\langle \boldsymbol{\gamma} - \boldsymbol{\gamma}', \boldsymbol{\gamma}'' - \boldsymbol{\gamma}'\rangle \\ &\geq |\boldsymbol{\gamma}' - \boldsymbol{\gamma}|^2 + |\boldsymbol{\gamma}'' - \boldsymbol{\gamma}'|^2 + |\boldsymbol{\gamma}' - \boldsymbol{\gamma}| \cdot |\boldsymbol{\gamma}'' - \boldsymbol{\gamma}'| \\ &\geq \left(|\boldsymbol{\gamma}' - \boldsymbol{\gamma}| + \frac{1}{2}|\boldsymbol{\gamma}'' - \boldsymbol{\gamma}'| \right)^2.\end{aligned}$$

Take square root on both sides and by $\delta_1 \leq \frac{1}{2}$, we have the desired result. $\square$

Next, we prove the first statement in Proposition 2.2.5.

Proof of Proposition 2.2.5 (1). For $\boldsymbol{\beta} \in V$, since Γ is closed, there exists a point $\boldsymbol{\gamma}(\boldsymbol{\beta}) = (\gamma(\boldsymbol{\beta})_1, \gamma(\boldsymbol{\beta})_2, \gamma(\boldsymbol{\beta})_3)$ on the singularity Γ such that $|\boldsymbol{\beta} - \boldsymbol{\gamma}(\boldsymbol{\beta})| = d_\Gamma(\boldsymbol{\beta})$. Suppose $\boldsymbol{\beta} \in W_{\boldsymbol{\gamma}'',t}$. First, we show that $\boldsymbol{\beta} \notin B(\boldsymbol{\gamma}, \delta_1 t)$. The argument for $\boldsymbol{\beta} \notin B(\boldsymbol{\gamma}', \delta_1 t)$ is the same. By (2.2.4) and (2.2.12),

$$|\boldsymbol{\gamma}'' - \boldsymbol{\gamma}| \leq \frac{1}{\delta_0}|\gamma''_j - \gamma_j| \leq (1 - \delta_1)t. \tag{2.6.7}$$

Then we have

$$|\boldsymbol{\beta} - \boldsymbol{\gamma}| \geq |\boldsymbol{\beta} - \boldsymbol{\gamma}''| - |\boldsymbol{\gamma}'' - \boldsymbol{\gamma}| \geq t - (1 - \delta_1)t = \delta_1 t.$$

Second, we show either $\delta_1|\boldsymbol{\beta} - \boldsymbol{\gamma}| \leq d_\Gamma(\boldsymbol{\beta})$ or $\delta_1|\boldsymbol{\beta} - \boldsymbol{\gamma}'| \leq d_\Gamma(\boldsymbol{\beta})$. According to the location of $\gamma(\boldsymbol{\beta})_j$, we divide into three cases: (1) : $\gamma_j \leq \gamma(\boldsymbol{\beta})_j \leq \gamma'_j$, (2) : $\gamma(\boldsymbol{\beta})_j < \gamma_j$, (3) : $\gamma'_j < \gamma(\boldsymbol{\beta})_j$. For case (1), via an augment similar to (2.6.7) with $\boldsymbol{\gamma}''$ replaced by $\boldsymbol{\gamma}(\boldsymbol{\beta})$ and the assumption $\boldsymbol{\beta} \in W_{\boldsymbol{\gamma}'',t}$, we obtain

$$|\boldsymbol{\beta} - \gamma| \leq |\boldsymbol{\beta} - \boldsymbol{\gamma}(\boldsymbol{\beta})| + |\boldsymbol{\gamma}(\boldsymbol{\beta}) - \boldsymbol{\gamma}| \leq d_\Gamma(\boldsymbol{\beta}) + (1 - \delta_1)t \leq \frac{1}{\delta_1} d_\Gamma(\boldsymbol{\beta}).$$

Case (2) and (3) are symmetric. Hence, we only prove the case (2). In this case, $\boldsymbol{\gamma}(\boldsymbol{\beta}), \boldsymbol{\gamma}, \boldsymbol{\gamma}''$ satisfy the relation

$$\gamma(\boldsymbol{\beta})_j \leq \gamma_j \leq \gamma''_j.$$

By Lemma 2.6.3,

$$\delta_1|\boldsymbol{\gamma}'' - \boldsymbol{\gamma}| \leq |\boldsymbol{\gamma}'' - \boldsymbol{\gamma}(\boldsymbol{\beta})| - |\boldsymbol{\gamma} - \boldsymbol{\gamma}(\boldsymbol{\beta})|.$$

Then by Lemma 2.6.1,

$$\left(\delta_1|\boldsymbol{\beta} - \boldsymbol{\gamma}''| \leq |\boldsymbol{\beta} - \boldsymbol{\gamma}(\boldsymbol{\beta})| \right) \implies \left(\delta_1|\boldsymbol{\beta} - \boldsymbol{\gamma}| \leq |\boldsymbol{\beta} - \boldsymbol{\gamma}(\boldsymbol{\beta})| \right).$$

Since $\boldsymbol{\beta} \in W_{\gamma'',t}$,

$$\delta_1|\boldsymbol{\beta} - \boldsymbol{\gamma}''| \leq d_\Gamma(\boldsymbol{\beta}) = |\boldsymbol{\beta} - \boldsymbol{\gamma}(\boldsymbol{\beta})|.$$

We then have

$$\delta_1|\boldsymbol{\beta} - \boldsymbol{\gamma}| \leq |\boldsymbol{\beta} - \boldsymbol{\gamma}(\boldsymbol{\beta})| = d_\Gamma(\boldsymbol{\beta})$$

and complete the proof of statement *(1)* in Proposition 2.2.5. $\square$

Before proving statement *(2)* in Proposition 2.2.5, we introduce a Lemma.

Lemma 2.6.4. *We have the inclusion*

$$W_{\boldsymbol{\gamma}',0} \setminus W_{\boldsymbol{\gamma},t} \subseteq B_t(\boldsymbol{\gamma}) \cup \left(W_{\boldsymbol{\gamma}',0} \setminus U^j_{\boldsymbol{\gamma}} \right)^{>\gamma_j}, \tag{2.6.8}$$

where $\left(W_{\boldsymbol{\gamma}',0} \setminus U^j_{\boldsymbol{\gamma}} \right)^{>\gamma_j}$ *consists of all the points* $\boldsymbol{\beta}$ *in* $W_{\boldsymbol{\gamma}',0} \setminus U^j_{\boldsymbol{\gamma}}$ *with* $\beta_j > \gamma_j$.

Proof. Let $\boldsymbol{\beta} \in W_{\boldsymbol{\gamma}',0} \setminus W_{\boldsymbol{\gamma},t}$, we may split into four cases.

Case 1: $\gamma(\boldsymbol{\beta})_j \leq \gamma_j$.
Since $\gamma(\boldsymbol{\beta})_j \leq \gamma_j \leq \gamma'_j$, by Lemma 2.6.1 and Lemma 2.6.3,

$$\left(\delta_1 |\boldsymbol{\beta} - \boldsymbol{\gamma}'| \leq |\boldsymbol{\beta} - \boldsymbol{\gamma}(\boldsymbol{\beta})| \right) \implies \left(\delta_1 |\boldsymbol{\beta} - \boldsymbol{\gamma}| \leq |\boldsymbol{\beta} - \boldsymbol{\gamma}(\boldsymbol{\beta})| \right). \tag{2.6.9}$$

Since $\boldsymbol{\beta} \in W_{\boldsymbol{\gamma}',0}$, we then have $\boldsymbol{\beta} \in W_{\boldsymbol{\gamma},0}$. By assumption, $\boldsymbol{\beta} \notin W_{\boldsymbol{\gamma},t}$. Hence $\boldsymbol{\beta} \in B_t(\boldsymbol{\gamma})$.

Case 2: $\boldsymbol{\beta} \in U^j_{\boldsymbol{\gamma}}$.
By Lemma 2.6.2, $\boldsymbol{\beta} \in W_{\boldsymbol{\gamma},0}$. By assumption, $\boldsymbol{\beta} \notin W_{\boldsymbol{\gamma},t}$. Hence $\boldsymbol{\beta} \in B_t(\boldsymbol{\gamma})$.

Case 3: $\gamma_j < \beta_j$.
Since, $\boldsymbol{\beta} \in W_{\boldsymbol{\gamma}',0} \setminus W_{\boldsymbol{\gamma},t} \subseteq W_{\boldsymbol{\gamma}',0} \setminus U^j_{\boldsymbol{\gamma}}$, we have $\boldsymbol{\beta} \in \left(W_{\boldsymbol{\gamma}',0} \setminus U^j_{\boldsymbol{\gamma}} \right)^{>\gamma_j}$.

Case 4: $\boldsymbol{\beta} \notin U^j_{\boldsymbol{\gamma}}$ and $\beta_j < \gamma_j < \gamma(\boldsymbol{\beta})_j$.
Notice that

$$W_{\boldsymbol{\gamma},0} = V \setminus \bigcup_{\widetilde{\boldsymbol{\gamma}} \in \Gamma} B_{\delta_1}(\widetilde{\boldsymbol{\gamma}}, \boldsymbol{\gamma}). \tag{2.6.10}$$

Now that $V \setminus U^j_{\boldsymbol{\gamma}}$ has two connected components. On the one hand, $\gamma_j < \gamma(\boldsymbol{\beta})_j$ and thus, $B_{\delta_1}(\boldsymbol{\gamma}(\boldsymbol{\beta}), \boldsymbol{\gamma})$ lies in the right component. On the other hand, $\beta_j < \gamma_j$ and thus, $\boldsymbol{\beta}$ lies in the left component. Hence $\boldsymbol{\beta} \notin B_{\delta_1}(\boldsymbol{\gamma}(\boldsymbol{\beta}), \boldsymbol{\gamma})$. Unpacking the definition of $B_{\delta_1}(\boldsymbol{\gamma}(\boldsymbol{\beta}), \boldsymbol{\gamma})$ and $W_{\boldsymbol{\gamma},0}$, we obtain $\boldsymbol{\beta} \in W_{\boldsymbol{\gamma},0}$. Together with $\boldsymbol{\beta} \notin W_{\boldsymbol{\gamma},t}$, we conclude $\boldsymbol{\beta} \in B_t(\boldsymbol{\gamma})$. $\square$

We finish this section with the proof of statement *(2)* in Proposition 2.2.5.

Proof of Proposition 2.2.5 (2). By Lemma 2.6.4,

$$W_{\boldsymbol{\gamma}',0} \setminus W_{\boldsymbol{\gamma},\delta_1 t} \subseteq B_{\delta_1 t}(\boldsymbol{\gamma}) \cup \left(W_{\boldsymbol{\gamma}',0} \setminus U^j_{\boldsymbol{\gamma}} \right)^{>\gamma_j}. \tag{2.6.11}$$

Suppose $\boldsymbol{\beta}' \in B_{\delta_1 t}(\boldsymbol{\gamma})$, we have the following estimate

$$|\beta'_j - \gamma_j| \leq |\boldsymbol{\beta}' - \boldsymbol{\gamma}| < \delta_1 t \leq \delta_1 |\boldsymbol{\beta} - \boldsymbol{\gamma}|.$$

With the assumption that $\boldsymbol{\beta} \in W_{\boldsymbol{\gamma},t} \setminus U^j_{\boldsymbol{\gamma}}$, we obtain

$$|\beta_j - \beta'_j| \geq |\beta_j - \gamma_j| - |\beta'_j - \gamma_j| \geq (\delta_2 - \delta_1) |\boldsymbol{\beta} - \boldsymbol{\gamma}|. \tag{2.6.12}$$

We then split the previous term and further estimate (2.6.12) by

$$\begin{aligned} &(\delta_2 - \delta_1) \frac{1}{1+\delta_1} |\boldsymbol{\beta} - \boldsymbol{\gamma}| + (\delta_2 - \delta_1) \frac{\delta_1}{1+\delta_1} |\boldsymbol{\beta} - \boldsymbol{\gamma}| \\ \geq\ & \frac{\delta_2 - \delta_1}{1+\delta_1} \left(|\boldsymbol{\beta} - \boldsymbol{\gamma}| + |\boldsymbol{\beta}' - \boldsymbol{\gamma}| \right) \geq \rho \left(d_\Gamma(\boldsymbol{\beta}) + d_\Gamma(\boldsymbol{\beta}') \right). \end{aligned}$$

Suppose $\boldsymbol{\beta}' \in \left(W_{\boldsymbol{\gamma}',0} \setminus U^j_{\boldsymbol{\gamma}} \right)^{>\gamma_j}$, then $\beta_j < \gamma_j < \beta'_j$. Hence

$$|\beta'_j - \beta_j| = |\beta'_j - \gamma_j| + |\gamma_j - \beta_j| \geq \delta_2 (|\boldsymbol{\beta}' - \boldsymbol{\gamma}| + |\boldsymbol{\beta} - \boldsymbol{\gamma}|) \geq \rho \left(d_\Gamma(\boldsymbol{\beta}') + d_\Gamma(\boldsymbol{\beta}) \right).$$

We complete the proof of Proposition 2.2.5. $\square$

2.7 Proof of Proposition 2.2.6: Selection Algorithm, L^∞ Component

Let $t_0 := \operatorname{dist}(P_V\Omega, \Gamma) = \inf_{\boldsymbol{\beta}\in P_V\Omega} d_\Gamma(\boldsymbol{\beta})$. By compactness, there's a positive distance between $P_V\Omega$ and Γ and therefore $t_0 > 0$. For $\boldsymbol{\beta} \in V$, $j \in \{1,2,3\}$, we write the projection map $P_j\boldsymbol{\beta} := P_{\mathbb{R}\cdot \boldsymbol{e}_j}\boldsymbol{\beta} = \beta_j$ for simplicity. Let $\mathcal{T}_0 = \varnothing$. Suppose we have $\mathcal{T}_k$ for $0 \le k \le k_0$ we define

$$\mathbf{P}_{k_0} := \left(\Omega \cap |F_j f|^{-1}(\lambda, 2\lambda] \cap \left\{(\alpha,\boldsymbol{\beta}) \in \mathbb{R}\times V\setminus\Gamma : 2^k t_0 \le d_\Gamma(\boldsymbol{\beta}) < 2^{k+1}t_0\right\}\right)$$

$$\setminus\Big(\bigcup_{k=0}^{k_0}\bigcup_{T\in\mathcal{T}_k} D_T\Big).$$

For next iteration, suppose we have $\mathbf{P}_k$ and $\mathcal{T}_k$ for $0\le k\le k_0$, we construct $\mathcal{T}_{k_0+1}$ through the following process. For $(\alpha,\boldsymbol{\beta})\in\mathbb{R}\times V$, $t>0$ define rectangles

$$R_{\alpha,\boldsymbol{\beta},t} := \left(\alpha + \frac{c_s}{t}\left[-\frac{1}{2},\frac{1}{2}\right]\right)\times\left(\gamma(\boldsymbol{\beta})_j + c_f t\left[-\frac{1}{2},\frac{1}{2}\right]\right). \tag{2.7.1}$$

where c_s and c_f are two constants to be determined later. Let $\mathbf{P}^*_{k_0}$ be a finite subset of $\mathbf{P}_{k_0}$ such that for distinct point $(\alpha,\boldsymbol{\beta}),(\alpha',\boldsymbol{\beta}')\in\mathbf{P}^*_{k_0}$

$$R_{\alpha,\boldsymbol{\beta},2^{k_0}t_0}\cap R_{\alpha',\boldsymbol{\beta}',2^{k_0}t_0} = \varnothing. \tag{2.7.2}$$

and maximal in the sense that for any $(\alpha,\boldsymbol{\beta})\in\mathbf{P}_{k_0}$, there exists a $(\alpha',\boldsymbol{\beta}')\in\mathbf{P}^*_{k_0}$ such that

$$R_{\alpha,\boldsymbol{\beta},2^{k_0}t_0}\cap R_{\alpha',\boldsymbol{\beta}',2^{k_0}t_0} \ne \varnothing. \tag{2.7.3}$$

The existence of such finite set $\mathbf{P}^*_{k_0}$ is guaranteed by the compactness of Ω and a greedy algorithm. Given $\boldsymbol{\beta}\in V\setminus\Gamma$, and for $i\in\mathbb{Z}$, $-M\le i\le M$ with M being the least integer greater than $\frac{3c_f}{2\delta_0(1-\delta_1)}$, we define

$$\Gamma^i_{\boldsymbol{\beta}} := \Gamma\cap P_j^{-1}\left(\gamma(\boldsymbol{\beta})_j + \delta_0(1-\delta_1)2^{k_0}t_0[i-1,i]\right). \tag{2.7.4}$$

If $\Gamma^i_{\boldsymbol{\beta}}\ne\varnothing$, by closedness of Γ^i, there exists $\boldsymbol{\gamma}^{(i)}_-(\boldsymbol{\beta}),\boldsymbol{\gamma}^{(i)}_+(\boldsymbol{\beta})\in\Gamma^i_{\boldsymbol{\beta}}$ such that $P_j(\Gamma^i_{\boldsymbol{\beta}})\subseteq[\gamma^{(i)}_-(\boldsymbol{\beta})_j,\gamma^{(i)}_+(\boldsymbol{\beta})_j]$. Define

$$\mathcal{T}_{k_0+1} := \bigcup_{i=-M}^{M}\left\{\left(\alpha + \frac{c}{2^{k_0}t_0}\left[-\frac{1}{2},\frac{1}{2}\right],\boldsymbol{\gamma}^{(i)}_\pm(\boldsymbol{\beta})\right) : (\alpha,\boldsymbol{\beta})\in P^*_{k_0}\right\}, \tag{2.7.5}$$

where $c = (3c_s\vee\frac{1}{\delta_1})$. We will show that $\mathbf{P} = \bigcup_{k\ge0}\mathbf{P}^*_k$ and $\mathcal{T} = \bigcup_{k\ge0}\mathcal{T}_k$ satisfy the desired properties in Proposition 2.2.6. To show the covering property (2.2.16), we first recall that by construction $\mathbf{P}^*_k\subset\Omega\cap|F_jf|^{-1}(\lambda,2\lambda]$, and thus $\mathbf{P}\subset\Omega\cap|F_jf|^{-1}(\lambda,2\lambda]$. On the other hand, for $(\alpha,\boldsymbol{\beta})\in\Omega\cap|F_jf|^{-1}(\lambda,2\lambda]$, there is k_0 such that $2^{k_0}t_0\le d_\Gamma(\boldsymbol{\beta}) < 2^{k_0+1}t_0$. By construction, either

$$(\alpha,\boldsymbol{\beta})\in\bigcup_{k=0}^{k_0}\bigcup_{T\in\mathcal{T}_k} D_T\subset\bigcup_{(I,\gamma)\in\mathcal{T}} D_T$$

and the covering property (2.2.16) is verified, or the alternative $(\alpha, \boldsymbol{\beta}) \in \mathbf{P}_{k_0}$ happens. In the case where $(\alpha, \boldsymbol{\beta}) \in \mathbf{P}_{k_0}$, the maximality of $\mathbf{P}^*_{k_0}$ guarantees the existence of a point $(\alpha', \boldsymbol{\beta}') \in \mathbf{P}^*_{k_0}$ such that

$$R_{\alpha,\boldsymbol{\beta},2^{k_0}t_0} \cap R_{\alpha',\boldsymbol{\beta}',2^{k_0}t_0} \neq \varnothing.$$

As a direct consequence

$$\left(\alpha, \gamma\left(\boldsymbol{\beta}\right)_j\right) \in \left(\alpha' + \frac{3c_s}{2^{k_0}t_0}\left[-\frac{1}{2}, \frac{1}{2}\right]\right) \times \left(\gamma\left(\boldsymbol{\beta}'\right)_j + 3c_f 2^{k_0} t_0 \left[-\frac{1}{2}, \frac{1}{2}\right]\right).$$

We now recall that $M \geq \frac{3c_f}{2\delta_0(1-\delta_1)}$. This implies that $\boldsymbol{\gamma}\left(\boldsymbol{\beta}\right) \in \Gamma^i_{\boldsymbol{\beta}'}$ for some $i \in [-M, M]$. In particular, $\gamma_-^{(i)}\left(\boldsymbol{\beta}\right)_j \leq \gamma\left(\boldsymbol{\beta}\right)_j \leq \gamma_+^{(i)}\left(\boldsymbol{\beta}\right)_j \leq \gamma_-^{(i)}\left(\boldsymbol{\beta}\right)_j + \delta_0\left(1-\delta_1\right)2^{k_0}t_0$ for the same i. Using statement *(1)* in Proposition 2.2.5 and the fact that $c \geq \frac{1}{\delta_1}$, we obtain

$$(\alpha, \boldsymbol{\beta}) \in \bigcup_{(I,\boldsymbol{\gamma}) \in \mathcal{T}_{k_0+1}} D_T \subset \bigcup_{(I,\boldsymbol{\gamma}) \in \mathcal{T}} D_T.$$

The estimate (2.2.17) holds directly by the construction of $\mathbf{P}$ and $\mathcal{T}$. We now verify the orthogonality property for $\mathbf{P}$. For $(\alpha, \boldsymbol{\beta}) \in P^*_k$ and $(\alpha', \boldsymbol{\beta}') \in \mathbf{P}^*_{k'}$, we split the argument into two cases.

case 1: $k = k'$. By (2.7.2), we have either

$$|\alpha - \alpha'| \geq c_s(2^k t_0)^{-1} \tag{2.7.6}$$

or

$$|\gamma(\boldsymbol{\beta})_j - \gamma(\boldsymbol{\beta}')_j| \geq c_f 2^k t_0. \tag{2.7.7}$$

In case (2.7.6), taking $c_s = 4$ gives

$$|\alpha - \alpha'| \geq \frac{c_s}{2}(d_\Gamma(\boldsymbol{\beta})^{-1} + d_\Gamma(\boldsymbol{\beta}')^{-1}) \geq 2(d_\Gamma(\boldsymbol{\beta})^{-1} + d_\Gamma(\boldsymbol{\beta}')^{-1}).$$

In case (2.7.7), taking $c_f = \frac{11}{\delta_2}$ gives $c_f \geq 8 \geq 4(\rho+1)$, and thus

$$|\beta_j - \beta_j'| \geq |\gamma(\boldsymbol{\beta})_j - \gamma(\boldsymbol{\beta}')_j| - |\beta_j - \gamma(\boldsymbol{\beta})_j| - |\beta_j' - \gamma(\boldsymbol{\beta}')_j|$$

$$\geq c_f 2^k t_0 - d_\Gamma(\boldsymbol{\beta}) - d_\Gamma(\boldsymbol{\beta}') \geq (\frac{c_f}{4} - 1)(d_\Gamma(\boldsymbol{\beta}) + d_\Gamma(\boldsymbol{\beta}')) \geq \rho(d_\Gamma(\boldsymbol{\beta}) + d_\Gamma(\boldsymbol{\beta}')).$$

case 2: $k < k'$. Either

$$\alpha' \notin \alpha + \frac{c}{2^k t_0}\left[-\frac{1}{2}, \frac{1}{2}\right], \tag{2.7.8}$$

or

$$\alpha' \in \alpha + \frac{c}{2^k t_0}\left[-\frac{1}{2}, \frac{1}{2}\right]. \tag{2.7.9}$$

In the case (2.7.8), since $c \geq 3c_s \geq 12$,

$$|\alpha - \alpha'| \geq \frac{1}{2} \cdot \frac{c}{2^k t_0} \geq \frac{c}{4}(\frac{1}{2^k t_0} + \frac{1}{2^{k'} t_0}) \geq 2(d_\Gamma(\boldsymbol{\beta})^{-1} + d_\Gamma(\boldsymbol{\beta}')^{-1}).$$

In the case (2.7.9), notice that

$$(\alpha', \boldsymbol{\beta}') \notin \Big(\bigcup_{k=0}^{k'} \bigcup_{T \in \mathcal{T}_k} D_T \Big).$$

In particular, since $k < k'$,

$$(\alpha', \boldsymbol{\beta}') \notin \bigcup_{i=-M}^{M} \Big(\alpha + \frac{c}{2^k t_0}[-\frac{1}{2}, \frac{1}{2}] \Big) \times W_{\boldsymbol{\gamma}_\pm^{(i)}(\boldsymbol{\beta}), \frac{2^k t_0}{c}}.$$

Together with (2.7.9), and by $c \geq \frac{1}{\delta_1}$ and Proposition 2.2.5, we obtain

$$\boldsymbol{\beta}' \notin \bigcup_{\substack{\boldsymbol{\gamma} \in \Gamma \\ |\gamma_j - \gamma(\boldsymbol{\beta})_j| \leq \frac{3}{2} c_f 2^k t_0}} W_{\boldsymbol{\gamma}, 2^k t_0}. \tag{2.7.10}$$

On the other hand,

$$\boldsymbol{\beta}' \in W_{\boldsymbol{\gamma}(\boldsymbol{\beta}'), 2^{k'} t_0} \subseteq W_{\boldsymbol{\gamma}(\boldsymbol{\beta}'), 2^k t_0}.$$

Together with (2.7.10), we have

$$|\boldsymbol{\gamma}(\boldsymbol{\beta}') - \boldsymbol{\gamma}(\boldsymbol{\beta})| \geq |\gamma(\boldsymbol{\beta}')_j - \gamma(\boldsymbol{\beta})_j| \geq \frac{3}{2} c_f 2^k t_0. \tag{2.7.11}$$

Note that $\boldsymbol{\beta}' \notin W_{\boldsymbol{\gamma}(\boldsymbol{\beta}), 2^k t_0}$ and

$$|\boldsymbol{\beta}' - \boldsymbol{\gamma}(\boldsymbol{\beta})| \geq d_\Gamma(\boldsymbol{\beta}') \geq d_\Gamma(\boldsymbol{\beta}) \geq 2^k t_0,$$

we have $\boldsymbol{\beta}' \notin W_{\boldsymbol{\gamma}(\boldsymbol{\beta}), 0}$. That is,

$$\delta_1 |\boldsymbol{\beta}' - \boldsymbol{\gamma}(\boldsymbol{\beta})| \geq d_\Gamma(\boldsymbol{\beta}') = |\boldsymbol{\beta}' - \boldsymbol{\gamma}(\boldsymbol{\beta}')|. \tag{2.7.12}$$

Combining (2.7.11) and (2.7.12),

$$\frac{3}{2} c_f 2^k t_0 \leq |\boldsymbol{\gamma}(\boldsymbol{\beta}') - \boldsymbol{\gamma}(\boldsymbol{\beta})| \leq |\boldsymbol{\beta}' - \boldsymbol{\gamma}(\boldsymbol{\beta})| + |\boldsymbol{\beta}' - \boldsymbol{\gamma}(\boldsymbol{\beta}')| \leq (1 + \delta_1) |\boldsymbol{\beta}' - \boldsymbol{\gamma}(\boldsymbol{\beta})|. \tag{2.7.13}$$

Since $\boldsymbol{\beta}' \notin W_{\boldsymbol{\gamma}(\boldsymbol{\beta}), 0}$, by Lemma 2.6.2, we also have $\boldsymbol{\beta}' \notin U^j_{\boldsymbol{\gamma}(\boldsymbol{\beta})}$. Therefore,

$$|\beta_j^{'} - \beta_j| \geq |\beta_j^{'} - \gamma(\boldsymbol{\beta})_j| - |\beta_j - \gamma(\boldsymbol{\beta})_j| \geq \delta_2 |\boldsymbol{\beta}' - \boldsymbol{\gamma}(\boldsymbol{\beta})| - 2^{k+1} t_0. \tag{2.7.14}$$

Together with (2.7.13) and the trivial estimate $|\boldsymbol{\beta}' - \boldsymbol{\gamma}(\boldsymbol{\beta})| \geq d_\Gamma(\boldsymbol{\beta}')$, (2.7.14) can be estimated from below by

$$\frac{\delta_2}{2} \frac{3 c_f}{2(1 + \delta_1)} 2^k t_0 + \frac{\delta_2}{2} d_\Gamma(\boldsymbol{\beta}') - 2^{k+1} t_0$$

$$\geq \Big(\frac{3 \delta_2 c_f}{8(1 + \delta_1)} - 1 \Big) d_\Gamma(\boldsymbol{\beta}) + \frac{\delta_2}{2} d_\Gamma(\boldsymbol{\beta}') \geq \rho(d_\Gamma(\boldsymbol{\beta}) + d_\Gamma(\boldsymbol{\beta}')).$$

This completes the proof of Proposition 2.2.6.

2.8 Proof of Proposition 2.2.7: Selection Algorithm, L^2 Component

Since Ω is compact, we may assume $\Omega \subseteq \mathbb{R} \times P_j^{-1}([-A, A])$. We set up an iteration algorithm. Let $\Omega_0 := \Omega$ and $t_0 = \frac{\lambda^2}{2C^2\|f\|_{L^2}^2}$, where $C = C(\theta_0)$ is the constant that realizes the estimate (2.5.6)

$$\|F_j f\|_{L^2_\nu\left(\mathbb{R}\times\left(W_{\boldsymbol{\gamma},0}\setminus U^j_{\boldsymbol{\gamma}}\right)\right)} \leq C\,\|f\|_{L^2}\,. \tag{2.8.1}$$

Suppose now Ω_{k-1} is given, we define a collection of intervals $\mathcal{I}_k$ for integer k in the range $0 \leq k \leq \frac{2A}{\delta_0(1-\delta_1)t_0} + 1$. The collection $\mathcal{I}_k$ consists of interval I with the following properties: there is a point $\boldsymbol{\gamma}$ in the strip

$$\Gamma^k := \Gamma \cap P_j^{-1}\Big(-A + \delta_0(1-\delta_1)t_0 \cdot [k-1, k]\Big) \subseteq V \tag{2.8.2}$$

such that

$$|I|^{-\frac{1}{2}}\|1_{\Omega_{k-1}} F_j f\|_{L^2_\nu\left(I\times(W_{\boldsymbol{\gamma},1/|I|}\setminus U^j_{\boldsymbol{\gamma}})^{<j}\right)} \geq \frac{\lambda}{\sqrt{2}}. \tag{2.8.3}$$

The relation (2.8.3) and (2.5.6) imply the following estimate

$$\frac{\lambda}{\sqrt{2}} \leq |I|^{-\frac{1}{2}}\,\|F_j f\|_{L^2_\nu\left(\mathbb{R}\times\left(W_{\boldsymbol{\gamma},0}\setminus U^j_{\boldsymbol{\gamma}}\right)\right)} \leq C\,|I|^{-\frac{1}{2}}\,\|f\|_{L^2}\,. \tag{2.8.4}$$

By the definition of t_0, $\|f\|_{L^2} = \frac{\lambda}{C\sqrt{2t_0}}$, and thus (2.8.4) is equivalent to $|I| \leq 1/t_0$. Once we verify that all intervals I in $\mathcal{I}_k$ have their diameter bounded by a fixed number $1/t_0$, we apply Vitali covering lemma on $5\mathcal{I}_k := \{5I : I \in \mathcal{I}_k\}$. As a result, there is a subcollection $\mathbf{J}_k \subseteq \mathcal{I}_k$ such that for all distinct $I, J \in \mathbf{J}_k$,

$$5I \cap 5J = \varnothing$$

and

$$\bigcup \mathcal{I}_k \subseteq \bigcup 5\mathcal{I}_k \subseteq \bigcup 25\mathbf{J}_k.$$

For $I \in \mathbf{J}_k$, let $\boldsymbol{\gamma}_I \in \Gamma^k$ be the point γ that realizes (2.8.3) and $\boldsymbol{\gamma}^k_+, \boldsymbol{\gamma}^k_- \in \Gamma^k$ be the two endpoints such that $P_j\left(\Gamma^k\right) \subseteq \left[\gamma^k_{-,j}, \gamma^k_{+,j}\right]$. Define

$$\mathbf{S}_k := \left\{ \left(I, \boldsymbol{\gamma}_I, \Omega_{k-1} \cap I \times \left(W_{\boldsymbol{\gamma},\frac{1}{|I|}} \setminus U^j_{\boldsymbol{\gamma}}\right)^{<j}\right) \;:\; I \in \mathbf{J}_k \right\} \tag{2.8.5}$$

and

$$\mathcal{T}_k := \left\{ \left(\frac{25}{\delta_1} I, \boldsymbol{\gamma}^k_\pm \right) \;:\; I \in \mathbf{J}_k \right\} \cup \left\{ \left(\frac{1}{\delta_1} I, \boldsymbol{\gamma}_I \right) : I \in \mathbf{J}_k \right\}. \tag{2.8.6}$$

For the next iteration, we set

$$\Omega_k := \Omega_{k-1} \setminus \left(\bigcup_{i=1}^{k} \left(\bigcup_{T \in \mathcal{T}_i} D_T \right) \right). \tag{2.8.7}$$

Eventually, we obtain

$$\mathbf{S} := \bigcup_{k=1}^{\infty} \mathbf{S}_k, \quad \mathcal{T} := \bigcup_{k=1}^{\infty} \mathcal{T}_k. \tag{2.8.8}$$

We will show that $\mathbf{S}$ and $\mathcal{T}$ satisfy the desired properties in Proposition 2.2.7. To show the covering property (2.2.22), we assume the alternative that there is a tent $(I, \boldsymbol{\gamma})$ that violates (2.2.22):

$$|I|^{-\frac{1}{2}} \left\| 1_{\Omega \setminus \bigcup_{T \in \mathcal{T}} D_T} F_j f \right\|_{L^2_\nu\left(I \times (W_{\boldsymbol{\gamma}, 1/|I|} \setminus U^j_{\boldsymbol{\gamma}})^{<j}\right)} > \frac{\lambda}{\sqrt{2}}. \tag{2.8.9}$$

By construction, there is a k such that $I \in \mathcal{I}_k$ and $\boldsymbol{\gamma} \in \Gamma^k$ realizes (2.8.3). As a result,

$$\gamma^k_{-,j} \leq \gamma_j \leq \gamma^k_{+,j} \leq \gamma^k_{-,j} + \delta_0 \left(1 - \delta_1\right) t_0 \leq \gamma^k_{-,j} + \delta_0 \left(1 - \delta_1\right) \frac{1}{|I|}.$$

By statement *(1)* in Proposition 2.2.5 and the construction of $\mathbf{J}_k$, there exist a $J \in \mathbf{J}_k$ such that

$$I \subseteq 5I \subseteq 25J \subseteq \frac{25}{\delta_1} J$$

and

$$W_{\boldsymbol{\gamma}, \frac{1}{|I|}} \subseteq W_{\boldsymbol{\gamma}^k_-, \frac{\delta_1}{|I|}} \cup W_{\boldsymbol{\gamma}^k_+, \frac{\delta_1}{|I|}} \subseteq W_{\boldsymbol{\gamma}^k_-, \frac{\delta_1}{|25J|}} \cup W_{\boldsymbol{\gamma}^k_+, \frac{\delta_1}{|25J|}}.$$

That is,

$$I \times (W_{\boldsymbol{\gamma}, 1/|I|} \setminus U^j_{\boldsymbol{\gamma}})^{<j} \subseteq D_{(I, \boldsymbol{\gamma})} \subseteq \bigcup_{T \in \mathcal{T}_k} D_T \subseteq \bigcup_{T \in \mathcal{T}} D_T.$$

This is a contradiction, and thus (2.2.22) must hold. The estimate (2.2.23) follows directly from the construction of $\mathbf{S}$ and $\mathcal{T}$. In the following, we check $\mathbf{S}$ satisfies the orthogonal property. Given $(I, \boldsymbol{\gamma}, S) \in \mathbf{S}_k, (\alpha, \boldsymbol{\beta}) \in S$ and $(I', \boldsymbol{\gamma}', S') \in \mathbf{S}_{k'}, (\alpha', \boldsymbol{\beta}') \in S'$, without loss of generality, we may assume $\boldsymbol{\gamma}_j \leq \boldsymbol{\gamma}_{j'}$. We split into two cases according to whether they are in the same strip.

case 1: $k = k'$. By construction, we have $S \subseteq I \times V \setminus \Gamma$, $S' \subseteq I' \times V \setminus \Gamma$, and $5I \cap 5I' = \varnothing$. Hence

$$|\alpha - \alpha'| \geq \frac{5-1}{2}(|I| + |I'|) \geq 2|I|.$$

case 2: $k < k'$. By the construction of $S_{k'}$,

$$S' \subseteq I' \times \left(W_{\boldsymbol{\gamma}', \frac{1}{|I'|}} \setminus U^j_{\boldsymbol{\gamma}'} \right)^{<j} \cap \Omega_{k'-1} \subseteq I' \times \left(W_{\boldsymbol{\gamma}', \frac{1}{|I'|}} \setminus U^j_{\boldsymbol{\gamma}'} \right)^{<j} \setminus \left(\frac{1}{\delta_1} I \times W_{\boldsymbol{\gamma}, \frac{\delta_1}{|I|}} \right). \tag{2.8.10}$$

We either have

$$|\alpha - \alpha'| \geq \frac{1/\delta_1 - 1}{2} |I| \geq 2\,|I|\,,$$

and the orthogonality property (2.2.24) is verified, or the alternative

$$|\alpha - \alpha'| < \frac{1/\delta_1 - 1}{2} |I|.$$

Since $\alpha \in I$, we have $\alpha' \in \frac{1}{\delta_1} I$. Then by (2.8.10), we have

$$\boldsymbol{\beta}' \in \left(W_{\boldsymbol{\gamma}', \frac{1}{|I'|}} \setminus U^j_{\boldsymbol{\gamma}'} \right)^{<j} \setminus W_{\boldsymbol{\gamma}, \frac{\delta_1}{|I|}}.$$

By applying the statement *(2)* of Proposition 2.2.5, we obtain (2.2.25). This completes the proof of Proposition 2.2.7.

2.9 Proof of Proposition 2.2.8: Bessel Type Estimate

Throughout the rest of the argument, we take ε to be the specific value $\frac{\delta_1 \rho^2}{2}$.

Lemma 2.9.1. *Let* $\mathbf{P}$ *and* f *be as in Proposition 2.2.6. We have*

$$\sum_{(\alpha,\boldsymbol{\beta}) \in \mathbf{P}} \frac{\left|F_j f(\alpha, \boldsymbol{\beta})\right|^2}{d_\Gamma(\boldsymbol{\beta})} \lesssim \|f\|^2_{L^2}.$$

Proof. Direct calculation and Cauchy-Schwarz yield

$$\sum_{(\alpha,\boldsymbol{\beta}) \in \mathbf{P}} \frac{\left|F_j f(\alpha, \boldsymbol{\beta})\right|^2}{d_\Gamma(\boldsymbol{\beta})} = \sum_{(\alpha,\boldsymbol{\beta}) \in \mathbf{P}} \frac{\left\langle f, \varphi^j_{\alpha,\boldsymbol{\beta}} \right\rangle \overline{F_j f(\alpha, \boldsymbol{\beta})}}{d_\Gamma(\boldsymbol{\beta})}$$

$$= \left\langle f, \sum_{(\alpha,\boldsymbol{\beta}) \in \mathbf{P}} \frac{F_j f(\alpha, \boldsymbol{\beta}) \varphi^j_{\alpha,\boldsymbol{\beta}}}{d_\Gamma(\boldsymbol{\beta})} \right\rangle \leq \|f\|_{L^2} \left\| \sum_{(\alpha,\boldsymbol{\beta}) \in \mathbf{P}} \frac{F_j f(\alpha, \boldsymbol{\beta}) \varphi^j_{\alpha,\boldsymbol{\beta}}}{d_\Gamma(\boldsymbol{\beta})} \right\|_{L^2}.$$

It suffices to show

$$\left\| \sum_{(\alpha,\boldsymbol{\beta}) \in \mathbf{P}} \frac{F_j f(\alpha, \boldsymbol{\beta}) \varphi^j_{\alpha,\boldsymbol{\beta}}}{d_\Gamma(\boldsymbol{\beta})} \right\|^2_{L^2} \lesssim \sum_{(\alpha,\boldsymbol{\beta}) \in \mathbf{P}} \frac{\left|F_j f(\alpha, \boldsymbol{\beta})\right|^2}{d_\Gamma(\boldsymbol{\beta})}.$$

We develop the L^2 norm and introduce asymmetry

$$\left\| \sum_{(\alpha,\boldsymbol{\beta}) \in \mathbf{P}} \frac{F_j f(\alpha, \boldsymbol{\beta}) \varphi^j_{\alpha,\boldsymbol{\beta}}}{d_\Gamma(\boldsymbol{\beta})} \right\|^2_{L^2} \leq \sum_{(\alpha,\boldsymbol{\beta}),(\alpha',\boldsymbol{\beta}') \in \mathbf{P}} \left|F_j f(\alpha, \boldsymbol{\beta})\right| \left|F_j f(\alpha', \boldsymbol{\beta}')\right| \frac{\left|\left\langle \varphi^j_{\alpha,\boldsymbol{\beta}}, \varphi^j_{\alpha',\boldsymbol{\beta}'} \right\rangle\right|}{d_\Gamma(\boldsymbol{\beta}) d_\Gamma(\boldsymbol{\beta}')}$$

$$\leq 2 \sum_{\substack{(\alpha,\boldsymbol{\beta}),(\alpha',\boldsymbol{\beta}') \in \mathbf{P} \\ d_\Gamma(\boldsymbol{\beta}) \leq d_\Gamma(\boldsymbol{\beta}')}} \left|F_j f(\alpha, \boldsymbol{\beta})\right| \left|F_j f(\alpha', \boldsymbol{\beta}')\right| \frac{\left|\left\langle \varphi^j_{\alpha,\boldsymbol{\beta}}, \varphi^j_{\alpha',\boldsymbol{\beta}'} \right\rangle\right|}{d_\Gamma(\boldsymbol{\beta}) d_\Gamma(\boldsymbol{\beta}')}.$$

We recall that $\mathbf{P} \subset \left|F_j f\right|^{-1}(\lambda, 2\lambda]$. In particular,

$$\forall (\alpha, \boldsymbol{\beta}), (\alpha', \boldsymbol{\beta}') \in \mathbf{P}, \quad \left|F_j f(\alpha', \boldsymbol{\beta}')\right| \sim \left|F_j f(\alpha, \boldsymbol{\beta})\right|.$$

Consequently, we may perform the following substitution

$$\left\| \sum_{(\alpha,\boldsymbol{\beta})\in\mathbf{P}} \frac{F_j f(\alpha,\boldsymbol{\beta})\,\varphi^j_{\alpha,\boldsymbol{\beta}}}{d_\Gamma(\boldsymbol{\beta})} \right\|^2_{L^2} \lesssim \sum_{(\alpha,\boldsymbol{\beta})\in\mathbf{P}} \frac{|F_j f(\alpha,\boldsymbol{\beta})|^2}{d_\Gamma(\boldsymbol{\beta})} \sum_{\substack{(\alpha',\boldsymbol{\beta}')\in\mathbf{P}\\ d_\Gamma(\boldsymbol{\beta})\le d_\Gamma(\boldsymbol{\beta}')}} \frac{\left|\left\langle \varphi^j_{\alpha,\boldsymbol{\beta}}, \varphi^j_{\alpha',\boldsymbol{\beta}'}\right\rangle\right|}{d_\Gamma(\boldsymbol{\beta}')}.$$

It remains to show, for $(\alpha,\boldsymbol{\beta})\in\mathbf{P}$,

$$\sum_{\substack{(\alpha',\boldsymbol{\beta}')\in\mathbf{P}\\ d_\Gamma(\boldsymbol{\beta})\le d_\Gamma(\boldsymbol{\beta}')}} \frac{\left|\left\langle \varphi^j_{\alpha,\boldsymbol{\beta}}, \varphi^j_{\alpha',\boldsymbol{\beta}'}\right\rangle\right|}{d_\Gamma(\boldsymbol{\beta}')} \lesssim 1. \tag{2.9.1}$$

We discard all terms with no contribution to the sum. Therefore, we may assume

$$\left\langle \varphi^j_{\alpha,\boldsymbol{\beta}}, \varphi^j_{\alpha',\boldsymbol{\beta}'}\right\rangle \neq 0.$$

This implies that the frequency supports of the two functions overlap. In other words,

$$\left|\beta_j - \beta'_j\right| \le \frac{4\varepsilon}{10}\left(d_\Gamma(\boldsymbol{\beta}) + d_\Gamma(\boldsymbol{\beta}')\right) \le \frac{8\varepsilon}{10} d_\Gamma(\boldsymbol{\beta}') < \rho d_\Gamma(\boldsymbol{\beta}').$$

As a result, for any other $(\alpha'',\boldsymbol{\beta}'')$ remained in the summand,

$$\left|\beta'_j - \beta''_j\right| \le \left|\beta_j - \beta'_j\right| + \left|\beta_j - \beta''_j\right| < \rho\left(d_\Gamma(\boldsymbol{\beta}') + d_\Gamma(\boldsymbol{\beta}'')\right).$$

This violates (2.2.19). Thus, for distinct $(\alpha',\boldsymbol{\beta}'),(\alpha'',\boldsymbol{\beta}'')$ remained in the summand, they must satisfy (2.2.18)

$$\left|\alpha' - \alpha''\right| \ge 2\left(d_\Gamma(\boldsymbol{\beta}')^{-1} + d_\Gamma(\boldsymbol{\beta}'')^{-1}\right).$$

Finally, we apply the standard wave-packet estimate as in *Lemma 2.1* of [93] and utilize the physical separation to complete the proof

$$\begin{aligned}
\sum_{\substack{(\alpha',\boldsymbol{\beta}')\in\mathbf{P}\\ d_\Gamma(\boldsymbol{\beta})\le d_\Gamma(\boldsymbol{\beta}')}} \frac{\left|\left\langle \varphi^j_{\alpha,\boldsymbol{\beta}}, \varphi^j_{\alpha',\boldsymbol{\beta}'}\right\rangle\right|}{d_\Gamma(\boldsymbol{\beta}')} &\lesssim \sum_{\substack{(\alpha',\boldsymbol{\beta}')\in\mathbf{P}\\ d_\Gamma(\boldsymbol{\beta})\le d_\Gamma(\boldsymbol{\beta}')}} \frac{d_\Gamma(\boldsymbol{\beta})}{d_\Gamma(\boldsymbol{\beta}')}\left(1 + d_\Gamma(\boldsymbol{\beta})\left|\alpha-\alpha'\right|\right)^{-N} \\
&\sim 1 + \sum_{\substack{(\alpha',\boldsymbol{\beta}')\in\mathbf{P}\setminus\{(\alpha,\boldsymbol{\beta})\}\\ d_\Gamma(\boldsymbol{\beta})\le d_\Gamma(\boldsymbol{\beta}')}} \int_{\alpha' + \frac{[-1,1]}{d_\Gamma(\boldsymbol{\beta}')}} d_\Gamma(\boldsymbol{\beta})\left(1 + d_\Gamma(\boldsymbol{\beta})|\alpha - x|\right)^{-N} dx \\
&\lesssim 1 + \int_{\mathbb{R}} d_\Gamma(\boldsymbol{\beta})\left(1 + d_\Gamma(\boldsymbol{\beta})|\alpha - x|\right)^{-N} dx = 1 + \int_{\mathbb{R}} (1+|x|)^{-N} dx \sim 1.
\end{aligned}$$

□

Lemma 2.9.2. *We define a constant*

$$c := \frac{\frac{4\varepsilon}{10} + \frac{1}{\delta_1}}{\delta_2 - \frac{4\varepsilon}{10}} < \frac{10\rho}{4\epsilon}. \tag{2.9.2}$$

Let $\boldsymbol{\gamma}, \boldsymbol{\gamma}' \in \Gamma$ satisfy $\gamma_j \leq \gamma_j'$. For $\boldsymbol{\beta} \in \left(W_{\boldsymbol{\gamma},0} \setminus U_{\boldsymbol{\gamma}}^j\right)^{<j}$ and $\boldsymbol{\beta}' \in W_{\boldsymbol{\gamma}',0}$, suppose

$$\left|\beta_j - \beta_j'\right| \leq \frac{4\varepsilon}{10}\left(d_\Gamma(\boldsymbol{\beta}) + d_\Gamma(\boldsymbol{\beta}')\right), \tag{2.9.3}$$

then

$$d_\Gamma(\boldsymbol{\beta}) \leq c d_\Gamma(\boldsymbol{\beta}'). \tag{2.9.4}$$

If additionally that $\boldsymbol{\beta}' \notin U_{\boldsymbol{\gamma}'}^j$,

$$\left(\delta_2 - \frac{4\varepsilon}{10}(1+c)\right) d_\Gamma(\boldsymbol{\beta}') \leq \left|\beta_j - \gamma_j'\right| \leq \left(\frac{1}{\delta_1} + \frac{4\varepsilon}{10}(1+c)\right) d_\Gamma(\boldsymbol{\beta}'). \tag{2.9.5}$$

Proof. We first show (2.9.4) via estimating $\left|\beta_j - \gamma_j'\right|$ from above and from below. On the one hand, the triangle inequality gives

$$\left|\beta_j - \gamma_j'\right| \leq \left|\beta_j - \beta_j'\right| + \left|\beta_j' - \gamma_j'\right|. \tag{2.9.6}$$

The point $\boldsymbol{\beta}' \in W_{\boldsymbol{\gamma}',0}$ satisfies

$$\left|\beta_j' - \gamma_j'\right| \leq \left|\boldsymbol{\beta}' - \boldsymbol{\gamma}'\right| \leq \frac{1}{\delta_1} d_\Gamma(\boldsymbol{\beta}'), \tag{2.9.7}$$

which controls the second term on the right-hand side of (2.9.6). We control the first term on the right-hand side of (2.9.6)

$$\left|\beta_j - \beta_j'\right| \leq \frac{4\varepsilon}{10}\left(d_\Gamma(\boldsymbol{\beta}) + d_\Gamma(\boldsymbol{\beta}')\right).$$

Hence, we obtain the following

$$\left|\beta_j - \gamma_j'\right| \leq \frac{4\varepsilon}{10} d_\Gamma(\boldsymbol{\beta}) + \left(\frac{4\varepsilon}{10} + \frac{1}{\delta_1}\right) d_\Gamma(\boldsymbol{\beta}'). \tag{2.9.8}$$

On the other hand, by assumption, the three points $\boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\gamma}'$ satisfy the ordering relation $\beta_j \leq \gamma_j \leq \gamma_j'$, and thus,

$$\left|\beta_j - \gamma_j'\right| = \left|\beta_j - \gamma_j\right| + \left|\gamma_j - \gamma_j'\right| \geq \left|\beta_j - \gamma_j\right|. \tag{2.9.9}$$

Since $\boldsymbol{\beta} \in \left(W_{\boldsymbol{\gamma},0} \setminus U_{\boldsymbol{\gamma}}^j\right)^{<j} \subseteq \left(U_{\boldsymbol{\gamma}}^j\right)^c$, the above quantity can be estimated from below by

$$\delta_2 \left|\boldsymbol{\beta} - \boldsymbol{\gamma}\right| \geq \delta_2 d_\Gamma(\boldsymbol{\beta}). \tag{2.9.10}$$

Combining (2.9.8), (2.9.9), and (2.9.10), we obtain

$$\delta_2 d_\Gamma(\boldsymbol{\beta}) \leq \left|\beta_j - \gamma_j'\right| \leq \frac{4\varepsilon}{10} d_\Gamma(\boldsymbol{\beta}) + \left(\frac{4\varepsilon}{10} + \frac{1}{\delta_1}\right) d_\Gamma(\boldsymbol{\beta}').$$

This completes the proof for (2.9.4). To show (2.9.5), we further assume $\boldsymbol{\beta}' \notin U_{\gamma'}^j$ and obtain a lower bound similar to (2.9.7)

$$\delta_2 d_\Gamma(\boldsymbol{\beta}') \leq \delta_2 \left|\boldsymbol{\beta}' - \boldsymbol{\gamma}'\right| \leq \left|\beta_j' - \gamma_j'\right|.$$

By triangle inequality

$$\left|\beta_j' - \gamma_j'\right| - \left|\beta_j - \beta_j'\right| \leq \left|\beta_j - \gamma_j'\right| \leq \left|\beta_j' - \gamma_j'\right| + \left|\beta_j - \beta_j'\right|.$$

Hence, we obtain

$$\delta_2 d_\Gamma\left(\boldsymbol{\beta}'\right) - \left|\beta_j - \beta_j'\right| \leq \left|\beta_j - \gamma_j'\right| \leq \frac{1}{\delta_1} d_\Gamma\left(\boldsymbol{\beta}'\right) + \left|\beta_j - \beta_j'\right|.$$

It remains to control $\left|\beta_j - \beta_j'\right|$ from above with a small constant multiple of $d_\Gamma\left(\boldsymbol{\beta}'\right)$, which can be achieved via utilizing the consequence of (2.9.4)

$$\left|\beta_j - \beta_j'\right| \leq \frac{4\varepsilon}{10}\left(d_\Gamma\left(\boldsymbol{\beta}\right) + d_\Gamma\left(\boldsymbol{\beta}'\right)\right) \leq \frac{4\varepsilon}{10} \cdot (1+c)\, d_\Gamma\left(\boldsymbol{\beta}'\right).$$

□

Lemma 2.9.3. *Let $\Omega, \lambda, \mathbf{S}$ and f be as in the Proposition 2.2.7. Assume $1_\Omega \left|F_j f\right| \leq \lambda$, then we have the following bound*

$$\sum_{(I,\boldsymbol{\gamma},S)\in\mathbf{S}} \|F_j f\|^2_{L^2_\nu(S)} \lesssim \|f\|^2_{L^2}. \tag{2.9.11}$$

Proof. The following argument is parallel to the first parts of the proof of Lemma 2.9.1

$$\begin{aligned}\sum_{(I,\boldsymbol{\gamma},S)\in\mathbf{S}} \|F_j f\|^2_{L^2_\nu(S)} &= \sum_{(I,\boldsymbol{\gamma},S)\in\mathbf{S}} \int_S \left\langle f, \varphi^j_{\alpha,\boldsymbol{\beta}} \right\rangle \overline{F_j f\left(\alpha,\boldsymbol{\beta}\right)} d\alpha d\mu\left(\boldsymbol{\beta}\right) \\ &= \left\langle f, \sum_{(I,\boldsymbol{\gamma},S)\in\mathbf{S}} \int_S F_j f\left(\alpha,\boldsymbol{\beta}\right) \varphi^j_{\alpha,\boldsymbol{\beta}} d\alpha d\mu\left(\boldsymbol{\beta}\right) \right\rangle \\ &\leq \|f\|_{L^2} \left\| \sum_{(I,\boldsymbol{\gamma},S)\in\mathbf{S}} \int_S F_j f\left(\alpha,\boldsymbol{\beta}\right) \varphi^j_{\alpha,\boldsymbol{\beta}} d\alpha d\mu\left(\boldsymbol{\beta}\right) \right\|_{L^2}.\end{aligned}$$

Again, it suffices to show

$$\left\| \sum_{(I,\boldsymbol{\gamma},S)\in\mathbf{S}} \int_S F_j f\left(\alpha,\boldsymbol{\beta}\right) \varphi^j_{\alpha,\boldsymbol{\beta}} d\alpha d\mu\left(\boldsymbol{\beta}\right) \right\|^2_{L^2} \lesssim \sum_{(I,\boldsymbol{\gamma},S)\in\mathbf{S}} \|F_j f\|^2_{L^2_\nu(S)}. \tag{2.9.12}$$

Developing the L^2 norm, we can estimate the left-hand side of (2.9.12) by

$$\leq \sum_{\substack{(I,\boldsymbol{\gamma},S)\in\mathbf{S} \\ (I',\boldsymbol{\gamma}',S')\in\mathbf{S}}} \int_S \left|F_j f\left(\alpha,\boldsymbol{\beta}\right)\right| \int_{S'} \left|F_j f\left(\alpha',\boldsymbol{\beta}'\right)\right| \cdot \left|\left\langle \varphi^j_{\alpha,\boldsymbol{\beta}}, \varphi^j_{\alpha',\boldsymbol{\beta}'} \right\rangle\right| d\alpha' d\mu\left(\boldsymbol{\beta}'\right) d\alpha d\mu\left(\boldsymbol{\beta}\right).$$

The summation can be further decomposed into diagonal terms

$$\sum_{(I,\boldsymbol{\gamma},S)\in\mathbf{S}} \int_S \int_S \left|F_j f\left(\alpha,\boldsymbol{\beta}\right)\right| \left|F_j f\left(\alpha',\boldsymbol{\beta}'\right)\right| \cdot \left|\left\langle \varphi^j_{\alpha,\boldsymbol{\beta}}, \varphi^j_{\alpha',\boldsymbol{\beta}'} \right\rangle\right| d\alpha' d\mu\left(\boldsymbol{\beta}'\right) d\alpha d\mu\left(\boldsymbol{\beta}\right). \tag{2.9.13}$$

and two copies of off-diagonal terms

$$\sum_{\substack{(I,\boldsymbol{\gamma},S)\in\mathbf{S}\\(I',\boldsymbol{\gamma}',S')\in\mathbf{S}\\(I,\boldsymbol{\gamma},S)\neq(I',\boldsymbol{\gamma}',S')\\\gamma_j\leq\gamma_j'}}\int_S |F_jf(\alpha,\boldsymbol{\beta})|\int_{S'}|F_jf(\alpha',\boldsymbol{\beta}')|\cdot\left|\left\langle\varphi^j_{\alpha,\boldsymbol{\beta}},\varphi^j_{\alpha',\boldsymbol{\beta}'}\right\rangle\right|d\alpha'd\mu(\boldsymbol{\beta}')\,d\alpha d\mu(\boldsymbol{\beta}). \tag{2.9.14}$$

We treat diagonal terms and off-diagonal terms differently. Starting with the diagonal terms (2.9.13), we rewrite the expression

$$\sum_{(I,\boldsymbol{\gamma},S)\in\mathbf{S}}\int_S\int_S|F_jf(\alpha,\boldsymbol{\beta})|\left|\left\langle\varphi^j_{\alpha,\boldsymbol{\beta}},\varphi^j_{\alpha',\boldsymbol{\beta}'}\right\rangle\right|^{\frac{1}{2}}$$

$$\cdot|F_jf(\alpha',\boldsymbol{\beta}')|\left|\left\langle\varphi^j_{\alpha,\boldsymbol{\beta}},\varphi^j_{\alpha',\boldsymbol{\beta}'}\right\rangle\right|^{\frac{1}{2}}d\alpha'd\mu(\boldsymbol{\beta}')\,d\alpha d\mu(\boldsymbol{\beta}).$$

Cauchy-Schwarz then gives

$$\leq\sum_{(I,\boldsymbol{\gamma},S)\in\mathbf{S}}\int_S|F_jf(\alpha,\boldsymbol{\beta})|^2\int_S\left|\left\langle\varphi^j_{\alpha,\boldsymbol{\beta}},\varphi^j_{\alpha',\boldsymbol{\beta}'}\right\rangle\right|d\alpha'd\mu(\boldsymbol{\beta}')\,d\alpha d\mu(\boldsymbol{\beta}).$$

Controlling (2.9.13) is reduced to showing that for all $(\alpha,\boldsymbol{\beta})\in S$,

$$\int_S\left|\left\langle\varphi^j_{\alpha,\boldsymbol{\beta}},\varphi^j_{\alpha',\boldsymbol{\beta}'}\right\rangle\right|d\alpha'd\mu(\boldsymbol{\beta}')\lesssim 1. \tag{2.9.15}$$

For the off-diagonal terms (2.9.14), we observe that

$$|F_jf(\alpha,\boldsymbol{\beta})|,|F_jf(\alpha',\boldsymbol{\beta}')|\leq\lambda\lesssim|I|^{-\frac{1}{2}}\|F_jf\|_{L^2_\nu(S)}.$$

After substitution, we can dominate (2.9.14) by

$$\sum_{(I,\boldsymbol{\gamma},S)\in\mathbf{S}}\|F_jf\|^2_{L^2_\nu(S)}\cdot\frac{1}{|I|}\int_S\sum_{\substack{(I',\boldsymbol{\gamma}',S')\in\mathbf{S}\\(I,\boldsymbol{\gamma},S)\neq(I',\boldsymbol{\gamma}',S')\\\gamma_j\leq\gamma_j'}}\int_{S'}\left|\left\langle\varphi^j_{\alpha,\boldsymbol{\beta}},\varphi^j_{\alpha',\boldsymbol{\beta}'}\right\rangle\right|d\alpha'd\mu(\boldsymbol{\beta}')\,d\alpha d\mu(\boldsymbol{\beta}).$$

Controlling (2.9.14) is reduced to showing for $(I,\boldsymbol{\gamma},S)\in\mathbf{S}$,

$$\frac{1}{|I|}\int_S\sum_{\substack{(I',\boldsymbol{\gamma}',S')\in\mathbf{S}\\(I,\boldsymbol{\gamma},S)\neq(I',\boldsymbol{\gamma}',S')\\\gamma_j\leq\gamma_j'}}\int_{S'}\left|\left\langle\varphi^j_{\alpha,\boldsymbol{\beta}},\varphi^j_{\alpha',\boldsymbol{\beta}'}\right\rangle\right|d\alpha'd\mu(\boldsymbol{\beta}')\,d\alpha d\mu(\boldsymbol{\beta})\lesssim 1. \tag{2.9.16}$$

To prove (2.9.15) and control the inner-most integral of (2.9.16), we first focus on a fixed $(\alpha,\boldsymbol{\beta})\in S$ and discard all points $(\alpha',\boldsymbol{\beta}')$ in S' (in (2.9.15), we take $S'=S$) that don't contribute to the integral. Therefore, we may assume

$$\left\langle\varphi^j_{\alpha,\boldsymbol{\beta}},\varphi^j_{\alpha',\boldsymbol{\beta}'}\right\rangle\neq 0.$$

This implies the frequency supports of the functions overlap, and thus,

$$\left|\beta_j-\beta_j'\right| \le \frac{4\varepsilon}{10}\left(d_\Gamma\left(\boldsymbol{\beta}\right)+d_\Gamma\left(\boldsymbol{\beta}'\right)\right).$$

Recall that $S\subseteq I\times W^{<j}_{\boldsymbol{\gamma},\frac{1}{|I|}}$ and $S'\subseteq I'\times W^{<j}_{\boldsymbol{\gamma}',\frac{1}{|I'|}}$. Since $\gamma_j\le\gamma_j'$, Lemma 2.9.2 guarantees the following relations in both (2.9.15) and (2.9.16)

$$d_\Gamma\left(\boldsymbol{\beta}\right)\le c d_\Gamma\left(\boldsymbol{\beta}'\right), \tag{2.9.17}$$

$$\left|\beta_j-\gamma_j'\right|\sim d_\Gamma\left(\boldsymbol{\beta}'\right)\sim\left|\boldsymbol{\beta}'-\boldsymbol{\gamma}'\right|. \tag{2.9.18}$$

On the one hand, due to the relation (2.9.17), the standard wave-packet estimate gives

$$\left|\left\langle\varphi^j_{\alpha,\boldsymbol{\beta}},\varphi^j_{\alpha',\boldsymbol{\beta}'}\right\rangle\right|\lesssim d_\Gamma\left(\boldsymbol{\beta}\right)\left(1+d_\Gamma\left(\boldsymbol{\beta}\right)\left|\alpha-\alpha'\right|\right)^{-N}.$$

On the other hand, the relation (2.9.18) implies that

$$\mu\left(P_V\left(S'\right)\right)\le\int_{\left\{\boldsymbol{\beta}'\in W_{\boldsymbol{\gamma}',0}\,\middle|\,\left|\beta_j-\gamma_j'\right|\sim\left|\boldsymbol{\beta}'-\boldsymbol{\gamma}'\right|\right\}}\frac{d\mathcal{H}^2\left(\boldsymbol{\beta}'\right)}{d_\Gamma\left(\boldsymbol{\beta}'\right)^2}\lesssim\fint_{\left\{\boldsymbol{\beta}'\in V\,\middle|\,\left|\beta_j-\gamma_j'\right|\sim\left|\boldsymbol{\beta}'-\boldsymbol{\gamma}'\right|\right\}}d\mathcal{H}^2\left(\boldsymbol{\beta}'\right)=1.$$

In combination, we obtain

$$\begin{aligned}\int_{S'}\left|\left\langle\varphi^j_{\alpha,\boldsymbol{\beta}},\varphi^j_{\alpha',\boldsymbol{\beta}'}\right\rangle\right|d\alpha' d\mu\left(\boldsymbol{\beta}'\right)&\lesssim\int_{P_V(S')}\int_{P_{\mathbb{R}}(S')}d_\Gamma\left(\boldsymbol{\beta}\right)\left(1+d_\Gamma\left(\boldsymbol{\beta}\right)\left|\alpha-\alpha'\right|\right)^{-N}d\alpha' d\mu\left(\boldsymbol{\beta}'\right)\\&\lesssim\int_{P_{\mathbb{R}}(S')}d_\Gamma\left(\boldsymbol{\beta}\right)\left(1+d_\Gamma\left(\boldsymbol{\beta}\right)\left|\alpha-\alpha'\right|\right)^{-N}d\alpha'\le\int_{\mathbb{R}}\left(1+|x|\right)^{-N}dx\sim 1\end{aligned}$$

In particular, this proves (2.9.15) by taking $S'=S$. To address the issue with summation involved in (2.9.16), we utilize the relation (2.9.17) and infer that all points in $(\alpha'',\boldsymbol{\beta}'')\in S''$ from any other $(I'',\boldsymbol{\gamma}'',S'')$ that contribute to the integral in (2.9.16) satisfy also the following relation

$$\begin{aligned}\left|\beta_j'-\beta_j''\right|\le\left|\beta_j-\beta_j'\right|+\left|\beta_j-\beta_j''\right|&\le\frac{4\varepsilon}{10}\left(2d_\Gamma\left(\boldsymbol{\beta}\right)+d_\Gamma\left(\boldsymbol{\beta}'\right)+d_\Gamma\left(\boldsymbol{\beta}''\right)\right)\\&\le\frac{4\varepsilon}{10}\cdot(1+c)\left(d_\Gamma\left(\boldsymbol{\beta}'\right)+d_\Gamma\left(\boldsymbol{\beta}''\right)\right)<\rho\left(d_\Gamma\left(\boldsymbol{\beta}'\right)+d_\Gamma\left(\boldsymbol{\beta}''\right)\right).\end{aligned}$$

This forces that all pairs of points $(\alpha',\boldsymbol{\beta}')\in S'$ and $(\alpha'',\boldsymbol{\beta}'')\in S''$ with distinct $(I',\boldsymbol{\gamma}',S')$ and $(I'',\boldsymbol{\gamma}'',S'')$ in (2.9.16) have their physical components separated in the following way

$$\left|\alpha-\alpha'\right|,\left|\alpha-\alpha''\right|\ge 2\left|I\right|,\quad\left|\alpha'-\alpha''\right|\ge 2\left(\left|I'\right|\wedge\left|I''\right|\right)>0.$$

In other words, the sets $3I$, $P_{\mathbb{R}}\left(S'\right)$, and $P_{\mathbb{R}}\left(S''\right)$ are disjoint from one another. Finally, we combine the result above,

$$\begin{aligned}
&\frac{1}{|I|}\int_S \sum_{\substack{(I',\boldsymbol{\gamma}',S')\in\mathbf{S}\\ (I,\boldsymbol{\gamma},S)\neq(I',\boldsymbol{\gamma}',S')\\ \gamma_j\leq\gamma_j'}} \int_{S'} \left|\left\langle \varphi^j_{\alpha,\boldsymbol{\beta}}, \varphi^j_{\alpha',\boldsymbol{\beta}'}\right\rangle\right| d\alpha' d\mu\left(\boldsymbol{\beta}'\right) d\alpha d\mu\left(\boldsymbol{\beta}\right)\\
\lesssim&\frac{1}{|I|}\int_S \sum_{\substack{(I',\boldsymbol{\gamma}',S')\in\mathbf{S}\\ (I,\boldsymbol{\gamma},S)\neq(I',\boldsymbol{\gamma}',S')\\ \gamma_j\leq\gamma_j'}} \int_{\pi_{\mathbb{R}}(S')} d_\Gamma\left(\boldsymbol{\beta}\right)\left(1+d_\Gamma\left(\boldsymbol{\beta}\right)\left|\alpha-\alpha'\right|\right)^{-N} d\alpha' d\alpha d\mu\left(\boldsymbol{\beta}\right)\\
\leq&\fint_I \int_{W_{\gamma,\frac{1}{|I|}}} \int_{3I^c} d_\Gamma\left(\boldsymbol{\beta}\right)\left(1+d_\Gamma\left(\boldsymbol{\beta}\right)\left|\alpha-\alpha'\right|\right)^{-N} d\alpha' d\mu\left(\boldsymbol{\beta}\right) d\alpha\\
\lesssim&\int_{W_{\gamma,\frac{1}{|I|}}} \left(1+d_\Gamma\left(\boldsymbol{\beta}\right)|I|\right)^{2-N} \fint_I \int_{3I^c} d_\Gamma\left(\boldsymbol{\beta}\right)\left(1+d_\Gamma\left(\boldsymbol{\beta}\right)\left|\alpha-\alpha'\right|\right)^{-2} d\alpha' d\alpha d\mu\left(\boldsymbol{\beta}\right)\\
\lesssim&\int_{W_{\gamma,\frac{1}{|I|}}} \left(d_\Gamma\left(\boldsymbol{\beta}\right)|I|\right)^{2-N} d\mu\left(\boldsymbol{\beta}\right)\\
\lesssim&\int_{B\left(\boldsymbol{\gamma},\frac{1}{|I|}\right)^c} |I|^{-N}\left|\boldsymbol{\beta}-\boldsymbol{\gamma}\right|^{-N}|I|^2\, d\mathcal{H}^2\left(\boldsymbol{\beta}\right)=\int_{B(0,1)^c}\left|\boldsymbol{\beta}\right|^{-N} d\mathcal{H}^2\left(\boldsymbol{\beta}\right)\sim 1.
\end{aligned}$$

□

We close this section with the proof of Proposition 2.2.8.

$$\Omega\cap|F_jf|^{-1}(\lambda,\infty)=\bigsqcup_{k\in\mathbb{N}}\Omega\cap|F_jf|^{-1}(2^{k-1}\lambda,2^k\lambda] \tag{2.9.19}$$

For each k, apply Proposition 2.2.6 to the set $\Omega\cap|F_jf|^{-1}(2^{k-1}\lambda,2^k\lambda]$ with the threshold $2^{k-1}\lambda$, we obtain a set of countable points $\mathbf{P}_k$ and a countable collection of tents $\mathcal{T}_k$ satisfying the properties stated in Proposition 2.2.6. Define $\mathbf{P}_\infty=\bigcup_{k\in\mathbb{N}}\mathbf{P}_k$ and $\mathcal{T}_\infty=\bigcup_{k\in\mathbb{N}}\mathcal{T}_k$. By (2.2.17),

$$\sum_{(I,\boldsymbol{\gamma})\in\mathcal{T}_\infty}|I|\sim\sum_{(\alpha,\boldsymbol{\beta})\in\mathbf{P}_\infty}d_\Gamma(\boldsymbol{\beta})^{-1}\lesssim\sum_{k\in\mathbb{N}}\sum_{(\alpha,\boldsymbol{\beta})\in\mathbf{P}_k}\frac{|F_jf(\alpha,\boldsymbol{\beta})|^2}{d_\Gamma(\boldsymbol{\beta})(2^{k-1}\lambda)^2}. \tag{2.9.20}$$

By Lemma 2.9.1, we can further bound (2.9.20) by

$$\sum_{k\in\mathbb{N}}2^{-2(k-1)}\lambda^{-2}\sum_{(\alpha,\boldsymbol{\beta})\in\mathbf{P}_k}\frac{|F_jf(\alpha,\boldsymbol{\beta})|^2}{d_\Gamma(\boldsymbol{\beta})}\lesssim\frac{\|f\|_{L^2}^2}{\lambda^2}. \tag{2.9.21}$$

As a direct consequence of (2.2.16), we have

$$\left\|1_{\Omega\setminus\bigcup_{T\in\mathcal{T}_\infty}D_T}F_jf\right\|_{L^\infty}\leq\lambda. \tag{2.9.22}$$

Next, define $\widetilde{\Omega}=\Omega\setminus\bigcup_{T\in\mathcal{T}_\infty}D_T\subseteq|F_jf|^{-1}[0,\lambda]$. Applying Proposition 2.2.7 to $\widetilde{\Omega}$, we obtain a countable collection $\mathbf{T}_{left}$ of tents and a countable collection $\mathbf{S}_{left}$ of the form $(I,\boldsymbol{\gamma},S)$ with

$(I,\boldsymbol{\gamma})$ a tent and S a measurable subset of $\Omega \cap I \times \left(W_{\boldsymbol{\gamma},\frac{1}{|I|}} \setminus U^j_{\boldsymbol{\gamma}}\right)^{<j}$ satisfying Properties stated in Proposition 2.2.7. By Lemma 2.9.3,

$$\sum_{(I,\boldsymbol{\gamma})\in\mathcal{T}_{left}} |I| \sim \sum_{(I,\boldsymbol{\gamma},S)\in\mathbf{S}_{left}} |I| \lesssim \sum_{(I,\boldsymbol{\gamma},S)\in\mathbf{S}_{left}} \frac{\|F_j f\|^2_{L^2_\nu(S)}}{\lambda^2} \lesssim \frac{\|f\|^2_{L^2}}{\lambda^2}. \tag{2.9.23}$$

Applying the dual version of Proposition 2.2.7 to $\left(\Omega \setminus \bigcup_{T\in\mathcal{T}_\infty} D_T\right) \setminus \bigcup_{T\in\mathcal{T}_{left}} D_T$ with $\left(W_{\boldsymbol{\gamma},\frac{1}{|I|}} \setminus U^j_{\boldsymbol{\gamma}}\right)^{<j}$ replaced by $\left(W_{\boldsymbol{\gamma},\frac{1}{|I|}} \setminus U^j_{\boldsymbol{\gamma}}\right)^{>j}$, we obtain another countable collection $\mathbf{T}_{right}$ of tents and a countable collection $\mathbf{S}_{right}$ of triples which satisfy equation similar to (2.9.23). Define $\mathcal{T} = \mathcal{T}_\infty \cup \mathcal{T}_{left} \cup \mathcal{T}_{right}$. Direct calculation gives (2.2.26)

$$\sum_{(I,\boldsymbol{\gamma})\in\mathcal{T}} |I| = \sum_{(I,\boldsymbol{\gamma})\in\mathcal{T}_\infty} |I| + \sum_{(I,\boldsymbol{\gamma})\in\mathcal{T}_{left}} |I| + \sum_{(I,\boldsymbol{\gamma})\in\mathcal{T}_{right}} |I| \lesssim \frac{\|f\|^2_{L^2}}{\lambda^2}. \tag{2.9.24}$$

Finally, notice that by Proposition 2.2.7, $\mathcal{T}$ satisfies (2.2.22)

$$|I|^{-\frac{1}{2}} \left\| 1_{\Omega\setminus\bigcup_{T\in\mathcal{T}} D_T} F_j f \right\|_{L^2_\nu\left(I\times(W_{\boldsymbol{\gamma},1/|I|}\setminus U^j_{\boldsymbol{\gamma}})^{<j}\right)} \leq \frac{\lambda}{\sqrt{2}}, \tag{2.9.25}$$

and the dual statement of (2.2.22)

$$|I|^{-\frac{1}{2}} \left\| 1_{\Omega\setminus\bigcup_{T\in\mathcal{T}} D_T} F_j f \right\|_{L^2_\nu\left(I\times(W_{\boldsymbol{\gamma},1/|I|}\setminus U^j_{\boldsymbol{\gamma}})^{>j}\right)} \leq \frac{\lambda}{\sqrt{2}}, \tag{2.9.26}$$

for all $(I,\boldsymbol{\gamma})$. Combining (2.9.22), (2.9.25), and (2.9.26) verifies that

$$\left\| 1_{\Omega\setminus\bigcup_{T\in\mathcal{T}} D_T} F_j f \right\|_{S^j} \leq \lambda \vee \sqrt{\left(\frac{\lambda}{\sqrt{2}}\right)^2 + \left(\frac{\lambda}{\sqrt{2}}\right)^2} = \lambda. \tag{2.9.27}$$

That is, $\mathcal{T}$ also satisfies (2.2.27). This completes the proof of Proposition 2.2.8.

2.10 Proof of Proposition 2.2.1: Bound of Model Form

Notice that there is a decomposition $V \setminus \Gamma = \bigsqcup_{\iota=1}^N V_\iota$ with V_ι defined iteratively

$$V_0 := \Gamma, \quad V_\iota := \left\{ \boldsymbol{\beta} \in V \setminus \bigcup_{j<\iota} V_j \;:\; d_\Gamma(\boldsymbol{\beta}) = d_{\Gamma_\iota}(\boldsymbol{\beta}) \right\}. \tag{2.10.1}$$

As a result, we obtain a decomposition on the trilinear form $\Lambda_m = \sum_{\iota=1}^N \Lambda_{m,\iota}$, where

$$\Lambda_{m,\iota}(f_1,f_2,f_3) := \int_{V_\iota} \int_{\mathbb{R}^3} K(\boldsymbol{\alpha},\boldsymbol{\beta}) \cdot \prod_{j=1}^3 (F_j f_j)(\alpha_j,\boldsymbol{\beta}) d\boldsymbol{\alpha} d\mu(\boldsymbol{\beta}). \tag{2.10.2}$$

Thus, by triangle inequality, it suffices to deal with a single piece $\Lambda_{m,\iota}$. Moreover, via a standard limiting argument, it suffices to perform the estimate with $\mathbb{R}^3$ replaced by a set of the form $[-A,A]^3$ and V replaced by a compact subset $V' \subseteq V \setminus \Gamma$. For simplicity, we may fix ι and assume $\Gamma = \Gamma_\iota$, $K = 1_{[-A,A]^3 \times (V_\iota \cap V')} K$, and $\Lambda_m = \Lambda_{m,\iota}$.

The remaining part is parallel to the restricted weak type estimate in *(4.2)* of [60]. Let $E_j \subseteq \mathbb{R}$ be measurable sets and $|f_j| \le 1_{E_j}$, it suffices to show

$$|\Lambda_m(f_1,f_2,f_3)| \lesssim a_2^{\frac{1}{2}} a_3^{\frac{1}{2}} (1 + \log \frac{a_1}{a_2}), \tag{2.10.3}$$

where $a_j = |E_{\sigma(j)}|$ is a decreasing rearrangement ($a_1 > a_2 > a_3$). Since for any $\varepsilon > 0$ and $x \ge 1$, we have the the asymptotic $1 + \log x = o(x^\varepsilon)$,

$$a_2^{\frac{1}{2}} a_3^{\frac{1}{2}} (1 + \log \frac{a_1}{a_2}) < C a_1^\varepsilon a_2^{\frac{1}{2}-\varepsilon} a_3^{\frac{1}{2}}.$$

Then, by interpolating the restricted weak type estimates, we obtain the strong bound in the local L^2 range. By L^2 normalization, $\widetilde{f}_j := \frac{f_j}{|E_j|^{\frac{1}{2}}}$, (2.10.3) reduces to

$$|\Lambda_m(\widetilde{f}_1, \widetilde{f}_2, \widetilde{f}_3)| \lesssim a_1^{\frac{1}{2}} (1 + \log \frac{a_1}{a_2}).$$

Note that by Proposition 2.2.4 (Global estimate),

$$\|F_j \widetilde{f}_j\|_{S^j} \lesssim \|\widetilde{f}_j\|_{L^\infty} \le \|E_j\|^{-\frac{1}{2}} = a_{\sigma^{-1}(j)}^{-\frac{1}{2}},$$

and by normalization, $\|\widetilde{f}_j\|_{L^2} \le 1$. Let n_j be the integer such that $2^{n_j - 1} < a_j^{-\frac{1}{2}} \le 2^{n_j}$. By design, $n_1 \le n_2 \le n_3$. We perform Proposition 2.2.8 (Bessel type estimate) iteratively. Let $\widetilde{\Omega}_{n_3} = [-A,A]^3 \times V'$ and $\pi_i : \mathbb{R}^3 \times V \to \mathbb{R} \times V$ be the following projection:

$$\pi_i(\boldsymbol{\alpha}, \boldsymbol{\beta}) := (\alpha_i, \boldsymbol{\beta}).$$

Given a compact set $\widetilde{\Omega}_n \subset [-A,A]^3 \times V'$ with the properties that for $i, i' \in \{1,2,3\}$,

$$\Omega_n := \pi_i \widetilde{\Omega}_n = \pi_{i'} \widetilde{\Omega}_n \tag{2.10.4}$$

and for all $j \in \{1,2,3\}$,

$$\left\| 1_{\Omega_n} F_j \widetilde{f}_j \right\|_{S^j} \lesssim 2^n, \tag{2.10.5}$$

we apply Proposition 2.2.8 with $\lambda = 2^{n-1}$ for each $j \in \{1,2,3\}$ and obtained a collection of tents $\mathbf{T}_{n,j}$ such that

$$\sum_{(I,\boldsymbol{\gamma}) \in \mathbf{T}_{n,j}} |I| \lesssim 2^{-2n}.$$

Let $\mathcal{T}_n := \bigcup_{j=1}^3 \mathcal{T}_{n,j}$. By triangle inequality,

$$\sum_{(I,\boldsymbol{\gamma}) \in \mathbf{T}_n} |I| \lesssim 2^{-2n}.$$

We can associate the data $T = (I, \boldsymbol{\gamma})$ with a region D_T^i defined by

$$D_T^i := (I\boldsymbol{e}_i \oplus \boldsymbol{e}_i^{\perp}) \times W_{\boldsymbol{\gamma},1/|I|}$$

and define the following compact subset:

$$\widetilde{\Omega}_{n-1} := \widetilde{\Omega}_n \setminus \bigcup_{T\in\mathcal{T}_n} \bigcup_{i=1}^{3} D_T^i.$$

By construction, the set $\widetilde{\Omega}_{n-1}$ satisfies (2.10.4) and (2.10.5) with n replaced by $n-1$. Through this iteration process, we obtain a nested sequence of compact sets

$$\widetilde{\Omega}_{n_3} \supseteq \cdots \supseteq \widetilde{\Omega}_n \supseteq \widetilde{\Omega}_{n-1} \supseteq \cdots$$

and a countable collection of tents $\mathcal{T} := \bigcup_{n\le n_3} \mathcal{T}_n$. Using the identity (2.10.2), we dominate $\Lambda_m\left(\widetilde{f}_1, \widetilde{f}_2, \widetilde{f}_3\right)$ in the following manner

$$\left|\Lambda_m\left(\widetilde{f}_1, \widetilde{f}_2, \widetilde{f}_3\right)\right| \le \sum_{n\le n_3} \sum_{T\in\mathcal{T}_n} \sum_{i=1}^{3} \left\| 1_{\widetilde{\Omega}_n} K(\boldsymbol{\alpha},\boldsymbol{\beta}) \cdot \prod_{j=1}^{3} (F_j\widetilde{f}_j)(\alpha_j,\boldsymbol{\beta}) \right\|_{L^1_{\boldsymbol{\alpha},\mu(\boldsymbol{\beta})}(D_T^i)}. \tag{2.10.6}$$

Apply Proposition 2.2.3 (Tent estimate), we obtain

$$|\Lambda_m(\widetilde{f}_1, \widetilde{f}_2, \widetilde{f}_3)| \lesssim \sum_{n\le n_3} \sum_{T\in\mathbf{T}_n} |I_T| \prod_{i=1}^{3} \|1_{\Omega_n} F_j\widetilde{f}_j\|_{S^j(T)} \lesssim \sum_{n\le n_3} 2^{-2n} \prod_{i=1}^{3} \min(2^n, 2^{n_i})$$

$$= \sum_{n\le n_1} 2^n + \sum_{n_1\le n<n_2} 2^{n_1} + \sum_{n_2\le n<n_3} 2^{n_1}\cdot 2^{n_2-n} \lesssim 2^{n_1} + (n_2-n_1)2^{n_1} + 2^{n_1}$$

$$= 2^{n_1}(2 + n_2 - n_1) \lesssim a_1^{-\frac{1}{2}}(2 + \log\frac{a_1}{a_2}).$$

which completes the proof of Proposition 2.2.1.

Chapter 3

A smoothing inequality related to triangular Hilbert transform along curves

3.1 Introduction

The third part of this thesis is a blueprint for the Lean project based on the paper [53]. We begin with a brief introduction to proof assistants, the Lean programming language, and the "blueprint" format. A proof assistant is a software tool designed to verify the correctness of formal proofs written by humans. Typically, it involves a specialized programming language based on logic and type theory. Since the 1960s, several proof assistants have been developed, including Mizar [72], Isabelle [80], and Lean [27].

Lean, a programming language and proof assistant developed by Leonardo de Moura at Microsoft Research in 2013. Over the past decade, Lean has gone through several updates. Lean features a robust mathematical library known as mathlib including many formalized definitions and theorems from modern mathematics.

The collaboration model behind Lean-based formalization works as follows. Starting from an original mathematics paper, a group of contributors writes a detailed "blueprint". This blueprint expands on the original work, spelling out every new definition not already found in mathlib and elaborating on steps that may have been omitted in the original paper. Collaborating with members of the Lean community, the team then works together to formalize the content into Lean code.

This model has led to several successful projects in recent years. One notable example is the formalization of Perfectoid spaces, sophisticated objects in arithmetic geometry introduced by Scholze in 2012 [88]. This project was led by Buzzard, Commelin, and Massot [10]. Another major achievement is the formalization of the Polynomial Freiman-Ruzsa (PFR) conjecture [40], originally proved by Gowers, Green, Manners, and Tao in 2023. Remarkably, the formalization was completed in just three weeks thanks to an extraordinary collaborative effort within the Lean community.

This model demonstrates another significant advantage. By breaking down a complex proof into smaller, manageable parts, it enables contributions from individuals who may not be experts in the specific area of mathematics involved.

An ongoing example of this approach is the formalization of Carleson's theorem in harmonic analysis. Originally proved by Carleson in 1966 [15], with alternative proofs later provided by Fefferman in 1973 [35] and by Lacey and Thiele in 2000 [65], this project is currently being led by Becker, van Doorn, Jamneshan, Srivastava, and Thiele [2].

We begin with some definitions.

Definition 3.1.1 (Exponential function)**.** Define the exponential function $e : \mathbb{R} \to \mathbb{C}$:

$$e(x) := e^{2\pi i x}\,. \tag{3.1.1}$$

Definition 3.1.2 (Translation and dilation operators)**.** Given a function $f : \mathbb{R} \to \mathbb{C}$ and parameters $a \in \mathbb{R}$, $b \in \mathbb{R}_{>0}$, $1 \leq p \leq \infty$, define the translation and dilation operator

$$\mathrm{Tr}_a\, f(x) := f(x-a)\,, \tag{3.1.2}$$

$$\mathrm{Dil}_b^p\, f(x) := b^{-\frac{1}{p}} f\left(\frac{x}{b}\right)\,. \tag{3.1.3}$$

Let $\mathcal{S}(\mathbb{R})$ denotes the space of Schwartz function on $\mathbb{R}$.

Definition 3.1.3 (Fourier transform)**.** Define the Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$

$$\mathcal{F}f(\xi) := \int_{\mathbb{R}} f(x)e(-x\xi)dx\,. \tag{3.1.4}$$

Define the partial Fourier transform

$$\mathcal{F}_{(k)}f(x_1, \cdots, x_{k-1}, \xi, x_{k+1}, \cdots, x_d) := \int_{\mathbb{R}} f(x)e(-x_k\xi)dx_k\,. \tag{3.1.5}$$

Definition 3.1.4 (Lebesgue norms)**.** For measurable function $f \in \mathbb{R}^d$, for exponent $1 \leq p, q \leq \infty$, define the L^p norm of f

$$\|f\|_{L^p} := \left(\int_{\mathbb{R}^d} |f(x)|^p dx\right)^{\frac{1}{p}}\,. \tag{3.1.6}$$

Define the partial L^p norm of f on variable x_k

$$\|f(x_1, \cdots, x_k, \cdots, x_d)\|_{L^p_{x_k}} := \left(\int_{\mathbb{R}} |f(x)|^p dx_k\right)^{\frac{1}{p}}\,. \tag{3.1.7}$$

Define the mixed L^pL^q norm of f on variables $x_k, x_{k'}$

$$\|f(x_1, \cdots, x_k, \cdots, x_{k'}, \cdots x_d)\|_{L^p_{x_k} L^q_{x_{k'}}} := \Big\| \|f(x_1, \cdots, x_k, \cdots, x_{k'}, \cdots x_d)\|_{L^p_{x_k}} \Big\|_{L^q_{x_{k'}}}\,. \tag{3.1.8}$$

Definition 3.1.5 (Multiplicative differences)**.** For function $f \in \mathbb{R}^d$, define the multiplicative differences

$$(\mathcal{D}_{(y_1, \cdots, y_d)} f)(x_1, \cdots, x_d) := f(x_1 + y_1, \cdots, x_d + y_d)\overline{f(x_1, \cdots, x_d)}\,. \tag{3.1.9}$$

Define the partial multiplicative differences

$$\mathcal{D}_y^{(x_k)}\left(f(x_1, \cdots, x_k, \cdots, x_d)\right) := f(x_1, \cdots, x_k + y, \cdots, x_d)\overline{f(x_1, \cdots, x_k, \cdots, x_d)}\,. \tag{3.1.10}$$

Definition 3.1.6 (Bump function)**.** Let ρ be the bump function

$$\rho(x) := e^{\frac{1}{x^2-1}}\,,\ |x| < 1 \quad \rho(x) := 0\,,\ |x| \geq 1\,. \tag{3.1.11}$$

Given a number $0 < \delta < 1$, we define the bump function

$$\psi(x) := (\mathrm{Tr}_1 \, \mathrm{Dil}_\delta^\infty \, \rho)(x)\,. \tag{3.1.12}$$

Note that ψ is a bump function which is smooth, with $\|\psi\|_{L^\infty} \leq 1$ and supported in $\mathcal{A} := [1-\delta, 1+\delta]$.

Definition 3.1.7 (Localized fiberwise bilinear Fourier multiplier)**.** For multiplier $m \in L^\infty(\mathbb{R})$, and $f, g, F, G \in \mathcal{S}(\mathbb{R}^2)$ define the localized fiberwise bilinear Fourier multiplier operator T_m and the associated bilinear operator $\mathcal{T}_m$ by

$$T_m(f,g)(x,y) := 1_{[0,1]^2}(x,y) \int_{\mathbb{R}^2} (\mathcal{F}_{(1)}f)(\xi,y)(\mathcal{F}_{(2)}g)(x,\eta) m(\xi,\eta) e(x\xi + y\eta) d\xi d\eta\,, \tag{3.1.13}$$

$$\mathcal{T}_m(F,G)(x,y) := 1_{[0,1]^2}(x,y) \int_{\mathbb{R}^2} F(\xi,y)G(x,\eta)m(\xi,\eta)e(x\xi + y\eta)d\xi d\eta\,. \tag{3.1.14}$$

Definition 3.1.8 (Main multiplier)**.** For a curve $\gamma \in C^5(\mathbb{R})$, and a scale $\lambda > 0$, define the multiplier $m_{\gamma,\lambda}$ by

$$m_{\gamma,\lambda}(\xi,\eta) := \psi\left(\frac{\xi}{\lambda}\right) \psi\left(\frac{\eta\gamma'(1)}{\lambda}\right) \int_{\mathbb{R}} e(-\xi t + \eta\gamma(t)) \psi\left(\frac{1}{\gamma'(1)} t\right) dt\,. \tag{3.1.15}$$

Definition 3.1.9 (Counting the sign change)**.** Let $\varphi : [a,b] \to \mathbb{R}$ be a C^k function. Suppose that for all $x \in [a,b]$, there exists a $j \in \mathbb{N}$ with $1 \leq j \leq k$ such that $\varphi^{(j)}(x) \neq 0$. Define

$$(J_k\varphi)(t) := \sup_{1 \leq j \leq k} |\varphi^{(j)}(t)|^{\frac{1}{j}}\,, \tag{3.1.16}$$

$$B_j := \left\{ t \in [a,b] \ :\ |\varphi^j(t)|^{\frac{1}{j}} > \frac{1}{2}(J_k\varphi)(t) \right\}\,, \quad 1 \leq j \leq k\,, \tag{3.1.17}$$

$$B_1^* := \left\{ t \in [a,b] \ :\ \varphi''(t) \neq 0 \right\}\,. \tag{3.1.18}$$

Iteratively define

$$\widetilde{B}_1 := B_1\,, \quad \widetilde{B}_j := B_j \setminus \bigcup_{l=1}^{j} \widetilde{B}_l\,. \tag{3.1.19}$$

For a set $B \subseteq \mathbb{R}$, let $C(B)$ be the number of connected components of B. Define a number

$$M_k(\varphi) := \sum_{j=2}^{k} C(\widetilde{B}_j) + C(\widetilde{B}_1 \cap B_1^*)\,. \tag{3.1.20}$$

Definition 3.1.10 (Sign change of γ)**.** Given $\gamma \in C^5(\mathbb{R})$, with

$$\inf_{t \in [\frac{1}{2}, \frac{3}{2}]} |\gamma''(t)| \neq 0\,, \tag{3.1.21}$$

we have the phase function:

$$\Phi(\xi,\eta) := -\xi t_0 + \eta\gamma(t_0)\,, \quad t_0 = t(\xi,\eta) := (\gamma')^{-1}\left(\frac{\xi}{\eta}\right). \tag{3.1.22}$$

Now we justify the function $t(\xi,\eta)$ is well-defined for ξ,η in the range (3.1.23). Note that since $\gamma'' \neq 0$ on $\left[\frac{1}{2},\frac{3}{2}\right]$ and γ'' is continuous, γ'' has the same sign on $\left[\frac{1}{2},\frac{3}{2}\right]$ and hence γ' is monotone on $\left[\frac{1}{2},\frac{3}{2}\right]$. On the other hand, for the multiplier $m_{\gamma,\lambda}$ defined in (3.1.8), we need to verified that for

$$\xi \in \lambda[1-\delta,1+\delta], \quad \gamma'(1)\eta \in \lambda[1-\delta,1+\delta], \tag{3.1.23}$$

we have $\frac{\xi}{\eta} \in \gamma'\left(\left[\frac{1}{2},\frac{3}{2}\right]\right)$. Note that for ξ,η in the range (3.1.23), we have

$$\frac{\xi}{\eta} \in \gamma'(1)\left[1-\frac{2\delta}{1+\delta}, 1+\frac{2\delta}{1-\delta}\right] =: A_5\,. \tag{3.1.24}$$

Also we have

$$A_4 := \gamma'(1) + \inf_{t\in[\frac{1}{2},\frac{3}{2}]}|\gamma''(t)| \cdot \left[-\frac{1}{2},\frac{1}{2}\right] \tag{3.1.25}$$

$$= \gamma'(1)\left(1-\frac{\inf_{t\in[\frac{1}{2},\frac{3}{2}]}|\gamma''(t)|}{2\gamma'(1)}, 1+\frac{\inf_{t\in[\frac{1}{2},\frac{3}{2}]}|\gamma''(t)|}{2\gamma'(1)}\right) \subseteq \gamma'\left(\left[\frac{1}{2},\frac{3}{2}\right]\right). \tag{3.1.26}$$

Comparing (3.1.24) and (3.1.26), we have that if

$$\delta \leq \frac{\inf_{t\in[\frac{1}{2},\frac{3}{2}]}|\gamma''(t)|}{4\gamma'(1) + \inf_{t\in[\frac{1}{2},\frac{3}{2}]}|\gamma''(t)|} \tag{3.1.27}$$

then $A_5 \subseteq A_4$ which together with the monotonicity of γ' imply that $t_0 = t(\xi,\eta) = (\gamma')^{-1}\left(\frac{\xi}{\eta}\right)$ is well-defined for ξ, η in the range (3.1.23).

Define functions

$$\varphi_{s,\eta,x}(\xi) := (\Delta_{(0,s)}\Phi)(\xi,\eta) + x\xi\,, \quad \widetilde{\varphi}_{s,u,v,\eta}(\xi) := (\Delta_{(0,s)}\Delta_{(u,v)}\Phi)(\xi,\eta)\,, \tag{3.1.28}$$

where Δ is the difference operator defined as

$$\Delta_{(a,b)}f(x,y) := f(x+a,y+b) - f(x,y) \tag{3.1.29}$$

for parameters $a,b\in\mathbb{R}$ and function $f:\mathbb{R}^2\to\mathbb{R}$. Define the number

$$M(\gamma) := \sup_{s,\eta,x} M_2(\varphi_{s,\eta,x})\,, \quad \widetilde{M}(\gamma) := \sup_{s,u,v,\eta} M_2(\widetilde{\varphi}_{s,u,v,\eta})\,. \tag{3.1.30}$$

Definition 3.1.11 (Constants)**.** Given $\gamma\in C^5(\mathbb{R})$, $\theta := \frac{\gamma'}{\gamma''}$, $\gamma''\neq 0$, $0<\delta<1$, we define the following constants:

$$C_{\psi,\gamma,\delta,1} := \left(\|\psi'\|_{L^\infty} + \|\psi''\|_{L^1} + \left\|\frac{\gamma'''}{\gamma''}\right\|_{L^\infty}\left(\|\psi\|_{L^\infty} + 2\|\psi'\|_{L^1} + \left\|\frac{\gamma'''}{\gamma''}\right\|_{L^\infty}\|\psi\|_{L^1}\right)\right.$$

$$+\left\|\frac{\gamma''''}{\gamma''}\right\|_{L^\infty}\|\psi\|_{L^1}\Bigg), \tag{3.1.31}$$

$$C_{\psi,\gamma,\delta,2} := 256\|\psi\|_{L^\infty}^4\left\|\frac{\gamma'(1)}{(1-\delta)\gamma''}\right\|_{L^\infty}\left(\|\psi\|_{L^\infty}+\|\psi'\|_{L^1}\right)^2, \tag{3.1.32}$$

$$C_{\psi,\gamma,\delta,3} := 1024\delta\Bigg\{\|\psi'\|_{L^\infty}\|\psi\|_{L^\infty}^3\left\|\frac{\gamma'(1)}{(1-\delta)\gamma''}\right\|_{L^\infty}\left(\|\psi\|_{L^\infty}+\|\psi'\|_{L^1}\right)^2$$

$$+\|\psi\|_{L^\infty}^4\left\|\frac{\gamma'(1)}{(1-\delta)\gamma''}\right\|_{L^\infty}^2\left(\|\psi\|_{L^\infty}+\|\psi'\|_{L^1}\right)\cdot C_{\psi,\gamma,\delta,1}\Bigg\}, \tag{3.1.33}$$

$$C_{\gamma,\delta,4} := \frac{1}{1+(1-\delta)\left\|\frac{1}{(\gamma'/\gamma'')'}\right\|_{L^\infty}}. \tag{3.1.34}$$

$$C_{\psi,\gamma,\delta,5} := 2^{16}\|\psi\|_{L^\infty}^8\left\|\frac{\gamma'(1)}{(1-\delta)\gamma''}\right\|_{L^\infty}^2\left(\|\psi\|_{L^\infty}+\|\psi'\|_{L^1}\right)^4, \tag{3.1.35}$$

$$C_{\psi,\gamma,\delta,6} := 2^{19}\delta\Bigg\{4\|\psi'\|_{L^\infty}\|\psi\|_{L^\infty}^7\left\|\frac{\gamma'(1)}{(1-\delta)\gamma''}\right\|_{L^\infty}^2\left(\|\psi\|_{L^\infty}+\|\psi'\|_{L^1}\right)^4$$

$$+\|\psi\|_{L^\infty}^8\left\|\frac{\gamma'(1)}{(1-\delta)\gamma''}\right\|_{L^\infty}^3\left(\|\psi\|_{L^\infty}+\|\psi'\|_{L^1}\right)^3 C_{\psi,\gamma,\delta,,1}\Bigg\}, \tag{3.1.36}$$

$$C_{\gamma,8} := \inf_{\xi,\eta}\left|\theta\cdot\theta''-\theta'(1+\theta')\right|(\xi,\eta), \tag{3.1.37}$$

$$C_{\gamma,9} := \inf_{\xi,\eta}\left\|\begin{pmatrix}\frac{1}{\gamma''}\cdot\theta' & -\theta\left(1+\theta'\right)\\ \frac{1}{\gamma''}\cdot\left(\frac{1}{\gamma''}\cdot\theta'\right)' & -\frac{1}{\gamma''}\left(\theta(1+\theta')\right)'\end{pmatrix}\right\|_{L^2\to L^2}^{-1}(\xi,\eta), \tag{3.1.38}$$

$$C_{\gamma,10:=}\left(\frac{2}{\|\gamma''\|_{L^\infty}^2}\frac{C_{\gamma,8}}{C_{\gamma,9}}\right)^{\frac{1}{4}}, \tag{3.1.39}$$

$$C_{\psi,\gamma,\delta,11} := 8\delta^2 C_{\psi,\gamma,\delta,5}+\frac{4C_{\psi,\gamma,\delta,5}}{C_{\gamma,10}^2}(\delta+\delta^{\frac{1}{2}})+4^4\left(\frac{C_{\psi,\gamma,\delta,5}+C_{\psi,\gamma,\delta,6}}{C_{\gamma,10}^2}\right)\widetilde{M}(\gamma)(1+\delta^{-\frac{1}{2}})$$

$$+4^3\left(\frac{C_{\psi,\gamma,\delta,5}+C_{\psi,\gamma,\delta,6}}{C_{\gamma,10}}\right)^{\frac{1}{2}}\widetilde{M}(\gamma)^{\frac{1}{2}}(\log(2\delta))^{\frac{1}{2}}, \tag{3.1.40}$$

$$C_{\psi,\gamma,\delta,12} := 4^4\left(\frac{C_{\psi,\gamma,\delta,5}+C_{\psi,\gamma,\delta,6}}{C_{\gamma,10}}\right)\cdot\widetilde{M}(\gamma). \tag{3.1.41}$$

Theorem 3.1.12 (Main theorem, smoothing inequality)**.** *Let $\gamma\in C^5([\frac{1}{2},\frac{3}{2}])$ be a curve satisfying the following conditions and $\theta=\frac{\gamma'}{\gamma''}$ as defined in Definition 3.1.11:*

$$\inf_{t\in[\frac{1}{2},\frac{3}{2}]}|\gamma'(t)|\neq 0, \quad \inf_{t\in[\frac{1}{2},\frac{3}{2}]}|\gamma''(t)|\neq 0, \tag{3.1.42}$$

$$\inf_{t\in[\frac{1}{2},\frac{3}{2}]}|\theta'(t)|\neq 0\,,\qquad \inf_{t\in[\frac{1}{2},\frac{3}{2}]}|\theta\cdot\theta''-\theta'(1+\theta')|(t)\neq 0\,, \tag{3.1.43}$$

$$M(\gamma)<+\infty\,,\quad \widetilde{M}(\gamma)<+\infty\,. \tag{3.1.44}$$

Let δ be a constant with $0<\delta<\delta_0$, where

$$\delta_0<\min\left\{\frac{1}{2}\,,\left\|\frac{2}{(\gamma'')^2}\right\|_{L^\infty}\frac{C_{\gamma,8}}{C_{\gamma,9}C_{\gamma,10}},\frac{\inf_{t\in[\frac{1}{2},\frac{3}{2}]}|\gamma''(t)|}{4\gamma'(1)+\inf_{t\in[\frac{1}{2},\frac{3}{2}]}|\gamma''(t)|}\right\}. \tag{3.1.45}$$

There exist constants $c>0$ and $C_{\psi,\gamma,\delta}>0$ such that for all $\lambda>1$, for functions $f,g\in\mathcal{S}(\mathbb{R}^2)$ we have the following smoothing inequality

$$\|T_{m_{\gamma,\lambda}}(f,g)\|_{L^1}\leq C_{\psi,\gamma,\delta}\lambda^{-c}\|f\|_{L^2}\cdot\|g\|_{L^2}\,. \tag{3.1.46}$$

Proposition 3.1.13 (Interpolation)**.** *Let $c_1,c_2,c_3>0$. Let T be a bilinear map. Assume that for all functions $F,G\in\mathcal{S}(\mathbb{R}^2)$ with*

$$\operatorname{supp}F\subseteq\mathbb{R}\times[0,1]\ , \tag{3.1.47}$$

$$\operatorname{supp}G\subseteq[0,1]\times\mathbb{R}\ , \tag{3.1.48}$$

we have the estimates:

$$\|T(F,G)\|_{L^1}\leq c_1\|F(\xi,y)\|_{L^2_\xi L^1_y}\cdot\|G(x,\eta)\|_{L^2_\eta L^1_x}\,, \tag{3.1.49}$$

$$\|T(F,G)\|_{L^1}\leq c_2\|F\|_{L^2}\cdot\|G(x,\eta)\|_{L^1_\eta L^2_x}\,, \tag{3.1.50}$$

$$\|T(F,G)\|_{L^2}\leq c_3\|F(\xi,y)\|_{L^2_\xi L^4_y}\cdot\|G(x,\eta)\|^{\frac{1}{2}}_{L^2_\eta L^4_x}\cdot\|G\|^{\frac{1}{2}}_{L^4}\,. \tag{3.1.51}$$

Then for all such F,G, we also have the estimate:

$$\|T(F,G)\|_{L^1}\leq(2^{13}c_1^{\frac{1}{2}}c_2^{\frac{1}{2}}+2^{18}c_1^{\frac{1}{2}}c_2^{\frac{1}{16}}c_3^{\frac{7}{16}})\cdot\|F\|_{L^2}\|G\|_{L^2}\,. \tag{3.1.52}$$

Definition 3.1.14 (Definition of variant of Gowers norm)**.** For $m\in L^\infty(\mathbb{R})$, we define the following three quasi-norms:

$$\|m\|_{(L^2\otimes L^2)^*}:=\sup_{\substack{f,g\in L^2\\ \|f\|_{L^2},\|g\|_{L^2}\leq 1}}\left|\int_{\mathbb{R}^2}f(\xi)g(\eta)m(\xi,\eta)d\xi d\eta\right|\,, \tag{3.1.53}$$

$$\|m\|_u:=\left\|(\mathcal{F}_{(1)}\mathcal{D}_{(0,s)}m)(x,\eta)\right\|^{\frac{1}{2}}_{L^\infty_{s\eta}L^2_x([-1,1])}\ , \tag{3.1.54}$$

$$\|m\|_U:=\left\|\int_{\mathbb{R}}\mathcal{D}_{(0,s)}\mathcal{D}_{(u,v)}m(\xi,\eta)d\xi\right\|^{\frac{1}{4}}_{L^\infty_{u\eta}L^1_sL^2_v}\,. \tag{3.1.55}$$

Proposition 3.1.15 (Control by variant of Gowers norm of the multiplier). *For all functions $F, G \in \mathcal{S}(\mathbb{R}^2)$ with*

$$\operatorname{supp} F \subseteq \mathbb{R} \times [0,1] \ , \tag{3.1.56}$$

$$\operatorname{supp} G \subseteq [0,1] \times \mathbb{R} \ , \tag{3.1.57}$$

and for multiplier $m \in L^\infty(\mathbb{R})$, we have for the bilinear operator $\mathcal{T}_m(F,G)$ as defined in (3.1.14) *the estimates:*

$$\|\mathcal{T}_m(F,G)\|_{L^1} \leq \|m\|_{(L^2 \otimes L^2)^*} \cdot \|F(\xi,y)\|_{L^2_\xi L^1_y} \cdot \|G(x,\eta)\|_{L^2_\eta L^1_x} \,, \tag{3.1.58}$$

$$\|\mathcal{T}_m(F,G)\|_{L^1} \leq \|m\|_u \cdot \|F\|_{L^2} \cdot \|G(x,\eta)\|_{L^1_\eta L^2_x} \,, \tag{3.1.59}$$

$$\|\mathcal{T}_m(F,G)\|_{L^2} \leq \|m\|_U \cdot \|F(\xi,y)\|_{L^2_\xi L^4_y} \cdot \|G(x,\eta)\|^{\frac{1}{2}}_{L^2_\eta L^4_x} \cdot \|G\|^{\frac{1}{2}}_{L^4} \,. \tag{3.1.60}$$

Proposition 3.1.16 (Oscillatory Integrals). *Suppose the number $M_k(\varphi)$ as defined in* (3.1.20) *is finite. Then we have the estimate*

$$\left| \int_a^b e(\varphi(t))\psi(t)dt \right| \leq 4^{k+1} \left(\|\psi\|_{L^\infty} + \|\psi'\|_{L^1} \right) \cdot \frac{M_k(\varphi)}{\inf\limits_{t \in [a,b]} (J_k \varphi)(t)} \,. \tag{3.1.61}$$

Proposition 3.1.17 (Decomposition of multiplier). *Let $\gamma : \mathbb{R} \to \mathbb{R}$ be a C^5 function with $\gamma''(t) \neq 0$ for all $t \in \mathbb{R}$. Let $\mathcal{A} = [1-\delta, 1+\delta]$. Let $\psi : \mathcal{A} \to \mathbb{R}$ be a C^1 function. We have the decomposition*

$$\int_{\mathbb{R}} e(-\xi t + \eta\gamma(t))\psi(t)dt = e\left(\Phi(\xi,\eta)\right) \cdot \Psi(\xi,\eta) \tag{3.1.62}$$

where

$$\Phi(\xi,\eta) := -\xi t_0 + \eta\gamma(t_0) \,, \quad t_0 = t(\xi,\eta) := (\gamma')^{-1}\left(\frac{\xi}{\eta}\right) \,, \tag{3.1.63}$$

as defined in (3.1.22) *and $\Psi(\xi,\eta)$ satisfies the estimates:*

$$|\Psi(\xi,\eta)| \leq \eta^{-\frac{1}{2}} \cdot 16 \left\| \frac{1}{\gamma''} \right\|^{\frac{1}{2}}_{L^\infty} \cdot \left(\|\psi\|_{L^\infty} + \|\psi'\|_{L^1} \right) \,, \tag{3.1.64}$$

$$|\partial_\xi \Psi(\xi,\eta)| \leq \eta^{-\frac{3}{2}} \cdot 16 \left\| \frac{1}{\gamma''} \right\|^{\frac{3}{2}}_{L^\infty} \cdot C_{\psi,\gamma,\delta,1} \,. \tag{3.1.65}$$

Proposition 3.1.18 (Gowers norm calculation for multiplier supported at high frequency). *Let $m_{\gamma,\lambda}$ be the multiplier defined in* (3.1.15) *and γ, δ satisfy the conditions in Theorem 3.1.12, then we have the following estimates of $m_{\gamma,\lambda}$:*

$$\|m_{\gamma,\lambda}\|_{(L^2 \otimes L^2)^*} \leq \|\psi\|^3_{L^\infty} \cdot \left\| \frac{1}{\gamma'} \right\|_{L^\infty} \,, \tag{3.1.66}$$

$$\|m_{\gamma,\lambda}\|_u \leq \lambda^{-\frac{1}{4}} \left(2^3 \delta^2 C^2_{\psi,\gamma,\delta,2} + \log\lambda \cdot 2^7 \left(\frac{C_{\psi,\gamma,\delta,2} + C_{\psi,\gamma,\delta,3}}{C_{\gamma,\delta,4}} \cdot M(\gamma) \right)^2 \right)^{\frac{1}{4}} \,, \tag{3.1.67}$$

$$\|m_{\gamma,\lambda}\|_U \leq \left(C_{\psi,\gamma,\delta,11} + C_{\psi,\gamma,\delta,12} (\log\lambda)^{\frac{1}{2}} \right)^{\frac{1}{4}} \,. \tag{3.1.68}$$

3.2 Proof of Theorem 3.1.12: Smoothing Inequality

For $f, g \in \mathcal{S}(\mathbb{R}^2)$, define the functions F, G:

$$F(\xi, y) := \rho(2y-1)\mathcal{F}_{(1)}f(\xi, y)\,, \quad G(x, \eta) := \rho(2x-1)\mathcal{F}_{(2)}g(x, \eta)\,. \tag{3.2.1}$$

The functions F, G then satisfy the conditions $F, G \in \mathcal{S}(\mathbb{R}^2)$ and

$$\operatorname{supp} F \subseteq \mathbb{R} \times [0,1]\ , \tag{3.2.2}$$

$$\operatorname{supp} G \subseteq [0,1] \times \mathbb{R}\ , \tag{3.2.3}$$

which is the assumptions in the *Proposition* 3.1.13, *Proposition* 3.1.15.

Combining *Proposition* 3.1.13, *Proposition* 3.1.15, *Proposition* 3.1.18, we obtain

$$\|T_{m_{\gamma,\lambda}}(f,g)\|_{L^1} = \|\mathcal{T}_{m_{\gamma,\lambda}}(F,G)\|_{L^1} \leq (2^{13}c_1^{\frac{1}{2}}c_2^{\frac{1}{2}} + 2^{18}c_1^{\frac{1}{2}}c_2^{\frac{1}{16}}c_3^{\frac{7}{16}}) \cdot \|F\|_{L^2}\|G\|_{L^2} \tag{3.2.4}$$

$$\leq C_{\psi,\gamma,\delta}\lambda^{-c}\|f\|_{L^2} \cdot \|g\|_{L^2} \tag{3.2.5}$$

for some $C_{\psi,\gamma,\delta} > 0$, $c > 0$, where

$$c_1 = \|m_{\gamma,\lambda}\|_{(L^2\otimes L^2)^*} \leq \|\psi\|_{L^\infty}^3 \cdot \left\|\frac{1}{\gamma'}\right\|_{L^\infty}\ , \tag{3.2.6}$$

$$c_2 = \|m_{\gamma,\lambda}\|_u \leq \lambda^{-\frac{1}{4}}\left(2^3\delta^2 C_{\psi,\gamma,\delta,2}^2 + \log\lambda \cdot 2^7\left(\frac{C_{\psi,\gamma,\delta,2}+C_{\psi,\gamma,\delta,3}}{C_{\gamma,\delta,4}} \cdot M(\gamma)\right)^2\right)^{\frac{1}{4}}\ , \tag{3.2.7}$$

$$c_3 = \|m_{\gamma,\lambda}\|_U \leq \left(C_{\psi,\gamma,\delta,11} + C_{\psi,\gamma,\delta,12}(\log\lambda)^{\frac{1}{2}}\right)^{\frac{1}{4}}\ . \tag{3.2.8}$$

The line (3.2.5) comes from Plancherel identity and that polynomial grows faster than logarithm.

3.3 Proof of Proposition 3.1.13: Interpolation

Let F, G be functions satisfying conditions (3.1.56), (3.1.57), (3.1.49), (3.1.50), (3.1.51).

For $j, k, l \in \mathbb{Z}$, we first define the following level sets:

$$P_j := \left\{y \in \mathbb{R} \,:\, 2^{j-1}\|F\|_{L^2} < \|F(\xi,y)\|_{L^2_\xi} \leq 2^j\|F\|_{L^2}\right\}\ , \tag{3.3.1}$$

$$Q_k := \left\{x \in \mathbb{R} \,:\, 2^{k-1}\|G\|_{L^2} < \|G(x,\eta)\|_{L^2_\eta} \leq 2^k\|G\|_{L^2}\right\}\ , \tag{3.3.2}$$

$$Q^l := \left\{(x,\eta) \in \mathbb{R}^2 \,:\, 2^{l-1}\|G(x,\eta')\|_{L^2_{\eta'}} < |G(x,\eta)| \leq 2^l\|G(x,\eta')\|_{L^2_{\eta'}}\right\}\ . \tag{3.3.3}$$

The family $\{P_j\}_{j\in\mathbb{Z}}$ forms a partition of $\mathbb{R}$. Namely, for each $y \in \mathbb{R}$ there is precisely one $j \in \mathbb{Z}$ such that $y \in P_j$. Similarly, the family $\{(Q_k \times \mathbb{R}) \cap Q^l\}_{k,l\in\mathbb{Z}}$ is a partition of $\mathbb{R}^2$.

We then decompose the functions F, G with respect to these level sets.

$$F = \sum_{j\in\mathbb{Z}} F_j\,, \quad F_j := 1_{\mathbb{R}\times P_j}F\,, \quad G = \sum_{k,l\in\mathbb{Z}} G_{k,l}\,, \quad G_{k,l} := 1_{((Q_k\times\mathbb{R})\cap Q^l)} \cdot G\,. \tag{3.3.4}$$

By Chebyshev inequality, we have

$$|P_j| \le \left(2^{j-1}\|F\|_{L^2}\right)^{-2} \cdot \int_{P_j} \left(\|F(\xi,y)\|_{L^2_\xi}\right)^2 dy \le 2^{2-2j}\,. \tag{3.3.5}$$

Since $\operatorname{supp} F \subseteq \mathbb{R}\times[0,1]$, we also have the trivial bound $|P_j| \le 1$. Combining this with (3.3.5), we have

$$|P_j| \le 2^{2-2j} \wedge 1\,. \tag{3.3.6}$$

Similarly, we have

$$|Q_k| \le 2^{2-2k} \wedge 1\ , \tag{3.3.7}$$

$$\|1_{Q^l}(x,\eta)\|_{L^1_\eta} \le 2^{2-2l}\,, \quad \|1_{Q^l}(x,\eta)\|_{L^4_\eta} \le 2^{\frac{1}{2}-\frac{l}{2}}\,. \tag{3.3.8}$$

Using above estimates, we now deduce a number of inequalities.

By construction (3.3.1) and by estimate (3.3.6), we have

$$\|F_j(\xi,y)\|_{L^2_\xi L^1_y} = \int_{P_j} \|F(\xi,y)\|_{L^2_\xi} dy \le |P_j|\cdot 2^j\|F\|_{L^2} \le 4(2^{-j}\wedge 2^j)\|F\|_{L^2}\,. \tag{3.3.9}$$

By construction (3.3.1) and by estimate (3.3.6), we have

$$\|F_j(\xi,y)\|_{L^2_\xi L^4_y} = \left(\int_{P_j} \|F(\xi,y)\|^4_{L^2_\xi} dy\right)^{\frac{1}{4}} \le |P_j|^{\frac{1}{4}}\cdot 2^j\|F\|_{L^2} \le 2(2^{\frac{j}{2}}\wedge 2^j)\|F\|_{L^2}\,. \tag{3.3.10}$$

By construction (3.3.2) and by estimate (3.3.7), we have

$$\|G_{k,l}(x,\eta)\|_{L^2_\eta L^1_x} \le \int_{Q_k} \|G(x,\eta)\|_{L^2_\eta} dx \le |Q_k|\cdot 2^k\|G\|_{L^2} \le 4(2^{-k}\wedge 2^k)\|G\|_{L^2}\,. \tag{3.3.11}$$

By construction (3.3.2) and by estimate (3.3.7), we have

$$\|G_{k,l}(x,\eta)\|_{L^2_\eta L^4_x} \le \left(\int_{Q_k} \|G(x,\eta)\|^4_{L^2_\eta} dx\right)^{\frac{1}{4}} \le |Q_k|^{\frac{1}{4}}\cdot 2^k\|G\|_{L^2} \le 2(2^{\frac{k}{2}}\wedge 2^k)\|G\|_{L^2}\,. \tag{3.3.12}$$

By construction (3.3.3) and by estimate (3.3.8), we have

$$\|G_{k,l}(x,\eta)\|_{L^1_\eta L^2_x} \le \left\| \|1_{Q^l}(x,\eta)\|_{L^1_\eta}\cdot \|1_{((Q_k\times\mathbb{R})\cap Q^l)}\cdot G\|_{L^\infty_\eta}\right\|_{L^2_x} \tag{3.3.13}$$

$$\le \left\| \|1_{Q^l}(x,\eta)\|_{L^1_\eta}\cdot 2^l\cdot \|G(x,l)\|_{L^2_\eta}\right\|_{L^2_x} \le 4\cdot 2^{-l}\|G\|_{L^2}\,. \tag{3.3.14}$$

By construction (3.3.3) and by estimate (3.3.8), (3.3.12), we have

$$\|G_{k,l}(x,\eta)\|_{L^4_\eta L^4_x} \le \left\| \|1_{Q^l}(x,\eta)\|_{L^4_\eta}\cdot \|1_{((Q_k\times\mathbb{R})\cap Q^l)}\cdot G\|_{L^\infty_\eta}\right\|_{L^4_x} \tag{3.3.15}$$

$$\le \left\| \|1_{Q^l}(x,\eta)\|_{L^4_\eta}\cdot 2^l\cdot \|G(x,l)\|_{L^2_\eta}\right\|_{L^4_x} \le 2\cdot 2^{\frac{l}{2}}\cdot \|G_{k,l}(x,\eta)\|_{L^2_\eta L^4_x} \tag{3.3.16}$$

$$\le 4\cdot 2^{\frac{l}{2}}\cdot (2^{\frac{k}{2}}\wedge 2^k)\|G\|_{L^2}\,. \tag{3.3.17}$$

Plug the above estimates (3.3.9), (3.3.10), (3.3.11), (3.3.12), (3.3.13), (3.3.15) into (3.1.49), (3.1.50), (3.1.51), we then have

$$\|T(F_j, G_{k,l})\|_{L^1} \leq 16c_1(2^{-j} \wedge 2^j)(2^{-k} \wedge 2^k) \cdot \|F\|_{L^2}\|G\|_{L^2}\,, \tag{3.3.18}$$

$$\|T(F_j, G_{k,l})\|_{L^1} \leq 4c_2 2^{-l} \cdot \|F\|_{L^2}\|G\|_{L^2}\,, \tag{3.3.19}$$

$$\|T(F_j, G_{k,l})\|_{L^1} \leq \|T(F_j, G_{k,l})\|_{L^2} \leq 4c_3(2^{\frac{j}{2}} \wedge 2^j)(2^{\frac{k}{2}} \wedge 2^k)2^{\frac{l}{4}} \cdot \|F\|_{L^2}\|G\|_{L^2}\,. \tag{3.3.20}$$

For $j, k \in \mathbb{Z}, l \in \mathbb{Z}_{\geq 0}$, we compute the geometric mean of (3.3.18), (3.3.19) with ratio $(\frac{1}{2}, \frac{1}{2})$,

$$\|T(F_j, G_{k,l})\|_{L^1} \leq 8c_1^{\frac{1}{2}}c_2^{\frac{1}{2}}(2^{-\frac{j}{2}} \wedge 2^{\frac{j}{2}})(2^{-\frac{k}{2}} \wedge 2^{\frac{k}{2}})2^{-\frac{l}{2}} \cdot \|F\|_{L^2}\|G\|_{L^2}\,. \tag{3.3.21}$$

Hence we have

$$\sum_{j,k\in\mathbb{Z}, l\in\mathbb{Z}_{\geq 0}} \|T(F_j, G_{k,l})\|_{L^1} \leq 4c_1^{\frac{1}{2}}c_2^{\frac{1}{2}} \cdot 4^2 \cdot 4^2 \cdot 8 \cdot \|F\|_{L^2}\|G\|_{L^2} \leq 2^{13}c_1^{\frac{1}{2}}c_2^{\frac{1}{2}} \cdot \|F\|_{L^2}\|G\|_{L^2}\,. \tag{3.3.22}$$

On the other hand, for $j, k \in \mathbb{Z}, l \in \mathbb{Z}_{<0}$, we compute the geometric mean of (3.3.18), (3.3.19) (3.3.20) with ratio $(\frac{1}{2}, \frac{1}{16}, \frac{1}{2} - \frac{1}{16})$, then we have

$$\|T(F_j, G_{k,l})\|_{L^1} \leq (16c_1)^{\frac{1}{2}}(4c_2)^{\frac{1}{16}}(4c_3)^{\frac{7}{16}}(2^{-j} \wedge 2^j)^{\frac{1}{2}}(2^{-k} \wedge 2^k)^{\frac{1}{2}}(2^{-l})^{\frac{1}{16}} \tag{3.3.23}$$

$$\cdot (2^{\frac{j}{2}} \wedge 2^j)^{\frac{7}{16}}(2^{\frac{k}{2}} \wedge 2^k)^{\frac{7}{16}}(2^{\frac{l}{4}})^{\frac{7}{16}} \cdot \|F\|_{L^2}\|G\|_{L^2} \tag{3.3.24}$$

$$\leq 8c_1^{\frac{1}{2}}c_2^{\frac{1}{16}}c_3^{\frac{7}{16}}(2^{-\frac{5}{32}j} \wedge 2^{\frac{15}{16}j})(2^{-\frac{5}{32}k} \wedge 2^{\frac{15}{16}k})2^{\frac{3}{64}l} \cdot \|F\|_{L^2}\|G\|_{L^2} \tag{3.3.25}$$

Hence we have

$$\sum_{j,k\in\mathbb{Z}, l\in\mathbb{Z}_{<0}} \|T(F_j, G_{k,l})\|_{L^1} \leq 8c_1^{\frac{1}{2}}c_2^{\frac{1}{16}}c_3^{\frac{7}{16}}(16 \cdot 4) \cdot (16 \cdot 4) \cdot 32 \cdot \|F\|_{L^2}\|G\|_{L^2} \tag{3.3.26}$$

$$\leq 2^{18}c_1^{\frac{1}{2}}c_2^{\frac{1}{16}}c_3^{\frac{7}{16}}\|F\|_{L^2}\|G\|_{L^2}\,. \tag{3.3.27}$$

Combining (3.3.22), (3.3.27), we have the desired result.

3.4 Proof of Proposition 3.1.15: Control by Gowers Norm

We start with proving the inequality (3.1.58). Multiply and divide $|\mathcal{T}_m(F, G)(x, y)|$ by the quantities $\|F(\xi, y)\|_{L^2_\xi}$ and $\|G(x, \eta)\|_{L^2_\eta}$ and notice that

$$\left\| \frac{F(\xi, y)}{\|F(\xi, y)\|_{L^2_\xi}} e(x\xi) \right\|_{L^2_\xi} = \left\| \frac{G(x, \eta)}{\|G(x, \eta)\|_{L^2_\eta}} e(y\eta) \right\|_{L^2_\eta} = 1\,. \tag{3.4.1}$$

Then by the definition (3.1.53), we have

$$\begin{aligned} |\mathcal{T}_m(F, G)(x, y)| \leq\; & \|F(\xi, y)\|_{L^2_\xi}\|G(x, \eta)\|_{L^2_\eta} \\ & \cdot \left| \int_{\mathbb{R}^2} \left(\frac{F(\xi, y)}{\|F(\xi, y)\|_{L^2_\xi}} e(x\xi) \right) \left(\frac{G(x, \eta)}{\|G(x, \eta)\|_{L^2_\eta}} e(y\eta) \right) \cdot m(\xi, \eta) d\xi d\eta \right| \\ \leq\; & \|m\|_{(L^2 \otimes L^2)^*}\|F(\xi, y)\|_{L^2_\xi}\|G(x, \eta)\|_{L^2_\eta}\,. \end{aligned}$$

Then we take the L^1 norm of $\mathcal{T}_m(F,G)$ and obtain the estimate

$$\|\mathcal{T}_m(F,G)\|_{L^1} \leq \|m\|_{(L^2\otimes L^2)^*}\|F(\xi,y)\|_{L^2_\xi L^1_y} \cdot \|G(x,\eta)\|_{L^2_\eta L^1_x}\,. \tag{3.4.2}$$

This proves inequality (3.1.58).

Next, we prove the inequality (3.1.59). We do Fubini to first integrate over ξ and then over η, then we do a trivial L^1 estimate in η to obtain

$$|\mathcal{T}_m(F,G)(x,y)| \leq \left\|\int_{\mathbb{R}} F(\xi,y)m(\xi,\eta)e(x\xi)d\xi \cdot G(x,\eta)\right\|_{L^1_\eta}\,. \tag{3.4.3}$$

By the conditions (3.1.56), (3.1.57), the function $\mathcal{T}_m(F,G)$ is supported on the square $[0,1]^2$. We take the $L^1(\mathbb{R}^2)$ norm of $\mathcal{T}_m(F,G)$, which is the same as the $L^1([0,1]^2)$ norm. By Fubini, we sort the iterated integral as

$$\|\mathcal{T}_m(F,G)\|_{L^1} \leq \left\|\left\|\left\|\int_{\mathbb{R}} F(\xi,y)m(\xi,\eta)e(x\xi)d\xi \cdot G(x,\eta)\right\|_{L^1_y([0,1])}\right\|_{L^1_\eta}\right\|_{L^1_x([0,1])}\,. \tag{3.4.4}$$

Pulling out the constant $G(x,\eta)$ of the innermost integral and using the nesting properties of the L^p norms on the space $[0,1]$ of measure 1, we estimate this by

$$\left\|\left\|\left\|\int_{\mathbb{R}} F(\xi,y)m(\xi,\eta)e(x\xi)d\xi\right\|_{L^2_y([0,1])} \cdot G(x,\eta)\right\|_{L^1_\eta}\right\|_{L^1_x([0,1])}\,. \tag{3.4.5}$$

Then by $L^1_\eta \times L^\infty_\eta \to L^1_\eta$ Hölder inequality, we can bound (3.4.5) by

$$\left\|\left\|\left\|\int_{\mathbb{R}} F(\xi,y)m(\xi,\eta(x))e(x\xi)d\xi\right\|_{L^2_y([0,1])}\right\|_{L^\infty_\eta} \cdot \|G(x,\eta)\|_{L^1_\eta}\right\|_{L^1_x([0,1])}\,. \tag{3.4.6}$$

We introduce a measurable function $\eta : [0,1] \to \mathbb{R}$ to linearize the norm $\|\cdot\|_{L^\infty_\eta}$ and equate (3.4.6) to

$$\left\|\left\|\int_{\mathbb{R}} F(\xi,y)m(\xi,\eta(x))e(x\xi)d\xi\right\|_{L^2_y([0,1])} \cdot \|G(x,\eta)\|_{L^1_\eta}\right\|_{L^1_x([0,1])}\,. \tag{3.4.7}$$

Then by $L^2_x \times L^2_x \to L^1_x$ Hölder inequality, we can bound (3.4.7) by

$$\left\|\int_{\mathbb{R}} F(\xi,y)m(\xi,\eta(x))e(x\xi)d\xi\right\|_{L^2_{xy}([0,1]^2)} \cdot \|G(x,\eta)\|_{L^1_\eta L^2_x}\,. \tag{3.4.8}$$

Let $\mathcal{M}(\mathbb{R}^n)$ be the space of measurable functions on $\mathbb{R}^n$. The following are some useful properties for multiplicative differences.

Lemma 3.4.1. *For $f \in \mathcal{M}(\mathbb{R}^2)$, $u \in \mathbb{R}$, we have*

$$\left|\int_{\mathbb{R}} f(x,y)dx\right|^2 = \int_{\mathbb{R}^2} \mathcal{D}^{(x)}_u(f(x,y))dudx\,. \tag{3.4.9}$$

Proof. By expanding the square and do a change of variable $x' = x + u$, we have

$$\left|\int_{\mathbb{R}} f(x,y)dx\right|^2 = \int_{\mathbb{R}} f(x',y)dx' \cdot \overline{\int_{\mathbb{R}} f(x,y)dx} \tag{3.4.10}$$

$$= \int_{\mathbb{R}^2} f(x+u,y)\overline{f(x,y)}dudx = \int_{\mathbb{R}^2} \mathcal{D}_u^{(x)}(f(x,y))dudx \tag{3.4.11}$$

□

Lemma 3.4.2. *For $f \in \mathcal{M}(\mathbb{R}^2)$, $1 \leq p \leq \infty$, we have*

$$\|f(x,y)\|_{L_x^p}^2 = \left\|D_u^{(x)}(f(x,y))\right\|_{L_{xu}^p} \tag{3.4.12}$$

Proof. Expanding the right hand side of (3.4.12) and with the fact that integration is invariant under translation , we have

$$\left\|D_u^{(x)}(f(x,y))\right\|_{L_{xu}^p} = \left\|f(x+u,y)\overline{f(x,y)}\right\|_{L_{xu}^p} \tag{3.4.13}$$

$$= \left\|\left\|f(x+u,y)\overline{f(x,y)}\right\|_{L_u^p}\right\|_{L^p(x)} \tag{3.4.14}$$

$$= \left\|\|f(x+u,y)\|_{L_u^p} \cdot f(x,y)\right\|_{L_x^p} = \left\|\|f(u,y)\|_{L_u^p} \cdot f(x,y)\right\|_{L_x^p} \tag{3.4.15}$$

$$= \|f(u,y)\|_{L_u^p} \cdot \|f(x,y)\|_{L_x^p} = \|f(x,y)\|_{L_x^p}^2 \,. \tag{3.4.16}$$

□

Lemma 3.4.3. *For $f \in \mathcal{M}(\mathbb{R}^2)$, $A \subseteq \mathbb{R}$, we have*

$$\left\|\int_{\mathbb{R}} f(x,y)dx\right\|_{L_y^2(A)}^2 = \int_A \int_{\mathbb{R}^2} \mathcal{D}_u^{(x)}(f(x,y))dudxdy = \int_A \int_{\mathbb{R}^2} (\mathcal{D}_{(u,0)}f)(x,y)dudxdy \,. \tag{3.4.17}$$

Proof. By developing the L^2 norm and by (3.4.9), we have

$$\left\|\int_{\mathbb{R}} f(x,y)dx\right\|_{L_y^2(A)}^2 = \int_A \left|\int_{\mathbb{R}} f(x,y)dx\right|^2 dy \tag{3.4.18}$$

$$= \int_A \int_{\mathbb{R}^2} \mathcal{D}_u^{(x)}(f(x,y))dudxdy = \int_A \int_{\mathbb{R}^2} (\mathcal{D}_{(u,0)}f)(x,y)dudxdy \,. \tag{3.4.19}$$

□

Lemma 3.4.4. *For $f \in \mathcal{M}(\mathbb{R}^2)$, $u \in \mathbb{R}$, we have*

$$\mathcal{D}_u^{(x)}(f(x,y)) = (\mathcal{D}_{(u,0)}f)(x,y) \,. \tag{3.4.20}$$

Proof. By the definitions of multiplicative derivative (3.1.9) and partial multiplicative derivative (3.1.10), we have

$$\mathcal{D}_u^{(x)}(f(x,y)) = f(x+u,y)\overline{f(x+u,y)} = f(x+u,y+0)\overline{f(x+u,y+0)} = (\mathcal{D}_{(u,0)}f)(x,y) \,. \tag{3.4.21}$$

□

Lemma 3.4.5. *For $f_k \in \mathcal{M}(\mathbb{R}^2)$, $u \in \mathbb{R}$, we have*

$$\mathcal{D}_u^{(x)} \left(\prod_{k=1}^{n} f_k(x,y) \right) = \prod_{k=1}^{n} \mathcal{D}_u^{(x)} (f_k(x,y)) \,. \tag{3.4.22}$$

Proof. By expanding the definition of multiplicative derivative, we have

$$\mathcal{D}_u^{(x)} \left(\prod_{k=1}^{n} f_k(x,y) \right) = \left(\prod_{k=1}^{n} f_k(x+u,y) \right) \overline{\left(\prod_{k=1}^{n} f_k(x,y) \right)} \tag{3.4.23}$$

$$= \prod_{k=1}^{n} f_k(x+u,y)\overline{f_k(x,y)} = \prod_{k=1}^{n} \mathcal{D}_u^{(x)} (f_k(x,y)) \,. \tag{3.4.24}$$

□

Lemma 3.4.6. *For $f \in \mathcal{M}(\mathbb{R}^2)$, $u, v \in \mathbb{R}$, we have*

$$\mathcal{D}_u^{(x)} \mathcal{D}_v^{(y)} f(x,y) = \mathcal{D}_v^{(y)} \mathcal{D}_u^{(x)} f(x,y) \,. \tag{3.4.25}$$

Proof. Expanding the definition of multiplicative derivative, we have

$$\mathcal{D}_u^{(x)} \mathcal{D}_v^{(y)} f(x,y) = \mathcal{D}_u^{(x)} \left(f(x,y+v)\overline{f(x,y)} \right) \tag{3.4.26}$$

$$= \left(f(x+u,y+v)\overline{f(x+u,y)} \right) \overline{\left(f(x,y+v)\overline{f(x,y)} \right)} \tag{3.4.27}$$

$$= \left(f(x+u,y+v)\overline{f(x,y+v)} \right) \overline{\left(f(x+u,y)\overline{f(x,y)} \right)} \tag{3.4.28}$$

$$= \mathcal{D}_v^{(y)} \left(f(x+u,y)\overline{f(x,y)} \right) = \mathcal{D}_v^{(y)} \mathcal{D}_u^{(x)} f(x,y) \,. \tag{3.4.29}$$

□

Lemma 3.4.7. *For $f \in \mathcal{M}(\mathbb{R}^2)$, $u \in \mathbb{R}$, $1 \le p \le \infty$, we have*

$$\left\| \mathcal{D}_u^{(x)} (f(x,y)) \right\|_{L_y^p} \le \mathcal{D}_u^{(x)} \left\| f(x,y) \right\|_{L_y^{2p}} \,. \tag{3.4.30}$$

Proof. Expand the definition of multiplicative derivative and by Hölder inequality, we have

$$\left\| \mathcal{D}_u^{(x)} (f(x,y)) \right\|_{L_y^p} = \| f(x+u,y)\overline{f(x,y)} \|_{L_y^p} \tag{3.4.31}$$

$$\le \| f(x+u,y) \|_{L^{2p}} \cdot \| f(x,y) \|_{L^{2p}} = \mathcal{D}_u^{(x)} \left\| f(x,y) \right\|_{L_y^{2p}} \,. \tag{3.4.32}$$

□

To show (3.1.59), suffice to show the following estimate

$$\left\| \int_{\mathbb{R}} F(\xi,y) m(\xi,\eta(x)) e(x\xi) d\xi \right\|_{L_{xy}^2([0,1]^2)} \le \|m\|_u \cdot \|F\|_{L^2} \,. \tag{3.4.33}$$

By (3.4.17), the square of the left hand side of (3.4.33) is equal to

$$\int_0^1 \int_0^1 \int_{\mathbb{R}^2} \mathcal{D}_u^{(\xi)} \left(F(\xi,y)m(\xi,\eta(x))e(x\xi)\right) d\xi du dx dy \tag{3.4.34}$$

By (3.4.22), we rewrite this as

$$= \int_0^1 \int_0^1 \int_{\mathbb{R}^2} \mathcal{D}_u^{(\xi)}(F(\xi,y)) \cdot \mathcal{D}_u^{(\xi)}(m(\xi,\eta(x))e(x\xi)) d\xi du dx dy \tag{3.4.35}$$

By Fubini's theorem, we rewrite this as

$$= \int_{\mathbb{R}^2} \left(\int_0^1 \mathcal{D}_u^{(\xi)}(F(\xi,y))dy \right) \cdot \left(\int_0^1 \mathcal{D}_u^{(\xi)}(m(\xi,\eta(x))e(x\xi))dx \right) d\xi du \,. \tag{3.4.36}$$

Then by $L^2_{\xi u} \times L^2_{\xi u} \to L^1_{\xi u}$ Hölder inequality, we bound (3.4.36) by

$$\left\| \int_0^1 \mathcal{D}_u^{(\xi)}(F(\xi,y))dy \right\|_{L^2_{\xi u}} \cdot \left\| \int_0^1 \mathcal{D}_u^{(\xi)}(m(\xi,\eta(x))e(x\xi))dx \right\|_{L^2_{\xi u}} . \tag{3.4.37}$$

By (3.4.30), the first term in (3.4.37) is bounded by

$$\left\| \mathcal{D}_u^{(\xi)} \left(\|f(\xi,y)\|_{L^2_y} \right) \right\|_{L^2_{\xi u}} . \tag{3.4.38}$$

Then by (3.4.17), the term (3.4.38) equals to

$$\left\| \|F(\xi,y)\|_{L^2_y} \right\|^2_{L^2_\xi} = \|F\|^2_{L^2} . \tag{3.4.39}$$

Again, by (3.4.17), the square of the second term in (3.4.37) equals to

$$\int_{\mathbb{R}^4} \mathcal{D}_v^{(x)} \left(1_{[0,1]}(x) \mathcal{D}_u^{(\xi)} \left(m(\xi,\eta(x))e(x\xi)\right) \right) dv dx d\xi du \tag{3.4.40}$$

$$= \int_{\mathbb{R}^4} \mathcal{D}_v^{(x)} \mathcal{D}_u^{(\xi)} \left(1_{[0,1]}(x) m(\xi,\eta(x))e(x\xi) \right) dv dx d\xi du \,. \tag{3.4.41}$$

Then by (3.4.25), the term (3.4.41) equals to

$$= \int_{\mathbb{R}^4} \mathcal{D}_u^{(\xi)} \mathcal{D}_v^{(x)} \left(1_{[0,1]}(x) m(\xi,\eta(x))e(x\xi) \right) dv dx d\xi du \tag{3.4.42}$$

$$= \left\| \int_{\mathbb{R}} \mathcal{D}_v^{(x)} \left(1_{[0,1]}(x) m(\xi,\eta(x))e(x\xi) \right) d\xi \right\|^2_{L^2_{vx}} \tag{3.4.43}$$

Expanding the multiplicative derivative, (3.4.43) equals to

$$\left\| \int_{\mathbb{R}} 1_{[0,1]}(x+v) 1_{[0,1]}(x) \cdot m(\xi,\eta(x+v)) \overline{m(\xi,\eta(x))} e((x+v)\xi - x\xi) d\xi \right\|^2_{L^2_{xv}} \tag{3.4.44}$$

$$= \left\| \int_{\mathbb{R}} (\mathcal{D}_{(0,\eta(x+v)-\eta(x))} m)(\xi,\eta(x)) e(v\xi) d\xi \right\|^2_{L^2_v([-x,1-x]) L^2_x([0,1])} \tag{3.4.45}$$

$$= \left\| (\mathcal{F}_{(1)}\mathcal{D}_{(0,\eta(x+v)-\eta(x))}m)(v,\eta(x)) \right\|^2_{L^2_v([-x,1-x])L^2_x([0,1])} \tag{3.4.46}$$

Then we can bound (3.4.46) by

$$\left\| \left\| (\mathcal{F}_{(1)}\mathcal{D}_{(0,s)}m)(v,\eta) \right\|_{L^\infty_{s\eta}} \right\|^2_{L^2_v([-x,1-x])L^2_x([0,1])} \tag{3.4.47}$$

$$\leq \left\| \left\| (\mathcal{F}_{(1)}\mathcal{D}_{(0,s)}m)(v,\eta) \right\|_{L^\infty_{s\eta}} \right\|^2_{L^2_v([-1,1])} = \|m\|^4_u \,. \tag{3.4.48}$$

This completes the proof of (3.1.59).

Last, we prove the inequality (3.1.60). By (3.4.17), (3.4.22), we have

$$\|\mathcal{T}_m(F,G)\|^2_{L^2} = \int_{\mathbb{R}^6} (\mathcal{D}_{(u,0)}F)(\xi,y)(\mathcal{D}_{(0,v)}G)(x,\eta)(\mathcal{D}_{(u,v)}m)(\xi,\eta)e(xu+yv)dxdyd\xi d\eta dudv \tag{3.4.49}$$

$$= \int_{\mathbb{R}^4} \left(\mathcal{F}^{-1}_{(2)}\mathcal{D}_{(u,0)}F\right)(\xi,v)\left(\mathcal{F}^{-1}_{(1)}\mathcal{D}_{(0,v)}G\right)(u,\eta)(\mathcal{D}_{(u,v)}m)(\xi,\eta)d\xi d\eta dudv \tag{3.4.50}$$

By $L^2_{\xi uv} \times L^2_{\xi uv} \to L^1_{\xi uv}$ Hölder inequality, we bound (3.4.50) by

$$\left\| \left(\mathcal{F}^{-1}_{(2)}\mathcal{D}_{(u,0)}F\right)(\xi,v) \right\|_{L^2_{\xi uv}} \cdot \left\| \int_{\mathbb{R}} \left(\mathcal{F}^{-1}_{(1)}\mathcal{D}_{(0,v)}G\right)(u,\eta)(\mathcal{D}_{(u,v)}m)(\xi,\eta)d\eta \right\|_{L^2_{\xi uv}} . \tag{3.4.51}$$

By Plancherel identity and (3.4.12), the first component in (3.4.51) equals

$$\left\| \left(\mathcal{D}_{(u,0)}F\right)(\xi,y) \right\|_{L^2_{\xi uy}} = \left\| \left\| \mathcal{D}^{(\xi)}_u(F(\xi,y)) \right\|_{L^2_{\xi u}} \right\|_{L^2_y} = \left\| \|F(\xi,y)\|^2_{L^2_\xi} \right\|_{L^2_y} = \|F(\xi,y)\|^2_{L^2_\xi L^4_y} \,. \tag{3.4.52}$$

Next, by (3.4.17), (3.4.22), the square of the second component in (3.4.51) equals

$$\int_{\mathbb{R}^5} \left(\mathcal{D}_{(0,s)}\mathcal{F}^{-1}_{(1)}\mathcal{D}_{(0,v)}G\right)(u,\eta) \cdot \left(\mathcal{D}_{(0,s)}\mathcal{D}_{(u,v)}m\right)(\xi,\eta)d\eta ds d\xi dudv \,. \tag{3.4.53}$$

Then by $L^1_{u\eta} \times L^\infty_{u\eta} \to L^1_{u\eta}$ Hölder inequality, (3.4.53) is bounded by

$$\left\| \left\| \left(\mathcal{D}_{(0,s)}\mathcal{F}^{-1}_{(1)}\mathcal{D}_{(0,v)}G\right)(u,\eta) \right\|_{L^1_{u\eta}} \cdot \left\| \int_{\mathbb{R}} \left(\mathcal{D}_{(0,s)}\mathcal{D}_{(u,v)}m\right)(\xi,\eta)d\xi \right\|_{L^\infty_{u\eta}} \right\|_{L^1_{vs}} \tag{3.4.54}$$

Note that by Cauchy-Schwarz inequality and Plancherel identity, the first component in (3.4.54) is bounded by

$$\left\| \left(\mathcal{D}_{(0,s)}\mathcal{F}^{-1}_{(1)}\mathcal{D}_{(0,v)}G\right)(u,\eta) \right\|_{L^1_{u\eta}} = \left\| \left(\mathcal{F}^{-1}_{(1)}\mathcal{D}_{(0,v)}G\right)(u+s,\eta) \cdot \overline{\left(\mathcal{F}^{-1}_{(1)}\mathcal{D}_{(0,v)}G\right)(u,\eta)} \right\|_{L^1_{u\eta}} \tag{3.4.55}$$

$$\leq \left\| \left(\mathcal{F}^{-1}_{(1)}\mathcal{D}_{(0,v)}G\right)(u+s,\eta) \right\|_{L^2_{u\eta}} \cdot \left\| \left(\mathcal{F}^{-1}_{(1)}\mathcal{D}_{(0,v)}G\right)(u,\eta) \right\|_{L^2_{u\eta}} \tag{3.4.56}$$

$$= \left\| \left(\mathcal{F}^{-1}_{(1)}\mathcal{D}_{(0,v)}G\right)(u,\eta) \right\|^2_{L^2_{u\eta}} = \left\| \left(\mathcal{D}_{(0,v)}G\right)(x,\eta) \right\|^2_{L^2_{x\eta}} \,. \tag{3.4.57}$$

Hence, (3.4.54) is bounded by

$$\left\| \left\| \left(\mathcal{D}_{(0,v)}G\right)(x,\eta) \right\|_{L^2_{x\eta}}^2 \cdot \left\| \int_{\mathbb{R}} \left(\mathcal{D}_{(0,s)}\mathcal{D}_{(u,v)}m\right)(\xi,\eta)d\xi \right\|_{L^\infty_{u\eta}L^1_s} \right\|_{L^1_v} . \tag{3.4.58}$$

Then by $L^2_v \times L^2_v \to L^1_v$ Hölder inequality, (3.4.58) is bounded by

$$\left\| \left\| \left(\mathcal{D}_{(0,v)}G\right)(x,\eta) \right\|_{L^2_{x\eta}}^2 \right\|_{L^2_v} \cdot \left\| \left\| \int_{\mathbb{R}} \left(\mathcal{D}_{(0,s)}\mathcal{D}_{(u,v)}m\right)(\xi,\eta)d\xi \right\|_{L^\infty_{u\eta}L^1_s} \right\|_{L^2_v} . \tag{3.4.59}$$

By the definition (3.1.55), the second term in (3.4.59) is $\|m\|_U^4$. By Minkovski inequality and $L^4_x \times L^4_x \to L^2_x$ Hölder inequality, the first term in (3.4.59) is bounded by

$$\left\| \left\| \left(\mathcal{D}_{(0,v)}G\right)(x,\eta) \right\|_{L^2_{x\eta}}^2 \right\|_{L^2_v} = \left\| \left\| G(x,\eta+v)\overline{G(x,\eta)} \right\|_{L^2_{x\eta}} \right\|_{L^4_v}^2 \tag{3.4.60}$$

$$\leq \left\| \left\| G(x,\eta+v) \right\|_{L^4_v} \cdot \overline{G(x,\eta)} \right\|_{L^2_{x\eta}}^2 \tag{3.4.61}$$

$$= \left\| \left\| G(x,v) \right\|_{L^4_v} \cdot \left\| G(x,\eta) \right\|_{L^2_\eta} \right\|_{L^2_x}^2 \tag{3.4.62}$$

$$\leq \left\| G(x,v) \right\|_{L^4_{xv}}^2 \cdot \left\| G(x,\eta) \right\|_{L^2_\eta L^4_x}^2 . \tag{3.4.63}$$

3.5 Proof of Proposition 3.1.16: Oscillatory Integrals

We first establish two fundamental lemma which are considered to be the standard Van der Corput lemma. The first one is without a bump function and the second one is with a bump function.

Lemma 3.5.1. *Let $\varphi : [a,b] \to \mathbb{R}$ be a C^k function and suppose that $\varphi^{(k)}(x) \neq 0$ for some $k \geq 1$ and for all $x \in [a,b]$. If $k = 1$, we further assume that φ' is monotonic. Then we have the estimate*

$$\left| \int_a^b e(\varphi(t))dt \right| \leq 4^k \left\| \frac{1}{\varphi^{(k)}} \right\|_{L^\infty}^{\frac{1}{k}} . \tag{3.5.1}$$

Proof. We prove (3.5.1) by induction. We begin with case $k = 1$. Since $\varphi' \neq 0$, we can multiply and divide $2\pi i\varphi'(t)$ to $e(\varphi(t))$. Then by integration by part, we have the identity

$$\int_a^b e(\varphi(t))dt = \int_a^b 2\pi i\varphi'(t)e(\varphi(t)) \cdot \frac{1}{2\pi i\varphi'(t)}dt \tag{3.5.2}$$

$$= \left(\frac{e(\varphi(b))}{2\pi i\varphi'(b)} - \frac{e(\varphi(a))}{2\pi i\varphi'(a)} \right) - \int_a^b e(\varphi(t)) \cdot \left(\frac{1}{2\pi i\varphi'(t)} \right)' dt . \tag{3.5.3}$$

Hence, by taking the absolute value, we have the estimate

$$\left| \int_a^b e(\varphi(t))dt \right| \leq \frac{1}{|2\pi\varphi'(b)|} + \frac{1}{|2\pi\varphi'(a)|} + \int_a^b \left| \left(\frac{1}{2\pi\varphi'(t)} \right)' \right| dt . \tag{3.5.4}$$

By the assumption that φ' is monotonic, for the third term of right hand side of (3.5.4), we take the absolute value inside the integral outside the integral. Then by fundamental theorem of calculus, we can bound the right hand side of (3.5.4) by

$$\frac{1}{|2\pi\varphi'(b)|} + \frac{1}{|2\pi\varphi'(a)|} + \left|\int_a^b \left(\frac{1}{2\pi\varphi'(t)}\right)' dt\right| \tag{3.5.5}$$

$$= \frac{1}{|2\pi\varphi'(b)|} + \frac{1}{|2\pi\varphi'(a)|} + \left|\frac{1}{2\pi\varphi'(b)} - \frac{1}{2\pi\varphi'(a)}\right| \tag{3.5.6}$$

$$\leq 2 \cdot \frac{1}{2\pi} \cdot \left(\frac{1}{|\varphi'(b)|} + \frac{1}{|\varphi'(a)|}\right) \tag{3.5.7}$$

$$\leq 4 \cdot \left\|\frac{1}{\varphi'}\right\|_{L^\infty} . \tag{3.5.8}$$

Now assume (3.5.1) holds for $k-1$, we will show (3.5.1) also holds for k. Since $\varphi^{(k)} \neq 0$ and $\varphi^{(k)}$ is continuous, by intermediate value theorem, we know that $\varphi^{(k)} > 0$ or $\varphi^{(k)} < 0$. In either cases $\varphi^{(k-1)}$ is monotonic. If for all $t \in [a,b]$ we have

$$|\varphi^{(k-1)}(t)| > \left\|\frac{1}{\varphi^{(k)}}\right\|_{L^\infty}^{-\frac{k-1}{k}} . \tag{3.5.9}$$

Then by induction hypothesis, we have

$$\left|\int_a^b e(\varphi(t))dt\right| \leq 4^{k-1}\left(1/\left\|\frac{1}{\varphi^{(k)}}\right\|_{L^\infty}^{-\frac{k-1}{k}}\right)^{\frac{1}{k-1}} \leq 4^k \left\|\frac{1}{\varphi^{(k)}}\right\|_{L^\infty}^{\frac{1}{k}} . \tag{3.5.10}$$

Suppose there is a point, say t_0 with the property that

$$|\varphi^{(k-1)}(t_0)| \leq \left\|\frac{1}{\varphi^{(k)}}\right\|_{L^\infty}^{-\frac{k-1}{k}} . \tag{3.5.11}$$

Then we divide the integral into three regions

$$\int_a^{t_0-2\left\|\frac{1}{\varphi^{(k)}}\right\|_{L^\infty}^{\frac{1}{k}}} e(\varphi(t))dt + \int_{t_0-2\left\|\frac{1}{\varphi^{(k)}}\right\|_{L^\infty}^{\frac{1}{k}}}^{t_0+2\left\|\frac{1}{\varphi^{(k)}}\right\|_{L^\infty}^{\frac{1}{k}}} e(\varphi(t))dt + \int_{t_0+2\left\|\frac{1}{\varphi^{(k)}}\right\|_{L^\infty}^{\frac{1}{k}}}^{b} e(\varphi(t))dt . \tag{3.5.12}$$

For $t \in \left[a, t_0 - 2\left\|\frac{1}{\varphi^{(k)}}\right\|_{L^\infty}^{\frac{1}{k}}\right] \cup \left[t_0 + 2\left\|\frac{1}{\varphi^{(k)}}\right\|_{L^\infty}^{\frac{1}{k}}, b\right]$, by fundamental theorem of calculus and φ^k has the same sign, we have the estimate

$$\left|\varphi^{(k-1)}(t_0) - \varphi^{(k-1)}(t)\right| = \left|\int_t^{t_0} \varphi^{(k)}(\xi)d\xi\right| \geq |t_0 - t| \cdot \left\|\frac{1}{\varphi^{(k)}}\right\|_{L^\infty}^{-1} \geq 2\left\|\frac{1}{\varphi^{(k)}}\right\|_{L^\infty}^{-\frac{k-1}{k}} . \tag{3.5.13}$$

By (3.5.11), (3.5.13) and triangle inequality, for $t \in \left[a, t_0 - 2\left\|\frac{1}{\varphi^{(k)}}\right\|_{L^\infty}^{\frac{1}{k}}\right] \cup \left[t_0 + 2\left\|\frac{1}{\varphi^{(k)}}\right\|_{L^\infty}^{\frac{1}{k}}, b\right]$, we have the estimate

$$|\varphi^{k-1}(t)| \geq \left\|\frac{1}{\varphi^{(k)}}\right\|_{L^\infty}^{-\frac{k-1}{k}} . \tag{3.5.14}$$

We then estimate the first and the third term of (3.5.12) by the induction hypothesis and put the absolute value inside the second term of (3.5.12) and estimate it by the measure of the underlying set. Hence the modulus of (3.5.12) is bounded by

$$2 \cdot 4^{k-1} \left(1 / \left\| \frac{1}{\varphi^{(k)}} \right\|_{L^\infty}^{-\frac{k-1}{k}} \right)^{\frac{1}{k-1}} + 4 \left\| \frac{1}{\varphi^{(k)}} \right\|_{L^\infty}^{\frac{1}{k}} \leq 4^k \left\| \frac{1}{\varphi^{(k)}} \right\|_{L^\infty}^{\frac{1}{k}} . \tag{3.5.15}$$

By induction on k, we then finish the proof of (3.5.1). □

Lemma 3.5.2. *Let $\varphi : [a,b] \to \mathbb{R}$ be a C^k function and suppose that $\varphi^{(k)}(x) \neq 0$ for some $k \geq 1$ and for all $x \in [a,b]$. If $k = 1$, we further assume that φ' is monotonic. Let $\psi : [a,b] \to \mathbb{R}$ be a C^1 function. Then we have the estimate*

$$\left| \int_a^b e(\varphi(t))\psi(t)dt \right| \leq 4^k \left(\|\psi\|_{L^\infty} + \|\psi'\|_{L^1} \right) \cdot \left\| \frac{1}{\varphi^{(k)}} \right\|_{L^\infty}^{\frac{1}{k}} . \tag{3.5.16}$$

Proof. Now we prove (3.5.16). Define $\Phi(t) := \int_a^t e(\varphi(\xi))d\xi$. Then by integration by part, we have

$$\left| \int_a^b e(\varphi(t))\psi(t)dt \right| = \left| \Phi(b)\psi(b) - \int_a^b \Phi(t)\psi'(t)dt \right| \tag{3.5.17}$$

$$\leq |\Phi(b)\psi(b)| + \sup_{t\in[a,b]} |\Phi(t)| \cdot \int_a^b |\psi'(t)|dt . \tag{3.5.18}$$

By (3.5.1), we have the estimate

$$|\Phi(b)| \leq \sup_{t\in[a,b]} |\Phi(t)| \leq 4^k \left\| \frac{1}{\varphi^{(k)}} \right\|_{L^\infty}^{\frac{1}{k}} . \tag{3.5.19}$$

Hence (3.5.18) is bounded by

$$4^k \left\| \frac{1}{\varphi^{(k)}} \right\|_{L^\infty}^{\frac{1}{k}} \cdot \left(|\psi(b)| + \int_a^b |\psi'(t)|dt \right) \leq 4^k \left(\|\psi\|_{L^\infty} + \|\psi'\|_{L^1} \right) \cdot \left\| \frac{1}{\varphi^{(k)}} \right\|_{L^\infty}^{\frac{1}{k}} . \tag{3.5.20}$$

Then we finish the proof of (3.5.16). □

Finally, with the two lemma above, we prove the inequality (3.1.61). By assumption and the definition of $(J_k\varphi)(t)$, for all $t \in [a,b]$, there exists a j with $1 \leq j \leq k$ such that

$$|\varphi^j(t)|^{\frac{1}{j}} > \frac{1}{2}(J_k\varphi)(t) . \tag{3.5.21}$$

Hence, we have

$$[a,b] = \bigcup_{j=1}^k B_j . \tag{3.5.22}$$

By the construction of $\widetilde{B}_j$ (3.1.19), we further have

$$[a,b] = \bigsqcup_{j=1}^k \widetilde{B}_j , \quad \widetilde{B}_j \subseteq B_j . \tag{3.5.23}$$

Notice that B_j is the preimage of an open set $(0,\infty)$ under a continuous map $|\varphi^j|^{\frac{1}{j}} - \frac{1}{2}J_k\varphi$. Hence B_j is open under the subset topology of $[a,b]$. With the fact that an open set in $\mathbb{R}$ is a disjoint union of countably many open intervals in the topology of $\mathbb{R}$, B_j is a disjoint union of countably many open intervals in the topology of $\mathbb{R}$ and at most with one interval containing the left endpoint a and one interval containing the right endpoint b. Hence $\widetilde{B}_j$ is a disjoint union of countably many intervals.

For $2 \le j \le l$, let $\{I_{j,l}\}_{l\in E_j}$ be the collection of connected components of $\widetilde{B}_j$ which is a collection of disjoint intervals. For $j = 1$, let $\{I_{1,l}\}_{l\in E_1}$ be the collection of connected components of $\widetilde{B}_j \cap B_1^*$. Notice that on $I_{j,l}$

$$\left\|\frac{1}{\varphi^{(j)}}\right\|_{L^\infty(I_{j,l})}^{\frac{1}{j}} = \left(\inf_{t\in I_{j,l}} |\varphi^{(j)}(t)|\right)^{-\frac{1}{j}} = \frac{1}{\left(\inf\limits_{t\in I_{j,l}} |\varphi^{(j)}(t)|^{\frac{1}{j}}\right)} < \frac{2}{\inf\limits_{t\in[a,b]} (J_k\varphi)(t)} . \tag{3.5.24}$$

Notice that for each $I_{1,l}$, $l \in E_1$, φ'' keeps the same sign and thus φ' is monotone.

We apply Vander Corput lemma (3.5.16) on each $I_{j,l}$ and by (3.5.24)

$$\left|\int_{I_{j,l}} e(\varphi(t))\psi(t)dt\right| \le 4^{k+1}\left(\|\psi\|_{L^\infty} + \|\psi'\|_{L^1}\right) \cdot \left\|\frac{1}{\varphi^{(k)}}\right\|_{L^\infty}^{\frac{1}{k}} \tag{3.5.25}$$

$$\le 4^{k+1}\left(\|\psi\|_{L^\infty} + \|\psi'\|_{L^1}\right) \cdot \frac{1}{\inf\limits_{t\in[a,b]} (J_k\varphi)(t)} . \tag{3.5.26}$$

Since $\{\widetilde{B}_j\}_{j=1}^k$ forms a partition of $[a,b]$ (3.5.23) and that $(B_1^*)^c$ is of measure zero, we decompose the domain into $\{I_{j,l}\}_{1\le j\le k,\, l\in E_j}$, then apply (3.5.26) on each $I_{j,l}$

$$\left|\int_a^b e(\varphi(t))\psi(t)dt\right| = \sum_{j=1}^k \sum_{l\in E_j} \left|\int_{I_{j,l}} e(\varphi(t))\psi(t)dt\right| \tag{3.5.27}$$

$$\le 4^{k+1}\left(\|\psi\|_{L^\infty} + \|\psi'\|_{L^1}\right) \cdot \left(\sum_{j=2}^k C(\widetilde{B}_j) + C(\widetilde{B}_1 \cap B_1^*)\right) \cdot \frac{1}{\inf\limits_{t\in[a,b]} (J_k\varphi)(t)} \tag{3.5.28}$$

$$= 4^{k+1}\left(\|\psi\|_{L^\infty} + \|\psi'\|_{L^1}\right) \cdot \frac{M_k(\varphi)}{\inf\limits_{t\in[a,b]} (J_k\varphi)(t)} . \tag{3.5.29}$$

Thus finish the proof of (3.1.61).

3.6 Proof of Proposition 3.1.17: Decomposition of multiplier

We prove the decomposition (3.1.62). We first analyze the phase. The critical point of the phase

$$t_0 = t(\xi,\eta) := (\gamma')^{-1}\left(\frac{\xi}{\eta}\right) \tag{3.6.1}$$

satisfies the equation

$$-\xi + \eta\gamma'(t_0) = 0 . \tag{3.6.2}$$

We can then pull out the main oscillation and obtain the decomposition

$$\int_{\mathbb{R}} e(-\xi t+\eta\gamma(t))\psi(t)dt = e(-\xi t_0+\eta\gamma(t_0))\cdot\int_{\mathbb{R}} e(-\xi(t-t_0)+\eta(\gamma(t)-\gamma(t_0)))\psi(t)dt \tag{3.6.3}$$

$$= e(\Phi(\xi,\eta))\cdot\Psi(\xi,\eta) \tag{3.6.4}$$

where we define

$$\Phi(\xi,\eta) := -\xi t_0+\eta\gamma(t_0)\,, \quad \Psi(\xi,\eta) := \int_{\mathbb{R}} e(-\xi(t-t_0)+\eta(\gamma(t)-\gamma(t_0)))\psi(t)dt\,. \tag{3.6.5}$$

By Vander Corput lemma (3.5.16), we have the estimate

$$|\Psi(\xi,\eta)| \le 16\left(\|\psi\|_{L^\infty}+\|\psi'\|_{L^1}\right)\cdot\left\|\frac{1}{\eta\gamma''}\right\|_{L^\infty}^{\frac{1}{2}} \tag{3.6.6}$$

$$= \eta^{-\frac{1}{2}}\cdot 16\left\|\frac{1}{\gamma''}\right\|_{L^\infty}^{\frac{1}{2}}\cdot\left(\|\psi\|_{L^\infty}+\|\psi'\|_{L^1}\right)\,. \tag{3.6.7}$$

Thus we finish the proof of (3.1.64).

Next, for estimate (3.1.65), through integration by part, we have the identity

$$\partial_\xi\Psi(\xi,\eta) = \int_{\mathbb{R}} 2\pi i\cdot e(-\xi(t-t_0)+\eta(\gamma(t)-\gamma(t_0)))\cdot\left(-(t-t_0)-\xi\cdot\frac{\partial}{\partial\xi}t_0+\eta\gamma'(t_0)\frac{\partial}{\partial\xi}t_0\right)\psi(t)dt \tag{3.6.8}$$

$$= -\int_{\mathbb{R}} 2\pi i\cdot e(-\xi(t-t_0)+\eta(\gamma(t)-\gamma(t_0)))\cdot(t-t_0)\psi(t)dt\,. \tag{3.6.9}$$

Then by integration by part and that t_0 is a critical point (3.6.2), we equate (3.6.9) to

$$-\int_{\mathbb{R}} 2\pi i(-\xi+\eta\gamma'(t))\cdot e(-\xi(t-t_0)+\eta(\gamma(t)-\gamma(t_0)))\cdot\frac{t-t_0}{-\xi+\eta\gamma'(t)}\psi(t)dt \tag{3.6.10}$$

$$= \int_{\mathbb{R}} e(-\xi(t-t_0)+\eta(\gamma(t)-\gamma(t_0)))\cdot\left(\frac{t-t_0}{-\xi+\eta\gamma'(t)}\psi(t)\right)'dt\,. \tag{3.6.11}$$

Notice that byt_0 is a critical point (3.6.2), we have

$$\frac{t-t_0}{-\xi+\eta\gamma'(t)}\psi(t) = \frac{t-t_0}{(-\xi+\eta\gamma'(t))-(-\xi+\eta\gamma'(t_0))}\psi(t) = \frac{t-t_0}{\eta(\gamma'(t)-\gamma'(t_0))}\psi(t)\,. \tag{3.6.12}$$

Define

$$R(t) = \left(\frac{t-t_0}{\gamma'(t)-\gamma'(t_0)}\psi(t)\right)'\,. \tag{3.6.13}$$

Apply Vander Corput lemma (3.5.16) to (3.6.11), we have the estimate

$$|\partial_\xi\Psi(\xi,\eta)| \le 16\left(\left\|\frac{1}{\eta}\cdot R\right\|_{L^\infty}+\left\|\frac{1}{\eta}\cdot R'\right\|_{L^1}\right)\cdot\left\|\frac{1}{\eta\gamma''}\right\|_{L^\infty}^{\frac{1}{2}} \tag{3.6.14}$$

$$= \eta^{-\frac{3}{2}}\cdot 16\left(\|R\|_{L^\infty}+\|R'\|_{L^1}\right)\cdot\left\|\frac{1}{\gamma''}\right\|_{L^\infty}^{\frac{1}{2}}\,. \tag{3.6.15}$$

Remain to estimate the two quantities $\|R\|_{L^\infty}, \|R'\|_{L^1}$.

By fundamental theorem of calculus and change of variable, we have

$$\gamma'(t)-\gamma'(t_0)=\int_{t_0}^{t}\gamma''(s)ds=\int_0^1\gamma''(t_0+\theta(t-t_0))d\theta\cdot(t-t_0)\,. \tag{3.6.16}$$

Place (3.6.16) into (3.6.13), we equate (3.6.13) to

$$R(t)=\left(\psi(t)/\frac{\gamma'(t)-\gamma'(t_0)}{t-t_0}\right)'=\left(\psi(t)\cdot\frac{1}{\int_0^1\gamma''(t_0+\theta(t-t_0))d\theta}\right)' \tag{3.6.17}$$

To do a further calculation on $R(t)$, we introduce the following integral Cauchy mean value theorem.

Lemma 3.6.1 (Integral Cauchy Mean Value Theorem)**.** *If $f:[a,b]\to\mathbb{R}$ is continuous and g is an integrable function which does not change sign on $[a,b]$, then there exists a point $c\in(a,b)$ such that*

$$\frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx}=f(c)\,. \tag{3.6.18}$$

Furthermore, if f/g is also continuous on $[a,b]$, then there exists a point $c\in(a,b)$ such that

$$\frac{\int_a^b f(x)dx}{\int_a^b g(x)dx}=\frac{\int_a^b(f(x)/g(x))\cdot g(x)dx}{\int_a^b g(x)dx}=\frac{f(c)}{g(c)}\,. \tag{3.6.19}$$

Continue the calculation of $R(t)$ (3.6.17), we have

$$R(t)=\psi'(t)\cdot\frac{1}{\int_0^1\gamma''(t_0+\theta(t-t_0))d\theta}-\psi(t)\cdot\frac{\int_0^1\gamma'''(t_0+\theta(t-t_0))\theta d\theta}{\left(\int_0^1\gamma''(t_0+\theta(t-t_0))d\theta\right)^2}\,. \tag{3.6.20}$$

By integral Cauchy mean value theorem (3.6.19), there exist $\theta_1,\theta_2\in(0,1)$ such that $R(t)$ (3.6.20) is equal to

$$\psi'(t)\cdot\frac{1}{\gamma''(t_0+\theta_1(t-t_0))}-\psi(t)\cdot\frac{1}{\gamma''(t_0+\theta_1(t-t_0))}\cdot\frac{\gamma'''(t_0+\theta_2(t-t_0))\theta_2}{\gamma''(t_0+\theta_2(t-t_0))}\,. \tag{3.6.21}$$

Hence we have the estimate

$$\|R\|_{L^\infty}\le\|\psi'\|_{L^\infty}\cdot\left\|\frac{1}{\gamma''}\right\|_{L^\infty}+\|\psi\|_{L^\infty}\cdot\left\|\frac{1}{\gamma''}\right\|_{L^\infty}\cdot\left\|\frac{\gamma'''}{\gamma''}\right\|_{L^\infty}\,. \tag{3.6.22}$$

Next, we calculate $R'(t)$. We have

$$R'(t)=\left(\psi(t)\cdot\frac{1}{\int_0^1\gamma''(t_0+\theta(t-t_0))d\theta}\right)'' \tag{3.6.23}$$

$$=\psi''(t)\cdot\frac{1}{\int_0^1\gamma''(t_0+\theta(t-t_0))d\theta}-2\psi'(t)\cdot\frac{\int_0^1\gamma'''(t_0+\theta(t-t_0))\theta d\theta}{\left(\int_0^1\gamma''(t_0+\theta(t-t_0))d\theta\right)^2} \tag{3.6.24}$$

$$+\psi''(t)\cdot\left(-\frac{\int_0^1\gamma''''(t_0+\theta(t-t_0))\theta^2d\theta}{\left(\int_0^1\gamma''(t_0+\theta(t-t_0))d\theta\right)^2}+\frac{2\left(\int_0^1\gamma'''(t_0+\theta(t-t_0))\theta d\theta\right)^2}{\left(\int_0^1\gamma''(t_0+\theta(t-t_0))d\theta\right)^3}\right). \tag{3.6.25}$$

By integral Cauchy mean value theorem (3.6.19), there exist $\theta_1,\theta_2,\theta_3\in(0,1)$ such that $R'(t)$ is equal to

$$\psi''(t)\cdot\frac{1}{\gamma''(t_0+\theta_1(t-t_0))}-2\psi'(t)\cdot\frac{1}{\gamma''(t_0+\theta_1(t-t_0))}\cdot\frac{\gamma'''(t_0+\theta_2(t-t_0))\theta_2}{\gamma''(t_0+\theta_2(t-t_0))} \tag{3.6.26}$$

$$+\psi(t)\cdot\frac{1}{\gamma''(t_0+\theta_1(t-t_0))}\cdot\left(\frac{-\gamma''''(t_0+\theta_3(t-t_0))\theta_3^2}{\gamma''(t_0+\theta_3(t-t_0))}+\left(\frac{\gamma'''(t_0+\theta_2(t-t_0))\theta_2}{\gamma''(t_0+\theta_2(t-t_0))}\right)^2\right). \tag{3.6.27}$$

Hence $\|R'\|_{L^1}$ is bounded by

$$\|\psi''\|_{L^1}\cdot\left\|\frac{1}{\gamma''}\right\|_{L^\infty}+2\|\psi'\|_{L^1}\cdot\left\|\frac{1}{\gamma''}\right\|_{L^\infty}\cdot\left\|\frac{\gamma'''}{\gamma''}\right\|_{L^\infty}+\|\psi\|_{L^1}\cdot\left\|\frac{1}{\gamma''}\right\|_{L^\infty}\cdot\left(\left\|\frac{\gamma''''}{\gamma''}\right\|_{L^\infty}+\left\|\frac{\gamma'''}{\gamma''}\right\|_{L^\infty}^2\right). \tag{3.6.28}$$

Thus, we finish the proof of estimate (3.1.65).

3.7 Proof of Proposition 3.1.18: Calculation of Gowers Norm

We start with the proof of (3.1.66). By separating the variables ξ,η and $L^2\times L^2\to L^1$ Hölder inequality, we have

$$\left|\int_{\mathbb{R}}f(\xi)g(\eta)m_{\gamma,\lambda}(\xi,\eta)d\xi d\eta\right|=\left|\int_{\mathbb{R}^3}f(\xi)g(\eta)e(-\xi t+\eta\gamma(t))\psi(t)\psi\left(\frac{\xi}{\lambda}\right)\psi\left(\frac{\eta\gamma'(1)}{\lambda}\right)dtd\xi d\eta\right| \tag{3.7.1}$$

$$=\left|\int_{\mathbb{R}}\mathcal{F}\left(f\cdot\mathrm{Dil}_\lambda^\infty\psi\right)(t)\cdot\mathcal{F}\left(g\cdot\mathrm{Dil}_{\lambda/\gamma'(1)}^\infty\psi\right)(-\gamma(t))\cdot\psi(t)dt\right| \tag{3.7.2}$$

$$\le\|\mathcal{F}\left(f\cdot\mathrm{Dil}_\lambda^\infty\psi\right)\|_{L^2}\cdot\left\|\left(\mathcal{F}\left(g\cdot\mathrm{Dil}_{\lambda/\gamma'(1)}^\infty\psi\right)\circ\gamma\right)\cdot\psi\right\|_{L^2}. \tag{3.7.3}$$

By Plancherel identity and $L^2\times L^\infty\to L^2$ Hölder inequality, the first term in (3.7.3) is bounded by

$$\|f\cdot\mathrm{Dil}_\lambda^\infty\psi\|_{L^2}\le\|f\|_{L^2}\cdot\|\mathrm{Dil}_\lambda^\infty\psi\|_{L^\infty}=\|f\|_{L^2}\cdot\|\psi\|_{L^\infty}. \tag{3.7.4}$$

By change of variable $s=\gamma'(t)$ and Plancherel identity and $L^2\times L^\infty\to L^2$ Hölder inequality, the second term in (3.7.3) is bounded by

$$\left\|\left(\mathcal{F}\left(g\cdot\mathrm{Dil}_{\lambda/\gamma'(1)}^\infty\psi\right)\right)\cdot\left(\left(\psi\cdot\frac{1}{\gamma'}\right)\circ\gamma^{-1}\right)\right\|_{L^2} \tag{3.7.5}$$

$$\le\left\|\mathcal{F}\left(g\cdot\mathrm{Dil}_{\lambda/\gamma'(1)}^\infty\psi\right)\right\|_{L^2}\cdot\left\|\left(\psi\cdot\frac{1}{\gamma'}\right)\circ\gamma^{-1}\right\|_{L^\infty} \tag{3.7.6}$$

$$=\left\|g\cdot\mathrm{Dil}_{\lambda/\gamma'(1)}^\infty\psi\right\|_{L^2}\cdot\left\|\psi\cdot\frac{1}{\gamma'}\right\|_{L^\infty} \tag{3.7.7}$$

$$\leq \|g\|_{L^2} \cdot \| \operatorname{Dil}^{\infty}_{\lambda/\gamma'(1)} \psi\|_{L^\infty} \cdot \|\psi\|_{L^\infty} \cdot \left\|\frac{1}{\gamma'}\right\|_{L^\infty} = \|g\|_{L^2} \cdot \|\psi\|^2_{L^\infty} \cdot \left\|\frac{1}{\gamma'}\right\|_{L^\infty} . \tag{3.7.8}$$

Combining (3.7.4), (3.7.8), we have the bound

$$\left|\int_{\mathbb{R}} f(\xi)g(\eta)m_{\gamma,\lambda}(\xi,\eta)d\xi d\eta\right| \leq \|f\|_{L^2}\|g\|_{L^2} \cdot \|\psi\|^3_{L^\infty} \left\|\frac{1}{\gamma'}\right\|_{L^\infty} . \tag{3.7.9}$$

Taking supreme on both sides of (3.7.9) over functions $\|f\|_{L^2} \leq 1$, $\|g\|_{L^2} \leq 1$, we obtain the desired result (3.1.66).

Next, we will prove inequality (3.1.67). We first obtain a pointwise estimate on

$$(\mathcal{F}_{(1)}\mathcal{D}_{(0,s)}m_{\gamma,\lambda})(x,\eta) . \tag{3.7.10}$$

By *Proposition* 3.1.17, we have the decomposition

$$m_{\gamma,\lambda}(x,\eta) = \psi\left(\frac{\xi}{\lambda}\right)\psi\left(\frac{\eta\gamma'(1)}{\lambda}\right)e\left(\Phi(\xi,\eta)\right)\cdot\Psi(\xi,\eta) \tag{3.7.11}$$

where by (3.1.63), Φ is of the form

$$\Phi(\xi,\eta) := -\xi t_0 + \eta\gamma(t_0) , \quad t_0 = t(\xi,\eta) := (\gamma')^{-1}\left(\frac{\xi}{\eta}\right) , \tag{3.7.12}$$

and Ψ satisfies the estimates (3.1.64), (3.1.65). Hence, by change of variable $\xi = \lambda\widetilde{\xi}$, $\eta = \lambda\widetilde{\eta}$, $s = \lambda\widetilde{s}$, we have

$$(\mathcal{F}_{(1)}\mathcal{D}_{(0,s)}m_{\gamma,\lambda})(x,\eta) = \int_{\mathbb{R}} \left(\mathcal{D}_{(0,s)}m_{\gamma,\lambda}\right)(\xi,\eta)e(x\xi)d\xi \tag{3.7.13}$$

$$= \int_{\lambda\mathcal{A}} e\left((\Delta_{(0,s)}\Phi)(\xi,\eta) + x\xi\right)\cdot\mathcal{D}^{(\eta)}_s\left(\psi\left(\frac{\xi}{\lambda}\right)\psi\left(\frac{\eta\gamma'(1)}{\lambda}\right)\Psi(\xi,\eta)\right)d\xi \tag{3.7.14}$$

$$= \int_{\mathcal{A}} e\left(\lambda\left[(\Delta_{(0,\widetilde{s})}\Phi)(\widetilde{\xi},\widetilde{\eta}) + x\widetilde{\xi}\right]\right)\cdot\lambda\mathcal{D}^{(\widetilde{\eta})}_{\widetilde{s}}\left(\psi(\widetilde{\xi})\psi(\widetilde{\eta}\gamma'(1))\Psi(\lambda\widetilde{\xi},\lambda\widetilde{\eta})\right)d\widetilde{\xi} \tag{3.7.15}$$

$$= \int_{\mathcal{A}} e(\widetilde{\Phi}(\lambda,x,\widetilde{s},\widetilde{\xi},\widetilde{\eta}))\cdot\widetilde{\Psi}(\lambda,\widetilde{s},\widetilde{\xi},\widetilde{\eta})d\widetilde{\xi} \tag{3.7.16}$$

where

$$\widetilde{\Phi}(\lambda,x,\widetilde{s},\widetilde{\xi},\widetilde{\eta}) = \lambda\left[(\Delta_{(0,\widetilde{s})}\Phi)(\widetilde{\xi},\widetilde{\eta}) + x\widetilde{\xi}\right] , \tag{3.7.17}$$

$$\widetilde{\Psi}(\lambda,\widetilde{s},\widetilde{\xi},\widetilde{\eta}) = \lambda\mathcal{D}^{(\widetilde{\eta})}_{\widetilde{s}}\left(\psi(\widetilde{\xi})\psi(\widetilde{\eta}\gamma'(1))\Psi(\lambda\widetilde{\xi},\lambda\widetilde{\eta})\right) . \tag{3.7.18}$$

We first obtain estimates of $\|\widetilde{\Psi}\|_{L^\infty}$ and $\|\partial_{\widetilde{\xi}}\widetilde{\Psi}\|_{L^1}$. By (3.1.64), we have

$$\|\widetilde{\Psi}\|_{L^\infty} \leq \lambda\|\psi\|^4_{L^\infty}\cdot\left\|1_{\mathcal{A}}((\widetilde{\eta}+\widetilde{s})\gamma'(1))1_{\mathcal{A}}(\widetilde{\eta}\gamma'(1))\Psi(\lambda\widetilde{\xi},\lambda(\widetilde{\eta}+\widetilde{s}))\overline{\Psi(\lambda\widetilde{\xi},\lambda\widetilde{\eta})}\right\|_{L^\infty} \tag{3.7.19}$$

$$\leq \lambda\|\psi\|^4_{L^\infty}\cdot\left[\lambda^{-\frac{1}{2}}\left(\frac{|\gamma'(1)|}{1-\delta}\right)^{\frac{1}{2}}\cdot 16\left\|\frac{1}{\gamma''}\right\|^{\frac{1}{2}}_{L^\infty}\left(\|\psi\|_{L^\infty} + \|\psi'\|_{L^1}\right)\right]^2 \tag{3.7.20}$$

$$= 256\|\psi\|_{L^\infty}^4 \left\|\frac{\gamma'(1)}{(1-\delta)\gamma''}\right\|_{L^\infty} \left(\|\psi\|_{L^\infty} + \|\psi'\|_{L^1}\right)^2 =: C_{\psi,\gamma,\delta,2}\,. \tag{3.7.21}$$

Next, by product rule and (3.1.65), we have

$$\left\|\partial_{\widetilde{\xi}}\widetilde{\Psi}\right\|_{L^1} \leq 2\delta \left\|\partial_{\widetilde{\xi}}\widetilde{\Psi}\right\|_{L^\infty} \leq 1024\delta \Bigg\{ \|\psi'\|_{L^\infty}\|\psi\|_{L^\infty}^3 \left\|\frac{\gamma'(1)}{(1-\delta)\gamma''}\right\|_{L^\infty} \left(\|\psi\|_{L^\infty} + \|\psi'\|_{L^1}\right)^2 \tag{3.7.22}$$

$$+\|\psi\|_{L^\infty}^4 \left\|\frac{\gamma'(1)}{(1-\delta)\gamma''}\right\|_{L^\infty}^2 \left(\|\psi\|_{L^\infty} + \|\psi'\|_{L^1}\right) \cdot C_{\psi,\gamma,\delta,1} \Bigg\} =: C_{\psi,\gamma,\delta,3}\,. \tag{3.7.23}$$

Hence, we finish the estimates for the amplitude function $\widetilde{\Psi}$. Next, we establish an estimate for the phase function $\widetilde{\Phi}$.

Lemma 3.7.1. *Suppose $|\lambda x| > 1$, then we have the following estimate for the function $\widetilde{\Phi}$ defined in* (3.7.17)

$$\max\left\{ \left|\partial_\xi \widetilde{\Phi}(\lambda, x, s, \xi, \eta)\right|, \left|\partial_\xi^2 \widetilde{\Phi}(\lambda, x, s, \xi, \eta)\right|^{\frac{1}{2}} \right\} \geq C_{\gamma,\delta,4}|\lambda x|^{\frac{1}{2}}\,. \tag{3.7.24}$$

Proof. Recall that by (3.6.2), $t(\xi,\eta)$ is the solution of the equation

$$-\xi + \eta\gamma'(t(\xi,\eta)) = 0\,. \tag{3.7.25}$$

Hence we have

$$\partial_\xi \Phi(\xi,\eta) = \partial_\xi\big(-\xi t(\xi,\eta) + \eta\gamma(t(\xi,\eta))\big) \tag{3.7.26}$$

$$= -t(\xi,\eta) - \xi\partial_\xi t(\xi,\eta) + \eta\gamma'(t(\xi,\eta)) \cdot \partial_\xi t(\xi,\eta) \tag{3.7.27}$$

$$= -t(\xi,\eta) + \partial_\xi t(\xi,\eta) \cdot (-\xi + \eta\gamma'(t(\xi,\eta))) = -t(\xi,\eta)\,. \tag{3.7.28}$$

Take the partial derivative ∂_ξ on both side of (3.6.2), we obtain

$$-1 + \eta\gamma''(t(\xi,\eta)) \cdot \partial_\xi t(\xi,\eta) = 0\,. \tag{3.7.29}$$

Hence we have

$$\partial_\xi t(\xi,\eta) = \frac{1}{\eta\gamma''(\xi,\eta)}\,. \tag{3.7.30}$$

We also obtain

$$\partial_\xi^2 \Phi(\xi,\eta) = -\partial_\xi t(\xi,\eta) = -\frac{1}{\eta\gamma''(\xi,\eta)}\,. \tag{3.7.31}$$

Notice that from (3.6.2), we have

$$\eta = \frac{\xi}{\gamma'(t(\xi,\eta))} \tag{3.7.32}$$

to freely change between ξ and η.

Let c be a constant to be determined later. Suppose

$$\left|\partial_\xi \widetilde{\Phi}((\lambda, x, s, \xi, \eta))\right| = \left|\lambda\left[(\Delta_{(0,s)}\partial_\xi \Phi)(\xi,\eta) + x\right]\right| \tag{3.7.33}$$

$$= \left|-\lambda\left[(\Delta_{(0,s)}t)(\xi,\eta)+x\right]\right| \le c|\lambda x|^{\frac{1}{2}}\,. \tag{3.7.34}$$

In order to prove (3.7.24), we will find a small enough c such that

$$\left|\partial_\xi^2\widetilde{\Phi}((\lambda,x,s,\xi,\eta))\right|^{\frac{1}{2}} = \left|-\lambda\Delta_s^{(\eta)}\left(\partial_\xi t(\xi,\eta)\right)\right|^{\frac{1}{2}} \tag{3.7.35}$$

$$= \left|\lambda\Delta_s^{(\eta)}\left(\frac{1}{\eta\gamma''(t(\xi,\eta))}\right)\right|^{\frac{1}{2}} = \left|\frac{\lambda}{\xi}\Delta_s^{(\eta)}\left(\frac{\gamma'}{\gamma''}(t(\xi,\eta))\right)\right|^{\frac{1}{2}} \tag{3.7.36}$$

$$= \left|\frac{\lambda}{\xi}\left(\frac{\gamma'}{\gamma''}\right)'(\widetilde{t})\cdot(\Delta_{(0,s)}t)(\xi,\eta)\right|^{\frac{1}{2}} \tag{3.7.37}$$

$$\ge \left(\frac{1}{1-\delta}\cdot\inf_{t\in\mathbb{R}}\left|\left(\frac{\gamma'}{\gamma''}\right)'(t)\right|\right)^{\frac{1}{2}}\cdot\left|(\Delta_{(0,s)}t)(\xi,\eta)\right|^{\frac{1}{2}} \ge c|\lambda x|^{\frac{1}{2}}\,. \tag{3.7.38}$$

By assumption (3.7.34) and that $|\lambda x| > 1$, we have

$$\left|\lambda\left[(\Delta_{(0,s)}t)(\xi,\eta)-x\right]\right| \le c|\lambda x|^{\frac{1}{2}} < c|\lambda x|\,. \tag{3.7.39}$$

Hence we have

$$\lambda(\Delta_{(0,s)}t)(\xi,\eta) \ge (1-c)|\lambda x|. \tag{3.7.40}$$

To show (3.7.38), suffice to find a small enough c such that

$$1-c \ge c^2(1-\delta)\cdot\frac{1}{\inf\limits_{t\in\mathbb{R}}\left|\left(\frac{\gamma'}{\gamma''}\right)'(t)\right|} = c^2(1-\delta)\left\|\frac{1}{(\gamma'/\gamma'')'}\right\|_{L^\infty}\,. \tag{3.7.41}$$

We may take

$$c = C_{\gamma,\delta,4} = \frac{1}{1+(1-\delta)\left\|\frac{1}{(\gamma'/\gamma'')'}\right\|_{L^\infty}}\,, \tag{3.7.42}$$

which is a constant that satisfies the above relation and thus finish the proof of this lemma. □

For $|\lambda x| \le 1$, we estimate (3.7.16) trivially by moving the modulus inside. By (3.7.21), we bound the modulus of (3.7.16) by

$$2\delta\cdot\|\widetilde{\Psi}\|_{L^\infty} = 2\delta C_{\psi,\gamma,\delta,2}\,. \tag{3.7.43}$$

Now suppose $|\lambda x| > 1$, we combine *Lemma* 3.7.1 which is an estimate of the phase $\widetilde{\Phi}$ and the estimate (3.7.21), (3.7.23) of the amplitude $\widetilde{\Psi}$ then apply (3.1.61) in *Proposition* 3.1.16 to bound the modulus of (3.7.16) by

$$4^3\left(\|\widetilde{\Psi}\|_{L^\infty}+\|\partial_\xi\widetilde{\Psi}\|_{L^1}\right)\cdot\frac{M_2(\widetilde{\Phi})}{\inf\limits_{t\in[a,b]}(J_2\widetilde{\Phi})(t)} \tag{3.7.44}$$

$$\le 4^3\frac{C_{\psi,\gamma,\delta,2}+C_{\psi,\gamma,\delta,3}}{C_{\gamma,\delta,4}}\cdot M(\gamma)\cdot|\lambda x|^{-\frac{1}{2}}\,. \tag{3.7.45}$$

Combining (3.7.43) and (3.7.45), we have

$$\|m_{\gamma,\lambda}\|_u = \left\|(\mathcal{F}_{(1)}\mathcal{D}_{(0,s)}m_{\gamma,\lambda})(x,\eta)\right\|^{\frac{1}{2}}_{L^\infty_{s\eta}L^2_x([-1,1])} \tag{3.7.46}$$

$$\leq \left(2\int_0^{\frac{1}{\lambda}}(2\delta C_{\psi,\gamma,\delta,2})^2dx + 2\int_{\frac{1}{\lambda}}^1\left(4^3\frac{C_{\psi,\gamma,\delta,2}+C_{\psi,\gamma,\delta,3}}{C_{\gamma,\delta,4}}\cdot M(\gamma)\cdot|\lambda x|^{-\frac{1}{2}}\right)^2dx\right)^{\frac{1}{4}} \tag{3.7.47}$$

$$= \lambda^{-\frac{1}{4}}\left(2^3\delta^2C^2_{\psi,\gamma,\delta,2} + \log\lambda\cdot 2^7\left(\frac{C_{\psi,\gamma,\delta,2}+C_{\psi,\gamma,\delta,3}}{C_{\gamma,\delta,4}}\cdot M(\gamma)\right)^2\right)^{\frac{1}{4}}. \tag{3.7.48}$$

Hence we finish the estimate (3.1.67).

Finally, we will prove (3.1.68). We first obtain a pointwise estimate on

$$\int_{\mathbb{R}}\mathcal{D}_{(0,s)}\mathcal{D}_{(u,v)}m_{\gamma,\lambda}(\xi,\eta)d\xi\,. \tag{3.7.49}$$

By *Proposition* 3.1.17, we have the decomposition

$$m_{\gamma,\lambda}(x,\eta) = \psi\left(\frac{\xi}{\lambda}\right)\psi\left(\frac{\eta\gamma'(1)}{\lambda}\right)e\left(\Phi(\xi,\eta)\right)\cdot\Psi(\xi,\eta) \tag{3.7.50}$$

where by (3.1.63), Φ is of the form

$$\Phi(\xi,\eta) := -\xi t_0 + \eta\gamma(t_0)\,,\quad t_0 = t(\xi,\eta) := (\gamma')^{-1}\left(\frac{\xi}{\eta}\right)\,, \tag{3.7.51}$$

and Ψ satisfies the estimates (3.1.64), (3.1.65). Hence, by change of variable $\xi = \lambda\widetilde{\xi}$, $\eta = \lambda\widetilde{\eta}$, $s = \lambda\widetilde{s}, u = \lambda\widetilde{u}, v = \lambda\widetilde{v}$, we have

$$\int_{\mathbb{R}}\mathcal{D}_{(0,s)}\mathcal{D}_{(u,v)}m_{\gamma,\lambda}(\xi,\eta)d\xi \tag{3.7.52}$$

$$= \int_{\lambda\mathcal{A}}e((\Delta_{(0,s)}\Delta_{(u,v)}\Phi)(\xi,\eta))\cdot\mathcal{D}^{(\xi,\eta)}_{(0,s)}\mathcal{D}^{(\xi,\eta)}_{(u,v)}\left(\psi\left(\frac{\xi}{\lambda}\right)\psi\left(\frac{\eta\gamma'(1)}{\lambda}\right)\Psi(\xi,\eta)\right)d\xi \tag{3.7.53}$$

$$= \int_{\mathcal{A}}e\left(\lambda(\Delta_{(0,\widetilde{s})}\Delta_{(\widetilde{u},\widetilde{v})}\Phi)(\widetilde{\xi},\widetilde{\eta})\right)\cdot\lambda\mathcal{D}^{(\widetilde{\xi},\widetilde{\eta})}_{(0,\widetilde{s})}\mathcal{D}^{(\widetilde{\xi},\widetilde{\eta})}_{(\widetilde{u},\widetilde{v})}\left(\psi(\widetilde{\xi})\psi(\widetilde{\eta}\gamma'(1))\Psi(\lambda\widetilde{\xi},\lambda\widetilde{\eta})\right)d\widetilde{\xi} \tag{3.7.54}$$

$$= \int_{\mathcal{A}}e\left(\widetilde{\Phi}(\lambda,\widetilde{s},\widetilde{u},\widetilde{v},\widetilde{\xi},\widetilde{\eta})\right)\cdot\widetilde{\Psi}(\lambda,\widetilde{s},\widetilde{u},\widetilde{v},\widetilde{\xi},\widetilde{\eta})d\widetilde{\xi} \tag{3.7.55}$$

where

$$\Delta_{(u,v)}f(\xi,\eta) = f(\xi+u,\eta+v) - f(\xi,\eta)\,, \tag{3.7.56}$$

$$\widetilde{\Phi}(\lambda,\widetilde{s},\widetilde{u},\widetilde{v},\widetilde{\xi},\widetilde{\eta}) = \lambda(\Delta_{(0,\widetilde{s})}\Delta_{(\widetilde{u},\widetilde{v})}\Phi)(\widetilde{\xi},\widetilde{\eta})\,, \tag{3.7.57}$$

$$\widetilde{\Psi}(\lambda,\widetilde{s},\widetilde{u},\widetilde{v},\widetilde{\xi},\widetilde{\eta}) = \lambda\mathcal{D}^{(\widetilde{\xi},\widetilde{\eta})}_{(0,\widetilde{s})}\mathcal{D}^{(\widetilde{\xi},\widetilde{\eta})}_{(\widetilde{u},\widetilde{v})}\left(\psi(\widetilde{\xi})\psi(\widetilde{\eta}\gamma'(1))\Psi(\lambda\widetilde{\xi},\lambda\widetilde{\eta})\right)\,. \tag{3.7.58}$$

We first obtain estimates of $\|\widetilde{\Psi}\|_{L^\infty}$ and $\|\partial_{\widetilde{\xi}}\widetilde{\Psi}\|_{L^1}$. By (3.1.64), we have

$$\|\widetilde{\Psi}\|_{L^\infty} \leq \lambda\|\psi\|^8_{L^\infty}\left[\lambda^{-\frac{1}{2}}\left(\frac{|\gamma'(1)|}{1-\delta}\right)^{\frac{1}{2}}\cdot 16\left\|\frac{1}{\gamma''}\right\|^{\frac{1}{2}}_{L^\infty}\left(\|\psi\|_{L^\infty}+\|\psi'\|_{L^1}\right)\right]^4 \tag{3.7.59}$$

$$= 2^{16}\frac{1}{\lambda}\|\psi\|_{L^\infty}^8 \left\|\frac{\gamma'(1)}{(1-\delta)\gamma''}\right\|_{L^\infty}^2 \left(\|\psi\|_{L^\infty} + \|\psi'\|_{L^1}\right)^4 = \frac{1}{\lambda}C_{\psi,\gamma,\delta,5}\,. \tag{3.7.60}$$

Next, by product rule and (3.1.65), we have

$$\left\|\partial_{\tilde{\xi}}\widetilde{\Psi}\right\|_{L^1} \leq 2\delta\lambda \left\|\partial_{\tilde{\xi}}\widetilde{\Psi}\right\|_{L^\infty} \leq 2\delta\lambda\Bigg\{4\|\psi'\|_{L^\infty}\|\psi\|_{L^\infty}^7 \cdot 2^{16}\lambda^{-2}\left\|\frac{\gamma'(1)}{(1-\delta)\gamma''}\right\|_{L^\infty}^2 \left(\|\psi\|_{L^\infty} + \|\psi'\|_{L^1}\right)^4 \tag{3.7.61}$$

$$+4\|\psi\|_{L^\infty}^8 \cdot \lambda \left[\lambda^{-\frac{1}{2}}\left(\frac{|\gamma'(1)|}{1-\delta}\right)^{\frac{1}{2}} \cdot 16\left\|\frac{1}{\gamma''}\right\|_{L^\infty}^{\frac{1}{2}} \left(\|\psi\|_{L^\infty} + \|\psi'\|_{L^1}\right)\right]^3 \tag{3.7.62}$$

$$\cdot\left(\lambda^{-\frac{3}{2}}\left(\frac{|\gamma'(1)|}{1-\delta}\right)^{\frac{3}{2}} \cdot 16\left\|\frac{1}{\gamma''}\right\|_{L^\infty}^{\frac{3}{2}} \cdot C_{\psi,\gamma,\delta,1}\right)\Bigg\} \tag{3.7.63}$$

$$= \frac{1}{\lambda}\cdot 2^{19}\delta\Bigg\{4\|\psi'\|_{L^\infty}\|\psi\|_{L^\infty}^7 \left\|\frac{\gamma'(1)}{(1-\delta)\gamma''}\right\|_{L^\infty}^2 \left(\|\psi\|_{L^\infty} + \|\psi'\|_{L^1}\right)^4 \tag{3.7.64}$$

$$+\|\psi\|_{L^\infty}^8 \left\|\frac{\gamma'(1)}{(1-\delta)\gamma''}\right\|_{L^\infty}^3 \left(\|\psi\|_{L^\infty} + \|\psi'\|_{L^1}\right)^3 C_{\psi,\gamma,\delta,1}\Bigg\} = \frac{1}{\lambda}C_{\psi,\gamma,\delta,6}\,. \tag{3.7.65}$$

Hence, we finish the estimates for the amplitude function $\widetilde{\Psi}$. Next, we establish an estimate for the phase function $\widetilde{\Phi}$.

Lemma 3.7.2. *Suppose $C_{\gamma,10}^2|\lambda vs| \geq 1$, for $u \in (-\delta,\delta)$, $s+v \in (-\delta,\delta)$, we have the following estimate for the function $\widetilde{\Phi}$ defined in* (3.7.57)*:*

$$\max\left\{\left|\partial_\xi\widetilde{\Phi}(\lambda,s,u,v,\xi,\eta)\right|, \left|\partial_\xi^2\widetilde{\Phi}(\lambda,s,u,v,\xi,\eta)\right|^{\frac{1}{2}}\right\} \geq \frac{1}{2}C_{\gamma,10}|\lambda vs|^{\frac{1}{2}} \tag{3.7.66}$$

where

Proof. We will first prove

$$\left|\begin{pmatrix}\partial_\xi\widetilde{\Phi} \\ \partial_\xi^2\widetilde{\Phi}\end{pmatrix}\right| = \left|\begin{pmatrix}\partial_\xi\widetilde{\Phi}(\lambda,s,u,v,\xi,\eta) \\ \partial_\xi^2\widetilde{\Phi}(\lambda,s,u,v,\xi,\eta)\end{pmatrix}\right| \geq C_{\gamma,10}^4|\lambda vs|\,. \tag{3.7.67}$$

Since derivative can commute with difference, we have

$$\left|\begin{pmatrix}\partial_\xi\widetilde{\Phi}(\lambda,s,u,v,\xi,\eta) \\ \partial_\xi^2\widetilde{\Phi}(\lambda,s,u,v,\xi,\eta)\end{pmatrix}\right| = \left|\begin{pmatrix}\partial_\xi\left[\lambda(\Delta_{(0,s)}\Delta_{(u,v)}\Phi)(\xi,\eta)\right] \\ \partial_\xi^2\left[\lambda(\Delta_{(0,s)}\Delta_{(u,v)}\Phi)(\xi,\eta)\right]\end{pmatrix}\right| \tag{3.7.68}$$

$$= \left|\begin{pmatrix}\lambda(\Delta_{(0,s)}\Delta_{(u,v)}\partial_\xi\Phi)(\xi,\eta) \\ \lambda(\Delta_{(0,s)}\Delta_{(u,v)}\partial_\xi^2\Phi)(\xi,\eta)\end{pmatrix}\right| \tag{3.7.69}$$

Lemma 3.7.3. *We have the following identity for iterated finite difference:*

$$\Delta_{(0,s)}\Delta_{(u,v)}f(\xi,\eta) = s\cdot\begin{pmatrix}\partial_\xi\partial_\eta f(\xi_1,\eta_1) & \partial_\eta^2 f(\xi_1,\eta_1)\end{pmatrix}\cdot\begin{pmatrix}u \\ v\end{pmatrix} \tag{3.7.70}$$

for some $\xi_1 \in [\xi,\xi+u]$, $\eta_1 \in [\eta,\eta+s+v]$.

Proof. By mean value theorem, we have

$$(\Delta_{(0,s)}\Delta_{(u,v)})f(\xi,\eta) = (\Delta_{(u,v)}f)(\xi,\eta+s) - (\Delta_{(u,v)}f)(\xi,\eta) \tag{3.7.71}$$

$$= s\cdot(\partial_\eta\Delta_{(u,v)}f)(\xi,\widetilde{\eta}) = s\cdot(\Delta_{(u,v)}\partial_\eta f)(\xi,\widetilde{\eta}) \tag{3.7.72}$$

for some $\widetilde{\eta}\in[\eta,\eta+s]$. Then by fundamental theorem of calculus and mean value theorem for integral, we can further equate the above expression to

$$s\cdot(\partial_\eta f(\xi+u,\widetilde{\eta}+v) - \partial_\eta f(\xi,\widetilde{\eta})) = s\cdot\int_0^1 \nabla\partial_\eta f(\xi+\theta u,\widetilde{\eta}+\theta v)\cdot\begin{pmatrix}u\\v\end{pmatrix}d\theta \tag{3.7.73}$$

$$= s\cdot\begin{pmatrix}\partial_\xi\partial_\eta f(\xi_1,\eta_1) & \partial_\eta^2 f(\xi_1,\eta_1)\end{pmatrix}\cdot\begin{pmatrix}u\\v\end{pmatrix} \tag{3.7.74}$$

for some $\xi_1\in[\xi,\xi+u]$, $\eta_1\in[\widetilde{\eta},\widetilde{\eta}+v]$. $\square$

By *Lemma* 3.7.3, the term (3.7.69) equals to

$$\lambda s\cdot\begin{pmatrix}\partial_\xi\partial_\eta(\partial_\xi\Phi)(\xi_1,\eta_1) & \partial_\eta^2(\partial_\xi\Phi)(\xi_1,\eta_1)\\ \partial_\xi\partial_\eta(\partial_\xi^2\Phi)(\xi_2,\eta_2) & \partial_\eta^2(\partial_\xi^2\Phi)(\xi_2,\eta_2)\end{pmatrix}\cdot\begin{pmatrix}u\\v\end{pmatrix} \tag{3.7.75}$$

Notice that

$$\partial_\eta\Phi(\xi,\eta) = -\xi\cdot\partial_\eta t(\xi,\eta) + \gamma(t(\xi,\eta)) + \eta\gamma'(t(\xi,\eta))\cdot\partial_\eta t(\xi,\eta) \tag{3.7.76}$$

$$= (-\xi+\eta\gamma'(t(\xi,\eta)))\partial_\eta\cdot t(\xi,\eta) + \gamma(t(\xi,\eta)) = \gamma(t(\xi,\eta))\,. \tag{3.7.77}$$

Hence, the term (3.7.75) equals to

$$\lambda s\cdot\begin{pmatrix}\partial_\xi^2(\partial_\eta\Phi)(\xi_1,\eta_1) & \partial_\xi\partial_\eta(\partial_\eta\Phi)(\xi_1,\eta_1)\\ \partial_\xi^3(\partial_\eta\Phi)(\xi_2,\eta_2) & \partial_\xi^2\partial_\eta(\partial_\eta\Phi)(\xi_2,\eta_2)\end{pmatrix}\cdot\begin{pmatrix}u\\v\end{pmatrix} \tag{3.7.78}$$

$$= \lambda s\cdot\begin{pmatrix}\partial_\xi^2\gamma(t(\xi_1,\eta_1)) & \partial_\xi\partial_\eta\gamma(t(\xi_1,\eta_1))\\ \partial_\xi^3\gamma(t(\xi_2,\eta_2)) & \partial_\xi^2\partial_\eta\gamma(t(\xi_2,\eta_2))\end{pmatrix}\cdot\begin{pmatrix}u\\v\end{pmatrix} \tag{3.7.79}$$

$$= \lambda s\cdot\begin{pmatrix}\partial_\xi^2\gamma(t(\xi_2,\eta_2)) & \partial_\xi\partial_\eta\gamma(t(\xi_2,\eta_2))\\ \partial_\xi^3\gamma(t(\xi_2,\eta_2)) & \partial_\xi^2\partial_\eta\gamma(t(\xi_2,\eta_2))\end{pmatrix}\cdot\begin{pmatrix}u\\v\end{pmatrix} \tag{3.7.80}$$

$$-\lambda s\cdot\begin{pmatrix}\partial_\xi^2\gamma(t(\xi_1,\eta_1)) - \partial_\xi^2\gamma(t(\xi_2,\eta_2)) & \partial_\xi\partial_\eta\gamma(t(\xi_1,\eta_1)) - \partial_\xi\partial_\eta\gamma(t(\xi_2,\eta_2))\\ 0 & 0\end{pmatrix}\cdot\begin{pmatrix}u\\v\end{pmatrix} \tag{3.7.81}$$

$$= \lambda s\cdot A\cdot\begin{pmatrix}u\\v\end{pmatrix} - \lambda s\cdot B\cdot\begin{pmatrix}u\\v\end{pmatrix}\,. \tag{3.7.82}$$

For simplicity, define a function

$$\theta(t) := \frac{\gamma'(t)}{\gamma''(t)}\,. \tag{3.7.83}$$

Note that we have the following calculation of higher order derivatives of γ:

$$\partial_\xi t = \frac{1}{\eta\gamma''},\quad \partial_\eta t = -\frac{\gamma'}{\eta\gamma''} = -\frac{1}{\eta}\cdot\theta\,, \tag{3.7.84}$$

$$\partial_\xi \gamma = \gamma' \cdot \partial_\xi t = \frac{1}{\eta} \cdot \theta, \quad \partial_\eta \gamma = \gamma' \cdot \partial_\eta t = -\frac{\gamma'}{\eta} \cdot \theta\,, \tag{3.7.85}$$

$$\partial_\xi^2 \gamma = \frac{1}{\eta} \cdot \theta' \cdot \frac{1}{\eta \gamma''} = \frac{1}{\eta^2 \gamma''} \cdot \theta'\,, \tag{3.7.86}$$

$$\partial_\xi \partial_\eta \gamma = -\frac{1}{\eta^2} \cdot \theta + \frac{1}{\eta} \cdot \theta' \cdot \left(-\frac{1}{\eta} \cdot \theta\right) = -\frac{1}{\eta^2} \theta \left(1 + \theta'\right)\,, \tag{3.7.87}$$

$$\partial_\xi^3 \gamma = \frac{1}{\eta^2} \cdot \left(\frac{1}{\gamma''} \cdot \theta'\right)' \cdot \frac{1}{\eta \gamma''} = \frac{1}{\eta^3 \gamma''} \cdot \left(\frac{1}{\gamma''} \cdot \theta'\right)'\,, \tag{3.7.88}$$

$$\partial_\xi^2 \partial_\eta \gamma = -\frac{1}{\eta^2} \left(\theta(1+\theta')\right)' \cdot \frac{1}{\eta\gamma''} = -\frac{1}{\eta^3 \gamma''} \left(\theta(1+\theta')\right)'\,, \tag{3.7.89}$$

$$\partial_\xi \partial_\eta^2 \gamma = \frac{2}{\eta^3} \theta(1+\theta') - \frac{1}{\eta^2} \left(\theta(1+\theta')\right)' \cdot \left(-\frac{1}{\eta}\theta\right) \tag{3.7.90}$$

$$= \frac{1}{\eta^3} \theta \left(2(1+\theta') + (\theta(1+\theta'))'\right)\,. \tag{3.7.91}$$

Let σ_1, σ_2 be the singular value of A. Hence, we have

$$\left| \lambda s \cdot A \cdot \begin{pmatrix} u \\ v \end{pmatrix} \right| \geq |\lambda s| \cdot \sigma_2 \cdot \left| \begin{pmatrix} u \\ v \end{pmatrix} \right| \tag{3.7.92}$$

$$= |\lambda s| \cdot \frac{|\det A|}{\sigma_1} \cdot \left| \begin{pmatrix} u \\ v \end{pmatrix} \right| = |\lambda s| \cdot \frac{|\det A|}{\|A\|_{L^2 \to L^2}} \cdot \left| \begin{pmatrix} u \\ v \end{pmatrix} \right|\,. \tag{3.7.93}$$

By the calculation of higher derivatives of γ, we have

$$|\det A| = \left| \det \begin{pmatrix} \frac{1}{\eta^2 \gamma''} \cdot \theta' & -\frac{1}{\eta^2} \theta \left(1+\theta'\right) \\ \frac{1}{\eta^3 \gamma''} \cdot \left(\frac{1}{\gamma''} \cdot \theta'\right)' & -\frac{1}{\eta^3 \gamma''} \left(\theta(1+\theta')\right)' \end{pmatrix} (\xi_2, \eta_2) \right| \tag{3.7.94}$$

$$= \left| \frac{1}{\eta^5 (\gamma'')^2} \cdot \left(\theta \cdot \theta'' - \theta'(1+\theta')\right) \right| (\xi_2, \eta_2) \tag{3.7.95}$$

$$\geq \frac{1}{(1-\delta)^5 \|\gamma''\|_{L^\infty}^2} \cdot \inf_{\xi, \eta} \left| \theta \cdot \theta'' - \theta'(1+\theta') \right| (\xi, \eta) \tag{3.7.96}$$

$$\geq \frac{2^5}{\|\gamma''\|_{L^\infty}^2} \cdot C_{\gamma,8}\,. \tag{3.7.97}$$

On the other hand, we also have

$$\frac{1}{\|A\|_{L^2 \to L^2}} \geq 2^{-3} \cdot \inf_{\xi, \eta} \left\| \begin{pmatrix} \frac{1}{\gamma''} \cdot \theta' & -\theta \left(1+\theta'\right) \\ \frac{1}{\gamma''} \cdot \left(\frac{1}{\gamma''} \cdot \theta'\right)' & -\frac{1}{\gamma''} \left(\theta(1+\theta')\right)' \end{pmatrix} \right\|_{L^2 \to L^2}^{-1} (\xi, \eta) = 2^{-3} \cdot C_{\gamma,9}\,. \tag{3.7.98}$$

Combining (3.7.93), (3.7.97), (3.7.98), we have

$$\left| \lambda s \cdot A \cdot \begin{pmatrix} u \\ v \end{pmatrix} \right| \geq |\lambda s| \frac{2^2}{\|\gamma''\|_{L^\infty}^2} \frac{C_{\gamma,8}}{C_{\gamma,9}} \left| \begin{pmatrix} u \\ v \end{pmatrix} \right|\,. \tag{3.7.99}$$

On the other hand,

$$\left|\lambda s\cdot B\cdot\begin{pmatrix}u\\v\end{pmatrix}\right|\le|\lambda s|\cdot\|B\|_{L^2\to L^2}\cdot\left|\begin{pmatrix}u\\v\end{pmatrix}\right|\,. \tag{3.7.100}$$

By fundamental theorem of calculus and mean value theorem for integral, B equals to

$$\begin{pmatrix}b_{11}&b_{12}\\b_{21}&b_{22}\end{pmatrix} \tag{3.7.101}$$

where

$$b_{11}=\left(\partial_\xi^3\gamma(t(\xi_3,\eta_3))\quad\partial_\xi^2\partial_\eta\gamma(t(\xi_3,\eta_3))\right)\cdot\begin{pmatrix}\xi_1-\xi_2\\\eta_1-\eta_2\end{pmatrix}\,, \tag{3.7.102}$$

$$b_{12}=\left(\partial_\xi^2\partial_\eta\gamma(t(\xi_3,\eta_3))\quad\partial_\xi\partial_\eta^2\gamma(t(\xi_3,\eta_3))\right)\cdot\begin{pmatrix}\xi_1-\xi_2\\\eta_1-\eta_2\end{pmatrix}\,, \tag{3.7.103}$$

$$b_{21}=0\,,\quad b_{22}=0\,. \tag{3.7.104}$$

Since

$$|\xi_1-\xi_2|<2\delta,\quad|\eta_1-\eta_2|<2\delta, \tag{3.7.105}$$

we have

$$\|B\|_{L^2\to L^2}\le4\delta\cdot2^3\sup_{\xi,\eta}\Bigg(\left|\frac{1}{\gamma''}\cdot\left(\frac{1}{\gamma''}\cdot\theta'\right)'\right|(\xi,\eta)+2\left|\frac{1}{\gamma''}\left(\theta(1+\theta')\right)'\right|(\xi,\eta) \tag{3.7.106}$$

$$+\left|\theta\left(2(1+\theta')+(\theta(1+\theta'))'\right)\right|(\xi,\eta)\Bigg)=\delta\cdot C_{\gamma,10} \tag{3.7.107}$$

where

$$C_{\gamma,10}:=2^5\sup_{\xi,\eta}\Bigg(\left|\frac{1}{\gamma''}\cdot\left(\frac{1}{\gamma''}\cdot\theta'\right)'\right|(\xi,\eta)+2\left|\frac{1}{\gamma''}\left(\theta(1+\theta')\right)'\right|(\xi,\eta) \tag{3.7.108}$$

$$+\left|\theta\left(2(1+\theta')+(\theta(1+\theta'))'\right)\right|(\xi,\eta)\Bigg) \tag{3.7.109}$$

Place (3.7.107) in (3.7.100), we obtain

$$\left|\lambda s\cdot B\cdot\begin{pmatrix}u\\v\end{pmatrix}\right|\le|\lambda s|\delta C_{\gamma,10}\left|\begin{pmatrix}u\\v\end{pmatrix}\right|\,. \tag{3.7.110}$$

Combine (3.7.99) and (3.7.110), then place in (3.7.82), we have

$$\left|\begin{pmatrix}\partial_\xi\widetilde{\Phi}(\lambda,s,u,v,\xi,\eta)\\\partial_\xi^2\widetilde{\Phi}(\lambda,s,u,v,\xi,\eta)\end{pmatrix}\right|\ge|\lambda s|\frac{2^2}{\|\gamma''\|_{L^\infty}^2}\frac{C_{\gamma,8}}{C_{\gamma,9}}\left|\begin{pmatrix}u\\v\end{pmatrix}\right|-|\lambda s|\delta C_{\gamma,10}\left|\begin{pmatrix}u\\v\end{pmatrix}\right| \tag{3.7.111}$$

$$\ge|\lambda vs|\left(\frac{2^2}{\|\gamma''\|_{L^\infty}^2}\frac{C_{\gamma,8}}{C_{\gamma,9}}-\delta C_{\gamma,10}\right)\ge|\lambda vs|\cdot\frac{2}{\|\gamma''\|_{L^\infty}^2}\frac{C_{\gamma,8}}{C_{\gamma,9}}=|\lambda vs|C_{\gamma,10}^4 \tag{3.7.112}$$

where the last inequality comes from the assumption

$$\delta\le\frac{2}{\|\gamma''\|_{L^\infty}^2}\frac{C_{\gamma,8}}{C_{\gamma,9}C_{\gamma,10}}\,. \tag{3.7.113}$$

Denote the left hand side of (3.7.66) by M:

$$M := \max\left\{\left|\partial_\xi\widetilde{\Phi}(\lambda,s,u,v,\xi,\eta)\right|, \left|\partial_\xi^2\widetilde{\Phi}(\lambda,s,u,v,\xi,\eta)\right|^{\frac{1}{2}}\right\} \tag{3.7.114}$$

Then we have

$$(M^2+M^4)^{\frac{1}{2}} \geq \left|\begin{pmatrix}\partial_\xi\widetilde{\Phi}(\lambda,s,u,v,\xi,\eta)\\ \partial_\xi^2\widetilde{\Phi}(\lambda,s,u,v,\xi,\eta)\end{pmatrix}\right| \geq |\lambda vs| C_{\gamma,10}^4 \,. \tag{3.7.115}$$

Solving this second order inequality, we obtain

$$M \geq \left(-\frac{1}{2}+\sqrt{\frac{1}{2}+C_{\gamma,10}^4|\lambda vs|^2}\right)^{\frac{1}{2}} \geq \left(-\frac{1}{2}+C_{\gamma,10}^2|\lambda vs|\right)^{\frac{1}{2}} \tag{3.7.116}$$

$$\geq \left(\frac{1}{2}C_{\gamma,10}^2|\lambda vs|\right)^{\frac{1}{2}} \geq \frac{1}{2}C_{\gamma,10}|\lambda vs|^{\frac{1}{2}} \,. \tag{3.7.117}$$

□

Back to the proof of (3.1.68). For $C_{\gamma,10}^2|\lambda\frac{v}{\lambda}\frac{s}{\lambda}| < 1$, we bound (3.7.55) trivially by

$$2\delta\|\widetilde{\Psi}\|_{L^\infty} = 2\delta\frac{1}{\lambda}C_{\psi,\gamma,\delta,5} \,. \tag{3.7.118}$$

For $C_{\gamma,10}^2|\lambda\frac{v}{\lambda}\frac{s}{\lambda}| \geq 1$, by the modulus of (3.1.61), (3.7.60), (3.7.65), and *Lemma* 3.7.2, we bound the modulus of (3.7.55) by

$$4^3\left(\|\widetilde{\Psi}\|_{L^\infty}+\|\partial_\xi\widetilde{\Psi}\|_{L^1}\right)\cdot\frac{M_2(\widetilde{\Phi})}{\inf\limits_{t\in[a,b]}(J_2\widetilde{\Phi})(t)} \tag{3.7.119}$$

$$\leq 4^3\left(\frac{1}{\lambda}C_{\psi,\gamma,\delta,5}+\frac{1}{\lambda}C_{\psi,\gamma,\delta,6}\right)\cdot\widetilde{M}(\gamma)\cdot\frac{1}{C_{\gamma,10}}\cdot\left|\lambda\frac{v}{\lambda}\frac{s}{\lambda}\right|^{-\frac{1}{2}} \tag{3.7.120}$$

$$= 4^3\left(\frac{C_{\psi,\gamma,\delta,5}+C_{\psi,\gamma,\delta,6}}{C_{\gamma,10}}\right)\cdot\widetilde{M}(\gamma)\cdot\frac{1}{\lambda}\left|\frac{vs}{\lambda}\right|^{-\frac{1}{2}} \,. \tag{3.7.121}$$

Hence, we have

$$\|m_{\gamma,\lambda}\|_U^4 = \left\|\int_{\mathbb{R}}\mathcal{D}_{(0,s)}\mathcal{D}_{(u,v)}m_{\gamma,\lambda}(\xi,\eta)d\xi\right\|_{L^\infty_{u\eta}L^1_sL^2_v} \tag{3.7.122}$$

$$\leq 2\left\|\int_{\mathbb{R}}\mathcal{D}_{(0,s)}\mathcal{D}_{(u,v)}m(\xi,\eta)d\xi\right\|_{L^\infty_{u\eta}L^1_s([0,2\delta\lambda])L^2_v([0,2\delta\lambda])} \tag{3.7.123}$$

$$\leq 2\left\|\int_0^{2\delta\lambda}2\delta\frac{1}{\lambda}C_{\psi,\gamma,\delta,5}\,ds\right\|_{L^2_v([0,1])}+2\left\|\int_0^{\frac{\lambda}{vC_{\gamma,10}^2}}2\delta\frac{1}{\lambda}C_{\psi,\gamma,\delta,5}\,ds\right. \tag{3.7.124}$$

$$\left.+\int_{\frac{\lambda}{vC_{\gamma,10}^2}}^{2\delta\lambda}4^3\left(\frac{C_{\psi,\gamma,\delta,5}+C_{\psi,\gamma,\delta,6}}{C_{\gamma,10}}\right)\cdot\widetilde{M}(\gamma)\cdot\frac{1}{\lambda}\left|\frac{vs}{\lambda}\right|^{-\frac{1}{2}}ds\right\|_{L^2_v([1,2\delta\lambda])} \tag{3.7.125}$$

After the integration in s, by triangle inequality, we move L_v^2 norm inside and equate the above term as

$$8\delta^2 C_{\psi,\gamma,\delta,5} + 2\left(\left\| \frac{2\delta C_{\psi,\gamma,\delta,5}}{C_{\gamma,10}^2} \cdot \frac{1}{v} \right\|_{L_v^2([1,2\delta\lambda])} + 4^3 \left(\frac{C_{\psi,\gamma,\delta,5} + C_{\psi,\gamma,\delta,6}}{C_{\gamma,10}} \right) \cdot \widetilde{M}(\gamma) \right. \tag{3.7.126}$$

$$\left. \cdot 2 \left\| \frac{1}{(2\delta v)^{\frac{1}{2}}} - \frac{1}{C_{\gamma,10} v} \right\|_{L_v^2([1,2\delta\lambda])} \right) \tag{3.7.127}$$

$$\leq 8\delta^2 C_{\psi,\gamma,\delta,5} + \frac{4\delta C_{\psi,\gamma,\delta,5}}{C_{\gamma,10}^2} + \frac{4\delta^{\frac{1}{2}} C_{\psi,\gamma,\delta,5}}{C_{\gamma,10}^2} \lambda^{-\frac{1}{2}} \tag{3.7.128}$$

$$+4^4 \left(\frac{C_{\psi,\gamma,\delta,5} + C_{\psi,\gamma,\delta,6}}{C_{\gamma,10}^2} \right) \cdot \widetilde{M}(\gamma) + 4^4 \left(\frac{C_{\psi,\gamma,\delta,5} + C_{\psi,\gamma,\delta,6}}{C_{\gamma,10}^2 \delta^{\frac{1}{2}}} \right) \cdot \widetilde{M}(\gamma) \lambda^{-\frac{1}{2}} \tag{3.7.129}$$

$$+4^4 \left(\frac{C_{\psi,\gamma,\delta,5} + C_{\psi,\gamma,\delta,6}}{C_{\gamma,10}} \right) \cdot \widetilde{M}(\gamma) (\log(4\delta^2\lambda))^{\frac{1}{2}} \tag{3.7.130}$$

Since $\lambda > 1$, we have $\lambda^{-\frac{1}{2}} \leq 1$ and can bound the above expression by

$$8\delta^2 C_{\psi,\gamma,\delta,5} + \frac{4C_{\psi,\gamma,\delta,5}}{C_{\gamma,10}^2} (\delta + \delta^{\frac{1}{2}}) + 4^4 \left(\frac{C_{\psi,\gamma,\delta,5} + C_{\psi,\gamma,\delta,6}}{C_{\gamma,10}^2} \right) \widetilde{M}(\gamma) (1 + \delta^{-\frac{1}{2}}) \tag{3.7.131}$$

$$+4^3 \left(\frac{C_{\psi,\gamma,\delta,5} + C_{\psi,\gamma,\delta,6}}{C_{\gamma,10}} \right)^{\frac{1}{2}} \widetilde{M}(\gamma)^{\frac{1}{2}} (\log(2\delta))^{\frac{1}{2}} + 4^4 \left(\frac{C_{\psi,\gamma,\delta,5} + C_{\psi,\gamma,\delta,6}}{C_{\gamma,10}} \right) \cdot \widetilde{M}(\gamma) (\log\lambda)^{\frac{1}{2}} \tag{3.7.132}$$

$$= C_{\psi,\gamma,\delta,11} + C_{\psi,\gamma,\delta,12} (\log\lambda)^{\frac{1}{2}} . \tag{3.7.133}$$

Bibliography

[1] Lars Becker, Polona Durcik, and Fred Yu-Hsiang Lin. On trilinear singular Brascamp-Lieb integrals. 2024. URL: `https://arxiv.org/abs/2411.00141`, `arXiv:2411.00141`.

[2] Lars Becker, Floris van Doorn, Asgar Jamneshan, Rajula Srivastava, and Christoph Thiele. Carleson operators on doubling metric measure spaces. 2024. URL: `https://arxiv.org/abs/2405.06423`, `arXiv:2405.06423`.

[3] David Beltran, Shaoming Guo, Jonathan Hickman, and Andreas Seeger. Sharp L^p bounds for the helical maximal function. *Amer. J. Math.*, 147(1):149–234, 2025.

[4] Jonathan Bennett, Anthony Carbery, Michael Christ, and Terence Tao. The Brascamp-Lieb inequalities: finiteness, structure and extremals. *Geom. Funct. Anal.*, 17(5):1343–1415, 2008. `doi:10.1007/s00039-007-0619-6`.

[5] Frédéric Bernicot. Fiber-wise Calderón-Zygmund decomposition and application to a bi-dimensional paraproduct. *Illinois J. Math.*, 56(2):415–422, 2012. URL: `http://projecteuclid.org/euclid.ijm/1385129956`.

[6] Ankit Bhojak, Surjeet Singh Choudhary, Saurabh Shrivastava, and Kalachand Shuin. Sharp endpoint L^p-estimates for bilinear spherical maximal functions. 2024. URL: `https://arxiv.org/abs/2310.00425`, `arXiv:2310.00425`.

[7] Tainara Borges, Benjamin Foster, and Yumeng Ou. Sobolev smoothing estimates for bilinear maximal operators with fractal dilation sets. *J. Funct. Anal.*, 288(2):Paper No. 110694, 53, 2025. `doi:10.1016/j.jfa.2024.110694`.

[8] J. Bourgain. Averages in the plane over convex curves and maximal operators. *J. Analyse Math.*, 47:69–85, 1986. `doi:10.1007/BF02792533`.

[9] Jean Bourgain. A nonlinear version of Roth's theorem for sets of positive density in the real line. *Journal d'Analyse Mathématique*, 50(1):169–181, 1988.

[10] Kevin Buzzard, Johan Commelin, and Patrick Massot. Formalising perfectoid spaces. In *Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs*, POPL '20, page 299–312. ACM, January 2020. URL: `http://dx.doi.org/10.1145/3372885.3373830`, `doi:10.1145/3372885.3373830`.

[11] A.-P. Calderón. Commutators of singular integral operators. *Proc. Nat. Acad. Sci. U.S.A.*, 53:1092–1099, 1965. `doi:10.1073/pnas.53.5.1092`.

[12] A. P. Calderon and A. Zygmund. On the existence of certain singular integrals. *Acta Math.*, 88:85–139, 1952. `doi:10.1007/BF02392130`.

[13] A. P. Calderón and A. Zygmund. On singular integrals. *Amer. J. Math.*, 78:289–309, 1956. `doi:10.2307/2372517`.

[14] Anthony Carbery, Michael Christ, James Vance, Stephen Wainger, and David K. Watson. Operators associated to flat plane curves: L^p estimates via dilation methods. *Duke Math. J.*, 59(3):675–700, 1989. `doi:10.1215/S0012-7094-89-05930-9`.

[15] Lennart Carleson. On convergence and growth of partial sums of Fourier series. *Acta Math.*, 116:135–157, 1966. `doi:10.1007/BF02392815`.

[16] Jiao Chen, Martin Hsu, and Fred Yu-Hsiang Lin. A sharp Hörmander condition for bilinear Fourier multipliers with Lipschitz singularities. 2024. URL: `https://arxiv.org/abs/2403.04721`, `arXiv:2403.04721`.

[17] Michael Christ. Hilbert transforms along curves. I. Nilpotent groups. *Ann. of Math. (2)*, 122(3):575–596, 1985. `doi:10.2307/1971330`.

[18] Michael Christ. Hilbert transforms along curves. II. A flat case. *Duke Math. J.*, 52(4):887–894, 1985. `doi:10.1215/S0012-7094-85-05246-9`.

[19] Michael Christ. Weak type $(1,1)$ bounds for rough operators. *Ann. of Math. (2)*, 128(1):19–42, 1988. `doi:10.2307/1971461`.

[20] Michael Christ, Polona Durcik, and Joris Roos. Trilinear smoothing inequalities and a variant of the triangular Hilbert transform. *Adv. Math.*, 390:Paper No. 107863, 60, 2021. `doi:10.1016/j.aim.2021.107863`.

[21] Michael Christ, Alexander Nagel, Elias M. Stein, and Stephen Wainger. Singular and maximal Radon transforms: analysis and geometry. *Ann. of Math. (2)*, 150(2):489–577, 1999. `doi:10.2307/121088`.

[22] Michael Christ and José Luis Rubio de Francia. Weak type $(1,1)$ bounds for rough operators. II. *Invent. Math.*, 93(1):225–237, 1988. `doi:10.1007/BF01393693`.

[23] Michael Christ and Zirui Zhou. A class of singular bilinear maximal functions. *J. Funct. Anal.*, 287(8):Paper No. 110572, 37, 2024. `doi:10.1016/j.jfa.2024.110572`.

[24] R. Coifman and Y. Meyer. Commutateurs d'intégrales singulières et opérateurs multilinéaires. *Ann. Inst. Fourier (Grenoble)*, 28(3):xi, 177–202, 1978. `doi:10.5802/aif.708`.

[25] R. R. Coifman, A. McIntosh, and Y. Meyer. L'intégrale de Cauchy définit un opérateur borné sur L^2 pour les courbes lipschitziennes. *Ann. of Math. (2)*, 116(2):361–387, 1982. `doi:10.2307/2007065`.

[26] Ronald R. Coifman and Yves Meyer. *Au delà des opérateurs pseudo-différentiels*, volume 57 of *Astérisque*. Société Mathématique de France, Paris, 1978. With an English summary.

[27] Leonardo de Moura, Soonho Kong, Jeremy Avigad, Floris van Doorn, and Jakob von Raumer. The lean theorem prover (system description). In *Automated deduction—CADE 25*, volume 9195 of *Lecture Notes in Comput. Sci.*, pages 378–388. Springer, Cham, 2015. URL: `https://doi.org/10.1007/978-3-319-21401-6_26`, `doi:10.1007/978-3-319-21401-6\_26`.

[28] Ciprian Demeter and Christoph Thiele. On the two-dimensional bilinear Hilbert transform. *Amer. J. Math.*, 132(1):201–256, 2010. `doi:10.1353/ajm.0.0101`.

[29] Geoff Diestel, Loukas Grafakos, Petr Honzík, Zengyan Si, and Erin Terwilleger Mullen. Method of rotations for bilinear singular integrals. *Commun. Math. Anal.*, pages 99–107, 2011.

[30] Yen Do and Christoph Thiele. L^p theory for outer measures and two themes of Lennart Carleson united. *Bull. Amer. Math. Soc. (N.S.)*, 52(2):249–296, 2015. `doi:10.1090/S0273-0979-2014-01474-0`.

[31] Georgios Dosidis, Bae Jun Park, and Lenka Slavikova. Boundedness criteria for bilinear Fourier multipliers via shifted square function estimates. 2024. URL: `https://arxiv.org/abs/2402.15785`, `arXiv:2402.15785`.

[32] Georgios Dosidis and Lenka Slavíková. Multilinear singular integrals with homogeneous kernels near L^1. *Math. Ann.*, 389(3):2259–2271, 2024. `doi:10.1007/s00208-023-02691-x`.

[33] Polona Durcik, Shaoming Guo, and Joris Roos. A polynomial Roth theorem on the real line. *Trans. Amer. Math. Soc.*, 371(10):6973–6993, 2019. `doi:10.1090/tran/7574`.

[34] Polona Durcik and Christoph Thiele. Singular Brascamp-Lieb: a survey. In *Geometric aspects of harmonic analysis*, volume 45 of *Springer INdAM Ser.*, pages 321–349. Springer, Cham, [2021] ©2021. URL: `https://doi.org/10.1007/978-3-030-72058-2_9`, `doi:10.1007/978-3-030-72058-2\_9`.

[35] Charles Fefferman. Pointwise convergence of Fourier series. *Ann. of Math. (2)*, 98:551–571, 1973. `doi:10.2307/1970917`.

[36] Marco Fraccaroli, Olli Saari, and Christoph Thiele. Phase space localizing operators. 2024. URL: `https://arxiv.org/abs/2210.16164`, `arXiv:2210.16164`.

[37] Robert Fraser, Shaoming Guo, and Malabika Pramanik. Polynomial Roth theorems on sets of fractional dimensions. *Int. Math. Res. Not. IMRN*, (10):7809–7838, 2022. `doi:10.1093/imrn/rnaa377`.

[38] Alejandra Gaitan and Victor Lie. Non-zero to zero curvature transition: Operators along hybrid curves with no quadratic (quasi-)resonances. 2024. URL: `https://arxiv.org/abs/2402.03451`, `arXiv:2402.03451`.

[39] John E. Gilbert and Andrea R. Nahmod. Boundedness of bilinear operators with nonsmooth symbols. *Math. Res. Lett.*, 7(5-6):767–778, 2000. `doi:10.4310/MRL.2000.v7.n6.a9`.

[40] W. T. Gowers, Ben Green, Freddie Manners, and Terence Tao. On a conjecture of Marton. *Ann. of Math. (2)*, 201(2):515–549, 2025. doi:10.4007/annals.2025.201.2.5.

[41] Loukas Grafakos, Danqing He, and Petr Honzík. The Hörmander multiplier theorem, II: The bilinear local L^2 case. *Math. Z.*, 289(3-4):875–887, 2018. doi:10.1007/s00209-017-1979-8.

[42] Loukas Grafakos, Danqing He, and Petr Honzík. Rough bilinear singular integrals. *Adv. Math.*, 326:54–78, 2018. doi:10.1016/j.aim.2017.12.013.

[43] Loukas Grafakos, Danqing He, and Lenka Slavíková. $L^2 \times L^2 \to L^1$ boundedness criteria. *Math. Ann.*, 376(1-2):431–455, 2020. doi:10.1007/s00208-018-1794-5.

[44] Loukas Grafakos and Xiaochun Li. Uniform bounds for the bilinear Hilbert transforms. I. *Ann. of Math. (2)*, 159(3):889–933, 2004. doi:10.4007/annals.2004.159.889.

[45] Loukas Grafakos and Xiaochun Li. The disc as a bilinear multiplier. *Amer. J. Math.*, 128(1):91–119, 2006. URL: http://muse.jhu.edu/journals/american_journal_of_mathematics/v128/128.1grafakos.pdf.

[46] Loukas Grafakos and Zengyan Si. The Hörmander multiplier theorem for multilinear operators. *J. Reine Angew. Math.*, 668:133–147, 2012. doi:10.1515/crelle.2011.137.

[47] Loukas Grafakos and Rodolfo H. Torres. Multilinear Calderón-Zygmund theory. *Adv. Math.*, 165(1):124–164, 2002. doi:10.1006/aima.2001.2028.

[48] Philip Gressman, Danqing He, Vjekoslav Kovac, Brian Street, Christoph Thiele, and Po-Lam Yung. On a trilinear singular integral form with determinantal kernel. *Proc. Amer. Math. Soc.*, 144(8):3465–3477, 2016. doi:10.1090/proc/13007.

[49] Danqing He and Bae Jun Park. Improved estimates for bilinear rough singular integrals. *Math. Ann.*, 386(3-4):1951–1978, 2023. doi:10.1007/s00208-022-02444-2.

[50] Kevin Henriot, Izabella Laba, and Malabika Pramanik. On polynomial configurations in fractal sets. *Anal. PDE*, 9(5):1153–1184, 2016. doi:10.2140/apde.2016.9.1153.

[51] Steve Hofmann. Weak $(1,1)$ boundedness of singular integrals with nonsmooth kernel. *Proc. Amer. Math. Soc.*, 103(1):260–264, 1988. doi:10.2307/2047563.

[52] Lars Hörmander. Estimates for translation invariant operators in L^p spaces. *Acta Math.*, 104:93–140, 1960. doi:10.1007/BF02547187.

[53] Martin Hsu and Fred Yu-Hsiang Lin. A short proof on the boundedness of triangular hilbert transform along curves. 2024. arXiv:2410.15791.

[54] Eunhee Jeong and Sanghyuk Lee. Maximal estimates for the bilinear spherical averages and the bilinear Bochner-Riesz operators. *J. Funct. Anal.*, 279(7):108629, 29, 2020. doi:10.1016/j.jfa.2020.108629.

[55] Roger L. Jones, Andreas Seeger, and James Wright. Strong variational and jump inequalities in harmonic analysis. *Trans. Amer. Math. Soc.*, 360(12):6711–6742, 2008. doi:10.1090/S0002-9947-08-04538-8.

[56] Carlos E. Kenig and Elias M. Stein. Multilinear estimates and fractional integration. *Math. Res. Lett.*, 6(1):1–15, 1999. doi:10.4310/MRL.1999.v6.n1.a1.

[57] Hyerim Ko, Sanghyuk Lee, and Sewook Oh. Maximal estimates for averages over space curves. *Invent. Math.*, 228(2):991–1035, 2022. doi:10.1007/s00222-021-01094-6.

[58] Hyerim Ko, Sanghyuk Lee, and Sewook Oh. Sharp smoothing properties of averages over curves. *Forum Math. Pi*, 11:Paper No. e4, 33, 2023. doi:10.1017/fmp.2023.2.

[59] Vjekoslav Kovac. Boundedness of the twisted paraproduct. *Rev. Mat. Iberoam.*, 28(4):1143–1164, 2012. doi:10.4171/RMI/707.

[60] Vjekoslav Kovac, Christoph Thiele, and Pavel Zorin-Kranich. Dyadic triangular Hilbert transform of two general functions and one not too general function. *Forum Math. Sigma*, 3:Paper No. e25, 27, 2015. doi:10.1017/fms.2015.25.

[61] Ben Krause, Mariusz Mirek, Sarah Peluse, and James Wright. Polynomial progressions in topological fields. *Forum Math. Sigma*, 12:Paper No. e106, 51, 2024. doi:10.1017/fms.2024.104.

[62] Izabella Laba and Malabika Pramanik. Arithmetic progressions in sets of fractional dimension. *Geom. Funct. Anal.*, 19(2):429–456, 2009. doi:10.1007/s00039-009-0003-9.

[63] Michael Lacey and Christoph Thiele. L^p estimates on the bilinear Hilbert transform for $2 < p < \infty$. *Ann. of Math. (2)*, 146(3):693–724, 1997. doi:10.2307/2952458.

[64] Michael Lacey and Christoph Thiele. On Calderón's conjecture. *Ann. of Math. (2)*, 149(2):475–496, 1999. doi:10.2307/120971.

[65] Michael Lacey and Christoph Thiele. A proof of boundedness of the Carleson operator. *Math. Res. Lett.*, 7(4):361–370, 2000. doi:10.4310/MRL.2000.v7.n4.a1.

[66] Jongho Lee, Yaryong Heo, Sunggeum Hong, Jin Bong Lee, Bae Jun Park, Yejune Park, and Chan Woo Yang. The Hörmander multiplier theorem for n-linear operators. *Math. Ann.*, 381(1-2):499–555, 2021. doi:10.1007/s00208-021-02162-1.

[67] Xiaochun Li. Uniform bounds for the bilinear Hilbert transforms. II. *Rev. Mat. Iberoam.*, 22(3):1069–1126, 2006. doi:10.4171/RMI/483.

[68] Xiaochun Li. Bilinear Hilbert transforms along curves I: The monomial case. *Anal. PDE*, 6(1):197–220, 2013. doi:10.2140/apde.2013.6.197.

[69] Victor Lie. On the boundedness of the bilinear Hilbert transform along "non-flat" smooth curves. *Amer. J. Math.*, 137(2):313–363, 2015. doi:10.1353/ajm.2015.0013.

[70] Victor Lie. On the boundedness of the bilinear Hilbert transform along "non-flat" smooth curves. The Banach triangle case (L^r, $1 \leq r < \infty$). *Rev. Mat. Iberoam.*, 34(1):331–353, 2018. doi:10.4171/RMI/987.

[71] Fred Yu-Hsiang Lin. On the family of singular Brascamp-Lieb inequalities with dimension datum $(1, 2, 2, 1)$. *Mathematika*, 71(1):Paper No. e70003, 13, 2025. doi:10.1112/mtk.70003.

[72] Roman Matuszewski and Piotr Rudnicki. Mizar: the first 30 years. *Mechanized mathematics and its applications*, 4(1):3–24, 2005.

[73] S. G. Mihlin. On the multipliers of Fourier integrals. *Dokl. Akad. Nauk SSSR (N.S.)*, 109:701–703, 1956.

[74] Camil Muscalu. Calderón commutators and the Cauchy integral on Lipschitz curves revisited: I. First commutator and generalizations. *Rev. Mat. Iberoam.*, 30(2):727–750, 2014. doi:10.4171/RMI/798.

[75] Camil Muscalu. Calderón commutators and the Cauchy integral on Lipschitz curves revisited II. The Cauchy integral and its generalizations. *Rev. Mat. Iberoam.*, 30(3):1089–1122, 2014. doi:10.4171/RMI/808.

[76] Camil Muscalu and Wilhelm Schlag. *Classical and multilinear harmonic analysis. Vol. I*, volume 137 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2013.

[77] Camil Muscalu, Terence Tao, and Christoph Thiele. Multi-linear operators given by singular multipliers. *J. Amer. Math. Soc.*, 15(2):469–496, 2002. doi:10.1090/S0894-0347-01-00379-4.

[78] Florin Camil Muscalu. *L(p) estimates for multilinear operators given by singular symbols*. ProQuest LLC, Ann Arbor, MI, 2000. Thesis (Ph.D.)–Brown University. URL: http://gateway.proquest.com/openurl?url_ver=Z39.88-2004&rft_val_fmt=info:ofi/fmt:kev:mtx:dissertation&res_dat=xri:pqdiss&rft_dat=xri:pqdiss:9987818.

[79] Alexander Nagel, James Vance, Stephen Wainger, and David Weinberg. Hilbert transforms for convex curves. *Duke Math. J.*, 50(3):735–744, 1983. doi:10.1215/S0012-7094-83-05036-6.

[80] Tobias Nipkow, Lawrence C. Paulson, and Markus Wenzel. *Isabelle/HOL*, volume 2283 of *Lecture Notes in Computer Science*. Springer-Verlag, Berlin, 2002. A proof assistant for higher-order logic. doi:10.1007/3-540-45949-9.

[81] Sarah Peluse. Three-term polynomial progressions in subsets of finite fields. *Israel J. Math.*, 228(1):379–405, 2018. doi:10.1007/s11856-018-1768-z.

[82] Sarah Peluse. On the polynomial Szemerédi theorem in finite fields. *Duke Math. J.*, 168(5):749–774, 2019. doi:10.1215/00127094-2018-0051.

[83] Sarah Peluse. Bounds for sets with no polynomial progressions. *Forum Math. Pi*, 8:e16, 55, 2020. doi:10.1017/fmp.2020.11.

[84] Sarah Peluse and Sean Prendiville. Quantitative bounds in the nonlinear Roth theorem. *Invent. Math.*, 238(3):865–903, 2024. doi:10.1007/s00222-024-01293-x.

[85] Sean Prendiville. Quantitative bounds in the polynomial Szemerédi theorem: the homogeneous case. *Discrete Anal.*, pages Paper No. 5, 34, 2017. doi:10.19086/da.1282.

[86] Marcel Riesz. Sur les maxima des formes bilinéaires et sur les fonctionnelles linéaires. *Acta Math.*, 49(3-4):465–497, 1927. doi:10.1007/BF02564121.

[87] Olli Saari and Christoph Thiele. Paraproducts for bilinear multipliers associated with convex sets. *Math. Ann.*, 385(3-4):2013–2036, 2023. doi:10.1007/s00208-022-02397-6.

[88] Peter Scholze. Perfectoid spaces. *Publ. Math. Inst. Hautes Études Sci.*, 116:245–313, 2012. doi:10.1007/s10240-012-0042-x.

[89] Andreas Seeger. Singular integral operators with rough convolution kernels. *J. Amer. Math. Soc.*, 9(1):95–105, 1996. doi:10.1090/S0894-0347-96-00185-3.

[90] Elias M. Stein. Maximal functions. I. Spherical means. *Proc. Nat. Acad. Sci. U.S.A.*, 73(7):2174–2175, 1976. doi:10.1073/pnas.73.7.2174.

[91] Elias M. Stein and Stephen Wainger. Problems in harmonic analysis related to curvature. *Bull. Amer. Math. Soc.*, 84(6):1239–1295, 1978. doi:10.1090/S0002-9904-1978-14554-6.

[92] Christoph Thiele. A uniform estimate. *Ann. of Math. (2)*, 156(2):519–563, 2002. doi:10.2307/3597197.

[93] Christoph Thiele. *Wave packet analysis*, volume 105 of *CBMS Regional Conference Series in Mathematics*. Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2006. doi:10.1090/cbms/105.

[94] Naohito Tomita. A Hörmander type multiplier theorem for multilinear operators. *J. Funct. Anal.*, 259(8):2028–2044, 2010. doi:10.1016/j.jfa.2010.06.010.

[95] Gennady Uraltsev. *Time-frequency analysis of the variational Carleson operator using outer-measure L p spaces.* PhD thesis, Universitäts-und Landesbibliothek Bonn, 2017.

[96] Gennady Uraltsev and Michał Warchalski. The full range of uniform bounds for the bilinear Hilbert transform. *arXiv preprint arXiv:2205.09851*, 2022.