

The Effects of Differential Constraints and Surface Energies on Scaling Laws for Singular Perturbation Problems

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Abstract

Motivated by mathematical models for phase transformations, arising, for example, in shape-memory alloys, and for micromagnetism in ferromagnetic materials, we study a class of $\mathcal{A}$ -free differential inclusions quantitatively. For this reason, we consider singular perturbation models of differential inclusions under an $\mathcal{A}$ -free constraint and determine the scaling in the singular perturbation parameter. The focus of this thesis is to characterize the influence of the order of the operator on the possible scaling laws and to show that the model is robust under changes of the surface energy. The first is achieved by establishing lower bounds for the compatible and incompatible two-well problem. It turns out that the scaling is determined by the maximal vanishing order on the sphere of the symbol of the differential operator applied to the compatible direction. These lower bounds are proven to be optimal for the divergence operator and for a higher order generalization of the curl and curl curl operators as an annihilator of symmetrized derivatives. The influence of the surface energy is studied by comparing sharp and diffuse interface models, as well as suitable interpolations of these. We deduce the lower bound for the diffuse model by estimating the energy from below through the sharp interface model and complement them with matching upper bounds for a model class of wells in the case of the curl operator. Furthermore, an N -well setting giving rise to higher order laminates for the curl operator is studied. If the energy penalizes only oscillations in a certain direction, we observe that for almost all directions this anisotropic energy scales like the full isotropic energy. The exceptional directions are those, where the anisotropic energy does not penalize the oscillations of the “inner-most” laminate. The scaling for these matches the ones of a lower order laminate. Furthermore, a non-algebraic scaling law for a divergence-free T_3 structure is discussed. This result quantifies the dichotomy between the rigidity of exact solutions and the flexibility of approximate solutions for the associated inclusion.

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Introduction

In this thesis we study scaling laws for a model class of differential inclusions. Scaling laws allow us to extract fine properties of different *microstructures* arising in certain problems in the calculus of variations and provide information on the involved length scales. A prominent application in research is given by gradient inclusions to understand the microstructures arising in shape-memory alloys. For these alloys the observable microstructures are associated to minimizers or minimizing sequences of the elastic energy corresponding to the gradient inclusion.

Here, we consider a variational model for the differential inclusions, that is we allow the values to deviate from the prescribed ones and measure the cost of this deviation by a suitable elastic energy. This energy is then singularly perturbed by a surface energy, and the scaling of the minimal energy in the perturbation parameter $\varepsilon > 0$ is deduced. Besides determining the involved length scales, scaling laws show the preference of certain microstructures in terms of the singularly perturbed energy. Moreover, they provide quantitative versions of (qualitative) rigidity and flexibility results. As a prominent example is given by shape-memory alloys we start with a short introduction on the mathematical model of these alloys and give a heuristic explanation of the *shape-memory effect*.

1.1 Shape-memory alloys

Shape-memory alloys are special materials that undergo a solid-solid phase transition when the temperature changes, that is their crystalline structure changes abruptly when a certain critical temperature is reached. The striking and intriguing property of the shape-memory alloys is that the body returns back to its original shape when deformed and then heated above the critical temperature [Bha03]. The material thus “remembers” that shape even when deformed. This is called *shape-memory effect* and has many possible applications as for example in medicine and aviation [MLSG14]. One advantage of those materials is that due to the shape-memory effect, they can be stored in a compact way for transport, and after heating the material returns to its desired shape. Alternatively, it is possible to use shape-memory alloys as actuators, where the start of the motion is temperature dependent. One famous example in applications for its physical properties is an alloy of Nickel and Titanium, called *Nitinol* or NiTi [MLSG14].

From a crystallographic point of view, the crystalline structure of the material changes in the phase transformation. In the high temperature regime, the so-called *austenite* phase,

the structure is highly symmetric and there is a unique variant of the crystalline structure. When cooled such that the solid-solid phase transition takes place, the material changes to the so-called *martensite* phase, in which the crystalline symmetry reduces [Bha03]. Due to this, there are multiple preferred lattice structures. With this it is possible to explain the shape-memory effect. Taking a material in the martensite phase and deforming it, there are regions of the different variants of the lattice structure. After heating the deformed body above the critical temperature the crystalline grid changes to the austenite phase, hence there is only one preferred variant. The material has to return back to its original shape to accommodate to the presence of only one variant, thus the shape-memory effect is observable.

Using the Cauchy-Born hypothesis [Bha03, Section 3.3], the mathematical analysis of this phenomenon is carried out using a continuum model. Assume that the material sample in its reference configuration is given by $\Omega \subset \mathbb{R}^3$ and assume for simplicity that the sample Ω has a spatially constant temperature $\theta \in [0, \infty)$. Denote by $\theta_{\text{crit}} > 0$ the critical temperature at which the phase transition from austenite to martensite is taking place. The material is subject to the deformation $y: \Omega \rightarrow \mathbb{R}^3$, where the vector $y(x)$ describes the position of the point $x \in \Omega$ in the deformed configuration. We model the martensite phase, i.e., the temperature is below the critical temperature $\theta < \theta_{\text{crit}}$, by defining a compact set of admissible deformation gradients $\mathcal{K}(\theta) \subset \mathbb{R}^{3 \times 3}$. For an exactly stress-free state, we impose $\nabla y(x) \in \mathcal{K}(\theta)$ pointwise for each $x \in \Omega$. In physical applications, the set $\mathcal{K}(\theta)$ fulfills *frame invariance*, i.e.,

$$R\mathcal{K}(\theta) \subset \mathcal{K}(\theta) \text{ for all } R \in SO(3),$$

and the *material symmetry* assumption

$$\mathcal{K}(\theta)H \subset \mathcal{K}(\theta) \text{ for all } H \in \mathcal{P},$$

where $\mathcal{P} \subset SO(3)$ is the (local) symmetry group of the material. In the austenite phase (without thermal expansion), there is only one variant which is given by $\mathcal{K}(\theta) = SO(3)$ for $\theta > \theta_{\text{crit}}$. The solid-solid phase transition causes a loss of symmetry and therefore in the martensite phase there are multiple variants $\mathcal{K}(\theta) = \bigcup_{j=1}^N SO(3)A_j$ for $\theta < \theta_{\text{crit}}$ and specific $A_j \in \mathbb{R}^{3 \times 3}$. At the critical temperature $\theta = \theta_{\text{crit}}$ both phases can be present and thus

$$\mathcal{K}(\theta_{\text{crit}}) = \bigcup_{j=1}^N SO(3)A_j \cup SO(3).$$

We refer to [Bha03; Mül99b] for a more detailed introduction of the mathematical model of shape-memory alloys.

In the literature, two methods are prominent to simplify the frame invariant model. The first is to neglect frame invariance, that is one assumes $\mathcal{K}(\theta)$ to be a discrete set of finite *wells* $\mathcal{K}(\theta) = \{A_1, A_2, \dots, A_N\}$. This simplified model helps to get a fundamental understanding

of differential inclusions and the corresponding microstructures and scaling laws without the rotational invariance. In particular, it helps to understand the phenomenology of the corresponding microstructures with the mathematical advantage of having a discrete set of wells. This type of model has for example been studied quantitatively in the famous articles by Kohn and Müller [KM92a; KM94]. The second simplification is to consider the *geometrically linearized model*. For this assume that $y(x) = x + \delta u(x)$ with $\delta > 0$ and the *displacement field* $u: \Omega \rightarrow \mathbb{R}^3$. Thus, for small δ the deformation gradient $\nabla y(x)$ is close to the identity. Using the fact that the tangent space of $SO(3)$ at the identity is given by $skew(3) = \{M \in \mathbb{R}^{3 \times 3} : M^T = -M\}$, we consider a $skew(3)$ invariance instead of the $SO(3)$ invariance as an approximation. By considering $\nabla^{\text{sym}} u = \frac{1}{2}(\nabla u + (\nabla u)^T)$ instead of the full gradient, we “remove” the skew-symmetric part. In other words, frame invariance is replaced by an invariance under infinitesimal rotations, i.e., a $skew(3)$ invariance. Therefore only the symmetrized gradient carries information on the state. For small deformations, it thus is reasonable to consider the inclusion $\nabla^{\text{sym}} u \in \mathcal{K}_{\text{sym}}(\theta)$, where $\mathcal{K}_{\text{sym}}(\theta)$ does not fulfill the frame invariance anymore. The geometrically linearized model likewise has the advantage of a discrete set of wells $\mathcal{K}_{\text{sym}}(\theta)$. The relation of the nonlinear model (with frame invariance) and the geometrically linear model (model with symmetrized gradient) has been studied for example in [Bha93; BJ92; Koh91; Sch08]. To simplify the notation, we will omit the temperature dependence in the following.

1.2 Differential inclusions

Building on the above described model for shape-memory alloys, and aiming for a more fundamental understanding of differential inclusions, we turn to the setting of *$\mathcal{A}$ -free differential inclusions*. This is a possible way of generalizing the gradient (and symmetrized gradient) inclusions by changing the constraint of being a gradient (or symmetrized gradient). In this way we gain a deeper analytical insight into (general) differential inclusions without frame invariance such as the existence of non-trivial solutions. In particular, it is possible to consider both cases $\nabla u \in \mathcal{K}$ or $\nabla^{\text{sym}} u \in \mathcal{K}_{\text{sym}}$ in this framework and find shared properties of solutions and identify differing behaviors.

1.2.1 $\mathcal{A}$ -free differential inclusions

The underlying concept of the aforementioned generalization is the Poincaré lemma. In $\mathbb{R}^3$ it is equivalent to impose the constraint that a map $v: \mathbb{R}^3 \rightarrow \mathbb{R}^{3 \times 3}$ is a gradient field, that is there exists $u: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that $v = \nabla u$, and that $\text{curl } v = 0$. Similarly it holds $v = \nabla^{\text{sym}} u$ if and only if $\text{curl curl } v = 0$ in $\mathbb{R}^3$, see for example [Rül16b, Lemma 1]. Therefore, instead of $v \in \mathcal{K}$ under the constraint $\text{curl } v = 0$, we consider differential inclusions of the type $v \in \mathcal{K}$ such that $\mathcal{A}(D)v = 0$ in $\mathbb{R}^d$ for $d \geq 2$, where the differential equation is to be understood in the distributional sense. For this we consider a homogeneous linear constant

coefficient differential operator $\mathcal{A}(D): C^\infty(\mathbb{R}^d; W') \rightarrow C^\infty(\mathbb{R}^d; W)$ of order $m \in \mathbb{N}$ for some finite dimensional vector spaces W' and W , defined for $v \in C^\infty(\mathbb{R}^d; W')$ by

$$\mathcal{A}(D)v := \sum_{|\alpha|=m} A_\alpha \partial^\alpha v. \quad (1.1)$$

Here, $A^\alpha: W' \rightarrow W$ are linear maps for $\alpha \in \mathbb{N}^d$ and we use the multi-index notation, see Section 1.7.2 below. By the identification $W' \cong \mathbb{R}^n$ with $n = \dim W'$, in the following we only consider the case $W' = \mathbb{R}^n$. Further, let $\mathcal{K} \subset \mathbb{R}^n$ be a compact set. The exact solutions/stress-free states in $\Omega \subset \mathbb{R}^d$ ($d \geq 2$) are given as the solutions $v: \mathbb{R}^d \rightarrow \mathbb{R}^n$ of the differential inclusion

$$\begin{cases} v \in \mathcal{K} & \text{in } \Omega, \\ \mathcal{A}(D)v = 0 & \text{in } \mathcal{D}'(\mathbb{R}^d), \end{cases} \quad (1.2)$$

see Section 1.7.2. Here and in the following, we will always assume Ω to be an open bounded Lipschitz domain, i.e., it is open, bounded, simply connected, non-empty, and has a Lipschitz boundary $\partial\Omega$.

Besides the study of microstructures for martensitic phase transitions, $\mathcal{A}$ -free differential inclusions can be used in the field of micromagnetism. There, following for example [DKMO06], the micromagnetic energy of the *magnetization* $m: \Omega \rightarrow \mathbb{S}^2$ in a ferromagnetic sample $\Omega \subset \mathbb{R}^3$ without an external field can be written as

$$E(m) = E_{\text{aniso}}(m) + \varepsilon E_{\text{surf}}(m) + E_{\text{stray-field}}(m), \quad \varepsilon > 0.$$

The different energy contributions are the anisotropic energy E_{aniso} favoring certain directions depending on the ferromagnetic material, the interfacial energy E_{surf} penalizing the oscillation of m , and the stray-field energy $E_{\text{stray-field}}$ which is proportional to

$$E_{\text{stray-field}}(m) \sim \int_{\mathbb{R}^3} |\nabla u|^2 dx, \quad \Delta u = \operatorname{div} m, \quad (1.3)$$

i.e., it favors divergence-free magnetizations m . For the equation $\Delta u = \operatorname{div} m$, the magnetization is extended by zero outside of Ω , and the equation is to be understood in the sense of distributions in $\mathbb{R}^3$. The stray-field energy models the influence of the magnetic field induced by the magnetization on itself. Thus, it favors charge free magnetizations, which corresponds directly to $\operatorname{div} m = 0$. Due to the three contributions there are competing effects. The anisotropic energy favors configurations m such that $m \in \mathcal{K}$, where the set $\mathcal{K} \subset \mathbb{S}^2$ is given by the favored directions in E_{aniso} . As oscillations are penalized by the surface energy, the first two energy contributions combined favor large regions of a constant magnetization with favorable direction. The stray-field energy on the other hand favors oscillation in the sense that it prefers divergence-free magnetizations. Thus, depending on $\mathcal{K}$, the stray-field energy is smaller for oscillating magnetizations. Therefore, the understanding of the differential inclusions $m \in \mathcal{K}$ under the constraint $\operatorname{div} m = 0$

provides a possible first step for the analysis of the total energy. Settings like this have for example be studied in [CK98; CKO99]. This motivates to consider the divergence as a second model operator in what follows below.

1.2.2 Exact solutions

Fundamental for the following analysis is an understanding of stress-free states, that is solutions to (1.2). An example of an exact solution is the constant function $v(x) = A \in \mathcal{K}$. A natural question then is whether non-trivial, i.e., non-constant solutions, exist. To give an answer to this, we have to introduce the associated *symbol* of the operator $\mathcal{A}(D)$, defined for $\xi \in \mathbb{R}^d$ by

$$\mathbb{A}(\xi) := \sum_{|\alpha|=m} A_\alpha \xi^\alpha, \quad (1.4)$$

and the *wave cone*

$$\Lambda_{\mathcal{A}} := \bigcup_{\xi \in \mathbb{S}^{d-1}} \ker \mathbb{A}(\xi) \subset \mathbb{R}^n. \quad (1.5)$$

For the two-well problem $\mathcal{K} = \{A, B\}$ there are non-trivial solutions if $A - B \in \Lambda_{\mathcal{A}}$. Taking such $A, B \in \mathbb{R}^n$, there is a $\xi \in \mathbb{S}^{d-1}$ such that $\mathbb{A}(\xi)(A - B) = 0$. Without loss of generality we can assume $\xi_1 \neq 0$. Defining then

$$v(x) := (A - B)h(x \cdot \xi) + B, \quad (1.6)$$

for $h: \mathbb{R} \rightarrow \{0, 1\}$, e.g., $h(t) = \chi_{(-\infty, 0)}(t)$, yields a solution to (1.2). Indeed, for this choice of h , we directly verify $\mathcal{A}(D)v = 0$, i.e., (1.27) below, that is for every $\varphi \in C_c^\infty(\mathbb{R}^d; W)$ it holds that

$$\begin{aligned} \int_{\mathbb{R}^d} v \cdot \mathcal{A}(D)^* \varphi \, dx &= \sum_{|\alpha|=m} (-1)^m \int_{\{x \in \mathbb{R}^d : x \cdot \xi < 0\}} A_\alpha (A - B) \cdot \partial^\alpha \varphi \, dx \\ &= \sum_{|\alpha|=m} (-1)^m \xi_1^{-m} \int_{\{x \in \mathbb{R}^d : x \cdot \xi < 0\}} A_\alpha (A - B) \xi^\alpha \cdot \partial_1^m \varphi \, dx \\ &= (-1)^m \xi_1^{-m} \int_{\{x \in \mathbb{R}^d : x \cdot \xi < 0\}} \mathbb{A}(\xi)(A - B) \cdot \partial_1^m \varphi \, dx = 0. \end{aligned}$$

Such one-dimensional functions are called *simple laminates* and form an important class of solutions to (1.2).

It turns out that this is an equivalent characterization.

Lemma 1.1 ([DPR18, Theorem 1.2(A)]). *Let $\Omega \subset \mathbb{R}^d$ be an open bounded Lipschitz domain, let $\mathcal{A}(D)$ be a differential operator as given in (1.1) and let $\mathcal{K} = \{A, B\}$ for $A, B \in \mathbb{R}^n$.*

Consider the wave cone $\Lambda_{\mathcal{A}}$ defined as in (1.5).

- (i) Compatible states: If $A - B \in \Lambda_{\mathcal{A}}$ there exists a solution $v \in L^\infty(\mathbb{R}^d; \mathbb{R}^n)$ to (1.2) that is not constant in Ω .
- (ii) Incompatible states: If $A - B \notin \Lambda_{\mathcal{A}}$ every solution $v \in L^\infty(\mathbb{R}^d; \mathbb{R}^n)$ to (1.2) is constant in Ω .

The differential inclusion (1.2) has a non-trivial solution if and only if $A - B \in \Lambda_{\mathcal{A}}$.

Here and in the following we give slightly adapted versions of the results from the literature to be consistent in the presentation and notation. We call two wells A and B *compatible* if $A - B \in \Lambda_{\mathcal{A}}$. For $\mathcal{A}(D) = \text{curl}$ the result recovers the Hadamard jump condition, cf. [BJ87, Proposition 1], where two matrices $A, B \in \mathbb{R}^{d \times d} = \mathbb{R}^n$ are compatible if $\text{rank}(A - B) = 1$, i.e.,

$$A - B = a \otimes \xi, \text{ for some } a, \xi \in \mathbb{R}^d \setminus \{0\}. \quad (1.7)$$

As observed in the case of compatible wells, a class of solutions is given by simple laminates, i.e., maps of the form as in (1.6). For any $\mu \in \Lambda_{\mathcal{A}}$, we define the set of possible directions of lamination (of μ and 0) by

$$V_{\mathcal{A}, \mu} := \{\xi \in \mathbb{R}^d : \mathbb{A}(\xi)\mu = 0\}. \quad (1.8)$$

We remark that depending on the differential operator $\mathcal{A}(D)$, there might be more types of solutions to (1.2) besides simple laminates. In particular, introducing the set of *super-compatible* states

$$I_{\mathcal{A}} := \bigcap_{\xi \in \mathbb{S}^{d-1}} \ker \mathbb{A}(\xi) = \bigcap_{|\alpha|=m} \ker A_{\alpha}, \quad (1.9)$$

we notice that any map $v \in L^\infty(\mathbb{R}^d; \{A, B\})$ with $A - B \in I_{\mathcal{A}}$ is a solution to (1.2) with $\mathcal{K} = \{A, B\}$, cf. [RRT23, Theorem 1(iii)] or the summary in Section 2.2 below. This shows that the $\mathcal{A}$ -free constraint is trivially fulfilled for functions taking values in super-compatible states. Operators such that there are no super-compatible states, i.e., such that $I_{\mathcal{A}} = \{0\}$, are called *cocanceling* operators. One example of a cocanceling operator is the curl operator.

For more than two wells, the question if there are non-trivial solutions becomes more challenging. Already for the case $\mathcal{A}(D) = \text{curl}$, i.e., for gradient inclusions, this is a non-trivial question that has been studied in the literature. As seen above in Lemma 1.1, the two-well problem for gradient inclusions with two incompatible wells is *rigid* for exact solutions, i.e., there are only trivial solutions. This also holds for up to four pairwise incompatible wells [CK02; Šve91], but for five wells the rigidity of exact solutions is lost. There are five matrices $T_5 = \{A_1, A_2, A_3, A_4, A_5\} \subset \mathbb{R}_{\text{sym}}^{2 \times 2}$ with $A_j - A_k \notin \Lambda_{\text{curl}}$ for $j \neq k$

and such that there is a non-affine Lipschitz function $u: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with $\nabla u \in T_5$ almost everywhere [Kir03]. In this instance, we say that exact solutions for these five wells are *flexible*. This field of research is connected to *convex integration*. In the case $\mathcal{A}(D) = \text{curl}$ for the above mentioned five wells, a non-trivial solution is given by a convex integration solution, see for example [MŠ99] and the references therein.

1.2.3 Approximate solutions

Turning now to a weaker notion of a solution to (1.2), we say a sequence $(v_j)_{j \in \mathbb{N}} \subset L^1_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^n)$ is an *approximate solution*, if for every $j \in \mathbb{N}$ we have $\mathcal{A}(D)v_j = 0$ in the distributional sense, and

$$\int_{\Omega} \text{dist}(v_j, \mathcal{K}) \, dx \rightarrow 0$$

as $j \rightarrow \infty$. As for exact solutions, we analyze rigidity and flexibility properties of approximate solutions. Given a compact set $\mathcal{K} \subset \mathbb{R}^n$, we say approximate solutions of the differential inclusion (1.2) are *rigid* if for every approximate solution $(v_j)_{j \in \mathbb{N}}$ there is an $A \in \mathcal{K}$ such that, up to a subsequence, $v_j \rightarrow A$ in L^1 as $j \rightarrow \infty$. As in the case of exact solutions, a rigidity result for the two-well problem for incompatible states was shown in [DPR18]. The result, slightly adapted to our notation, is as follows.

Lemma 1.2 ([DPR18, Theorem 1.2(B)]). *Let $\Omega \subset \mathbb{R}^d$ be an open bounded Lipschitz domain, let $\mathcal{A}(D)$ be given as in (1.1). Consider the wave cone $\Lambda_{\mathcal{A}}$ defined as in (1.5) and $\mathcal{K} = \{A, B\} \subset \mathbb{R}^n$ with $A - B \notin \Lambda_{\mathcal{A}}$. Let $(v_j)_{j \in \mathbb{N}} \subset L^1_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^n)$ be such that $\mathcal{A}(D)v_j = 0$ in $\mathcal{D}'(\mathbb{R}^d)$ for all $j \in \mathbb{N}$ and*

$$\int_{\Omega} \text{dist}(v_j(x), \{A, B\}) \, dx \rightarrow 0 \text{ as } j \rightarrow \infty.$$

Then, up to a subsequence, in the limit $j \rightarrow \infty$ it holds either

$$\int_{\Omega} |v_j(x) - A| \, dx \rightarrow 0 \quad \text{or} \quad \int_{\Omega} |v_j(x) - B| \, dx \rightarrow 0.$$

To complement the above discussion of exact solutions of the gradient inclusions ($\mathcal{A}(D) = \text{curl}$), we note that approximate solutions for three pairwise incompatible wells are rigid [Šve91], see also [Rin18, Theorem 8.11]. In contrast to exact solutions rigidity of approximate solutions is not guaranteed for four pairwise incompatible wells. These four wells given in (1.20), named after Tartar, have been studied intensively in the literature [AH86; BFJK94; CK02; FS08; KMŠ03; Tar93]. The approximate solution $(\nabla u_j)_{j \in \mathbb{N}}$ is constructed as an infinite order laminate on different length scales vanishing in the limit $j \rightarrow \infty$. Starting with a simple laminate of one of the four wells and a suitable auxiliary matrix, we replace the parts in Ω where we used the auxiliary matrix by a second order

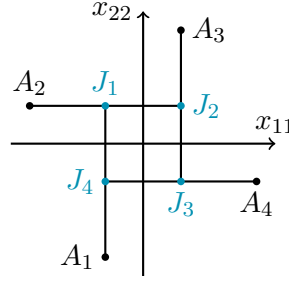


Figure 1.1: The four diagonal wells A_1, A_2, A_3, A_4 of the Tartar square with the auxiliary matrices J_1, J_2, J_3, J_4 in blue. The compatible directions are shown as lines connecting the wells. More details on its structure are provided in Section 1.5.2.

laminate, formed by a different well and a new auxiliary matrix. Continuity of u_j can be guaranteed by a cut-off argument similar to (1.15) below. See Figure 1.1 for an illustration of the Tartar square and the auxiliary matrices.

Iterating the procedure, we get an infinite order laminate, where in the limit $j \rightarrow \infty$ we only take values in the wells but we defined a non-affine deformation. Below in Section 1.5.2 we will also introduce wells giving rise to higher order laminates, but not to an infinite order.

1.3 Variational model

The above observations regarding rigidity and flexibility of exact and approximate solutions are of *qualitative* nature, i.e., approximate solutions do not quantify the error of their deviation to (1.2). In this thesis a singularly perturbed model is used to make these results *quantitative* in the sense of a scaling law. For this we consider an L^2 -based elastic energy with a higher order perturbation by a surface energy to penalize oscillation. Furthermore, to rule out trivial solutions and enforce oscillation, we impose exterior data $F \in \mathbb{R}^n$ on v . In this model we have to use exterior data of the form $v = F$ outside of Ω , as in general we do not have a trace of v . As the differential constraint $\mathcal{A}(D)v = 0$ is on the whole space $\mathbb{R}^d$, we therefore ensure that only “compatible” jumps at the boundary are admissible. Taking for example $\mathcal{A}(D) = \text{curl}$ with an open bounded Lipschitz domain Ω , we know that $v = \nabla u$ for some function u and as $v = F$ outside Ω , we have $u(x) = Fx$ outside Ω . In particular it satisfies the Dirichlet boundary data $u(x) = Fx$ on $\partial\Omega$. We also adapt the notion of exact and approximate solutions accordingly to fulfill the condition $v = F$ outside Ω . In particular, for the compatible two-well problem and an incompatible boundary data $F \notin \mathcal{K}$, we might lose the existence of exact solutions satisfying the condition $v = F$ outside Ω .

1.3.1 Singularly perturbed energy

For a quantitative study of rigidity and flexibility properties of the $\mathcal{A}$ -free differential inclusion (1.2) let $\Omega \subset \mathbb{R}^d$ be an open bounded Lipschitz domain for $d \geq 2$, $\mathcal{K} \subset \mathbb{R}^n$ be compact and $F \in \mathbb{R}^n$. We define the L^2 -based elastic energy as

$$E_{\text{el}}^{\mathcal{A}}(v, \chi) := \int_{\Omega} |v - \chi|^2 dx, \quad (1.10)$$

with the sets of admissible functions

$$\begin{aligned} v &\in \mathcal{D}_F^{\mathcal{A}} := \{v \in L_{\text{loc}}^2(\mathbb{R}^d; \mathbb{R}^n) : \mathcal{A}(D)v = 0 \text{ in } \mathcal{D}'(\mathbb{R}^d), v = F \text{ in } \mathbb{R}^d \setminus \overline{\Omega}\}, \\ \chi &\in L^2(\Omega; \mathcal{K}). \end{aligned} \quad (1.11)$$

That is, the elastic energy measures the L^2 -distance from an $\mathcal{A}$ -free map v , to the set of states $\mathcal{K}$. The advantage of this formulation is that we remove the non-convexity of $v \mapsto \text{dist}(v, \mathcal{K})$ by introducing the *phase indicator* χ . With a slight abuse of terms, we call F the *boundary data*.

For the quantitative study of rigidity and flexibility, we singularly perturb the elastic energy by a surface energy. With the phase indicator in hand, we penalize each change of phase in χ by setting

$$E_{\text{surf}}(\chi) := \|D\chi\|_{TV(\Omega)}, \quad \chi \in BV(\Omega; \mathcal{K}), \quad (1.12)$$

and define for $\varepsilon > 0$ the total energy by

$$E_{\varepsilon}^{\mathcal{A}}(v, \chi) := E_{\text{el}}^{\mathcal{A}}(v, \chi) + \varepsilon E_{\text{surf}}(\chi). \quad (1.13)$$

We refer to Section 1.7.2 for the definition of the total variation norm $\|D\chi\|_{TV(\Omega)}$.

The minimization of the total energy $E_{\varepsilon}^{\mathcal{A}}$ instead of the elastic energy $E_{\text{el}}^{\mathcal{A}}$ selects microstructure. Indeed, due to the (general) lack of exact solutions for prescribed boundary data, in the minimization of $E_{\text{el}}^{\mathcal{A}}$ we can not distinguish between simple laminates and *branching structures*, cf. Figure 1.4 and the discussion in Section 1.3.3. In $E_{\varepsilon}^{\mathcal{A}}$, taking the regularizing effect of the higher order term in the surface energy into account, we can differ between these types of microstructures. In particular, minimizing $E_{\varepsilon}^{\mathcal{A}}$ stops arbitrary fine oscillations on a length scale depending on ε . Moreover, the higher order term E_{surf} , due to its regularizing effect, implies better compactness properties of the energy functional.

Turning back to the micromagnetic energy model (1.3), we notice that for “strong” stray-field energies it is a reasonable approximation to assume $\text{div } m = 0$. This implies that the energy then resembles the energy introduced here in (1.13) with an additional difficulty arising due to the (non-convex) constraint $|m| = 1$ almost everywhere in Ω .

1.3.2 Boundary data

Before we discuss scaling laws in Section 1.3.3 below, let us comment on the boundary data F . It is (usually) chosen from the $\mathcal{A}$ -quasi-convex hull of the compact set $\mathcal{K} \subset \mathbb{R}^n$, defined by

$$\mathcal{K}^{\text{qc}} := \{F \in \mathbb{R}^n : f(F) \leq \sup f(\mathcal{K}) \text{ for all } \mathcal{A}\text{-quasi-convex } f: \mathbb{R}^n \rightarrow \mathbb{R}\}.$$

We postpone the definition of $\mathcal{A}$ -quasi-convex functions to Section 1.7.2. As every convex function is $\mathcal{A}$ -quasi-convex, we have $\mathcal{K}^{\text{qc}} \subset \mathcal{K}^{\text{conv}}$ with the (closed) convex hull of $\mathcal{K}$ denoted by $\mathcal{K}^{\text{conv}}$. For simplicity, in the instance of two wells $\mathcal{K} = \{A, B\}$ we will use $F \in \mathcal{K}^{\text{conv}} \setminus \mathcal{K}$. This way, we include incompatible boundary data for incompatible wells. Indeed, for $\mathcal{A}(D) = \text{curl}$, we have for example $\{0, \text{Id}\}^{\text{qc}} = \{0, \text{Id}\} \neq \{\lambda \text{Id} : \lambda \in [0, 1]\} = \{0, \text{Id}\}^{\text{conv}}$.

We further define the $\Lambda_{\mathcal{A}}$ -convex hull, or *lamination convex hull*

$$\mathcal{K}^{\text{lc}} := \bigcup_{j \in \mathbb{N}} \mathcal{K}^{(j)}, \quad (1.14)$$

with

$$\mathcal{K}^{(1)} := \mathcal{K}, \quad \mathcal{K}^{(j+1)} := \mathcal{K}^{(j)} \cup \{\lambda A + (1 - \lambda)B : A, B \in \mathcal{K}^{(j)}, A - B \in \Lambda_{\mathcal{A}}, \lambda \in (0, 1)\}.$$

The elements of $\mathcal{K}^{(j)} \setminus \mathcal{K}^{(j-1)}$ are called *laminates of order j* .

Note that in the notation of the sets $\mathcal{K}^{\text{qc}}$ and $\mathcal{K}^{\text{lc}}$ we omit the dependence on the operator $\mathcal{A}(D)$. For $\mathcal{A}(D) = \text{curl}$, we have the following well known chain of inclusions for compact sets $\mathcal{K}$:

$$\mathcal{K}^{\text{lc}} \subset \mathcal{K}^{\text{qc}} \subset \mathcal{K}^{\text{conv}},$$

see for example [Mül99b].

The set $\mathcal{K}^{\text{qc}}$ is the natural choice for the boundary data, as it consists of those affine deformations F that are “macroscopically stress-free”. For $\mathcal{A}(D) = \text{curl}$, the minimal energy $\inf_{\nabla u \in \mathcal{D}_F^{\text{curl}}} \inf_{\chi \in L^2(\Omega; \mathcal{K})} E_{\text{el}}^{\text{curl}}(\nabla u, \chi)$ vanishes for $F \in \mathcal{K}^{\text{qc}}$. This can be seen by analyzing the relaxed energy functional of

$$\inf_{\chi \in L^2(\Omega; \mathcal{K})} E_{\text{el}}^{\text{curl}}(\nabla \cdot, \chi) = \int_{\Omega} \text{dist}^2(\nabla \cdot, \mathcal{K}) \, dx,$$

for more details we refer to [Rin18, Chapter 8.3].

As mentioned above, due to the boundary condition we might lose the existence of exact solutions to our differential inclusion with prescribed boundary data. For $\mathcal{A}(D) = \text{curl}$ we now give an example of an approximate solution for the compatible two-well problem

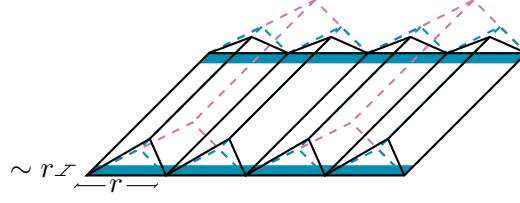


Figure 1.2: Illustration of a cut-off simple laminate on scale r , with the region of the cut-off highlighted in blue. The simple laminate without cut-off is shown by dashed lines, where a laminate on scale $2r$ is depicted in purple.

satisfying the boundary data. Let $\mathcal{K} = \{A, B\}$ be a set of two compatible wells $A - B \in \Lambda_{\text{curl}}$ with the boundary data $F_\lambda = \lambda A + (1 - \lambda)B$ for some $\lambda \in (0, 1)$. We define the simple laminate $v = \nabla u$ as in (1.6) by using a potential u . Assume that we can laminate in direction $\xi \in \mathbb{S}^{d-1}$, i.e., $A - B = a \otimes \xi$ for some $a \in \mathbb{R}^d \setminus \{0\}$. We fix the laminate of $A - F_\lambda$ and $B - F_\lambda$ in direction ξ on the scale $r > 0$ by setting

$$\tilde{u}_r(x) := \begin{cases} (1 - \lambda)(A - B)x & \text{for } x \cdot \xi \in [0, \lambda r), \\ -\lambda(A - B)x + \lambda r a & \text{for } x \cdot \xi \in [\lambda r, r), \end{cases}$$

where we extend the function r -periodically in direction ξ . This function is Lipschitz continuous with $|\tilde{u}_r(x)| \leq \lambda(1 - \lambda)|a|$ and satisfies $\nabla \tilde{u}_r(x) + F_\lambda \in \mathcal{K}$ almost everywhere, but it is not admissible in the sense of $\nabla \tilde{u}_r + F_\lambda \notin \mathcal{D}_{F_\lambda}^{\text{curl}}$, cf. (1.11). Defining the *cut-off simple laminate* $v_r := \nabla u_r$, where

$$u_r(x) := \min \left\{ \frac{\text{dist}(x, \mathbb{R}^d \setminus \Omega)}{r}, 1 \right\} \tilde{u}_r(x) + F_\lambda x \in H^1(\Omega; \mathbb{R}^d), \quad (1.15)$$

we indeed have $v_r \in \mathcal{D}_{F_\lambda}^{\text{curl}}$, cf. Figure 1.2.

Thus, for two compatible wells, we construct a cut-off simple laminate with volume fractions of the two phases corresponding to the boundary data. This in turn defines an approximate solution and minimizing sequence of the elastic energy $E_{\text{el}}^{\text{curl}}$ (for suitable chosen phase indicator χ), see (1.10) for its definition. More precisely, if $\chi_r(x) := \nabla \tilde{u}_r(x) + F_\lambda \in \mathcal{K}$, it holds that $E_{\text{el}}^A(v_r, \chi_r) \leq Cr$ for some constant $C > 0$. Furthermore, with this we motivate the definition of laminates of higher order.

In the case $\mathcal{A}(D) = \text{curl}$, consider for a suitable choice of $\mathcal{K}$ a boundary data $F \in \mathcal{K}^{(2)} \setminus \mathcal{K}^{(1)} \neq \emptyset$. Then there are four (not necessarily different) wells $A_1, A_2, A_3, A_4 \in \mathcal{K}$ such that

$$F = \mu(\lambda_1 A_1 + (1 - \lambda_1)A_2) + (1 - \mu)(\lambda_2 A_3 + (1 - \lambda_2)A_4), \quad \lambda_1, \lambda_2, \mu \in (0, 1).$$

Therefore it is possible to first construct a cut-off simple laminate of $\lambda_1 A_1 + (1 - \lambda_1)A_2$ and $\lambda_2 A_3 + (1 - \lambda_2)A_4$ satisfying the boundary condition, and then replace the cells of the corresponding phase by cut-off simple laminates of A_1 and A_2 , and A_3 and A_4 , respectively.

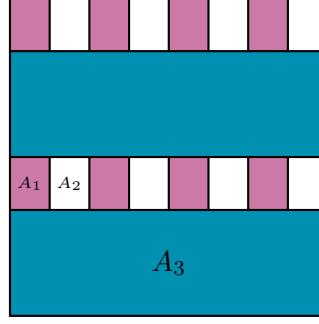


Figure 1.3: Schematic picture of a second order laminate. Shown is a possible choice of χ for a laminate of A_1 and A_2 inside the laminate of $\frac{1}{2}A_1 + \frac{1}{2}A_2$ and A_3 for $F = \frac{1}{3}(\frac{1}{2}A_1 + \frac{1}{2}A_2) + \frac{2}{3}A_3$.

Hence, we have constructed a laminate within a laminate, or in other words a second order laminate, justifying the name for the elements of $\mathcal{K}^{(2)} \setminus \mathcal{K}^{(1)}$, cf. Figure 1.3.

Related to the analysis of these macroscopically stress-free states is the theory of relaxation of the energy functional. Prominent tools are the relaxation of the energy using the $\mathcal{A}$ -quasi-convex envelope of the integrand or the so-called (*gradient*) *Young measures*, which are a probabilistic description of the asymptotic behavior of minimizing sequences. As these are not content of this thesis, we refer to [FM99; Mül99b; Rin18] for an introduction and examples of applications.

1.3.3 Scaling laws

As discussed in the previous sections, we consider the energy of the form $E_\varepsilon^\mathcal{A}(v, \chi)$ as in (1.13) with $v \in \mathcal{D}_F^\mathcal{A}$, $F \in \mathcal{K}^{\text{qc}} \setminus \mathcal{K}$, and $\chi \in BV(\Omega; \mathcal{K})$, cf. (1.11). Understanding the behavior of the optimal energy $\inf_{\chi \in BV(\Omega; \mathcal{K})} \inf_{v \in \mathcal{D}_F^\mathcal{A}} E_\varepsilon^\mathcal{A}(v, \chi)$ in the limit $\varepsilon \rightarrow 0$ allows us to draw conclusions about the microstructures arising in the corresponding $\mathcal{A}$ -free differential inclusion (1.2).

One way to achieve this understanding is by showing a scaling law of the form

$$C^{-1}s(\varepsilon) \leq \inf_{\chi \in BV(\Omega; \mathcal{K})} \inf_{v \in \mathcal{D}_F^\mathcal{A}} E_\varepsilon^\mathcal{A}(v, \chi) \leq Cs(\varepsilon) \quad (1.16)$$

with a scaling $s: [0, \infty) \rightarrow [0, \infty)$ and a constant $C > 0$ independent of ε . To elaborate on an example of application, let us consider the compatible two-well problem for $\mathcal{A}(D) = \text{curl}$, i.e., we aim at analyzing

$$\nabla u \in \{A, B\}$$

for the two compatible matrices $A, B \in \mathbb{R}^{d \times d}$ with the boundary data $F_\lambda = \lambda A + (1 - \lambda)B$ quantitatively.

As we have seen above in (1.15), a particular approximate solution is given by a (cut-off) simple laminate. Plugging this simple laminate on scale $r > 0$ into the energy, and choosing $\chi_r = \nabla \tilde{u}_r + F_\lambda \in \{A, B\}$, yields for some constant $C > 0$ independent of ε

$$E_\varepsilon^{\text{curl}}(\nabla u_r, \chi_r) \geq C(r + \varepsilon r^{-1}).$$

Hence, after choosing the ε -dependent length scale $r \sim \varepsilon^{1/2}$, we get a lower bound on the ε behavior of this simple laminate as

$$E_\varepsilon^{\text{curl}}(\nabla u_r, \chi_r) \geq C\varepsilon^{\frac{1}{2}}.$$

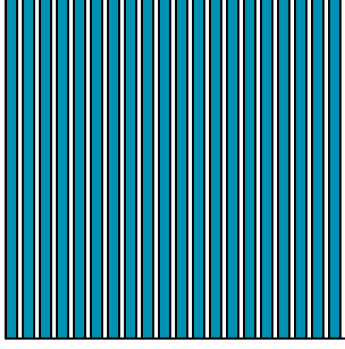
In the seminal works by Kohn and Müller [KM92a; KM94] it was shown that the (scalar-valued) compatible two-well problem scales like $\varepsilon^{2/3}$, i.e., there is a constant $C > 0$, independent of ε , such that

$$C^{-1}\varepsilon^{\frac{2}{3}} \leq \inf_{\chi \in BV(\Omega; \{A, B\})} \inf_{v \in \mathcal{D}_{F_\lambda}^{\text{curl}}} E_\varepsilon^{\text{curl}}(v, \chi) \leq C\varepsilon^{\frac{2}{3}}.$$

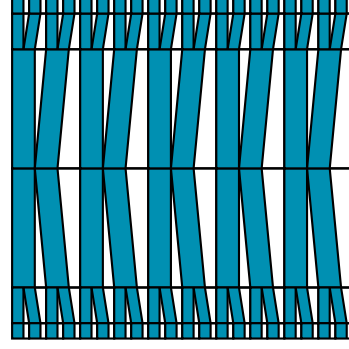
Therefore, simple laminates are *not* optimal microstructures. The upper bound on the optimal energy is derived by an explicit construction, which then is a good indicator on how an optimal microstructure might look like. A word of caution: as the constants C and C^{-1} in the scaling law are not matching, we do not know how the optimal microstructure looks like. Nonetheless with the scaling law, we rule out certain structures, e.g., simple laminates, as the corresponding energy has a non-optimal scaling behavior in ε . The elastic energy favors a high oscillation at the boundary as we have seen in the example of the cut-off simple laminate. Therefore, a simple laminate costs a lot of surface energy away from the boundary, cf. Figure 1.4(A). The idea of the upper scaling bound is to construct a microstructure refining towards the boundary and having less interfaces in the bulk of Ω , cf. Figure 1.4(B). By this it still complies with the high oscillation at the boundary but has a smaller interfacial energy in Ω . This comes at the expense that the normal vector between the different phases slightly changes. As the difference of the matrices at an interface needs to be a rank-one matrix, see (1.7) above, the deformation gradient needs to be adjusted and some elastic energy is produced. The scaling law shows that the balance of elastic energy and surface energy for a branching construction is preferred in contrast to the simple laminate. Branching constructions are also justified by physical observations in shape-memory alloys, see for example [Sei+20]. The observations by Kohn and Müller are complemented by Conti, who showed that asymptotically the minimizer of the energy is self-similar [Con00].

1.3.4 Quantitative rigidity and flexibility

Besides the explicit information about microstructures as discussed in the previous section, scaling laws offer quantitative results about the rigidity or flexibility of exact and approx-



(A) Simple laminate construction.



(B) Branching construction, see [RTT25, Figure 6(B)].

Figure 1.4: Schematic pictures for a laminate and branching construction with the same oscillation at the boundary. In the bulk the branching construction has less interfaces than the simple laminate. Both pictures show a suitable choice of the phase indicator χ with different colors representing different phases.

imate solutions to the differential inclusion (1.2). Contrary, the results by De Philippis, Palmieri, and Rindler [DPR18] in Section 1.2, concerning rigidity of exact and approximate solutions are qualitative. They do not measure the deviation (of approximate solutions) from the wells $\mathcal{K}$ or the amount of phase transitions. Quantitative versions of these results can be derived by analyzing the minimum energy of (1.13). The elastic energy (1.10) measures the deviation from the wells. Further, the surface energy penalizes the complexity of (approximate) solutions by measuring the rate of oscillation. By prescribing the boundary data $F \notin \mathcal{K}$, we ensure (for not super-compatible wells) that there are no trivial solutions, i.e., constant exact solutions or approximate solutions converging to a constant state.

Notice that if there exists an exact solution to (1.2) satisfying the boundary data, the minimal elastic energy is achieved and is equal to zero. For non-constant solutions there is at least an energy contribution of the order ε , due to the surface energy. If this exact solution $v = \chi$ were in BV , we could further deduce $E_\varepsilon(v, \chi) \leq C\varepsilon$ for some constant $C > 0$ independent of ε . In contrast, if approximate solutions do not exist, the infimum of the elastic energy is not zero, in particular for the total energy in (1.13) it holds that $E_\varepsilon(v, \chi) \geq C > 0$ uniformly in ε . Moreover, if there is an approximate solution $(v_j)_{j \in \mathbb{N}}$ satisfying the boundary data such that $\|v_j\|_{L^\infty(\Omega)} \leq C < \infty$, it holds that

$$\inf_{v \in \mathcal{D}_F^A} \inf_{\chi \in L^2(\Omega; \mathcal{K})} E_{\text{el}}^A(v, \chi) = 0.$$

Thus, a scaling law as in (1.16), for example in [KM92a; KM94], measures the rigidity of the differential inclusion. If the scaling $s(\varepsilon) > 0$ vanishes in the limit, it implies the existence of an approximate solution satisfying the boundary data. Moreover, for $s(\varepsilon) > \varepsilon$, we can exclude the existence of exact solutions in BV . By measuring deviations from the wells and penalizing oscillation, which are necessary to fulfill the boundary data, we “quantify the flexibility/rigidity” of the solution. Therefore, we say that a scaling law

provides a quantitative description of the rigidity of the differential inclusion.

As we discussed, if the scaling is greater than ε , there cannot be an exact BV -solution to (1.2). To this end, it is an intriguing question if more information on the regularity of exact solutions can be inferred from a scaling law. This is content of [RTZ19], where the *threshold regularity* of exact solutions is discussed. In [RTZ19, Theorem 1], it is proven that for a $C^{1,1}$ domain Ω if there is an exact solution to the gradient inclusion

$$\begin{aligned}\nabla u &\in \bigcup_{j=1}^N SO(d)A_j \text{ almost everywhere in } \Omega, \\ \nabla u &= F \text{ almost everywhere outside } \Omega,\end{aligned}$$

with $\tilde{u}(x) := u(x) - Fx - b$ for some $b \in \mathbb{R}^d$ satisfying $\tilde{u} \in H^{1+s}(\mathbb{R}^d; \mathbb{R}^d)$ and $\text{supp}(\tilde{u}) \subset \overline{\Omega}$ for some $s \in \mathbb{R}$, it holds that

$$\inf_{\nabla u \in \mathcal{D}_F^{\text{curl}}} \int_{\Omega} \text{dist}^2(\nabla u, \bigcup_{j=1}^N SO(d)A_j) \, dx + \varepsilon^2 \int_{\Omega} |\nabla^2 u|^2 \, dx \leq C\varepsilon^{2s}. \quad (1.17)$$

In particular, if in this framework a (lower) scaling law is known, there cannot exist exact solutions of higher regularity corresponding to the scaling law. Notice that in most of the problems arising in this thesis, the problems are rigid in the sense that there are no solutions (satisfying the boundary condition) with zero elastic energy. The $SO(d)$ invariance makes the problem less rigid and exact solutions exist for some $F \in \mathcal{K}^{\text{qc}}$, see for example [DR20; MŠ99; RZZ18]. Still, it motivates the connection between the regularity of solutions to (1.2), hence also the rigidity of the problem, and the scaling law.

1.4 Guiding questions

Based on these observations, the main focus of this thesis is to discuss the following two guiding questions:

- (Q1) What is the influence of the order of the differential operator on the possible scaling behaviors?
- (Q2) How does the choice of surface energy influence the scaling law?

Let us elaborate on these questions. Motivated by the article by Chan and Conti [CC15] for the two-well problem with $\mathcal{A}(D) = \text{curl curl}$, the scaling of the minimum energy of (1.13) can be either $\varepsilon^{2/3}$, as in [KM92a; KM94], or it can be $\varepsilon^{4/5}$. For the gradient inclusion, i.e., $\mathcal{A}(D) = \text{curl}$ a first order operator, the scaling can only be $\varepsilon^{2/3}$, see the discussion in Section 2.2 below. This already suggests that the order of the operator plays a crucial role in the possible scaling laws that can occur. In the following, we will analyze the role of the differential operator for the (compatible) two-well problem. It turns out, see [RRT23;

RRTT24], that the maximal vanishing order on the sphere of the multiplier $|\mathbb{A}(\xi)(A - B)|$ determines the scaling, where $\mathbb{A}$ is defined in (1.4). As the order of the operator determines the possible vanishing orders, it particularly classifies the possible scaling laws. We refer to Chapters 2 and 3 for more details.

Above in (1.13), we penalize changes of phase by the total variation of the derivative of the phase indicator. Other natural choices would be a surface energy depending on ∇v , e.g., the L^2 -norm in Ω of the gradient squared for $v \in H_{\text{loc}}^1(\mathbb{R}^d; \mathbb{R}^n)$ (with a factor ε^2) as in (1.17) or the TV -norm of the distributional derivative Dv for $v \in BV(\Omega; \mathbb{R}^n)$. It is therefore natural to ask whether the choice of surface energy influences the scaling law, which is content of question (Q2).

It is convenient for our analysis to consider the surface energy defined in (1.12), as in contrast to $\|\nabla v\|_{L^2(\Omega)}^2$, it does not introduce a second length scale for the transition layers of v . Another advantage of this choice of surface energy is that the inclusion $v \in \mathcal{K}$ is decoupled from the two constraints $\mathcal{A}(D)v = 0$ and the boundary condition.

1.5 Model settings

Now we introduce the different model scenarios which we will analyze in the following sections below.

1.5.1 Operators

The main focus of the analysis in this thesis are the two differential operators (acting row-wise on matrix fields)

$$\begin{aligned} (\text{curl } v(x))_{kj\ell} &= \frac{1}{2}(\partial_\ell(v_{kj})(x) - \partial_j(v_{k\ell})(x)) \in \mathbb{R}, & v: \mathbb{R}^d &\rightarrow \mathbb{R}^{d \times d}, \quad k, j, \ell = 1, \dots, d \\ (\text{div } v(x))_k &= \sum_{j=1}^d \partial_j(v_{kj}) \in \mathbb{R}, & v: \mathbb{R}^d &\rightarrow \mathbb{R}^{\bar{n} \times d}, \quad k = 1, \dots, \bar{n}. \end{aligned}$$

Here and in the following, we denote by $\{e_1, \dots, e_d\} \subset \mathbb{R}^d$ the canonical basis of $\mathbb{R}^d$.

We study the curl operator for the reasons highlighted in Section 1.1, and the divergence operator for its applications in micromagnetism and as a very flexible operator as explained below. Furthermore, the divergence operator always gives rise to lower bounds for the $\mathcal{A}$ -free setting, as we will discuss in Section 2.4. For both operators, we consider the m -th

order generalized operator. For the curl operator, we define for $v: \mathbb{R}^d \rightarrow \text{Sym}(\mathbb{R}^d; m) := \{M \in (\mathbb{R}^d)^{\otimes m} : M \text{ is symmetric}\}$

$$\begin{aligned} \text{curl}^m: C^\infty(\mathbb{R}^d; \text{Sym}(\mathbb{R}^d; m)) &\rightarrow C^\infty(\mathbb{R}^d; (\mathbb{R}^d)^{\otimes 2m}), \\ [\text{curl}^m v]_{i_1 j_1 i_2 j_2 \dots i_m j_m} &:= \alpha_{i_1 j_1} \circ \alpha_{i_2 j_2} \circ \dots \circ \alpha_{i_m j_m} (\partial_{j_1 \dots j_m}^m v_{i_1 \dots i_m}), \quad i_k, j_k \in \{1, 2, \dots, d\}, \\ &\quad k \in \{1, 2, \dots, m\}, \end{aligned}$$

with the *alternation operator* in the indices j_k, j_ℓ defined by

$$\alpha_{j_k j_\ell} (M_{j_1 \dots j_k \dots j_\ell \dots j_{2m}}) := \frac{1}{2} (M_{j_1 \dots j_{2m}} - M_{j_1 \dots j_\ell \dots j_k \dots j_{2m}}). \quad (1.18)$$

The tensor notation is introduced in Section 1.7.2 below.

Here a comment on the case $m = 1$ is required. The operator $\mathcal{A}(D) = \text{curl}^1$ acts on functions $v: \mathbb{R}^d \rightarrow \mathbb{R}^d$ instead of $v: \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ as defined above. In the following, when we refer to the curl operator, we are referring to the curl operator on matrix fields, acting on the rows like curl^1 . For $m = 2$ we have $\text{curl}^2 v = 0$ if and only if $\text{curl} \text{curl} v = 0$, and furthermore, it holds $\text{curl}^m v = 0$ for $v \in C_c^\infty(\mathbb{R}^d; \text{Sym}(\mathbb{R}^d; m))$ if and only if there exists $u \in C_c^\infty(\mathbb{R}^d; \text{Sym}(\mathbb{R}^d; m-1))$ with $v = D^{\text{sym}} u$ [Sha94, Theorem 2.2.1].

The higher order divergence operator is the adjoint of the m -th derivative (with a suitable choice of L^2 -scalar product). It is defined by

$$\begin{aligned} \text{div}^m: C^\infty(\mathbb{R}^d; \mathbb{R}^{\bar{n}} \otimes \text{Sym}(\mathbb{R}^d; m)) &\rightarrow C^\infty(\mathbb{R}^d; \mathbb{R}^{\bar{n}}) \\ (\text{div}^m v)_j &:= \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_m \leq d} \partial_{i_1 i_2 \dots i_m}^m v_{j i_1 \dots i_m}, \quad j \in \{1, 2, \dots, \bar{n}\}. \end{aligned}$$

To compare the two operators $\mathcal{A}(D) = \text{div}$ and $\mathcal{A}(D) = \text{curl}$, we first notice that both operators are cocanceling, that is there are no super-compatible states for both operators, i.e., $I_{\mathcal{A}} = \{0\}$. Moreover, both have a potential: On the one hand, for the curl operator, curl-free maps are given by gradients due to the Poincaré lemma. In terms of explicit constructions, this has the advantage of a straightforward method of defining curl-free functions satisfying the boundary condition. On the other hand, for div-free functions the potentials are more involved. To give an example, we consider $d = 3$ and $\bar{n} = 1$ for the divergence operator. In this case, divergence-free functions are given by $\nabla \times u$ for some function $u: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ with

$$\nabla \times u := 2 \begin{pmatrix} (\text{curl}^1 u)_{32} \\ (\text{curl}^1 u)_{13} \\ (\text{curl}^1 u)_{21} \end{pmatrix} = \begin{pmatrix} \partial_2 u_3 - \partial_3 u_2 \\ \partial_3 u_1 - \partial_1 u_3 \\ \partial_1 u_2 - \partial_2 u_1 \end{pmatrix}.$$

To be more precise, for both operators it holds that the rank of $\mathbb{A}(\xi)$ is constant for all $\xi \neq 0$, they are *constant-rank operators*. This already implies the existence of a potential under suitable conditions, see [Rai19, Theorem 1, Lemma 2]. Secondly, the divergence operator

is more flexible than the curl operator. For the curl operator the only possible solutions to (1.2) with $\mathcal{K} = \{A, B\}$ (without boundary data) are locally simple laminates [BJ87, Proposition 1]. In contrast to that, for the divergence operator $\mathcal{A}(D) = \text{div}$, two wells can be compatible in more than one direction and thus different solutions are possible. To elaborate, let us consider the div-free two-well problem ($d = 3, \bar{n} = 1$)

$$v(x) \in \{0, e_1\} \subset \mathbb{R}^3, \quad \text{div } v = 0 \text{ in } \mathcal{D}'(\mathbb{R}^3),$$

where the equation is to be understood as in (1.27) below. For every bounded domain $U \subset \mathbb{R}^2$ with smooth boundary the function $v(x) = \chi_U(x_2, x_3)e_1 \otimes e_1$ is a solution. Indeed, as the divergence is the adjoint operator of the gradient, i.e., $\text{div}^* = \nabla$, for any $\varphi \in C_c^\infty(\mathbb{R}^3; \mathbb{R})$ we have

$$\begin{aligned} \int_{\mathbb{R}^3} v(x) \cdot \text{div}^* \varphi(x) \, dx &= \int_{\{x \in \mathbb{R}^3 : (x_2, x_3) \in U\}} e_1 \cdot \nabla \varphi(x) \, dx \\ &= \int_{\mathbb{R}} \int_{\partial U} \varphi(x_1, x') (n_U(x') \cdot e_1) \, d\mathcal{H}^1(x') \, dx_1 = 0. \end{aligned}$$

Here we denote by $n_U \in \mathbb{R}^3$ the unit normal of $\mathbb{R} \times U$, which satisfies $n_U(x') \cdot e_1 = 0$ for $\mathcal{H}^1$ -almost every $x' \in \partial U$. This is an advantage of the divergence operator over the curl as it is possible to have more degrees of freedom. As a final remark, below in Section 2.4 we will discuss that all m -th order operators can be transformed to the case $\mathcal{A}(D) = \text{div}^m$ by a linear transformation and adaptation of the wells in $\mathcal{K}$.

To conclude this section, let us give two examples for the set $V_{\mathcal{A}, \mu}$, defined in (1.8). For $\mathcal{A}(D) = \text{curl}$, we have for $\mu \in \mathbb{R}^{d \times d}$

$$(\mathbb{A}(\xi)\mu)_{kj\ell} = \frac{1}{2}(\xi_\ell \mu_{kj} - \xi_j \mu_{k\ell}) = \frac{1}{2} \begin{pmatrix} \xi_\ell \\ \xi_j \end{pmatrix} \times \begin{pmatrix} \mu_{k\ell} \\ \mu_{kj} \end{pmatrix} = 0,$$

where the last equality holds if and only if the two-vectors $(\xi_\ell, \xi_j)^T$ and $(\mu_{k\ell}, \mu_{kj})^T$ are linearly dependent. In particular this implies $\mathbb{A}(\xi)\mu = 0$ if and only if the rows of μ are parallel to ξ , i.e., there is $a \in \mathbb{R}^d$ such that $\mu = a \otimes \xi$. With this, we deduce that

$$\Lambda_{\text{curl}} = \{a \otimes \xi : a \in \mathbb{R}^d, \xi \in \mathbb{S}^{d-1}\}.$$

For $\mu = a \otimes \xi \in \Lambda_{\text{curl}}$ it holds

$$V_{\text{curl}, \mu} = \text{span}(\xi).$$

Similarly for $\mathcal{A}(D) = \text{div}$, we have for any $\mu \in \mathbb{R}^{\bar{n} \times d}$

$$(\mathbb{A}(\xi)\mu)_k = \sum_{j=1}^d \xi_j \mu_{kj} = (\mu\xi)_k = 0$$

if and only if $\xi \in \ker \mu$ and therefore

$$\Lambda_{\text{div}} = \{\mu \in \mathbb{R}^{\bar{n} \times d} : \dim(\ker \mu) \geq 1\}.$$

For $\mu \in \Lambda_{\text{div}}$ it holds

$$V_{\text{div}, \mu} = \ker \mu.$$

Comparing these observations, we again notice the difference in flexibility of those two model operators. Whereas the set of possible directions of lamination for the curl operator is always one-dimensional, for the divergence it is up to $d - 1$ -dimensional.

1.5.2 Wells

Besides these model operators, we also consider the following model wells. Most prominent is the compatible (but not super-compatible) two-well problem $A - B \in \Lambda_{\mathcal{A}} \setminus I_{\mathcal{A}}$. This is the first step of understanding scaling laws for more involved microstructures. Building on the two-well problem, we consider the “staircase” structure for gradient inclusions giving rise to higher order laminates that is studied in [RT23b]. For $\mathcal{A}(D) = \text{curl}$ and $N \leq d + 1$ consider the N -well structure $\mathcal{K}_N \subset \mathbb{R}_{\text{diag}}^{d \times d}$ defined by

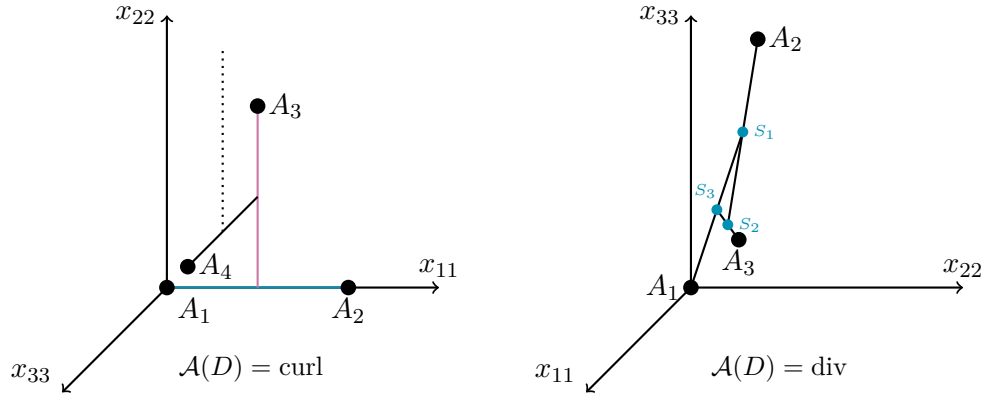
$$\begin{aligned} \mathcal{K}_N &:= \{A_1, A_2, \dots, A_N\}, \\ A_1 &= 0, & A_2 &= \text{diag}(1, 0, \dots, 0), \\ A_3 &= \text{diag}\left(\frac{1}{2}, 1, 0, \dots, 0\right), & A_4 &= \text{diag}\left(\frac{1}{2}, \frac{1}{2}, 1, 0, \dots, 0\right), \\ A_j &= \text{diag}\left(\frac{1}{2}, \dots, \frac{1}{2}, 1, 0, \dots, 0\right), & j &= 5, 6, \dots, N. \end{aligned} \tag{1.19}$$

The wells are depicted in Figure 1.5(A). These wells are chosen such that the Λ_{curl} -convex hull $\mathcal{K}_N^{\text{lc}}$, cf. Section 1.3.2, is a finite union of line segments. Each line segment corresponds to a higher order laminate and thus, to an increasing complexity of the microstructure. To be more precise, the laminates of order $j \in \{1, 2, \dots, N - 1\}$ are given by

$$\mathcal{K}^{(j)} \setminus \mathcal{K}^{(j-1)} = \left\{ \text{diag}\left(\frac{1}{2}, \dots, \frac{1}{2}, \lambda, 0, \dots, 0\right) : \lambda \in (0, 1) \right\}.$$

There are $j - 1$ many entries $\frac{1}{2}$. We already mentioned another example for $\mathcal{A}(D) = \text{curl}$: the Tartar square in $\mathbb{R}_{\text{diag}}^{2 \times 2}$, defined by

$$\begin{aligned} T_4 &:= \{A_1, A_2, A_3, A_4\}, \\ A_1 &= \text{diag}(-1, -3), \quad A_2 = \text{diag}(-3, 1), \quad A_3 = \text{diag}(1, 3), \quad A_4 = \text{diag}(3, -1). \end{aligned} \tag{1.20}$$



(A) Structure of the N diagonal wells in (1.19). The set $\mathcal{K}^{(1)}$ is highlighted in blue and $\mathcal{K}^{(2)} \setminus \mathcal{K}^{(1)}$ in purple. The dotted line illustrates the connection to A_5 in the fourth dimension. (B) The three wells in T_3 with the auxiliary matrices S_1, S_2, S_3 highlighted in blue.

Figure 1.5: The N wells in (1.19) for $\mathcal{A}(D) = \text{curl}$ and the three wells in (1.21) for $\mathcal{A}(D) = \text{div}$. Compatible connections in the wave cone are shown as solid lines.

In T_4 there are no rank-one connections, hence we have $T_4^{\text{lc}} = T_4$, still there are observable “macroscopic” affine transformations as

$$T_4^{\text{qc}} = \{J_1, J_2, J_3, J_4\}^{\text{conv}} \cup \bigcup_{j=1}^4 [A_j, J_j],$$

with the auxiliary matrices

$$J_1 = \text{diag}(-1, 1), \quad J_2 = \text{diag}(1, 1), \quad J_3 = \text{diag}(1, -1), \quad J_4 = \text{diag}(-1, -1),$$

see [Rin18, Proposition 9.4]. These wells are illustrated in Figure 1.1.

Finally, for $\mathcal{A}(D) = \text{div}$, highlighting the flexibility of the operator compared to $\mathcal{A}(D) = \text{curl}$, we consider a T_3 structure due to [GN04], which is an analogous structure to the Tartar square for $\mathcal{A}(D) = \text{curl}$. It is given by

$$\begin{aligned} T_3 &:= \{A_1, A_2, A_3\} \subset \mathbb{R}_{\text{diag}}^{3 \times 3}, \\ A_1 &= \text{diag}(0, 0, 0), \quad A_2 = \text{diag}\left(-\frac{1}{2}, \frac{2}{3}, 3\right), \quad A_3 = \text{diag}(1, 1, 1). \end{aligned} \tag{1.21}$$

Similarly to the Tartar square T_4 , we have $T_3^{\text{lc}} = T_3$, but the div-quasi-convex hull is given by

$$T_3^{\text{qc}} = \{S_1, S_2, S_3\}^{\text{conv}} \cup \bigcup_{j=1}^3 [A_j, S_j],$$

with the auxiliary matrices

$$S_1 = \text{diag}(0, \frac{2}{3}, 2), \quad S_2 = \text{diag}(\frac{1}{2}, \frac{2}{3}, 1), \quad S_3 = \text{diag}(0, \frac{1}{3}, 1).$$

This is shown in [PS09, Theorem 2] and the matrices are depicted in Figure 1.5(B).

For three divergence-free wells that are pairwise incompatible it is shown in [PP04] that exact solutions (without boundary conditions) are rigid, in particular this applies to the set T_3 . In [GN04] flexibility of approximate solutions (without boundary conditions) for the T_3 structure is shown. The scaling law associated to the T_3 structure in [RRT23] relies on both these results. The lower bound resembles the rigidity of the exact solutions and the upper bound uses the construction from [GN04]. We refer to Section 2.3 below for a discussion of the latter scaling law.

1.5.3 Modifications of the variational model

To answer question (Q2), we study different types of singularly perturbed energies. In Chapter 4 below, we consider an anisotropic version of E_ε^A for $\mathcal{A}(D) = \text{curl}$, defined for $\nu \in \mathbb{S}^{d-1}$ by

$$E_{\varepsilon, \nu}(u, \chi) := E_{\varepsilon, \nu}^{\text{curl}}(\nabla u, \chi) := \int_{\Omega} |\nabla u - \chi|^2 dx + \varepsilon \|D_\nu \chi\|_{TV(\Omega)}, \quad (1.22)$$

for $u \in H^1(\Omega; \mathbb{R}^{d \times d})$ with $u(x) = Fx$ on $\partial\Omega$ for some $F \in \mathcal{K}^{\text{qc}} \setminus \mathcal{K}$ and $\chi \in BV_\nu(\Omega; \mathcal{K})$. Here, $BV_\nu(\Omega; \mathcal{K})$ denotes an anisotropic BV space, see Section 1.7.2 for the definition of the space and the total variation norm $\|D_\nu \chi\|_{TV(\Omega)}$. By studying this anisotropic energy as an interesting prototypical modification of the energy (1.13), we are able to understand the robustness of the model in terms of choices of a sharp interface surface energy. Furthermore, instead of a sharp interface model, we analyze diffuse energy models of the form

$$E_{\varepsilon, p, q}^A(v, \chi) := \int_{\Omega} |v - \chi|^p + \varepsilon^q |\nabla v|^q dx, \quad p, q \in [1, \infty)$$

for $\chi \in L^p(\Omega; \mathcal{K})$ and $v \in L_{\text{loc}}^p(\mathbb{R}^d; \mathbb{R}^n)$ such that

$$\mathcal{A}(D)v = 0 \text{ in } \mathcal{D}'(\mathbb{R}^d), \quad v = F \text{ outside } \Omega, \quad \nabla v \in L^q(\Omega; \mathbb{R}^{n \times d}).$$

Besides passing from the L^2 -based framework to an L^p -based one in the elastic energy, also the structure of the interfaces changes. As the phase indicator χ is not necessarily a function of bounded variation anymore, a minimization in $\chi \in L^p(\Omega; \mathcal{K})$ yields the energy

$$E_{\varepsilon, p, q}^A(v) := \inf_{\chi \in L^p(\Omega; \mathcal{K})} E_{\varepsilon, p, q}^A(v, \chi) = \int_{\Omega} \text{dist}^p(v, \mathcal{K}) + \varepsilon^q |\nabla v|^q dx. \quad (1.23)$$

The interpretation changes for $q > 1$ as sharp interfaces are ruled out by the higher regularity assumption on v . In this model the changes of phases that are penalized by the surface energy are in a *transition layer* of a certain width. For $q = 1$ we also allow $v \in BV_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^n)$ with $v = F$ outside Ω and therefore the transition layers are concentrated on lines. In this case the L^1 -norm is replaced by a TV -norm of the measure Dv .

Another possible regularization to stop the increasing oscillation of minimizing sequences at a certain length scale is to discretize the domain and thus the functions. This can be interpreted as a surface energy as fine oscillations are penalized (by a restriction of the domain). Moreover, as we use a triangulation of the domain, this energy is related to the numerical analysis of the elastic energy and was already studied in related models for example in [Chi99; CM99; Lor09]. In two dimensions and for $\mathcal{A}(D) = \text{curl}$ we change the set of admissible functions to model the discretized energy. For this we define the two reference triangles for small $h \in (0, 1)$ by

$$T_h := \{x \in [0, h]^2 : x_2 < h - x_1\}, \quad T'_h := \{x \in [0, h]^2 : x_2 \geq h - x_1\}.$$

With this in hand, we fix the rotated triangulation for $R \in SO(2)$ as

$$\mathcal{T}_h^R := R\mathcal{T}_h = \{R\tau : \tau \in \mathcal{T}_h\}, \quad \mathcal{T}_h := \{T_h + z : z \in h\mathbb{Z}^2\} \cup \{T'_h + z : z \in h\mathbb{Z}^2\}. \quad (1.24)$$

The set of admissible functions is chosen such that ∇u and χ are constant on the triangles. More specifically, we define for $p \in [1, \infty)$

$$\begin{aligned} \mathcal{D}_{h,F}^{p,R} &:= \{u \in W_{\text{loc}}^{1,p}(\mathbb{R}^2; \mathbb{R}^2) : u \text{ is affine on each triangle } \tau \in \mathcal{T}_h^R, u(x) = Fx \text{ outside } \Omega\}, \\ \mathcal{C}_h^R &:= \{\chi \in L^\infty(\mathbb{R}^2; \mathcal{K}) : \chi \text{ is constant on each triangle } \tau \in \mathcal{T}_h^R\}. \end{aligned} \quad (1.25)$$

For the discretized model we only consider the L^p -based elastic energy

$$E_{\text{el},h}^p(u, \chi) := E_{\text{el}}^{p,\text{curl}}(\nabla u, \chi) := \int_{\Omega} |\nabla u - \chi|^p dx, \quad (1.26)$$

for $u \in \mathcal{D}_{h,F}^{p,R}$, $\chi \in \mathcal{C}_h^R$ for some $F \in \mathcal{K}^{\text{qc}} \setminus \mathcal{K}$.

1.6 Relation to the literature

As energies of the form (1.13) play an important role in the study of shape-memory alloys, there is a vast literature on scaling laws for singularly perturbed models. With regards to our model examples, there are already several known related scaling laws. In the articles by Kohn and Müller [KM92a; KM94] the scaling of the compatible two-gradient problem was shown to be $\varepsilon^{2/3}$. As was hinted by the results of Chan and Conti [CC15], that even

include the frame invariance, the two-symmetrized-gradient problem scales either like $\varepsilon^{2/3}$ or $\varepsilon^{4/5}$ depending on the number of compatible directions in the set $\mathcal{K}$, cf. Section 3.2.1. For the Tartar square, upper scaling bounds were derived in [Chi99; Win97] and an almost matching lower bound was recently shown by Růland and Tribuzio [RT22]. To be more precise, consider T_4 as given in (1.20), then for $\mathcal{A}(D) = \text{curl}$ and $F \in T_4^{\text{qc}} \setminus T_4$ for all $\gamma \in (0, 1)$ there are constants $C, c_\gamma, c > 0$ where C and c are independent of γ such that for ε sufficiently small

$$C^{-1} \exp(-c_\gamma |\log \varepsilon|^{\frac{1}{2} + \gamma}) \leq \inf_{\chi \in BV((0,1)^2; T_4)} \inf_{\nabla u \in \mathcal{D}_F^{\text{curl}}} E_\varepsilon^{\text{curl}}(\nabla u, \chi) \leq C \exp(-c |\log \varepsilon|^{\frac{1}{2}}),$$

see [RT22, Theorem 1]. The question whether γ can be set to zero is still an open question.

Contrary, the scaling of the N -well problem $\mathcal{K}_N$ in (1.19) with boundary data for an ℓ -th order laminate $F \in \mathcal{K}_N^{(\ell)} \setminus \mathcal{K}_N^{(\ell-1)}$ was deduced in [RT23b, Theorem 1.4], i.e.,

$$C^{-1} \varepsilon^{\frac{2}{\ell+2}} \leq \inf_{\chi \in BV((0,1)^d; \mathcal{K}_N)} \inf_{\nabla u \in \mathcal{D}_F^{\text{curl}}} E_\varepsilon^{\text{curl}}(\nabla u, \chi) \leq C \varepsilon^{\frac{2}{\ell+2}}.$$

These results show that higher order laminates give rise to larger scaling bounds. Furthermore, the scaling of the Tartar square is larger than any order of lamination, resembling again the infinite order of lamination for the flexibility of approximate solutions:

$$\varepsilon^{\frac{2}{3}} \ll \varepsilon^{\frac{1}{2}} \ll \varepsilon^{\frac{2}{\ell+2}} \ll \exp^{-c_\gamma |\log \varepsilon|^{\frac{1}{2} + \gamma}} \quad \text{for } \ell \geq 3, \gamma \in (0, \frac{1}{2}).$$

This shows that scaling laws are also a useful tool to measure the complexity of the observable microstructures.

Moreover, in [RT23b], the scaling of the compatible two-well problem for $\mathcal{A}(D) = \text{curl}$ was studied for an L^p -based elastic energy using localization methods in real space as in [CC15]. For two wells $\mathcal{K} = \{A, B\} \subset \mathbb{R}^{2 \times 2}$ with $A - B = e_1 \otimes e_1$ and $F \in \mathcal{K}^{\text{qc}} \setminus \mathcal{K}$, it is shown in [RT23b, Theorem 1.1] that there is a constant $C > 0$ such that for every $\varepsilon \in (0, 1)$ it holds that

$$C^{-1} \varepsilon^{\frac{p}{p+1}} \leq \inf_{\substack{u \in W_{\text{loc}}^{1,p}(\mathbb{R}^2; \mathbb{R}^2): \nabla u \in BV((0,1)^2; \mathbb{R}^{2 \times 2}), \\ u(x) = Fx \text{ outside } (0,1)^2}} E_{\varepsilon,p,1}^{\text{curl}}(\nabla u) \leq C \varepsilon^{\frac{p}{p+1}}.$$

Other areas of research related to the study of $\mathcal{A}$ -free differential inclusions are the methods of compensated compactness [DiP85; GR22; GRS22; MT97; Rai24; Tar79; Tar83], generalized Korn-type inequalities [GLN23; GLN24; GRV24], the study of the Aviles-Giga functional [LLP20; LLP22], results regarding $\mathcal{A}$ -quasi-convexity and lower semicontinuity of functionals [ADR20; BMS17; CG22; FM99; Rai19; SW21], and properties of such operators $\mathcal{A}(D)$ [ADHR19; BDG20; DPR18; DR16] to name some examples. As discussed above in Sections 1.1 and 1.2.1 our models are related to shape-memory alloys. For the

modeling of the latter we refer to [Bha03; BJ87; BJ92; MLSG14] and to [Mül99b; Rül25] for a further introduction into its mathematical analysis. An evolutionary model has for example been studied in [KK16; KMR05; Mai04]. The framework of micromagnetism is studied in [BK23; DKMO06; DKO06; KMN19; KS21; Ott02]. We complement these references by the following non exhaustive lists of qualitative references related to rigidity and flexibility properties for gradient inclusions with frame invariance [CDK07; MŠ99] or in the context of linearized elasticity [RS23; Rül16b]. Quantitative statements can be found in [CM04; DM95; DS06; FJM02; LLP24] (with frame invariance) and in [CO12; Lew23] (linearized elasticity). Similar results without an (infinitesimal) frame invariance assumption for different operators $\mathcal{A}(D)$ can be found in [BJ87; CK02; DPR18; FS18; MŠ03; PP04; ST23]. Besides the above mentioned results on scaling laws [CC15; KM92a; KM94; RT22; RT23b], similar results were discussed in the framework of nucleation [AKKR24; CDMZ20; KK11; KKO13; KO19; RT23a; TZ25], micromagnetism [CK98; CKO99; KM11], compliance minimization [KW14; KW16] and other related models [CDZ17; Chi99; Cho01; CM99; CO09; Lor01; Lor09; RT24; Zwi14].

1.7 Outline and Notation

1.7.1 Outline of the thesis

First in Chapter 2 we discuss the results of [RRT23]. There the general scaling of the two-well problem for first order differential operators $\mathcal{A}(D)$ is studied and further, the scaling law for the T_3 structure from (1.21) for the divergence operator $\mathcal{A}(D) = \text{div}$ is deduced. The scaling for the T_3 structure is the corresponding analogous result to the scaling of the Tartar square in [RT22] with the additional difficulties due to the higher flexibility of the divergence operator.

Afterwards, in Chapter 3 we summarize [RRTT24], in particular we will discuss the generalized scaling in the context of the two-well problem for higher order differential operators with the assumption that $V_{\mathcal{A}, A-B}$, cf. (1.8), is a finite union of vector spaces. The results are applied to the operators curl^m and div^m introduced in Section 1.5.1 and matching upper bounds are deduced. Therefore the optimality of the lower bounds are shown in the considered cases.

In Chapter 4 we turn to [RTTZ25], where the role of the choice of the surface energy is discussed. In particular, we will characterize assumptions on anisotropic surface energies of the form $\|D_\nu \chi\|_{TV(\Omega)}$ such that they generate the same “full” scaling as in the isotropic version. Moreover, a class of diffuse or fractional surface energies is considered. For the fractional surface energies, we use the same Fourier based methods as for the anisotropic surface energies. Furthermore, it is shown that the lower bounds on the sharp interface model as in (1.13) provide lower bounds for the diffuse variants of the energy. As a final

version of penalization of oscillation, the discretization from Section 1.5.3 is considered. To conclude, in Chapter 5 the questions (Q1) and (Q2) are linked to the content of Chapters 2 to 4 and some open problems are discussed.

1.7.2 Notation

As mentioned above, the differential constraint is defined in a weak form, asking for $\mathcal{A}(D)v = 0$ in the distributional sense, i.e.,

$$\int_{\mathbb{R}^d} v \cdot \mathcal{A}(D)^* \varphi \, dx = 0 \quad (1.27)$$

for all $\varphi \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^n)$, where the adjoint operator of $\mathcal{A}(D)$, cf. (1.1), is given by

$$\mathcal{A}(D)^* = (-1)^m \sum_{|\alpha|=m} A_\alpha^* \partial^\alpha.$$

In Section 1.3.2, $\mathcal{K}^{\text{qc}}$ is introduced in duality to $\mathcal{A}$ -quasi-convex functions. We denote by $\mathbb{T}^d := \mathbb{R}^d / \mathbb{Z}^d$ the d -dimensional torus. Following [FM99], we say a function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is $\mathcal{A}$ -quasi-convex if

$$f(F) \leq \int_{\mathbb{T}^d} f(F + w(x)) \, dx,$$

for all $F \in \mathbb{R}^n$ and all one-periodic functions $w \in C^\infty(\mathbb{T}^d; \mathbb{R}^n)$ such that $\mathcal{A}(D)w = 0$ and $\int_{\mathbb{T}^d} w(x) \, dx = 0$. In particular, for $\mathcal{A}(D) = \text{curl}$, we recover the well-known definition of quasi-convexity.

The arguments in this thesis heavily rely on the *Fourier transform*. On the one hand, for functions $v \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^n)$ it is given by

$$\hat{v}(\xi) := \mathcal{F}[v](\xi) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} v(x) e^{-i\xi \cdot x} \, dx, \quad \xi \in \mathbb{R}^d,$$

and is extended to functions $v \in L^2(\mathbb{R}^d; \mathbb{R}^n)$ by the density of $C_c^\infty(\mathbb{R}^d; \mathbb{R}^n)$ in $L^2(\mathbb{R}^d; \mathbb{R}^n)$. On the other hand, for one-periodic functions $v \in L^2(\mathbb{T}^d; \mathbb{R}^n)$ it is defined by

$$\hat{v}(k) := \mathcal{F}[v](k) := \int_{\mathbb{T}^d} v(x) e^{-2\pi i k \cdot x} \, dx, \quad k \in \mathbb{Z}^d.$$

With a slight abuse of notation, we use the same notation for both Fourier transformations. For a general overview of Fourier theory, we refer to [Gra14].

The surface energy in (1.12) is chosen as the total variation norm of the distributional

derivative of a function $\chi \in BV(\Omega; \mathbb{R}^n)$. It is given by

$$\|D\chi\|_{TV(\Omega)} := \sup \left\{ \int_{\Omega} \chi \cdot \operatorname{div} \varphi \, dx : \varphi \in C_c^1(\Omega; \mathbb{R}^{n \times d}), \|\varphi\|_{\infty} \leq 1 \right\}.$$

Similarly, for the anisotropic surface energies discussed in Section 4.1 and a direction $\nu \in \mathbb{S}^{d-1}$, we denote the directional distributional derivative of $\chi: \Omega \rightarrow \mathbb{R}^n$ by $D_{\nu}\chi$, i.e., it holds

$$\int_{\Omega} \chi \partial_{\nu} \varphi \, dx = - \int_{\Omega} \varphi \, d(D_{\nu}\chi) \quad \text{for all } \varphi \in C_c^{\infty}(\Omega; \mathbb{R}).$$

The total variation norm of this measure is given by

$$\|D_{\nu}\chi\|_{TV(\Omega)} := \sup \left\{ \int_{\Omega} \chi \cdot \partial_{\nu} \varphi \, dx : \varphi \in C_c^1(\Omega; \mathbb{R}^n), \|\varphi\|_{\infty} \leq 1 \right\}.$$

We denote by $BV_{\nu}(\Omega; \mathbb{R}^n)$ the space of functions $\chi \in L^1(\Omega; \mathbb{R}^n)$ for which the distributional derivative in direction ν is a finite $\mathbb{R}^n$ -valued Radon measure satisfying $\|D_{\nu}\chi\|_{TV(\Omega)} < \infty$. For more details on functions of bounded variation we refer to [AFP00].

In the analysis of higher order operators, we also use the multi-index notation. For $l \in \mathbb{N}^d$ the absolute value is given by $|l| = \sum_{j=1}^d l_j$, the factorial by $l! = \prod_{j=1}^d l_j!$, the multinomial coefficient by $\binom{|l|}{l} = \frac{|l|!}{l!}$, and the power of a vector $\xi \in \mathbb{R}^d$ by $\xi^l = \prod_{j=1}^d \xi_j^{l_j}$, analogously $\partial^{\alpha} = \partial_1^{l_1} \dots \partial_d^{l_d}$.

For the discussion of the operator $\mathcal{A}(D) = \operatorname{curl}^m$, we use the following notation for tensors. We write $e_1, \dots, e_d \in \mathbb{R}^d$ for the standard basis vectors in $\mathbb{R}^d$. Let $M \in (\mathbb{R}^d)^{\otimes m}$ be an m -tensor over $\mathbb{R}^d$, $m \geq 1$. We denote its components for $j_1, j_2, \dots, j_m \in \{1, \dots, d\}$ by

$$M_{j_1 j_2 \dots j_m} := M[e_{j_1}, e_{j_2}, \dots, e_{j_m}] \in \mathbb{R}.$$

We call an m -tensor symmetric, and write $M \in \operatorname{Sym}(\mathbb{R}^d; m)$ if

$$M_{j_1 \dots j_m} = M_{j_{\tau(1)} \dots j_{\tau(m)}},$$

for all permutations $\tau \in S_m$. We recall the definition of the alternation operators $\alpha_{j_k j_{\ell}}$ from (1.18) and similarly introduce the *symmetrization operator* $\sigma_{j_1 \dots j_m}$ in the indices $j_1, \dots, j_m$ by

$$\sigma_{j_1 \dots j_m}(M_{j_1 \dots j_m}) = \frac{1}{m!} \sum_{\tau \in S_m} M_{j_{\tau(1)} \dots j_{\tau(m)}}, \quad \text{for } M \in (\mathbb{R}^d)^{\otimes m}.$$

With this we set the *symmetrized tensor product* of vectors $a^1, \dots, a^m \in \mathbb{R}^d$ to be given by

$$a^1 \odot \dots \odot a^m := \sigma_{1 \dots m}(a^1 \otimes \dots \otimes a^m)$$

with the tensor product $(a^1 \otimes \dots \otimes a^m)_{j_1 \dots j_m} = \prod_{\ell=1}^m a_{j_{\ell}}^{\ell}$. We denote by $a^{\odot k}$ the k times

symmetrized tensor product of $a \in \mathbb{R}^d$ with itself. Furthermore, for two vectors $a, b \in \mathbb{R}^d$ we define the *antisymmetric tensor product* by

$$a \ominus b := \frac{1}{2}(a \otimes b - b \otimes a),$$

and with a slight abuse of notation, we introduce the *symmetric tensor product of antisymmetric tensors* as

$$(a_1 \ominus b_1) \odot \cdots \odot (a_m \ominus b_m) := \frac{1}{m!} \sum_{\tau \in \mathcal{S}_m} (a_{\tau(1)} \ominus b_{\tau(1)}) \otimes \cdots \otimes (a_{\tau(m)} \ominus b_{\tau(m)}). \quad (1.28)$$

The two-state problem and a T_3 structure for the divergence

Contribution of author

In this chapter we summarize the results of [RRT23], which is a joint project of Angkana Rüland, Bogdan Raiță, and the author of this thesis. The article is reproduced in Appendix A and is published as:

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In [RRT23] a first step to answering question (Q1) is accomplished. There the scaling behavior of the singularly perturbed energy (1.13) corresponding to the $\mathcal{A}$ -free inclusion (1.2) for two wells is determined. For two compatible but not super-compatible states it is further assumed that the operator $\mathcal{A}(D)$ is a first order operator, i.e., $m = 1$ in (1.1).

The rigidity of exact solutions shown in [DPR18] for incompatible wells is made quantitative by showing a lower scaling bound in the volume of the domain Ω . For super-compatible states the minimal energy is zero as due to the super-compatibility jumps of the wells (and the boundary condition $F \in \mathcal{K}^{\text{conv}} \setminus \mathcal{K}$) are admissible. Moreover, it is shown that for finitely many wells that are not pairwise super-compatible, the model can be reduced to that of a cocanceling operator, i.e., $I_{\mathcal{A}} = \{0\}$, cf. (1.9). Thus, without loss of generality for not super-compatible wells we can always assume $I_{\mathcal{A}} = \{0\}$. For compatible but not super-compatible states it is shown that the lower scaling bound of $\varepsilon^{\frac{2}{3}}$ for $\mathcal{A}(D) = \text{curl}$, deduced in [KM92a; KM94], is generic for general first order differential operators $\mathcal{A}(D)$. In the particular case of $\mathcal{A}(D) = \text{div}$ this lower bound is complemented by a matching upper bound using a branching construction. It is further observed that the order of the differential operator $\mathcal{A}(D)$ plays a crucial role in the possible scaling. Building on the scaling laws in [CC15] for the curl curl operator $\mathcal{A}(D) = \text{curl curl}$ and for a particular choice of two wells, the $\varepsilon^{\frac{4}{5}}$ -lower scaling bound of [CC15] is recovered. This in particular shows that for second order operators we can not always expect a lower scaling bound of $\varepsilon^{\frac{2}{3}}$. Besides the scaling of the two-well problem for first order differential operators, the T_3 structure in (1.21), introduced in [GN04], for the divergence operator $\mathcal{A}(D) = \text{div}$ is analyzed and an

(almost matching) scaling law similar to that in [RT22] is shown. Moreover, the role of the divergence operator for the deduction of lower bounds is explained.

2.1 Fourier characterization of the elastic energy

The fundamental ingredient of the Fourier based analysis of the two-well problem is given by a Fourier representation of the elastic energy. It turns out that for any $\chi \in L^2(\Omega; \mathcal{K})$, extended by F outside of Ω , we have

$$\inf_{v \in \mathcal{D}_F^A} E_{\text{el}}^A(v, \chi) \geq c \int_{\mathbb{R}^d} \left| \mathbb{A}\left(\frac{\xi}{|\xi|}\right)(\hat{\chi} - F\hat{\chi}_\Omega) \right|^2 d\xi, \quad (2.1)$$

for a constant $c > 0$, see [RRT23, Lemma 3.1]. Here, we denote by χ_Ω the indicator function of the set Ω . For the sake of exposition, here and in the following we will not state the explicit dependencies of constants. They depend on various quantities such as $\Omega, \mathcal{K}, F$ but are independent of ε and the functions v and χ . The quantities they depend on can be found in Appendices A to C or the given references.

The estimate (2.1) is shown by a projection argument in Fourier space, where we choose $\hat{v}(\xi)$ as the pointwise projection of $\hat{\chi}(\xi)$ onto $\ker \mathbb{A}(\xi)$, potentially not complying with the boundary data anymore. For two wells $\mathcal{K} = \{A, B\} \subset \mathbb{R}^n$ with boundary data $F_\lambda = \lambda A + (1-\lambda)B \in \mathcal{K}^{\text{conv}} \setminus \mathcal{K}$ for $\lambda \in (0, 1)$, we write $\chi = \chi_A A + \chi_B B$ with $\chi_A + \chi_B = \chi_\Omega$. Then (2.1) implies that

$$\inf_{v \in \mathcal{D}_{F_\lambda}^A} E_{\text{el}}^A(v, \chi) \geq c \int_{\mathbb{R}^d} \left| \mathbb{A}\left(\frac{\xi}{|\xi|}\right)(A - B) \right|^2 |(1-\lambda)\hat{\chi}_A - \lambda\hat{\chi}_B|^2 d\xi. \quad (2.2)$$

The estimate (2.2) is stated in [RRT23, Corollary 3.2]. This “characterization” of the elastic energy has the advantage of the visibility of the multiplier structure of $\rho(\xi) := |\mathbb{A}(\xi)(A - B)|^2$: Whenever $\rho(\xi)$ restricted to the unit sphere is bounded away from zero, the energy is coercive. The lower scaling bounds for the two-well problem then are based on a careful analysis of this multiplier.

2.2 The two-well problem

The lower scaling bounds for the three cases of the two-well problem, incompatible, compatible but not super-compatible, and super-compatible, are an immediate consequence of the Fourier characterization in (2.2). Indeed, for incompatible wells, the multiplier $\rho(\xi)$

only vanishes in the origin, thus there is a constant $C > 0$ such that $\rho(\xi) \geq C > 0$ for all $\xi \in \mathbb{S}^{d-1}$. In particular, for a new constant $c' > 0$ we get

$$\inf_{v \in \mathcal{D}_{F_\lambda}^A} E_{\text{el}}^A(v, \chi) \geq c' \int_{\mathbb{R}^d} |(1-\lambda)\chi_A - \lambda\chi_B|^2 dx \geq c' \min\{(1-\lambda)^2, \lambda^2\} |\Omega|,$$

shown in [RRT23, Corollary 3.2].

For compatible but not super-compatible wells A and B the multiplier has a non-trivial zero set given by $V_{A,A-B}$, cf. (1.8). In [RRT23, Theorem 1(ii)] the lower bound for the total energy E_ε^A under the assumption that $\mathcal{A}(D)$ is a first order operator is shown to be

$$\inf_{\chi \in BV(\Omega; \{A, B\})} \inf_{v \in \mathcal{D}_{F_\lambda}^A} E_\varepsilon^A(v, \chi) \geq C\varepsilon^{\frac{2}{3}}.$$

This estimate uses a splitting argument in Fourier space as used in [CKO99; KW16].

A crucial ingredient is the linearity of the map $\xi \mapsto \mathbb{A}(\xi)$, as then $V_{A,A-B}$ is a vector space. We split $\mathbb{R}^d$ into

$$\mathbb{R}^d = V_{A,A-B} \oplus V_{A,A-B}^\perp.$$

Then, the elastic energy provides a control over $|(1-\lambda)\hat{\chi}_A - \lambda\hat{\chi}_B|$ for frequencies “close” to $V_{A,A-B}$ and the surface energy provides a high frequency control, cf. [KKO13]. For $\mathcal{A}(D) = \text{div}$ and $\Omega = (0, 1)^d$ this lower bound is complemented by a matching upper bound using a branching construction as done for example in [KM92a; KM94] for the operator $\mathcal{A}(D) = \text{curl}$. Here the higher flexibility of the divergence operator in contrast to general first order operators is convenient, as it ensures that the branching construction still defines a divergence-free function.

For two super-compatible wells A and B with boundary data $F_\lambda = \lambda A + (1-\lambda)B$, we notice that the map defined by

$$v(x) := \begin{cases} A, & x \in \Omega, \\ F_\lambda, & x \notin \Omega, \end{cases}$$

is admissible, i.e., $v \in \mathcal{D}_{F_\lambda}^A$. Therefore the optimal energy vanishes [RRT23, Theorem 1(iii)]. The reduction to cocanceling operators for finitely many wells, that are pairwise not super-compatible, then is achieved by a projection of the images of v and χ onto $I_{\mathcal{A}}^\perp$ combined with the observation that for the super-compatible two-well problem the problem is essentially unconstrained.

Besides the scaling of compatible wells for first order operators, the special case for a two-well problem for the second order operator $\mathcal{A}(D) = \text{curl curl}$ is considered. In this instance a lower $\varepsilon^{\frac{4}{5}}$ -scaling bound motivated by [CC15] is derived. In this particular example $V_{A,A-B}$ still is a vector space, thus our arguments are applicable. The difference

in scaling arises due to the different vanishing order of the multiplier $\rho(\xi)$. This is discussed in more detail in Chapter 3.

2.3 A T_3 structure for the divergence operator

Besides the two-well problem, to elaborate on the higher flexibility of the divergence operator, a quantitative version of a T_3 structure is considered. The methods used are an adaptation of the arguments in [RT22], where instead of $\mathcal{A}(D) = \text{curl}$ for the Tartar square $\mathcal{K} = T_4$, cf. (1.20), we analyze $\mathcal{A}(D) = \text{div}$ using the wells $\mathcal{K} = T_3$ in (1.21). We recall that the divergence-free inclusion for T_3 is rigid for exact solution but flexible for approximate solutions [GN04; PP04]. The scaling resembles the scaling for the Tartar square shown in [RT22], cf. Section 1.6.

Theorem 2.1 ([RRT23, Theorem 2]). *Let $\Omega = (0, 1)^3$, let T_3 be the three-well set given in (1.21), and let $F \in T_3^{\text{qc}} \setminus T_3$. We consider the energy $E_\varepsilon^{\text{div}}$ as above in (1.13) with the set of admissible functions $\mathcal{D}_F^{\text{div}}$ in (1.11) for the divergence operator $\mathcal{A}(D) = \text{div}$. Then, there are constants $C = C(F) > 1$ and $c = c(F) > 0$ such that for every $\gamma \in (0, \frac{1}{2})$ there are $\varepsilon_0 = \varepsilon_0(\gamma, F) > 0$ and $c_\gamma > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ it holds that*

$$C^{-1} \exp(-c_\gamma |\log \varepsilon|^{\frac{1}{2}+\gamma}) \leq \inf_{\chi \in BV(\Omega; T_3)} \inf_{v \in \mathcal{D}_F^{\text{div}}} E_\varepsilon^{\text{div}}(v, \chi) \leq C \exp(-c |\log \varepsilon|^{\frac{1}{2}}).$$

Moreover, we have the following related result.

Proposition 2.2 ([RRT23, Proposition 1.2]). *Let $\Omega = (0, 1)^3$ and let T_3 be the three-well set given in (1.21), and let $F \in T_3^{\text{qc}} \setminus T_3$. We denote by χ_{jj} the diagonal entries of the matrix field $\chi \in BV(\Omega; T_3)$ and by $E_\varepsilon^{\text{div}}$ the singularly perturbed energy as in (1.13) for the operator $\mathcal{A}(D) = \text{div}$ with $\mathcal{D}_F^{\text{div}}$ defined as in (1.11). Then, there is $\varepsilon_0 > 0$ such that for every $\gamma \in (0, \frac{1}{2})$ there is a constant $c_\gamma > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ it holds*

$$\sum_{j=1}^3 \|\chi_{jj} - \int_\Omega \chi_{jj}(y) \, dy\|_{L^2(\Omega)}^2 \leq \exp(c_\gamma |\log \varepsilon|^{\frac{1}{2}+\gamma}) \left(\inf_{v \in \mathcal{D}_F^{\text{div}}} E_\varepsilon^{\text{div}}(v, \chi) \right)^{\frac{1}{2}}.$$

This result measures both the rigidity of exact solutions and the flexibility of approximate solutions quantitatively. The left-hand side measures the deviation of χ to a constant, reflecting the rigidity. In particular, if $\inf_{v \in \mathcal{D}_F^{\text{div}}} E_\varepsilon^{\text{div}}(v, \chi) = 0$, χ needs to be constant. Notice however that due to the lack of exact solutions satisfying the boundary data the energy never vanishes. On the right-hand side the “cost” of this deviation is controlled in terms of the energy with a multiplicative factor. This large factor reflects the flexibility of the problem as it allows large deviations of χ from a constant state while the energy on the right-hand side remains small.

The proofs of these results rely on a similar Fourier characterization of the elastic energy as for the two-well problem in Section 2.2. For its derivation, the explicit operator $\mathcal{A}(D) = \text{div}$ is plugged into (2.1). Instead of exploiting the two-well structure, it is used that the wells are diagonal matrices. Moreover, in contrast to the continuous Fourier transform, we extend the functions v and χ one-periodically and use the discrete Fourier transform.

The diagonal entries of the phase indicator in its Fourier representation then concentrate in truncated cones around planes, whereas the Fourier mass outside these cones can be controlled in terms of the energy. By the structure of the three matrices, i.e., that each diagonal entry determines the others, the size of the cones can iteratively be reduced. This cone reduction argument is an adaptation of the argument introduced by R  land and Tribuzio [RT22; RT23b] in the curl-free setting, where the cones are around one-dimensional vector space. After a certain number of iterations only the mass in the zero frequency is not controlled in terms of the energy, yielding Proposition 2.2. Controlling the mean of χ_{jj} then gives the lower bound. The upper bound construction is based on an infinite order lamination, quantifying the flexibility of approximate solutions.

2.4 Role of the divergence operator

As a final part in [RRT23, Appendix B] the role of the divergence operator is studied. The special property is that lower bounds for the m -th order divergence operator provide corresponding lower bounds for any m -th order differential operator. More precisely, for any constant coefficient, homogeneous, linear differential operator $\mathcal{A}(D)$ of order m there is a linear transformation

$$\omega_m: \mathbb{R}^n \rightarrow W \otimes \text{Sym}(\mathbb{R}^d; m), \quad (\omega_m(x))_{ji_1 \dots i_m} := (A \sum_{\ell=1}^m e_{i_\ell} x)_j.$$

such that

$$\mathcal{A}(D) = \sum_{|\alpha|=m} A_\alpha \partial^\alpha = \text{div}^m \circ \omega_m,$$

with $\alpha = \sum_{\ell=1}^m e_{i_\ell} \in \mathbb{N}^d$ understood as a multi-index as in Section 1.7.2. For any $v \in \mathcal{D}_F^A$ it holds $\omega_m \circ v \in \mathcal{D}_{\omega_m(F)}^{\text{div}^m}$. Moreover, for cocanceling operators the transformation ω_m is injective as $\ker \omega_m = I_{\mathcal{A}}$. Using this transformation, we deduce the lower bounds for any differential operator [RRT23, Proposition B.2].

The compatible two-well problem for higher order operators

Contribution of author

In this chapter we summarize the results of [RRTT24], which is a joint project of Angkana Rüland, Bogdan Raiță, Antonio Tribuzio, and the author of this thesis. The article is reproduced in Appendix B and is published as:

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All authors contributed equally to this project.

Building on the results of [RRT23], as discussed in Chapter 2, in [RRTT24] possible scaling laws for higher order operators are derived. In Section 2.2 above, we observed that the $\varepsilon^{\frac{2}{3}}$ -lower scaling bound is generic for first order operators. Furthermore, for $\mathcal{A}(D) = \text{curl curl}$ the compatible two-well problem can also give rise to an $\varepsilon^{\frac{4}{5}}$ -scaling law. In [RRTT24] this observation is generalized to a class of higher order operators. In particular, for $\mathcal{A}(D) = \text{curl}^m$ scaling laws of the form $\varepsilon^{\frac{2L}{2L+1}}$ are shown for $L \in \{1, 2, \dots, m\}$, where the precise scaling depends on the structure of $A - B$. Additionally, the same behavior for $\mathcal{A}(D) = \text{div}^m$ is observed. For this the underlying structure relevant for the scaling behavior is made explicit and the key factor for understanding the lower bound using Fourier (localization) methods is identified.

3.1 Lower scaling bound for higher order operators

Using similar ideas as in Section 2.2 the lower scaling bound for the two-well problem for an m -th order differential operator is shown under the assumption that $V_{\mathcal{A}, \mathcal{A}-B}$ is a finite union of linear spaces. The scaling then is determined by the vanishing order of the multiplier that arises when rewriting the elastic energy in its Fourier representation. The *maximal vanishing order on the unit sphere (vanishing order)* $L[\rho]$ of a $2m$ homogeneous

polynomial $\rho \in \mathbb{R}[\xi]$ with zero set $V = \rho^{-1}(\{0\})$ is defined by, cf. [RRTT24, Definition 1.3],

$$L[\rho] := \min \left\{ \ell \in \mathbb{N} : \inf_{\xi \in \mathbb{S}^{d-1} \setminus V} \frac{\rho(\xi)}{\text{dist}_V(\xi)^{2\ell}} > 0 \right\}.$$

For $A - B \in \Lambda_A \setminus I_A$ assume that $\rho(\xi) := |\mathbb{A}(\xi)(A - B)|^2$ has vanishing order $L := L[\rho] \in \mathbb{N}$ and that $V := V_{A,A-B} = \rho^{-1}(\{0\})$ is a finite union of linear spaces, then for $F_\lambda = \lambda A + (1 - \lambda)B$ with $\lambda \in (0, 1)$ there are constants $\varepsilon_0 > 0$ and $C > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ it holds

$$\inf_{\chi \in BV(\Omega; \{A, B\})} \inf_{v \in \mathcal{D}_{F_\lambda}^A} E_\varepsilon^A(v, \chi) \geq C \min\{(1 - \lambda)^2, \lambda^2\} \varepsilon^{\frac{2L}{2L+1}}, \quad (3.1)$$

see [RRTT24, Theorem 1.4]. This result hence gives rise to a new class of scaling laws and quantifies the relation of the scaling and the order of the differential operator $\mathcal{A}(D)$.

While for first order operators, as seen in Section 2.2, the multiplier carries a linear structure for higher order operators the vanishing order can be higher and thus the scaling might be different. This complements the observation in [RRT23], where for $\mathcal{A}(D) = \text{curl curl}$ the $\varepsilon^{\frac{4}{5}}$ -lower scaling bound is shown.

Similar to Section 2.2, the proof is based on the Fourier representation (2.2) of the elastic energy from Section 2.1. This characterization of the elastic energy, combined with the definition of the vanishing order of the multiplier yields

$$\inf_{v \in \mathcal{D}_{F_\lambda}^A} E_{\text{el}}^A(v, \chi) \geq C \int_{\mathbb{R}^d} \frac{\text{dist}_V(\xi)^{2L}}{|\xi|^{2L}} |(1 - \lambda)\hat{\chi}_A - \lambda\hat{\chi}_B|^2 d\xi, \quad (3.2)$$

where we write $\chi = A\chi_A + B\chi_B$ as in Sections 2.1 and 2.2. The proof of the lower scaling bound then is based on a splitting argument in Fourier space. We decompose $\mathbb{R}^d$ into a high frequency region, a region for small frequencies “far away” from the zero set V , and the remaining frequencies “close” to V , see Figure 3.1. In the high frequency region, we control the Fourier mass of $|(1 - \lambda)\hat{\chi}_A - \lambda\hat{\chi}_B|$ in terms of the surface energy, cf. [KKO13]. For the small frequencies away from V the elastic energy is elliptic and thus, it provides a corresponding frequency control. The Fourier mass of $|(1 - \lambda)\hat{\chi}_A - \lambda\hat{\chi}_B|$ in the remaining frequencies is controlled by an absorption argument using the uniform support of the functions χ in Ω .

3.2 Applications

With the two model examples curl^m and div^m in mind, the above scaling estimate is applied to these operators for certain choices of A and B . This in particular generalizes

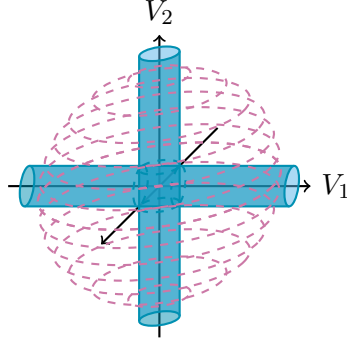


Figure 3.1: Splitting of Fourier space into regions of different energy estimates for the two-well problem. Here, $V_{A,A-B} = V_1 \cup V_2$ is a union of two linear spaces. Inside the blue cylinders an absorption argument is applied. Outside the purple ball (dashed) we use a high frequency estimate in terms of the surface energy. Inside the purple ball and outside the cylinders we have control over the frequencies in terms of the elastic energy using (3.2).

the observation made in [RRT23] on the lower scaling bound for $\mathcal{A}(D) = \text{curl curl}$ based on the results by Chan and Conti [CC15].

3.2.1 Application to curl^m

To show the lower bounds for $\mathcal{A}(D) = \text{curl}^m$, we aim to apply (3.1). The novelty of the remaining steps required to show the lower bound therefore lies in calculating the vanishing order of the associated multiplier and verifying that the zero set is indeed a finite union of linear spaces. We do not consider any A, B such that $A - B \in \Lambda_{\text{curl}^m}$ but restrict ourselves to $A - B = e_1^{\odot l_1} \odot \cdots \odot e_d^{\odot l_d} \in \Lambda_{\text{curl}^m}$ for $l \in \mathbb{N}^d$ such that $|l| = m$, where the notation is introduced in Section 1.7.2. These form a basis of the wave cone (1.5), cf. [RRTT24], and already give rise to a large class of scaling laws.

As an auxiliary result as in [RRTT24, Lemma 2.2] we make the form of $\mathbb{A}(\xi)(e_1^{\odot l_1} \odot \cdots \odot e_d^{\odot l_d})$ explicit as

$$\mathbb{A}(\xi)(e_1^{\odot l_1} \odot \cdots \odot e_d^{\odot l_d}) = (e_1 \odot \xi)^{\odot l_1} \odot \cdots \odot (e_d \odot \xi)^{\odot l_d},$$

where the symmetric tensor product of antisymmetric tensors is defined in (1.28). This implies that $V = \bigcup_{j: l_j \neq 0} \text{span}(e_j)$ is indeed a finite union of linear spaces. Moreover, we use this representation to determine the vanishing order, and in particular show

$$|\mathbb{A}(\frac{\xi}{|\xi|})(e_1^{\odot l_1} \odot \cdots \odot e_d^{\odot l_d})|^2 \geq C \frac{\text{dist}_V(\xi)^{2L}}{|\xi|^{2L}}$$

for a constant $C > 0$ and $L = \max_{j=1, \dots, d} l_j$, see [RRTT24, Lemma 3.4]. Thus, we apply the lower bound in (3.1) to $\mathcal{A}(D) = \text{curl}^m$ with the vanishing order $L = \max_{j=1, \dots, d} l_j$ to

show the desired lower scaling of the form as in Theorem 3.1 below:

$$\inf_{\chi \in BV(\Omega; \{A, B\})} \inf_{v \in \mathcal{D}_{F_\lambda}^{\text{curl}^m}} E_\varepsilon^{\text{curl}^m}(v, \chi) \geq C \varepsilon^{\frac{2L}{2L+1}}.$$

The matching upper bound is provided in two dimensions on the unit cube based on the branching construction in [CC15]. For $m > 2$, a careful adaptation is required to accommodate for the boundary data. The construction gives an explicit potential, i.e., a map $u: \Omega \rightarrow \text{Sym}(\mathbb{R}^2; m-1)$ fulfilling the boundary condition $u(x) = Fx$ on $\partial\Omega$, and then we set $v := D^{\text{sym}}u$ which is curl^m -free by construction. The idea is to set all but one components in the energy density to zero.

Starting from the components of $D^{\text{sym}}u - F_\lambda - (\chi - F_\lambda)$ corresponding to $A - B = e_1^{\odot l_1} \odot e_2^{\odot l_2}$, i.e., those for which the components of $A - B$ are not vanishing, we set all components to zero, except the one for the index $(1, \dots, 1)$ or $(2, \dots, 2)$. This yields a system of differential equations on the level of the potential u , which are then solved iteratively. The non-vanishing energy contribution is carried by the last component, i.e., the one which is not set to zero. Depending on l this is either the energy contribution of $u_{1\dots 1}$ or $u_{2\dots 2}$. To ensure that this iteration is possible while still ensuring to satisfy the boundary condition, the assumption $F = F_{1/2} = \frac{1}{2}(A + B)$ is made. The scaling then is determined by how many times the above mentioned iteration can be carried out, i.e., l_1 or l_2 .

Together these results yield the scaling for a class of two-well problems for $\mathcal{A}(D) = \text{curl}^m$. This in particular verifies the optimality of (3.1) in the considered cases.

Theorem 3.1 ([RRTT24, Theorem 1.2]). *Let $d, m \in \mathbb{N}$, $d \geq 2$, and let $l \in \mathbb{N}^d$ with $|l| = m$. Let $\Omega \subset \mathbb{R}^d$ be an open, bounded Lipschitz domain. Consider $A, B \in \text{Sym}(\mathbb{R}^d; m)$ such that $A - B = e_1^{\odot l_1} \odot \dots \odot e_d^{\odot l_d}$, see Section 1.7.2 for the notation, and $F_\lambda = \lambda A + (1 - \lambda)B$ for $\lambda \in (0, 1)$. For the operator $\mathcal{A}(D) = \text{curl}^m$ let $E_\varepsilon^{\text{curl}^m}$ be as above in (1.13) with the set of admissible functions $\mathcal{D}_{F_\lambda}^{\text{curl}^m}$ given in (1.11). We set $L := \max_{j=1, \dots, d} l_j$. Then there are constants $C = C(\Omega, d, m) > 0$ and $\varepsilon_0 = \varepsilon_0(\Omega, d, m, \lambda) > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ it holds*

$$\inf_{\chi \in BV(\Omega; \{A, B\})} \inf_{v \in \mathcal{D}_{F_\lambda}^{\text{curl}^m}} E_\varepsilon^{\text{curl}^m}(v, \chi) \geq C \min\{(1 - \lambda)^2, \lambda^2\} \varepsilon^{\frac{2L}{2L+1}}.$$

Moreover, for $d = 2$, $\Omega = (0, 1)^2$ and $\lambda = \frac{1}{2}$ there exists $C' = C'(m) > 1$ such that for $L = \max\{l_1, l_2\}$ and for any $\varepsilon \in (0, \varepsilon_0)$ it holds

$$C'^{-1} \varepsilon^{\frac{2L}{2L+1}} \leq \inf_{\chi \in BV(\Omega; \{A, B\})} \inf_{v \in \mathcal{D}_{F_{1/2}}^{\text{curl}^m}} E_\varepsilon^{\text{curl}^m}(v, \chi) \leq C' \varepsilon^{\frac{2L}{2L+1}}.$$

3.2.2 Application to div^m

As a second application, recalling Section 2.4 the higher order divergence operator is analyzed. Following the above ideas, we first explicitly determine the symbol $\mathbb{A}$ and the vanishing order, allowing us to apply the estimate in (3.1).

For $A - B = v \otimes e_1^{\odot l_1} \odot \cdots \odot e_d^{\odot l_d}$ for some $v \in \mathbb{R}^{\bar{n}}$ and $l \in \mathbb{N}^d$ with $|l| = m$, it holds for $\mathcal{A}(D) = \operatorname{div}^m$ that

$$\mathbb{A}(\xi)(A - B) = \binom{m}{l}^{-1} \xi^l v,$$

and hence $V = V_{\operatorname{div}^m, A-B} = \bigcup_{j: l_j \neq 0} \operatorname{span}(e_j)^\perp$, cf. [RRTT24, Lemma 2.3]. The vanishing order is given by $L = m - \min_{j=1, \dots, d} l_j$. Plugging this into (3.1) then yields the lower scaling bound

$$\inf_{\chi \in BV(\Omega; \{A, B\})} \inf_{v \in \mathcal{D}_{F_\lambda}^{\operatorname{div}^m}} E_\varepsilon^{\operatorname{div}^m}(v, \chi) \geq C \min\{(1 - \lambda)^2, \lambda^2\} \varepsilon^{\frac{2L}{2L+1}},$$

see [RRTT24, Lemma 3.7].

In two dimensions the upper bound for $\mathcal{A}(D) = \operatorname{curl}^m$ can be transformed to provide a corresponding upper bound for $\mathcal{A}(D) = \operatorname{div}^m$. This is a consequence of the fact that the two operators are related by a change of coordinates, e.g., for $m = 1$ the two operators only differ by a rotation. Thus, the upper bound construction for $\mathcal{A}(D) = \operatorname{curl}^m$ also implies the optimality of the lower bound in two dimensions for $\mathcal{A}(D) = \operatorname{div}^m$.

The effect of surface energies

Contribution of author

In this chapter we summarize the results of [RTTZ25], which is a joint project of Angkana Rüland, Antonio Tribuzio, Christian Zillinger, and the author of this thesis. The article is reproduced in Appendix C and is available as a preprint at:

A. Rüland, C. Tissot, A. Tribuzio, and C. Zillinger. *On surface energies in scaling laws for singular perturbation problems for martensitic phase transitions*. 2025. arXiv: 2507.06773 [math.AP]

All authors contributed equally to this project.

In [RTTZ25] the role of the surface energy is analyzed. In the first part, anisotropic surface energies for $\mathcal{A}(D) = \text{curl}$ of the form (1.22) are considered. These only penalize an oscillation in a certain direction. For the model wells in $\mathcal{K}_N$, cf. (1.19), we provide a characterization of the directions, which yield the same scaling for the anisotropic energy as for the isotropic case in [RT23b]. It turns out that only the “inner-most” lamination needs to be penalized to observe the same scaling. For a finite number of degenerate directions the scaling resembles that of a lower order laminate, depending on the relation of the anisotropic direction ν and the wells $\mathcal{K}_N$. These observations then are used to also show lower bounds for an (anisotropic) fractional surface energy penalization, where for $s \in (0, \frac{1}{2})$ the surface energy is given by the H^s -seminorm of χ instead of the total variation norm of $D\chi$.

In the second part, diffuse surface energies are considered. Based on a Modica-Mortola type argument [MM77], similar to [KK11], it is shown that an energy of the form as in (1.23), i.e.,

$$E_{\varepsilon,p,q}^{\mathcal{A}}(v, \chi) = \int_{\Omega} \text{dist}^p(v, \mathcal{K}) + \varepsilon^q |\nabla v| \, dx,$$

is bounded from below by the sharp interface model with a surface energy as in (1.12). In particular, we use the lower scaling bounds from Chapters 2 and 3 to also directly deduce the same lower bounds for the diffuse energy model. For $\mathcal{A}(D) = \text{curl}$ and $\mathcal{K}_3$ as in (1.19) for three wells in two dimensions $d = 2$, these lower bounds are complemented by matching upper bounds. This shows that also the diffuse surface energies give rise to the same energy scaling as the sharp interface model analyzed in [RT23b]. Analogous results

hold when the isotropic surface energy $\int_{\Omega} |\nabla v|^q dx$ is replaced by the anisotropic version $\int_{\Omega} |\partial_{\nu} v|^q dx$ for some $\nu \in \mathbb{S}^{d-1}$.

For $\mathcal{A}(D) = \text{curl}$ and in two dimensions $d = 2$, in the last part, oscillations on a scale finer than h are prevented by a discretization instead of an additional surface energy. For this the triangulation in (1.24) is used and ∇u and χ are assumed to be constant on the triangles, cf. (1.25). The scaling of the wells in (1.19) for $N = 3$ is determined, where the precise scaling depends on the orientation of the triangulation. In particular the scaling bounds from the anisotropic energy are recovered. The lower bounds are complemented by upper bounds, which are a piecewise affine interpolation of the already known upper bounds for the sharp interface model.

In addition to the model example of $\mathcal{K}_N$, and in particular $\mathcal{K}_3$, also the Tartar square, cf. (1.20) is used for a better understanding of the different surface energies.

4.1 Anisotropic surface energies

Let us recall the anisotropic energy from (1.22) that is we only consider $\mathcal{A}(D) = \text{curl}$ and the anisotropic surface energy as defined (1.22).

As mentioned above only a penalization of the “inner-most” laminate is required to preserve the isotropic scaling. Heuristically this can be explained by the following observation in terms of the upper bound. For simplicity, let us consider (cut-off) simple laminates instead of branching constructions as discussed in (1.15). In the case of a second order laminate where only the inner, i.e., second order laminate, is penalized by the surface energy, we first construct the outer laminate on a certain scale r . This scale r is not penalized by the surface energy as the lamination is in the degenerate direction. Inside this laminate, we now construct the inner laminate on a scale $r_2 < r$ for which we have to do a cut-off towards the boundaries of the cells. As the oscillation of the second laminate is penalized, we assume that its scale is not finer than ε , hence, the cut-off area has a width of order $r_2 > \varepsilon$. Indeed, as explained in (1.15) the size of the cut-off area is proportional to the scale on which we are laminating. In particular the elastic energy due to the cut-off is of an order larger than ε times the length of the side of the cells, i.e., the length of the interfaces of the first order laminate, see Figure 4.1. This suggests that a second order laminate always gives rise to an energy contribution which is larger than the (isotropic) surface energy contribution of the first order laminate.

In [RTTZ25] these observations are made explicit for the wells in (1.19) and for $\Omega = (0, 1)^d$. Taking $F \in \mathcal{K}_N^{(\ell)} \setminus \mathcal{K}_N^{(\ell-1)}$, cf. (1.14), and $\nu \cdot e_1 \neq 0$, we recover the isotropic scaling, else, the scaling depends on ν according to the following result.

Theorem 4.1 ([RTTZ25, Theorem 2]). *Let $d \geq 2$, $N \leq d + 1$, $\Omega = (0, 1)^d$, $\nu \in \mathbb{S}^{d-1}$, and $\ell \in \{1, 2, \dots, N - 1\}$. Consider the set $\mathcal{K}_N \subset \mathbb{R}^{d \times d}$ given in (1.19) and $F \in \mathcal{K}_N^{(\ell)} \setminus \mathcal{K}_N^{(\ell-1)}$,*

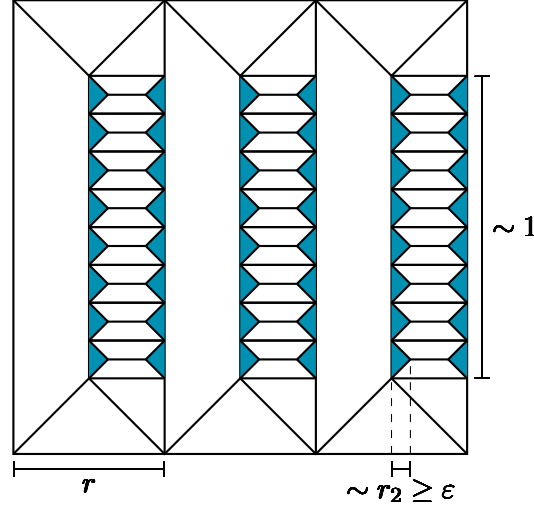


Figure 4.1: Illustration of the second order laminate in the anisotropic case. The elastic energy contribution of the cut-off in the second order laminate (highlighted in blue) amounts to a larger order than the interfacial energy of the first order laminate with an isotropic surface energy, i.e., $\frac{r_2}{r} \geq \frac{\varepsilon}{r}$.

and let $E_{\varepsilon, \nu}$ be the energy given in (1.22) with the corresponding set of admissible functions $\mathcal{D}_F^{\text{curl}}$ in (1.11) for $\mathcal{A}(D) = \text{curl}$. Then, there are constants $C = C(d, F, \ell) > 0$ and $\varepsilon_0 = \varepsilon_0(d, F, \ell, \nu) > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ it holds

$$\inf_{\chi \in BV_\nu(\Omega; \mathcal{K}_N)} \inf_{u \in \mathcal{D}_F^{\text{curl}}} E_{\varepsilon, \nu}(u, \chi) \geq C \sum_{j=0}^{\ell-1} |\nu_{j+1}|^{\frac{2}{\ell-j+2}} \varepsilon^{\frac{2}{\ell-j+2}}.$$

The lower bound is based on [RT23b] and uses similar Fourier localization methods as the proof of the two-well problem, cf. Sections 2.2 and 3.1. Due to a non-linear relation between the wells in $\mathcal{K}_N$ the localization argument is improved by a cone reduction argument similar to the cone reduction for the divergence-free T_3 structure in Section 2.3. In two dimensions for $N = 3$, upper bounds are provided, where the construction relies on the one in [RT23b] and the ν dependence is made explicit. Moreover, for the Tartar square it is observed that for any choice of $\nu \in \mathbb{S}^1$ in the anisotropic energy the scaling from [RT22] is recovered.

In [RTTZ25], using the discrete Fourier transform, cf. Section 1.7.2, on $\Omega = (0, 1)^d$ for the one-periodic extension of $\chi \in BV(\Omega; \mathcal{K})$, it is further observed that for (anisotropic) fractional surface energies of the form $E_{\text{surf}, s, \nu}(\chi)^{2s} := \sum_{k \in \mathbb{Z}^d} |k \cdot \nu|^{2s} |\hat{\chi}(k)|^2$ for $s \in (0, \frac{1}{2})$ the same methods are applicable. Thus, the scaling also holds for this type of surface penalization. The upper bound in two dimensions is a consequence of an interpolation inequality, as the surface energy is bounded from above by the sharp interface variant $E_{\text{surf}, s, \nu}(\chi) \leq C \|D_\nu \chi\|_{TV(\Omega)}$ for $\chi \in BV_\nu(\Omega; \mathcal{K})$ for some constant $C > 0$ independent of s . The precise statements are given in [RTTZ25, Theorem 4].

4.2 Diffuse surface energies

Based on an argument by Knüpfer and Kohn [KK11] it is proven that the diffuse energy given in (1.23) is bounded from below by the sharp interface energy for a suitable choice of the phase indicator χ . For this let $\mathcal{K} = \{A_1, \dots, A_N\} \subset \mathbb{R}^n$ be a discrete set and take $F \in \mathcal{K}^{\text{qc}} \setminus \mathcal{K}$. Then we have, as shown in [RTTZ25, Proposition 4.3], that for any $v \in L^p(\Omega; \mathbb{R}^n)$ such that $\mathcal{A}(D)v = 0$ in $\mathcal{D}'(\mathbb{R}^d)$, $v = F$ outside Ω , and $\nabla v \in L^q(\Omega; \mathbb{R}^{n \times d})$ there is $\chi \in BV(\Omega; \mathcal{K})$ such that

$$\int_{\Omega} \text{dist}^p(v, \mathcal{K}) + \varepsilon^q |\nabla v|^q dx \geq C \left(\int_{\Omega} |v - \chi|^p dx + \varepsilon \|D\chi\|_{TV(\Omega)} \right).$$

An analogous result for an anisotropic version is shown in [RTTZ25, Theorem 3].

This is achieved by projecting the image of v on a particular direction ζ such that the number of wells in $\mathcal{K} \cdot \zeta$ is the same as in $\mathcal{K}$. For the projected energy with one-dimensional image, by a Modica-Mortola type trick and the co-area formula, we quantify the energy contribution of changes of phases in v . We then define $\chi \cdot \zeta$ via a projection of $v \cdot \zeta$ onto $\mathcal{K} \cdot \zeta$. The remaining components of χ then are fixed by the one-to-one correspondence of $\mathcal{K}$ and $\mathcal{K} \cdot \zeta$.

Similar upper bounds for the diffuse energy are derived. For given $v \in \mathcal{D}_F^A \cap BV_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^n) \cap L^\infty(\mathbb{R}^d; \mathbb{R}^n)$ and $\chi \in BV(\Omega; \mathcal{K})$ there is a function $v_\varepsilon \in L^p(\Omega; \mathbb{R}^n)$ with $\mathcal{A}(D)v_\varepsilon = 0$ in $\mathcal{D}'(\mathbb{R}^d)$, $v_\varepsilon = F$ outside Ω , and $\nabla v_\varepsilon \in L^q(\Omega; \mathbb{R}^{n \times d})$, such that

$$\begin{aligned} \int_{\Omega} \text{dist}^p(v_\varepsilon, \mathcal{K}) + \varepsilon^q |\nabla v_\varepsilon|^q dx &\leq C \left(\int_{\Omega} |v - \chi|^p dx + \varepsilon \|D\chi\|_{TV(\Omega)} \right. \\ &\quad \left. + \varepsilon \|Dv - D\chi\|_{TV(\Omega)} + \varepsilon \text{Per}(\Omega) \right). \end{aligned} \quad (4.1)$$

This result can be found in [RTTZ25, Lemma 4.4]. In particular, for our model application, i.e., $\mathcal{A}(D) = \text{curl}$, $p = 2$, and $\mathcal{K} = \mathcal{K}_N$, we can apply both estimates. Moreover, the functions used as competitors for the upper bounds ($d = 2$) in the scaling laws satisfy

$$\|Dv - D\chi\|_{TV(\Omega)} \leq C \|D\chi\|_{TV(\Omega)}.$$

Hence, these results provide matching upper and lower bounds for the diffuse surface penalization for $\mathcal{K} = \mathcal{K}_3$, see [RTTZ25, Corollary 1.5].

4.3 Discretization

For the discretization of the elastic energy (for $\mathcal{A}(D) = \text{curl}$ and $d = 2$) we consider the triangulation $\mathcal{T}_h^R$ for some rotation $R \in SO(2)$ defined in (1.24), the energy $E_{\text{el},h}^p$ in (1.26), and the sets of admissible functions $u \in \mathcal{D}_{h,F}^{p,R}$ and $\chi \in \mathcal{C}_h^R$ defined for $p \geq 1$ and $F \in \mathcal{K}^{\text{qc}} \setminus \mathcal{K}$ in (1.25).

As it was for example observed in [CM99], there is a non-trivial energy contribution in the bulk of Ω if the grid is misaligned with the potentially possible direction of lamination. If the direction of lamination is aligned with the grid of the discretization, i.e., if the direction of lamination is orthogonal to an edge of the triangles, there is no energy contribution besides the cut-off area. For higher order laminates, the same phenomenon as observed for the anisotropic surface penalization in Section 4.1 is present. Thus only the “inner-most” laminate needs to be misaligned with the triangulation to observe the same scaling as in the sharp interface model. To prove this, a lower bound in terms of an anisotropic singular perturbation energy is shown, where the grid size h is taking the role of the (small) singular perturbation parameter. Settings of finitely many wells $\mathcal{K} = \{A_1, \dots, A_N\}$ with boundary data $F \in \mathcal{K}^{\text{qc}} \setminus \mathcal{K}$ such that there is at most one possible direction of lamination $w \in \mathbb{S}^{d-1}$ are studied. Then, as shown in [RTTZ25, Theorem 5], for any $u \in \mathcal{D}_{h,F}^{p,R}$ and $\chi \in \mathcal{C}_h^R$, setting $\nu \in \mathbb{S}^{d-1}$ with $\nu \cdot w = 0$, it holds for sufficiently small h that

$$E_{\text{el},h}^p(u, \chi) \geq C \left(\int_{\Omega} |\nabla u - \chi|^p dx + h \|D_{\nu} \chi\|_{TV(\Omega)} + h \right).$$

The additional term h is a consequence of the incompatibility of the boundary data with the wells and the triangulation. To be more precise, due to the boundary data, any admissible function contributes to the energy in a non-vanishing part of the h -neighborhood of the boundary.

Turning back to the three-well setting $\mathcal{K}_3$ (see (1.19) for $N = 3$ and $d = 2$), we notice that the only rank-one connection present is in direction e_1 , thus for rotations R such that e_1 is not orthogonal to any edge of the triangle RT_h , we get

$$E_{\text{el},h}^p(u, \chi) \geq \begin{cases} Ch^{\frac{2}{3}}, & F \in \mathcal{K}_3^{(1)} \setminus \mathcal{K}_3, \\ Ch^{\frac{1}{2}}, & F \in \mathcal{K}_3^{(2)} \setminus \mathcal{K}_3^{(1)}. \end{cases}$$

For rotations such that e_1 is orthogonal to an edge of the triangle RT_h , we have the scaling as in the degenerate anisotropic case

$$E_{\text{el},h}^p(u, \chi) \geq \begin{cases} Ch, & F \in \mathcal{K}_3^{(1)} \setminus \mathcal{K}_3, \\ Ch^{\frac{2}{3}}, & F \in \mathcal{K}_3^{(2)} \setminus \mathcal{K}_3^{(1)}, \end{cases}$$

see [RTTZ25, Corollary 1.6]. Notice that the scaling for first order laminates is not zero, but h .

The (matching) upper bounds are following the same ideas as in [Chi99; CM99; Lor09]. For this we take the competitors for the corresponding upper bounds in the sharp interface model (1.13) and define their affine interpolation on the triangles. For the phase indicators we take the constant interpolations on the triangles. This yields the upper bounds in the discrete setting.

Conclusion

Let us summarize the observations and results of this thesis. We presented different scaling laws for general $\mathcal{A}$ -free differential inclusions and explicit examples of scaling laws for both $\mathcal{A}(D) = \operatorname{curl}^m$ and $\mathcal{A}(D) = \operatorname{div}^m$. In particular, lower scaling bounds for the two-well problem for general operators $\mathcal{A}(D)$ were investigated and the optimality of those scaling estimates for $\mathcal{A}(D) = \operatorname{div}^m$ and $\mathcal{A}(D) = \operatorname{curl}^m$ were shown. Moreover, we derived the scaling for a T_3 structure for the divergence operator as given in (1.21). We characterized the energy scaling for anisotropic surface energies for an N -well problem that gives rise to higher order laminates and showed that the scaling remains the same if the sharp interface energy is replaced by a diffuse (or fractional) surface energy. It is even possible to replace the surface energy by a discreteness assumption to deduce the same scaling behavior. These results show that the model we consider is robust under changes of the surface energy and the scaling stays unchanged even for highly degenerate anisotropic surface energies. Besides the explicit scaling laws, we discussed a useful Fourier characterization of the elastic energy, general relations between diffuse and sharp interface energies, and a relation of the discrete energy to the (anisotropic) sharp interface energy.

The lower bounds in the scaling laws rely on Fourier methods. The first step to deduce the lower bound is the Fourier characterization of the form (2.1) followed by a splitting argument in Fourier space to control the total energy from below. For higher order laminates we apply an iterative reduction argument in Fourier space to improve the scaling. The optimal scaling then corresponds to the maximal possible number of iterations of this reduction argument, which is given by the order of lamination of the boundary data. In the case of the T_3 structure discussed in Section 2.3 a similar reduction argument yields the lower scaling bound where the number of iterations is not determined by the boundary data but chosen to be optimal for the behavior in ε . The upper bounds (with the exception of the infinite order laminate for the T_3 structure in the case of $\mathcal{A}(D) = \operatorname{div}$) are given by branching constructions. These branching constructions compared to laminates, provide a better balance of the elastic energy and the surface energy. Due to the incompatibility of the boundary data and the wells, the elastic energy favors a high oscillation of simple laminates. This high oscillation results in a large contribution of the surface energy. Branching structures enable us to preserve that high oscillation close to the boundary whereas they “save” surface energy away from the boundary which becomes small for less changes of phases. The scaling laws particularly imply that a branching structure is favorable over the simple laminate structure, but they do not imply that these are the optimal structures. However, in [Con00] a further step to understand the microstructures of the singularly perturbed energy was accomplished by showing that the minimizers of the

energy in [KM92a; KM94] for $\mathcal{A}(D) = \text{curl}$ with two compatible wells are self-similar.

The upper bounds for the fractional surface energies are a consequence of an interpolation argument, bounding the fractional surface energy from above by the sharp surface energy. For the diffuse setting, the upper bounds are given by the upper bound v of the sharp interface setting, where the function v is mollified on the scale ε . The explicit estimates then consist of a careful analysis of those mollified functions and can then be reduced to the known cases of the sharp interface model. The discrete upper bounds, analogously, are an affine interpolation of the upper bounds for the sharp interface model. Therefore this shows the robustness of the model under changes of the surface energy to determine the scaling law. In particular, it is possible to choose one specific convenient model for the analysis. For the methods used here this is the case for the sharp interface energy given in (1.13) as Fourier methods are available and it does not introduce a second length scale for the transition layers.

5.1 Discussion of the guiding questions

With these results, we formulate answers to the questions (Q1) and (Q2).

- **Ad (Q1):** The order of the operator plays a major role for the possible scaling laws. For higher orders m of the operator, there is a larger class of possible scaling behaviors for the singularly perturbed energy. The scaling of the two-well problem is determined by the vanishing order of the multiplier $|\mathbb{A}(\xi)(A - B)|$. In particular, the possible scaling laws are determined by the order of the operator $\mathcal{A}(D)$ and $A - B$. With this, at least for the two-well problem, we obtained a deep understanding of the influence of the order of the operator on the scaling behavior. By this result, we can deduce lower bounds by determining the vanishing order of the corresponding multiplier. The key ingredient for this observation is the Fourier representation of the elastic energy in Section 2.1. The optimality of these lower bounds was proven for certain model cases. For general operators $\mathcal{A}(D)$ and wells A, B the optimality of the lower bound has to be analyzed separately. The upper bounds for $\mathcal{A}(D) = \text{curl}^m$ suggest that the scaling is optimal in the cases relevant for applications.
- **Ad (Q2):** The choice of the surface energy plays a minor role. In particular, by the results discussed in Chapter 4, we explained that a large class of natural choices of the surface energy yields the same scaling as the energy defined in (1.13). Thus, it is justified to consider the sharp energy model. We discussed that diffuse surface energies and discrete energies can be bounded from below by the corresponding sharp interface model. Moreover, by considering explicit examples, we expect that in the most cases the resulting lower scaling bounds for the diffuse and discrete energies are optimal. We can adapt the constructions of the upper bounds for the sharp interface model to define an upper bound for the diffuse or discrete energies.

If we have a control over the deviation of interfaces of the function v to the phase indicator χ , one can provide a corresponding upper bound. Moreover, a large class of anisotropic surface energies is expected to give rise to the same scaling laws as the analogous isotropic model. To justify this claim, we studied an explicit N -well structure for $\mathcal{A}(D) = \text{curl}$. There, as long as the anisotropic surface energy penalizes the inner-most laminate, the energy scaling of the isotropic surface energy is recovered. In terms of the lower bound this is based on the cone reduction argument as introduced in [RT22; RT23b] with particular care in the explicit choice of the non-linear relation of the diagonal entries. A similar behavior is expected to be valid for other choices of wells and operators.

5.2 Follow-up questions

To conclude this thesis, let us comment on some open questions that could be studied in the future.

5.2.1 Higher order laminates for higher order operators

One natural problem to consider is to combine the results for higher order operators in [RRTT24], discussed in Chapter 3, with the observations regarding higher order laminates made in [RT23b] and [RTTZ25], discussed in Chapter 4. For a more detailed answer to question (Q1), it is of interest to determine the scaling of higher order laminates for higher order operators as for example $\mathcal{A}(D) = \text{curl curl}$. One difficulty is that the Fourier multipliers for the diagonal components of χ might have different vanishing orders and thus a more careful argument may be required. Furthermore, the $\Lambda_{\mathcal{A}}$ -convex/ $\mathcal{A}$ -quasi-convex hull might look different and thus more complicated for higher order operators. To elaborate on this, consider $\mathcal{A}(D) = \text{curl curl}$ and $\mathcal{K} = \mathcal{K}_3$ as in (1.19) in two dimensions. In this case it holds

$$A_3 - A_2 = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 1 \end{pmatrix} \odot \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 1 \end{pmatrix},$$

and thus $A_3 - A_2 \in \Lambda_{\text{curl curl}}$, therefore $\lambda A_3 + (1 - \lambda) A_2 \in \mathcal{K}^{(1)}$ for all $\lambda \in [0, 1]$. This shows that the lamination convex hull looks more complicated than in the case $\mathcal{A}(D) = \text{curl}$ and extra care is needed. For a similar staircase structure as for the curl operator, it might be necessary to adapt the wells.

It is suspected that for a similar staircase structure in $\mathcal{K}$ the different vanishing orders of the diagonal entries play a crucial role, but the order of the different vanishing orders does not. To be more precise, assume that the Fourier multiplier corresponding to the diagonal entry χ_{jj} has vanishing order L_j . Any reordering of the wells such that the set of vanishing

orders remains the same (taking into account how often they occur) may give rise to the same scaling law. We motivate this conjecture by commenting on the strategy for the derivation of lower bounds in Fourier space. Here the cones are iteratively truncated as done in the proof of Theorem 3.1. The new truncation parameter in the reduction of the cones in Fourier space depends on the vanishing order. Thus, the “final” cut-off parameter, i.e., the one relevant for the scaling, may depend on the number of the different vanishing orders but not on the order in which they appear.

Building on this and the results presented in Chapter 4 one can ask whether in this setting anisotropic surface energies provide the same scaling. Also here it would be interesting to transfer the results obtained in the gradient case to higher order differential operators. We conjecture that also in this setting only the inner-most laminate is required to be penalized for the same (isotropic) scaling laws to hold. As for the curl curl operator, and similar for other operators, there can be more than one compatible direction for the inner-most laminate, it may be required that the anisotropic energy provides control in more than one direction. Thus, the degenerate setting, where the inner-most laminate is not penalized (in all directions), needs to be analyzed more carefully. Moreover, another difficulty arises as for this case certain vanishing orders might not influence the scaling. In contrast to the conjecture above, the order of the vanishing orders might play a role as only certain vanishing orders are influencing the anisotropic scaling law.

For the upper bounds additional difficulties arise if we have to construct higher order branching structures as in [RT23b]. Already for the curl curl operator particular care is required as the cells of the second order branching are not given by rectangles or linear deformations of such but have a curved boundary, cf. [CC15; RRT24].

5.2.2 Generalizations of results

Besides this problem that further elaborates on the influence of the order of the operator, we could aim for more generality trying to remove some of the assumptions in the above presented results. The lower bound in (3.1) relies on the assumption that the zero set is given by a finite union of linear spaces. Without this assumption, instead of the union of vector spaces the zero set is a projective variety and therefore we could lose the linear structure. In the presented arguments the distance to the zero set is used, therefore, for a projective variety the argument is possibly more involved. Additionally, without the assumption on the zero set, we may require extra care for the absorption argument in the neighborhood of $V_{A,A-B}$ used for the low frequency control.

Another interesting problem would be to show the L^2 -based diffuse energy scaling law using similar Fourier methods as for the sharp interface energy. In [RTT25], cf. Section 4.2, the diffuse energy is compared to the sharp version of the energy. One difficulty for a treatment directly with the Fourier methods is that due to the diffuse interfaces a second length scale arising from the second gradient is present. This length scale is not

present when penalizing the energy with the total variation of the phase indicator. As the minimization in $v = \nabla u$ not only happens on the level of the elastic energy but also the surface energy, the multiplier is not given by $|\mathbb{A}(\xi)(A - B)|$. The analysis of this new multiplier may be more challenging due to the second length scale which needs to be considered in the Fourier methods.

The results discussed in Chapter 4 could also be generalized. Anisotropic surface energies have been studied for the model class of wells defined in (1.19) (and the Tartar square (1.20)). An interesting follow-up question is whether the same results hold for other structures and operators. It is thus natural to aim at showing similar results in a more general fashion. Similarly, the optimality of the lower bound for the diffuse energies has only been shown in the instances of $\mathcal{A}(D) = \text{curl}$ and $\mathcal{K} = \mathcal{K}_3$. To show similar estimates for other choices of $\mathcal{K}$, we could invoke (4.1) to relate upper bounds of the sharp interface model to the diffuse interface model.

A possible generalization regarding the model is to consider a soft boundary condition instead of the hard boundary data $v = F$ outside Ω . This was already done in [KM92a; KM94], where instead of prescribing the exact value of v outside of Ω (or on $\partial\Omega$), deviations from F are admissible but penalized by a suitable energy. Further, a systematic study of periodic functions with a prescribed average condition on v might be an intriguing research question.

Furthermore, building on the micromagnetic energy as introduced in Section 1.2.1, it is interesting to incorporate strict non-convex constraints on the admissible functions. In the case of micromagnetism this corresponds to $|v| = 1$ almost everywhere in Ω . Our Fourier based methods would still be applicable, as there the minimization is considered over a larger set of functions, the optimality of the lower bound, however, might be lost due to that constraint. For the upper bounds a more careful construction might be required.

5.2.3 Related models

Instead of generalizing the results for the models introduced in this thesis, also certain modifications of the model give rise to many interesting problems. The models discussed in this thesis neglect frame invariance whereas frame invariance is an important physical concept for large deformations. As discussed in Section 1.1 for gradient inclusions a possible approximation for small deformations is to consider $\mathcal{A}(D) = \text{curl curl}$, i.e., the symmetrized gradient instead of the gradient. Nonetheless understanding settings with full $SO(d)$ invariance, as done in [CC15], is of great significance for research. The above models are a good step towards that understanding, still are not yet the complete picture. It is not clear whether the methods presented here allow for a proof of scaling laws for models with frame invariance. The Fourier methods as discussed in this thesis heavily relied on the fact that χ only attains values in a discrete set, therefore an adaptation of the methods to this case seems to be challenging. Still, especially the results for the anisotropic

surface energies in Section 4.1, suggest the advantages of these Fourier methods. Moreover, the arguments to relate diffuse and sharp interface models in Section 4.2 rely on the discreteness of the set $\mathcal{K}$. Showing a similar result for sets $\mathcal{K}$ with $SO(d)$ invariance therefore is still subject to research. The results in [CC15] rely on localization methods in real space instead of Fourier space. An additional question would be to extend these techniques to frame invariant models for more than two wells.

A further related model is that of nucleation. In terms of shape-memory alloys this models a nucleus of martensite inside austenite, thus we still study energies of the form (1.13). The set Ω models the martensite phase and hence, the set in which the inclusion $\nabla u \in \mathcal{K}$ has to hold. In contrast to the model we considered above, Ω is not fixed but variable with fixed volume $|\Omega| = V$. We then aim to determine the scaling in the prescribed volume V of the martensite phase. The shape of the nucleus has to be chosen suitable to accommodate for the compatibility or incompatibility to the austenite phase. This has already been studied in some instances as mentioned in Section 1.6, e.g., in [KK11; TZ25], also in connection with Fourier methods [KKO13; RT23a]. It would be an interesting task to analyze nucleation problems in the context of $\mathcal{A}$ -free differential inclusions and for different types of surface energies.

Bibliography

- [ADHR19] A. Arroyo-Rabasa, G. De Philippis, J. Hirsch, and F. Rindler. “Dimensional estimates and rectifiability for measures satisfying linear PDE constraints”. In: *Geometric and Functional Analysis* 29.3 (2019), pp. 639–658. DOI: 10.1007/s00039-019-00497-1.
- [ADR20] A. Arroyo-Rabasa, G. De Philippis, and F. Rindler. “Lower semicontinuity and relaxation of linear-growth integral functionals under PDE constraints”. In: *Advances in Calculus of Variations* 13.3 (2020), pp. 219–255. DOI: 10.1515/acv-2017-0003.
- [AFP00] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. New York, NY: Oxford University Press, 2000. DOI: 10.1093/oso/9780198502456.001.0001.
- [AH86] R. J. Aumann and S. Hart. “Bi-convexity and bi-martingales”. In: *Israel Journal of Mathematics* 54.2 (1986), pp. 159–180. DOI: 10.1007/BF02764940.
- [AKKR24] I. Akramov, H. Knüpfer, M. Kružík, and A. Růland. “Minimal energy for geometrically nonlinear elastic inclusions in two dimensions”. In: *Proceedings of the Royal Society of Edinburgh Section A: Mathematics* 154.3 (2024), pp. 769–792. DOI: 10.1017/prm.2023.36.
- [BDG20] D. Breit, L. Diening, and F. Gmeineder. “On the trace operator for functions of bounded $\mathbb{A}$ -variation”. In: *Analysis & PDE* 13.2 (2020), pp. 559–594. DOI: 10.2140/apde.2020.13.559.
- [BFJK94] K. Bhattacharya, N. B. Firoozye, R. D. James, and R. V. Kohn. “Restrictions on microstructure”. In: *Proceedings of the Royal Society of Edinburgh: Section A Mathematics* 124.5 (1994), pp. 843–878. DOI: 10.1017/S0308210500022381.
- [Bha03] K. Bhattacharya. *Microstructure of Martensite: Why it forms and how it gives rise to the shape-memory effect*. Oxford Series on Materials Modelling. Oxford: Oxford University Press, 2003. DOI: 10.1093/oso/9780198509349.001.0001.
- [Bha93] K. Bhattacharya. “Comparison of the geometrically nonlinear and linear theories of martensitic transformation”. In: *Continuum Mechanics and Thermodynamics* 5.3 (1993), pp. 205–242. DOI: 10.1007/BF01126525.
- [BJ87] J. M. Ball and R. D. James. “Fine phase mixtures as minimizers of energy”. In: *Archive for Rational Mechanics and Analysis* 100.1 (1987), pp. 13–52. DOI: 10.1007/BF00281246.

- [BJ92] J. M. Ball and R. D. James. “Proposed experimental tests of a theory for fine microstructures and the two-well problem”. In: *Philosophical Transactions of the Royal Society of London. Series A: Physical and Engineering Sciences* 338.1650 (1992), pp. 389–450. DOI: 10.1098/rsta.1992.0013.
- [BK23] B. Brietzke and H. Knüpfer. “Onset of pattern formation in thin ferromagnetic films with perpendicular anisotropy”. In: *Calculus of Variations and Partial Differential Equations* 62.133 (2023). DOI: 10.1007/s00526-023-02459-w.
- [BMS17] A. C. Barroso, J. Matias, and P. M. Santos. “Differential inclusions and $\mathcal{A}$ -quasiconvexity”. In: *Mediterranean Journal of Mathematics* 14.116 (2017). DOI: 10.1007/s00009-017-0917-7.
- [CC15] A. Chan and S. Conti. “Energy scaling and branched microstructures in a model for shape-memory alloys with $SO(2)$ invariance”. In: *Mathematical Models and Methods in Applied Sciences* 25.6 (2015), pp. 1091–1124. DOI: 10.1142/S0218202515500281.
- [CDK07] S. Conti, G. Dolzmann, and B. Kirchheim. “Existence of Lipschitz minimizers for the three-well problem in solid-solid phase transitions”. In: *Annales de l’Institut Henri Poincaré. Analyse Non Linéaire* 24.6 (2007), pp. 953–962. DOI: 10.1016/j.anihpc.2006.10.002.
- [CDMZ20] S. Conti, J. Diermeier, D. Melching, and B. Zwicknagl. “Energy scaling laws for geometrically linear elasticity models for microstructures in shape memory alloys”. In: *ESAIM. Control, Optimisation and Calculus of Variations* 26.115 (2020). DOI: 10.1051/cocv/2020020.
- [CDZ17] S. Conti, J. Diermeier, and B. Zwicknagl. “Deformation concentration for martensitic microstructures in the limit of low volume fraction”. In: *Calculus of Variations and Partial Differential Equations* 56.16 (2017). DOI: 10.1007/s00526-016-1097-1.
- [CG22] S. Conti and F. Gmeineder. “ $\mathcal{A}$ -quasiconvexity and partial regularity”. In: *Calculus of Variations and Partial Differential Equations* 61.215 (2022). DOI: 10.1007/s00526-022-02326-0.
- [Chi99] M. Chipot. “The appearance of microstructures in problems with incompatible wells and their numerical approach”. In: *Numerische Mathematik* 83.3 (1999), pp. 325–352. DOI: 10.1007/s002110050452.
- [Cho01] R. Choksi. “Scaling laws in microphase separation of diblock copolymers”. In: *Journal of Nonlinear Science* 11.3 (2001), pp. 223–236. DOI: 10.1007/s00332-001-0456-y.
- [CK02] M. Chlebík and B. Kirchheim. “Rigidity for the four gradient problem”. In: *Journal für die Reine und Angewandte Mathematik* 551 (2002), pp. 1–9. DOI: 10.1515/crll.2002.082.

- [CK98] R. Choksi and R. V. Kohn. “Bounds on the micromagnetic energy of a uniaxial ferromagnet”. In: *Communications on Pure and Applied Mathematics* 51.3 (1998), pp. 259–289. DOI: 10.1002/(SICI)1097-0312(199803)51:3<259::AID-CPA3>3.0.CO;2-9.
- [CKO99] R. Choksi, R. V. Kohn, and F. Otto. “Domain branching in uniaxial ferromagnets: a scaling law for the minimum energy”. In: *Communications in Mathematical Physics* 201.1 (1999), pp. 61–79. DOI: 10.1007/s002200050549.
- [CM04] N. Chaudhuri and S. Müller. “Rigidity estimate for two incompatible wells”. In: *Calculus of Variations and Partial Differential Equations* 19.4 (2004), pp. 379–390. DOI: 10.1007/s00526-003-0220-2.
- [CM99] M. Chipot and S. Müller. “Sharp energy estimates for finite element approximations of non-convex problems”. In: *IUTAM Symposion on Variations of domain and free-boundary problems in solid mechanics*. Ed. by P. Argoul, M. Frémond, and Q. S. Nguyen. Vol. 66. Solid Mechanics and its Applications. Dordrecht: Springer, 1999, pp. 317–325. DOI: 10.1007/978-94-011-4738-5_38.
- [CO09] A. Capella and F. Otto. “A rigidity result for a perturbation of the geometrically linear three-well problem”. In: *Communications on Pure and Applied Mathematics* 62.12 (2009), pp. 1632–1669. DOI: 10.1002/cpa.20297.
- [CO12] A. Capella and F. Otto. “A quantitative rigidity result for the cubic-to-tetragonal phase transition in the geometrically linear theory with interfacial energy”. In: *Proceedings of the Royal Society of Edinburgh: Section A. Mathematics* 142.2 (2012), pp. 273–327. DOI: 10.1017/S0308210510000478.
- [Con00] S. Conti. “Branched microstructures: scaling and asymptotic self-similarity”. In: *Communications on Pure and Applied Mathematics* 53.11 (2000), pp. 1448–1474. DOI: 10.1002/1097-0312(200011)53:11<1448::AID-CPA6>3.0.CO;2-C.
- [DiP85] R. J. DiPerna. “Compensated compactness and general systems of conservation laws”. In: *Transactions of the American Mathematical Society* 292.2 (1985), pp. 383–420. DOI: 10.2307/2000221.
- [DKMO06] A. DeSimone, R. V. Kohn, S. Müller, and F. Otto. “Chapter 4 - Recent Analytical Developments in Micromagnetics”. In: *The Science of Hysteresis*. Ed. by G. Bertotti and I. D. Mayergoyz. Vol. II. Oxford: Academic Press, 2006, pp. 269–381.
- [DKO06] A. DeSimone, H. Knüpfer, and F. Otto. “2-d stability of the Néel wall”. In: *Calculus of Variations and Partial Differential Equations* 27.2 (2006), pp. 233–253. DOI: 10.1007/s00526-006-0019-z.
- [DM95] G. Dolzmann and S. Müller. “Microstructures with finite surface energy: the two-well problem”. In: *Archive for Rational Mechanics and Analysis* 132.2 (1995), pp. 101–141. DOI: 10.1007/BF00380505.

- [DPR18] G. De Philippis, L. Palmieri, and F. Rindler. “On the two-state problem for general differential operators”. In: *Nonlinear Analysis* 177 (2018), pp. 387–396. DOI: 10.1016/j.na.2018.03.015.
- [DR16] G. De Philippis and F. Rindler. “On the structure of $\mathcal{A}$ -free measures and applications”. In: *Annals of Mathematics* 184.3 (2016), pp. 1017–1039. DOI: 10.4007/annals.2016.184.3.10.
- [DR20] F. Della Porta and A. Růland. “Convex integration solutions for the geometrically nonlinear two-well problem with higher Sobolev regularity”. In: *Mathematical Models and Methods in Applied Sciences* 30.03 (2020), pp. 611–651. DOI: 10.1142/S0218202520500153.
- [DS06] C. De Lellis and L. Székelyhidi Jr. “Simple proof of two-well rigidity”. In: *Comptes Rendus Mathématique* 343.5 (2006), pp. 367–370. DOI: 10.1016/j.crma.2006.07.008.
- [FJM02] G. Friesecke, R. D. James, and S. Müller. “A theorem on geometric rigidity and the derivation of nonlinear plate theory from three-dimensional elasticity”. In: *Communications on Pure and Applied Mathematics* 55.11 (2002), pp. 1461–1506. DOI: 10.1002/cpa.10048.
- [FM99] I. Fonseca and S. Müller. “ $\mathcal{A}$ -quasiconvexity, lower semicontinuity, and Young measures”. In: *SIAM Journal on Mathematical Analysis* 30.6 (1999), pp. 1355–1390. DOI: 10.1137/S0036141098339885.
- [FS08] D. Faraco and L. Székelyhidi Jr. “Tartar’s conjecture and localization of the quasiconvex hull in $\mathbb{R}^{2 \times 2}$ ”. In: *Acta Mathematica* 200.2 (2008), pp. 279–305. DOI: 10.1007/s11511-008-0028-1.
- [FS18] C. Förster and L. Székelyhidi Jr. “ T_5 -configurations and non-rigid sets of matrices”. In: *Calculus of Variations and Partial Differential Equations* 57.19 (2018). DOI: 10.1007/s00526-017-1293-7.
- [GLN23] F. Gmeineder, P. Lewintan, and P. Neff. “Optimal incompatible Korn-Maxwell-Sobolev inequalities in all dimensions”. In: *Calculus of Variations and Partial Differential Equations* 62.182 (2023). DOI: 10.1007/s00526-023-02522-6.
- [GLN24] F. Gmeineder, P. Lewintan, and P. Neff. “Korn-Maxwell-Sobolev inequalities for general incompatibilities”. In: *Mathematical Models and Methods in Applied Sciences* 34.3 (2024), pp. 523–570. DOI: 10.1142/S0218202524500088.
- [GN04] A. Garroni and V. Nesi. “Rigidity and lack of rigidity for solenoidal matrix fields”. In: *Proceedings of The Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences* 460.2046 (2004), pp. 1789–1806. DOI: 10.1098/rspa.2003.1249.
- [GR22] A. Guerra and B. Raită. “Quasiconvexity, null Lagrangians, and Hardy space integrability under constant rank constraints”. In: *Archive for Rational Mechanics and Analysis* 245.1 (2022), pp. 279–320. DOI: 10.1007/s00205-022-01775-3.

- [Gra14] L. Grafakos. *Classical Fourier analysis*. Third. Vol. 249. Graduate Texts in Mathematics. New York, NY: Springer, 2014. DOI: 10.1007/978-1-4939-1194-3.
- [GRS22] A. Guerra, B. Raiță, and M. R. I. Schrecker. “Compensated compactness: continuity in optimal weak topologies”. In: *Journal of Functional Analysis* 283.7 (2022), p. 109596. DOI: 10.1016/j.jfa.2022.109596.
- [GRV24] F. Gmeineder, B. Raiță, and J. Van Schaftingen. “Boundary ellipticity and limiting L^1 -estimates on halfspaces”. In: *Advances in Mathematics* 439.109490 (2024). DOI: 10.1016/j.aim.2024.109490.
- [Kir03] B. Kirchheim. *Rigidity and Geometry of Microstructures*. MPI-MIS Lecture notes. Issue 16. 2003. URL: https://www.mis.mpg.de/publications/preprint-repository/lecture_note/2003/issue-16.
- [KK11] H. Knüpfer and R. V. Kohn. “Minimal energy for elastic inclusions”. In: *Proceedings of The Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences* 467.2127 (2011), pp. 695–717. DOI: 10.1098/rspa.2010.0316.
- [KK16] H. Knüpfer and M. Kružík. “A sharp interface evolutionary model for shape memory alloys”. In: *ZAMM - Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte Mathematik und Mechanik* 96.11 (2016), pp. 1347–1355. DOI: 10.1002/zamm.201500209.
- [KKO13] H. Knüpfer, R. V. Kohn, and F. Otto. “Nucleation barriers for the cubic-to-tetragonal phase transformation”. In: *Communications on Pure and Applied Mathematics* 66.6 (2013), pp. 867–904. DOI: 10.1002/cpa.21448.
- [KM11] H. Knüpfer and C. B. Muratov. “Domain structure of bulk ferromagnetic crystals in applied fields near saturation”. In: *Journal of Nonlinear Science* 21.6 (2011), pp. 921–962. DOI: 10.1007/s00332-011-9105-2.
- [KM92a] R. V. Kohn and S. Müller. “Branching of twins near an austenite—twinned-martensite interface”. In: *Philosophical Magazine A* 66.5 (1992), pp. 697–715. DOI: 10.1080/01418619208201585.
- [KM94] R. V. Kohn and S. Müller. “Surface energy and microstructure in coherent phase transitions”. In: *Communications on Pure and Applied Mathematics* 47.4 (1994), pp. 405–435. DOI: 10.1002/cpa.3160470402.
- [KMN19] H. Knüpfer, C. B. Muratov, and F. Nolte. “Magnetic domains in thin ferromagnetic films with strong perpendicular anisotropy”. In: *Archive for Rational Mechanics and Analysis* 232.2 (2019), pp. 727–761. DOI: 10.1007/s00205-018-1332-3.
- [KMR05] M. Kružík, A. Mielke, and T. Roubíček. “Modelling of microstructure and its evolution in shape-memory-alloy single-crystals, in particular in CuAlNi”. In: *Meccanica* 40 (2005), pp. 389–418. DOI: 10.1007/s11012-005-2106-1.

- [KMŠ03] B. Kirchheim, S. Müller, and V. Šverák. “Studying nonlinear pde by geometry in matrix space”. In: *Geometric analysis and nonlinear partial differential equations*. Ed. by S. Hildebrandt and H. Karcher. Berlin, Heidelberg: Springer, 2003, pp. 347–395. DOI: 10.1007/978-3-642-55627-2_19.
- [KO19] H. Knüpfer and F. Otto. “Nucleation barriers for the cubic-to-tetragonal phase transformation in the absence of self-accommodation”. In: *ZAMM - Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte Mathematik und Mechanik* 99.2 (2019), e201800179. DOI: 10.1002/zamm.201800179.
- [Koh91] R. V. Kohn. “The relaxation of a double-well energy”. In: *Continuum Mechanics and Thermodynamics* 3.3 (1991), pp. 193–236. DOI: 10.1007/BF01135336.
- [KS21] H. Knüpfer and W. Shi. “ Γ -limit for two-dimensional charged magnetic zigzag domain walls”. In: *Archive for Rational Mechanics and Analysis* 239.3 (2021), pp. 1875–1923. DOI: 10.1007/s00205-021-01606-x.
- [KW14] R. V. Kohn and B. Wirth. “Optimal fine-scale structures in compliance minimization for a uniaxial load”. In: *Proceedings of The Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences* 470.2170 (2014), p. 20140432. DOI: 10.1098/rspa.2014.0432.
- [KW16] R. V. Kohn and B. Wirth. “Optimal fine-scale structures in compliance minimization for a shear load”. In: *Communications on Pure and Applied Mathematics* 69.8 (2016), pp. 1572–1610. DOI: 10.1002/cpa.21589.
- [Lew23] M. Lewicka. *Calculus of variations on thin prestressed films—asymptotic methods in elasticity. Asymptotic Methods in Elasticity*. Vol. 101. Progress in Nonlinear Differential Equations and their Applications. Cham: Birkhäuser, 2023. DOI: 10.1007/978-3-031-17495-7.
- [LLP20] X. Lamy, A. Lorent, and G. Peng. “Rigidity of a non-elliptic differential inclusion related to the Aviles-Giga conjecture”. In: *Archive for Rational Mechanics and Analysis* 238.1 (2020), pp. 383–413. DOI: 10.1007/s00205-020-01545-z.
- [LLP22] X. Lamy, A. Lorent, and G. Peng. “On a generalized Aviles-Giga functional: compactness, zero-energy states, regularity estimates and energy bounds”. In: *Communications in Partial Differential Equations* 47.11 (2022), pp. 2270–2308. DOI: 10.1080/03605302.2022.2118609.
- [LLP24] X. Lamy, A. Lorent, and G. Peng. “Quantitative rigidity of differential inclusions in two dimensions”. In: *International Mathematics Research Notices* 2024.8 (2024), pp. 6325–6349. DOI: 10.1093/imrn/rnad108.
- [Lor01] A. Lorent. “An optimal scaling law for finite element approximations of a variational problem with non-trivial microstructure”. In: *ESAIM: Mathematical Modelling and Numerical Analysis* 35.5 (2001), pp. 921–934. DOI: 10.1051/m2an:2001143.

- [Lor09] A. Lorent. “The regularisation of the N -well problem by finite elements and by singular perturbation are scaling equivalent in two dimensions”. In: *ESAIM. Control, Optimisation and Calculus of Variations* 15.2 (2009), pp. 322–366. DOI: 10.1051/cocv:2008039.
- [Mai04] A. Mainik. “A rate-independent model for phase transformations in shape-memory alloys”. PhD thesis. Universität Stuttgart, 2004. DOI: 10.18419/opus-4749.
- [MLSG14] J. Mohd Jani, M. Leary, A. Subic, and M. A. Gibson. “A review of shape memory alloy research, applications and opportunities”. In: *Materials & Design* 56 (2014), pp. 1078–1113. DOI: 10.1016/j.matdes.2013.11.084.
- [MM77] L. Modica and S. Mortola. “Un esempio di Γ^- -convergenza”. In: *Unione Matematica Italiana. Bollettino. B. Serie V* 14.1 (1977), pp. 285–299.
- [MŠ03] S. Müller and V. Šverák. “Convex integration for Lipschitz mappings and counterexamples to regularity”. In: *Annals of Mathematics* 157.3 (2003), pp. 715–742. DOI: 10.4007/annals.2003.157.715.
- [MŠ99] S. Müller and V. Šverák. “Convex integration with constraints and applications to phase transitions and partial differential equations”. In: *Journal of the European Mathematical Society* 1.4 (1999), pp. 393–422. DOI: 10.1007/s100970050012.
- [MT97] F. Murat and L. Tartar. “ H -convergence”. In: *Topics in the mathematical modelling of composite materials*. Ed. by A. Cherkaev and R. V. Kohn. Vol. 31. Progress in Nonlinear Differential Equations and Their Applications. Boston, MA: Birkhäuser Boston, 1997, pp. 21–43. DOI: 10.1007/978-1-4612-2032-9_3.
- [Mül99b] S. Müller. “Variational models for microstructure and phase transitions”. In: *Calculus of variations and geometric evolution problems*. Vol. 1713. Lecture Notes in Mathematics. Berlin, Heidelberg: Springer, 1999, pp. 85–210. DOI: 10.1007/BFb0092670.
- [Ott02] F. Otto. “Cross-over in scaling laws: a simple example from micromagnetics”. In: *Proceedings of the International Congress of Mathematicians*. Ed. by L. Tatsien. Vol. III: Invited Lectures. Beijing: Higher Education Press, 2002, pp. 829–838.
- [PP04] M. Palombaro and M. Ponsiglione. “The three divergence free matrix fields problem”. In: *Asymptotic Analysis* 40.1 (2004), pp. 37–49. DOI: 10.3233/asy-2004-643.
- [PS09] M. Palombaro and V. P. Smyshlyaev. “Relaxation of three solenoidal wells and characterization of extremal three-phase H -measures”. In: *Archive for Rational Mechanics and Analysis* 194.3 (2009), pp. 775–722. DOI: 10.1007/s00205-008-0204-7.

- [Rai19] B. Raiță. “Potentials for $\mathcal{A}$ -quasiconvexity”. In: *Calculus of Variations and Partial Differential Equations* 58.105 (2019). DOI: 10.1007/s00526-019-1544-x.
- [Rai24] B. Raiță. “A simple construction of potential operators for compensated compactness”. In: *The Quarterly Journal of Mathematics* 75.2 (2024), pp. 451–456. DOI: 10.1093/qmath/haae008.
- [Rin18] F. Rindler. *Calculus of variations*. Universitext. Cham: Springer, 2018. DOI: 10.1007/978-3-319-77637-8.
- [RRT23] B. Raiță, A. Rüland, and C. Tissot. “On scaling properties for two-state problems and for a singularly perturbed T_3 structure”. In: *Acta Applicandae Mathematicae* 184.5 (2023). DOI: 10.1007/s10440-023-00557-7.
- [RRTT24] B. Raiță, A. Rüland, C. Tissot, and A. Tribuzio. “On scaling properties for a class of two-well problems for higher order homogeneous linear differential operators”. In: *SIAM Journal on Mathematical Analysis* 56.3 (2024), pp. 3720–3758. DOI: 10.1137/23M1588287.
- [RS23] A. Rüland and T. M. Simon. “On rigidity for the four-well problem arising in the cubic-to-trigonal phase transformation”. In: *Journal of Elasticity* 153.3 (2023), pp. 455–475. DOI: 10.1007/s10659-023-10011-2.
- [RT22] A. Rüland and A. Tribuzio. “On the energy scaling behaviour of a singularly perturbed Tartar square”. In: *Archive for Rational Mechanics and Analysis* 243.1 (2022), pp. 401–431. DOI: 10.1007/s00205-021-01729-1.
- [RT23a] A. Rüland and A. Tribuzio. “On scaling laws for multi-well nucleation problems without gauge invariances”. In: *Journal of Nonlinear Science* 33.25 (2023). DOI: 10.1007/s00332-022-09879-6.
- [RT23b] A. Rüland and A. Tribuzio. “On the energy scaling behaviour of singular perturbation models with prescribed Dirichlet data involving higher order laminates”. In: *ESAIM: Control, Optimisation and Calculus of Variations* 29.68 (2023). DOI: 10.1051/cocv/2023047.
- [RT24] A. Rüland and A. Tribuzio. “On the scaling of the cubic-to-tetragonal phase transformation with displacement boundary conditions”. In: *Journal of Elasticity* 156.3 (2024), pp. 837–875. DOI: 10.1007/s10659-024-10075-8.
- [RTTZ25] A. Rüland, C. Tissot, A. Tribuzio, and C. Zillinger. *On surface energies in scaling laws for singular perturbation problems for martensitic phase transitions*. 2025. arXiv: 2507.06773 [math.AP].
- [RTZ19] A. Rüland, J. M. Taylor, and C. Zillinger. “Convex integration arising in the modelling of shape-memory alloys: some remarks on rigidity, flexibility and some numerical implementations”. In: *Journal of Nonlinear Science* 29.5 (2019), pp. 2137–2184. DOI: 10.1007/s00332-019-09540-9.

- [Rül16b] A. Růland. “The cubic-to-orthorhombic phase transition: rigidity and non-rigidity properties in the linear theory of elasticity”. In: *Archive for Rational Mechanics and Analysis* 221.1 (2016), pp. 23–106. DOI: 10.1007/s00205-016-0971-5.
- [Rül25] A. Růland. “Microstructures in the Modelling of Shape-Memory Alloys: Rigidity, Flexibility and Scaling”. In: *Variational and PDE Methods in Nonlinear Science: Cetraro, Italy 2023*. Ed. by F. Bethuel, G. Orlandi, and B. Stroffolini. Vol. 2366. Lecture Notes in Mathematics. Cham: Springer, 2025, pp. 83–144. DOI: 10.1007/978-3-031-87202-0_2.
- [RZZ18] A. Růland, C. Zillinger, and B. Zwicknagl. “Higher Sobolev regularity of convex integration solutions in elasticity: the Dirichlet problem with affine data in $\text{int}(K^{lc})$ ”. In: *SIAM Journal on Mathematical Analysis* 50.4 (2018), pp. 3791–3841. DOI: 10.1137/17M1149687.
- [Sch08] B. Schmidt. “Linear Γ -limits of multiwell energies in nonlinear elasticity theory”. In: *Continuum Mechanics and Thermodynamics* 20.6 (2008), pp. 375–396. DOI: 10.1007/s00161-008-0087-8.
- [Sei+20] H. Seiner, P. Plucinsky, V. Dabade, B. Benešová, and R. D. James. “Branching of twins in shape memory alloys revisited”. In: *Journal of the Mechanics and Physics of Solids* 141.103961 (2020). DOI: 10.1016/j.jmps.2020.103961.
- [Sha94] V. A. Sharafutdinov. *Integral geometry of tensor fields*. Inverse and Ill-posed Problems Series. Berlin, New York: De Gruyter, 1994. DOI: 10.1515/9783110900095.
- [ST23] M. Sorella and R. Tione. “The four-state problem and convex integration for linear differential operators”. In: *Journal of Functional Analysis* 284.4 (2023), Paper No. 109785. DOI: 10.1016/j.jfa.2022.109785.
- [Šve91] V. Šverák. *On Regularity for the Monge-Ampère equation without convexity assumptions*. Tech. rep. Heriot-Watt University, 1991.
- [SW21] J. Skipper and E. Wiedemann. “Lower semi-continuity for $\mathcal{A}$ -quasiconvex functionals under convex restrictions”. In: *ESAIM. Control, Optimisation and Calculus of Variations* 27.107 (2021). DOI: 10.1051/cocv/2021105.
- [Tar79] L. Tartar. “Compensated compactness and applications to partial differential equations”. In: *Nonlinear analysis and mechanics: Heriot-Watt Symposium*. Vol. 4. Research Notes in Mathematics 39. London: Pitman, 1979, pp. 136–212.
- [Tar83] L. Tartar. “The compensated compactness method applied to systems of conservation laws”. In: *Systems of nonlinear partial differential equations*. Ed. by J. M. Ball. Vol. 111. NATO Science Series C: Mathematical and Physical Sciences. Dordrecht, Netherlands: Springer, 1983, pp. 263–285. DOI: 10.1007/978-94-009-7189-9_13.

- [Tar93] L. Tartar. “Some remarks on separately convex functions”. In: *Microstructure and phase transition*. Ed. by D. Kinderlehrer, R. James, L. Mitchell, and J. L. Ericksen. Vol. 54. The IMA Volumes in Mathematics and its Applications. New York, NY: Springer, 1993, pp. 191–204. DOI: 10.1007/978-1-4613-8360-4_12.
- [TZ25] A. Tribuzio and K. Zemas. “Energy barriers for boundary nucleation in a two-well model without gauge invariances”. In: *Calculus of Variations and Partial Differential Equations* 64.11 (2025). DOI: 10.1007/s00526-024-02850-1.
- [Win97] M. Winter. “An example of microstructure with multiple scales”. In: *European Journal of Applied Mathematics* 8.2 (1997), pp. 185–207. DOI: 10.1017/S0956792597003021.
- [Zwi14] B. Zwicknagl. “Microstructures in low-hysteresis shape memory alloys: scaling regimes and optimal needle shapes”. In: *Archive for Rational Mechanics and Analysis* 213 (2014), pp. 355–421. DOI: 10.1007/s00205-014-0736-y.

Further literature

The references listed below are not cited in Chapters 1 to 5 above. They appear in the articles [RRT23; RRTT24; RTTZ25] reproduced in Appendices A to C.

- [ALP17] R. Alicandro, G. Lazzaroni, and M. Palombaro. “On the effect of interactions beyond nearest neighbours on non-convex lattice systems”. In: *Calculus of Variations and Partial Differential Equations* 56.42 (2017). DOI: 10.1007/s00526-017-1129-5.
- [AM01] G. Alberti and S. Müller. “A new approach to variational problems with multiple scales”. In: *Communications on Pure and Applied Mathematics* 54.7 (2001), pp. 761–825. DOI: 10.1002/cpa.1013.
- [BC07] A. Braides and M. Cicalese. “Surface energies in nonconvex discrete systems”. In: *Mathematical Models and Methods in Applied Sciences* 17.07 (2007), pp. 985–1037. DOI: 10.1142/S0218202507002182.
- [BGS23] L. Behn, F. Gmeineder, and S. Schiffer. “On symmetric div-quasiconvex hulls and divsym-free L^∞ -truncations”. In: *Annales de l’Institut Henri Poincaré C. Analyse Non Linéaire* 40.6 (2023), pp. 1267–1317. DOI: 10.4171/aihpc/66.
- [BM09] J. M. Ball and C. Mora-Corral. “A variational model allowing both smooth and sharp phase boundaries in solids”. In: *Communications on Pure and Applied Analysis* 8.1 (2009), pp. 55–81. DOI: 10.3934/cpaa.2009.8.55.
- [BM18] H. Brezis and P. Mironescu. “Gagliardo-Nirenberg inequalities and non-inequalities: The full story”. In: *Annales de l’Institut Henri Poincaré. Analyse Non Linéaire* 35.5 (2018), pp. 1355–1376. DOI: 10.1016/j.anihpc.2017.11.007.
- [BO13] Á. Bényi and T. Oh. “The Sobolev inequality on the torus revisited”. In: *Publicationes Mathematicae Debrecen* 83.3 (2013), pp. 359–374. DOI: 10.5486/PMD.2013.5529.
- [BP04] S. Bartels and A. Prohl. “Multiscale resolution in the computation of crystalline microstructure”. In: *Numerische Mathematik* 96 (2004), pp. 641–660. DOI: 10.1007/s00211-003-0483-8.
- [BVA86] I. Baele, G. Van Tendeloo, and S. Amelinckx. “A striking high-resolution imaging phenomenon in NiMn”. In: *Ultramicroscopy* 19.2 (1986), pp. 201–204. DOI: 10.1016/0304-3991(86)90207-X.

- [CC14] A. Chan and S. Conti. “Energy scaling and domain branching in solid-solid phase transitions”. In: *Singular phenomena and scaling in mathematical models*. Ed. by M. Griebel. Cham: Springer, 2014, pp. 243–260. DOI: 10.1007/978-3-319-00786-1_11.
- [CCK95] M. Chipot, C. Collins, and D. Kinderlehrer. “Numerical analysis of oscillations in multiple well problems”. In: *Numerische Mathematik* 70.3 (1995), pp. 259–282. DOI: 10.1007/s002110050119.
- [CDS12] S. Conti, C. De Lellis, and L. Székelyhidi Jr. “ h -Principle and Rigidity for $C^{1,\alpha}$ Isometric Embeddings”. In: *Nonlinear Partial Differential Equations*. Ed. by H. Holden and K. H. Karlsen. Vol. 7. Abel Symposia. Berlin, Heidelberg: Springer, 2012, pp. 83–116. DOI: 10.1007/978-3-642-25361-4_5.
- [Ces+20] P. Cesana, F. Della Porta, A. Růland, C. Zillinger, and B. Zwicknagl. “Exact constructions in the (non-linear) planar theory of elasticity: from elastic crystals to nematic elastomers”. In: *Archive for Rational Mechanics and Analysis* 237.1 (2020), pp. 383–445. DOI: 10.1007/s00205-020-01511-9.
- [CKM22] S. Conti, R. V. Kohn, and O. Misiats. “Energy minimizing twinning with variable volume fraction, for two nonlinear elastic phases with a single rank-one connection”. In: *Mathematical Models and Methods in Applied Sciences* 32.8 (2022), pp. 1671–1723. DOI: 10.1142/S0218202522500397.
- [CKZ17] S. Conti, M. Klar, and B. Zwicknagl. “Piecewise affine stress-free martensitic inclusions in planar nonlinear elasticity”. In: *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* 473.2203 (2017), p. 20170235. DOI: 10.1098/rspa.2017.0235.
- [CS13] I. V. Chenchiah and A. Schlömerkemper. “Non-laminate microstructures in monoclinic-I martensite”. In: *Archive for Rational Mechanics and Analysis* 207.1 (2013), pp. 39–74. DOI: 10.1007/s00205-012-0552-1.
- [CT05] S. Conti and F. Theil. “Single-slip elastoplastic microstructures”. In: *Archive for Rational Mechanics and Analysis* 178.1 (2005), pp. 125–148. DOI: 10.1007/s00205-005-0371-8.
- [CZ16] S. Conti and B. Zwicknagl. “Low volume-fraction microstructures in martensites and crystal plasticity”. In: *Mathematical Models and Methods in Applied Sciences* 26.7 (2016), pp. 1319–1355. DOI: 10.1142/S0218202516500317.
- [Dac07] B. Dacorogna. *Direct methods in the calculus of variations*. Second. Vol. 78. Applied Mathematical Sciences. New York, NY: Springer, 2007. DOI: 10.1007/978-0-387-55249-1.
- [Dac82] B. Dacorogna. *Weak continuity and weak lower semicontinuity of non-linear functionals*. Vol. 922. Lecture Notes in Mathematics. Berlin, Heidelberg: Springer, 1982. DOI: 10.1007/BFb0096144.
- [Die13] J. Diermeier. *Domain Branching in Geometrically Linear Elasticity*. Master’s thesis. 2013.

- [DM12] B. Dacorogna and P. Marcellini. *Implicit partial differential equations*. Vol. 37. Progress in Nonlinear Differential Equations and their Applications. Boston, MA: Birkhäuser, 2012. DOI: 10.1007/978-1-4612-1562-2.
- [Dol03] G. Dolzmann. *Variational methods for crystalline microstructure—analysis and computation*. Vol. 1803. Lecture Notes in Mathematics. Berlin, Heidelberg: Springer, 2003. DOI: 10.1007/b10191.
- [DS09] C. De Lellis and L. Székelyhidi Jr. “The Euler equations as a differential inclusion”. In: *Annals of Mathematics. Second Series* 170.3 (2009), pp. 1417–1436. DOI: 10.4007/annals.2009.170.1417.
- [DS12] C. De Lellis and L. Székelyhidi Jr. “The h -principle and the equations of fluid dynamics”. In: *Bulletin of the American Mathematical Society* 49.3 (2012), pp. 347–375. DOI: 10.1090/S0273-0979-2012-01376-9.
- [DS15] C. De Lellis and L. Székelyhidi Jr. “On h -principle and Onsager’s conjecture”. In: *European Mathematical Society. Newsletter* 95 (2015), pp. 19–24.
- [GRS24] A. Guerra, B. Raită, and M. R. I. Schrecker. “Compensation phenomena for concentration effects via nonlinear elliptic estimates”. In: *Ars Inveniendi Analytica* (2024), Paper No. 1. DOI: 10.15781/7187-xq59.
- [GRTZ24] J. Ginster, A. Rüland, A. Tribuzio, and B. Zwicknagl. *On the Effect of Geometry on Scaling Laws for a Class of Martensitic Phase Transformations*. 2024. arXiv: 2405.05927 [math.AP].
- [GZ23] J. Ginster and B. Zwicknagl. “Energy scaling law for a singularly perturbed four-gradient problem in helimagnetism”. In: *Journal of Nonlinear Science* 33.20 (2023). DOI: 10.1007/s00332-022-09847-0.
- [IM19] J. Ilmavirta and F. Monard. “Integral geometry on manifolds with boundary and applications”. In: *The Radon transform—the first 100 years and beyond*. Vol. 22. Radon Series on Computational Applied Mathematics. Berlin: Walter de Gruyter, 2019, pp. 43–113. DOI: 10.1515/9783110560855-004.
- [Ind+24] R. Indergand, D. Kochmann, A. Rüland, A. Tribuzio, and C. Zillinger. *On a T_3 -Structure in Geometrically Linearized Elasticity: Qualitative and Quantitative Analysis and Numerical Simulations*. 2024. arXiv: 2408.13110 [math.AP].
- [KM92b] R. V. Kohn and S. Müller. “Relaxation and regularization of nonconvex variational problems”. In: *Rendiconti del Seminario Matematico e Fisico di Milano* 62.1 (1992), pp. 89–113. DOI: 10.1007/BF02925437.
- [Koh07] R. V. Kohn. “Energy-driven pattern formation”. In: *International Congress of Mathematicians*. Vol. I. European Mathematical Society, 2007, pp. 359–383. DOI: 10.4171/022-1/15.
- [KR20] J. Kristensen and B. Raită. *An introduction to generalized Young measures*. MPI-MIS Lecture notes. Issue 45. 2020. URL: https://www.mis.mpg.de/publications/preprint-repository/lecture_note/2020/issue-45.

- [KR22] J. Kristensen and B. Raită. “Oscillation and concentration in sequences of PDE constrained measures”. In: *Archive for Rational Mechanics and Analysis* 246.2 (2022), pp. 823–875. DOI: 10.1007/s00205-022-01828-7.
- [Kui55] N. H. Kuiper. “On C^1 -isometric imbeddings. II”. In: *Indagationes Mathematicae (Proceedings)*. Vol. 58. Elsevier, 1955, pp. 683–689. DOI: 10.1016/S1385-7258(55)50093-X.
- [MSZ03] J. Malý, D. Swanson, and W. P. Ziemer. “The co-area formula for Sobolev mappings”. In: *Transactions of the American Mathematical Society* 355.2 (2003), pp. 477–492. DOI: 10.1090/S0002-9947-02-03091-X.
- [Mül93] S. Müller. “Singular perturbations as a selection criterion for periodic minimizing sequences”. In: *Calculus of Variations and Partial Differential Equations* 1.2 (1993), pp. 169–204. DOI: 10.1007/BF01191616.
- [Mül99a] S. Müller. “Rank-one convexity implies quasiconvexity on diagonal matrices”. In: *International Mathematics Research Notices* 1999.20 (1999), pp. 1087–1095. DOI: 10.1155/S1073792899000598.
- [Mur81] F. Murat. “Compacité par compensation: condition nécessaire et suffisante de continuité faible sous une hypothèse de rang constant”. In: *Annali della Scuola Normale Superiore di Pisa — Classe di Scienze. Serie 4* 8.1 (1981), pp. 69–102. URL: http://www.numdam.org/item?id=ASNSP_1981_4_8_1_69_0.
- [MVA86] C. Manolikas, G. Van Tendeloo, and S. Amelinckx. “The “local” structure of domain boundaries in ferroelastic lead orthovanadate”. In: *Solid State Communications* 58.12 (1986), pp. 851–855. DOI: 10.1016/0038-1098(86)90245-0.
- [Nas54] J. Nash. “ C^1 isometric imbeddings”. In: *Annals of Mathematics* 60.3 (1954), pp. 383–396. DOI: 10.2307/1969840.
- [OV10] F. Otto and T. Viehmann. “Domain branching in uniaxial ferromagnets: asymptotic behavior of the energy”. In: *Calculus of Variations and Partial Differential Equations* 38.1 (2010), pp. 135–181. DOI: 10.1007/s00526-009-0281-y.
- [Pal10] M. Palombaro. “Rank- $(n - 1)$ convexity and quasiconvexity for divergence free fields”. In: *Advances in Calculus of Variations* 3.3 (2010), pp. 279–285. DOI: 10.1515/ACV.2010.010.
- [Ped97] P. Pedregal. *Parametrized measures and variational principles*. Vol. 30. Progress in Nonlinear Differential Equations and their Applications. Basel: Birkhäuser, 1997. DOI: 10.1007/978-3-0348-8886-8.
- [PSU14] G. P. Paternain, M. Salo, and G. Uhlmann. “Tensor tomography: progress and challenges”. In: *Chinese Annals of Mathematics. Series B* 35.3 (2014), pp. 399–428. DOI: 10.1007/s11401-014-0834-z.

- [PSU23] G. P. Paternain, M. Salo, and G. Uhlmann. *Geometric inverse problems: With emphasis on two dimensions*. Vol. 204. Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press, 2023. DOI: 10.1017/9781009039901.
- [PW22] J. Potthoff and B. Wirth. “Optimal fine-scale structures in compliance minimization for a uniaxial load in three space dimensions”. In: *ESAIM. Control, Optimisation and Calculus of Variations* 28.27 (2022). DOI: 10.1051/cocv/2022023.
- [Rül16a] A. Rüland. “A rigidity result for a reduced model of a cubic-to-orthorhombic phase transition in the geometrically linear theory of elasticity”. In: *Journal of Elasticity* 123.2 (2016), pp. 137–177. DOI: 10.1007/s10659-015-9553-2.
- [RZZ20] A. Rüland, C. Zillinger, and B. Zwicknagl. “Higher Sobolev regularity of convex integration solutions in elasticity: the planar geometrically linearized hexagonal-to-rhombic phase transformation”. In: *Journal of Elasticity* 138 (2020), pp. 1–76. DOI: 10.1007/s10659-018-09719-3.
- [Sim21a] T. M. Simon. “Quantitative aspects of the rigidity of branching microstructures in shape memory alloys via H-measures”. In: *SIAM Journal on Mathematical Analysis* 53.4 (2021), pp. 4537–4567. DOI: 10.1137/18M1220017.
- [Sim21b] T. M. Simon. “Rigidity of branching microstructures in shape memory alloys”. In: *Archive for Rational Mechanics and Analysis* 241.3 (2021), pp. 1707–1783. DOI: 10.1007/s00205-021-01679-8.
- [Szé04] L. Székelyhidi Jr. “The regularity of critical points of polyconvex functionals”. In: *Archive for Rational Mechanics and Analysis* 172.1 (2004), pp. 133–152. DOI: 10.1007/s00205-003-0300-7.
- [Tar05] L. Tartar. “Compensation effects in partial differential equations”. In: *Rendiconti Accademia Nazionale delle Scienze detta dei XL. Serie 5. Memorie di Matematica e Applicazioni. Parte I* 29 (2005), pp. 395–453.
- [Van13] J. Van Schaftingen. “Limiting Sobolev inequalities for vector fields and canceling linear differential operators”. In: *Journal of the European Mathematical Society (JEMS)* 15.3 (2013), pp. 877–921. DOI: 10.4171/JEMS/380.
- [WT03] M. Warner and E. M. Terentjev. *Liquid crystal elastomers*. Vol. 120. Oxford University Press, 2003. DOI: 10.1093/oso/9780198527671.001.0001.

On scaling properties for two-state problems and for a singularly perturbed T_3 structure

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A summary of the article is given in Chapter 2.



On Scaling Properties for Two-State Problems and for a Singularly Perturbed T_3 Structure

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Abstract

In this article we study quantitative rigidity properties for the compatible and incompatible two-state problems for suitable classes of $\mathcal{A}$ -free differential inclusions and for a singularly perturbed T_3 structure for the divergence operator. In particular, in the compatible setting of the two-state problem we prove that all homogeneous, first order, linear operators with affine boundary data which enforce oscillations yield the typical $\epsilon^{\frac{2}{3}}$ -lower scaling bounds. As observed in Chan and Conti (Math. Models Methods Appl. Sci. 25(06):1091–1124, 2015) for higher order operators this may no longer be the case. Revisiting the example from Chan and Conti (Math. Models Methods Appl. Sci. 25(06):1091–1124, 2015), we show that this is reflected in the structure of the associated symbols and that this can be exploited for a new Fourier based proof of the lower scaling bound. Moreover, building on Rüland and Tribuzio (Arch. Ration. Mech. Anal. 243(1):401–431, 2022); Garroni and Nesi (Proc. R. Soc. Lond., Ser. A, Math. Phys. Eng. Sci. 460(2046):1789–1806, 2004, <https://doi.org/10.1098/rspa.2003.1249>); Palombaro and Ponsiglione (Asymptot. Anal. 40(1):37–49, 2004), we discuss the scaling behavior of a T_3 structure for the divergence operator. We prove that as in Rüland and Tribuzio (Arch. Ration. Mech. Anal. 243(1):401–431, 2022) this yields a non-algebraic scaling law.

Keywords $\mathcal{A}$ -Free inclusions · Two-well problem · Divergence T_3 · Phase transformation

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1 Introduction

Rigidity and flexibility properties associated with (nonlinear) differential inclusions for the gradient have been objects of intensive study. They arise in a variety of applications, including the analysis of PDEs, e.g. the study of regularity of elliptic systems [5–7], fluid dynamics [8, 9], geometry [10–13] and various settings in the materials sciences, e.g. the study of patterns in shape-memory alloys [14–19]. Motivated by applications of microstructures in composites [20, 21], optimal design problems [22–24] and micromagnetics [25, 26] as well as by recent developments on more general differential inclusion problems [27–29], in this article we study two instances of quantitative rigidity and flexibility properties of differential inclusions for more general operators. On the one hand, we consider *constant-coefficient, homogeneous, linear differential operators* for which we discuss quantitative versions of the compatible and incompatible two-state problems. On the other hand, we investigate quantitative properties of a T_3 structure for the *divergence operator*.

1.1 On Quantitative Results for the Two-Well Problem for $\mathcal{A}$ -Free Differential Inclusions

$\mathcal{A}$ -free differential inclusions arise in many different settings, including linearized elasticity [30, 31], liquid crystal elastomers [32, 33] and the study of the Aviles-Giga functional [34, 35] to name just a few examples. They have been systematically investigated in the context of compensated compactness theory in classical works such as [36–39] but also in more recent literature on compensated compactness theory [40–46], in truncation results [47], in classical minimization and regularity questions in the calculus of variations [48–51] and in the context of fine properties of such operators in borderline spaces [27, 52–54]. In the recent articles [27, 28] general $\mathcal{A}$ -free versions of the incompatible two-well problem in borderline spaces and the study of T_N structures have been initiated. Motivated by these applications, in the first part of this article, we seek to study *quantitative* versions of the compatible and incompatible two-state problems.

Before turning to the setting of general $\mathcal{A}$ -free differential inclusions, let us recall the analogous “classical” setting for the gradient: Inspired by problems from materials science and phase transformations, the exact and approximate rigidity properties of differential inclusion problems for the gradient [14] (see also [18, 55–57]) with and without gauge invariance have been considered. For two energy wells without gauge invariances this amounts to the study of the differential inclusion

$$\nabla v \in \{A, B\} \text{ in } \Omega \quad (1)$$

for $A, B \in \mathbb{R}^{d \times d}$, $A \neq B$ and $\Omega \subset \mathbb{R}^d$ a bounded Lipschitz domain. It is well-known that depending on the *compatibility* of the wells, a dichotomy arises:

- On the one hand, for *incompatible* wells, i.e. if $B - A \in \mathbb{R}^{d \times d}$ is not a rank-one matrix, the differential inclusion (1) is *rigid* both for exact and approximate solutions: Indeed, if A, B are *incompatible*, (1) only permits solutions with constant deformation gradient, which is in the following referred to as the rigidity of the two-state problem for *exact* solutions. Moreover, in this setting one has that for sequences ∇u_k with $\text{dist}(\nabla u_k, \{A, B\}) \rightarrow 0$ in measure, it necessarily holds that along a subsequence $\nabla u_k \rightarrow A$ or $\nabla u_k \rightarrow B$ in measure. We will refer to this as rigidity of the two-state problem for *approximate* solutions.

We remark that various far-reaching generalizations of these results have been obtained: For settings with $SO(d)$ symmetries a quantitative version of such a result was

deduced in [58]; a further proof was found in [59]. The one-state problem with continuous symmetry group was studied in [60, 61].

The article [27] investigates a similar problem for two incompatible wells for a *general constant coefficient, homogeneous, linear differential operator* $\mathcal{A}(D)$, providing qualitative rigidity results for the associated exact and approximate differential inclusions, including L^1 -based frameworks.

- On the other hand, if the wells are *compatible*, i.e. if $B - A \in \mathbb{R}^{d \times d}$ is a rank-one matrix, then *simple laminate solutions* of (1) exist, in which the deformation gradient oscillates between the two fixed values A, B and is a one-dimensional function depending only on the direction determined by the difference $B - A$. Due to the failure of rigidity on the exact level, also rigidity on the approximate level cannot be expected without additional regularization terms.

In the first part of this article we seek to consider *quantitative, L^2 -based* variants of these type of results for more general, linear differential operators $\mathcal{A}(D)$. In this context, we will consider the following two guiding questions:

- **Quantitative incompatible rigidity.** For a *constant coefficient, homogeneous, linear differential operator* $\mathcal{A}(D)$ and two *incompatible* wells, i.e. $A, B \in \mathbb{R}^n$ such that $B - A \notin \Lambda_{\mathcal{A}}$, cf. (5) for the definition of the wave cone, do we have a quantitative rigidity result in terms of domain scaling for prescribed boundary data which are a convex combination of the two states? Here the dimension n depends on the operator $\mathcal{A}(D)$; for $\mathcal{A}(D) = \text{curl}$ (which corresponds to the gradient setting from (1) above) we would for instance consider $n = d \times d$.

More precisely, we seek to study the following question: Let $A, B \in \mathbb{R}^n$ be such that $B - A \notin \Lambda_{\mathcal{A}}$. Is it true that

$$E_{el}(u, \chi) = \int_{\Omega} |u - \chi_A A - \chi_B B|^2 dx \geq C(A, B, \lambda) |\Omega|,$$

if $u : \mathbb{R}^d \rightarrow \mathbb{R}^n$, $\mathcal{A}(D)u = 0$ in $\mathbb{R}^d$, $\chi := A\chi_A + B\chi_B \in \{A, B\}$ in Ω and $\chi_A, \chi_B \in \{0, 1\}$ with $\chi_A + \chi_B = 1$ and if for some $\lambda \in (0, 1)$ we have that $u = F_{\lambda} := \lambda A + (1 - \lambda)B$ in $\mathbb{R}^d \setminus \overline{\Omega}$?

- **Quantitative compatible rigidity.** Let us next consider two *compatible* wells $A, B \in \mathbb{R}^n$, i.e. let $A, B \in \mathbb{R}^n$ be such that $B - A \in \Lambda_{\mathcal{A}}$, see (5) below, and let us again consider boundary data $F_{\lambda} := \lambda A + (1 - \lambda)B$ for some $\lambda \in (0, 1)$ as above and with the set of admissible functions given by $\mathcal{D}_{F_{\lambda}}$ in (10). For a singularly perturbed energy similarly as in (15) is it true that as in [62, 63] also in the setting of a *more general constant coefficient, homogeneous, linear differential operator* $\mathcal{A}(D)$ the following bound holds

$$\inf_{\chi \in BV(\Omega; \{A, B\})} \inf_{u \in \mathcal{D}_{F_{\lambda}}} (E_{el}(u, \chi) + \epsilon \int_{\Omega} |\nabla \chi|) \geq C\epsilon^{2/3}?$$

In what follows, we will formulate the set-up, the relevant operator classes and our results on these questions.

1.2 Formulation of the Two-State Problem for $\mathcal{A}$ -Free Operators in Bounded Domains

Following [27, 28, 48], we consider a particular class of *linear, homogeneous, constant-coefficient operators*. The operator $\mathcal{A}(D) : C^{\infty}(\mathbb{R}^d; \mathbb{R}^n) \rightarrow C^{\infty}(\mathbb{R}^d; \mathbb{R}^m)$ of order $k \in \mathbb{N}$ is

given in the form

$$\mathcal{A}(D) := \sum_{|\alpha|=k} A_\alpha \partial^\alpha, \quad (2)$$

where $\alpha \in \mathbb{N}^d$ denotes a multi-index of length $|\alpha| := \sum_{j=1}^d \alpha_j$ and $A_\alpha \in \mathbb{R}^{m \times n}$ are constant matrices. Seeking to study microstructure, in the sequel we are particularly interested in *non-elliptic* operators. Here the operator $\mathcal{A}(D)$ is said to be *elliptic* if its symbol

$$\mathbb{A}(\xi) := \sum_{|\alpha|=k} A_\alpha \xi^\alpha \quad (3)$$

is injective for all $\xi \neq 0$. If $\mathcal{A}(D)$ is not elliptic, there exist vectors $\xi \in \mathbb{R}^d \setminus \{0\}$ and $\mu \in \mathbb{R}^n \setminus \{0\}$ such that

$$\mathbb{A}(\xi)\mu = 0. \quad (4)$$

The collection of these vectors $\mu \in \mathbb{R}^n \setminus \{0\}$ forms the *wave cone* associated with the operator $\mathcal{A}(D)$:

$$\Lambda_{\mathcal{A}} := \bigcup_{\xi \in \mathbb{S}^{d-1}} \ker(\mathbb{A}(\xi)). \quad (5)$$

The relevance of the wave cone $\Lambda_{\mathcal{A}}$ for compensated compactness and the existence of microstructure is well-known. For instance, for any pair (μ, ξ) as in (4) it is possible to obtain $\mathcal{A}$ -free *simple laminate solutions*. These are one-dimensional functions $u(x) := \mu h(x \cdot \xi)$, where $h : \mathbb{R} \rightarrow \{0, 1\}$, which obey the differential constraint $\mathcal{A}(D)u = 0$ due to the choice of μ, ξ and $u \in \{0, \mu\}$. More generally, if $k = 1$, and for $\mu \in \Lambda_{\mathcal{A}}$ (see, for instance, [64]) it holds that

$$u(x) := \mu h(x \cdot \xi_1, \dots, x \cdot \xi_\ell)$$

is a solution to the differential equation $\mathcal{A}(D)u = 0$ for vectors $\xi_1, \dots, \xi_\ell \in \mathbb{R}^d \setminus \{0\}$ forming a basis of the vectorspace

$$V_{\mathcal{A}, \mu} := \{\xi \in \mathbb{R}^d : \mathbb{A}(\xi)\mu = 0\}. \quad (6)$$

For the row-wise curl operator ($n = d \times d$) this is an at most one-dimensional space, while for the row-wise divergence operator ($n = m \times d$), it is a space of possibly higher dimension as $V_{\text{div}, \mu} = \ker \mu$, leading to substantially more flexible solutions of the associated differential inclusions than for the curl.

In order to study microstructures arising as solutions to the two-state problem, for the above specified class of operators, analogously as in the gradient setting, we consider the following $\mathcal{A}$ -free differential inclusion with prescribed boundary values:

$$\begin{aligned} u &\in \mathcal{K} \text{ in } \Omega, \\ \mathcal{A}(D)u &= 0 \text{ in } \mathbb{R}^d, \end{aligned} \quad (7)$$

with $\mathcal{K} \subset \mathbb{R}^n$, and $\Omega \subset \mathbb{R}^d$ an open, bounded, simply connected domain, with appropriately prescribed boundary data. For the two-state problem we consider $\mathcal{K} = \{A, B\} \subset \mathbb{R}^n$.

Now, in analogy to the gradient setting, on the one hand, we call the differential inclusion for the two-state problem (7) *incompatible* if it is elliptic in the sense that $B - A \notin \Lambda_{\mathcal{A}}$. In this case it is proved in [27] that both the exact and approximate differential inclusion (7) are rigid. We emphasize that incompatibility in particular excludes the presence of simple laminates. On the other hand, the differential inclusion (7) is said to be *compatible* if $B - A \in \Lambda_{\mathcal{A}}$. In this case, also in the setting of more general operators, a particular class of solutions to (7) consists of (generalized) simple laminates. Moreover, in the compatible setting, we further distinguish a particular case: We consider the subspace

$$I_{\mathcal{A}} := \bigcap_{\xi \in \mathbb{R}^d} \ker(\mathbb{A}(\xi)) = \bigcap_{|\alpha|=k} \ker(A_{\alpha}). \quad (8)$$

This is the space of values that are (algebraically) unconstrained by $\mathcal{A}(D)$, meaning that for all $u \in L^2(\mathbb{R}^d; I_{\mathcal{A}})$, we have that $\mathcal{A}(D)u = 0$ without taking any regularity constraints on u . In the case of $I_{\mathcal{A}} = \{0\}$, the operator $\mathcal{A}(D)$ belongs to the class of *cocanceling* operators, introduced in [65].

We seek to study both settings and the resulting microstructures *quantitatively* in the spirit of scaling results as, for instance, in the following non-exhaustive list involving different physical applications [2, 22, 62, 63, 66–77]. To this end, for $\Omega \subset \mathbb{R}^d$ an open, bounded, Lipschitz set, we introduce elastic and surface energies and consider their minimization for prescribed, not globally compatible boundary data $F_{\lambda} = \lambda A + (1 - \lambda)B$ for some $\lambda \in (0, 1)$, where again the set of states is given by $\mathcal{K} = \{A, B\}$.

Motivated by the applications from materials science, we study the following “elastic energy”

$$E_{el}(u, \chi) := \int_{\Omega} |u - \chi|^2 dx, \quad (9)$$

which we minimize in the following admissible class of deformations

$$u \in \mathcal{D}_{F_{\lambda}} := \{u \in L^2_{loc}(\mathbb{R}^d; \mathbb{R}^n) : \mathcal{A}(D)u = 0 \text{ in } \mathbb{R}^d, u = F_{\lambda} \text{ in } \mathbb{R}^d \setminus \overline{\Omega}\}, \chi \in L^2(\Omega; \mathcal{K}). \quad (10)$$

For ease of notation, here and in what follows, we often use the convention that $\chi := \chi_A A + \chi_B B$ with $\chi_A, \chi_B \in L^2(\Omega; \{0, 1\})$ and $\chi_A + \chi_B = 1$ in Ω . Moreover, we further use the notation

$$E_{el}(\chi; F_{\lambda}) := \inf_{u \in \mathcal{D}_{F_{\lambda}}} E_{el}(u, \chi).$$

In addition to the “elastic” energy contributions, we also introduce a surface energy contribution of the form

$$E_{surf}(\chi) := \int_{\Omega} |\nabla \chi|, \quad \chi \in BV(\Omega; \mathcal{K}) \quad (11)$$

and consider the following singularly perturbed elastic energy for $\epsilon > 0$

$$E_{\epsilon}(u, \chi) := E_{el}(u, \chi) + \epsilon E_{surf}(\chi), \quad (12)$$

and correspondingly

$$E_{\epsilon}(\chi; F_{\lambda}) := E_{el}(\chi; F_{\lambda}) + \epsilon E_{surf}(\chi) = \inf_{u \in \mathcal{D}_{F_{\lambda}}} E_{\epsilon}(u, \chi).$$

We note that this can be defined for an arbitrary set of states $\mathcal{K}$ and suitable boundary data F_λ .

With these quantities in hand, we can formulate the following quantitative rigidity results for the two-state problems:

Theorem 1 *Let $d, n, m, k \in \mathbb{N}$, $n > 1$. Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, and let $\mathcal{A}(D)$ be as in (2) with the wave cone $\Lambda_{\mathcal{A}}$ given in (5) and let $I_{\mathcal{A}}$ be given in (8). Let $A, B \in \mathbb{R}^n$ and let $\chi = \chi_A A + \chi_B B \in L^2(\Omega; \{A, B\})$. Further, let the elastic and surface energies E_{el} , E_{surf} be given as in (9) and (11), respectively. For $\lambda \in (0, 1)$ set $F_\lambda = \lambda A + (1 - \lambda)B \in \mathbb{R}^n$ and consider $\mathcal{D}_{F_\lambda}$ as in (10). The following results hold:*

- (i) Incompatible case: Assume that $B - A \notin \Lambda_{\mathcal{A}}$. Then there is a constant $C = C(A, B) > 0$ such that,

$$\inf_{\chi \in L^2(\Omega; \{A, B\})} \inf_{u \in \mathcal{D}_{F_\lambda}} E_{el}(u, \chi) \geq C(\min\{\lambda, 1 - \lambda\})^2 |\Omega|.$$

- (ii) Compatible case: Assume that $\mathcal{A}(D)$ is one-homogeneous, i.e. $k = 1$ in (2), and that $B - A \in \Lambda_{\mathcal{A}} \setminus I_{\mathcal{A}}$. Then, there exist $C = C(\mathcal{A}(D), A, B, \Omega, d, \lambda) > 0$ and $\epsilon_0 = \epsilon_0(\mathcal{A}(D), A, B, \Omega, d, \lambda) > 0$ such that for $\epsilon \in (0, \epsilon_0)$

$$\inf_{\chi \in BV(\Omega; \{A, B\})} \inf_{u \in \mathcal{D}_{F_\lambda}} (E_{el}(u, \chi) + \epsilon E_{surf}(\chi)) \geq C\epsilon^{2/3}.$$

Furthermore, if we assume $\mathcal{A}(D) = \text{div}$ and $\Omega = [0, 1]^d$, then there exists a constant $c = c(A, B, \lambda) > 0$ such that for $\epsilon > 0$ we also have the matching upper bound

$$\inf_{\chi \in BV(\Omega; \{A, B\})} \inf_{u \in \mathcal{D}_{F_\lambda}} (E_{el}(u, \chi) + \epsilon E_{surf}(\chi)) \leq c\epsilon^{2/3}.$$

- (iii) Super-compatible case: Assume that $A - B \in I_{\mathcal{A}}$. Then,

$$\inf_{\chi \in BV(\Omega; \{A, B\})} \inf_{u \in \mathcal{D}_{F_\lambda}} (E_{el}(u, \chi) + \epsilon E_{surf}(\chi)) = 0.$$

We highlight that in our discussion of the compatible case, we have restricted ourselves to operators of order one. This is due to the fact that for higher order operators it is expected that more complicated microstructures may arise. This is also reflected in the Fourier space properties of the symbol $\mathbb{A}$. We refer to Sect. 3.5, see Proposition 3.10, for a brief discussion of this, illustrating that the scaling may, in general, be no longer of the order $\epsilon^{\frac{2}{3}}$ in the higher order setting.

Let us discuss a prototypical example of the above results:

Example 1.1 As an example of the above differential inclusion, we consider the case in which $\mathcal{A}(D) = \text{div} : C^\infty(\mathbb{R}^d; \mathbb{R}^{m \times d}) \rightarrow C^\infty(\mathbb{R}^d; \mathbb{R}^m)$ row-wise for matrix fields $u : \mathbb{R}^d \rightarrow \mathbb{R}^{m \times d}$. In this case the boundary value problem under consideration turns into the following differential inclusion:

$$\begin{aligned} u &\in \{A, B\} \text{ in } \Omega, \\ \text{div } u &= 0 \text{ in } \mathbb{R}^d, \\ u &= F_\lambda \text{ in } \mathbb{R}^d \setminus \overline{\Omega}, \end{aligned}$$

for some $F_\lambda := \lambda A + (1 - \lambda)B$, $\lambda \in (0, 1)$. Such differential inclusions are related to applications in shape-optimization as, for instance, in [22].

The divergence is applied row-wise to matrix fields $u : \mathbb{R}^d \rightarrow \mathbb{R}^{m \times d}$, i.e. $\mathcal{A}(D)u = \sum_{j=1}^d (\partial_j u) e_j$ with the matrix vector product. Hence $\mathbb{A}(\xi)M = M\xi$ for $M \in \mathbb{R}^{m \times d}$ and thus the wave cone is given by

$$\Lambda_{\text{div}} = \{M \in \mathbb{R}^{m \times d} : \text{there is } \xi \in \mathbb{R}^d \text{ with } M\xi = 0\}.$$

Moreover the divergence is a cocanceling operator as $I_{\text{div}} = \{0\}$.

We emphasize that Example 1.1 is indeed a prototypical example and plays a central role in the study of first order operators in that all first order operators can be reduced to this model operator by a suitable linear transformation, see [28, Appendix] and also Sect. B below. We emphasize that this reduction is particularly useful if the differential inclusion is *incompatible* or if the boundary data are in $\Lambda_{\mathcal{A}} \setminus I_{\mathcal{A}}$. As a consequence, quantitative *lower bound estimates* for *incompatible* differential inclusions for first order operators, e.g. for T_N structures as qualitatively studied in [28], or for *compatible, but not super-compatible boundary data* can be deduced from the ones of the divergence operator (see Proposition B.2 and, in general, the discussion in Sect. B). A reduction to an equivalent problem for a modified operator and modified boundary data to the setting involving a cocanceling operator will be discussed in Sect. 3.4, see Proposition 3.8 and Corollary 3.9.

1.3 Quantitative Rigidity of a T_3 Structure for the Divergence Operator

In the second part of the article, building on the works [2–4] and motivated by the highlighted considerations on the role of the divergence operator, we study the quantitative rigidity of the T_3 configuration

$$u \in \{A_1, A_2, A_3\} \text{ a.e. in } \Omega, \quad \text{div } u = 0 \text{ in } \mathbb{R}^3, \quad (13)$$

where $\Omega = [0, 1]^3$, $u : \mathbb{R}^3 \rightarrow \mathbb{R}^{3 \times 3}$ and

$$A_1 = 0_{3 \times 3}, A_2 = \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & 3 \end{pmatrix}, A_3 = Id_{3 \times 3}. \quad (14)$$

By virtue of the results from [3, 4] this problem is flexible for approximate solutions but rigid on the level of exact solutions:

- More precisely, on the level of *exact solutions* to (13), only constant solutions $u \equiv A_j$ for $j \in \{1, 2, 3\}$ obey the differential inclusion.
- Considering however *approximate solutions*, i.e. sequences $(u_k)_{k \in \mathbb{N}}$ such that

$$\text{dist}(u_k, \{A_1, A_2, A_3\}) \rightarrow 0 \text{ in measure as } k \rightarrow \infty, \quad \text{div } u_k = 0 \text{ for all } k \in \mathbb{N},$$

there exists a sequence of approximate solutions $(u_k)_{k \in \mathbb{N}}$ for (13) such that there is no subsequence which converges in measure to one of the constant deformations $\{A_1, A_2, A_3\}$.

Compared to the setting of the gradient, for the divergence operator rigidity for approximate solutions is already lost for the three-state problem (while this arises only for four or more states for the gradient [78, 79], see also [18] for further instances in which the Tartar square was found and used).

As in [2] we here study a *quantitative* version of the dichotomy between rigidity and flexibility: We consider the singularly perturbed variant of (13) as in Sect. 1.2

$$E_\epsilon(u, \chi) := \int_{\Omega} |u - \chi|^2 dx + \epsilon \int_{\Omega} |\nabla \chi| \quad (15)$$

under the constraint $u \in \mathcal{D}_F$, cf. (10), with $\mathcal{A}(D) = \operatorname{div}$ and where $F \in \{A_1, A_2, A_3\}^{qc}$ (see Sect. 2 for the definition of the $\mathcal{A}$ -quasi-convexification of a compact set). Here the function $\chi \in BV(\Omega; \{A_1, A_2, A_3\})$ denotes the phase indicator of the “phases” A_1, A_2, A_3 , respectively.

Adapting the ideas from [2] to the divergence operator in three dimensions, we prove the following scaling result:

Theorem 2 *Let $\Omega = [0, 1]^3$ and let $\mathcal{K} = \{A_1, A_2, A_3\}$ with $A_j, j \in \{1, 2, 3\}$ given in (14), $F \in \mathcal{K}^{qc} \setminus \mathcal{K}$, E_ϵ be as in (15) above for the divergence operator $\mathcal{A}(D) = \operatorname{div}$, and consider $\mathcal{D}_F$ given analogously as (10). Then, there exist constants $c = c(F) > 0$ and $C = C(F) > 1$ such that for any $v \in (0, \frac{1}{2})$ there is $\epsilon_0 = \epsilon_0(v, F) > 0$ and $c_v > 0$ such that for $\epsilon \in (0, \epsilon_0)$ we have*

$$C^{-1} \exp(-c_v |\log(\epsilon)|^{\frac{1}{2}+v}) \leq \inf_{\chi \in BV(\Omega; \mathcal{K})} \inf_{u \in \mathcal{D}_F} E_\epsilon(u, \chi) \leq C \exp(-c |\log(\epsilon)|^{\frac{1}{2}}).$$

Let us comment on this result: As in [2] we obtain essentially matching upper and lower scaling bounds with less than algebraic decay behavior as $\epsilon \rightarrow 0$, reflecting the infinite order laminates underlying the T_3 structure and the fact that the problem is “nearly” rigid. As in [2] a key step is the analysis of a “quantitative chain rule in a negative Sobolev space” which results from the interaction of Riesz type transforms and a nonlinearity originating from the “ellipticity” of the differential inclusion. Both in the upper and the lower bound, these estimates for the divergence operator however require additional care due to the three-dimensionality of the problem. In the upper bound construction this is manifested in the use of careful cut-off arguments; in the lower bound, a more involved iterative scheme has to be used to reduce the possible regions of concentration in Fourier space.

Similarly, as in [76] the scaling law from Theorem 2 is obtained as a consequence of a rigidity estimate encoding both the *rigidity* and *flexibility* of the T_3 differential inclusion (in analogy to the T_4 case from [76, Proposition 3]).

Proposition 1.2 *Let $\Omega = [0, 1]^3$ and let $\mathcal{K} = \{A_1, A_2, A_3\}$ with $A_j, j \in \{1, 2, 3\}$ given in (14), $F \in \mathcal{K}^{qc} \setminus \mathcal{K}$. Let χ_{jj} denote the diagonal entries of the matrix field $\chi \in BV(\Omega; \mathcal{K})$ and denote by $E_\epsilon^{per}(\chi; F)$ the periodic singularly perturbed energy (see (32) and Sect. 4.3.2). Then, there exists $\epsilon_0 > 0$ such that for any $v \in (0, 1)$ there is a constant $c_v > 0$ such that for $\epsilon \in (0, \epsilon_0)$ we have*

$$\sum_{j=1}^3 \|\chi_{jj} - \langle \chi_{jj} \rangle\|_{L^2([0,1]^3)}^2 \leq \exp(c_v |\log(\epsilon)|^{\frac{1}{2}+v}) E_\epsilon^{per}(\chi; F)^{\frac{1}{2}}.$$

We emphasize that in parallel to the setting of the Tartar square, this estimate quantitatively encodes both rigidity and flexibility of the differential inclusion, as it measures the distance to the constant state (and thus reflects *rigidity* of the exact differential inclusion) but also quantifies the “price” for this in terms of a “high energy” scaling law (and thus reflects the underlying flexibility of the approximate problem). Moreover, due to the *flexibility* of the differential inclusion, we stress that such an estimate can only be inferred for a combination of elastic *and* surface energies.

1.4 Outline of the Article

The remainder of the article is structured as follows: After briefly recalling relevant notation and facts on convex hulls related to the operator $\mathcal{A}(D)$, we first discuss the compatible and incompatible two-state problems in Sect. 3. Here we begin by discussing the incompatible setting, which is a consequence of direct elliptic estimates in Sect. 3.2 and then turn to the lower bounds for the compatible setting in Sect. 3.3. The super-compatible case is then treated in Sect. 3.4. It is also in this section that we discuss a reduction to cocanceling operators. We revisit the scaling of a prototypical higher order operator from [1] in Sect. 3.5 and explain how our scheme of deducing lower bounds also yields a quick Fourier based proof of the lower scaling bound from [1].

In Sect. 4 we then turn to the T_3 differential inclusion, for which we first prove upper bounds in Sect. 4.1 and then adapt the ideas from [2] to infer essentially matching lower bounds in Sect. 4.2.

In the Appendix, we complement the general lower bounds for first order differential operators with upper bounds for the specific case of the divergence operator (Sect. A). Moreover, in Sect. B.1 we discuss the reduction to the divergence operator.

2 Notation

In this section we collect the notation which is used throughout the article.

- For a set U , we denote by d_U the (possibly smoothed-out) distance to this set: $d_U(x) = \text{dist}(x, U) = \inf_{y \in U} |x - y|$ and denote by χ_U the (in some places smoothed-out) indicator function of this set.
- For a set U with finite measure and a function $f : U \rightarrow \mathbb{R}^n$ we denote the mean by $\langle f \rangle = \frac{1}{|U|} \int_U f dx$.
- For a function $f \in L^2(\mathbb{T}^d)$ or $f \in L^2(\mathbb{R}^d)$ we denote the Fourier transform by

$$\mathcal{F}(f)(k) = \hat{f}(k) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{T}^d} e^{-ik \cdot x} f(x) dx \quad (k \in \mathbb{Z}^d)$$

or

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-i\xi \cdot x} f(x) dx \quad (\xi \in \mathbb{R}^d).$$

For a function $h \in L^\infty(\mathbb{R}^d)$, we denote by $h(D)$ the corresponding Fourier multiplier

$$h(D)f = \mathcal{F}^{-1}(h(\cdot)\hat{f}(\cdot)).$$

- We usually denote the phase indicator by $\chi \in L^2(\Omega; \mathcal{K})$, with the component functions given by $\chi = (\chi_i)$, $i \in \{1, \dots, n\}$. Moreover, we use $F \in \mathcal{K}^{qc}$ (where $\mathcal{K}^{qc}$ is introduced below) as the exterior data.
- The set of admissible 'deformations' u for an operator of order $k \geq 1$ is given by $\mathcal{D}_F := \mathcal{D}_F^A := \{u \in L_{loc}^2(\mathbb{R}^d; \mathbb{R}^n) : \mathcal{A}(D)u = 0 \text{ in } \mathbb{R}^d, u = F \text{ in } \mathbb{R}^d \setminus \bar{\Omega}\}$. Here the equation $\mathcal{A}(D)u = 0$ is considered in a distributional sense.
- As introduced in Sect. 1.2, for $F \in \mathcal{K}^{qc}$ we consider the elastic energy with $u \in \mathcal{D}_F$, $\chi \in L^2(\Omega; \mathcal{K})$:

$$E_{el}(u, \chi) = \int_{\Omega} |u - \chi|^2 dx, \quad E_{el}(\chi; F) = \inf_{u \in \mathcal{D}_F} E_{el}(u, \chi), \quad (16)$$

and, for $\chi \in BV(\Omega; \mathcal{K})$, the surface energy as the total variation norm:

$$E_{surf}(\chi) = \|\nabla \chi\|_{TV(\Omega)} = \int_{\Omega} |\nabla \chi|. \quad (17)$$

- The total energy with $\epsilon > 0$ is given by $E_{\epsilon}(u, \chi) = E_{el}(u, \chi) + \epsilon E_{surf}(\chi)$.
- We write $f \sim g$ if there are constants $c, C > 0$ such that $cf \leq g \leq Cf$.
- We use the notation $\|\cdot\|_{\dot{H}^{-1}}$ for the homogeneous H^{-1} semi-norm for $f \in H^{-1}(\mathbb{T}^3; \mathbb{R})$:

$$\|f\|_{\dot{H}^{-1}}^2 = \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \frac{1}{|k|^2} |\hat{f}(k)|^2.$$

We further recall the notions of the $\Lambda_{\mathcal{A}}$ -convex hull of a set (see [64]) and the $\mathcal{A}$ -quasi-convex hull of a set:

- Let $\Lambda_{\mathcal{A}} \subset \mathbb{R}^n$ be the wave cone from (5) and let $K \subset \mathbb{R}^n$ be a compact set. For $j \in \mathbb{N}$, we then define $K^{(j)}$ as follows:

$$K^{(0)} := K,$$

$$K^{(j)} := \{M \in \mathbb{R}^n : M = \lambda A + (1 - \lambda)B : A, B \in K^{(j-1)}, B - A \in \Lambda_{\mathcal{A}}, \lambda \in [0, 1]\}.$$

Moreover, we define the $\Lambda_{\mathcal{A}}$ -convex hull K^{lc} :

$$K^{lc} := \bigcup_{j=0}^{\infty} K^{(j)}.$$

We define the *order of lamination* of a matrix $M \in \mathbb{R}^n$ to be the minimal $j \in \mathbb{N}$ such that $M \in K^{(j)}$. In analogy to the gradient case, we will also refer to K^{lc} as the *laminar convex hull*.

- We recall that the $\mathcal{A}$ -quasi-convex hull K^{qc} of a compact set $K \subset \mathbb{R}^n$ is defined by duality to $\mathcal{A}$ -quasi-convex functions [48]. We recall that the div-quasi-convex hull of the T_3 matrices from (13), (14) has been explicitly characterized in [21, Theorem 2] to consist of the union of the closed triangle formed by the matrices S_1, S_2, S_3 and the three “legs” formed by the line segments $A_j S_j$. The matrices $S_1, S_2, S_3 \in \mathbb{R}^{3 \times 3}$ are introduced in Sect. 4 below. Given the set $\{A_1, A_2, A_3\} \subset \mathbb{R}^{3 \times 3}$ from (13), (14), as in the gradient case, we denote its $\mathcal{A}$ -quasi-convex hull by $\{A_1, A_2, A_3\}^{qc}$.

3 Quantitative Results on the Two-State Problem

In this section, we study quantitative versions of the two-state problem for general, linear, constant coefficient, homogeneous operators, always considering the divergence operator as a particular model case.

Building on the precise formulation of the problem from (7), in Sect. 3.1, we first characterize the elastic energies in terms of the operator $\mathcal{A}(D)$. With this characterization in hand, using ellipticity, we next prove the quantitative L^2 bounds in the *incompatible* two-well case (Sect. 3.2) and *lower* $\epsilon^{\frac{2}{3}}$ -scaling bounds for the *compatible* case with first order, linear operators (Sect. 3.3). In Sect. 3.4, we prove Theorem 1(iii) and deduce a reduction to cocanceling operators. In Sect. 3.5 we briefly discuss the role of degeneracies in the symbol of the elastic

energies which may arise for higher order, linear differential operators and which may thus lead to an alternative scaling behavior different from the $\epsilon^{\frac{2}{3}}$ -scaling bound.

Finally, in Sect. A we complement the lower bounds from this section with matching *upper* bounds for the special case of $\mathcal{A}(D) = \text{div}$. Similar constructions are also known for the gradient, the symmetrized gradient and lower dimensional problems from micromagnetics [1, 66, 73, 74, 80].

3.1 Elastic Energy Characterization

We begin by recalling an explicit lower bound for the elastic energy in terms of the operator $\mathcal{A}(D)$ (see, for instance also [49, discussion before Lemma 1.17]) in an, for us, convenient form. In everything that follows, we will assume $d > 1$.

Lemma 3.1 (Fourier characterization of the elastic energy) *Let $d, n, k \in \mathbb{N}$. Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain with associated indicator function χ_Ω and let $\mathcal{K} \subset \mathbb{R}^n$. Let $\mathcal{A}(D) = \sum_{|\alpha|=k} A_\alpha \partial^\alpha$ be as in (2) with the symbol $\mathbb{A}$, cf. (3), and E_{el} be as in (16) and $F \in \mathcal{K}^{qc}$. Then there is a constant $c = c(\mathcal{A}(D)) > 0$ such that for any $\chi \in L^2(\Omega; \mathcal{K})$, extended by zero outside of Ω ,*

$$\begin{aligned} E_{el}(\chi; F) &\geq \inf_{\mathcal{A}(D)u=0} \int_{\mathbb{R}^d} |(u - F) - (\chi - F\chi_\Omega)|^2 dx \\ &= \int_{\mathbb{R}^d} \left| \mathbb{A}(\xi)^* (\mathbb{A}(\xi) \mathbb{A}(\xi)^*)^{-1} \mathbb{A}(\xi) (\hat{\chi} - F\hat{\chi}_\Omega) \right|^2 d\xi \\ &\geq c \int_{\mathbb{R}^d} \left| \mathbb{A}\left(\frac{\xi}{|\xi|}\right) (\hat{\chi} - F\hat{\chi}_\Omega) \right|^2 d\xi, \end{aligned}$$

where the infimum is taken over all $u \in L^2_{loc}(\mathbb{R}^d; \mathbb{R}^n)$ that fulfill $\mathcal{A}(D)u = 0$ in $\mathbb{R}^d$.

Proof The proof follows from a projection argument in Fourier space.

Step 1: Whole space extension, Fourier and pseudoinverse of the differential operator. We begin by transforming our problem to one on the whole space $\mathbb{R}^d$, introducing the whole space extension $w := u - F$:

$$\begin{aligned} E_{el}(u, \chi) &= \int_{\Omega} |u - \chi|^2 dx = \int_{\Omega} |u - F - (\chi - F)|^2 dx \\ &= \int_{\mathbb{R}^d} |w - (\chi - F\chi_\Omega)|^2 dx, \end{aligned}$$

where we have extended all functions in the integrand by zero outside of Ω .

In the following, we write $\tilde{\chi} := \chi - F\chi_\Omega$. Fourier transforming the expression for the elastic energy then leads to

$$E_{el}(u, \chi) = \int_{\mathbb{R}^d} |\hat{w} - \hat{\tilde{\chi}}|^2 d\xi.$$

Further, seeking to deduce a lower bound, we relax the boundary data for w , obtaining

$$E_{el}(\chi; F) = \inf_{u \in \mathcal{D}_F} \int_{\Omega} |u - \chi|^2 dx = \inf_{w \in \mathcal{D}_0} \int_{\mathbb{R}^d} |w - \tilde{\chi}|^2 dx \geq \inf_{\mathcal{A}(D)w=0} \int_{\mathbb{R}^d} |\hat{w} - \hat{\tilde{\chi}}|^2 d\xi.$$

Minimizing the integrand $\hat{w}$ (and still denoting the minimizer by $\hat{w}$) for each fixed mode $\xi \in \mathbb{R}^d \setminus \{0\}$, we infer that

$$\begin{aligned} |\hat{w}(\xi) - \hat{\chi}(\xi)| &= |\Pi_{\ker \mathbb{A}(\xi)} \hat{\chi}(\xi) - \hat{\chi}(\xi)| = |\Pi_{\text{ran } \mathbb{A}(\xi)^*} \hat{\chi}(\xi)| \\ &= |\mathbb{A}(\xi)^* (\mathbb{A}(\xi) \mathbb{A}(\xi)^*)^{-1} \mathbb{A}(\xi) \hat{\chi}(\xi)|, \end{aligned}$$

where we view $\mathbb{A}(\xi) \mathbb{A}(\xi)^* : \text{ran } \mathbb{A}(\xi) \rightarrow \text{ran } \mathbb{A}(\xi)$ as an isomorphism. This directly implies the claimed lower bound for $E_{el}(\chi; F)$ in terms of the symbol $\mathbb{A}$ and its pseudoinverse.

Step 2: Proof of the final estimate. Now to show the final estimate for the elastic energy, we seek to bound $|\mathbb{A}(\xi)^* (\mathbb{A}(\xi) \mathbb{A}(\xi)^*)^{-1} \mathbb{A}(\xi) x|$ for every $\xi \in \mathbb{R}^d$, $x \in \mathbb{R}^n$ from below in terms of $|\mathbb{A}(\xi) x|$. As the projection operator is zero-homogeneous, we can reduce to $\xi \in \mathbb{S}^{d-1}$, and thus can use the continuity of $\mathbb{S}^{d-1} \ni \xi \mapsto \mathbb{A}(\xi)$ and the compactness of $\mathbb{S}^{d-1}$ for the desired bound: For any $x \in \mathbb{R}^n$, $\xi \in \mathbb{S}^{d-1}$ it hence holds

$$\begin{aligned} |\mathbb{A}(\xi) x| &= |\mathbb{A}(\xi) \mathbb{A}(\xi)^* (\mathbb{A}(\xi) \mathbb{A}(\xi)^*)^{-1} \mathbb{A}(\xi) x| \\ &\leq |\mathbb{A}(\xi)^* (\mathbb{A}(\xi) \mathbb{A}(\xi)^*)^{-1} \mathbb{A}(\xi) x| \sup_{\xi \in \mathbb{S}^{d-1}} (|\mathbb{A}(\xi)|). \end{aligned}$$

As $\mathcal{A}(D) \neq 0$, we have that $0 < \sup_{\zeta \in \mathbb{S}^{d-1}} |\mathbb{A}(\zeta)| \leq C < \infty$. Dividing by this and plugging this into the expression with the pseudoinverse, we obtain

$$\begin{aligned} E_{el}(u, \chi) &\geq \int_{\mathbb{R}^d} |\mathbb{A}(\xi)^* (\mathbb{A}(\xi) \mathbb{A}(\xi)^*)^{-1} \mathbb{A}(\xi) \hat{\chi}|^2 d\xi \\ &\geq \frac{1}{\sup_{\zeta \in \mathbb{S}^{d-1}} |\mathbb{A}(\zeta)|^2} \int_{\mathbb{R}^d} \left| \mathbb{A}\left(\frac{\xi}{|\xi|}\right) \hat{\chi} \right|^2 d\xi, \end{aligned}$$

which concludes the argument. $\square$

We emphasize that we are relaxing the boundary conditions for $\hat{w}(\xi) := \Pi_{\ker \mathbb{A}(\xi)} \hat{\chi}(\xi)$ as we do not calculate the projection of χ onto $\mathcal{D}_F$, hence the above Fourier bounds only provide lower bounds for the elastic energy.

We apply the lower bound from Lemma 3.1 to the two-well problem:

Corollary 3.2 *Let $d, n \in \mathbb{N}$. Let Ω , $\mathcal{A}(D)$, $\mathbb{A}$, E_{el} be as in Lemma 3.1. Consider $\mathcal{K} = \{A, B\} \subset \mathbb{R}^n$ with $F_\lambda = \lambda A + (1 - \lambda)B$ for $\lambda \in (0, 1)$. Then there exists a constant $C = C(\mathcal{A}(D)) > 0$ such that for any $\chi = \chi_A A + \chi_B B \in L^2(\Omega; \mathcal{K})$, extended to $\mathbb{R}^d$ by zero, it holds*

$$E_{el}(\chi; F_\lambda) \geq C \int_{\mathbb{R}^d} \left| ((1 - \lambda) \hat{\chi}_A - \lambda \hat{\chi}_B) \mathbb{A}\left(\frac{\xi}{|\xi|}\right) (A - B) \right|^2 d\xi.$$

Proof Using the expression of F_λ in terms of A, B, λ yields $A - F_\lambda = (1 - \lambda)(A - B)$ and $B - F_\lambda = -\lambda(A - B)$. Thus, the fact that $\chi = \chi_A A + \chi_B B$ and Lemma 3.1 imply

$$\begin{aligned} E_{el}(\chi; F_\lambda) &\geq C \int_{\mathbb{R}^d} \left| \mathbb{A}\left(\frac{\xi}{|\xi|}\right) (\hat{\chi} - F_\lambda \hat{\chi}_\Omega) \right|^2 d\xi \\ &= C \int_{\mathbb{R}^d} \left| ((1 - \lambda) \hat{\chi}_A - \lambda \hat{\chi}_B) \mathbb{A}\left(\frac{\xi}{|\xi|}\right) (A - B) \right|^2 d\xi. \quad \square \end{aligned}$$

Remark 3.3 (The divergence operator) As seen in Example 1.1, in the case of $\mathcal{A}(D) = \text{div}$, we have for $u \in C^\infty(\mathbb{R}^d; \mathbb{R}^{d \times d})$ (note that we chose square matrices out of simplicity)

$$\mathcal{A}(D)u = \sum_{i=1}^d (\partial_i u) e_i \in C^\infty(\mathbb{R}^d; \mathbb{R}^d).$$

With this we can calculate

$$\mathbb{A}(\xi)M = \sum_{i=1}^d \xi_i M e_i = M\xi.$$

This, in particular, shows that the adjoint operator is given by $\mathbb{A}(\xi)^* : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ with

$$\mathbb{A}(\xi)^* x = x \otimes \xi.$$

Therefore $\mathbb{A}(\xi)\mathbb{A}(\xi)^* x = (x \otimes \xi)\xi = |\xi|^2 x$, and the projection in the lower bound for the elastic energy of Lemma 3.1 takes the desired form

$$\mathbb{A}(\xi)^* \left(\mathbb{A}(\xi)\mathbb{A}(\xi)^* \right)^{-1} \mathbb{A}(\xi)M = \frac{1}{|\xi|^2} (M\xi \otimes \xi).$$

Furthermore it holds

$$\left| \mathbb{A}(\xi)^* \left(\mathbb{A}(\xi)\mathbb{A}(\xi)^* \right)^{-1} \mathbb{A}(\xi)M \right| = \left| \mathbb{A}\left(\frac{\xi}{|\xi|}\right)M \right|.$$

3.2 The Incompatible Two-Well Problem and Scaling

As a first application of the Fourier characterizations from the previous section, we prove a quantitative lower bound for the incompatible two-well problem. We emphasize that – as in [27] – this argument is an elliptic argument and thus can be applied to all linear, constant coefficient homogeneous operators. Indeed, the following result holds:

Proposition 3.4 *Let $d, n \in \mathbb{N}$. Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, let $\mathcal{A}(D)$ be given in (2) and $A, B \in \mathbb{R}^n$ with $B - A \notin \Lambda_{\mathcal{A}}$, cf. (5), further let $F_\lambda = \lambda A + (1 - \lambda)B$ for some $\lambda \in (0, 1)$, and let E_{el} be as in (16) with $\mathcal{D}_{F_\lambda}$ given in (10). Then there is $C = C(A, B, \mathcal{A}(D)) > 0$, such that for any $\chi \in L^2(\Omega; \{A, B\})$*

$$\inf_{u \in \mathcal{D}_{F_\lambda}} E_{el}(u, \chi) \geq C \min\{\lambda, 1 - \lambda\}^2 |\Omega|.$$

Proof By virtue of Corollary 3.2, we have the lower bound

$$E_{el}(\chi; F_\lambda) \geq C \int_{\mathbb{R}^d} \left| ((1 - \lambda)\hat{\chi}_A - \lambda\hat{\chi}_B)\mathbb{A}\left(\frac{\xi}{|\xi|}\right)(A - B) \right|^2 d\xi,$$

where the constant only depends on the operator $\mathcal{A}(D)$.

As $(A - B) \notin \ker \mathbb{A}(\xi)$ for all $\xi \in \mathbb{R}^d$ and thus $|\mathbb{A}(\xi)(A - B)| > 0$ for any $\xi \in \mathbb{R}^d$, by continuity of $\xi \mapsto \mathbb{A}(\xi)$ and compactness of $\mathbb{S}^{d-1}$, this implies $|\mathbb{A}(\frac{\xi}{|\xi|})(A - B)| \geq C(A, B, \mathcal{A}(D)) > 0$. Hence,

$$E_{el}(u, \chi) \geq C^2 \int_{\mathbb{R}^d} |(1 - \lambda)\hat{\chi}_A - \lambda\hat{\chi}_B|^2 d\xi = C^2 \int_{\mathbb{R}^d} |(1 - \lambda)\chi_A - \lambda\chi_B|^2 dx$$

$$\geq C^2 \left(\int_{\Omega_A} (1-\lambda)^2 dx + \int_{\Omega_B} \lambda^2 dx \right) \geq C^2 \min\{\lambda, 1-\lambda\}^2 |\Omega|,$$

where $\Omega_A := \text{supp}(\chi_A) \subset \overline{\Omega}$ and $\Omega_B := \text{supp}(\chi_B) \subset \overline{\Omega}$. $\square$

Remark 3.5 We emphasize that this result can be viewed as an incompatible nucleation bound.

3.3 The Compatible Two-Well Case and Scaling

We next turn to the setting of two compatible wells and restrict our attention to *first order operators*. In this case, we claim the following $\epsilon^{\frac{2}{3}}$ -lower scaling bound. This is in analogy to the situation for the gradient which had first been derived in the seminal works [62, 63].

Proposition 3.6 *Let $d, n \in \mathbb{N}$. Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and let $\mathcal{A}(D)$ be a first order operator as in (2). Let $A, B \in \mathbb{R}^n$ be such that $B - A \in \Lambda_{\mathcal{A}} \setminus I_{\mathcal{A}}$ where $\Lambda_{\mathcal{A}}$ and $I_{\mathcal{A}}$ are given in (5) and (8), and define $F_{\lambda} := \lambda A + (1-\lambda)B$ for some $\lambda \in (0, 1)$. Let $E_{\epsilon} := E_{el} + \epsilon E_{surf}$ be given in (12) with $\mathcal{D}_{F_{\lambda}}$ defined in (10). Then, there exist $C = C(\mathcal{A}(D), A, B, \Omega, d, \lambda) > 0$ and $\epsilon_0 = \epsilon_0(\mathcal{A}(D), A, B, \Omega, d, \lambda) > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ we have*

$$\epsilon^{2/3} \leq C \inf_{\chi \in BV(\Omega; \{A, B\})} \inf_{u \in \mathcal{D}_{F_{\lambda}}} E_{\epsilon}(u, \chi).$$

In order to deal with the compatible case, we invoke the following (slightly generalized) auxiliary results from [2], see also [23, 71], which we formulate for a general Fourier multiplier m :

Lemma 3.7 (Elastic, surface and low frequency cut-off) *Let $d, n, N \in \mathbb{N}$. Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Let $m : \mathbb{R}^d \rightarrow \mathbb{R}^N$ be a linear map and denote by $V := \ker m \subsetneq \mathbb{R}^d$ its kernel and by $\Pi_V : \mathbb{R}^d \rightarrow V$ the orthogonal projection onto V . Let $f \in BV(\mathbb{R}^d; \{-\lambda, 0, 1-\lambda\})$ for $\lambda \in (0, 1)$ with $f = 0$ outside Ω and $f \in \{-\lambda, 1-\lambda\}$ in Ω . Consider the elastic and surface energies given by*

$$\tilde{E}_{el}(f) := \int_{\mathbb{R}^d} |m(\frac{\xi}{|\xi|}) \hat{f}(\xi)|^2 d\xi, \quad \tilde{E}_{surf}(f) := \int_{\Omega} |\nabla f|.$$

Then the following results hold:

- (a) Low frequency elastic energy control. *Let $\mu > 1$, then there exists $C = C(m, \Omega) > 0$ with*

$$\|\hat{f}\|_{L^2(\{\xi \in \mathbb{R}^d : |\Pi_V(\xi)| \leq \mu\})}^2 \leq C \mu^2 \tilde{E}_{el}(f).$$

- (b) High frequency surface energy control. *There exists $C = C(d, \lambda) > 0$ such that for $\mu > 0$ it holds*

$$\|\hat{f}\|_{L^2(\{\xi \in \mathbb{R}^d : |\xi| \geq \mu\})}^2 \leq C \mu^{-1} (\tilde{E}_{surf}(f) + \text{Per}(\Omega)).$$

Proof of Lemma 3.7 Since the property (b) is directly analogous to the one from [2, Lemma 2], we only discuss the proof of (a) which requires some (slight) modifications with respect

to [2] (and [23]). We thus present the argument for this for completeness. We split $\xi = \xi' + \xi''$, where $\xi' \in V^\perp \neq \{0\}$, $\xi'' \in V = \ker m$. With this in hand and by the linearity of m it holds

$$m\left(\frac{\xi}{|\xi|}\right) = m\left(\frac{\xi' + \xi''}{|\xi|}\right) = m\left(\frac{\xi'}{|\xi|}\right) = M \frac{\xi'}{|\xi|}$$

for some matrix representation M of m . Hence, using $\xi' \perp \ker m$, there is $c = c(m) > 0$ such that

$$\tilde{E}_{el}(f) = \int_{\mathbb{R}^d} |m\left(\frac{\xi}{|\xi|}\right) \hat{f}(\xi)|^2 d\xi \geq c \int_{\mathbb{R}^d} \left|\frac{\xi'}{|\xi|}\right| \hat{f}(\xi)|^2 d\xi.$$

With this in hand, we argue similarly as in [23] and [2]: For $a, b \in \mathbb{R}$, $a, b \geq 0$ and the orthogonal splitting $\xi = \xi' + \xi''$ from above, it holds that

$$\begin{aligned} \frac{1}{c} \tilde{E}_{el}(f) &\geq \int_{\mathbb{R}^d} \left| \frac{\xi'}{|\xi|} \hat{f} \right|^2 d\xi = \int_{\mathbb{R}^d} \frac{|\xi'|^2}{|\xi'|^2 + |\xi''|^2} |\hat{f}|^2 d\xi \\ &\geq \frac{1}{\frac{a^2}{b^2} + 1} \int_{\{|\xi'| \geq \frac{1}{a}, |\xi''| \leq \frac{1}{b}\}} |\hat{f}|^2 d\xi \\ &= \frac{1}{\frac{a^2}{b^2} + 1} \int_{\{|\xi''| \leq \frac{1}{b}\}} \left(\int_{V^\perp} |\hat{f}|^2 d\xi' - \int_{\{|\xi'| < \frac{1}{a}\}} |\hat{f}|^2 d\xi' \right) d\xi'' \\ &\geq \frac{1}{\frac{a^2}{b^2} + 1} \int_{\{|\xi''| \leq \frac{1}{b}\}} \left(\int_{V^\perp} |\hat{f}|^2 d\xi' - \left(\frac{2}{a}\right)^{\dim V^\perp} \sup_{\xi' \in V^\perp} |\hat{f}|^2 \right) d\xi''. \end{aligned} \quad (18)$$

Using the notation $f(\xi) = f(\xi', \xi'')$, setting

$$a^{\dim V^\perp} := 2^{\dim V^\perp + 1} \sup_{\xi'' \in V} \frac{\sup_{\xi' \in V^\perp} |\hat{f}(\xi', \xi'')|^2}{\int_{V^\perp} |\hat{f}(\xi', \xi'')|^2 d\xi'},$$

and using Plancherel's identity, the $L^\infty - L^1$ bounds for the Fourier transform and Hölder's inequality, we obtain that

$$a^{\dim V^\perp} \leq 2^{\dim V^\perp + 1} \sup_{\xi'' \in V} \frac{\|\mathcal{F}_{\xi''} f(\cdot, \xi'')\|_{L^1(V^\perp)}^2}{\|\mathcal{F}_{\xi''} f(\cdot, \xi'')\|_{L^2(V^\perp)}^2} \leq C(\Omega) 2^{\dim V^\perp + 1}.$$

In particular, the constant a is well-defined. Returning to (18), we consequently deduce that for $b \in (0, 1)$

$$\tilde{E}_{el}(f) \geq C(\Omega, m) b^2 \int_{\{|\xi'| \leq \frac{1}{b}\}} |\hat{f}|^2 d\xi.$$

Choosing $b = \mu^{-1} < 1$ and noting that $\{\xi \in \mathbb{R}^d : |\Pi_V(\xi)| \leq \mu\} = \{\xi \in \mathbb{R}^d : |\xi''| \leq \frac{1}{b}\}$ implies the claim. $\square$

With Lemma 3.7 in hand, we turn to the proof of the lower bound in Proposition 3.6:

Proof of the lower bound in Proposition 3.6 Since $B - A \in \Lambda_{\mathcal{A}} \setminus I_{\mathcal{A}}$, there exists $\xi \in \mathbb{R}^d \setminus \{0\}$ such that $\mathbb{A}(\xi)(B - A) = 0$. As $\mathcal{A}(D)$ is a first order operator, we have that $\mathbb{A}(\xi)$ is linear in $\xi \in \mathbb{R}^d$. In particular, we have that the set, cf. (6),

$$V_{B-A} := V_{\mathcal{A}, B-A} = \{\xi \in \mathbb{R}^d : \mathbb{A}(\xi)(B - A) = 0\} \neq \{0\}$$

is a linear space. Rewriting $\mathbb{R}^d \ni \xi = \xi' + \xi''$ with $\xi' \in V_{B-A}^\perp$ and $\xi'' \in V_{B-A}$ as in the proof of Lemma 3.7, then Corollary 3.2 implies that

$$\begin{aligned} E_{el}(\chi; F_\lambda) &\geq C \int_{\mathbb{R}^d} |((1-\lambda)\hat{\chi}_A - \lambda\hat{\chi}_B)\mathbb{A}\left(\frac{\xi}{|\xi|}\right)(A-B)|^2 d\xi, \\ E_{surf}(\chi) &= \int_{\Omega} |\nabla(\chi - F_\lambda)| = \int_{\Omega} |\nabla((1-\lambda)\chi_A - \lambda\chi_B)(A-B)| dx \\ &\geq C \int_{\Omega} |\nabla[(1-\lambda)\chi_A - \lambda\chi_B]|, \end{aligned} \quad (19)$$

for a constant C depending on the operator $\mathcal{A}(D)$ and on $A - B$.

Now, setting $m(\xi) := \mathbb{A}(\xi)(A - B)$ and $f := (1-\lambda)\chi_A - \lambda\chi_B$, yields the applicability of Lemma 3.7 with $V = V_{B-A} \subsetneq \mathbb{R}^d$. This is the only place where we use the assumption that $A - B \notin I_{\mathcal{A}}$. We deduce that by (19) and the decomposition of $\mathbb{R}^d$ into the two regions from Lemma 3.7 we have for $\mu > 1$

$$\begin{aligned} \|f\|_{L^2}^2 &\leq 2\left(\| \chi_{\{|\xi| \geq \mu\}}(D) f \|_{L^2}^2 + \| \chi_{\{|\xi| \leq \mu\}}(D) f \|_{L^2}^2\right) \\ &\leq C\left(\mu^2 E_{el}(\chi; F_\lambda) + (\mu^{-1}\epsilon^{-1})\epsilon E_{surf}(\chi) + \mu^{-1} \text{Per}(\Omega)\right), \end{aligned}$$

where the constant C depends on $\mathcal{A}(D)$, A , B , Ω , d , λ . Now choosing $\mu = \epsilon^{-\frac{1}{3}} > 1$, noting that then $\mu^{-1}\epsilon^{-1} = \epsilon^{-\frac{2}{3}}$ for $\epsilon < 1$, we obtain

$$\|f\|_{L^2}^2 \leq C\left(\epsilon^{-\frac{2}{3}} E_\epsilon(\chi; F_\lambda) + \epsilon^{\frac{1}{3}} \text{Per}(\Omega)\right),$$

where $E_\epsilon(\chi; F_\lambda) := E_{el}(\chi; F_\lambda) + \epsilon E_{surf}(\chi)$. Using the lower bound

$$\|f\|_{L^2}^2 = \int_{\mathbb{R}^d} |(1-\lambda)\hat{\chi}_A + \lambda\hat{\chi}_B|^2 d\xi = \int_{\mathbb{R}^d} |(1-\lambda)\chi_A + \lambda\chi_B|^2 dx \geq (\min\{\lambda, 1-\lambda\})^2 |\Omega|,$$

then implies that

$$\epsilon^{\frac{2}{3}} \leq C(E_\epsilon(\chi; F_\lambda) + \epsilon \text{Per}(\Omega)).$$

Finally, for $\epsilon \in (0, \epsilon_0)$ and $\epsilon_0 = \epsilon_0(\mathcal{A}(D), A, B, \Omega, d, \lambda) > 0$ sufficiently small, the perimeter contribution on the right hand side can be absorbed into the left hand side, which yields the desired result. $\square$

3.4 The Super-Compatible Setting: Proof of Theorem 1(iii) and Reduction to Cocanceling Operators

In this subsection we will show that if we are in the setting in which the estimates of the previous subsection degenerate, i.e., $\mathbb{A}(\xi)(A - B) = 0$ for all $\xi \in \mathbb{R}^d$, then in fact there can be no non-trivial bound from below. We will however also show that, in general for pairwise not super-compatible wells, it is possible to reduce to an equivalent minimization problem in the setting of cocanceling operators for suitably modified boundary data.

Proof of the super-compatible case in Theorem 1 It suffices to give an upper bound construction with zero total energy. To this end, we consider $\chi = A\chi_\Omega$ and $u = A\chi_\Omega + F_\lambda \chi_{\mathbb{R}^d \setminus \bar{\Omega}}$ and observe that

$$\mathcal{A}(D)u = \mathcal{A}(D)(u - B) = \mathcal{A}(D)[(\chi_\Omega + \lambda \chi_{\mathbb{R}^d \setminus \bar{\Omega}})(A - B)] = 0.$$

As a consequence, u is admissible in the definition of the elastic energy and the elastic energy vanishes. Moreover, since $\chi = A$ in Ω we also have the vanishing of the surface energy. This concludes the argument. $\square$

We will show that for two not super-compatible wells, we can always assume that $I_A = \{0\}$, in which case we work in the class of *cocanceling* operators introduced by Van Schaftingen in [65].

Proposition 3.8 *Let $n, d, k \in \mathbb{N}$, let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $\mathcal{A}(D)$ a differential operator of order k as in (2) with I_A given in (8). For $\chi \in L^2(\Omega; \mathcal{K})$ for some compact set of states $\mathcal{K} \subset \mathbb{R}^n$ let $E_{el}(\chi; F)$ be as in (9) for $F \in \mathcal{K}^{qc}$, cf. Sect. 2. Then for the restricted operator $\tilde{\mathcal{A}}(D) : C^\infty(\mathbb{R}^d; I_A^\perp) \rightarrow C^\infty(\mathbb{R}^d; \mathbb{R}^m)$ there exists a function $\chi_\perp \in L^2(\Omega, \Pi_{I_A^\perp} \mathcal{K})$ such that*

$$E_{el}(\chi; F) = E_{el}^{\tilde{\mathcal{A}}}(\chi_\perp; F_\perp) = \inf_{u_\perp \in \mathcal{D}_{F_\perp}^{\tilde{\mathcal{A}}}} \int_{\Omega} |u_\perp - \chi_\perp|^2 dx.$$

Here we denote the orthogonal projection of F onto $I_A^\perp$ by $F_\perp$.

Proof We use the orthogonal decomposition $\mathbb{R}^n = I_A^\perp \oplus I_A$ to write

$$u = u_\perp + u_I, \quad \chi = \chi_\perp + \chi_I,$$

with $u_\perp : \mathbb{R}^d \rightarrow I_A^\perp$, $u_I : \mathbb{R}^d \rightarrow I_A$, $\chi_\perp : \Omega \rightarrow I_A^\perp$, $\chi_I : \Omega \rightarrow I_A$. By orthogonality we can also split the elastic energy

$$E_{el}(u, \chi) = \int_{\Omega} |u - \chi|^2 dx = \int_{\Omega} |u_\perp - \chi_\perp|^2 dx + \int_{\Omega} |u_I - \chi_I|^2 dx.$$

Defining the restricted operator $\tilde{\mathcal{A}}(D) : C^\infty(\mathbb{R}^d; I_A^\perp) \rightarrow C^\infty(\mathbb{R}^d; \mathbb{R}^m)$, $\tilde{\mathcal{A}}(D)u_\perp := \mathcal{A}(D)u_\perp$ and the restricted space of admissible functions as in (10)

$$\mathcal{D}_{F_\perp}^{\tilde{\mathcal{A}}} := \{u_\perp \in L_{loc}^2(\mathbb{R}^d; I_A^\perp) : \tilde{\mathcal{A}}(D)u_\perp = 0 \text{ in } \mathbb{R}^d, u_\perp = F_\perp \text{ in } \mathbb{R}^d \setminus \bar{\Omega}\},$$

we see that for $u \in \mathcal{D}_F$ it holds $u_\perp \in \mathcal{D}_{F_\perp}^{\tilde{\mathcal{A}}}$, with $F_\perp = \Pi_{I_A^\perp} F$, and $u_I = F - F_\perp$ outside Ω .

Thus, after minimizing the elastic energy in u , it holds

$$\inf_{u \in \mathcal{D}_F} E_{el}(u, \chi) = \inf_{u_\perp \in \mathcal{D}_{F_\perp}^{\tilde{A}}} \inf_{u_I \in L_{loc}^2(\mathbb{R}^d; I_{\mathcal{A}}), u_I = F - F_\perp \text{ in } \tilde{\Omega}^c} \int_{\Omega} |u_\perp - \chi_\perp|^2 dx + \int_{\Omega} |u_I - \chi_I|^2 dx.$$

As we have seen in the proof of Theorem 1(iii), the second term involving u_I vanishes and hence,

$$E_{el}(\chi; F) = E_{el}^{\tilde{A}}(\chi_\perp; F_\perp).$$

This reduces the elastic energy to the case of a cocanceling operator as indeed $I_{\tilde{A}} = \{0\}$. $\square$

As the surface energy does not depend on the operator $\mathcal{A}(D)$, this result yields:

$$E_\epsilon(\chi; F) = E_{el}^{\tilde{A}}(\chi_\perp; F_\perp) + \epsilon E_{surf}(\chi).$$

As a corollary, we apply this to the N -well problem:

Corollary 3.9 (Finitely many, pairwise not super-compatible wells) *Under the same assumptions as in Proposition 3.8, with $\chi \in BV(\Omega; \mathcal{K})$, for the special case that $\mathcal{K} := \{B_1, \dots, B_N\} \subset \mathbb{R}^n$ for $N \in \mathbb{N}$ such that B_j , $j \in \{1, \dots, N\}$, are pairwise not super-compatible, i.e. $B_i - B_j \notin I_{\mathcal{A}}$ for $i \neq j$, there exists a constant $C = C(B_1, \dots, B_N) > 1$ such that*

$$C^{-1} E_{surf}(\chi_\perp) \leq E_{surf}(\chi) \leq C E_{surf}(\chi_\perp).$$

In particular, it hence holds that

$$E_\epsilon(\chi; F) \sim E_\epsilon^{\tilde{A}}(\chi_\perp; F_\perp).$$

Proof Writing $\chi = \sum_{j=1}^N B_j \chi_{\Omega_j}$ with $\chi_{\Omega_j} \in BV(\Omega; \{0, 1\})$, $\sum_{j=1}^N \chi_{\Omega_j} = 1$ in Ω , we can calculate

$$\begin{aligned} |\nabla \chi| &= \sum_{i < j} |B_i - B_j| \mathcal{H}^{d-1}(\partial^* \Omega_i \cap \partial^* \Omega_j), \\ |\nabla \chi_\perp| &= \sum_{i < j} |(B_i - B_j)_\perp| \mathcal{H}^{d-1}(\partial^* \Omega_i \cap \partial^* \Omega_j), \end{aligned}$$

where we denote the reduced boundary of a set E with finite perimeter by $\partial^* E$ and used the notation from above for $B \in \mathbb{R}^n$ to write $B_\perp = \Pi_{I_{\mathcal{A}}^\perp} B$.

By assumption, for $i < j$ it holds $B_i - B_j \notin I_{\mathcal{A}}$, and therefore also the projection satisfies $(B_i - B_j)_\perp \neq 0$. This implies $|(B_i - B_j)_\perp| > 0$ for all tuples (i, j) such that $i < j$ and hence there is a constant $0 < C^{-1} < \frac{|B_i - B_j|}{|(B_i - B_j)_\perp|} < C$ such that

$$0 < C^{-1} |\nabla \chi_\perp| \leq |\nabla \chi| \leq C |\nabla \chi_\perp|.$$

This together with Proposition 3.8 concludes the proof. $\square$

We emphasize that Corollary 3.9 in particular holds in the context of Theorem 1. Therefore in the statement of Theorem 1(ii) we can assume without loss of generality that $I_{\mathcal{A}} = \{0\}$. In fact, note that in the crucial bound of Corollary 3.2 we have

$$\mathbb{A}(\xi)(A - B) = \mathbb{A}(\xi)(A_{\perp} - B_{\perp}) = \tilde{\mathbb{A}}(\xi)(A_{\perp} - B_{\perp})$$

for $|\xi| = 1$.

3.5 Some Remarks on the Compatible Two-State Problem for Higher Order Operators

We conclude our discussion of lower scaling bounds by commenting on the case of the compatible two-well problem for higher order operators. Here the situation is still less transparent, yet some remarks are possible.

Indeed, on the one hand, it is known that, in general, for operators of order $k \geq 2$ the two-well problem does *not* have to scale with $\epsilon^{\frac{2}{3}}$. In order to illustrate this, we consider the specific operator $\mathcal{A}(D) := \text{curl curl}$. This operator is the annihilator of the symmetrized gradient $e(v) := \frac{1}{2}(\nabla v + (\nabla v)^t)$. We consider the following quantitative two-state problem (for $d = 2$)

$$\begin{aligned} \mathcal{E}_{\epsilon}(v, \chi) &:= \mathcal{E}_{el}(v, \chi) + \epsilon \mathcal{E}_{surf}(\chi) \\ &:= \int_{[0,1]^2} \left| e(v) - \begin{pmatrix} 1 & 0 \\ 0 & 1 + \alpha(1 - 2\chi) \end{pmatrix} \right|^2 dx + \epsilon \int_{[0,1]^2} |\nabla \chi| \end{aligned} \quad (20)$$

with $e(v) \in \mathcal{D}_F^A$, $\chi \in BV([0, 1]^2; \{0, 1\})$, $\alpha \in (0, 1)$ and study the corresponding minimization problem with prescribed boundary data $F := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Proposition 3.10 *Let $\mathcal{E}_{\epsilon}(v, \chi)$ and F be as in (20). Then there exists $\epsilon_0 = \epsilon_0(\alpha) > 0$ such that for $\epsilon \in (0, \epsilon_0)$ it holds that*

$$\inf_{\chi \in BV([0,1]^2; \{0,1\})} \inf_{e(v) \in \mathcal{D}_F^A} \mathcal{E}_{\epsilon}(v, \chi) \sim \epsilon^{\frac{4}{3}}.$$

We remark that this observation is not new; indeed, a geometrically nonlinear version of this had earlier been derived in [1, Theorem 1.2]. As observed in [1] the reason for the different scaling in Proposition 3.10 and [1, Theorem 1.2], compared to the more standard $\epsilon^{\frac{2}{3}}$ behavior from Theorem 1, consists of the higher degeneracy of the multiplier associated with the energy which is manifested in the presence of only one possible normal in the (symmetrized) rank-one condition. On the level of the multiplier this can be seen as $\mathbb{A}(\xi)(e_2 \otimes e_2) = \xi_1^2$, opposed to $\mathbb{A}(\xi)(\frac{e_1 \otimes e_2 + e_2 \otimes e_1}{2}) = -\xi_1 \xi_2$, has only one root of multiplicity two instead of two roots of single multiplicity on the unit sphere. For convenience of the reader and in order to illustrate the robustness of the above approach within geometrically linear theories, we present an alternative short proof (of the lower bound) of Proposition 3.10 based on our Fourier theoretic framework. We note that in the geometrically linear setting this provides an alternative to the approach from [1] in which the lower bound for the energy is deduced by a local “averaging” argument, considering the energy on representative domain patches with the expected scaling behavior.

Proof *Step 1: Lower bound.* We note that the lower bound for this setting directly follows from our arguments above: Indeed, for $A - B = 2\alpha e_2 \otimes e_2$, we obtain that

$$\mathbb{A}(\xi)(B - A) = 2\alpha \xi \times (\xi \times (e_2 \otimes e_2))^t = 2\alpha \xi_1^2.$$

With this in hand, an analogous argument as in Lemma 3.7 and, in particular, in (18) implies that

$$\|\hat{\chi}\|_{L^2(\{\xi \in \mathbb{R}^2: |k_2| \leq \mu\})}^2 \leq C\mu^4 \mathcal{E}_{el}(\chi; F), \quad (21)$$

where we have used that in this situation the multiplier is given by $m(\xi) = \mathbb{A}(\xi)(B - A) \sim \xi_1^2$. The different exponent of μ in (21) (compared to the one from Lemma 3.7(a)) is a consequence of the degeneracy of the symbol $m(\xi)$ and the higher order of the operator curl curl (or put, more concretely, the quadratic dependence ξ_1^2). Hence, replacing the bound from Lemma 3.7(a) by the one from (21) and carrying out the splitting as in the proof of Theorem 1, we obtain the following optimization problem: For $f := 1 + \alpha(1 - 2\chi)$

$$\begin{aligned} (1 - \alpha)^2 &\leq \|f\|_{L^2}^2 \leq \|\chi_{\{|\xi| \geq \mu\}}(D)f\|_{L^2}^2 + \|\chi_{\{|\xi| \leq \mu\}}(D)f\|_{L^2}^2 \\ &\leq C(\mu^4 \mathcal{E}_{el}(\chi; F) + (\mu^{-1}\epsilon^{-1})\epsilon \mathcal{E}_{surf}(\chi) + \mu^{-1} \text{Per}(\Omega)). \end{aligned}$$

Choosing $\mu \sim \epsilon^{-\frac{1}{3}}$ and rearranging the estimates then imply the claim.

Step 2: Upper bound. The associated improved upper bound makes use of the vectorial structure of the problem in contrast to the essentially scalar “standard $\epsilon^{2/3}$ construction” (see the arguments below). Recalling that the nonlinear construction from the proof of [1, Lemma 2.1] also yields a construction with the desired scaling for the geometrically linearized problem, we do not carry out the details of this but refer to [1, Lemma 2.1] for these. $\square$

On the other hand, the arguments from [1, 73, 74, 81] show that still for $\mathcal{A}(D) = \text{curl curl}$ if $A - B = \gamma(e_1 \otimes e_2 + e_2 \otimes e_1)$ for $\gamma \in \mathbb{R} \setminus \{0\}$, then one recovers the $\epsilon^{2/3}$ scaling for the symmetrized gradient differential inclusion. In this case, the symbol reads $m(\xi) = \mathbb{A}(\xi)(B - A) \sim \xi_1 \xi_2$ and the operator is “less degenerate”.

We expect that the scaling behavior of general higher order operators is in many interesting settings directly linked to the degeneracy of the symbol $\mathbb{A}(\xi)(B - A)$. We plan to explore this in future work.

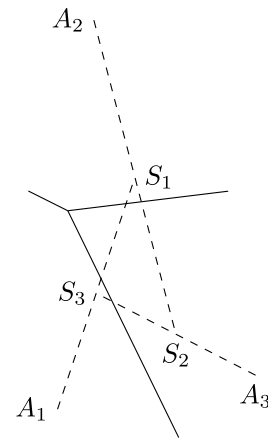
4 Quantitative Rigidity of the T_3 Structure from (13), (14) for $\mathcal{A}(D) = \text{div}$

In this section, we consider the T_3 structure for the divergence operator introduced in (13), (14). The upper bound construction is given by an approximate solution of the type described in the introduction. The lower bound is motivated by the rigidity of exact solutions as outlined in Sect. 4.3.1.

4.1 The Upper Bound Construction – an Infinite Order Laminate

To begin with, we construct an infinite order laminate similar to the one for the Tartar square (cf. [2, 82, 83] for quantitative versions of this). This is based on [3] and will yield the

Fig. 1 The diagonal matrices $A_1, A_2, A_3, S_1, S_2, S_3$ with the dashed lines depicting the connections in the wave cone for $\mathcal{A}(D) = \text{div}$ and the Voronoi-regions of A_i shown by the lines. As shown in [21] the set $\mathcal{K}^{qc}$ is given by the inner triangle formed by $S_1 S_2 S_3$ and the “legs” connecting S_j and A_j for $j \in \{1, 2, 3\}$



upper bound estimate from Theorem 2. We recall that for the divergence operator, instead of requiring rank-one connectedness for the existence of a laminate as for the curl, in our three-dimensional set-up we need rank-one or rank-two connectedness as can be seen from the wave cone for the divergence operator, cf. (5). As a consequence, for two matrices $A, B \in \mathbb{R}^{3 \times 3}$ such that $\text{rank}(B - A) \leq 2$ there exists a piecewise constant map $u : \mathbb{R}^3 \rightarrow \mathbb{R}^{3 \times 3}$ such that $u \in \{A, B\}$ a.e. in Ω and $\text{div } u = 0$. The lamination can be done in any direction of the kernel $\ker(B - A) \neq \{0\}$.

Considering now the matrices A_1, A_2, A_3 given in (14), we observe that $\text{rank}(A_i - A_j) = 3$ for $i \neq j$. Following [3], we introduce auxiliary matrices $S_1, S_2, S_3 \in \mathbb{R}^{3 \times 3}$.

$$S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad S_2 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (22)$$

It then holds for $i = 1, 2, 3$ that

$$\ker(S_i - A_i) = \text{span}(e_i), \quad S_i = \frac{1}{2}(A_{i+1} + S_{i+1}),$$

where $A_4 = A_1, S_4 = S_1$. As proved in [21, Theorem 2], the $\mathcal{A}$ -quasi-convex hull $\{A_1, A_2, A_3\}^{qc}$ can be explicitly characterized as the convex hull of the matrices S_1, S_2, S_3 together with the “legs” given by the line segments $A_j S_j$ for $j \in \{1, 2, 3\}$, cf. Fig. 1.

For simplicity and definiteness, we first assume, that $u = F = S_3$ outside $\Omega = [0, 1]^3$. In this setting we prove the following energy estimate:

Proposition 4.1 *Let $\Omega = [0, 1]^3, \mathcal{K} = \{A_1, A_2, A_3\}$ for the particular choice of matrices in (14) and let E_ϵ be as in (12) and $\epsilon \in (0, 1)$. Then for any $r \in (0, \frac{1}{4})$ with $r^{-1} \in 4\mathbb{N}$ there are sequences $u^{(m)} \in W^{1,\infty}(\mathbb{R}^3; \mathbb{R}^{3 \times 3})$, such that $\text{div } u^{(m)} = 0$, $u^{(m)} = F := S_3$ outside $[0, 1]^3$, and $\chi^{(m)} \in BV(\mathbb{R}^3; \mathcal{K})$, and a constant $C = C(F) > 1$ with*

$$E_\epsilon(u^{(m)}, \chi^{(m)}) \leq C \left(2^{-m} + \sum_{k=1}^m 2^{-k} r + r + \epsilon \frac{1}{r^m} \right).$$

In order to achieve this, in the next subsections, we iteratively construct a higher and higher order laminate (depending on $\epsilon > 0$). As in the setting of the Tartar square, we keep track of the surface and elastic energy contributions which arise in this process.

4.2 Proof of the Upper Bound from Theorem 2

We split the proof of the upper bound from Theorem 2 into several steps which we will carry out in the next sections and then combine in Sect. 4.2.4.

First, we start by a simple lamination of A_1 and S_1 to obtain regions in which $u \in \mathcal{K}$ holds and to satisfy the exterior data condition $u = F = S_3 = \frac{1}{2}(A_1 + S_1)$ outside Ω . This is followed by a similar construction replacing S_1 by a lamination of A_2 and S_2 achieving a second order laminate. Iterating the procedure of replacing S_j by a lamination of A_{j+1} and S_{j+1} (with the convention that $A_4 = A_1$, $S_4 = S_1$) yields Proposition 4.1. Finally, we optimize the parameter r and the number of iterations depending on ϵ in order to show the desired upper bound estimate in Proposition 4.3.

As we will use a potential for the laminates, we will define a “profile-function” once in a more general form and will then refer to this in our construction for the higher order laminates.

Lemma 4.2 *Let $R = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \subset \mathbb{R}^3$ be an axis-parallel cuboid, then for any direction e_j with $j \in \{1, 2, 3\}$ and any scale $r > 0$ such that $\frac{b_j - a_j}{r} \in \mathbb{N}$ there is a continuous function $f_j(\cdot; R, r) : R \rightarrow \mathbb{R}^{3 \times 3}$ satisfying the following properties:*

- The function $f_j(\cdot; R, r)$ only depends on the j -th coordinate x_j and is r -periodic.
- It holds $f_j(x; R, r) = 0$ if $x \in R$ is such that $x = (x_1, x_2, x_3)$ lies in one of the planes characterized by $x_j \in a_j + r\mathbb{N}_0$.
- The matrix-valued function $\text{curl } f_j(x; R, r)$ only attains the values $\pm \frac{S_j - A_j}{2}$, where A_j, S_j is given as in (14) and in (22), respectively. Furthermore, the volumes of the level sets are equal, i.e. $|\{x \in R : f_j(x; R, r) = S_j\}| = |\{x \in R : f_j(x; R, r) = A_j\}| = \frac{1}{2}|R|$.

Proof Without loss of generality by a translation, we may assume that $R = [0, b_1] \times [0, b_2] \times [0, b_3]$ for $b_1, b_2, b_3 > 0$. We consider the continuous one-periodic extension of the function

$$h : [0, 1) \rightarrow \mathbb{R}, \quad h(t) := \begin{cases} \frac{1}{2}t & t \in [0, \frac{1}{2}), \\ \frac{1}{2}(1-t) & t \in [\frac{1}{2}, 1). \end{cases}$$

Furthermore, we define the matrices

$$M_1 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{2}{3} \\ 0 & 2 & 0 \end{pmatrix}, \quad M_2 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \quad M_3 := \begin{pmatrix} 0 & 1 & 0 \\ -\frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

satisfying $e_j \times M_j = S_j - A_j$. With this at hand we define $f_j(\cdot; R, r) : R \rightarrow \mathbb{R}^{3 \times 3}$:

$$f_j(x_1, x_2, x_3; R, r) := rh\left(\frac{x_j}{r}\right)M_j.$$

It follows directly, that $f_j(\cdot; R, r)$ is continuous, only depends on x_j , is r -periodic, and vanishes for $x_j \in r\mathbb{N}_0$. Lastly, we note that $\text{curl } f_j(x; R, r) = h'\left(\frac{x_j}{r}\right)e_j \times M_j \in \{\pm \frac{S_j - A_j}{2}\}$ and that indeed $|\{x \in R : f_j(x; R, r) = S_j\}| = |\{x \in R : f_j(x; R, r) = A_j\}| = \frac{1}{2}|R|$. $\square$

4.2.1 First Order Laminates

We use a potential $v : \mathbb{R}^3 \rightarrow \mathbb{R}^{3 \times 3}$ to construct our laminates attaining the prescribed exterior data, i.e. we consider the row-wise curl: $u = \text{curl } v$.

As $S_3 = \frac{1}{2}A_1 + \frac{1}{2}S_1$ and $(S_1 - A_1)e_1 = 0$ the first order lamination is in the e_1 -direction.

We seek to use Lemma 4.2 to construct v , but have to adapt the boundary condition. For this we define a mapping $\tilde{S}_3 : \mathbb{R}^3 \rightarrow \mathbb{R}^{3 \times 3}$ with $\text{curl } \tilde{S}_3 = S_3 = F$. A possible choice for $\tilde{S}_3$ is given by the following matrix-valued function

$$\tilde{S}_3(x) := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{3}x_1 \\ 0 & x_1 & 0 \end{pmatrix}.$$

Furthermore, we also define the cut-off function

$$\phi : \mathbb{R} \rightarrow [0, 1], \quad \phi(t) = \begin{cases} 0 & t < \frac{1}{8}, \\ 4t - \frac{1}{2} & t \in [\frac{1}{8}, \frac{3}{8}], \\ 1 & t > \frac{3}{8}. \end{cases}$$

With this, we define the (continuous) potential for $r \in (0, \frac{1}{4})$, $r^{-1} \in 4\mathbb{N}$ by using Lemma 4.2 for $j = 1$, $R = [0, 1]^3$:

$$v^{(1)} : \Omega \rightarrow \mathbb{R}^{3 \times 3},$$

$$v^{(1)}(x) = \phi\left(\frac{1}{r}d_{\partial\Omega}(x)\right)f_1(x; \Omega, r) + \tilde{S}_3(x),$$

where $d_{\partial\Omega}(x)$ denotes a smoothed-out distance function to the boundary $\partial\Omega$. Without change of notation, we consider the (continuous) extension of $v^{(1)}$ to $\mathbb{R}^3$ by $\tilde{S}_3(x)$, which is possible, as $v^{(1)}(x) = \tilde{S}_3(x)$ on $\partial\Omega$.

We then set

$$u^{(1)} := \text{curl } v^{(1)},$$

and note that in Ω it holds

$$\begin{aligned} u^{(1)}(x) &= \text{curl } v^{(1)}(x) \\ &= \phi'\left(\frac{d_{\partial\Omega}(x)}{r}\right)h\left(\frac{x_1}{r}\right)\nabla d_{\partial\Omega}(x) \times M_1 + \phi\left(\frac{d_{\partial\Omega}(x)}{r}\right)\text{curl } f_1(x; \Omega, r) + F \\ &= \phi'\left(\frac{d_{\partial\Omega}(x)}{r}\right)h\left(\frac{x_1}{r}\right)\nabla d_{\partial\Omega}(x) \times M_1 + \phi\left(\frac{d_{\partial\Omega}(x)}{r}\right)h'\left(\frac{x_1}{r}\right)(S_1 - A_1) + \frac{1}{2}(S_1 + A_1). \end{aligned}$$

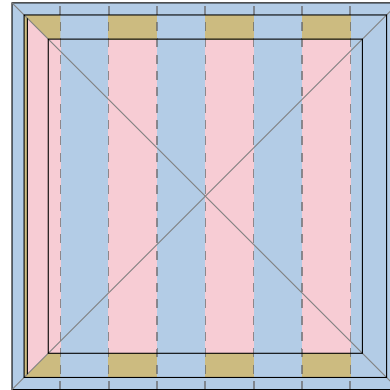
With these considerations, we have obtained the following properties: It holds $\text{div } u^{(1)} = \text{div}(\text{curl } v^{(1)}) = 0$ in $\mathbb{R}^3$, $u^{(1)}(x) = F$ in $\mathbb{R}^3 \setminus \overline{\Omega}$ and for $x \in \Omega \setminus d_{\partial\Omega}^{-1}([0, \frac{3}{8}r])$ we have $u^{(1)} \in \{A_1, S_1\}$. In other words, our deformation $u^{(1)}$ is a divergence-free function satisfying the desired boundary conditions and which, outside of the cut-off region, is a solution to the differential inclusion $u^{(1)} \in \{A_1, S_1\}$.

With the higher order laminates in mind we rephrase this using the decomposition of Ω into the three disjoint parts consisting of the S_1 -cells, the A_1 -cells and the cut-off region. To be more precise, we define

$$R^{(1)} := \{x \in \Omega : \phi\left(\frac{d_{\partial\Omega}(x)}{r}\right) = 1, u^{(1)} = S_1\},$$

$$Q^{(1)} := \{x \in \Omega : \phi\left(\frac{d_{\partial\Omega}(x)}{r}\right) = 1, u^{(1)} = A_1\},$$

Fig. 2 The $x_3 = \frac{1}{2}$ slice of the projection $\chi^{(1)}$, blue represents A_1 , red A_2 and green A_3 . (Color figure online)



$$C^{(1)} := \{x \in \Omega : \phi\left(\frac{d_{\partial\Omega}(x)}{r}\right) < 1\}.$$

Indeed it holds $\Omega = R^{(1)} \cup Q^{(1)} \cup C^{(1)}$ and $|C^{(1)}| = 6\frac{3}{8}r$ and by Lemma 4.2 we know $|R^{(1)}| \leq \frac{1}{2}|\Omega| = \frac{1}{2}$.

Choosing $\chi^{(1)} = \Pi_{\mathcal{K}} u^{(1)}$ as the pointwise orthogonal (with a fixed choice for the not uniquely defined points) projection of $u^{(1)}$ onto $\mathcal{K}$, see Fig. 2 for an illustration, up to a uniformly bounded constant, the elastic energy can be bounded by the measure of the region in which $u^{(1)}(x) = S_1$ and the cut-off region:

$$\begin{aligned} E_{el}(u^{(1)}, \chi^{(1)}) &= \int_{\Omega} |u^{(1)} - \chi^{(1)}|^2 dx \leq C(|R^{(1)}| + |C^{(1)}|) \\ &\leq C\left(\frac{1}{2}|\Omega| + 6\frac{3r}{8}\right) \leq C\left(\frac{1}{2} + 3r\right). \end{aligned}$$

Furthermore, the surface energy is bounded by counting the interfaces at which $\chi^{(1)}$ may jump. This consists of at most $\frac{2}{r}$ interfaces in the interior and at most $\frac{2}{r} + 2 \cdot 6 \leq \frac{8}{r}$ new interfaces in the cut-off region. For $r \in (0, \frac{1}{4})$, the surface area is thus controlled by

$$E_{surf}(\chi^{(1)}) = \int_{\Omega} |\nabla \chi^{(1)}| \leq C \frac{10}{r}.$$

4.2.2 Second Order Lamination

After this first order lamination, the differential inclusion $u \in \mathcal{K}$ with $\operatorname{div} u = 0$ holds only in $Q^{(1)}$. In order to further reduce the energy, we now replace each of the $\frac{1}{r}$ many cuboids in $R^{(1)}$ for which $u^{(1)} = S_1 = \frac{1}{2}A_2 + \frac{1}{2}S_2$ by a lamination in the e_2 -direction. For this, we modify the potential $v^{(1)}$ in these regions:

For $x \in R^{(1)}$ and for $r^2 \in (0, \frac{r}{2})$, $r^{-2} + \frac{3}{4r} \in \mathbb{N}$, we define with the help of Lemma 4.2

$$v^{(2)}(x) = \phi\left(\frac{d_{\partial R^{(1)}}(x)}{r^2}\right) f_2(x; R^{(1)}, r^2) + f_1(x; \Omega, r) + \tilde{S}_3(x)$$

and $v^{(2)}(x) = v^{(1)}(x)$ else. Here, and in the following, we use the notation $f_j(x; R^{(1)}, r)$ for $R^{(1)}$ which is not a cuboid but an union of disjoint cuboids and mean depending on x the corresponding connected cuboid. By this $v^{(2)}$ defines a continuous map, as inside $R^{(1)}$

the cut-off attains the constant value one and therefore $v^{(1)}(x) = f_1(x; \Omega, r) + \tilde{S}_3(x)$ for $x \in \partial R^{(1)}$. By construction, the map $u^{(2)} := \text{curl } v^{(2)}$ is divergence free, i.e. the interfaces are compatible.

As in the construction for first order laminates we set $\chi^{(2)} = \Pi_{\mathcal{K}} u^{(2)}$, and define the sets

$$R^{(2)} := \{x \in R^{(1)} : \phi(\frac{d_{\partial R^{(1)}}(x)}{r^2}) = 1, u^{(2)} = S_2\},$$

$$Q^{(2)} := \{x \in R^{(1)} : \phi(\frac{d_{\partial R^{(1)}}(x)}{r^2}) = 1, u^{(2)} = A_2\},$$

$$C^{(2)} := \{x \in R^{(1)} : \phi(\frac{d_{\partial R^{(1)}}(x)}{r^2}) < 1\}.$$

Then indeed again we have the decomposition $R^{(1)} = R^{(2)} \cup Q^{(2)} \cup C^{(2)}$ with $|R^{(2)}| \leq \frac{1}{2}|R^{(1)}|$ and $|C^{(2)}| \leq \frac{1}{r} \frac{3}{8} r^2 6$ as the volume of the cut-off region can be bounded by the number of cells in $R^{(1)}$ times $\frac{3}{8} r^2$ times six times the area of the biggest face.

As the elastic energy vanishes in $Q^{(2)}$, we obtain

$$\begin{aligned} E_{el}(u^{(2)}, \chi^{(2)}) &= \int_{\Omega} |u^{(2)} - \chi^{(2)}|^2 dx \leq C(|R^{(2)}| + |C^{(2)}| + |C^{(1)}|) \\ &\leq C\left(\frac{1}{4} + 3r + \frac{3}{2}r\right). \end{aligned}$$

Indeed, this follows from the fact that we have improved our deformation in half the volume of the region in which $u^{(1)} \notin \mathcal{K}$ but have added a new cut-off region in each cuboid in which we do the second order lamination. For the surface energy it holds

$$E_{surf}(\chi^{(2)}) \leq C\left(\frac{10}{r} + \frac{1}{r} \frac{10}{r^2} r\right) = C\left(\frac{10}{r} + \frac{5}{r^2}\right),$$

as we add at most $\frac{4}{r^2} + 12 \leq \frac{10}{r^2}$ many new faces in each one of the $\frac{1}{r}$ many cuboids and as each surface has a surface area of size at most $\frac{r}{2}$.

4.2.3 Iteration: (m+1)-th Order

Without loss of generality, we assume that the $(m+1)$ -th order lamination will be in e_1 -direction, i.e. $m = 3j$ for some $j \in \mathbb{N}$. Else, we only have to adapt the corresponding roles of the directions.

We define iteratively the $(m+1)$ -th potential with the help of the sets $R^{(m)}$, for this we set (for given $v^{(m)}, u^{(m)} = \text{curl } v^{(m)}$)

$$R^{(m)} := \{x \in R^{(m-1)} : \phi(\frac{d_{\partial R^{(m-1)}}(x)}{r^m}) = 1, u^{(m)} = S_3\},$$

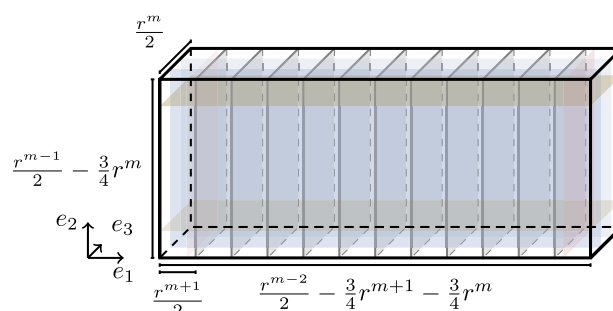
$$Q^{(m)} := \{x \in R^{(m-1)} : \phi(\frac{d_{\partial R^{(m-1)}}(x)}{r^m}) = 1, u^{(m)} = A_3\},$$

$$C^{(m)} := \{x \in R^{(m-1)} : \phi(\frac{d_{\partial R^{(m-1)}}(x)}{r^m}) < 1\}.$$

Inside $R^{(m)}$ we then define $v^{(m+1)}$ by

$$v^{(m+1)}(x) = \phi(\frac{d_{\partial R^{(m)}}(x)}{r^{m+1}}) f_1(x; R^{(m)}, r^{m+1}) + \sum_{k=1}^m f_{[k]}(x; R^{(k-1)}, r^k) + \tilde{S}_3(x),$$

Fig. 3 One S_3 -cell with the corresponding lamination in e_1 -direction. In green/red/orange the inner boundary of the cut-off region is depicted. (Color figure online)



$$[k] = \begin{cases} 1 & k \equiv 1 \pmod{3}, \\ 2 & k \equiv 2 \pmod{3}, \\ 3 & k \equiv 0 \pmod{3}, \end{cases}$$

and $v^{(m+1)}(x) = v^{(m)}(x)$ else for $x \notin R^{(m)}$. By induction we see that $v^{(m+1)}$ is continuous, that for $x \in R^{(m)}$ such that $\phi(\frac{d_{\partial R^{(m)}}(x)}{r^{m+1}}) = 1$ it holds that $u^{(m+1)} := \text{curl } v^{(m+1)} \in \{A_1, S_1\}$ and that we have the decomposition $R^{(m-1)} = R^{(m)} \cup Q^{(m)} \cup C^{(m)}$ with $|R^{(m)}| \leq \frac{1}{2}|R^{(m-1)}|$, $|C^{(m)}| \leq C2^{-m} \frac{r^m}{r^{m-1}} = C2^{-m}r$ for a universal constant $C > 0$ independent of m, r . This follows from the fact, that the number of cuboids in $R^{(m)}$ is bounded by $C \frac{2^{-m}}{r^{m-1}r^{m-2}r^{m-3}}$ and that the biggest face of each cuboid has an area of at most $r^{m-2}r^{m-3}$.

As the previous cut-offs still contribute to the total energy, we obtain the following elastic energy bound ($\chi^{(m+1)} := \Pi_{\mathcal{K}} u^{(m+1)}$)

$$E_{el}(u^{(m+1)}, \chi^{(m+1)}) \leq C(|R^{(m+1)}| \cup \bigcup_{k=1}^{m+1} |C^{(k)}|) = C \left(2^{-m} + r + \sum_{j=0}^{m+1} 2^{-j}r \right).$$

This bound resembles the (iterative) decomposition of $\Omega = R^{(1)} \cup Q^{(1)} \cup C^{(1)} = R^{(m+1)} \cup \bigcup_{k=1}^{m+1} Q^{(k)} \cup \bigcup_{k=1}^{m+1} C^{(k)}$ and the fact that $u^{m+1} = \chi^{(m+1)}$ in $\bigcup_{k=1}^{m+1} Q^{(k)}$.

For the surface energy, we again calculate the new contribution of the next order lamination and then sum over all previous ones. Each new face has a surface area of at most $\frac{r^m}{2} \frac{r^{m-1}}{2}$. In each S_3 -cell, we add $\frac{r^{m-2}}{r^{m+1}}$ new faces in the lamination and at most $\frac{r^{m-2}}{r^{m+1}} + 12$ ones for the cut-off, cf. Fig. 3. Since we have at most $C \frac{2^{-m-1}}{r^m r^{m-1} r^{m-2}}$ new cells the surface energy is increased by

$$C2^{-m} \frac{8}{r^m r^{m-1} r^{m-2}} \left(\frac{r^{m-2}}{r^{m+1}} + (\text{surf. in cut-off}) \right) \frac{r^m}{2} \frac{r^{m-1}}{2} \leq C2^{-m} \frac{1}{r^{m+1}},$$

yielding the following overall surface energy bound:

$$E_{surf}(\chi^{(m+1)}) \leq C \left(\sum_{j=1}^{m+1} 2^{-j}r^{-j} \right) \leq \frac{C}{r^{m+1}}.$$

For the total energy this implies

$$E_{\epsilon}(u^{(m+1)}, \chi^{(m+1)}) \leq C \left(2^{-m} + r + \sum_{j=1}^m 2^{-j}r + \epsilon \frac{1}{r^{m+1}} \right).$$

With this we have shown the claimed upper bound and have thus concluded the proof of Proposition 4.1. $\square$

4.2.4 Combining the Estimates: Proof of the Upper Bound in Theorem 2

With the previous construction in hand, we conclude the upper estimate from Theorem 2:

Proposition 4.3 *Let $\Omega = [0, 1]^3$, $\mathcal{K} = \{A_1, A_2, A_3\}$ for $A_1, A_2, A_3 \in \mathbb{R}^{3 \times 3}$ be given in (14), let $\epsilon \in (0, 1)$, and let E_ϵ be as in (12), $F \in \mathcal{K}^{qc} \setminus \mathcal{K}$ and $\mathcal{D}_F$ given in (10). Then there are constants $c > 0$ and $C > 1$, only depending on the boundary data F , such that*

$$\inf_{u \in \mathcal{D}_F} \inf_{\chi \in BV(\Omega; \mathcal{K})} E_\epsilon(u, \chi) \leq C \exp(-c |\log \epsilon|^{\frac{1}{2}}).$$

Proof *Step 1: Conclusion of the argument for $F = S_3$.* Using the sequences constructed in the construction from Proposition 4.1, implies

$$E_\epsilon(u^{(m)}, \chi^{(m)}) \leq C \left(2^{-m} + \sum_{k=1}^m 2^{-k} r + r + \epsilon r^{-m} \right) \leq C (2^{-m} + r + \epsilon r^{-m}).$$

Optimizing the value of r depending on $\epsilon > 0$, we require that $r \sim \epsilon^{\frac{1}{m+1}}$. Finally, we balance the resulting contributions and seek for the optimal number of iterations m . This is given by $2^{-m} \sim \epsilon^{\frac{1}{m+1}}$, that is $m \sim |\log \epsilon|^{\frac{1}{2}}$.

Plugging this into the upper bound results in

$$E_\epsilon(u^{(m)}, \chi^{(m)}) \leq C \exp\left(-\log(2) |\log(\epsilon)|^{\frac{1}{2}}\right).$$

This concludes the proof for the special case $F = S_3$.

Step 2: Conclusion of the argument for a general boundary datum. The situation of other boundary data $F \in \mathcal{K}^{qc}$ can be reduced to the one from Proposition 4.3 by at most two further iterations. Indeed, a general matrix $F \in \mathbb{R}^{3 \times 3}$ in the convex hull of S_1, S_2, S_3 , can be represented as

$$F = \lambda F_1 + (1 - \lambda) F_2 = \lambda(v_1 A_j + (1 - v_1) S_j) + (1 - \lambda)(v_2 A_k + (1 - v_2) S_k)$$

with $\lambda, v_1, v_2 \in [0, 1]$, $j, k \in \{1, 2, 3\}$ and $k \neq j$. Hence, after two additional iterations compared to the argument from above, we arrive at similar iterative procedures as in the previous subsections. In case that F is an element of one of the legs $S_j A_j$ a single iteration suffices to reduce the situation to the above argument. This proves the result for a general boundary condition $F \in \mathcal{K}^{qc} \setminus \mathcal{K}$. $\square$

4.3 Proof of the Lower Bound

In this section, we present the proof of the lower bound from Theorem 2. To this end, similarly as in [2], we mimic and quantify the analogous argument from the stress-free setting which we briefly recall in the following Sect. 4.3.1 and for which we will provide a number of auxiliary results in Sect. 4.3.2. The main argument, given in Sect. 4.3.3, will then consist of a bootstrap strategy, similar to [2], in which we iteratively reduce the possible regions of mass concentration in Fourier space.

Contrary to the previous section, in what follows we will work in a periodic set-up. Since the energy contributions on periodic functions provides a lower bound on the energy contributions of functions with prescribed Dirichlet boundary conditions, we hence also obtain the desired lower bound for the setting of Dirichlet boundary conditions. Indeed, for the elastic energy this is immediate; for the surface energy there is at most an increase by a fixed factor (see the discussion in Lemma 4.5 in Sect. 4.3.2).

4.3.1 The Stress-Free Argument

We begin by recalling the argument for the rigidity of the exact inclusion, as we will mimic this on the energetic level.

Proposition 4.4 *Let $u : [0, 1]^3 \rightarrow \mathbb{R}^{3 \times 3}$ be a solution to the differential inclusion (13)–(14). Then, there exists $j \in \{1, 2, 3\}$ such that $u \equiv A_j$ in $[0, 1]^3$.*

Proof In the exactly stress-free setting in which the differential inclusion is satisfied exactly, i.e. $u \in \{A_1, A_2, A_3\}$, the observation that $\operatorname{div} u = 0$ and that u is a diagonal matrix leads to the following three equations

$$\partial_1 u_{11} = 0, \quad \partial_2 u_{22} = 0, \quad \partial_3 u_{33} = 0,$$

where u_{jj} denote the diagonal components of the matrix u . As a consequence,

$$u_{11} = f_1(x_2, x_3), \quad u_{22} = f_2(x_1, x_3), \quad u_{33} = f_3(x_1, x_2).$$

Next we note that the values of u_{jj} determine the ones for u_{kk} if $j \neq k$, i.e. there are functions $h_{k,j}$ such that $h_{k,j}(u_{jj}) = u_{kk}$. Hence, comparing the functions u_{11} and u_{22} , we first obtain that u_{11} and u_{22} can only be functions of x_3 . Indeed it holds

$$\partial_2 u_{11}(x) = \partial_2(h_{1,2}(u_{22}(x))) = \partial_2(h_{1,2}(f_2(x_1, x_3))) = 0, \quad (23)$$

and analogously $\partial_1 u_{22}(x) = 0$. Comparing this to u_{33} , we obtain that all three functions must be constant. Hence, any solution to the (exact) differential inclusion must be constant and u is equal to one of the three matrices A_1, A_2, A_3 globally. The exact problem is hence rigid. $\square$

Using the ideas from [2], we seek to turn this into a corresponding scaling result. The main difference that arises can be seen in the qualitative rigidity argument above: Instead of comparing only two diagonal entries like in [2], we have to compare twice to deduce that the map is constant. This will be seen in the quantitative argument for the lower bound below. Whereas in [2] there are cones around a single axis (the diagonal entries only depend on one variable), we consider cones around a plane (the diagonal entries depend on two variables). Furthermore, the bootstrap argument will be slightly modified as it resembles the comparison of the diagonal entries in the qualitative argument given above.

4.3.2 Reduction to the Periodic Setting and Auxiliary Results for the Elastic Energy

In this subsection, we provide a number of auxiliary results which we will exploit in the following bootstrap arguments for deducing the lower bound. As a first step, we reduce to the situation of periodic deformations.

Lemma 4.5 Let $\Omega = [0, 1]^3$, let $\mathcal{K} := \{A_1, A_2, A_3\}$ be as in (14) and $F \in \mathcal{K}^{qc} \setminus \mathcal{K}$. Let $E_\epsilon(u, \chi)$ be given by (15) and set

$$E_\epsilon^{per}(u, \chi) := \int_{\mathbb{T}^3} |u - \chi|^2 dx + \epsilon \int_{\mathbb{T}^3} |\nabla \chi|.$$

Let further $\mathcal{D}_F^{per} := \{u : \mathbb{R}^3 \rightarrow \mathbb{R}^{3 \times 3} : \operatorname{div} u = 0 \text{ in } \mathbb{R}^3, \langle u \rangle = F\}$, where $\langle u \rangle := \int_{\mathbb{T}^3} u(x) dx$. Assume that $E_\epsilon(u, \chi) \leq 1$ and that there is $\epsilon_0 > 0$ such that for any $v \in (0, \frac{1}{2})$ there is $c_v > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ it holds that

$$E_\epsilon^{per}(u, \chi) \geq \exp(-c_v |\log(\epsilon)|^{\frac{1}{2}+v}). \quad (24)$$

Then, there exists a constant $C > 1$ such that for $\tilde{\epsilon}_0 = \tilde{\epsilon}_0(v) > 0$ sufficiently small

$$C^{-1} \exp(-c_v |\log(\epsilon)|^{\frac{1}{2}+v}) \leq \inf_{\chi \in BV([0,1]^3; \mathcal{K})} \inf_{u \in \mathcal{D}_F} E_\epsilon(u, \chi)$$

for all $\epsilon \in (0, \tilde{\epsilon}_0)$.

Proof In order to infer the lower bound, we show that any function $u : [0, 1]^3 \rightarrow \mathbb{R}^3$ with constant boundary data can be associated with a suitable periodic function which has the boundary data of u as its mean value and satisfies related energy estimates. Indeed, for given $u \in \mathcal{D}_F$, we view it as a function on $\mathbb{T}^3$ by restriction. By the prescribed boundary data it still satisfies the differential constraint and further the mean value property. Moreover,

$$\inf_{u \in \mathcal{D}_F^{per}} E_{el}^{per}(u, \chi) := \inf_{u \in \mathcal{D}_F^{per}} \int_{\mathbb{T}^3} |u - \chi|^2 dx \leq \inf_{u \in \mathcal{D}_F} E_{el}(u, \chi).$$

Next, viewing $\chi : [0, 1]^3 \rightarrow \mathcal{K}$ as a periodic function $\tilde{\chi} : \mathbb{T}^3 \rightarrow \mathcal{K}$, we infer that

$$\int_{\mathbb{T}^3} |\nabla \tilde{\chi}| \leq \int_{[0,1]^3} |\nabla \chi| + C.$$

Now, due to (24), we obtain that

$$\exp(-c_v |\log(\epsilon)|^{\frac{1}{2}+v}) \leq \inf_{\chi \in BV(\mathbb{T}^3; \mathcal{K})} \inf_{u \in \mathcal{D}_F^{per}} E_\epsilon^{per}(u, \chi) \leq \inf_{\chi \in BV([0,1]^3; \mathcal{K})} \inf_{u \in \mathcal{D}_F} E_\epsilon(u, \chi) + C\epsilon.$$

For $\tilde{\epsilon}_0(v) > 0$ sufficiently small, the last right hand side term may thus be absorbed into the left hand side, yielding the desired result. $\square$

With this result in hand it suffices to consider the periodic set-up in the remainder of this section. This will, in particular, allow us to rely on the periodic Fourier transform in deducing lower bounds for the elastic energy. In what follows all (semi-)norms will thus be considered on the torus. With a slight abuse of notation, we will often omit this dependence.

Lemma 4.6 Let $F \in \mathcal{K}^{qc} \setminus \mathcal{K}$, where $\mathcal{K} = \{A_1, A_2, A_3\}$ with A_j in (14), and E_{el}^{per} and $\mathcal{D}_F^{per}$ be as in Lemma 4.5 and as in (10). Then, it holds for any $\chi \in L^2(\mathbb{T}^3; \mathcal{K})$ and for $E_{el}^{per}(\chi; F) := \inf_{u \in \mathcal{D}_F^{per}} \int_{\mathbb{T}^3} |u - \chi|^2 dx$

$$E_{el}^{per}(\chi; F) = \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \sum_{i=1}^3 \frac{k_i^2}{|k|^2} |\hat{\chi}_{i,i}|^2 + |\hat{\chi}(0) - F|^2,$$

where $\chi_{i,i}$ are the diagonal entries of χ and $\hat{\chi}$ is the (discrete) Fourier transform of χ .

Proof We first calculate $E_{el}^{per}(u, \chi)$ in Fourier space

$$E_{el}^{per}(u, \chi) = \int_{\mathbb{T}^3} |u - \chi|^2 dx = \sum_{k \in \mathbb{Z}^3} |\hat{u} - \hat{\chi}|^2,$$

which allows us to characterize minimizers of the elastic energy.

In order to minimize this elastic energy in $u \in \mathcal{D}_F^{per}$, $\hat{u}$ has to be the (pointwise) orthogonal projection of $\hat{\chi}$ onto the orthogonal complement of k , as the differential constraint $\operatorname{div} u = 0$ reads $i\hat{u}k = 0$ in Fourier space. Noting that the row-wise orthogonal projection of a matrix M onto $\operatorname{span}(k)$ can be written as $\Pi_k(M) = (M \frac{k}{|k|}) \otimes \frac{k}{|k|}$, the optimal $\hat{u}$ is given by

$$\hat{u} = \Pi_{k^\perp} \hat{\chi} = (\operatorname{Id} - \Pi_k) \hat{\chi} = \hat{\chi} - (\hat{\chi} \frac{k}{|k|}) \otimes \frac{k}{|k|}$$

for any $k \in \mathbb{Z}^3 \setminus \{0\}$.

Returning to our energy, this yields

$$\begin{aligned} E_{el}^{per}(\chi; F) &= E_{el}^{per}(u, \chi) = \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} |\hat{\chi} - (\hat{\chi} \frac{k}{|k|}) \otimes \frac{k}{|k|} - \hat{\chi}|^2 + |F - \hat{\chi}(0)|^2 \\ &= \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} |\hat{\chi} \frac{k}{|k|}|^2 + |\hat{\chi}(0) - F|^2 \\ &= \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \sum_{i=1}^3 \frac{k_i^2}{|k|^2} |\hat{\chi}_{i,i}|^2 + |\hat{\chi}(0) - F|^2 \end{aligned}$$

and shows the claim. $\square$

Next, following the ideas from [2], for $\mu, \lambda > 0$, $j = 1, 2, 3$, we introduce the cones

$$C_{j,\mu,\lambda} = \{k \in \mathbb{Z}^3 : |k_j| \leq \mu|k|, |k| \leq \lambda\}, \quad (25)$$

and their corresponding cut-off functions $m_{j,\mu,\lambda}(k) \in C^\infty(C_{j,2\mu,2\lambda} \setminus \{0\}; [0, 1])$ fulfilling $m_{j,\mu,\lambda} = 1$ on $C_{j,\mu,\lambda}$, $\operatorname{supp}(m_{j,\mu,\lambda}(k)) \subset C_{j,2\mu,2\lambda}$ and the decay properties in Marcinkiewicz's multiplier theorem (see, for instance, [84, Corollary 6.2.5]). The corresponding cut-off multiplier is thus defined by

$$m_{j,\mu,\lambda}(D)f = \mathcal{F}^{-1}(m_{j,\mu,\lambda}(\cdot)\hat{f}(\cdot)). \quad (26)$$

Furthermore, we use the following results which are shown in [2, Lemma 2, Lemma 3, and Corollary 1]. Following the conventions from [75], with slight abuse of notation compared to our setting in the first part of the article, in the whole following section, we now use d to denote the degree of some suitable polynomials and no longer the dimension of the ambient space which in the whole section is simply fixed to be equal to three.

Lemma 4.7 *Let $\beta, \delta, \mu, \lambda > 0$. Let $m_{i,\mu,\lambda}(D)$ denote the Fourier multipliers associated with the cones $C_{i,\mu,\lambda}$ for $i \in \{1, 2, 3\}$ as defined in (25) with the corresponding multipliers given*

in (26). Let $f_i \in L^\infty(\mathbb{T}^3) \cap BV(\mathbb{T}^3)$ for $i = 1, 2, 3$ and let $h_{j,i} : \mathbb{R} \rightarrow \mathbb{R}$ be nonlinear polynomials (of degree d) with $h_{j,i}(0) = 0$ such that $h_{j,i}(f_i) = f_j$ for $i \neq j$. If

$$\sum_{i=1}^3 \|\partial_i f_i\|_{\dot{H}^{-1}}^2 \leq \delta, \quad \sum_{i=1}^3 \|\nabla f_i\|_{TV} \leq \beta,$$

then there exist constants $C = C(h_{i,j}, \|f_i\|_\infty)$, $C' = C'(h_{i,j}, \|f_i\|_\infty, d)$, $C_0 = C_0(h_{i,j}, \|f_i\|_\infty, d) > 0$ such that for any $\gamma \in (0, 1)$ we have for any $i \neq j$

$$\sum_{k=1}^3 \|f_k - m_{k,\mu,\lambda}(D)f_k\|_{L^2}^2 \leq C(\mu^{-2}\delta + \lambda^{-1}\beta),$$

$$\|f_i - h_{i,j}(m_{j,\mu,\lambda}(D)f_j)\|_{L^2} \leq \frac{C'}{\gamma^{12d}} \|f_j - m_{j,\mu,\lambda}(D)f_j\|_{L^2}^{1-\gamma},$$

$$\|h_{i,j}(m_{j,\mu,\lambda}(D)f_j) - m_{i,\mu,\lambda}(D)f_i\|_{L^2}^2 \leq \frac{C_0}{\gamma^{24d}} \max\{(\mu^{-2}\delta + \lambda^{-1}\beta)^{1-\gamma}, \mu^{-2}\delta + \lambda^{-1}\beta\}.$$

Here we choose the constants such that $C_0 > 2C + 2C'^2 + 3$.

Let us comment on these bounds: The functions f_i are representing the diagonal entries of the phase indicator $\chi \in BV([0, 1]^3; \mathcal{K})$. Thus the first estimate corresponds to a first frequency localization by exploiting the surface energy control for the high frequencies and the ellipticity of the elastic energy away from the cones $C_{j,\mu,\lambda}$. It can be viewed as a quantified version of the statement that u_{11} is a function only depending on x_2, x_3 in Proposition 4.4. The second estimate is a commutator bound that arises from the nonlinear relation $h_{i,j}(f_j) = f_i$ for $i \neq j$. The third estimate combines the first two bounds. The second and third estimate will form the core tool to iteratively decrease the Fourier support of the characteristic functions of our phase indicators. We will detail this in the remainder of the article.

Remark 4.8 It is possible to make the mappings $h_{j,i}$ for $i, j \in \{1, 2, 3\}$, $i \neq j$ explicit for the choice of matrices A_1, A_2, A_3 , cf. (14). To this end, we may, for instance, consider

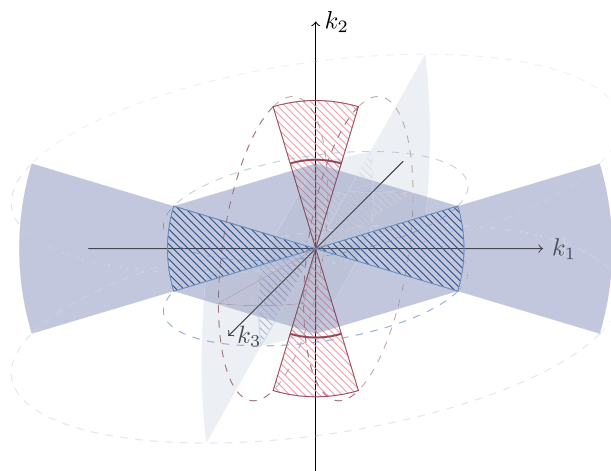
$$\begin{aligned} h_{1,2}(x) &= \frac{14}{9}x^2 - \frac{5}{9}x, & h_{1,3}(x) &= \frac{14}{3}x^2 - \frac{11}{3}x, \\ h_{2,1}(x) &= \frac{21}{4}x^2 - \frac{17}{4}x, & h_{2,3}(x) &= -\frac{21}{2}x^2 + \frac{23}{2}x, \\ h_{3,1}(x) &= -\frac{7}{12}x^2 + \frac{19}{12}x, & h_{3,2}(x) &= -\frac{7}{18}x^2 + \frac{25}{18}x. \end{aligned}$$

4.3.3 Comparison Argument in Fourier Space

In this section, we carry out the iterative bootstrap argument which allows us to deduce the final rigidity result.

As a first step of the bootstrap argument, we invoke the results from above which allow us to decrease the region of potential Fourier concentration from the cone $C_{1,\mu,\lambda}$ to a cone $C_{1,\mu,\lambda'}$ with $\lambda' < \lambda$. This resembles (23) in the exactly stress-free setting in a quantified version. As f_1 is determined by f_2 and f_3 with the help of $h_{1,j}$ we can reduce λ , similarly as in our reduction of the dependences of u_{11} in Proposition 4.4.

Fig. 4 Illustration of the cones $C_{1,\mu,\lambda}$ (red) and $C_{2,\mu,\lambda}$ (blue), and also of $C_{2,\mu,\lambda} + C_{2,\mu,\lambda}$ in Fourier space. (Color figure online)



Lemma 4.9 Under the same conditions as in Lemma 4.7 and for

$$\lambda > 0, \mu \in (0, \frac{1}{2\sqrt{2d^2+1}}), \lambda' \in (\frac{1}{\sqrt{2}} \frac{4d\lambda\mu}{\sqrt{1-4\mu^2}}, \lambda)$$

it holds

$$\begin{aligned} & \|f_1 - m_{1,\mu,\lambda'}(D)f_1\|_{L^2}^2 + \|f_2 - m_{2,\mu,\lambda}(D)f_2\|_{L^2}^2 + \|f_3 - m_{3,\mu,\lambda}(D)f_3\|_{L^2}^2 \\ & \leq 10 \frac{C_0}{\gamma^{24d}} \max\{(\mu^{-2}\delta + \lambda^{-1}\beta)^{1-\gamma}, \mu^{-2}\delta + \lambda^{-1}\beta\}. \end{aligned}$$

Here the constant C_0 is chosen to be the same as in Lemma 4.7.

Remark 4.10 We remark that the interval for λ' is chosen such that it holds $C_{1,2\mu,2\lambda} \setminus C_{1,2\mu,2\lambda'} \subset \{\max\{|k_2|, |k_3|\} > 4d\mu\lambda\} \cap C_{1,2\mu,2\lambda}$ and the one for μ such that $\frac{1}{\sqrt{2}} \frac{4d\lambda\mu}{\sqrt{1-4\mu^2}} < \lambda$, i.e. the interval for λ' is non-empty.

Proof We first observe that the Fourier transform of $h_{i,j}(m_{j,\mu,\lambda}(D)f_j)$ is given by a convolution of the function $m_{j,\mu,\lambda}\hat{f}_j$ with itself. Hence, the support of $h_{i,j}(m_{j,\mu,\lambda}(D)f_j)$ is contained in the d -fold Minkowski-sum of $C_{j,2\mu,2\lambda}$ with itself. For $j = 2, 3$ it therefore holds that

$$\mathcal{F}(h_{1,j}(m_{j,\mu,\lambda}(D)f_j))(k) = 0 \quad \text{for } |k_j| > 4d\mu\lambda.$$

Further we introduce the sets $K_2 = \{|k_2| > 4d\mu\lambda\}$, $K_3 = \{|k_3| > 4d\mu\lambda\}$ and consider the corresponding Fourier multipliers of the smoothed-out indicator functions $\chi_{K_2}(D)$, $\chi_{K_3}(D)$, $\chi_{K_2 \cup K_3}(D)$. This implies for $i = 2, 3$

$$\chi_{K_i}(D)h_{1,i}(m_{i,\mu,\lambda}(D)f_i) = 0. \quad (27)$$

With this we can show that the Fourier mass concentrates in a cone with smaller truncation parameter, depicted in Fig. 4: Indeed, in Fourier space $\chi_{K_2 \cup K_3} \leq \chi_{K_2} + \chi_{K_3}$. Thus,

$$\begin{aligned} \|\chi_{K_2 \cup K_3}(D)m_{1,\mu,\lambda}(D)f_1\|_{L^2}^2 &\leq \|(\chi_{K_2}(D) + \chi_{K_3}(D))m_{1,\mu,\lambda}(D)f_1\|_{L^2}^2 \\ &\leq 2\|\chi_{K_2}(D)m_{1,\mu,\lambda}(D)f_1\|_{L^2}^2 + 2\|\chi_{K_3}(D)m_{1,\mu,\lambda}(D)f_1\|_{L^2}^2. \end{aligned} \quad (28)$$

Invoking (27) together with the bounds from Lemma 4.7, then implies for $i = 2, 3$

$$\begin{aligned} \|\chi_{K_i}(D)m_{1,\mu,\lambda}(D)f_1\|_{L^2}^2 &= \|\chi_{K_i}(D)(m_{1,\mu,\lambda}(D)f_1 - h_{1,i}(m_{i,\mu,\lambda}(D)f_i))\|_{L^2}^2 \\ &\leq \|m_{1,\mu,\lambda}(D)f_1 - h_{1,i}(m_{i,\mu,\lambda}(D)f_i)\|_{L^2}^2 \\ &\leq \frac{C_0}{\gamma^{24d}} \max\{(\mu^{-2}\delta + \lambda^{-1}\beta)^{1-\gamma}, \mu^{-2}\delta + \lambda^{-1}\beta\}. \end{aligned}$$

Combined with (28) this yields

$$\|\chi_{K_2 \cup K_3}(D)m_{1,\mu,\lambda}(D)f_1\|_{L^2}^2 \leq 4 \frac{C_0}{\gamma^{24d}} \max\{(\mu^{-2}\delta + \lambda^{-1}\beta)^{1-\gamma}, \mu^{-2}\delta + \lambda^{-1}\beta\}.$$

We observe that by the choice of the parameters $C_{1,2\mu,2\lambda} \setminus C_{1,2\mu,2\lambda'} \subset (K_2 \cup K_3) \cap C_{1,2\mu,2\lambda}$, and thus $|m_{1,\mu,\lambda}(k) - m_{1,\mu,\lambda'}(k)| \leq \chi_{K_2 \cup K_3}(k)m_{1,\mu,\lambda}(k)$. Therefore,

$$\begin{aligned} \|m_{1,\mu,\lambda}(D)f_1 - m_{1,\mu,\lambda'}(D)f_1\|_{L^2}^2 &\leq \|\chi_{K_2 \cup K_3}(D)m_{1,\mu,\lambda}(D)f_1\|_{L^2}^2 \\ &\leq 4 \frac{C_0}{\gamma^{24d}} \max\{(\mu^{-2}\delta + \lambda^{-1}\beta)^{1-\gamma}, \mu^{-2}\delta + \lambda^{-1}\beta\}. \end{aligned}$$

In conclusion, (for $C_0 \geq C$)

$$\begin{aligned} &\|f_1 - m_{1,\mu,\lambda'}(D)f_1\|_{L^2}^2 + \|f_2 - m_{2,\mu,\lambda}(D)f_2\|_{L^2}^2 + \|f_3 - m_{3,\mu,\lambda}(D)f_3\|_{L^2}^2 \\ &\leq \|f_2 - m_{2,\mu,\lambda}(D)f_2\|_{L^2}^2 + \|f_3 - m_{3,\mu,\lambda}(D)f_3\|_{L^2}^2 \\ &\quad + 2\|f_1 - m_{1,\mu,\lambda}(D)f_1\|_{L^2}^2 + 2\|m_{1,\mu,\lambda}(D)f_1 - m_{1,\mu,\lambda'}(D)f_1\|_{L^2}^2 \\ &\leq 2C(\mu^{-2}\delta + \lambda^{-1}\beta) + 8 \frac{C_0}{\gamma^{24d}} \max\{(\mu^{-2}\delta + \lambda^{-1}\beta)^{1-\gamma}, \mu^{-2}\delta + \lambda^{-1}\beta\} \\ &\leq 10 \frac{C_0}{\gamma^{24d}} \max\{(\mu^{-2}\delta + \lambda^{-1}\beta)^{1-\gamma}, \mu^{-2}\delta + \lambda^{-1}\beta\}. \end{aligned} \quad \square$$

Let us stress that the decomposition into K_2 and K_3 is in analogy to the two comparisons from Proposition 4.4 in order to show that u_{11} is constant. The set K_2 resembles the comparison of u_{11} and u_{22} to show that u_{11} is constant in x_2 and the set K_3 resembles the comparison of u_{11} and u_{33} to show that u_{11} does not depend on x_3 .

Applying the previous result for all three directions simultaneously then yields the following corollary which will serve as the induction basis for the subsequent inductive bootstrap argument.

Corollary 4.11 (Induction Basis) *Let $\beta, \delta, \mu, \lambda > 0$ and let $C_{i,\mu,\lambda}$ be the cones in (25) with corresponding multiplier $m_{i,\mu,\lambda}(D)$, cf. (26), for $i = 1, 2, 3$. Further let $f_i, h_{j,i}$ be functions*

as in Lemma 4.7. Let $C_0 > 0$, $\gamma \in (0, 1)$ and $d \geq 0$ be the constants from Lemma 4.7. For $\lambda_0 = \lambda > 0$, $\mu \in (0, \frac{1}{2\sqrt{2d^2+1}})$, $\lambda_1 \in (\frac{4d\mu\lambda_0}{\sqrt{2}\sqrt{1-4\mu^2}}, \lambda_0)$ it holds that

$$\begin{aligned} & \|f_1 - m_{1,\mu,\lambda_1}(D)f_1\|_{L^2}^2 + \|f_2 - m_{2,\mu,\lambda_1}(D)f_2\|_{L^2}^2 + \|f_3 - m_{3,\mu,\lambda_1}(D)f_3\|_{L^2}^2 \\ & \leq \frac{30C_0}{\gamma^{24d}} \max\{(\mu^{-2}\delta + \lambda^{-1}\beta)^{1-\gamma}, \mu^{-2}\delta + \lambda^{-1}\beta\}. \end{aligned}$$

With Corollary 4.11 in hand, we now iteratively further decrease the Fourier supports. To this end, we will invoke the commutator bounds from Lemma 4.7.

Lemma 4.12 (Iteration process) *Let $\beta, \delta > 0$ and let $C_{i,\mu,\lambda}$ be the cones in (25) with corresponding multiplier $m_{i,\mu,\lambda}(D)$, cf. (26), for $i = 1, 2, 3$. Further let $\mu \in (0, \frac{1}{2\sqrt{2d^2+1}})$, $\lambda > 0$, and let $f_i, h_{j,i}$ be functions as in Lemma 4.7. Let $C_0 > 0$, $\gamma \in (0, 1)$ and $d \geq 0$ be the constants from Lemma 4.7. Let $\lambda_k > 0$ be a sequence for $k \in \mathbb{N}$ with $\lambda_0 = \lambda$ and $\lambda_k \in (\frac{4d\mu\lambda_{k-1}}{\sqrt{2}\sqrt{1-4\mu^2}}, \lambda_{k-1})$. It then holds for every $k \in \mathbb{N} \setminus \{0\}$*

$$\begin{aligned} & \|f_1 - m_{1,\mu,\lambda_k}(D)f_1\|_{L^2}^2 + \|f_2 - m_{2,\mu,\lambda_k}(D)f_2\|_{L^2}^2 + \|f_3 - m_{3,\mu,\lambda_k}(D)f_3\|_{L^2}^2 \\ & \leq \left(\frac{30C_0}{\gamma^{24d}}\right)^k \max\{(\mu^{-2}\delta + \lambda^{-1}\beta)^{(1-\gamma)^k}, \mu^{-2}\delta + \lambda^{-1}\beta\}. \end{aligned}$$

Proof We prove the statement by induction on k with the induction basis given by Corollary 4.11. Assume that for some arbitrary but fixed $k \in \mathbb{N}$ it holds

$$\begin{aligned} & \|f_1 - m_{1,\mu,\lambda_k}(D)f_1\|_{L^2}^2 + \|f_2 - m_{2,\mu,\lambda_k}(D)f_2\|_{L^2}^2 + \|f_3 - m_{3,\mu,\lambda_k}(D)f_3\|_{L^2}^2 \\ & \leq \left(\frac{30C_0}{\gamma^{24d}}\right)^k \max\{(\mu^{-2}\delta + \lambda^{-1}\beta)^{(1-\gamma)^k}, \mu^{-2}\delta + \lambda^{-1}\beta\}. \end{aligned} \quad (29)$$

Now for the induction step $k \mapsto k+1$ we carry out the same argument as above to show

$$\begin{aligned} & \|f_1 - m_{1,\mu,\lambda_{k+1}}(D)f_1\|_{L^2}^2 + \|f_2 - m_{2,\mu,\lambda_{k+1}}(D)f_2\|_{L^2}^2 + \|f_3 - m_{3,\mu,\lambda_{k+1}}(D)f_3\|_{L^2}^2 \\ & \leq \left(\frac{30C_0}{\gamma^{24d}}\right)^{k+1} \max\{(\mu^{-2}\delta + \lambda^{-1}\beta)^{(1-\gamma)^{k+1}}, \mu^{-2}\delta + \lambda^{-1}\beta\}. \end{aligned} \quad (30)$$

We argue as in the induction basis and present the calculations only for the first term on the left hand side of (30). By the triangle inequality,

$$\begin{aligned} & \|f_1 - m_{1,\mu,\lambda_{k+1}}(D)f_1\|_{L^2}^2 \\ & \leq 2\|f_1 - m_{1,\mu,\lambda_k}(D)f_1\|_{L^2}^2 + 2\|m_{1,\mu,\lambda_k}(D)f_1 - m_{1,\mu,\lambda_{k+1}}(D)f_1\|_{L^2}^2. \end{aligned}$$

The first contribution is already of the desired form. It thus remains to consider the second contribution. For this we consider an analogous argument as before: Let $K_2^k := \{|k_2| > 4d\mu\lambda_k\}$, $K_3^k := \{|k_3| > 4d\mu\lambda_k\}$. Then, since $\chi_{K_j^k}(D)h_{1,j}(m_{j,\mu,\lambda_k}(D)f_j) = 0$ on K_j^k ,

$$\begin{aligned} & \|m_{1,\mu,\lambda_k}(D)f_1 - m_{1,\mu,\lambda_{k+1}}(D)f_1\|_{L^2}^2 \leq \|\chi_{K_2^k \cup K_3^k}(D)m_{1,\mu,\lambda_k}(D)f_1\|_{L^2}^2 \\ & \leq 2 \sum_{j=2}^3 \|\chi_{K_j^k}(D)(m_{1,\mu,\lambda_k}(D)f_1 - h_{1,j}(m_{j,\mu,\lambda_k}(D)f_j))\|_{L^2}^2 \end{aligned}$$

$$\begin{aligned}
&\leq 2 \sum_{j=2}^3 \|m_{1,\mu,\lambda_k}(D)f_1 - h_{1,j}(m_{j,\mu,\lambda_k}(D)f_j)\|_{L^2}^2 \\
&\leq 4 \sum_{j=2}^3 [\|m_{1,\mu,\lambda_k}(D)f_1 - f_1\|_{L^2}^2 + \|f_1 - h_{1,j}(m_{j,\mu,\lambda_k}(D)f_j)\|_{L^2}^2] \\
&= 8\|m_{1,\mu,\lambda_k}(D)f_1 - f_1\|_{L^2}^2 + 4 \sum_{j=2}^3 \|f_1 - h_{1,j}(m_{j,\mu,\lambda_k}(D)f_j)\|_{L^2}^2. \quad (31)
\end{aligned}$$

Again for the second right hand side contribution in (31) we use Lemma 4.7:

$$\|f_1 - h_{1,j}(m_{j,\mu,\lambda_k}(D)f_j)\|_{L^2}^2 \leq \frac{C'^2}{\gamma^{24d}} (\|f_j - m_{j,\mu,\lambda_k}(D)f_j\|_{L^2}^2)^{1-\gamma}.$$

Using the inductive hypothesis (29), overall, we arrive at the following upper bound

$$\begin{aligned}
&\|f_1 - m_{1,\mu,\lambda_{k+1}}(D)f_1\|_{L^2}^2 \\
&\leq 2\|f_1 - m_{1,\mu,\lambda_k}(D)f_1\|_{L^2}^2 + 16\|m_{1,\mu,\lambda_k}(D)f_1 - f_1\|_{L^2}^2 \\
&\quad + 8\frac{C'^2}{\gamma^{24d}} \sum_j (\|f_j - m_{j,\mu,\lambda_k}(D)f_j\|_{L^2}^2)^{1-\gamma} \\
&\leq 18\|f_1 - m_{1,\mu,\lambda_k}(D)f_1\|_{L^2}^2 \\
&\quad + 16\frac{C'^2}{\gamma^{24d}} \left(\left(\frac{30C_0}{\gamma^{24d}} \right)^k \max\{(\mu^{-2}\delta + \lambda^{-1}\beta)^{(1-\gamma)^k}, \mu^{-2}\delta + \lambda^{-1}\beta\} \right)^{1-\gamma}.
\end{aligned}$$

Arguing symmetrically for f_2 and f_3 then yields:

$$\begin{aligned}
&\|f_1 - m_{1,\mu,\lambda_{k+1}}(D)f_1\|_{L^2}^2 + \|f_2 - m_{2,\mu,\lambda_{k+1}}(D)f_2\|_{L^2}^2 + \|f_3 - m_{3,\mu,\lambda_{k+1}}(D)f_3\|_{L^2}^2 \\
&\leq 18(\|f_1 - m_{1,\mu,\lambda_k}(D)f_1\|_{L^2}^2 + \|f_2 - m_{2,\mu,\lambda_k}(D)f_2\|_{L^2}^2 + \|f_3 - m_{3,\mu,\lambda_k}(D)f_3\|_{L^2}^2) \\
&\quad + 48\frac{C'^2}{\gamma^{24d}} \left(\left(\frac{30C_0}{\gamma^{24d}} \right)^k \max\{(\mu^{-2}\delta + \lambda^{-1}\beta)^{(1-\gamma)^k}, \mu^{-2}\delta + \lambda^{-1}\beta\} \right)^{1-\gamma} \\
&\leq 18 \left(\frac{30C_0}{\gamma^{24d}} \right)^k \max\{(\mu^{-2}\delta + \lambda^{-1}\beta)^{(1-\gamma)^k}, \mu^{-2}\delta + \lambda^{-1}\beta\} \\
&\quad + 48\frac{C'^2}{\gamma^{24d}} \left(\left(\frac{30C_0}{\gamma^{24d}} \right)^k \max\{(\mu^{-2}\delta + \lambda^{-1}\beta)^{(1-\gamma)^k}, \mu^{-2}\delta + \lambda^{-1}\beta\} \right)^{1-\gamma}.
\end{aligned}$$

Using that $2C'^2 \leq C_0$, $1 - \gamma \in (0, 1)$ and that $\frac{C_0}{\gamma^{24d}} \geq 3$, leads to

$$\begin{aligned}
&\|f_1 - m_{1,\mu,\lambda_{k+1}}(D)f_1\|_{L^2}^2 + \|f_2 - m_{2,\mu,\lambda_{k+1}}(D)f_2\|_{L^2}^2 + \|f_3 - m_{3,\mu,\lambda_{k+1}}(D)f_3\|_{L^2}^2 \\
&\leq 18 \left(\frac{30C_0}{\gamma^{24d}} \right)^k \max\{(\mu^{-2}\delta + \lambda^{-1}\beta)^{(1-\gamma)^k}, \mu^{-2}\delta + \lambda^{-1}\beta\} \\
&\quad + 24\frac{C_0}{\gamma^{24d}} \left(\frac{30C_0}{\gamma^{24d}} \right)^k \max\{(\mu^{-2}\delta + \lambda^{-1}\beta)^{(1-\gamma)^{k+1}}, (\mu^{-2}\delta + \lambda^{-1}\beta)^{1-\gamma}\}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{6C_0}{\gamma^{24d}} \left(\frac{30C_0}{\gamma^{24d}} \right)^k \max\{(\mu^{-2}\delta + \lambda^{-1}\beta)^{(1-\gamma)^{k+1}}, \mu^{-2}\delta + \lambda^{-1}\beta\} \\
&\quad + \frac{24C_0}{\gamma^{24d}} \left(\frac{30C_0}{\gamma^{24d}} \right)^k \max\{(\mu^{-2}\delta + \lambda^{-1}\beta)^{(1-\gamma)^{k+1}}, \mu^{-2}\delta + \lambda^{-1}\beta\} \\
&= \left(\frac{30C_0}{\gamma^{24d}} \right)^{k+1} \max\{(\mu^{-2}\delta + \lambda^{-1}\beta)^{(1-\gamma)^{k+1}}, \mu^{-2}\delta + \lambda^{-1}\beta\}.
\end{aligned}$$

This concludes the proof. $\square$

Now with the inductive procedure of reducing regions of Fourier space concentration in hand, we turn to the proof of the lower bound in Theorem 2 and the argument for Proposition 1.2.

Proof of the lower bound in Theorem 2 and proof of Proposition 1.2 We argue in two steps, first fixing the free parameters and then exploiting the boundary conditions.

Step 1: Choice of parameters and proof of Proposition 1.2. We seek to invoke Lemma 4.12 expressing the bounds in terms of our energies, i.e. setting $\delta = E_{el}^{per}(\chi; F) := \inf_{u \in \mathcal{D}_F} E_{el}^{per}(u, \chi)$, $\beta = E_{surf}^{per}(\chi)$, and $\lambda^{-1} = \mu^{-2}\epsilon$. Moreover, we use the notation

$$E_{\epsilon}^{per}(\chi; F) := E_{el}^{per}(\chi; F) + \epsilon E_{surf}^{per}(\chi). \quad (32)$$

By virtue of Lemma 4.6 we obtain that for $f_i := \chi_{i,i}$ indeed $\sum_{j=1}^3 \|\partial_i f_i\|_{H^{-1}}^2 \leq \delta$. It follows directly that also $\sum_{i=1}^3 \|\nabla f_i\|_{TV} \leq \beta$ and therefore the conditions in Lemma 4.7 are fulfilled. As a consequence, the above iteration in Lemma 4.12 is applicable. With this in mind, we choose $\mu = \epsilon^{\alpha}$ for some $\alpha > 0$ to be specified. Therefore,

$$\lambda = \lambda_0 = \epsilon^{2\alpha-1}$$

and thus since without loss of generality $E_{\epsilon}^{per}(\chi; F) := \inf_{u \in \mathcal{D}_F} E_{el}^{per}(u, \chi) + \epsilon E_{surf}^{per}(\chi) \leq 1$ (having the upper bound from Proposition 4.3 in mind), we deduce

$$\sum_{j=1}^3 \|f_j - m_{j,\mu,\lambda_k}(D)f_j\|_{L^2}^2 \leq \left(\frac{30C_0}{\gamma^{24d}} \right)^k \epsilon^{-2\alpha} E_{\epsilon}^{per}(\chi; F)^{(1-\gamma)^k}. \quad (33)$$

We further choose

$$\lambda_k = (2\sqrt{2d^2+1}\mu)^k \lambda_0 = M^k \epsilon^{(2+k)\alpha-1},$$

where $M = M(d) = 2\sqrt{2d^2+1} > 2$. This is admissible in the sense of the assumptions in Lemma 4.9 since $\frac{\lambda_k}{\lambda_{k-1}} = 2\sqrt{2d^2+1}\mu \in (\frac{1}{\sqrt{2}} \frac{4d\mu}{\sqrt{1-4\mu^2}}, 1)$.

Next, for $\alpha = \alpha(\epsilon) \in (0, 1)$ and $\epsilon > 0$ we choose $k \in \mathbb{N}$ to be given by

$$k := \left\lceil \frac{(1-2\alpha)|\log \epsilon| + \log 2}{\alpha|\log \epsilon| - \log M} \right\rceil \leq \left\lceil \frac{1 + \frac{\log M}{|\log \epsilon|}}{\alpha - \frac{\log M}{|\log \epsilon|}} \right\rceil.$$

This ensures that $\lambda_k \leq \frac{1}{2}$. In what follows, we will choose the parameters ϵ, α such that $\frac{\log M}{|\log \epsilon|} \leq \frac{\alpha}{2}$ which implies $k \leq \frac{4}{\alpha}$. Exploiting the discrete Fourier transform, then yields

$$\|f_j - m_{j,\mu,\lambda_k}(D)f_j\|_{L^2}^2 = \sum_{\xi \in \mathbb{Z}^3 \setminus \{0\}} |\hat{f}_j(\xi)|^2 = \|f_j - \langle f_j \rangle\|_{L^2}^2,$$

and hence, by (33) results in the estimate

$$\sum_{j=1}^3 \|f_j - \langle f_j \rangle\|_{L^2}^2 \leq \left(\frac{30C_0}{\gamma^{24d}}\right)^{\frac{4}{\alpha}} \epsilon^{-2\alpha} E_{\epsilon}^{per}(\chi; F)^{(1-\gamma)\frac{4}{\alpha}}.$$

Using that $(1-\gamma)^{\frac{4}{\alpha}} \geq 1 - \frac{4}{\alpha}\gamma$, we set $\gamma := \frac{\alpha}{8} \in (0, 1)$ which leads to the bound

$$\sum_{j=1}^3 \|f_j - \langle f_j \rangle\|_{L^2}^2 \leq (30C_0 8^{24d})^{\frac{4}{\alpha}} \alpha^{-\frac{96d}{\alpha}} \epsilon^{-2\alpha} E_{\epsilon}^{per}(\chi; F)^{\frac{1}{2}}.$$

Next, we fix the parameter $\alpha > 0$: Observing that for any $\nu > 0$ there exists $\alpha_0 > 0$ such that for $\alpha \in (0, \alpha_0)$ it holds that

$$\alpha^{-\frac{96d}{\alpha}} = \exp(96d \log(\alpha^{-1}) \alpha^{-1}) \leq \exp\left(\frac{96d}{\nu e} \alpha^{-1-\nu}\right) = \exp(C(\nu) \alpha^{-1-\nu}), \quad (34)$$

we choose $\alpha = |\log \epsilon|^{-\frac{1}{2+\nu}}$ for all $\epsilon \in (0, \epsilon_0)$ with $\epsilon_0 > 0$ still to be chosen. In particular, for $\epsilon_0 > 0$ sufficiently small, such that $|\log \epsilon|^{\frac{1}{2}} \geq 2 \log M$, (34) holds and also $\frac{\log M}{|\log \epsilon|} \leq \frac{\alpha}{2}$ holds as required above.

As a consequence, for $\nu' := \frac{\nu}{4+2\nu} \in (0, \frac{1}{2})$ we arrive at

$$\sum_{j=1}^3 \|f_j - \langle f_j \rangle\|_{L^2}^2 \leq \exp(c_{\nu'} |\log \epsilon|^{\frac{1}{2}+\nu'}) E_{\epsilon}^{per}(\chi; F)^{\frac{1}{2}}.$$

Step 2: Conclusion. In order to conclude the estimate, we derive a lower bound for $\sum_{j=1}^3 \|f_j - \langle f_j \rangle\|_{L^2}^2$. For this we recall that $f_j = \chi_{j,j}$ is the j -th diagonal entry of the phase indicator χ and hence $\sum_{j=1}^3 \|f_j - \langle f_j \rangle\|_{L^2}^2 = \|\chi - \langle \chi \rangle\|_{L^2}^2$. Thus, by the mean value condition in $\mathcal{D}_F^{per}$,

$$|\langle \chi \rangle - F|^2 = |\langle \chi \rangle - \langle u \rangle|^2 \leq \int_{\mathbb{T}^3} |u - \chi|^2 dx \leq E_{el}^{per}(u, \chi),$$

and, furthermore, as the left hand side is independent of u ,

$$|\langle \chi \rangle - F|^2 \leq E_{el}^{per}(\chi; F).$$

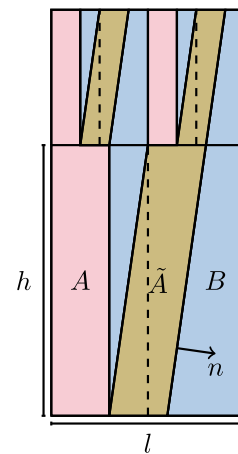
Overall this implies for the total energy $E_{\epsilon}^{per}(\chi; F) = E_{el}^{per}(\chi; F) + \epsilon E_{surf}^{per}(\chi)$

$$\begin{aligned} \text{dist}^2(F, \mathcal{K}) &\leq \int_{\mathbb{T}^3} |\chi - F|^2 dx \leq 2 \int_{\mathbb{T}^3} |\chi - \langle \chi \rangle|^2 dx + 2 \int_{\mathbb{T}^3} |\langle \chi \rangle - F|^2 dx \\ &\leq 2 \exp(c_{\nu'} |\log \epsilon|^{\frac{1}{2}+\nu'}) E_{\epsilon}^{per}(\chi; F)^{\frac{1}{2}} + 2 E_{el}^{per}(\chi; F) \\ &\leq 2 \exp(c_{\nu'} |\log \epsilon|^{\frac{1}{2}+\nu'}) E_{\epsilon}^{per}(\chi; F)^{\frac{1}{2}} + 2 E_{\epsilon}^{per}(\chi; F)^{\frac{1}{2}} \\ &\leq 4 \exp(c_{\nu'} |\log \epsilon|^{\frac{1}{2}+\nu'}) E_{\epsilon}^{per}(\chi; F)^{\frac{1}{2}}. \end{aligned}$$

Finally, solving for $E_{\epsilon}^{per}(\chi; F)$ shows the desired estimate

$$E_{\epsilon}^{per}(\chi; F) \geq 2^{-4} \exp(-2c_{\nu'} |\log \epsilon|^{\frac{1}{2}+\nu'}) \text{dist}^4(F, \mathcal{K}).$$

Fig. A.1 Self-similar construction in Proposition A.3.
(Color figure online)



The desired claim follows by an application of Lemma 4.5. $\square$

Appendix A: Branching, Upper Bound for the Two-State Problem for the Divergence Operator

We complement our lower bounds for the compatible two-well problem from Sect. 3 by an upper bound in the case of the divergence operator acting on matrix fields as introduced in Example 1.1. For simplicity, we only consider square matrices, i.e. $m = d$, and like before in Sect. 3 only consider the case $d > 1$ in the following. For earlier, closely related, three-dimensional constructions in the context of compliance minimization problems we refer to [24]. While the $\epsilon^{\frac{2}{3}}$ construction is by now rather “standard” [1, 62, 63, 80], our argument does provide a slightly different perspective, in that, in arbitrary dimension, we can ensure boundary conditions on *all* faces of the domain $\Omega = [0, 1]^d$ (see the upper bound construction for [76, Theorem 3] for a similar construction for the gradient).

In deducing the upper bound for the divergence operator, we first provide a construction in a unit cell (Lemma A.1) and iterate this construction (Proposition A.3). This yields a construction which attains the boundary data in two directions. In order to attain these also on the remaining sides we use the flexibility of the wave cone for the divergence operator (Proposition A.5). We remark that in two dimensions there would be no modification with respect to the gradient construction since there the curl and divergence only differ by a rotation of 90 degrees.

In our unit cell branching construction, we do not work on the level of the potential, but directly consider the problem on the level of the wells. In this context, we recall the compatibility conditions for laminates formed by the divergence operator which is determined by the associated wave cone: For $M \in \mathbb{R}^{d \times d}$, we have that $M \in \ker \mathbb{A}(\xi)$ if and only if it holds $M\xi = 0$.

With this in hand, we introduce an auxiliary matrix which will play an important role in our construction: Let $A, B \in \mathbb{R}^{d \times d}$ be such that $(B - A)e_1 = 0$ and let $n = e_1 + \gamma_2 v$ for a unit vector v perpendicular to e_1 and for some $\gamma_2 \in \mathbb{R}$, $\gamma_2 \neq 0$. We then define E_v by

$$E_v = \gamma_2(B - A)v \otimes e_1 \quad (\text{A.1})$$

and the associated “perturbed” matrix $\tilde{A}_v = A + E_v$. By construction, this matrix obeys the identities

$$(\tilde{A}_v - A)v = 0, \quad (B - \tilde{A}_v)n = 0. \quad (\text{A.2})$$

This in particular allows for interfaces of $\tilde{A}_v$ and A with normal v and of $\tilde{A}_v$ and B with normal n which we will use in our branching construction below.

Lemma A.1 *For $0 < l < h \leq 1$, we define $\omega = [0, l] \times [0, h] \times [0, 1]^{d-2}$. Let $A, B \in \mathbb{R}^{d \times d}$ be such that $(B - A)e_1 = 0$ and let $F_\lambda = \lambda A + (1 - \lambda)B$ for some $\lambda \in (0, 1)$. Then there exists $u : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ such that*

$$\begin{aligned} \operatorname{div} u &= 0 \text{ in } \mathbb{R}^d, \\ u &= F_\lambda \text{ for } x_1 \in (-\infty, 0) \cup (l, \infty). \end{aligned}$$

Furthermore, there exist $\chi \in BV(\omega; \{A, B\})$ and a constant $C = C(A, B) > 0$ such that for any $\epsilon > 0$ the localized energy can be bounded by

$$E_\epsilon(u, \chi; \omega) := \int_\omega |u - \chi|^2 dx + \epsilon \int_\omega |\nabla \chi| \leq C(1 - \lambda)^2 \frac{l^3}{h} + 5\epsilon h.$$

Proof We consider the following partition of the domain ω into subdomains:

$$\begin{aligned} \omega_1 &= \{x_1 \in (0, \frac{\lambda l}{2})\}, & \omega_2 &= \{x_1 \in (\frac{\lambda l}{2}, \frac{\lambda l}{2} + \frac{(1 - \lambda)l}{2h}x_2)\}, \\ \omega_3 &= \{x_1 \in (\frac{\lambda l}{2} + \frac{(1 - \lambda)l}{2h}x_2, \lambda l + \frac{(1 - \lambda)l}{2h}x_2)\}, & \omega_4 &= \{x_1 \in (\lambda l + \frac{(1 - \lambda)l}{2h}x_2, l)\}. \end{aligned}$$

Based on this we define

$$u(x) = \begin{cases} A & x \in \omega_1, \\ B & x \in \omega_2 \cup \omega_4, \\ A + E_{e_2} & x \in \omega_3, \end{cases} \quad \chi(x) = \begin{cases} A & x \in \omega_1 \cup \omega_3, \\ B & x \in \omega_2 \cup \omega_4, \end{cases}$$

where E_{e_2} is given in (A.1) for $n = e_1 - \frac{(1 - \lambda)l}{2h}e_2$. We highlight that u is independent of x_k for $k \geq 3$. By definition of E_{e_2} , the characterization of the wave cone (5) for the divergence operator and the remarks on laminates in (A.2), this defines an divergence-free mapping. Further, as $(B - F_\lambda)e_1 = (A - F_\lambda)e_1 = 0$, the exterior data are attained in $x_1 \in (-\infty, 0) \cup (l, \infty)$.

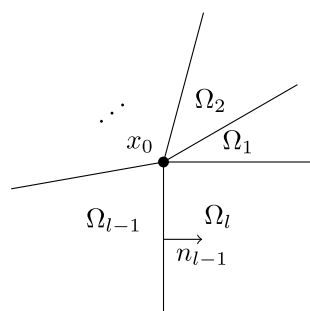
To calculate the energy, we observe, that the only contribution to the elastic energy is given in ω_3 . Hence,

$$\begin{aligned} E_{el}(u, \chi; \omega) &:= \int_\omega |u - \chi|^2 dx = \int_{\omega_3} |A + E_{e_2} - A|^2 dx \\ &= |E_{e_2}|^2 |\omega_3| = |(B - A)e_2|^2 \frac{(1 - \lambda)^2 \lambda l^3}{8h}. \end{aligned}$$

As the surface energy is determined by the interfaces between ω_j and ω_k , we obtain ($l < h$)

$$E_{surf}(\chi; \omega) := \int_\omega |\nabla \chi| = h + 2\sqrt{\frac{(1 - \lambda)^2 l^2}{4} + h^2} \leq h + 2h\sqrt{\frac{(1 - \lambda)^2}{4} + 1} \leq 5h.$$

Fig. A.2 Setting of Lemma A.2



This shows the claim. $\square$

As for analogous constructions for the gradient, we will use this unit cell as a building block in order to achieve a self-similar construction attaining the boundary data on two directions. For this to be admissible in the sense of an $\mathcal{A}$ -free map, we rely on the following lemma. It shows that for first order operators corners in which interfaces meet do not give rise to singularities.

Lemma A.2 *Let $d, n, m \in \mathbb{N}$ and let $\mathcal{A}(D) = \sum_{j=1}^d A_j \partial_j : C^\infty(\mathbb{R}^d; \mathbb{R}^n) \rightarrow C^\infty(\mathbb{R}^d; \mathbb{R}^m)$ be a homogeneous, linear operator of degree one with symbol $\mathbb{A}$ given in (3) and let Ω_j , $j = 1, \dots, l$, be a polygonal set (the set is defined as the intersection of half spaces) with outer unit normal n_j such that*

- $\mathbb{R}^d = \bigcup_{j=1}^l \Omega_j$,
- the two sets Ω_j, Ω_{j+1} have one common face ($l+1=1$),
- and such that they meet in one point, i.e. $\bigcap_{j=1}^l \Omega_j = \{x_0\}$,

cf. Fig. A.2. Assume further that $B_j \in \mathbb{R}^n$, $j = 1, \dots, l$, are such that $B_j - B_{j+1} \in \ker \mathbb{A}(n_j)$. Then the map

$$u(x) = B_j \text{ for } x \in \Omega_j$$

is $\mathcal{A}$ -free.

Proof First we note that u is indeed well-defined by the properties of Ω_j , further we notice that for $M \in \mathbb{R}^n$ and $U \subset \mathbb{R}^d$ a Lipschitz domain it holds for $\varphi \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^m)$

$$\begin{aligned} \int_U M \cdot (\mathcal{A}(D))^* \varphi dx &= \sum_{k=1}^d \int_U M \cdot \partial_k (A_k^t \varphi) dx = \sum_{k=1}^d \int_{\partial U} M \cdot (A_k^t \varphi) n_k d\mathcal{H}^{d-1} \\ &= \int_{\partial U} \mathbb{A}(n) M \cdot \varphi d\mathcal{H}^{d-1}. \end{aligned}$$

With this it holds

$$\int_{\mathbb{R}^d} u \cdot (\mathcal{A}(D))^* \varphi dx = \sum_{j=1}^l \int_{\Omega_j} B_j \cdot (\mathcal{A}(D))^* \varphi dx = \sum_{j=1}^l \int_{\partial \Omega_j} \mathbb{A}(n_j) B_j \cdot \varphi d\mathcal{H}^{d-1}.$$

Moreover on $\partial\Omega_j \cap \partial\Omega_{j+1}$ it holds that $n_{j+1} = -n_j$ and thus by the assumptions on B_j

$$\int_{\mathbb{R}^d} u \cdot (A(D))^* \varphi dx = \sum_{j=1}^l \int_{\partial\Omega_j \cap \partial\Omega_{j+1}} \mathbb{A}(n_j)(B_j - B_{j+1}) \cdot \varphi d\mathcal{H}^{d-1} = 0.$$

As φ was arbitrary the claim follows. $\square$

With Lemma A.2 in hand, we now iterate the unit cell-construction from Lemma A.1, as illustrated in Fig. A.1.

Proposition A.3 *Let $d, n \in \mathbb{N}$. Let $\Omega = [0, 1]^d$, let $A, B \in \mathbb{R}^n$ be such that $B - A \in \Lambda_{\text{div}}$, cf. (5), and let $F_\lambda = \lambda A + (1 - \lambda)B$ for some $\lambda \in (0, 1)$. Let E_ϵ be as in (12). Then there exist $u : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ and $\chi \in BV(\Omega; \{A, B\})$ with $\text{div } u = 0$ in $\mathbb{R}^d$ and $u = F_\lambda$ for $(x_1, x_2) \notin [0, 1]^2$ such that for any $\epsilon \in (0, 1)$ and any $N \in \mathbb{N}$*

$$E_\epsilon(u, \chi) \leq C\left(\frac{1}{N^2} + \epsilon N\right)$$

for some constant $C = C(A, B, \lambda) > 0$.

Proof Without loss of generality we may assume $(B - A)e_1 = 0$, i.e. $B - A \in \ker \mathbb{A}(e_1)$ for $\mathcal{A}(D) = \text{div}$. Let $\theta \in (\frac{1}{4}, \frac{1}{2})$. We argue symmetrically in the upper and lower half of the cube, i.e. we give the construction of u on $[0, 1] \times [\frac{1}{2}, 1] \times [0, 1]^{d-2}$ and define u on the lower half by symmetry. We define for $N \in \mathbb{N}$ and for $j \in \mathbb{N}_0$

$$y_j = 1 - \frac{\theta^j}{2}, l_j = \frac{1}{2^j N}, h_j = y_{j+1} - y_j = \theta^j \frac{1 - \theta}{2}.$$

Furthermore, let $j_0 \in \mathbb{N}$ be the maximal $j \in \mathbb{N}$ such that $l_j < h_j$. We set

$$\omega_{j,k} = \left((kl_j, y_j) + [0, l_j] \times [0, h_j] \right) \times [0, 1]^{d-2},$$

for $k \in \{0, 1, \dots, N2^j - 1\}$, $j \in \{0, 1, \dots, j_0\}$; for $k \in \{0, 1, \dots, N2^{j_0+1} - 1\}$, $j = j_0 + 1$ we set

$$\omega_{j_0+1,k} = \left((kl_{j_0+1}, y_{j_0+1}) + [0, l_{j_0+1}] \times [0, \frac{\theta^{j_0+1}}{2}] \right) \times [0, 1]^{d-2}.$$

Let u_j, χ_j in $[0, l_j] \times [0, h_j] \times [0, 1]^{d-2}$ be given by Lemma A.1 for $j = 1, \dots, j_0$. Further, in the layer $j = j_0 + 1$ we interpolate with the desired boundary data by a cut-off argument: To this end, we introduce the cut-off function $\phi : [0, \infty) \rightarrow [0, 1]$ and the profile $h : [0, 1] \rightarrow [0, \infty)$ by setting

$$\phi(t) = \begin{cases} 1 & t \in [0, \frac{1}{2}], \\ -4t + 3 & t \in (\frac{1}{2}, \frac{3}{4}), \\ 0 & t \geq \frac{3}{4}, \end{cases} \quad h(t) = \begin{cases} (1 - \lambda)t & t \in [0, \lambda), \\ \lambda(1 - t) & t \in [\lambda, 1]. \end{cases}$$

We consider the function $\tilde{u}_{j_0+1} : [0, l_{j_0+1}] \times [0, \frac{\theta^{j_0+1}}{2}] \times [0, 1]^{d-2}$ defined via

$$\tilde{u}_{j_0+1}(x) = -\frac{2l_{j_0+1}}{\theta^{j_0+1}} \phi\left(\frac{2x_2}{\theta^{j_0+1}}\right) h\left(\frac{x_1}{l_{j_0+1}}\right) ((A - B)e_2) \otimes e_1$$

$$+ \phi\left(\frac{2x_2}{\theta^{j_0+1}}\right)h'\left(\frac{x_1}{l_{j_0+1}}\right)(A - B) + F_\lambda.$$

The associated phase indicator is defined by

$$\chi_{j_0+1}(x) = \chi_{(0, \lambda l_{j_0})}(x_1)A + \chi_{(\lambda l_{j_0}, l_{j_0})}(x_1)B.$$

We note that $\chi_{j_0+1}(x) = h'\left(\frac{x_1}{l_{j_0+1}}\right)(A - B) + F_\lambda$ and moreover for $x_2 < \frac{1}{2}$ it holds $\tilde{u}_{j_0+1}(x) = \chi_{j_0+1}(x)$ and for $x_2 > \frac{3}{4}$ correspondingly $\tilde{u}_{j_0+1}(x) = F_\lambda$. Furthermore, for $x_1 \in \{0, l_{j_0+1}\}$, we know $h\left(\frac{x_1}{l_{j_0+1}}\right) = 0$ and thus $(\tilde{u}_{j_0+1}(x) - F_\lambda)e_1 = \phi\left(\frac{2x_2}{\theta^{j_0+1}}\right)h'\left(\frac{x_1}{l_{j_0+1}}\right)(A - B)e_1 = 0$.

With the help of this construction we meet the prescribed data for $x_2 \geq 1$, and we can define u in the upper half of the full cube:

$$u(x) = \begin{cases} u_j(x - (kl_j, y_j)) & x \in \omega_{j,k}, \\ \tilde{u}_{j_0+1}(x - (kl_{j_0}, y_{j_0+1})) & x \in \omega_{j_0+1,k}. \end{cases}$$

For the lower half of the cube we argue similarly, mirroring the unit cell construction of Lemma A.1, i.e. instead of E_{e_2} we consider E_{-e_2} . We define χ in $[0, 1]^d$ analogously.

We note, that this defines a divergence free mapping, as all the laminations are in compatible directions as $(B - A)e_1 = 0$ and by the choice of E_{e_2} in (A.1). Lemma A.2 shows, that this function is divergence-free even though interfaces meet in corners. Moreover, we can bound the energy in the $\omega_{j_0+1,k}$ cells for any $k \in \{1, \dots, N2^{j_0+1} - 1\}$:

$$\begin{aligned} |\tilde{u}_{j_0+1}(x) - \chi_{j_0+1}(x)|^2 &= \left| \frac{2l_{j_0+1}}{\theta^{j_0+1}} \phi'\left(\frac{2x_2}{\theta^{j_0+1}}\right)h\left(\frac{x_1}{l_{j_0+1}}\right)((A - B)e_2 \otimes e_1) \right. \\ &\quad \left. - \left(\phi\left(\frac{2x_2}{\theta^{j_0+1}}\right) - 1\right)h'\left(\frac{x_1}{l_{j_0+1}}\right)(A - B) \right|^2 \\ &\leq C(A, B, \lambda) \left(\frac{l_{j_0+1}^2}{\theta^{2j_0+2}} \phi'^2\left(\frac{2x_2}{\theta^{j_0+1}}\right) + 1 \right), \end{aligned}$$

and, since $l_{j_0+1} \geq h_{j_0+1}$ and $\theta^{j_0+1} \sim h_{j_0+1}$,

$$\begin{aligned} \int_{[0, l_{j_0+1}] \times [0, \frac{\theta^{j_0+1}}{2}] \times [0, 1]^{d-2}} |\tilde{u}_{j_0+1} - \chi_{j_0+1}(x)|^2 dx &\leq Cl_{j_0+1} \left(\int_0^1 \frac{l_{j_0+1}^2}{\theta^{j_0+1}} \phi'^2(t) dt + \theta^{j_0+1} \right) \\ &\leq C \left(\frac{l_{j_0+1}^3}{\theta^{j_0+1}} + l_{j_0+1} \theta^{j_0+1} \right) \\ &\leq C \frac{l_{j_0+1}^3}{h_{j_0+1}}. \end{aligned}$$

Furthermore, the surface energy is bounded by $E_{surf}(\chi_{j_0+1}; \omega_{j_0+1,k}) \leq C(A, B, \lambda)h_{j_0+1}$.

Overall, we have a function defined on $[0, 1]^d$ and can extend it to be F_λ for $(x_1, x_2) \notin [0, 1]^2$. The energy then can be bounded by

$$E_\epsilon(u, \chi) \leq 2 \sum_{j=0}^{j_0+1} \sum_{k=0}^{N2^j-1} E_\epsilon(u_j, \chi_j; \omega_{j,k}) \leq C \sum_{j=0}^{j_0+1} N2^j \left(\frac{l_j^3}{h_j} + \epsilon h_j \right)$$

$$\begin{aligned}
&\leq C \sum_{j=0}^{j_0+1} \left(\frac{l_j^2}{h_j} + \epsilon \frac{h_j}{l_j} \right) = C \sum_{j=0}^{j_0+1} \frac{1}{N^2} \left(\frac{1}{4\theta} \right)^j + \epsilon N (2\theta)^j \\
&\leq C \left(\frac{1}{N^2} + \epsilon N \right). \quad \square
\end{aligned}$$

Remark A.4 We remark that in the situation of the divergence operator, there are situations with substantially more flexibility than for the gradient: If for the two wells $A, B \in \mathbb{R}^{d \times d}$ it does not only hold that $(B - A)e_1 = 0$ but also that $(B - A)e_2 = 0$, there would not be any elastic energy contribution involved. In this situation, for the above construction, we would only have contributions to the surface energy, as then $E_{e_2} = 0$ and hence $A + E_{e_2} = A \in \{A, B\}$. In particular, for boundary data which are only attained on two directions this would yield a linear scaling law in ϵ . For curl free mappings as in gradient inclusions, this is not possible, as the direction of lamination is unique in that case, i.e. $V_{\text{rot}, \lambda}$ is at most one-dimensional.

As a last auxiliary step towards the upper bound construction from Theorem 1, in order to achieve the exterior data on *all* sides of the unit cube, we adapt the branching construction similarly as in [75], as the construction from Proposition A.3 does not yet satisfy F_λ at, e.g., $x_3 = 0$. Thus, we combine Proposition A.3 with a further domain splitting for which we split $[0, 1]^d$ into different regions. In each region, we prescribe a different direction for the branching construction from Proposition A.3. To this end, we use that the choice of e_2 in the above results was arbitrary and we also can choose any other direction e_j for $j \in \{2, \dots, d\}$. Combined with compatibility conditions at the resulting interfaces, this will allow us to deduce the desired branching construction.

Proposition A.5 *Under the same assumptions as in Proposition A.3 there exist $u : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$, $\chi \in BV(\Omega; \{A, B\})$ and a constant $C = C(A, B, \lambda) > 0$ such that $u \in \mathcal{D}_{F_\lambda}$ for $F_\lambda = \lambda A + (1 - \lambda)B$ ($\lambda \in (0, 1)$) and for any $\epsilon \in (0, 1)$ and $N \in \mathbb{N}$ it holds*

$$E_\epsilon(u, \chi) \leq C \left(\frac{1}{N^2} + \epsilon N \right).$$

Proof For simplicity, we first carry out the details for the case $d = 3$ and then only comment on the changes in the case of arbitrary dimension. We split $[0, 1]^3$ into the following four parts and use different branching directions in each part: Let

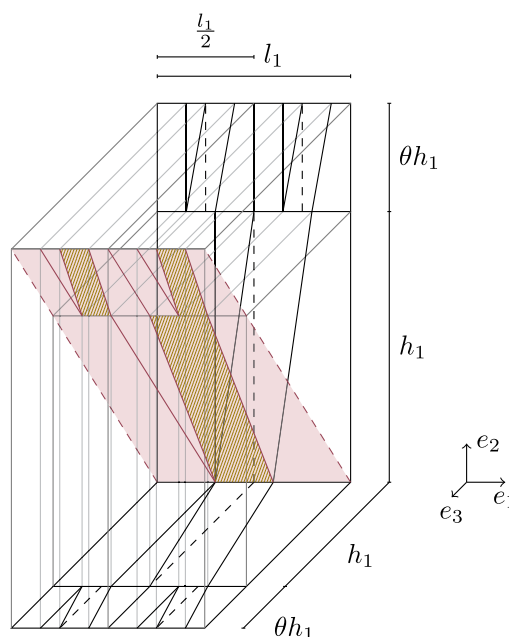
$$\begin{aligned}
\Omega_2^\pm &= \{x \in [0, 1]^3 : \pm(x_2 - \frac{1}{2}) \geq 0, \frac{1}{2} - |x_2 - \frac{1}{2}| \leq \frac{1}{2} - |x_3 - \frac{1}{2}|\}, \\
\Omega_3^\pm &= \{x \in [0, 1]^3 : \pm(x_3 - \frac{1}{2}) \geq 0, \frac{1}{2} - |x_3 - \frac{1}{2}| \leq \frac{1}{2} - |x_2 - \frac{1}{2}|\},
\end{aligned}$$

and consider the upper (Ω_2^+) and lower (Ω_2^-) halves separately as in the proof of Proposition A.3.

Next, we define u_2 in Ω_2^+ and u_3 in Ω_3^+ using Proposition A.3: The function u_2 is given by the function from Proposition A.3 above, whereas u_3 is obtained from u_2 by exchanging roles of e_2 and e_3 , i.e. the branching is done in e_3 direction and we use E_{e_3} instead of E_{e_2} . For $v \in \{e_2, e_3\}$ the error matrix E_v is given in (A.1). We then define the overall deformation u in $\Omega_2^+ \cup \Omega_3^+$ by

$$u(x) = \begin{cases} u_2(x) & x \in \Omega_2^+, \\ u_3(x) & x \in \Omega_3^+. \end{cases}$$

Fig. A.3 The three-dimensional branching construction to achieve the boundary data on all sides. The shaded regions are the diagonal interface at $x_2 = x_3$ with the interfaces of $A + E_{e_2}$ and $A + E_{e_3}$ marked in orange



This construction is depicted in Fig. A.3. As above u is defined in the lower halves Ω_2^- , Ω_3^- by symmetry.

We claim that this overall construction is divergence-free. In the individual regions Ω_2^+ and Ω_3^+ this follows by Proposition A.3. It thus remains to discuss the compatibility at the interface $x_2 = x_3$. Since all other values of u are given by (matching domains in which) $u \in \{A, B\}$, it suffices to discuss the compatibility of the error matrices E_{e_2} and E_{e_3} at this interface. To this end, we however note that $(E_{e_3} - E_{e_2})\zeta = 0$ for all $\zeta \in \text{span}(e_2, e_3)$, i.e. also this interface is admissible. This shows that u indeed defines a divergence free map.

The upper bound for the elastic and surface energies from Proposition A.3 remains valid, thus yielding the claimed estimate which concludes the proof of the proposition.

In order to show the d -dimensional result, we split $[0, 1]^d$ into $2d - 2$ regions $\Omega_j^\pm := \{x \in [0, 1]^d : \pm(x_j - \frac{1}{2}) \geq 0, \frac{1}{2} - |x_j - \frac{1}{2}| = \min_{2 \leq k \leq d} \frac{1}{2} - |x_k - \frac{1}{2}|\}$ for $j = 2, \dots, d$ and argue as above. $\square$

Finally, with Proposition A.5 in hand, we immediately obtain the proof of the upper bound construction from Theorem 1 for the divergence operator.

Proof of the upper bound in Theorem 1 In order to deduce the upper bound of Theorem 1, we choose $N \sim \epsilon^{-\frac{1}{3}}$, which shows the claim. $\square$

Remark A.6 (Generalizations) Building on the ideas from the gradient case and the ones from above one can formulate (rather restrictive) conditions, allowing for similar constructions for more general linear, constant coefficient differential operators. A key difficulty here consists in the “hard form” of the prescribed boundary conditions. When considering “softer forms” of these, as for instance in [66], constructions for general constant coefficient operators with the desired boundary conditions would be feasible under much more general conditions by using Fourier theoretic arguments as in [74]. We do not pursue these ideas here but postpone this to possible future work.

Appendix B: On the Role of the Divergence Operator

Following [28], in this section we highlight the relevance of the divergence operator which also partially motivates our discussion of the scaling law for the T_3 problem from Sect. 1.3.

To this end, we first recall that considering any first order homogeneous constant coefficient differential operator $\mathcal{A}(D) = \sum_{j=1}^d A^j \partial_j : C^\infty(\mathbb{R}^d; \mathbb{R}^n) \rightarrow C^\infty(\mathbb{R}^d; \mathbb{R}^m)$, we can rewrite $\mathcal{A}(D)u = \operatorname{div} \omega_1(u)$ with a linear map $\omega_1 : \mathbb{R}^n \rightarrow \mathbb{R}^{m \times d}$. Indeed, let us define

$$\omega_1 : \mathbb{R}^n \rightarrow \mathbb{R}^{m \times d}, \quad \omega_1(x) = \left(\sum_{k=1}^n A_{ik}^j x_k \right)_{\substack{i=1, \dots, m, \\ j=1, \dots, d}}.$$

Let $u \in C^\infty(\mathbb{R}^d; \mathbb{R}^n)$, then it holds $\mathcal{A}(D)u = (\operatorname{div} \circ \omega_1)(u)$ for the row-wise divergence.

Indeed this can be generalized for higher order operators $\mathcal{A}(D)u = \sum_{|\alpha|=k} A^\alpha \partial^\alpha u$, where $A^\alpha \in \mathbb{R}^{m \times n}$ are coefficient matrices, as follows. For this we denote the space of symmetric k tensors on $\mathbb{R}^d$ by $\operatorname{Sym}(\mathbb{R}^d, k)$. Let the k -th order divergence be given as

$$\begin{aligned} \operatorname{div}^k : C^\infty(\mathbb{R}^d; \mathbb{R}^m \otimes \operatorname{Sym}(\mathbb{R}^d, k)) &\rightarrow C^\infty(\mathbb{R}^d; \mathbb{R}^m), \\ (\operatorname{div}^k u)_j &:= \sum_{1 \leq i_1 \leq \dots \leq i_k \leq d} \partial_{i_1} \dots \partial_{i_k} u_{ji_1 \dots i_k}. \end{aligned} \quad (\text{B.3})$$

For $k = 1$ this is exactly the row-wise divergence as mentioned above.

Remark B.1 This definition is natural in that sense that this operator (up to a sign) is the adjoint of the k -th derivative D^k .

The linear map $\omega_k : \mathbb{R}^n \rightarrow \mathbb{R}^m \otimes \operatorname{Sym}(\mathbb{R}^d, k)$ then takes the form

$$(\omega_k(x))_{ji_1 \dots i_k} := \left(A^{\sum_{l=1}^k e_{i_l}} x \right)_j \quad (\text{B.4})$$

and by this choice it holds $\mathcal{A}(D)u = (\operatorname{div}^k \circ \omega_k)(u)$ for any $u \in C^\infty(\mathbb{R}^d; \mathbb{R}^n)$ and $\ker \omega_k = I_{\mathcal{A}} = \bigcap_{|\alpha|=k} \ker A^\alpha$. In what follows, we omit the k dependence of $\omega = \omega_k$ in the notation.

With this in hand, it is possible to bound the energy for a general homogeneous linear operator $\mathcal{A}(D)$ (of order k) by the corresponding energy for the (k -th order) divergence (cf. [28, Appendix] for the corresponding qualitative result in the case $k = 1$).

Proposition B.2 *Let $d, n, k \in \mathbb{N}$, $\mathcal{K} \subset \mathbb{R}^n$, and let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Let $\mathcal{A}(D)$ be a k -th order homogeneous linear differential as in (2) and the elastic and surface energies be given by (16) and (17). Moreover let $\omega = \omega_k$ be the linear transformation in (B.4) and div^k the generalized k -th order divergence in (B.3) with the corresponding energies $E_{el}^{\operatorname{div}^k}$, $E_{surf}^{\operatorname{div}^k}$. Then there exist constants $C_1, C_2 > 0$ such that for any $\chi \in BV(\Omega; \mathcal{K})$, $F \in \mathbb{R}^n$*

$$E_{el}(\chi; F) \geq C_1 E_{el}^{\operatorname{div}^k}(\omega(\chi); \omega(F)),$$

$$E_{surf}(\chi) \geq C_2 E_{surf}^{\operatorname{div}^k}(\omega(\chi)).$$

Moreover if $\mathcal{A}(D)$ is cocanceling (and thus ω is injective), it also holds for all $u \in \mathcal{D}_F$, $\chi \in BV(\Omega; \mathcal{K})$ (cf. (10))

$$E_{el}(u, \chi) + \epsilon E_{surf}(\chi) \sim E_{el}^{\operatorname{div}^k}(\omega(u), \omega(\chi)) + \epsilon E_{surf}^{\operatorname{div}^k}(\omega(\chi)).$$

Proof In order to obtain the desired result, we use the pointwise bound $|u - \chi| \geq C|\omega(u) - \omega(\chi)|$ and consider the adapted boundary data: For any $u \in \mathcal{D}_F$ with $\mathcal{D}_F$ denoting the set from (10), the composition $\omega(u)$ satisfies $\operatorname{div}^k \omega(u) = \mathcal{A}(D)u = 0$ in $\mathbb{R}^d$ and $\omega(u) = \omega(F)$ in $\mathbb{R}^d \setminus \overline{\Omega}$. In other words, it holds that $\omega(u) \in \mathcal{D}_{\omega(F)}^{\operatorname{div}^k}$ for the divergence operator and boundary data $\omega(F)$. For the elastic energy (denoting by $E_{el}^{\operatorname{div}^k}$, $\mathcal{D}_{\omega(F)}^{\operatorname{div}^k}$ the energy and domain for the divergence operator) this implies

$$E_{el}(u, \chi) = \int_{\Omega} |u - \chi|^2 dx \geq C \int_{\Omega} |\omega(u) - \omega(\chi)|^2 dx = C E_{el}^{\operatorname{div}^k}(\omega(u), \omega(\chi)), \quad (\text{B.5})$$

$$E_{el}(\chi; F) \geq C \inf_{u \in \mathcal{D}_F} \int_{\Omega} |\omega(u) - \omega(\chi)|^2 dx \geq C \inf_{u: \omega(u) \in \mathcal{D}_{\omega(F)}^{\operatorname{div}^k}} \int_{\Omega} |\omega(u) - \omega(\chi)|^2 dx.$$

We emphasize that, in general, this only yields lower bound *inequalities* since ω is possibly not injective and thus there may be deformations u with $\mathcal{A}(D)u = 0$ and $u \neq F$ outside Ω but still fulfilling $\omega(u) = \omega(F)$ outside Ω (see the example in Sect. B.1 below). Replacing now $\omega(u)$ by a general function $w : \mathbb{R}^d \rightarrow \mathbb{R}^m \otimes \operatorname{Sym}(\mathbb{R}^d, k)$ such that $w \in \mathcal{D}_{\omega(F)}^{\operatorname{div}^k}$ yields

$$E_{el}(\chi; F) \geq C \inf_{w \in \mathcal{D}_{\omega(F)}^{\operatorname{div}^k}} \int_{\Omega} |w - \omega(\chi)|^2 dx = C E_{el}^{\operatorname{div}^k}(\omega(\chi); \omega(F)).$$

Furthermore, as $|\nabla \chi| \geq c|\nabla(\omega(\chi))|$, we can also bound the surface energy

$$E_{surf}(\chi) = \int_{\Omega} |\nabla \chi| \geq c \int_{\Omega} |\nabla(\omega(\chi))| = c E_{surf}^{\operatorname{div}^k}(\omega(\chi)). \quad (\text{B.6})$$

In the case of a cocanceling operator ω is injective and we also have the bounds $|u - \chi| \leq C|\omega(u) - \omega(\chi)|$, $|\nabla \chi| \leq C|\nabla(\omega(\chi))|$, thus in (B.5) and (B.6) also the matching upper bounds hold, which concludes the proof. $\square$

As a consequence, lower bounds for the divergence operator often also imply lower bounds for more general operators. A particular setting (see [28]) for instance arises in the three state problem with $\mathcal{K} = \{A_1, A_2, A_3\}$ being such that $A_j - A_k \notin \Lambda_{\mathcal{A}}$. In this case also $\omega(\mathcal{K})$ consists of three states which is a result of the fact that the kernel of ω is given by

$$\ker(\omega) = \bigcap_{j=1}^d \ker A^j = \bigcap_{j=1}^d \ker \mathbb{A}(e_j) = I_{\mathcal{A}}.$$

In particular, if we find a T_3 structure for a general linear, homogeneous, constant coefficient, first order differential operator $\mathcal{A}(D)$ such that it is mapped to the T_3 structure in Sect. 4, we can exploit the same lower bound as for the divergence operator. In addition to the relevance of the divergence operator for applications, this argument serves as an additional motivation for focusing particularly on the divergence operator in this article.

Moreover with Proposition 3.8 in mind, also for pairwise non super-compatible wells, we can assume without loss of generality that $I_{\mathcal{A}} = \{0\}$ and thus ω is injective.

B.1 Comparison of the Two-State Problem for the Divergence Operator

In this section, we discuss the comparison between the general two-state problem for linear, homogeneous differential operators and the one for the (k -th order) divergence operator. In particular, this yields yet another proof of the compatible case in Theorem 1.

In the calculations from the first part of Sect. B, we notice that for an injective map ω we can also bound the quantities $E_{el}(u, \chi)$, $E_{surf}(\chi)$ from above with the corresponding term in which u, χ are replaced by $\omega(u)$, $\omega(\chi)$; hence for $I_{\mathcal{A}} = \{0\}$ it holds

$$E_{\epsilon}(u, \chi) \sim E_{\epsilon}^{\text{div}^k}(\omega(u), \omega(\chi)).$$

We here emphasize that this only holds on the level of *fixed* u, χ and that after the minimization in u , this does not necessarily yield a two-sided comparison of the energies any more. Indeed, while the lower bound estimates always hold (cf. Proposition B.2), this may not be true for the upper bound estimates. In fact, even if $I_{\mathcal{A}} = \{0\}$, we can at the moment not exclude that there may be $w \in \mathcal{D}_{\omega(F)}^{\text{div}^k} \setminus \omega(\mathcal{D}_F)$. We postpone a further discussion of this to future work.

The advantage of $I_{\mathcal{A}} = \{0\}$ is that we do not lose wells in that sense that for $I_{\mathcal{A}} = \{0\}$ also $\ker \omega = \{0\}$ and thus, ω is injective. As seen above in Sect. 3.4 for two wells $A, B \in \mathbb{R}^n$, $A - B \notin I_{\mathcal{A}}$ we can restrict to $\tilde{A}(D)$ which fulfills $I_{\tilde{A}} = \{0\}$. This implies that for two compatible wells, which are not super-compatible, in deducing lower scaling bounds, we can use the corresponding lower bounds of the divergence operator as we do not lose information.

Example B.3 In concluding this section, we give an example of an operator which is not cocanceling. Considering $d = 2$, $n = 3$, $m = 1$ and

$$\mathcal{A}(D)u = \partial_1 u_2 + \partial_2 u_3,$$

implies that $\omega : \mathbb{R}^3 \rightarrow \mathbb{R}^{1 \times 2}$, $\omega(x) = (x_2, x_3)$ and $\ker(\omega) = \text{span}(e_1) = I_{\mathcal{A}}$.

The reduced operator $\tilde{\mathcal{A}}(D)$ would act on mappings taking values only in $\{0\} \times \mathbb{R}^2 \subset \mathbb{R}^3$.

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Declarations

Competing Interests The authors have no competing interests to declare that are relevant to the content of this article.

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References

1. Chan, A., Conti, S.: Energy scaling and branched microstructures in a model for shape-memory alloys with $SO(2)$ invariance. *Math. Models Methods Appl. Sci.* **25**(06), 1091–1124 (2015)
2. Růland, A., Tribuzio, A.: On the energy scaling behaviour of a singularly perturbed Tartar square. *Arch. Ration. Mech. Anal.* **243**(1), 401–431 (2022)
3. Garroni, A., Nesi, V.: Rigidity and lack of rigidity for solenoidal matrix fields. *Proc. R. Soc. Lond., Ser. A, Math. Phys. Eng. Sci.* **460**(2046), 1789–1806 (2004). <https://doi.org/10.1098/rspa.2003.1249>
4. Palombaro, M., Ponsiglione, M.: The three divergence free matrix fields problem. *Asymptot. Anal.* **40**(1), 37–49 (2004)
5. Müller, S., Šverák, V.: Convex integration for Lipschitz mappings and counterexamples to regularity. *Ann. Math.* **157**(3), 715–742 (2003)
6. Kirchheim, B., Müller, S., Šverák, V.: Studying nonlinear PDE by geometry in matrix space. In: *Geometric Analysis and Nonlinear Partial Differential Equations*, pp. 347–395. Springer, Berlin (2003)
7. Székelyhidi, L. Jr.: The regularity of critical points of polyconvex functionals. *Arch. Ration. Mech. Anal.* **172**(1), 133–152 (2004)
8. De Lellis, C., Székelyhidi, L. Jr.: The Euler equations as a differential inclusion. *Ann. Math.* **170**(3), 1417–1436 (2009)
9. De Lellis, C., Székelyhidi, L. Jr.: The h -principle and the equations of fluid dynamics. *Bull. Am. Math. Soc.* **49**(3), 347–375 (2012)
10. Kuiper, N.H.: On C^1 -isometric imbeddings. II. In: *Indagationes Mathematicae (Proceedings)*, vol. 58, pp. 683–689. Elsevier, Amsterdam (1955)
11. Nash, J.: C^1 isometric imbeddings. *Ann. Math.* **60**(3), 383–396 (1954)
12. Conti, S., De Lellis, C., Székelyhidi, L. Jr.: h -principle and rigidity for $C^{1,\alpha}$ isometric embeddings. In: *Nonlinear Partial Differential Equations*, pp. 83–116. Springer, Berlin (2012)
13. De Lellis, C., Székelyhidi, L. Jr.: On h -principle and Onsager's conjecture. *News. - Eur. Math. Soc.* **95**, 19–24 (2015)
14. Ball, J.M., James, R.D.: Fine phase mixtures as minimizers of energy. In: *Analysis and Continuum Mechanics*, pp. 647–686. Springer, Berlin (1989)
15. Ball, J.M., James, R.D.: Proposed experimental tests of a theory of fine microstructure and the two-well problem. *Philos. Trans. R. Soc. Lond. A, Math. Phys. Eng. Sci.* **338**(1650), 389–450 (1992)
16. Bhattacharya, K., Firoozye, N.B., James, R.D., Kohn, R.V.: Restrictions on microstructure. *Proc. R. Soc. Edinb., Sect. A, Math.* **124**(5), 843–878 (1994)
17. Müller, S., Šverák, V.: Convex integration with constraints and applications to phase transitions and partial differential equations. *J. Eur. Math. Soc.* **1**(4), 393–422 (1999)
18. Müller, S.: Variational models for microstructure and phase transitions. In: *Calculus of Variations and Geometric Evolution Problems*, pp. 85–210. Springer, Berlin (1999)
19. Růland, A.: The cubic-to-orthorhombic phase transition: rigidity and non-rigidity properties in the linear theory of elasticity. *Arch. Ration. Mech. Anal.* **221**(1), 23–106 (2016)
20. Palombaro, M.: Rank- $(n-1)$ convexity and quasiconvexity for divergence free fields. *Adv. Calc. Var.* **3**(3), 279–285 (2010)
21. Palombaro, M., Smyshlyaev, V.P.: Relaxation of three solenoidal wells and characterization of extremal three-phase H -measures. *Arch. Ration. Mech. Anal.* **194**(3), 775–822 (2009)
22. Kohn, R.V., Wirth, B.: Optimal fine-scale structures in compliance minimization for a uniaxial load. *Proc. R. Soc. A, Math. Phys. Eng. Sci.* **470**(2170), 20140432 (2014)
23. Kohn, R.V., Wirth, B.: Optimal fine-scale structures in compliance minimization for a shear load. *Commun. Pure Appl. Math.* **69**(8), 1572–1610 (2016)
24. Potthoff, J., Wirth, B.: Optimal fine-scale structures in compliance minimization for a uniaxial load in three space dimensions (2021). [arXiv:2111.06910](https://arxiv.org/abs/2111.06910)
25. Kohn, R.V., DeSimone, A., Otto, F., Müller, S.: Recent analytical developments in micromagnetics. *Sci. Hyst.* **2**, 269–381 (2006)
26. DeSimone, A., Knüpfer, H., Otto, F.: 2-d stability of the Néel wall. *Calc. Var. Partial Differ. Equ.* **27**(2), 233–253 (2006)
27. De Philippis, G., Palmieri, L., Rindler, F.: On the two-state problem for general differential operators. *Nonlinear Anal.* **177**, 387–396 (2018)
28. Sorella, M., Tione, R.: The four-state problem and convex integration for linear differential operators (2021). [arXiv:2107.10785](https://arxiv.org/abs/2107.10785)
29. Skipper, J.W.D., Wiedemann, E.: Lower semi-continuity for $\mathcal{A}$ -quasiconvex functionals under convex restrictions (2019). [arXiv:1909.11543](https://arxiv.org/abs/1909.11543)
30. Bhattacharya, K.: *Microstructure of Martensite*. Oxford Series on Materials Modelling. Oxford University Press, London (2003)

31. Bhattacharya, K.: Comparison of the geometrically nonlinear and linear theories of martensitic transformation. *Contin. Mech. Thermodyn.* **5**(3), 205–242 (1993)
32. Warner, M., Terentjev, E.M.: *Liquid Crystal Elastomers*, vol. 120. Oxford University Press, London (2007)
33. Cesana, P., Della Porta, F., Rüland, A., Zillinger, C., Zwicknagl, B.: Exact constructions in the (non-linear) planar theory of elasticity: from elastic crystals to nematic elastomers. *Arch. Ration. Mech. Anal.* **237**(1), 383–445 (2020)
34. Lamy, X., Lorent, A., Peng, G.: Rigidity of a non-elliptic differential inclusion related to the Aviles–Giga conjecture. *Arch. Ration. Mech. Anal.* **238**(1), 383–413 (2020)
35. Lamy, X., Lorent, A., Peng, G.: On a generalized Aviles–Giga functional: compactness, zero-energy states, regularity estimates and energy bounds (2022). [arXiv:2203.05418](https://arxiv.org/abs/2203.05418)
36. Tartar, L.: Compensated compactness and applications to partial differential equations. In: *Nonlinear Analysis and Mechanics: Heriot-Watt Symposium*, vol. 4, pp. 136–212 (1979)
37. DiPerna, R.J.: Compensated compactness and general systems of conservation laws. *Trans. Am. Math. Soc.* **292**(2), 383–420 (1985)
38. Tartar, L.: The compensated compactness method applied to systems of conservation laws. In: *Systems of Nonlinear Partial Differential Equations*, pp. 263–285. Springer, Berlin (1983)
39. Murat, F., Tartar, L.: H-convergence. In: *Topics in the Mathematical Modelling of Composite Materials*, pp. 21–43. Springer, Berlin (2018)
40. Müller, S.: Rank-one convexity implies quasiconvexity on diagonal matrices. *Int. Math. Res. Not.* **1999**(20), 1087–1095 (1999)
41. Faraco, D., Székelyhidi, L. Jr.: Tartar’s conjecture and localization of the quasiconvex hull in $\mathbb{R}^{2 \times 2}$. *Acta Math.* **200**(2), 279–305 (2008)
42. Guerra, A., Raiță, B., Schrecker, M.R.I.: Compensated compactness: continuity in optimal weak topologies. *J. Funct. Anal.* **283**(7), 109596 (2022)
43. Arroyo-Rabasa, A., De Philippis, G., Rindler, F.: Lower semicontinuity and relaxation of linear-growth integral functionals under PDE constraints. *Adv. Calc. Var.* **13**(3), 219–255 (2020)
44. Raiță, B.: Potentials for $\mathcal{A}$ -quasiconvexity. *Calc. Var. Partial Differ. Equ.* **58**(3), 105 (2019)
45. Guerra, A., Raiță, B., Schrecker, M.: Compensation phenomena for concentration effects via nonlinear elliptic estimates (2021). [arXiv:2112.10657](https://arxiv.org/abs/2112.10657)
46. Raiță, B.: A simple construction of potential operators for compensated compactness (2021). [arXiv:2112.11773](https://arxiv.org/abs/2112.11773)
47. Behn, L., Gmeineder, F., Schiffer, S.: On symmetric div-quasiconvex hulls and divsym-free L^∞ -truncations (2021). [arXiv:2108.05757](https://arxiv.org/abs/2108.05757)
48. Fonseca, I., Müller, S.: $\mathcal{A}$ -Quasiconvexity, lower semicontinuity, and young measures. *SIAM J. Math. Anal.* **30**(6), 1355–1390 (1999)
49. Kristensen, J., Raiță, B.: An introduction to generalized Young measures. *Max-Planck-Institut für Mathematik in den Naturwissenschaften Leipzig* **45** (2020)
50. Conti, S., Gmeineder, F.: $\mathcal{A}$ -Quasiconvexity and Partial Regularity (2020). [arXiv:2009.13820](https://arxiv.org/abs/2009.13820)
51. Gmeineder, F., Lewintan, P., Neff, P.: Optimal incompatible Korn–Maxwell–Sobolev inequalities in all dimensions (2022). [arXiv:2206.10373](https://arxiv.org/abs/2206.10373)
52. De Philippis, G., Rindler, F.: On the structure of $\mathcal{A}$ -free measures and applications. *Ann. Math.* **184**(3), 1017–1039 (2016)
53. Arroyo-Rabasa, A., De Philippis, G., Hirsch, J., Rindler, F.: Dimensional estimates and rectifiability for measures satisfying linear PDE constraints. *Geom. Funct. Anal.* **29**(3), 639–658 (2019)
54. Breit, D., Diening, L., Gmeineder, F.: On the trace operator for functions of bounded $\mathbb{A}$ -variation. *Anal. PDE* **13**(2), 559–594 (2020)
55. Dacorogna, B.: *Direct Methods in the Calculus of Variations*, vol. 78. Springer, Berlin (2007)
56. Dacorogna, B., Marcellini, P.: *Implicit Partial Differential Equations*, vol. 37. Springer, Berlin (2012)
57. Pedregal, P.: *Parametrized Measures and Variational Principles*. Springer, Berlin (1997)
58. Chaudhuri, N., Müller, S.: Rigidity estimate for two incompatible wells. *Calc. Var. Partial Differ. Equ.* **19**(4), 379–390 (2004)
59. De Lellis, C., Székelyhidi, L. Jr.: Simple proof of two-well rigidity. *C. R. Math.* **343**(5), 367–370 (2006)
60. Friesecke, G., James, R.D., Müller, S.: A theorem on geometric rigidity and the derivation of nonlinear plate theory from three-dimensional elasticity. *Commun. Pure Appl. Math.* **55**(11), 1461–1506 (2002)
61. Lamy, X., Lorent, A., Peng, G.: Quantitative rigidity of differential inclusions in two dimensions (2022). [arXiv:2208.08526](https://arxiv.org/abs/2208.08526)
62. Kohn, R.V., Müller, S.: Branching of twins near an austenite—twinned-martensite interface. *Philos. Mag. A* **66**(5), 697–715 (1992)
63. Kohn, R.V., Müller, S.: Surface energy and microstructure in coherent phase transitions. *Commun. Pure Appl. Math.* **47**(4), 405–435 (1994)

64. Barroso, A.C., Matias, J., Santos, P.M.: Differential inclusions and $\mathcal{A}$ -quasiconvexity. *Mediterr. J. Math.* **3**(14), 1–14 (2017)
65. Van Schaftingen, J.: Limiting sobolev inequalities for vector fields and canceling linear differential operators. *J. Eur. Math. Soc.* **15**(3), 877–921 (2013)
66. Choksi, R., Kohn, R.V., Otto, F.: Domain branching in uniaxial ferromagnets: a scaling law for the minimum energy. *Commun. Math. Phys.* **201**(1), 61–79 (1999)
67. Conti, S., Diermeier, J., Melching, D., Zwicknagl, B.: Energy scaling laws for geometrically linear elasticity models for microstructures in shape memory alloys. *ESAIM Control Optim. Calc. Var.* **26**, 115 (2020)
68. Kohn, R.V.: Energy-driven pattern formation. In: *International Congress of Mathematicians*, vol. 1, pp. 359–383 (2007)
69. Conti, S., Kohn, R.V., Misiats, O.: Energy minimizing twinning with variable volume fraction, for two nonlinear elastic phases with a single rank-one connection. *Math. Models Methods Appl. Sci.* **32**(08), 1671–1723 (2022). <https://doi.org/10.1142/S0218202522500397>
70. Knüpfer, H., Kohn, R.V.: Minimal energy for elastic inclusions. *Proc. R. Soc. A, Math. Phys. Eng. Sci.* **467**(2127), 695–717 (2011)
71. Knüpfer, H., Kohn, R.V., Otto, F.: Nucleation barriers for the cubic-to-tetragonal phase transformation. *Commun. Pure Appl. Math.* **66**(6), 867–904 (2013)
72. Růland, A.: A rigidity result for a reduced model of a cubic-to-orthorhombic phase transition in the geometrically linear theory of elasticity. *J. Elast.* **123**(2), 137–177 (2016)
73. Capella, A., Otto, F.: A quantitative rigidity result for the cubic-to-tetragonal phase transition in the geometrically linear theory with interfacial energy. *Proc. R. Soc. Edinb., Sect. A, Math.* **142**(2), 273–327 (2012)
74. Capella, A., Otto, F.: A rigidity result for a perturbation of the geometrically linear three-well problem. *Commun. Pure Appl. Math.* **62**(12), 1632–1669 (2009)
75. Růland, A., Tribuzio, A.: On the energy scaling behaviour of singular perturbation models involving higher order laminates (2021). [arXiv:2110.15929](https://arxiv.org/abs/2110.15929)
76. Růland, A., Tribuzio, A.: On scaling laws for multi-well nucleation problems without gauge invariances (2022). [arXiv:2206.05164](https://arxiv.org/abs/2206.05164)
77. Růland, A., Taylor, J.M., Zillinger, C.: Convex integration arising in the modelling of shape-memory alloys: some remarks on rigidity, flexibility and some numerical implementations. *J. Nonlinear Sci.* **29**(5), 2137–2184 (2019)
78. Tartar, L.: Some remarks on separately convex functions. In: *Microstructure and Phase Transition*, pp. 191–204. Springer, Berlin (1993)
79. Chlebík, M., Kirchheim, B.: Rigidity for the four gradient problem. *J. Reine Angew. Math.* **2002**(551), 1–9 (2002)
80. Otto, F., Viehmann, T.: Domain branching in uniaxial ferromagnets: asymptotic behavior of the energy. *Calc. Var. Partial Differ. Equ.* **38**(1), 135–181 (2010)
81. Diermeier, J.: Domain branching in geometrically linear elasticity (2013)
82. Winter, M.: An example of microstructure with multiple scales. *Eur. J. Appl. Math.* **8**(2), 185–207 (1997)
83. Chipot, M.: The appearance of microstructures in problems with incompatible wells and their numerical approach. *Numer. Math.* **83**(3), 325–352 (1999)
84. Grafakos, L.: *Classical Fourier Analysis*. Graduate Texts in Mathematics, vol. 249. Springer, Berlin (2014)

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On scaling properties for a class of two-well problems for higher order homogeneous linear differential operators

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A summary of the article is given in Chapter 3.

ON SCALING PROPERTIES FOR A CLASS OF TWO-WELL PROBLEMS FOR HIGHER ORDER HOMOGENEOUS LINEAR DIFFERENTIAL OPERATORS*

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Abstract. We study the scaling behavior of a class of compatible two-well problems for higher order, homogeneous linear differential operators. To this end, we first deduce general lower scaling bounds which are determined by the vanishing order of the symbol of the operator on the unit sphere in the direction of the associated element in the wave cone. We complement the lower bound estimates by a detailed analysis of the two-well problem for generalized (tensor-valued) symmetrized derivatives with the help of the (tensor-valued) Saint-Venant compatibility conditions. In two spatial dimensions for highly symmetric boundary data (but arbitrary tensor order $m \in \mathbb{N}$) we provide upper bound constructions matching the lower bound estimates. This illustrates that for the two-well problem for higher order operators new scaling laws emerge which are determined by the Fourier symbol in the direction of the wave cone. The scaling for the symmetrized gradient from [A. Chan and S. Conti, *Math. Models Methods Appl. Sci.*, 25 (2015), pp. 1091–1124] which was also discussed in [B. Raiță, A. Rüland, and C. Tissot, *Acta Appl. Math.*, 184 (2023), 5] provides an example of this family of new scaling laws.

Key words. $\mathcal{A}$ -free inclusions, two-well problem, Saint-Venant condition, phase transformation, singular perturbation model, twinning

MSC codes. 35Q74, 35G05, 74N05, 74G99

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1. Introduction. It is the objective of this article to quantitatively study the two-well problem for a class of *higher order*, constant coefficient, linear differential operators generalizing the curl and curl curl as the annihilators of the gradient and of the symmetrized gradient, respectively. We seek to illustrate that for this class of operators and a suitable class of wells the *maximal vanishing order* of the associated symbols on the unit sphere determines the scaling behavior of corresponding singular perturbation problems. In particular, we show that such higher order operators may lead to a scaling behavior which no longer satisfies the typical $\epsilon^{\frac{2}{3}}$ scaling behavior, first obtained in [30, 31].

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1.1. The classical two-well problem for the gradient. In order to put our results into perspective, let us first recall the classical (compatible) two-well problem for the gradient. The qualitative and quantitative compatible two-well problem for the gradient without gauge invariance is a well-studied problem motivated by questions from materials science [3, 34, 4]. In its quantitative forms it is a prototypical problem in the vector-valued calculus of variations, giving rise to pattern formation problems.

As a first *qualitative* observation, one notes that the only exact solutions to the differential inclusion corresponding to the compatible two-well problem

$$(1) \quad \nabla v \in \{A, B\} \text{ a.e. in } \Omega, \quad v \in W_{loc}^{1,\infty}(\mathbb{R}^d; \mathbb{R}^d),$$

with $A, B \in \mathbb{R}^{d \times d}$ and $\text{rank}(A - B) = 1$, are so-called simple laminates [3, 34]. These solutions are locally of the form $v(x) = f(n \cdot x) + \text{affine function}$, where $f : \mathbb{R} \rightarrow \mathbb{R}^d$ and $n \in \mathbb{S}^{d-1}$ is (up to its sign) determined by the relation $A - B = a \otimes n$ for some $a \in \mathbb{R}^d$. In particular, for $\lambda \in (0, 1)$ and $F_\lambda := \lambda A + (1 - \lambda)B$, there are no solutions to (1) with $\nabla v = F_\lambda$ in $\mathbb{R}^d \setminus \Omega$ if Ω is, e.g., bounded.

When viewing the two-well problem energetically, by minimizing elastic energies of the form

$$E_{el}(v, \chi) := \int_{\Omega} |\nabla v - \chi_A A - \chi_B B|^2 dx,$$

where for every $x \in \Omega$ we let $\chi(x) := \chi_A(x)A + \chi_B(x)B \in \{A, B\}$, $\chi_A(x), \chi_B(x) \in \{0, 1\}$, among

$$\mathcal{D}_{F_\lambda} := \left\{ v \in W_{loc}^{1,2}(\mathbb{R}^d; \mathbb{R}^d) : \nabla v = F_\lambda \text{ in } \mathbb{R}^d \setminus \overline{\Omega} \right\},$$

a rather different behavior emerges: Although no exact solutions to (1) exist, due to the lack of lower-semicontinuity it still holds that $\inf_{\chi \in L^2(\Omega; \{A, B\})} \inf_{v \in \mathcal{D}_{F_\lambda}} E_{el}(v, \chi) = 0$. The boundary conditions enforce oscillations and thus result in infinitely fine-scale structure [3, 34, 4]. A relaxation leads to the notion of gradient Young measure solutions, a type of generalized solutions (parametrized measures) which describe the oscillatory behavior of minimizing sequences.

Motivated by the discrepancy between the exact differential inclusion and its energetically quantified version and seeking to study finer properties of the two-well problem, an important class of models consists of *singular perturbation models*, penalizing fine oscillations. Instead of only minimizing the elastic energy, one here also considers additional (regularizing) surface energies

$$E_{surf}(\chi) := \int_{\Omega} |\nabla \chi|,$$

and

$$E_\epsilon(v, \chi) := E_{el}(v, \chi) + \epsilon E_{surf}(\chi).$$

Here $\int_{\Omega} |\nabla \chi|$ denotes the total variation norm of $\nabla \chi$, the distributional gradient of $\chi \in BV(\Omega; \{A, B\})$. Due to the higher order regularization term, the energy no longer permits arbitrarily fine oscillations but introduces a length scale depending on $\epsilon > 0$ and thus selects microstructure, e.g., it can distinguish between simple laminate and branching type structures [30, 31]. Comparing the regularized singular perturbation problem and the nonregularized “elastic” energies, it is particularly interesting to

investigate the limit $\epsilon \rightarrow 0$ and the scaling behavior of the singular perturbation problem in $\epsilon > 0$ as $\epsilon \rightarrow 0$. In this context, the celebrated results [30, 31] assert that the minimal energy does *not* display the scaling behavior of simple laminates but that of branching type structures (at least in generic, nondegenerate domain geometries); see also [8]. In [11] this observation is further strengthened by proving that minimizers are asymptotically self-similar. The scaling behavior of these energies thus encodes important information on the interaction of the differential constraint (i.e., the condition of dealing with gradients) and the nonlinearity (i.e., the two-well nature of the problem). Motivated by problems from materials science, similar $\epsilon^{\frac{2}{3}}$ results have been obtained for generalizations of the differential constraint, including, for instance, divergence and symmetrized gradient constraints [5, 6, 32, 10]. However, it was pointed out in the work [8] that the scaling behavior does not *always* have to be of the order $\epsilon^{\frac{2}{3}}$. Indeed, in [8] in a “degenerate” setting (with only one rank-one direction, compared to the generic setting of two rank-one directions) a scaling of the order $\epsilon^{\frac{4}{5}}$ was observed.

1.2. The compatible two-well problem for constant coefficient, linear differential operators. Motivated by the outlined problems from materials science and the qualitative study of the incompatible two-well problem from [19] for homogeneous linear differential operators, in [39], we started to systematically study the scaling properties of the compatible two-well problem depending on the class of differential operators at hand. To this end, we considered general, homogeneous, constant coefficient, linear differential operators

$$\mathcal{A}(D)u := \sum_{|\alpha|=m} A_\alpha \partial^\alpha u$$

for $u : \mathbb{R}^d \rightarrow \mathbb{R}^n$, $A_\alpha \in \mathbb{R}^{k \times n}$. The two-well problem from (1) then turns into: Find $u \in L^2_{loc}(\mathbb{R}^d; \mathbb{R}^n)$ such that

$$(2) \quad \begin{aligned} u &\in \{A, B\} \text{ in } \Omega, \\ \mathcal{A}(D)u &= 0 \text{ in } \mathbb{R}^d \text{ distributionally,} \\ u &= F_\lambda \text{ in } \mathbb{R}^d \setminus \overline{\Omega}, \end{aligned}$$

where $A - B \in \Lambda_{\mathcal{A}} \setminus I_{\mathcal{A}}$ and $F_\lambda = \lambda A + (1 - \lambda)B$ with $\lambda \in (0, 1)$. Here

$$(3) \quad \Lambda_{\mathcal{A}} := \bigcup_{\xi \in \mathbb{S}^{d-1}} \ker \mathbb{A}(\xi)$$

denotes the *wave cone* introduced in [48, 35] and

$$(4) \quad I_{\mathcal{A}} := \bigcap_{\xi \in \mathbb{S}^{d-1}} \ker \mathbb{A}(\xi)$$

is the set of *supercompatible states* introduced in this context in [39], where we denote the *symbol* of $\mathcal{A}(D)$ by

$$(5) \quad \mathbb{A}(\xi) := \sum_{|\alpha|=m} A_\alpha \xi^\alpha.$$

It is known that the wave cone generalizes the presence of rank-one connections for the curl operator and that it is possible to construct generalized simple laminates from it.

Indeed, for $h : \mathbb{R} \rightarrow \{0, 1\}$, $A - B \in \Lambda_{\mathcal{A}}$, and $\xi \in \mathbb{R}^d$ such that $A - B \in \ker \mathbb{A}(\xi)$, the function

$$u(x) := (A - B)h(\xi \cdot x) + B$$

is a one-dimensional solution to $u \in \{A, B\}$, $\mathcal{A}(D)u = 0$.

Similarly as in the case of the gradient inclusion, it is possible to associate a singular perturbation problem to (2). To this end, we consider

$$(6) \quad E_{\epsilon}^{\mathcal{A}}(\chi; F_{\lambda}) := E_{el}^{\mathcal{A}}(\chi; F_{\lambda}) + \epsilon E_{surf}^{\mathcal{A}}(\chi) := \inf_{u \in \mathcal{D}_{F_{\lambda}}^{\mathcal{A}}} \int_{\Omega} |u - \chi|^2 dx + \epsilon \int_{\Omega} |\nabla \chi|,$$

where $\chi \in BV(\Omega; \{A, B\})$ and

$$(7) \quad \mathcal{D}_{F_{\lambda}}^{\mathcal{A}} := \left\{ u \in L_{loc}^2(\mathbb{R}^d; \mathbb{R}^n) : \mathcal{A}(D)u = 0 \text{ in } \mathcal{D}'(\mathbb{R}^d), u = F_{\lambda} \text{ in } \mathbb{R}^d \setminus \overline{\Omega} \right\}.$$

In the following we will often omit the superscript $\mathcal{A}$ in the notation of the energy $E_{\epsilon}^{\mathcal{A}}$ and the set $\mathcal{D}_{F_{\lambda}}^{\mathcal{A}}$. We remark that the outlined setting easily generalizes to operators acting on fields $u : \mathbb{R}^d \rightarrow V$ with V being a (real) vector space of dimension n . In particular, in what follows we will consider the cases $V = \text{Sym}(\mathbb{R}^d; m)$ and $V = \mathbb{R}^k \otimes \text{Sym}(\mathbb{R}^d; m)$ (see section 2.2).

Within this setting, in the article [39], as one of the main results, the first three authors proved that for *first order*, constant coefficient differential operators, the lower $\epsilon^{\frac{2}{3}}$ scaling behavior is generic, provided that the wells are chosen to be *compatible* but not supercompatible.

THEOREM 1.1 (see [39, Theorem 1]). *Let $d, n \in \mathbb{N}$. Let $\Omega \subset \mathbb{R}^d$ be open, bounded, and Lipschitz. Let $\mathcal{A}(D)$ be a homogeneous, constant coefficient, linear, first order differential operator and $A, B \in \mathbb{R}^n$ such that $A - B \in \Lambda_{\mathcal{A}} \setminus I_{\mathcal{A}}$; see (3) and (4). Let $\lambda \in (0, 1)$, $F_{\lambda} = \lambda A + (1 - \lambda)B$ and let $E_{\epsilon}^{\mathcal{A}}$ be as in (6) above. Then, there exist constants $C = C(\mathcal{A}(D), \Omega, d, \lambda, A, B) > 0$ and $\epsilon_0 = \epsilon_0(\mathcal{A}(D), \Omega, d, \lambda, A, B) > 0$ such that for any $\epsilon \in (0, \epsilon_0)$*

$$\inf_{\chi \in BV(\Omega; \{A, B\})} E_{\epsilon}^{\mathcal{A}}(\chi; F_{\lambda}) \geq C \epsilon^{\frac{2}{3}}.$$

While our result illustrates that the lower bounds from the two-well problem for the gradient persist for general *first order* operators, our argument strongly relied on the *linearity* of the associated first order symbols and the linear structure of the null set of

$$(8) \quad \xi \in \mathbb{R}^d \mapsto \mathbb{A}(\xi)(A - B).$$

Revisiting the example from [8] (which involves a second order differential operator) we could prove that the vanishing order of (8) gives rise to a *different*, non- $\epsilon^{\frac{2}{3}}$ -scaling behavior which had first been identified in [8]. It however remained an open problem to investigate the behavior of *more general, higher order operators*. In this context, central questions are the following:

- (Q1) What scaling behavior can emerge for higher order, homogeneous linear operators for their associated singularly perturbed compatible two-well problems?
- (Q2) Can one identify structural conditions (for $\mathcal{A}(D)$ and $A - B \in \Lambda_{\mathcal{A}} \setminus I_{\mathcal{A}}$) giving rise to certain scaling behavior?

In the present article, we begin to systematically investigate these questions by considering a rather general class of homogeneous, linear differential operators and by deducing lower bound estimates for these. These lead to nonstandard, non- $\epsilon^{\frac{2}{3}}$ -lower bound scaling behavior which is directly associated with the structure of the symbol (8). For a particular class of operators generalizing the symmetrized gradient and a class of particular wells and boundary data we complement these lower bounds with matching upper bounds proving their optimality.

1.3. Outline of the main results. Continuing our investigation from [39], it is our objective to investigate the scaling properties of *higher order compatible two-well problems*. To this end, on the one hand, we study *specific families* of such operators in detail. On the other hand, we systematically investigate the lower bound scaling behavior for a rather general class of constant coefficient, homogeneous, linear differential operators. Our results and in particular the specific example classes illustrate that, in general, for higher order, homogeneous, linear differential operators *different scaling behavior* may arise than for first order operators. More precisely, we show that the scaling behavior for our classes of operators depends in a precise way on the *maximal vanishing order* of the associated symbol restricted to the unit sphere. This generalizes and systematizes the example from [39, section 3.5] which was based on the results from [8] and provides further, new scaling laws of higher order.

In order to explain this, let us describe the precise set-up of our problem. In what follows, we will first focus on a family of homogeneous, higher order linear differential operators, generalizing the curl (and curl curl) operators as well as their associated potentials. In a second step, we will then discuss a rather general class of lower scaling bounds for which the estimates will be determined by the *maximal vanishing order* of the associated symbol (8) restricted to the unit sphere.

1.4. Scaling results for generalized symmetrized gradients. Let us begin by considering the case of the curl (and curl curl) operator and its generalizations. It is the content of Poincaré's lemma that for a simply connected domain a *one-tensor field* (i.e., a vector field) is a *gradient* if and only if its curl vanishes. Similarly, it is well-known and often used in geometrically linearized elasticity that a *symmetric two-tensor field* (i.e., a symmetric matrix field) on a simply connected domain is a *symmetrized gradient* if and only if it satisfies the Saint-Venant compatibility conditions. More precisely, on the whole space, a necessary and sufficient condition for a tensor field $u : \mathbb{R}^d \rightarrow \mathbb{R}_{\text{sym}}^{d \times d}$ to be a symmetrized derivative associated with some $v : \mathbb{R}^d \rightarrow \mathbb{R}^d$, i.e.,

$$u = e(v) := D^{\text{sym}} v := \frac{1}{2}(\nabla v + (\nabla v)^t),$$

is given by the vanishing of the curlcurl operator, or equivalently the validity of the system of differential equations given by

$$\partial_{kl}^2 u_{ij} + \partial_{ij}^2 u_{kl} - \partial_{jk}^2 u_{il} - \partial_{il}^2 u_{jk} = 0, \quad i, j, k, l \in \{1, \dots, d\}.$$

Higher order symmetric tensors play a major role in inverse problems and tensor tomography. In the Euclidean setting, the ray transform is given by

$$\begin{aligned} I_m : C_c^\infty(\mathbb{R}^d; \text{Sym}(\mathbb{R}^d; m)) &\rightarrow C^\infty(T\mathbb{S}^{d-1}), \\ I_m f(x, \xi) &:= \int_{\mathbb{R}} \sum_{i_1, \dots, i_m=1}^d f_{i_1 \dots i_m}(x + t\xi) \xi^{i_1} \dots \xi^{i_m} dt. \end{aligned}$$

Seeking to recover the higher rank tensor f from measurements of $I_m f$, a (generalized) Helmholtz type decomposition into a potential and solenoidal part plays a major role (see, for instance, [36, 25, 37, 46] dealing with geometric versions of the ray transform). Indeed, it is at best the solenoidal part of f which can be reconstructed from the knowledge of I_m , while the potential part is characterized by the generalized Saint-Venant compatibility conditions and cannot be inferred from the measurements of $I_m f$. Hence, additional structural conditions, such as the two-valuedness, of the potential part may be imposed and investigated.

Following [46, Chapter 2], in this article as a model class of higher order differential operators we will study such generalizations of the Saint-Venant compatibility condition for *higher rank tensors fields*

$$u : \mathbb{R}^d \rightarrow \text{Sym}(\mathbb{R}^d; m) := \left\{ M \in (\mathbb{R}^d)^{\otimes m} : M \text{ is symmetric} \right\}$$

and their interaction with the nonlinear constraint given by the two-well problem. More precisely, for $u : \mathbb{R}^d \rightarrow \text{Sym}(\mathbb{R}^d; m)$, we consider the m th order differential operator $\mathcal{A}(D)$ with

$$(9) \quad [\mathcal{A}(D)u]_{i_1 j_1 i_2 j_2 \dots i_m j_m} := \alpha_{i_1 j_1} \circ \alpha_{i_2 j_2} \circ \dots \circ \alpha_{i_m j_m} (\partial_{j_1 \dots j_m}^m u_{i_1 \dots i_m}).$$

Here, by $\alpha_{i_k i_l}$ we denote the *alternation operator* in the indices i_k, i_l by

$$\alpha_{i_k i_l} (M_{i_1 \dots i_k \dots i_l \dots i_{2m}}) := \frac{1}{2} (M_{i_1 \dots i_{2m}} - M_{i_1 \dots i_l \dots i_k \dots i_{2m}}).$$

Now the differential operator $\mathcal{A}(D)$ generalizes the Saint-Venant compatibility conditions in the sense that it holds (for compactly supported maps) that $\mathcal{A}(D)u = 0$ for $u : \mathbb{R}^d \rightarrow \text{Sym}(\mathbb{R}^d; m)$ if and only if u is the symmetrized derivative of a tensor valued map [46, Theorem 2.2.1]. In this context, we say that a mapping $u : \mathbb{R}^d \rightarrow \text{Sym}(\mathbb{R}^d; m)$ is a *symmetrized derivative* if there is $v : \mathbb{R}^d \rightarrow \text{Sym}(\mathbb{R}^d; m-1)$ such that $u = D^{\text{sym}} v$; cf. (12) in section 2.1. We consider some specific examples of this set-up in Example 2.1 in section 2.2.

Generalizing the $\epsilon^{\frac{2}{3}}$ -scaling result for the gradient inclusion from [30, 31] and the observations made in [8] that for $\mathcal{A}(D) = \text{curl curl}$ there are instances of an $\epsilon^{\frac{4}{5}}$ -scaling, we prove the following analogous result to Theorem 1.1 for this model class of operators.

THEOREM 1.2 (symmetrized derivative). *Let $d, m \in \mathbb{N}, d \geq 2$, and $l \in \mathbb{N}^d$. Let $\Omega \subset \mathbb{R}^d$ be an open, bounded Lipschitz domain. Let $E_\epsilon^{\mathcal{A}}(\chi; F)$ be as above in (6) with the operator $\mathcal{A}(D)$ given in (9). Then the following scaling results hold:*

- Sharp scaling bounds for $d = 2, \lambda = \frac{1}{2}$. Let $d = 2, A - B = e_1^{\odot l_1} \odot e_2^{\odot l_2}$ (see (13) for the symmetric tensor notation) such that $l_1 + l_2 = m$ and $F = \frac{1}{2}A + \frac{1}{2}B$. Moreover let $\Omega = (0, 1)^2$. Then there exist constants $C > 1$ and $\epsilon_0 > 0$ (depending on m) such that for $L := \max\{l_1, l_2\}$ and for any $\epsilon \in (0, \epsilon_0)$

$$C^{-1} \epsilon^{\frac{2L}{2L+1}} \leq \inf_{\chi \in BV(\Omega; \{A, B\})} E_\epsilon^{\mathcal{A}}(\chi; F) \leq C \epsilon^{\frac{2L}{2L+1}}.$$

- Lower scaling bounds for $d \geq 2$. Let $d \geq 2, A - B = e_1^{\odot l_1} \odot e_2^{\odot l_2} \odot \dots \odot e_d^{\odot l_d}$ such that $\sum_{j=1}^d l_j = m$ and $F_\lambda := \lambda A + (1 - \lambda)B$ for some $\lambda \in (0, 1)$. Then there exist $C > 0$ and $\epsilon_0 > 0$ (depending on d, m, Ω and ϵ_0 also depending on λ) such that for $L := \max_{j \in \{1, 2, \dots, d\}} l_j$ and for any $\epsilon \in (0, \epsilon_0)$

$$C \min\{1 - \lambda, \lambda\}^2 \epsilon^{\frac{2L}{2L+1}} \leq \inf_{\chi \in BV(\Omega; \{A, B\})} E_\epsilon^{\mathcal{A}}(\chi; F_\lambda).$$

Let us comment on the assertions of the theorem. First, we remark that the scaling laws in Theorem 1.2 represent a *new* class of scaling laws for the two-well problem for higher order, linear differential operators. Second, this class of operators systematizes the observations from [8] and [39, section 3.4] in which a non- $\epsilon^{\frac{2}{3}}$ -scaling behavior emerges and which is determined by the *vanishing order of the associated symbols* on the unit sphere. We also mention that for this class of differential operators modeled on generalizations of the gradient and symmetrized gradient, our results from Theorem 1.2 do not cover all possible settings of A, B such that $A - B \in \Lambda_A$. Indeed, our choices of $A - B$ form a basis of the wave cone and yield the scaling behavior on these basis vectors. This, however, *does not* yield the scaling behavior of a general element in the wave cone. We seek to return to this in future work. Finally, let us comment on the constraints in the upper bound constructions in Theorem 1.2. The condition that $\lambda = \frac{1}{2}$ provides strong symmetry properties. In particular, it allows for both odd and even reflections of certain building block constructions (see Lemma 4.2). For general $\lambda \in (0, 1) \setminus \{\frac{1}{2}\}$ only weaker replacements (of odd reflections) are available which do not allow for an immediate generalization to an arbitrary choice of $\lambda \in (0, 1)$ and general tensor order. We further remark that providing matching upper bound constructions in higher dimensions $d > 2$ leads to new technical difficulties even in the case of the model operators. Indeed, in this case one would need to ensure the validity of the prescribed Dirichlet data on *all* sides of the cube. The construction given in this article only achieves the boundary datum on four sides, and would thus have to be modified correspondingly to achieve an admissible deformation in higher dimensions. To account for this, “rotation-type” arguments have been introduced in [39, 45] to match the Dirichlet data on all faces. It is expected that—at the expense of additional technicalities—similar ideas could also be of relevance in our context for the model operators under consideration. Finding matching upper bound constructions under the given strong Dirichlet conditions for general operators *beyond* model settings, however, remains a major challenge.

We show in Lemma 3.7 that a similar result as stated in Theorem 1.2 holds for the m th order divergence, which is defined in (19).

1.5. Lower bounds for a class of linear, homogeneous differential operators. Building on the specific example of the higher order curl and its potential, the generalized symmetrized derivative, we seek to study the scaling behavior of more general two-well problems for homogeneous, constant coefficient, linear differential operators. To this end, we systematically deduce lower scaling bounds for a rather large class of linear operators. As in the previous section, we consider singular perturbation problems as in (6), (7). A crucial role to determine lower bounds is played by the polynomial $p(\xi) = |A(\xi)(A - B)|^2$; cf. [39, Corollary 3.2]. To that end, we introduce the *maximal vanishing order on the unit sphere* of the nonnegative, homogeneous polynomial $p: \mathbb{R}^d \rightarrow \mathbb{R}$.

DEFINITION 1.3 (maximal vanishing order on the unit sphere). *Let $p \in \mathbb{R}[\xi]$ be a nonnegative, $2m$ homogeneous polynomial. Let V denote the zero set of p . We then define the maximal vanishing order $L[p]$ of p as*

$$L[p] := \min \left\{ \ell \in \mathbb{N} : \inf_{\xi \in \mathbb{S}^{d-1} \setminus V} \frac{p(\xi)}{\text{dist}_V(\xi)^{2\ell}} > 0 \right\},$$

where we denote the distance function to V by

$$(10) \quad \text{dist}_V(\xi) := \inf \{ |\xi - \zeta| : \zeta \in V \}.$$

With this notion in hand, we prove corresponding lower bound scaling estimates which hold for a large class of homogeneous, constant coefficient, linear differential operators $\mathcal{A}(D)$.

THEOREM 1.4. *Let $d, m \in \mathbb{N}, d \geq 2$. Let $\Omega \subset \mathbb{R}^d$ be an open and bounded Lipschitz domain. Let $\mathcal{A}(D)$ be a homogeneous, constant coefficient, linear differential operator and $A, B \in \mathbb{R}^n$ such that $A - B \in \Lambda_{\mathcal{A}} \setminus I_{\mathcal{A}}$. Let $p(\xi) = |\mathbb{A}(\xi)(A - B)|^2$ have the maximal vanishing order equal to $L \leq m$ (cf. Definition 1.3) and further assume $V = p^{-1}(0)$ to be a finite union of vector spaces. For $\lambda \in (0, 1)$ consider $F_\lambda = \lambda A + (1 - \lambda)B$ and let $E_\epsilon^{\mathcal{A}}$ be as in (6) above. Then, there exist constants $C = C(\mathcal{A}(D), A, B, d, m, \Omega, V) > 0$ and $\epsilon_0 = \epsilon_0(\mathcal{A}(D), \lambda, A, B, d, m, \Omega, V) > 0$ such that for any $\epsilon \in (0, \epsilon_0)$*

$$\inf_{\chi \in BV(\Omega; \{A, B\})} E_\epsilon^{\mathcal{A}}(\chi; F_\lambda) \geq C \min\{1 - \lambda, \lambda\}^2 \epsilon^{\frac{2L}{2L+1}}.$$

Let us comment on this. First, we highlight that for the special case of the higher order curl and its potential, the generalized symmetrized gradient, the lower bound from Theorem 1.4 coincides with the lower bounds from Theorem 1.2. Given the matching upper bounds for $d = 2$, $\lambda = \frac{1}{2}$ and arbitrary tensor order $m \in \mathbb{N}$ these are indeed optimal. The estimates from Theorem 1.4 are obtained by a combination of general high frequency bounds, quantitative coercivity estimates away from the zero set of p , and low frequency bounds for which we use careful localization arguments. Apart from the setting of the higher order curl and $d = 2$, $\lambda = \frac{1}{2}$, we do not know whether the bounds from Theorem 1.4 are optimal. We plan to study associated upper bound constructions in future work. Second, we point out that the assumption on V is such that the theorem is applicable for the model operators curl, curl curl, div (and their higher order generalizations). From a technical point of view, it allows for an easier splitting argument in Fourier space. Without the assumption that V is a finite union of linear spaces, more complicated structures may arise. It is feasible that covering arguments can be used to reduce these to similar settings as for the vector space case. For clarity of presentation, however, we do not discuss this in the present article.

1.6. Relation to the literature. The two-well problem and more generally the N -well problem are intensively studied questions in the nonconvex calculus of variations [34, 18, 26, 27, 9, 40]. They are closely related to questions on pattern formation in various materials, including, for instance, shape-memory alloys. By now there is a large literature also on quantitative results for associated singular perturbation problems which build on the seminal works of [30, 31], including, for instance, the articles [1, 8, 12, 16, 15, 14, 7, 6, 5, 21, 28, 29, 41, 42, 43, 45, 44]. While generalizations of the differential inclusions from materials science to more general linear differential operators were already studied in the context of compensated compactness [48, 49, 50, 35, 20, 17], renewed interest in the associated problems in the calculus of variations has recently arisen in the context of investigating structure conditions, e.g., in the context of Korn-type inequalities [22], compensated compactness [24, 23, 33, 38], and regularity results [13] and also in the context of N -well problems [19, 39, 47].

1.7. Outline of the article. The remainder of the article is structured as follows. After briefly recalling and fixing notation in section 2, in section 3 we turn to the derivation of lower scaling bounds. To this end, we first deduce lower bounds for the symbol and then translate these into scaling behavior, identifying the maximal vanishing order of the associated symbol on the unit sphere as the determining ingredient for these estimates. In section 4 we complement these bounds with upper

scaling bounds in the case of the generalized symmetrized gradient in two dimensions and specific boundary data but with general tensor order.

2. Notation and preliminaries. In this section we collect some background on the tensors under consideration. In particular, we recall a characterization for the higher order curl by the Saint-Venant conditions and compute the wave cone for the higher order curl and divergence.

2.1. Tensor notation. We denote the space of rank m -tensors by $T^m(\mathbb{R}^d) = (\mathbb{R}^d)^{\otimes m}$; the elements are thus multilinear maps $M : \prod_{j=1}^m \mathbb{R}^d \rightarrow \mathbb{R}$ with components given by $M_{i_1 i_2 \dots i_m} = M(e_{i_1}, e_{i_2}, \dots, e_{i_m})$, where $i_1, \dots, i_m \in \{1, \dots, d\}$ and with $e_1, \dots, e_d$ being the canonical basis of $\mathbb{R}^d$. The subspace of symmetric m -tensors is denoted by $\text{Sym}(\mathbb{R}^d; m) \subset T^m(\mathbb{R}^d)$, i.e., they satisfy $M(v_1, \dots, v_m) = M(v_{\tau(1)}, \dots, v_{\tau(m)})$ for any permutation $\tau \in \mathfrak{S}_m$ and $v_1, \dots, v_m \in \mathbb{R}^d$.

By $\sigma_{i_1 \dots i_m}$ we denote the *symmetrization operator* in the indices $i_1, \dots, i_m$ which is defined as

$$\sigma_{i_1 \dots i_m}(M_{i_1 \dots i_m}) = \frac{1}{m!} \sum_{\tau \in \mathfrak{S}_m} M_{i_{\tau(1)} \dots i_{\tau(m)}}, \quad M \in T^m(\mathbb{R}^d).$$

Similarly, we define the *alternation operators* $\alpha_{i_k i_l}$ as

$$(11) \quad \alpha_{i_k i_l}(M_{i_1 \dots i_m}) = \frac{1}{2} (M_{i_1 \dots i_m} - M_{i_1 \dots i_{k-1} i_l i_{k+1} \dots i_{l-1} i_k i_{l+1} \dots i_m}),$$

where, without loss of generality, we have assumed that $i_k < i_l$.

We say a mapping $u : \mathbb{R}^d \rightarrow \text{Sym}(\mathbb{R}^d; m)$ is a *symmetrized derivative* if there is $v : \mathbb{R}^d \rightarrow \text{Sym}(\mathbb{R}^d; m-1)$ such that

$$(12) \quad u_{i_1 \dots i_m} = [D^{\text{sym}} v]_{i_1 \dots i_m} := \sigma_{i_1 \dots i_m}(\partial_{i_1} v_{i_2 \dots i_m}) = \frac{1}{m!} \sum_{\tau \in \mathfrak{S}_m} \partial_{i_{\tau(1)}} v_{i_{\tau(2)} \dots i_{\tau(m)}}.$$

We refer to u as the *symmetrized derivative* of v .

Using the tensor product of vectors given by $[v^1 \otimes \dots \otimes v^m]_{i_1 \dots i_m} = \prod_{k=1}^m v_{i_k}^k$ for $v^1, \dots, v^m \in \mathbb{R}^d$, $i_1, \dots, i_m \in \{1, \dots, d\}$, we introduce the *symmetric tensor product* of vectors by setting

$$(13) \quad v^1 \odot \dots \odot v^m = \sigma_{1 \dots m}(v^1 \otimes \dots \otimes v^m)$$

for $v^1, \dots, v^m \in \mathbb{R}^d$. Both definitions can be adapted for tensor products of tensors instead of vectors, as these elementary tensors form a spanning set; thus for $M, N \in T^m(\mathbb{R}^d)$, we can write

$$M = \sum_{i_1, \dots, i_m=1}^d M_{i_1 \dots i_m} e_{i_1} \otimes \dots \otimes e_{i_m}, \quad N = \sum_{j_1, \dots, j_m=1}^d N_{j_1 \dots j_m} e_{j_1} \otimes \dots \otimes e_{j_m}$$

and therefore we have

$$M \otimes N = \sum_{i_1, \dots, i_m, j_1, \dots, j_m=1}^d M_{i_1 \dots i_m} N_{j_1 \dots j_m} e_{i_1} \otimes \dots \otimes e_{i_m} \otimes e_{j_1} \otimes \dots \otimes e_{j_m}.$$

An analogous operation is defined for symmetric tensors (for which an additional symmetrization is necessary). Moreover this can also be defined for tensors of different order. For convenience of notation, for $e_k \in \mathbb{R}^d$ and $j \in \mathbb{N}$ we also set

$$e_k^{\odot j} := e_k \odot \dots \odot e_k,$$

where the symmetrized product on the right-hand side is j times with itself.

In order to simplify the notation, we will use standard notation for multi-indices $l \in \mathbb{N}^d$. For given $l = (l_1, \dots, l_d) \in \mathbb{N}^d$ the absolute value is given by $|l| = \sum_{j=1}^d l_j$, the factorial by $l! = \prod_{j=1}^d l_j!$, and the multinomial coefficient by $\binom{|l|}{l} = \frac{|l|!}{l!}$. Moreover, for a given vector $\xi \in \mathbb{R}^d$ we use the convention that $\xi^l := \prod_{j=1}^d \xi_j^{l_j}$ and $\partial^l = \partial_1^{l_1} \dots \partial_d^{l_d}$.

2.2. Example operators. Using the alternation operators α_{ij} from (11), we consider the generalized Saint-Venant compatibility operator as the first example of an m th order operator $\mathcal{A}(D) : C^\infty(\mathbb{R}^d; \text{Sym}(\mathbb{R}^d; m)) \rightarrow C^\infty(\mathbb{R}^d; T^{2m}(\mathbb{R}^d))$ defined by

$$(14) \quad [\mathcal{A}(D)u]_{i_1 j_1 \dots i_m j_m} = \alpha_{i_1 j_1} \circ \dots \circ \alpha_{i_m j_m} (\partial_{j_1 \dots j_m}^m u_{i_1 \dots i_m}).$$

We provide the explicit formulas for this operator in the case $m \in \{1, 2, 3\}$ and $d = 2$.

Example 2.1 ($d = 2$). Fixing the spatial dimension $d = 2$, the compatibility conditions, which are given by a system of equations for general dimension d , become particularly transparent. More precisely, due to symmetry (see (15) below), they simplify to the single equation $[\mathcal{A}(D)u]_{12\dots 12} = 0$. Considering tensors of order $m = 1, 2, 3$, we obtain the following compatibility conditions.

- $m = 1$: In this case we compute that $[\mathcal{A}(D)u]_{ij} = \frac{1}{2}(\partial_j u_i - \partial_i u_j)$ and thus

$$\mathcal{A}(D)u = 0 \text{ if and only if } \partial_1 u_2 - \partial_2 u_1 = 0.$$

This is the well-known case of the characterization of gradients by means of Poincaré's lemma.

- $m = 2$: In the case of second order tensors we observe that

$$[\mathcal{A}(D)u]_{ikjl} = \frac{1}{4}(\partial_{kl}^2 u_{ij} + \partial_{ij}^2 u_{kl} - \partial_{il}^2 u_{kj} - \partial_{kj}^2 u_{il}).$$

Hence,

$$\mathcal{A}(D)u = 0 \text{ if and only if } \partial_{11}^2 u_{22} + \partial_{22}^2 u_{11} - 2\partial_{12}^2 u_{12} = \text{curl curl } u = 0.$$

This corresponds to the classical characterization of the symmetrized gradient by means of the Saint-Venant conditions.

- $m = 3$: For third order tensors, also only one independent equation remains:

$$\mathcal{A}(D)u = 0 \text{ if and only if}$$

$$[\mathcal{A}(D)u]_{121212} = \frac{1}{8}(\partial_{222}^3 u_{111} - \partial_{111}^3 u_{222} + 3\partial_{211}^3 u_{122} - 3\partial_{221}^3 u_{112}) = 0.$$

This is the characterization of being a symmetrized derivative. We emphasize that analogous characterizations can be obtained for tensors of arbitrary order and dimension.

Also beyond the case $d = 2$, the Saint-Venant operator characterizes symmetrized derivatives: A function $f \in C_c^\infty(\mathbb{R}^d; \text{Sym}(\mathbb{R}^d; m))$ fulfills $\mathcal{A}(D)f = 0$ if and only if f is a symmetrized derivative [46, Thm. 2.2.1, eq. (2.4.6), (2.4.7)]. In what follows, we will therefore also refer to the Saint-Venant operator as a higher order curl operator.

By definition of $\mathcal{A}(D)$, we have the following (anti)symmetries:

$$(15) \quad \begin{aligned} [\mathcal{A}(D)u]_{i_{\tau(1)}j_{\tau(1)}i_{\tau(2)}j_{\tau(2)}\dots i_{\tau(m)}j_{\tau(m)}} &= [\mathcal{A}(D)u]_{i_1 j_1 \dots i_m j_m} \text{ for all } \tau \in \mathfrak{S}_m, \\ [\mathcal{A}(D)u]_{i_1 j_1 \dots i_m j_m} &= -[\mathcal{A}(D)u]_{j_1 i_1 j_2 i_2 \dots j_m i_m}. \end{aligned}$$

In what follows, we will make use of these symmetries to further determine the symbol of $\mathcal{A}(D)$, as defined in (5). For $\xi \in \mathbb{R}^d$, componentwise, it is given by

$$\begin{aligned} [\mathbb{A}(\xi)M]_{i_1 j_1 \dots i_m j_m} &= \alpha_{i_1 j_1} \circ \dots \circ \alpha_{i_m j_m} (\xi_{j_1} \dots \xi_{j_m} M_{i_1 \dots i_m}) \\ (16) \quad &= \alpha_{i_1 j_1} \circ \dots \circ \alpha_{i_m j_m} (M(\xi_{j_1} e_{i_1}, \dots, \xi_{j_m} e_{i_m})) \\ &= 2^{-m} M(\xi_{j_1} e_{i_1} - \xi_{i_1} e_{j_1}, \dots, \xi_{j_m} e_{i_m} - \xi_{i_m} e_{j_m}) \end{aligned}$$

for $M \in \text{Sym}(\mathbb{R}^d; m)$. Moreover, by multilinearity for any orthonormal basis $v_1, \dots, v_d$ it holds that

$$\begin{aligned} (17) \quad [\mathbb{A}(\xi)M](v_{i_1}, v_{j_1}, \dots, v_{i_m}, v_{j_m}) \\ = 2^{-m} M((\xi \cdot v_{j_1})v_{i_1} - (\xi \cdot v_{i_1})v_{j_1}, \dots, (\xi \cdot v_{j_m})v_{i_m} - (\xi \cdot v_{i_m})v_{j_m}). \end{aligned}$$

Using these observations, we rewrite the symbol of the higher order curl in a concise way.

LEMMA 2.2. *Let $M = a_1 \odot \dots \odot a_m$ for $a_1, \dots, a_m \in \mathbb{R}^d$. For $\xi \in \mathbb{R}^d$ and $\mathbb{A}$ as in (16), we have*

$$(18) \quad \mathbb{A}(\xi)M = (a_1 \odot \xi) \odot \dots \odot (a_m \odot \xi).$$

Here we use the symbol $\odot$ to denote the antisymmetric tensor product of two vectors, i.e., $a \odot b = \frac{1}{2}(a \otimes b - b \otimes a)$ for $a, b \in \mathbb{R}^d$, and consider the symmetrized tensor product of tensors:

$$\begin{aligned} (a_1 \odot \xi) \odot \dots \odot (a_m \odot \xi) &:= \frac{1}{m!} \sum_{\tau \in \mathfrak{S}_m} (a_{\tau(1)} \odot \xi) \otimes \dots \otimes (a_{\tau(m)} \odot \xi) \\ &= \frac{1}{m!} 2^{-m} \sum_{\tau \in \mathfrak{S}_m} (a_{\tau(1)} \otimes \xi - \xi \otimes a_{\tau(1)}) \otimes \dots \otimes (a_{\tau(m)} \otimes \xi - \xi \otimes a_{\tau(m)}). \end{aligned}$$

Proof. By (16), we have

$$\begin{aligned} [\mathbb{A}(\xi)M]_{i_1 j_1 \dots i_m j_m} &= \frac{1}{m!} \sum_{\tau \in \mathfrak{S}_m} \left(\prod_{k=1}^m \frac{1}{2} a_{\tau(k)} \cdot (\xi_{j_k} e_{i_k} - \xi_{i_k} e_{j_k}) \right) \\ &= \frac{1}{m!} \sum_{\tau \in \mathfrak{S}_m} \left(\prod_{k=1}^m \frac{1}{2} ([a_{\tau(k)} \otimes \xi]_{i_k j_k} - [\xi \otimes a_{\tau(k)}]_{i_k j_k}) \right) \\ &= \frac{1}{m!} \sum_{\tau \in \mathfrak{S}_m} \left(\prod_{k=1}^m [a_{\tau(k)} \odot \xi]_{i_k j_k} \right) \\ &= \frac{1}{m!} \sum_{\tau \in \mathfrak{S}_m} [(a_{\tau(1)} \odot \xi) \otimes \dots \otimes (a_{\tau(m)} \odot \xi)]_{i_1 j_1 \dots i_m j_m}. \end{aligned}$$

This shows the claim. $\square$

Motivated by [39, Appendix B], i.e., by the fact that lower bounds on E_ϵ^A for any m th order homogeneous, constant coefficient, linear differential operator $\mathcal{A}(D)$ can be deduced by lower bounds for the m th order divergence, as a second model example of an m th order operator we consider the m th order divergence

$$\begin{aligned} \mathcal{B}(D) &= \text{div}^m : C^\infty(\mathbb{R}^d; \mathbb{R}^k \otimes \text{Sym}(\mathbb{R}^d; m)) \rightarrow C^\infty(\mathbb{R}^d; \mathbb{R}^k), \\ (19) \quad [\text{div}^m u]_j &:= \sum_{1 \leq i_1 \leq \dots \leq i_m \leq d} \partial_{i_1 \dots i_m}^m u_{j i_1 \dots i_m}, \quad j \in \{1, \dots, k\}, \end{aligned}$$

for some integer $k \geq 1$. It is straightforward to extend this definition (and all the results obtained in what follows for div^m) to m th order divergence-type operators acting on fields $u: \mathbb{R}^d \rightarrow W \otimes \text{Sym}(\mathbb{R}^d; m)$, where W is a (real) k -dimensional vector space. We denote the symbol of $\mathcal{B}(D)$ by $\mathbb{B}$.

Similarly as in Lemma 2.2, we give a precise formulation for the symbol of $\mathcal{B}(D)$ on a basis of $\mathbb{R}^k \otimes \text{Sym}(\mathbb{R}^d; m)$.

LEMMA 2.3. *Let $v \in \mathbb{R}^k$ and let $M = v \otimes e_1^{\odot l_1} \odot \cdots \odot e_d^{\odot l_d}$ for a partition $\sum_{j=1}^d l_j = m$. The symbol of $\mathcal{B}(D)$ is given by*

$$\mathbb{B}(\xi)M = \frac{l_1!l_2!\cdots l_d!}{m!} \prod_{j=1}^d \xi_j^{l_j} v = \binom{m}{l}^{-1} \xi^l v$$

with $l = (l_1, l_2, \dots, l_d) \in \mathbb{N}^d$, $\xi \in \mathbb{R}^d$.

Proof. Let $M = v \otimes e_1^{\odot l_1} \odot \cdots \odot e_d^{\odot l_d}$ for some $v \in \mathbb{R}^k$. Using that for $1 \leq i_1 \leq i_2 \leq \cdots \leq i_m \leq d$

$$[M]_{ji_1 \dots i_m} = \begin{cases} v_j \frac{l_1!l_2!\cdots l_d!}{m!}, & i_1, \dots, i_{l_1} = 1, i_{l_1+1}, \dots, i_{l_1+l_2} = 2, \dots, \\ & i_{m-l_d+1}, \dots, i_m = d, \\ 0 & \text{else,} \end{cases}$$

we obtain

$$\begin{aligned} [\mathbb{B}(\xi)M]_j &= \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_m \leq d} \xi_{i_1} \xi_{i_2} \cdots \xi_{i_m} M_{ji_1 \dots i_m} = \xi_1^{l_1} \xi_2^{l_2} \cdots \xi_d^{l_d} v_j \frac{l_1! \cdots l_d!}{m!} \\ &= v_j \binom{m}{l}^{-1} \xi^l. \end{aligned} \quad \square$$

In what follows, we will consider the extensions of $\mathcal{A}(D), \mathcal{B}(D)$ to distributional derivatives by duality.

2.3. Computation of the wave cones for symmetrized derivatives and the higher order divergence. With the above discussion in mind, in what follows, we consider the differential operators given by (14) and (19). We next identify the associated wave cones.

LEMMA 2.4 (higher order curl). *For $\xi \in \mathbb{R}^d$ let $\mathbb{A}(\xi)$ be given by (16), where the operator $\mathcal{A}(D)$ is given in (14). The kernel of $\mathbb{A}(\xi)$ is given by*

$$\begin{aligned} \ker \mathbb{A}(\xi) &= \text{span} \left\{ \sigma_{1\dots m}(v_1 \otimes \cdots \otimes v_m) : v_i \in \text{span}(\xi) \text{ for some } i \in \{1, \dots, m\} \right\} \\ &= \text{span} \left\{ \xi \odot a_2 \odot \cdots \odot a_m : a_2, \dots, a_m \in \mathbb{R}^d \right\}. \end{aligned}$$

Proof. We prove the claim of the lemma by showing that being in the span of $\{\xi \odot a_2 \odot \cdots \odot a_m : a_2, \dots, a_m \in \mathbb{R}^d\}$ is both necessary and sufficient for being an element of the kernel.

For simplicity we may assume that $\xi = e_1$ as by the homogeneity of $\mathbb{A}$ we have $\ker \mathbb{A}(\xi) = \ker \mathbb{A}(\frac{\xi}{|\xi|})$ and further we can choose an orthonormal basis $v_1, \dots, v_d$ such that $v_1 = \frac{\xi}{|\xi|}$, and thus

$$[\mathbb{A}(\xi)M](v_{i_1}, v_{j_1}, \dots, v_{i_m}, v_{j_m}) = |\xi|^m [\mathbb{A}(v_1)M](v_{i_1}, v_{j_1}, \dots, v_{i_m}, v_{j_m}).$$

This change of basis can be seen in (17).

Having fixed this, we seek to show that

$$\ker \mathbb{A}(e_1) = \text{span}\{e_1 \odot a_2 \odot \cdots \odot a_m : a_2, \dots, a_m \in \mathbb{R}^d\}.$$

First, let $M = e_1 \odot a_2 \odot \cdots \odot a_m$. The inclusion $\text{span}\{e_1 \odot a_2 \odot \cdots \odot a_m : a_2, \dots, a_m \in \mathbb{R}^d\} \subset \ker \mathbb{A}(e_1)$ is then immediate. Indeed, by (18), we see that $\mathbb{A}(\xi)M = 0$, since $e_1 \odot e_1 = 0$.

For the converse inclusion, we assume that $M \in \ker \mathbb{A}(e_1)$. Using (16), we consider the components given by $i_1, \dots, i_m \neq 1$:

$$[\mathbb{A}(e_1)M]_{i_1 1 \dots i_m 1} = M(e_{i_1}, \dots, e_{i_m}) = M_{i_1 \dots i_m} = 0.$$

Furthermore, as $e_{k_1} \odot \cdots \odot e_{k_m}$ for $1 \leq k_1 \leq k_2 \leq \cdots \leq k_m \leq d$ forms a basis of $\text{Sym}(\mathbb{R}^d; m)$ and as we have seen, the only nonvanishing components of M are those with at least one 1 in the index, we can write

$$\begin{aligned} M &= \sum_{i_1, i_2, \dots, i_m=1}^d M_{i_1 i_2 \dots i_m} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_m} \\ &= \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_m \leq d} \binom{m}{\sum_{p=1}^m e_{i_p}} M_{i_1 \dots i_m} e_{i_1} \odot \cdots \odot e_{i_m} \\ &= \sum_{1 \leq i_2 \leq \dots \leq i_m \leq d} \binom{m}{e_1 + \sum_{p=2}^m e_{i_p}} M_{1 i_2 \dots i_m} e_1 \odot e_{i_2} \odot \cdots \odot e_{i_m}. \end{aligned}$$

This shows that indeed $M \in \text{span}\{e_1 \odot a_2 \odot \cdots \odot a_m : a_2, \dots, a_m \in \mathbb{R}^d\}$. $\square$

In concluding this section, we also consider the m th order divergence and compute the structure of its wave cone.

LEMMA 2.5 (higher order divergence). *Let $\mathcal{B}(D)$ be the m th order divergence given by (19). Then the wave cone of $\mathcal{B}(D)$ is given by $(\xi \in \mathbb{R}^d)$*

$$\ker \mathbb{B}(\xi) = \text{span}\{v \otimes (a_1 \odot \cdots \odot a_m) : v \in \mathbb{R}^k, a_j \cdot \xi = 0 \text{ for some } j \in \{1, \dots, m\}\}.$$

Proof. We show the claim by using that $\ker \mathbb{B}(\xi) = (\text{ran } \mathbb{B}(\xi)^*)^\perp$, as the adjoint has a simple structure.

Indeed, considering the scalar product on symmetric tensors $S, T \in \text{Sym}(\mathbb{R}^d; m)$ given by $S \cdot T = \sum_{1 \leq i_1 \leq \dots \leq i_m \leq d} S_{i_1 \dots i_m} T_{i_1 \dots i_m}$, the adjoint is given by

$$\mathbb{B}(\xi)^* w = w \otimes \xi \otimes \xi \otimes \cdots \otimes \xi = w \otimes \xi^{\odot m} \in \mathbb{R}^k \otimes \text{Sym}(\mathbb{R}^d; m), \quad w \in \mathbb{R}^k.$$

Thus the kernel is given by

$$\begin{aligned} \ker \mathbb{B}(\xi) &= \{w \otimes \xi^{\odot m} : w \in \mathbb{R}^k\}^\perp \\ &= \text{span}\{v \otimes (a_1 \odot \cdots \odot a_m) : v \in \mathbb{R}^k, a_j \cdot \xi = 0 \text{ for some } j\}. \end{aligned}$$

This shows the statement. $\square$

3. Lower bound scaling results. With the characterization of the wave cones for the higher order curl and the higher order divergence in hand, in this section we turn to the proof of (general) lower scaling bounds. The core of this consists of an adaptation of the lower bound argument from [39] allowing us to deal with rather general zero sets consisting of a union of linear subspaces; see section 3.1.

To be more precise, we have the following lower bound of the energy for a general homogeneous, constant coefficient, linear operator $\mathcal{A}(D)$ and two wells $A, B \in \mathbb{R}^n$ with $F_\lambda = \lambda A + (1 - \lambda)B$: Writing $\chi = f(A - B) + F_\lambda \in BV(\Omega; \{A, B\})$ with $f \in BV(\Omega; \{1 - \lambda, -\lambda\})$ (extended to $\mathbb{R}^d$ by zero), [39, Corollary 3.2] states

$$(20) \quad E_\epsilon^{\mathcal{A}}(\chi; F_\lambda) \geq C \left(\int_{\mathbb{R}^d} \left| \mathbb{A} \left(\frac{\xi}{|\xi|} \right) (A - B) \right|^2 |\hat{f}|^2 d\xi + \epsilon \int_{\Omega} |\nabla f| \right).$$

Here and in the following, for every $f \in L^2(\mathbb{R}^d)$ we denote its Fourier transform by

$$\hat{f}(\xi) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-i\xi \cdot x} f(x) dx.$$

Moreover, we recall the definition of the distance function (cf. (10) in Definition 1.3) and note that dist_V is positively 1-homogeneous for V being a finite union of vector spaces. Now for the polynomial $p(\xi) = |\mathbb{A}(\xi)(A - B)|^2$ having the maximal vanishing order L (cf. Definition 1.3), we can further bound

$$E_\epsilon^{\mathcal{A}}(\chi; F_\lambda) \geq C \left(\int_{\mathbb{R}^d} \frac{\text{dist}_V(\xi)^{2L}}{|\xi|^{2L}} |\hat{f}|^2 d\xi + \epsilon \int_{\Omega} |\nabla f| \right).$$

Thus, to control the lack of coercivity near the zero set V of p , we can consider the multiplier to be given by $\text{dist}_V(\frac{\xi}{|\xi|})^{2L}$.

3.1. Scaling results. In this subsection, we provide the central estimates for our lower scaling bounds in the case that the symbol of the operator vanishes on a union of vector spaces. Let $\Omega \subset \mathbb{R}^d$ be a bounded set. We will work with functions $f \in L^2(\Omega)$ which we identify with their extensions by zero to the full space without mention.

As a central result in this section, we prove the following bounds.

PROPOSITION 3.1. *Let $d \in \mathbb{N}$, $d \geq 2$. Let L be a positive integer and let $\Omega \subset \mathbb{R}^d$ be an open, bounded Lipschitz domain. Suppose that $V \subset \mathbb{R}^d$, $V \neq 0$ is a union of finitely many linear spaces of dimension at most $d - 1$. Then the following estimates hold for every $\eta > 1$:*

(i) *For any $\delta \in (0, 1)$ there exists $\alpha = \alpha(\delta, \Omega, V) \in (0, 1)$ such that*

$$\int_{\text{dist}_V(\xi) \leq \alpha} |\hat{f}|^2 d\xi \leq \delta \int_{\mathbb{R}^d} |\hat{f}|^2 d\xi \quad \text{for } f \in L^2(\Omega).$$

(ii) *For all $\alpha > 0$*

$$\int_{\text{dist}_V(\xi) \geq \alpha, |\xi| \leq \eta} |\hat{f}|^2 d\xi \leq \left(\frac{\eta}{\alpha} \right)^{2L} \int_{\mathbb{R}^d} \frac{\text{dist}_V(\xi)^{2L}}{|\xi|^{2L}} |\hat{f}|^2 d\xi \quad \text{for } f \in L^2(\Omega).$$

(iii) *We have that*

$$\int_{|\xi| \geq \eta} |\hat{f}|^2 d\xi \leq C(d) \|f\|_\infty \eta^{-1} \left(\int_{\Omega} |\nabla f| + \|f\|_\infty \text{Per}(\Omega) \right) \quad \text{for } f \in L^\infty(\Omega) \cap BV(\Omega).$$

Summing the three estimates, with the constant α from (i), and absorbing the right-hand side of (i), we obtain that for any $f \in L^\infty(\Omega) \cap BV(\Omega)$,

$$\int_{\mathbb{R}^d} |\hat{f}|^2 d\xi \leq C \left(\frac{\eta}{\alpha} \right)^{2L} \int_{\mathbb{R}^d} \frac{\text{dist}_V(\xi)^{2L}}{|\xi|^{2L}} |\hat{f}|^2 d\xi + C\eta^{-1} \|f\|_\infty \left(\int_{\Omega} |\nabla f| + \|f\|_\infty \text{Per}(\Omega) \right),$$

where the constant $C > 0$ depends on δ, d .

The proof of (ii) is immediate from the definition of the domain of integration on the left-hand side. The proof of (iii) is known from [29]. The proof of (i) requires more attention, so we extract a relevant slicing lemma.

LEMMA 3.2. *Let $f \in L^2(\Omega)$, $1 \leq s \leq d$, be a positive integer. Write $\xi = (\xi', \xi'')$ for $\xi' \in \mathbb{R}^s, \xi'' \in \mathbb{R}^{d-s}$. We have that for $\mathcal{H}^{d-s}$ -a.e. $\xi'' \in \mathbb{R}^{d-s}$*

$$(21) \quad \text{ess sup}_{\xi' \in \mathbb{R}^s} |\hat{f}(\xi', \xi'')|^2 \leq \left(\frac{\text{diam } \Omega}{2\pi} \right)^s \int_{\mathbb{R}^s} |\hat{f}(\xi', \xi'')|^2 d\xi'.$$

Proof. It is instructive to first cover the case $s = d$. Then we have that

$$(22) \quad \begin{aligned} \|\hat{f}\|_{L^\infty} &\leq (2\pi)^{-\frac{d}{2}} \|f\|_{L^1} \leq (2\pi)^{-\frac{d}{2}} \mathcal{L}^d(\Omega)^{1/2} \|f\|_{L^2} \\ &\leq (2\pi)^{-\frac{d}{2}} (\text{diam } \Omega)^{d/2} \|f\|_{L^2} = \left(\frac{\text{diam } \Omega}{2\pi} \right)^{d/2} \|\hat{f}\|_{L^2}. \end{aligned}$$

To show the claim for $s \leq d$, we first notice that by Fubini's theorem, the right-hand side in (21) is finite for $\mathcal{H}^{d-s}$ a.e. ξ'' . The required estimate follows from (22) applied to $g = \mathcal{F}_{\xi''} f(\cdot, \xi'')$, provided we show that g has compact support in $\mathbb{R}^s$ of diameter at most $\text{diam } \Omega$. Let ω be the projection of Ω on $\mathbb{R}^s$, so that $\text{diam } \omega \leq \text{diam } \Omega$. Moreover, $f(x', x'') = 0$ for all $x' \in \mathbb{R}^s \setminus \omega$. Therefore,

$$g(x') = (2\pi)^{-\frac{d-s}{2}} \int_{\mathbb{R}^{d-s}} f(x', x'') e^{-ix'' \cdot \xi''} dx'' = 0 \quad \text{for } x' \in \mathbb{R}^s \setminus \omega$$

so g is supported inside ω . The proof is complete. $\square$

We can now return to the proof of Proposition 3.1 above.

Proof of Proposition 3.1. It remains to prove (i). To do this, we first assume that V is a linear space, which we identify with $\mathbb{R}^{d-s}$ for some $1 \leq s \leq d-1$. Writing coordinates $\mathbb{R}^d \ni \xi = (\xi', \xi'')$ with $\xi' \in \mathbb{R}^s = V^\perp, \xi'' \in \mathbb{R}^{d-s}$, we control, with $\alpha > 0$ to be determined,

$$\begin{aligned} \int_{|\xi'| \leq \alpha} |\hat{f}|^2 d\xi &= \int_{\mathbb{R}^{d-s}} \int_{|\xi'| \leq \alpha} |\hat{f}(\xi', \xi'')|^2 d\xi' d\xi'' \\ &\leq C(s) \alpha^s \int_{\mathbb{R}^{d-s}} \text{ess sup}_{\xi' \in \mathbb{R}^s} |\hat{f}(\xi', \xi'')|^2 d\xi'' \\ &\leq C(s) \left(\frac{1}{2\pi} \alpha \text{diam } \Omega \right)^s \int_{\mathbb{R}^{d-s}} \int_{\mathbb{R}^s} |\hat{f}(\xi', \xi'')|^2 d\xi' d\xi'', \end{aligned}$$

where to obtain the last inequality we use Lemma 3.2; here $C(s)$ denotes the area of the $s-1$ dimensional unit sphere. This is enough to conclude the proof of this case by taking $\alpha = \alpha(\delta, \Omega, V)$ small enough.

In the general case of a finite union of linear spaces, i.e., $V = \bigcup_{j=1}^N V_j$, where each V_j is a linear space, by the previous step, there exists $\alpha = \alpha(\delta, \Omega, V) > 0$ such that

$$\int_{\text{dist}_{V_j}(\xi) \leq \alpha} |\hat{f}|^2 d\xi \leq \frac{\delta}{N} \int_{\mathbb{R}^d} |\hat{f}|^2 d\xi.$$

It follows that

$$\int_{\text{dist}_V(\xi) \leq \alpha} |\hat{f}|^2 d\xi \leq \sum_{j=1}^N \int_{\text{dist}_{V_j}(\xi) \leq \alpha} |\hat{f}|^2 d\xi \leq \sum_{j=1}^N \frac{\delta}{N} \int_{\mathbb{R}^d} |\hat{f}|^2 d\xi,$$

which concludes the proof. $\square$

With the results of Proposition 3.1 in hand, we turn to the first lower scaling bounds.

PROPOSITION 3.3 (lower scaling bounds). *Let $d, L \in \mathbb{N}, d \geq 2$. Let $\Omega \subset \mathbb{R}^d$ be an open, bounded Lipschitz domain. Let $V \subset \mathbb{R}^d, V \neq \emptyset$ be a union of finitely many linear spaces of dimension at most $d-1$. For $f \in BV(\mathbb{R}^d; \{-\lambda, 0, 1-\lambda\})$ for $\lambda \in (0, 1)$ with $f = 0$ in $\mathbb{R}^d \setminus \overline{\Omega}$, $f \in \{1-\lambda, -\lambda\}$ in Ω , we consider the energies given by*

$$\tilde{E}_{el}(f) := \int_{\mathbb{R}^d} \frac{\text{dist}_V(\xi)^{2L}}{|\xi|^{2L}} |\hat{f}|^2 d\xi, \quad \tilde{E}_{surf}(f) := \int_{\Omega} |\nabla f|.$$

Then there exist $\epsilon_0 = \epsilon_0(d, \lambda, L, \Omega, V) > 0, C = C(d, L, \Omega, V) > 0$ such that for $\epsilon \in (0, \epsilon_0)$ we have the following lower bound:

$$\tilde{E}_{\epsilon}(f) := \tilde{E}_{el}(f) + \epsilon \tilde{E}_{surf}(f) \geq C \min\{1-\lambda, \lambda\}^2 \epsilon^{\frac{2L}{2L+1}}.$$

Proof. As V is a union of finitely many linear spaces of dimension at most $d-1$, we can apply Proposition 3.1 for any $\eta > 1$. Thus, as $\|f\|_{\infty} \leq 1$, there exist a constant $C = C(d) > 0$ independent of η and $\alpha = \alpha(\Omega, V) \in (0, 1)$ such that

$$\begin{aligned} \int_{\mathbb{R}^d} |\hat{f}|^2 d\xi &\leq C \left(\left(\frac{\eta}{\alpha} \right)^{2L} \tilde{E}_{el}(f) + \eta^{-1} \tilde{E}_{surf}(f) + \eta^{-1} \text{Per}(\Omega) \right) \\ &\leq C \alpha^{-2L} \left(\eta^{2L} \tilde{E}_{el}(f) + (\eta\epsilon)^{-1} \epsilon \tilde{E}_{surf}(f) + \eta^{-1} \text{Per}(\Omega) \right). \end{aligned}$$

Taking now $\eta = \epsilon^{-\frac{1}{2L+1}} > 1$, it follows that

$$\int_{\mathbb{R}^d} |\hat{f}|^2 d\xi \leq C \epsilon^{-\frac{2L}{2L+1}} \tilde{E}_{\epsilon}(f) + C \epsilon^{\frac{1}{2L+1}} \text{Per}(\Omega)$$

for some constant $C = C(d, L, \Omega, V) > 0$. As $f \in L^2(\Omega; \{1-\lambda, \lambda\})$ we can bound the L^2 norm from below, thus by Plancherel's identity we infer

$$\int_{\mathbb{R}^d} |\hat{f}|^2 d\xi = \int_{\mathbb{R}^d} |f|^2 dx \geq \min\{1-\lambda, \lambda\}^2 |\Omega|.$$

Choosing now $\epsilon_0 = \epsilon_0(d, \lambda, L, \Omega, V)$ such that $C \epsilon_0^{\frac{1}{2L+1}} \text{Per}(\Omega) \leq \frac{1}{2} \min\{1-\lambda, \lambda\}^2 |\Omega|$, we obtain

$$C^{-1} \frac{1}{2} \min\{1-\lambda, \lambda\}^2 |\Omega| \epsilon^{\frac{2L}{2L+1}} \leq \tilde{E}_{\epsilon}(f),$$

which is the desired inequality. $\square$

Proposition 3.3 directly leads to the proof of Theorem 1.4.

Proof of Theorem 1.4. By definition of the maximal vanishing order and from (20), there exists a constant $C = C(\mathcal{A}(D), A, B) > 0$ such that

$$E_{\epsilon}(\chi; F_{\lambda}) \geq C \left(\int_{\mathbb{R}^d} \frac{\text{dist}_V(\xi)^{2L}}{|\xi|^{2L}} |\hat{f}|^2 d\xi + \epsilon \int_{\Omega} |\nabla f| \right).$$

Here $f \in BV(\Omega; \{1-\lambda, -\lambda\})$ is determined by $\chi = (A-B)f + F_{\lambda}$ and extended to $\mathbb{R}^d$ by zero. For this we can apply Proposition 3.3 as by assumption V is a finite union of vector spaces and $V \neq \emptyset$ as $A-B \in \Lambda_{\mathcal{A}}$. This shows the desired claim. $\square$

3.2. Applications. As consequences of the estimates from the previous section, we can deduce lower scaling bounds for the higher order curl and the higher order divergence operators. Both fall into the class of operators for which Proposition 3.1 is applicable. In order to infer this, we begin by providing lower bound estimates for the associated multipliers.

LEMMA 3.4 (the higher order curl). *Let $d, m \in \mathbb{N}, d \geq 2$. Let $\mathcal{A}(D) : C^\infty(\mathbb{R}^d; \text{Sym}(\mathbb{R}^d; m)) \rightarrow C^\infty(\mathbb{R}^d; T^{2m}(\mathbb{R}^d))$ be the operator from (14) with its symbol $\mathbb{A}(\xi)$ given in (16) and for $\sum_{j=1}^d l_j = m$ let $M = e_1^{\odot l_1} \odot \dots \odot e_d^{\odot l_d}$, $V := \{\xi \in \mathbb{R}^d : \mathbb{A}(\xi)M = 0\}$, and $L = \max_{j=1, \dots, d} l_j \leq m$. Then,*

$$(23) \quad V = \bigcup_{j: l_j \neq 0} \text{span}(e_j),$$

and there exists a constant $C = C(d, m) > 0$ such that

$$(24) \quad \left| \mathbb{A} \left(\frac{\xi}{|\xi|} \right) M \right|^2 \geq C \frac{\text{dist}_V(\xi)^{2L}}{|\xi|^{2L}}.$$

Proof. We note that the equality in (23) follows from the characterization of the kernel of $\mathbb{A}(\xi)$ in Lemma 2.4. Indeed, by the assertion of Lemma 2.4 for $M = e_1^{\odot l_1} \odot \dots \odot e_d^{\odot l_d}$ the roots of $\mathbb{A}(\xi)M = 0$ are given by $\xi \in \bigcup_{j: l_j \neq 0} \text{span}(e_j)$.

In order to deduce (24) it suffices to prove that

$$\inf_{\xi \in \mathbb{S}^{d-1} \setminus V} \frac{|\mathbb{A}(\xi)M|^2}{\text{dist}_V(\xi)^{2L}} > C > 0$$

for some constant $C > 0$.

To show this claim, we fix $\delta \in (0, \frac{1}{2})$ such that $\text{dist}_V(\xi) \geq \delta$ for $\xi \in \mathbb{S}^{d-1}$ implies $|\mathbb{A}(\xi)M|^2 \geq C$ for some constant $C = C(\delta, \mathbb{A}) > 0$. We consider two cases for $\xi \in \mathbb{S}^{d-1} \setminus V$.

First, if $\text{dist}_V(\xi) \geq \delta$, then it holds that

$$\frac{|\mathbb{A}(\xi)M|^2}{\text{dist}_V(\xi)^{2L}} \geq |\mathbb{A}(\xi)M|^2 \geq C > 0.$$

It thus remains to consider the case that $\text{dist}_V(\xi) < \delta$. In this case there exists $j \in \{1, \dots, m\}$ such that $l_j \neq 0$ with $\text{dist}_V(\xi)^2 = 1 - \xi_j^2$ and therefore also $\min\{|\xi \pm e_j|\} < \varepsilon(\delta)$ with $\varepsilon(\delta) \rightarrow 0$ as $\delta \rightarrow 0$. Without loss of generality $|\xi - e_j|^2 < \varepsilon$. Thus, there exists $w \perp e_j, |w| = 1$, and $\rho \in (0, \varepsilon)$ such that $\xi = \frac{e_j + \rho w}{\sqrt{1 + \rho^2}}$. In the following we write $\tilde{\xi} = e_j + \rho w$.

Using the structure of $M = e_1^{\odot l_1} \odot \dots \odot e_d^{\odot l_d}$, (18), and that $e_j \odot \tilde{\xi} = \rho(e_j \odot w)$, we calculate

$$\mathbb{A}(\tilde{\xi})M = \rho^{l_j} (e_1 \odot \tilde{\xi})^{\odot l_1} \odot \dots \odot (e_j \odot w)^{\odot l_j} \odot \dots \odot (e_d \odot \tilde{\xi})^{\odot l_d}.$$

Moreover, as $\rho < 1$ and as $l_j \neq 0$, we know $\text{dist}_V(\tilde{\xi}) = \rho$ and therefore

$$(25) \quad \frac{|\mathbb{A}(\tilde{\xi})M|^2}{\text{dist}_V(\tilde{\xi})^{2L}} = (1 + \rho^2)^{L-m} \rho^{2l_j-2L} \left| (e_1 \odot \tilde{\xi})^{\odot l_1} \odot \dots \odot (e_j \odot w)^{\odot l_j} \odot \dots \odot (e_d \odot \tilde{\xi})^{\odot l_d} \right|^2.$$

Due to the convergence of $1 + \rho^2$ and $|(e_1 \odot \tilde{\xi})^{\odot l_1} \odot \dots \odot (e_j \odot w)^{\odot l_j} \odot \dots \odot (e_d \odot \tilde{\xi})^{\odot l_d}|^2$ to nonzero numbers for $\delta \rightarrow 0$ (and thus $\varepsilon, \rho \rightarrow 0$), these are uniformly bounded from below for $\delta > 0$ sufficiently small. Furthermore, $l_j - L < 0$ and therefore $\rho^{2l_j - 2L} \rightarrow \infty$ if $l_j \neq L$ or $\rho^{2l_j - 2L} \rightarrow 1$ if $l_j = L$, hence also this factor is bounded from below. In conclusion, also in the second case we have the existence of some constant $C = C(\delta, \mathbb{A}) > 0$ such that

$$\frac{|\mathbb{A}(\xi)M|^2}{\text{dist}_V(\xi)^{2L}} \geq C > 0.$$

Combining both cases for small enough $\delta > 0$ shows the claim. $\square$

Similarly as for the higher order curl, we deduce a lower order bound for the higher order divergence.

LEMMA 3.5 (the higher order divergence). *Let $\mathcal{B}(D)$ be given as in (19) and for $l \in \mathbb{N}^d$ with $|l| = m$ let $M = v \otimes e_1^{\odot l_1} \odot \dots \odot e_d^{\odot l_d} \in \mathbb{R}^k \otimes \text{Sym}(\mathbb{R}^d; m)$. We set $V = \{\xi \in \mathbb{R}^d : \mathbb{B}(\xi)M = 0\}$ and let $L = m - \min_{j=1, \dots, d} l_j \leq m$. Then*

$$V = \bigcup_{j: l_j \neq 0} \text{span}(e_j)^\perp,$$

and there exists a constant $C = C(d, m, v) > 0$ such that

$$\left| \mathbb{B} \left(\frac{\xi}{|\xi|} \right) M \right|^2 \geq C \frac{\text{dist}_V(\xi)^{2L}}{|\xi|^{2L}}.$$

Proof. By virtue of Lemma 2.3 the symbol $\mathbb{B}(\frac{\xi}{|\xi|})M$ is given by $\mathbb{B}(\xi)(v \otimes e_1^{\odot l_1} \odot \dots \odot e_d^{\odot l_d}) = \binom{m}{l}^{-1} \xi^l v$ and therefore $V = \bigcup_{j: l_j \neq 0} \text{span}(e_j)^\perp$. Thus the distance to the zero set is given by

$$\text{dist}_V(\xi)^2 = \min_{j: l_j \neq 0} |\xi_j|^2.$$

Moreover, we use that for any $\xi \in \mathbb{S}^{d-1}$ there is k such that $|\xi_k|^2 \geq \frac{1}{d}$ and that $L \geq m - l_k$ and therefore

$$\begin{aligned} \min_{j: l_j \neq 0} |\xi_j|^{2L} &\leq \min_{j: l_j \neq 0} |\xi_j|^{2(m-l_k)} = \prod_{p: l_p \neq 0, p \neq k} \min_{j: l_j \neq 0} |\xi_j|^{2l_p} \leq \prod_{p: l_p \neq 0, p \neq k} |\xi_p|^{2l_p} = \frac{\xi^{2l}}{\xi_k^{2l_k}} \\ &\leq d^m \xi^{2l}. \end{aligned}$$

Using the chain of inequalities above together with the fact that $|\mathbb{B}(\xi)M|^2 \geq C(v)\xi^{2l}$, for $\xi \in \mathbb{S}^{d-1}$ there holds

$$|\mathbb{B}(\xi)M|^2 \geq C(d, m, v) \text{dist}_V(\xi)^{2L}$$

and the claim follows. $\square$

As these two lemmas will be used to derive lower scaling bounds in section 3.2, we comment on the choice of L .

Remark 3.6. The values of L in Lemmas 3.4 and 3.5 are indeed the maximal vanishing order; cf. Definition 1.3. Let us elaborate on this statement:

- (i) For the higher order curl, we obtain the characterization of the constant L as follows. By Lemma 3.4, we immediately infer that $L = \max_{j=1,\dots,d} l_j$ is an upper bound for the maximal vanishing order. To prove that L coincides with the maximal vanishing order, consider $L' < L$. Working as in the proof of Lemma 3.4 with L' in place of L , by (25) and the fact that there exists $j \in \{1, \dots, d\}$ such that $l_j - L' > 0$, we obtain

$$\inf_{\xi \in \mathbb{S}^{d-1} \setminus V} \frac{|\mathbb{A}(\xi)M|^2}{\text{dist}_V(\xi)^{2L'}} = 0.$$

- (ii) Similarly as above, for the higher order divergence, Lemma 3.5 yields that $L = m - \min_{j=1,\dots,d} l_j$ is an upper bound for the maximal vanishing order. Now, let $L' < L$. Then it holds that

$$\inf_{\xi \in \mathbb{S}^{d-1} \setminus V} \frac{|\mathbb{B}(\xi)M|^2}{\text{dist}_V(\xi)^{2L'}} = 0.$$

Indeed, choosing $\xi_j^{(k)} = \frac{1}{k}$ for $j \neq j_0$ (where j_0 is an index such that $L = m - l_{j_0}$) and $\xi_{j_0}^{(k)} = \sqrt{1 - \frac{d-1}{k^2}}$ yields

$$\frac{(\xi^{(k)})^{2L}}{\text{dist}_V(\xi^{(k)})^{2L'}} \leq \frac{k^{-2(m-l_{j_0})} \left(1 - \frac{d-1}{k^2}\right)^{l_{j_0}}}{k^{-2(m-l_{j_0}-1)}} = \frac{\left(1 - \frac{d-1}{k^2}\right)^{l_{j_0}}}{k^2} \rightarrow 0.$$

This indeed proves that our choices of L in Lemmas 3.4 and 3.5 correspond to the maximal vanishing orders for these operators.

With the above observations in hand, for the higher order curl and divergence we then obtain the following lower bound estimates.

Proof of lower bounds in Theorem 1.2. By (20) and Lemma 3.4 it holds, with V and L as in Lemma 3.4, that

$$E_{el}(\chi; F_\lambda) \geq C \int_{\mathbb{R}^d} \left| \mathbb{A} \left(\frac{\xi}{|\xi|} \right) (A - B) \hat{f} \right|^2 d\xi \geq C \int_{\mathbb{R}^d} \text{dist}_V \left(\frac{\xi}{|\xi|} \right)^{2L} |\hat{f}|^2 d\xi$$

with $f = (1 - \lambda)\chi_A - \lambda\chi_B$ and a constant $C = C(d, L) > 0$. Extending f outside of Ω by zero, we can apply Proposition 3.3 and infer that

$$E_\epsilon(\chi; F_\lambda) \geq C \min\{1 - \lambda, \lambda\}^2 \epsilon^{\frac{2L}{2L+1}}.$$

This concludes the proof. $\square$

Analogously, in the setting of the higher order divergence we infer the following lower bounds.

LEMMA 3.7. *Let $d, m \in \mathbb{N}, d \geq 2$, and $l \in \mathbb{N}^d$. Let $\Omega \subset \mathbb{R}^d$ be an open, bounded Lipschitz domain. Let $E_\epsilon^{\mathcal{B}}(\chi; F)$ be as above in (6) with the operator $\mathcal{B}(D) = \text{div}^m$ given in (19). Then the following scaling results hold: Let $A - B = v \otimes e_1^{\odot l_1} \odot e_2^{\odot l_2} \odot \dots \odot e_d^{\odot l_d}$ for some $v \in \mathbb{R}^k$ and such that $\sum_{j=1}^d l_j = m$ and let $F_\lambda := \lambda A + (1 - \lambda)B$ for some $\lambda \in (0, 1)$. Then there exists $C > 0$ and $\epsilon_0 > 0$ (depending on d, m, Ω, v , and ϵ_0 also depending on λ) such that for $L := m - \min_{j \in \{1, 2, \dots, d\}} l_j \leq m$ and for any $\epsilon \in (0, \epsilon_0)$*

$$C \min\{1 - \lambda, \lambda\}^2 \epsilon^{\frac{2L}{2L+1}} \leq \inf_{\chi \in BV(\Omega; \{A, B\})} E_\epsilon^{\mathcal{B}}(\chi; F_\lambda).$$

As in Theorem 1.2, for $d = 2$ this lower bound is optimal for $\lambda = \frac{1}{2}$. In two dimensions the curl and divergence operators only differ by a rotation; therefore this is to be expected once the result for the symmetrized derivative is proved (see Theorem 1.2). In section 4 we will further comment on this.

Proof of Lemma 3.7. This is a direct consequence of applying Theorem 1.4 with $L = m - \min_{j=1,\dots,d} l_j$; cf. Lemma 3.5. $\square$

Remark 3.8 (comparison of m th order curl and divergence). Comparing the results from Lemmas 3.4 and 3.7 and noting that the exponents indeed originate from the maximal vanishing order (cf. Remark 3.6), we observe that since the function $\mathbb{R} \ni t \mapsto \frac{2t}{2t+1} \in \mathbb{R}$ is monotone increasing, the bounds for the higher order curl are, in general, substantially tighter than for the divergence. Indeed, denoting by $\mathcal{A}(D)$ the higher order curl from (14) and by $\mathcal{B}(D)$ the higher order divergence from (19) for $m \in \mathbb{N}$ fixed, $l \in \mathbb{N}^d$ with $|l| = m$ and $v \in \mathbb{R}^k$, with the notation from Definition 1.3, it follows that

$$\begin{aligned} L \left[|\mathbb{B}(\xi)v \otimes e_1^{l_1} \odot \dots \odot e_d^{l_d}|^2 \right] &:= m - \min_{j=1,\dots,d} l_j \\ &\geq \max_{j=1,\dots,d} l_j =: L \left[|\mathbb{A}(\xi)e_1^{l_1} \odot \dots \odot e_d^{l_d}|^2 \right]. \end{aligned}$$

We highlight that this is consistent with the fact that the higher order divergence yields lower bounds for general symbols (cf. [39, Appendix B]).

4. Upper bound constructions. In this section we provide the arguments for the upper bounds in Theorem 1.2 in the setting in which $d = 2$, $\lambda = \frac{1}{2}$ and with $m \in \mathbb{N}$ general. We emphasize that for $m \in \{1, 2\}$ (and general $\lambda \in (0, 1)$) these results are known (cf. [8]). In order to deduce these in the case of higher order tensors we mimic the construction from the setting in [8] and adapt it correspondingly. We split our discussion into first dealing with the highest possible maximal vanishing order in which $M := A - B = e_1^{\odot m}$ and then make use of this construction to also infer the result for the intermediate cases $M := A - B = e_1^{\odot l} \odot e_2^{\odot(m-l)}$, $l \in \{1, \dots, m-1\}$.

In what follows, for convenience, we introduce the following notational convention.

CONVENTION 4.1. *In order to simplify the notation for $d = 2$, we use the symmetries of the tensors and define for $M \in \text{Sym}(\mathbb{R}^2; m)$*

$$\tilde{M}_0 = M_{1\dots 1}, \quad \tilde{M}_1 = M_{1\dots 12}, \quad \tilde{M}_k = M_{1\dots 12\dots 2}, \quad \tilde{M}_m = M_{2\dots 2},$$

where for $\tilde{M}_k$ there are k many twos and $m - k$ many ones in the index. Thus, the “new” index counts how many twos appear in the index as by symmetry of M the order of the indices 1 and 2 does not matter. Analogously, we will use this notation for symmetric tensor fields, in particular, for the map $u : \mathbb{R}^2 \rightarrow \text{Sym}(\mathbb{R}^2; m)$ and the potential $v : \mathbb{R}^2 \rightarrow \text{Sym}(\mathbb{R}^2; m-1)$; cf. (12). With this notation it holds for $k = 1, \dots, m-1$ that

$$\tilde{u}_0 = \partial_1 \tilde{v}_1, \quad \tilde{u}_k = \frac{m-k}{m} \partial_1 \tilde{v}_k + \frac{k}{m} \partial_2 \tilde{v}_{k-1}, \quad \tilde{u}_m = \partial_2 \tilde{v}_{m-1}.$$

Indeed, here we have used that for $(i_1 \dots i_m) = (1 \dots 12 \dots 2)$

$$\begin{aligned}
\tilde{u}_k &= u_{1\dots 12\dots 2} = u_{i_1\dots i_m} = \frac{1}{m!} \sum_{\tau \in \mathfrak{S}_m} \partial_{i_{\tau(1)}} v_{i_{\tau(2)}\dots i_{\tau(m)}} \\
&= \frac{1}{m!} \sum_{\tau \in \mathfrak{S}_m: i_{\tau(1)}=1} \partial_1 \tilde{v}_k + \frac{1}{m!} \sum_{\tau \in \mathfrak{S}_m: i_{\tau(1)}=2} \partial_2 \tilde{v}_{k-1} \\
&= \frac{|\{\tau \in \mathfrak{S}_m : \tau(1) \in \{1, 2, \dots, m-k\}\}|}{m!} \partial_1 \tilde{v}_k \\
&\quad + \frac{|\{\tau \in \mathfrak{S}_m : \tau(1) \in \{m-k+1, m-k+2, \dots, m\}\}|}{m!} \partial_2 \tilde{v}_{k-1} \\
&= \frac{(m-k)(m-1)!}{m!} \partial_1 \tilde{v}_k + \frac{k(m-1)!}{m!} \partial_2 \tilde{v}_{k-1} = \frac{m-k}{m} \partial_1 \tilde{v}_k + \frac{k}{m} \partial_2 \tilde{v}_{k-1}.
\end{aligned}$$

As it will be of relevance in our constructions below, we recall that, for $d = 2$, by the antisymmetry properties of $\mathcal{A}(D)u$, it holds that $\mathcal{A}(D)u = 0$ if and only if $[\mathcal{A}(D)u]_{12\dots 12} = 0$. Moreover it holds that

$$(26) \quad [\mathcal{A}(D)u]_{12\dots 12} = \sum_{k=0}^m \left((-1)^k 2^{-m} \binom{m}{k} \partial_1^k \partial_2^{m-k} \tilde{u}_k \right),$$

as every time we switch an index one and an index two we multiply by a factor of (-1) and there are exactly $\binom{m}{k}$ possibilities to switch k distinct twos and ones.

4.1. Cell construction. In this section, we consider the case $M = e_1^{\odot m}$, which has the largest possible maximal vanishing order. Following [8], we begin with a unit cell construction (Lemma 4.2) in which the higher vanishing order of our symbols will be turned into higher order scaling properties. This construction requires more careful considerations than for the case of first or second order tensors, as for $m \geq 3$, in general, one cell does not suffice to achieve the desired boundary conditions. Next, we will introduce a suitable cut-off procedure and combine these ingredients into a branching construction in section 4.2. The construction will be carried out on the level of a potential, i.e., a map $v : \mathbb{R}^2 \rightarrow \text{Sym}(\mathbb{R}^2; m-1)$, and then we set $u = D^{\text{sym}} v$. To ensure that $\tilde{v}_k$ (see Convention 4.1 for the notation) attains the desired boundary conditions, we will use a suitable reflection argument.

LEMMA 4.2 (unit cell construction). *Let $m \in \mathbb{N}$ and $\mathcal{A}(D)$ be as in (14) for $d = 2$. For $0 < l \leq h \leq 1$ let $\omega = (0, 2^m l) \times (0, h)$. Let $A, B \in \text{Sym}(\mathbb{R}^2; m)$ be such that $A - B = e_1^{\odot m}$ and let $F = \frac{1}{2}A + \frac{1}{2}B$. Define for $u \in L_{loc}^2(\mathbb{R}^2; \text{Sym}(\mathbb{R}^2; m))$, $\chi \in BV(\omega; \{A, B\})$*

$$E_{el}(u, \chi; \omega) := \int_{\omega} |u - \chi|^2 dx, \quad E_{surf}(\chi; \omega) := \int_{\omega} |\nabla \chi|.$$

Then, there exist a potential $v : \omega \rightarrow \text{Sym}(\mathbb{R}^2; m-1)$, a function $f \in BV(\omega; \{\pm \frac{1}{2}\})$, and a constant $C = C(m) > 0$ such that with $u := D^{\text{sym}} v + F$ and $\chi := (A - B)f + F \in BV(\omega; \{A, B\})$ it holds that

$$E_{el}(u, \chi; \omega) \leq C \frac{l^{2m+1}}{h^{2m-1}}, \quad E_{surf}(\chi) \leq Ch.$$

Furthermore it holds that $\partial_1 \tilde{v}_0 = f$ and the following boundary conditions are satisfied:

$$\begin{aligned}
v(0, y) &= v(2^m l, y) = 0, & y &\in [0, h], \\
v(x, 0) &= \left(-\frac{1}{2} x \chi_{[0, \frac{l}{2}]}(x) + \frac{1}{2} (x - l) \chi_{[\frac{l}{2}, l]}(x) \right) e_1^{\odot(m-1)}, & x &\in [0, l], \\
v(x, 0) &= -v(x - 2^j l, 0), & x &\in [2^j l, 2^{j+1} l], \\
& & j &\in \{0, 1, \dots, m-1\}, \\
v(x, h) &= -\frac{1}{2} v(2x, 0), & x &\in [0, 2^{m-1} l], \\
v(x, h) &= \frac{1}{2} v(2x - 2^m l, 0), & x &\in [2^{m-1} l, 2^m l], \\
\tilde{v}_k(x, 0) &= \tilde{v}_k(x, h) = 0, & x &\in [0, 2^m l], \\
& & k &\in \{1, \dots, m-1\}.
\end{aligned}$$

Proof. We adopt the ideas from the upper bound construction in [8] to our context. To achieve this, we first reduce to the setting of $F = 0$ by subtracting the boundary data:

$$E_{el}(u, \chi; \omega) = \int_{\omega} |(u - F) - (\chi - F)|^2 dx.$$

Hence, without loss of generality we may assume $F = 0$ and therefore $\frac{1}{2}A + \frac{1}{2}B = 0$.

Using this we introduce a function $f : \omega \rightarrow \{\pm \frac{1}{2}\}$ such that the phase indicator reads $\chi = (A - B)f = f e_1^{\odot m}$. Plugging this into the elastic energy and recalling Convention 4.1, we obtain

$$\begin{aligned}
E_{el}(u, \chi; \omega) &\leq C(m) \int_{\omega} |\tilde{u}_0 - f|^2 + \sum_{j=1}^m |\tilde{u}_j|^2 dx \\
&= C(m) \int_{\omega} |\tilde{u}_0 - f|^2 + \sum_{j=1}^{m-1} |\tilde{u}_j|^2 + |\tilde{u}_m|^2 dx.
\end{aligned}$$

Let us next outline the idea of constructing the tensor u . To this end, we will first fix $\tilde{u}_0$ using the construction from [8], which leads to $|\tilde{u}_0 - f| = 0$. Iteratively, we will then define the remaining $\tilde{u}_k$ by setting them to zero except for $\tilde{u}_m$. This will lead to an energy bound of the form

$$E_{el}(u, \chi; \omega) \leq C \int_{\omega} |\tilde{u}_m|^2 dx,$$

where $\tilde{u}_m$ is determined by all the other components through the constraint $\mathcal{A}(D)u = 0$. In comparison to the cases of $m = 1, 2$, i.e., of the curl and curl curl operator, for higher m the argument to achieve the boundary condition is more involved. To ensure this, we rely on an (iterative) reflection-type argument. We split the proof into several substeps carrying this out successively.

Step 1: Preliminary definitions and outline of the strategy. In order to implement the outlined strategy, we first define a monotone function $\gamma \in C^\infty(\mathbb{R}; [0, 1])$ such that

$$\gamma(t) = 0 \text{ for } t \leq \delta, \quad \gamma(t) = 1 \text{ for } t \geq 1 - \delta \text{ for some } \delta \in \left(0, \frac{1}{4}\right).$$

We start by giving the arguments in a smaller cell $\omega_0 := [0, l] \times [0, h]$ which we will reflect suitably to achieve zero boundary values for the potential. We split ω_0 into three subregions given by (cf. [8]):

$$\begin{aligned}\omega_1 &:= \left\{ (x, y) \in \omega_0 : x \in \left[0, \gamma \left(\frac{y}{h} \right) \frac{l}{4} \right] \right\}, \\ \omega_2 &:= \left\{ (x, y) \in \omega_0 : x \in \left[\gamma \left(\frac{y}{h} \right) \frac{l}{4}, \frac{l}{2} + \gamma \left(\frac{y}{h} \right) \frac{l}{4} \right] \right\}, \\ \omega_3 &:= \left\{ (x, y) \in \omega_0 : x \in \left[\frac{l}{2} + \gamma \left(\frac{y}{h} \right) \frac{l}{4}, l \right] \right\}.\end{aligned}$$

We define

$$f(x, y) := \begin{cases} \frac{1}{2}, & (x, y) \in \omega_1 \cup \omega_3, \\ -\frac{1}{2}, & (x, y) \in \omega_2. \end{cases}$$

This is illustrated in Figure 1.

Our next goal is to set $\tilde{u}_0 = f$ and then to iteratively fix the other components such that $\tilde{u}_k = 0$ for $k \in \{1, \dots, m-1\}$. To this end, we use a potential $v : \omega \rightarrow \text{Sym}(\mathbb{R}^2; m-1)$ and define $\tilde{v}_0$ by integration of $\tilde{u}_0 = \partial_1 \tilde{v}_0$ in ω_0 . We then fix $\tilde{v}_0$ by a reflection-type argument in the remainder of $\omega = [0, 2^m l] \times [0, h]$. More precisely, we define

$$\tilde{v}_0(x, y) := \begin{cases} \frac{1}{2}x, & (x, y) \in \omega_1, \\ -\frac{1}{2}x + \gamma \left(\frac{y}{h} \right) \frac{l}{4}, & (x, y) \in \omega_2, \\ \frac{1}{2}(x-l), & (x, y) \in \omega_3. \end{cases}$$

Therefore we have in ω_0

$$\begin{aligned}\partial_1 \tilde{v}_0(x, y) &= \begin{cases} \frac{1}{2}, & (x, y) \in \omega_1 \cup \omega_3, \\ -\frac{1}{2}, & (x, y) \in \omega_2, \end{cases} \\ \partial_2 \tilde{v}_0(x, y) &= \begin{cases} 0, & (x, y) \in \omega_1 \cup \omega_3, \\ \gamma' \left(\frac{y}{h} \right) \frac{l}{4h}, & (x, y) \in \omega_2. \end{cases}\end{aligned}$$

Implementing the above outlined idea, we seek to define the $\tilde{v}_k$ by setting $\tilde{u}_k = 0$ for $k \in \{1, \dots, m-1\}$. As a consequence, we iteratively solve the equations

$$\begin{cases} \frac{1}{m} \partial_2 \tilde{v}_0 + \frac{m-1}{m} \partial_1 \tilde{v}_1 = 0, \\ \frac{k}{m} \partial_2 \tilde{v}_{k-1} + \frac{m-k}{m} \partial_1 \tilde{v}_k = 0, & k \in \{2, \dots, m-2\}, \\ \frac{m-1}{m} \partial_2 \tilde{v}_{m-2} + \frac{1}{m} \partial_1 \tilde{v}_{m-1} = 0, \end{cases}$$

with the boundary condition $\tilde{v}_k(0, y) = 0$. The function $\tilde{v}_k$ is then defined in terms of $\tilde{v}_{k-1}$ by

$$(27) \quad \tilde{v}_k(x, y) := -\frac{k}{m-k} \int_0^x \partial_2 \tilde{v}_{k-1}(t, y) dt.$$

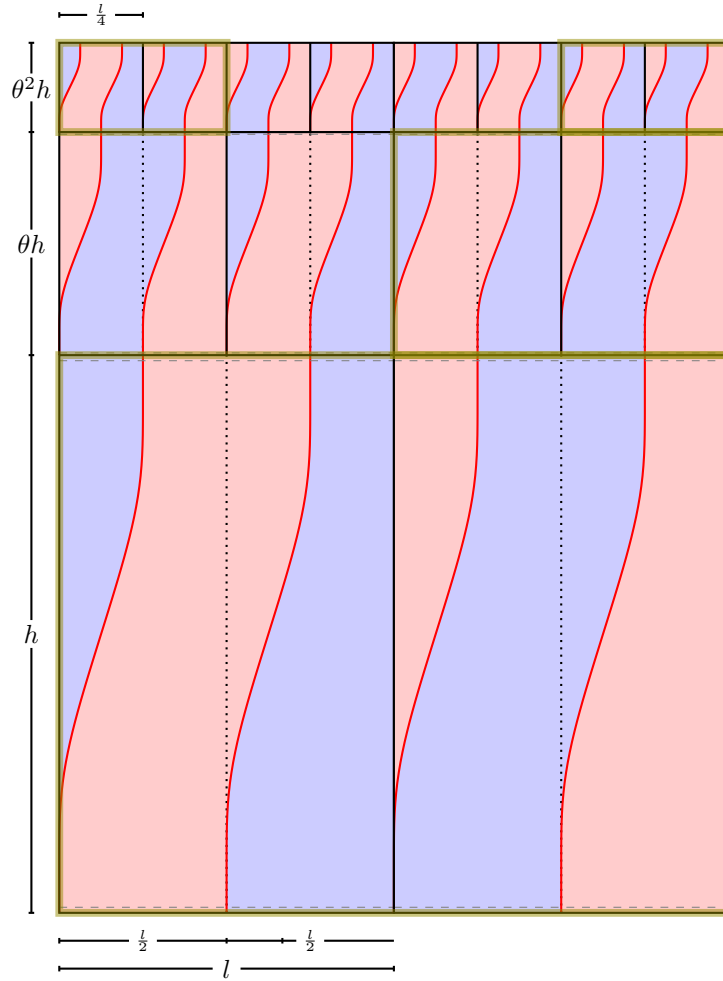


FIG. 1. An illustration of the branching construction used in Lemma 4.5. The individual unit cell constructions from Lemma 4.2 are iteratively combined into a construction refining in the e_2 direction. In blue is the region where $\tilde{u}_0 = \tilde{A}_0$, and red corresponds to $\tilde{u}_0 = \tilde{B}_0$. The dashed horizontal lines depict the region in which we have a simple laminate. As in [8] for $m \geq 2$ we need the curves separating the domains to be of a sufficiently high regularity (see the discussion in Remark 4.3). The unit cell and its copies are highlighted with a green box. Moreover, for $m \geq 3$, we need to do the reflection-type argument outlined in Lemma 4.5 to ensure zero boundary values at the left and right. (Color online.)

Notice that $\tilde{v}_0(l, y) = 0$, but for larger $k \geq 1$ this in general fails. Thus, we use appropriate reflection arguments in order to attain a zero right-boundary condition. Indeed, using the fact that $-\tilde{v}_0(x, y)$ fulfills $\partial_1(-\tilde{v}_0) \in \{\pm \frac{1}{2}\}$, we implement a reflection-type argument in the cell $[0, 2l] \times [0, h]$, i.e., for $x \in [l, 2l]$, we set

$$\tilde{v}_0(x, y) = -\tilde{v}_0(x - l, y).$$

This, by the iterative definition (27), immediately implies that $\tilde{v}_1(2l, y) = 0$. By exploiting yet another reflection argument, i.e., for $x \in [2l, 4l]$

$$\tilde{v}_0(x, y) = -\tilde{v}_0(x - 2l, y),$$

and the fact that $\tilde{v}_1(2l, y) = 0$, this “reflection” property carries over to $\tilde{v}_1$ in the sense that for $x \in [2l, 4l]$, we then have $\tilde{v}_1(x, y) = -\tilde{v}_1(x - 2l, y)$. By this, it then also follows that

$$\int_0^{4l} \tilde{v}_1(x, y) dx = 0,$$

which in turn ensures that $\tilde{v}_2(4l, y) = 0$. As this still does not suffice to ensure zero boundary conditions for $\tilde{v}_k$ for $k \geq 3$, we iterate further this reflection in the next steps.

Step 2: Definition of $\tilde{v}_0 : \omega \rightarrow \mathbb{R}$. Building on the outlined reflection idea, we define $\tilde{v}_0 : \omega \rightarrow \mathbb{R}$ to be given by

$$\tilde{v}_0(x, y) := \begin{cases} \frac{1}{2}x, & (x, y) \in \omega_1, \\ -\frac{1}{2}x + \gamma\left(\frac{y}{h}\right)\frac{l}{4}, & (x, y) \in \omega_2, \\ \frac{1}{2}(x - l), & (x, y) \in \omega_3, \end{cases} \quad \text{for } x \in [0, l],$$

$$\tilde{v}_0(x, y) := -\tilde{v}_0(x - 2^j l, y) \quad \text{for } x \in [2^j l, 2^{j+1} l], j \in \{0, 1, \dots, m-1\}.$$

As $\tilde{v}_0(0, y) = \tilde{v}_0(l, y)$ this function is continuous and well-defined and $\tilde{v}_0(2^j l, y) = 0$ for all $j \in \{1, \dots, m\}$. Furthermore, it holds that $\partial_1 \tilde{v}_0(x, y) \in \{\pm \frac{1}{2}\}$ and for $y \in [0, \delta h]$

$$\tilde{v}_0(x, y) = \begin{cases} -\frac{1}{2}x, & x \in \left[0, \frac{l}{2}\right), \\ \frac{1}{2}(x - l), & x \in \left[\frac{l}{2}, l\right), \\ \frac{1}{2}(x - l), & x \in \left[l, \frac{3}{2}l\right), \\ -\frac{1}{2}(x - 2l), & x \in \left[\frac{3}{2}l, 2l\right]. \end{cases}$$

For $y \in [(1 - \delta)h, h]$ and $x \in [0, l]$ we have

$$\tilde{v}_0(x, y) = \begin{cases} \frac{1}{2}x, & x \in \left[0, \frac{l}{4}\right), \\ -\frac{1}{2}\left(x - \frac{l}{2}\right), & x \in \left[\frac{l}{4}, \frac{3l}{4}\right), \\ \frac{1}{2}(x - l), & x \in \left[\frac{3l}{4}, l\right], \end{cases}$$

which, for $x \in [0, l]$, can be written as

$$\tilde{v}_0(x, y) = -\frac{1}{2}\tilde{v}_0(2x, 0).$$

Iteratively, by using that $\tilde{v}_0(x, y) = -\tilde{v}_0(x - 2^j l, y)$ for $x \in [2^j l, 2^{j+1} l]$, $j \in \{0, \dots, m-1\}$, this carries on, and hence

$$\begin{aligned} \tilde{v}_0(x, h) &= -\frac{1}{2}\tilde{v}_0(2x, 0), & x \in [0, 2^{m-1}l], \\ \tilde{v}_0(x, h) &= \frac{1}{2}\tilde{v}_0(2x - 2^m l, 0), & x \in [2^{m-1}l, 2^m l]. \end{aligned}$$

In the end, we also define $f : \omega \rightarrow \{\pm \frac{1}{2}\}$ to be given as $f(x, y) = \partial_1 \tilde{v}_0(x, y)$.

Step 3: Definition of $\tilde{v}_k : \omega \rightarrow \mathbb{R}$. With $\tilde{v}_0$ defined on ω , we iteratively define $\tilde{v}_1, \tilde{v}_2, \dots, \tilde{v}_{m-1}$ on ω by using the equation for the potential. We define iteratively for $k \in \{1, 2, \dots, m-1\}$

$$\tilde{v}_1(x, y) := -\frac{1}{m-1} \int_0^x \partial_2 \tilde{v}_0(t, y) dt, \quad \tilde{v}_k(x, y) := -\frac{k}{m-k} \int_0^x \partial_2 \tilde{v}_{k-1}(t, y) dt.$$

As shown in Step 4 below these functions are Lipschitz continuous. By this definition, it holds for a.e. $(x, y) \in \omega$

$$(28) \quad \frac{k}{m} \partial_2 \tilde{v}_{k-1}(x, y) + \frac{m-k}{m} \partial_1 \tilde{v}_k(x, y) = 0, \quad k \in \{1, 2, \dots, m-1\}.$$

We now claim that for $k \in \{1, \dots, m-1\}$ the following properties are satisfied:

$$(29) \quad \int_0^{2^{k+1}l} \tilde{v}_k(t, y) dt = 0,$$

$$(30) \quad \tilde{v}_k(x, y) = -\tilde{v}_k(x - 2^{j+k}l, y) \quad \text{for } x \in [2^{j+k}l, 2^{j+1+k}l], \\ j \in \{0, 1, \dots, m-1-k\},$$

$$(31) \quad \tilde{v}_k(2^{j+k}l, y) = 0 = \tilde{v}_k(0, y) \quad \text{for } j \in \{0, 1, \dots, m-k\}.$$

Properties (30) and (31) will be proved by finite induction in Steps 5 and 6 below. Property (29) then directly follows from (30).

Step 4: Regularity of $\tilde{v}_k$. We claim that each $\tilde{v}_k$ is Lipschitz continuous. We will first discuss this only in $\omega_0 := [0, l] \times [0, h]$ and will then comment on how this immediately implies Lipschitz continuity on the full cell ω .

Let $G : \omega_0 \rightarrow \mathbb{R}$ be of the form $G(x, y) = G_1(x, y)\chi_{\omega_1}(x, y) + G_2(x, y)\chi_{\omega_2}(x, y) + G_3(x, y)\chi_{\omega_3}(x, y)$ with G_j being Lipschitz functions. Notice that both f and $\partial_2 \tilde{v}_0$ have this form.

Claim 1: Let $g(x, y) = \int_0^x G(t, y) dt$; then g is Lipschitz and $\|\nabla g\|_{L^\infty(\omega_0)} \leq C(\|G\|_{L^\infty(\omega_0)} + \sum_{j=1}^3 \|\partial_2 G_j\|_{L^\infty(\omega_j)})$, where $C > 0$ depends on $l, \frac{l}{h}, \|\gamma'\|_{L^\infty}$.

Indeed: For every $(x_1, y_1), (x_2, y_2) \in \omega_0$ we have

$$|g(x_2, y_2) - g(x_1, y_1)| = \left| \int_0^{x_2} G(t, y_2) dt - \int_0^{x_1} G(t, y_1) dt \right| \\ \leq \left| \int_0^{x_2} G(t, y_2) - G(t, y_1) dt + \int_{x_1}^{x_2} G(t, y_1) dt \right|.$$

For a.e. $t \in (0, l)$, $G(t, \cdot) \in BV((0, h))$ and by the representation theorem of one-dimensional BV functions (see, e.g., [2, Theorem 3.28 and Corollary 3.33])

$$\int_0^{x_2} G(t, y_2) - G(t, y_1) dt = \int_0^{x_2} \left(\int_{y_1}^{y_2} \partial_2 G(t, s) ds + \int_{y_1}^{y_2} [G](t, y_t) d\delta_{y_t}(s) \right) dt,$$

where $\partial_2 G$ denotes the absolutely continuous part of the derivative of G , $[G](t, y_t)$ is the amplitude of the jump of $G(t, \cdot)$ in y_t , and $\{(t, y_t)\} = (\cup_{j=1}^3 \partial \omega_j) \cap (\{t\} \times (y_1, y_2))$ which is unique for fixed $t \in (0, l) \setminus \{\frac{l}{4}, \frac{l}{2}, \frac{3l}{4}\}$ by definition of the sets ω_j (see Figure 1). Without loss of generality we assume $y_1 \leq y_2$ and obtain that the last term above can be controlled as

$$\begin{aligned} \left| \int_0^{x_2} \int_{y_1}^{y_2} [G](t, y_t) d\delta_{y_t}(s) dt \right| &= \left| \int_0^{x_2} [G](t, y_t) \chi_{(y_1, y_2)}(y_t) dt \right| \\ &\leq 4 \|G\|_{L^\infty(\omega_0)} \frac{l}{4} \left| \gamma\left(\frac{y_2}{h}\right) - \gamma\left(\frac{y_1}{h}\right) \right|. \end{aligned}$$

Combining the three formulas above, by regularity of γ , we obtain

$$\begin{aligned} |g(x_2, y_2) - g(x_1, y_1)| &\leq \left| \int_0^{x_2} \int_{y_1}^{y_2} \partial_2 G(t, s) ds dt \right| + \left| \int_0^{x_2} [G](t, y_t) \chi_{(y_1, y_2)}(y_t) dt \right| \\ &\quad + \left| \int_{x_1}^{x_2} G(t, y_1) dt \right| \\ &\leq l |y_2 - y_1| \sum_{j=1}^3 \|\partial_2 G_j\|_{L^\infty(\omega_j)} + \frac{l}{h} \|\gamma'\|_{L^\infty} \|G\|_{L^\infty(\omega_0)} |y_2 - y_1| \\ &\quad + |x_2 - x_1| \|G\|_{L^\infty(\omega_0)}, \end{aligned}$$

which yields the claim by recalling the condition $0 < l \leq h \leq 1$.

Since we seek to iterate this, we need to prove that $\partial_2 g$ has the same structure as G if we start with G_j sufficiently regular.

Claim 2: Let G be as above with $G_j \in C^\infty(\overline{\omega_j})$. Then there exist functions $\tilde{g}_j \in C^\infty(\overline{\omega_j})$, $j \in \{1, 2, 3\}$, such that $\partial_2 g(x, y) = \tilde{g}_1(x, y) \chi_{\omega_1}(x, y) + \tilde{g}_2(x, y) \chi_{\omega_2}(x, y) + \tilde{g}_3(x, y) \chi_{\omega_3}(x, y)$.

Indeed: We have

$$g(x, y) = \int_0^x G(t, y) dt = \sum_{j=1}^3 \int_0^x G_j(t, y) \chi_{\omega_j}(t, y) dt = \sum_{j=1}^3 \int_{((0, x) \times \{y\}) \cap \omega_j} G_j(t, y) dt.$$

By definition $((0, x) \times \{y\}) \cap \omega_j = (\min\{a_j(y), x\}, \min\{b_j(y), x\}) \times \{y\}$ for $a_1 = 0$, $b_1(y) = a_2(y) = \frac{l}{4} \gamma(\frac{y}{h})$, $b_2(y) = a_3(y) = \frac{l}{2} + \frac{l}{4} \gamma(\frac{y}{h})$, $b_3(y) = l$. Note that $a_j, b_j \in C^\infty$. Denoting

$$g_j(x, y) = \int_{a_j(y)}^x G_j(t, y) dt,$$

which are C^∞ functions in y , we get

$$g(x, y) = \sum_{j=1}^3 (g_j(x, y) \chi_{\omega_j}(x, y) + g_j(b_j(y), y) \chi_{\tilde{\omega}_j}(x, y)),$$

where $\tilde{\omega}_1 := \omega_2 \cup \omega_3$, $\tilde{\omega}_2 := \omega_3$, and $\tilde{\omega}_3 := \emptyset$. By Claim 1 we know that g is Lipschitz and therefore $\partial_2 g(x, y)$ is given by calculating the derivative almost everywhere, thus

$$\partial_2 g(x, y) = \sum_{j=1}^3 \left(\partial_2 g_j(x, y) \chi_{\omega_j}(x, y) + \left(\frac{d}{dy} g_j(b_j(y), y) \right) \chi_{\tilde{\omega}_j}(x, y) \right).$$

Since $\tilde{\omega}_j$ can be written as suitable unions of ω_k , $k \in \{1, 2, 3\}$, the claim follows by recalling the higher regularity of g_j and b_j . Without assuming Claim 1, it is possible to calculate the distributional derivative, in which the singular measures cancel out. Thus this calculation would provide a second argument for Claim 1, together with the fact that $\partial_1 g \in L^\infty$ by the fundamental theorem of calculus.

Applying iteratively the two claims above, we conclude the Lipschitz regularity of $\tilde{v}_k$. We highlight that the L^∞ bounds below are only a priori bounds and will be improved later in Step 8.

Claim 3: For every $k \in \{0, 1, \dots, m-1\}$, $\tilde{v}_k : \omega \rightarrow \mathbb{R}$ is Lipschitz continuous and $\|\tilde{v}_k\|_{L^\infty(\omega)}, \|\nabla \tilde{v}_k\|_{L^\infty(\omega)} \leq C(m, \|\gamma\|_{C^m}, l, h)$.

Indeed: We reason by finite induction. By Claim 1, applied to $G = f$, $\tilde{v}_0$ complies with Claim 3, and by Claim 2 it has the desired structure.

Assume, by induction, that $\tilde{v}_{k-1}$ complies with Claim 3 and is such that $\partial_2 \tilde{v}_{k-1} = \sum_{j=1}^3 G_j^{(k-1)} \chi_{\omega_j}$. Then applying Claim 1 with $G_j = -\frac{k}{m-k} \partial_2 \tilde{v}_{k-1} \chi_{\omega_j}$, we obtain that $\tilde{v}_k$ complies with Claim 3 and has the desired structure.

In the end we notice that if $(x, y) \in \omega_0 + (l, 0)$, then

$$\tilde{v}_k(x, y) = \tilde{v}_k(l, y) - \frac{k}{m-k} \int_l^x \partial_2 \tilde{v}_{k-1}(t, y) dt,$$

from which we immediately infer that $\tilde{v}_k$ is Lipschitz also in $[0, 2l] \times [0, h]$ and, again by finite induction, we obtain the claim in the full domain ω .

Step 5: Induction basis; properties for $\tilde{v}_1$. Properties (29)–(31) are shown by induction, with the induction basis being given by $k = 1$. We note that $\partial_2 \tilde{v}_0(x, y) = \gamma'(\frac{y}{h}) \frac{l}{4h} \chi_{\omega_2}(x, y)$ for $x \in [0, l]$. Spelling out the definition

$$\tilde{v}_1(x, y) = -\frac{1}{m-1} \int_0^x \partial_2 \tilde{v}_0(t, y) dt$$

yields for $x \in [0, 2l]$

$$\tilde{v}_1(x, y) = \begin{cases} 0, & (x, y) \in \omega_1, \\ -\frac{1}{m-1} \gamma'(\frac{y}{h}) \frac{l}{4h} \left(x - \gamma(\frac{y}{h}) \frac{l}{4} \right), & (x, y) \in \omega_2, \\ -\frac{1}{m-1} \gamma'(\frac{y}{h}) \frac{l^2}{8h}, & (x, y) \in \omega_3 \cup \omega_1 + (l, 0), \\ \frac{1}{m-1} \gamma'(\frac{y}{h}) \frac{l}{4h} \left(x - \frac{3l}{2} - \gamma(\frac{y}{h}) \frac{l}{4} \right), & (x, y) \in \omega_2 + (l, 0), \\ 0, & (x, y) \in \omega_3 + (l, 0). \end{cases}$$

Thus, indeed it holds that $\tilde{v}_1(0, y) = \tilde{v}_1(2l, y) = 0$.

Let us show (30) and (31) for $k = 1$, i.e., $\tilde{v}_1(x, y) = -\tilde{v}_1(x - 2^j 2l, y)$ for $x \in [2^{j+1}l, 2^{j+2}l]$ and $\tilde{v}_1(2^{j+1}l, y) = 0$ with $j \in \{0, \dots, m-2\}$. To this end, let $x \in [2l, 4l]$, then

$$\begin{aligned} \tilde{v}_1(x, y) &= -\frac{1}{m-1} \int_0^x \partial_2 \tilde{v}_0(t, y) dt = \tilde{v}_1(2l, y) - \frac{1}{m-1} \int_{2l}^x \partial_2 \tilde{v}_0(t, y) dt \\ &= -\frac{1}{m-1} \int_{2l}^x -\partial_2 \tilde{v}_0(t - 2l, y) dt = \frac{1}{m-1} \int_0^{x-2l} \partial_2 \tilde{v}_0(t, y) dt \\ &= -\tilde{v}_1(x - 2l, y). \end{aligned}$$

This also implies $\tilde{v}_1(4l, y) = -\tilde{v}_1(2l, y) = 0$. Therefore, assuming by induction that for some $j_0 \in \{0, 1, \dots, m-2\}$ it holds that $\tilde{v}_1(2^{j+1}l, y) = 0$ for $j \in \{0, 1, \dots, j_0\}$ and also (30) holds for $x \in [0, 2^{j_0+1}l]$, then for $x \in [2^{j_0+1}l, 2^{j_0+2}l]$ we have

$$\begin{aligned}
\tilde{v}_1(x, y) &= -\frac{1}{m-1} \int_0^x \partial_2 \tilde{v}_0(t, y) dt = \tilde{v}_1(2^{j_0+1}l, y) - \frac{1}{m-1} \int_{2^{j_0+1}l}^x \partial_2 \tilde{v}_0(t, y) dt \\
&= \frac{1}{m-1} \int_{2^{j_0+1}l}^x \partial_2 \tilde{v}_0(t - 2^{j_0+1}l, y) dt = \frac{1}{m-1} \int_0^{x-2^{j_0+1}l} \partial_2 \tilde{v}_0(t, y) dt \\
&= -\tilde{v}_1(x - 2^{j_0+1}l, y).
\end{aligned}$$

This shows that (30) holds also in $[2^{j_0+1}l, 2^{j_0+2}l]$ and, moreover, as a consequence we have $\tilde{v}_1(2^{j_0+2}l, y) = -\tilde{v}_1(2^{j_0+1}l, y) = 0$.

Step 6: Induction step; properties for $\tilde{v}_k$. Assuming now that (29)–(31) hold for $1, \dots, k-1$, we seek to prove that they also hold for k . By definition, we have

$$\tilde{v}_k(x, y) = -\frac{k}{m-k} \int_0^x \partial_2 \tilde{v}_{k-1}(t, y) dt.$$

As $\int_0^{2^k l} \tilde{v}_{k-1}(t, y) dt = 0$ by (29) for $k-1$, we deduce $\tilde{v}_k(2^k l, y) = 0 = \tilde{v}_k(0, y)$.

As for $k=1$, we next deduce (30) and (31) inductively. We start by giving the argument for (30) for $j=0$: Let $x \in [2^k l, 2^{k+1}l]$, then

$$\begin{aligned}
\tilde{v}_k(x, y) &= -\frac{k}{m-k} \int_0^x \partial_2 \tilde{v}_{k-1}(t, y) dt = \tilde{v}_k(2^k l, y) - \frac{k}{m-k} \int_{2^k l}^x -\partial_2 \tilde{v}_{k-1}(t - 2^k l, y) dt \\
&= \frac{k}{m-k} \int_0^{x-2^k l} \partial_2 \tilde{v}_{k-1}(t, y) dt = -\tilde{v}_k(x - 2^k l, y).
\end{aligned}$$

Again, as in the case $k=1$, this can be continued iteratively. Assume $\tilde{v}_k(2^{j+k}l, y) = 0$ for $j \in \{0, 1, \dots, j_0\}$ and also that (30) holds in $[0, 2^{j_0+k}l]$; then we have for $x \in [2^{j_0+k}l, 2^{j_0+k+1}l]$

$$\begin{aligned}
\tilde{v}_k(x, y) &= \tilde{v}_k(2^{j_0+k}l, y) - \frac{k}{m-k} \int_{2^{j_0+k}l}^x \partial_2 \tilde{v}_{k-1}(t, y) dt \\
&= -\frac{k}{m-k} \int_{2^{j_0+k}l}^x -\partial_2 \tilde{v}_{k-1}(t - 2^{j_0+k}l, y) dt \\
&= \frac{k}{m-k} \int_0^{x-2^{j_0+k}l} \partial_2 \tilde{v}_{k-1}(t, y) dt = -\tilde{v}_k(x - 2^{j_0+k}l, y).
\end{aligned}$$

Thus (30) holds in $[0, 2^{j_0+1+k}l]$ and, moreover, by an application of (30), we have $\tilde{v}_k(2^{j_0+1+k}l, y) = -\tilde{v}_k(2^{j_0+k}l, y) = 0$, i.e., (31) also holds for $j = j_0 + 1$. By induction, and the above arguments, we have shown (29)–(31) for $k \in \{1, \dots, m-1\}$, where we stop at order $m-1$ to obtain $v : \omega \rightarrow \text{Sym}(\mathbb{R}^2; m-1)$.

Step 7: Conclusion of the construction. As $\gamma'(t) = 0$ for $t \in [0, \delta] \cup [1 - \delta, 1]$, we have

$$\partial_2 \tilde{v}_0(x, y) = 0 \quad \text{for } y \in [0, \delta h] \cup [(1 - \delta)h, h],$$

and therefore for $k \in \{1, \dots, m-1\}$ it holds that

$$\tilde{v}_k(x, y) = 0 \quad \text{for } y \in [0, \delta h] \cup [(1 - \delta)h, h].$$

Considering (31) for $k \in \{1, \dots, m-1\}$ and $j = m-k$, we see that $\tilde{v}_k(2^{j+k}l, y) = \tilde{v}_k(2^m l, y) = 0$. In particular, due to the size of ω , it holds that $\tilde{v}_{m-1}(2^m l, y) = 0$.

Step 8: Energy estimates. In conclusion, we have $v : \omega \rightarrow \text{Sym}(\mathbb{R}^2; m-1)$ with $v(x, y) = 0$ for $x \in \{0, 2^m l\}$ and, moreover, $\tilde{v}_k(x, y) = 0$ for $k \in \{1, \dots, m-1\}$ at $y = 0$ and $y = h$. Setting now $u = D^{\text{sym}} v$ it holds that

$$u(x, y) = f(x, y)e_1^{\odot m} + \partial_2 \tilde{v}_{m-1}(x, y)e_2^{\odot m}, \quad (x, y) \in \omega,$$

by the iterative definition of $\tilde{v}_k$. Choosing $\chi(x, y) := f(x, y)e_1^{\odot m} \in BV(\omega; \{A, B\})$ we therefore have

$$(32) \quad E_{el}(u, \chi; \omega) \leq C(m) \int_{\omega} |\partial_2 \tilde{v}_{m-1}|^2 dx.$$

It remains to provide the bound for $\partial_2 \tilde{v}_{m-1}$. To this end, we consider $\tilde{w}_k : [0, 2^m] \times [0, 1] \rightarrow \mathbb{R}$ defined as $\tilde{w}_k(x', y') := \frac{h^k}{l^{k+1}} \tilde{v}_k(lx', hy')$. Notice that w coincides with v when $l = h = 1$; it is indeed just a rescaled version of v . Since w is a particular choice of v (when $l = h = 1$), it has the same structure, i.e., it solves the same system (with rescaled f). Indeed, for $k \in \{1, \dots, m-2\}$ we obtain

$$\begin{aligned} & k \partial_2 \tilde{w}_{k-1}(x', y') + (m-k) \partial_1 \tilde{w}_k(x', y') \\ &= k \partial_{y'} \left(\frac{h^{k-1}}{l^k} \tilde{v}_{k-1}(lx', hy') \right) + (m-k) \partial_{x'} \left(\frac{h^k}{l^{k+1}} \tilde{v}_k(lx', hy') \right) \\ &= (k \partial_2 \tilde{v}_{k-1}(lx', hy') + (m-k) \partial_1 \tilde{v}_k(lx', hy')) \frac{h^k}{l^k} = 0. \end{aligned}$$

We also have

$$\partial_1 \tilde{w}_0(x', y') = \partial_{x'} \frac{1}{l} \tilde{v}_0(lx', hy') = \partial_1 \tilde{v}_0(lx', hy') = f(lx', hy').$$

By Claim 3 in Step 4, applied to $l = h = 1$, we get $\|\tilde{w}_k\|_{L^\infty}, \|\nabla \tilde{w}_k\|_{L^\infty} \leq C(m, \|\gamma\|_{C^m})$ for every $k \in \{0, 1, \dots, m-1\}$. Finally,

$$|\partial_2 \tilde{v}_k(x, y)| = \frac{l^{k+1}}{h^k} |\partial_y \tilde{w}(x/l, y/h)| = \frac{l^{k+1}}{h^{k+1}} |\partial_2 \tilde{w}(x/l, y/h)| \leq C(m, \|\gamma\|_{C^m}) \frac{l^{k+1}}{h^{k+1}}.$$

Plugging this for $k = m-1$ into the above estimate (32) yields

$$E_{el}(u, \chi; \omega) \leq C(m) \int_{\omega} \frac{l^{2m}}{h^{2m}} d(x, y) = C(m) 2^m \frac{l^{2m+1}}{h^{2m-1}}.$$

To bound the surface energy, we note that $f = \partial_1 \tilde{v}_0$ has only two interfaces in the interior of each $([0, l] + jl) \times [0, h]$ cell. Thus, we can bound the surface energy, using $l < h$, by

$$\begin{aligned} E_{surf}(\chi; \omega) &= \int_{\omega} |\nabla \chi| \leq C 2^m \left(2 \int_0^h \sqrt{1 + \left(\gamma' \left(\frac{y}{h} \right) \frac{l}{4h} \right)^2} dy + 2h \right) \\ &\leq C(m, \|\gamma'\|_{\infty}) h. \end{aligned} \quad \square$$

Remark 4.3. Let us comment on three technical aspects of the previous construction:

- *The function γ .* A possible choice for the function γ as in the proof of Lemma 4.2 is given by

$$\gamma(t) = \frac{h(t - \delta)}{h(1 - \delta - t) + h(t - \delta)},$$

where $h : \mathbb{R} \rightarrow \mathbb{R}$ is the smooth function defined by

$$h(t) = \begin{cases} e^{-\frac{1}{t}}, & t > 0, \\ 0, & t \leq 0. \end{cases}$$

Moreover, we emphasize that for $A - B$ as above, it would be sufficient to require $\gamma \in C^m(\mathbb{R}; [0, 1])$ instead of $\gamma \in C^\infty(\mathbb{R}; [0, 1])$.

- *The off-set δ .* We point out that the off-set δ on the top and bottom layers in our unit cell construction is not necessary. It is, however, convenient as by virtue of this off-set, we immediately obtain that the “corner” arising at the meeting point of ω_1 and ω_2 does not result in losing the $\mathcal{A}$ -freeness of u . Thus, later when we combine the unit cell constructions into a self-similarly refining construction, we will automatically obtain compatibility at these corners. If we would choose $\delta = 0$, we would need to require $\gamma^{(k)}(t) = 0$ for $t \in \{0, 1\}$ for $k \in \{1, \dots, m\}$; cf. [8].

Moreover, by this δ off-set, we can directly observe that $\tilde{v}_k(x, 0) = \tilde{v}_k(x, h) = 0$ for $k \in \{1, \dots, m-1\}$.

- In Lemma 4.2, we could reduce the size of ω to $[0, 2^{m-1}l] \times [0, h]$, as we do not require $\int_0^{2^{m-1}l} \tilde{v}_{m-1}(t, y) dt = 0$, but it is sufficient for our construction to have $\tilde{v}_{m-1}(2^{m-1}l, y) = 0$, as we are interested in Dirichlet boundary data.

With the unit cell construction in hand, we proceed to the definition of a suitable cut-off layer which will be used in the top and bottom boundary regions of our self-similarly refining construction in the next section.

LEMMA 4.4 (cut-off layer). *Let $m \in \mathbb{N}$ and $\mathcal{A}(D), A, B, F$ be as in Lemma 4.2. For $0 < h \leq 2l \leq 1$, let $\omega = (0, 2^m l) \times (0, h)$ and let $E_{el}(\cdot, \cdot; \omega), E_{surf}(\cdot; \omega)$ be as in Lemma 4.2. Then there exists a potential $v : \omega \rightarrow \text{Sym}(\mathbb{R}^2; m-1)$ such that*

$$\begin{aligned} v(0, y) &= v(2^m l, y) = 0 & \text{for } y \in [0, h], \\ v(x, h) &= 0 & \text{for } x \in [0, 2^m l], \\ v(x, 0) &= \begin{cases} -\frac{1}{2} x e_1^{\odot(m-1)}, & x \in \left[0, \frac{l}{2}\right), \\ \frac{1}{2} (x - l) e_1^{\odot(m-1)}, & x \in \left[\frac{l}{2}, l\right], \end{cases} & \text{for } x \in [0, l], \\ v(x, 0) &= -v(x - 2^j l, 0) & \text{for } x \in [2^j l, 2^{j+1} l], j \in \{0, 1, \dots, m-1\}. \end{aligned}$$

Moreover, there exist $f \in BV(\omega; \{\pm \frac{1}{2}\})$ and a constant $C = C(m) > 0$ such that $\chi = (A - B)f + F$, $u = D^{\text{sym}} v + F$, and

$$E_{el}(u, \chi; \omega) \leq C \frac{l^{2m+1}}{h^{2m-1}}, \quad E_{surf}(\chi; \omega) \leq Ch.$$

Proof. We consider a smooth cut-off function $\phi : [0, \infty) \rightarrow [0, 1]$ such that $\phi(t) = 1$ for $0 \leq t < \frac{1}{2}$ and $\phi(t) = 0$ for $t > \frac{3}{4}$, e.g., a function of a similar form as in Remark 4.3. Let

$$\begin{aligned} v : \omega &\rightarrow \text{Sym}(\mathbb{R}^2; m-1), \\ v(x, y) &= \phi\left(\frac{y}{h}\right) \tilde{f}(x) e_1^{\odot(m-1)}, \end{aligned}$$

where for $x \in [0, l]$

$$\tilde{f}(x) = \begin{cases} -\frac{1}{2}x, & x \in \left[0, \frac{l}{2}\right), \\ \frac{1}{2}x - \frac{1}{2}l, & x \in \left[\frac{l}{2}, l\right), \end{cases}$$

$\tilde{f}(x) = -\tilde{f}(x - 2^j l)$ for $x \in [2^j l, 2^{j+1} l]$ for $j \in \{0, 1, \dots, m-1\}$, and $\tilde{f}(x) = 0$ for $x \notin [0, 2^m l]$. Notice that $\tilde{f}$ is continuous in $[0, 2^m l]$ since $\tilde{f}(0, y) = \tilde{f}(l, y) = 0$. It is direct that $v(x, y) = 0$ for $y > \frac{3}{4}h$ and that the properties for $y = 0$ hold. Moreover, we have $|\tilde{f}(x)| \leq \frac{l}{4}$. The phase indicator is given by

$$\chi(x, y) = D^{\text{sym}} \left(\tilde{f}(x) e_1^{\odot(m-1)} \right) + F = f(A - B) + F,$$

where $f = \tilde{f}' \in \{\pm \frac{1}{2}\}$. As $\phi(\frac{y}{h})\tilde{f}(x)e_1^{\odot(m-1)} = 0$ for $y \geq \frac{3}{4}h$ or for $x \notin [0, 2^m l]$ the boundary data condition is fulfilled.

It thus remains to provide the estimate for the energy contribution in ω . For the elastic energy, we notice that with $u = D^{\text{sym}} v + F$

$$\begin{aligned} |u - \chi|^2 &= \left| D^{\text{sym}} \left(\left(\phi\left(\frac{y}{h}\right) - 1 \right) \tilde{f}(x) e_1^{\odot(m-1)} \right) \right|^2 \\ &\leq C(m) \left(\frac{1}{4} \left| 1 - \phi\left(\frac{y}{h}\right) \right|^2 + \left| \tilde{f}(x) \frac{1}{h} \phi'\left(\frac{y}{h}\right) \right|^2 \right) \\ &\leq C(m) \left(1 + \frac{l^2}{h^2} \right). \end{aligned}$$

Thus,

$$E_{el}(u, \chi; \omega) \leq C(m) \left(1 + \frac{l^2}{h^2} \right) lh \leq C(m) \frac{l^3}{h} \leq C(m) \frac{l^{2m+1}}{h^{2m-1}},$$

as $h \leq 2l$. Moreover, since the interfaces are given by straight lines and since χ is bounded, we also have

$$E_{surf}(\chi; \omega) \leq C(m)h.$$

Combining the two bounds hence yields the desired result. $\square$

4.2. Highest vanishing order. With the unit cell construction and the cut-off function the two central ingredients of our construction are in place. We now combine these into the usual branching construction. For the construction to work, we make use of the fact that by the choice $\lambda = \frac{1}{2}$ we have that for v given in Lemma 4.2 the function $-v$ still satisfies the desired properties, most importantly $\partial_1(-\tilde{v}_0) = \tilde{u}_0 \in \{\pm \frac{1}{2}\}$.

LEMMA 4.5. *Let $m \in \mathbb{N}$, $\Omega = (0, 1)^2$ and A, B be as in Lemma 4.2 and let $F = \frac{1}{2}A + \frac{1}{2}B$. Consider the operator $\mathcal{A}(D)$ given in (14) and let the energy $E_\epsilon(u, \chi)$ for $u \in \mathcal{D}_F$, defined as in (7), and $\chi \in BV(\Omega; \{A, B\})$, be given by*

$$E_\epsilon(u, \chi) = \int_{\Omega} |u - \chi|^2 dx + \epsilon \int_{\Omega} |\nabla \chi|.$$

Then for every $N \in 2^m \mathbb{N}$, $N \geq 4$ there exist $u : \mathbb{R}^2 \rightarrow \text{Sym}(\mathbb{R}^2; m)$, $f \in BV(\Omega; \{\pm \frac{1}{2}\})$ with $\mathcal{A}(D)u = 0$ in $\mathbb{R}^2$, $u = F$ outside Ω , and a constant $C = C(m) > 0$ such that for any $\epsilon > 0$ it holds that

$$E_\epsilon(u, \chi) \leq C(N^{-2m} + \epsilon N),$$

where $\chi = f(A - B) + F \in BV(\Omega; \{A, B\})$.

Proof. We argue as in [8, 39, 43]. Let $\theta \in (2^{-\frac{2m}{2m-1}}, 2^{-1})$ and consider the splitting $\Omega = \Omega_+ \cup \Omega_-$ with $\Omega_+ = [0, 1] \times [\frac{1}{2}, 1]$, $\Omega_- = [0, 1] \times [0, \frac{1}{2}]$. In what follows, we give the construction of u and χ on Ω_+ . For given $N \in 2^m \mathbb{N}$, $N \geq 4$, we define

$$y_j = 1 - \frac{\theta^j}{2}, \quad l_j = \frac{1}{2^j N}, \quad h_j = y_{j+1} - y_j, \quad j \in \mathbb{N} \cup \{0\},$$

and set $j_0 \in \mathbb{N}$ as the maximal j satisfying $l_j < h_j$. This is possible as, due to $N \geq 4$, we have $l_0 < h_0$. In what follows, we use Lemma 4.2 in the cells $\omega_{j,k}$ (see the definition below) to achieve a refinement toward $y = 1$. For the sake of clarity of exposition, we define a construction on $[0, +\infty) \times [\frac{1}{2}, 1]$ and then restrict it to Ω_+ .

For this, let $v^{(j)}$ be the map defined on $[0, 2^m l_j] \times [0, h_j]$ according to Lemma 4.2 (with off-set at top and bottom fixed by $\delta = \frac{1}{10}$) for $j \in \{0, 1, \dots, j_0\}$; in particular $v^{(j)} : [0, 2^m l_j] \times [0, h_j] \rightarrow \text{Sym}(\mathbb{R}^2; m-1)$ is such that it satisfies the boundary conditions from Lemma 4.2. Moreover, let $v^{(j_0+1)}$ be the map defined on $[0, 2^m l_{j_0+1}] \times [0, \frac{\theta^{j_0+1}}{2}]$ according to Lemma 4.4, which is applicable due to $\frac{\theta^{j_0+1}}{2} = (1 - \theta)^{-1} h_{j_0+1} \leq (1 - \theta)^{-1} l_{j_0+1} < 2l_{j_0+1}$. In the following we will write $h_{j_0+1} = \frac{\theta^{j_0+1}}{2}$.

We extend $v^{(j)}$ for $j \in \{0, 1, \dots, j_0 + 1\}$ (without relabeling) onto $[0, +\infty) \times [0, h_j]$ as

$$v^{(j)}(x, y) := -v^{(j)}\left(x - 2^{m+m'} l_j, y\right) \quad \text{for } x \in [2^{m+m'} l_j, 2^{m+m'+1} l_j], \quad m' \in \mathbb{N}.$$

Notice that $v^{(j)}(\alpha 2^m l_j, y) = v^{(j)}(2^m l_j, y) = 0$ for every $\alpha \in \mathbb{N}$. Thus $v^{(j)}(\beta, y) = 0$ for every $\beta \in \mathbb{N}$, in particular $v^{(j)}(1, y) = 0$.

With this we define v on the upper half Ω_+ as follows:

$$v(x, y) = (-1)^j v^{(j)}(x, y - y_j) \quad \text{for } (x, y) \in [0, 1] \times [y_j, y_{j+1}].$$

We have that $v^{(j)}(x, 0) = -v^{(j-1)}(x, h_{j-1})$ for $x \in [0, 1]$. Indeed, by Lemmas 4.2 and 4.4, this is true for $x \in [0, 2^m l_j]$. Assuming that it is true for $x \in [0, 2^{m+m'} l_j]$, by definition of $v^{(j)}$, and using the fact that $l_{j-1} = 2l_j$, we infer that for $x \in [2^{m+m'} l_j, 2^{m+m'+1} l_j]$ we have

$$\begin{aligned} v^{(j-1)}(x, h_{j-1}) &= -v^{(j-1)}\left(x - 2^{m+m'-1} l_{j-1}, h_{j-1}\right) = -v^{(j-1)}\left(x - 2^{m+m'} l_j, h_{j-1}\right) \\ &= v^{(j)}\left(x - 2^{m+m'} l_j, 0\right) = -v^{(j)}(x, 0). \end{aligned}$$

Moreover, as already pointed out, it holds that $v^{(j)}(0, y) = v^{(j)}(1, y) = 0$; thus we can extend $v(x, y) = 0$ for $x < 0$ and $x > 1$. Finally, as $v^{(j_0+1)}(x, h_{j_0+1}) = 0$, we can deduce $v(x, 1) = (-1)^{j_0+1} v^{(j_0+1)}(x, h_{j_0+1}) = 0$, and thus we can set $v(x, y) = 0$ for $y \geq 1$.

Now let us turn to proving the energy estimate. We denote now

$$\omega_{j,k} := (k 2^m l_j, y_j) + [0, 2^m l_j] \times [0, h_j], \quad k \in \{0, 1, \dots, N 2^{j-m} - 1\}, \quad j \in \{0, 1, \dots, j_0\},$$

and

$$\omega_{j_0+1,k} := (k2^m l_{j_0+1}, y_{j_0+1}) + [0, 2^m l_{j_0+1}] \times [0, h_{j_0+1}].$$

Setting $f = \partial_1 \tilde{v}_0$, $u = D^{\text{sym}}v + F$, which is $\mathcal{A}$ -free, and $\chi = f(A - B) + F$, we have

$$\begin{aligned} E_\epsilon(u, \chi; \omega_{j,k}) &= E_\epsilon\left((-1)^j D^{\text{sym}}v^{(j)} + F, (-1)^j \partial_1 \tilde{v}_0^{(j)} e_1^{\odot m} + F; [0, 2^m l_j] \times [0, h_j]\right) \\ &= E_\epsilon\left(D^{\text{sym}}v^{(j)} + F, \partial_1 \tilde{v}_0^{(j)} e_1^{\odot m} + F; [0, 2^m l_j] \times [0, h_j]\right) \\ &\leq C(m) \left(\frac{l_j^{2m+1}}{h_j^{2m-1}} + \epsilon h_j \right), \end{aligned}$$

where we used the estimates in Lemmas 4.2 and 4.4 for the corresponding index j .

By symmetry we can repeat the same construction in Ω_- by replacing e_2 with $-e_2$. Therefore we obtain an $\mathcal{A}$ -free map u attaining the exterior data $u = F$ outside Ω and an associated phase indicator χ . Moreover, it holds that

$$\begin{aligned} E_\epsilon(u, \chi) &\leq 2 \sum_{j=0}^{j_0+1} \sum_{k=0}^{N2^{j-m}-1} \left(E_\epsilon(u, \chi; \omega_{j,k}) + \epsilon \text{Per}(\omega_{j,k}) \right) \\ &\leq 2C \sum_{j=0}^{j_0+1} \sum_{k=0}^{N2^{j-m}-1} \left(\frac{l_j^{2m+1}}{h_j^{2m-1}} + \epsilon h_j \right) = 2C \sum_{j=0}^{j_0+1} N2^{j-m} \left(\frac{l_j^{2m+1}}{h_j^{2m-1}} + \epsilon h_j \right) \\ &= 2C(m) \sum_{j=0}^{j_0+1} \left(\frac{l_j^{2m}}{h_j^{2m-1}} + \epsilon \frac{h_j}{l_j} \right) \leq C(m) \sum_{j=0}^{j_0+1} \left(N^{-2m} (2^{2m} \theta^{2m-1})^{-j} + N(2\theta)^j \right) \\ &\leq C(m, \theta) (N^{-2m} + \epsilon N), \end{aligned}$$

which concludes the argument. $\square$

4.3. Intermediate cases and proof of the upper bounds in Theorem 1.2.

In this section, we use the construction from section 4.2 in order to deduce an analogous construction for the case of intermediate vanishing orders for which $M := A - B = e_1^{\odot l} \odot e_2^{\odot(m-l)}$, where $l, m-l \neq 0$.

In contrast to the highest vanishing order setting, mimicking the construction from above, we now are confronted with the fact that the term involving χ is paired with two different components of the potential v , i.e.,

$$E_{el}(D^{\text{sym}}v, \chi) \sim \int_{\Omega} \left| \frac{l}{m} \partial_1 \tilde{v}_{m-l} + \frac{m-l}{m} \partial_2 \tilde{v}_{m-l-1} - \tilde{\chi}_{m-l} \right|^2 + H(D^{\text{sym}}v) dx,$$

since now χ is nondiagonal.

In order to deal with this, we make the ansatz that $\tilde{v}_{m-l-1} = 0$ or $\tilde{v}_{m-l} = 0$ (depending on whether l or $m-l$ is larger) and apply the construction from Lemma 4.2 to define the other components in the energy. Using then the corresponding equations as in the proof of Lemma 4.2, we set all but the last of the components for a smaller or bigger index, respectively, to zero.

LEMMA 4.6. *Let $m, l \in \mathbb{N}$ with $0 \leq l \leq m$, $\Omega = (0, 1)^2$ and let A, B be such that $A - B = e_1^{\odot l} \odot e_2^{\odot(m-l)}$ and $F = \frac{1}{2}A + \frac{1}{2}B$. Consider the operator $\mathcal{A}(D)$ given in (14) and let the energy E_ϵ be given as in Lemma 4.5. We define $L := \max\{l, m-l\}$. Then for every $N \in 2^L \mathbb{N}$, $N \geq 4$, there exist a deformation $u : \mathbb{R}^2 \rightarrow \text{Sym}(\mathbb{R}^2; m)$, a phase*

indicator $\chi : \mathbb{R}^2 \rightarrow \{A, B\}$ with $\mathcal{A}(D)u = 0$ in $\mathbb{R}^2$, $u = F$ outside Ω , and a constant $C = C(m) > 0$ such that for any $\epsilon > 0$ it holds that

$$E_\epsilon(u, \chi) \leq C(N^{-2L} + \epsilon N).$$

Proof. The idea is to reduce the order of the tensor such that effectively we can use Lemma 4.5. As before, without loss of generality, $F = 0$ and therefore $A = \frac{1}{2}e_1^{\odot l} \odot e_2^{\odot(m-l)}, B = -\frac{1}{2}e_1^{\odot l} \odot e_2^{\odot(m-l)}$. In order to reduce the order, for $\chi = f(A - B)$, we estimate

$$E_{el}(u, \chi) \leq C(m) \left(\sum_{k=0}^{m-l-1} \int_{\Omega} |\tilde{u}_k|^2 dx + \int_{\Omega} |\tilde{u}_{m-l} - f|^2 dx + \sum_{k=m-l+1}^m \int_{\Omega} |\tilde{u}_k|^2 dx \right),$$

where $f(x) \in \{\pm \frac{1}{2}\}$. Since the case $l = m$ (and symmetrically $l = 0$) has already been considered in Lemma 4.5, we only present the construction for the setting $0 < l < m$. In this case, the ansatz is to set $\tilde{u}_k = 0$ for either $k \leq m-l-1$ or for $k \geq m-l+1$ (depending on whether $l > m-l$ or $m-l > l$) and to use Lemma 4.5 for the remaining components.

Without loss of generality let $l \geq m-l$; else change the roles of x and y and adapt the coefficients accordingly. Denoting by $\bar{\mathcal{A}}(D)$ the operator for l th order tensors, we invoke Lemma 4.5 for $m = l \neq 0$ to define $w : \mathbb{R}^2 \rightarrow \text{Sym}(\mathbb{R}^2; l)$ with $\bar{\mathcal{A}}(D)w = 0$ in $\mathbb{R}^2$, $w = 0$ outside Ω , and $f : \mathbb{R}^2 \rightarrow \{\pm \frac{1}{2}\}$ such that $f(A' - B') : \mathbb{R}^2 \rightarrow \{A', B'\}$ with $A' = \frac{1}{2}e_1^{\odot l}, B' = -\frac{1}{2}e_1^{\odot l}$. We then define $\tilde{u}_k := (-1)^{l-m} 2^{m-l} \frac{l!k!}{m!(k-(m-l))!} \tilde{w}_{k-(m-l)}$ for $k \geq m-l$ and $\tilde{u}_k = 0$ else. Moreover, we set $\chi := f(A - B) = f e_1^{\odot l} \odot e_2^{\odot(m-l)} : \mathbb{R}^2 \rightarrow \{A, B\}$. Then u defined by $u_{1\dots 12\dots 2} = \tilde{u}_k$ fulfills $u = 0$ outside Ω and defines indeed an $\mathcal{A}$ -free map as by (26):

$$\begin{aligned} [\mathcal{A}(D)u]_{1212\dots 12} &= \sum_{k=0}^m \left((-1)^k 2^{-m} \binom{m}{k} \partial_1^k \partial_2^{m-k} \tilde{u}_k \right) \\ &= \sum_{k=m-l}^m (-1)^k 2^{-m} \binom{m}{k} \partial_1^k \partial_2^{m-k} (-1)^{l-m} 2^{m-l} \frac{l!k!}{m!(k-(m-l))!} \tilde{w}_{k-(m-l)} \\ &= \sum_{k=0}^l (-1)^k 2^{-l} \frac{l!}{(m-(k+m-l))!k!} \partial_1^{k+m-l} \partial_2^{m-(k+m-l)} \tilde{w}_k \\ &= \partial_1^{m-l} \sum_{k=0}^l (-1)^k 2^{-l} \binom{l}{k} \partial_1^k \partial_2^{l-k} \tilde{w}_k = \partial_1^{m-l} [\bar{\mathcal{A}}(D)w]_{12\dots 12} = 0. \end{aligned}$$

Furthermore, it holds that

$$\begin{aligned} E_\epsilon(u, \chi) &\leq C \left(\int_{\Omega} |\tilde{w}_0 - f|^2 dx + \sum_{k=1}^l \int_{\Omega} |\tilde{w}_k|^2 dx + \epsilon \int_{\Omega} |\nabla f| \right) = C \tilde{E}_\epsilon(w, f e_1^{\odot l}) \\ &\leq C(N^{-2l} + \epsilon N), \end{aligned}$$

where $\tilde{E}_\epsilon(w, f e_1^{\odot l})$ denotes the corresponding energy for l tensors. Since $l \geq m-l$, this yields that

$$E_\epsilon(u, \chi) \leq C \min\{N^{-2l} + \epsilon N, N^{-2(m-l)} + \epsilon N\} = C(N^{-2L} + \epsilon N),$$

which therefore concludes the proof. $\square$

We combine the estimates from Lemmas 4.5 and 4.6 into the proof of the upper bound scaling result.

Proof of upper bounds in Theorem 1.2. To show the upper bounds, we use Lemmas 4.5 and 4.6 and optimize in N ; thus we choose $N \sim \epsilon^{-\frac{1}{2L+1}}$ and therefore

$$E_\epsilon(\chi; F) \leq E_\epsilon(u, \chi) \leq C(N^{-2L} + \epsilon N) \leq C\epsilon^{\frac{2L}{2L+1}}. \quad \square$$

Last but not least, we remark that the results for the higher order curl also imply a corresponding result for the higher order divergence in $d = 2$.

COROLLARY 4.7 (higher order divergence). *Let $m, k \in \mathbb{N}$. Let $\Omega = (0, 1)^2 \subset \mathbb{R}^2$ and E_ϵ^B as in (6) for the m th order divergence $B(D) = \operatorname{div}^m$ as in (19). Consider $A, B \in \mathbb{R}^k \otimes \operatorname{Sym}(\mathbb{R}^2; m)$ such that $A - B = v \otimes e_1^{\odot l} \odot e_2^{\odot(m-l)}$ for $l \in \{0, 1, \dots, m\}$, $v \in \mathbb{R}^k$, and $F = \frac{1}{2}A + \frac{1}{2}B$. Then there exist $\chi \in BV(\Omega; \{A, B\})$ and $C = C(m) > 1$ such that for $L := m - \min\{l, m - l\} = \max\{l, m - l\}$*

$$E_\epsilon(\chi; F) \leq C\epsilon^{\frac{2L}{2L+1}}.$$

Proof. First we notice that, without loss of generality, we can assume $k = 1$ by working componentwise and moreover, using the notation of Convention 4.1, $\operatorname{div}^m u = \sum_{j=0}^m \partial_1^j \partial_2^{m-j} \tilde{u}_{m-j}$. The idea is to use the $\mathcal{A}$ -free setting for $\mathcal{A}$ as in Lemma 4.6 and then transform u', χ' (defined below) such that we are in the divergence-free setting. To simplify the notation, set

$$\alpha(m, j) = (-1)^j 2^{-m} \binom{m}{j} \neq 0$$

such that $[\mathcal{A}(D)u']_{12\dots 12} = \sum_{j=0}^m \alpha(m, j) \partial_1^j \partial_2^{m-j} \tilde{u}'_j$. By an application of Lemma 4.6, for $N \in 2^L \mathbb{N}$, $A', B' \in \operatorname{Sym}(\mathbb{R}^2; m)$ defined componentwise by $\tilde{A}'_j = \alpha(m, j)^{-1} \tilde{A}_{m-j}$, $\tilde{B}'_j = \alpha(m, j)^{-1} \tilde{B}_{m-j}$, and for $F' = \frac{1}{2}A' + \frac{1}{2}B'$ there exists $u' : \mathbb{R}^2 \rightarrow \operatorname{Sym}(\mathbb{R}^2; m)$, $\chi' : \mathbb{R}^2 \rightarrow \{A', B'\}$ such that $u' = F'$ outside Ω and $\mathcal{A}(D)u' = 0$. We can apply Lemma 4.6 after a rescaling, as for the above definition of A', B' we have $A' - B' = \alpha(m, l)^{-1} e_1^{\odot(m-l)} \odot e_2^{\odot l}$. This can be seen by considering $(A' - B')_j = \tilde{A}'_j - \tilde{B}'_j = \alpha(m, j)^{-1} \cdot (\tilde{A} - \tilde{B})_{m-j}$ and using that $A - B = e_1^{\odot l} \odot e_2^{\odot(m-l)}$. Moreover it holds that

$$E_\epsilon^A(u', \chi') \leq C(N^{-2L} + \epsilon N)$$

with a constant $C > 0$ only depending on m . Setting now

$$\tilde{u}_{m-j} := \alpha(m, j) \tilde{u}'_j, \quad \tilde{\chi}_{m-j} := \alpha(m, j) \tilde{\chi}'_j,$$

we observe that

$$\operatorname{div}^m u = \sum_{j=0}^m \partial_1^j \partial_2^{m-j} \tilde{u}_{m-j} = [\mathcal{A}(D)u']_{12\dots 12} = 0$$

and that outside Ω we have

$$\tilde{u}_{m-j} = \alpha(m, j) \tilde{u}'_j = \alpha(m, j) \tilde{F}'_j = \alpha(m, j) \left(\frac{1}{2} \tilde{A}'_j + \frac{1}{2} \tilde{B}'_j \right) = \tilde{F}_{m-j}.$$

Thus u is admissible and $\tilde{\chi}_{m-j} = \alpha(m, j) \tilde{\chi}'_j \in \{\tilde{A}_{m-j}, \tilde{B}_{m-j}\}$. To bound the energy we notice that

$$|u - \chi|^2 = \sum_{j=0}^m \binom{m}{j} |\tilde{u}_j - \tilde{\chi}_j|^2 \leq C(m) |u' - \chi'|^2, \quad |\nabla \chi| \leq C(m) |\nabla \chi'|,$$

and therefore $E_\epsilon^B(u, \chi) \leq C(m) E_\epsilon^A(u', \chi') \leq C(m) (N^{-2L} + \epsilon N)$. Choosing $N \sim \epsilon^{-\frac{1}{2L+1}}$ concludes the proof. $\square$

REFERENCES

- [1] I. AKRAMOV, H. KNÜPFER, M. KRUŽÍK, AND A. RÜLAND, *Minimal energy for geometrically nonlinear elastic inclusions in two dimensions*, Proc. Roy. Soc. Edinburgh Sect. A (2023), pp. 1–24, <https://doi.org/10.1017/prm.2023.36>.
- [2] L. AMBROSIO, N. FUSCO, AND D. PALLARA, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford Math. Monogr., Oxford University Press, Oxford, UK, 2000.
- [3] J. M. BALL AND R. D. JAMES, *Proposed experimental tests of a theory of fine microstructure and the two-well problem*, Philos. Trans. A, 338 (1992), pp. 389–450, <https://doi.org/10.1098/rsta.1992.0013>.
- [4] K. BHATTACHARYA, *Microstructure of Martensite: Why It Forms and How It Gives Rise to the Shape-memory Effect*, Oxford Ser. Materials Modelling, Oxford University Press, Oxford, UK, 2003.
- [5] A. CAPELLA AND F. OTTO, *A rigidity result for a perturbation of the geometrically linear three-well problem*, Comm. Pure Appl. Math., 62 (2009), pp. 1632–1669, <https://doi.org/10.1002/cpa.20297>.
- [6] A. CAPELLA AND F. OTTO, *A quantitative rigidity result for the cubic-to-tetragonal phase transition in the geometrically linear theory with interfacial energy*, Proc. Roy. Soc. Edinburgh Sect. A, 142 (2012), pp. 273–327, <https://doi.org/10.1017/S0308210510000478>.
- [7] P. CESANA, F. DELLA PORTA, A. RÜLAND, C. ZILLINGER, AND B. ZWICKNAGL, *Exact constructions in the (non-linear) planar theory of elasticity: From elastic crystals to nematic elastomers*, Arch. Ration. Mech. Anal., 237 (2020), pp. 383–445, <https://doi.org/10.1007/s00205-020-01511-9>.
- [8] A. CHAN AND S. CONTI, *Energy scaling and branched microstructures in a model for shape-memory alloys with $SO(2)$ invariance*, Math. Models Methods Appl. Sci., 25 (2015), pp. 1091–1124, <https://doi.org/10.1142/S0218202515500281>.
- [9] M. CHLEBÍK AND B. KIRCHHEIM, *Rigidity for the four gradient problem*, J. Reine Angew. Math., 551 (2002), pp. 1–9, <https://doi.org/10.1515/crll.2002.082>.
- [10] R. CHOKSI, R. V. KOHN, AND F. OTTO, *Domain branching in uniaxial ferromagnets: A scaling law for the minimum energy*, Comm. Math. Phys., 201 (1999), pp. 61–79, <https://doi.org/10.1007/s002200050549>.
- [11] S. CONTI, *Branched microstructures: Scaling and asymptotic self-similarity*, Comm. Pure Appl. Math., 53 (2000), pp. 1448–1474, [https://doi.org/10.1002/1097-0312\(200011\)53:11<1448::AID-CPA6>3.0.CO;2-C](https://doi.org/10.1002/1097-0312(200011)53:11<1448::AID-CPA6>3.0.CO;2-C).
- [12] S. CONTI, J. DIERMEIER, D. MELCHING, AND B. ZWICKNAGL, *Energy scaling laws for geometrically linear elasticity models for microstructures in shape memory alloys*, ESAIM Control Optim. Calc. Var., 26 (2020), 115, <https://doi.org/10.1051/cocv/2020020>.
- [13] S. CONTI AND F. GMEINER, *$\mathcal{A}$ -quasiconvexity and partial regularity*, Calc. Var. Partial Differential Equations, 61 (2022), 215, <https://doi.org/10.1007/s00526-022-02326-0>.
- [14] S. CONTI, M. KLAR, AND B. ZWICKNAGL, *Piecewise affine stress-free martensitic inclusions in planar nonlinear elasticity*, Proc. A, 473 (2017), 20170235, <https://doi.org/10.1098/rspa.2017.0235>.
- [15] S. CONTI AND F. THEIL, *Single-slip elastoplastic microstructures*, Arch. Ration. Mech. Anal., 178 (2005), pp. 125–148, <https://doi.org/10.1007/s00205-005-0371-8>.
- [16] S. CONTI AND B. ZWICKNAGL, *Low volume-fraction microstructures in martensites and crystal plasticity*, Math. Models Methods Appl. Sci., 26 (2016), pp. 1319–1355, <https://doi.org/10.1142/S0218202516500317>.
- [17] B. DACOROGNA, *Weak Continuity and Weak Lower Semicontinuity of Non-linear Functionals*, Lecture Notes in Math. 922, Springer, New York, 1982, <https://doi.org/10.1007/BFb0096144>.
- [18] B. DACOROGNA AND P. MARCELLINI, *Implicit Partial Differential Equations*, Progr. Nonlinear Differential Equations Appl. 37, Birkhäuser, Boston, 2012, <https://doi.org/10.1007/978-1-4612-1562-2>.
- [19] G. DE PHILIPPIS, L. PALMIERI, AND F. RINDLER, *On the two-state problem for general differential operators*, Nonlinear Anal., 177 (2018), pp. 387–396, <https://doi.org/10.1016/j.na.2018.03.015>.
- [20] I. FONSECA AND S. MÜLLER, *$\mathcal{A}$ -quasiconvexity, lower semicontinuity, and Young measures*, SIAM J. Math. Anal., 30 (1999), pp. 1355–1390, <https://doi.org/10.1137/S0036141098339885>.
- [21] J. GINSTER AND B. ZWICKNAGL, *Energy scaling law for a singularly perturbed four-gradient problem in helimagnetism*, J. Nonlinear Sci., 33 (2023), 20, <https://doi.org/10.1007/s00332-022-09847-0>.

- [22] F. GMEINER, P. LEWINTAN, AND P. NEFF, *Optimal incompatible Korn-Maxwell-Sobolev inequalities in all dimensions*, Calc. Var. Partial Differential Equations, 62 (2023), 182, <https://doi.org/10.1007/s00526-023-02522-6>.
- [23] A. GUERRA, B. RAIȚĂ, AND M. R. I. SCHRECKER, *Compensated compactness: Continuity in optimal weak topologies*, J. Funct. Anal., 283 (2022), 109596, <https://doi.org/10.1016/j.jfa.2022.109596>.
- [24] A. GUERRA AND B. RAIȚĂ, *Quasiconvexity, null Lagrangians, and Hardy space integrability under constant rank constraints*, Arch. Ration. Mech. Anal., 245 (2022), pp. 279–320, <https://doi.org/10.1007/s00205-022-01775-3>.
- [25] J. ILMAVIRTA AND F. MONARD, *Integral geometry on manifolds with boundary and applications*, in The Radon Transform—The First 100 Years and Beyond, Radon Ser. Comput. Appl. Math. 22, De Gruyter, Berlin, 2019, pp. 43–113, <https://doi.org/10.1515/9783110560855-004>.
- [26] B. KIRCHHEIM, *Rigidity and Geometry of Microstructures*, 2003, <https://www.mis.mpg.de/publications/other-series/ln/lecturenote-1603.html>.
- [27] B. KIRCHHEIM, S. MÜLLER, AND V. ŠVERÁK, *Studying nonlinear PDE by geometry in matrix space*, in Geometric Analysis and Nonlinear Partial Differential Equations, Springer, New York, 2003, pp. 347–395, https://doi.org/10.1007/978-3-642-55627-2_19.
- [28] H. KNÜPFER AND R. V. KOHN, *Minimal energy for elastic inclusions*, Proc. A, 467 (2011), pp. 695–717, <https://doi.org/10.1098/rspa.2010.0316>.
- [29] H. KNÜPFER, R. V. KOHN, AND F. OTTO, *Nucleation barriers for the cubic-to-tetragonal phase transformation*, Comm. Pure Appl. Math., 66 (2013), pp. 867–904, <https://doi.org/10.1002/cpa.21448>.
- [30] R. V. KOHN AND S. MÜLLER, *Relaxation and regularization of nonconvex variational problems*, Rend. Semin. Mat. Fisico Milano, 62 (1992), pp. 89–113, <https://doi.org/10.1007/BF02925437>.
- [31] R. V. KOHN AND S. MÜLLER, *Surface energy and microstructure in coherent phase transitions*, Comm. Pure Appl. Math., 47 (1994), pp. 405–435, <https://doi.org/10.1002/cpa.3160470402>.
- [32] R. V. KOHN AND B. WIRTH, *Optimal fine-scale structures in compliance minimization for a uniaxial load*, Proc. A, 470 (2014), 20140432, <https://doi.org/10.1098/rspa.2014.0432>.
- [33] J. KRISTENSEN AND B. RAIȚĂ, *Oscillation and concentration in sequences of PDE constrained measures*, Arch. Ration. Mech. Anal., 246 (2022), pp. 823–875, <https://doi.org/10.1007/s00205-022-01828-7>.
- [34] S. MÜLLER, *Variational models for microstructure and phase transitions*, in Calculus of Variations and Geometric Evolution Problems, Lecture Notes in Math. 1713, Springer, New York, 1999, pp. 85–210, <https://doi.org/10.1007/BFb0092670>.
- [35] F. MURAT, *Compacité par compensation: Condition nécessaire et suffisante de continuité faible sous une hypothèse de rang constant*, Ann. Sc. Norm. Super. Pisa Cl. Sci., 4 (1981), pp. 8–102.
- [36] G. P. PATERNAIN, M. SALO, AND G. UHLMANN, *Tensor tomography: Progress and challenges*, Chinese Ann. Math. Ser. B, 35 (2014), pp. 399–428, <https://doi.org/10.1007/s11401-014-0834-z>.
- [37] G. P. PATERNAIN, M. SALO, AND G. UHLMANN, *Geometric Inverse Problems: With Emphasis on Two Dimensions*, Cambridge Stud. Adv. Math. 204, Cambridge University Press, Cambridge, UK, 2023, <https://doi.org/10.1017/9781009039901>.
- [38] B. RAIȚĂ, *Potentials for $\mathcal{A}$ -quasiconvexity*, Calc. Var. Partial Differential Equations, 58 (2019), 105, <https://doi.org/10.1007/s00526-019-1544-x>.
- [39] B. RAIȚĂ, A. RÜLAND, AND C. TISSOT, *On scaling properties for two-state problems and for a singularly perturbed T_3 structure*, Acta Appl. Math., 184 (2023), 5, <https://doi.org/10.1007/s10440-023-00557-7>.
- [40] F. RINDLER, *Calculus of Variations*, Universitext, Springer, Cham, 2018, <https://doi.org/10.1007/978-3-319-77637-8>.
- [41] A. RÜLAND, *A rigidity result for a reduced model of a cubic-to-orthorhombic phase transition in the geometrically linear theory of elasticity*, J. Elasticity, 123 (2016), pp. 137–177, <https://doi.org/10.1007/s10659-015-9553-2>.
- [42] A. RÜLAND, J. M. TAYLOR, AND C. ZILLINGER, *Convex integration arising in the modelling of shape-memory alloys: Some remarks on rigidity, flexibility and some numerical implementations*, J. Nonlinear Sci., 29 (2018), pp. 2137–2184, <https://doi.org/10.1007/s00332-019-09540-9>.
- [43] A. RÜLAND AND A. TRIBUZIO, *On the energy scaling behaviour of a singularly perturbed Tartar square*, Arch. Ration. Mech. Anal., 243 (2022), pp. 401–431, <https://doi.org/10.1007/s00205-021-01729-1>.

- [44] A. RÜLAND AND A. TRIBUZIO, *On scaling laws for multi-well nucleation problems without gauge invariances*, J. Nonlinear Sci., 33 (2023), <https://doi.org/10.1007/s00332-022-09879-6>.
- [45] A. RÜLAND AND A. TRIBUZIO, *On the energy scaling behaviour of singular perturbation models with prescribed Dirichlet data involving higher order laminates*, ESAIM Control Optim. Calc. Var., 29 (2023), <https://doi.org/10.1051/cocv/2023047>.
- [46] V. A. SHARAFUTDINOV, *Integral Geometry of Tensor Fields*, Inverse Ill-posed Probl. Ser., De Gruyter, Berlin, 1994, <https://doi.org/10.1515/9783110900095>.
- [47] M. SORELLA AND R. TIONE, *The four-state problem and convex integration for linear differential operators*, J. Funct. Anal., 284 (2023), 109785, <https://doi.org/10.1016/j.jfa.2022.109785>.
- [48] L. TARTAR, *Compensated compactness and applications to partial differential equations*, in Non-linear Analysis and Mechanics: Heriot-Watt Symposium, Research Notes in Mathematics 4, Pitman, London, 1979, pp. 136–212.
- [49] L. TARTAR, *Some remarks on separately convex functions*, in Microstructure and Phase Transition, IMA Vol. Math. Appl. 54, Springer, New York, 1993, pp. 191–204, https://doi.org/10.1007/978-1-4613-8360-4_12.
- [50] L. TARTAR, *Compensation effects in partial differential equations*, Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. (5), 29 (2005), pp. 395–453, <https://media.academixl.it/memorie/S5-VXXIX-P1-2-2005/Tartar395-453.pdf>.

On surface energies in scaling laws for singular perturbation problems for martensitic phase transitions

This chapter contains the (unpublished) article [RTTZ25]. Reproduced is the arXiv version A. Rüland, C. Tissot, A. Tribuzio, and C. Zillinger. *On surface energies in scaling laws for singular perturbation problems for martensitic phase transitions*. 2025. arXiv: 2507.06773 [math.AP]. The article is reproduced with the permissions of the authors.

A summary of the article is given in Chapter 4.

ON SURFACE ENERGIES IN SCALING LAWS FOR SINGULAR PERTURBATION PROBLEMS FOR MARTENSITIC PHASE TRANSITIONS

ANGKANA RÜLAND, CAMILLO TISSOT, ANTONIO TRIBUZIO, AND CHRISTIAN ZILLINGER

ABSTRACT. The objective of this article is to compare different surface energies for multi-well singular perturbation problems associated with martensitic phase transformations involving higher order laminates. We deduce scaling laws in the singular perturbation parameter which are *robust* in the choice of the surface energy (e.g., diffuse, sharp, an interpolation thereof or discrete). Furthermore, we show that these scaling laws do not require the presence of isotropic surface energies but that generically also *highly anisotropic surface energies* yield the same scaling results. More precisely, the presence of essentially generic partial directional derivatives in the regularization terms suffices to produce the same scaling behaviour as in the isotropic setting. The only sensitive directional dependences are directly linked to the *lamination directions* of the well structure – and even for these only the “inner-most” lamination direction is of significance in determining the scaling law. In view of experimental applications, this shows that also for higher-order laminates, the precise structure of the surface energies – which is often very difficult to determine experimentally – does not have a crucial impact on the scaling behaviour of the investigated structures but only enters when considering finer properties.

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1. INTRODUCTION

Surface energies play an important role in singular perturbation models for solid-solid phase transformations. Combined with elastic energies, they introduce a natural length scale into the models. Thus, the combination of elastic and surface energies provides important information on the different length scales present in the experimentally observed microstructures. However, from an experimental point of view, surface energies are notoriously difficult to measure and are often highly anisotropic. It is thus of particular significance to investigate the robustness of singular perturbation models with respect to different choices of surface energy regularizations. The purpose of this article is to prove that the scaling behaviour is robust with respect to a rather large class of modifications of the surface energies and that the resulting scaling laws do not depend on the fine structure of the singular perturbation term. In fact, only minimal requirements are necessary, which only depend on very basic information on the model, even if higher order laminates are involved.

We are particularly interested in martensitic phase transformations in shape-memory alloys. These materials are typically metal alloys such as CuAlNi or NiTi which undergo a first order, diffusionless, solid-solid phase transformation. In this transition symmetry is reduced from the high to the low temperature phase which gives rise to multiple energy wells and complex material behaviour.

In this article, we will consider simplified models without gauges for the formation of microstructures in these materials which are governed by energies of the following form [Bha03, Mü99]

$$(1) \quad E_\epsilon(u) := E_{el}(u) + \epsilon E_{surf}(u).$$

Here $u : \Omega \subset \mathbb{R}^d \rightarrow \mathbb{R}^d$ denotes the deformation with respect to the reference configuration Ω and $E_{el}(u) := \int_{\Omega} \text{dist}^2(\nabla u, \mathcal{K}) dx$ models the elastic energy. Typically, the energy density is of

multi-well nature with $\mathcal{K} \subset \mathbb{R}^{d \times d}$ a prescribed set in matrix space corresponding to the energy wells of the respective model. The main focus of this article is on the structure of the second energy contribution in (1), $E_{surf}(u)$, which models a surface energy. The parameter $\epsilon > 0$ is material specific and typically small. Mathematically, the surface energy provides a higher order regularizing contribution which penalizes fine-scale oscillations in ∇u . In what follows below, we will consider different models for $E_{surf}(u)$. Our objective is to prove that for our model class of martensitic phase transformations, the investigated scaling laws typically do not depend on the precise structure of the surface energies. On the contrary, relatively “rough” information suffices to produce equivalent scaling laws. In order to illustrate this, we discuss different prototypical model classes for microstructures in shape-memory alloys.

1.1. Sharp surface energies. We begin by considering sharp interface models. In particular, we focus on settings involving anisotropy. Here, we specify the set-up from (1) as follows

$$(2) \quad E_\epsilon(u, \chi) := E_{el}(u, \chi) + \epsilon E_{surf}(\chi) := \int_{\Omega} |\nabla u - \chi|^2 dx + \epsilon \sum_{\nu \in \mathcal{N}} \|D_\nu \chi\|_{TV(\Omega)}.$$

For given $F \in \mathbb{R}^{d \times d}$, $u \in \mathcal{A}_F$ models the deformation, where

$$(3) \quad \mathcal{A}_F := \{v \in H^1(\Omega; \mathbb{R}^d) : v(x) = Fx \text{ on } \partial\Omega\},$$

while $\chi : \Omega \rightarrow \mathcal{K} \subset \mathbb{R}^{d \times d}$ represents the phase indicator.

As all the considered quantities are translation invariant, we could also consider a boundary condition of the form $u(x) = Fx + b$ on $\partial\Omega$ for some $b \in \mathbb{R}^d$. For the sake of simplicity, we assume $b = 0$.

In our study below, the set $\mathcal{K}$ will represent the wells of phase transformation models with a discrete set of minima for the energy density (i.e., we do not consider typical gauges such as $SO(d)$ or $Skew(d)$ invariances in our model). The surface energy under consideration $\|D_\nu \chi\|_{TV(\Omega)}$ is of sharp-interface nature and highly anisotropic, depending only on specified linearly independent directions $\nu \in \mathcal{N} \subset \mathbb{S}^{d-1}$ with $\#\mathcal{N} \leq d$ (see Section 2.2 for further discussion and definitions). In what follows below, we will discuss minimal conditions on the choice of the directions $\nu \in \mathcal{N}$ in order to ensure “generic” behaviour in the scaling laws – thus proving their robustness in this choice of surface energy. In particular, in many instances it suffices that $\mathcal{N}$ consists of a single, non-degenerate direction $\mathcal{N} = \{\nu\}$. We will investigate scaling laws for energies of the type (2) for various possible choices of the set $\mathcal{K}$ and discuss microstructures of different complexities.

1.1.1. The two-well problem. We begin with an essentially scalar setting by considering $\mathcal{K} = \{A, B\}$ with $\text{rank}(A - B) = 1$. In this setting, the expected microstructure consists of a branched version of twinning [KM94, KM92]. Such structures play an important role in austenite-martensite interfaces [KM94, KM92, CO12, CO09, Sim21b, Sim21a]. As already highlighted in the seminal works [KM94, KM92], in order to observe this phenomenon, it is not necessary to include all directional derivatives in the singular perturbation term. As expected from the experimental microstructure and the almost one-dimensional character of the problem, it suffices to regularize in the direction of oscillation. Hence, the scaling of the fully surface regularized and of the only in direction of oscillation regularized model behave analogously. We recall this in the following proposition (cf. [KM94, KM92]).

Proposition 1.1. *Let $A, B \in \mathbb{R}^{d \times d}$ be such that $A - B = a \otimes e_1$ for some $a \in \mathbb{R}^d \setminus \{0\}$. Let $\Omega = (0, 1)^d$ and $F_\alpha = \alpha A + (1 - \alpha)B$ for some $\alpha \in (0, 1)$. Let $\nu \in \mathbb{S}^{d-1}$ be such that $\nu \cdot e_1 \neq 0$. Consider for $\mathcal{N} = \{\nu\}$, $u \in \mathcal{A}_{F_\alpha}$, cf. (3), and $\chi \in BV_\nu(\Omega; \{A, B\})$ the energy $E_\epsilon(u, \chi)$ as defined in (2). There are constants $C = C(d, \alpha, |A - B|) > 0$ and $\epsilon_0 = \epsilon_0(d, \alpha, |A - B|, |\nu_1|) > 0$ such that for any $\epsilon \in (0, \epsilon_0)$*

$$\inf_{\chi \in BV_\nu(\Omega; \{A, B\})} \inf_{u \in \mathcal{A}_{F_\alpha}} E_\epsilon(u, \chi) \geq C^{-1} |\nu_1|^{\frac{2}{3}} \epsilon^{\frac{2}{3}}.$$

Here ν_1 denotes the e_1 component of the vector ν .

In two dimensions the matching upper bounds hold, i.e. for $d = 2$ and every $\epsilon \in (0, \epsilon_0)$ there are $u \in \mathcal{A}_{F_\alpha}$ and $\chi \in BV_\nu(\Omega; \{A, B\})$ such that

$$E_\epsilon(u, \chi) \leq C |\nu_1|^{\frac{2}{3}} \epsilon^{\frac{2}{3}}.$$

Moreover (in d dimensions), if $\nu \cdot e_1 = 0$, then under the above assumptions for all $\epsilon > 0$

$$\inf_{\chi \in BV_\nu(\Omega; \{A, B\})} \inf_{u \in \mathcal{A}_{F_\alpha}} E_\epsilon(u, \chi) = 0.$$

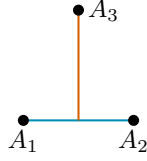


FIGURE 1. The Lorent three-well setting. The first order laminates are shown in blue, the second order laminates in orange.

Here and below the space $BV_\nu(\Omega; \{A, B\})$ denotes an anisotropic version of the space BV in which BV regularity is only required in the direction $\nu \in \mathbb{S}^{d-1}$. We refer to Section 2.2 for the precise definition of it.

Remark 1.2. *The matching upper bound can also be generalized to hold in higher dimensions. For the sake of simplicity we do not discuss this in this article; the construction for an isotropic surface penalization can, for instance, be found in [RT23b, Section 6.2].*

1.1.2. *The three-well problem of Lorent.* With the almost one-dimensional two-well problem in mind, we turn to models involving higher order laminates. Here a model problem is given by the three-well configuration of Lorent [Lor01]. In this setting, we have (up to normalization)

$$(4) \quad \mathcal{K}_3 := \{A_1, A_2, A_3\} \text{ with } A_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}.$$

These three matrices are chosen such that $\text{rank}(A_1 - A_2) = 1$, but neither A_1 nor A_2 are rank-one connected with the well A_3 . The lamination convex hull $\mathcal{K}_3^{lc}$ of $\mathcal{K}_3$ and, hence, the observable microstructure, consists of laminates up to second order (we recall notions such as lamination convexity in Section 2.1). The first order laminates $\mathcal{K}_3^1$ consist of convex combinations of the wells A_1, A_2 while the second order laminates $\mathcal{K}_3^2$ are obtained by a convex combination of the well A_3 with the auxiliary matrix $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}$ (see Figure 1). More precisely,

$$(5) \quad \mathcal{K}_3^1 := \mathcal{K}_3^{(1)} \setminus \mathcal{K}_3 = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} : \alpha \in (0, 1) \right\}, \quad \mathcal{K}_3^2 := \mathcal{K}_3^{(2)} \setminus \mathcal{K}_3^{(1)} = \left\{ \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \alpha \end{pmatrix} : \alpha \in (0, 1) \right\}.$$

In [RT23b] in the setting with isotropic surface energies, the different scaling behaviour of microstructures with affine boundary conditions was deduced with the behaviour depending only on the complexity of the boundary data encoded in their lamination order. Here we prove that this remains valid, if in (2) we consider generic regularization directions. Indeed, due to the one-dimensional structure of the lamination convex hull, it suffices to choose $\mathcal{N} = \{\nu\}$ such that $\nu \cdot e_1 \neq 0$ in order to recover the scaling from [RT23b] in which we considered the full gradient in the surface energy. If, however, $\nu = \pm e_2$, then, the setting indeed changes and the scaling behaviour becomes that of a laminate of one order less.

Theorem 1. *Let $\mathcal{K}_3$ be given by (4), $\Omega = (0, 1)^2$, and $\nu \in \mathbb{S}^1$. Consider the energy $E_\epsilon(u, \chi)$ as in (2) with $\mathcal{N} = \{\nu\}$, $u \in \mathcal{A}_F$, cf. (3), for $F \in \mathcal{K}_3^{lc} \setminus \mathcal{K}_3$, and $\chi \in BV_\nu(\Omega; \mathcal{K}_3)$. Let $\mathcal{K}_3^1$ and $\mathcal{K}_3^2$ be as in (5). We then have the following scaling laws:*

- (i) First order laminates: *For $F \in \mathcal{K}_3^1$ there are constants $C = C(F) > 0$ and $\epsilon_0 = \epsilon_0(F, \nu) > 0$ such that for any $\epsilon \in (0, \epsilon_0)$*

$$C^{-1} |\nu_1|^{\frac{2}{3}} \epsilon^{\frac{2}{3}} \leq \inf_{\chi \in BV_\nu(\Omega; \mathcal{K}_3)} \inf_{u \in \mathcal{A}_F} E_\epsilon(u, \chi) \leq C |\nu_1|^{\frac{2}{3}} \epsilon^{\frac{2}{3}}.$$

- (ii) Second order laminates: For $F \in \mathcal{K}_3^2$ there are constants $C = C(F) > 0$ and $\epsilon_0 = \epsilon_0(F, \nu) > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ we have

$$C^{-1} \left(|\nu_1|^{\frac{1}{2}\epsilon^{\frac{1}{2}}} + |\nu_2|^{\frac{2}{3}\epsilon^{\frac{2}{3}}} \right) \leq \inf_{\chi \in BV_\nu(\Omega; \mathcal{K}_3)} \inf_{u \in \mathcal{A}_F} E_\epsilon(u, \chi) \leq C \left(|\nu_1|^{\frac{1}{2}\epsilon^{\frac{1}{2}}} + |\nu_2|^{\frac{2}{3}\epsilon^{\frac{2}{3}}} \right).$$

Remark 1.3. We highlight that the estimate from case (ii) for the second order laminates in fact includes matching bounds for both the non-degenerate case in which $\nu \cdot e_1 \neq 0$ and the degenerate case in which $\nu \cdot e_1 = 0$. Indeed, by a case distinction, for $F \in \mathcal{K}_3^2$, on the one hand, if $\nu \cdot e_1 \neq 0$ and if $\epsilon_0 > 0$ is sufficiently small we have that

$$C^{-1} |\nu_1|^{\frac{1}{2}\epsilon^{\frac{1}{2}}} \leq \inf_{\chi \in BV_\nu(\Omega; \mathcal{K}_3)} \inf_{u \in \mathcal{A}_F} E_\epsilon(u, \chi) \leq C |\nu_1|^{\frac{1}{2}\epsilon^{\frac{1}{2}}}.$$

On the other hand, if $\nu \cdot e_1 = 0$, then

$$C^{-1} |\nu_2|^{\frac{2}{3}\epsilon^{\frac{2}{3}}} \leq \inf_{\chi \in BV_\nu(\Omega; \mathcal{K}_3)} \inf_{u \in \mathcal{A}_F} E_\epsilon(u, \chi) \leq C |\nu_2|^{\frac{2}{3}\epsilon^{\frac{2}{3}}}.$$

1.1.3. Settings involving a higher number of wells. In order to illustrate that the above phenomenon is no coincidence, we show that it persists for a certain class of diagonal wells $\mathcal{K}_N$ having one-dimensional lamination convex hulls, which had also been discussed with a full surface energy regularization in [RT23b]. Indeed, for $N \leq d + 1$ we consider

$$\mathcal{K}_N := \{A_1, A_2, \dots, A_N\} \subset \mathbb{R}_{\text{diag}}^{d \times d},$$

with

$$(6) \quad \begin{aligned} A_1 &= 0, & A_2 &= \text{diag}(1, 0, \dots, 0), \\ A_3 &= \text{diag}\left(\frac{1}{2}, 1, 0, \dots, 0\right), & A_4 &= \text{diag}\left(\frac{1}{2}, \frac{1}{2}, 1, \dots, 0\right), \\ A_j &= \text{diag}\left(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}, 1, 0, \dots, 0\right), \end{aligned}$$

for $j = 5, 6, \dots, N$, where for A_j the entry 1 is at the $(j - 1)$ -th diagonal entry. For $d = 2$ and $N = 3$ the set $\mathcal{K}_3$ is exactly the one defined in (4) above. As in Section 1.1.2, we again obtain a structure such that the lamination convex hull of the set $\mathcal{K}_N$ consists of one-dimensional segments (see Figure 2): For $2 \leq \ell \leq N - 1$

$$\begin{aligned} \mathcal{K}_N^1 &:= \mathcal{K}_N^{(1)} \setminus \mathcal{K}_N = \{\text{diag}(\alpha, 0, 0, \dots, 0) \in \mathbb{R}^{d \times d} : \alpha \in (0, 1)\}, \\ \mathcal{K}_N^2 &:= \mathcal{K}_N^{(2)} \setminus \mathcal{K}_N^{(1)} = \{\text{diag}\left(\frac{1}{2}, \alpha, 0, \dots, 0\right) \in \mathbb{R}^{d \times d} : \alpha \in (0, 1)\}, \\ \mathcal{K}_N^3 &:= \mathcal{K}_N^{(3)} \setminus \mathcal{K}_N^{(2)} = \{\text{diag}\left(\frac{1}{2}, \frac{1}{2}, \alpha, 0, \dots, 0\right) \in \mathbb{R}^{d \times d} : \alpha \in (0, 1)\}, \\ \mathcal{K}_N^\ell &:= \mathcal{K}_N^{(\ell)} \setminus \mathcal{K}_N^{(\ell-1)} = \{\text{diag}\left(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}, \alpha, 0, \dots, 0\right) \in \mathbb{R}^{d \times d} : \alpha \in (0, 1)\}, \end{aligned}$$

where $\mathcal{K}_N^\ell$ then has $(\ell - 1)$ entries $1/2$, cf. Section 2.1 for the definition of $\mathcal{K}_N^{(\ell)}$. Also in this setting, we show that in terms of lower bounds only the direction of the “inner-most” lamination is relevant for the scaling of the singularly perturbed model (2), i.e. to obtain the “classical” isotropic scaling only $\nu \cdot e_1 \neq 0$ is necessary.

Theorem 2. Let $d \geq 2$, $N \leq d + 1$, $\Omega = (0, 1)^d$, $\nu \in \mathbb{S}^{d-1}$, $\mathcal{K}_N$ be given by (6), and $\ell \in \{1, 2, \dots, N - 1\}$. Consider the energy $E_\epsilon(u, \chi)$ given in (2) with $\mathcal{N} = \{\nu\}$, $u \in \mathcal{A}_F$ for some $F \in \mathcal{K}_N^\ell$, cf. (3), and $\chi \in BV_\nu(\Omega; \mathcal{K}_N)$. Then, there are $C = C(d, F, \ell) > 0$ and

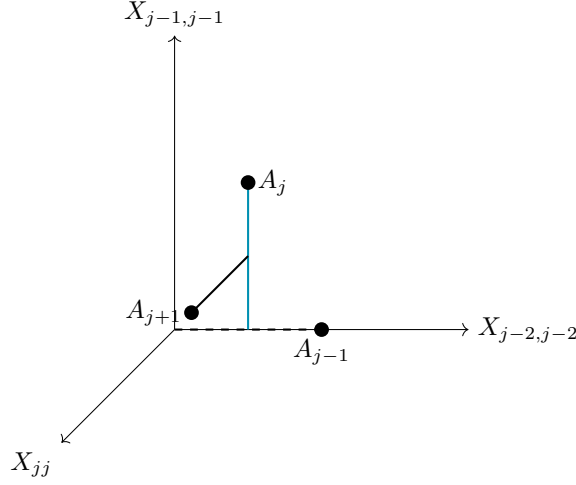


FIGURE 2. Illustration of the relation between $A_{j-1}, A_j, A_{j+1} \in \mathcal{K}_N$ for $3 \leq j \leq N-1$. The lines show rank one connections, where the dashed line is connected to the structure spanned by the previous wells. The set $\mathcal{K}_N^{j-1}$ is highlighted in blue.

$\epsilon_0 = \epsilon_0(d, F, \ell, \nu) > 0$ such that for any $\epsilon \in (0, \epsilon_0)$

$$\inf_{\chi \in BV_\nu(\Omega; \mathcal{K}_N)} \inf_{u \in \mathcal{A}_F} E_\epsilon(u, \chi) \geq C \sum_{j=0}^{\ell-1} |\nu_{j+1}|^{\frac{2}{\ell-j+2}} \epsilon^{\frac{2}{\ell-j+2}}.$$

Remark 1.4. As above, the bound in Theorem 2 includes various individual estimates which arise depending on the dominating degree of degeneracy of the data with respect to the direction $\nu \in \mathbb{S}^{d-1}$. Indeed, by considering different cases for ν , we can also state the following lower scaling estimates:

- (i) For $F \in \mathcal{K}_N^\ell$ and $\nu \cdot e_1 \neq 0$ there are $C = C(d, F, \ell) > 0$ and $\epsilon_0 = \epsilon_0(d, F, \ell, \nu) > 0$ such that for any $\epsilon \in (0, \epsilon_0)$

$$\inf_{\chi \in BV_\nu(\Omega; \mathcal{K}_N)} \inf_{u \in \mathcal{A}_F} E_\epsilon(u, \chi) \geq C |\nu_1|^{\frac{2}{\ell+2}} \epsilon^{\frac{2}{\ell+2}}.$$

- (ii) For $F \in \mathcal{K}_N^\ell$, $0 < k < \ell$ and $\nu \cdot e_1 = \nu \cdot e_2 = \dots = \nu \cdot e_k = 0$ and $\nu \cdot e_{k+1} \neq 0$ there are $C = C(d, F, \ell) > 0$ and $\epsilon_0 = \epsilon_0(d, F, \ell, \nu) > 0$ such that for any $\epsilon \in (0, \epsilon_0)$

$$\inf_{u \in \mathcal{A}_F} \inf_{\chi \in BV_\nu(\Omega; \mathcal{K}_N)} E_\epsilon(u, \chi) \geq C |\nu_{k+1}|^{\frac{2}{\ell-k+2}} \epsilon^{\frac{2}{\ell-k+2}}.$$

- (iii) For $F \in \mathcal{K}_N^\ell$ and $\nu \cdot e_1 = \nu \cdot e_2 = \dots = \nu \cdot e_\ell = 0$ we have $\inf_{\chi \in BV_\nu(\Omega; \mathcal{K}_N)} \inf_{u \in \mathcal{A}_F} E_\epsilon(u, \chi) = 0$.

We expect that this behaviour is sharp in the sense that there are matching upper bound constructions. As, however, already the upper bounds in [RT23b] were rather involved, we do not discuss these here.

1.1.4. Main ideas for the sharp interface anisotropic surface energies. Let us briefly comment on the ideas for the derivation of the above scaling laws with anisotropic surface energy contributions. As in [RT23b] these results rely on a Fourier space perspective on the elastic and surface energies

[Koh91, CO09, CO12, RT23a, RT22]. In all these results, the role of the surface energy is to control high frequencies. While in the previous works this was done in an isotropic way with a frequency cut-off in all directions, we here leverage on the fact that the (diagonal) components of the phase indicator are already controlled by the elastic energy in most directions. In fact, in all the above models, the elastic energy provides strong control outside of certain cones along one-dimensional axes. Hence, only for these directions high frequency control becomes necessary. As a second ingredient, we exploit that the components of the phase indicator in the respective cones are not independent but are all functions of the inner-most one. This allows us to reduce the high frequency control even further and to require a singular perturbation regularization only in a single direction which is given by the direction of the inner-most component.

1.2. Diffuse L^q -based surface energies. Next, we turn to L^q -based diffuse surface energies with microstructures governed by energy functionals of the following form:

$$(7) \quad E_{\epsilon,q}(u, \chi) := \int_{\Omega} |\nabla u - \chi|^2 dx + \epsilon^q \sum_{\nu \in \mathcal{N}} \int_{\Omega} |\partial_{\nu}(\nabla u)|^q dx, \quad q \geq 1,$$

with $\mathcal{N} \subset \mathbb{S}^{d-1}$. As above, we will be particularly interested in anisotropic variants of this energy with $\#\mathcal{N} \leq d$ a finite set of linearly independent directions. Here, due to the regularizing property of the surface energy, a novel phenomenon arises compared to the sharp structures which renders their analysis more challenging: due to the diffuse regularization, a second length scale emerges, which regularizes the “zig-zag” structures. This is already observed in one-dimensional models [Mül93], see also [AM01]. In contrast to the sharp setting from above, we hence do not discuss these energies in Fourier space (although we believe that such a strategy should be possible) but deduce direct lower bounds in terms of the sharp energies.

Moreover, as the same arguments apply if the gradient is replaced by the symmetrized gradient $\nabla^{\text{sym}} u = 1/2(\nabla u + \nabla u^T)$ we prove the results in a more general framework in order to be applicable also for this case. For this let $p, q \in [1, \infty)$, $F \in \mathbb{R}^n$, $\nu \in \mathbb{S}^{d-1}$, $r \in \mathbb{S}^{n-1}$, and $L(D)$ a differential operator. We consider functions

$$U \in \{U \in L^p_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^n) : U = F \text{ outside } \Omega, \partial_{\nu}(U \cdot r) \in L^q(\mathbb{R}^d; \mathbb{R}), L(D)U = 0 \text{ in } \mathbb{R}^d\},$$

where the equation $L(D)U = 0$ holds in the distributional sense in $\mathbb{R}^d$. For $L(D) = \text{curl}$, this translates directly to $U = \nabla u$ for some $u \in W^{1,p}(\Omega; \mathbb{R}^d)$ such that $u(x) = Fx$ on $\partial\Omega$.

Since this is not the main focus point of our article, in the following we will consider $L(D) \in \{\text{curl}, \text{curl curl}\}$ but the same result holds for a more general class of operators, as for instance those considered in [RRT24].

Theorem 3. *Let $d \geq 2$, $n \geq 1$, let $p, q \in [1, \infty)$, $\nu \in \mathbb{S}^{d-1}$, $r \in \mathbb{S}^{n-1}$, and $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and let $\mathcal{K} = \{A_1, \dots, A_N\} \subset \mathbb{R}^n$. For any $U \in L^p(\mathbb{R}^d; \mathbb{R}^n)$ with $U = 0$ outside Ω and $\partial_{\nu}(U \cdot r) \in L^q(\mathbb{R}^d; \mathbb{R})$, $\chi \in L^{\infty}(\Omega; \mathcal{K})$, there exists $\tilde{\chi} \in L^{\infty}(\Omega; \mathcal{K})$ with $\tilde{\chi} \cdot r \in BV_{\nu}(\Omega; \mathcal{K})$ and a constant $C = C(\mathcal{K}, p, q) > 0$ such that for any $\epsilon > 0$*

$$\int_{\Omega} |U - \chi|^p + \epsilon^q |\partial_{\nu}(U \cdot r)|^q dx \geq C \epsilon \|D_{\nu}(\tilde{\chi} \cdot r)\|_{TV(\Omega)} \quad \text{and} \quad \int_{\Omega} |U - \chi|^p dx \geq C \int_{\Omega} |U - \tilde{\chi}|^p dx.$$

In particular,

$$\int_{\Omega} |U - \chi|^p + \epsilon^q |\partial_{\nu}(U \cdot r)|^q dx \geq \frac{C}{2} \left(\int_{\Omega} |U - \tilde{\chi}|^p dx + \epsilon \|D_{\nu}(\tilde{\chi} \cdot r)\|_{TV(\Omega)} \right).$$

This relies on Modica-Mortola type arguments which have been discussed in various contexts in the literature [MM77, KK11, Zwi14, CC14], in particular, we follow the argument from [KK11]. Here the surface energy is even more degenerate than in the previous section in the sense that we only consider a single component of U or $\tilde{\chi}$. We further elaborate on this additional degeneracy

in Remark 3.4. We emphasize that the same estimates remain valid for the full data in which $U \cdot r$ and $\tilde{\chi} \cdot r$ are replaced by U and $\tilde{\chi}$, respectively.

Together with the associated upper bound constructions, as a corollary, in our setting involving higher order laminates, we directly obtain that the scaling behaviour of the microstructures from the previous subsection is unchanged. As an example, we formulate this for the Lorent three-well problem from above where we consider $U = \nabla u$:

Corollary 1.5. *Let $q \in [1, \infty)$, $\mathcal{K}_3$ be given by (4), $\Omega = (0, 1)^2$, $\nu \in \mathbb{S}^1$, and consider for $\epsilon > 0$, $F \in \mathcal{K}_3^{lc} \setminus \mathcal{K}_3$, $u \in \mathcal{A}_F$, cf. (3), such that $\partial_\nu \nabla u \in L^q(\Omega; \mathbb{R}^{2 \times 2})$, and $\chi \in L^\infty(\Omega; \mathcal{K}_3)$ the energy $E_{\epsilon, q}$ given by (7) with $\mathcal{N} = \{\nu\}$. The following scaling laws hold:*

- (i) First order laminates: *For $F \in \mathcal{K}_3^1$ there is a constant $C = C(F, q) > 0$ and $\epsilon_0 = \epsilon_0(F, q, \nu) > 0$ such that for any $\epsilon \in (0, \epsilon_0)$*

$$C^{-1} |\nu_1|^{\frac{2}{3}} \epsilon^{\frac{2}{3}} \leq \inf_{\chi \in L^2(\Omega; \mathcal{K}_3)} \inf_{\substack{u \in \mathcal{A}_F \\ \partial_\nu \nabla u \in L^q(\Omega; \mathbb{R}^{2 \times 2})}} E_{\epsilon, q}(u, \chi) \leq C |\nu_1|^{\frac{2}{3}} \epsilon^{\frac{2}{3}}.$$

- (ii) Second order laminates: *For $F \in \mathcal{K}_3^2$ there is a constant $C = C(F, q) > 0$ and $\epsilon_0 = \epsilon_0(F, q, \nu) > 0$ such that for any $\epsilon \in (0, \epsilon_0)$*

$$C^{-1} \left(|\nu_1|^{\frac{1}{2}} \epsilon^{\frac{1}{2}} + |\nu_2|^{\frac{2}{3}} \epsilon^{\frac{2}{3}} \right) \leq \inf_{\chi \in L^2(\Omega; \mathcal{K}_3)} \inf_{\substack{u \in \mathcal{A}_F \\ \partial_\nu \nabla u \in L^q(\Omega; \mathbb{R}^{2 \times 2})}} E_{\epsilon, q}(u, \chi) \leq C \left(|\nu_1|^{\frac{1}{2}} \epsilon^{\frac{1}{2}} + |\nu_2|^{\frac{2}{3}} \epsilon^{\frac{2}{3}} \right).$$

1.3. Fractional L^2 -based surface energies. Building on the Fourier analysis from the sharp interface setting, we also consider fractional models for surface energies. To this end, for $\mathcal{K} \subset \mathbb{R}^{d \times d}$ and $s \in (0, \frac{1}{2})$, we define the fractional (directional) Sobolev space on the torus using the Fourier transform, cf. (19),

$$(8) \quad H_\nu^s(\mathbb{T}^d; \mathbb{R}^{d \times d}) = \{u \in L^2(\mathbb{T}^d; \mathbb{R}^{d \times d}) : \sum_{k \in \mathbb{Z}^d} (1 + 4\pi^2 |k \cdot \nu|^2)^s |\hat{u}(k)|^2 < \infty\}$$

and set for $\chi \in H_\nu^s(\mathbb{T}^d; \mathcal{K})$

$$E_{surf}^s(\chi) := \left(\sum_{k \in \mathbb{Z}^d \setminus \{0\}} |k \cdot \nu|^{2s} |\hat{\chi}(k)|^2 \right)^{\frac{1}{2s}}.$$

For $s < 1/2$ it then holds $BV_\nu(\mathbb{T}^d; \mathcal{K}) \subset H_\nu^s(\mathbb{T}^d; \mathcal{K})$, see the proof of Theorem 4. Moreover, in the following by extending $\chi \in L^2((0, 1)^d; \mathcal{K})$ one-periodically, we view it as a function on the torus $\chi \in L^2(\mathbb{T}^d; \mathcal{K})$ and define the space $H_\nu^s((0, 1)^d; \mathbb{R}^{d \times d})$ analogously. The power of $1/2s$ is chosen such that the surface energy admits the correct behaviour in the length of interfaces. In addition, by the choice of $\nu \in \mathbb{S}^{d-1}$ we also focus in this nonlocal context on anisotropic settings. The full energy is then defined to be

$$(9) \quad E_{\epsilon, s}(u, \chi) := \int_{\Omega} |\nabla u - \chi|^2 dx + \epsilon E_{surf}^s(\chi).$$

We prove that also in this setting, the scaling laws from Section 1.1 remain valid.

Theorem 4. *Let $d \geq 2$, $N \leq d + 1$, $s \in (0, \frac{1}{2})$, $\Omega = (0, 1)^d$, $\nu \in \mathbb{S}^{d-1}$, and let $\mathcal{K}_N$ be defined in (6), and $\ell \in \{1, \dots, N - 1\}$. For $F \in \mathcal{K}_N^\ell$, cf. Section 2.1, consider the energy $E_{\epsilon, s}(u, \chi)$ defined in (9) for $u \in \mathcal{A}_F$, cf. (3), and $\chi \in H_\nu^s(\Omega; \mathcal{K}_N)$. Then there are $C = C(d, F, s, \ell) > 0$ and*

$\epsilon_0 = \epsilon_0(d, F, s, \ell, \nu) > 0$ such that for any $\epsilon \in (0, \epsilon_0)$

$$\inf_{\chi \in H_\nu^s(\Omega; \mathcal{K}_N)} \inf_{u \in \mathcal{A}_F} E_{\epsilon, s}(u, \chi) \geq C \left(\sum_{j=0}^{\ell-1} |\nu_{j+1}|^{\frac{2}{\ell-j+2}} \epsilon^{\frac{2}{\ell-j+2}} \right).$$

Moreover, for $d = 2$ and $N = 3$ a matching upper bound holds

$$\inf_{\chi \in H_\nu^s(\Omega; \mathcal{K}_3)} \inf_{u \in \mathcal{A}_F} E_{\epsilon, s}(u, \chi) \leq C^{-1} \left(\sum_{j=0}^{\ell-1} |\nu_{j+1}|^{\frac{2}{\ell-j+2}} \epsilon^{\frac{2}{\ell-j+2}} \right).$$

As above, this shows the robustness of the scaling bounds with respect to possible choices of the (anisotropic) surface energies. While we only discuss the upper bound in the case of the Lorent three-well problem, our interpolation strategy of Section 3.4 shows that the fractional energy bounds are sharp whenever they are sharp in the sharp interface settings.

1.4. Discrete models. As a last prototypical setting we turn to discrete models as regularized versions of continuum models. It is well-known [Dol03, Lor01, Lor09, CM99, ALP17] in the context of shape-memory alloys and, more generally, in phase transition problems [BC07] that under suitable conditions, discretizations lead to surface-type energies in a first order expansion. In fact, [Lor09] proved the equivalence of scaling laws for continuous singular perturbation models with isotropic surface energy regularization and a class of associated discrete models. The article [Lor09] even considered settings in the presence of gauges, i.e., with the full $SO(2)$ symmetry. These results, hence, prove that also for shape-memory type models, one expects that – provided that the lattice structures and the rank-one geometry of the wells match – the scaling behaviour in the discretization parameter corresponds to that of the associated singular perturbation model.

Similarly as above, in this section, we seek to extend these observations to *anisotropic* scenarios. As above, we search for minimal conditions on the lattice with respect to the geometry of the wells to guarantee this behaviour. For a fixed class of lattice structures, we here show that discrete energies with an “anisotropic lattice structure” can be compared to singular perturbation models with anisotropic surface regularization contributions.

1.4.1. The discrete set-up. In order to outline our result, let us introduce some notation. We consider a specific choice of lattice structure. Here the triangulation is given by using the following “upper” and “lower” triangles (see Figure 3(A))

$$(10) \quad T_h := \{x \in [0, h]^2 : x_2 < h - x_1\}, \quad T'_h := \{x \in [0, h]^2 : x_2 \geq h - x_1\},$$

and, hence,

$$\mathcal{T}_h := \{T_h + z : z \in h\mathbb{Z}^2\} \cup \{T'_h + z : z \in h\mathbb{Z}^2\}.$$

We also consider rotated variants of this triangulation (see Figure 3(B)), that is for $R \in SO(2)$, consider

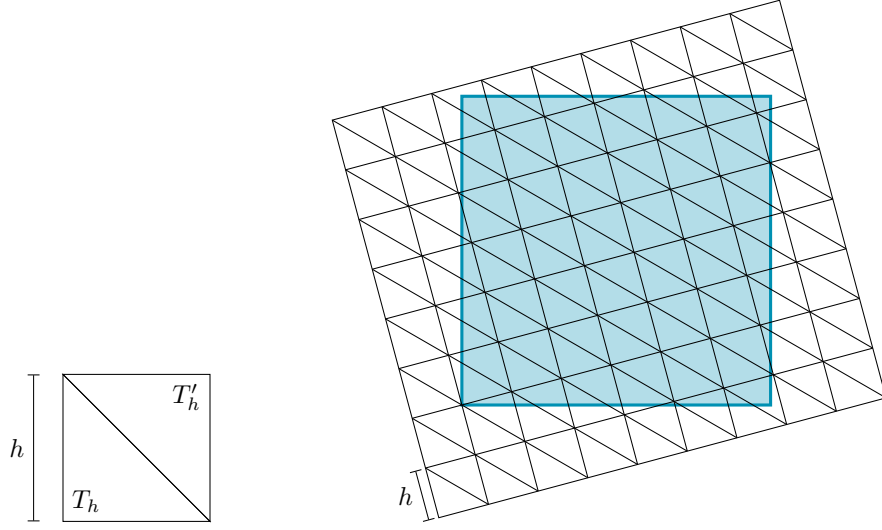
$$(11) \quad \mathcal{T}_h^R := R\mathcal{T}_h = \{R(T_h + z) : z \in h\mathbb{Z}^2\} \cup \{R(T'_h + z) : z \in h\mathbb{Z}^2\}.$$

Given the lattice structure, we associate an energy to it: For $\Omega \subset \mathbb{R}^2$ bounded and polygonal, $F \notin \mathcal{K}$, and $p \in [1, \infty)$, we define the sets of admissible functions

$$(12) \quad \begin{aligned} \mathcal{A}_{h, F}^{p, R} &:= \{u \in W_{\text{loc}}^{1, p}(\mathbb{R}^2; \mathbb{R}^2) : u \text{ is affine on each triangle } \tau \in \mathcal{T}_h^R, u(x) = Fx \text{ outside } \Omega\}, \\ \mathcal{C}_h^R &:= \{\chi \in L^\infty(\mathbb{R}^2; \mathcal{K}) : \chi \text{ is constant on each triangle } \tau \in \mathcal{T}_h^R\}. \end{aligned}$$

We consider the energy given by

$$(13) \quad E_{\epsilon, h}^p(u, \chi) := \int_{\Omega} |\nabla u - \chi|^p dx, \quad \text{for } u \in \mathcal{A}_{h, F}^{p, R}, \chi \in \mathcal{C}_h^R.$$



(A) Illustration of the upper and lower triangles as defined in (10). (B) Triangulation $\mathcal{T}_h^R$ of $(0,1)^2$ (highlighted in blue) with $h = 1/6$ and R a rotation by 15° .

FIGURE 3. Illustration of the definition of triangulations used in this section.

We highlight that by the choice of the admissible deformations this corresponds to a discrete energy. Upper bounds for multi-well energies of this type have been systematically investigated in [Chi99] for a rich class of well structures. The two-well problem had been studied with matching upper and lower bounds in [CM99]. We also refer to [CCK95, BP04] for further upper bound constructions and matching lower bounds on the two-well problem.

For ease of notation, in this section we consider the row-wise cross product of a vector and a matrix defined by

$$(v \times M)_j := v_1 M_{j2} - v_2 M_{j1}, \quad v \in \mathbb{R}^2, M \in \mathbb{R}^{2 \times 2}, j = 1, 2.$$

In particular, using this notation, there is a rank-one connection between two matrices $A, B \in \mathbb{R}^{2 \times 2}$, i.e., $A - B = a \otimes n$ for $a, n \in \mathbb{R}^2 \setminus \{0\}$, if and only if $A \neq B$ and

$$n \times (A - B) = 0.$$

Given this set-up, our main result is given by the following comparison estimate.

Theorem 5. *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded polygonal domain. Let $R \in SO(2)$. Let $\mathcal{K} = \{A_1, A_2, \dots, A_N\} \subset \mathbb{R}^{2 \times 2}$ and $F \notin \mathcal{K}$. Let $p \in [1, \infty)$ and consider $\mathcal{T}_h^R$ defined in (11), and the discrete energy $E_{el,h}^p(u, \chi)$ for $u \in \mathcal{A}_{h,F}^{p,R}$ and $\chi \in \mathcal{C}_h^R$, cf. (12), defined in (13). Then the following results hold:*

- (i) Isotropic setting: Assume that for any $A_j, A_k \in \mathcal{K}$ with $j \neq k$ it holds
- $$(14) \quad (Re_1) \times (A_j - A_k) \neq 0, (Re_2) \times (A_j - A_k) \neq 0, \left(R \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \right) \times (A_j - A_k) \neq 0.$$

Then, there are constants $C = C(\mathcal{K}, R, p, F, \Omega) > 0$ and $h_0 = h_0(\Omega, R, p) > 0$ such that for all $h \in (0, h_0)$

$$E_{el,h}^p(u, \chi) \geq C \left(\int_{\Omega} |\nabla u - \chi|^p dx + h \|D\chi\|_{TV(\Omega)} + h \right).$$

In particular, we have

$$\inf_{u \in \mathcal{A}_{h,F}^{p,R}} \inf_{\chi \in \mathcal{C}_h^R} E_{el,h}^p(u, \chi) \geq C \left(\inf_{\chi \in BV(\Omega; \mathcal{K})} \inf_{\substack{u \in W^{1,p}(\Omega; \mathbb{R}^2) \\ u(x) = Fx \text{ on } \partial\Omega}} \int_{\Omega} |\nabla u - \chi|^p dx + h \|D\chi\|_{TV(\Omega)} + h \right).$$

(ii) Anisotropic setting: Assume that there exists $v \in \left\{ e_1, e_2, \left(\frac{1/\sqrt{2}}{1/\sqrt{2}} \right) \right\}$ such that for any $j \neq k$

$$(15) \quad (Rw) \times (A_j - A_k) \neq 0, \quad \text{for every } w \in \left\{ e_1, e_2, \left(\frac{1/\sqrt{2}}{1/\sqrt{2}} \right) \right\} \setminus \{v\}.$$

Then, there are constants $C = C(\mathcal{K}, R, p, F, \Omega, v) > 0$ and $h_0 = h_0(\Omega, R, p) > 0$ such that for all $h \in (0, h_0)$

$$E_{el,h}^p(u, \chi) \geq C \left(\int_{\Omega} |\nabla u - \chi|^p + h \|D_{R\nu} \chi\|(\Omega) + h \right),$$

with $\nu \in \mathbb{S}^1$ such that $\nu \cdot v = 0$.

We emphasize that the result on the *isotropic* case is not new. It only recovers the setting already analysed in [Lor09] for the general N -well problem in two-dimensions which in [Lor09] is even considered with $SO(2)$ symmetry. In the two-well case, it recovers the result from [CM99]. Our main contribution is the bound in the *anisotropic* setting. Its proof relies on similar ideas as in [Lor09].

1.4.2. *Applications.* We will consider two prototypical examples – the Lorent three-well setting and the Tartar square. In both settings, it is interesting to trace the effects of anisotropy.

We begin by discussing the discrete version of the Lorent three-well problem, introduced in Section 1.1.2. It admits laminates up to order two.

Corollary 1.6. Let $\Omega = (0, 1)^2$, $R \in SO(2)$, $\mathcal{T}_h^R$ be the triangulation defined in (11), $\mathcal{K}_3$ be given in (4), and $F \in \mathcal{K}_3^{lc} \setminus \mathcal{K} = \mathcal{K}_3^1 \cup \mathcal{K}_3^2$, cf. (5). Let $E_{el,h}(u, \chi) := E_{el,h}^2(u, \chi)$ be defined in (13) for $p = 2$ and $u \in \mathcal{A}_{h,F}^{2,R}$, $\chi \in \mathcal{C}_h^R$, cf. (12). Then there are constants $C = C(R, F) > 0$ and $h_0 = h_0(R, F) > 0$ such that one of the following applies.

(i) Isotropic setting. If $R \in SO(2)$ is such that $R^T e_1 \notin \{\pm e_1, \pm e_2, \pm 2^{-1/2}(e_1 + e_2)\}$ we have for any $h \in (0, h_0)$

$$\inf_{u \in \mathcal{A}_{h,F}^{2,R}} \inf_{\chi \in \mathcal{C}_h^R} E_{el,h}(u, \chi) \geq \begin{cases} Ch^{\frac{2}{3}}, & F \in \mathcal{K}_3^1, \\ Ch^{\frac{1}{2}}, & F \in \mathcal{K}_3^2. \end{cases}$$

(ii) Anisotropic setting. If $R^T e_1 \in \{\pm e_1, \pm e_2, \pm 2^{-1/2}(e_1 + e_2)\}$ it holds for any $h \in (0, h_0)$

$$\inf_{u \in \mathcal{A}_{h,F}^{2,R}} \inf_{\chi \in \mathcal{C}_h^R} E_{el,h}(u, \chi) \geq \begin{cases} Ch, & F \in \mathcal{K}_3^1, \\ Ch^{\frac{2}{3}}, & F \in \mathcal{K}_3^2. \end{cases}$$

Moreover, the matching upper bounds hold, that is for every $h \in (0, h_0)$

$$\inf_{u \in \mathcal{A}_{h,F}^{2,R}} \inf_{\chi \in \mathcal{C}_h^R} E_{el,h}(u, \chi) \leq \begin{cases} C^{-1} h^{\frac{2}{3}}, & F \in \mathcal{K}_3^1, \\ C^{-1} h^{\frac{1}{2}}, & F \in \mathcal{K}_3^2, \end{cases}$$

in the isotropic setting, and

$$\inf_{u \in \mathcal{A}_{h,F}^{2,R}} \inf_{\chi \in \mathcal{C}_h^R} E_{el,h}(u, \chi) \leq \begin{cases} C^{-1} h, & F \in \mathcal{K}_3^1, \\ C^{-1} h^{\frac{2}{3}}, & F \in \mathcal{K}_3^2, \end{cases}$$

in the anisotropic setting.

Let us comment on these results. In the isotropic setting, we recover the lower scaling bounds from the continuous setting. A discretized version of the upper bounds from the continuous setting shows that these estimates are sharp. The more interesting setting arises in the anisotropic situation. Here, for second order laminate boundary conditions, the lower bounds also match the anisotropic ones in the Lorent three-well problem in Theorem 1. For boundary data consisting of first order laminates, we observe a difference to the continuum – due to the limitation of the size of the lengths scales, a scaling bound of the order h emerges in the discrete setup, while it vanishes completely in the continuum. Again, a discretization of the upper bounds from the continuous setting yields the sharpness of these estimates. We expect that similar behaviour is observed for other well configurations.

As a second example and as an extreme case, we consider the setting of the Tartar square. This is a setting of four diagonal matrices which play a prominent role both in inner-mathematical settings [Tar93, MŠ03, FS08] and materials [CS13, BFJK94, IKR⁺24]. Indeed, while these four diagonal matrices are pairwise incompatible, they still admit microstructure and display a first loss of rigidity in that approximate solutions become flexible. They are given by $T_4 := \{A_1, A_2, A_3, A_4\}$ with

$$(16) \quad A_1 = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}.$$

As is well-known in the case of the Tartar square, we have that $T_4^{lc} = T_4$, but its quasiconvex hull is given by $T_4^{qc} := \{J_1, J_2, J_3, J_4\}^{\text{conv}} \cup \bigcup_{j=1}^4 [A_j, J_j]$ (see [Rin18, Theorem 9.4]), where for a set $M \subset \mathbb{R}^{d \times d}$, the notation M^{conv} denotes the convex hull of M . Here the auxiliary matrices $J_1, \dots, J_4$ are given as

$$(17) \quad J_1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad J_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

With this notation in hand, we prove the following scaling law for the discrete version of the Tartar energy.

Corollary 1.7. *Let $\Omega = (0, 1)^2$, $R \in SO(2)$, and let $T_4 = \{A_1, A_2, A_3, A_4\}$ be the Tartar square given in (16) and let $F \in T_4^{qc} \setminus T_4$. Consider the discrete energy $E_{el,h}(u, \chi) := E_{el,h}^2(u, \chi)$ for $u \in \mathcal{A}_{h,F}^{2,R}$, $\chi \in \mathcal{C}_h^R$, cf. (12), defined in (13). Then, for any $R \in SO(2)$ and any $\eta > 0$ there exist constants $C_\eta > 0$, $C = C(F, R) > 0$ and $h_0 = h_0(F, R) > 0$ such that for all $h \in (0, h_0)$*

$$\inf_{u \in \mathcal{A}_{h,F}^{2,R}} \inf_{\chi \in \mathcal{C}_h^R} E_{el,h}(u, \chi) \geq C \exp(-C_\eta |\log h|^{\frac{1}{2} + \eta}).$$

Let us comment on this result. Contrary to the setting from Corollary 1.6, the lower scaling bound holds for all lattice structures in our class of lattices, independently of the choice of the rotation R . This is due to the fact that the Tartar square is extremely rigid, in the sense that no rank-one connections are present. Hence, independently of the precise lattice structure, the lower bound directly follows from an application of the isotropic setting from Theorem 5 for any rotation $R \in SO(2)$. We remark that the lower bound is essentially sharp. An upper bound of the form $C^{-1} \exp(-C' |\log h|^{\frac{1}{2}})$ (with a constant $C' > 0$) had been deduced in [Chi99]. The loss of $\eta > 0$ in the exponent in Corollary 1.7 is expected to be a technical artifact, which had already been present in the discussion of its continuous counterpart in [RT22].

1.5. Relation to the literature. Let us connect our results to the literature on martensitic phase transformations. In the literature, martensitic phase transformations are considered with various types of surface energies, including sharp [KM94, KM92, CO12, CO09], diffuse [Zwi14],

mixed ones [BMC09] and discrete regularizations [Dol03, Lor01, Lor09, BC07]. In fact, Lorent [Lor09] proved that under additional technical assumptions on the finite element discretization, for a model with frame indifference, the discrete and continuous, diffuse energies are scaling equivalent. We refer to [KK11, CC14, CC15, CDZ17, KO19, RTZ18, RRT23, RT24, GRTZ24, TZ25, AKKR24, IKR⁺24] for further results on scaling laws for shape-memory alloys with various types of surface energy regularizations. Also experimentally, one observes different transitions: In [BVTa86] one observes atomistically sharp interfaces while in [MVTa86] rather diffuse boundaries are observed. Measurements of surface energies are experimentally notoriously difficult and, hence, the precise structure is often not known. Thus, it is particularly important to deduce mathematical results *independently* of the precise form of the surface energies. By discussing the scaling laws from [RT23b, RT22] in the context of various surface regularizations, it is our objective to illustrate and prove that these results are rather robust in the choice of the surface energies and to identify for an interesting class of martensitic phase transformations minimal anisotropic conditions recovering the known scaling laws for isotropic singular perturbation contributions.

1.6. Outline of the article. The remainder of the article is structured as follows. We begin by discussing some preliminary results in Section 2. In particular, we recall the relevant *BV* set-up and the associated high-frequency bounds and the Fourier analysis from [RT22, RT23b]. In Section 3 we turn to the setting of anisotropic, sharp surface energies. We present both upper and lower bound results in the outlined highly anisotropic settings and thus provide the proofs of Theorems 1 and 2. In Section 4 we turn to the discussion of diffuse energies. Here we provide comparison results such as Theorem 3 which relate diffuse surface energies with the sharp energies and also present the lower bounds in the setting of anisotropic fractional energies. Finally, we consider discrete (anisotropic) situations in Section 5 and present the proof of Theorem 5 and the applications from Corollaries 1.6 and 1.7.

2. PRELIMINARIES

In this section, we collect various auxiliary results which we will use in the following sections.

2.1. On the lamination convex hull. For the convenience of the reader we recall the lamination convex hull of a set $\mathcal{K} \subset \mathbb{R}^{d \times d}$.

Definition 2.1. Let $\mathcal{K} \subset \mathbb{R}^{d \times d}$. The lamination convex hull $\mathcal{K}^{lc}$ of $\mathcal{K}$ is given by

$$\mathcal{K}^{lc} := \bigcup_{j=0}^{\infty} \mathcal{K}^{(j)},$$

where

$$\mathcal{K}^{(0)} := \mathcal{K}, \quad \mathcal{K}^{(j)} := \{\lambda A + (1 - \lambda)B : A, B \in \mathcal{K}^{(j-1)}, \lambda \in [0, 1], \text{rank}(A - B) = 1\}.$$

For $j \geq 1$, we refer to the elements of $\mathcal{K}^j := \mathcal{K}^{(j)} \setminus \mathcal{K}^{(j-1)}$ as laminates of order j .

In what follows below, we will prove that in our geometric settings with anisotropic surface energies, the interaction between the directional dependences in the anisotropy (in the surface energies) and the lamination orders of the boundary data will determine the scaling behaviour of the investigated microstructures.

2.2. Directional derivative in a BV sense. Given $\Omega \subseteq \mathbb{R}^d$ open, for a function $f \in BV(\Omega; \mathbb{R}^n)$ we denote the total variation norm of the measure Df by $\|Df\|_{TV(\Omega)}$. Building on the definition of BV functions, we consider functions for which only a single directional derivative exists as a measure. Let $f \in L^1(\Omega; \mathbb{R}^n)$ and $\nu \in \mathbb{S}^{d-1}$. We write $f \in BV_\nu(\Omega; \mathbb{R}^n)$ if the distributional derivative of f in direction ν is a $\mathbb{R}^n$ -valued Radon-measure, denoted by $D_\nu f$, i.e. for every $\phi \in C_c^\infty(\Omega; \mathbb{R}^n)$ it holds that

$$\int_{\Omega} f \cdot \partial_\nu \phi dx = - \int_{\Omega} \phi \cdot d(D_\nu f).$$

The total variation norm of this measure, again denoted by $\|D_\nu f\|_{TV(\Omega)}$, is given as

$$\|D_\nu f\|_{TV(\Omega)} = \sup \left\{ \int_{\Omega} f \cdot \partial_\nu \phi dx : \phi \in C_c^1(\Omega; \mathbb{R}^n), \|\phi\|_\infty \leq 1 \right\}.$$

As shown in [AFP00, Thm 3.103], for $f \in BV_\nu(\Omega; \mathbb{R}^n)$ it holds

$$(18) \quad \|D_\nu f\|_{TV(\Omega)} = \int_{\Omega_\nu} \|Df_y^\nu\|_{TV(\Omega_y^\nu)} dy,$$

where we introduce the notation

$$\Omega_\nu := \Pi_{\nu^\perp} \Omega \subset \mathbb{R}^d$$

as the orthogonal projection of Ω onto $\nu^\perp$, and for $y \in \Omega_\nu$, we set

$$\Omega_y^\nu := \{t \in \mathbb{R} : y + t\nu \in \Omega\} \subset \mathbb{R}.$$

Hence, the total variation norm of $D_\nu f$ is given by integration of the one-dimensional total variation norms of the distributional derivatives of

$$f_y^\nu : \Omega_y^\nu \rightarrow \mathbb{R}^n, \quad f_y^\nu(t) := f(y + t\nu).$$

In particular, for $f \in BV_\nu(\Omega; \mathbb{R}^n)$ for almost every $y \in \Omega_\nu$ we have $f_y^\nu \in BV(\Omega_y^\nu; \mathbb{R}^n)$.

As a remark, let $E \subset \Omega$ be a set of finite perimeter, then it holds

$$\|D_\nu \chi_E\|_{TV(\Omega)} = \int_{\partial E \cap \Omega} |n_{\partial E} \cdot \nu| d\mathcal{H}^{d-1},$$

where $n_{\partial E}$ is the outer unit normal of E . We will often exploit this in the sequel.

2.3. Directional high frequency control. As a preparation for our discussion of sharp energies, we deduce lower bounds for the surface energies. Here we follow similar ideas as in [CO12, CO09, RT22, RRT23] but with only directional control for the surface energies. For the convenience of the reader, we hence recall the arguments. These arguments use Fourier methods, thus let us recall the definition of the Fourier transform of one-periodic functions $u \in L^2(\mathbb{T}^d; \mathbb{R}^n)$ as $\mathcal{F}[u] \in \ell^2(\mathbb{Z}^d; \mathbb{C}^n)$

$$(19) \quad \mathcal{F}[u](k) := \hat{u}(k) := \int_{\mathbb{T}^d} u(x) e^{-2\pi i k \cdot x} dx, \quad k \in \mathbb{Z}^d.$$

In the following, we often consider the one-periodic extension of a function $u \in L^2((0, 1)^d; \mathbb{R}^n)$ without changing the notation and hence also define the Fourier transform of such functions as above.

In order to motivate our arguments in the following sections, we recall part of the strategy from the isotropic setting from [CO12, CO09, RT22, RRT23] in two dimensions. As a central

ingredient for the lower bound, we use the surface energy to control high frequencies. More precisely, viewing $\chi \in BV((0,1)^2; \mathcal{K})$ as a periodic function, we have

$$\sum_{|k| \geq \lambda} |\hat{\chi}(k)|^2 \leq C\lambda^{-1} (\|D\chi\|_{TV((0,1)^2)} + \text{Per}((0,1)^2)).$$

As a complementary step, the elastic energy is used to control the frequencies of the associated “multiplier” in the form of a coercivity bound in the complement of certain non-elliptic regions. The analysis of these multipliers shows that the only regions without coercivity are given by the complement of cones (cf. the discussion below). As a consequence, it seems natural to conjecture that it suffices to have only control of high frequencies in direction of the axes of these cones instead of requiring control in all directions.

Our first step towards the anisotropic setting, thus, is showing an analogous high frequency control as above, but using only a single direction.

Lemma 2.2. *Let $d \geq 2$, $n \geq 1$, let $\Omega = (0,1)^d$, $\nu \in \mathbb{S}^{d-1}$, and $f \in BV_\nu(\Omega; \mathbb{R}^n) \cap L^\infty(\Omega; \mathbb{R}^n)$, then there is a constant $C = C(d) > 0$ independent of ν and f , such that for any $\lambda > 0$ it holds*

$$\sum_{k \in \mathbb{Z}^d: |k \cdot \nu| \geq \lambda} |\hat{f}(k)|^2 \leq C\lambda^{-1} \|f\|_{L^\infty} (\|D_\nu f\|_{TV(\Omega)} + \|f\|_{L^\infty} \text{Per}(\Omega)),$$

where we extend f one-periodically.

Proof. We preliminarily notice that, in the case $\lambda \leq 1$, the statement is straightforward. Indeed,

$$\sum_{k \in \mathbb{Z}^d: |k \cdot \nu| \geq \lambda} |\hat{f}(k)|^2 \leq \|f\|_{L^2(\Omega)}^2 \leq \|f\|_{L^\infty(\Omega)}^2 \leq \lambda^{-1} \|f\|_{L^\infty(\Omega)}^2,$$

which yields the statement by taking $C \geq \text{Per}(\Omega)^{-1}$. We are then left to prove the result in the case $\lambda > 1$.

We use the notation introduced in Section 2.2, i.e. we set $\Omega_\nu = \Pi_{\nu^\perp} \Omega$ the orthogonal projection of Ω onto $\nu^\perp$, and for $y \in \Omega_\nu$ we set $\Omega_y^\nu = \{t \in \mathbb{R} : y + t\nu \in \Omega\}$. For given $f \in BV_\nu(\Omega; \mathbb{R}^n)$, we consider the one-periodic extension of f . Thus, let us first consider $f \in BV_\nu(\mathbb{T}^d; \mathbb{R}^n)$.

Step 1: Estimate on a difference quotient. We claim that, for every $|h| < 1$

$$(20) \quad \int_{\mathbb{T}^d} \frac{|f(x + h\nu) - f(x)|}{|h|} dx \leq \|D_\nu f\|_{TV(\mathbb{T}^d)}.$$

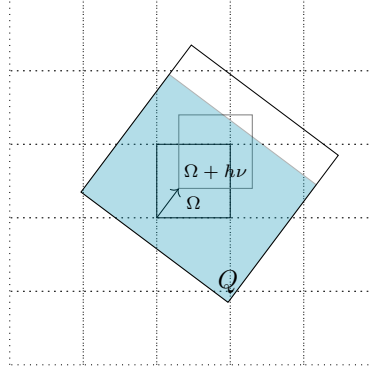
To show (20), we begin by noting that for a given connected domain of integration $Q \subset \mathbb{R}^d$ and for $f_y^\nu \in BV(Q_y^\nu; \mathbb{R}^n)$, by the Fundamental Theorem (and approximation), we obtain that

$$\int_{Q_y^\nu \cap (Q_y^\nu - h)} |f_y^\nu(t + h) - f_y^\nu(t)| dt \leq |h| \|Df_y^\nu\|_{TV(Q_y^\nu)}.$$

We choose $Q \subset \mathbb{R}^d$ as a rotated cube with one face in direction ν , such that $\Omega \subset Q \cap (Q - h\nu)$, see Figure 4. Thus, by (18) and the periodicity of f

$$\begin{aligned} \int_{\mathbb{T}^d} |f(x + h\nu) - f(x)| dx &\leq \int_{Q \cap (Q - h\nu)} |f(x + h\nu) - f(x)| dx \\ &= \int_{Q_\nu} \int_{Q_y^\nu \cap (Q_y^\nu - h)} |f_y^\nu(t + h) - f_y^\nu(t)| dt dy \\ &\leq |h| \int_{Q_\nu} \|Df_y^\nu\|_{TV(Q_y^\nu)} dy = |h| \|D_\nu f\|_{TV(Q)} \leq |h| C(Q, d) \|D_\nu f\|_{TV(\mathbb{T}^d)}, \end{aligned}$$

where the constant $C(Q, d) \geq \#\{z \in \mathbb{Z}^d : (\bar{\Omega} + z) \cap Q \neq \emptyset\} > 0$ is chosen large enough and independent of ν such that it is larger than the number of copies of $\bar{\Omega}$ required to cover Q .

FIGURE 4. Illustration of the choice Q , with $Q \cap (Q - h\nu)$ highlighted in blue.

Step 2: Fourier estimate. Let $f \in BV_\nu(\Omega; \mathbb{R}^n) \cap L^\infty(\Omega; \mathbb{R}^n)$ be as in the statement. Then by step 1, the one-periodic extension, still denoted by f , satisfies

$$\int_{\mathbb{T}^d} \frac{|f(x + h\nu) - f(x)|^2}{|h|} dx \leq 4\|f\|_{L^\infty} \int_{\mathbb{T}^d} \frac{|f(x + h\nu) - f(x)|}{|h|} dx \leq 4C\|f\|_{L^\infty} \|D_\nu f\|_{TV(\mathbb{T}^d)}.$$

With this estimate in hand, we can reformulate the difference quotient in Fourier space which turns the estimate into

$$4C\|f\|_{L^\infty} \|D_\nu f\|_{TV(\mathbb{T}^d)} \geq |h|^{-1} \sum_{k \in \mathbb{Z}^d} |(e^{2\pi i h k \cdot \nu} - 1) \hat{f}(k)|^2.$$

Integrating this inequality for $L < 1$ over $h \in (-L, L)$, we get

$$\begin{aligned} 8LC\|f\|_{L^\infty} \|D_\nu f\|_{TV(\mathbb{T}^d)} &\geq L^{-1} \int_{-L}^L \sum_{k \in \mathbb{Z}^d} |e^{2\pi i h k \cdot \nu} - 1|^2 |\hat{f}(k)|^2 dh \\ &= L^{-1} \sum_{k \in \mathbb{Z}^d} |\hat{f}(k)|^2 \int_{-L}^L |e^{2\pi i h k \cdot \nu} - 1|^2 dh. \end{aligned}$$

Calculating the integral on the right-hand side yields for $k \cdot \nu \neq 0$

$$\begin{aligned} \int_{-L}^L |e^{2\pi i h k \cdot \nu} - 1|^2 dh &= \int_{-L}^L (2 - 2\cos(2\pi h k \cdot \nu)) dh = \left[2h - \frac{2\sin(2\pi h k \cdot \nu)}{2\pi k \cdot \nu} \right]_{h=-L}^L \\ &= 4L - \frac{4\sin(2\pi L k \cdot \nu)}{2\pi k \cdot \nu} \geq 4L - \frac{2}{\pi |k \cdot \nu|}. \end{aligned}$$

Hence, after restricting the series in k to $|k \cdot \nu| \geq L^{-1}$

$$\begin{aligned} 8LC\|f\|_{L^\infty} \|D_\nu f\|_{TV(\mathbb{T}^d)} &\geq L^{-1} \sum_{|k \cdot \nu| \geq L^{-1}} |\hat{f}(k)|^2 \left(4L - \frac{2}{\pi |k \cdot \nu|} \right) \\ &\geq L^{-1} \sum_{|k \cdot \nu| \geq L^{-1}} |\hat{f}(k)|^2 \left(4L - \frac{2}{\pi L^{-1}} \right) \\ &= \frac{4\pi - 2}{\pi} \sum_{|k \cdot \nu| \geq L^{-1}} |\hat{f}(k)|^2. \end{aligned}$$

Setting $L = \lambda^{-1} < 1$, and observing that $\|D_\nu f\|_{TV(\mathbb{T}^d)} \leq \|D_\nu f\|_{TV(\Omega)} + \|f\|_{L^\infty} \text{Per}(\Omega)$, shows the statement. $\square$

2.4. Fourier localization. In this section we recall some of the relevant tools to derive our Fourier-based lower bounds in Section 1.1. They build on the strategy from [RT23b], see also [CO09, KW16]. For this we introduce the following truncated cone for $j \in \{1, 2, \dots, d\}$ and $\mu, \lambda > 0$

$$(21) \quad C_{j,\mu,\lambda} := \{k \in \mathbb{Z}^d : |k|^2 - k_j^2 \leq \mu^2 |k|^2, |k_j| \leq \frac{2}{|\nu_1|} \lambda\},$$

and the following (infinitely extended) cones

$$(22) \quad C_{j,\mu} := \{k \in \mathbb{Z}^d : |k|^2 - k_j^2 \leq \mu^2 |k|^2\}.$$

For these cones let $m_{j,\mu,\lambda}(D), m_{j,\mu}(D)$ be the corresponding Fourier multipliers acting on $f \in L^2(\mathbb{T}^d; \mathbb{R})$ as

$$m_{j,\mu,\lambda}(D)f(x) = \sum_{k \in \mathbb{Z}^d} m_{j,\mu,\lambda}(k) \hat{f}(k) e^{2\pi i k \cdot x}.$$

They are given (for instance) by

$$(23) \quad \begin{aligned} m_{j,\mu,\lambda}(k) &= (1 - \varphi(2|k|)) \varphi\left(\frac{\sqrt{|k|^2 - k_j^2}}{\mu|k|}\right) \varphi\left(\frac{|\nu_1||k_j|}{2\lambda}\right) + \varphi(2|k|) \in C^\infty(\mathbb{R}^d), \\ m_{j,\mu}(k) &= (1 - \varphi(2|k|)) \varphi\left(\frac{\sqrt{|k|^2 - k_j^2}}{\mu|k|}\right) + \varphi(2|k|) \in C^\infty(\mathbb{R}^d), \end{aligned}$$

where

$$\varphi \in C^\infty([0, +\infty); [0, 1]), \quad \varphi(x) = 1, \varphi'(x) \leq 0, x \in [0, 1], \quad \varphi(x) = 0, x \notin (0, 2).$$

Thus the multipliers satisfy for $k \in \mathbb{Z}^d$

$$\begin{aligned} m_{j,\mu,\lambda}(k) &= 1, \quad k \in C_{j,\mu,\lambda}, & m_{j,\mu,\lambda}(k) &= 0, \quad k \notin C_{j,2\mu,2\lambda}, \\ m_{j,\mu}(k) &= 1, \quad k \in C_{j,\mu} & m_{j,\mu}(k) &= 0, \quad k \notin C_{j,2\mu}. \end{aligned}$$

With this choice of multipliers we are able to apply Marcinkiewicz's multiplier theorem [Gra08, Cor. 6.2.5] (combined with the transference principle [Gra08, Thm. 4.3.7]). These multipliers have been considered in the context of shape-memory alloys in [RT22, RT23b].

We start by giving a Fourier interpretation of the elastic energy. For notational convenience, here and in what follows below, we will use the notation $\mathbb{R}_{\text{diag}}^{d \times d}$ to denote the diagonal matrices.

Lemma 2.3. *Let $d \geq 2$ and $\Omega = (0, 1)^d$. For $u \in H_0^1(\Omega; \mathbb{R}^d)$ and $\chi \in L^2(\Omega; \mathbb{R}_{\text{diag}}^{d \times d})$ let the elastic energy $E_{el}(u, \chi)$ be given by*

$$E_{el}(u, \chi) = \int_{\Omega} |\nabla u - \chi|^2 dx.$$

Then,

$$E_{el}(u, \chi) \geq |\hat{\chi}(0)|^2 + \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \sum_{j=1}^d \frac{|k|^2 - k_j^2}{|k|^2} |\hat{\chi}_{jj}(k)|^2,$$

where we consider the one-periodic extensions of u and χ without changing the notation.

Proof. We consider the one-periodic extensions of u and χ and switch to Fourier space

$$E_{el}(u, \chi) = |\hat{\chi}(0)|^2 + \sum_{k \in \mathbb{Z}^d \setminus \{0\}} |2\pi i \hat{u} \otimes k - \hat{\chi}|^2.$$

For any given χ , we solve the related Euler-Lagrange equation. That is, we choose $\hat{u}(k)$ such that it fulfils

$$2\pi i(\hat{u} \otimes k)k = \hat{\chi}k, \quad k \neq 0.$$

Plugging this choice of $\hat{u}(k)$ into the above Fourier representation and using that χ is diagonal then gives the desired result. $\square$

Combining Lemmata 2.2 and 2.3, we can deduce a first Fourier localization argument.

Lemma 2.4. *Let $d \geq 2$, $\Omega = (0, 1)^d$, and $\nu \in \mathbb{S}^{d-1}$ be such that $\nu \cdot e_1 \neq 0$. Consider the energy to be given for $\epsilon > 0$ and $\chi \in BV_\nu(\Omega; \mathbb{R}_{\text{diag}}^{d \times d}) \cap L^\infty(\Omega; \mathbb{R}_{\text{diag}}^{d \times d})$ by*

$$E_\epsilon(\chi) = \inf_{u \in H_0^1(\Omega; \mathbb{R}^d)} \int_\Omega |\nabla u - \chi|^2 dx + \epsilon \|D_\nu \chi\|_{TV(\Omega)}.$$

Extending χ one-periodically and considering the cones $C_{1,\mu,\lambda}$ and $C_{j,\mu}$ for $\mu \in (0, \frac{|\nu_1|}{2})$, $\lambda > 0$, and $j = 2, 3, \dots, d$ defined in (21) and (22) it holds

$$\sum_{k \notin C_{1,\mu,\lambda}} |\hat{\chi}_{11}(k)|^2 + \sum_{j=2}^d \sum_{k \notin C_{j,\mu}} |\hat{\chi}_{jj}(k)|^2 \leq C(\mu^{-2} + \|\chi\|_{L^\infty}(\lambda\epsilon)^{-1}) E_\epsilon(\chi) + C\|\chi\|_{L^\infty}^2 \lambda^{-1} \text{Per}(\Omega),$$

where $C = C(d) > 0$ is a constant independent of ν, μ, λ and χ .

Proof. By Lemma 2.3, we get

$$\inf_{u \in H_0^1(\Omega; \mathbb{R}^d)} \int_\Omega |\nabla u - \chi|^2 dx \geq \sum_{j=1}^d \sum_{k \in \mathbb{Z}^d} \frac{|k|^2 - k_j^2}{|k|^2} |\hat{\chi}_{jj}(k)|^2,$$

where we set the multiplier to be equal to one for $k = 0$. For $j \in \{1, \dots, d\}$ and $k \notin C_{j,\mu}$ it therefore holds

$$\frac{|k|^2 - k_j^2}{|k|^2} > \mu^2,$$

and thus

$$(24) \quad \sum_{j=1}^d \sum_{k \notin C_{j,\mu}} |\hat{\chi}_{jj}(k)|^2 \leq \mu^{-2} \sum_{j=1}^d \sum_{k \in \mathbb{Z}^d} \frac{|k|^2 - k_j^2}{|k|^2} |\hat{\chi}_{jj}(k)|^2 \leq \mu^{-2} \inf_{u \in H_0^1(\Omega; \mathbb{R}^d)} \int_\Omega |\nabla u - \chi|^2 dx.$$

To improve the estimate on χ_{11} to a bound outside of the truncated cone $C_{1,\mu,\lambda}$ instead of in the complement of the infinitely extended cone $C_{1,\mu}$, we note that due to the assumptions that $\nu \cdot e_1 \neq 0$ and $\mu < \frac{|\nu_1|}{2}$

$$\{k \in \mathbb{Z}^d : |k|^2 - k_1^2 \leq \mu^2 |k|^2, |k \cdot \nu| \leq \lambda\} \subset C_{1,\mu,\lambda}.$$

Indeed, for $k = (k_1, k')$ and $\nu = (\nu_1, \nu') \neq \pm e_1$, using that $\mu < \frac{|\nu_1|}{2}$, we have $\mu^2/(1 - \mu^2) \leq |\nu_1|^2/(4|\nu'|^2)$. Hence, for k such that $|k \cdot \nu| \leq \lambda$ and $|k'|^2 = |k|^2 - k_1^2 \leq \mu^2 |k|^2$, by exploiting the reverse triangle inequality we observe that

$$(25) \quad |k \cdot \nu| = |k_1 \nu_1 + k' \cdot \nu'| \geq |k_1| |\nu_1| - \mu |k| |\nu'| \geq |k_1| \left(|\nu_1| - \frac{\mu |\nu'|}{\sqrt{1 - \mu^2}} \right) \geq |k_1| \frac{|\nu_1|}{2}.$$

Hence, it holds that

$$|k_1| \leq \frac{2}{|\nu_1|} \lambda.$$

Note, that here it is of importance that $\nu_1 \neq 0$.

By (24) and Lemma 2.2 it follows that

$$\begin{aligned} \sum_{k \notin C_{1,\mu,\lambda}} |\hat{\chi}_{11}(k)|^2 &\leq \sum_{k \notin C_{1,\mu}} |\hat{\chi}_{11}(k)|^2 + \sum_{|k \cdot \nu| > \lambda} |\hat{\chi}_{11}(k)|^2 \\ &\leq \mu^{-2} \sum_{k \in \mathbb{Z}^d} \frac{|k|^2 - k_1^2}{|k|^2} |\hat{\chi}_{11}(k)|^2 \\ &\quad + C(d) \lambda^{-1} \|\chi_{11}\|_{L^\infty} (\|D_\nu \chi_{11}\|_{TV(\Omega)} + \|\chi_{11}\|_{L^\infty} \text{Per}(\Omega)). \end{aligned}$$

Summing this estimate and (24) for $j = 2, 3, \dots, d$, we get for $\chi \in L^\infty(\Omega; \mathbb{R}_{\text{diag}}^{d \times d})$

$$\begin{aligned} \sum_{k \notin C_{1,\mu,\lambda}} |\hat{\chi}_{11}(k)|^2 + \sum_{j=2}^d \sum_{k \notin C_{j,\mu}} |\hat{\chi}_{jj}(k)|^2 &\leq C(d) (\mu^{-2} + \|\chi\|_{L^\infty} (\lambda \epsilon)^{-1}) E_\epsilon(\chi) \\ &\quad + C(d) \|\chi\|_{L^\infty}^2 \lambda^{-1} \text{Per}(\Omega). \end{aligned}$$

□

With this result we have a combination of an ellipticity estimate of the form (24) and a high frequency control in one direction using Lemma 2.2. The next result provides a low frequency control, with similar methods as in [RT23b, Lem. 4.2].

Lemma 2.5. *Assume that the same conditions hold as in Lemma 2.3 and further assume that $\inf_{u \in H_0^1(\Omega; \mathbb{R}^d)} E_{el}(u, \chi) > 0$. Viewing $\chi \in L^2(\Omega; \mathbb{R}_{\text{diag}}^{d \times d})$ as a function on $\mathbb{T}^d$, there is a constant $C > 0$ such that for any $\bar{\lambda} > 1$*

$$\sum_{|k_1| \leq \bar{\lambda}} |\hat{\chi}_{11}(k)|^2 \leq C \bar{\lambda}^2 \inf_{u \in H_0^1(\Omega; \mathbb{R}^d)} E_{el}(u, \chi).$$

Proof. The proof relies on the diagonal structure of χ and the zero boundary data which we impose on the functions u . For $u \in H_0^1(\Omega; \mathbb{R}^d)$, seen as a function on $\mathbb{T}^d$, it holds by Poincaré's inequality in x_2 and as $\chi_{12} = 0$

$$\begin{aligned} \sum_{|k_1| \leq \bar{\lambda}} |\hat{u}_1(k)|^2 &\leq \int_{\Omega} |u_1(x)|^2 dx \leq \int_{\Omega} |\partial_2 u_1(x)|^2 dx \\ &= \int_{\Omega} |\partial_2 u_1 - \chi_{12}|^2 dx \leq E_{el}(u, \chi). \end{aligned}$$

Choosing $v \in H_0^1(\Omega; \mathbb{R}^d)$ such that $E_{el}(v, \chi) \leq 2 \inf_{u \in H_0^1(\Omega; \mathbb{R}^d)} E_{el}(u, \chi)$, where we use that $\inf_{u \in H_0^1(\Omega; \mathbb{R}^d)} E_{el}(u, \chi) > 0$, implies for $\bar{\lambda} > 1$

$$\begin{aligned} \sum_{|k_1| \leq \bar{\lambda}} |\hat{\chi}_{11}(k)|^2 &\leq 2 \sum_{|k_1| \leq \bar{\lambda}} \left(|\widehat{\partial_1 v_1}(k) - \hat{\chi}_{11}(k)|^2 + |2\pi i k_1 \hat{v}_1(k)|^2 \right) \\ &\leq 2 E_{el}(v, \chi) + 8\pi^2 \bar{\lambda}^2 \sum_{|k_1| \leq \bar{\lambda}} |\hat{v}_1(k)|^2 \\ &\leq C \bar{\lambda}^2 E_{el}(v, \chi) \leq 2C \bar{\lambda}^2 \inf_{u \in H_0^1(\Omega; \mathbb{R}^d)} E_{el}(u, \chi) \end{aligned}$$

which yields the result. □

As a final ingredient, for more than two wells, we rely on a commutator estimate. With this we can use the truncation of one cone, cf. Lemma 2.4, and carry it over to another one and reduce its size in the process. This gives rise to an iterative procedure, where the number of iterations determines the scaling in Theorems 1 and 2.

Lemma 2.6 ([RT22, Lem. 3], [RT23b, Prop. 4.6]). *Let $d \geq 2$, $\Omega = (0, 1)^d$, $\nu \in \mathbb{S}^{d-1}$ with $\nu \cdot e_1 \neq 0$. For $\mu \in (0, \frac{|\nu_1|}{2})$ and $\lambda > 0$, let $m_{1,\mu,\lambda}(D)$ be given by (23). Moreover, for $t > 0$ let $\psi_t(x) = \max\{|x|, |x|^t\}$. Let $f_1, f_2 \in BV_\nu(\Omega; \mathbb{R}) \cap L^\infty(\Omega; \mathbb{R})$ and let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a polynomial of degree two with $f_2 = g(f_1)$. Then for any $\gamma \in (0, 1)$ there is a constant $C = C(g, \gamma, \|f_1\|_{L^\infty}) > 0$ such that*

$$\|f_2 - g(m_{1,\mu,\lambda}(D)f_1)\|_{L^2} \leq C\psi_{1-\gamma}(\|f_1 - m_{1,\mu,\lambda}(D)f_1\|_{L^2}).$$

Proof. By the triangle inequality we can assume without loss of generality that $g(x) = x^2$. Invoking Hölder's inequality, we get

$$\begin{aligned} \|f_2 - g(m_{1,\mu,\lambda}(D)f_1)\|_{L^2} &= \|f_1^2 - (m_{1,\mu,\lambda}(D)f_1)^2\|_{L^2} \\ &\leq \|f_1 - m_{1,\mu,\lambda}(D)f_1\|_{L^{2+2\gamma}} \|f_1 + m_{1,\mu,\lambda}(D)f_1\|_{L^{\frac{2+2\gamma}{\gamma}}}. \end{aligned}$$

By virtue of the interpolation inequality of L^p spaces and the L^p - L^p multiplier bounds from Marcinkiewicz's theorem and the transference principle [Gra08, Cor. 6.2.5, Thm. 4.3.7], we get the desired estimate. For details we refer to [RT23b, Prop. 4.6]. $\square$

Corollary 2.7. *Let $d \geq 2$, $\Omega = (0, 1)^d$, $\nu \in \mathbb{S}^{d-1}$ with $\nu \cdot e_1 \neq 0$, $\lambda > 0$, and $\mu \in (0, \frac{|\nu_1|}{16})$. Let $\chi \in BV_\nu(\Omega; \mathbb{R}^{d \times d}_{\text{diag}}) \cap L^\infty(\Omega; \mathbb{R}^{d \times d}_{\text{diag}})$, extended one-periodically, and suppose that*

$$\sum_{j=2}^d \alpha_j \chi_{jj} = g(\chi_{11}),$$

for a polynomial $g : \mathbb{R} \rightarrow \mathbb{R}$ of degree two and coefficients $\alpha_j \in \mathbb{R}$. Let $m_{j,\mu,\lambda}$ and $m_{j,\mu}$ be as in (23). Then for any $\gamma \in (0, 1)$ there is $C = C(d, g, \gamma, \|\chi_{11}\|_{L^\infty}) > 0$ such that

$$\begin{aligned} (26) \quad \left\| \sum_{j=2}^d \alpha_j m_{j,\mu}(D) \chi_{jj} - g(m_{1,\mu,\lambda}(D) \chi_{11}) \right\|_{L^2} &\leq C\psi_{1-\gamma}(\|\chi_{11} - m_{1,\mu,\lambda}(D) \chi_{11}\|_{L^2}) \\ &\quad + \sum_{j=2}^d |\alpha_j| \|\chi_{jj} - m_{j,\mu}(D) \chi_{jj}\|_{L^2}. \end{aligned}$$

Moreover, there is $M > 0$ such that for $\lambda_2 = M\mu\lambda < \lambda$ it holds that

$$\begin{aligned} \sum_{j=2}^d |\alpha_j| \|\chi_{jj} - m_{j,\mu,\lambda_2}(D) \chi_{jj}\|_{L^2} &\leq C\psi_{1-\gamma}(\|\chi_{11} - m_{1,\mu,\lambda}(D) \chi_{11}\|_{L^2}) \\ &\quad + 2 \sum_{j=2}^d |\alpha_j| \|\chi_{jj} - m_{j,\mu}(D) \chi_{jj}\|_{L^2}. \end{aligned}$$

Proof. The first statement is a direct consequence of Lemma 2.6 and the triangle inequality:

$$\begin{aligned} &\left\| \sum_{j=2}^d \alpha_j m_{j,\mu}(D) \chi_{jj} - g(m_{1,\mu,\lambda}(D) \chi_{11}) \right\|_{L^2} \\ &\leq \left\| \sum_{j=2}^d \alpha_j m_{j,\mu}(D) \chi_{jj} - \sum_{j=2}^d \alpha_j \chi_{jj} \right\|_{L^2} + \left\| \sum_{j=2}^d \alpha_j \chi_{jj} - g(m_{1,\mu,\lambda}(D) \chi_{11}) \right\|_{L^2} \\ &\leq \sum_{j=2}^d |\alpha_j| \|\chi_{jj} - m_{j,\mu}(D) \chi_{jj}\|_{L^2} + \|f_2 - g(m_{1,\mu,\lambda}(D) f_1)\|_{L^2}, \end{aligned}$$

with $f_1 = \chi_{11}$ and $f_2 = \sum_{j=2}^d \alpha_j \chi_{jj}$.

To see the second claim, we note that, since the support of $\mathcal{F}[g(m_{1,\mu,\lambda}(D)\chi_{11})]$ is contained in $C_{1,2\mu,2\lambda} + C_{1,2\mu,2\lambda}$ (in the sense of the Minkowski sum), there is an $M > 0$, independent of μ, λ, ν , such that with $\lambda_2 = M\mu\lambda$

$$(27) \quad \mathcal{F}[g(m_{1,\mu,\lambda}(D)\chi_{11})](k) = 0, \quad \text{for } |k_j| > \frac{2}{|\nu_1|} \lambda_2, \quad \text{for any } j \in \{2, \dots, d\}.$$

Furthermore, we use that $|m_{j,\mu}(k) - m_{j,\mu,\lambda_2}(k)| \leq \chi_{\{|k_j| \geq \frac{2}{|\nu_1|} \lambda_2\}}(k) m_{j,\mu}(k)$, thus

$$\|m_{j,\mu}(D)\chi_{jj} - m_{j,\mu,\lambda_2}(D)\chi_{jj}\|_{L^2} \leq \|\chi_{\{|k_j| \geq \frac{2}{|\nu_1|} \lambda_2\}}(D)m_{j,\mu}(D)\chi_{jj}\|_{L^2},$$

and after an application of the triangle inequality

$$(28) \quad \begin{aligned} \|\chi_{jj} - m_{j,\mu,\lambda_2}(D)\chi_{jj}\|_{L^2} &\leq \|\chi_{jj} - m_{j,\mu}(D)\chi_{jj}\|_{L^2} \\ &\quad + \|\chi_{\{|k_j| \geq \frac{2}{|\nu_1|} \lambda_2\}}(D)m_{j,\mu}(D)\chi_{jj}\|_{L^2}. \end{aligned}$$

Here, with slight abuse of notation, we define $\chi_{\{|k_j| \geq \frac{2}{|\nu_1|} \lambda_2\}}(D)$ as the Fourier multiplier associated with the function $\chi_{\{|k_j| \geq \frac{2}{|\nu_1|} \lambda_2\}}(k)$ in Fourier space.

Using the fact that as $\mu < \frac{1}{2\sqrt{2}}$ we have $m_{j,\mu}(k)m_{\ell,\mu}(k) = \delta_0(k)$ for $j \neq \ell$, we see that $m_{j,\mu}(D)\chi_{jj}$ and $m_{\ell,\mu}(D)\chi_{\ell\ell}$ have disjoint Fourier support away from zero. In particular the functions $\alpha_j \chi_{\{|k_j| \geq 2|\nu_1|^{-1}\lambda_2\}}(D)m_{j,\mu}(D)\chi_{jj}$ have pairwise disjoint Fourier support, hence, after summing (28) over $j = 2, \dots, d$ with the weights $|\alpha_j|$, we get

$$(29) \quad \begin{aligned} \sum_{j=2}^d |\alpha_j| \|\chi_{jj} - m_{j,\mu,\lambda_2}(D)\chi_{jj}\|_{L^2} &\leq \sum_{j=2}^d |\alpha_j| \|\chi_{jj} - m_{j,\mu}(D)\chi_{jj}\|_{L^2} \\ &\quad + C(d) \left\| \sum_{j=2}^d \alpha_j \chi_{\{|k_j| \geq \frac{2}{|\nu_1|} \lambda_2\}}(D)m_{j,\mu}(D)\chi_{jj} \right\|_{L^2}. \end{aligned}$$

In the following, we write $k = (k_1, k')$ with $k' = (k_2, \dots, k_d) \in \mathbb{Z}^{d-1}$ and also use $|k'|_\infty = \max\{|k_2|, \dots, |k_d|\}$. Since $\mu < \frac{1}{4}$, we remark that $\chi_{\{|k_j| \geq \frac{2}{|\nu_1|} \lambda_2\}}(k)m_{j,\mu}(k) = \chi_{\{|k'|_\infty \geq \frac{2}{|\nu_1|} \lambda_2\}}(k)m_{j,\mu}(k)$ for $j = 2, \dots, d$ and moreover by (27), we can further control the second term in (29) as follows

$$(30) \quad \begin{aligned} &\left\| \sum_{j=2}^d \alpha_j \chi_{\{|k_j| \geq \frac{2}{|\nu_1|} \lambda_2\}}(D)m_{j,\mu}(D)\chi_{jj} \right\|_{L^2} \\ &= \left\| \chi_{\{|k'|_\infty \geq \frac{2}{|\nu_1|} \lambda_2\}}(D) \left(\sum_{j=2}^d \alpha_j m_{j,\mu}(D)\chi_{jj} - g(m_{1,\mu,\lambda}(D)\chi_{11}) \right) \right\|_{L^2} \\ &\leq \left\| \sum_{j=2}^d \alpha_j m_{j,\mu}(D)\chi_{jj} - g(m_{1,\mu,\lambda}(D)\chi_{11}) \right\|_{L^2}. \end{aligned}$$

Thus, gathering (29), (30) and the first statement (26), we can conclude

$$\begin{aligned} \sum_{j=2}^d |\alpha_j| \|\chi_{jj} - m_{j,\mu,\lambda_2}(D)\chi_{jj}\|_{L^2} &\leq (C(d) + 1) \sum_{j=2}^d |\alpha_j| \|\chi_{jj} - m_{j,\mu}(D)\chi_{jj}\|_{L^2} \\ &\quad + C(d) \psi_{1-\gamma}(\|\chi_{11} - m_{1,\mu,\lambda}(D)\chi_{11}\|_{L^2}). \end{aligned}$$

□

Let us compare the use of the nonlinear relation with the one from [RT23b]. In [RT23b] the nonlinear relation is exploited in the form $\chi_{kk} = g(\sum_{j \neq k} \alpha_j \chi_{jj})$. This strategy hence requires working with truncated cones in *all but one* direction. In contrast, in the formulation of Corollary 2.7, since the polynomial relation is inverted, it is clear why there is only a single truncation necessary in order to “propagate” the truncation to the other diagonal entries.

Remark 2.8. *In the sequel, when dealing with higher order laminates, we need to iterate the cone localization from above. For this reason, we will exploit the following variant of Corollary 2.7, whose proof is identical.*

In the context of Corollary 2.7, let $1 \leq k \leq d$ be the component we want to (further) localize and let $\lambda_k > 0$ denote the localization length of χ_{kk} . If χ_{kk} has the nonlinear relation

$$\sum_{j=k+1}^d \alpha_j \chi_{jj} = g(\chi_{kk})$$

then, for any $\gamma \in (0, 1)$ and given $\lambda_{k+1} = M\mu\lambda_k$ it holds that

$$\begin{aligned} \sum_{j=k+1}^d |\alpha_j| \|\chi_{jj} - m_{j,\mu,\lambda_{k+1}}(D)\chi_{jj}\|_{L^2} &\leq C\psi_{1-\gamma}(\|\chi_{kk} - m_{k,\mu,\lambda_k}(D)\chi_{kk}\|_{L^2}) \\ &\quad + 2 \sum_{j=k+1}^d |\alpha_j| \|\chi_{jj} - m_{j,\mu}(D)\chi_{jj}\|_{L^2}. \end{aligned}$$

We will also apply Corollary 2.7 and the above variant by replacing $\nu \cdot e_1$ with $\nu \cdot e_n$ for some $1 \leq n \leq d$, with the definitions of $C_{j,\mu,\lambda}$, $C_{j,\mu}$, $m_{j,\mu,\lambda}$ and $m_{j,\mu}$ modified accordingly.

2.5. Fractional surface energies in L^2 -based settings. For the proofs of the lower bound of Theorems 1 and 2 we invoke Fourier methods, relying on Lemma 2.2 for the surface energy. Therefore we can also replace the sharp surface energy by a fractional L^2 -based surface energy, giving rise to a similar high frequency control as in Lemma 2.2.

By extending $\chi \in L^\infty((0, 1)^d; \mathbb{R}_{\text{diag}}^{d \times d})$ one-periodically, we consider for $s \in (0, \frac{1}{2})$

$$E_{surf}^s(\chi) := \left(\sum_{k \in \mathbb{Z}^d \setminus \{0\}} |k \cdot \nu|^{2s} |\hat{\chi}(k)|^2 \right)^{\frac{1}{2s}}.$$

It then follows directly that it holds

$$\left(\sum_{k \in \mathbb{Z}^d: |k \cdot \nu| \geq \lambda} |\hat{\chi}(k)|^2 \right)^{\frac{1}{2s}} \leq \lambda^{-1} E_{surf}^s(\chi).$$

Based on this observation, we have an analogous result as in Lemma 2.4.

Lemma 2.9. *Let $d \geq 2$, $\Omega = (0, 1)^d$, and $\nu \in \mathbb{S}^{d-1}$ such that $\nu \cdot e_1 \neq 0$. Consider for $\epsilon > 0$, $s \in (0, \frac{1}{2})$ and $\chi \in H_\nu^s(\Omega; \mathbb{R}_{\text{diag}}^{d \times d})$, cf. (8), the following energy*

$$E_{\epsilon,s}(\chi) = \inf_{u \in H_0^1(\Omega; \mathbb{R}^d)} \int_{\Omega} |\nabla u - \chi|^2 dx + \epsilon E_{surf}^s(\chi).$$

Then, it holds

$$\sum_{k \notin C_{1,\mu,\lambda}} |\hat{\chi}_{11}(k)|^2 + \sum_{j=2}^d \sum_{k \notin C_{j,\mu}} |\hat{\chi}_{jj}(k)|^2 \leq \psi_{2s}((\mu^{-2} + (\lambda\epsilon)^{-1})E_{\epsilon,s}(\chi)),$$

where, for $\mu \in (0, \frac{|\nu_1|}{2})$, $\lambda > 0$, the cones $C_{1,\mu,\lambda}, C_{j,\mu}$ are given as in (21) and (22), and $\psi_t(x) = \max\{|x|, |x|^t\}$ for $t > 0$.

Proof. We argue as in the proof of Lemma 2.4. By Lemma 2.3 we have

$$E_{\epsilon,s}(\chi) \geq \sum_{j=1}^d \sum_{k \in \mathbb{Z}^d} \frac{|k|^2 - k_j^2}{|k|^2} |\hat{\chi}_{jj}(k)|^2 + \epsilon \left(\sum_{k \in \mathbb{Z}^d \setminus \{0\}} |k \cdot \nu|^{2s} |\hat{\chi}(k)|^2 \right)^{\frac{1}{2s}}.$$

Consequently, for $j \in \{1, \dots, d\}$ and $k \notin C_{j,\mu}$ we have $|k|^2 - k_j^2 \geq \mu^2 |k|^2$ and, hence,

$$\sum_{j=1}^d \sum_{k \notin C_{j,\mu}} |\hat{\chi}_{jj}(k)|^2 \leq \mu^{-2} \sum_{j=1}^d \sum_{k \in \mathbb{Z}^d} \frac{|k|^2 - k_j^2}{|k|^2} |\hat{\chi}_{jj}(k)|^2 \leq \mu^{-2} E_{\epsilon,s}(\chi).$$

For the surface energy it is immediate that it holds

$$\sum_{|k \cdot \nu| \geq \lambda} |\hat{\chi}(k)|^2 \leq \lambda^{-2s} \sum_{k \in \mathbb{Z}^d \setminus \{0\}} |k \cdot \nu|^{2s} |\hat{\chi}(k)|^2 \leq (\lambda\epsilon)^{-2s} E_{\epsilon,s}(\chi)^{2s}.$$

Moreover, if we combine both the above estimates we arrive at

$$\begin{aligned} \sum_{k \notin C_{1,\mu,\lambda}} |\hat{\chi}_{11}(k)|^2 + \sum_{j=2}^d \sum_{k \notin C_{j,\mu}} |\hat{\chi}_{jj}(k)|^2 &\leq \sum_{j=1}^d \sum_{k \notin C_{j,\mu}} |\hat{\chi}_{jj}(k)|^2 + \sum_{|k \cdot \nu| \geq \lambda} |\hat{\chi}_{11}(k)|^2 \\ &\leq \mu^{-2} E_{\epsilon,s}(\chi) + (\lambda\epsilon)^{-2s} E_{\epsilon,s}(\chi)^{2s} \\ &\leq 2 \max\{(\mu^{-2} + (\lambda\epsilon)^{-1}) E_{\epsilon,s}(\chi), ((\mu^{-2} + (\lambda\epsilon)^{-1}) E_{\epsilon,s}(\chi))^{2s}\} \\ &= 2\psi_{2s}((\mu^{-2} + (\lambda\epsilon)^{-1}) E_{\epsilon,s}(\chi)) \end{aligned}$$

and the result is proven. $\square$

As we will see below in Section 3 an estimate of this form is sufficient to deduce the lower scaling estimates, consequently, we will be able to generalize Theorem 2 to fractional surface energies as stated in Theorem 4.

3. SHARP SURFACE ENERGIES – PROOFS OF THEOREMS 1 AND 2

In this section we focus on the sharp anisotropic surface energies introduced in Section 1.1 and seek to identify minimal assumptions on the surface energy in order to ensure the same scaling as with isotropic surface energy penalizations.

3.1. Two-well problem. As motivation, we begin by considering a simple two-well gradient inclusion $\nabla u \in \{0, e_1 \otimes e_1\}$. In the standard models in the literature a quantification of this problem is most often considered with an isotropic surface penalization. There are also instances where only the oscillation in e_1 direction is penalized, as for instance in the seminal works [KM94, KM92, Con00]. As we view the two-well problem as a prototypical model set-up which we will then, in the following sections, generalize to more complex microstructures, we briefly present the proof of Proposition 1.1.

We use similar Fourier methods as in [RT22, RT23b] and argue in three steps. We first give the arguments for the lower and upper bounds in the case of $\nu \cdot e_1 \neq 0$, afterwards we present the upper bound construction for $\nu \cdot e_1 = 0$. With this we have a full characterisation of which directions are required in the surface energy to have the same scaling as the isotropic surface penalization.

Proof of Proposition 1.1 for $\nu \cdot e_1 \neq 0$. Step 1: Lower bound. We first assume that $F_\alpha = 0$ and $A - B = e_1 \otimes e_1$, namely $A = (1 - \alpha)e_1 \otimes e_1$, $B = -\alpha e_1 \otimes e_1$. The general case will be recovered at the end of the proof. We define for $\chi \in BV_\nu(\Omega; \{A, B\})$

$$E_\epsilon(\chi) := \inf_{u \in \mathcal{A}_{F_\alpha}} E_\epsilon(u, \chi) = \inf_{u \in \mathcal{A}_{F_\alpha}} \int_{\Omega} |\nabla u - \chi|^2 dx + \epsilon \|D_\nu \chi\|_{TV(\Omega)}.$$

Thus, after extending χ and ∇u one-periodically, we can apply Lemma 2.3 to obtain

$$\int_{\Omega} |\nabla u - \chi|^2 dx \geq \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \frac{|k|^2 - k_1^2}{|k|^2} |\hat{f}|^2 + |\hat{f}(0)|^2,$$

where we wrote $f = \chi_{11} \in BV_\nu(\Omega; \{1 - \alpha, -\alpha\})$.

Lemma 2.4 then implies for $\mu \in (0, \frac{|\nu_1|}{2})$, $\lambda > 1$

$$\sum_{k \notin C_{1,\mu,\lambda}} |\hat{f}(k)|^2 \leq C(d, \alpha) \left((\mu^{-2} + (\lambda\epsilon)^{-1}) E_\epsilon(\chi) + \lambda^{-1} \text{Per}(\Omega) \right),$$

where the truncated cone $C_{1,\mu,\lambda}$ is defined in (21).

Moreover, by an application of Lemma 2.5 for $\bar{\lambda} = \frac{2}{|\nu_1|} \lambda > 1$, for $\lambda > 1 > \frac{|\nu_1|}{2}$,

$$(31) \quad \sum_{|k_1| \leq 2\lambda/|\nu_1|} |\hat{f}(k)|^2 \leq C \frac{\lambda^2}{|\nu_1|^2} E_\epsilon(\chi).$$

With this, we have control over the Fourier mass of f in the whole space partitioned into $\mathbb{Z}^d = (C_{1,\mu,\lambda}^c \cap \mathbb{Z}^d) \cup \{k \in \mathbb{Z}^d : |k_1| \leq \frac{2}{|\nu_1|} \lambda\}$, cf. Figure 5, as follows

$$\begin{aligned} \sum_{k \in \mathbb{Z}^d} |\hat{f}(k)|^2 &\leq \sum_{|k_1| \leq 2\lambda/|\nu_1|} |\hat{f}(k)|^2 + \sum_{k \notin C_{1,\mu,\lambda}} |\hat{f}(k)|^2 \\ &\leq C \left(\frac{\lambda^2}{|\nu_1|^2} + \mu^{-2} + (\lambda\epsilon)^{-1} \right) E_\epsilon(\chi) + C\lambda^{-1} \text{Per}(\Omega). \end{aligned}$$

To balance the first two terms $\lambda^2 \nu_1^{-2} + \mu^{-2}$, the optimal choice of μ is $\mu \sim \lambda^{-1} |\nu_1|$, e.g. $\mu = \frac{|\nu_1|}{2} \lambda^{-1} < \frac{|\nu_1|}{2}$. This yields

$$\sum_{k \in \mathbb{Z}^d} |\hat{f}(k)|^2 \leq C \left(\frac{\lambda^2}{\nu_1^2} + (\lambda\epsilon)^{-1} \right) E_\epsilon(\chi) + C\lambda^{-1} \text{Per}(\Omega),$$

which again we optimize in λ , by choosing $\lambda = |\nu_1|^{\frac{2}{3}} \epsilon^{-\frac{1}{3}} > 1$ for $\epsilon < |\nu_1|^2$.

By this choice, and as $\sum_{k \in \mathbb{Z}^d} |\hat{f}(k)|^2 = \int_{\Omega} |f(x)|^2 dx \geq \min\{1 - \alpha, \alpha\}^2$, after an absorption of the perimeter term into the left-hand side, we derive the lower scaling bound for $\epsilon < \epsilon_0(|\nu_1|, \alpha, d)$

$$E_\epsilon(\chi) \geq C |\nu_1|^{\frac{2}{3}} \epsilon^{\frac{2}{3}}.$$

Fixing $\epsilon_0 < |\nu_1|^2$ we can also ensure $\lambda > 1$ as required above.

For the general case $F_\alpha \neq 0$ and $A - B = a \otimes e_1$ for $a \in \mathbb{R}^d \setminus \{0\}$ we consider the functions $\tilde{u} = R(u - F_\alpha x)$, $\tilde{\chi} = R(\chi - F_\alpha)$ with a rotation $R \in SO(d)$ such that $Ra = |a|e_1$, then $\tilde{\chi} = |a|f e_1 \otimes e_1$, and we can apply the above arguments as $E_\epsilon(\tilde{\chi}) \sim E_\epsilon(\chi)$.

Step 2: Upper bound. Since simple branching constructions are well-understood (cf. [KM94, CC15]), our sole goal in this step is to make the ν_1 dependence of the prefactor explicit. For this, our proof is only a minor adaptation of the “usual” branching construction (see, for instance, [RT23b, Lemma 3.2]), and we work only in two dimensions for simplicity. For later use (cf.

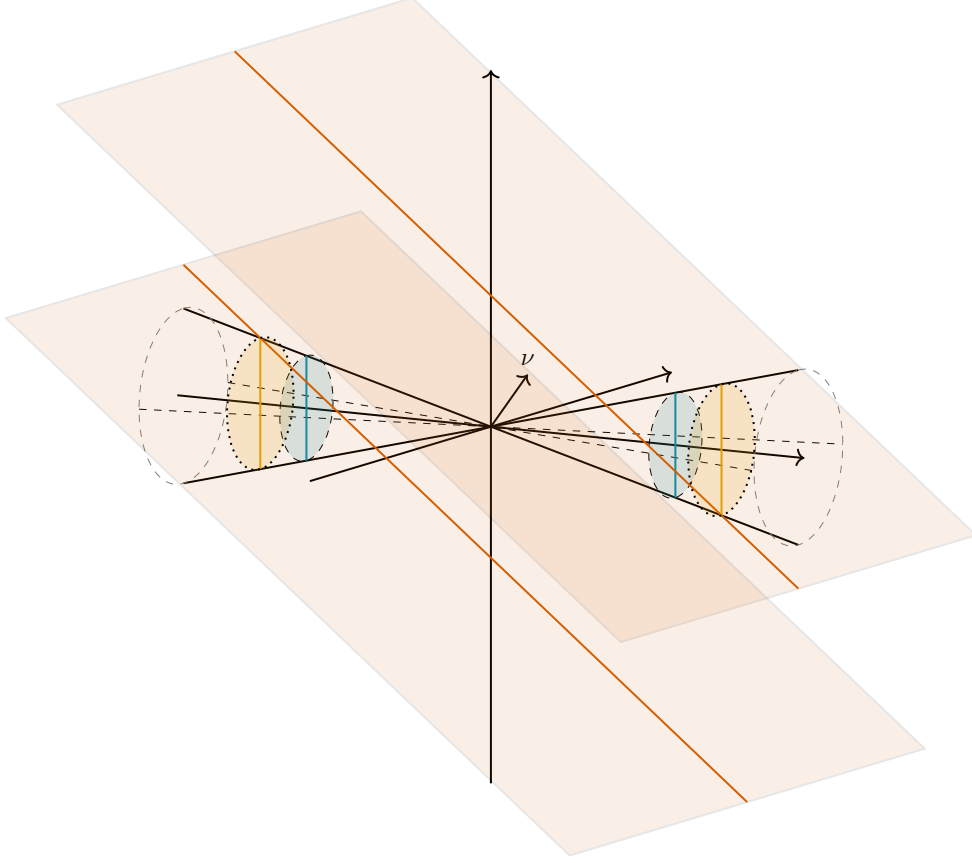


FIGURE 5. Illustration of regions of different Fourier mass control in the two-well setting. Choosing $\bar{\lambda} = \bar{\lambda}(\lambda)$ such that the blue (dashed) and orange (dotted) circles coincide we control the Fourier mass everywhere.

proof of Lemma 3.3 below) we provide the main estimate on a general rectangular domain $Q = (0, L) \times (0, H)$, which in particular includes the case of the unit square.

For the reader's convenience, we recall that the domain Q is subdivided in cells $\{\omega_{j,k} : j = 0, \dots, j_0 + 1, k = 1, \dots, 2^j N\}$ for some $N \in \mathbb{N}$ sufficiently large, where the cells $\omega_{j,k}$ coincide (up to translations) with $(0, \ell_j) \times (0, h_j)$, where

$$(32) \quad \ell_j := \frac{L}{N2^j}, \quad h_j := \frac{(1-\theta)H}{2}\theta^j,$$

for some $\theta \in (1/4, 1/2)$. We refer to Figure 6 for an illustration of this. One produces a lamination (which doubles the frequency from the bottom to the top) in a reference rectangular cell $\omega = (0, \ell) \times (0, h)$, cf. Figure 6(A). This lamination is transferred on every $\omega_{j,k}$ via rescaling, then obtaining the global construction by attaching all the self-similar copies together, see Figure 6(B).

In particular for the constructed functions $u \in W^{1,\infty}(\Omega; \mathbb{R}^2)$, $\chi \in BV(\Omega; \{A, B\})$ it holds

$$(33) \quad \nabla u \in BV(\Omega; \mathbb{R}^{2 \times 2}), \quad \|\nabla u\|_{L^\infty} \leq C(\alpha, \Omega), \quad \|D\nabla u\|_{TV(\Omega)} \leq C(\alpha, \Omega)(\|D\chi\|_{TV(\Omega)} + \text{Per}(\Omega)).$$

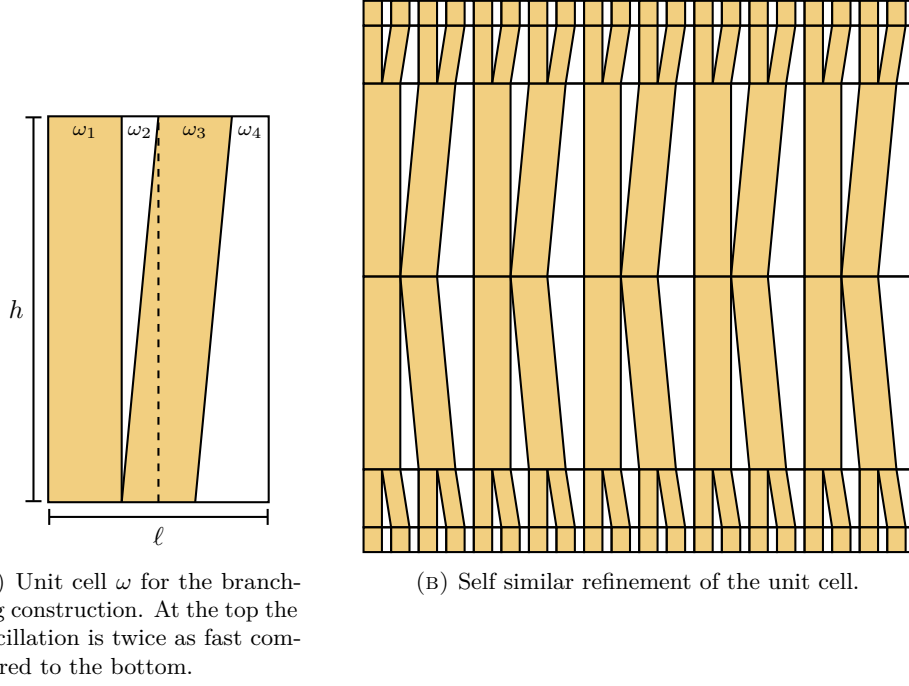


FIGURE 6. Branching construction in the unit cube.

In what follows, we now make the ν dependence explicit by considering the unit-cell construction in more detail. The interfaces of the optimal (in the sense of scaling) construction in ω (see [RT23b, Lemma 3.1] for details) are either given by a straight line with unit normal e_1 , or normal in direction $(-h, \frac{1-\alpha}{2}\ell)^T$, and thus, denoting the normal by n , in the first case it holds that $|n \cdot \nu| = |\nu_1|$ and in the second case

$$|n \cdot \nu| \leq C(\alpha)(|\nu_1| + \frac{\ell}{h}|\nu_2|).$$

Hence, through a scaling argument and by summing all the self-similar contributions of $\omega_{j,k}$, we get for sufficiently large $N \in \mathbb{N}$

$$(34) \quad \int_Q |\nabla u - \chi|^2 dx + \epsilon \|D_\nu \chi\|_{TV(Q)} \leq C(\alpha) \left(\frac{L^3}{N^2 H} + \epsilon H N |\nu_1| + \epsilon L j_0 |\nu_2| \right),$$

for

$$j_0 + 1 \sim \log \left(\frac{HN}{L} \right).$$

We now turn to the case $Q = \Omega$. Hence, for $H = L = 1$, after optimizing the first two terms in N , i.e. choosing $N \sim |\nu_1|^{-1/3} \epsilon^{-1/3}$, we get

$$\int_\Omega |\nabla u - \chi|^2 dx + \epsilon \|D_\nu \chi\|_{TV(\Omega)} \leq C(\alpha) \left(|\nu_1|^{\frac{2}{3}} \epsilon^{\frac{2}{3}} + \epsilon |\log(|\nu_1| \epsilon)| |\nu_2| \right).$$

Using that $\epsilon |\log(|\nu_1| \epsilon)| |\nu_2| \leq C |\nu_1|^{2/3} \epsilon^{2/3}$ for small $\epsilon < \epsilon_0(|\nu_1|)$, we deduce the desired upper bound. $\square$

Proof of Proposition 1.1 for $\nu \cdot e_1 = 0$. In the case of $\nu \cdot e_1 = 0$ the argument uses that for a simple laminate in e_1 direction, we do not pay surface energy, and thus can do an infinitely fine oscillation of the two phases. Indeed, $E_\epsilon(u, \chi) \geq 0$ is direct. To obtain a suitable upper bound on the energy, we now choose for any $N \in \mathbb{N}$ the functions

$$\tilde{u}_N(x_1) = \begin{cases} (1 - \alpha)x_1 a & x_1 \in (0, \frac{\alpha}{N}), \\ -\alpha x_1 a + \frac{\alpha}{N} a & x_1 \in [\frac{\alpha}{N}, \frac{1}{N}), \end{cases}$$

$$\chi_N(x) = \begin{cases} A & x_1 \in (0, \frac{\alpha}{N}), \\ B & x_1 \in [\frac{\alpha}{N}, \frac{1}{N}), \end{cases}$$

and extend both $\frac{1}{N}$ -periodically. We fix $u_N \in W^{1,\infty}(\Omega; \mathbb{R}^d)$ as

$$u_N(x_1, x') = \tilde{u}_N(x_1) \psi(N \operatorname{dist}(x', \partial(0, 1)^{d-1})) + F_\alpha x$$

for some bump function $\psi \in C^\infty(\mathbb{R}; [0, 1])$ such that $\psi(t) = 1$ for $t \geq 1$ and $\psi(t) = 0$ for $t \leq \frac{1}{2}$. Here $a \in \mathbb{R}^d \setminus \{0\}$ is given by the relation $A - B = a \otimes e_1$. We have that $\chi_N \in BV(\Omega; \{A, B\})$ and $u_N \in W^{1,\infty}(\Omega; \mathbb{R}^d)$ fulfils the Dirichlet data. Moreover, as $\nu \cdot e_1 = 0$, it holds

$$\int_{\Omega} \chi_N(x) \partial_\nu \phi(x) dx = \int_{(0,1)} \chi_N\left(x_1, \frac{1}{2}\right) \int_{(0,1)^{d-1}} \operatorname{div}'(\phi(x_1, x') \nu) dx' dx_1 = 0$$

for every $\phi \in C_c^1(\Omega; \mathbb{R})$, thus $\|D_\nu \chi\|_{TV(\Omega)} = 0$. Hence, we infer that

$$E_\epsilon(u_N, \chi_N) = \int_{\Omega} |\nabla u_N(x) - \chi_N(x)|^2 dx \leq \frac{C}{N}.$$

As the functions u and χ are admissible for any $N \in \mathbb{N}$, passing to the limit yields the optimal energy $\inf_{\chi \in BV_\nu(\Omega; \{A, B\})} \inf_{u \in \mathcal{A}_{F_\alpha}} E_\epsilon(u, \chi) = 0$. $\square$

3.2. The three-well problem of Lorent. In this section we now consider the three-well setting due to Lorent. As outlined in Section 1.1.2, in this situation one obtains both first and second order laminates, depending on the boundary condition.

We split the proof of Theorem 1 into several steps. We start by showing the scaling for $F \in \mathcal{K}_3^1$ for both $\nu \cdot e_1 \neq 0$ and $\nu \cdot e_1 = 0$, both of which are essentially covered by Proposition 1.1. In contrast to the argument given above, due to our specific choices of the possible boundary data, we do not only work with one of the entries of the matrix $\chi - F$ (see the comments in Remark 3.2 below).

In a second step, we will consider $F \in \mathcal{K}_3^2$ where we can exploit the determinedness of χ_{22} in terms of χ_{11} to obtain the scaling of second order laminates. Here as long as $\nu \cdot e_1 \neq 0$, we obtain a scaling law of the order $\epsilon^{\frac{1}{2}}$ as in the isotropic setting. If, however, $\nu \cdot e_1 = 0$, the scaling does not change with respect to the one for first order laminates, as we can construct an infinitely fine simple laminate within a branching construction, without paying surface energy for the simple laminate. This then yields the same scaling as in the two-well setting in Proposition 1.1, cf. Figure 7.

Lemma 3.1. *Under the same assumptions as in Theorem 1, let $F_\alpha = \operatorname{diag}(\alpha, 0) \in \mathcal{K}_3^1$ for $\alpha \in (0, 1)$. If $\nu \cdot e_1 \neq 0$ there is a constant $C = C(\alpha) > 0$ and $\epsilon_0 = \epsilon_0(\alpha, |\nu_1|) > 0$ such that*

$$C^{-1} |\nu_1|^{\frac{2}{3}} \epsilon^{\frac{2}{3}} \leq \inf_{\chi \in BV_\nu(\Omega; \mathcal{K}_3)} \inf_{u \in \mathcal{A}_{F_\alpha}} E_\epsilon(u, \chi) \leq C |\nu_1|^{\frac{2}{3}} \epsilon^{\frac{2}{3}}.$$

If instead $\nu \cdot e_1 = 0$, we have

$$\inf_{\chi \in BV_\nu(\Omega; \mathcal{K}_3)} \inf_{u \in \mathcal{A}_{F_\alpha}} E_\epsilon(u, \chi) = 0.$$

Proof. Upper bounds: As A_1 and A_2 are rank-one connected in e_1 direction, we directly note that the upper bound construction in the proof of Proposition 1.1 also yields an upper bound construction in this setting (after an adaptation of parameters). Thus, we have

$$\inf_{\chi \in BV_\nu(\Omega; \mathcal{K}_3)} \inf_{u \in \mathcal{A}_{F_\alpha}} E_\epsilon(u, \chi) \leq \begin{cases} C(\alpha) |\nu_1|^{\frac{2}{3}} \epsilon^{\frac{2}{3}}, & \nu \cdot e_1 \neq 0, \\ 0, & \nu \cdot e_1 = 0. \end{cases}$$

Lower bound for $\nu \cdot e_1 \neq 0$: For any $u \in \mathcal{A}_{F_\alpha}$, $\chi \in BV_\nu(\Omega; \mathcal{K}_3)$, by considering $\nabla u - F_\alpha$ and $\chi - F_\alpha$, we can reduce to the case $F_\alpha = 0$ (cf. proof of Proposition 1.1). With a slight abuse of notation, we still write u and χ for the modified functions, hence $u \in H_0^1(\Omega; \mathbb{R}^2)$ and $\chi \in \mathcal{K}_3 - F_\alpha$. For the function χ_{11} we exploit the same ideas as in the two-well case but will make use of both the χ_{11} and the χ_{22} components. To be more precise by Lemma 2.3 it holds

$$\int_{\Omega} |\nabla u - \chi|^2 dx \geq \sum_{k \in \mathbb{Z}^2} \left(\frac{|k|^2 - k_1^2}{|k|^2} |\hat{\chi}_{11}(k)|^2 + \frac{|k|^2 - k_2^2}{|k|^2} |\hat{\chi}_{22}(k)|^2 \right) \geq \sum_{k \in \mathbb{Z}^2} \frac{|k|^2 - k_1^2}{|k|^2} |\hat{\chi}_{11}(k)|^2,$$

where we fix the multipliers to be equal to one in $k = 0$. Moreover, it holds

$$\|D_\nu \chi\|_{TV(\Omega)} \geq \|D_\nu \chi_{11}\|_{TV(\Omega)},$$

and thus, following the ideas of Proposition 1.1, by Lemma 2.4, we deduce for $\lambda > 1$ and $\mu \in (0, \frac{|\nu_1|}{16})$, that

$$\begin{aligned} \|\chi_{11} - m_{1,\mu,\lambda}(D)\chi_{11}\|_{L^2}^2 + \|\chi_{22} - m_{1,\mu}(D)\chi_{22}\|_{L^2}^2 &\leq \sum_{k \notin C_{1,\mu,\lambda}} |\hat{\chi}_{11}(k)|^2 + \sum_{k \notin C_{2,\mu}} |\hat{\chi}_{22}(k)|^2 \\ &\leq C(\mu^{-2} + (\lambda\epsilon)^{-1})E_\epsilon(u, \chi) + C\lambda^{-1}\text{Per}(\Omega), \end{aligned}$$

where we consider the smooth cut-off multipliers as in (23). Applying moreover Corollary 2.7 with $\chi_{22} = -4(\chi_{11} + \alpha)(\chi_{11} - 1 + \alpha)$, yields for $\lambda_2 = M\mu\lambda < \lambda$

$$\begin{aligned} \|\chi_{22} - m_{2,\mu,\lambda_2}(D)\chi_{22}\|_{L^2}^2 &\leq C(\alpha)\psi_{1-\gamma}(\|\chi_{11} - m_{1,\mu,\lambda}(D)\chi_{11}\|_{L^2}^2) + 4\|\chi_{22} - m_{2,\mu}(D)\chi_{22}\|_{L^2}^2 \\ &\leq C(\alpha)\psi_{1-\gamma}\left((\mu^{-2} + (\lambda\epsilon)^{-1})E_\epsilon(u, \chi) + \lambda^{-1}\text{Per}(\Omega)\right), \end{aligned}$$

with $\psi_{1-\gamma}(x) = \max\{|x|, |x|^{1-\gamma}\}$.

In conclusion, using that $1 - m_{2,\mu,\lambda_2}(k) \geq 1 - m_{2,\mu,\lambda}(k) \geq 0$,

$$\begin{aligned} \|\chi_{11} - m_{1,\mu,\lambda}(D)\chi_{11}\|_{L^2}^2 + \|\chi_{22} - m_{2,\mu,\lambda}(D)\chi_{22}\|_{L^2}^2 \\ \leq C\psi_{1-\gamma}\left((\mu^{-2} + (\lambda\epsilon)^{-1})E_\epsilon(u, \chi) + \lambda^{-1}\text{Per}(\Omega)\right). \end{aligned}$$

Applying Lemma 2.5 for both χ_{11} and χ_{22} with the frequency cut-off given by $\bar{\lambda} = \frac{4}{|\nu_1|}\lambda$ we get

$$\begin{aligned} \|\chi_{11}\|_{L^2}^2 + \|\chi_{22}\|_{L^2}^2 &\leq 2\|\chi_{11} - m_{1,\mu,\lambda}(D)\chi_{11}\|_{L^2}^2 + 2 \sum_{|k_1| \leq \frac{4}{|\nu_1|}\lambda} |\hat{\chi}_{11}(k)|^2 + \\ &\quad + 2\|\chi_{22} - m_{2,\mu,\lambda}(D)\chi_{22}\|_{L^2}^2 + 2 \sum_{|k_2| \leq \frac{4}{|\nu_1|}\lambda} |\hat{\chi}_{22}(k)|^2 \\ &\leq C\psi_{1-\gamma}\left((\mu^{-2} + (\lambda\epsilon)^{-1})E_\epsilon(u, \chi) + \lambda^{-1}\text{Per}(\Omega)\right) \\ &\quad + C|\nu_1|^{-2}\lambda^2 E_\epsilon(u, \chi) \\ &\leq C\psi_{1-\gamma}\left((|\nu_1|^{-2}\lambda^2 + \mu^{-2} + (\lambda\epsilon)^{-1})E_\epsilon(u, \chi) + \lambda^{-1}\text{Per}(\Omega)\right). \end{aligned}$$

Fixing $\mu^{-2} \sim |\nu_1|^{-2}\lambda^2$ and $\lambda \sim |\nu_1|^{2/3}\epsilon^{-1/3}$ (which are compatible with the constraints $\mu < \frac{|\nu_1|}{16}$ and $\bar{\lambda} > 1$) yields

$$(35) \quad \|\chi_{11}\|_{L^2}^2 + \|\chi_{22}\|_{L^2}^2 \leq C\psi_{1-\gamma} \left(|\nu_1|^{-\frac{2}{3}}\epsilon^{-\frac{2}{3}}E_\epsilon(u, \chi) + |\nu_1|^{-\frac{2}{3}}\epsilon^{\frac{1}{3}}\text{Per}(\Omega) \right).$$

By the fact that $\|\chi_{11}\|_{L^2}^2 + \|\chi_{22}\|_{L^2}^2 = \|\chi\|_{L^2}^2 \geq c > 0$, we can show the desired lower bound for $\epsilon < \epsilon_0(|\nu_1|, \alpha)$. Indeed, we have either

$$C \leq |\nu_1|^{-\frac{2}{3}}\epsilon^{-\frac{2}{3}}E_\epsilon(u, \chi) + |\nu_1|^{-\frac{2}{3}}\epsilon^{\frac{1}{3}}\text{Per}(\Omega),$$

or

$$C^{\frac{1}{1-\gamma}} \leq |\nu_1|^{-\frac{2}{3}}\epsilon^{-\frac{2}{3}}E_\epsilon(u, \chi) + |\nu_1|^{-\frac{2}{3}}\epsilon^{\frac{1}{3}}\text{Per}(\Omega),$$

depending on the case distinction in $\psi_{1-\gamma}$. In conclusion, after absorbing the perimeter term, we arrive at

$$E_\epsilon(u, \chi) \geq \frac{1}{2} \min\{C, C^{\frac{1}{1-\gamma}}\} |\nu_1|^{\frac{2}{3}}\epsilon^{\frac{2}{3}}.$$

□

Remark 3.2. Let us comment on a technical aspect which is specific to our anisotropic surface energies and which does not arise in this form in the isotropic setting. We observe that for specific choices of $F_\alpha \in \mathcal{K}_3^1$ (both in the isotropic and anisotropic settings) it may happen that χ_{11} vanishes. Indeed, we recall that $\chi \in \mathcal{K}_3 - F_\alpha = \{\text{diag}(-\alpha, 0), \text{diag}(1-\alpha, 0), \text{diag}(1/2-\alpha, 1)\}$ for $\alpha \in (0, 1)$ and, hence, $\chi_{11} = 0$ can occur for $\alpha = 1/2$. For this reason, in the above proof, we also used χ_{22} in the lower bound, in order to deduce a uniform lower bound in (35). To this end, in the above argument, we used a commutator estimate already for first order laminates (while in the isotropic setting commutators only enter for second and higher order laminates). In the setting of [RT23b] (while also here $\chi_{11} = 0$ may arise) it is not necessary to use a commutator estimate of the form Lemma 2.6 for first order laminates, as – due to the isotropy of the surface energy – in that article the first high frequency localization truncates the cones in all directions. In our setting this truncation is not present due to the anisotropy of the surface energy and we, hence, require an extra step to show the lower bound.

As an alternative argument, also in our anisotropic setting, one could have avoided an application of the commutator estimate at the expense of using further information on the χ_{22} component. Indeed, one could have restricted to a bound of the form

$$(36) \quad \|\chi_{11}\|_{L^2}^2 \leq C\psi_{1-\gamma} \left(|\nu_1|^{-\frac{2}{3}}\epsilon^{-\frac{2}{3}}E_\epsilon(u, \chi) + |\nu_1|^{-\frac{2}{3}}\epsilon^{\frac{1}{3}}\text{Per}(\Omega) \right),$$

and then, in a second step, invoked information on the average of χ_{22} . To this end, we note that by Jensen's inequality, it holds that

$$|\langle \nabla u \rangle_\Omega - \langle \chi \rangle_\Omega|^2 \leq E_{el}(u, \chi),$$

where $\langle \cdot \rangle_\Omega$ denotes the average on Ω . In particular, we here use that by the imposed boundary conditions and the fundamental theorem of calculus, $\langle \nabla u \rangle_\Omega = F_\alpha$ and that $F_{22} = 0$ for all $F \in \mathcal{K}_3^1$.

Hence, considering the second component we have

$$|\{x \in \Omega : \chi(x) = A_3\}|^2 \leq |\langle \chi_{22} \rangle_\Omega|^2 \leq |F_\alpha - \langle \chi \rangle_\Omega|^2 \leq E_{el}(u, \chi).$$

In particular,

$$\begin{aligned} \|\chi_{11}\|_{L^2}^2 &\geq \min\{\alpha^2, (1-\alpha)^2\} |\{x \in \Omega : \chi \neq A_3\}| \geq C(\alpha)(|\Omega| - |\{x \in \Omega : \chi = A_3\}|) \\ &\geq C(\alpha)(|\Omega| - E_{el}(u, \chi)^{\frac{1}{2}}). \end{aligned}$$

Returning to (36) with this additional information and rearranging the inequality, one then infers that

$$c \leq \|\chi_{11}\|_{L^2}^2 + CE_{el}(u, \chi)^{\frac{1}{2}} \leq C\psi_{1-\gamma} \left(|\nu_1|^{-\frac{2}{3}} \epsilon^{-\frac{2}{3}} E_\epsilon(u, \chi) + |\nu_1|^{-\frac{2}{3}} \epsilon^{\frac{1}{3}} \text{Per}(\Omega) \right) + CE_{el}(u, \chi)^{\frac{1}{2}},$$

which also concludes the argument after an absorption of the perimeter and additional elastic energy terms.

Turning now to the second order laminates, we consider

$$F = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \alpha \end{pmatrix} \in \mathcal{K}_3^2$$

for some $\alpha \in (0, 1)$ and note that $\chi_{22} - F_{22} \in \{-\alpha, 1 - \alpha\} \not\equiv 0$. Thus, we now aim to control the Fourier mass of χ_{22} .

Lemma 3.3. *Under the same assumptions as in Theorem 1, let $F_\alpha = \text{diag}(\frac{1}{2}, \alpha) \in \mathcal{K}_3^2$ for $\alpha \in (0, 1)$. If $\nu \cdot e_1 \neq 0$ there are constants $C = C(\alpha) > 0$ and $\epsilon_0 = \epsilon_0(\alpha, |\nu_1|) > 0$ such that for any $\epsilon \in (0, \epsilon_0)$*

$$C^{-1} |\nu_1|^{\frac{1}{2}} \epsilon^{\frac{1}{2}} \leq \inf_{u \in \mathcal{A}_{F_\alpha}} \inf_{\chi \in BV_\nu(\Omega; \mathcal{K}_3)} E_\epsilon(u, \chi) \leq C |\nu_1|^{\frac{1}{2}} \epsilon^{\frac{1}{2}}.$$

If $\nu \cdot e_1 = 0$, we have with $C = C(\alpha) > 0$ and $\epsilon_0 = \epsilon_0(\alpha, |\nu_2|) > 0$ for all $\epsilon \in (0, \epsilon_0)$

$$C^{-1} |\nu_2|^{\frac{2}{3}} \epsilon^{\frac{2}{3}} \leq \inf_{u \in \mathcal{A}_{F_\alpha}} \inf_{\chi \in BV_\nu(\Omega; \mathcal{K}_3)} E_\epsilon(u, \chi) \leq C |\nu_2|^{\frac{2}{3}} \epsilon^{\frac{2}{3}}.$$

Again, we split the proof into two parts, first for $\nu \cdot e_1 \neq 0$ and the second part for $\nu \cdot e_1 = 0$.

Proof for $\nu \cdot e_1 \neq 0$. Upper bound: As for the first-order branching construction, the claimed upper scaling bound in ϵ is already known (cf. [KW16, RT23b]). We just need to focus on the dependence of the prefactor on ν . In particular, we will follow the strategy of the proof of [RT23b, Thm. 1.2 (ii)]. This consists in concatenating two orders of branching constructions; an outer one between gradients A_3 and $\text{diag}(1/2, 0)$, and then replacing the regions in which $\nabla u \approx \text{diag}(1/2, 0)$ with an inner branched lamination between A_1 and A_2 .

Let $\{\omega_{j,k}\}$ be the (outer) first-order branching covering as in the Step 2 of the proof of Proposition 1.1 (now with switched roles between x_1 and x_2 because of the structure of the wells). In each cell $\omega_{j,k}$ we can produce an inner branching construction (see [RT23b] for details) so that estimate (34) applies as follows

$$\int_{\omega_{j,k}} |\nabla u - \chi|^2 dx + \epsilon \|D_\nu \chi\|_{TV(\omega_{j,k})} \leq C(\alpha) \left(\frac{h_j^3}{M^2 \ell_j} + \epsilon \ell_j (M |\nu_1| + k_0 |\nu_2|) \right),$$

where $M \in \mathbb{N}$, $M \sim (2\theta)^j N^2$ denotes the number of oscillations of the zeroth generation of this inner branching construction (the dependence of M on j is dropped for notational simplicity) and $k_0 \sim \log(N)$. Summing this for every k and every (outer) generation j and by also adding the surface energy term of the outer branching construction (which comes by attaching the $\omega_{j,k}$ -cells together), by the relations (32) we obtain

$$\int_\Omega |\nabla u - \chi|^2 dx + \epsilon \|D_\nu \chi\|_{TV(\Omega)} \leq C(\alpha) \left(\frac{1}{N^2} + \epsilon (N^2 + j_0) |\nu_1| + \epsilon (k_0 + N) |\nu_2| \right).$$

Optimizing as $N \sim (\epsilon |\nu_1|)^{-\frac{1}{4}}$ we deduce the claimed upper scaling bound. Due to the construction, the bounds from (33) still hold for u and χ .

Lower bound: Analogously as in the previous proofs, by subtracting the boundary conditions we can assume that $u \in H_0^1(\Omega; \mathbb{R}^2)$ and $\chi \in \mathcal{K}_3 - F_\alpha$. For the readers' convenience we recall the (truncated) cones from (21) and (22): for $\mu \in (0, \frac{|\nu_1|}{16})$, $\lambda > 0$, $j = 1, 2$

$$C_{j,\mu,\lambda} = \{k \in \mathbb{Z}^d : |k|^2 - k_j^2 \leq \mu^2 |k|^2, |k_j| \leq \frac{2}{|\nu_1|} \lambda\}, \quad C_{2,\mu} = \{k \in \mathbb{Z}^d : |k|^2 - k_2^2 \leq \mu^2 |k|^2\}.$$

For the multipliers defined in (23) we infer by Lemma 2.4

$$\begin{aligned} \|\chi_{11} - m_{1,\mu,\lambda}(D)\chi_{11}\|_{L^2}^2 + \|\chi_{22} - m_{2,\mu}(D)\chi_{22}\|_{L^2}^2 &\leq \sum_{k \notin C_{1,\mu,\lambda}} |\hat{\chi}_{11}(k)|^2 + \sum_{k \notin C_{2,\mu}} |\hat{\chi}_{22}(k)|^2 \\ &\leq C(\mu^{-2} + (\lambda\epsilon)^{-1})E_\epsilon(u, \chi) + C\lambda^{-1}\text{Per}(\Omega). \end{aligned}$$

Here the constant $C > 0$ only depends on α .

Using that $\chi_{22} = 1 - \alpha - 4\chi_{11}^2$, we deduce from an application of Corollary 2.7 with $\lambda_2 = M\mu\lambda$ for any $\gamma \in (0, 1)$

$$(37) \quad \|\chi_{22} - m_{2,\mu,\lambda_2}(D)\chi_{22}\|_{L^2}^2 \leq C(\alpha)\psi_{1-\gamma}(\|\chi_{11} - m_{1,\mu,\lambda}(D)\chi_{11}\|_{L^2}^2) + 4\|\chi_{22} - m_{2,\mu}(D)\chi_{22}\|_{L^2}^2,$$

with $\psi_t(x) = \max\{|x|^t, |x|\}$ for $t > 0$.

Now we use this estimate on $\chi_{22} - m_{2,\mu,\lambda_2}(D)\chi_{22}$ to improve the lower bound in comparison to the two-well case. Using Lemma 2.5 for χ_{22} , we get for $\bar{\lambda} = \frac{4}{|\nu_1|}\lambda_2$

$$(38) \quad \sum_{|k_2| \leq 4\lambda_2/|\nu_1|} |\hat{\chi}_{22}(k)|^2 \leq C \frac{\lambda_2^2}{|\nu_1|^2} E_{el}(u, \chi).$$

Thus, as in the proof of the two-well problem in Proposition 1.1, the idea is to combine (37) and (38) to have an estimate of $\|\chi_{22}\|_{L^2}^2$ in terms of the energy depending on the parameters μ, λ_2 and then to optimize in these parameters. To be precise, we have

$$\begin{aligned} \|\chi_{22}\|_{L^2}^2 &\leq 2 \sum_{k \in C_{2,2\mu,2\lambda_2}} |\hat{\chi}_{22}(k)|^2 + 2\|\chi_{22} - m_{2,\mu,\lambda_2}(D)\chi_{22}\|_{L^2}^2 \\ &\leq 2 \sum_{|k_2| \leq 4\lambda_2/|\nu_1|} |\hat{\chi}_{22}(k)|^2 + 2\|\chi_{22} - m_{2,\mu,\lambda_2}(D)\chi_{22}\|_{L^2}^2. \end{aligned}$$

Thus, plugging in (37) and (38), yields with $\lambda_2 = M\mu\lambda$ and a constant $C = C(\alpha) > 0$

$$\begin{aligned} \|\chi_{22}\|_{L^2}^2 &\leq C|\nu_1|^{-2}\lambda_2^2 E_{el}(u, \chi) + C\psi_{1-\gamma}(\|\chi_{11} - m_{1,\mu,\lambda}(D)\chi_{11}\|_{L^2}^2) + C\|\chi_{22} - m_{2,\mu}(D)\chi_{22}\|_{L^2}^2 \\ &\leq C \left(|\nu_1|^{-2}\lambda_2^2 E_\epsilon(u, \chi) + \psi_{1-\gamma} \left((\mu^{-2} + (\lambda\epsilon)^{-1})E_\epsilon(u, \chi) + \lambda^{-1}\text{Per}(\Omega) \right) \right) \\ &\leq C\psi_{1-\gamma} \left((|\nu_1|^{-2}\mu^2\lambda^2 + \mu^{-2} + (\lambda\epsilon)^{-1})E_\epsilon(u, \chi) + \lambda^{-1}\text{Per}(\Omega) \right). \end{aligned}$$

As $\|\chi_{22}\|_{L^2}^2 \geq C \min\{\alpha^2, (1-\alpha)^2\} > 0$, fixing $\mu^{-2} \sim |\nu_1|^{-1}\lambda$ and $\lambda \sim |\nu_1|^{1/2}\epsilon^{-1/2}$, we argue as in the proof of Lemma 3.1, i.e. considering the two cases for $\psi_{1-\gamma}$ and carrying out an absorption argument for the perimeter term, we arrive at

$$E_\epsilon(u, \chi) \geq \frac{1}{2} \min\{C, C^{\frac{1}{1-\gamma}}\} |\nu_1|^{\frac{1}{2}} \epsilon^{\frac{1}{2}}.$$

□

Proof for $\nu \cdot e_1 = 0$. Upper bound: We construct a (cut-off) simple laminate of A_1 and A_2 within a branching construction using A_3 . See Figure 7 for an illustration of a laminate within a branching

construction. As a first-order branching construction has been already explained in the proof of Proposition 1.1, we only give an outline of the argument.

Considering a reference cell $\omega = (0, h) \times (0, \ell)$ with $0 < \ell < h \leq 1$, we decompose this into further subdomains given by

$$\begin{aligned}\omega_1 &= \{(x_1, x_2) \in \omega : x_2 \in (0, (1-\alpha)\frac{\ell}{2})\}, \\ \omega_2 &= \{(x_1, x_2) \in \omega : x_2 \in [(1-\alpha)\frac{\ell}{2}, (1-\alpha)\frac{\ell}{2} + \alpha\frac{\ell}{2}\frac{x_1}{h}]\}, \\ \omega_3 &= \{(x_1, x_2) \in \omega : x_2 \in [(1-\alpha)\frac{\ell}{2} + \alpha\frac{\ell}{2}\frac{x_1}{h}, (1-\alpha)\ell + \alpha\frac{\ell}{2}\frac{x_1}{h}]\}, \\ \omega_4 &= \{(x_1, x_2) \in \omega : x_2 \in [(1-\alpha)\ell + \alpha\frac{\ell}{2}\frac{x_1}{h}, \ell]\}.\end{aligned}$$

For $r < h$ such that $\frac{h}{r} \in \mathbb{N}$ and a bump function $\phi \in C^\infty(\mathbb{R}; [0, 1])$ with $\phi(t) = 0$ for $t \leq 0$ and $\phi(t) = 1$ for $t \geq 1$, we then define the continuous function

$$\tilde{u}(x_1, x_2) = \begin{cases} \begin{pmatrix} \text{Lam}_r(x_1)\phi(\frac{x_2}{r})\phi(\frac{(1-\alpha)\frac{\ell}{2}-x_2}{r}) \\ -\alpha x_2 \end{pmatrix}, & (x_1, x_2) \in \omega_1, \\ \begin{pmatrix} 0 \\ (1-\alpha)x_2 - (1-\alpha)\frac{\ell}{2} \end{pmatrix}, & (x_1, x_2) \in \omega_2, \\ \begin{pmatrix} \text{Lam}_r(x_1)\phi(\frac{x_2-(1-\alpha)\frac{\ell}{2}-\alpha\frac{\ell}{2}\frac{x_1}{h}}{r})\phi(\frac{(1-\alpha)\ell+\alpha\frac{\ell}{2}\frac{x_1}{h}-x_2}{r}) \\ -\alpha x_2 + \alpha\frac{\ell}{2}\frac{x_1}{h} \end{pmatrix}, & (x_1, x_2) \in \omega_3, \\ \begin{pmatrix} 0 \\ (1-\alpha)x_2 - (1-\alpha)\ell \end{pmatrix}, & (x_1, x_2) \in \omega_4. \end{cases}$$

Here we used

$$\text{Lam}_r(t) = \begin{cases} -\frac{1}{2}t, & t \in [0, \frac{r}{2}), \\ \frac{1}{2}t - \frac{r}{2}, & t \in [\frac{r}{2}, r), \end{cases}$$

and extended it r -periodically. Setting also

$$\tilde{\chi}(x_1, x_2) = \begin{cases} \begin{pmatrix} \text{Lam}'_r(x_1) & 0 \\ 0 & -\alpha \end{pmatrix}, & (x_1, x_2) \in \omega_1 \cup \omega_3, \\ \begin{pmatrix} 0 & 0 \\ 0 & 1-\alpha \end{pmatrix}, & (x_1, x_2) \in \omega_2 \cup \omega_4, \end{cases}$$

we can calculate the energy contribution of $u(x) = \tilde{u}(x) + F_\alpha x$ and $\chi = \tilde{\chi} + F_\alpha$. Note that $F_\alpha = \text{diag}(1/2, \alpha)$. We have

$$\int_\omega |\nabla u(x) - \chi(x)|^2 dx = \int_\omega |\nabla \tilde{u}(x) - \tilde{\chi}(x)|^2 dx \leq C(\alpha)(rh + \frac{\ell^3}{h}),$$

where the first term is determined by the size of the cut-off areas in ω_1 and ω_2 (i.e., by the inner laminate), and the second term is due to the error we make by adjusting the interfaces away from e_2 to achieve the refinement in e_1 direction (i.e., the usual elastic energy originating from branching). Moreover, as we do not penalize oscillation in the e_1 direction, it holds

$$\|D_{e_2}\chi\|_{TV(\omega)} + \text{Per}(\omega) = \|D_{e_2}\tilde{\chi}\|_{TV(\omega)} + \text{Per}(\omega) \leq C(\alpha)h,$$

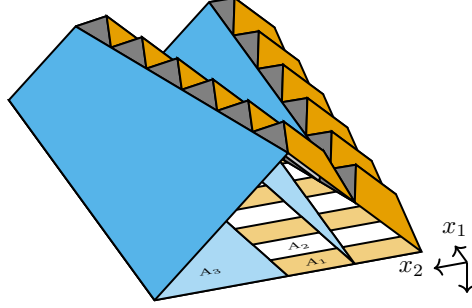


FIGURE 7. Illustration of a laminate within a branching with boundary condition $F_\alpha = 0$. This does not depict the exact same situation as in Theorem 1 for $\nu = e_2$, as here we only show a scalar-valued map.

and thus in total

$$\int_{\omega} |\nabla u(x) - \chi(x)|^2 dx + \epsilon \|D_{e_2} \chi\|_{TV(\omega)} + \epsilon \text{Per}(\omega) \leq C(\alpha)(rh + \frac{\ell^3}{h} + \epsilon h).$$

As for any $h > 0$ we can choose $r < \epsilon$ the contribution of the rh term is negligible. Hence,

$$\int_{\omega} |\nabla u(x) - \chi(x)|^2 dx + \epsilon \|D_{e_2} \chi\|_{TV(\omega)} + \epsilon \text{Per}(\omega) \leq C(\alpha)(\frac{\ell^3}{h} + \epsilon h).$$

This is the same energy contribution as for the standard branching construction, thus concluding as in the proof of Proposition 1.1 (see also [RT23b, Sec. 3]), we get

$$\inf_{\chi \in BV_\nu(\Omega; \mathcal{K}_3)} \inf_{u \in \mathcal{A}_{F_\alpha}} \int_{(0,1)^2} |\nabla u - \chi|^2 dx + \epsilon \|D_{e_2} \chi\|_{TV((0,1)^2)} \leq C(\alpha)\epsilon^{\frac{2}{3}}.$$

As it is used below in the proof of Corollary 1.5, we emphasize that the above constructed function satisfies that $u \in W^{1,\infty}(\Omega; \mathbb{R}^2)$, $\chi \in BV(\Omega; \mathcal{K}_3)$ with

$$\nabla u \in BV(\Omega; \mathbb{R}^{2 \times 2}), \quad \|D \nabla u\|_{TV(\Omega)} \leq C(\alpha)(\|D \chi\|_{TV(\Omega)} + \text{Per}(\Omega)),$$

and thus the bounds from (33) also hold in this setting.

Lower bound: Again, by considering $\mathcal{K}_3 - F_\alpha$, we reduce to the case of zero boundary conditions. In the case of $\nu \cdot e_1 = 0$ we have $\nu \cdot e_2 \neq 0$, instead of the high frequency control in k_1 , we consider a high frequency control in k_2 for χ_{22} . Indeed, let $\mu < \frac{|\nu_2|}{16}$, $\lambda > 0$, as in (25) we have

$$C_{2,\mu,\lambda} := \{k \in \mathbb{Z}^2 : |k|^2 - k_2^2 \leq \mu^2 |k|^2, |k_2| \leq \frac{2}{|\nu_2|} \lambda\} \supset \{k \in \mathbb{Z}^2 : |k|^2 - k_1^2 \leq \mu^2 |k|^2, |k \cdot \nu| \leq \lambda\}$$

and thus we can bound the Fourier mass of χ_{22} outside of $C_{2,\mu,\lambda}$ in terms of the energy, cf. Lemma 2.4,

$$\sum_{k \notin C_{2,\mu,\lambda}} |\hat{\chi}_{22}(k)|^2 \leq C(\mu^{-2} + (\lambda\epsilon)^{-1}) E_\epsilon(u, \chi) + C\lambda^{-1} \text{Per}(\Omega),$$

with a constant $C = C(\alpha) > 0$. Notice that the definition of the cone is analogous to that in (21), but different as the second direction plays the role of the first. This is not the cone given in (21) for $j = 2$, as there the truncation is dependent on the parameter $|\nu_1|$, here on $|\nu_2|$. Following now the proof of Proposition 1.1 for $f = \chi_{22}$ with a suitable change of coordinates, we get

$$E_\epsilon(u, \chi) \geq C|\nu_2|^{\frac{2}{3}} \epsilon^{\frac{2}{3}}.$$

□

With Lemmata 3.1 and 3.3 proved, we combine their estimates to deduce the desired lower bound in Theorem 1.

Conclusion of Theorem 1. We finally combine the above discussion by rewriting the lower bound estimate in a concise way. We notice that for $\nu_1 = 0$ there is nothing to prove as cases (i) and (ii) are given by Lemmata 3.1 and 3.3 respectively. If $\nu_1 \neq 0$, we choose ϵ_0 to be small depending on ν , i.e. fix ϵ_0 to fulfil

$$|\nu_2|^{\frac{2}{3}} \epsilon_0^{\frac{2}{3}} \leq |\nu_1|^{\frac{1}{2}} \epsilon_0^{\frac{1}{2}}.$$

With this choice, for every $\epsilon < \epsilon_0$, it holds

$$\inf_{u \in \mathcal{A}_F} \inf_{\chi \in BV_\nu(\Omega; \mathcal{K}_3)} E_\epsilon(u, \chi) \geq C |\nu_1|^{\frac{1}{2}} \epsilon^{\frac{1}{2}} \geq \frac{C}{2} \left(|\nu_1|^{\frac{1}{2}} \epsilon^{\frac{1}{2}} + |\nu_2|^{\frac{2}{3}} \epsilon^{\frac{2}{3}} \right).$$

The upper bound follows by adding the two upper bounds from Lemmata 3.1 and 3.3. $\square$

Remark 3.4. *As can be seen in the above proof of Lemma 3.3 (and later analogously in the proof of Theorem 2), it would be possible to deduce the above scaling behaviour for $\nu \cdot e_1 \neq 0$ also for an even more degenerate anisotropic surface energy of the following type*

$$E_{\text{surf}}^r(\chi) = \|D_\nu(\chi : r)\|_{TV(\Omega)},$$

for $r \in \mathbb{R}^{d \times d}$ such that $|r| = 1$ and

$$\|D_\nu(\chi : r)\|_{TV(\Omega)} \geq C(\mathcal{K}, d, r) \|D_\nu \chi_{11}\|_{TV(\Omega)}.$$

Here we denoted the Frobenius scalar product of matrices by $\chi : r = \sum_{i,j=1}^d \chi_{ij} r_{ij}$. A sufficient condition on $r \in \mathbb{R}^{d \times d}$ is the following:

$$(A_j)_{11} \neq (A_k)_{11} \text{ if and only if } A_j : r \neq A_k : r.$$

For the three-wells from $\mathcal{K}_3$ in (4) this translates into the condition that

$$r \notin \{r_{11} = 0\} \cup \{r_{11} = 2r_{22}\} \cup \{r_{11} = -2r_{22}\}.$$

3.3. Higher order laminates. We next turn to the scaling behaviour of higher order laminates. Due to the anisotropic energies, we cannot immediately invoke the argument from [RT23b]. As an important technical novelty, we have to treat the relevant nonlinear relations between the components of the phase indicator substantially more carefully. This is due to the fact that, initially, our surface energy (potentially) only controls high frequencies in the k_1 direction. Thus, particularly the first step in the localization is crucial, as this will iteratively allow us the further high frequency reduction steps in the other conical directions.

As in the setting of the three-well problem of Lorent in the previous section, the key idea is to use that χ_{11} determines the other diagonal entries. In [RT23b] χ_{22} is written as a nonlinear polynomial of a combination of the remaining diagonal entries $\chi_{11}, \chi_{33}, \dots, \chi_{N-1, N-1}$. In the case of $\nu = e_1$, if only such a relation were available, this would lead to an issue with our argument as none of the cones in the $e_3, \dots, e_{N-1}$ directions is initially localized in the high frequencies (due to our anisotropic energies). The central novel idea is to rely on additional structure: More precisely, we will instead use that χ_{22} is given by a *nonlinear* polynomial of χ_{11} plus a *linear* one in $\chi_{33}, \dots, \chi_{N-1, N-1}$.

To elaborate on the strategy of the proof of Theorem 2, let us analyse the four well setting in three dimensions in the case of $\nu \cdot e_1 \neq 0$ first. Let $N = 4$ and $d = 3$, and thus consider $\mathcal{K}_4 = \{A_1, A_2, A_3, A_4\}$ with

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We will subsequently generalize the lower bound to the family of wells from Section 1.1.3.

Proposition 3.5. *Let $\alpha \in (0, 1)$ and $F_\alpha = \text{diag}(1/2, 1/2, \alpha) \in \mathcal{K}_4^3$. Let $\nu \in \mathbb{S}^2$ with $\nu \cdot e_1 \neq 0$. There exist constants $C = C(\alpha) > 0$ and $\epsilon_0 = \epsilon_0(\alpha, |\nu_1|) > 0$ such that for any $\epsilon \in (0, \epsilon_0)$*

$$\inf_{u \in \mathcal{A}_{F_\alpha}} \inf_{\chi \in BV_\nu(\Omega; \mathcal{K}_4)} E_\epsilon(u, \chi) \geq C |\nu_1|^{\frac{2}{5}} \epsilon^{\frac{2}{5}},$$

where $\mathcal{A}_{F_\alpha}$ is given in (3).

Let us outline the strategy of proof for the derivation of the lower bound in Proposition 3.5. To show this result, we first use the nonlinear relation for $\tilde{\chi} = \chi - F_\alpha : \Omega \rightarrow \mathcal{K}_3 - F_\alpha$

$$(39) \quad 2\tilde{\chi}_{22} + \tilde{\chi}_{33} = 1 - 8\tilde{\chi}_{11}^2 - \alpha.$$

After a first application of the first localization Lemma 2.4 with

$$C_{1,\mu,\lambda} = \{k \in \mathbb{Z}^3 : |k|^2 - k_1^2 \leq \mu^2 |k|^2, |k_1| \leq \frac{2}{|\nu_1|} \lambda\}, \quad C_{j,\mu} = \{k \in \mathbb{Z}^3 : |k|^2 - k_j^2 \leq \mu^2 |k|^2\},$$

for $\mu < \frac{|\nu_1|}{16}$, $\lambda > 0$ and $j = 2, 3$ and their smoothed out multipliers, cf. (23), we have

$$\begin{aligned} & \|\tilde{\chi}_{11} - m_{1,\mu,\lambda}(D)\tilde{\chi}_{11}\|_{L^2}^2 + \sum_{j=2,3} \|\tilde{\chi}_{jj} - m_{j,\mu}(D)\tilde{\chi}_{jj}\|_{L^2}^2 \\ & \leq C(\alpha)(\mu^{-2} + (\lambda\epsilon)^{-1})E_\epsilon(u, \chi) + C(\alpha)\lambda^{-1}\text{Per}(\Omega). \end{aligned}$$

Using the above nonlinear relation (39), we aim to truncate $C_{2,\mu}$ and $C_{3,\mu}$ by means of Corollary 2.7.

Proof of Proposition 3.5. Without loss of generality, by subtracting F_α , we assume $F_\alpha = 0$ and $\chi \in \mathcal{K}_4 - F_\alpha$. We observe that differently from the situation in Remark 3.4, it suffices to only deduce a lower bound for χ_{33} . We aim to use Corollary 2.7. After an application of the first localization Lemma 2.4 for $\mu < \frac{|\nu_1|}{16}$, $\lambda > 0$ and a constant $C = C(\alpha) > 0$

$$(40) \quad \begin{aligned} & \|\chi_{11} - m_{1,\mu,\lambda}(D)\chi_{11}\|_{L^2}^2 + \sum_{j=2,3} \|\chi_{jj} - m_{j,\mu}(D)\chi_{jj}\|_{L^2}^2 \\ & \leq C(\mu^{-2} + (\lambda\epsilon)^{-1})E_\epsilon(u, \chi) + C\lambda^{-1}\text{Per}(\Omega). \end{aligned}$$

To achieve the energy bound, we use Corollary 2.7 to truncate the cone corresponding to χ_{22} , which in turn is used to reduce the size of $C_{3,\mu}$ even further. In detail we apply Corollary 2.7 compounded with (40) for $2\chi_{22} + \chi_{33} = 1 - \alpha - 8\chi_{11}^2$ and get for $\lambda_2 = M\mu\lambda$

$$(41) \quad \|\chi_{22} - m_{2,\mu,\lambda_2}(D)\chi_{22}\|_{L^2}^2 \leq C\psi_{1-\gamma}\left((\mu^{-2} + (\lambda\epsilon)^{-1})E_\epsilon(u, \chi) + C\lambda^{-1}\text{Per}(\Omega)\right).$$

Here we used the truncated cone C_{2,μ,λ_2} defined in (21) with the corresponding smooth multiplier defined in (23).

Iterating now the comparison argument, using $\chi_{33} = 1 - \alpha - 4\chi_{22}^2$ we can improve the estimate on the Fourier mass of χ_{33} in the sense that we can truncate the corresponding cone $C_{3,\mu}$ on a scale $\lambda_3 = M\mu\lambda_2 = M^2\mu^2\lambda$. Indeed, by a variant of the commutation estimate from Corollary 2.7 (see Remark 2.8) and (41)

$$\begin{aligned} \|\chi_{33} - m_{3,\mu,\lambda_3}(D)\chi_{33}\|_{L^2}^2 & \leq C\psi_{1-\gamma}\left(\|\chi_{22} - m_{2,\mu,\lambda_2}(D)\chi_{22}\|_{L^2}^2\right) + 4\|\chi_{33} - m_{3,\mu}(D)\chi_{33}\|_{L^2}^2 \\ & \leq C\psi_{(1-\gamma)^2}\left((\mu^{-2} + (\lambda\epsilon)^{-1})E_\epsilon(u, \chi) + \lambda^{-1}\text{Per}(\Omega)\right). \end{aligned}$$

As above, the multiplier is of the form as in (23) for the truncated cone C_{3,μ,λ_3} defined in (21).

By Lemma 2.5 with $\bar{\lambda} = 4\lambda_3/|\nu_1|$ we can control the “missing” (low frequency) region and infer

$$\begin{aligned} \|\chi_{33}\|_{L^2}^2 &\leq 2\|\chi_{33} - m_{3,\mu,\lambda_3}\chi_{33}\|_{L^2}^2 + 2 \sum_{|k_3| \leq 4\lambda_3/|\nu_1|} |\hat{\chi}_{33}(k)|^2 \\ &\leq C\psi_{(1-\gamma)^2} \left((\mu^{-2} + (\lambda\epsilon)^{-1})E_\epsilon(u, \chi) + \lambda^{-1}\text{Per}(\Omega) \right) + C \frac{\lambda_3^2}{|\nu_1|^2} E_\epsilon(u, \chi). \end{aligned}$$

We now choose the parameters μ and λ_3 , that is, λ , in an optimal way. To be more precise, we fix $\mu^{-1} \sim \frac{\lambda_3}{|\nu_1|} \sim \frac{\mu^2 \lambda}{|\nu_1|}$, i.e. $\mu \sim |\nu_1|^{1/3} \lambda^{-1/3}$ and get

$$\begin{aligned} \|\chi_{33}\|_{L^2}^2 &\leq C\psi_{(1-\gamma)^2} \left((|\nu_1|^{-2/3} \lambda^{2/3} + (\lambda\epsilon)^{-1})E_\epsilon(u, \chi) + \lambda^{-1}\text{Per}(\Omega) \right) + |\nu_1|^{-2/3} \lambda^{2/3} E_\epsilon(u, \chi) \\ &\leq C\psi_{(1-\gamma)^2} \left((|\nu_1|^{-2/3} \lambda^{2/3} + (\lambda\epsilon)^{-1})E_\epsilon(u, \chi) + \lambda^{-1}\text{Per}(\Omega) \right). \end{aligned}$$

Optimizing the energy contributions in λ by choosing $\lambda \sim |\nu_1|^{2/5} \epsilon^{-3/5}$ yields, after absorbing the perimeter term,

$$E_\epsilon(u, \chi) \geq C|\nu_1|^{\frac{2}{5}} \epsilon^{\frac{2}{5}}.$$

Here we used that $\|\chi_{33}\|_{L^2}^2 \geq C \min^2\{\alpha, 1 - \alpha\}$ and the fact that the function $\psi_{(1-\gamma)^2}$ does not influence the scaling behaviour, cf. the proof of Lemma 3.1. $\square$

Proof of Theorem 2. With the previous results in hand, the remainder of the proof of Theorem 2 is exactly the same as in [RT23b] with the same modifications as above for four wells. For the convenience of the reader we recall the main ideas. Without loss of generality, we reduce to the case of $F = 0$ by considering $\chi - F \in \mathcal{K}_N - F$. We first clarify the nonlinear relations. Each component determines the following ones, that is

$$\sum_{n=j+1}^{N-1} \alpha_k \chi_{nn} = g_j(\chi_{jj}).$$

The relations are given as follows

$$\sum_{n=j+1}^{N-1} 2^{-n+j+1} \chi_{nn} = 4F_{jj} - 4F_{jj}^2 - \sum_{n=j+1}^{N-1} 2^{-n+j+1} F_{nn} + 4\chi_{jj} - 8F_{jj}\chi_{jj} - 4\chi_{jj}^2 =: g_j(\chi_{jj}).$$

The case $\nu \cdot e_1 \neq 0$: By Lemma 2.4 we obtain for a constant $C = C(d, F) > 0$

$$\begin{aligned} (42) \quad &\|\chi_{11} - m_{1,\mu,\lambda}(D)\chi_{11}\|_{L^2}^2 + \sum_{j=2}^d \|\chi_{jj} - m_{j,\mu}(D)\chi_{jj}\|_{L^2}^2 \\ &\leq C(\mu^{-2} + (\lambda\epsilon)^{-1})E_\epsilon(u, \chi) + C\lambda^{-1}\text{Per}(\Omega). \end{aligned}$$

To facilitate the reading, we recall the definition of the truncated cones C_{j,μ,λ_2} in (21)

$$C_{j,\mu,\lambda_2} = \{k \in \mathbb{Z}^d : |k|^2 - k_j^2 \leq \mu^2 |k|^2, |k_j| \leq \frac{2}{|\nu_1|} \lambda_2\},$$

and that $m_{j,\mu,\lambda_2}(D)$ denote their smooth Fourier multipliers as in (23). By Corollary 2.7, in combination with (42), we have for $\lambda_2 = M\mu\lambda$

$$\sum_{j=2}^d \|\chi_{jj} - m_{j,\mu,\lambda_2}(D)\chi_{jj}\|_{L^2}^2 \leq C2^d \psi_{1-\gamma} \left((\mu^{-2} + (\lambda\epsilon)^{-1})E_\epsilon(u, \chi) + \lambda^{-1}\text{Per}(\Omega) \right).$$

Here we exploited that $\alpha_k \in [2^{-d}, 1]$ for $k \in \{2, 3, \dots, d\}$. Then an iterative application of the higher-order variant of Corollary 2.7 (Remark 2.8) yields, in combination with (42), after $\ell - 1$ many iterations

$$(43) \quad \sum_{j=\ell}^d \|\chi_{jj} - m_{j,\mu,\lambda_\ell}(D)\chi_{jj}\|_{L^2}^2 \leq C\psi_{(1-\gamma)^{\ell-1}} \left((\mu^{-2} + (\lambda\epsilon)^{-1})E_\epsilon(u, \chi) + \lambda^{-1}\text{Per}(\Omega) \right),$$

with $\lambda_\ell = M\mu\lambda_{\ell-1} = M^{\ell-1}\mu^{\ell-1}\lambda$ and a constant $C = C(d, F, \ell) > 0$.

As we have already seen in the proof of Theorem 1 for $F \in \mathcal{K}_3^1$, it might happen, that we need information on more than one χ_{jj} , cf. Remark 3.2. Depending on $\ell \in \{1, \dots, N-2\}$ we have for some $\alpha \in (0, 1)$, given by the boundary data,

$$(\chi_{\ell\ell}, \chi_{\ell+1,\ell+1}) \in \{(-\alpha, 0), (1-\alpha, 0), (\frac{1}{2}-\alpha, 1), (\frac{1}{2}-\alpha, \frac{1}{2})\}.$$

In particular, $(\chi_{\ell\ell}, \chi_{\ell+1,\ell+1}) \neq 0$. For $\ell = N-1$, we already have $\chi_{N-1,N-1} \in \{-\alpha, 1-\alpha\}$ with $\alpha \in (0, 1)$, it holds $\chi_{N-1,N-1} \neq 0$.

Thus, if $N = d+1$, setting $\chi_{NN} \equiv 0$, we obtain that if $F \in \mathcal{K}_N^\ell$ for some $\ell \in \{1, 2, \dots, N-1\}$, then it holds $(\chi_{\ell\ell}, \chi_{\ell+1,\ell+1}) \neq 0$. Hence, we note that the control for two diagonal components χ_{jj} is sufficient for deducing the desired lower bound.

By (43) we have

$$\begin{aligned} & \|\chi_{\ell\ell} - m_{\ell,\mu,\lambda_\ell}(D)\chi_{\ell\ell}\|_{L^2}^2 + \|\chi_{\ell+1,\ell+1} - m_{\ell+1,\mu,\lambda_\ell}(D)\chi_{\ell+1,\ell+1}\|_{L^2}^2 \\ & \leq C\psi_{(1-\gamma)^{\ell-1}} \left((\mu^{-2} + (\lambda\epsilon)^{-1})E_\epsilon(u, \chi) + \lambda^{-1}\text{Per}(\Omega) \right). \end{aligned}$$

As in the proof of Proposition 3.5 we use Lemma 2.5 to gain control over $\|m_{\ell,\mu,\lambda_\ell}(D)\chi_{\ell\ell}\|_{L^2}^2$ and $\|m_{\ell+1,\mu,\lambda_\ell}(D)\chi_{\ell+1,\ell+1}\|_{L^2}^2$. For this let $\bar{\lambda} = 4\lambda_\ell/|\nu_1|$, then

$$\begin{aligned} & \|m_{\ell,\mu,\lambda_\ell}(D)\chi_{\ell\ell}\|_{L^2}^2 + \|m_{\ell+1,\mu,\lambda_\ell}(D)\chi_{\ell+1,\ell+1}\|_{L^2}^2 \\ & \leq \sum_{|k_\ell| \leq 4\lambda_\ell/|\nu_1|} |\hat{\chi}_{\ell\ell}(k)|^2 + \sum_{|k_{\ell+1}| \leq 4\lambda_\ell/|\nu_1|} |\hat{\chi}_{\ell+1,\ell+1}(k)|^2 \\ & \leq C \frac{\lambda_\ell^2}{|\nu_1|^2} E_\epsilon(u, \chi). \end{aligned}$$

These two estimates combined give

$$\begin{aligned} & \|\chi_{\ell\ell}\|_{L^2}^2 + \|\chi_{\ell+1,\ell+1}\|_{L^2}^2 \leq 2\|\chi_{\ell\ell} - m_{\ell,\mu,\lambda_\ell}(D)\chi_{\ell\ell}\|_{L^2}^2 + 2\|\chi_{\ell+1,\ell+1} - m_{\ell+1,\mu,\lambda_\ell}(D)\chi_{\ell+1,\ell+1}\|_{L^2}^2 \\ & \quad + 2\|m_{\ell,\mu,\lambda_\ell}(D)\chi_{\ell\ell}\|_{L^2}^2 + 2\|m_{\ell+1,\mu,\lambda_\ell}(D)\chi_{\ell+1,\ell+1}\|_{L^2}^2 \\ & = C\psi_{(1-\gamma)^{\ell-1}} \left((\mu^{-2} + (\lambda\epsilon)^{-1})E_\epsilon(u, \chi) + \lambda^{-1}\text{Per}(\Omega) \right) + C \frac{\lambda_\ell^2}{|\nu_1|^2} E_\epsilon(u, \chi) \\ & \leq C\psi_{(1-\gamma)^{\ell-1}} \left(\left(\frac{\lambda_\ell^2}{|\nu_1|^2} + \mu^{-2} + (\lambda\epsilon)^{-1} \right) E_\epsilon(u, \chi) + \lambda^{-1}\text{Per}(\Omega) \right). \end{aligned}$$

Arguing as in the proof of Proposition 3.5, we optimize this in μ and λ by fixing $\mu \sim |\nu_1|^{1/\ell} \lambda^{-1/\ell}$ and $\lambda \sim |\nu_1|^{2/(\ell+2)} \epsilon^{-\ell/(\ell+2)}$ (such a choice of parameters is compatible with the constraints on μ and $\bar{\lambda}$) and thus

$$\|\chi_{\ell\ell}\|_{L^2}^2 + \|\chi_{\ell+1,\ell+1}\|_{L^2}^2 \leq C\psi_{(1-\gamma)^{\ell-1}} \left(|\nu_1|^{-2/(\ell+2)} \epsilon^{-2/(\ell+2)} E_\epsilon(u, \chi) + |\nu_1|^{2/(\ell+2)} \epsilon^{\ell/(\ell+2)} \text{Per}(\Omega) \right).$$

Using that

$$\|\chi_{\ell\ell}\|_{L^2}^2 + \|\chi_{\ell+1,\ell+1}\|_{L^2}^2 \geq C(F) > 0,$$

after arguing similarly as in the proof of Proposition 1.1, that is considering the two cases for $\psi_{(1-\gamma)^{\ell-1}}$ and absorbing the perimeter, we obtain with $C = C(d, F, \ell) > 0$ and for $\epsilon < \epsilon_0(d, F, \ell, |\nu_1|)$

$$(44) \quad E_\epsilon(u, \chi) \geq C|\nu_1|^{\frac{2}{\ell+2}} \epsilon^{\frac{2}{\ell+2}}.$$

The case $\nu \cdot e_1 = \dots = \nu \cdot e_n = 0$ and $\nu \cdot e_{n+1} \neq 0$, with $0 < n < \ell$: We argue similarly, but we start the iterative application of Corollary 2.7 for the $n+1$ diagonal entry $\chi_{n+1, n+1}$.

Note that due to the change of roles of the coordinate directions, the truncated cones in this setting are given by

$$C_{j, \mu, \lambda} = \{k \in \mathbb{Z}^d : |k|^2 - k_j^2 \leq \mu^2 |k|^2, |k_j| \leq \frac{2}{|\nu_n|} \lambda\},$$

i.e. the truncation depends on $|\nu_n|$ instead of $|\nu_1|$ as above in Theorem 1 in the case $\nu \cdot e_1 = 0$.

By Lemma 2.4, with the roles of the axes changed such that $n+1$ is the first coordinate direction, we get

$$\begin{aligned} & \|\chi_{n+1, n+1} - m_{n+1, \mu, \lambda}(D)\chi_{n+1, n+1}\|_{L^2}^2 + \sum_{j=2}^d \|\chi_{n+j, n+j} - m_{n+j, \mu}(D)\chi_{n+j, n+j}\|_{L^2}^2 \\ & \leq C(\mu^{-2} + (\lambda\epsilon)^{-1})E_\epsilon(\chi) + C\lambda^{-1}\text{Per}(\Omega). \end{aligned}$$

Applying an analogous iteration of a variant of Corollary 2.7 as above, now starting at $j = n+1$ instead of $j = 1$, we deduce

$$\begin{aligned} & \sum_{j=\ell-n}^d \|\chi_{n+j, n+j} - m_{k+j, \mu, \lambda_{\ell-n}}(D)\chi_{n+j, n+j}\|_{L^2}^2 \\ & \leq C\psi_{(1-\gamma)^{\ell-n-1}} \left((\mu^{-2} + (\lambda\epsilon)^{-1})E_\epsilon(u, \chi) + \lambda^{-1}\text{Per}(\Omega) \right). \end{aligned}$$

Concluding as above for $\nu \cdot e_1 \neq 0$ to get (44), while taking the off-set in the index into account, i.e. having $\ell - n$ instead of ℓ , yields (where the constant can be chosen to be independent of n)

$$E_\epsilon(u, \chi) \geq C|\nu_{n+1}|^{\frac{2}{\ell-n+2}} \epsilon^{\frac{2}{\ell-n+2}}.$$

Here the scaling depends on ν_{n+1} instead of ν_1 as above in (44) as we start our arguments with $\chi_{n+1, n+1}$ instead of χ_{11} and with the corresponding high frequency control in the direction e_{n+1} . This is due to the assumption that $\nu \cdot e_{n+1} \neq 0$.

The case $\nu \cdot e_1 = \dots = \nu \cdot e_\ell = 0$: If $\nu \cdot e_j = 0$ for $j = 1, \dots, \ell$, we consider an ℓ -th order simple laminate of arbitrary fine oscillations, as those directions are not penalized, yielding a minimizing sequence, with energies converging to zero. We refer to the proof of Proposition 1.1 for $\nu \cdot e_1 = 0$ for a similar setting.

Conclusion of the proof. As above, to combine the derived estimates we choose ϵ sufficiently small depending on ν , that is for $0 \leq n < \ell$ being the index such that $\nu_{n+1} \neq 0$, $\nu_j = 0$ for $j \leq n$, we fix ϵ such that for all $n < j < \ell \leq d$

$$|\nu_{j+1}|^{\frac{2}{\ell-j+2}} \epsilon^{\frac{2}{\ell-j+2}} \leq |\nu_{n+1}|^{\frac{2}{\ell-n+2}} \epsilon^{\frac{2}{\ell-n+2}},$$

which is possible as $2/(\ell - j + 2) > 2/(\ell - n + 2)$. Hence, we arrive at

$$|\nu_{n+1}|^{\frac{2}{\ell-n+2}} \epsilon^{\frac{2}{\ell-n+2}} \geq \frac{1}{\ell-n} \left(\sum_{j=n}^{\ell-1} |\nu_{j+1}|^{\frac{2}{\ell-j+2}} \epsilon^{\frac{2}{\ell-j+2}} \right) \geq \frac{1}{\ell} \left(\sum_{j=0}^{\ell-1} |\nu_{j+1}|^{\frac{2}{\ell-j+2}} \epsilon^{\frac{2}{\ell-j+2}} \right),$$

where we used that $\nu_{j+1} = 0$ for $0 \leq j < n$. $\square$

We conclude our discussion of sharp interface models by highlighting that the above arguments also allow us to treat situations in which the set T_4 is given by the Tartar square (see Section 1.4.2). This is of particular interest as it is an instance of an extremely rigid phase transition involving laminates of infinite order.

Remark 3.6 (On the Tartar square with anisotropic sharp interface energies). *With the same arguments as above, it is possible to generalize the scaling of the Tartar square, cf. (16), that was derived in [RT22] to a setting involving anisotropic surface energies. To be precise, for any $F \in T_4^{qc} \setminus T_4$, $\nu \in \mathbb{S}^1$ and $\eta \in (0, \frac{1}{2})$ it holds*

$$\inf_{u \in \mathcal{A}_F} \inf_{\chi \in BV_\nu((0,1)^2; T_4)} \int_{\Omega} |\nabla u - \chi|^2 dx + \epsilon \|D_\nu \chi\|_{TV((0,1)^2)} \geq C \exp(-c_\eta |\log(\epsilon)|^{\frac{1}{2}+\eta}),$$

for some constant $c_\eta > 0$. Compared to the settings from above, there are no degenerate directions for the lower scaling bound in the Tartar square. This is due to the fact, that there are no rank-one connections present in the Tartar square and that, hence, each diagonal entry determines the corresponding other entry uniquely. As a consequence, if $\nu \cdot e_1 \neq 0$, we start by truncating the cone in e_1 direction, else we have $\nu \cdot e_2 \neq 0$ and thus can start by truncating the cone in e_2 direction.

3.4. Proof of Theorem 4. Building on the observations from Section 2.5, we conclude the proof of Theorem 4 analogously as in the (local) sharp interface arguments.

Proof of Theorem 4. Lower bound. We note that for $t_1, t_2 \in (0, 1)$

$$\psi_{t_1} \circ \psi_{t_2}(x) = \psi_{t_1 t_2}(x),$$

and thus

$$\psi_{1-\gamma} \circ \psi_{2s} = \psi_{(1-\gamma)2s}.$$

Setting $\gamma_s = 1 - 2s + 2s\gamma \in (0, 1)$, we combine Corollary 2.7 and Lemma 2.9 and get with a constant $C = C(d, F, s) > 0$

$$\begin{aligned} \sum_{j=2}^d |\alpha_j| \|\chi_{jj} - m_{j,\mu,\lambda}(D)\chi_{jj}\|_{L^2}^2 &\leq C \psi_{1-\gamma} \circ \psi_{2s}((\mu^{-2} + (\lambda\epsilon)^{-1})E_{\epsilon,s}(\chi)) \\ &= C \psi_{1-\gamma_s}((\mu^{-2} + (\lambda\epsilon)^{-1})E_{\epsilon,s}(\chi)). \end{aligned}$$

Therefore, the remainder of the proof follows as the proof of Theorem 2 also for the fractional surface energy.

Upper bound. Thanks to an interpolation argument, we prove that we can directly exploit the BV -regular upper bounds χ of Theorem 2. For this, we argue similarly to [BO13, Prop. 1.3] but taking into account the anisotropy of the surface energy. Let $\chi \in BV_\nu(\Omega; \mathcal{K}_N)$. We first claim that for χ seen as a function on the torus $\mathbb{T}^d$ it holds

$$\sum_{k \in \mathbb{Z}^d \setminus \{0\}} |k \cdot \nu|^{2s} |\hat{\chi}(k)|^2 \leq C(s) \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\mathbb{T}^d} \frac{|\chi(x + h\nu) - \chi(x)|^2}{|h|^{1+2s}} dx dh + \|\chi\|_{L^2}^2.$$

Indeed, this follows from observing that by Plancherel's theorem

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\mathbb{T}^d} \frac{|\chi(x + h\nu) - \chi(x)|^2}{|h|^{1+2s}} dx dh = \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{k \in \mathbb{Z}^d} \frac{|e^{2\pi i h k \cdot \nu} - 1|^2}{|h|^{1+2s}} |\hat{\chi}(k)|^2 dh,$$

and by noting that for $k \in \mathbb{Z}^d$ with $|k \cdot \nu| > 1$ we have $(2|k \cdot \nu|)^{-1} \leq 2^{-1}$ and hence

$$B_\nu(k) := |k \cdot \nu|^{-2s} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{|e^{2\pi i h k \cdot \nu} - 1|^2}{|h|^{1+2s}} dh \geq 8|k \cdot \nu|^{-2s} \int_0^{\frac{1}{2|k \cdot \nu|}} \frac{\sin^2(\pi h |k \cdot \nu|)}{|h|^{1+2s}} dh \geq C(s)^{-1} > 0$$

for a constant $C(s)$ independent of k . With this in hand, by monotone convergence we have

$$\begin{aligned} \sum_{k \in \mathbb{Z}^d \setminus \{0\}} |k \cdot \nu|^{2s} |\hat{\chi}(k)|^2 &= \sum_{|k \cdot \nu| > 1} |k \cdot \nu|^{2s} |\hat{\chi}(k)|^2 + \sum_{|k \cdot \nu| \leq 1} |k \cdot \nu|^{2s} |\hat{\chi}(k)|^2 \\ &\leq C(s) \sum_{|k \cdot \nu| > 1} |k \cdot \nu|^{2s} B_\nu(k) |\hat{\chi}(k)|^2 + \sum_{|k \cdot \nu| \leq 1} |\hat{\chi}(k)|^2 \\ &\leq C(s) \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\mathbb{T}^d} \frac{|\chi(x + h\nu) - \chi(x)|^2}{|h|^{1+2s}} dx dh + \|\chi\|_{L^2}^2. \end{aligned}$$

We now fix $\tilde{\chi} : \mathbb{R}^d \rightarrow \mathcal{K}_N$ by setting

$$\tilde{\chi}(x) := \begin{cases} \chi(x) & x \in (-1, 2)^d, \\ 0 & \text{else.} \end{cases}$$

Noting that for all $x \in (0, 1)^d$ and $h \in [-\frac{1}{2}, \frac{1}{2}]$ it holds that $x + h\nu \in (-1, 2)^d$, by the definition of $\tilde{\chi}$ we infer that

$$\sum_{k \in \mathbb{Z}^d \setminus \{0\}} |k \cdot \nu|^{2s} |\hat{\chi}(k)|^2 \leq C(s) \int_{\mathbb{R}} \int_{\mathbb{R}^d} \frac{|\tilde{\chi}(x + h\nu) - \tilde{\chi}(x)|^2}{|h|^{1+2s}} dx dh + \|\chi\|_{L^2}^2.$$

Using a slicing argument together with a Gagliardo-Nirenberg type inequality [BM18, Theorem 1] (see also [ZZ19, Theorem 2])

$$\begin{aligned} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \frac{|\tilde{\chi}(x + h\nu) - \tilde{\chi}(x)|^2}{|h|^{1+2s}} &= \int_{\nu^\perp} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|\tilde{\chi}_y^\nu(t + h) - \tilde{\chi}_y^\nu(t)|^2}{|h|^{1+2s}} dh dt dy \\ &\leq C(s, d, \mathcal{K}) \int_{\nu^\perp} (\|\tilde{\chi}_y^\nu\|_{L^1(\mathbb{R})} + \|D\tilde{\chi}_y^\nu\|_{TV(\mathbb{R})})^{2s} dy. \end{aligned}$$

By the uniformly compact support of the functions $\tilde{\chi}$ on each slice and as $2s < 1$, we can further bound this by Jensen's inequality

$$\begin{aligned} \int_{\nu^\perp} (\|\tilde{\chi}_y^\nu\|_{L^1(\mathbb{R})} + \|D\tilde{\chi}_y^\nu\|_{TV(\mathbb{R})})^{2s} dy &\leq C \left(\int_{\nu^\perp} \|\tilde{\chi}_y^\nu\|_{L^1(\mathbb{R})} + \|D\tilde{\chi}_y^\nu\|_{TV(\mathbb{R})} dy \right)^{\frac{1}{2s}} \\ &= C \left(\|\tilde{\chi}\|_{L^1(\mathbb{R}^d)} + \|D_\nu \tilde{\chi}\|_{TV(\mathbb{R}^d)} \right)^{\frac{1}{2s}}. \end{aligned}$$

Thus, by the relation between χ and $\tilde{\chi}$, we conclude

$$\sum_{k \in \mathbb{Z}^d \setminus \{0\}} |k \cdot \nu|^{2s} |\hat{\chi}(k)|^2 \leq C(s, d, \mathcal{K}) (1 + \text{Per}(\Omega) + \|D_\nu \chi\|_{TV(\Omega)})^{\frac{1}{2s}}.$$

In particular, this implies

$$E_{\epsilon, s}(\chi) \leq C(s, d, \mathcal{K}) E_\epsilon(\chi) + C(s, d, \mathcal{K}) \epsilon (1 + \text{Per}(\Omega)).$$

Thus, the matching upper bounds for the sharp interface model in Theorem 2 are still applicable which yields the desired result. $\square$

4. DIFFUSE SURFACE ENERGIES

4.1. Diffuse to sharp interface model – the lower bound. Following the arguments of [KK11], we show that in the continuous model, the minimal diffuse energy can be controlled by the lower bound for the sharp interface energy. Hence, the lower scaling estimates for the sharp interface model which had been deduced in the previous section for various anisotropic situations also give rise to lower scaling estimates for the diffuse energy. We show this lower bound for

any *discrete set* of wells and also for a more general framework covering both the gradient and symmetrized gradient settings. Combined with the upper bounds from the next subsection, this will lead to sharp scaling results in our model problems also for anisotropic, diffuse interface energies.

4.1.1. The scalar-valued setting. We first show the lower bound in the scalar-valued case, and afterwards will reduce the general vector-valued setting to the one-dimensional one by a projection argument.

Lemma 4.1. *Let $p, q \in [1, \infty)$ and $\Omega \subset \mathbb{R}^d$ be bounded and let $\mathcal{K} = \{A_1, \dots, A_N\} \subset \mathbb{R}$. For any $U \in L^p(\Omega; \mathbb{R}) \cap W^{1,q}(\Omega; \mathbb{R})$, $f \in L^\infty(\Omega; \mathcal{K})$ there exist $\tilde{f} \in BV(\Omega; \mathcal{K})$ and a constant $C = C(\mathcal{K}, p, q) > 0$ such that for any $\epsilon > 0$*

$$(45) \quad \int_{\Omega} |U - f|^p + \epsilon^q |\nabla U|^q dx \geq C\epsilon \|D\tilde{f}\|_{TV(\Omega)} \quad \text{and} \quad |U - f| \geq C|U - \tilde{f}|.$$

For $q = 1$, the same result holds with $\|DU\|_{TV(\Omega)}$ instead of $\|\nabla U\|_{L^1(\Omega)}$ (on the left-hand side of (45)), for $U \in BV(\Omega; \mathbb{R})$ instead of $U \in W^{1,1}(\Omega; \mathbb{R})$.

Proof. Up to relabelling, we can assume that $-\infty < A_1 < A_2 < \dots < A_N < +\infty$. We start by giving the argument for the first inequality in (45). We assume for simplicity that U coincides with its Lebesgue representative. In this case, for scalar-valued functions, the validity of the coarea formula for Sobolev functions is known (cf. [MSZ03]). We, hence, invoke Young's inequality for $q > 1$ and the coarea formula to get

$$\begin{aligned} \int_{\Omega} |U - f|^p + \epsilon^q |\nabla U|^q dx &\geq \epsilon \int_{\Omega} |U - f|^{\frac{p(q-1)}{q}} |\nabla U| dx = \epsilon \int_{\mathbb{R}} \int_{U^{-1}(t) \cap \Omega} |t - f|^{\frac{p(q-1)}{q}} d\mathcal{H}^{d-1}(x) dt \\ &\geq \epsilon \sum_{k=1}^{N-1} \int_{A_k + \frac{c}{4}}^{A_{k+1} - \frac{c}{4}} \int_{U^{-1}(t) \cap \Omega} |t - f|^{\frac{p(q-1)}{q}} d\mathcal{H}^{d-1}(x) dt, \end{aligned}$$

where in the last inequality we introduced $c = \min\{A_{k+1} - A_k : k = 1, \dots, N-1\}$.

As for $t \in (A_k + \frac{c}{4}, A_{k+1} - \frac{c}{4})$ we have $\text{dist}(t, \mathcal{K}) \geq \frac{c}{4}$, we can control $|t - f(x)|^{\frac{p(q-1)}{q}} \geq C$ with a constant $C = C(\mathcal{K}, p, q) > 0$. Plugging this lower bound into the above inequality yields

$$\int_{\Omega} |U - f|^p + \epsilon^q |\nabla U|^q dx \geq C\epsilon \sum_{k=1}^{N-1} \int_{A_k + \frac{c}{4}}^{A_{k+1} - \frac{c}{4}} \mathcal{H}^{d-1}(\{x \in \Omega : U(x) = t\}) dt.$$

For every $k \in \{1, \dots, N-1\}$ there is $t_k \in (A_k + \frac{c}{4}, A_{k+1} - \frac{c}{4})$, which, without loss of generality, we may assume to be a Lebesgue point for the function $t \mapsto \mathcal{H}^{d-1}(\{x \in \Omega : U(x) = t\})$, such that

$$\begin{aligned} \int_{A_k + \frac{c}{4}}^{A_{k+1} - \frac{c}{4}} \mathcal{H}^{d-1}(\{x \in \Omega : U(x) = t\}) dt &\geq (A_{k+1} - A_k - \frac{c}{2}) \mathcal{H}^{d-1}(\{x \in \Omega : U(x) = t_k\}) \\ &\geq \frac{c}{2} \mathcal{H}^{d-1}(\{x \in \Omega : U(x) = t_k\}). \end{aligned}$$

Therefore, combined with the previous bound, we obtain

$$\int_{\Omega} |U - f|^p + \epsilon^q |\nabla U|^q dx \geq C\epsilon \sum_{k=1}^{N-1} \mathcal{H}^{d-1}(\{x \in \Omega : U(x) = t_k\}).$$

In particular, we obtain a sequence of t_k such that $t_k < t_{k+1}$ for all $k \in \{1, \dots, N-2\}$.

The idea now is to define a new phase indicator with BV-seminorm which is exactly determined by these measures. To this end let $h : \mathbb{R} \rightarrow \mathbb{R}$ be given as $h(t) = \sum_{k=1}^N A_k \chi_{(t_{k-1}, t_k]}(t) \in \mathcal{K}$ and

with slight abuse of notation, we set $t_0 := -\infty, t_N := \infty$. We define $\tilde{f} = (h \circ U) : \Omega \rightarrow \mathcal{K}$, and claim that $\tilde{f} \in BV(\Omega; \mathcal{K})$ and that it satisfies

$$\|D\tilde{f}\|_{TV(\Omega)} \leq C \sum_{k=1}^{N-1} \mathcal{H}^{d-1}(\{x \in \Omega : U(x) = t_k\}).$$

The claim will be shown by an approximation argument. To this end, let $h_j \in C^\infty(\mathbb{R}; \mathbb{R})$ be a mollification of h fulfilling $\text{supp}(h - h_j) \subset \bigcup_{k=1}^{N-1} (t_k - \frac{1}{j}, t_k + \frac{1}{j})$, $|h'_j| \leq Cj\chi_{\text{supp}(h-h_j)}$, and $|h_j| \leq \max \mathcal{K}$. By construction, it holds that $h_j(t) \rightarrow h(t)$ for every $t \neq t_k, k = 1, \dots, N-1$. Since the sets $\{U = t_k\}$ have Hausdorff dimension $d-1$, by the boundedness of Ω , and boundedness of h_j , the dominated convergence theorem implies $h_j \circ U \rightarrow h \circ U = \tilde{f}$ in $L^1(\Omega)$. As the BV-seminorm is lower semicontinuous with respect to the (strong) L^1 convergence, we deduce

$$\|D\tilde{f}\|_{TV(\Omega)} \leq \liminf_{j \rightarrow \infty} \|D(h_j \circ U)\|_{TV(\Omega)}.$$

As h_j is smooth, we can use the chain rule to further bound the total variation norm. Indeed, by the coarea formula

$$\begin{aligned} \|D(h_j \circ U)\|_{TV(\Omega)} &= \int_{\Omega} |h'_j(U(x))| |\nabla U(x)| dx = \int_{\mathbb{R}} |h'_j(t)| \mathcal{H}^{d-1}(\{x \in \Omega : U(x) = t\}) dt \\ &\leq Cj \sum_{k=1}^{N-1} \int_{t_k - \frac{1}{j}}^{t_k + \frac{1}{j}} \mathcal{H}^{d-1}(\{x \in \Omega : U(x) = t\}) dt \\ &\rightarrow \frac{C}{2} \sum_{k=1}^{N-1} \mathcal{H}^{d-1}(\{x \in \Omega : U(x) = t_k\}). \end{aligned}$$

Here we used that the t_k are Lebesgue points of the function $t \mapsto \mathcal{H}^{d-1}(\{x \in \Omega : U(x) = t\})$.

Thus, with this definition, $\tilde{f} \in BV(\Omega; \mathcal{K})$ and also the desired upper bound from (45) follows.

If $q = 1$, we do not use Young's inequality, but neglect the first term. We give the argument for $U \in BV(\Omega; \mathbb{R})$ and note that the statement for $U \in W^{1,1}(\Omega; \mathbb{R})$ is also covered by this. By the coarea formula for BV functions [AFP00, Thm 3.40]

$$\begin{aligned} \int_{\Omega} |U - f|^p dx + \epsilon \|DU\|_{TV(\Omega)} &\geq \epsilon \|DU\|_{TV(\Omega)} = \epsilon \int_{\mathbb{R}} \|D\chi_{\{U > t\}}\|_{TV(\Omega)} dt \\ &\geq \epsilon \sum_{k=1}^{N-1} \int_{A_k}^{A_{k+1}} \|D\chi_{\{U > t\}}\|_{TV(\Omega)} dt \geq \epsilon \sum_{k=1}^{N-1} |A_{k+1} - A_k| \|D\chi_{\{U > t_k\}}\|_{TV(\Omega)}, \end{aligned}$$

for some $t_k \in (A_k, A_{k+1})$. Considering the function $h : \mathbb{R} \rightarrow \mathbb{R}$ and $\tilde{f} = h \circ U$ as above, we write h as

$$h(t) = A_1 + \sum_{k=1}^{N-1} (A_{k+1} - A_k) \chi_{(t_k, \infty)}(t),$$

and thus it holds

$$\tilde{f}(x) = h \circ U(x) = A_1 + \sum_{k=1}^{N-1} (A_{k+1} - A_k) \chi_{\{U > t_k\}}(x).$$

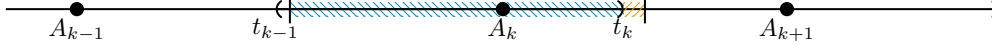


FIGURE 8. Picture of the two cases which arise in our proof of the second estimate in (45). The blue hashed region depicts the first case, where $\tilde{f}(x) = A_k$, the orange region the second case, in which $\text{dist}(U(x), \mathcal{K}) \geq \frac{c}{4}$. The vertical dashes are the center points between A_{k-1} and A_k and A_k and A_{k+1} respectively.

In particular, we have

$$\begin{aligned} \epsilon \|D\tilde{f}\|_{TV(\Omega)} &\leq \epsilon \sum_{k=1}^{N-1} |A_{k+1} - A_k| \|D\chi_{\{U > t_k\}}\|_{TV(\Omega)} \\ &\leq \int_{\Omega} |U - f|^p dx + \epsilon \|DU\|_{TV(\Omega)}. \end{aligned}$$

Now we prove the second inequality from (45). For this, let $x \in \Omega$ and $k \in \{1, \dots, N-1\}$ be such that $U(x) \in (\frac{A_k + A_{k-1}}{2}, \frac{A_k + A_{k+1}}{2}]$, where we set $A_0 = -\infty$ and $A_{N+1} = \infty$. In particular, $|U(x) - A_k| = \text{dist}(U(x), \mathcal{K})$. We distinguish two cases, which are illustrated in Figure 8.

On the one hand, if $U(x) \in (t_{k-1}, t_k]$, we have $\tilde{f}(x) = A_k$ and thus $|U(x) - \tilde{f}(x)| = \text{dist}(U(x), \mathcal{K}) \leq |U(x) - f(x)|$.

On the other hand, if $U(x) \notin (t_{k-1}, t_k]$, we have either

$$U(x) > t_k > A_k + \frac{c}{4} \quad \text{or} \quad U(x) \leq t_{k-1} < A_k - \frac{c}{4},$$

and hence $\text{dist}(U, \mathcal{K}) = |U(x) - A_k| > \frac{c}{4}$. This means that $|U(x) - f(x)| \geq \frac{c}{4}$ and furthermore

$$|U(x) - \tilde{f}(x)| \leq |U(x) - f(x)| + |f(x) - \tilde{f}(x)| \leq |U(x) - f(x)| + \text{diam}(\mathcal{K}) \leq C|U(x) - f(x)|.$$

This concludes the proof. $\square$

Seeking to deduce the same scaling laws from Section 1.1 for diffuse surface energies, we also generalize Lemma 4.1 to anisotropic surface energies:

Corollary 4.2. *Let $p, q \in [1, \infty)$, $\nu \in \mathbb{S}^{d-1}$ and $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and let $\mathcal{K} = \{A_1, \dots, A_N\} \subset \mathbb{R}$. For any $U \in L^p(\Omega; \mathbb{R})$ such that the weak directional derivative satisfies $\partial_{\nu} U \in L^q(\Omega; \mathbb{R})$ and $f \in L^{\infty}(\Omega; \mathcal{K})$ there exist $\tilde{f} \in BV_{\nu}(\Omega; \mathcal{K})$ and a constant $C = C(\mathcal{K}, p, q) > 0$ such that for any $\epsilon > 0$*

$$\int_{\Omega} |U - f|^p + \epsilon^q |\partial_{\nu} U|^q dx \geq C\epsilon \|D_{\nu} \tilde{f}\|_{TV(\Omega)}, \quad \text{and} \quad \int_{\Omega} |U - f|^p dx \geq C \int_{\Omega} |U - \tilde{f}|^p dx.$$

Proof. The proof uses similar methods as above, with the addition of a splitting in direction ν and $\nu^{\perp}$. We start by considering $U \in W^{1,p}(\mathbb{R}^d; \mathbb{R})$. For almost every $y \in \Omega_{\nu}$ (see Section 2.2 for the notation), we follow the arguments from the previous proof and obtain

$$\begin{aligned} \int_{\Omega} |U - f|^p + \epsilon^q |\partial_{\nu} U|^q dx &\geq \epsilon \int_{\Omega_{\nu}} \int_{\Omega_y^{\nu}} |U_y^{\nu}(t) - f_y^{\nu}(t)|^{\frac{p(q-1)}{q}} |(U_y^{\nu})'(t)| dt dy \\ &\geq C\epsilon \int_{\Omega_{\nu}} \sum_{k=1}^{N-1} \mathcal{H}^0(\{s \in \Omega_y^{\nu} : U_y^{\nu}(s) = t_k^y\}) dy, \end{aligned}$$

where, for a.e. $y \in \Omega_{\nu}$, $t_k^y \in (A_k + \frac{c}{4}, A_{k+1} - \frac{c}{4})$, with c defined as in the proof of Lemma 4.1.

The main difference with respect to the isotropic case, is that here we use a slicing argument, hence in the definition of the modified phase indicator $\tilde{f}$, the thresholds t_k^y change from slice to

slice, giving in principle measurability issues. To avoid this, we define $\tilde{f}$ to be piecewise constant in the variable y as follows.

Given $\delta > 0$, we define $\mathcal{L}_\delta := \{z \in \delta\mathbb{Z}^{d-1} : Q_\delta(z) \subset \Omega_\nu\}$, where $Q_\delta(z) = (0, \delta)^{d-1} + z$, and we use the notation $\mathcal{L}_\delta = \{z_j\}_{j=1}^M$, with $M = \#\mathcal{L}_\delta$ (the dependence on ν is omitted for the sake of clarity of exposition). From the above estimate, by an average argument, we get

$$(46) \quad \begin{aligned} \int_{\Omega} |U - f|^p + \epsilon^q |\partial_\nu U|^q dx &\geq C\epsilon \sum_{j=1}^M \int_{Q_\delta(z_j)} \sum_{k=1}^{N-1} \mathcal{H}^0(\{s \in \Omega_{y_j}^\nu : U_{y_j}^\nu(s) = t_k^{y_j}\}) dy \\ &\geq C\epsilon\delta^{d-1} \sum_{j=1}^M \sum_{k=1}^{N-1} \mathcal{H}^0(\{s \in \Omega_{y_j}^\nu : U_{y_j}^\nu(s) = t_k^{y_j}\}) \end{aligned}$$

with $y_j \in Q_\delta(z_j)$. In every cube, we now work as in the proof of Lemma 4.1, namely we define $h_j = \sum_{k=1}^N A_k \chi_{(t_{k-1}^{y_j}, t_k^{y_j}]}$ and $\tilde{f}_j^\nu := h_j \circ U_{y_j}^\nu$. Notice that as $U \in W^{1,p}(\mathbb{R}^d; \mathbb{R})$ the functions $\tilde{f}_j^\nu$ are defined on the whole $\mathbb{R}$. Working as in the previous proof, for every j we have

$$(47) \quad \|D\tilde{f}_j^\nu\|_{TV(\Omega_{y_j}^\nu)} \leq C \sum_{k=1}^{N-1} \mathcal{H}^0(\{s \in \Omega_{y_j}^\nu : U_{y_j}^\nu(s) = t_k^{y_j}\}).$$

Defining

$$\tilde{f}(x) := \begin{cases} \tilde{f}_j^\nu(t), & \Pi_{\nu^\perp} x = y \in Q_\delta(z_j), x = y + t\nu, j \in \{1, \dots, M\} \\ A_1 & \text{otherwise,} \end{cases}$$

by slicing $\tilde{f} \in BV_\nu(\Omega; \mathcal{K})$ and combining (46) and (47) we get

$$\begin{aligned} \epsilon \|D_\nu \tilde{f}\|_{TV(\Omega)} &= \epsilon \sum_{j=1}^M \|D_\nu \tilde{f}\|_{TV(C_\delta(z_j))} = \epsilon\delta^{d-1} \sum_{j=1}^M \|D\tilde{f}_j^\nu\|_{TV(\Omega_{y_j}^\nu)} \\ &\leq C \int_{\Omega} |U - f|^p + \epsilon^q |\partial_\nu U|^q dx, \end{aligned}$$

where $C_\delta(z_j) := \{x \in \Omega : \Pi_{\nu^\perp} x \in Q_\delta(z_j)\}$.

For $U \in BV_\nu(\mathbb{R}^d; \mathbb{R})$ we can then also follow the arguments in the proof of Lemma 4.1 combined with the methods above to show the estimate.

We now prove the estimate regarding elastic energies. Notice preliminarily that we can assume that $U \neq f$ in L^p otherwise we can simply choose $\tilde{f} = U$.

By definition of $\tilde{f}_j^\nu$, arguing as in the proof of Lemma 4.1 the pointwise estimate holds $|U_{y_j}^\nu(t) - \tilde{f}_j^\nu(t)| \leq C \text{dist}(U_{y_j}^\nu(t), \mathcal{K})$ a.e. in $\mathbb{R}$. Hence, applying the triangle inequality twice, we may write

$$\begin{aligned} &\int_{Q_\delta(z_j)} \int_{\Omega_{y_j}^\nu} |U_y^\nu(t) - (\tilde{f})_y^\nu(t)|^p dt dy \\ &\leq 2^{p-1} \int_{Q_\delta(z_j)} \int_{\Omega_{y_j}^\nu} |U_y^\nu(t) - U_{y_j}^\nu(t)|^p dt dy + 2^{p-1} \int_{Q_\delta(z_j)} \int_{\Omega_{y_j}^\nu} |U_{y_j}^\nu(t) - \tilde{f}_j^\nu(t)|^p dt dy \\ &\leq 2^p \int_{Q_\delta(z_j)} \int_{\Omega_{y_j}^\nu} |U_y^\nu(t) - U_{y_j}^\nu(t)|^p dt dy + 2^{p-1} \int_{Q_\delta(z_j)} \int_{\Omega_{y_j}^\nu} \text{dist}^p(U_y^\nu(t), \mathcal{K}) dt dy \end{aligned}$$

Eventually, since $U \in W^{1,p}(\mathbb{R}^d; \mathbb{R})$ by assumption, we infer

$$\begin{aligned} \int_{Q_\delta(z_j)} \int_{\Omega_y^\nu} |U_y^\nu(t) - U_{y_j}^\nu(t)|^p dt dy &= \int_{Q_\delta(z_j)} \int_{\Omega_y^\nu} |U(y + t\nu) - U(y_j + t\nu)|^p dt dy \\ &\leq \sup_{|h| \leq \delta\sqrt{d-1}} \int_{C_\delta(z_j) \cap \Omega_\delta} |U(x) - U(x+h)|^p dx + C \int_{C_\delta(z_j) \setminus \Omega_\delta} |U(x)|^p dx \\ &\leq C\delta^p \|\nabla U\|_{L^p(\Omega_\delta^j)}^p + C\|U\|_{L^p(C_\delta(z_j) \setminus \Omega_\delta)}^p, \end{aligned}$$

where $\Omega_\delta := \{x \in \Omega : \text{dist}(x, \partial\Omega) > 2\sqrt{d-1}\delta\}$, and where we used the notation $\Omega_\delta^j = (B_{\frac{\delta}{2\sqrt{d-1}}}(z_j) \times \mathbb{R}\nu) \cap \Omega$ which intersect a finite number of times. Gathering the two above inequalities, and summing over j we obtain

$$\int_\Omega |U(x) - \tilde{f}(x)|^p dx \leq C \int_\Omega \text{dist}^p(U(x), \mathcal{K}) dx + C\delta^p \|\nabla U\|_{L^p(\Omega)}^p + C \int_{\Omega \setminus \Omega_\delta} |U(x)|^p dx + C|\Omega \setminus \Omega_\delta|.$$

Taking δ sufficiently small, the result follows for every $U \in W^{1,p}(\Omega; \mathbb{R})$. By an approximation argument, let $U_n \in W^{1,p}(\mathbb{R}^d; \mathbb{R})$ converge to U strongly in $L^p(\Omega)$ and $\partial_\nu U_n$ converge to $\partial_\nu U$ strongly in $L^q(\Omega)$ and let $\tilde{f}_n$ be their phase indicators (defined as above), we have that

$$\int_\Omega |U - \tilde{f}_n|^p dx \leq C \int_\Omega |U - U_n|^p dx + C \int_\Omega |U_n - \tilde{f}_n|^p dx.$$

By taking n sufficiently large (and $\delta > 0$ sufficiently small such that $C\delta^p \|\nabla U\|_{L^p(\Omega)}^p + C\|U\|_{L^p(\Omega \setminus \Omega_\delta)}^p + C|\Omega \setminus \Omega_\delta| < \frac{\eta}{2}$), the result follows:

$$\begin{aligned} \int_\Omega |U - \tilde{f}_n|^p dx &\leq \eta + C \int_\Omega \text{dist}^p(U_n(x), \mathcal{K}) dx \\ &\leq \eta + C \int_\Omega \text{dist}^p(U(x), \mathcal{K}) dx + C \int_\Omega |U - U_n|^p dx \leq 2\eta + C \int_\Omega \text{dist}^p(U(x), \mathcal{K}) dx, \end{aligned}$$

after choosing $\eta \leq \int_\Omega \text{dist}^p(U(x), \mathcal{K}) dx$. Analogously, for n sufficiently large we get

$$\|D_\nu \tilde{f}_n\|_{TV(\Omega)} \leq C \int_\Omega |U_n - f|^p + \epsilon^q |\partial_\nu U_n|^q dx \leq C \int_\Omega |U - f|^p + \epsilon^q |\partial_\nu U|^q dx + C\eta$$

which gives the result. $\square$

4.1.2. The vector-valued setting. In this section, we translate the results from the previous section to the vector-valued setting.

Proposition 4.3. *Let $p, q \in [1, \infty)$ and $\Omega \subset \mathbb{R}^d$ be bounded and let $\mathcal{K} = \{A_1, \dots, A_N\} \subset \mathbb{R}^n$. For any $U \in L^p(\Omega; \mathbb{R}^n) \cap W^{1,q}(\Omega; \mathbb{R}^n)$ and $\chi \in L^\infty(\Omega; \mathcal{K})$ there exist $\tilde{\chi} \in BV(\Omega; \mathcal{K})$ and a constant $C = C(\mathcal{K}, p, q) > 0$ such that for any $\epsilon > 0$*

$$\int_\Omega |U - \chi|^p + \epsilon^q |\nabla U|^q dx \geq C\epsilon \|D\tilde{\chi}\|_{TV(\Omega)} \quad \text{and} \quad |U - \chi| \geq C|U - \tilde{\chi}|.$$

In particular,

$$\int_\Omega |U - \chi|^p + \epsilon^q |\nabla U|^q dx \geq C \left(\int_\Omega |U - \tilde{\chi}|^p dx + \epsilon \|D\tilde{\chi}\|_{TV(\Omega)} \right).$$

Proof. The main idea is to invoke Lemma 4.1 for a suitable projection of the wells onto a one-dimensional subspace in which there are still N distinct wells. For this we choose

$$\zeta \in \mathbb{S}^{n-1} \setminus \left(\bigcup_{i \neq j} (A_i - A_j)^\perp \right) \neq \emptyset.$$

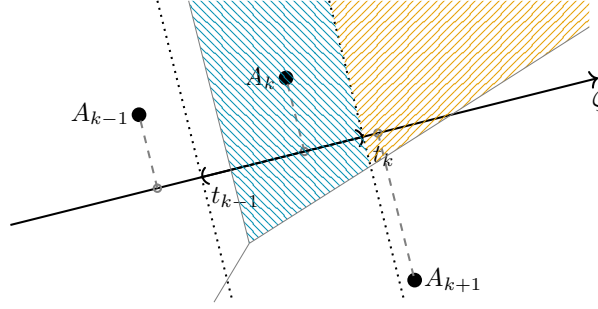


FIGURE 9. Picture of the two cases to show the estimate on the elastic energy in Proposition 4.3. The hashed region is the set in which $U(x)$ lies within. In particular, the blue region illustrates the first case, in which $\tilde{\chi} = A_k$. The orange region illustrates the second case, in which $\text{dist}(U(x), \mathcal{K}) \geq C$. The smaller circles denote the projections $A_{k-1} \cdot \zeta, A_k \cdot \zeta, A_{k+1} \cdot \zeta$. The thin lines are the boundaries of the Voronoi-regions.

By this choice of ζ , we know that $A_i \cdot \zeta \neq A_j \cdot \zeta$ for any $i, j \in \{1, \dots, N\}, i \neq j$, hence there is a one-to-one correspondence of $\mathcal{K}' = \{A_1 \cdot \zeta, A_2 \cdot \zeta, \dots, A_N \cdot \zeta\}$ and $\mathcal{K}$. For further use in what follows below, we note that due to the discreteness of $\mathcal{K}$ and $\mathcal{K}'$ we can define a Lipschitz map

$$(48) \quad h \in C^1(\mathbb{R}; \mathbb{R}^n) \text{ with } h(A_j \cdot \zeta) = A_j \text{ for all } j \in \{1, 2, \dots, N\}.$$

Now we project the energy onto the direction ζ in the image:

$$\int_{\Omega} |U - \chi|^p + \epsilon^q |\nabla U|^q dx \geq \int_{\Omega} |U \cdot \zeta - \chi \cdot \zeta|^p + \epsilon^q |\nabla(U \cdot \zeta)|^q dx.$$

The functions $U \cdot \zeta : \mathbb{R}^d \rightarrow \mathbb{R}$ and $\chi \cdot \zeta : \mathbb{R}^d \rightarrow \mathbb{R}$ are admissible for Lemma 4.1, and thus there is $\tilde{f} \in BV(\Omega; \mathcal{K}')$ and $C = C(\mathcal{K}, p, q) > 0$ such that

$$(49) \quad \int_{\Omega} |U - \chi|^p + \epsilon^q |\nabla U|^q dx \geq C\epsilon \|D\tilde{f}\|_{TV(\Omega)}, \quad |U \cdot \zeta - \chi \cdot \zeta| \geq C|U \cdot \zeta - \tilde{f}|.$$

Using the determinedness of $\mathcal{K}$ in terms of $\mathcal{K}'$, i.e. using the function $h : \mathbb{R} \rightarrow \mathbb{R}^n$ from (48), and the chain rule in BV [AFP00, Thm 3.96] hence shows that $\tilde{\chi} = h \circ \tilde{f} \in BV(\Omega; \mathcal{K})$ and

$$\|D\tilde{\chi}\|_{TV(\Omega)} \leq C(\mathcal{K}) \|D\tilde{f}\|_{TV(\Omega)}.$$

Combining this bound with (49) yields

$$\int_{\Omega} |U - \chi|^p + \epsilon^q |\nabla U|^q dx \geq C\epsilon \|D\tilde{\chi}\|_{TV(\Omega)}.$$

The pointwise bound follows with the same argument as in the one-dimensional case. Let $x \in \Omega$ and $k \in \{1, \dots, N\}$ be such that $\text{dist}(U(x), \mathcal{K}) = |U(x) - A_k|$. If $\tilde{\chi}(x) = A_k$, it is direct that $|U(x) - \chi(x)| \geq |U(x) - A_k| = |U(x) - \tilde{\chi}(x)|$. If $\tilde{\chi}(x) \neq A_k$, also $\tilde{f}(x) \neq A_k \cdot \zeta$, and we can deduce that $|U(x) \cdot \zeta - A_k \cdot \zeta| \geq C$ and thus also $\text{dist}(U(x), \mathcal{K}) \geq C$ and hence

$$\begin{aligned} |U(x) - \tilde{\chi}(x)| &\leq |U(x) - \chi(x)| + |\chi(x) - \tilde{\chi}(x)| \leq |U(x) - \chi(x)| + \text{diam}(\mathcal{K}) \\ &\leq |U(x) - \chi(x)| + c \text{dist}(U(x), \mathcal{K}) \leq C|U(x) - \chi(x)|. \end{aligned}$$

The two cases are illustrated in Figure 9. □

Also this result can be translated to anisotropic surface energies which proves Theorem 3.

Proof of Theorem 3. We split the energy into the r component and the remainder. By an application of Corollary 4.2 it holds

$$\begin{aligned} \int_{\Omega} |U - \chi|^p + \epsilon^q |\partial_{\nu}(U \cdot r)|^q dx &\geq \int_{\Omega} |U \cdot r - \chi \cdot r|^p + \epsilon^q |\partial_{\nu}(U \cdot r)|^q dx \\ &\geq C \left(\int_{\Omega} |U \cdot r - f_r|^p dx + \epsilon \|D_{\nu} f_r\|_{TV(\Omega)} \right), \end{aligned}$$

for some $f_r : \Omega \rightarrow \{A_1 \cdot r, A_2 \cdot r, \dots, A_N \cdot r\}$. We now define $\tilde{\chi} \in BV_{\nu}(\Omega, \mathcal{K})$ by setting $\tilde{\chi} \cdot r = f_r$. To fix the part perpendicular to r , we choose $\Pi_{r^{\perp}} \tilde{\chi} = \tilde{\chi} - (\tilde{\chi} \cdot r)r$ as the projection of $\Pi_{r^{\perp}} U$ onto $\Pi_{r^{\perp}} \mathcal{K} = \{A_1 - (A_1 \cdot r)r, \dots, A_N - (A_N \cdot r)r\}$ with the additional constraint that $\tilde{\chi} = f_r r + \Pi_{r^{\perp}} \tilde{\chi} \in \mathcal{K}$. That is $\tilde{\chi}$ is chosen such that

$$|U(x) - \tilde{\chi}(x)| = \min\{|U(x) - A_k| : A_k \cdot r = f_r(x)\}, \quad \text{for a.e. } x \in \Omega.$$

Then we have

$$\int_{\Omega} |U - \chi|^p dx \geq c \int_{\Omega} |U - \tilde{\chi}|^p dx.$$

Indeed, by Corollary 4.2 and the pointwise estimate in the orthogonal part we get

$$\begin{aligned} \int_{\Omega} |U - \chi|^p dx &= \int_{\Omega} |U \cdot r - \chi \cdot r|^p + |\Pi_{r^{\perp}} U - \Pi_{r^{\perp}} \chi|^p dx \\ &\geq \int_{\Omega} |U \cdot r - f_r|^p + \text{dist}^p(\Pi_{r^{\perp}} U, \Pi_{r^{\perp}} \mathcal{K}) dx = \int_{\Omega} |U - \tilde{\chi}|^p dx. \end{aligned}$$

In conclusion, by Corollary 4.2, we obtain

$$\begin{aligned} \int_{\Omega} |U - \chi|^p + \epsilon^q |\partial_{\nu}(U \cdot r)|^q dx &\geq C \int_{\Omega} |U - \tilde{\chi}|^p + |U \cdot r - \chi \cdot r|^p + \epsilon^q |\partial_{\nu}(U \cdot r)|^q dx \\ &\geq C \int_{\Omega} |U - \tilde{\chi}|^p dx + C \epsilon \|D_{\nu} f_r\|_{TV(\Omega)} \\ &\geq C \left(\int_{\Omega} |U - \tilde{\chi}|^p dx + \epsilon \|D_{\nu}(\tilde{\chi} \cdot r)\|_{TV(\Omega)} \right). \end{aligned}$$

□

4.2. Diffuse to sharp interface model – the upper bound. Based on the estimates in Proposition 4.3 and Theorem 3 in this section we provide a complementary upper bound. Due to the presence of “lower order errors”, on its own, this upper bound does not show that the diffuse and sharp interface models display the same scaling behaviour. However, with the knowledge of known upper bounds (e.g. in our prototypical model scenarios), it implies that for all our applications the diffuse and sharp interface models admit the same ϵ scaling.

Lemma 4.4. *Let $\Omega = (0, 1)^d$, $p, q \in [1, \infty)$. For any $U \in L^p(\mathbb{R}^d; \mathbb{R}^n) \cap BV(\mathbb{R}^d; \mathbb{R}^n)$ with $U = 0$ outside Ω , $\chi \in L^{\infty}(\mathbb{R}^d; \mathbb{R}^n) \cap BV(\mathbb{R}^d; \mathbb{R}^n)$, and $\epsilon > 0$ there exist $U_{\epsilon} \in C_0^{\infty}(\Omega; \mathbb{R}^n)$ and a constant $C = C(\|\chi\|_{L^{\infty}}, p, q, d) > 0$ such that*

$$\begin{aligned} \int_{\Omega} |U_{\epsilon} - \chi|^p + \epsilon^q |\nabla U_{\epsilon}|^q dx &\leq C \left(\int_{\Omega} |U - \chi|^p dx + \epsilon (1 + \max\{\|U\|_{L^{\infty}}^{p-1}, \|U\|_{L^{\infty}}^{q-1}\}) \|D\chi\|_{TV(\Omega)} \right. \\ &\quad \left. + \epsilon \max\{\|U\|_{L^{\infty}}^{p-1}, \|U\|_{L^{\infty}}^{q-1}\} (\|DU - D\chi\|_{TV(\mathbb{R}^d)} + \text{Per}(\Omega)) \right). \end{aligned}$$

Moreover, if we have $\text{curl } U = 0$ or $\text{curl curl } U = 0$ in $\mathbb{R}^d$ it also holds $\text{curl } U_{\epsilon} = 0$ or $\text{curl curl } U_{\epsilon} = 0$ in $\mathbb{R}^d$, respectively.

We highlight that similar arguments would also work for more general $\mathcal{A}$ -free differential inclusions as in [RRT24].

Proof. We split the proof into two parts, first presenting the construction of the main estimate without adjusting the boundary conditions of the function U_ϵ and then correcting for this.

Step 1: The estimate without the boundary conditions. Let us start by giving the argument for $U \in L^p(\mathbb{R}^d; \mathbb{R}^n) \cap BV(\mathbb{R}^d; \mathbb{R}^n)$ with $U = 0$ outside Ω and $\chi \in L^\infty(\mathbb{R}^d; \mathbb{R}^n) \cap BV(\mathbb{R}^d; \mathbb{R}^n)$, but without preserving the “boundary condition”. We use this to show the full result with the preserved boundary conditions in what follows below. We set $U_\epsilon = U * \psi_\epsilon$, where ψ_ϵ is a radially symmetric mollifier with $\psi_\epsilon(x) = \epsilon^{-d} \psi(\epsilon^{-1}x)$, $\int_{\mathbb{R}^d} \psi_\epsilon dx = 1$, $\text{supp}(\psi_\epsilon) \subset B_\epsilon(0)$.

First let us consider some auxiliary results. By Young’s convolution inequality, it holds for any $f \in L^r(\mathbb{R}^d; \mathbb{R}^n)$, $1 \leq r \leq \infty$, that

$$(50) \quad \|\nabla(f * \psi_\epsilon)\|_{L^r} \leq \|\nabla \psi_\epsilon\|_{L^1} \|f\|_{L^r} \leq \epsilon^{-1} \|\nabla \psi\|_{L^1} \|f\|_{L^r},$$

where we used

$$\int_{\mathbb{R}^d} |\nabla \psi_\epsilon(x)| dx = \int_{\mathbb{R}^d} \epsilon^{-d-1} |\nabla \psi(x/\epsilon)| dx = \epsilon^{-1} \|\nabla \psi\|_{L^1}.$$

Next, since $U \in BV(\mathbb{R}^d; \mathbb{R}^n)$,

$$\begin{aligned} \int_{\mathbb{R}^d} U_\epsilon(x) \cdot \text{div} \phi(x) dx &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} U(x) \psi_\epsilon(y) \cdot \text{div} \phi(x+y) dy dx \\ &= \int_{\mathbb{R}^d} U(x) \cdot \text{div}(\phi * \psi_\epsilon)(x) dx \leq \|DU\|_{TV(\mathbb{R}^d)}, \end{aligned}$$

and thus

$$(51) \quad \|\nabla U_\epsilon\|_{L^1} \leq \|DU\|_{TV(\mathbb{R}^d)}.$$

The same then holds for χ , and $U - \chi$. As a last ingredient, we observe that

$$(52) \quad \|\chi * \psi_\epsilon - \chi\|_{L^1} \leq C\epsilon \|D\chi\|_{TV(\mathbb{R}^d)}.$$

With this we can prove the claim. Starting with the diffuse elastic energy, we have

$$\begin{aligned} \int_{\mathbb{R}^d} |U_\epsilon - \chi|^p dx &\leq C(p) (\|(U - \chi) * \psi_\epsilon\|_{L^p}^p + \|\chi * \psi_\epsilon - \chi\|_{L^p}^p) \\ &\leq C(p) \left(\|U - \chi\|_{L^p}^p + \|\chi * \psi_\epsilon - \chi\|_{L^\infty}^{p-1} \|\chi * \psi_\epsilon - \chi\|_{L^1} \right) \\ &\leq C(p) \left(\|U - \chi\|_{L^p}^p + \epsilon \|\chi\|_{L^\infty}^{p-1} \|D\chi\|_{TV(\mathbb{R}^d)} \right), \end{aligned}$$

where in the last step we applied (52).

For the diffuse surface energy, by (50) and (51) we obtain that

$$\begin{aligned} \int_{\mathbb{R}^d} |\nabla U_\epsilon|^q dx &\leq \|\nabla U_\epsilon\|_{L^\infty}^{q-1} \left(\|\nabla(U - \chi) * \psi_\epsilon\|_{L^1} + \|\nabla \chi * \psi_\epsilon\|_{L^1} \right) \\ &\leq \epsilon^{1-q} \|\nabla \psi\|_{L^1}^{q-1} \|U\|_{L^\infty}^{q-1} \left(\|DU - D\chi\|_{TV(\mathbb{R}^d)} + \|D\chi\|_{TV(\mathbb{R}^d)} \right). \end{aligned}$$

As a consequence of the two inequalities above we obtain that

$$(53) \quad \begin{aligned} \int_{\mathbb{R}^d} |U_\epsilon - \chi|^p + \epsilon^q |\nabla(U_\epsilon)|^q dx &\leq C(\|\chi\|_{L^\infty}, p, q) \left(\int_{\mathbb{R}^d} |U - \chi|^p dx + \epsilon \|D\chi\|_{TV(\mathbb{R}^d)} \right. \\ &\quad \left. + \epsilon \|U\|_{L^\infty}^{q-1} \|D\chi\|_{TV(\mathbb{R}^d)} + \epsilon \|U\|_{L^\infty}^{q-1} \|DU - D\chi\|_{TV(\mathbb{R}^d)} \right). \end{aligned}$$

Step 2: Adjusting the boundary conditions.

Now we turn to the construction preserving the exterior data, i.e. we aim to construct $U_\epsilon \in L^p(\mathbb{R}^d; \mathbb{R}^n) \cap W^{1,q}(\mathbb{R}^d; \mathbb{R}^n)$ such that $U_\epsilon = 0$ outside Ω . Let $U \in L^p(\mathbb{R}^d; \mathbb{R}^n) \cap BV(\mathbb{R}^d; \mathbb{R}^n)$ such that $U = 0$ outside Ω and $\chi \in L^\infty(\mathbb{R}^d; \mathbb{R}^n) \cap BV(\mathbb{R}^d; \mathbb{R}^n)$.

By a translation we consider $\Omega' = (-1/2, 1/2)^d$, $V := U(\cdot + 1/2(1, \dots, 1)) : \Omega' \rightarrow \mathbb{R}^n$, and $\chi' := \chi(\cdot + 1/2(1, \dots, 1)) : \Omega' \rightarrow \mathbb{R}^n$ in the following. We fix

$$\Omega'_\epsilon := (1 - 2\epsilon)\Omega' = (-\frac{1}{2} + \epsilon, \frac{1}{2} - \epsilon)^d \subset \Omega'.$$

In particular, we have

$$\text{dist}(\partial\Omega', \partial\Omega'_\epsilon) \geq \epsilon.$$

We now choose $V_\epsilon = V^{(\epsilon)} * \psi_\epsilon$, where

$$V^{(\epsilon)}(y) = V\left(\frac{1}{1-2\epsilon}y\right), \quad y \in \mathbb{R}^d.$$

As $V = 0$ outside Ω' , we can infer $V_\epsilon = 0$ outside Ω' . We now compare the energy for V_ϵ to that of $V * \psi_\epsilon$, which we can control with the arguments from step 1, as follows

$$\begin{aligned} \int_{\Omega'} |V_\epsilon - \chi'|^p + \epsilon^q |\nabla V_\epsilon|^q dy &\leq C(p, q) \left(\int_{\Omega'} |V * \psi_\epsilon - \chi'|^p + \epsilon^q |\nabla(V * \psi_\epsilon)|^q dy \right. \\ &\quad \left. + \|V_\epsilon - V * \psi_\epsilon\|_{L^p}^p + \epsilon^q \|\nabla V_\epsilon - \nabla(V * \psi_\epsilon)\|_{L^q}^q \right). \end{aligned}$$

Hence, it remains to control

$$\begin{aligned} \|(V^{(\epsilon)} - V) * \psi_\epsilon\|_{L^p}^p + \epsilon^q \|\nabla(V^{(\epsilon)} - V) * \psi_\epsilon\|_{L^q}^q &\leq \|V^{(\epsilon)} - V\|_{L^p}^p + C\|V^{(\epsilon)} - V\|_{L^q}^q \\ &\leq C \max\{\|U\|_{L^\infty}^{p-1}, \|U\|_{L^\infty}^{q-1}\} \|V^{(\epsilon)} - V\|_{L^1}, \end{aligned}$$

where we invoked Young's convolution inequality and (50). Thus, using that $V, V^{(\epsilon)} = 0$ outside Ω' it holds

$$\begin{aligned} \int_{\mathbb{R}^d} |V^{(\epsilon)}(x) - V(x)| dx &\leq C \int_1^{\frac{1}{1-2\epsilon}} t^{-d} dt \|DV\|_{TV(\mathbb{R}^d)} \leq C(1 - (1-2\epsilon)^d) \|DV\|_{TV(\mathbb{R}^d)} \\ &\leq C(d)\epsilon \|DU\|_{TV(\mathbb{R}^d)}, \end{aligned}$$

and, therefore, for $U_\epsilon(x) = V_\epsilon(x - 1/2(1, \dots, 1))$

$$\begin{aligned} \int_{\Omega} |U_\epsilon - \chi|^p + \epsilon^q |\nabla U_\epsilon|^q dx &\leq C(\|\chi\|_{L^\infty}, p, q, d) \left(\int_{\Omega} |U - \chi|^p dx \right. \\ &\quad \left. + \epsilon(1 + \max\{\|U\|_{L^\infty}^{p-1}, \|U\|_{L^\infty}^{q-1}\}) \|D\chi\|_{TV(\Omega)} \right. \\ &\quad \left. + \epsilon \max\{\|U\|_{L^\infty}^{p-1}, \|U\|_{L^\infty}^{q-1}\} (\|DU - D\chi\|_{TV(\mathbb{R}^d)} + \text{Per}(\Omega)) \right). \end{aligned}$$

As U_ϵ is a combination of a rescaling and convolution of U , the differential constraint is preserved. $\square$

For completeness, we also derive an estimate of the above type for the anisotropic setting.

Corollary 4.5. *Let $p, q \in [1, \infty)$, $\nu \in \mathbb{S}^{d-1}$, $\Omega = (0, 1)^d$. For any $U, \chi \in L^p(\mathbb{R}^d; \mathbb{R}^n) \cap BV_\nu(\mathbb{R}^d; \mathbb{R}^n)$, with $U = 0$ outside Ω and $\epsilon > 0$ there exists $U_\epsilon \in L^p(\mathbb{R}^d; \mathbb{R}^n)$ such that $\partial_\nu U_\epsilon \in L^q(\mathbb{R}^d; \mathbb{R}^n)$, $U_\epsilon = 0$ outside Ω , and a constant $C = C(\|\chi\|_{L^\infty}, p, q, d) > 0$ such that*

$$\begin{aligned} \int_{\Omega} |U_\epsilon - \chi|^p + \epsilon^q |\partial_\nu U_\epsilon|^q dx &\leq C \left(\int_{\Omega} |U - \chi|^p dx + \epsilon(1 + \max\{\|U\|_{L^\infty}^{p-1}, \|U\|_{L^\infty}^{q-1}\}) \|D_\nu \chi\|_{TV(\Omega)} \right. \\ &\quad \left. + \epsilon \max\{1, \|U\|_{L^\infty}^{p-1}, \|U\|_{L^\infty}^{q-1}\} (\|D_\nu U - D_\nu \chi\|_{TV(\Omega)} + \text{Per}(\Omega)) \right) \end{aligned}$$

Moreover, if we have $\text{curl } U = 0$ or $\text{curl curl } U = 0$ in $\mathbb{R}^d$ it also holds $\text{curl } U_\epsilon = 0$ or $\text{curl curl } U_\epsilon = 0$, respectively.

Proof. For given $U \in L^p(\mathbb{R}^d; \mathbb{R}^n)$, we define $U_\epsilon(x) = \int_{\mathbb{R}} U(x - t\nu)\psi_\epsilon(t)dt$ with a one-dimensional mollifier ψ_ϵ . That is, by U_ϵ we denote the function which is obtained by mollifying U in direction ν on a scale ϵ ; we omit the ν dependence in the notation. Due to the fact that $(U_\epsilon)_y^\nu(t) = (U_y^\nu * \psi_\epsilon)(t)$, using the notation introduced in Section 2.2, all the required estimates (50) to (52) have their respective analogous inequalities:

$$\begin{aligned}\|\partial_\nu U_\epsilon\|_{L^r} &\leq \epsilon^{-1} \|\psi'\|_{L^1(\mathbb{R})} \|U\|_{L^r}, \\ \|\partial_\nu U_\epsilon\|_{L^1} &\leq \|D_\nu U\|_{TV(\mathbb{R}^d)}, \\ \|U_\epsilon - U\|_{L^1} &\leq C\epsilon \|D_\nu U\|_{TV(\mathbb{R}^d)}.\end{aligned}$$

Hence, the proof works as that of Lemma 4.4. In particular, the changes required to not change the exterior data outside Ω are done by the same methods.

As the function U_ϵ is essentially defined via a convolution and scaling, the differential constraint of the form $\text{curl } U_\epsilon = 0$ or $\text{curl curl } U_\epsilon = 0$ is still fulfilled in $\mathbb{R}^d$. $\square$

Remark 4.6. *We want to conclude this general analysis with two remarks. First, the same results as in Sections 4.1 and 4.2 hold if we consider periodic functions instead of prescribing the exterior data outside of Ω . Second, we could also consider other differential constraints besides $\text{curl } U = 0$ or $\text{curl curl } U = 0$ in $\mathbb{R}^d$. In general, in Lemma 4.4 and Corollary 4.5 we can preserve the constraint $L(D)U = 0$ in $\mathbb{R}^d$ for any linear, homogeneous, constant coefficient differential operator $L(D)$, e.g. we could also consider $\text{div } U = 0$ in $\mathbb{R}^d$. The role of the divergence operator for lower scaling bounds was discussed in [RRT23].*

4.3. Applications – proof of Corollary 1.5. We now use the derived comparison results of the diffuse and sharp interface models to show Corollary 1.5.

Proof of Corollary 1.5. Let us begin by recalling the upper bound constructions from Section 3 for Theorem 1 for $U = \nabla u$. We recall that, as highlighted in (33) and in the proofs of Proposition 1.1 and Theorem 1, we always have

$$\begin{aligned}\|DU - D\chi\|_{TV(\Omega)} &\leq C(\mathcal{K}, F)(\|D\chi\|_{TV(\Omega)} + \text{Per}(\Omega)), \\ \|U\|_{L^\infty} &\leq C(\mathcal{K}, F).\end{aligned}$$

Thus, considering the functions $u_\epsilon, \chi_\epsilon$ defined in the respective upper bound, we use Theorem 3 and Corollary 4.5 (with $U = \nabla u_\epsilon, \chi = \chi_\epsilon$) to derive the scaling bounds. Indeed, we obtain that

$$\begin{aligned}&^c \inf_{\tilde{\chi} \in BV_\nu(\Omega; \mathcal{K}_3)} \inf_{u \in \mathcal{A}_F} \left(\int_{\Omega} |\nabla u - \tilde{\chi}|^2 dx + \epsilon \|D_\nu \tilde{\chi}\|_{TV(\Omega)} \right) \\ &\leq \inf_{\chi \in L^2(\Omega; \mathcal{K}_3)} \inf_{\substack{u \in \mathcal{A}_F \\ \partial_\nu \nabla u \in L^q(\Omega; \mathbb{R}^{2 \times 2})}} \left(\int_{\Omega} |\nabla u - \chi|^2 + \epsilon^q |\partial_\nu \nabla u|^q dx \right) \\ &\leq C \left(\int_{\Omega} |\nabla u_\epsilon - \chi_\epsilon|^2 dx + \epsilon \|D_\nu \chi_\epsilon\|_{TV(\Omega)} + \epsilon \text{Per}(\Omega) + \epsilon \|D_\nu \nabla u_\epsilon - D_\nu \chi_\epsilon\|_{TV(\Omega)} \right) \\ &\leq C \left(\int_{\Omega} |\nabla u_\epsilon - \chi_\epsilon|^2 dx + \epsilon \|D_\nu \chi_\epsilon\|_{TV(\Omega)} + \epsilon \text{Per}(\Omega) \right),\end{aligned}$$

where C may depend on $\mathcal{K}$ and F . In particular, after an absorption of the perimeter term, both the upper and lower bounds from Corollary 1.5 are proven by inserting the upper and lower bounds for the associated sharp interface models. $\square$

5. DISCRETE MODELS AND ANISOTROPIC SURFACE ENERGIES

As a final prototypical example, we complement our discussion on possible regularizations by now considering the case of discrete models. Indeed, discrete systems prevent oscillations at scales smaller than the grid size of the lattice. As already known in the literature (see e.g. [CM99, Lor09]), this produces a regularization effect comparable to the addition of a singular surface energy term in the continuous case. In our discussion, we recover these results in our model settings. Our main focus and novelty here, however, is on the analysis of anisotropic situations. In this context, the orientation of the lattice naturally introduces an anisotropy which may affect the scaling depending on the geometry of $\mathcal{K}$. This effect can be tracked into its continuous counterpart. In particular, this will allow us to invoke the scaling results from the previous sections.

5.1. Quantitative surface penalization in discrete models. In this section we consider discrete energies in the sense that, for $h \in (0, 1)$ and $R \in SO(2)$, we fix a triangulation $\mathcal{T}_h^R$ of Ω on the scale h and assume that ∇u and χ are constant on the triangles $\tau \in \mathcal{T}_h^R$. By doing this we hence rule out – and, in particular, “penalize” – oscillations on a scale finer than h .

For this reason, in certain instances, this discrete energy can be bounded from below by the continuous elastic energy contribution singularly perturbed by a sharp (anisotropic) surface penalization of the form given in (2).

To observe this, we will make the energy contribution of three adjacent triangles explicit, and note that when a change of phase occurs, the “middle” triangle T'_h has to pay elastic energy, giving rise to a contribution which resembles the surface energy $\|D_\nu \chi\|_{TV(\Omega)}$. We recall the notation for the lattice structures from (10), (11), the admissible deformations from (12) and the discrete energy from (13). We begin by deducing “interior” estimates. In the subsequent result, we will also incorporate associated boundary conditions.

In what follows, we focus on the anisotropic setting, more precisely, on the case in which we assume that there is a compatible direction of the wells which is perpendicular to one of the sides of the triangles. We allow for rank-one connections which are in exactly one of the directions Re_1 , Re_2 , or $R(e_1 + e_2)$ for a fixed matrix $R \in SO(2)$. We note that this cannot yield a full surface penalization for the associated continuum model but will give rise to an anisotropic surface penalization.

Lemma 5.1. *Let $\mathcal{K} = \{A_1, A_2, \dots, A_N\} \subset \mathbb{R}^{2 \times 2}$. Let T_h, T'_h denote the triangles from (10). Assume that there exists $v \in \{e_1, e_2, 2^{-1/2}(e_1 + e_2)\}$ such that for any $A_j, A_k \in \mathcal{K}$ with $j \neq k$ it holds*

$$(54) \quad (Rw) \times [A_j - A_k] \neq 0, \quad \text{for every } w \in \{e_1, e_2, 2^{-1/2}(e_1 + e_2)\} \setminus \{v\}.$$

Let $p \in [1, \infty)$, $z \in h\mathbb{Z}^2$ and consider

$$(55) \quad \Omega_j := (RT_h \cup RT'_h \cup (RT_h + hRe_j)) + Rz,$$

for $j = 1, 2$, illustrated in Figure 10. Then there is a constant $C = C(\mathcal{K}, R, p, v) > 0$ such that for any $u \in W^{1,p}(\Omega_1 \cup \Omega_2; \mathbb{R}^2)$ and any $\chi \in L^\infty(\Omega_1 \cup \Omega_2; \mathcal{K})$ which are affine or, respectively, constant on the triangles $R(T_h + z), R(T'_h + z), R(T_h + z + he_1), R(T_h + z + he_2)$, it holds

$$\int_{\Omega_1 \cup \Omega_2} |\nabla u - \chi|^p dx \geq Ch \|D_{R\nu} \chi\|_{TV(\Omega_1 \cup \Omega_2)},$$

where $\nu \in \mathbb{S}^1$ is such that $\nu \cdot v = 0$.

Moreover, defining

$$\Omega'_j := (RT_h \cup RT'_h \cup (RT'_h - hRe_j)) + Rz,$$

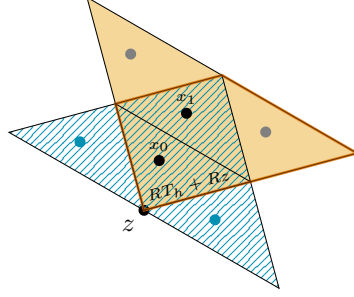


FIGURE 10. Illustration of the sets in Lemma 5.1 for a rotation R by 15° . The set $\Omega_1 \cup \Omega_2$ is filled in orange. The fixed points x_0, x_1 are marked in black and their translations $x_0 + he_j$ are shown in gray. The set Ω_1 is highlighted with a red border. The corresponding setting for $\Omega'_1 \cup \Omega'_2$ is shown in blue.

for $j = 1, 2$, the same result holds with Ω'_j in place of Ω_j .

Proof. Without loss of generality, we can assume $z = 0$. We also start by considering $R = Id$; the general case will be reduced to this one below. We fix $x_0 \in T_h$ and $x_1 \in T'_h$ (see Figure 3). For simplicity of exposition we prove the estimate for $\Omega_1 \cup \Omega_2$ only, with the estimate on $\Omega'_1 \cup \Omega'_2$ being completely analogous.

Step 1: The case of $R = Id$. We first quantify the energy contribution of a possible change of phase in Ω_j . Its main contribution concentrates on T'_h and amounts to

$$(56) \quad \int_{T'_h} |\nabla u - \chi|^p dx \geq ch^2 |e_j \times (\chi(x_0 + he_j) - \chi(x_1))| - 2^p \int_{T_h \cup (T_h + he_j)} |\nabla u - \chi|^p dx,$$

for some $c = c(\mathcal{K}, p) > 0$. To prove (56), we first work on Ω_1 , i.e. with $j = 1$. The estimate for Ω_2 follows by switching the roles of the coordinates. Since u is affine on each triangle in $\Omega_1 \cup \Omega_2$, by tangential continuity along the edges of T'_h it holds that

$$\partial_1 u(x_1) = \partial_1 u(x_0 + he_2), \quad \partial_2 u(x_1) = \partial_2 u(x_0 + he_1), \quad \partial_1 u(x_1) - \partial_2 u(x_1) = \partial_1 u(x_0) - \partial_2 u(x_0),$$

and, hence,

$$(57) \quad \begin{aligned} \partial_1 u(x_1) &= \partial_2 u(x_0 + he_1) + \partial_1 u(x_0) - \partial_2 u(x_0) = \partial_1 u(x_0 + he_2), \\ \partial_2 u(x_1) &= \partial_2 u(x_0 + he_1) = \partial_1 u(x_0 + he_2) + \partial_2 u(x_0) - \partial_1 u(x_0). \end{aligned}$$

By exploiting the fact that $A^{\frac{p}{2}} + B^{\frac{p}{2}} \leq \sqrt{2}(A + B)^{\frac{p}{2}}$ for $A, B \geq 0$, relation (57), by Jensen's inequality (in the form $|A - B|^p - 2^{p-1}|B|^p \leq 2^{p-1}|A|^p$), and writing $\chi_j = \chi e_j$ for the j th column of χ , we obtain

$$\begin{aligned} & \int_{T'_h} |\nabla u - \chi|^p dx \\ &= \frac{h^2}{2} (|\partial_2 u(x_0 + he_1) + \partial_1 u(x_0) - \partial_2 u(x_0) - \chi_1(x_1)|^2 + |\partial_2 u(x_0 + he_1) - \chi_2(x_1)|^2)^{\frac{p}{2}} \\ &\geq \frac{h^2}{2} \frac{1}{2^p \sqrt{2}} (|\partial_1 u(x_0) - \partial_2 u(x_0) - (\chi_1(x_1) - \chi_2(x_1))|^p + |\partial_2 u(x_0 + he_1) - \chi_2(x_1)|^p) \\ &\geq \frac{h^2}{2} \frac{1}{2^p \sqrt{2}} \left(2^{1-p} |\chi_1(x_0) - \chi_1(x_1) - (\chi_2(x_0) - \chi_2(x_1))|^p + 2^{1-p} |\chi_2(x_0 + he_1) - \chi_2(x_1)|^p \right. \\ &\quad \left. - |\partial_1 u(x_0) - \partial_2 u(x_0) - (\chi_1(x_0) - \chi_2(x_0))|^p - |\partial_2 u(x_0 + he_1) - \chi_2(x_0 + he_1)|^p \right). \end{aligned}$$

Let $C_0 = \min \{ |w \times (A_j - A_k)| : j \neq k, w \in \{e_1, e_2, 2^{-1/2}(\frac{1}{1})\} \setminus \{v\} \} > 0$ which is strictly positive thanks to (54). For simplicity of exposition, in the next two estimates we assume $v = 2^{-1/2}(e_1 + e_2)$. Hence, we deduce

$$\begin{aligned}
& \int_{T'_h} |\nabla u - \chi|^p dx \\
& \geq 2^{-2p-1} h^2 \left(\left| \left(\frac{1}{1} \right) \times (\chi(x_0) - \chi(x_1)) \right|^p + |e_1 \times (\chi(x_0 + he_1) - \chi(x_1))|^p \right) \\
& \quad - \frac{h^2}{2} \left(|\partial_1 u(x_0) - \partial_2 u(x_0) - (\chi_1(x_0) - \chi_2(x_0))|^p + |\partial_2 u(x_0 + he_1) - \chi_2(x_0 + he_1)|^p \right) \\
& \geq 2^{-2p-1} h^2 \left(\left| \left(\frac{1}{1} \right) \times (\chi(x_0) - \chi(x_1)) \right|^p + |e_1 \times (\chi(x_0 + he_1) - \chi(x_1))|^p \right) \\
& \quad - \frac{h^2}{2} \left(2^p |\nabla u(x_0) - \chi(x_0)|^p + |\nabla u(x_0 + he_1) - \chi(x_0 + he_1)|^p \right) \\
& \geq 2^{-2p-1} C_0^{p-1} h^2 |e_1 \times (\chi(x_0 + he_1) - \chi(x_1))| - 2^p \int_{T_h \cup T_h + he_1} |\nabla u - \chi|^p dx.
\end{aligned}$$

Here, we used $(\frac{1}{1}) \times \nabla u = (\frac{-1}{1}) \cdot \nabla u = \partial_2 u - \partial_1 u$ to denote the tangential components for the edges with normal $(\frac{1}{1})$, see the notation introduced in Section 1.4. This proves (56) with $c = 2^{-2p-1} C_0^{p-1} > 0$ depending only on $\mathcal{K}$, v , and p .

We now combine (56) with the analogous inequality with e_1 and e_2 switched. We thus have

$$\begin{aligned}
& \int_{\Omega_1 \cup \Omega_2} |\nabla u - \chi|^p dx \\
& \geq \int_{T_h \cup (T_h + he_1) \cup (T_h + he_2)} |\nabla u - \chi|^p dx + 2^{-p-1} \int_{T'_h} |\nabla u - \chi|^p dx + 2^{-p-1} \int_{T'_h} |\nabla u - \chi|^p dx \\
& \geq ch^2 2^{-p-1} \sum_{j=1}^2 \left| e_j \times [\chi(x_0 + he_j) - \chi(x_1)] \right| \\
& \quad + (1 - 2^{-1} - 2^{-1}) \int_{T_h} |\nabla u - \chi|^p dx + (1 - 2^{-1}) \int_{(T_h + he_1) \cup (T_h + he_2)} |\nabla u - \chi|^p dx \\
& \geq c 2^{-p-1} h^2 \sum_{j=1}^2 \left| e_j \times [\chi(x_0 + he_j) - \chi(x_1)] \right|.
\end{aligned}$$

For general $v \in \{e_1, e_2, 2^{-1/2}(e_1 + e_2)\}$ we follow the same arguments as above and get

$$\begin{aligned}
(58) \quad \int_{\Omega_1 \cup \Omega_2} |\nabla u - \chi|^p dx & \geq c 2^{-p-1} h^2 \left(\sum_{j=1}^2 \left| e_j \times [\chi(x_0 + he_j) - \chi(x_1)] \right| + \left| \left(\frac{1/\sqrt{2}}{1/\sqrt{2}} \right) \times [\chi(x_0) - \chi(x_1)] \right| \right. \\
& \quad \left. - \left| v \times [\chi(x_v) - \chi(x_1)] \right| \right),
\end{aligned}$$

where $x_v = x_0$ if $v = 2^{-1/2}(e_1 + e_2)$, $x_v = x_0 + he_j$ if $v = e_j$, such that exactly the term with v cancels.

To relate this expression to the desired surface energy, we note that for $\nu \in \mathbb{S}^1$ such that $\nu \cdot v = 0$, it follows that

$$\begin{aligned} \|D_\nu \chi\|_{TV(\Omega_1 \cup \Omega_2)} &= h|\nu \cdot 2^{-1/2}(e_1 + e_2)| |\chi(x_0) - \chi(x_1)| + \sum_{j=1}^2 h|\nu \cdot e_j| |\chi(x_1) - \chi(x_0 + he_j)| \\ &\leq Ch \left(|\chi(x_0) - \chi(x_1)| + \sum_{j=1}^2 |\chi(x_1) - \chi(x_0 + he_j)| - |\chi(x_1) - \chi(x_v)| \right). \end{aligned}$$

From (58), exploiting the boundedness of χ , we can also infer that

$$\begin{aligned} \int_{\Omega_1 \cup \Omega_2} |\nabla u - \chi|^p dx &\geq c2^{-p-1} \frac{h^2}{\text{diam}(\mathcal{K})} \left(\left| \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \times [\chi(x_0) - \chi(x_1)] \right| |\chi(x_0) - \chi(x_1)| \right. \\ &\quad + \sum_{j=1}^2 \left| e_j \times [\chi(x_0 + he_j) - \chi(x_1)] \right| |\chi(x_0 + he_j) - \chi(x_1)| \\ &\quad \left. - \left| v \times [\chi(x_1) - \chi(x_v)] \right| |\chi(x_1) - \chi(x_v)| \right). \end{aligned}$$

Hence, gathering the two inequalities above and again by (54) for a constant $C = C(\mathcal{K}, p, v) > 0$ we get

$$\begin{aligned} \int_{\Omega_1 \cup \Omega_2} |\nabla u - \chi|^p dx &\geq Ch \left(h|\chi(x_0) - \chi(x_1)| + \sum_{j=1}^2 h|\chi(x_0 + he_j) - \chi(x_1)| - h|\chi(x_1) - \chi(x_v)| \right) \\ &\geq Ch \|D_\nu \chi\|_{TV(\Omega_1 \cup \Omega_2)}. \end{aligned}$$

Step 2: General rotations $R \in SO(2)$. Now we turn to the general (rotated) case. For given $R \in SO(2)$, consider $\tilde{\Omega}_j = R^T \Omega_j = (T_h \cup T'_h \cup (T_h + he_j)) + z$ and the functions

$$\begin{aligned} \tilde{u} : \tilde{\Omega}_1 \cup \tilde{\Omega}_2 &\rightarrow \mathbb{R}^2, & \tilde{u}(x) &= u(Rx), \\ \tilde{\chi} : \tilde{\Omega}_1 \cup \tilde{\Omega}_2 &\rightarrow \mathcal{K}R, & \tilde{\chi}(x) &= \chi(Rx)R. \end{aligned}$$

For those functions it holds that $\nabla \tilde{u}(x) = \nabla u(Rx)R$, and thus

$$\begin{aligned} \int_{\Omega_1 \cup \Omega_2} |\nabla u - \chi|^p dx &= \int_{R^T(\Omega_1 \cup \Omega_2)} |\nabla u(Rx) - \chi(Rx)|^p dx = \int_{\tilde{\Omega}_1 \cup \tilde{\Omega}_2} |\nabla \tilde{u} - \tilde{\chi}|^p dx, \\ \|D_{R\nu} \chi\|_{TV(\Omega_1 \cup \Omega_2)} &= \|D_\nu \tilde{\chi}\|_{TV(\tilde{\Omega}_1 \cup \tilde{\Omega}_2)}. \end{aligned}$$

For the wells $A_j, A_k \in \mathcal{K}$, $j \neq k$, from (54) it holds

$$e_j \times (A_j R - A_k R) = (R e_j) \times (A_j - A_k), \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix} \times (A_j R - A_k R) = (R \begin{pmatrix} 1 \\ 1 \end{pmatrix}) \times (A_j - A_k).$$

Therefore the set $\tilde{\mathcal{K}} = \mathcal{K}R$ fulfils the compatibility conditions (54) for $R = \text{Id}$. Hence, the claim follows by applying the argument for $R = \text{Id}$ to the functions $\tilde{u}$, $\tilde{\chi}$, and thus

$$\begin{aligned} \int_{\Omega_1 \cup \Omega_2} |\nabla u - \chi|^p dx &= \int_{\tilde{\Omega}_1 \cup \tilde{\Omega}_2} |\nabla \tilde{u} - \tilde{\chi}|^p dx \geq C(\tilde{\mathcal{K}}, p, v) h \|D_\nu \tilde{\chi}\|_{TV(\tilde{\Omega}_1 \cup \tilde{\Omega}_2)} \\ &= C(\mathcal{K}, R, p, v) h \|D_{R\nu} \chi\|_{TV(\Omega_1 \cup \Omega_2)}. \end{aligned}$$

□

With the above auxiliary result on “interior estimates” in hand, we now cover Ω with copies of $\Omega_1 \cup \Omega_2$ to relate the discretized setting to a sharp interface model. This, together with an

estimate of the energy contributions which arise at the boundary allows us to provide the proof of Theorem 5.

Proof of Theorem 5 in the anisotropic setting. Only the proof for the anisotropic setting is given, for the full isotropic surface penalization the arguments are analogous with the corresponding changes for the full derivative. By possibly replacing Ω by $R^T\Omega$, in the following we assume $R = \text{Id}$. The general result follows by a rotation as done in the proof of Lemma 5.1. Without loss of generality and for brevity of exposition, we assume that $v = 2^{-\frac{1}{2}}(1, 1)^T$ and thus $\nu = 2^{-\frac{1}{2}}(-1, 1)^T$. We work in multiple steps, separating the analysis in the interior of the domain and close to the boundary.

Step 1: Estimate on the interior. We cover Ω by sets of the form $\Omega_1^z \cup \Omega_2^z$ as in (55), i.e.

$$\Omega_1^z \cup \Omega_2^z = (T_h \cup T_h' \cup (T_h + he_1) \cup (T_h + e_2)) + z,$$

where we now keep track of the dependence on $z \in h\mathbb{Z}^2$ in the notation. We define the discrete set $\mathcal{J} := \{z \in h\mathbb{Z}^2 : (\Omega_1^z \cup \Omega_2^z) \subset \Omega\}$ and the set $\Omega_h := \cup_{z \in \mathcal{J}} \Omega_1^z \cup \Omega_2^z$. The interior set $\Omega_h \subset \Omega$ approximates Ω from the inside as $h \downarrow 0$ and may leave an h -neighbourhood of the boundary $\partial\Omega$ not yet covered, which will be addressed in Step 2.

As $\Omega_1^z \cup \Omega_2^z$ overlap at most six times for $z \in \mathcal{J}$, by an application of Lemma 5.1 we obtain the lower bound

$$(59) \quad \begin{aligned} \int_{\Omega} |\nabla u - \chi|^p dx &\geq \frac{1}{6} \sum_{z \in \mathcal{J}} \int_{(\Omega_1^z \cup \Omega_2^z) \cap \Omega} |\nabla u - \chi|^p dx \\ &\geq \sum_{z \in \mathcal{J}} C' h \|D_\nu \chi\|_{TV(\Omega_1^z \cup \Omega_2^z)} \geq C' h \|D_\nu \chi\|_{TV(\Omega_h)}, \end{aligned}$$

with $C' > 0$ depending on $\mathcal{K}$, v , and p .

Step 2: Boundary layer. To estimate the energy contribution of the boundary layer, we introduce the set of boundary triangles

$$\mathcal{I} := \{\tau \in \mathcal{T}_h : |\tau \cap \Omega| > 0, \mathcal{H}^1(\bar{\tau} \cap \partial\Omega) > 0\}.$$

For a constant $\sigma = \sigma(\Omega) \in (0, \frac{1}{2})$ which will be fixed below, we decompose the set of boundary triangles into $\mathcal{I} = \mathcal{I}_{\text{small}} \cup \mathcal{I}_1 \cup \mathcal{I}_2$, where

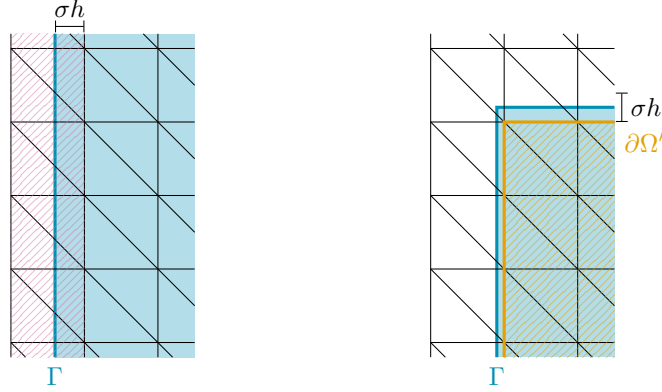
$$\begin{aligned} \mathcal{I}_{\text{small}} &:= \{\tau \in \mathcal{I} : |\tau \cap \Omega| < \frac{1}{2} \sigma h^2\}, \\ \mathcal{I}_1 &:= \{\tau \in \mathcal{I} \setminus \mathcal{I}_{\text{small}} : |\tau \setminus \Omega| > 0\}, \\ \mathcal{I}_2 &:= \{\tau \in \mathcal{I} \setminus \mathcal{I}_{\text{small}} : |\tau \setminus \Omega| = 0\}. \end{aligned}$$

The triangles in $\mathcal{I}_1$ and $\mathcal{I}_2$ have non-degenerate area inside Ω and, in particular, those in $\mathcal{I}_2$ have at least one edge that has a common line segment with $\partial\Omega$.

We will show that the energy on this boundary layer produces a contribution of order (at least) h . To prove this, the case in which the edges of Ω are aligned with the grid necessitates a careful treatment, since (a priori) in this case $u \in \mathcal{A}_{h,F}^{p,\text{Id}}$ could laminate with the boundary condition without paying any energy. We thus split this analysis into two substeps, depending on the geometry of the domain.

Step 2.1: Domains aligned with the grid. Consider first the case in which Ω is a polygon with edges parallel to the directions of the triangulation, i.e. such that $n_{\partial\Omega} \in \{\pm e_1, \pm e_2, \pm v\}$ $\mathcal{H}^1$ -a.e. on $\partial\Omega$, where $n_{\partial\Omega}$ is the outer unit normal vector to $\partial\Omega$. Notice that there exist $h_0 = h_0(\Omega) > 0$ and $C_0 > 0$ a universal constant such that, for every edge Γ of Ω and for every $h < h_0$ it holds

$$(60) \quad \#\{\tau \in \mathcal{I} : \bar{\tau} \cap \Gamma \neq \emptyset\} \geq \frac{C_0}{h} \mathcal{H}^1(\Gamma).$$



(A) Case 1: The distance of Γ to the (interior) edges of the triangulation is at least σh . The boundary triangles (hashed in purple) have uniform area inside Ω .
 (B) Case 2: The two edges have distance less than σh to the edges. We replace Ω with Ω' , which is highlighted in orange (hashed).

FIGURE 11. Illustration for cases 1, 2 in step 2. The interior of Ω is highlighted in blue with its boundary segment Γ parallel to the edges of the triangles.

We now split our discussion in two cases.

Case 1: Assume that there exists an edge Γ of Ω whose distance (towards the interior of Ω , i.e. oriented in direction $-n_\Gamma$) from the lattice lines orthogonal to n_Γ is larger than σh , see Figure 11(A). In formulas, we assumed that $d_h(\Gamma) \geq \sigma h$, where

$$d_h(\Gamma) := \min\{t \geq 0 : \Gamma - tn_\Gamma \subset \mathbb{L}_h(n_\Gamma)\}, \quad \mathbb{L}_h(n_\Gamma) := \begin{cases} h\mathbb{Z} \times \mathbb{R} & n_\Gamma = \pm e_1, \\ \mathbb{R} \times h\mathbb{Z} & n_\Gamma = \pm e_2, \\ \{(t, z - t) : t \in \mathbb{R}, z \in h\mathbb{Z}\} & n_\Gamma = \pm v. \end{cases}$$

In this case

$$\#\{\tau \in \mathcal{I}_1 : \bar{\tau} \cap \Gamma \neq \emptyset\} \geq \frac{1}{2} \#\{\tau \in \mathcal{I} : \bar{\tau} \cap \Gamma \neq \emptyset\}.$$

Since $u \in \mathcal{A}_{h,F}^{p,\text{Id}}$ and $\nabla u \equiv F$ outside Ω , for every $\tau \in \mathcal{I}_1$ we have that $u(x) = Fx$ on τ and hence, using $F \notin \mathcal{K}$,

$$\int_{\tau \cap \Omega} |\nabla u - \chi|^p dx = \int_{\tau \cap \Omega} |F - \chi|^p dx \geq \sigma C' h^2.$$

Consequently, by (60) and the estimates above, we get

$$(61) \quad \int_{\Omega} |\nabla u - \chi|^p dx \geq \sum_{\tau \in \mathcal{I}_1} \int_{\tau \cap \Omega} |\nabla u - \chi|^p dx \geq \sum_{\tau \in \mathcal{I}_1} \sigma C' h^2 \geq \frac{1}{2} C' C_0 \sigma h \mathcal{H}^1(\Gamma).$$

Case 2: If case 1 does not take place, the edges of Ω have distance (towards the interior) from the lattice lines (parallel to that edge) smaller than σh , i.e. $d_h(\Gamma) < \sigma h$. If there is one edge Γ for which this distance is not zero, we define $\Omega' \subset \Omega$ as the polygon obtained by replacing Γ with the closest lattice line orthogonal to n_Γ . Notice that, $u(x) = Fx$ also outside of Ω' . Up to repeating this procedure a finite number of times, we obtain that all the edges of Ω' lie on lattice lines, depicted in Figure 11(B). With a slight abuse of notation, we still denote Ω' as Ω , in particular we have that $\mathcal{I} = \mathcal{I}_2$.

In this case, due to the anisotropy, there might be an energy-free phase transition between F and $\mathcal{K}$ on the boundary triangles. We separate the cases in which this phase transition does not happen (case 2(a)) with the one in which the transition takes place, case 2(b). In case 2(b) we need global arguments to prove that the lower bound is non-degenerate.

For $w \in \{-e_1, -e_2, v\}$ we introduce the set

$$\mathcal{I}_2^w := \{\tau \in \mathcal{I}_2 : E_w(\tau) \subset \partial\Omega\},$$

where by $E_w(\tau)$ we denote the edge of τ with normal w . These are the sets of lattice triangle touching the edges of Ω that are orthogonal to w .

Case 2(a): First let us assume that there is one edge Γ of Ω with normal $w \in \{-e_1, -e_2, v\}$ such that in the majority of triangles at this edge χ is in an incompatible well to the boundary condition F , i.e.

$$\#\{\tau \in \mathcal{I}_2^w : \bar{\tau} \cap \Gamma \neq \emptyset, w \times [\chi - F] \neq 0 \text{ on } \tau\} \geq \frac{C_0}{2h} \mathcal{H}^1(\Gamma).$$

For any such triangle τ it then holds

$$\int_{\tau \cap \Omega} |\nabla u - \chi|^p dx \geq \int_{\tau} |w \times (F - \chi)|^p dx \geq C' h^2.$$

Here we have used that, as there is a line segment in $\partial\tau \cap \Gamma$ perpendicular to w , by the boundary condition and the fact that ∇u is constant on τ , we have

$$w \times \nabla u = w \times F.$$

This yields that

$$(62) \quad \int_{\Omega} |\nabla u - \chi|^p dx \geq \frac{1}{2} C' C_0 h \mathcal{H}^1(\Gamma).$$

This case can either happen, if all the wells are incompatible to F in direction w , or if χ is “badly” chosen.

Case 2(b): We are then left to consider the case in which for every edge of Ω (orthogonal to w) there exists (at least) one A_w compatible to F in direction w , i.e. $w \times (A_w - F) = 0$, and such that $\chi = A_w$ in the majority of the boundary triangles. Hence, there exist at least two (different) vectors $w_1, w_2 \in \{-e_1, -e_2, v\}$, two edges $\Gamma_1, \Gamma_2 \subset \partial\Omega$ (orthogonal to w_1 and w_2 respectively) and two matrices (up to relabelling) $A_1, A_2 \in \mathcal{K}$ such that $w_1 \times (F - A_1) = 0$, $w_2 \times (F - A_2) = 0$ and

$$\begin{aligned} \#\{\tau \in \mathcal{I}_2^{w_1} : \bar{\tau} \cap \Gamma_1 \neq \emptyset, \chi = A_1 \text{ on } \tau\} &\geq \frac{C_0}{2hN} \mathcal{H}^1(\Gamma_1), \\ \#\{\tau \in \mathcal{I}_2^{w_2} : \bar{\tau} \cap \Gamma_2 \neq \emptyset, \chi = A_2 \text{ on } \tau\} &\geq \frac{C_0}{2hN} \mathcal{H}^1(\Gamma_2). \end{aligned}$$

From the fact that $F \notin \mathcal{K}$ we also infer that $A_1 \neq A_2$. The factor $\frac{1}{N}$ here accounts for the fact that there might be multiple possible choices of A_1, A_2 .

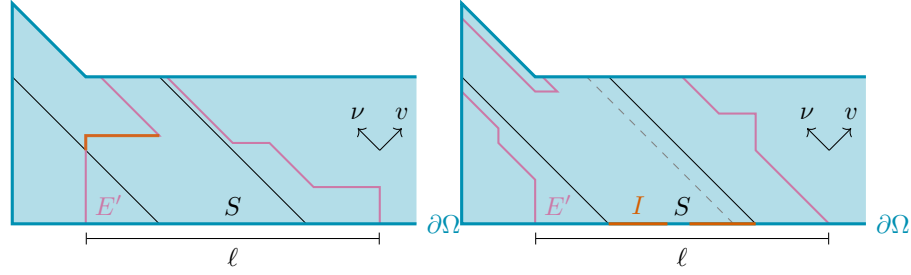
As the edges of Ω lie on the lattice lines, we can cover the whole Ω by copies of $\Omega_1^z \cup \Omega_2^z$ and its flipped version $(\Omega_1')^z \cup (\Omega_2')^z$. Hence, by reasoning as in (59) of Step 1, we have

$$\int_{\Omega} |\nabla u - \chi|^p dx \geq C' h \|D_\nu \chi\|_{TV(\Omega)}.$$

We now prove that, either there exists a non-degenerate interface oriented in a direction not orthogonal to v or we have a non-vanishing elastic energy contribution in the bulk.

As $w_1 \neq w_2$, we can assume, without loss of generality, that $w_2 = -e_2$. Let

$$E := \bigcup_{\tau \in \mathcal{T}_h : \chi \equiv A_2 \text{ on } \tau} \tau$$



(A) The case $\int_{\gamma_{\text{in}} \cap S} |\nu \cdot n_{\partial E'}| d\mathcal{H}^1 \geq \ell/(4\sqrt{2})$. The line segment relevant for the energy contribution in (63) is highlighted in orange.

(B) The case $\int_{\gamma_{\text{in}} \cap S} |\nu \cdot n_{\partial E'}| d\mathcal{H}^1 < \ell/4$. The line segment of length $f(x_1)$ in S with direction ν is shown as the dashed line. The set I for which we consider these line segments is highlighted in orange.

FIGURE 12. The two cases for the sets S and E' in case 2(b).

and let E' be a connected component of E whose closure intersects Γ_2 and let $\ell := \mathcal{H}^1(\partial E' \cap \Gamma_2)$. For simplicity, assume also $\partial E' \cap \Gamma_2 = (0, \ell) \times \{0\}$.

Denote $\gamma = \partial E' \setminus \Gamma_2$, which is a connected piecewise affine path. We decompose it into the part inside Ω and the part on its boundary, that is, we write $\gamma = \gamma_{\text{in}} \cup \gamma_{\text{b}} := (\gamma \setminus \partial\Omega) \cup (\gamma \cap \partial\Omega)$. Consider the stripe

$$S := \left\{ (x_1, x_2) \in \Omega : x_2 > 0, \frac{\ell}{4} < x_1 + x_2 < \frac{3\ell}{4} \right\},$$

and let $f(x_1)$ be the length of the segment in direction ν originating from $(x_1, 0)$ and ending on $\partial\Omega$ (but not necessarily in E'). Here, we recall that $v = 2^{-1/2}(1, 1)^T$ and $\nu \cdot v = 0$. Notice that, as Ω is a polygonal domain and $S \cap \Gamma_2$ is well-contained in Γ_2 , $f(x_1)$ is bounded from below, hence up to reducing the value of σ , $\sigma < f(x_1) < \text{diam}(\Omega)$ for every $x_1 \in (\ell/4, 3\ell/4)$. By connectedness of γ , we notice that

$$\int_{\gamma \cap \bar{S}} |\nu \cdot n_{\partial E'}| d\mathcal{H}^1 = \int_{\gamma_{\text{in}} \cap S} |\nu \cdot n_{\partial E'}| d\mathcal{H}^1 + \int_{\gamma_{\text{b}} \cap \bar{S}} |\nu \cdot n_{\partial E'}| d\mathcal{H}^1 \geq \frac{\ell}{2\sqrt{2}}.$$

Indeed, as E' is connected γ is a continuous path from $(0, 0)$ to $(\ell, 0)$ hence intersects ∂S both in $\{(\ell/4, 0) + t\nu : t \in \mathbb{R}\}$ and $\{(3\ell/4, 0) + t\nu : t \in \mathbb{R}\}$. By the choice of S , that is, $\partial S \cap \Omega$ has normal v , and the fact that the segment γ of $\partial E'$ consists of line segments with possible directions e_1, e_2 , and ν , $\gamma \cap S$ has to move a distance larger than $\ell/2$ in the directions e_1 or e_2 .

Since we have a control only of the interfaces inside Ω , if

$$\int_{\gamma_{\text{in}} \cap S} |\nu \cdot n_{\partial E'}| d\mathcal{H}^1 \geq \frac{\ell}{4\sqrt{2}}$$

then we infer that

$$(63) \quad \|D_\nu \chi\|_{TV(\Omega)} \geq C \int_{\gamma_{\text{in}} \cap S} |\nu \cdot n_{\partial E'}| d\mathcal{H}^2 \geq C_1 \ell,$$

with $C_1 > 0$ depending on $\mathcal{K}$.

Consider now the opposite case, i.e. that

$$\int_{\gamma_{\text{in}} \cap S} |\nu \cdot n_{\partial E'}| d\mathcal{H}^1 \leq \frac{\ell}{4\sqrt{2}} \leq \frac{\ell}{4}.$$

In this case, loosely speaking, we can prove that there is a union of stripes (contained in S and oriented in its same direction) fully contained in E' , in which we can perform an integration argument which provides a lower bound (see below). To be precise, let $I := \{x_1 \in (\ell/4, 3\ell/4) : ((x_1, 0) + \mathbb{R}\nu) \cap \gamma_{\text{in}} = \emptyset\}$ be the set of starting points on $\partial S \cap \Gamma_2$ such that the ray in direction ν does not intersect γ_{in} . Then, with a slicing argument

$$\begin{aligned} \frac{\ell}{4} &\geq \int_{\gamma_{\text{in}} \cap S} |\nu \cdot n_{\partial E'}| d\mathcal{H}^1 \\ &= \int_{\frac{\ell}{4}}^{\frac{3\ell}{4}} \mathcal{H}^0(\{x_2 \in (0, f(x_1)) : (x_1 - x_2, x_2) \in \gamma_{\text{in}}, n_{\partial E'}(x_1 - x_2, x_2) \neq \nu\}) dx_1 \geq \frac{\ell}{2} - |I|. \end{aligned}$$

Hence, after a suitable change of variables and an application of Jensen's inequality and the fundamental theorem of calculus, we get

$$\begin{aligned} \int_{E'} |\nabla u - \chi|^p dx &\geq \int_I \int_0^{f(x_1)} |\partial_\nu \tilde{u}(x_1 - x_2, x_2) - A_2 \nu|^p dx_2 dx_1 \\ &\geq \sigma \int_I \left| \frac{1}{f(x_1)} \int_0^{f(x_1)} (\partial_\nu u(x_1 - x_2, x_2) - A_2 \nu) dx_2 \right|^p dx_1 \\ &= \sigma \int_I \left| \frac{u(x_1 - f(x_1), f(x_1)) - u(x_1, 0)}{\sqrt{2} f(x_1)} - A_2 \nu \right|^p dx_1. \end{aligned}$$

Since, by the boundary conditions, $u(x_1 - f(x_1), f(x_1)) = F(x_1 - f(x_1), f(x_1))^T$, $u(x_1, 0) = F(x_1, 0)^T$, and $F - A_2 = a \otimes e_2$ for some $a \in \mathbb{R}^2 \setminus \{0\}$ we infer that

$$(64) \quad \int_{E'} |\nabla u - \chi|^p dx \geq \sigma \frac{\ell}{4} |(a \otimes e_2) \nu|^p \geq \sigma C_1 \ell,$$

up to reducing the value of C_1 if needed. Gathering (63) and (64) and repeating it for all the connected components of E we infer that also in this case

$$(65) \quad \int_{\Omega} |\nabla u - \chi|^p dx \geq C_1 \min\{h, \sigma\} \frac{C_0}{2} \mathcal{H}^1(\Gamma_2).$$

Step 2.2: Domains not aligned with the grid. Consider now a general polygonal domain not admissible in Step 2.1, namely a polygon with an edge Γ such that

$$(66) \quad \min \{|n_\Gamma - w| : w \in \{\pm e_1, \pm e_2, \pm v\}\} \geq \sigma_0,$$

for some $\sigma_0 \in (0, \frac{1}{2})$ depending on the domain, see Figure 13.

Due to the misalignment with the grid there are no boundary triangles around Γ in the set $\mathcal{I}_2$, namely, denoting $\mathcal{I}_\Gamma := \{\tau \in \mathcal{I} : \bar{\tau} \cap \Gamma \neq \emptyset\}$, we have

$$\mathcal{I}_\Gamma = (\mathcal{I}_\Gamma \cap \mathcal{I}_1) \cup (\mathcal{I}_\Gamma \cap \mathcal{I}_{\text{small}}).$$

We now show that, for every $h < h_0$, $\#(\mathcal{I}_\Gamma \cap \mathcal{I}_1) \geq C_0 h^{-1} \mathcal{H}^1(\Gamma)$, which then leads to the desired lower bound by arguing as in Case 1 of Step 3. By (66), consider the case $0 < |(n_\Gamma)_2| \leq |(n_\Gamma)_1| \leq 1 - \frac{\sigma_0^2}{2}$ (the case with switched coordinates works analogously). Loosely speaking, Γ has “slope” larger than 1 but is not vertical. Hence, for h sufficiently small, it intersects h^{-1} many vertical lattice lines, and (as proved in the following) for each line intersected there is a non-degenerate boundary triangle.

Indeed, let $z_1 \in h\mathbb{Z}$ such that $\Gamma \cap (\{z_1\} \times \mathbb{R}), \Gamma \cap (\{z_1 + h\} \times \mathbb{R}) \neq \emptyset$ and consider the rectangle E with vertices $\{(z_1, \zeta_\ell)\} := \Gamma \cap (\{z_1\} \times \mathbb{R})$ and $\{(z_1, \zeta_r)\} := \Gamma \cap (\{z_1 + h\} \times \mathbb{R})$. Note that, as

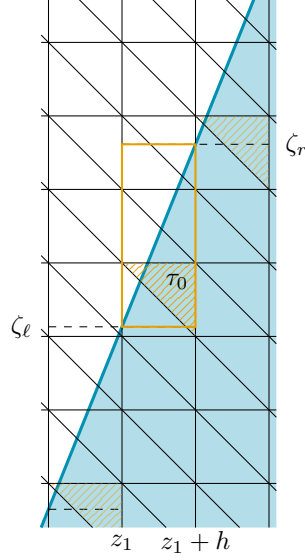


FIGURE 13. In the case of an edge Γ misaligned with the triangulations, there are always “many” triangles with a majority of the area inside Ω . The rectangle E is highlighted in orange, with the part inside Ω in blue. The triangle $\tau_0 \in \mathcal{I}_\Gamma \cap \mathcal{I}_1$ is hashed in orange. For the adjacent vertical segments the corresponding triangles in $\mathcal{I}_\Gamma \cap \mathcal{I}_1$ are also hashed in orange.

$|\zeta_\ell - \zeta_r| = h \left| \frac{(n_\Gamma)_1}{(n_\Gamma)_2} \right|$, there are at most $2 \left\lfloor \left| \frac{(n_\Gamma)_1}{(n_\Gamma)_2} \right| \right\rfloor + 4$ triangles intersecting E which all belong to $\mathcal{I}_\Gamma$. Now we have

$$\left| \frac{(n_\Gamma)_1}{(n_\Gamma)_2} \right| \frac{h^2}{2} = |E \cap \Omega| = \sum_{\substack{\tau \in \mathcal{I}_\Gamma \\ \tau \cap E \neq \emptyset}} |\tau \cap \Omega| \leq |\tau_0 \cap \Omega| \left(2 \left\lfloor \left| \frac{(n_\Gamma)_1}{(n_\Gamma)_2} \right| \right\rfloor + 4 \right)$$

where $\tau_0 \in \mathcal{I}_\Gamma$ is such that $|\tau_0 \cap \Omega| \geq |\tau \cap \Omega|$ for every $\tau \in \mathcal{I}_\Gamma$ with $\tau \cap E \neq \emptyset$. Since $|(n_\Gamma)_2| \leq |(n_\Gamma)_1|$, the previous inequality yields that $|\tau_0 \cap \Omega| \geq \frac{h^2}{16}$. By $|(n_\Gamma)_1| \leq 1 - \frac{\sigma_0^2}{2}$, Γ intersect $\cup_{z \in h\mathbb{Z}} \{z\} \times \mathbb{R}$ at least

$$\left\lfloor h^{-1} |(n_\Gamma)_2| \mathcal{H}^1(\Gamma) \right\rfloor \geq \left\lfloor \frac{\sigma_0^2}{2h} \mathcal{H}^1(\Gamma) \right\rfloor$$

many times. Thus, up to reducing the value of σ we have $\tau_0 \in \mathcal{I}_\Gamma \cap \mathcal{I}_1$, and repeating the argument above, we find $\frac{\sigma_0^2}{4} h^{-1} \mathcal{H}^1(\Gamma)$ many triangles in $\mathcal{I}_\Gamma \cap \mathcal{I}_1$. Eventually, arguing as in (61) we infer

$$(67) \quad \int_{\Omega} |\nabla u - \chi|^p dx \geq C' C_0 \sigma h \mathcal{H}^1(\Gamma),$$

here we possibly reduced the value of C_0 (e.g. by multiplying it with a universal constant) if needed.

Step 3: Conclusion. Gathering (59), (61), (62), (65) and (67) we get

$$(68) \quad \int_{\Omega} |\nabla u - \chi|^p dx \geq \frac{1}{3} \int_{\Omega} |\nabla u - \chi|^p dx + \frac{1}{3} C h \|D_{R\nu} \chi\|_{TV(\Omega_h)} + \frac{1}{3} \sigma C'' h$$

for suitable constants $C'', C > 0$ depending on Ω , R , p , v , F , and $\mathcal{K}$.

Furthermore, since

$$\|D_{R\nu}\chi\|_{TV(\Omega\setminus\Omega_h)} \leq C(\mathcal{K}, d, \Omega)h^{-1}\text{Per}(\tau) \leq C(\mathcal{K}, d, \Omega),$$

we get that

$$\|D_{R\nu}\chi\|_{TV(\Omega)} + 1 \leq C(\|D_{R\nu}\chi\|_{TV(\Omega_h)} + 1)$$

which then gives the result together with (68). $\square$

With this we can deduce lower scaling bounds for a discrete model by using the corresponding lower bounds for the sharp interface model, provided we do not argue on scales smaller than h . We will comment on this in more detail below.

Remark 5.2. *The proof of Theorem 5 in the isotropic setting follows the same idea as for the anisotropic setting, hence we omit it at this point.*

We expect the same result to hold in higher dimensions, where the triangulation has to be replaced by the corresponding higher-dimensional generalization.

5.2. Applications. In this last section we apply the result of Theorem 5 to derive scaling laws for the discretized Lorent and Tartar settings.

5.2.1. The discrete Lorent three-well problem. Consider $\mathcal{K}_3 \subset \mathbb{R}^{2 \times 2}$ given in (4). For the (anisotropic, continuous) sharp interface model the scaling is given by Theorem 1. By Theorem 5 we expect the scaling law to carry over to the discretized model as soon as the rank-one connection in $\mathcal{K}_3$ is misaligned with the triangulation $\mathcal{T}_h^R$. Building on these observations, we present the proof of Corollary 1.6.

Proof of the lower bounds in Corollary 1.6. We recall that the only rank-one connection within the wells forming the set $\mathcal{K}_3$ is between A_1 and A_2 and is given by the vector e_1 .

In case (i) we have, $e_1 \notin \{\pm Re_1, \pm Re_2, \pm 2^{-1/2}R(e_1 + e_2)\}$. Hence, since e_1 is the only available rank-one direction,

$$(Re_1) \times [A_j - A_k] \neq 0, (Re_2) \times [A_j - A_k] \neq 0, (R\begin{pmatrix} 1 \\ 1 \end{pmatrix}) \times [A_j - A_k] \neq 0,$$

for all $A_j, A_k \in \mathcal{K}_3$, $j \neq k$. As a consequence, by Theorem 5 we have

$$\int_{\Omega} |\nabla u - \chi|^2 dx \geq C \left(\int_{\Omega} |\nabla u - \chi|^2 dx + h \|D\chi\|_{TV(\Omega)} \right).$$

The statement then follows by an application of Theorem 1.

In the second case, that is in the case that the compatible direction is in one of the “bad” directions, we use the anisotropic version of Theorem 5. Thus, there is $v \in \{e_1, e_2, 2^{-1/2}(e_1 + e_2)\}$ such that $Rv = \pm e_1$ and, hence, $Rv \times [A_1 - A_2] = 0$. By Theorem 5, we get with $\nu \in \mathbb{S}^1$, $\nu \cdot v = 0$

$$\int_{\Omega} |\nabla u - \chi|^2 dx \geq C \left(\int_{\Omega} |\nabla u - \chi|^2 dx + h \|D_{R\nu}\chi\|_{TV(\Omega)} + h \right).$$

As $\nu \cdot v = R\nu \cdot Rv = \pm R\nu \cdot e_1 = 0$ it holds, without loss of generality, that $R\nu = e_2$. As a consequence, we infer that

$$\int_{\Omega} |\nabla u - \chi|^2 dx \geq C \left(\int_{\Omega} |\nabla u - \chi|^2 dx + h \|D_{e_2}\chi\|_{TV(\Omega)} + h \right).$$

The desired lower bound then follows by Theorem 1. $\square$

We can use our upper bound constructions from Section 3 and interpolate them on the triangulation to get matching upper bounds.

Proof of upper bounds in Corollary 1.6. We follow the arguments of [Lor09, Chi99, CM99]. Let us start by considering the isotropic setting.

Taking the corresponding upper bound construction u, χ from Theorem 1 with $\epsilon = h$, we define $u_h \in \mathcal{A}_{h,F}^{2,R}$ as the piecewise affine approximation of u , preserving the boundary condition. For this it holds

$$\int_{\Omega} \text{dist}^2(\nabla u_h, \mathcal{K}_3) dx \leq \int_{\Omega} |\nabla u_h - \nabla u|^2 dx + E_{el}(u, \chi).$$

As ∇u is bounded for each of our constructions (see (33)), we get

$$\int_{\Omega} \text{dist}^2(\nabla u_h, \mathcal{K}_3) dx \leq C(\mathcal{K}_3) |\{u_h \neq u\}| + E_{el}(u, \chi).$$

We now seek to bound the volume $|\{u_h \neq u\}|$. For this we note that, firstly, u is piecewise affine in our constructions (but not on the triangulation), and, secondly, that when we are at least at distance $2h$ from the jump set of ∇u and $\partial\Omega$, then it holds $u_h = u$. By this we conclude

$$|\{u_h \neq u\}| \leq Ch(\|D(\nabla u)\|_{TV(\Omega)} + \text{Per}(\Omega)).$$

Moreover, by construction of the continuous upper bound deformation we also have, see (33) and the proof of Theorem 1,

$$\|D(\nabla u)\|_{TV(\Omega)} \leq C(\|D\chi\|_{TV(\Omega)} + \text{Per}(\Omega)).$$

In conclusion this yields

$$\int_{\Omega} \text{dist}^2(\nabla u_h, \mathcal{K}_3) dx \leq C(E_{el}(u, \chi) + h\|D\chi\|_{TV(\Omega)} + h\text{Per}(\Omega)).$$

Thus, after choosing χ_h pointwise as the projection of ∇u_h onto $\mathcal{K}_3$ (with an arbitrary choice where the projection is not well-defined) the upper bound follows also in the discrete set-up.

Compared to the above isotropic argument, additional care is needed for the anisotropic setting. If $F \in \mathcal{K}_3^1$ in the anisotropic case, the oscillation of the upper bound construction is finer than h . Thus, an unmodified variant of the above argument would yield the trivial bound

$$\int_{\Omega} \text{dist}^2(\nabla u_h, \mathcal{K}_3) dx \leq C|\Omega|.$$

To achieve the desired upper bound which displays an h scaling behaviour, we consider a simple laminate of A_1 and A_2 on a scale h with a suitable cut-off. Then, in the bulk of Ω we have $\nabla u_h \in \mathcal{K}_3$ due to the compatibility and as we can laminate exactly on the boundaries of triangles. Therefore, the energy concentrates in a boundary region, and we obtain

$$\int_{\Omega} \text{dist}^2(\nabla u_h, \chi) dx \leq C|\{ \text{dist}(x, \partial\Omega) \leq 2h \}| \leq Ch.$$

Again choosing χ_h as the pointwise projection of ∇u_h onto $\mathcal{K}_3$ yields the desired result.

For $F \in \mathcal{K}_3^2$ in the anisotropic case, we remark that the rescaled unit cells always have lengths and heights such that $\ell_j, h_j \gg \epsilon$. In this case, we can therefore directly translate the continuous upper bound construction (laminate within branching) by discretization. More precisely, we fix the laminate on the scale $\epsilon = h$ and apply analogous arguments as in the isotropic setting above which allow us to transfer the bound from the continuum to the discrete framework. $\square$

5.2.2. *The discrete Tartar square.* Last but not least, we turn to the proof of the bounds for the Tartar square.

Proof of Corollary 1.7. Recalling that there are no rank-one connections in the Tartar square, the proof follows from applying the isotropic version of Theorem 5 and invoking the lower scaling result from [RT22]. $\square$

The almost matching upper bound can be found in [Chi99].

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REFERENCES

- [AFP00] Luigi Ambrosio, Nicola Fusco, and Diego Pallara. *Functions of Bounded Variation and Free Discontinuity Problems*. Oxford University Press, March 2000.
- [AKKR24] Ibromkhimbek Akramov, Hans Knüpfer, Martin Kružík, and Angkana Rüland. Minimal energy for geometrically nonlinear elastic inclusions in two dimensions. *Proc. Roy. Soc. Edinburgh Sect. A*, 154(3):769–792, 2024.
- [ALP17] Roberto Alicandro, Giuliano Lazzaroni, and Mariapia Palombaro. On the effect of interactions beyond nearest neighbours on non-convex lattice systems. *Calc. Var.*, 56(42), 2017.
- [AM01] Giovanni Alberti and Stefan Müller. A new approach to variational problems with multiple scales. *Comm. Pure Appl. Math.*, 54(7):761–825, 2001.
- [BC07] Andrea Braides and Marco Cicalese. Surface energies in nonconvex discrete systems. *Mathematical Models and Methods in Applied Sciences*, 17(07):985–1037, 2007.
- [BFJK94] Kaushik Bhattacharya, Nikan B. Firoozye, Richard D. James, and Robert V. Kohn. Restrictions on microstructure. *Proc. Roy. Soc. Edinburgh Sect. A*, 124(5):843–878, 1994.
- [Bha03] Kaushik Bhattacharya. *Microstructure of martensite: why it forms and how it gives rise to the shape-memory effect Oxford series on materials modeling*. Oxford University Press, 2003.
- [BM18] Haïm Brezis and Petru Mironescu. Gagliardo-Nirenberg inequalities and non-inequalities: The full story. *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, 35(5):1355–1376, 2018.
- [BMC09] John M Ball and Carlos Mora-Corral. A variational model allowing both smooth and sharp phase boundaries in solids. *Commun. Pure Appl. Anal.*, 8(1):55–81, 2009.
- [BO13] Árpád Bényi and Tadahiro Oh. The Sobolev inequality on the torus revisited. *Publicationes Mathematicae Debrecen*, 83(3):359–374, 2013.
- [BP04] Sören Bartels and Andreas Prohl. Multiscale resolution in the computation of crystalline microstructure. *Numerische Mathematik*, 96:641–660, 2004.
- [BVT86] I Baele, Gustaaf Van Tendeloo, and Severin Amelinckx. A striking high-resolution imaging phenomenon in NiMn. *Ultramicroscopy*, 19(2):201–204, 1986.
- [CC14] Allan Chan and Sergio Conti. Energy scaling and domain branching in solid-solid phase transitions. In *Singular phenomena and scaling in mathematical models*, pages 243–260. Springer, 2014.
- [CC15] Allan Chan and Sergio Conti. Energy scaling and branched microstructures in a model for shape-memory alloys with $SO(2)$ invariance. *Mathematical Models and Methods in Applied Sciences*, 25(06):1091–1124, 2015.
- [CCK95] Michel Chipot, Charles Collins, and David Kinderlehrer. Numerical analysis of oscillations in multiple well problems. *Numer. Math.*, 70(3):259–282, 1995.
- [CDZ17] Sergio Conti, Johannes Diermeier, and Barbara Zwicknagl. Deformation concentration for martensitic microstructures in the limit of low volume fraction. *Calc. Var. Partial Differential Equations*, 56(1):Paper No. 16, 24, 2017.
- [Chi99] Michel Chipot. The appearance of microstructures in problems with incompatible wells and their numerical approach. *Numer. Math.*, 83(3):325–352, 1999.
- [CM99] Michel Chipot and Stefan Müller. Sharp energy estimates for finite element approximations of non-convex problems. In *IUTAM Symposium on Variations of Domain and Free-Boundary Problems in Solid Mechanics*, pages 317–325. Springer, 1999.

- [CO09] Antonio Capella and Felix Otto. A rigidity result for a perturbation of the geometrically linear three-well problem. *Communications on Pure and Applied Mathematics*, 62(12):1632–1669, 2009.
- [CO12] Antonio Capella and Felix Otto. A quantitative rigidity result for the cubic-to-tetragonal phase transition in the geometrically linear theory with interfacial energy. *Proceedings of the Royal Society of Edinburgh: Section A Mathematics*, 142, pp 273–327 doi:10.1017/S0308210510000478, 2012.
- [Con00] Sergio Conti. Branched microstructures: scaling and asymptotic self-similarity. *Comm. Pure Appl. Math.*, 53(11):1448–1474, 2000.
- [CS13] Isaac Vikram Chenchiah and Anja Schlömerkemper. Non-laminate microstructures in monoclinic-I martensite. *Archive for Rational Mechanics and Analysis*, 207(1):39–74, 2013.
- [Dol03] Georg Dolzmann. *Variational methods for crystalline microstructure-analysis and computation*, volume 1803. Springer Science & Business Media, 2003.
- [FS08] Daniel Faraco and László Székelyhidi. Tartar’s conjecture and localization of the quasicontinuous hull in $\mathbb{R}^{2 \times 2}$. *Acta mathematica*, 200(2):279–305, 2008.
- [Gra08] Loukas Grafakos. *Classical Fourier analysis*, volume 2. Springer, 2008.
- [GRTZ24] Janusz Ginster, Angkana Rüland, Antonio Tribuzio, and Barbara Zwicknagl. On the effect of geometry on scaling laws for a class of martensitic phase transformations. *arXiv preprint arXiv:2405.05927*, 2024.
- [IKR⁺24] Roman Indergand, Dennis Kochmann, Angkana Rüland, Antonio Tribuzio, and Christian Zillinger. On a T_3 -structure in geometrically linearized elasticity: Qualitative and quantitative analysis and numerical simulations. *arXiv preprint arXiv:2408.13110*, 2024.
- [KK11] Hans Knüpfer and Robert V. Kohn. Minimal energy for elastic inclusions. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 467(2127):695–717, 2011.
- [KM92] Robert V Kohn and Stefan Müller. Branching of twins near an austenite—twinned-martensite interface. *Philosophical Magazine A*, 66(5):697–715, 1992.
- [KM94] Robert V Kohn and Stefan Müller. Surface energy and microstructure in coherent phase transitions. *Communications on Pure and Applied Mathematics*, 47(4):405–435, 1994.
- [KO19] Hans Knüpfer and Felix Otto. Nucleation barriers for the cubic-to-tetragonal phase transformation in the absence of self-accommodation. *ZAMM Z. Angew. Math. Mech.*, 99(2):e201800179, 12, 2019.
- [Koh91] Robert V Kohn. The relaxation of a double-well energy. *Continuum Mechanics and Thermodynamics*, 3(3):193–236, 1991.
- [KW16] Robert V Kohn and Benedikt Wirth. Optimal fine-scale structures in compliance minimization for a shear load. *Communications on Pure and Applied Mathematics*, 69(8):1572–1610, 2016.
- [Lor01] Andrew Lorent. An optimal scaling law for finite element approximations of a variational problem with non-trivial microstructure. *ESAIM: Mathematical Modelling and Numerical Analysis-Modélisation Mathématique et Analyse Numérique*, 35(5):921–934, 2001.
- [Lor09] Andrew Lorent. The regularisation of the N-well problem by finite elements and by singular perturbation are scaling equivalent in two dimensions. *ESAIM: Control, Optimisation and Calculus of Variations*, 15(2):322–366, 2009.
- [MM77] Luciano Modica and Stefano Mortola. Un esempio di Γ^- -convergenza. *Boll. Un. Mat. Ital. B (5)*, 14(1):285–299, 1977.
- [MŠ03] Stefan Müller and Vladimír Šverák. Convex integration for Lipschitz mappings and counterexamples to regularity. *Annals of mathematics*, 157(3):715–742, 2003.
- [MSZ03] Jan Malý, David Swanson, and William P. Ziemer. The co-area formula for Sobolev mappings. *Transactions of the American Mathematical Society*, 355(2):477–492, 2003.
- [Mül93] Stefan Müller. Singular perturbations as a selection criterion for periodic minimizing sequences. *Calc. Var. Partial Differential Equations*, 1(2):169–204, 1993.
- [Mül99] Stefan Müller. Variational models for microstructure and phase transitions. In *Calculus of variations and geometric evolution problems*, pages 85–210. Springer, 1999.
- [MVTA86] Constantinos Manolikas, Gustaaf Van Tendeloo, and Severin Amelinckx. The “local” structure of domain boundaries in ferroelastic lead orthovanadate. *Solid state communications*, 58(12):851–855, 1986.
- [Rin18] Filip Rindler. *Calculus of variations*. Springer, 2018.
- [RRT23] Bogdan Raiță, Angkana Rüland, and Camillo Tissot. On scaling properties for two-state problems and for a singularly perturbed T_3 structure. *Acta Applicandae Mathematicae*, 184(1):5, 2023.
- [RRTT24] Bogdan Raiță, Angkana Rüland, Camillo Tissot, and Antonio Tribuzio. On scaling properties for a class of two-well problems for higher order homogeneous linear differential operators. *SIAM J. Math. Anal.*, 56(3):3720–3758, 2024.
- [RT22] Angkana Rüland and Antonio Tribuzio. On the energy scaling behaviour of a singularly perturbed Tartar square. *Archive for Rational Mechanics and Analysis*, 243(1):401–431, 2022.

- [RT23a] Angkana Rüland and Antonio Tribuzio. On scaling laws for multi-well nucleation problems without gauge invariances. *J. Nonlinear Sci.*, 33(2):Paper No. 25, 41, 2023.
- [RT23b] Angkana Rüland and Antonio Tribuzio. On the energy scaling behaviour of singular perturbation models with prescribed Dirichlet data involving higher order laminates. *ESAIM: Control, Optimisation and Calculus of Variations*, 29(68), 2023.
- [RT24] Angkana Rüland and Antonio Tribuzio. On the scaling of the cubic-to-tetragonal phase transformation with displacement boundary conditions. *J. Elasticity*, 156(3):837–875, 2024.
- [RTZ18] Angkana Rüland, Jamie M Taylor, and Christian Zillinger. Convex integration arising in the modelling of shape-memory alloys: some remarks on rigidity, flexibility and some numerical implementations. *Journal of Nonlinear Science*, pages 1–48, 2018.
- [RZZ19] Angkana Rüland, Christian Zillinger, and Barbara Zwicknagl. Higher Sobolev regularity of convex integration solutions in elasticity: The planar geometrically linearized hexagonal-to-rhombic phase transformation. *Journal of Elasticity*, 2019.
- [Sim21a] Theresa M Simon. Quantitative aspects of the rigidity of branching microstructures in shape memory alloys via h-measures. *SIAM Journal on Mathematical Analysis*, 53(4):4537–4567, 2021.
- [Sim21b] Theresa M Simon. Rigidity of branching microstructures in shape memory alloys. *Archive for Rational Mechanics and Analysis*, 241(3):1707–1783, 2021.
- [Tar93] Luc Tartar. Some remarks on separately convex functions. In *Microstructure and phase transition*, pages 191–204. Springer, 1993.
- [TZ25] Antonio Tribuzio and Konstantinos Zemas. Energy barriers for boundary nucleation in a two-well model without gauge invariances. *Calc. Var. Partial Differential Equations*, 64(1):Paper No. 11, 35, 2025.
- [Zwi14] Barbara Zwicknagl. Microstructures in low-hysteresis shape memory alloys: scaling regimes and optimal needle shapes. *Archive for Rational Mechanics and Analysis*, 213:355–421, 2014.

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