Periplectic and Isomeric Lie superalgebras, KLR algebras and Categorification

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Jonas Nehme
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Gutachterin/Betreuerin: Prof. Dr. Catharina Stroppel
Gutachterin: Prof. Dr. Vera Serganova

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Summary

In this thesis, we will study the representation theory of the periplectic Lie superalgebra $\mathfrak{p}(n)$ and the isomeric Lie superalgebra $\mathfrak{q}(n)$ in great detail and obtain explicit descriptions of the endomorphism rings of a projective generator. These are the two families of finite dimensional classical Lie superalgebras that are not widely understood (in contrast to $\mathfrak{gl}(m|n)$ and $\mathfrak{osp}(r|2n)$).
This thesis comprises two parts, one for each of the families, which can also be read independently, although many ideas and tools are very similar in both approaches (the difficulties, however, are not).
We tackle this problem via Schur–Weyl duality and the theory of KLR algebras. Namely, every projective representation appears as a direct summand of $V^{\otimes d} \otimes (V^*)^{\otimes d'}$, where V is the natural representation. We introduce respectively recall (for $\mathfrak{p}(n)$ resp. $\mathfrak{q}(n)$) certain KLR algebras, whose cyclotomic quotients can pick out the projective representations under Schur–Weyl duality. A very important step in this process is the proof of a basis theorem for these cyclotomic quotients. In principle, this basis theorem can be used to extract a basis for the endomorphism ring of a projective generator. However, the multiplication rules for these basis elements are very complicated, and it is impossible to give even a rough description of the result.
To overcome this issue, we introduce another type of diagrammatic algebra, which we call *Khovanov algebra of type P/Q*. These resemble the other versions of Khovanov algebras that were shown to describe the finite dimensional representation theory of $\mathfrak{gl}(m|n)$ and $\mathfrak{osp}(r|2n)$. They come equipped with a distinguished basis together with an explicit multiplication for these basis elements. We show that these are equivalent to the cyclotomic KLR algebras and prove that certain subquotients are isomorphic to the endomorphism ring of a projective generator.
This gives rise to diagrammatic descriptions of translation functors, and we study their effect on important classes of modules, i.e. projective, simple (and (co)standard for $\mathfrak{p}(n)$) representations.
We conclude both parts with some categorification results. For $\mathfrak{p}(n)$, we show that the respective KLR algebra with its cyclotomic quotients categorifies a Fock space representation of the quantum electrical algebra. This algebra is a rather newly introduced and strange object that has not been studied at all. In particular, much work had to be done to even define a Fock space, and we do this by realizing the quantum electrical Lie algebra as a coideal. For $\mathfrak{q}(n)$, it is already known that the KLR algebra in question categorifies a quantum supergroup of type $B_{0|\infty}$. We extend this result by showing that the here considered cyclotomic quotient categorifies a tensor product of a spin representation with its dual. This tensor product is not irreducible anymore, and we show that its Jordan–Hölder series is categorified by $\mathrm{rep}(\mathfrak{q}(n))$ for the various n.

Contents

Introduction

Lie superalgebras are $\mathbb{Z}_2$-graded Lie algebras which were originally introduced by Gerstenhaber in [Ger63] in the context of ring theory and Hochschild cohomology.
Ever since the introduction of Lie superalgebras, their representation theory has proven to be a source of fruitful mathematics with applications e.g. in the fields of mathematical physics [CNS75], deformation theory [Gin05], and geometry [DM99].
In this thesis, we lay our focus on the representation theory of two families of finite dimensional Lie superalgebras: the periplectic Lie superalgebra $\mathfrak{p}(n)$ and the isomeric Lie superalgebra $\mathfrak{q}(n)$. Even though these Lie superalgebras were defined already in the 1970s by [Kac77b], their representation theory appears to be mysterious and not well understood. These two Lie superalgebras are also called *strange* Lie superalgebras, as some of the classical properties of Lie algebras do not hold for them. For instance, not every root for $\mathfrak{p}(n)$ has a negative root, and for $\mathfrak{q}(n)$ there exist irreducible representations whose endomorphism ring is not 1-dimensional, i.e. Schur's lemma does not hold.
We construct explicit descriptions of the endomorphism ring of a projective generator of the respective category of finite dimensional representations via the use of diagram algebras and Schur–Weyl duality.
In the following, we will elaborate on these results in further detail. First, we give a brief overview on the representation theory of Lie superalgebras. For a general introduction to this topic see [CW12a, Mus12, Ser17].
Any Lie superalgebra $\mathfrak{g}$ comes with a decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ into an even part $\mathfrak{g}_0$ and an odd part $\mathfrak{g}_1$. The even part $\mathfrak{g}_0$ is a Lie algebra and the odd part $\mathfrak{g}_1$ is a representation of $\mathfrak{g}_0$.
The first milestone in the theory of Lie superalgebras was the classification of simple Lie superalgebras by Kac in [Kac77b]. The classification of the simple Lie superalgebras $\mathfrak{g}$ comprises two main classes of Lie superalgebras depending on whether the Lie algebra $\mathfrak{g}_0$ is reductive or not. If $\mathfrak{g}_0$ is reductive, $\mathfrak{g}$ is called a *classical* Lie superalgebra. Independent of Kac's work, a classification of classical Lie superalgebras was also obtained in [SNR76]. Partial results for classical Lie superalgebras were previously achieved in [PR75, Djo76, FK76, NRS76] under different restrictions.
The classical Lie superalgebras consist of four infinite families and a few exceptional cases. There are (subquotients of) the general linear Lie superalgebras $\mathfrak{gl}(n|m)$, the orthosymplectic Lie superalgebras $\mathfrak{osp}(r|2n)$, the isomeric Lie superalgebras $\mathfrak{q}(n)$ and the periplectic Lie superalgebras $\mathfrak{p}(n)$. The exceptionals consist of two finite dimensional ones (types $F(4)$ and $G(3)$) and a family of deformations of $\mathfrak{osp}(4|2)$.
By definition, the classical Lie superalgebras have reductive even part and thus are the only Lie superalgebras where the tools from Lie theory could be applied. For instance, as for semisimple Lie algebras, finite-dimensional irreducible representations of classical

Lie superalgebras can be parametrized by their highest weight. Sufficient and necessary conditions have been derived in [Kac77b]. Understanding the characters of the irreducible representations turns out to be very difficult. In particular, some difficulties arose from the fact that the finite dimensional representation theory of Lie superalgebras is not semisimple, in contrast to Lie algebras.

A basic measurement of the non-semisimplicity is the notion of atypicality, already introduced in [Kac77a]. The degree of atypicality is a number associated to the highest weight of an irreducible representation and the higher this number, the more difficult the representation theory of this irreducible representation becomes. Typical representations are those, which do not have non-trivial extensions. These behave as representations of semisimple Lie algebras. For typical representations of $\mathfrak{gl}(n|m)$ and $\mathfrak{osp}(r|2n)$, character formulae were obtained in [Kac77a]. For these, the resulting character formula is close to Weyl's character formula.

For characters of atypical representations, first advancements were achieved in [BL80] for $\mathfrak{gl}(1|n)$. These methods were extended to $\mathfrak{osp}(2|2n)$ and singly atypical representations of $\mathfrak{gl}(m|n)$ in [VdJ91] and [VHKTM90] respectively.

For $\mathfrak{gl}(m|n)$, Serganova obtained in [Ser96] character formulae for all irreducible representations with geometric methods. The same methods were used in [GS10] to obtain character formulae for $\mathfrak{osp}(r|2n)$, which were also shown in [CLW11] via super duality. A purely algebraic approach was given in [Bru03] for $\mathfrak{gl}(m|n)$.

For the isomeric Lie superalgebra $\mathfrak{q}(n)$, character formulae were first obtained for typical representations in [Ser84]. For all representations, formulae were discovered in [PS97] via geometric means. An algebraic proof later appeared in [Bru04].

For the periplectic Lie superalgebra, the question of finding character formulas remains open, though through the works of [BDEA$^+$19], at least an algorithm to compute the characters should be in reach.

Although the task of finding character formulas is almost complete, until rather recently only little could be said about the respective representation theory. Small rank examples were studied in [Ger98] for $\mathfrak{sl}(1|n)$, where all indecomposable representations were classified. In [GQS07], the Lie superalgebras $\mathfrak{gl}(1|1)$ and $\mathfrak{sl}(2|1)$ were studied and in particular explicit tensor product decompositions for all finite dimensional representations were obtained. For $\mathfrak{osp}(1|2n)$ a deep connection with the representation theory of $\mathfrak{so}(2n+1)$ was discovered in [RS82], including connections between their respective characters and tensor product decompositions.

A description of $\mathfrak{q}(2)$ as a quiver with relations was given in [MM12] and extended to $\mathfrak{q}(3)$ in [GS20].

A major breakthrough in understanding the representation theory of $\mathfrak{gl}(m|n)$ was made in [BS12b], when the authors showed that (a limiting version of) Khovanov's diagram algebra describes the endomorphism ring of a projective generator of $\mathfrak{gl}(m|n)$. Khovanov's diagram algebra arose originally in the context of knot theory and categorification of the Jones polynomial, see [Kho00, Kho02]. A special feature of this algebra is its diagrammatic description, which is well-suited for direct computations. The main idea of [BS12b] is to understand the representation theory of $\mathfrak{gl}(m|n)$ via translation functors. Translation functors are given by direct summands of the functors $\_ \otimes V$, where V denotes the natural representation, and provide a powerful tool to study the representation theory of Lie (super)algebras. Via repeatedly applying these functors to certain special representations, one can obtain a projective generator of the category of finite dimensional representations of $\mathfrak{gl}(m|n)$. If one additionally understands the natural transformations between these functors, one can obtain a complete description of the endomorphism ring of a projective generator. Via (higher) Schur–Weyl duality, translation functors and their natural transformations are governed by KLR algebras, see [CW08], [BK09]. The KLR algebras were independently introduced in [KL09] and [Rou08] and arose originally in the context of categorification of quantum groups. The methodology of [BS12b] relies then on comparing Khovanov's diagram algebra with a cyclotomic quotient of the KLR algebra. Together with the results of [BS10], this allows computing the effect of translations functors on projective, standard and simple modules via easy combinatorics.

In [BR87] and [Ser84], Schur–Weyl duality for $\mathfrak{gl}(m|n)$ on $V^{\otimes d}$ was already considered. In the language from above, this means repeatedly applying translation functors to the trivial representation. As for the Lie algebra $\mathfrak{gl}(n)$, not every indecomposable projective representation (for $\mathfrak{gl}(n)$ these are also irreducible) appears as a direct summand of $V^{\otimes d}$, where V is the natural representation. In particular, in [BS12b], they needed to consider $M \otimes V^{\otimes d}$ for some (more complicated) representation M.

Alternatively (and more conceptually), one could consider mixed tensor powers $V^{\otimes d} \otimes (V^*)^{\otimes d'}$. Now, every indecomposable projective representation appears as a direct summand of $V^{\otimes d} \otimes (V^*)^{\otimes d'}$ for some d and d'. In particular, these mixed tensor powers contain all information about the abelian category of finite dimensional representations of $\mathfrak{gl}(m|n)$.

In [Del07], an interpolation category $\mathrm{Rep}(\mathrm{GL}_\delta)$ was introduced, which describes all these mixed tensor powers for different m and n. This is a category build around the *walled Brauer algebras*, which were independently introduced in [Tur89] and [Koi89]. From the universal property of $\mathrm{Rep}(\mathrm{GL}_\delta)$ in [Del07], there is an obvious monoidal functor from $\mathrm{Rep}(\mathrm{GL}_\delta)$ to $\mathrm{rep}(\mathfrak{gl}(m|n))$. In [CW12b], the authors showed that this functor is full by extending the classical Schur–Weyl duality to the case of mixed tensor powers. They also classified indecomposable summands of the mixed tensor powers $V^{\otimes d} \otimes (V^*)^{\otimes d'}$ and gave decomposition rules for their tensor products. However, this approach gives no direct access to the Jordan–Hölder filtrations of the indecomposable summands, in contrast to [BS12b]. In [BS12a], the authors showed that the walled Brauer algebras are Morita equivalent to (an idempotent truncation of) Khovanov's diagram algebra. In principle, this says that the afore-mentioned description of $\mathrm{rep}(\mathfrak{gl}(m|n))$ via Khovanov's diagram algebra from [BS12b] can also be obtained via $\mathrm{Rep}(\mathrm{GL}_\delta)$.

Similar results were then obtained for $\mathfrak{osp}(r|2n)$. Here, the Deligne category $\mathrm{Rep}(O_\delta)$, also introduced by Deligne in [Del07], is used, which is a category built around the *Brauer algebras* from [Bra37]. This category was studied e.g. in [CH17], where classifications of thick tensor ideals and indecomposable objects were obtained as well as tensor product decomposition formulae. In [Cou18b], it all tensor ideals were classified, and the results were used to describe the kernel of Schur–Weyl duality. In [ES16], a version of Khovanov's diagram algebra adapted to type B was introduced and shown to be equivalent to $\mathrm{Rep}(O_\delta)$ in [ES21]. There is again a full monoidal functor from $\mathrm{Rep}(O_\delta)$ to $\mathrm{rep}(\mathfrak{osp}(r|2n))$, see [DLZ18]. A diagrammatic description of the endomorphism ring of a projective generator of $\mathfrak{osp}(r|2n)$ as a particular subquotient of Khovanov's diagram algebra of type B was also constructed in [ES21]. In [HNS24], it was proven that this description is compatible with translation functors and using the same methods as in [BS10] explicit descriptions of the effect of translation functors on projective and irreducible modules were obtained.

This thesis contains similar considerations for the periplectic and isomeric Lie superalgebras, completing this approach for (non-exceptional) classical Lie superalgebras. We obtain explicit descriptions of the endomorphism ring of a projective generator of $\mathfrak{p}(n)$ and $\mathfrak{q}(n)$, which resemble the diagrammatics of Khovanov's diagram algebras in types A and B. In the same manner as for $\mathfrak{gl}(m|n)$ and $\mathfrak{osp}(r|2n)$, we approach this via Schur–Weyl duality and translation functors. In the periplectic and isomeric case, the counterparts for Schur–Weyl duality are taken by the *super Brauer algebra*, studied in [Cou18a, CE18, CE21, KT17, Moo03], and the *oriented Brauer–Clifford algebra*, see e.g. [Ser84, HKS11, BCK19, GRS24], respectively. Idempotents in these algebras play a crucial role for understanding translation functors, i.e. direct summands of the functors $\_ \otimes V$ and $\_ \otimes V^*$. In type A, the idempotents can be obtained via identifying the degenerate affine Hecke algebra with the KLR algebra, see [BK09]. In type B, the idempotent version is described in [Li14]. For $\mathfrak{q}(n)$, the *quiver Hecke superalgebras* from [KKT16] play the role of the idempotent version. These algebras describe the natural transformations between the translation functors coming from $\_ \otimes V$. The appropriate object that also considers the translation functors coming from $\_ \otimes V^*$ was defined in [BE17b]. For $\mathfrak{p}(n)$, we define the respective idempotent replacement, called *electric KLR algebra*, motivated by the *fake Casimir* introduced in [BDEA+19]. In both cases, we study certain cyclotomic quotients and obtain basis theorems.

Afterward, we introduce two further diagrammatic algebras resembling Khovanov's diagram algebras of types A and B. We will prove that these are equivalent to the studied cyclotomic quotients and obtain explicit descriptions of the endomorphism ring of a projective generator as subquotients of these diagrammatic algebras. With the methodology of [BS10], we obtain descriptions of translations functors on interesting classes of modules.

This establishes explicit descriptions of the categories of finite-dimensional representations of classical Lie superalgebras as well as diagrammatic interpretations of translation functors in the two missing cases.

This thesis contains two parts, one for each of the Lie superalgebras. The first part is devoted to the periplectic Lie superalgebra $\mathfrak{p}(n)$, whereas the second part is concerned with the isomeric Lie superalgebra $\mathfrak{q}(n)$. Both parts can (and probably should) be read independently. However, many ideas and the used tools are very similar, but difficulties arise in different places. Both parts have an individual introduction (more specialized to the respective Lie superalgebra) and are structured similarly.

We being by recalling some facts about the respective Lie superalgebras, including e.g. the precise statements of Schur–Weyl duality. Then, the respective idempotent versions are

defined (or recalled), and we prove a basis theorem for certain cyclotomic quotients. Afterward, we go about defining a new diagrammatic algebra resembling Khovanov's diagram algebras of types A and B. We study geometric bimodules for these algebras and show that they are isomorphic to the cyclotomic quotients that were considered before. We then prove that certain subquotients of these diagrammatic algebras are isomorphic to the endomorphism ring of a projective generator for $\mathrm{rep}(\mathfrak{p}(n))$ and $\mathrm{rep}(\mathfrak{q}(n))$. For $\mathfrak{p}(n)$, we also include some direct applications that have not yet appeared in the literature. For instance, we present $\mathfrak{p}(1)$ and $\mathfrak{p}(2)$ as a quiver with relations.
In both cases, we close with some categorification results. For $\mathfrak{q}(n)$, we obtain a categorification of the tensor product of a spin representation (for $U(B_{0|\infty})$) with its dual, including an interpretation for the canonical basis. These results are new, but not surprising, considering the results of [Bru04, KKO13, BE17b, GRS24]. For $\mathfrak{p}(n)$, the story is more complicated and rather surprising. We show that the cyclotomic quotients of the electric KLR algebras categorify a Fock space representation of the quantum electrical algebra. This quantum electrical algebra is a quantization of the electrical Lie algebras in [BGG24]. Its defining relations are a deformation of the positive part of $\mathfrak{gl}(\infty)$, but generically $q^{\pm 4}$ appears in the defining relations. Categorified these strange q-powers must give rise to a rather unnatural grading with generic generators having degree ± 4. It is rather surprising that this categorification works out.

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I would like to extend my thanks to all my wonderful friends and colleagues who accompanied me during my PhD time. In particular, I am grateful to my office mates Viktória Klász and Markus Kleinau for showing great patience and understanding during the final stages of this thesis.
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Part I.

The periplectic Lie superalgebra

1. Introduction

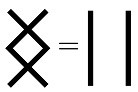
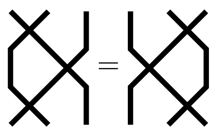
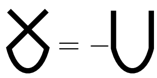
In this part, we consider the periplectic Lie superalgebra $\mathfrak{p}(n)$. It is the Lie subsuperalgebra of $\mathfrak{gl}(n|n)$ preserving a non-degenerate odd bilinear form on $(n|n)$-dimensional vector superspace V. Fixing a basis for V, we can also describe $\mathfrak{p}(n)$ explicitly in terms of matrices

$$\mathfrak{p}(n) = \left\{ \left(\begin{array}{c|c} A & B \\ \hline C & -A^t \end{array} \right) \mid B = B^t, C = -C^t \right\}. \tag{1.1}$$

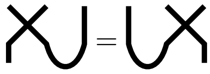
We restrict ourselves to finite dimensional representations of $\mathfrak{p}(n)$ which are integrable with respect to $\mathrm{GL}(n)$, the algebraic group corresponding to $\mathfrak{p}(n)_0 \cong \mathfrak{gl}(n)$. And from now on a finite dimensional representation means finite dimensional *integrable* representation. The main reason for this restriction is that we can endow the category of such representations with a highest weight structure (see [Che15] and [BDEA+19]). Namely, the supergrading on $\mathfrak{p}(n)$ comes from a $\mathbb{Z}$-grading $\mathfrak{p}(n) \cong \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ (where $\mathfrak{g}_{-1}$ are the matrices with $A = B = 0$ and $\mathfrak{g}_1$ those with $A = C = 0$), which allows defining *thick* and *thin Kac modules* via

$$\Delta(\lambda) = \mathrm{Ind}_{\mathfrak{g}_0 \oplus \mathfrak{g}_{-1}}^{\mathfrak{p}(n)} \mathcal{L}^{\mathfrak{g}_0}(\lambda) \qquad \nabla(\lambda) = \mathrm{Coind}_{\mathfrak{g}_0 \oplus \mathfrak{g}_1}^{\mathfrak{p}(\mathfrak{n})} \mathcal{L}^{\mathfrak{g}_0}(\lambda),$$

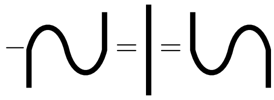
where $\mathcal{L}^{\mathfrak{g}_0}(\lambda)$ denotes the irreducible $\mathfrak{g}_0 \cong \mathfrak{gl}(n)$-module with ($\rho$-shifted) highest weight $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_n)$ and $\lambda_i \in \mathbb{Z}$. These are called thick respectively thin as they have different dimensions (cf. (1.1)), and these give the standard and costandard modules for the highest weight structure. They have a unique irreducible head respectively socle $\mathcal{L}(\lambda)$ with highest weight λ and this describes all the irreducible modules up to parity shift. Every $\mathcal{L}(\lambda)$ admits a projective cover $\mathcal{P}(\lambda)$ and every projective is also injective, see [BDEA+19] for more details.
Question: Can we describe the category of finite dimensional representations explicitly, for instance by describing the endomorphism ring of a projective generator?
To tackle this problem, we will study the representation theory via Schur–Weyl duality. For this recall from [Moo03], see also [BDEA+19], the super Brauer category sBR which provides exactly the counterpart in Schur–Weyl duality for $\mathfrak{p}(n)$. This is the $\mathbb{C}$-linear strict monoidal supercategory generated by one object $*$ and odd morphisms $\flat =$ 
and $\flat^* =$  as well as the even morphism $s =$  subject to the relations

(s𝕎-1)    (s𝕎-2)    (s𝕎-3) 

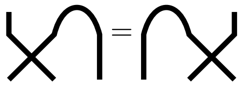
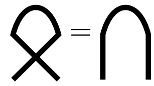
(s$\mathbb{W}$-4)



(s$\mathbb{W}$-5)



From (s$\mathbb{W}$-4) and (s$\mathbb{W}$-3) using (s$\mathbb{W}$-5) it is easy to deduce the following additional relations (keeping in mind that $\flat$ and $\flat^*$ are odd).

(1.2)  (1.3) 

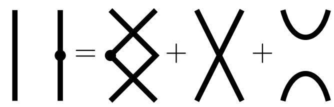
We write $\mathrm{Rep}(P)$ for the Karoubian closure of sBR, this is also known as *Deligne category of type P* and was introduced in [KT17] and studied e.g. in [Cou18a, CE21], see also [EAS21] for the perspective of an abelian envelope.
Denoting by $\mathrm{Fund}(\mathfrak{p}(n))$ the category with objects direct sums of direct summands of $V^{\otimes d}$ and *all* morphisms (not necessarily degree preserving), there exists a full monoidal functor

$$\mathrm{SW}_n\colon \mathrm{Rep}(P) \to \mathrm{Fund}(\mathfrak{p}(n)), \tag{SW}$$

see e.g. [Cou18a, Theorem 8.3.1]. The bilinear form on V provides an isomorphism $V \cong \Pi V^*$, and SW_n is given by mapping the object $*$ to V, the morphism s to the braiding and $\flat$ as well as $\flat^*$ to the morphisms induced by the bilinear form. This functor maps s to the braiding, and the odd bilinear form on V induces the images of $\flat$ and $\flat^*$. In particular, we obtain an algebra morphism $\mathrm{sBR}_d \to \mathrm{End}_{\mathfrak{p}(n)}(V^{\otimes d})$, where $\mathrm{sBR}_d \coloneqq \mathrm{End}_{\mathrm{sBR}}(*^{\otimes d})$. This was shown in [Cou18a, Theorem 8.3.1] to be an isomorphism for $n \gg 0$.
There also exists a degenerate affine version of sBR. The *super VW-category* introduced in [BDEA$^+$20] is the $\mathbb{C}$-linear strict monoidal supercategory s$\mathbb{W}$ generated by a single object $*$ and morphisms $\flat =$ , $\flat^* =$ , $s =$  as above and an additional even morphism $y =$  subject to the relations (s$\mathbb{W}$-1)–(s$\mathbb{W}$-5) together with two additional relations:

(s$\mathbb{W}$-6)

(s$\mathbb{W}$-7)



Using (s$\mathbb{W}$-6) and (s$\mathbb{W}$-5), it is very easy to deduce

(1.4) 

In [BDEA$^+$20, Proposition 22], it was shown that there is a monoidal functor s$\mathbb{W} \to \mathrm{End}(\mathrm{Rep}(\mathfrak{p}(n)))$, which sends the object $*$ to the endofunctor $\_ \otimes V$. The maps s, $\flat$ and $\flat^*$ are mapped to the "same" morphisms as in (SW). The final generating morphism y is sent to a *fake Casimir*.
This upgrades (SW) to a homomorphism of s$\mathbb{W}$-module categories. In this case, y is also given by *Jucys–Murphy elements*, see e.g. [BDEA$^+$20, Remark 20] and [Cou18b, §6].

The existence of this fake Casimir was a main insight of [BDEA$^+$19] and implies that $\_ \otimes V$ can be refined to $\_ \otimes V \cong \bigoplus_{i\in\mathbb{Z}} \Theta_i$ by projecting onto generalized eigenspaces. The summand Θ_i corresponds to a refinement i-ind on the super Brauer side.
We want to use this refinement to understand $V^{\otimes d}$ and its decomposition into indecomposable summands. An abstract classification can be obtained via understanding indecomposable objects of $\mathrm{Rep}(P)$ (using primitive idempotents in sBR_d). This was done in [CE21] giving rise to a combinatorial bijection

$$\left\{ {\text{indec. objects} \atop \text{in } \mathrm{Rep}(P) \text{ up to iso.}} \right\} \overset{1:1}{\leftrightarrow} \{\text{partitions}\}. \tag{1.5}$$

Moreover, the non-zero images of these indecomposable objects under SW_n give a complete classification of indecomposable summands in $V^{\otimes d}$, see [CE21] for an explicit description of the corresponding partitions. Unfortunately, these classification results provide no further description of the structure of the indecomposable summands in $V^{\otimes d}$. To overcome this issue, we will define an idempotent version of sBR called $\mathrm{sR}^{\mathrm{cyc}}$.
We even go a step further and introduce the *electric KLR-category* sR. The reason for this naming will only become apparent in Chapter 10, when we talk about categorification. This category should be seen as an idempotent replacement of $\mathrm{s}\mathbb{W}$, similar to the relationship between the KLR algebra from [KL09] and the degenerate affine Hecke algebra, see e.g. [BK09]. We make this connection precise for *generic* cyclotomic quotients and show

Theorem A. *There is a fully faithful functor* $\Phi\colon \mathrm{sR}^{\ell} \to \mathrm{Kar}(\mathrm{s}\mathbb{W}^{\ell})$. *It is an isomorphism after passing to additive envelopes.*

The notation sR^{ℓ} and $\mathrm{s}\mathbb{W}^{\ell}$ refers to cyclotomic quotients of level ℓ. The above-mentioned $\mathrm{sR}^{\mathrm{cyc}}$ is a special case of sR^{ℓ} for $\ell = 1$. On the way, we also show

Theorem B. *The category* sR^{ℓ} *has a basis indexed by pairs of multi-up-down-tableaux of the same shape and this endows* sR^{ℓ} *with the structure of an upper finite based quasi-hereditary algebra in the sense of* [BS24].

As a special case, we obtain an isomorphism $\Psi^{\ell}_{-}\colon (\mathrm{sR}^{\mathrm{cyc}})^{\oplus} \to \mathrm{Rep}(P)$. This isomorphism gives rise to the following commutative diagram:

$$\begin{tikzcd}
(\mathrm{sR}^{\mathrm{cyc}})^{\oplus} \arrow[loop left, "\theta_i"] \arrow[rr, "\Psi^{\ell}_{-}", "\sim"'] \arrow[dr, "\Psi_{-,n}"'] & & \mathrm{Rep}(P) \arrow[loop right, "i\text{-ind}"] \arrow[dl, "\mathrm{SW}_n"] \\
& \mathrm{Fund}(\mathfrak{p}(n)) \arrow[loop below, "\Theta_i"] &
\end{tikzcd},$$

where θ_i correspond to adding a strand labelled i in $\mathrm{sR}^{\mathrm{cyc}}$ (as in the classical case, $\mathrm{sR}^{\mathrm{cyc}}$ is defined diagrammatically with strands labelled by integers). In this case, the basis is indexed by pairs of “normal” up-down-tableaux of the same shape.

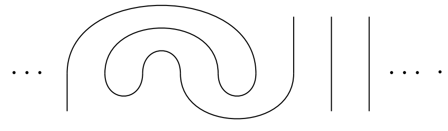
Recall that by the mere definition of $\mathfrak{p}(n)$, we have *odd* morphisms in $\mathrm{Fund}(\mathfrak{p}(n))$. For our purposes of describing the endomorphism ring of a projective generator, we rather want to work with *even* morphisms. If we denote by $\mathrm{Fund}^{\oplus}_{\Pi}(\mathfrak{p}(n))$ the category of direct sums of direct summands of $V^{\otimes d}$ with parity shifts and *all* morphisms and by $\mathrm{Fund}^{\oplus}_{\mathrm{even}}(\mathfrak{p}(n))$ its restriction to even morphisms, we have the following situation:

$$\begin{tikzcd}
& \begin{matrix}\mathrm{Fund}^{\oplus}_{\Pi}(\mathfrak{p}(n)) \\ \text{(enriched over } \mathbb{Z}/2\mathbb{Z})\end{matrix} \arrow[loop above, "\Pi"] \arrow[dl, squiggly, "\deg 0 \text{ morphisms}"'] \arrow[dr, squiggly, "\text{forget } \Pi\text{-action}"] & \\
\begin{matrix}\mathrm{Fund}^{\oplus}_{\mathrm{even}}(\mathfrak{p}(n)) \\ \text{(underlying category)}\end{matrix} \arrow[loop left, "\Pi"] & \simeq & \begin{matrix}\mathrm{Fund}(\mathfrak{p}(n)) \\ \text{(enriched over } \mathbb{Z}/2\mathbb{Z}),\end{matrix}
\end{tikzcd}$$

where the lower two are essentially the same. Now, $\mathrm{Fund}(\mathfrak{p}(n))$ arises via choosing representatives for the Π-action in $\mathrm{Fund}^{\oplus}_{\Pi}(\mathfrak{p}(n))$ (namely those that appear as a direct summand of $V^{\otimes d}$). A different choice, called $\mathrm{Fund}^{\oplus}_{+}(\mathfrak{p}(n))$, will be presented in Section 3.5, and we obtain a full functor

$$\Psi_{+,n} \colon \hat{\mathrm{s}}\mathrm{R}^{\mathrm{cyc}} \to \mathrm{Fund}^{\oplus}_{+}(\mathfrak{p}(n)),$$

where $\Psi_{+,n}$ and $\hat{\mathrm{s}}\mathrm{R}^{\mathrm{cyc}}$ are essentially the same as $\Psi_{-,n}$ and $\mathrm{sR}^{\mathrm{cyc}}$. The main purpose for introducing $\mathrm{Fund}^{\oplus}_{+}(\mathfrak{p}(n))$ is that the graded homomorphism spaces of $\mathrm{Fund}^{\oplus}_{+}(\mathfrak{p}(n))$ (and $\hat{\mathrm{s}}\mathrm{R}^{\mathrm{cyc}}$) are all concentrated in even degree, which is more feasible for our purposes.
Now, every indecomposable projective $\mathfrak{p}(n)$-representation is a direct summand of $V^{\otimes d}$ for some d (or equivalently in $\mathrm{Fund}^{\oplus}_{+}(\mathfrak{p}(n))$), see e.g. [BDEA$^+$19].
On the other hand, $\hat{\mathrm{s}}\mathrm{R}^{\mathrm{cyc}}$ was built exactly in the way that all these indecomposable projective modules arise as the image of some explicit object in $\hat{\mathrm{s}}\mathrm{R}^{\mathrm{cyc}}$ (and not as an abstract direct summand as for $\mathrm{Rep}(P)$). For our endeavor to describe explicitly the endomorphism ring of a projective generator, Theorem B can be used to extract a basis of this endomorphism ring. Thus, it can be viewed as an analog of the well-known arc algebras from [BS11a, ES16, ES21] describing the finite dimensional representations of $\mathrm{GL}(m|n)$ and $OSp(r|2n)$. But the multiplication of the basis elements in $\hat{\mathrm{s}}\mathrm{R}^{\mathrm{cyc}}$ is very complicated and there is no way to describe explicitly the multiplication using these basis elements.
Hence, we will introduce yet another combinatorial approach in Chapter 4. We will define the *Khovanov algebra* $\mathbb{K}$ *of type* P. These algebras will have a distinguished basis and an explicit multiplication procedure for these basis elements. It is a locally unital locally finite dimensional algebra and the distinguished basis elements look like



For $\hat{\mathrm{s}}\mathrm{R}^{\mathrm{cyc}}$, we considered the endofunctor θ_i by adding an i-labelled strand. We want to mimic this for $\mathbb{K}$, and we introduce an endofunctor $\hat{\theta}_i$ of $\mathbb{K}$-mod given by tensoring with a certain $\mathbb{K}$-$\mathbb{K}$-bimodule $\hat{G}_i$. This gives all the ingredients for

Theorem C. *There exists an isomorphism Φ from $\hat{\mathrm{s}}\mathrm{R}^{\mathrm{cyc}}$ to the full subcategory of $\mathbb{K}$-mod containing $\hat{\theta}_{i_k} \ldots \hat{\theta}_{i_1} \hat{P}(\bar{\iota})$ intertwining $\hat{\theta}_i$ and θ_i.*

The main difference between $\hat{\mathrm{s}}\mathrm{R}^{\mathrm{cyc}}$ and $\mathbb{K}$ is that $\hat{\mathrm{s}}\mathrm{R}^{\mathrm{cyc}}$ is defined via generators and relations whereas $\mathbb{K}$ comes with a distinguished basis and an explicit multiplication rule for this. This difference between an explicit basis and generators with relations appears very often in representations theory, most prominently in diagram algebras (like Temperley–Lieb, Brauer algebras and versions thereof).
For our ultimate goal of an explicit description of the endomorphism ring of a projective generator for $\mathfrak{p}(n)$, we define a quotient $\mathbb{K}_n$ of an idempotent truncation of $\mathbb{K}$ in Section 6.1 (where the idempotents are labeled by dominant integral weights for $\mathfrak{p}(n)$) and show that its finite dimensional representation category is upper finite highest weight. In Section 6.3 we analyze the endofunctors θ_i of $\mathbb{K}_n$-mod induced by $\hat{\theta}_i$. We will show the following.

Theorem D. *We have an adjunction (θ_i, θ_{i-1}).*

Afterward, we will also study thoroughly the effect of θ_i on projective, standard, costandard and irreducible modules in Propositions 6.26, 6.33 and 6.34 and Theorem 6.35.
In Section 6.3, we will show that $\mathbb{K}_n$ is isomorphic to the endomorphism ring of a projective generator of $\mathrm{rep}(\mathfrak{p}(n))$. We construct a projective generator by applying translation functors to the trivial module. This gives us two different ways to associate a cup diagram to an indecomposable module, we can either use Lemma 4.2 or realize this module as the image of an object in $\hat{\mathrm{s}}\mathrm{R}^{\mathrm{cyc}}$ and use Theorem C.

$$\begin{tikzcd}
& \{\text{indec. proj. } \mathfrak{p}(n)\text{-modules}\} \arrow[dl, "\text{hw of its head}"'] & \\
\{\lambda_1 > \cdots > \lambda_n\} \arrow[dr, "\text{Lemma 4.2}"'] & & \{\text{partitions}\} \arrow[ul, "\text{Schur–Weyl duality}"'] \arrow[dl, "\text{Theorem C}"] \\
& \{\text{cap diagrams}\} &
\end{tikzcd} \tag{1.6}$$

We will show in Proposition 7.3 that this diagram in fact commutes. This gives us all the ingredients for our main theorem.

Theorem E (Main theorem)**.** *There is an equivalence of categories*

$$\Psi \colon \mathbb{K}_n\text{-mod} \to \mathrm{rep}(\mathfrak{p}(n)) \tag{1.7}$$

identifying the highest weight structures and intertwining θ_i and Θ_i.

In particular, we obtain explicit results on the action of (iterated) translation functors on projective, standard, costandard and irreducible modules.

Applications and consequences

- We show that the dual of an irreducible $\mathfrak{p}(n)$-module can be computed by just rotating the cap diagram associated to its highest weight by $180°$, providing a much easier formula than the combinatorial procedure from [BDEA+19].

- In [BGS96], it was shown, that category $\mathcal{O}$ of a semisimple Lie algebra admits a Koszul grading. This also holds for $\mathfrak{gl}(m|n)$ by [BS12a] and is still conjectural for $\mathfrak{osp}(r|2n)$ (see e.g. [ES21] and [HNS24]). However, we will show that $\mathrm{rep}(\mathfrak{p}(n))$ does not admit a Koszul grading, we even prove:

  Theorem F. *There does not exist a non-negative grading on $\mathbb{K}_n$ with semisimple degree 0 part, that is generated in degree 1 for $n \geq 2$. In particular, $\mathrm{rep}(\mathfrak{p}(n))$ does not admit a Koszul grading.*

- There exist exactly n irreducible summands of $V^{\otimes d}$, one for each block (except for the one containing $\mathcal{L}(n-2, n-4, \dots, -n)$).

- We will give a simple combinatorial criterion to compute extensions between irreducible $\mathfrak{p}(n)$-modules.

  Theorem G. *The dimension of Ext^1 is given by*

$$\dim \mathrm{Ext}^1_{\mathfrak{p}(n)}(\mathcal{L}(\lambda), \mathcal{L}(\mu)) = \begin{cases} 1 & \text{if } \underline{\mu}\overline{\lambda} \text{ satisfies Def. 8.5,} \\ 0 & \text{otherwise.} \end{cases}$$

  Utilizing this, we give explicit description of $\mathrm{rep}(\mathfrak{p}(1))$ and $\mathrm{rep}(\mathfrak{p}(2))$ as a quiver with relations, which also explicitly shows that $\mathfrak{p}(2)$ does not admit a Koszul grading.

The contents of Chapters 4–8 are based on [Neh24].

Categorification

After having intensively studied the periplectic Lie superalgebra, we will turn our attention once again towards the electric KLR category. This will also give some context to the name “electric” KLR category. The contents of this are also available as a preprint version in [NS25].
We can ask what the category of projective sR_ϵ-modules categorify. This is the content of Chapter 10, which sloppily formulated reads as follows:

Theorem H. *The analog of the positive half of the quantum group for $\mathfrak{gl}_\infty$ from the KLR setting is, in the electric KLR setting, the quantum electrical algebra $\mathfrak{el}_q^\epsilon$.*

Electrical Lie algebras arose from the study of electrical networks. An electrical network comprises a directed graph with labelled edges (conductivity) and special boundary vertices. Its response matrix (sometimes also called Dirichlet-to-Neuman map) encodes the linear operator sending boundary electric potentials to the induced boundary currents.

This response matrix contains all the information about the network that can be observed on the boundary. Hence, it makes sense to consider electrical networks as equivalent if they have the same response matrix.
In [CIM98], certain combinatorial operations on the space of electrical networks (i.e. up to equivalence given by equal response matrices) have been studied. Loosely speaking, these correspond to adding edges or vertices to a network. These have been assembled in [LP15] into an *electrical Lie group* that acts on the space of electrical networks. The corresponding electrical Lie algebras are obtained by deforming the Serre relations of a semisimple Lie algebra in way suggested by the Y-Δ-transformation on electrical networks, [Ken99].
Electrical Lie algebras appeared only recently in the mathematical literature, see [LP15, Su14, BGG24, Geo24, Lam24] to name just a few.
Rather surprising, in type A the electrical Lie algebra is isomorphic to the symplectic Lie algebra, [LP15]. As was already observed by Serganova, the refined translation functors Θ_i for $\mathfrak{p}(n)$ satisfy the defining relations of a symplectic Lie algebra of infinite rank.
There are essentially two different ways to define a grading on the electric KLR categories, see Remark 3.6. This allows us to define a quantization $\mathfrak{el}_q^\epsilon$ of the electrical Lie algebras of type A, see Definition 9.1, depending on a sign ϵ reflecting the two different gradings. The grading on sR_ϵ is very unusual and induces generically $q^{\pm 4}$ in the relations of $\mathfrak{el}_q^\epsilon$.
We construct a quantum electric Hopf algebra U_q in Definition 9.22 and show the following:

Theorem I. *The quantum electric algebra $\mathfrak{el}_q^\epsilon$ is a coideal subalgebra of the quantum electric Hopf algebra U_q.*

We then define a natural representation V and its dual $V^\circledast$ for U_q and its restriction to $\mathfrak{el}_q^\epsilon$ and introduce Fock spaces $\mathscr{F}_\delta$ and $\mathscr{F}_\delta^\circledast$. These constructions are surprisingly involved. We can define exterior powers of V and $V^\circledast$ but the naive limit to semiinfinite wedges is not compatible with the U_q-action. We mix two different comultiplication to ensure a well-defined limit and finally obtain:

Theorem J. *There is an electric Fock space $\mathscr{F}_\delta$, i.e. the space of semiinfinite wedges can be equipped with an action of $\mathfrak{el}_q^\epsilon$.*

The generators $\mathfrak{el}_q^\epsilon$ act on $\mathscr{F}_\delta$ (up to q-powers) via the usual combinatorics of adding and removing boxes to partitions. However, these generators are a mixture of creation and annihilation operators.
In analogy to the classical KLR setting, we show a categorification result for cyclotomic quotients given by a charge vector as in Notation 2.8:

Theorem K. *The categories* $\mathrm{sR}_\epsilon^\ell(\boldsymbol{\delta})$-proj *of projective modules over the level ℓ cyclotomic quotient categorify the level ℓ Fock space $\mathscr{F}_{\boldsymbol{\delta},\ell}$. The action of $\mathfrak{el}_q^\epsilon$ is given by an action of the electric KLR category.*

By passing to right modules, we also obtain a categorification of the dual Fock space $\mathscr{F}_{\boldsymbol{\delta},\ell}^\circledast$ and the pairing between them, see Theorem 10.35, We categorify several involutions, including a bar involution, that might be interesting in their own right.

2. Combinatorics of multi-up-down-tableaux

2.1. Some preliminaries and notation

We denote by $\mathfrak{S}_n$ the symmetric group of order $n!$, generated by the simple transpositions $s_1 \coloneqq (1,2), \dots s_{n-1} \coloneqq (n-1,n)$.

Notation 2.1. We denote by $\mathbb{R}$ a subset of a ring with unit 1 (there is no harm to take for $\mathbb{R}$ the real numbers) such that for any $r \in \mathbb{R}$, $r + m1 \in \mathbb{R}$ for any $m \in \mathbb{Z}$.

Definition 2.2. A *standard subsequence* of $\boldsymbol{i} = (i_1, \dots, i_m) \in \mathbb{R}^m$ is given by some $\boldsymbol{j} = (i_j, i_{j+1}, \dots, i_{j+n-1}) \in \mathbb{R}^n$ obtained from $\boldsymbol{i}$ by taking a connected sequence of entries. By an *admissible permutation* of $\boldsymbol{i}$ we mean a permutation of the entries which involves only simple transpositions that swap entries a, b with $a \neq b \pm 1$. By a *subsequence* of $\boldsymbol{i}$ we mean any standard subsequence of an admissible permutation of $\boldsymbol{i}$. Moreover, $\boldsymbol{i}$ is *braid avoiding* if $(a, a \pm 1, a)$ is not a subsequence of $\boldsymbol{i}$.

2.2. Partitions and residues

Throughout this article we fix a *charge* $\delta \in \mathbb{R}$. A *partition* λ is a sequence $\lambda_1 \geq \lambda_2 \geq \dots \geq \dots$ of weakly decreasing non-negative integers. The *length* of λ is the maximal ℓ such that $\lambda_\ell > 0$. We call $|\lambda| \coloneqq \sum_{i=1}^{\ell} \lambda_i$ the *size* of λ. We will not distinguish between λ and the finite sequence $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell$. We also identify λ with its *Young diagram* built from $|\lambda|$ boxes with λ_i boxes (left-adjusted) in row i.

For every box $\square = (r, c)$ in the Young diagram of λ, specified by its row r and its column c, we define its *charged content* as $\mathrm{cont}(\square) \coloneqq \delta + r - c$.

We denote by $\mathrm{Add}(\lambda)$ and by $\mathrm{Rem}(\lambda)$ the set of boxes of λ that can be added to respectively removed from λ such that the result is again a Young diagram. These sets refine to the union of the sets $\mathrm{Add}_i(\lambda) \coloneqq \{\square \in \mathrm{Add}(\lambda) \mid \mathrm{cont}(\square) = i\}$ respectively $\mathrm{Rem}_i(\lambda) \coloneqq \{\square \in \mathrm{Rem}(\lambda) \mid \mathrm{cont}(\square) = i\}$ with i running through $\delta + \mathbb{Z}$.

If μ can be obtained from λ by adding a box we write $\lambda \to \mu$ or $\lambda \xrightarrow{\square} \mu$ encoding additionally the box $\square$ which was added. We also write in this case $\lambda \xrightarrow{-\square} \mu$, i.e. μ is obtained by removing $\square$ from λ. We moreover use the notation $\lambda \oplus \square$ for μ and $\mu \ominus \square$ for λ. The abbreviation $\lambda \leftrightarrow \mu$ means $\lambda \to \mu$ or $\mu \to \lambda$.

Next, we extend the notion of charged contents to treat box addition and box removal $\lambda \xrightarrow{\blacksquare} \mu$ for $\blacksquare = \pm\square = \pm(r,c)$ in parallel. We introduce two different extensions, the *residue* $\operatorname{res}(\blacksquare)$ and the *dual residue* $\operatorname{res}^{\circledast}(\blacksquare)$ of $\blacksquare$ as follows:

$$
\begin{aligned}
\operatorname{res}(\blacksquare) = \operatorname{res}(\lambda \xrightarrow{\blacksquare} \mu) &:= \begin{cases} \delta + c - r & \text{if } \blacksquare = (r,c), \\ \delta + c - r + 1 & \text{if } \blacksquare = -(r,c), \end{cases} \\
\operatorname{res}^{\circledast}(\blacksquare) = \operatorname{res}^{\circledast}(\lambda \xrightarrow{\blacksquare} \mu) &:= \begin{cases} \delta + c - r & \text{if } \blacksquare = (r,c), \\ \delta + c - r - 1 & \text{if } \blacksquare = -(r,c). \end{cases}
\end{aligned}
\tag{2.1}
$$

An *up-down-tableau* of length k is a sequence $(\mathfrak{t}_0, \mathfrak{t}_1, \ldots, \mathfrak{t}_k)$ of partitions such that $|\mathfrak{t}_0| = 0$ and $\mathfrak{t}_i \leftrightarrow \mathfrak{t}_{i+1}$. The *shape* $\operatorname{Shape}(\mathfrak{t})$ *of* $\mathfrak{t}$ is $\mathfrak{t}_k$. To each up-down-tableau we can associate the two residue sequences $\boldsymbol{i} := \boldsymbol{i}_{\mathfrak{t}} := (\operatorname{res}(\blacksquare_1), \ldots, \operatorname{res}(\blacksquare_k))$ and $\boldsymbol{i}^{\circledast} := \boldsymbol{i}^{\circledast}_{\mathfrak{t}} := (\operatorname{res}^{\circledast}(\blacksquare_1), \ldots, \operatorname{res}^{\circledast}(\blacksquare_k))$, where $\mathfrak{t}_i \xrightarrow{\blacksquare_i} \mathfrak{t}_{i+1}$. If $\mathfrak{t}_{i+1} = \mathfrak{t}_i \oplus \square_i$, then $\blacksquare_i = \square_i$ and $\operatorname{res}(\blacksquare_i) = \operatorname{cont}(\square_i) = \operatorname{res}^{\circledast}(\blacksquare_i)$. Thus, we recover the charged contents.

2.2.1. Combinatorics of multipartitions

We now consider multi-partitions and multi-up-down-tableaux. These are straightforward generalizations obtained by replacing every partition by a tuple of partitions. Namely, an *ℓ-multi-partition* is an ℓ-tuple $\lambda = (\lambda^1, \ldots, \lambda^\ell)$ of partitions λ^i. We identify λ with the corresponding tuple of Young diagrams and call $|\lambda| = \sum_{i=1}^n |\lambda^i|$ the *size* of λ. The set of all ℓ-multi-partitions is denoted Par^ℓ. We identify Par^1 with the set Par of partitions. Every box $\square = (r,c,k)$ in the Young diagram of $\lambda \in \operatorname{Par}^\ell$ has now a third coordinate k that indexes the component λ^k, $1 \leq k \leq \ell$ containing $\square$.
To distinguish the components of a multi-partition, we use a *charge sequence* $\boldsymbol{\delta}(\infty) \in \mathbb{R}^{\mathbb{N}}$. It determines a charge vector $\boldsymbol{\delta} := \boldsymbol{\delta}(\ell) := (\delta_1, \ldots, \delta_\ell) \in \mathbb{R}^\ell$ for any fixed $\ell \in \mathbb{N}$. For $\lambda \in \operatorname{Par}^\ell$ we define the *charged content* of a box $\square = (r,c,k)$ as $\operatorname{cont}(\square) := \delta_k + r - c$. We denote by $\operatorname{Add}(\lambda)$ and $\operatorname{Rem}(\lambda)$ the set of boxes that can be added to respectively removed from λ. As for partitions, these sets are the union of the sets $\operatorname{Add}_i(\lambda)$ (and of $\operatorname{Rem}_i(\lambda)$) of addable (respectively removable) boxes of charged content $i \in \mathbb{R}$.

Remark 2.3. Note that if $\delta_i - \delta_j \notin \mathbb{Z}1 \subseteq \mathbb{R}$ for all $i \neq j$, then the charged content of a box (r,c,k) in $\lambda \in \operatorname{Par}^\ell$ uniquely determines this component.

We again use the arrow notation $\lambda \xrightarrow{\blacksquare} \mu$ if μ can be obtained from λ by adding or removing a box $\square$. If $\blacksquare = \square = (r,c,k)$, we have $\lambda^k \xrightarrow{(r,c)} \mu^k$ and $\lambda^i = \mu^i$ for $i \neq k$. We also extend the notion of (dual) residues involving boxes $\square = (r,c,k)$ in $\lambda \in \operatorname{Par}^\ell$:

$$
\begin{aligned}
\operatorname{res}(\blacksquare) &:= \begin{cases} \delta_k + c - r & \text{if } \blacksquare = (r,c,k), \\ \delta_k + c - r + 1 & \text{if } \blacksquare = -(r,c,k), \end{cases} \\
\operatorname{res}^{\circledast}(\blacksquare) &:= \begin{cases} \delta_k + c - r & \text{if } \blacksquare = (r,c,k), \\ \delta_k + c - r - 1 & \text{if } \blacksquare = -(r,c,k). \end{cases}
\end{aligned}
\tag{2.2}
$$

Obviously $\ell = 1$, $\delta_1 = \delta$ recovers the case (2.1) of partitions.

Notation 2.4. An *ℓ-multi-up-down-tableau* $\mathfrak{t}$ of length m is a sequence $(\mathfrak{t}_0, \mathfrak{t}_1, \dots, \mathfrak{t}_m)$ of ℓ-multi-partitions $\mathfrak{t}_i$ such that $\mathfrak{t}_0$ has size $|\mathfrak{t}_0| = 0$ and $\mathfrak{t}_i \leftrightarrow \mathfrak{t}_{i+1}$. We call $\mathfrak{t}_m$ the *shape* $\mathrm{Shape}(\mathfrak{t})$ *of* $\mathfrak{t}$. By $\mathfrak{t}|_n$ for $n < m$ we denote the ℓ-multi-up-down-tableau $(\mathfrak{t}_0, \mathfrak{t}_1, \dots, \mathfrak{t}_n)$ of length n which is the restriction to the first $n+1$ multi-partitions.

We can draw an ℓ-multi-up-down-tableau by drawing the tuple of Young diagrams of the partitions and arrows between consecutive ℓ-multi-partitions. Observe, that any ℓ-multi-up-down-tableau $\mathfrak{t}$ necessarily has $\mathfrak{t}_0 = (\emptyset, \dots, \emptyset)$.
As above, we associate to $\mathfrak{t}$ two residue sequences $\boldsymbol{i} = (\mathrm{res}(\blacksquare_1), \dots, \mathrm{res}(\blacksquare_m))$ and $\boldsymbol{i}^{\circledast} = (\mathrm{res}^{\circledast}(\blacksquare_1), \dots, \mathrm{res}^{\circledast}(\blacksquare_m))$ if $\mathfrak{t}_i \xrightarrow{\blacksquare_i} \mathfrak{t}_{i+1}$.

Notation 2.5. Denote by $\mathcal{T}^{\mathrm{ud},\ell}_m(\lambda)$ the set of all ℓ-multi-up-down-tableaux of shape λ and length m, and by $\mathcal{T}^{\mathrm{ud},\ell}$ (resp. $\mathcal{T}^{\mathrm{ud},\ell}_m$, $\mathcal{T}^{\mathrm{ud},\ell}(\lambda)$) the set of ℓ-multi-up-down-tableaux (of fixed length m and of fixed shape λ). For each ℓ-multi-partition λ there exists the *canonical up-down-tableaux* $\mathfrak{t}^\lambda$ *of shape* λ which is obtained by first adding the boxes for λ^ℓ row by row, then the boxes of $\lambda^{\ell-1}$ row by row, and so on.

Example 2.6. Here is an example of $\lambda \in \mathrm{Par}^2$ its $\mathfrak{t}^\lambda$ and the charged contents:

$$\lambda = \left(\begin{array}{|c|c|c|}\hline & & \\ \hline & \\ \cline{1-2} \end{array}\ , \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array} \right) \qquad \mathfrak{t}^\lambda = \left(\begin{array}{|c|c|c|}\hline 5 & 6 & 7 \\ \hline 8 & 9 \\ \cline{1-2} \end{array}\ , \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \right) \qquad \left(\begin{array}{|c|c|c|}\hline \delta_1 & \delta_1+1 & \delta_1+2 \\ \hline \delta_1-1 & \delta_1 \\ \cline{1-2} \end{array}\ , \begin{array}{|c|c|}\hline \delta_2 & \delta_2+1 \\ \hline \delta_2-1 & \delta_2 \\ \hline \end{array} \right).$$

Definition 2.7. Define a partial ordering on Par^ℓ by setting $\lambda > \mu$ if $|\lambda| < |\mu|$.

Notation 2.8. For the remainder of the article we fix a charge sequence $\boldsymbol{\delta}^\infty \in \mathbb{R}^{\mathbb{N}}$ (with charge vectors $\boldsymbol{\delta}(\ell)$) which is *generic* that is $\delta_i - \delta_j \notin \mathbb{Z}1 \subseteq \mathbb{R}$ for all $i \neq j$.

3. The electric KLR-category sR_ϵ

The goal of this section is to introduce a new monoidal supercategory, the electric KLR-category, by generators and relations and describe some basic properties. The morphism spaces assemble into electric KLR algebras which should be seen as analogues of the KLR algebras from [KL09], [Rou08].

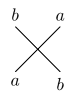
Notation 3.1. For this section we fix a ground field $\Bbbk$ and denote by $\mathcal{SV}\mathrm{ec}^\circ$ the symmetric monoidal category of $\Bbbk$-vector superspaces with (super) degree preserving morphisms, and we write $\mathcal{SV}\mathrm{ec}$ if we allow all morphisms. For $V \in \mathcal{SV}\mathrm{ec}^\circ$ we denote by $|v| \in \{0,1\}$ the degree of $v \in V$ implicitly assuming v to be homogeneous.

Thus, in $\mathcal{SV}\mathrm{ec}^\circ$, the braiding morphisms are $v \otimes w \mapsto (-1)^{|v||w|} w \otimes v$. By a *supercategory* we mean a $\mathcal{SV}\mathrm{ec}^\circ$-category, i.e. a category enriched in $\mathcal{SV}\mathrm{ec}^\circ$, in the sense of [Kel05]. Moreover, $\mathcal{SV}\mathrm{ec}^\circ$ has a symmetric braiding and we can consider monoidal supercategories. Morphisms in these satisfy $(f \otimes 1)(1 \otimes g) = (-1)^{|f||g|}(1 \otimes g)(1 \otimes f)$.

3.1. Basic definition and formulation of main theorems

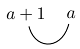
For basics on monoidal supercategories we refer for instance to [BE17a], [CE21]. We denote by $\mathbb{1}$ the monoidal unit in a given monoidal (super)category.

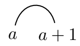
Definition 3.2. Let $\mathrm{sR}(\mathbb{Z})$ be the $\Bbbk$-linear strict monoidal supercategory freely generated on the level of objects by $a \in \mathbb{Z}$ and on the level of morphisms by

even generators: 

$: a \otimes b \to b \otimes a,$

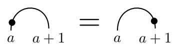
$: a \to a,$

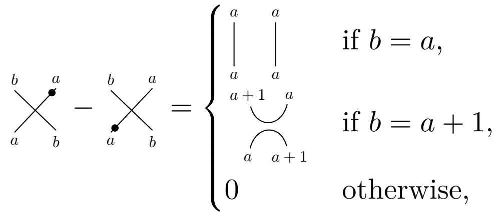
odd generators: 

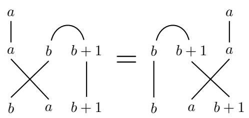
$: \mathbb{1} \to (a+1) \otimes a,$ 

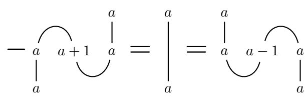
$: a \otimes (a+1) \to \mathbb{1},$

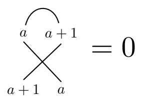
modulo the following (local) relations (sR-1)–(sR-7):

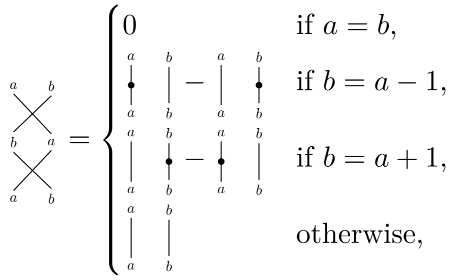
(sR-1)


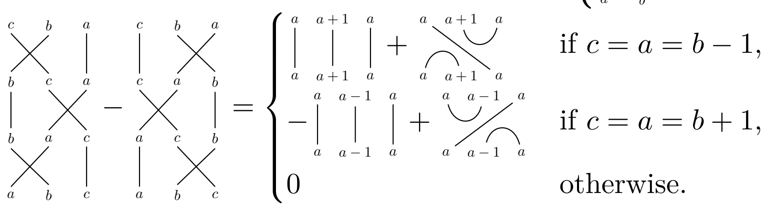
(sR-2)


(sR-3)


(sR-4)


(sR-5)


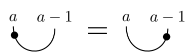
(sR-6)


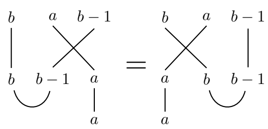
(sR-7)


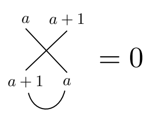
Remark 3.3. For simplicity, we work over $\Bbbk$ and not an arbitrary ground ring or $\mathbb{Z}$.

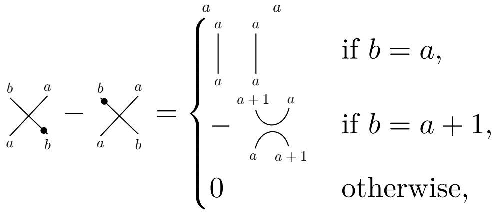
This definition makes also sense when we replace the set of objects/of labels of the strands by any set $\mathbb{R}'$ with an automorphism $(+1)\colon \mathbb{R}' \to \mathbb{R}'$. In particular $\mathbb{R}' = \mathbb{R}$ as in Notation 2.1 works. Objects in $\mathrm{sR}(\mathbb{R}')$ are then (possibly empty) finite sequences $\mathbf{a}$ of elements in $\mathbb{R}'$. We will denote the resulting category $\mathrm{sR}(\mathbb{R}')$, but mostly work from now on with $\mathrm{sR} \coloneqq \mathrm{sR}(\mathbb{R})$.

Lemma 3.4. *The defining relations* (sR-1)–(sR-7) *imply the following equalities:*

(3.1)


(3.2)


(3.3)


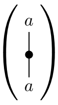
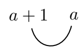
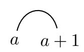
(3.4)


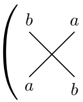
Proof. The relations (3.1) and (3.2) follow from (sR-1) respectively (sR-3) using (sR-4). After adding a snake (sR-4) to the left-hand side of (3.3), (3.3) follows with (3.2), (sR-3) from (sR-5). Finally, (3.4) follows from (sR-2) by rotation, i.e. by adding a cup to the bottom and a cap to the top and then applying (sR-3) and (sR-4). ☐

We denote by $\mathcal{GSV}\mathrm{ec}^\circ$ (and $\mathcal{GSV}\mathrm{ec}$) the monoidal category of $\mathbb{Z}$-graded vector superspaces with supergrading preserving morphisms which preserve (respectively not necessarily preserve) the $\mathbb{Z}$-degree. The braiding morphisms are the flip maps adjusted by signs only with respect to the super grading and not the $\mathbb{Z}$-grading.

Proposition 3.5. *Let $\epsilon \in \{\pm 1\}$. Then $\mathrm{sR}(\mathbb{Z})$, or more generally $\mathrm{sR}(\mathbb{R})$, can be viewed as a monoidal $\mathcal{GSV}\mathrm{ec}^\circ$-category sR_ϵ by setting*

(3.5)

deg () $= 2$, deg () $= -\epsilon$, deg () $= \epsilon$,

deg () $= \begin{cases} -2 & \text{if } b = a, a+1, \\ 0 & \text{if } b-a \notin \mathbb{Z}, \\ 4\,\mathrm{sgn}(b-a)(-1)^{b-a} & \text{otherwise.} \end{cases}$

Proof. It suffices to check that (3.5) is compatible with (sR-1)-(sR-7). □

Remark 3.6. The (surprisingly difficult) degrees for the crossings are forced on us if we require for $\mathrm{sR}_\epsilon(\mathbb{Z})$ the dot generator to be of degree 2, i.e. compatible with the usual KLR convention. First, by (sR-2) the crossings labelled (a, a) must have degree -2. Then (sR-3) forces the crossing labelled $(a, a+1)$ at the bottom to be of degree -2. Finally, by (sR-6), the crossing labelled $(a+1, a)$ must have degree 4. The arguments from [Neh24, Definition 1.6, Lemma 1.8] imply then that the degrees of the other crossings are also forced. The degree for a $(a, a+1)$ cap can be an arbitrary integer ϵ_a as long as the $(a+1, a)$ cup has degree $-\epsilon_a$. If we require independent of a, our choices for ϵ are unique up to an overall positive scaling.

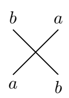
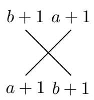
Definition 3.7. The monoidal $\mathcal{GSV}\mathrm{ec}^\circ$-category sR_ϵ is the *electric KLR category.*

Definition 3.8. The category sR_ϵ can be viewed as a locally unital algebra sR_ϵ with set of idempotents labelled by $\mathbb{R}$, namely $\mathrm{sR}_\epsilon = \bigoplus_{\mathbf{a},\mathbf{b}\in\mathbb{R}} 1_{\mathbf{a}}\mathrm{sR}_\epsilon 1_{\mathbf{b}}$, where $1_{\mathbf{a}}\mathrm{sR}_\epsilon 1_{\mathbf{b}}$ is the $\mathbb{Z}$-graded vector superspace of all morphisms from $\mathbf{a}$ to $\mathbf{b}$ in sR_ϵ. More precisely, sR_ϵ is a $\mathbb{Z}$-graded superalgebra (that is an algebra object in $\mathcal{GSV}\mathrm{ec}^\circ$). We call this algebra the *electric KLR (super)algebra.*

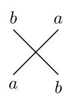
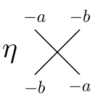
For a supercategory $\mathcal{C}$ we denote by $\mathcal{C}^{\mathrm{op}}$ its opposite supercategory. If $\mathcal{C}$ is moreover monoidal, let $\mathcal{C}^{\mathrm{rev}}$ the category $\mathcal{C}$ with the opposite monoidal structure $a \otimes_{\mathrm{rev}} b = b \otimes a$ on objects and $f \otimes_{\mathrm{rev}} g = (-1)^{|f||g|} g \otimes f$. Denote $\mathcal{C}^{\mathrm{oprev}} = (\mathcal{C}^{\mathrm{op}})^{\mathrm{rev}} \cong (\mathcal{C}^{\mathrm{rev}})^{\mathrm{op}}$.

Lemma 3.9. *There are equivalences of monoidal $\mathcal{GSV}\mathrm{ec}$-categories*

$\Sigma\colon \mathrm{sR}_\epsilon^{\mathrm{oprev}} \to \mathrm{sR}_\epsilon$, $\quad a \mapsto a+1$,

 $\mapsto -$  ,

$\mathcal{T}\colon \mathrm{sR}_\epsilon^{\mathrm{op}} \to \mathrm{sR}_\epsilon$, $\quad a \mapsto -a$,

 $\mapsto \eta$  ,

where $a, b \in \mathbb{R}$, *and* $\eta = -1$ *if* $b \neq a, a+1$ *and* $\eta = 1$ *if* $b = a+1, a$.

Proof. This is straightforward bearing in mind that $f \circ_{\mathrm{op}} g = (-1)^{|f||g|} g \circ f$. $\square$

3.1.1. Cyclotomic quotients $\mathrm{sR}^\ell_\epsilon$

For a general overview about cyclotomic quotients in the context of (quiver) Hecke algebras we refer to [Mat15].

Definition 3.10. Given a natural number ℓ, called the *level*, we define the *cyclotomic quotients* sR^ℓ and $\mathrm{sR}^\ell_\epsilon$, of charge $\boldsymbol{\delta} = \boldsymbol{\delta}(\ell)$, as the quotients of sR and sR_ϵ respectively by the right tensor ideal generated by

(3.6) 

$$, \quad \text{where} \quad n = \begin{cases} 1 & \text{if } a = \delta_i,\ 1 \leq i \leq \ell, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 3.11. The *cyclotomic polynomial of level* ℓ (and charge $\boldsymbol{\delta}$) is defined as

$$\Omega^\ell(x) = \prod_{i=1}^{\ell} (x - \delta_i). \tag{3.7}$$

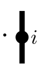
Definition 3.12. We denote by $\theta_i^k \colon \mathrm{sR}^\ell_\epsilon \to \mathrm{sR}^\ell_\epsilon$ the endofunctor given by adding a strand labelled i with k dots on the right. If $k = 0$, we abbreviate $\theta_i \coloneqq \theta_i^0$.

Notation 3.13. The case, that $\ell = 1$ and the corresponding $\delta = 0$, will appear very often in the following. In this case we will also write $\mathrm{sR}^{\mathrm{cyc}}$ for sR^ℓ. The cyclotomic polynomial is in this case given by $\Omega^1(x) = x$.
This will be important when relating sR to representations of $\mathfrak{p}(n)$.

3.1.2. Cyclotomic quotients in the affine VW-supercategory $\mathrm{s}\mathbb{W}$

Recall from the introduction the affine VW-supercategory $\mathrm{s}\mathbb{W}$.

Definition 3.14. The *level* ℓ *cyclotomic quotient* $\mathrm{s}\mathbb{W}^\ell$ is the cyclotomic quotient of $\mathrm{s}\mathbb{W}$ by the cyclotomic polynomial $\Omega^\ell(x)$ of level ℓ from (3.7). This is the quotient of $\mathrm{s}\mathbb{W}$ by the right tensor ideal generated by

(3.8) $\sum_{i=0}^{n} a_i \cdot$ 

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Notation 3.15. As for $\mathrm{sR}^\ell_\epsilon$, we also write $\mathrm{s}\mathbb{W}^{\mathrm{cyc}}$ for the cyclotomic quotient of $\mathrm{s}\mathbb{W}$ by the cyclotomic polynomial x (i.e. $\ell = 1$ and $\delta = 0$). Observe that $\mathrm{s}\mathbb{W}^{\mathrm{cyc}} \cong \mathrm{sBR}$.

Given an object, say $*^{\otimes m}$, its endomorphism algebra $\mathrm{End}_{\mathrm{s}\mathbb{W}^\ell}(m)$ is a finite dimensional algebra and the y_j, (i.e. identities with a dot on the j-th strand) for $1 \leq j \leq m$ form a family of pairwise commuting elements.

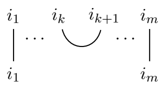
Notation 3.16. Denote by $e_{\mathbf{i}} = e_{i_1,\dots,i_m}$ the idempotents projecting onto the simultaneous generalized i_j-eigenspaces for the y_j's, in particular $y_j e_{\mathbf{i}} = e_{\mathbf{i}} y_j = i_j e_{\mathbf{i}}$.

3.2. The Isomorphism Theorem and Cyclotomic Equivalence

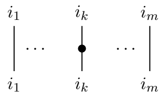
We finally formulate the *Isomorphism Theorem* from the introduction:

Theorem 3.17 (Isomorphism Theorem)**.** *For any level ℓ, the following assignments define a fully faithful functor to the Karoubian envelope* $\mathrm{Kar}(\mathrm{s}\mathbb{W}^\ell)$ *of* $\mathrm{s}\mathbb{W}^\ell$*:*

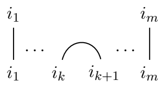
$$\Phi\colon \mathrm{sR}^\ell \to \mathrm{Kar}(\mathrm{s}\mathbb{W}^\ell), \quad \mathbf{i} = (i_1, \dots, i_m) \mapsto e_{\mathbf{i}},$$



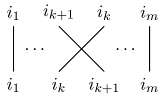
$\mapsto e_{\mathbf{i}} \flat_k^* e_{\mathbf{i}'}$,



$\mapsto e_{\mathbf{i}}(y_k - i_k)$,



$\mapsto e_{\mathbf{i}'} \flat_k e_{\mathbf{i}}$,



$$\mapsto \begin{cases} e_{\mathbf{i}} \eta_{i_{k+1}, i_k}((i_{k+1} - i_k)s_k + 1) & \textit{if } i_{k+1} \notin \{i_k, i_k + 1\}, \\ 0 & \textit{otherwise,} \end{cases}$$

where $\mathbf{i}' = (i_1, \dots, \hat{i}_k, \hat{i}_{k+1}, \dots, i_m)$ *and* $\eta_{b,a}$ *is any choice of scalars, such that*

(i) $\eta_{a,b}\eta_{b,a} = \frac{1}{1-(a-b)^2}$ *for all* a, $b \in \mathbb{R}$ *such that* $a - b \notin \{0, \pm 1\}$ *and*

(ii) $\eta_{b,a}(b - a) = \eta_{a,b+1}(a - b - 1)$ *for all* a, $b \in \mathbb{R}$ *such that* $a \neq b, b + 1$.

As a consequence of the Isomorphism Theorem in the special case of $\ell = 1$ and $\delta = 0$ we obtain an idempotent version of the *periplectic Brauer algebras*, [Cou18a].
For the proof we will introduce elements $\Psi_{\mathfrak{t}}^{\mathfrak{s}} \in \mathrm{sR}_\epsilon^\ell$ and show a Basis Theorem.

Remark 3.18. Note that the functor is not an equivalence, but it will become an equivalence after additive completion by the Cyclotomic equivalence below.

Recall that the Karoubian closure of a category is the idempotent completion of a category. We could also take its additive envelope (which means we allow also finite direct sums of objects and morphisms). In general, taking additive closure and taking Karoubian closure does not commute, but since we have finite dimensional morphism spaces these procedures in fact do commute.

Theorem 3.19 (Cyclotomic equivalence)**.** *For any level ℓ, the additive closure of* sR^ℓ *is equivalent as* $\mathcal{SV}\mathrm{ec}^\circ$*-category to the additive closure of* $\mathrm{Kar}(\mathrm{s}\mathbb{W}^\ell)$ *of* $\mathrm{s}\mathbb{W}^\ell$*.*

Remark 3.20. We expect that the Isomorphism Theorem holds for any (not necessarily generic) charge sequence, but our formulation and proof of the Basis Theorem requires the charge to be generic.

Remark 3.21. As a consequence of the Isomorphism Theorem we obtain in particular an idempotent version of cyclotomic quotients of the *periplectic Brauer categories* from [CE21] or the marked Brauer categories from [KT17].

3.3. The Basis Theorem and applications

In this section we formulate the Basis Theorem and show some important consequences. Let $\mathfrak{t} = (\mathfrak{t}_0, \mathfrak{t}_1, \dots, \mathfrak{t}_m) \in \mathcal{T}^{\mathrm{ud}}$ with $\mathrm{Shape}(\mathfrak{t}) = \lambda$. We start by defining morphisms

$$\Psi_{\mathfrak{t}}^{\mathfrak{t}^\lambda} \colon \boldsymbol{i}_{\mathfrak{t}} \to \boldsymbol{i}_{\mathfrak{t}^\lambda} \quad \text{and} \quad \Psi_{\mathfrak{t}^\lambda}^{\mathfrak{t}} \colon \boldsymbol{i}_{\mathfrak{t}^\lambda}^{\circledast} \to \boldsymbol{i}_{\mathfrak{t}}^{\circledast} \quad \text{in } \mathrm{sR}_\epsilon^\ell. \tag{3.9}$$

Construction of $\Psi_{\mathfrak{t}}^{\mathfrak{t}^\lambda}$ and $\Psi_{\mathfrak{t}^\lambda}^{\mathfrak{t}}$

Case 1: If $|\lambda| = m$, then boxes were only added in $\mathfrak{t}$ and $\boldsymbol{i}_{\mathfrak{t}}$ differs from $\boldsymbol{i}_{\mathfrak{t}^\lambda}$ by a permutation. Let $d_{\mathfrak{t}} \in \mathfrak{S}_n$ be the unique such permutation of minimal length. Pick a reduced expression $d_{\mathfrak{t}} = s_{r_\ell} \cdots s_{r_1}$. This defines a corresponding composition $\boldsymbol{i}_{\mathfrak{t}} \to \boldsymbol{i}_{\mathfrak{t}^\lambda}$ of ℓ generating morphisms, where each simple transposition s_k is sent to a diagram where the kth and $k+1$th strand cross. Note that, by construction and by assumption on the charge, the labels, say a and b, at these two strands are distant in the sense that they satisfy $a \notin \{b, b \pm 1\}$. But then (sR-7) implies that the construction is independent of the choice of reduced expression. Thus, we get a well-defined morphism $\Psi_{\mathfrak{t}}^{\mathfrak{t}^\lambda} \colon \boldsymbol{i}_{\mathfrak{t}} \to \boldsymbol{i}_{\mathfrak{t}^\lambda}$. Analogously, define $\Psi_{\mathfrak{t}^\lambda}^{\mathfrak{t}} \colon \boldsymbol{i}_{\mathfrak{t}^\lambda}^{\circledast} \to \boldsymbol{i}_{\mathfrak{t}}^{\circledast}$ using the dual residue sequences. In both constructions $\Psi_{\mathfrak{t}^\lambda}^{\mathfrak{t}^\lambda}$ is the identity on $\boldsymbol{i}_{\mathfrak{t}^\lambda}$.

Case 2: If $|\lambda| < m$ then consider the minimal r such that $\mathfrak{t}_r$ is obtained from $\mathfrak{t}_{r-1}$ by removing a box. Denote by $l < k$ the index where this box was added to $\mathfrak{t}$. Draw a cap from i_l to i_k in $\boldsymbol{i}_{\mathfrak{t}}$. By adding vertical strands at the remaining residues we obtain a diagram representing a morphism from $\boldsymbol{i}_{\mathfrak{t}}$ to the subsequence of $\boldsymbol{i}_{\mathfrak{t}}$ given by the residues not involved in the cap. (We leave it to the reader to verify using (sR-3) that the diagram can be written as a product of crossings and caps and that any such product defines up to sign the same morphism.)

Repeat this procedure for all boxes that were removed in $\mathfrak{t}$ working with the residue sequence treated by caps already. This results in a composite morphism from $\boldsymbol{i}_{\mathfrak{t}}$ to the subsequence $\boldsymbol{i}'_{\mathfrak{t}}$ of $\boldsymbol{i}_{\mathfrak{t}}$ where all residues connected with cups are removed. The length of $\boldsymbol{i}'_{\mathfrak{t}}$ equals $|\lambda|$, and we can construct, as in Case 1, a morphism $\boldsymbol{i}'_{\mathfrak{t}} \to \boldsymbol{i}_{\mathfrak{t}^\lambda}$. Composing provides a morphism $\boldsymbol{i}_{\mathfrak{t}} \to \boldsymbol{i}_{\mathfrak{t}^\lambda}$ which is up to an overall sign independent of choices on the way. Similarly, we can construct a morphism $\boldsymbol{i}_{\mathfrak{t}^\lambda}^{\circledast} \to \boldsymbol{i}_{\mathfrak{t}}^{\circledast}$ by using cups instead of caps.

The constructed morphisms are only unique up to signs, since caps and cups have odd degree and thus height moves create signs. To fix this we adjust our construction by height moves so that they satisfy the following *height requirement*: We assume that if two caps (or cups) connect the positions (k,l) and (k',l') with $l<l'$, then (k,l) is lower (resp. higher) than (k',l'). With this we constructed the desired morphisms (3.9) in $\mathrm{sR}^\ell_\epsilon$. Recalling that $\boldsymbol{i}_{\mathfrak{t}^\lambda} = \boldsymbol{i}^{\circledast}_{\mathfrak{t}^\lambda}$ we can define the following compositions:

Definition 3.22. For $\mathfrak{t},\mathfrak{s}\in\mathcal{T}^{\mathrm{ud}}$ with $\mathrm{Shape}(\mathfrak{t})=\lambda=\mathrm{Shape}(\mathfrak{s})$, define $\Psi^{\mathfrak{s}}_{\mathfrak{t}} = \Psi^{\mathfrak{s}}_{\mathfrak{t}^\lambda}\Psi^{\mathfrak{t}^\lambda}_{\mathfrak{t}} \in \mathrm{sR}^\ell_\epsilon$. In particular, $\Psi^{\mathfrak{t}}_{\mathfrak{t}}$ is the identity on $\boldsymbol{i}_{\mathfrak{t}^\lambda}$.

Theorem 3.23 (Basis Theorem)**.** *The set* $\mathcal{B}:=\{\Psi^{\mathfrak{s}}_{\mathfrak{t}} \mid \mathfrak{t},\mathfrak{s}\in\mathcal{T}^{\mathrm{ud}}, \mathrm{Shape}(\mathfrak{t})=\mathrm{Shape}(\mathfrak{s})\}$ *is a basis, the* up-down-tableaux basis, *of* $\mathrm{sR}^\ell_\epsilon$.

Before the proof we show some nice properties of the basis elements.

Proposition 3.24. *Let* $\mathfrak{t},\mathfrak{s}\in\mathcal{T}^{\mathrm{ud}}(\lambda)$ *and* $\lambda\xrightarrow{\square}\mu$, $\mathrm{res}(\square)=i$*. Then* $\theta_i(\Psi^{\mathfrak{s}}_{\mathfrak{t}}) = \Psi^{\mathfrak{s}\frown\mu}_{\mathfrak{t}\frown\mu}$, *where* $\mathfrak{u}\frown\mu=(\mathfrak{u}_0,\dots,\mathfrak{u}_n,\mu)\in\mathcal{T}^{\mathrm{ud}}(\mu)$ *for* $\mathfrak{u}=(\mathfrak{u}_0,\dots,\mathfrak{u}_n)\in\mathcal{T}^{\mathrm{ud}}(\lambda)$.

Proof. Assume first that $\mathfrak{s}=\mathfrak{t}^\lambda$. By definition, $\Psi:=\Psi^{\mathfrak{t}^\lambda\frown\mu}_{\mathfrak{t}\frown\mu} = \Psi^{\mathfrak{t}^\lambda\frown\mu}_{\mathfrak{t}^\mu}\Psi^{\mathfrak{t}^\mu}_{\mathfrak{t}\frown\mu}$. By assumption, $\boldsymbol{i}_{\mathfrak{t}^\lambda\frown\mu}$ and $\boldsymbol{i}_{\mathfrak{t}^\mu}$ are obtained from $\boldsymbol{i}_{\mathfrak{t}}$ by adding i at the end respectively at the position, say p, corresponding to $\square$. In a diagram describing Ψ, this last entry in $\boldsymbol{i}^\lambda_{\mathfrak{t}}\frown\mu$ connects (via the right factor of Ψ) to the residue at position p and then (via the left factor) back to the last entry in $\boldsymbol{i}_{\mathfrak{t}^\lambda\frown\mu}$. Since the involved crossings carry distant labels, one can straighten this strand using (sR-6) and (sR-7) to obtain $\Psi^{\mathfrak{s}}_{\mathfrak{t}}$ with an additional vertical strand labelled i on the right. Thus, $\Psi=\theta_i(\Psi^{\mathfrak{s}}_{\mathfrak{t}})$. Similarly, the claim holds for $\Psi^{\mathfrak{t}}_{\mathfrak{t}^\lambda}$ and thus for $\Psi^{\mathfrak{s}}_{\mathfrak{t}} = \Psi^{\mathfrak{s}}_{\mathfrak{t}^\lambda}\Psi^{\mathfrak{t}^\lambda}_{\mathfrak{t}}$, since θ_i is a functor. $\square$

For the next application we consider Par^ℓ for fixed level ℓ with its partial order from Definition 2.7 as subset of $I:=\bigcup_{m\in\mathbb{Z}_{\geq 0}}\mathbb{R}^m$ by identifying $\lambda\in\mathrm{Par}^\ell$ with $\boldsymbol{i}_{\mathfrak{t}^\lambda}=\boldsymbol{i}^{\circledast}_{\mathfrak{t}^\lambda}$.

Theorem 3.25 (Highest weight)**.** *Consider* $\mathrm{sR}^\ell_\epsilon$ *with up-down-tableaux basis* $\mathcal{B}$*. For* $\lambda\in\mathrm{Par}^\ell$ *and* $\boldsymbol{i}\in\mathbb{R}^m$ *set* $Y(\boldsymbol{i},\lambda)=\{\Psi^{\mathfrak{s}}_{\mathfrak{t}^\lambda}\mid \boldsymbol{i}^{\circledast}_{\mathfrak{s}}=\boldsymbol{i}\}$ *and* $X(\lambda,\boldsymbol{i})=\{\Psi^{\mathfrak{t}^\lambda}_{\mathfrak{s}}\mid \boldsymbol{i}_{\mathfrak{s}}=\boldsymbol{i}\}$.
This data endows $A:=\bigoplus_{m,n\in\mathbb{N}_0}\bigoplus_{\mathbf{i}\in\mathbb{R}^m,\mathbf{j}\in\mathbb{R}^n}\mathrm{Hom}_{\mathrm{sR}^\ell_\epsilon}(\mathbf{i},\mathbf{j})$ *with the structure of an upper finite based quasi-hereditary (super-)algebra in the sense of* [BS24].

Proof. Writing $Y(\lambda):=\bigcup_{\boldsymbol{i}\in I}Y(\boldsymbol{i},\lambda)$ and $X(\lambda):=\bigcup_{\boldsymbol{i}\in I}X(\lambda,\boldsymbol{i})$, it follows by Definition 3.22 directly from Theorem 3.23 that $\bigcup_{\lambda\in\mathrm{Par}^\ell}Y(\lambda)\times X(\lambda)$ is a basis of A. The set $Y(\mu,\lambda)$ can only be nonempty if $\lambda=\mu$ or $|\mu|>|\lambda|$, and thus $\mu\leq\lambda$, similarly for $X(\lambda,\mu)$. It is also clear from Definition 3.22 that $X(\lambda,\lambda)=Y(\lambda,\lambda)=\{e_\lambda\}$ for each $\lambda\in\mathrm{Par}^\ell$. $\square$

3.3.1. The spanning set $\mathcal{B}$

We next show that the proposed basis $\mathcal{B}$ in Theorem 3.23 spans. For this fix the filtration $\{0\}=F_{\leq -1}\subseteq F_{\leq 0}\subseteq F_{\leq 1}\subseteq\dots$ on $\mathcal{T}^{\mathrm{ud}}$ given by $F_{\leq b}=\bigcup_{|\lambda|\leq b}\mathcal{T}^{\mathrm{ud}}(\lambda)$. This induces a filtration on the $\Bbbk$-span B of $\mathcal{B}$ with pieces $B_{\leq i}$ spanned by all $\Psi^{\mathfrak{s}}_{\mathfrak{t}}$ with $\mathrm{Shape}(\mathfrak{t})=\mathrm{Shape}(\mathfrak{s})\in F_{\leq i}$. Let $R\supseteq R_{\leq b}$ be the two-sided ideals in $\mathrm{sR}^\ell_\epsilon$ generated by $\mathcal{B}$ respectively $B_{\leq b}$. Thus, $R_{\leq b}$ defines a filtration on R by ideals which we use to show $B=R$. Abbreviate $B_{<b}:=B_{\leq(b-1)}$, $R_{<b}:=R_{\leq(b-1)}$.

We show now some properties of $\mathrm{sR}_\epsilon^\ell$ in the following situation for fixed $b \in \mathbb{N}$:

$$(\mathrm{Ass}{<}b) \qquad\qquad B_{\leq b'} = R_{\leq b'} \text{ for all } b' < b.$$

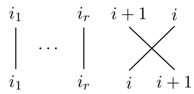
Proposition 3.26. *Assume* $(\mathrm{Ass}{<}b)$ *and let* $\lambda \in \mathrm{Par}^\ell$ *with* $|\lambda| = b$. *Then the following holds in* $\mathrm{sR}_\epsilon^\ell$ *for any* $i, j \in \mathbb{R}$ *with* $\mathrm{Add}_i(\lambda) = \emptyset$.
(3.10) $\theta_i(\Psi^{\mathfrak{t}^\lambda}_{\mathfrak{t}^\lambda}) \in B_{<|\lambda|}$, (3.11) $\theta^1_j(\Psi^{\mathfrak{t}^\lambda}_{\mathfrak{t}^\lambda}) = 0$, (3.12) $\theta_i(\Psi^{\mathfrak{s}}_{\mathfrak{t}}) \in B_{<|\lambda|}$ *for* $\mathfrak{t}, \mathfrak{s} \in \mathcal{T}^{\mathrm{ud}}(\lambda)$.
In particular, any diagram with a dot is zero in $\mathrm{sR}_\epsilon^\ell$ *by* (3.11).

The proof of Proposition 3.26 will show inductively the following refinements:

Corollary 3.27. *Assume* $(\mathrm{Ass}{<}b)$. *Then the following holds in* $\mathrm{sR}_\epsilon^\ell$.

(*a*) *Any object* $\boldsymbol{i}$ *such that* $\mathrm{id}_{\boldsymbol{i}} \in R_{\leq b+1}$ *which has a subsequence of the form* (a, a) *is zero.*

(*b*) *For any* $\boldsymbol{i} = (i_1, \ldots, i_r) \in R_{<b-1}$,

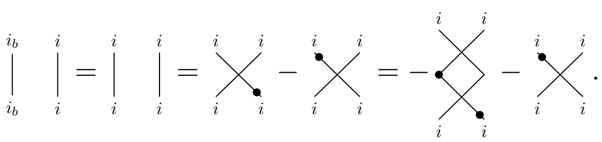
 $= 0.$

(*c*) *Let* $\lambda \in \mathrm{Par}^\ell$ *with* $|\lambda| \leq b$ *and assume* $\mathrm{Add}_i(\lambda) = \emptyset$. *If* $\theta_i(\Psi^{\mathfrak{t}^\lambda}_{\mathfrak{t}^\lambda}) \neq 0$ *then there exists a subsequence of the form* $(i, i \pm 1, i)$ *in* $\mathrm{res}(\mathfrak{t}^\lambda i)$.

Remark 3.28. In Corollary 3.27 the subsequence can in be chosen to involve the i at the end of $\mathrm{res}(\mathfrak{t}^\lambda i)$. The statement holds even for any $\mathfrak{t} \in \mathcal{T}^{\mathrm{ud}}(\lambda)$.

Proof of Proposition 3.26 (with Corollary 3.27 and Remark 3.28). The assumption and (3.10) directly imply (3.12). We prove (3.10) and (3.11) parallel via induction on $b := |\lambda|$. Let $(i_1, \ldots, i_b) := \boldsymbol{i}_{\mathfrak{t}^\lambda} = \boldsymbol{i}^{\circledast}_{\mathfrak{t}^\lambda}$, thus $\Psi^{\mathfrak{t}^\lambda}_{\mathfrak{t}^\lambda} = \mathrm{id}_{(i_1,\ldots,i_b)}$.
If $b = 0$, then $\mathrm{Add}_i(\lambda) = \emptyset$ implies $i \neq \delta_j$ for all j and both, (3.10) and (3.11), follow from (3.6).
Assume the claims hold for all $b' < b$. Via induction and Proposition 3.24, $\theta_i|_{B_{\leq b-1}}$ is a filtered map of degree 1.
We consider four different cases.

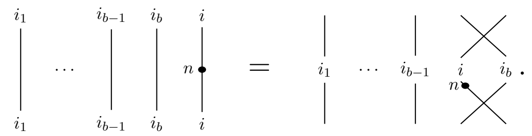
(i) If $i = i_b$, then we have, by (sR-2) and by (sR-2) with (sR-6),



Therefore, $\theta_i^n(\Psi^{\mathfrak{t}^\lambda}_{\mathfrak{t}^\lambda}) = 0$ for $n \in \{0, 1\}$ by induction. This also shows Corollary 3.27(*a*) and (*c*) in this case.

(ii) If $i \neq i_b$, $|i - i_b| \neq 1$, then $\mathrm{Add}_i(\mathfrak{t}^\lambda_{b-1}) = \mathrm{Add}_i(\lambda)$ and (sR-6), (sR-2) give

$$\theta_i^n(\Psi^{\mathfrak{t}^\lambda}_{\mathfrak{t}^\lambda}) =$$

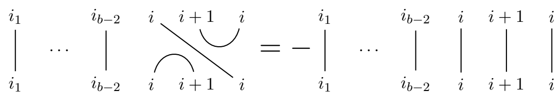


By induction, $\theta_i^n(\mathrm{id}_{(i_1,\dots,i_{b-1})}) \in B_{<b-1}$. Thus, $\mathrm{id}_{(i_1,\dots,i_{b-1},i,i_b)} \in B_{<b}$ as $\theta_i|_{B_{\leq b-1}}$ is filtered of degree 1. Corollary 3.27(a) and (c) follow also immediately in this case. Remark 3.28 holds, since $\mathrm{Add}_i(\mathfrak{t}^\lambda_{b-1}) = \mathrm{Add}_i(\lambda)$

(iii) Suppose that $i_b = i+1$. By definition of $\mathfrak{t}^\lambda$, i_b is the residue of the last box in the last row of λ. As there is no addable box with residue i, the last row of λ has more than one box and then $i_{b-1} = i$. Thus, $(i_{b-1}, i_b) = (i, i+1)$. This shows Corollary 3.27 (c) and Remark 3.28 in this case.

Define multi-up-down-tableaux $\mathfrak{u}$ and $\mathfrak{v}$ of length $b+1$ such that[1] $\mathfrak{u}_k = \mathfrak{t}^\lambda_k$ for $k < b$, $\mathfrak{u}_{b+1} = \mathfrak{u}_{b-1}$ and $\mathfrak{u}_b = \mathfrak{u}_{b-2}$ respectively $\mathfrak{v}_k = \mathfrak{t}^\lambda_k$ for $k \leq b$ and $\mathfrak{v}_{b+1} = \mathfrak{v}_{b-1}$. By construction, we have $\mathrm{res}(\mathfrak{u}) = (i_1, \dots, i_b, i) = \mathrm{res}^{\circledast}(\mathfrak{v})$. Now we can compute

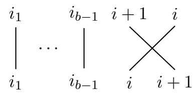
(3.13) $\Psi^{\mathfrak{v}}_{\mathfrak{u}} =$



$= -\theta_i(\Psi^{\mathfrak{t}^\lambda}_{\mathfrak{t}^\lambda})$.

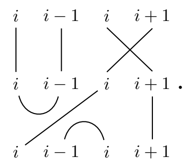
The first and last equalities here hold by definition, and the second equality used (sR-7). The reader might expect two more summands from this relation, but we proved in (i) that $\theta_i \circ \theta_i|_{R^{\leq b}} = 0$ and thus these terms vanish. We see that the number of propagating strands in $\Psi^{\mathfrak{v}}_{\mathfrak{u}}$ is exactly one less than the one in $\Psi^{\mathfrak{t}^\lambda}_{\mathfrak{t}^\lambda}$. Thus, $\theta_i(\Psi^{\mathfrak{t}^\lambda}_{\mathfrak{t}^\lambda}) \in R^{<b}$ and (3.10) holds.

We also need to show (3.11). By (sR-6) and the induction hypothesis it suffices to show Corollary 3.27(b), i.e.



$= 0.$

Observe that $\mathrm{Add}_i(\mathfrak{t}^\lambda_{b-1}) = \emptyset$ as $\mathrm{Add}_{i+1}(\mathfrak{t}^\lambda_{b-1}) = \mathrm{Add}_{i_b}(\mathfrak{t}^\lambda_{b-1}) \neq \emptyset$. Thus, by induction, and from the arguments given so far, we see that either $(i_1, \dots, i_{b-1}, i, i+1) = 0$ (in which case we are done) or we find a subsequence of the form $(i, i-1, i)$. For this subsequence we can apply the argument as in (3.13) and obtain



If we apply a height move to the cup and the crossing, the statement follows from Corollary 3.27(b) (for a shorter sequence).

(iv) Suppose that $i_b = i-1$. By definition of $\mathfrak{t}^\lambda$, i_b is the residue of the last box in the last row of λ. As there is no addable box with residue i, the second last row has a box with residue i but not with higher residues. (Note moreover that there are at least two rows). Now this case is similar to (iii), but one also has to use (sR-6) to move a value i to the position $b-1$. Here, for the proof of (3.10), a subsequence $(i, i-1, i)$ is obtained.

This shows (3.10) and (3.11) and hence also Proposition 3.26 and Remark 3.28. $\square$

Corollary 3.29. *Assume* (Ass<b). *Then* $\theta_i(B_{\leq b}) \subseteq B_{\leq b+1}$ *and* $\theta_i(B_{<b}) \subseteq B_{<b+1}$ *for all* $i \in \mathbb{R}$, $b \in \mathbb{N}_0$. *In particular* θ_i *is a filtered map of degree* 1.

Proof. This follows directly from Proposition 3.26 using Proposition 3.24. $\square$

Corollary 3.30. *Assume* (Ass<b). *Then* $\mathrm{id}_{\boldsymbol{i}} \in B_{\leq b}$ *for any object* $\boldsymbol{i} = (i_1, \ldots, i_b)$.

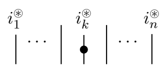
Proof. Since $\mathrm{id}_{i_1} \in R^{\leq 1}$, this follows directly from Corollary 3.29 $\square$

We next want to show that $B_{\leq b} = R_{\leq b}$ for all b.
We fix more notation for the rest of this subsection:

Notation 3.31. Consider multi-up-down-tableaux $\mathfrak{t}$ and $\mathfrak{s}$ of shape λ. We define $b := |\lambda|$ so that $\Psi^{\mathfrak{s}}_{\mathfrak{t}} \in B_{\leq b}$. Let $(i_1, \ldots, i_m) := \boldsymbol{i}_{\mathfrak{t}}$ and $(i^{\circledast}_1, \ldots, i^{\circledast}_n) := \boldsymbol{i}^{\circledast}_{\mathfrak{s}}$.

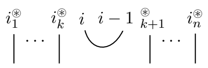
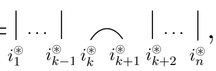
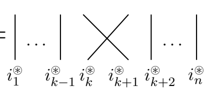
We formulate more properties for $\mathrm{sR}^\ell_\epsilon$ in case (Ass<b) holds:

Corollary 3.32. *Assume* (Ass<b). *Then* $d_k \Psi^{\mathfrak{s}}_{\mathfrak{t}} = 0 = \Psi^{\mathfrak{s}}_{\mathfrak{t}} d_k$ *for* $1 \leq k \leq n$, *where*

$d_k =$  .

Proof. This follows directly from Proposition 3.26. $\square$

Proposition 3.33. *Assume* (Ass<b) *and define for* $1 \leq k \leq n$, $i \in \mathbb{R}$ *the morphisms*

$x_{k,i} =$  , $y_{k,i} =$  , $z_{k,i} =$  .

Then $x_{k,i}\Psi^{\mathfrak{s}}_{\mathfrak{t}}$, $y_{k,i}\Psi^{\mathfrak{s}}_{\mathfrak{t}}$, $z_{k,i}\Psi^{\mathfrak{s}}_{\mathfrak{t}} \in B_{\leq b}$ *holds. Similarly,* $\Psi^{\mathfrak{s}}_{\mathfrak{t}} x_{k,i}$, $\Psi^{\mathfrak{s}}_{\mathfrak{t}} y_{k,i}$, $\Psi^{\mathfrak{s}}_{\mathfrak{t}} z_{k,i} \in B_{\leq b}$.

Proof of the case $x_{k,i}$ *in Proposition 3.33.* If $\square \in \mathrm{Add}_i(\mathfrak{s}_k)$, let $\mathfrak{u} \in \mathcal{T}^{\mathrm{ud}}(\lambda)$ with $\mathfrak{u}|_k = \mathfrak{s}|_k$, $\mathfrak{u}_{k+1} = \mathfrak{u}_k \oplus \square$, $\mathfrak{u}_{j+2} = \mathfrak{s}_j$ for $k \leq j \leq n$. Then, $x_{k,i}\Psi^{\mathfrak{s}}_{\mathfrak{t}} = \pm \Psi^{\mathfrak{u}}_{\mathfrak{t}} \in R^{\leq b}$. Otherwise, we have $\mathrm{Add}_i(\mathfrak{s}_k) = \emptyset$ and may also assume that $|\mathfrak{s}_k| = k$ (as removing boxes would correspond to cups commuting with the cup of $x_{k,i}$ up to sign). By Corollary 3.27, the object $(i_1, \ldots, i_k, i, i-1)$ is either 0 or we find a subsequence of the form $(i, i \pm 1, i)$. If the subsequence is $(i, i-1, i)$, applying (sR-7) gives the diagram according to removing with $a-1$ the box with residue a and then adding two boxes.
On the other hand if the subsequence is $(i, i+1, i)$, we get a valid multi-up-down tableau $\mathfrak{v}$ with $\mathrm{res}^{\circledast}(\mathfrak{v}) = (i_1, \ldots, i_k, i, i-1)$ where the last two entries remove the boxes

corresponding to the subsequence. If $\mathfrak{v}$ can be extended to a multi-up-down tableau of shape μ by $(i_{k+1},\dots,i_m)$, then $|\mu|<b$ and $x_{k,i}\Psi^{\mathfrak{t}}_{\mathfrak{s}} = a\Psi^{\mathfrak{v}}_{\mathfrak{t}^\mu} b \in R^{<b}$ by construction. If it cannot be extended, we either find a subsequence $(i-1,i-1)$ which is 0 by Corollary 3.27(a) or we try to add a box of residue $i+2$. But this strand can be moved to the left using (sR-6), where we then either get 0 by Corollary 3.27(a) or (3.6) or we find a subsequence $(i+2,i+3,i+2)$. In the last case we can apply (sR-7) and then use the same argument as above and end up eventually with 0. □

Proof of the case $z_{k,i}$ in Proposition 3.33. Suppose first that $i_{k+1}\notin\{i_k,i_k\pm 1\}$. Then we claim that $\mathfrak{s}s_k$ is an up-down-tableau. If in steps k and $k+1$ we only add respectively remove boxes, then this is clear as the boxes neither appear in the same row nor column of the same partition.

In the other two cases let i be the residue of the removal. This means that we removed a box □ with residue $i+1$. So this actually swaps with all residues which are $\leq i-1$ and $\geq i+3$.

The only case left to consider is, when the added box β has residue $i+2$. But note that after the removal of □, □ is addable again. As □ has residue $i+1$, no box with residue $i+2$ can be addable. And vice versa, if we add β after adding □, the box β lies directly to the right of □ in the same row. Thus, we cannot remove □ afterward. Therefore, this case cannot appear.

It remains to show the statement for $i_k = i_{k+1}\pm 1$. If steps k and $k+1$ consist out of adding boxes □ and β, these two boxes appear in the same row respectively column. This means that β cannot be added to $\mathfrak{s}_{k-1}$. By Proposition 3.26 and Corollary 3.29 we know that $z_{k,i}\Psi^{\mathfrak{s}}_{\mathfrak{t}}\in R^{<b}$.

Suppose that we remove a box in step k and add a box in step $k+1$. Let further l be the step in which the box was added which was removed in step k. Without loss of generality we may assume that $l=k-1$. Using (sR-6), we can move the i_l pass every distant entry and every neighbored entry has to be removed prior step k, which results in a cup that does not interact with the crossing z, meaning that we can swap these two as well. We then either have the subsequence (i_k+1,i_k,i_k+1), in which case applying z gives 0 by Corollary 3.27(a). Or we have (i_k+1,i_k,i_k-1), in which case step $\mathfrak{s}s_k$ is an up-down-tableau and $z_{k,i}\Psi^{\mathfrak{s}}_{\mathfrak{t}} = \Psi^{\mathfrak{s}s_k}_{\mathfrak{t}}$.

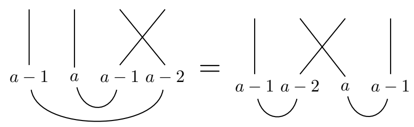
Suppose that we add a box in step k and remove a box in step $k+1$. Then $z_{k,i}\Psi^{\mathfrak{s}}_{\mathfrak{t}}=0$ by (sR-6) and Corollary 3.32 or (sR-5) respectively.

Suppose that we remove in both steps a box.

Let l denote the index of adding the box for step k and l' the one for $k+1$. Without loss of generality we may assume that $l=k-1$ and $l'=k-2$. Note that l' has to appear before l as $\mathfrak{s}$ is an up-down-tableau.

Then we either have a subsequence $(a+1,a,a-1,a)$ or $(a-1,a,a-1,a-2)$. In the first case, applying z gives 0 by Corollary 3.27(a). In the second case we have the equality displayed in (3.14) by (locally) using (sR-6) and (3.2). Now removing steps $k-1$ and k from $\mathfrak{s}$ gives a valid up-down-tableau $\mathfrak{u}$ of shape λ. Thus, looking at the diagrams we see that $z_{k,i}\Psi^{\mathfrak{s}}_{\mathfrak{t}}$ is obtained from $\Psi^{\mathfrak{u}}_{\mathfrak{t}}$ via left multiplication with a cup and a distant

(3.14)



crossing. Now the claim follows from Proposition 3.33 for $x_{k,i}$ and the first paragraph about distant crossings. ☐

Proof of the case $y_{k,i}$ in Proposition 3.33. If in the k-th step of $\mathfrak{s}$ a box is removed and in the $k+1$-th one is added, using (sR-4) we get $y_{k,i}\Psi^{\mathfrak{s}}_{\mathfrak{t}} = \pm\Psi^{\mathfrak{u}}_{\mathfrak{t}}$, where $\mathfrak{u}$ is obtained from $\mathfrak{s}$ via deleting steps k and $k+1$.
If the k-th step adds a box and the $k+1$-th removes one, then $y_{k,i}\Psi^{\mathfrak{s}}_{\mathfrak{t}} = 0$ by (sR-5).
If in the k-th and $k+1$-th steps boxes are removed from $\mathfrak{s}$, let $\mathfrak{s}'$ be the up-down tableaux, that is obtained from $\mathfrak{s}$ by removing all the "cups" of $\mathfrak{s}$, i.e. it is the same as $\mathfrak{s}$ but whenever we would remove a box in $\mathfrak{s}$ or add a box that later would be removed we skip this step. Now $\Psi^{\mathfrak{s}}_{\mathfrak{t}} = c\cdot\Psi^{\mathfrak{s}'}_{\mathfrak{t}}$, where c is a diagram consisting of cups (which might intersect). As the k-th and $k+1$th step both remove boxes, we see that $y_{k,i}\cdot\Psi^{\mathfrak{s}}_{\mathfrak{t}} = c'\cdot\Psi^{\mathfrak{s}'}_{\mathfrak{t}}$ by (sR-4), where c' also consists only of cups. The statement then follows from Proposition 3.33 for $x_{k,i}$ and $z_{k,i}$.
The remaining case to consider is when two boxes are added. But this case immediately follows from (Ass$<b$), as $y_{k,i}\Psi^{\mathfrak{s}}_{\mathfrak{t}} \in R^{<b}$. ☐

We directly obtain from Corollary 3.32 and Proposition 3.33:

Corollary 3.34. *Assume* (Ass$<b$)*, then $B_{\leq b} = R_{\leq b}$ holds in* $\mathrm{sR}^\ell_\epsilon$.

Proposition 3.35. *The set $\mathcal{B}$ of up-down-basis elements is a spanning set for* $\mathrm{sR}^\ell_\epsilon$.

Proof. By Corollary 3.34 we know that all $B_{\leq b}$ form two-sided ideals and by Corollary 3.30 we see that all identities lie in some $B_{\leq b}$ for some b. These two facts together imply that the $\Psi^{\mathfrak{s}}_{\mathfrak{t}} \in \mathcal{B}$ span $\mathrm{sR}^\ell_\epsilon$. ☐

Corollary 3.36. *In* $\mathrm{sR}^\ell_\epsilon$*, any object $\boldsymbol{i}$ with a subsequence of the form (a,a) is zero.*

Proof. This follows now directly from Proposition 3.35 and Corollary 3.27(a). ☐

Corollary 3.37. *Any nonzero object of* $\mathrm{sR}^\ell_\epsilon$ *is isomorphic to $\boldsymbol{i}_{\mathfrak{t}^\lambda}$ for some $\lambda \in \mathrm{Par}^\ell$.*

Proof. Let $\boldsymbol{i}$ be a nonzero object in $\mathrm{sR}^\ell_\epsilon$. If it is the residue sequence of some up-tableau, the statement follows. This is as all residue sequences for up-tableaux of the same shape differ only by distant crossings (which are isomorphisms by (sR-6)).
Otherwise, we can find a subsequence $(i, i\pm1, i)$ in $\boldsymbol{i}$ by Corollary 3.27(c). By Corollary 3.36 and (sR-7), we can do the following replacement

In conjunction with (sR-4), we see that $\boldsymbol{i}$ is isomorphic to $\boldsymbol{i'}$, where $\boldsymbol{i'}$ is obtained from $\boldsymbol{i}$ by replacing the subsequence $(i, i \pm 1, i)$ with i. We can repeat this argument until we end up with a residue sequence of an up-tableau. $\square$

Remark 3.38. After the proof of the Cyclotomic equivalence we see, for instance using Theorem 3.25, that the $\boldsymbol{i}_{\mathfrak{t}^\lambda}$ for $\lambda \in \mathrm{Par}^\ell$ are indeed all nonzero.

3.4. Proofs of the Isomorphism Theorem and the Basis Theorem

Now we are going to prove Theorem 3.17. We will begin by outlining our strategy.
Consider the functor $\Phi\colon \mathrm{sR}_\epsilon^\ell \to \mathrm{Kar}(\mathrm{s}\mathbb{W}^\ell)$. This functor is filtered by the number of strands, and we can consider its restriction $\Phi^{\leq k}$ to at most k strands (on either side). We then will prove Theorems 3.17 and 3.23 by induction on k. For $k = 1$ this is an easy calculation. Given the theorem for all $k' \leq k$, we will show that y_{k+1} act diagonalisably and use this to check the relations involving the $k+1$-st strand. For both calculations we will use the basis of $\mathrm{sR}_\epsilon^\ell$ (on the first $k-1$ strands) to exclude and simplify many cases in the calculations.
From these considerations it will also follow that the functor is full and by arguments from [AMR06] it follows that the spanning set of $\mathrm{sR}_\epsilon^\ell$ has the same size as a basis for $\mathrm{s}\mathbb{W}^\ell$ (up to this filtration degree).
We first examine how generators of $\mathrm{s}\mathbb{W}^\ell$ interact with the generalized eigenspaces:

Lemma 3.39. *Let a, b, c, $d \in \mathbb{R}$. In $\mathrm{s}\mathbb{W}^\ell$ the following holds:*

(*a*) *If $\flat e_{a,b} \neq 0$ then $b = a+1$, and if $e_{c,d}\flat^* \neq 0$ then $d = c-1$.*

(*b*) *If $e_{c,d} s_k e_{a,b} \neq 0$ then $(b,d) = (a+1, c-1)$ or $(c,d) = (a,b)$ or $(c,d) = (b,a)$.*

Proof. Part (*a*) holds by ($\mathrm{s}\mathbb{W}$-6) respectively (1.4). For (*b*) we deduce from ($\mathrm{s}\mathbb{W}$-1) and ($\mathrm{s}\mathbb{W}$-7) the two equations $e_{c,d}(s_k y_k + s_k y_{k+1})e_{a,b} = e_{c,d}(y_{k+1}s_k + y_k s_k + 2\flat\flat^*)e_{a,b}$ and $e_{c,d}(s_k y_k - s_k y_{k+1})e_{a,b} = e_{c,d}(y_{k+1}s_k - y_k s_k - 2)e_{a,b}$. If $(b,d) \neq (a+1, c-1)$, then in the first equation the cup-cap part vanishes and we get $a + b = c + d$. If $(a,b) \neq (c,d)$ then the second equation implies $a - b = d - c$, since the last term there vanishes. Thus, $e_{c,d}s_k e_{a,b} \neq 0$ implies $(b,d) \neq (a+1, c-1)$ or $(b,d) \neq (a+1, c-1)$ are satisfied, or the two conditions $a+b = c+d$, $a-b = d-c$ hold, that is $(a,b) = (d,c)$. $\square$

Next we show diagonalizability of y_{k-1} induces diagonalizability of y_k for any $k \in \mathbb{N}_0$:

Proposition 3.40. *Assume $\Phi^{\leq k-1}$ is an isomorphism. Then y_k acts diagonalisably on $\mathrm{s}\mathbb{W}^\ell_{\leq k}$.*

Remark 3.41. If $\Phi^{\leq k-1}$ is an isomorphism, then y_{k-1} acts diagonalisably for $\mathrm{s}\mathbb{W}^\ell_{\leq k-1}$ (since its preimage does so by Proposition 3.26), and $e_{i_1,\dots,i,i,\dots,i_{k-1}} = 0$.

Throughout the following proofs we will need the following technical result:

Lemma 3.42. *Assume $\Phi^{\leq k-1}$ is an isomorphism. Then we have for any $i \in \mathbb{R}$, $x \in s\mathbb{W}^\ell$, $(i_1, \cdots, i_{k-2}) \in \mathbb{R}^{k-2}$ that $e_{i_1,\cdots,i_{k-2},i+1,i} x e_{i_1,\cdots,i_{k-2},i,i+1} = 0$.*

Proof. By Corollary 3.37, we know that $e_{i_1,\cdots,i_{k-2}}$ is isomorphic to $e_{j_1,\ldots,j_l}$ with $l \leq k-2$, where $j_1, \ldots, j_l$ is the residue sequence of some $\mathfrak{t}^\lambda$, $\lambda \in \mathrm{Par}^\ell$. As $\mathfrak{t}^\lambda$ cannot have addable boxes of residue i and $i+1$ at the same time, one of the two idempotents must be conjugate to some $e_{j'_1,\cdots,j'_{l'}}$ with $l' < k-2$ (or zero which directly implies the claim). Adding a snake to $e_{i_1,\cdots,i_{k-2},i+1,i} x e_{i_1,\cdots,i_{k-2},i,i+1}$ and using the above conjugate idempotent, we obtain an idempotent of the form $e_{j'_1,\cdots,j'_{l'},a,a}$ with $a \in \{i, i+1\}$ (i if the first idempotent is conjugate to a shorter one, $i+1$ if the second one is). In particular, $e_{j'_1,\cdots,j'_{l'},a,a} =$ by Corollary 3.36 as $\Phi^{\leq k-1}$ is an isomorphism. The statement follows. $\square$

Proof of Proposition 3.40. It suffices to show the claim:

$$e_{i_1,\ldots,i_k} y_k = i_k e_{i_1,\ldots,i_k} \quad \text{for any} \quad e_{i_1,\ldots,i_k}. \tag{3.15}$$

For $k = 1$, this holds by definition of $s\mathbb{W}^\ell$ and the minimal polynomial (3.7) of y_1. Thus, let $k > 1$. We abbreviate $e_{(a,b]} := e_{i_1,\ldots i_{k-2},a,b}$ and set $j := i_{k-1}$, $i := i_k$. From ($s\mathbb{W}$-7) we get with $e = \sum_{a',b'} e_{(a',b']}$ the formula

$$e_{(c,d]} y_k e_{(a,b]} = e_{(c,d]} s_{k-1} y_{k-1} e s_{k-1} e_{(a,b]} + e_{(c,d]} s_{k-1} e_{(a,b]} + e_{(c,d]} \flat^* \flat e_{(a,b]}. \tag{3.16}$$

Note that the last summand vanishes in case $(a, b) = (c, d)$ by Lemma 3.39.
Case $j = i$. If we take $(a, b) = (c, d) = (i, i)$ in (3.16), only $(a', b') = (i, i)$ matters by Lemma 3.39, and we get $e_{(i,i]}(y_k - i) = e_{(i,i]} s_{k-1} e_{(i,i]}$, since y_{k-1} acts diagonalisably by assumption. Now, $(e_{(i,i]}(y_k - i)e_{(i,i]})^{2n} = 0$ for $n \gg 0$ whereas $(e_{(i,i]} s_{k-1} e_{(i,i]})^{2n} = e_{(i,i]}$ by Lemma 3.39 and ($s\mathbb{W}$-1). Thus, $e_{(i,i]} = 0$.
Case $j = i + 1$. Consider (3.16) with $(a, b) = (c, d) = (i + 1, i)$. By Lemma 3.39, only the terms $e_{(i,i+1]}$ and $e_{(i+1,i]}$ matter for e. But by Lemma 3.42, actually the term for $e_{(i,i+1]}$ vanishes as well and only $e_{(i+1,i]}$ remains. Then, we can use the same argument as for the case $j = i$.
Case $j = i - 1$. If we take now $(a, b) = (c, d) = (i - 1, i)$ in (3.16), only $(a', b') = (i - 1, i)$ matters by Lemmas 3.39 and 3.42, and we can argue as for the above two cases.
Case $j \notin \{i, i \pm 1\}$. Let $(a, b) \in \{(i, j), (j, i)\}$ and set $z_{(a,b)} := (s_{k-1} + \frac{1}{a-y_k})e_{(a,b]}$. Since the action of y_{k-1} is diagonalizable, ($s\mathbb{W}$-7) implies that $y_k z_{(a,b)} = z_{(a,b)} y_{k-1} = a z_{(a,b)}$ and $b z_{(a,b)} = z_{(a,b)} y_k = y_{k-1} z_{(a,b)}$. In particular, $e_{(b,a]} z_{(a,b)} = z_{(a,b)}$. We get $z_{(b,a)} z_{(a,b)} = (s_{k-1} + \frac{1}{b-a})(s_{k-1} + \frac{1}{a-b})e_{(a,b]} = (1 - \frac{1}{(a-b)^2})e_{(a,b]}$. Since $a - b \neq \pm 1$ and $z_{(a,b)}(y_k - b)e_{(a,b]} = 0$, we get $(y_k - b)e_{(a,b]} = 0$ for $(a, b) \in \{(i, j), (j, i)\}$.
We showed that y_k is diagonalizable. $\square$

The following two results follow directly from the proof of Proposition 3.40.

Corollary 3.43. *If y_{k-1} acts diagonalisably on $s\mathbb{W}^\ell_{\leq k-1}$, then $e_{i_1,\ldots,i_{k-2},i,i} = 0$.*

Corollary 3.44. *If y_{k-1} acts diagonalisably on $s\mathbb{W}^\ell_{\leq k-1}$, then $e_{a,b}((b - a)s_k + 1) = ((b - a)s_k + 1)e_{b,a}$ given that $a \notin \{b, b - 1\}$.*

Proposition 3.45. *Let $k \in \mathbb{N}_0$ and assume $\Phi^{\leq k-1}$ is an isomorphism of algebras. Then $\Phi^{\leq k}$ is a well-defined algebra homomorphism.*

Proof. If $k = 1$, the only relations are the cyclotomic relations (3.6) for $\mathrm{sR}_\epsilon^\ell$ and (3.8) with (3.7) for $\mathrm{s}\mathbb{W}^\ell$ which exactly correspond to each other, and thus the functor is well-defined (in this case the assumption is vacuous).
If $k > 1$ it suffices by assumption to verify the compatibility with the relations involving the last strand. Recall from (3.11) that all dots in $\mathrm{sR}_\epsilon^\ell$ become zero which fits with the fact that y_k acts diagonalizable by Proposition 3.40. We can ignore all terms involving dots in the relations, since they are zero and sent to zero. Again we abbreviate $e_{(a,b]} := e_{i_1,\dots i_{k-2},a,b}$, $e_{(a,b,c]} := e_{i_1,\dots a,b,c}$.
Relation (sR-1): Both sides are zero and are sent to zero.
Relation (sR-2): The right-hand sides are sent to zero by Corollary 3.43 respectively by Lemma 3.42 using that $\Phi^{\leq k-1}$ is an isomorphism.
Relation (sR-3): We may assume that $a \neq b$, $b+1$ as otherwise both sides are sent to zero by definition. The LHS of the relation is (using Corollary 3.44) sent to

$$\eta_{a,b}\flat_{k-1}((a-b)s_{k-2}+1)e_{(a,b+1]} = \eta_{a,b}\flat_{k-1}((a-b)s_{k-2})e_{(a,b+1]}. \tag{3.17}$$

Here, the second summand vanishes by Lemma 3.39. Similarly, the RHS is sent to

$$\eta_{b+1,a}\flat_{k-2}((b+1-a)s_k+1)e_{(a,b+1]} = \eta_{b+1,a}\flat_{k-2}((b+1-a)s_k)e_{(a,b+1]}. \tag{3.18}$$

Now (3.17)=(3.18) holds by the defining property (ii) of the η's and (1.2).
Relation (sR-4): By Lemma 3.39 the middle idempotent in the image is uniquely determined by the outer idempotents and the compatibility follows from ($\mathrm{s}\mathbb{W}$-5).
Relation (sR-5): The image is zero by ($\mathrm{s}\mathbb{W}$-3) noting that $i_k - i_{k-1} = a-(a+1) = -1$.
Relation (sR-6): The first case is clear by Corollary 3.43, the second and third case follow from Lemma 3.42. For the remaining one note that the image of the LHS is $\eta_{a,b}\eta_{b,a}((b-a)s_k+1)((a-b)s_k+1)e_{(a,b]}$ by Corollary 3.44. This equals $\eta_{a,b}\eta_{b,a}(1-(a-b)^2)e_{(a,b]}$ by ($\mathrm{s}\mathbb{W}$-1). By the first defining property of the η's, $\eta_{a,b}\eta_{b,a}(1-(a-b)^2) = 1$ and the desired compatibility holds.
Relation (sR-7): First assume $(a,b,c) \neq (a, a\pm 1, a)$. Then the RHS is zero and sent to zero. The left-hand side is mapped to zero if any of the pairs (a,b), (a,c), (b,c) are of the form (i,i) or $(i,i+1)$ by definition of Φ. Otherwise, the image of the first term is

$$\begin{aligned}
&\eta_{a,b}((b-a)s_{k-1}+1)\eta_{a,c}((c-a)s_k+1)\eta_{b,c}((c-b)s_{k-1}+1)\\
=\ &\eta_{a,b}\eta_{a,c}\eta_{b,c}\Big(1+(b-a)(c-b)s_{k-1}^2+(c-a)s_k+(b-a)s_{k-1}+(c-b)s_{k-1}\\
&+(c-a)(c-b)s_ks_{k-1}+(b-a)(c-a)s_{k-1}s_k+(b-a)(c-a)(c-b)s_{k-1}s_ks_{k-1}\Big),
\end{aligned}$$

whereas the image of the second term equals

$$\begin{aligned}
&\eta_{b,c}((c-b)s_k+1)\circ\eta_{a,c}((c-a)s_{k-1}+1)\circ\eta_{a,b}((b-a)s_k+1)\\
=\ &\eta_{b,c}\eta_{a,c}\eta_{a,b}\Big(1+(c-b)(b-a)s_k^2+(b-a)s_k+(c-a)s_{k-1}+(c-b)s_k\\
&+(c-a)(b-a)s_{k-1}s_k+(c-b)(c-a)s_ks_{k-1}+(c-b)(c-a)(b-a)s_ks_{k-1}s_k\Big).
\end{aligned}$$

The two images agree in all expressions involving one or two s_i's. The other terms match by ($\mathrm{s}\mathbb{W}$-1) and ($\mathrm{s}\mathbb{W}$-2).
Next assume that $(a, b, c) = (a, a \pm 1, a)$. We need to show that

$$e_{(a,a+1,a]} = -e_{(a,a+1,a]}\flat^*_{k-1}\flat_{k-1}e_{(a,a-1,a]}\flat^*_{k-2}\flat_{k-2}e_{(a,a+1,a]}. \tag{3.19}$$

We first rewrite $e_{(a,a+1,a]}$ by plugging in the relation ($\mathrm{s}\mathbb{W}$-7) three times. We always simplify using that the y_j's act by scalars (and thus the double cross can be straightened by ($\mathrm{s}\mathbb{W}$-1)) and that certain cups or caps vanish because of Lemma 3.39.

$$\begin{aligned} e_{(a,a+1,a]} &= e_{(a,a+1,a]}s_{k-2}e_{(a,a+1,a]} = e_{(a,a+1,a]}(-s_{k-1} - \flat^*_{k-1}\flat_{k-1})s_{k-2}e_{(a,a+1,i]} \\ &= -e_{(a,a+1,a]}s_{k-1}s_{k-2}e_{(a,a+1,a]} - e_{(a,a+1,a]}s_{k-1}\flat^*_{k-1}\flat_{k-1}\flat^*_{k-2}\flat_{k-2}e_{(a,a+1,a]}. \end{aligned}$$

Only with the idempotent $e_{(a,a-1a]}$ in the middle, the last term is nonzero. Thus, (3.19) follows if we show that $e_{(a,a+1,a]}s_{k-1}s_{k-2}e_{(a,a+1,a]} = 0$.
For this we observe that (again by ($\mathrm{s}\mathbb{W}$-7), diagonalizability and Lemma 3.39)

$$e_{(a,a+1,a]}s_{k-1}e_{(a,a+1,a]} = (a - (a+1))e_{(a,a+1,a]} = -e_{(a,a+1,a]}, \tag{3.20}$$

$$e_{(a,a+1,a]}s_{k-2}e_{(a,a+1,a]} = (a + 1 - a)e_{(a,a+1,a]} = e_{(a,a+1,a]}, \tag{3.21}$$

and compute (using Corollary 3.43 and Lemma 3.39 in the second and fourth step)

$$\begin{aligned} e_{(a,a+1,a]}s_{k-1}s_{k-2}e_{(a,a+1,a]} &\overset{(3.20)}{=} -e_{(a,a+1,a]}s_{k-1}s_{k-2}e_{(a,a+1,a]}s_{k-1}e_{(a,a+1,a]} \\ = -\, e_{(a,a+1,a]}s_{k-1}s_{k-2}s_{k-1}e_{(a,a+1,a]} &\overset{(\mathrm{s}\mathbb{W}\text{-}2)}{=} -e_{(a,a+1,a]}s_{k-2}s_{k-1}s_{k-2}e_{(a,a+1,a]} \\ = -\, e_{(a,a+1,a]}s_{k-2}e_{(a,a+1,a]}s_{k-1}s_{k-2}e_{(a,a+1,a]} &\overset{(3.21)}{=} -e_{(a,a+1,a]}s_{k-1}s_{k-2}e_{(a,a+1,a]}. \end{aligned}$$

Therefore, $e_{(a,a+1,a]}s_{k-1}s_{k-2}e_{(a,a+1,a]} = 0$ and (3.19) is proven.
The case $(a, b, c) = (a, a-1, a)$ is treated analogously. □

Proof of Theorem 3.17. We prove that $\Phi^{\leq k}$ is an isomorphism by induction on k. For $k = 0$ there is nothing to show. Now assume the statement for $k-1$. Then $\Phi^{\leq k}$ is well-defined by Proposition 3.45. Furthermore, the spanning set $\mathcal{B}$ for $\mathrm{sR}^\ell_\epsilon$ has the same size as a basis of $\mathrm{s}\mathbb{W}^\ell$, see [AMR06, Lemma 5.1]. Hence, it suffices to show that $\Phi^{\leq k}$ is full. For this let $\boldsymbol{i} \in \mathbb{R}^m$, $e_{\boldsymbol{j}} \in \mathbb{R}^n$, $m, n \leq k$. It is clear that $e_{\boldsymbol{i}}y_je_{\boldsymbol{j}} \in \operatorname{im}\Psi^{\leq}$ for all $1 \leq j \leq k$. By Lemma 3.39, we also have $e_{\boldsymbol{i}}\flat_ke_{\boldsymbol{j}}$ and $e_{\boldsymbol{i}}\flat^*_ke_{\boldsymbol{j}} \in \operatorname{im}\Psi^{\leq k}$ whenever they make sense. We claim that $e_{\boldsymbol{j}}s_{k-1}e_{\boldsymbol{i}} \in \operatorname{im}\Phi^{\leq k}$. By induction, it suffices to show that $e_{\boldsymbol{i}}s_ke_{\boldsymbol{j}} \in \operatorname{im}\Phi^{\leq k}$ for $m = n = k-1$. If $i_k \notin \{i_{k-1}, i_{k-1}-1\}$ this is clear by definition of $\Phi^{\leq k}$. If $i_k = i_{k-1}$ then $e_{\boldsymbol{i}} = 0$ by Corollary 3.43 and there is nothing to do.
Thus, assume $i_k+1 = i_{k-1} =: i$. By Lemma 3.39, we have $e_{\boldsymbol{i}}s_{k-1}e_{\boldsymbol{j}} = 0$ unless $(j_k, j_{k-1}) = (i_{k-1}, i_k)$ or $(j_{k-1}, j_k) = (i_{k-1}, i_k)$. For the former, we have $e_{(i+1,i]}s_{k-1}e_{(i,i+1]} = 0$ by Lemma 3.42. For the latter, we get $e_{\boldsymbol{i}}s_ke_{\boldsymbol{j}} = e_{\boldsymbol{i}}e_{\boldsymbol{j}}$ by ($\mathrm{s}\mathbb{W}$-7). Therefore, $e_{\boldsymbol{j}}s_ke_{\boldsymbol{i}} \in \operatorname{im}\Phi^{\leq k}$ as claimed. Similarly, if $i_k = i_{k-1}+1$ we have $e_{\boldsymbol{i}}s_{k-1}e_{\boldsymbol{j}} = 0$ by Lemma 3.42 and thus $e_{\boldsymbol{i}}s_ke_{\boldsymbol{j}} = 0$.
Altogether, $\operatorname{im}\Psi^{\leq k}$ contains a generating set for the morphism and thus $\Psi^{\leq k}$ is full. It follows that Ψ is an isomorphism. □

3.4.1. Proof of the Basis Theorem and the Cyclotomic Equivalence

Proof of Theorem 3.23. Since the cardinality of $\mathcal{B}$ equals the cardinality, see [AMR06, Lemma 5.1], of a basis of $\mathrm{s}\mathbb{W}^{\ell}$, the Basis Theorem follows from the Isomorphism Theorem 3.17 and Proposition 3.35. □

Proof of Theorem 3.19. By the Isomorphism Theorem 3.17 it is enough to show that the functor is essentially surjective. Write $1 = \sum e_{\boldsymbol{i}}$ for pairwise orthogonal nonzero idempotents. We claim that $e_{\boldsymbol{i}}$ is primitive for all $\boldsymbol{i}$. If the claim holds we are done, since then the image contains (up to equivalence) all primitive idempotents. By Corollary 3.37 we can restrict ourselves to the case $\boldsymbol{i} = \boldsymbol{i}^{\lambda}$ for $\lambda \in \mathrm{Par}^{\ell}$. Then the claim follows from Theorem 3.25. (One could also directly use Theorem 3.25.) □

3.5. Refined Schur–Weyl duality

In this section, we apply the result of Theorem 3.17 to obtain a refined Schur–Weyl duality for $\mathfrak{p}(n)$ using the KLR algebra $\mathrm{sR}^{\mathrm{cyc}}$.

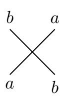
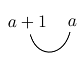
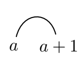
The following is immediate from (SW) and Theorem 3.17:

Corollary 3.46. *There is an essentially surjective full homomorphism* $\Psi_{-,n} \colon (\mathrm{sR}^{\mathrm{cyc}})^{\oplus} \to \mathrm{Fund}(\mathfrak{p}(n))$ *of module categories over* sR.

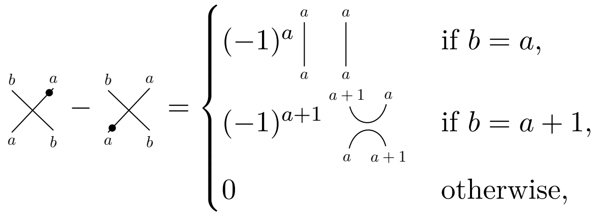
The main disadvantage of $\mathrm{Fund}(\mathfrak{p}(n))$ is that it contains odd morphisms. But as we consider all morphisms, $\mathrm{Fund}(\mathfrak{p}(n))$ depends on a choice of representatives under the parity switch. We chose this representative such that it appears as a direct summand of $V^{\otimes d}$. We proceed by fixing a new choice of representatives to obtain the category $\mathrm{Fund}_{+}^{\oplus}(\mathfrak{p}(n))$ where miraculously only even morphisms are left.

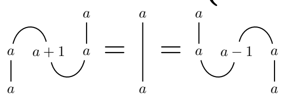
For this define the endofunctor $\Theta_k^+ := \Pi^k \Theta_k^-$ and the category $\mathrm{Fund}_{+}^{\oplus}(\mathfrak{p}(n))$ as the full additive subcategory of $\mathrm{Rep}(\mathfrak{p}(n))$ generated by $\Theta_{i_d}^+ \cdots \Theta_{i_1}^+ \mathbb{C}$. We also define an analog of sR that we call $\hat{\mathrm{s}}\mathrm{R}$.

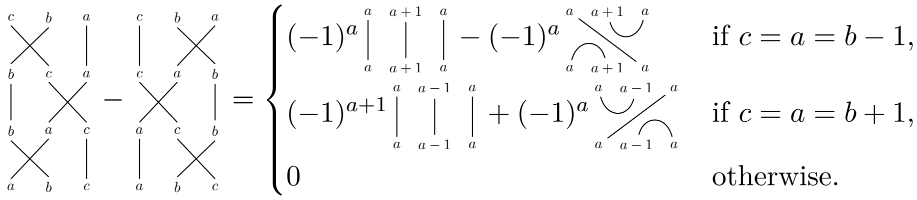
Definition 3.47. Let $\hat{\mathrm{s}}\mathrm{R}$ be the $\Bbbk$-linear strict monoidal category freely generated on the level of objects by $a \in \mathbb{Z}$ and on the level of morphisms by

 $: a\otimes b \to b\otimes a,$ $: a \to a,$  $: \mathbb{1} \to (a+1)\otimes a,$  $: a\otimes(a+1) \to \mathbb{1},$

modulo the relations (sR-1), (sR-3), (sR-5) and (sR-6) and the following modified relations:

(ŝR-2) 

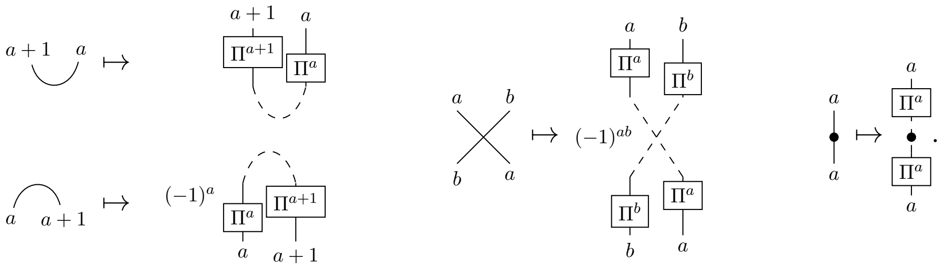
(ŝR-4) 

(ŝR-7) 

As before, we also define $\hat{\mathrm{s}}\mathrm{R}^{\mathrm{cyc}}$ as the quotient of $\hat{\mathrm{s}}\mathrm{R}$ by the right tensor ideal generated by

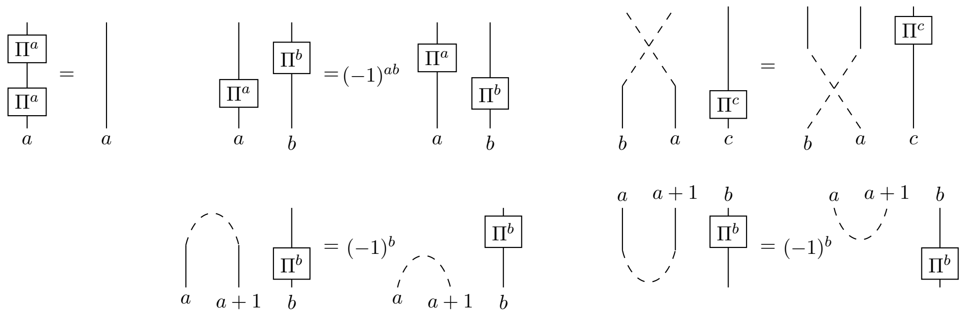
(3.22)  , where $n = \begin{cases} 1 & \text{if } a = 0, \\ 0 & \text{otherwise.} \end{cases}$

Theorem 3.48. *The category* $\mathrm{Fund}_+^{\oplus}(\mathfrak{p}(n))$ *is a module category over* $\hat{\mathrm{s}}\mathrm{R}$ *via (the dotted diagrams correspond to the action of* sR *on* $\mathrm{Fund}(\mathfrak{p}(n))$*)*



Furthermore, evaluation at $\mathbb{C}$ *factors through* $\hat{\mathrm{s}}\mathrm{R}^{\mathrm{cyc}}$*, and we obtain the homomorphism* $\Psi_{+,n} \colon \hat{\mathrm{s}}\mathrm{R}^{\mathrm{cyc}} \to \mathrm{Fund}_+^{\oplus}(\mathfrak{p}(n))$ *of module categories over* $\hat{\mathrm{s}}\mathrm{R}$.

Proof. The following equalities regarding the parity shift hold (all follow from the Koszul sign rule for height moves in supercategories):



Using these, it is straightforward to verify the relations of $\hat{\mathrm{s}}\mathrm{R}^{\mathrm{cyc}}$ using $\Psi_{-,n}\mathrm{sR}^{\mathrm{cyc}} \to \mathrm{Fund}(\mathfrak{p}(n))$.
Furthermore, we have $\Theta_i^+\mathbb{C} \neq 0$ if and only if $i = 0$, and thus we obtain the homomorphism $\Psi_{+,n}$ of module categories as claimed. □

Remark 3.49. If we only consider $\hat{\mathrm{s}}\mathrm{R}^{\mathrm{cyc}}$, many of the $(-1)^a$ in the defining relation disappear. Namely, in this case, one of the two sides in the defining relations is always zero, and we can rescale the relation.

Remark 3.50. We also get the "same" basis for $\hat{\mathrm{s}}\mathrm{R}^{\mathrm{cyc}}$ as in Theorem 3.23. By the "same", we mean the obvious analogues of the basis elements by replacing all generating morphisms with their respective analogues.

We conclude this section by stating some results about the functor of $\Psi_{+,n}$. All of these were already proven by Coulembier and Ehrig for the super Brauer category, and we translate their results to $\hat{\mathrm{s}}\mathrm{R}^{\mathrm{cyc}}$.

Definition 3.51. We abbreviate the partition $(k, k-1, \dots, 1)$ by δ_k.

Proposition 3.52. *Let $\boldsymbol{i} = (i_1, \dots, i_k)$ be the residue sequence of an up-tableau of shape λ. Then $\Psi_{+,n}(\boldsymbol{i}) = 0$ if and only if $\delta_{n+1} \subseteq \lambda$.*

Proof. The object $\boldsymbol{i}$ gets mapped under Ψ_-^ℓ to the indecomposable object associated to λ as this maps to a generalized eigenspace for the action of the Jucys–Murphy elements (see the paragraph after (1.4) and [Cou18a]). The statement then follows from [CE21, Theorem 6.2.1]. □

Theorem 3.53. *Let $\boldsymbol{i} = (i_1, \dots, i_k)$ be the residue sequence of an up-tableau of shape λ. The module $\Psi_{+,n}(\boldsymbol{i})$ is projective if and only if $\delta_n \subseteq \lambda$.*

Proof. This is [CE21, Theorem 6.3.1]. □

4. The Khovanov algebra of type P

In this chapter we aim at giving a different description of $\hat{s}R^{cyc}$, similar to the Khovanov algebras of type A or B.

4.1. Combinatorics of cup and cap diagrams

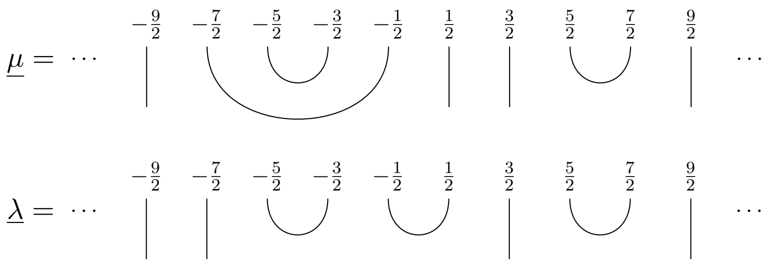
Definition 4.1. A cup diagram $\underline{\lambda}$ is a partitioning P of $\mathbb{Z} + \frac{1}{2}$ into subsets of size at most 2 such that there are only finitely many size 2 subsets. Furthermore, if $\{i, j\} \in P$ and $i < k < j$, then k is part of a subset of size two $\{k, k'\} \in P$ with $i < k' < j$.
We draw such a cup diagram (without crossings) by attaching a vertical ray to a one element subset and a cup with endpoints given by two element subsets.
The set of all cup diagrams with n cups is denoted by Λ_n and by $\Lambda = \bigcup \Lambda_n$ we denote the set of all cup diagrams.

The following is immediate from the definitions.

Lemma 4.2. *We have a bijection between Λ_n and strictly decreasing sequences of integers $\lambda_1 > \cdots > \lambda_n$ by sending such a sequence to the unique cup diagram with endpoints $\{\lambda_1 + \frac{3}{2}\}$. In particular, we can identify the set Λ with strictly decreasing sequences of integers of arbitrary length.*

Definition 4.3. Let $\underline{\lambda}$ and $\underline{\mu}$ be cup diagrams. Denote by $\lambda_1 > \lambda_2 > \cdots > \lambda_k$ the right endpoints of cups in $\underline{\lambda}$ and similar define $\mu_1 > \cdots > \mu_r$. We define a partial order on Λ by declaring $\underline{\lambda} \leq \underline{\mu}$ if either $k > r$ or $k = r$ and $\lambda_i \geq \mu_i$ for $1 \leq i \leq k$.

Example 4.4. The following are cup diagrams.

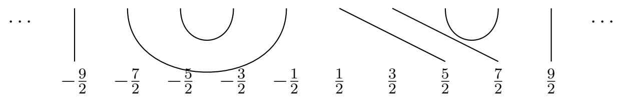


The right endpoints of cups of $\underline{\lambda}$ are given by $\frac{7}{2} > \frac{1}{2} > -\frac{3}{2}$ and those of $\underline{\mu}$ by $\frac{7}{2} > -\frac{1}{2} > -\frac{3}{2}$, and thus $\underline{\lambda} < \underline{\mu}$.

Definition 4.5. A cap diagram $\overline{\lambda} = \underline{\lambda}^*$ is the horizontal mirror image of a cup diagram $\underline{\lambda}$ shifted one to the right, i.e. if $\underline{\lambda}$ has a cup connecting a and b then $\overline{\lambda}$ has a cap connecting $a+1$ and $b+1$.
A circle diagram $\underline{\lambda}\overline{\mu}$ is a cup diagram $\underline{\lambda}$ drawn under a cap diagram $\overline{\mu}$.

Definition 4.6. A circle diagram is called *orientable* if it contains no circle and any non-propagating line has one endpoint to each side of 0.
To check this in practice, we first have to redraw cup and cap diagrams using the following rule. For a cup diagram with k cups, we draw the bottom endpoints of rays in the unique way such that $-k-\frac{1}{2}, -k+\frac{1}{2}, \ldots, k-\frac{3}{2}$ have no endpoints of rays. For a cap diagram with k caps, we draw it in the unique way such that there are no top endpoints of rays at $-k+\frac{1}{2}, -k+\frac{3}{2}, \ldots, k-\frac{1}{2}$. If we redraw the cup and the cap diagram in a circle diagram in this way, *one endpoint to each side of* 0 means that, any non-propagating line ending at the top, is centered around 0 and a non-propagating line ending at the bottom around -1.

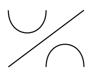
Example 4.7. For the cup diagram from Example 4.4 we would think of the rays as the following.



Remark 4.8. We could have also chosen to define cup and cap diagrams with the skew rays, which would have made the orientability criterion easier but the definition of cup and cap diagrams more involved. We chose to do this differently as all these technicalities fade away, when we pass to representations of $\mathfrak{p}(n)$.

4.2. Definition of $\mathbb{K}$

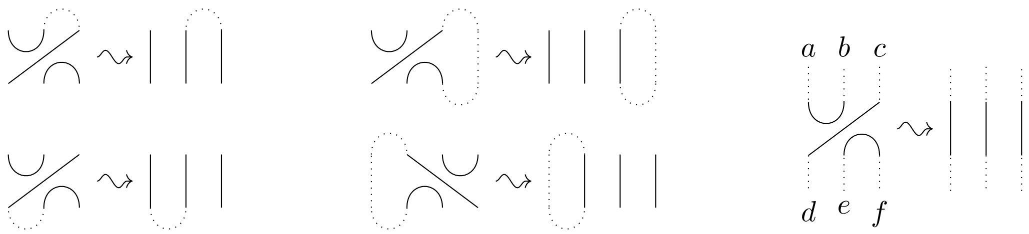
Definition 4.9. We define the *Khovanov algebra* $\mathbb{K}$ *of type* P to be the vector space with basis all orientable circle diagrams $\underline{\lambda}\overline{\mu}$. We define a multiplication by setting $\underline{\lambda}\overline{\mu} \cdot \underline{\mu'}\overline{\nu}$ to 0 if $\mu \neq \mu'$, and otherwise we draw $\underline{\lambda}\overline{\mu}$ below $\underline{\mu'}\overline{\nu}$ and connect endpoints of rays from $\overline{\mu}$ to endpoints of rays of $\underline{\mu}$ in the following way: If this endpoint is not directly to the left of a cap we connect it with a straight line to the top. If it is to the left of a cap we connect it to the first free endpoint right of the corresponding cup in $\underline{\mu}$. This gives us a middle section consisting of subpictures of the form



We then apply surgery procedures to remove this middle section. This is done by replacing these subpictures iteratively by

.

If any of these intermediate results is not orientable, we define the result to be 0. Otherwise, we set it to be $\underline{\lambda}\overline{\nu}$ (which is exactly the result of the surgery procedure). This turns $\mathbb{K}$ into an associative algebra by Theorem 4.11 below.



(a) Straightening (b) Split (c) Reconnect

Figure 4.1.: Different cases of the surgery procedure

Remark 4.10. We chose to call these replacements surgery procedures (even though they are, strictly speaking, not) as they play the same role as the surgery procedures from [BS11a] and [ES16] in the definition of the Khovanov algebra of types A and B.

4.2.1. Detailed analysis of surgery procedures

Before we are going to prove associativity, we inspect the surgery procedure in more detail. Every surgery procedure falls into one of the following three categories.

4.2.2. Straightening

This appears if we are in the situation of Figure 4.1a. In this case the surgery procedure does not change orientability and just "straightens" the kink.

4.2.3. Split

In this situation we split off a circle from a line, and thus the surgery procedure gives 0. Figure 4.1b shows this in detail.

4.2.4. Reconnect

The last case is that none of the endpoints of the surgery are connected with one another as in Figure 4.1c. In this situation we only reconnect three lines.
If all three lines are propagating, then the two right endpoints (as well as the two left endpoints) end either both on the top or both on the bottom. If these endpoints lie on different sides of 0, the surgery procedure is nonzero and produces two non-propagating lines.
In all other cases, we claim that the surgery procedure will produce 0. If all three lines are propagating, but some right (or left) endpoints do not lie on the same side of 0, then the result is not orientable, and thus 0. Now suppose that there is at least one non-propagating line. We may assume without loss of generality that it corresponds to the endpoints a and b and without loss of generality these end on the top. As our diagram is orientable, we know that a and b lie on the same side of 0. Thus, (if the result is also orientable) d has to end on the bottom. Furthermore, c and f cannot both end on

the top and e, f and c cannot all end on the bottom by the same argument as for a, b and d. Hence, a, b and c end on the top and d, e and f on the bottom. Thus, we have two non-propagating lines, one associated to the cap and one to the cup.
Then e and f have to be at positions $-k-\frac{3}{2}$ and $k-\frac{1}{2}$ where k denotes the number of cups as the diagram is orientable (see also Definition 4.6). On the other hand observe that a and b have to be at positions $-k-\frac{1}{2}$ and $k+\frac{1}{2}$. But this can only be achieved if there are more cups than caps to the left of this surgery procedure, which means that there has to be a non-propagating line ending at the top that is not nested with the one coming from the cup, thus this is not orientable.

Theorem 4.11. *The multiplication on $\mathbb{K}$ is well-defined and associative.*

Proof. That it is well-defined follows from Theorem 4.17 by horizontal stacking of surgery procedures and associativity by vertical stacking. □

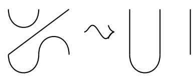
4.2.5. Properties of $\mathbb{K}$

Definition 4.12. For a decreasing sequence $\lambda = (\lambda_1 > \cdots > \lambda_n)$ of integers let $e_\lambda = \underline{\lambda}\overline{\lambda}$.

Lemma 4.13. *The following hold true.*

$$e_\lambda \cdot \underline{\nu}\overline{\mu} = \delta_{\lambda\nu}\underline{\lambda}\overline{\mu} \qquad\qquad \underline{\lambda}\overline{\nu} \cdot e_\mu = \delta_{\nu\mu}\underline{\lambda}\overline{\mu}$$

Proof. We only prove the first equation. If $\nu \neq \lambda$ the statement is clear by definition of the multiplication. Otherwise, as $e_\lambda = \underline{\lambda}\overline{\lambda}$, every surgery procedure is given locally by



which clearly does not change orientability. This means that any surgery procedure gives a non-zero result, and thus we obtain $\underline{\lambda}\overline{\mu}$ in the end. □

Corollary 4.14. *The algebra $\mathbb{K}$ is a locally unital locally finite dimensional algebra with $\mathbb{K} = \bigoplus_{\lambda,\mu\in\Lambda} e_\lambda \mathbb{K} e_\mu$.*

Proof. By Lemma 4.13 we know that the set of all e_λ form a set of mutually orthogonal idempotents and the direct sum decomposition is then immediate from Lemma 4.2. It is locally finite dimensional as $e_\lambda \mathbb{K} e_\mu$ is non-zero if and only if $\underline{\lambda}\overline{\mu}$ is orientable in which case it gives a basis. □

Definition 4.15. For each $\lambda \in \Lambda$ we have a one dimensional irreducible module $\hat{L}(\lambda)$ with action given by

$$\underline{\mu}\overline{\nu} \cdot v = \begin{cases} v & \text{if } \underline{\mu}\overline{\nu} = e_\lambda, \\ 0 & \text{otherwise.} \end{cases}$$

Each of these has a locally finite dimensional projective cover denoted by $\hat{P}(\lambda) = \mathbb{K}e_\lambda$.

4.3. Proof of associativity

In order to show that $\mathbb{K}$ is an associative algebra, we will look at a slightly more general situation. We consider multiple circle diagrams stacked over one another, i.e. we take a sequence $\nu_1, \dots \nu_k$ such that $\underline{\nu_i}\overline{\nu_{i+1}}$ is orientable for all i. Drawing all these beneath each other, as for the multiplication procedure, we get a big orientable diagram. We will prove that any two surgery procedures commute with each other.
Before we prove this, we will first analyze, when a surgery procedure gives 0.

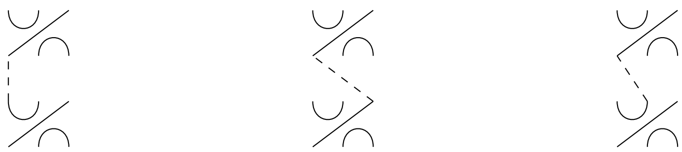
Lemma 4.16. *A surgery procedure gives* 0 *if and only if we are in one of the following two situations (up to rotational symmetry):*

In the second picture the two rays end on the same side of 0*, but they may also end on the bottom.*

Proof. One easily checks that these are all the possible cases. □

Theorem 4.17. *Given two potential surgeries D and D', we have $D \circ D' = D' \circ D$.*

Proof. Assume we are given two potential surgeries. If both surgeries produce in the first step orientable diagrams, the surgery procedures commute. So we may assume that one of them produces a non-orientable diagram. Lemma 4.16 describes how this has to look locally. But now note that if the dashed cup does not interact with the other surgery procedure, then the overall result will be zero, independent of the order. So we may assume that the dashed cup does interact with the other surgery. We will make a case distinction on how this cup connects to the other surgery. By rotating the bottom surgery, we can reduce to three different main cases as follows:



We make a further case distinction on how the other end of the dashed cup connects to the other surgery. As these need to be connected in the total picture, we are left with three more cases for each main case (see Figure 4.2 for this). Observe that in each of these cases the rays at the top may also end on the bottom or may be joined to form a cap. If they are not joined to a cap, then the two endpoints lie on the same side of 0.

(1–3a) In these three cases note that after performing both surgeries the result is not orientable as we have either a circle or a non-propagating line with two endpoints at the same side of 0.

(1a) (1b) (1c) (2a) (2b) (2c) (3a) (3b) (3c)

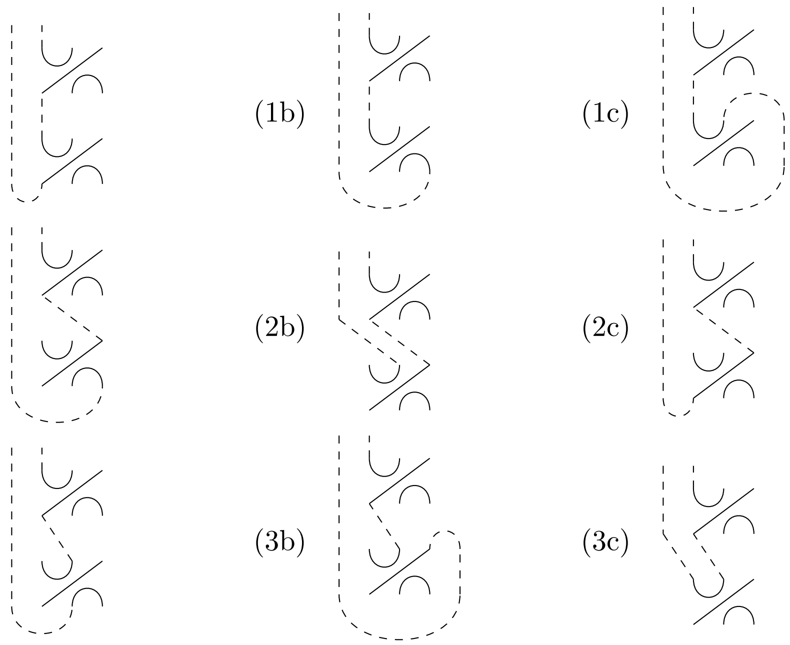
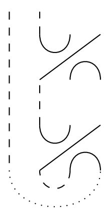


Figure 4.2.: First case distinction in the proof of associativity

(1b) By assumption the dashed lines have to be connected, so it has to look like the following:



But in this case, the bottom surgery procedure only "straightens" the cup-cap at the bottom and this commutes with the other surgery procedure.

(1–2c) These diagrams are not orientable as the components inside the dashed lines will either give circle or non-propagating lines. As these non-propagating lines would have both endpoints between the dashed ones, they would lie on the same side of 0 by assumption on the dashed lines.

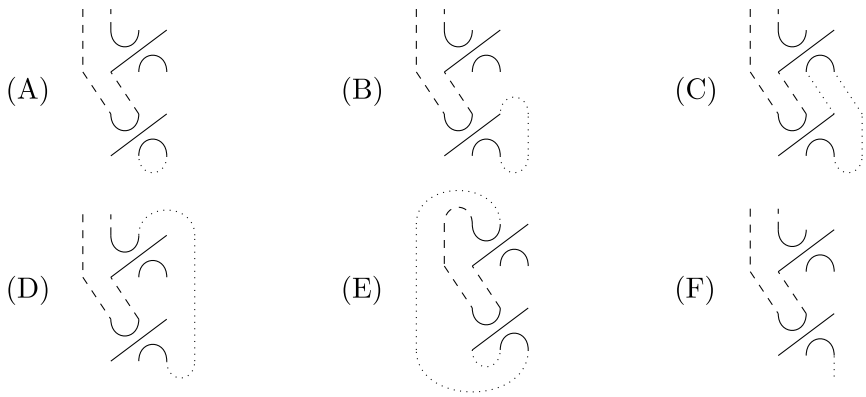
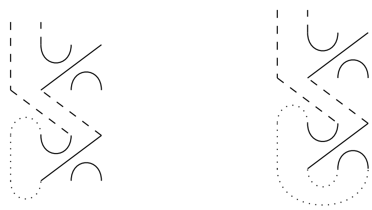


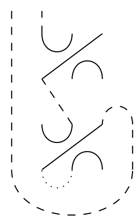
Figure 4.3.: Subdivision of the case (3c) in the proof of associativity

(2b) Observe that by our assumption the two dashed lines have to be connected. For this we have two possibilities.



In the left diagram, the second surgery procedure produces 0 as well by Lemma 4.16, so they commute. For the second picture note that the second surgery procedure “straightens” the cup-cap at the bottom, which commutes with the other surgery procedure.

(3b) Look at the left endpoint of the bottom cap. It cannot be a ray or form a circle as then the diagram would not be orientable, so we are in the following situation.



Similarly, to before, the bottom surgery only “straightens” the cup-cap at the bottom, and thus commutes with the other surgery.

(3c) This is the hardest case of all. In Figure 4.3 we make another case distinction depending on how the bottom right vertex is connected to the rest.

(A) This diagram is not orientable.

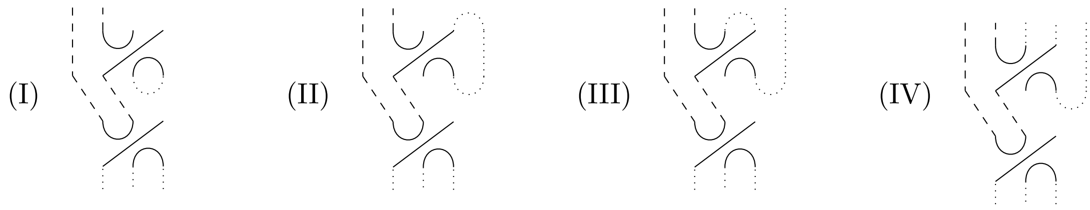
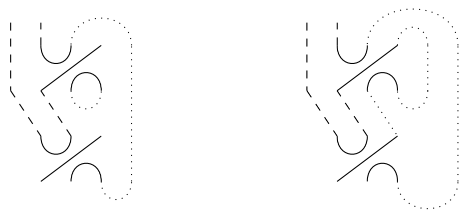


Figure 4.4.: Subdivision of case (f) in the proof of associativity

(B) The bottom surgery procedure gives 0 as well by Lemma 4.16.

(C) Again both surgery procedures give 0, so they commute.

(D) We distinguish two cases



The left picture is not orientable, whereas the right one gives 0 after applying both.

(E) The bottom surgery "straightens" only the cup-cap, so it commutes with the other one.

(F) We have two different possibilities.

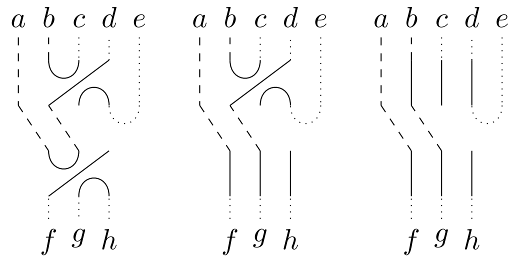
In the left diagram, the bottom surgery only "straightens" the cup-cap, and thus commutes with the other one. For the right diagram, we distinguish how the right endpoint of the upper cap is connected. All possibilities for this are listed in Figure 4.4.

(I) The diagram is not orientable.

(II) The upper surgery produces two circles, where one does not interact with the other one, so the overall product is 0 and hence they commute.

(III) The diagram is not orientable by assumption on the dashed lines.

(IV) In this case we claim that not all the following three pictures are orientable.



If we can show this claim, this means that applying the second surgery procedure before the first will also give 0, thus they commute.

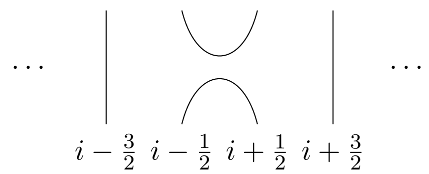
First note that not all of $\{b, c, d, e\}$ can end on the same side, as then one of the pictures would not be orientable (the non-propagating lines would not all wrap around the same spot). Similarly, not all of $\{a, f, g, h\}$ can end on the same side. Therefore, e and h end on the same side.

Now look at the middle picture. The two loose ends in the middle have to be now connected as otherwise e and h would contribute to two non-propagating lines next to one another, which is not orientable.

So $\{e, g, h\}$ (resp. $\{d, e, h\}$) cannot end on the same side for the same reasons as before. But this means that $\{a, b, c, f\}$ end on the same side, which produces two non-propagating lines in the middle picture that are not nested. Thus, there always exists a non-orientable picture and the overall surgery procedure result is 0. □

4.4. Geometric bimodules

Definition 4.18. A *crossingless matching* is a diagram t given by drawing a cap diagram c underneath a cup diagram d and connecting the rays from c to d via an order preserving bijection such that outside some finite strip we only have straight vertical rays. For $i \in \mathbb{Z}$ we can consider special crossingless matchings t^i given as



A *generalized crossingless matching* $\boldsymbol{t}$ is a sequence $t_k \cdots t_1$ of crossingless matchings. Given a cup diagram a and a cap diagram b we can form a *generalized circle diagram* $a\boldsymbol{t}b$. This is called *orientable* if it contains neither a circle nor a non-propagating such that both endpoints lie on the same side of 0.

Definition 4.19. Given a generalized crossingless matching $\boldsymbol{t} = t_k \cdots t_1$ we define $\hat{G}_{\boldsymbol{t}}$ to be the vector space with basis all orientable generalized circle diagrams $\underline{\lambda}\boldsymbol{t}\overline{\mu}$.
Given another generalized crossingless matching $\boldsymbol{u} = u_l \cdots u_1$ we define $\boldsymbol{ut}$ to be the generalized crossingless matching $u_l \cdots u_1 t_k \cdots t_1$. We define a map $m\colon \hat{G}_{\boldsymbol{u}} \otimes \hat{G}_{\boldsymbol{t}} \to \hat{G}_{\boldsymbol{ut}}$ by defining the product $(a\boldsymbol{u}b)(c\boldsymbol{t}d)$ of two basis vectors as follows. If $c^* \neq b$, we declare $(a\boldsymbol{u}b)(c\boldsymbol{t}d) = 0$. Otherwise, we draw $(a\boldsymbol{u}b)$ underneath $(c\boldsymbol{t}d)$ to create a new diagram. We now can use the surgery procedures two remove the middle section c^*c to obtain a vector in $\hat{G}_{\boldsymbol{ut}}$.

Given a third generalized crossingless matching $\boldsymbol{s}$, the following diagram commutes (by Theorem 4.17).

$$\begin{tikzcd} \hat{G}_{\boldsymbol{u}} \otimes \hat{G}_{\boldsymbol{t}} \otimes \hat{G}_{\boldsymbol{s}} \arrow[r, "1\otimes m"] \arrow[d, "m\otimes 1"] & \hat{G}_{\boldsymbol{u}} \otimes \hat{G}_{\boldsymbol{ts}} \arrow[d, "m"] \\ \hat{G}_{\boldsymbol{ut}} \otimes \hat{G}_{\boldsymbol{s}} \arrow[r, "m"] & \hat{G}_{\boldsymbol{uts}} \end{tikzcd} \tag{4.1}$$

Remark 4.20. Using (4.1) with $\boldsymbol{s}$ and $\boldsymbol{u}$ being the empty generalized crossingless matchings, we endow $\hat{G}_{\boldsymbol{t}}$ with the structure of a $\mathbb{K}$-$\mathbb{K}$-bimodule.

Definition 4.21. The *reduction* $\mathrm{red}(\boldsymbol{t})$ of a generalized crossingless matching $\boldsymbol{t}$ is the unique crossingless matching (i.e. only one layer) that is topologically equivalent. Similarly, given a crossingless matching t and a cup diagram a (resp. cap diagram b) we define the *lower reduction* $\mathrm{low}(at)$ of at (resp. *upper reduction* $\mathrm{upp}(tb)$ of tb) to be the unique cup (resp. cap) diagram that is topologically equivalent to at (resp. tb).

The following is clear from the definitions.

Corollary 4.22. *There exists a* $\mathbb{K}$-$\mathbb{K}$*-bimodule isomorphism* $\hat{G}_{\boldsymbol{t}} \to \hat{G}_{\mathrm{red}(\boldsymbol{t})}$ *via sending* $\underline{\lambda}\boldsymbol{t}\overline{\mu}$ *to* $\underline{\lambda}\,\mathrm{red}(\boldsymbol{t})\overline{\mu}$.

Remark 4.23. Corollary 4.22 allows us to reduce to crossingless matchings. In the following, we will consider everything only for the non-generalized version.

Lemma 4.24. *The map* $m\colon \hat{G}_u \otimes \hat{G}_t \to \hat{G}_{ut}$ *descends to an isomorphism* $\overline{m}\colon \hat{G}_u \otimes_{\mathbb{K}} \hat{G}_t \to \hat{G}_{ut}$.

Proof. The map m induces a map $\overline{m}\colon \hat{G}_u \otimes_{\mathbb{K}} \hat{G}_t \to \hat{G}_{ut}$ by the associativity (4.1). Note that the left-hand side is generated by elements of the form $aub \otimes ctd$ where a and c are cup diagrams and b and d cap diagrams. By definition of the multiplication and reduction we have $au(\mathrm{low}(au))^* \cdot \mathrm{low}(au)b = aub$. But this means that the domain of $\overline{m}$ is in fact generated by

$$au(\mathrm{low}\, au)^* \otimes \mathrm{low}(au)td. \tag{4.2}$$

Again by the definition of the multiplication, $\overline{m}(au(\mathrm{low}\, au)^* \otimes \mathrm{low}(au)td) = autd$. Thus, the generating set (4.2) maps to a basis of $\hat{G}_{ut}$. Hence, $\overline{m}$ is an isomorphism. □

Definition 4.25. Given a crossingless matching t, we define $\hat{\theta}_t$ to be the endofunctor of $\mathbb{K}$-mod given by tensoring with the $\mathbb{K}$-$\mathbb{K}$-bimodule $\hat{G}_t$ over $\mathbb{K}$. If $t = t^i$ we abbreviate $\hat{\theta}_i := \hat{\theta}_{t^i}$.

One should think of $\hat{\theta}_i$ as equivalents of the labelled strands in $\hat{\mathrm{s}}\mathrm{R}^{\mathrm{cyc}}$.

Proposition 4.26. *We have*

$$\hat{\theta}_t \hat{P}(\overline{\gamma}) = \begin{cases} 0 & \text{if } t\overline{\gamma} \text{ is not orientable,} \\ \hat{P}(\nu) & \text{otherwise, where } \overline{\nu} = \mathrm{upp}(t\overline{\gamma}). \end{cases}$$

Proof. We clearly have $\hat{\theta}_t \hat{P}(\overline{\gamma}) = \hat{G}_t e_\gamma$. But now if $t\overline{\gamma}$ is not orientable, then the right-hand side is 0 by definition. Otherwise, we denote by ν its upper reduction. Then we have a linear map

$$f\colon \hat{\theta}_t \hat{P}(\overline{\gamma}) \to \hat{P}(\nu), \qquad \underline{\lambda} t \overline{\gamma} \mapsto \underline{\lambda} \overline{\nu}.$$

This is an isomorphism as the process of upper reduction does not change orientability. Furthermore, it is $\mathbb{K}$-linear because the multiplication procedure depends only topologically on the diagram. ☐

Corollary 4.27. *The* $\mathbb{K}$-$\mathbb{K}$*-bimodule* $\hat{G}_t$ *is sweet, i.e. projective as a right and as a left* $\mathbb{K}$*-module.*

Proof. It is projective as left $\mathbb{K}$-module by Proposition 4.26 as $\hat{G}_t = \bigoplus_{\lambda \in \Lambda} \hat{G}_t e_\lambda$. For projectivity as right $\mathbb{K}$-module use the right-hand analog of Proposition 4.26. ☐

Our next goal is to prove an adjunction theorem between the $\hat{\theta}_t$.

Definition 4.28. Given a crossingless matching t we define $t^\ddagger$ to be the horizontal mirror image of t shifted one to the right.
Furthermore, we define $\phi\colon \hat{G}_{t^\ddagger} \otimes \hat{G}_t \to \mathbb{K}$ on basis vectors as

$$\phi(\underline{\lambda} t^\ddagger \overline{\nu} \otimes \underline{\nu'} t \overline{\mu}) = \delta_{\nu\nu'} (\underline{\lambda}\, \mathrm{upp}(t^\ddagger \overline{\nu})) \cdot (\mathrm{low}(\underline{\nu'} t) \overline{\mu}).$$

Lemma 4.29. *The map* ϕ *is a homogeneous* $\mathbb{K}$-$\mathbb{K}$*-bimodule homomorphism that is also* $\mathbb{K}$*-balanced.*

Proof. We can also realize ϕ as the following composition

$$\hat{G}_{t^\ddagger} \otimes \hat{G}_t \xrightarrow{m} \hat{G}_{t^\ddagger t} \xrightarrow{\cong} \hat{G}_{\mathrm{red}(t^\ddagger t)} \xrightarrow{\omega} \mathbb{K}$$

where ω is given by applying the surgery procedures to eliminate the middle section $\mathrm{red}(t^\ddagger t)$. This is possible because $t^\ddagger$ was defined as the horizontal mirror image shifted one to the right.
Note that this composition is a bimodule map as each of the composites is by (4.1), Corollary 4.22 and Theorem 4.17. Furthermore, m is balanced by (4.1), and thus ϕ is $\mathbb{K}$-balanced as well. ☐

Theorem 4.30. *There is a homogeneous* $\mathbb{K}$-$\mathbb{K}$*-bimodule isomorphism.*

$$\hat{\phi}\colon \hat{G}_t \to \operatorname{Hom}_{\mathbb{K}}(\hat{G}_{t^\ddagger}, \mathbb{K})$$

given by sending $y \in \hat{G}_t$ *to* $\hat{\phi}(y)\colon \hat{G}_{t^\ddagger} \to \mathbb{K}, x \mapsto \phi(x \otimes y)$.

Proof. First, note that this is well-defined as $\phi(\_ \otimes y)$ is a left $\mathbb{K}$-module homomorphism by Lemma 4.29.
To show that this is a $\mathbb{K}$-$\mathbb{K}$-bimodule homomorphism let $u \in \mathbb{K}$, $x \in \hat{G}_{t^\ddagger}$ and $y \in \hat{G}_t$. Then we have

$$\begin{aligned}(u\hat{\phi}(y))(x) &= \hat{\phi}(y)(xu) = \phi(xu \otimes y) = \phi(x \otimes uy) = (\hat{\phi}(uy))(x),\\ (\hat{\phi}(y)u)(x) &= (\hat{\phi}(y)(x))u = \phi(x \otimes y)u = \phi(x \otimes yu) = (\hat{\phi}(yu))(x),\end{aligned}$$

and thus $\hat{\phi}(uy) = u\hat{\phi}(y)$ and $\hat{\phi}(yu) = \hat{\phi}(y)u$.
It remains to show that $\hat{\phi}$ is a vector space isomorphism. For this it suffices to show that the restriction

$$\hat{\phi}\colon e_\lambda\hat{G}_t \to \operatorname{Hom}_{\mathbb{K}}(\hat{G}_{t^\ddagger}e_\lambda, \mathbb{K})$$

is an isomorphism. Now as $t^\ddagger$ is the mirror image and shifted one to the right with respect to t we see that $e_\lambda\hat{G}_t \neq 0$ if and only if $\hat{G}_{t^\ddagger}e_\lambda \neq 0$ (recall that cap diagrams arise by shifting the horizontal mirror image of cup diagrams one to the right). Thus, we may assume that $e_\lambda\hat{G}_t \neq 0$. In this case we have by the mirrored version of Proposition 6.26 $e_\lambda\hat{G}_t \cong e_\nu\mathbb{K}$ and $\hat{G}_{t^\ddagger}e_\lambda = \mathbb{K}e_\nu$ for some ν (which is the same for both). Under this isomorphism ϕ translates to multiplication. Thus, we need to show that $e_\nu\mathbb{K} \to \operatorname{Hom}_{\mathbb{K}}(\mathbb{K}e_\nu, \mathbb{K}), y \mapsto (x \mapsto xy)$ is an isomorphism, but this is obvious. ☐

Corollary 4.31. *We have an adjunction* $(\hat{\theta}_{t^\ddagger}, \hat{\theta}_t)$*. In particular,* $\hat{\theta}_{i+1}$ *is left adjoint to* $\hat{\theta}_i$.

Proof. We have the usual adjunction $(\hat{G}_{t^\ddagger} \otimes_{\mathbb{K}} \_, \operatorname{Hom}_{\mathbb{K}}(\hat{G}_{t^\ddagger}, \_))$. As $\hat{G}_{t^\ddagger}$ is projective as a left $\mathbb{K}$-module by Corollary 4.27, there exists a natural isomorphism $\operatorname{Hom}_{\mathbb{K}}(\hat{G}_{t^\ddagger}, \mathbb{K}) \otimes_{\mathbb{K}} M \to \operatorname{Hom}_{\mathbb{K}}(\hat{G}_{t^\ddagger}, M)$. Precomposing this with the isomorphism from Theorem 4.30 gives the desired adjunction. ☐

5. Relation to $\hat{\mathrm{s}}\mathrm{R}^{\mathrm{cyc}}$

In this section, we will relate $\hat{\mathrm{s}}\mathrm{R}$ with $\mathbb{K}$-mod. Namely, the object i of $\hat{\mathrm{s}}\mathrm{R}$ will correspond to the endofunctor $\hat{\theta}_i$ of $\mathbb{K}$. We will then define natural transformation between the $\hat{\theta}_i$ that satisfy the same relations as the relations in $\hat{\mathrm{s}}\mathrm{R}$. In particular, we will show that the $\hat{\mathrm{s}}\mathrm{R}$-module category generated by $\hat{P}(\bar{\iota}) \in \mathbb{K}$-mod is isomorphic to $\hat{\mathrm{s}}\mathrm{R}^{\mathrm{cyc}}$.
We begin by defining natural transformations $\eta_i \colon \hat{\theta}_{i+1}\hat{\theta}_i \to \mathbb{K}$, $\epsilon_i \colon \mathbb{K} \to \hat{\theta}_{i-1}\hat{\theta}_i$ and $\psi_{i,j} \colon \hat{\theta}_i\hat{\theta}_j \to \hat{\theta}_j\hat{\theta}_i$ that satisfy the same relations as sR. This will be the first step in relating $\mathbb{K}$ with $\hat{\mathrm{s}}\mathrm{R}^{\mathrm{cyc}}$.
Instead of directly defining the natural transformations, we first define bimodule homomorphisms $\hat{\eta}_i \colon \hat{G}_{t^{i+1}t^i} \to \mathbb{K}$, $\hat{\epsilon}_i \colon \mathbb{K} \to \hat{G}_{t^{i-1}t^i}$ and $\hat{\psi}_{i,j} \colon \hat{G}_{t^it^j} \to \hat{G}_{t^jt^i}$. Using Lemma 4.24, this induces then the natural transformations η_i, ϵ_i and $\psi_{i,j}$.

Definition 5.1. We set $\hat{\psi}_{i,j} = 0$ for $j = i$, $i-1$. Then, we define $\hat{\eta}_i$, $\hat{\epsilon}_i$ and $\hat{\psi}_{i,j}$ for $j \neq i$, $i \pm 1$ to be the linear maps

$$\hat{\eta}_i \colon \hat{G}_{t^{i+1}t^i} \to \mathbb{K} \qquad at^{i+1}t^ib \mapsto \begin{cases} ab & \text{if } ab \text{ orient.,} \\ 0 & \text{otherwise,} \end{cases}$$

$$\hat{\epsilon}_i \colon \mathbb{K} \to \hat{G}_{t^{i-1}t^i} \qquad ab \mapsto \begin{cases} at^{i-1}t^ib & \text{if } at^{i-1}t^ib \text{ orient.,} \\ 0 & \text{otherwise,} \end{cases}$$

$$\hat{\psi}_{i,j} \colon \hat{G}_{t^it^j} \to \hat{G}_{t^jt^i} \qquad at^it^jb \mapsto at^jt^ib$$

$$\hat{\psi}_{i,i+1} \colon \hat{G}_{t^it^{i+1}} \to \hat{G}_{t^{i+1}t^i} \qquad at^it^{i+1}b \mapsto \begin{cases} at^it^{i+1}b & \text{if } at^it^{i+1}b \text{ orient.,} \\ 0 & \text{otherwise.} \end{cases}$$

See also Figure 5.1 to see a pictorial description of these maps.

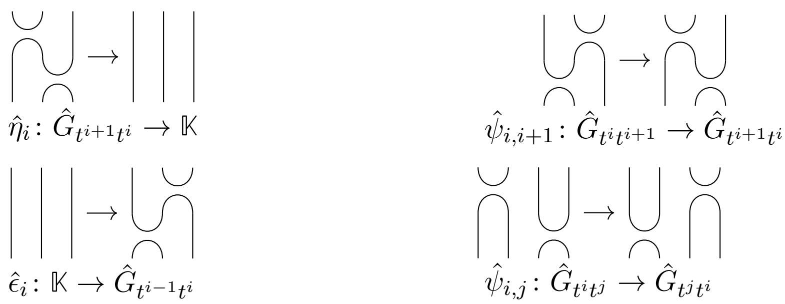


Figure 5.1.: Pictorial description of the maps $\hat{\eta}_i$, $\hat{\epsilon}_i$ and $\hat{\psi}_{i,j}$

Remark 5.2. Observe that the order of the matchings t^i is reversed in comparison to the strands in $\hat{s}\mathrm{R}^{\mathrm{cyc}}$, as we tensor from the left and add strands from the right.

Lemma 5.3. *The maps from Definition 5.1 are $\mathbb{K}$-$\mathbb{K}$-bimodule homomorphisms.*

Proof. For $\hat{\psi}_{i,j}$ with $j \neq i+1$ this is clear from the definition and Figure 5.1 as it is either 0 or does not change anything regarding the orientability. Regarding $\hat{\eta}_i$, $\hat{\epsilon}_i$ and $\hat{\psi}_{i,i+1}$, we remark that we can interpret all these maps as a rotated surgery procedure, see Figure 5.2 for details. But using Theorem 4.17 we see that these are actually bimodule homomorphism. $\square$

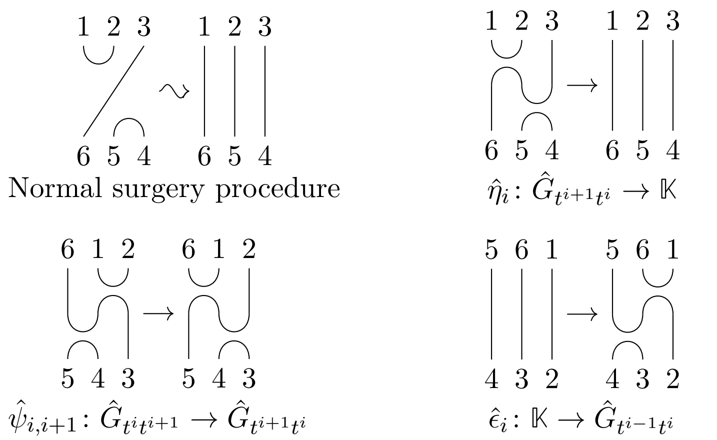


Figure 5.2.: How to interpret $\hat{\eta}_i$, $\hat{\epsilon}_i$ and $\hat{\psi}_{i,i+1}$ as rotated surgery procedures

Definition 5.4. A *stretched cup diagram* is a sequence $a\boldsymbol{t}$, where a is a cup diagram and $\boldsymbol{t} = t_k \cdots t_1$ is a generalized crossingless matching such that $t_j = t^{i_j}$ are special crossingless matchings for all j. A *stretched cap diagram* $\boldsymbol{t}'a'$ is the horizontal mirror image of a stretched cup diagram $a\boldsymbol{t}$ shifted one to the right. A *stretched circle diagram* $a\boldsymbol{u} \wr \boldsymbol{t}'b'$ is a stretched cup diagram $a\boldsymbol{u}$ glued underneath a stretched cap diagram $\boldsymbol{t}'b'$. A *stretched circle diagram* $a\boldsymbol{u} \wr \boldsymbol{t}'b'$ is called orientable if it contains no circle and no non-propagating line such that both endpoints are on the same side of 0.

Definition 5.5. Denote by $\bar{\iota}$ the unique cap diagram that has no caps.
Let $\mathcal{F}$ be the full subcategory of $\mathbb{K}$-mod with objects $\hat{\theta}_{i_k} \ldots \hat{\theta}_{i_1}\hat{P}(\bar{\iota})$. We can consider this category also as a locally finite dimensional locally unital algebra

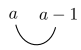
$$A := \bigoplus_{\boldsymbol{i}\in\mathbb{Z}^k, \boldsymbol{j}\in\mathbb{Z}^l} \mathrm{Hom}_{\mathbb{K}}(\hat{\theta}_{\boldsymbol{i}}\hat{P}(\bar{\iota}), \hat{\theta}_{\boldsymbol{j}}\hat{P}(\bar{\iota})) \tag{5.1}$$

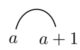
By Lemma 4.24 and Corollary 4.31 the definition of $\hat{\theta}_i$ respectively $\hat{G}_{\boldsymbol{t}}$, the algebra A has a basis given by all orientable stretched circle diagrams of the form $\underline{\iota}\boldsymbol{u} \wr \boldsymbol{t}'\bar{\iota}$, where $\boldsymbol{u}$ and $\boldsymbol{t}$ are generalized crossingless matchings build from the special ones. And the composition $\underline{\iota}\boldsymbol{u} \wr \boldsymbol{t}'\bar{\iota} \cdot \underline{\iota}\boldsymbol{r} \wr \boldsymbol{s}'\bar{\iota}$ is given by 0 if $\boldsymbol{t} \neq \boldsymbol{r}$, and otherwise we draw $\underline{\iota}\boldsymbol{u} \wr \mathrm{upp}(\boldsymbol{t}'\bar{\iota})$ underneath $\mathrm{low}(\underline{\iota}\boldsymbol{t}) \wr \boldsymbol{s}'\bar{\iota}$ and apply the surgery procedure to eliminate the middle section $\mathrm{upp}(\boldsymbol{t}'\bar{\iota})\,\mathrm{low}(\underline{\iota}\boldsymbol{t})$.

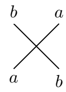
All the definitions above culminate in the following theorem.

Theorem 5.6. *There is an isomorphism of* $\hat{\mathrm{s}}\mathrm{R}$*-module categories* $\Phi\colon \hat{\mathrm{s}}\mathrm{R}^{\mathrm{cyc}} \to \mathcal{F}$*, given by*

$$(i_1, \dots i_k) \mapsto \hat{\theta}_{i_k} \dots \hat{\theta}_{i_1} \hat{P}(\bar{\iota})$$

$\mapsto 0$


$\mapsto \epsilon_a$


$\mapsto \eta_a$


$\mapsto \psi_{b,a}$.

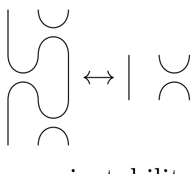
The subsequent sections will prove that this is functor is well-defined and faithful. Assuming these results, we can prove the theorem.

Proof. The functor is well-defined by Proposition 5.10 and faithful by Lemma 5.12. On the other hand, Proposition 5.11 and the adjunction $(\hat{\theta}_i, \hat{\theta}_{i-1})$ from Corollary 4.31 imply $\dim_{\hat{\mathrm{s}}\mathrm{R}^{\mathrm{cyc}}}(\boldsymbol{i}, \boldsymbol{j}) = \dim_{\mathcal{F}}(\hat{\theta}_{\boldsymbol{i}}\hat{P}(\bar{\iota}), \hat{\theta}_{\boldsymbol{j}}\hat{P}(\bar{\iota}))$. Furthermore, Φ is a bijection on objects given by $\boldsymbol{i} \mapsto \hat{\theta}_{\boldsymbol{i}}\hat{P}(\bar{\iota})$, so Φ is an isomorphism of categories. $\square$

5.1. Well-definedness of Φ

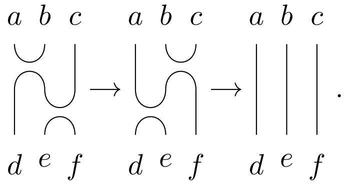
Lemma 5.7. *The functor* Φ *respects the relations* (sR-1)*,* ($\hat{\mathrm{s}}$R-2)*,* ($\hat{\mathrm{s}}$R-4)*,* (sR-5)*,* (sR-6) *and* (3.6)*.*

Proof.
- By definition of $\hat{\theta}_i$, we see that $\hat{\theta}_i P(\bar{\iota}) = G_i e_\iota$ has a basis given by all orientable generalized circle diagrams $at^i\bar{\iota}$. But by assumption $\bar{\iota}$ has no caps, so this is only orientable if $i = 0$, thus (3.6) holds.
- Equation (sR-1) trivially holds as any dot is mapped to 0.
- For ($\hat{\mathrm{s}}$R-2), we have to show that $\hat{\theta}_a \circ \hat{\theta}_a = 0$ and $\epsilon_{a+1} \circ \eta_a = 0$. Note that the former implies the latter by Theorem 6.29. We easily observe that $\hat{G}_{t^it^i} = 0$ as it contains a circle and so no generalized circle diagram of this form is orientable, hence $\hat{\theta}_a \circ \hat{\theta}_a = 0$.
- For ($\hat{\mathrm{s}}$R-4) observe that $\hat{\theta}_a\eta_a$, $\eta_a\hat{\theta}_{a+1}$, $\hat{\theta}_a\epsilon_a$ and $\epsilon_a\hat{\theta}_{a-1}$ all look as follows (up to vertical mirror image).

  Thus, these morphisms do not change orientability, and thus are isomorphisms. Equation (sR-4) follows.

- Next we want to show (sR-5). Note that the left-hand side is locally given by



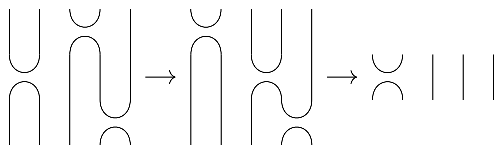
  The only possibility such that this composition is not 0 is if we can find a diagram such that all of these local changes preserve its orientability.

  For this note that none of these six endpoints can be connected to another one, as there exists a picture that creates a circle (also observe that e.g. a cannot be connected to e as there is nothing left for d). So we may assume that there are only rays connected to every endpoint. If the first two pictures are orientable, then neither all of a, b and c (resp. d, e and f) end on the top nor all three on the bottom as then one of these pictures has a non-propagating line, where both ends lie on the same side of 0. Orientability of the first and third picture implies that not all of a, b and d (resp. c, e and f) can end on the same half. Similarly, considering the second and third picture not all of a, d and e (resp. b, c and f) can end on the same half. This means that if all three pictures are orientable, no three consecutive (considering the letters as lying on a circle) rays can end on the same half. But we have six rays, and thus necessarily three consecutive ones have to end on the same half. This is a contradiction, and thus (sR-5) holds.

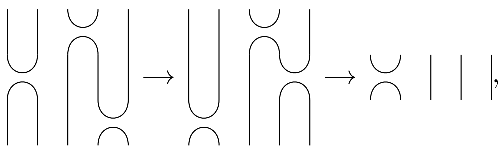
- Finally, for (sR-6) observe that if i and j are distant $\Psi_{i,j}$ is by definition an isomorphism and its inverse is given by $\Psi_{j,i}$, the other cases hold by definition. □

Lemma 5.8. *The functor* Φ *respects also* (sR-3)*.*

Proof. Both sides of (sR-3) are mapped to 0 if $a \in \{b, b+1\}$ by definition of Φ. Next assume that $a \neq b-1, b+2$. In this case the left-hand side of (sR-3) in terms of geometric bimodules looks like

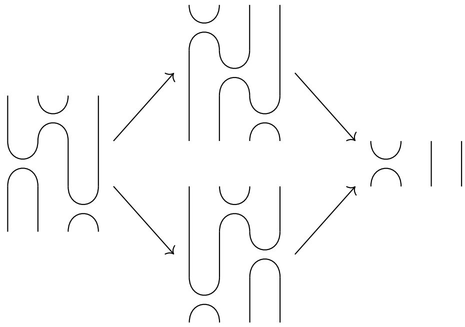


whereas the right-hand side is given by



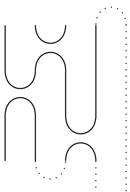
and it is easy to see that the first step is an isomorphism and does not influence orientability for the second.

Now assume that $a = b - 1$ (the case $a = b + 2$ is handled similarly). Now the situation looks like



Now note that the bottom left map is an isomorphism. This means that this diagram cannot commute only if the top left map sends something to zero, which is nonzero under the bottom composition. By Lemma 4.16 and Figure 5.2 this means that we are in one of the following two cases

where the dotted lines are either joined or end on the same side of 0. Now observe that the first case produces a not orientable diagram in the end. Furthermore, for the second case at least one of the dotted lines has to connect to the bottom right cap, as the picture is otherwise not orientable. If the right dotted line connects to this cap, then we create a circle in the end, so it is left to consider the following case.

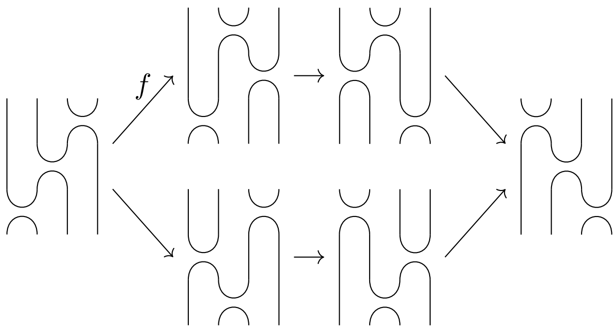


But as both dotted endpoints either join or lie on the same side of 0, this means that the resulting diagram in the end will also not be orientable. $\square$

Lemma 5.9. *The functor* Φ *respects also* ($\hat{\mathrm{s}}$R-7).

Proof. We split this proof into two parts. First assume that $(a, b, c) = (a, a \pm 1, a)$. We have $\hat{\theta}_a \circ \hat{\theta}_a = 0$ ($G_{t^a t^a} = 0$ as it contains a circle), and thus the left-hand side of ($\hat{\mathrm{s}}$R-7) is 0. The argument for the right-hand side is similar to checking (sR-4). Recall from the proof of Lemma 5.7, that $\hat{\theta}_a \eta_a$, $\eta_a \hat{\theta}_{a+1}$, $\hat{\theta}_a \epsilon_a$ and $\epsilon_a \hat{\theta}_{a-1}$ are isomorphisms. With this at hand, it is easy to check that the right-hand side of ($\hat{\mathrm{s}}$R-7) is also 0.

In the remaining cases, the RHS of (ŝR-7) is 0, so we have to show that the terms on the LHS amount to 0. If any of the pairs (a, b), (b, c) or (a, c) is of the form (i, i) or $(i, i + 1)$, then the left-hand side of (ŝR-7) is 0 (as the corresponding crossings are mapped to 0). In the remaining case suppose first that $|a - b| > 1$ and $|a - c| > 1$. This means in terms of generalized crossingless matchings that the cup cap pair corresponding to a does not interact with b and c. So swapping a with b (resp. c) is just given by changing the order of the cup cap pairs, which does not change anything regarding the orientability. Therefore, the left-hand side of (ŝR-7) is given by first swapping b and c and then moving a to the top, whereas the right-hand side first moves a to the top and then swapping b and c. Because a is distant to b and c both give the same result.
A similar argumentation proves the cases, where b (resp. c) is distant to a and c (resp. a and b).
So the only remaining case is that $c = b + 1 = a + 2$. In this case the situation looks as follows.



These two compositions do not agree only if there is an orientable generalized circle diagram that is mapped to 0 under one composition and something nonzero via the other one. Now the two middle arrows are isomorphisms, thus this situation can only occur if one of the first maps sends such a generalized circle diagram to 0. We may assume that f maps such a generalized circle diagram to 0 as the other case is obtained by rotational symmetry.
Using Figure 5.2 and Lemma 4.16 we are in one of the following two cases

and either the two dotted lines are connected or these are both rays ending on the same side of 0.
In the first case note that the diagram in the end is not orientable. Thus, either composition produces 0.
So we are left to look at the second case. We make a case distinction on how the top right endpoint of the cup connects to the rest of the diagram. We have essentially four different cases, which are presented in Figure 5.3.

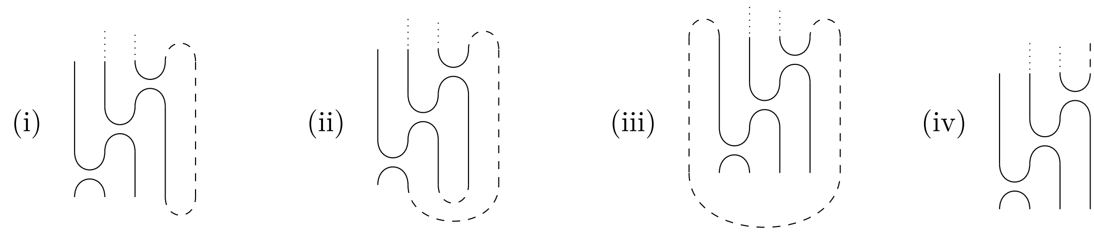


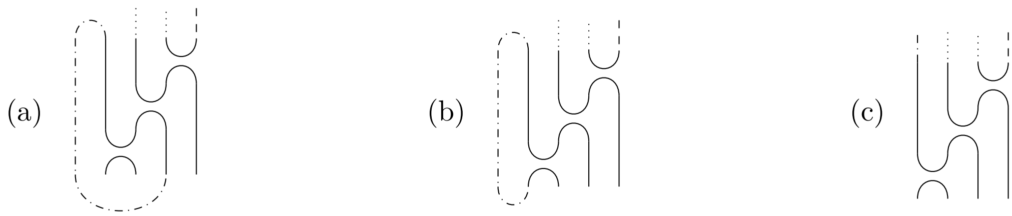
Figure 5.3.: Case distinction for the proof of the braid relation

(i) This diagram is not orientable by assumption on the dotted lines.

(ii) The small dashed cup forms a circle in the resulting diagram, thus both compositions have to be 0.

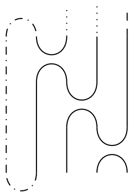
(iii) This diagram is not orientable as the two dotted lines will necessarily be connected.

(iv) In this case we make a case distinction to where the top left endpoint connects. We distinguish another three cases.



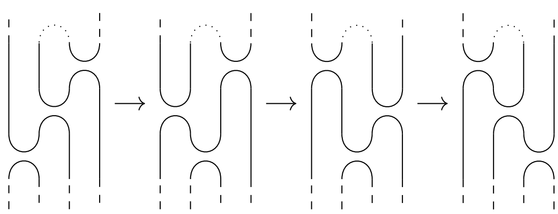
(a) This diagram is not orientable.

(b) In this case the end result would look like



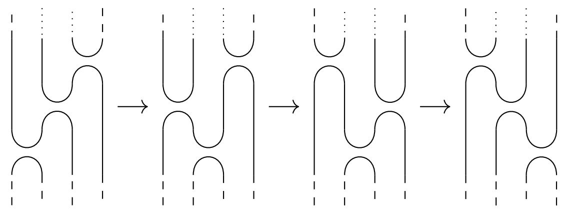
which is not orientable by the assumption on the dotted lines.

(c) Note that any configuration of cups below this diagram produce a non-orientable diagram along the bottom composition. So in these cases, the compositions would both give 0. And now we distinguish whether the dotted lines are connected or not. First assume that they are connected.



And now we have to show that not all of these picture can be orientable. Observe that for every two consecutive lines there is one diagram, where these are joined by a line. Assume there exists a configuration such that all four diagrams are orientable. There has to be a half where at least three strands end. But this means that two of these strands have to lie on the same side of 0. So there is a diagram that has a non-propagating line, where both endpoints lie on the same side of 0, in contradiction to orientability.

The last case is that we only have rays at the top and bottom.



Again we have to show that there is no possible configuration that all these pictures are orientable. Similar to before, there cannot be three of the bottom strands ending on the same half. Also, the four top rays cannot end on the same half using the same reasoning. But this means that the left two bottom rays end for example at the bottom and the right two bottom ones at the top. Also observe that there is a diagram such that each of these pairs are connected. Furthermore, there is a diagram such that the left two (resp. right two) top strands are connected. This means that the bottom pairs cannot be joined by two more rays and so there cannot be a configuration such that all four diagrams are orientable. So both compositions give 0. ☐

Proposition 5.10. *The functor Φ is a well-defined morphism of* $\hat{\mathrm{s}}\mathrm{R}$*-module categories.*

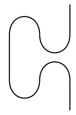
Proof. This is Lemmas 5.7–5.9. ☐

5.2. The functor Φ is faithful

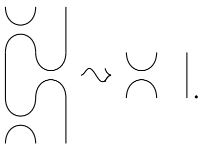
In order to show that Φ is faithful, we need the following technical result.

Proposition 5.11. *The sequence $(i_1, \dots, i_k)$ is a dual residue sequence of an up-down tableau of shape $\emptyset$ if and only if $\underline{\iota} t^{i_k} \cdots t^{i_1} \overline{\iota}$ is an orientable generalized circle diagram.*

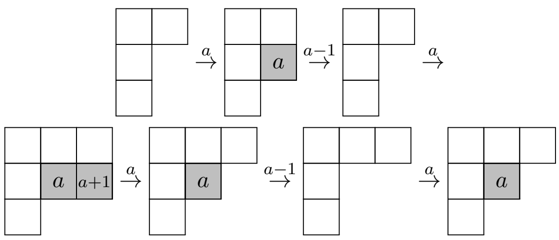
Proof. Let $(i_1, \dots, i_k)$ be a dual residue sequence of an up-down tableau of shape $\emptyset$. First assume that it does not contain a subsequence $(a, a \pm 1, a)$ (i.e. a consecutive subsequence up to swapping entries of difference > 1). Suppose furthermore that $\underline{\iota} t^{i_k} \cdots t^{i_1} \overline{\iota}$ is not orientable. This means that it either has a circle or a non-propagating line ending on both sides of 0. But as we assumed that we have no subsequence $(a, a \pm 1, a)$ we do not have a subpicture of the form (or its vertical mirror image)

,

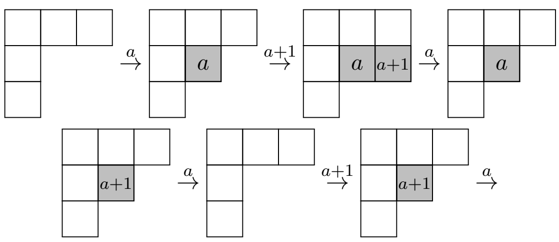
so a circle would consist of only one cup and cap and the non-propagating line only of one cup or cap. Now the circle would give a subsequence (a, a) which cannot happen for residue sequences of up-down tableaux. For the non-propagating line we can move the entry corresponding to the cup (resp. cap) to the front (resp. back) but as both endpoints lie on the same side of 0 the sequence then either does not start with 0 or does not end with -1. But every dual residue sequence of an up-down tableau of shape $\emptyset$ has to start with 0 and end with -1. Therefore, $\underline{\iota}t^{i_k}\cdots t^{i_1}\overline{\iota}$ is orientable.
Now we do an induction on the number of subsequences of the form $(a, a \pm 1, a)$. The paragraph before established the base case. Now we will take such a residue sequence $(i_1, \ldots, i_k)$ with a subsequence $(a, a \pm 1, a)$. We will show that after replacing $(a, a \pm 1, a)$ by (a) we are still left with a residue sequence of an up-down tableau of shape $\emptyset$. Note that this reduction process (up to vertical mirror image) looks like



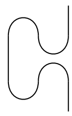
So this does not change orientability of the associated stacked circle diagram. Therefore, once we proved that the reduced sequence is a residue sequence of $\emptyset$ we know by induction that the associated stacked circle diagram is orientable and then reversing the reduction process there gives still an orientable diagram. Hence, it suffices to show that replacing a subsequence of the form $(a, a \pm 1, a)$ by (a) gives the dual residue sequence of an up-tableau of shape $\emptyset$. The following are the possibilities how a subsequence $(a, a-1, a)$ can look in terms of up-down tableau.



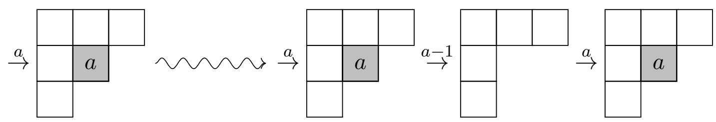
We easily see that in each case we have that $a-1$ removes a box that was either added by the a before or is added again by the a afterward. Thus, replacing this sequence by a we still have a residue sequence of an up-down tableau of shape $\emptyset$. Given a subsequence $(a, a+1, a)$ we have the following possibilities.



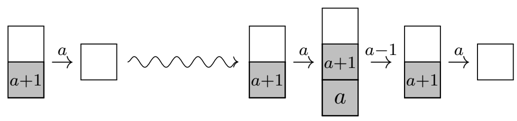
Similar to before $a+1$ adds a box which is removed by one of the a's. Thus, the reduced sequence is also a dual residue sequence of an up-down tableau of shape $\emptyset$. This proves then that such a residue sequence gives rise to an oriented stacked circle diagram.
Now assume that $\underline{\iota}t^{i_k}\cdots t^{i_1}\overline{\iota}$ is orientable. We will prove that any diagram without subpictures of the form



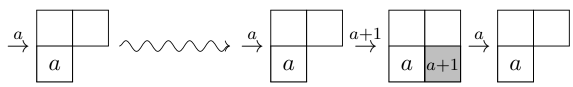
or its vertical mirror image gives rise to a dual residue sequence of shape $\emptyset$. Furthermore, we will prove that given a dual residue sequence of an up-down tableau, replacing any entry (a) by $(a, a\pm 1, a)$ still gives a dual residue sequence of the same shape.
Now given any diagram, we can use the reduction process from the second paragraph to obtain a picture without these subpictures. This is then a dual residue sequence by our first claim, and then we can reverse the reduction process and add the sequences $(a, a\pm 1, a)$ again and by the second claim this stays in the desired form.
We will first prove the second claim for $(a, a-1, a)$. Suppose we are given such a residue sequence $(i_1, \dots, i_k)$ and let $i_l = a$ be some entry. If there exists an entry a that corresponds to adding a box that is not removed until step l we can remove this with $a-1$ and add it again with a.



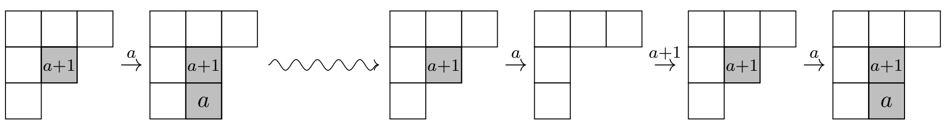
If no such entry exists, this means that a removes a box of residue $a+1$ in the first column. But in this case we could also add this a and then remove the two boxes with $a-1$ and a.



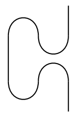
Next we prove the second claim for $(a, a+1, a)$. Suppose we are given such a residue sequence $(i_1, \dots, i_k)$ and let $i_l = a$ be some entry. If after step l there exists an addable box of content $a+1$, we can add this with $a+1$ and remove this with a.



If there is no addable box of content $a+1$, this means that a added a box and the situation looks as follows.



Hence, we have proved the second claim.
It remains to prove that any orientable diagram $\underline{\iota}t^{i_k}\cdots t^{i_1}\overline{\iota}$ without subpictures of the form



or its vertical mirror image gives rise to a dual residue sequence of an up-down tableau of shape $\emptyset$. In this case we have the following properties of the integer sequence $(i_1,\dots i_k)$.

(i) If $i_r = a > 0$, then there exists $r' < r$ with $i_{r'} = a - 1$.

(ii) If $i_r = a < 0$, then there exists $r' < r$ with $i_{r'} = a + 1$.

(iii) If $i_{r'} = i_r = a$ for $r' < r$, then there exists $r' < l, l' < r$ such that $i_l = a + 1$ and $i_{l'} = a - 1$.

(iv) The first two properties hold for the sequence $(i_k + 1, i_{k-1} + 1, \dots, i_1 + 1)$.

If the first two would not be satisfied we could create a non-propagating line ending at the top with both endpoints on the same side of 0. The last one ensures the same for non-propagating lines ending at the bottom (remember the shift by -1 for cup diagrams). And the third condition prevents the existence of the subpictures from the top as well as circles.
All these together imply that $\boldsymbol{i}$ is the residue sequence of an $(n+1)\times n$ rectangle for which it is easy to see that this can be interpreted as a residue sequence of an up-down tableau of shape $\emptyset$. This is done by adding boxes to reach the partition $(n, n-1, n-2, \dots 1)$ and then removing boxes until one ends up at $\emptyset$. □

Lemma 5.12. *The functor* Φ *is faithful.*

Proof. The category $\hat{\mathrm{s}}\mathrm{R}^{\mathrm{cyc}}$ has a basis given by $\Psi_{\mathfrak{s}\mathfrak{t}}$, where $\mathfrak{s}$ and $\mathfrak{t}$ are two up-down-tableaux of the same shape. By adjunction and using that Φ is compatible with this adjunction, we can assume that $\mathfrak{s} = \emptyset$ is the trivial up-down-tableaux. All these basis vectors are build by applying KLR-cups (to distinguish them from the cups in $\mathbb{K}$). Using (3.2), we can get rid of the non-distant crossings that might be involved in the KLR-cups. So $\Psi_{\emptyset\mathfrak{t}}$ can be cut into a sequence of distant crossings and KLR-cups on neighbored strands. In terms of the stretched circle diagram basis for A from (5.1) this means that we start with the diagram $\underline{\iota}\wr\overline{\iota}$ and successively apply the crossings and KLR-cups. But in every step we reach a pair $(\emptyset, \mathfrak{t}')$, where $\mathfrak{t}'$ is an up-down-tableau of shape $\emptyset$. By Proposition 5.11 this means that in every step we get an orientable stretched circle diagram, and thus by definition of ϵ_i and $\psi_{i,j}$ the result is nonzero in every step. But this means that $\Psi_{\emptyset\mathfrak{t}}$ is mapped to something nonzero and by adjunction this holds for all basis vectors.
Using that the dimension of morphism spaces in $\hat{\mathrm{s}}\mathrm{R}^{\mathrm{cyc}}$ is ≤ 1 we conclude that Φ is faithful. □

6. Khovanov algebras for $\mathfrak{p}(n)$

6.1. Definition of the algebra $\mathbb{K}_n$

Overall, we want to relate the algebra $\mathbb{K}$ with finite dimensional representations of $\mathfrak{p}(n)$. As $\mathbb{K}$ is Morita equivalent to $\hat{\mathrm{sR}}^{\mathrm{cyc}}$, this algebra is a bit too big to be equivalent to $\mathrm{rep}(\mathfrak{p}(n))$. Therefore, we introduce a new algebra $\mathbb{K}_n$ that will turn out to satisfy our purpose. This algebra is defined as a quotient of an idempotent truncation of $\mathbb{K}$. The general idea is to first pick those idempotents that correspond to projective representations of $\mathfrak{p}(n)$ and then quotient out by all the morphisms that vanish for the Lie superalgebra. With Lemma 4.2 we can associate to every (ρ-shifted) dominant integral weight λ a cup (or cap) diagram with n cups, and we obtain a bijection between dominant integral weights and Λ_n. Recall, that a weight is called typical if $\lambda_i - \lambda_{i+1} > 1$. This means for the associated cup (or cap) diagram that there are no nested cups (or caps).

Definition 6.1. We then define $e\mathbb{K}e$ to be the idempotent truncation of $\mathbb{K}$ at Λ_n, i.e. at all cup diagrams with n cups. The algebra $e\mathbb{K}e$ has a basis given by orientable circle diagrams with exactly n cups and caps.
We denote by $\mathbb{I}_n$ the subspace spanned by all orientable circle diagrams $\underline{\lambda}\overline{\mu}$ with n cups and caps such that there exists at least one non-propagating line.

Lemma 6.2. *The space $\mathbb{I}_n$ is a two-sided ideal of $\mathbb{K}$.*

Proof. From Section 4.2.1 it is clear that every surgery procedure either preserves non-propagating lines or produces 0. $\square$

Lemma 6.3. *Given $f \in e\mathbb{K}e$ the following holds:*

$$f \in \mathbb{I}_n \Rightarrow f \text{ factors through an object } \nu \in I_{n+1} \tag{6.1}$$

Proof. It suffices to prove this statement for all circle diagrams $\underline{\lambda}\overline{\mu} \in e\mathbb{K}e$ that contain a non-propagating line. As the number of cups in $\underline{\lambda}$ is the same as the number of caps in $\overline{\mu}$ we see that we have as many non-propagating lines ending at the bottom as we have at the top. So consider a non-propagating line L ending at the bottom, and we choose L such that its left endpoint is minimal. This has one cap more than cup. Now let $\underline{\lambda'}$ be the cup diagram that is the same as $\underline{\lambda}$ except that the left endpoint of the non-propagating line and the next ray to its left form a cup instead of two rays (see also Figure 6.1). As L was chosen minimal, $\underline{\lambda'}\overline{\mu}$ is orientable and it has one more cup than caps. Furthermore, $\underline{\lambda}\overline{\lambda'}$ is also orientable as the additional cap connects exactly the same endpoints as L in $\underline{\lambda}\overline{\mu}$ by construction. We claim that $\underline{\lambda}\overline{\lambda'} \cdot \underline{\lambda'}\overline{\mu} = \underline{\lambda}\overline{\mu}$. As $\underline{\lambda}$ and $\underline{\lambda'}$ agree up to one cup, we

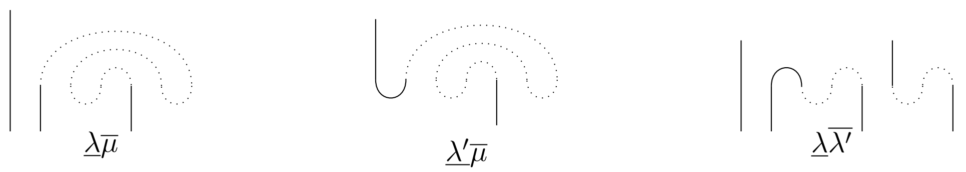


Figure 6.1.: Example of how λ' is constructed.

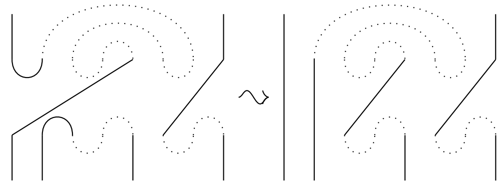


Figure 6.2.: Example for the first surgery procedure of $\underline{\lambda}\overline{\lambda'} \cdot \underline{\lambda'}\overline{\mu}$

see that $\underline{\lambda}\overline{\lambda'}$ is very close to the idempotent e_λ. We may choose the surgery procedure coming from the additional cup as the first one. But this surgery procedure produces by definition of $\underline{\lambda'}$ the diagram $\underline{\lambda}\overline{\lambda}$ drawn underneath $\underline{\lambda}\overline{\mu}$ which is clearly orientable (see also Figure 6.2). After this we essentially compute $e_\lambda \cdot \underline{\lambda}\overline{\mu}$ which gives clearly $\underline{\lambda}\overline{\mu}$. As $\underline{\lambda'} \in I_{n+1}$, this proves the lemma. □

Definition 6.4. We define the algebra $\mathbb{K}_n$ to be $e\mathbb{K}e/\mathbb{I}_n$.
When talking about orientable circle diagrams in the context of $\mathbb{K}_n$ we mean orientable circle diagrams for $\mathbb{K}$ without any non-propagating lines.

Remark 6.5. For $\mathbb{K}_n$ the multiplication is given exactly as for $\mathbb{K}$, but we declare the result to be 0 whenever a non-propagating line appears. Thus, comparing with Section 4.2.1 we see that the split and reconnect produce 0 in every case and straightening produces something non-zero.

Lemma 6.6. *The algebra $\mathbb{K}_n$ is an essentially finite locally unital algebra $\mathbb{K}$ with idempotents e_λ indexed by $\lambda \in \Lambda_n$.*

Proof. By Corollary 4.14 K is a locally unital algebra, so clearly $\mathbb{K}_n$ is as well. It is essentially finite dimensional, i.e. $\dim \mathbb{K}_n e_\lambda < \infty$, as we do not allow non-propagating lines. This means that if we fix a cap diagram $\overline{\lambda}$ and $\underline{\mu}\overline{\lambda}$ is orientable, then all the cups of $\underline{\mu}$ are close to the caps of $\overline{\lambda}$. In particular every cup diagram of this form fits into a finite strip where it is non-trivial, thus $\dim \mathbb{K}_n e_\lambda < \infty$. And similarly also $\dim e_\lambda \mathbb{K}_n < \infty$. □

Definition 6.7. We define an anti-involution $*$ on $\mathbb{K}_n$, which is given by rotating a circle diagram around $\frac{1}{2}$.

From now on we will assume that any $\mathbb{K}_n$-module M is compatible with the locally unital structure, i.e. $M = \bigoplus_{\lambda \in \Lambda_n} e_\lambda M$.

6.2. A triangular basis and quasi-hereditaryness

In this section we explore the structure of $\mathbb{K}_n$ further. In particular, we will prove that $\mathbb{K}_n$ is an essentially finite based quasi-hereditary algebra in the sense of [BS24].

6.2.1. Δ and ∇ orientations

We introduce orientations of circle diagrams that allow us to split our diagrams and provide then some cellular like properties. These additional orientations behave very much like orientations for the Khovanov algebras of type A respectively B, see [BS11a] for type A and [ES16] for type B. The main difference is that for type P the orientations are not needed to define the algebra structure, they are only important for the more involved structure.

Definition 6.8. A Δ-orientation of a cap diagram $\overline{\lambda}$ associates to each cap with endpoints $a < b \in \mathbb{Z} + \frac{1}{2}$ an integer k such that $a < k < b$ and for every other cap with endpoints $a' < k < b'$ we have $a' < a < b < b'$.
A ∇-orientation of a cup diagram $\overline{\lambda}$ associates to each cup with endpoints $a < b \in \mathbb{Z} + \frac{1}{2}$ an integer $k \in \{a - \frac{1}{2}, b + \frac{1}{2}\}$.
Furthermore, we require for both orientations that the integers are pairwise distinct. We draw this as n black dots at the corresponding positions in the cap (resp. cup) diagram.

Remark 6.9. One might argue that "orientation" is not a descriptive name for this. However, we chose this wording (similar to surgery procedures) as these mimic the orientations in [BS11b] and [ES16] for the Khovanov algebras of other types. In their setup, these orientations provide filtrations of indecomposable projective modules via standard and costandard modules. Our orientations play the exact same role as shown in Corollary 6.18 below.

Remark 6.10. We can think of Δ- and ∇-orientations also as integral dominant weights for $\mathfrak{p}(n)$. If the set $\{k_1 > \cdots > k_n\}$ describes the orientation we can associate the integral dominant weight $k_1 - 1 > \cdots > k_n - 1$.
So cup diagrams are shifted by $\frac{1}{2}$, orientations by 1 and cap diagrams by $\frac{3}{2}$.

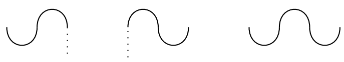
Remark 6.11. Spelling out the definition of Δ-orientation, we associate to each cap a point that lies below this cap but not below any other cap inside this cap. Informally speaking, we move the point from the cap somewhere to the inside. Especially if we have a small cap there is only one point we can associate.
So for a typical cap diagram there is only one Δ-orientation.

Remark 6.12. Instead of associating the integer $a - \frac{1}{2}$ or $b + \frac{1}{2}$ to a cup, we could also demand for an integer k such that $k \notin [a, b]$ and for another cup (here two neighbored rays are also considered as a cup) with endpoints $a' < a < b < b'$ we have $a' < k < b'$. This allows a priori for more possibilities for each cup diagram, but the interested reader can easily verify that these two definitions are equivalent. This other definition resembles more the definition of a Δ-orientation meaning that we can move every point to the outside but not outside another cup, but the original one is more practical.

Definition 6.13. An orientation of an orientable circle diagram $\underline{\lambda}\overline{\mu}$ is a Δ-orientation of $\overline{\mu}$ and a ∇-orientation $\underline{\lambda}$ such that the sets of the associated integers agree. Using Remark 6.10 we also write $\underline{\lambda}\nu\overline{\mu}$ if the orientation of $\underline{\lambda}\overline{\mu}$ is exactly ν.

Lemma 6.14. *Every orientable circle diagram admits a unique orientation.*

Proof. We prove this via induction on the number k of caps, with $k = 0$ being trivial. So let c be an orientable circle diagram. If $k > 0$, there exists a small cap, which is connected in one of the following ways.



The dotted ray indicates that it is either a ray or a cup that contains the undotted cup. Any orientation of c necessarily needs to have a black dot at the unique position inside this small cap, which needs to be associated to one of the undotted cups in the above picture. In the first two cases we can remove the cup/cap pair and in the last we replace the picture with a cup. In all three cases we obtain an orientable diagram with one cap less. By induction, we now that this admits a unique orientation.
In the first two cases we directly get a unique orientation for c by putting the cup/cap pair back and adding a dot below the small cap. In the third case note that the new cup that we added necessarily has a dot either to its left or right. If it is on the left, we associate this with the left cup and the dot inside the small cap with the right cup and vice versa. In any case, we can build a unique orientation out of the smaller diagram. $\square$

6.2.2. A triangular basis

Definition 6.15. For $\lambda, \mu \in \Lambda_n$ we define the sets $X(\lambda,\mu)$ and $Y(\lambda,\mu)$ as

$$X(\lambda,\mu) \coloneqq \begin{cases} \{\underline{\lambda}\lambda\overline{\mu}\} & \text{if } \lambda \text{ is a valid } \Delta\text{-orientation of } \overline{\mu}, \\ \emptyset & \text{otherwise,} \end{cases}$$

$$Y(\lambda,\mu) \coloneqq \begin{cases} \{\underline{\lambda}\mu\overline{\mu}\} & \text{if } \mu \text{ is a valid } \nabla\text{-orientation of } \underline{\lambda}, \\ \emptyset & \text{otherwise.} \end{cases}$$

We also set $Y(\lambda) \coloneqq \bigcup_{\mu\in\Lambda_n} Y(\mu,\lambda)$ and $X(\lambda) \coloneqq \bigcup_{\mu\in\Lambda_n} X(\lambda,\mu)$.

Lemma 6.16. *If ν is a valid Δ-orientation of $\overline{\mu}$ and a valid ∇-orientaion of $\underline{\lambda}$, then $\underline{\lambda}\overline{\mu}$ is orientable.*

Proof. Suppose that $\underline{\lambda}\overline{\mu}$ is not orientable. Then it contains either a non-propagating line ending at the bottom or a circle. In both cases look at the k caps of this component. As ν is a valid Δ-orientation of $\overline{\mu}$ we have k dots in between these k caps. But this means that we also have k dots in k cups if we have a circle or k dots for $k-1$ cups bounded by rays if we have a non-propagating line ending at the bottom. But in either case ν is not a valid ∇-orientation of $\underline{\lambda}$. $\square$

Theorem 6.17. *The following properties hold.*

(i) *The products (yx) for $(y,x) \in \sqcup_{\lambda\in\Lambda_n} Y(\lambda) \times X(\lambda)$ form a basis of $\mathbb{K}_n$.*

(ii) *For $\lambda, \mu \in \Lambda_n$, the sets $Y(\mu,\lambda)$ and $X(\lambda,\mu)$ are empty unless $\mu \leq \lambda$.*

(iii) *We have that $Y(\lambda,\lambda) = X(\lambda,\lambda) = \{e_\lambda\}$ for each $\lambda \in \Lambda_n$.*

Proof. For (ii) note that $Y(\mu,\lambda)$ (resp. $X(\lambda,\mu)$) is non-empty only if λ is a ∇-orientation (resp. Δ-orientation) of $\underline{\mu}$ (resp. $\overline{\mu}$). Observe that if one always puts the dot at the rightmost position possible one obtains the ∇-orientation (resp. Δ-orientation) μ. All the others are obtained from this by moving single dots to the left, which makes the weight bigger in our order from Definition 4.3.
For (iii) observe that λ is clearly a Δ-orientation of $\overline{\lambda}$ and a ∇-orientation of $\underline{\lambda}$.
Thus, it only remains to proof (i). We will prove that if $\underline{\lambda}\nu\overline{\mu}$ is an orientation, then $\underline{\lambda}\overline{\mu} = \underline{\lambda}\overline{\nu} \cdot \underline{\nu}\overline{\mu}$. We clearly have $\underline{\lambda}\overline{\nu} \in Y(\lambda,\nu)$ and $\underline{\nu}\overline{\mu} \in X(\nu,\mu)$ and together with Lemma 6.14 this proves (i).
Denote by $\underline{\nu_k'}$ the cup diagram, where we remove the k cups of $\underline{\lambda}$ which are associated to the k leftmost dots of ν. Then let $\underline{\nu_k}$ be obtained from $\underline{\nu_k'}$ by adding k cups such that their endpoints correspond to the k leftmost dots of ν. In other words $\nu_0 = \lambda$ and $\nu_n = \nu$. By construction, we know that ν is a ∇-orientation of $\underline{\nu_k}$. Therefore, $\underline{\nu_k}\overline{\mu}$ is orientable by Lemma 6.16. Furthermore, by construction we also have that ν_k is a ∇-orientation of $\underline{\lambda}$ and hence $\underline{\lambda}\overline{\nu_k}$ is orientable as well.
Now we compute some surgery procedures for $\underline{\lambda}\overline{\nu_k} \cdot \underline{\nu_k}\overline{\mu}$. Refer for this also to Section 6.2.3, where we describe explicitly how orientations behave under multiplication. When multiplying these diagrams we have n surgery procedures in total, $n-k$ of them come from λ and the last k from ν. By construction, we may choose to apply the $n-k$ procedures for λ first. As stated before ν_k is an orientation of $\underline{\lambda}\overline{\nu_k}$ and the $n-k$ rightmost dots in ν_k all also appear in λ. But this means that the first $n-k$ surgery procedures are similar to multiplying with e_{ν_k} instead of $\underline{\lambda}\overline{\nu_k}$ and each of these looks just as in the proof of Lemma 4.13, i.e. all theses are straightenings. Thus, none of these change orientability and after $n-k$ steps we have reached an orientable diagram. But this diagram is exactly the diagram that we obtain after $n-k$ surgery procedures (for one specific order) of the multiplication $\underline{\lambda}\overline{\nu} \cdot \underline{\nu}\overline{\mu}$. Therefore, with the correct order (namely from right to left) of surgery procedures we see that after applying each of them we receive an orientable diagram. Thus, by definition of the multiplication we get $\underline{\lambda}\overline{\mu} = \underline{\lambda}\overline{\nu} \cdot \underline{\nu}\overline{\mu}$. $\square$

Corollary 6.18. *The algebra $\mathbb{K}_n$ is an essentially finite based quasi-hereditary algebra in the sense of* [BS24].

Proof. This is immediate from Lemma 6.6 and Theorem 6.17. $\square$

6.2.3. An alternative description of $\mathbb{K}_n$ using orientations

Alternatively we can describe $\mathbb{K}_n$ as the algebra with a basis given by all oriented circle diagrams $\underline{\lambda}\nu\overline{\mu}$. And the multiplication is given by exactly the same procedure we just have to define what happens with the orientation during a surgery procedure.

In order to describe this, we first define what such an orientation ought to be.

Definition 6.19. Suppose we are given a generalized stacked circle diagram $\underline{\lambda}\boldsymbol{t}\overline{\mu}$ with $t = t_k \cdots t_1$. An orientation of this diagram is a sequence of weights $\nu_k, \ldots, \nu_0$ such that

(i) ν_k is a ∇-orientation of $\underline{\lambda}$,

(ii) ν_0 is a Δ-orientation of $\overline{\mu}$,

(iii) and every of $\nu_i t_i \nu_{i-1}$ is oriented (where we think of the orientations ν_i as dots), by which we mean
   - every cap has associated a unique dot of λ_i inside it that is not contained in any other cap,
   - every cup has associated a unique dot of λ_{i-1} directly to its left or right,
   - and every other dot of λ_i that is not contained in any cap lies in a region bounded by two rays. For each of these dots, there exists a unique dot of λ_{i-1} that is in the same region and not associated to a cup.

Remark 6.20. Associating the dots to cups and or caps is not part of the data. We require only the existence of such a pairing.
In Figure 6.3 it is easy to see that there are many possibilities to define this pairing, but we consider this to be only one orientation.

Example 6.21. Figure 6.3 provides examples for some orientations of crossingless matchings.

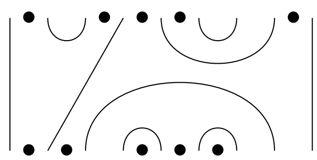


Figure 6.3.: Examples of orientations of crossingless matchings

Lemma 6.22. *Any orientable diagram of the form $\underline{\lambda}\boldsymbol{t}\overline{\mu}$ admits a unique orientation (and if it admits an orientation it is orientable).*

Proof. This is proven by exactly the same argument as for Lemma 6.14. ☐

Observe that all diagrams that are obtained using surgery procedures are of the form $\underline{\lambda}t\overline{\mu}$ for some crossingless matching t. We have now all the ingredients to describe how orientations behave under surgeries. If the surgery we apply is a split or a reconnect the result is 0, so we only have to look at straightening. Note that by definition of orientation these are all the cases that appear. In every case we move the dots as depicted in Figure 6.4 (the white dots may or may not be there)
This preserves the orientation as the resulting dots lie in the same region.
After every surgery procedure we have a trivial middle section and the two orientations agree. Then we collapse the middle section and identify the two orientations.

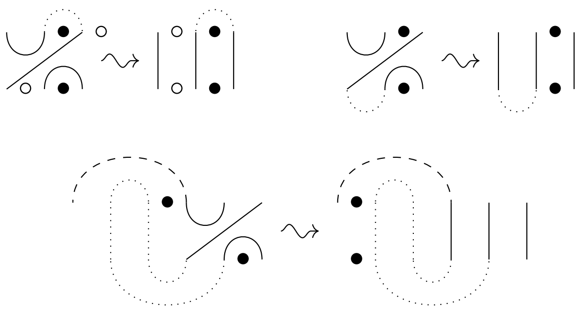


Figure 6.4.: How orientations behave under straightening

Example 6.23. Let $n = 3$ and $\lambda = (1, 0, -1)$, $\mu = (4, 3, -1)$ and $\nu = (3, 2, -1)$. We then compute in Figure 6.5 $\underline{\nu}\overline{\mu} \cdot \underline{\mu}\overline{\lambda}$ including the orientations.

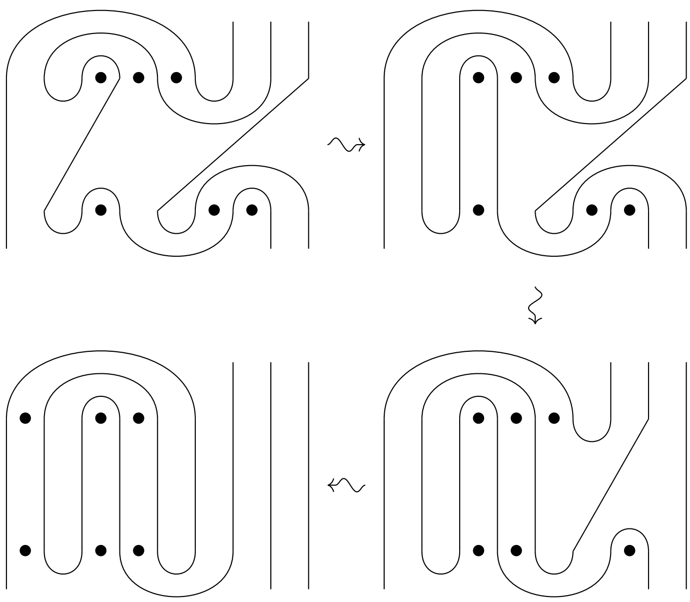


Figure 6.5.: Example of multiplication including orientations

6.3. Geometric bimodules and adjunction

This section adapts the definition of geometric bimodules for $\mathbb{K}$ to the subquotient $\mathbb{K}_n$ and proving similar results as in Section 4.4.

6.3.1. Definition

Definition 6.24. For a generalized crossingless matching $\boldsymbol{t}$ we can define the $\mathbb{K}_n$-$\mathbb{K}_n$-bimdoule $G_{\boldsymbol{t}} := e\hat{G}_{\boldsymbol{t}}e/\mathbb{I}_{\boldsymbol{t}}$ where $\mathbb{I}_{\boldsymbol{t}}$ is the vector subspace spanned by all generalized circle diagrams with non-propagating lines. That this is indeed a two-sided submodule can be verified analogously to Lemma 6.3.
We then define $\theta_{\boldsymbol{t}}$ as tensoring over $\mathbb{K}_n$ with $G_{\boldsymbol{t}}$ and if $\boldsymbol{t} = t^i$ we also write θ_i for θ_{t^i} and call these $\theta_{\boldsymbol{t}}$ *projective functors.*

Remark 6.25. As in Corollary 4.22, we still have $G_{\boldsymbol{t}} \cong G_{\mathrm{red}(\boldsymbol{t})}$, thus we restrict ourselves from now on to crossingless matchings of one layer.

Observe that for these bimodules the appropriate replacement for (4.1) also holds. On the other hand given a generalized crossingless matching $\boldsymbol{t}$ we define $\boldsymbol{t}^*$ as the rotation of $\boldsymbol{t}$ around $\frac{1}{2}$. And we can define a linear map $*\colon \hat{G}_{\boldsymbol{t}} \to \hat{G}_{\boldsymbol{t}^*}, a\boldsymbol{t}b \mapsto b^*\boldsymbol{t}^*a^*$, i.e. $*$ is given by rotating the total picture around $\frac{1}{2}$. Then the map $*$ is anti-multiplicative in the sense that

$$
\begin{tikzcd}
\hat{G}_{\boldsymbol{t}} \otimes \hat{G}_{\boldsymbol{u}} \arrow[r, "P\circ(*\otimes*)"] \arrow[d, "m"] & \hat{G}_{\boldsymbol{u}^*} \otimes \hat{G}_{\boldsymbol{t}^*} \arrow[d, "m"] \\
\hat{G}_{\boldsymbol{tu}} \arrow[r, "*"] & \hat{G}_{\boldsymbol{u}^*\boldsymbol{t}^*}
\end{tikzcd}
\tag{6.2}
$$

commutes, where P is the normal flip map.

6.3.2. Adjunction

The next proposition states the equivalent of Proposition 4.26 for $\mathbb{K}_n$.

Proposition 6.26. *We have*

$$\theta_t P(\overline{\gamma}) = \begin{cases} 0 & \text{if } t\overline{\gamma} \text{ is not orientable,} \\ P(\overline{\nu}) & \text{otherwise, where } \overline{\nu} = \mathrm{upp}(t\overline{\gamma}). \end{cases}$$

Proof. This is proven in the same way as Proposition 4.26 by noticing that now any non-propagating line is killed by definition of θ_t. $\square$

Corollary 6.27. *The $\mathbb{K}_n$-$\mathbb{K}_n$-bimodule G_t is sweet, i.e. projective as a left and as a right $\mathbb{K}_n$-module.*

Proof. We have $G_t = \bigoplus_{\lambda\in\Lambda_n} G_t e_\lambda$ is projective as a left $\mathbb{K}_n$-module by Proposition 6.26. Furthermore, using (6.2) we see that G_t is projective as a right $\mathbb{K}_n$-module if and only if G_{t^*} is projective as a left $\mathbb{K}_n$-module which we know. $\square$

Corollary 6.28. *Projective functors are exact and preserve finitely generated modules.*

Proof. Use Proposition 6.26 and Corollary 6.27. $\square$

Our next goal is to prove an adjunction theorem between the θ_t.

Theorem 6.29. *We have an adjunction $(\theta_{t^\ddagger}, \theta_t)$. Thus, θ_{i+1} is left adjoint to θ_i.*

Proof. This is proven exactly as Corollary 4.31. □

Theorem 6.30. *Given any crossingless matching t and any finite dimensional $\mathbb{K}_n$-module M there exists a natural isomorphism $\theta_{t^\dagger} M^\circledast \cong (\theta_t M)^\circledast$, where $t^\dagger$ is the vertical reflection of t at 0. In particular, $(\theta_i M)^\circledast \cong \theta_{-i} M^\circledast$.*

Proof. It suffices to construct a natural isomorphism $G_{t^\dagger} \otimes_{\mathbb{K}_n} M^\circledast \cong (G_t \otimes_{\mathbb{K}_n} M)^\circledast$. For this we define the auxiliary map

$$s\colon G_{t^\dagger} \otimes M^\circledast \otimes G_t \otimes M \to \mathbb{C}$$

by sending $x \otimes f \otimes y \otimes m$ to $f(\phi(x^* \otimes y)m)$. Observe that $(t^\dagger)^*$ is the rotation around $\frac{1}{2}$ of the vertical mirror of t at 0. But this is the same as the horizontal mirror image shifted one to the right, i.e. $(t^\dagger)^* = t^\ddagger$. Hence, we can apply ϕ in the definition of s, and thus s is well-defined.
Now let $u \in \mathbb{K}_n$. Using the linearity properties from Lemma 4.29 (or its $\mathbb{K}_n$-equivalent) we have

$$\begin{aligned} s(xu \otimes f \otimes y \otimes m) &= f(\phi((xu)^* \otimes y)m) = f(u^*\phi(x^* \otimes y)m) = s(x \otimes uf \otimes y \otimes m), \\ s(x \otimes f \otimes yu \otimes m) &= f(\phi(x^* \otimes yu)m) = f(\phi(x^* \otimes y)um) = s(x \otimes uf \otimes y \otimes um). \end{aligned}$$

This means that s factors over

$$s\colon G_{t^\dagger} \otimes_{\mathbb{K}_n} M^\circledast \otimes G_t \otimes_{\mathbb{K}_n} M \to \mathbb{C},$$

and thus we get an induced map

$$\tilde{s}\colon G_{t^\dagger} \otimes_{\mathbb{K}_n} M^\circledast \to (G_t \otimes_{\mathbb{K}_n} M)^\circledast$$

by sending $x \otimes f$ to $(y \otimes \mapsto s(x \otimes f \otimes y \otimes m))$.
It remains to show that $\tilde{s}$ is $\mathbb{K}_n$-linear and a vector space isomorphism.
For linearity let $u \in \mathbb{K}_n$.

$$\begin{aligned} (u\tilde{s}(x \otimes f))(y \otimes m) = \tilde{s}(x \otimes f)(u^* y \otimes m) &= f(\phi(x^* \otimes u^* y)m) = f(\phi((ux)^* \otimes y)m) \\ &= \tilde{s}(ux \otimes f)(y \otimes m) \end{aligned}$$

where we used that ϕ is $\mathbb{K}_n$-balanced.
In order to show the vector space isomorphism it suffices to restrict for each $\lambda \in \Lambda_n$ at

$$e_\lambda G_{t^\dagger} \otimes_{\mathbb{K}_n} M^\circledast \to e_\lambda (G_t \otimes_{\mathbb{K}_n} M)^\circledast.$$

We can identify $e_\lambda (G_t \otimes_{\mathbb{K}_n} M)^\circledast$ with $(e_\lambda^* G_t \otimes_{\mathbb{K}_n} M)^\circledast$. Now observe that e_λ^* is the same as vertical mirroring e_λ at 0. Therefore, we have $e_\lambda G_{t^\dagger} \neq 0$ if and only if $e_\lambda^* G_t \neq 0$, so we may assume that these are non-zero.

By Proposition 6.26 we have that $e_\lambda G_{t^\dagger} = e_\nu \mathbb{K}_n$ for some $\nu \in \Lambda_n$ and for the same ν we also have $e_\lambda^* G_t = e_\nu^* \mathbb{K}_n$.
Therefore, it suffices to (see also the proof of Theorem 4.30) show that

$$e_\nu M^{\circledast} \to (e_\nu^* M)^*, \qquad e_\nu f = f(e_\nu^* \cdot \_) \mapsto (e_\nu^* m \mapsto f(e_\nu^* m))$$

is an isomorphism, but this is clear. □

Remark 6.31. One should observe that the index shift in the theorem (i.e. θ_i turning into θ_{-i} under the duality) agrees exactly with Lemma 3.9.

6.3.3. Action on special classes of modules

We already have proven in Proposition 6.26 how geometric bimodules act on projective modules. In this section we are going to determine the effect of geometric bimodules on standard, costandard and irreducible modules. But first, we use Corollary 6.18 to define standard and costandard modules.

Definition 6.32. Let $\lambda \in \Lambda_n$ and define $\mathbb{K}_n^{\leq \lambda}$ to be the quotient of $\mathbb{K}_n$ by the two-sided ideal generated by all e_μ for $\mu \not\leq \lambda$. We will also write $\bar{x}$ for the image of $x \in \mathbb{K}_n$ in this quotient. We then define the left $\mathbb{K}_n$-modules $\Delta_n(\lambda) := \mathbb{K}_n^{\leq \lambda} \bar{e}_\lambda$ and $\nabla_n(\lambda) := (\bar{e}_\lambda \mathbb{K}_n^{\leq \lambda})^*$ where $(\_)^*$ denotes the usual vector space dual. The module $\Delta(\lambda)$ has a basis given by $\{(y\bar{e}_\lambda) \mid y \in Y(\lambda)\}$. Furthermore, the vectors $(\bar{e}_\lambda x)$ for $x \in X(\lambda)$ give a basis for $\bar{e}_\lambda \mathbb{K}_n^{\leq \lambda}$ and its dual basis gives a basis for $\nabla(\lambda)$.

Proposition 6.33. *Let t be a crossingless matching and $\gamma \in \Lambda_n$.*

(i) *The $\mathbb{K}_n$-module $\theta_t \Delta(\overline{\gamma})$ has a filtration*

$$\{0\} = M_0 \subset M_1 \subset \cdots \subset M_r = \theta_t \Delta(\overline{\gamma})$$

such that $M_i/M_{i-1} \cong \Delta(\overline{\mu_i})$. In this case $\mu_1, \ldots, \mu_r$ denote the elements of the set $\{\mu \in \Lambda_n \mid \mu t \gamma \text{ oriented}\}$ ordered such that $\mu_j > \mu_i$ implies $j < i$.

(ii) *The module is nonzero if and only if no cup of $t\gamma$ contains more dots than cups and for every cup there has to be a dot directly to its left or right (these are chosen pairwise distinct for all cups).*

(iii) *Assuming* (ii), *the module $\theta_t \Delta(\overline{\gamma})$ is indecomposable with irreducible head $L(\overline{\lambda})$, where $\overline{\lambda}$ is given by the upper reduction of $t\overline{\gamma}$.*

Proof. The module $\Delta(\overline{\gamma})$ is the quotient of $P(\overline{\gamma})$ by the subspace spanned by all oriented circle diagrams $\underline{\nu}\eta\overline{\gamma}$ with $\eta \neq \gamma$. Therefore, $\theta_t \Delta(\overline{\gamma}) = G_t \otimes_{\mathbb{K}_n} \Delta(\overline{\gamma})$ is obtained as the quotient of $G_t e_\gamma$ by the subspace spanned by all circle diagrams $\underline{\nu}\mu t \eta \overline{\gamma}$ with $\eta \neq \gamma$. Hence, $\theta_t \Delta(\overline{\gamma})$ has a basis given by the images of $\underline{\nu}\mu t \gamma \overline{\gamma}$ under the quotient map.

Now let $M_0 = \{0\}$ and define inductively M_i to be generated by M_{i-1} and $\{\underline{\nu}\mu_i t\gamma\overline{\gamma} \mid$ for all oriented cup diagrams $\underline{\nu}\mu_i\}$. Every M_i is a $\mathbb{K}_n$-submodule by [BS24, Lemma 5.5]. Furthermore, the map

$$M_i/M_{i-1} \to \Delta(\overline{\mu_i}), \quad \underline{\nu}\mu_i t\gamma\overline{\gamma} \mapsto \underline{\nu}\mu_i\overline{\mu_i}$$

defines an isomorphism of $\mathbb{K}_n$-modules (in the definition of the map we mean the images of the elements under the quotient maps rather than the elements themselves). This proves (i).

For (ii) observe that if any of the mentioned conditions is not satisfied then there exists no oriented diagram of the form $\mu t\gamma$. Thus, in this case $\theta_t\Delta(\lambda) = 0$. For the converse observe that $\lambda t\gamma$ is oriented with λ defined as in (iii).

Now the functors θ_t are exact by Corollary 4.27, and thus $\theta_t\Delta\overline{\gamma}$ is a quotient of $\theta_t P(\overline{\gamma})$. Thus, it is either 0 or indecomposable with irreducible head $L(\overline{\lambda})$ (see Proposition 6.26). □

Proposition 6.34. *Let t be a crossingless matching and $\gamma \in \Lambda_n$.*

(i) *The $\mathbb{K}_n$-module $\theta_t\nabla(\overline{\gamma})$ has a filtration*

$$\{0\} = M_0 \subset M_1 \subset \cdots \subset M_r = \theta_t\nabla(\overline{\gamma})$$

such that $M_i/M_{i-1} \cong \nabla(\overline{\mu_i})$. In this case $\mu_1, \ldots, \mu_r$ denote the elements of the set $\{\mu \in \Lambda_n \mid \gamma t^{\ddagger}\mu \text{ oriented}\}$ ordered such that $\mu_j > \mu_i$ implies $j > i$.

(ii) *The module is nonzero if and only if each cap of $\gamma t^{\ddagger}$ contains exactly as many dots as caps and for every cap there cannot be a dot directly to its left and right.*

(iii) *Assuming* (ii)*, the module $\theta_t\nabla(\overline{\gamma})$ is indecomposable with irreducible socle $L(\overline{\lambda})$, where $\underline{\lambda}$ is given by the lower reduction of $\underline{\gamma} t^{\ddagger}$.*

Proof. First we look at the right $\mathbb{K}_n$-modules $\nabla^*(\nu) = \bar{e}_\lambda \mathbb{K}_n^{\leq\lambda}$ as in the notation of Definition 6.32. With verbatim the same proof as Proposition 6.33 we can prove the following.

(i) The right $\mathbb{K}_n$-module $\nabla^*(\overline{\gamma}) \otimes_{\mathbb{K}_n} G_{t^{\ddagger}}$ has a filtration

$$\{0\} = M_0 \subset M_1 \subset \cdots \subset M_r = \theta_t\nabla(\overline{\gamma})$$

such that $M_i/M_{i-1} \cong \nabla^*(\overline{\mu_i})$. In this case $\mu_1, \ldots, \mu_r$ denote the elements of the set $\{\mu \in \Lambda_n \mid \gamma t^{\ddagger}\mu \text{ oriented}\}$ ordered such that $\mu_j > \mu_i$ implies $j < i$.

(ii) The module is nonzero if and only if each cap of $\gamma t^{\ddagger}$ contains exactly as many dots as caps and for every cap there cannot be a dot directly to its left and right.

(iii) Assuming (ii), the right module $\nabla^*(\overline{\gamma}) \otimes_{\mathbb{K}_n} G_{t^{\ddagger}}$ is indecomposable with irreducible head $L(\overline{\lambda})$, where $\underline{\lambda}$ is given by the lower reduction of $\underline{\gamma} t^{\ddagger}$.

Now using the proof and arguments as in the proof of Theorem 6.30, we can show that for a right $\mathbb{K}_n$-module M we have

$$G_t \otimes_{\mathbb{K}_n} M^* \cong (M \otimes_{\mathbb{K}_n} G_{t^\ddagger})^*$$

where $*$ denotes the usual vector space dual that turns the right module M into a left module via $(xf)(m) = f(mx)$.
By definition, we have that $\nabla(\overline{\nu})^* = \nabla^*(\overline{\nu})$, and thus $\theta_t\nabla(\overline{\nu}) = (\nabla^*(\overline{\nu}) \otimes_{\mathbb{K}_n} G_{t^\ddagger})^*$. Under this duality the filtration is turned upside down, and thus we have proven the proposition. □

Theorem 6.35. *Let t be a generalized crossingless matching and $\lambda \in \Lambda_n$. Then*

(i) *in the Grothendieck group of* $\mathbb{K}_n$ -mod

$$[\theta_t L(\overline{\lambda})] = \sum_{\mu} [L(\overline{\mu})]$$

where we sum over all μ such that

   (i) *$t^\ddagger\overline{\mu}$ contains neither circles nor non-propagating lines ending at the top,*

   (ii) *$\overline{\lambda}$ is the upper reduction of $t^\ddagger\overline{\mu}$.*

(ii) *$\theta_t L(\overline{\lambda})$ is nonzero if and only if $t\overline{\lambda}$ has neither circles nor non-propagating lines ending at the top and $\overline{\lambda}$ is the upper reduction of $t^\ddagger t\overline{\lambda}$.*

(iii) *Under the assumptions from* (ii) *define $\overline{\nu}$ to be the upper reduction of $t\overline{\lambda}$. In this case $\theta_t L(\overline{\lambda})$ is an indecomposable module with irreducible head $L(\overline{\nu})$.*

(iv) *Under the assumptions from* (ii) *define $\underline{\nu'}$ to be the lower reduction of $\underline{\lambda}t^\ddagger$. In this case $\theta_t L(\overline{\lambda})$ is an indecomposable module with irreducible socle $L(\overline{\nu'})$.*

Proof. In order to prove (i) we compute (using the adjunction $(\theta_{t^\ddagger}, \theta_t)$ from Theorem 6.29)

$$\mathrm{Hom}_{\mathbb{K}_n}(P(\overline{\mu}), \theta_t L(\overline{\lambda})) = \mathrm{Hom}_{\mathbb{K}_n}(\theta_{t^\ddagger} P(\overline{\mu}), L(\overline{\lambda})) \tag{6.3}$$

Now $\theta_{t^\dagger} P(\overline{\mu}) \neq 0$ if and only if (i) is satisfied, in which case it is isomorphic to $P(\overline{\beta})$ where $\overline{\beta}$ denotes the upper reduction of $t^\dagger\overline{\mu}$. Thus, (6.3) is nonzero if and only if (i) and (ii) are satisfied and in which case it is isomorphic to $\mathbb{C}\langle k\rangle$.
For (ii) and (iii) note that $\theta_t L(\overline{\lambda})$ is a quotient of $\theta_t P(\lambda)$ as θ_t is exact by Corollary 6.28. But $\theta_t P(\lambda)$ is non-zero if and only if $t\overline{\lambda}$ has neither circles nor non-propagating lines ending at the top and in which case it is isomorphic to $P(\overline{\nu})$ by Proposition 6.26. Thus, $\theta_t L(\overline{\lambda})$ is either zero or has irreducible head $L(\overline{\nu})$. But $L(\overline{\nu})$ can only appear if λ is the upper reduction of $t^\ddagger\overline{\nu}$ or equivalently $t^\ddagger t\overline{\lambda}$. For (iv) note that $L(\overline{\lambda})$ is the socle of $\nabla(\lambda)$. Again using Corollary 6.28 we see that $\theta_t L(\overline{\lambda})$ is a submodule of $\theta_t\nabla(\lambda)$, which is indecomposable and has irreducible socle $L(\overline{\nu'})$ (ν' as in the statement of the lemma). So if $\theta_t L(\overline{\lambda}) \neq 0$, it has the same socle. □

7. Equivalence between $\mathbb{K}_n$-mod and $\operatorname{rep}(\mathfrak{p}(n))$

In this section we will prove the main equivalence between $\mathbb{K}_n$-mod and $\mathfrak{p}(n)$-mod. We will achieve this by identifying $\mathbb{K}_n$ with the endomorphism ring of a projective generator for $\mathfrak{p}(n)$-mod.

7.1. Generating projective objects using translation functors

Recall that we can associate to each $\mathcal{P}(\lambda)$ a cap diagram with n caps via Lemma 4.2. On the other hand $\mathcal{P}(\lambda)$ also arises as the direct summand of some (shift of) $V^{\otimes d}$, and thus as $\Theta^+_{i_k} \cdots \Theta^+_{i_1}\mathbb{C}$. Theorems 3.48 and 5.6 tell us that $\mathcal{P}(\lambda)$ is actually the image of $\hat{\theta}_{i_k} \cdots \hat{\theta}_{i_1}\hat{P}(\overline{\iota})$ in $\mathcal{F}$. But $\hat{\theta}_{i_k} \cdots \hat{\theta}_{i_1}\hat{P}(\iota)$ is isomorphic to some $\hat{P}(\overline{\nu})$ for some cap diagram $\overline{\nu}$ by Proposition 4.26. This gives us two different ways to associate a cup diagram to an integral dominant weight λ and our next goal is to show that these two notions agree (cf. (1.6)).
The next proposition will prove that the action of Θ_i on $\mathcal{P}(\lambda)$ follows the same rule as Proposition 4.26.

Proposition 7.1. *Given a dominant integral weight λ, we have*

$$\Theta_i\mathcal{P}(\lambda) = \begin{cases} 0 & \textit{if } t^i\overline{\lambda} \textit{ contains a circle or a non-propagating line,} \\ \mathcal{P}(\nu) & \textit{otherwise, where } \nu \textit{ denotes the upper reduction of } t^i\overline{\lambda}. \end{cases}$$

Proof. It suffices to match the combinatorics from Proposition 4.26 with the black dot combinatorics from [BDEA+19] For this observe that our cap diagrams are obtained from their weight diagrams by connecting pairs of $\circ\bullet$ by a cap and then shifting the total diagram $\frac{3}{2}$ to the right. The statement then follows from [BDEA+19, Lemma 7.2.1 + Lemma 7.2.3]. $\square$

Lemma 7.2. *Given a residue sequence $(i_1, \dots i_k)$ of an up-tableau of shape δ_n, then*

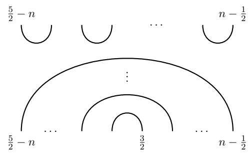
$$\Theta_{i_k} \dots \Theta_{i_1}\mathbb{C} = \mathcal{P}(n-2, n-4, \dots, -n).$$

Proof. We prove this only for the residue sequence $(0, 1, \dots, n-2, -1, \dots, n-4, \dots, -n+2, n-1, n+1, \dots, -n+1)$. This is the residue sequence that first builds δ_{n-1} and then adds the missing boxes. All the other ones are obtained via swapping entries i and j with $|i-j| > 1$, which gives isomorphic $\mathfrak{p}(n)$-modules.

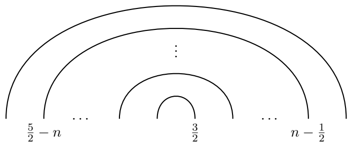
First look at the residue sequence $\boldsymbol{j} = (0, 1, \dots, n-2, -1, \dots, n-4, \dots, -n+2)$ of δ_{n-1}. We claim that $\Theta_{\boldsymbol{j}}\mathbb{C} = \nabla(0, -1, \dots, -n+1) = \mathcal{L}(n-1, n-3, \dots, -n+1)$. Using the adjunction (Θ_{i+1}, Θ_i) we find

$$\begin{aligned}[\Theta_{\boldsymbol{j}}\mathbb{C} : \mathcal{L}(\mu)] &= \dim \operatorname{Hom}_{\mathfrak{p}(n)}(\mathcal{P}(\mu), \Theta_{\boldsymbol{j}}\mathbb{C}) \\ &= \dim \operatorname{Hom}_{\mathfrak{p}(n)}(\Theta_{\boldsymbol{j}'}\mathcal{P}(\mu), \mathcal{L}(n-1, n-2, \dots, 0))\end{aligned}$$

where $\boldsymbol{j}' = (-n+3, -n+5, -n+4, -n+7, -n+6, -n+5, \dots, n-1, \dots, 1)$. This means that we need to find all μ such that $\Theta_{\boldsymbol{j}'}\mathcal{P}(\mu) \cong \mathcal{P}(n-1, n-2, \dots, 0)$. Note that the (reduction of the) generalized crossingless matching looks like:



The cap diagram associated to $\mathcal{P}(n-1, n-2, \dots, 0)$ has right endpoints of caps at positions $\{n+\frac{1}{2}, n-\frac{1}{2}, \dots, \frac{3}{2}\}$, hence its cap diagram looks like



Thus, there exists only one μ such that the upper reduction process returns the cap diagram of $(n-1, n-2, \dots, 0)$. Namely, the cap diagram where the right endpoints of caps are given by $\{n+\frac{1}{2}, n-\frac{1}{2}, \dots, -n+\frac{5}{2}\}$. And this is associated with $\mu = (n-1, n-3, \dots, -n+1)$ so $\Theta_{\boldsymbol{j}}\mathbb{C} = \mathcal{L}(n-1, n-3, \dots, -n+1) = \nabla(n-1, n-3, \dots, -n+1)$ as claimed.
For the next step we prove that

$$\Theta_{-n+1}\Theta_{-n+3}\cdots\Theta_{n-1}\nabla(n-1, n-3, \dots, -n+1) = \mathcal{P}(n-2, n-4, \dots, -n).$$

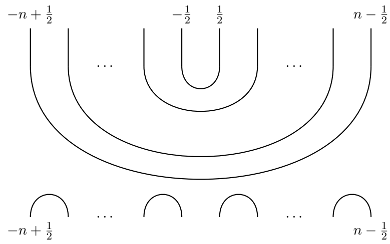
By [BDEA+19, Prop. 5.2.2] we know that

$$\Theta_{n-1-2i}\nabla(n-2, n-4, \dots, n-2i, n-2i-1, n-2i-3, \dots, -n+1)$$

has a quotient isomorphic to $\nabla(n-2, n-4, \dots, n-2i, n-2i-2, n-2i-3, \dots, -n+1)$. But repeating this step we see that $\Theta_{-n+1}\Theta_{-n+3}\cdots\Theta_{n-1}\nabla(n-1, n-3, \dots, -n+1)$ has a quotient isomorphic to $\nabla(n-2, n-4, \dots, -n) = \mathcal{L}(n-2, n-4, \dots, -n)$.
Furthermore, $\Theta_{-n+1}\Theta_{-n+3}\dots\Theta_{n-1}\nabla(0, -1, \dots, -n+1)$ is projective by Theorem 3.53. Hence, it has to be isomorphic to $\mathcal{P}(n-2, n-4, \dots, -n)$. □

Proposition 7.3. *The diagram* (1.6) *from the introduction commutes.*

Proof. Observe that the cap diagram associated to the dominant integral weight $(n-2, n-4, \ldots, -n)$ has n caps and the positions of their right endpoints are $n - 2i + \frac{3}{2}$ for $1 \leq i \leq n$. On the other hand to a residue sequence $(i_1, \ldots, i_k)$ of $\delta_n = (n, n-1, \ldots, 1)$ we associate the cap diagram that is obtained by the upper reduction $\overline{\nu}$ of $t^{i_k} \cdots t^{i_1} \overline{\iota}$. Now $t^{i_k} \cdots t^{i_1} \overline{\iota}$ looks (topologically) like the following.



Hence, the right endpoints of the caps in $\overline{\nu}$ are exactly at positions $-n+\frac{3}{2}, -n+\frac{5}{2}, \ldots, n-\frac{1}{2}$, which agrees with the cup diagram associated to $(n-2, n-4, \ldots, -n)$. Therefore, for this particular weight the two notions agree and using Proposition 7.1 we see that these notions agree for all dominant integral weights. Hence, we obtain the commutativity of (1.6) from the introduction. $\square$

7.2. Dimension of homomorphism spaces

When proving the main theorem we will need some estimate on the dimensions of homomorphism spaces to conclude that the ideal $\mathbb{I}_n$ is big enough. The next theorem provides this estimate. Note that we only state one inequality here as it suffices for the proof of the main theorem but from the main theorem it is clear that it is actually an equality.

Theorem 7.4. *We have the following equality.*

$$\dim \operatorname{Hom}_{\mathfrak{p}(n)}(\mathcal{P}(\lambda), \mathcal{P}(\mu)) \geq \dim e_\lambda \mathbb{K}_n e_\mu \tag{7.1}$$

Proof. Note that the right-hand side is given by circle diagrams with n cups and caps that have no non-propagating line. First assume that μ is very typical, i.e. $\mu_i \geq \mu_{i+1} + 4$. This means that the associated cap diagram contains no nested caps and all caps have by assumption at least two rays between them. Therefore, any orientable circle diagram necessarily looks locally like one of the following two

and as all caps are far apart from one another these local picture do not interact. This gives rise to 2^n valid orientable circle diagrams and the cup diagrams are associated to all weights of the form $(\mu_1 + 2\varepsilon_1, \ldots, \mu_n + 2\varepsilon_n)$ where $(\varepsilon_1, \ldots, \varepsilon_n) \in \{0,1\}^n$.
On the other hand by the proof of [BDEA+19, Proposition 8.1.1] we know that

$$\dim \operatorname{Hom}_{\mathfrak{p}(n)}(\mathcal{P}(\lambda), \mathcal{P}(\mu)) = |\blacktriangle(\mu) \cap \blacktriangledown(\lambda)| \tag{7.2}$$

in their notation. As λ is also a typical weight, it is clear from their definition that $\blacktriangledown(\lambda)$ contains all weights of the form $(\lambda_1 - 2\varepsilon_1, \ldots, \lambda_n - 2\varepsilon_n)$ as well as $\mu \in \blacktriangle(\mu)$. So we proved in this case that the inequality holds.
If μ is not very typical, we can use translation functors and [BDEA+19, Theorem 7.1.1] to obtain $\mathcal{P}(\mu)$ as $\Theta_{i_k} \cdots \Theta_{i_1} \mathcal{P}(\mu')$ where μ' is a very typical weight. Then we have

$$\begin{aligned}
&\dim \operatorname{Hom}_{\mathfrak{p}(n)}(\mathcal{P}(\lambda), \mathcal{P}(\mu)) \\
&= \dim \operatorname{Hom}_{\mathfrak{p}(n)}(\mathcal{P}(\lambda), \Theta_{i_k} \cdots \Theta_{i_1} \mathcal{P}(\mu')) \\
&= \dim \operatorname{Hom}_{\mathfrak{p}(n)}(\Theta_{i_1+1} \cdots \Theta_{i_k+1} \mathcal{P}(\lambda), \mathcal{P}(\mu')) \\
&= \dim \operatorname{Hom}_{\mathfrak{p}(n)}(\mathcal{P}(\lambda'), \mathcal{P}(\mu')) \\
&\geq \dim \operatorname{Hom}_{\mathbb{K}_n}(P(\overline{\lambda'}), P(\overline{\mu'})) \\
&= \dim \operatorname{Hom}_{\mathbb{K}_n}(\theta_{i_1+1} \cdots \theta_{i_k+1} P(\overline{\lambda}), P(\overline{\mu'})) \\
&= \dim \operatorname{Hom}_{\mathbb{K}_n}(P(\overline{\lambda}), \theta_{i_k} \cdots \theta_{i_1} P(\overline{\mu'}))
\end{aligned}$$

where we used the adjunction (Θ_i, Θ_{i-1}) (resp. (θ_i, θ_{i-1})) and that translation functor act topologically on projective modules by Proposition 7.1 and Proposition 6.26. $\square$

Remark 7.5. In fact, in Theorem 7.4 equality holds. This will follow from Theorem 7.6 below. We will however use this inequality in the proof of Theorem 7.6 to argue that the ideal $\mathbb{I}_n$ is "big enough".

7.3. The equivalence

Theorem 7.6 (Main theorem)**.** *There is an equivalence of categories*

$$\Psi\colon \mathbb{K}_n\text{-mod} \to \operatorname{rep}(\mathfrak{p}(n)) \tag{7.3}$$

such that $\Psi \circ \theta_i \cong \Theta_i \circ \Psi$*. Furthermore,* Ψ *identifies the highest weight structures on both sides.*

Proof. By Theorem 5.6 we know that the categories $\mathcal{F}$ and $\mathrm{sR}^\ell_\epsilon$ are isomorphic. Thus, we get an induced functor $\mathcal{F} \to \mathrm{Fund}^{\oplus}_{+}(\mathfrak{p}(n))$, where $\mathrm{Fund}^{\oplus}_{+}(\mathfrak{p}(n))$ was defined as the category with objects $\Theta_{i_k} \cdots \Theta_{i_1} \mathbb{C}$. This functor is full and by definition essentially surjective on objects, and it intertwines the translation functors. Now $\Theta_{i_k} \cdots \Theta_{i_1} \mathbb{C}$ is a direct summand of $V^{\otimes k}$ and V is a projective generator for $\mathfrak{p}(n)$. This means that every indecomposable projective module $\mathcal{P}(\lambda)$ appears as $\Theta_{i_k} \cdots \Theta_{i_1} \mathbb{C}$. By Theorem 3.25 we may assume that $(i_1, \ldots, i_k)$ is the residue sequence of some up-tableau of shape Γ. Using Proposition 3.52 and Theorem 3.53 we see that $\delta_n \subseteq \Gamma \subsetneq \delta_{n+1}$. By Proposition 4.26 we see that this corresponds to the module $\hat{P}(\overline{\nu})$ where $\overline{\nu}$ is the upper reduction of $t^{i_k} \ldots t^{i_1} \overline{\iota}$. But this is equal to $\overline{\lambda}$ (which has n caps) by Proposition 7.3, and thus we have a surjective algebra homomorphism

$$e\mathbb{K}e \cong \bigoplus_{\lambda,\mu} \operatorname{Hom}_{\mathbb{K}}(\hat{P}(\overline{\lambda}), \hat{P}(\overline{\mu})) \twoheadrightarrow \bigoplus_{\lambda,\mu} \operatorname{Hom}_{\mathfrak{p}(n)}(\mathcal{P}(\lambda), \mathcal{P}(\mu)), \tag{7.4}$$

where the sum runs over all λ and μ which are dominant integral weights for $\mathfrak{p}(n)$.
The left-hand side has a basis given by all orientable circle diagrams with exactly n cups and caps. Now by Lemma 6.3 we see that every circle diagram that has a non-propagating line factors through an object with more than n caps (which means that the shape of the corresponding partition contains δ_{n+1}). This means that its image in $\mathrm{rep}(\mathfrak{p}(n))$ is 0 by Proposition 3.52. Therefore, we get an induced surjective morphism

$$\Psi\colon \mathbb{K}_n = e\mathbb{K}e/\mathbb{I}_n \twoheadrightarrow \bigoplus_{\lambda,\mu} \mathrm{Hom}_{\mathfrak{p}(n)}(\mathcal{P}(\lambda), \mathcal{P}(\mu)). \tag{7.5}$$

Using Theorem 7.4, we see that this is actually an isomorphism. The right-hand side is now equivalent to $\mathrm{rep}(\mathfrak{p}(n))$.
The functor $\mathcal{F} \to \mathrm{Fund}_{+}^{\oplus}(\mathfrak{p}(n))$ was compatible with the translation functors by definition. But the inclusion $e\mathbb{K}e \to \mathbb{K}$ is not compatible with translation functors as these can create new cups in $\mathbb{K}$ but not in $e\mathbb{K}e$. But from Proposition 4.26 it is clear that if $\hat{\theta}_i\hat{P}(\overline{\lambda}) = \hat{P}(\overline{\mu})$, then $\overline{\mu}$ has the same number of caps as $\overline{\lambda}$ or one more. This last case is exactly the reason why the inclusion $e\mathbb{K}e \to \mathbb{K}$ is not compatible with translation functors as $e\hat{\theta}_i e$ would produce 0. But if the number of caps increases, the image in the right-hand side of (7.4) is 0, therefore (7.4) is still compatible with translation functors. So we see especially that $\Psi \circ \theta_i \cong \Theta_i \circ \Psi$. Finally, it is compatible with the highest weight structures as the combinatorics describing multiplicities of standard and costandard modules in projectives agree. $\square$

8. Applications and consequences

8.1. Duals of irreducible modules

Theorem 8.1. *Up to parity shift we have $\mathcal{L}(\lambda)^* \cong \mathcal{L}(\lambda^\#)$, where $\underline{\lambda^\#} = (\overline{\lambda}^*)$, i.e. it is obtained from λ by rotating its cap diagram around $\frac{1}{2}$.*

Proof. We begin by recalling a combinatorial procedure from [BDEA$^+$19, Proposition 5.3.1]. Namely, they look at a (ρ-shifted) dominant integral weight $\lambda_1 > \lambda_2 > \cdots > \lambda_n$. Then they ordered the set (i,j) for i,j lexicographically and then look iteratively at the pairs (i,j), starting with $(1,2)$ and ending with $(n-1,n)$. In each step one creates a new weight as follows. If $|\lambda_{k+1} - \lambda_k| > 1$ for $k = i,j$, we increase λ_i and λ_j by 1, otherwise we change nothing. Using this iterative procedure we receive a new weight diagram $\lambda^\dagger$. They prove then that $\mathcal{L}(\lambda)^*$ has highest weight given by the reflection of $\lambda^\dagger$ at $\frac{n-1}{2}$.
We will prove now that this procedure gives the same result as rotating the associated cap diagram around $\frac{1}{2}$.
We will prove this via induction on n. If $n = 1$, the procedure from [BDEA$^+$19] sends a weight (k) to $(-k)$. We would associate to (k) the cap diagram with one cap that has endpoints $k + \frac{1}{2}$ and $k + \frac{3}{2}$. Rotating this around $\frac{1}{2}$ gives the cup diagram with one cup that has endpoints at $-k + \frac{1}{2}$ and $-k - \frac{1}{2}$. The right endpoint of this cup is at $-k + \frac{1}{2}$ and when translating back into weight diagrams we have to shift by $-\frac{1}{2}$, which results in $(-k)$. Observe here that our cup diagrams are shifted with respect to the cap diagrams, i.e. when translating a weight into a cap diagram we shift by $\frac{3}{2}$ and by $\frac{1}{2}$ for cup diagrams.
Now let $n > 1$ and assume that the statement has been proven for all smaller values. We have two different cases to distinguish, either we have one big outer cap containing all smaller caps or we have multiple outer caps. Suppose first that we have multiple outer caps. We denote the rightmost outer cap and all the caps contained in it by C_1 and the rest by C_2 and let d_1 be the number of caps in C_1 and d_2 the same for C_2. Then the rotation is the same as rotating each of these components separately around $\frac{1}{2}$. We claim now that the same holds true for the procedure of [BDEA$^+$19]. For the procedure of [BDEA$^+$19] we have to evaluate all the pairs (i,j). We first only have pairs where both belong to C_1, then mixed terms and in the end all the ones from C_2. We claim that any of the mixed terms changes the weight non-trivially. This holds true as C_1 is given by an outer cap. This means that for every right endpoint there is also one left endpoint before C_2. Hence, in terms of weights that there is enough space in between so that every of the mixed pairs is non-trivial. But therefore we first shift everything according to C_1, then we have the non-trivial mixed pairs, which shift C_1 to the right d_2 steps and C_2 by d_1. And then we shift everything according to C_2. After this we reflect at $\frac{n-1}{2} = \frac{d_1+d_2-1}{2}$.

But shifting first by d_2 and then reflecting at $\frac{d_1+d_2-1}{2}$ is the same as reflecting at $\frac{d_1-1}{2}$ and similar for d_1. But this means that the procedure of [BDEA+19] splits this into two, applies their procedure there and glues them back together. Buy induction we know that for the shorter weights this procedure is the same as rotating around $\frac{1}{2}$ and so it also agrees for the long one.
The last case is that we have one big outer cap, and we want to relate the procedure here to the one for the big cap removed. So we compare the procedure for the weight $(\lambda_1, \ldots, \lambda_n)$ with $(\lambda'_2, \ldots, \lambda'_n)$ We denote the pairs for the smaller component by (i', j') starting from $(2', 3')$. We first look at the pairs $(1, i)$. Note first that $(1, 2)$ is trivial. But this means that the pair $(1, i)$ is non-trivial if and only if the pair $(2', i')$ is. Therefore, the λ_1 is in the end exactly one higher as the integer obtained for λ'_2. Furthermore, we know that $1 \leq |\lambda_2 - \lambda_3| \leq 2$ as the cap diagram is completely nested. If $|\lambda_2 - \lambda_3| = 2$ the pair $(1, 3)$ non-trivial, and thus after executing all pairs $(1, i)$ we have $|\lambda_2 - \lambda_3| = 1$. But then we see that $(2, 3)$ is trivial and the other $(2, i)$ are non-trivial if and only if $(3', i')$ is non-trivial. Therefore, λ_2 ends up exactly one higher than λ'_3 would. Repeating this argument over and over we see that λ_i for $i < n$ end up one bigger than λ'_{i+1} and λ_n lies directly next to λ_{n-1}. Reflecting λ_i for $i < n$ at $\frac{n-1}{2}$ is the same as reflecting λ'_{i+1} at $\frac{n-2}{2}$ and the additional λ_n is after reflection directly to the right of λ_{n-1}. On the other hand rotating this cap diagram means rotating the inner part and adding a right endpoint of a cup directly to the right of everything. Thus, we can apply our induction hypothesis and see that both procedures give the same result. □

Remark 8.2. There is no contravariant duality d on $\mathfrak{p}(n)$ preserving irreducibles and satisfying $d\theta_i \cong \theta_i d$. For instance in $\mathfrak{p}(1)$:

$$\begin{aligned} \theta_1\theta_0 d\mathbb{C} &= \theta_1\theta_0\mathbb{C} = \mathcal{P}(0) \\ d\theta_1\theta_0\mathbb{C} &= d\mathcal{P}(0) = \mathcal{I}(0) = \mathcal{P}(-2), \end{aligned}$$

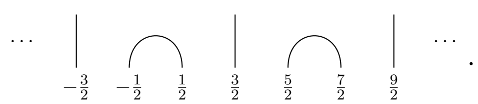
which are not isomorphic.

Remark 8.3. We do not expect a direct geometric realization of $\mathrm{rep}(\mathfrak{p}(n))$ in contrast to ordinary Brauer algebras (see e.g. [SW19]) since there one would expect such a (Verdier) duality.

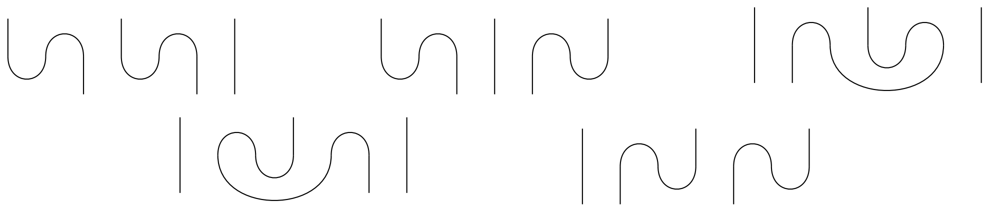
In many related representation theoretic contexts, a geometric interpretation implies Koszulity, [BGS96], which we will see now also fails for $\mathfrak{p}(n)$.

8.2. Gradings and Non-Koszulity

Let $n = 2$ and consider the indecomposable projective $\mathfrak{p}(2)$-module P corresponding to the cap diagram



By Theorem 7.6 we can carry out the computation inside $\mathbb{K}_n$ and $P \cong \mathcal{P}(2,-1) \cong \Delta(2,-1)$ has a basis given by



Using this and the multiplication rule inside $\mathbb{K}_n$, it is not hard to deduce the structure of P. Namely, one gets for the radical and socle filtration

$$\begin{array}{ccc} & \mathcal{L}(2,-1) & \\ \mathcal{L}(4,-1) & & \mathcal{L}(3,2) \\ & & \mathcal{L}(2,1) \\ & \mathcal{L}(4,1) & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} & \mathcal{L}(2,-1) & \\ & & \mathcal{L}(3,2) \\ \mathcal{L}(4,-1) & & \mathcal{L}(2,1) \\ & \mathcal{L}(4,1) & \end{array}$$

respectively. In particular, these do not agree. By [BGS96, Proposition 2.4.1], we see that $\mathbb{K}_n$ cannot be non-negatively graded such that $(\mathbb{K}_n)_0$ is semisimple and $\mathbb{K}_n$ is generated by $(\mathbb{K}_n)_1$. For $n > 2$, the same phenomenon occurs when considering e.g. $\mathcal{P}(2,-1,-5,-9,-13,\dots)$.

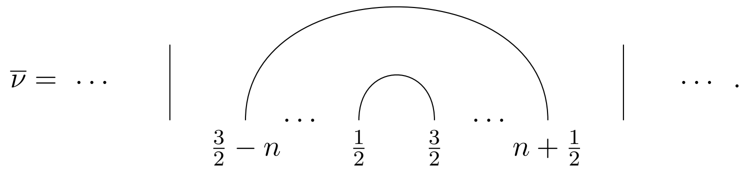
Hence, $\mathfrak{p}(n)$ does not admit a Koszul grading for $n \geq 2$.

8.3. Irreducible summands of $V^{\otimes d}$

We want to defer our attention now to irreducible modules appearing as direct summands of $V^{\otimes d}$. For $\mathfrak{gl}(m|n)$) and $\mathfrak{osp}(r|2n)$, there is a rather rich class of irreducible summands, the so-called *Kostant modules*, see e.g. [Hei17] and [HNS24] respectively. In particular, there is exactly one irreducible summand for every block. The situation is similar for $\mathfrak{p}(n)$ in the sense that we will show that there exist exactly n irreducible summands, each belonging to a unique one of the $n+1$ blocks. However, representations of $\mathfrak{gl}(m|n)$) and $\mathfrak{osp}(r|2n)$ decompose into very small blocks, giving rise to many Kostant modules, contrasting the n for $\mathfrak{p}(n)$.

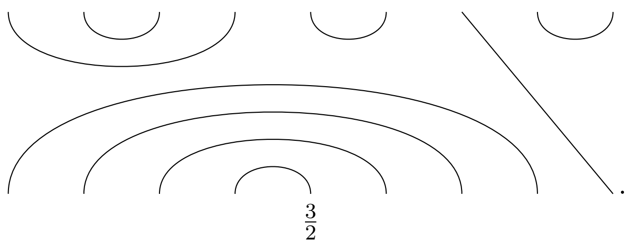
Theorem 8.4. *The irreducible $\mathfrak{p}(n)$-module $\mathcal{L}(\mu)$ appears as a summand of $V^{\otimes d}$ for some d if and only if $\mu = (n-1, n-2, \dots, k, k-2, k-4, \dots, -k)$ for some $0 \leq k \leq n-1$. Moreover, this summand appears as the image of the indecomposable object δ_k of* $\mathrm{Rep}(P)$ *under Schur–Weyl duality.*

Proof. Denote by $\overline{\nu}$ the cap diagram corresponding to the (ρ-shifted) highest weight $(n-1, n-2, \ldots, 0)$ of the trivial module. Then,



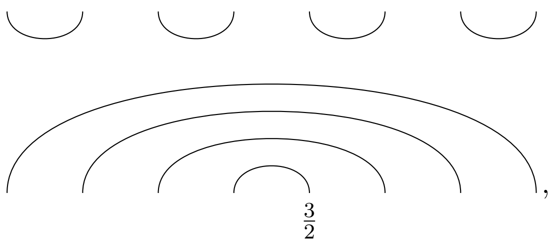
We have to find all generalized crossingless matchings t such that $\theta_t L(\overline{\nu})$ is irreducible. By means of Theorem 6.35, for any of these t there exists exactly one $\overline{\mu}$ such that $t^{\ddagger}\overline{\mu}$ is orientable and $\overline{\nu} = \mathrm{upp}(t^{\ddagger}\overline{\mu})$.

Hence, $t^{\ddagger}$ has to look something like



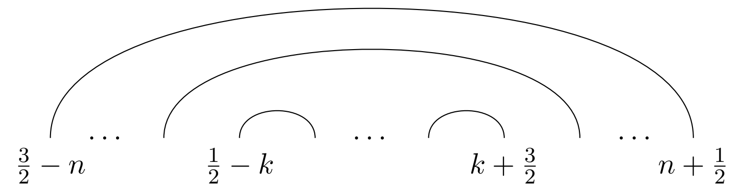
The condition $\overline{\nu} = \mathrm{upp}(t^{\ddagger}\overline{\mu})$ implies that all the caps are nested with the right endpoint of the innermost cap at $\frac{3}{2}$.

If there are any nested cups at the top, there cannot exist exactly one $\overline{\mu}$ because we could either connect the two left or right endpoints with a cup in $\overline{\mu}$ and can extend these to cap diagrams satisfying the conditions above. We also cannot have a line in between two cups, as we can connect this line either to the left or right cup with a cap in $\overline{\mu}$ and extend to two different admissible cap diagrams. But this means that $t^{\ddagger}$ is of the form



where the only freedom is given by the number of nested caps k. If $k > n$ the result is $\theta_t L(\overline{\nu}) = 0$ as $t^{\ddagger}\overline{\mu}$ will always contain a non-propagating line. If $k = n$, Lemma 7.2

computed $\theta_t L(\overline{\nu})$ to be an indecomposable projective module and not irreducible. In all other cases (i.e. $0 \leq k \leq n-1$) the result is irreducible and given by the cap diagram

$\overline{\mu_k} =$ 

corresponding to the weight $(n-1, n-2, \ldots, k, k-2, k-4, \ldots, -k)$ as claimed. $\square$

8.4. Extensions between irreducibles and Ext-quivers

The main goal of this section is to compute the dimension of

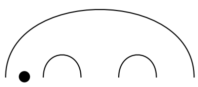
$$\mathrm{Ext}_{\mathfrak{p}(n)}(\mathcal{L}(\lambda), \mathcal{L}(\mu)).$$

Using a projective resolution of $\mathcal{L}(\lambda)$, it is easy to see that it is at most one dimensional as $\mathcal{P}(\lambda)$ is multiplicity free. Furthermore, it is one dimensional if and only if $\mathcal{L}(\mu)$ appears in the head of the radical of $\mathcal{P}(\lambda)$. The computation of the head of the radical of $\mathcal{P}(\lambda)$ can be done in $\mathbb{K}_n$-mod by Theorem 7.6, meaning that we need to determine all $\underline{\mu}\overline{\lambda}$ (with $\mu \neq \lambda$) such that

$$\underline{\mu}\overline{\lambda} = \underline{\mu}\overline{\kappa} \cdot \underline{\kappa}\overline{\lambda} \quad \Longrightarrow \quad \kappa \in \{\mu, \lambda\}. \tag{8.1}$$

Definition 8.5. An orientable circle diagram $\underline{\mu}\overline{\lambda}$ with $\mu \neq \lambda$ is called Δ-*primitive* if $\underline{\mu}\overline{\lambda}$ a Δ-orientation such that

(Δ1) There is at most one cap in $\overline{\lambda}$ such that its corresponding dot is not at the rightmost possible position,

(Δ2) it is  avoiding, and

(Δ3) it is  avoiding.

It is called ∇-primitive if $\underline{\mu}\lambda$ is a ∇-orientation such that

(∇1) There is at most one cup in $\overline{\lambda}$ such that its corresponding dot is not at the rightmost possible position, and

(∇2) it is  avoiding, where the outer cup is not contained in any other cup.

We call $\underline{\mu}\overline{\lambda}$ *primitive* if it is either Δ- or ∇-primitive.

Theorem 8.6. *Let $\underline{\mu}\overline{\lambda}$ be an orientable circle diagram. Then $\underline{\mu}\overline{\lambda}$ satisfies* (8.1) *if and only if it is primitive.*

Proof. First we prove that every primitive circle diagram satisfies (8.1) by contraposition. For this consider an oriented circle diagram

$$\underline{\mu}\delta\overline{\lambda} = \underline{\mu}\nu\overline{\eta} \cdot \underline{\eta}\kappa\overline{\lambda}$$

that violates (8.1). We have to show that $\underline{\mu}\delta\overline{\lambda}$ is neither Δ- nor ∇-primitive.
By Theorem 6.17(i) we know that

$$\underline{\mu}\delta\overline{\lambda} = \underline{\mu}\nu\overline{\nu} \cdot \underline{\nu}\kappa'\overline{\lambda}$$

for some κ'. Now using Theorem 6.17(ii) we know that $\kappa' \geq \lambda$, $\nu \geq \mu$ and $\delta \geq \kappa', \nu$ (this last inequality uses [BS24, Lemma 5.5]). If $\kappa' \neq \lambda$ and $\nu \neq \mu$ we have $\delta \neq \mu, \lambda$, and thus it can neither be Δ- nor ∇-primitive as any orientation is unique by Lemma 6.14. So either $\kappa' = \lambda$ or $\nu = \mu$.

- Suppose that $\kappa' = \lambda$ and $\nu \neq \mu$. This means we have

  $$\underline{\mu}\delta\overline{\lambda} = \underline{\mu}\nu\overline{\nu} \cdot \underline{\nu}\lambda\overline{\lambda}.$$

  Observe furthermore that λ,ν,and μ are pairwise distinct by assumption, and thus $\delta \geq \lambda > \nu > \mu$ by Theorem 6.17(ii).

  As $\delta > \mu$ this cannot be Δ-primitive. If δ and μ differ by at least two dots it cannot be ∇-primitive as it violates (∇1). If they differ by exactly one dot, observe that this dot is uniquely defined (there will be a sequence of neighbored cups that have their associated dots in between them (so one is missing), the additional dot has to be to the left of this sequence). In case this sequence contains more than one cup, we violate (∇1). If this sequence contains exactly one cup, we arrive at a contradiction as there cannot exist a $\mu < \nu < \delta$.

So we may assume that $\nu = \mu$, i.e. we are reduced to the case

$$\underline{\mu}\delta\overline{\lambda} = \underline{\mu}\mu\overline{\eta} \cdot \underline{\eta}\kappa\overline{\lambda}$$

Using again Theorem 6.17(i) we have

$$\underline{\mu}\delta\overline{\lambda} = \underline{\mu}\eta'\overline{\kappa} \cdot \underline{\kappa}\kappa\overline{\lambda}$$

for some η'. By the same reasoning as above we have $\delta \geq \eta', \kappa$, $\eta' \geq \mu$ and $\kappa \geq \lambda$. Thus, if $\eta' \geq \mu$ and $\kappa \geq \lambda$ the circle diagram cannot be primitive. So either $\eta' = \mu$ or $\kappa = \lambda$.

- Suppose that $\eta' = \mu$ and $\kappa \neq \lambda$. This means we have

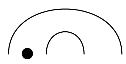
$$\underline{\mu}\delta\overline{\lambda} = \underline{\mu}\mu\overline{\kappa} \cdot \underline{\kappa}\kappa\overline{\lambda}$$

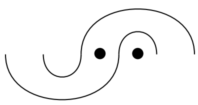
  Observe furthermore that λ, κ and μ are pairwise distinct, and thus $\delta \geq \mu > \kappa > \lambda$ by Theorem 6.17(ii). Therefore, this cannot be ∇-primitive. If δ and λ differ by at least two dots, it violates (Δ1), so it cannot be Δ-primitive. If they differ by exactly one dot look at the cap in $\overline{\lambda}$ that corresponds to this dot. As $\delta > \kappa > \lambda$, this dot has to move two times along a valid Δ-orientaion, and thus it violates (Δ2) and cannot be Δ-primitive.

Thus, we can also assume that $\kappa = \lambda$, meaning that we are reduced to showing that

$$\underline{\mu}\delta\overline{\lambda} = \underline{\mu}\mu\overline{\eta} \cdot \underline{\eta}\lambda\overline{\lambda}$$

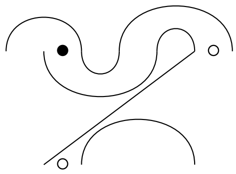
can be neither Δ- nor ∇-primitive. Note that by assumption $\eta \neq \lambda, \mu$ and furthermore $\mu \neq \lambda$ as otherwise it is not primitive by definition. Furthermore, if $\delta \neq \lambda, \mu$ it cannot be primitive by definition.

- Suppose that $\delta = \lambda > \mu$. In this case $\underline{\mu}\overline{\lambda}$ cannot be Δ-primitive. We may also assume that it satisfies (∇1) as otherwise it cannot be ∇-primitive either. This means that in the multiplication process no dot in the top number line moves. As $\mu \neq \eta$ we find a situation  in $\underline{\mu}\overline{\eta}$. This means that considering the total diagram $\underline{\mu}\mu\overline{\eta}$ it looks like



  We may assume that this outer cup is not contained in any other cup, as otherwise we would find the same pattern for the cup that contains this. Now consider the surgery procedure involving the outer cap. This cannot be of the second kind from Figure 6.4 as the inner cap afterward would be of the first kind with the additional white dots. This would mean that a dot in the top number line moves which contradicts our assumption. Therefore, it has to be of the third kind, which means that the left dot moves to the other side of the inner cup (and stays there). But then $\underline{\mu}\lambda\overline{\lambda}$ violates (∇2).

- Suppose that $\delta = \mu > \lambda$. In this case $\underline{\mu}\overline{\lambda}$ cannot be ∇-primitive. We may also assume that it satisfies (Δ1) and (Δ2) as otherwise it cannot be ∇-primitive either. This means that in the multiplication process no dot in the bottom number line moves. As $\lambda \neq \eta$ there exists a subpicture like  where there appears not dot to the right of this cup. Observe now that no surgery procedure of the third kind from Figure 6.4 can occur, as no dot in the bottom number line can move by assumption. But the other two kinds of straightenings require a dot to the right of the cup. This means that the one dot difference between λ and μ has to move via a surgery procedure of the first kind (as a white dot) and move to the right of this cup. Therefore, our situation looks like



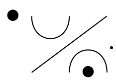
Now the surgery procedure of the first kind moves the white dot to the other side of the cap (and the dot stays there), so that $\underline{\mu}\mu\overline{\lambda}$ violates (Δ3), therefore it cannot be Δ-primitive.

This finishes the first half of the proof. We are left to show that any circle diagram satisfying (8.1) is primitive. We prove this also by contraposition, i.e. we show that any non-primitive diagram does not satisfy (8.1). For this, we split the proof into three different cases.

(i) Suppose that λ is not a ∇-orientation of $\underline{\mu}$ and μ is not a Δ-orientation of $\overline{\lambda}$. Now let η be the orientation of $\underline{\mu}\eta\overline{\lambda}$. By our assumption we have $\eta \neq \lambda, \mu$. By Theorem 6.17(i) we have
$$\underline{\mu}\eta\overline{\lambda} = \underline{\mu}\eta\overline{\eta} \cdot \underline{\eta}\eta\overline{\lambda}$$
violating (8.1).

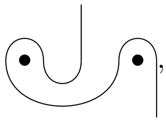
(ii) Suppose that λ is a valid ∇-orientation of $\underline{\mu}$.

(a) Suppose first that this violates (∇1). Let κ be the same weight as λ except that we replace the rightmost dot of λ that corresponds to a left endpoint of a cup with the dot to the right of the same cup. By construction $\underline{\kappa}\lambda\overline{\lambda}$ is oriented (and only one dot in κ and λ differ). Furthermore, $\underline{\mu}\kappa\overline{\kappa}$ is also oriented and κ and μ have one dot more in common than λ and μ. We claim that
$$\underline{\mu}\kappa\overline{\kappa} \cdot \underline{\kappa}\lambda\overline{\lambda} = \underline{\mu}\lambda\overline{\lambda}$$
For this it suffices to show that any appearing surgery procedure is a straightening. First look at the surgery procedures that involve a cup that has a dot to its right. This means there has to be a cap above this dot and because the orientation of $\underline{\kappa}\overline{\lambda}$ is λ, we have to have a straightening. There is exactly one surgery procedure without a dot to the right of the cup. The situation then looks like

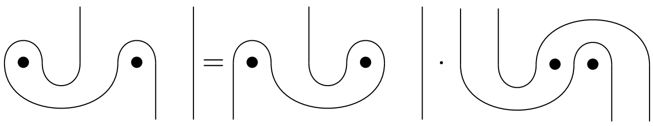


By construction κ and λ agree up to one dot (which is exactly the one depicted above), and thus all the other dots agree. But now there cannot be a dot between the line and the cap as it would have to correspond to one to the right of the line at the top number line, which would mean that κ and λ differ by at least two dots. Additionally, by construction (as the dot on the top is to the left of the cup) the dot on the bottom has to be to the right of a cup. As there is no dot between the cap and the line, this means that this cup connects the line with the cap, meaning that we have a straightening. Thus, the multiplication is nonzero, therefore violating (8.1).

(b) Suppose now that (∇1) holds but (∇2) is violated. Then our situation looks like



and it is easy to check that



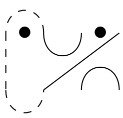
provides a counterexample to (8.1).

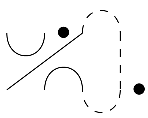
(iii) Suppose now that μ is a valid Δ-orientation of $\overline{\lambda}$

(a) First assume that (Δ1) is violated. As (Δ1) does not hold, there exists a dot in μ that is not at the right end below the corresponding cap C in $\overline{\lambda}$. Of these dots we choose the rightmost one such that any caps contained in C have their dot at the rightmost position. Let κ be the same as μ except that we replace this dot by the rightmost position in C. By (Δ1) we have $\kappa \neq \lambda, \mu$. The weight κ is a Δ-orienation of $\overline{\lambda}$ by construction, and thus $\underline{\kappa}\kappa\overline{\lambda}$ is oriented. Furthermore, our choice ensures that μ is a valid Δ-orientation of $\overline{\kappa}$, and thus $\underline{\mu}\mu\overline{\kappa}$ is oriented. We claim now that

$$\underline{\mu}\mu\overline{\kappa} \cdot \underline{\kappa}\kappa\overline{\lambda} = \underline{\mu}\mu\overline{\lambda}$$

For this we show that no split or reconnect can occur as a surgery procedure. If a split occurred, this would look like one of the following two possibilities:

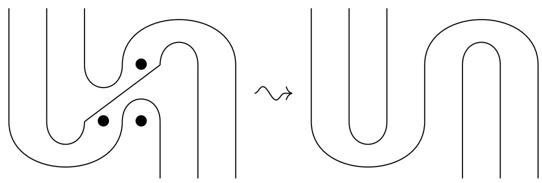




The first case cannot appear as every dot is to the right of a cup by assumption. This means that the left dot in the first case cannot be matched with the cup to its right, so it has to be matched to a dot at the bottom number line. This would be contained in the dashed cup, which is not possible by definition of orientation.

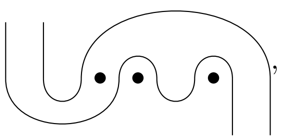
The second case also cannot appear, as the bottom dot is not contained in a cap, and thus has to be matched to a dot at the top. But then these two dots at the top need to lie inside the same cap, which contradicts the definition of orientation. Therefore, there cannot appear any split.

Now if a reconnect would appear the situation would look like

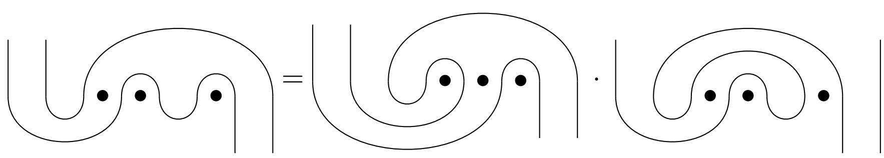


But any further surgery procedure that we could apply (even though it is already not orientable) would preserve at least one of the non-propagating lines (and as this number has to be even), this would imply that $\underline{\mu}\overline{\lambda}$ contain non-propagating lines which contradicts our assumption. Therefore, no reconnect can appear. Thus, only straightenings can occur, and thus the multiplication is nonzero.

(b) Suppose now that (Δ1) holds but (Δ2) is violated. Then our situation looks like

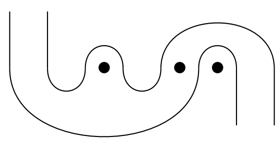


and it is easy to check that

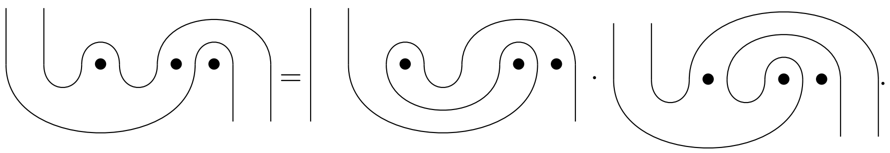


provides a counterexample to (8.1)

(c) Lastly suppose that (Δ1) and (Δ2) hold but (Δ3) does not. Then the circle diagram looks like



and a counterexample for (8.1) is given by



□

This explicit description of diagrams satisfying (8.1) gives us immediately the following corollary.

Corollary 8.7. *We have*

$$\dim \operatorname{Ext}_{\mathfrak{p}(n)}(\mathcal{L}(\lambda), \mathcal{L}(\mu)) = \begin{cases} 1 & \text{if } \underline{\mu}\overline{\lambda} \text{ is primitive,} \\ 0 & \text{otherwise.} \end{cases}$$

8.4.1. $\mathfrak{p}(1)$ as a quiver with relations

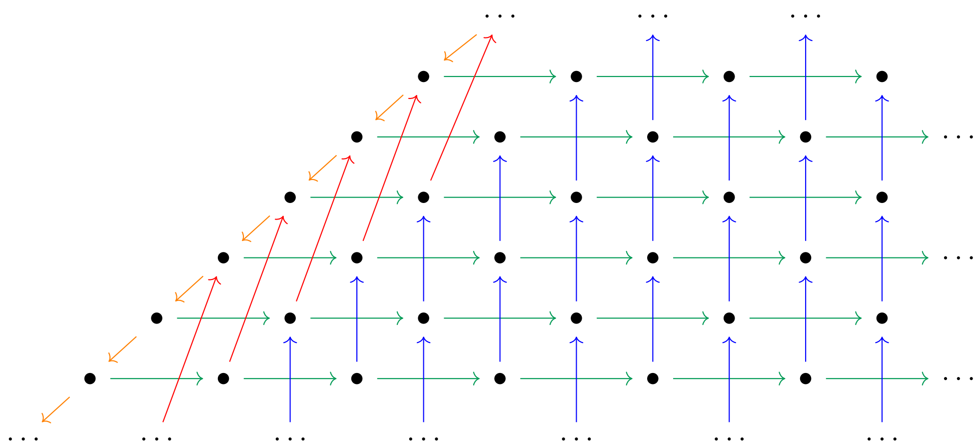
The irreducible modules of rep($\mathfrak{p}(1)$) are labelled by $\mathbb{Z}$. The category decomposes into 2 blocks B_0, B_1, where $\mathcal{L}(i) \in B_0$ if i is even and $\mathcal{L}(i) \in B_1$ if i is odd. The module $\mathcal{P}(i)$ has irreducible head $\mathcal{L}(i)$ and irreducible socle $\mathcal{L}(i+2)$ and nothing else. Both blocks can be described by the same quiver

$$\cdots \quad \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \quad \cdots$$

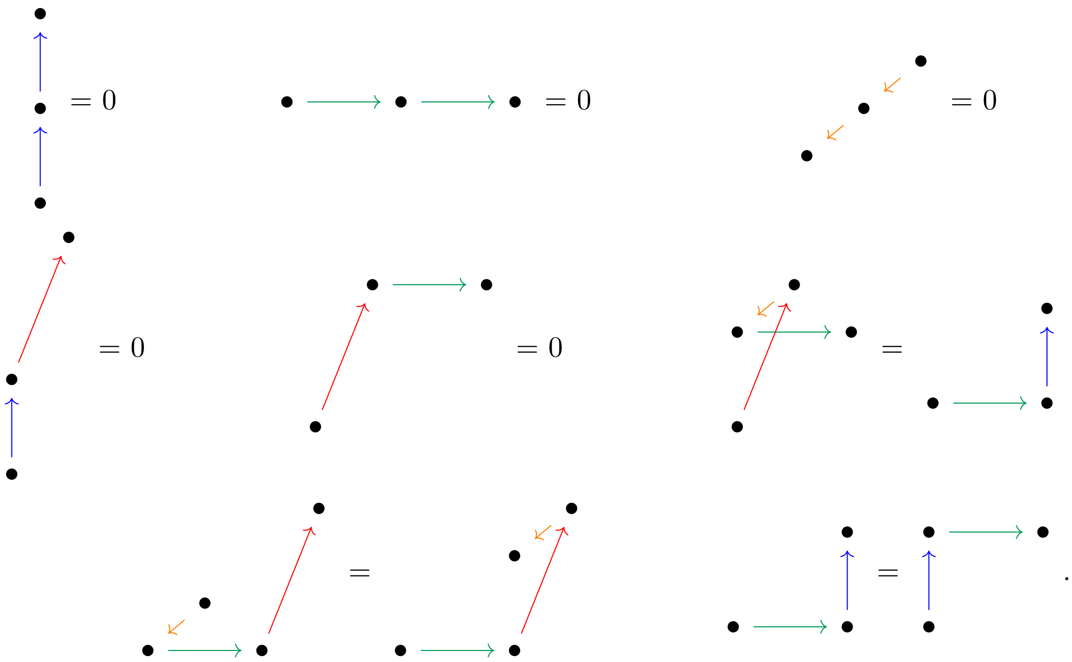
with relation $\longrightarrow \bullet \longrightarrow \; = 0.$

8.4.2. $\mathfrak{p}(2)$ as a quiver with relations

The irreducibles for $\mathfrak{p}(2)$ are labelled by two integers (i, j) with $i > j$. The category decomposes into three blocks B_0, B_1 and B_2 depending on how many odd entries there are. We only consider the block B_1 as it is the most irregular one (B_0 and B_1 can be obtained by removing the leftmost diagonal). It can be described as the quiver



with relations



Here, we also see that the last relation in the second row is not homogeneous, which reflects again the problems with the grading that occurred in Section 8.2.

9. The quantum ϵlectrical algebras and their Fock spaces

We fix from now on as ground field $\mathbb{Q}(q)$, the field of rational functions over $\mathbb{Q}$ in a variable q with its $\mathbb{Q}$-algebra involution $^{-}$ given by $q \mapsto \overline{q} := q^{-1}$. The quantum integer $[m]$ for $m \in \mathbb{Z}$ is the polynomial $[m] = \frac{q^m - q^{-m}}{q - q^1} = q^{m-1} + q^{m-3} \cdots + q^{1-m} \in \mathbb{Q}(q)$.

9.1. The quantum ϵlectrical algebras $\mathfrak{el}_q^\epsilon$

We next define a main player, the quantum ϵlectrical (or short q-ϵlectrical) algebras.

Definition 9.1. Let $\epsilon \in \{\pm 1\}$. We define the corresponding *q-ϵlectrical algebra* $\mathfrak{el}_q^\epsilon$ to be the algebra generated by $\mathcal{E}_i$, for $i \in \mathbb{Z}$, subject to the relations

$$\mathcal{E}_i \mathcal{E}_j = q^{b_{ij}} \mathcal{E}_j \mathcal{E}_i \quad \text{if } |i - j| > 1, \tag{ϵl-1}$$

$$q^3 \mathcal{E}_i^2 \mathcal{E}_{i+1} - [2] \mathcal{E}_i \mathcal{E}_{i+1} \mathcal{E}_i + q^{-3} \mathcal{E}_{i+1} \mathcal{E}_i^2 = -q^\epsilon [2] \mathcal{E}_i, \tag{ϵl-2}$$

$$q^{-3} \mathcal{E}_i^2 \mathcal{E}_{i-1} - [2] \mathcal{E}_i \mathcal{E}_{i-1} \mathcal{E}_i + q^3 \mathcal{E}_{i-1} \mathcal{E}_i^2 = -q^\epsilon [2] \mathcal{E}_i, \tag{ϵl-3}$$

where $b_{ij} = \begin{cases} -2 & \text{if } j = i, i+1, \\ 4 \cdot \operatorname{sgn}(j - i)(-1)^{j-i} & \text{otherwise.} \end{cases}$

Remark 9.2. The $b_{i,j}$ are *shift invariant*, i.e. $b_{i,j} = b_{i+1,j+1}$ and also $b_{i-1,j} = b_{j,i}$. Moreover, we have $b_{j,i} = -b_{i,j}$ if $|i - j| > 1$.

Remark 9.3. The q-ϵlectrical algebras $\mathfrak{el}_q^\epsilon$ should be seen as an analog of a special example of an electrical Lie algebra as defined e.g. in [BGG24]. In informal discussions with Azat Gainutdinov and Vassily Gorbounov we were informed that they are also working on quantum versions. Our example should arise as a special example of their construction.

Next, we define the bar involution and shift isomorphism connecting $\mathfrak{el}_q^\epsilon$ and $\mathfrak{el}_{q^{-1}}^\epsilon$.

Lemma 9.4 (Bar involution)**.** *There exists a unique q-antilinear isomorphism*

$$\overline{} : \mathfrak{el}_q^\epsilon \to \mathfrak{el}_{q^{-1}}^\epsilon, \quad \overline{\mathcal{E}_i} = \mathcal{E}_i$$

of $\mathbb{Q}$-algebras. Here, q-antilinear means $\overline{f\mathcal{E}} = \overline{f} \cdot \overline{\mathcal{E}}$ for any $f \in \mathbb{Q}(q)$, $\mathcal{E} \in \mathfrak{el}_q^\epsilon$.

Proof. Clearly, it suffices to show that the assignments are compatible with the defining relations of $\mathfrak{el}_q^\epsilon$, since then the statements follow from the definitions.
For $1 \leq i, j \leq n$ with $|i-j| > 1$ (such that the expressions make sense) we have

$$\begin{aligned}
\overline{\mathcal{E}_i\mathcal{E}_j} = \mathcal{E}_i\mathcal{E}_j = q^{-b_{ij}}\mathcal{E}_j\mathcal{E}_i = q^{-b_{ij}}\overline{\mathcal{E}_j\mathcal{E}_i} = \overline{q^{b_{ij}}\mathcal{E}_j\mathcal{E}_i},& \\
\overline{q^3\mathcal{E}_i^2\mathcal{E}_{i+1} - [2]\mathcal{E}_i\mathcal{E}_{i+1}\mathcal{E}_i + q^{-3}\mathcal{E}_{i+1}\mathcal{E}_i^2} =& q^{-3}\mathcal{E}_i^2\mathcal{E}_{i+1} - [2]\mathcal{E}_i\mathcal{E}_{i+1}\mathcal{E}_i + q^3\mathcal{E}_{i+1}\mathcal{E}_i^2 \\
=& - q^{-\epsilon}[2]\mathcal{E}_i = \overline{-q^{\epsilon}[2]\mathcal{E}_i}, \\
\overline{q^{-3}\mathcal{E}_i^2\mathcal{E}_{i-1} - [2]\mathcal{E}_i\mathcal{E}_{i-1}\mathcal{E}_i + q^3\mathcal{E}_{i-1}\mathcal{E}_i^2} =& q^3\mathcal{E}_i^2\mathcal{E}_{i-1} - [2]\mathcal{E}_i\mathcal{E}_{i-1}\mathcal{E}_i + q^{-3}\mathcal{E}_{i-1}\mathcal{E}_i^2 \\
=& - q^{-\epsilon}[2]\mathcal{E}_i = \overline{-q^{\epsilon}[2]\mathcal{E}_i}.
\end{aligned}$$

Thus, we obtain a well-defined antilinear algebra homomorphism $\mathfrak{el}_q^\epsilon \to \mathfrak{el}_{q^{-1}}^\epsilon$. □

Lemma 9.5 (Shift isomorphism)**.** *There exists a unique $\mathbb{Q}(q)$-algebra anti-isomorphism*

$$\sigma\colon \mathfrak{el}_{q^{-1}}^\epsilon \to \mathfrak{el}_q^\epsilon, \quad \sigma(\mathcal{E}_i) = q^{-\epsilon}\mathcal{E}_{i+1}.$$

Proof. Since the $\mathcal{E}_i$ are algebra generators of $\mathfrak{el}_{q^{-1}}^\epsilon$, there is at most one such anti-homomorphism which is then also an isomorphism, since $\sigma'\colon \mathfrak{el}_q^\epsilon \to \mathfrak{el}_{q^{-1}}^\epsilon$, $\mathcal{E}_i \mapsto q^\epsilon\mathcal{E}_{i-1}$ provides an inverse to σ. We however need to verify well-definedness, that is the compatibility with the defining relations of $\mathfrak{el}_q^\epsilon$ and $\mathfrak{el}_{q^{-1}}^\epsilon$.
To see (ϵl-1) we calculate for $|i-j|>1$ using Remark 9.2,

$$\sigma(\mathcal{E}_i\mathcal{E}_j) = q^{-2\epsilon}\mathcal{E}_{j+1}\mathcal{E}_{i+1} = q^{-2\epsilon+b_{j+1,i+1}}\mathcal{E}_{i+1}\mathcal{E}_{j+1} = \sigma(q^{-b_{i,j}}\mathcal{E}_j\mathcal{E}_i) = q^{b_{i,j}}\sigma(\mathcal{E}_j\mathcal{E}_i).$$

To see (ϵl-2) let $j = i+1$ and calculate

$$\begin{aligned}
\sigma(q^{-3}\mathcal{E}_i^2\mathcal{E}_j - [2]\mathcal{E}_i\mathcal{E}_j\mathcal{E}_i + q^3\mathcal{E}_j\mathcal{E}_i^2) =& q^{-3\epsilon}(q^{-3}\mathcal{E}_{j+1}\mathcal{E}_{i+1}^2 - [2]\mathcal{E}_{i+1}\mathcal{E}_{j+1}\mathcal{E}_{i+1} + q^3\mathcal{E}_{i+1}^2\mathcal{E}_{j+1}) \\
=& - q^{-2\epsilon}[2]\mathcal{E}_{i+1} = \sigma(-q^{-\epsilon}[2]\mathcal{E}_i).
\end{aligned}$$

To see (ϵl-3) let $j = i-1$ and calculate

$$\begin{aligned}
\sigma(q^3\mathcal{E}_i^2\mathcal{E}_j - [2]\mathcal{E}_i\mathcal{E}_j\mathcal{E}_i + q^{-3}\mathcal{E}_j\mathcal{E}_i^2) =& q^{-3\epsilon}(q^3\mathcal{E}_{j+1}\mathcal{E}_{i+1}^2 - [2]\mathcal{E}_{i+1}\mathcal{E}_{j+1}\mathcal{E}_{i+1} + q^{-3}\mathcal{E}_{i+1}^2\mathcal{E}_{j+1}) \\
=& - q^{-2\epsilon}[2]\mathcal{E}_{i+1} = \sigma(-q^{-\epsilon}[2]\mathcal{E}_i).
\end{aligned}$$

Therefore, the assignments give a well-defined q-linear anti-isomorphism σ. □

Lemma 9.6. *There exists a unique isomorphism of $\mathbb{Q}(q)$-algebras*

$$\tau\colon \mathfrak{el}_q^\epsilon \to \mathfrak{el}_{q^{-1}}^\epsilon, \quad \tau(\mathcal{E}_i) = q^\epsilon\mathcal{E}_{-i}.$$

Proof. This is similar to the proof of Lemma 9.5. □

9.1.1. The associated graded of the ϵlectrical algebras

The algebra $\mathfrak{el}_q^\epsilon$ becomes filtered by putting the monomials $\mathcal{E}_{i_1}\cdots\mathcal{E}_{i_k}$ in degree k. We directly obtain:

Lemma 9.7. *The associated graded algebra* $\operatorname{gr}\mathfrak{el}_q^\epsilon$ *of* $\mathfrak{el}_q^\epsilon$ *is the algebra with generators* $\mathcal{E}_i$, $i \in \mathbb{Z}$*, and relations*

$$\begin{aligned} \mathcal{E}_i\mathcal{E}_j &= q^{b_{ij}}\mathcal{E}_j\mathcal{E}_i \quad \text{if } |i-j|>1, \\ q^3\mathcal{E}_i^2\mathcal{E}_{i+1} - [2]\mathcal{E}_i\mathcal{E}_{i+1}\mathcal{E}_i + q^{-3}\mathcal{E}_{i+1}\mathcal{E}_i^2 &= 0, \\ q^{-3}\mathcal{E}_i^2\mathcal{E}_{i-1} - [2]\mathcal{E}_i\mathcal{E}_{i-1}\mathcal{E}_i + q^{3}\mathcal{E}_{i-1}\mathcal{E}_i^2 &= 0. \end{aligned} \tag{9.1}$$

We obtain a basis for $\mathfrak{el}_q^\epsilon$ (resembling a quantum group basis at $q=0$ from [Rei01]):

Corollary 9.8. *The subalgebra of* $\mathfrak{el}_q^\epsilon$ *generated by* $\mathcal{E}_i$, $1 \le i-a \le n-1$*, has basis*

$$\mathcal{E}_{a+1}^{m_1}(\mathcal{E}_{a+2}\mathcal{E}_{a+1})^{m_2}\mathcal{E}_{a+2}^{m_3}(\mathcal{E}_{a+3}\mathcal{E}_{a+2}\mathcal{E}_{a+1})^{m_4}(\mathcal{E}_{a+3}\mathcal{E}_{a+2})^{m_5}\mathcal{E}_{a+3}^{m_6}\cdots\mathcal{E}_{a+n-1}^{m_N}.$$

Here, $a \in \mathbb{Z}$, $n \in \mathbb{N}$ *are fixed arbitrarily and* $m_i \in \mathbb{N}_0$, $N = \frac{(n-1)n}{2}$.

Proof. For $a=0$, the corresponding polynomials (in the usual generators E_i) form a basis of its positive part of the quantum group $U_q(\mathfrak{sl}_n)$, see [Rin96, Theorem 2], [Jan96, 8.21]. The result then follows from the definitions and Lemma 9.7. ☐

9.2. The quantum electric Hopf algebra U_q

The goal of this section is to realize the q-ϵlectric algebras as coideal subalgebras of some Hopf algebra which is reminiscent of a quantized universal enveloping algebra. We define this Hopf algebra using a quantum double construction from a pairing between two Hopf algebras U_q^+ and U_q^+. We start by defining the ingredients of the construction.

Definition 9.9. Consider the free $\mathbb{Z}$-module $\mathfrak{h}_\mathbb{Z}$ with basis e_i, $i \in \mathbb{Z}$ and, via pointwise addition, the $\mathbb{Z}$-module $X := \operatorname{Hom}_\mathbb{Z}(\mathfrak{h}_\mathbb{Z}, \mathbb{Z})$. Let $\langle \_, \_ \rangle \colon X \otimes_\mathbb{Z} \mathfrak{h}_\mathbb{Z} \to \mathbb{Z}$ be the evaluation. We write $\alpha_i^\vee \in \mathfrak{h}_\mathbb{Z}$ for the element $e_{i+1}-e_i$ and denote by ε_i the dual element to e_i and set $\alpha_i := \varepsilon_{i+1} - \varepsilon_i$. In particular, the α_i (and $\alpha_i^\vee$) form the (dual) roots of a root system of type A_∞. Furthermore, let $\beta_i \in X$ be defined by

$$\langle \beta_i, e_j \rangle = \begin{cases} (-1)^i 2 & \text{if } j = i, \\ (-1)^i 4 & \text{if } (-1)^j j > (-1)^j i, \\ 0 & \text{otherwise.} \end{cases} \tag{9.2}$$

Finally, we define for $i \in \mathbb{Z}$, $\gamma_i \in X$ by $\langle \gamma_i, e_j \rangle = -\langle \beta_{i+1}, e_j \rangle$, that is $\gamma_i = -\beta_{i+1}$.

Notation 9.10. We denote by $X^{\mathrm{fsupp}} \subseteq X$ the set of all $\lambda \in X$ with *finite support*, i.e. $\langle \lambda, e_j \rangle \neq 0$ for only finitely many j. (Note that $\beta_i, \gamma_i \notin X^{\mathrm{fsupp}}$, $\alpha_i \in X^{\mathrm{fsupp}}$.)

Lemma 9.11. *For $i,j \in \mathbb{Z}$ we have $\langle \beta_i, \alpha_j^\vee \rangle = b_{ji}$, $\langle \gamma_i, \alpha_j^\vee \rangle = -b_{ij}$ and $b_{j,i+1} = b_{i,j}$.*

Proof. This follows by plugging in the definitions. □

Definition 9.12. Define the algebra U_q^- as the $\mathbb{Q}(q)$-algebra generated by F_i for $i \in \mathbb{Z}$ and by K_λ for $\lambda \in X$, subject to the relations

$$
\begin{array}{llll}
(1^-) & K_\lambda K_\mu = K_{\lambda+\mu}, & (4^-) & F_iF_j = q^{b_{ij}} F_jF_i \quad \text{if } |i-j|>1, \\
(2^-) & K_0 = 1, & (5^-) & q^3F_i^2F_{i+1} - [2]F_iF_{i+1}F_i + q^{-3}F_{i+1}F_i^2 = 0, \\
(3^-) & K_\lambda F_i = q^{-\langle \lambda, \alpha_i^\vee \rangle} F_i K_\lambda, & (6^-) & q^{-3}F_i^2F_{i-1} - [2]F_iF_{i-1}F_i + q^3F_{i-1}F_i^2 = 0.
\end{array}
$$

Lemma 9.13. *The following assignments define (anti-)algebra homomorphism*

$$
\begin{array}{lll}
\Delta\colon U_q^- \to U_q^- \otimes U_q^- & \varepsilon\colon U_q^- \to \mathbb{Q}(q) & S\colon U_q^- \to U_q^- \\
F_i \mapsto F_i \otimes K_{\beta_i} + 1 \otimes F_i, & F_i \mapsto 0, & F_i \mapsto -F_iK_{-\beta_i}, \\
K_\lambda \mapsto K_\lambda \otimes K_\lambda, & K_\lambda \mapsto 1, & K_\lambda \mapsto K_{-\lambda},
\end{array}
$$

which endow U_q^- with the structure of a Hopf algebra.

Proof. The proof is a standard calculation. For details see Section 11.1. □

In analogy to the universal enveloping algebra of a finite dimensional simple complex Lie algebra we call U_q^- the *negative Borel part*, and define a *positive Borel part* U_q^+:

Definition 9.14. Define the algebra U_q^+ as the $\mathbb{Q}(q)$-algebra generated by E_i and K_λ for $i \in \mathbb{Z}$ and $\lambda \in X$ subject to the relations

$$
\begin{array}{llll}
(1^+) & K_\lambda K_\mu = K_{\lambda+\mu}, & (4^+) & E_iE_j = q^{b_{ij}} E_jE_i \quad \text{if } |i-j|>1, \\
(2^+) & K_0 = 1, & (5^+) & q^3E_i^2E_{i+1} - [2]E_iE_{i+1}E_i + q^{-3}E_{i+1}E_i^2 = 0, \\
(3^+) & K_\lambda E_i = q^{\langle \lambda, \alpha_i^\vee \rangle} E_i K_\lambda, & (6^+) & q^{-3}E_i^2E_{i-1} - [2]E_iE_{i-1}E_i + q^3E_{i-1}E_i^2 = 0.
\end{array}
$$

Not very surprisingly, the positive Borel can also be turned into a Hopf algebra:

Lemma 9.15. *The following assignments define (anti-)algebra homomorphism*

$$
\begin{array}{lll}
\Delta\colon U_q^+ \to U_q^+ \otimes U_q^+ & \varepsilon\colon U_q^+ \to \mathbb{Q}(q) & S\colon U_q^+ \to U_q^+ \\
E_i \mapsto K_{\alpha_i} \otimes E_i + E_i \otimes K_{\alpha_i-\gamma_i} & E_i \mapsto 0 & E_i \mapsto -K_{-\alpha_i}E_iK_{\gamma_i-\alpha_i} \\
K_\lambda \mapsto K_\lambda \otimes K_\lambda & K_\lambda \mapsto 1 & K_\lambda \mapsto K_{-\lambda}
\end{array}
$$

which endow U_q^+ with the structure of a Hopf algebra.

Proof. The proof is totally analogous to the one of Lemma 9.13. □

Remark 9.16. The slight asymmetry between U_q^- and U_q^+ is chosen on purpose and motivated by the categorification results obtained later. It encodes the extra data, namely the $b_{i,j}$, appearing in the definition of $\mathfrak{el}_q^\epsilon$ via Lemma 9.11. A rescaling of E_i by $K_{-\alpha_i}$ would indeed provide formulas similar to those for the F_i's.

Both, in U_q^- and in U_q^+, the K_λ for $\lambda \in X$ generate a commutative subalgebra U^0 which is a Hopf subalgebra. We call these the *Cartan parts* of U_q^- and U_q^+.

Remark 9.17. The Cartan parts have basis K_λ, $\lambda \in X$ and multiplication as in (1^+). For simplicity, we write U_q^- and U_q^+ instead of more suggestively $U_q^{\geq}$ and $U_q^{\leq}$.

We now want to construct from U_q^- and U_q^+ a Hopf algebra via the usual Drinfeld double construction, see e.g. [Kas95, IX.4] for the general concept.
Fix now a bilinear pairing $(\_,\_)\colon X \times X \to \mathbb{Z}$ such that for all $i \in \mathbb{Z}$ we have $(\beta_i, \mu) = -\langle \mu, \alpha_i^\vee \rangle$ and $(\lambda, \gamma_i) = \langle \lambda, \alpha_i^\vee \rangle$. Note for this that $(\beta_i, \gamma_j) = -\langle \gamma_j, \alpha_i^\vee \rangle = b_{ji}$ is consistent with $(\beta_i, \gamma_i) = \langle \beta_i, \alpha_j^\vee \rangle = b_{ji}$ by Lemma 9.11.

Proposition 9.18. *There exists a unique Hopf pairing*

$$\langle \_,\_ \rangle \colon U_q^- \otimes U_q^+ \to \mathbb{Q}(q)$$

such that for all $i, j \in \mathbb{Z}$ and $\lambda, \mu \in X$ the following holds:

$$\langle K_\lambda, K_\mu \rangle = q^{(\lambda,\mu)}, \quad \langle F_i, K_\mu \rangle = 0, \quad \langle F_i, E_j \rangle = \delta_{ij} \frac{1}{q - q^{-1}}, \quad \langle K_\lambda, E_j \rangle = 0.$$

Proof. We need to verify that $\langle \_,\_ \rangle$ extends uniquely to a pairing which satisfies the Hopf pairing conditions

(i) $\langle a, 1 \rangle = \varepsilon(a)$ and $\langle 1, b \rangle = \varepsilon(b)$ for all $a \in U_q^-$ and $b \in U_q^+$.

(ii) $\langle aa', b \rangle = \langle a \otimes a', \Delta^{op}(b) \rangle$ for all $a, a' \in U_q^-$ and $b \in U_q^+$.

(iii) $\langle a, bb' \rangle = \langle \Delta(a), b \otimes b' \rangle$ for all $a \in U_q^-$ and $b, b' \in U_q^+$.

(iv) $\langle S(a), b \rangle = \langle a, S^{-1}(b) \rangle$ for all $a \in U_q^-$ and $b \in U_q^+$.

The proof is analogous to [Lus10, 1.2], see also [Xia97, Proposition 2.9.3, Proposition 2.9.4] for a summary. If one uses (in the notation of the latter) the slightly adjusted functionals $\xi_i(K_\lambda \Theta_i^-) = \frac{q^{(\lambda,\alpha) - \langle \lambda, \check{\alpha}_i \rangle}}{q - q^{-1}}$, $T_\mu(K_\lambda) = q^{(\lambda,\mu)}$, the arguments can be copied. $\square$

This Hopf pairing endows $U_q^- \otimes U_q^+$ with the structure of a Hopf algebra, see e.g. [Jos95, §3.2] for the construction and [Xia97, Proposition 2.4] for the explicit formulas:

Corollary 9.19. *There is a unique Hopf algebra structure on $U_q^- \otimes U_q^+$ such that U_q^- and U_q^+ are Hopf subalgebras via the canonical embeddings and*

(a) *the multiplication in Sweedler notation is given by*

$$(a \otimes b)(a' \otimes b') = \sum_{(a'),(b)} \langle S^{-1}(a'_{(1)}), b_{(1)} \rangle a a'_{(2)} \otimes b_{(2)} b' \langle a'_{(3)}, b_{(3)} \rangle, \tag{9.3}$$

(b) *the comultiplication is given by $\Delta(a \otimes b) = \sum_{(a),(b)} a_{(1)} \otimes b_{(1)} \otimes a_{(2)} \otimes b_{(2)}$ with counit $\varepsilon(a \otimes b) = \varepsilon(a)\varepsilon(b)$, and* (*c*) *the antipode is* $S(a \otimes b) = (1 \otimes S(b))(S(a) \otimes 1)$.

Remark 9.20. In $U_q^- \otimes U_q^+$, the product $(1 \otimes E_i)(F_j \otimes 1)$ is by (9.3) equal to

$$\begin{aligned}
&\langle S^{-1}(F_j), E_i\rangle K_{\beta_j} \otimes K_{\alpha_i-\beta_j}\langle K_{\beta_j}, K_{\alpha_i-\beta_j}\rangle + \langle S^{-1}(F_j), K_{\alpha_i}\rangle K_{\beta_j} \otimes E_i\langle K_{\beta_j}, K_{\alpha_i-\beta_j}\rangle \\
&\quad + \langle S^{-1}(F_j), K_{\alpha_i}\rangle K_{\beta_j} \otimes K_{\alpha_i}\langle K_{\beta_j}, E_i\rangle + \langle S^{-1}(1), E_i\rangle F_j \otimes K_{\alpha_i-\gamma_i}\langle K_{\beta_j}, K_{\alpha_i-\gamma_i}\rangle \\
&\quad + \langle S^{-1}(1), K_{\alpha_i}\rangle F_j \otimes E_i\langle K_{\beta_j}, K_{\alpha_i-\gamma_i}\rangle + \langle S^{-1}(1), K_{\alpha_i}\rangle F_j \otimes K_{\alpha_i}\langle K_{\beta_j}, E_i\rangle \\
&\quad + \langle S^{-1}(1), E_i\rangle 1 \otimes K_{\alpha_i-\gamma_i}\langle F_j, K_{\alpha_i-\gamma_i}\rangle + \langle S^{-1}(1), K_{\alpha_i}\rangle 1 \otimes E_i\langle F_j, K_{\alpha_i-\gamma_i}\rangle \\
&\quad + \langle S^{-1}(1), K_{\alpha_i}\rangle 1 \otimes K_{\alpha_i}\langle F_j, E_i\rangle \\
&= q^{\langle \beta_j-\alpha_i, \alpha_j^\vee\rangle} F_j \otimes E_i + \delta_{ij}\frac{1}{q-q^{-1}} 1 \otimes K_{\alpha_i} + q^{\langle \beta_j-\alpha_i, \alpha_j^\vee\rangle}\langle -K_{-\beta_j}F_j, E_i\rangle K_{\beta_j} \otimes K_{\alpha_i-\beta_j},
\end{aligned}$$

since the other summands vanish. Now the last summand can be simplified using $\langle -K_{-\beta_j}F_j, E_i\rangle = -\langle K_{-\beta_j}, K_{\alpha_i-\beta_j}\rangle\langle F_j, E_i\rangle = -\frac{q^{(-\beta_j,\alpha_i-\gamma_i)}}{q-q^{-1}} = -\frac{q^{\langle \alpha_i-\gamma_i,\alpha_j^\vee\rangle}}{q-q^{-1}}$.
Altogether, $(1 \otimes E_i)(F_j \otimes 1) = q^{\langle \beta_j-\alpha_i,\alpha_j^\vee\rangle} F_j \otimes E_i + \delta_{ij}\frac{1\otimes K_{\alpha_i} - K_{\beta_i}\otimes K_{\alpha_i-\beta_j}}{q-q^{-1}}$.

In analogy to the universal enveloping algebra of a simple complex Lie algebra we like to identify the Cartan parts U^0, see Remark 9.17, from the two Borel parts.

Proposition 9.21. *The maps* m, Δ, ε, S *defining the Hopf algebra structure on* $U_q^- \otimes U_q^+$ *are* U^0*-balanced. Thus,* $U_q^- \otimes_{U^0} U_q^+$ *inherits a Hopf algebra structure.*

Proof. This is proven in Section 11.2. ☐

Definition 9.22. We call $U_q \coloneqq U_q^- \otimes_{U^0} U_q^+$ the *quantum electric Hopf algebra.*

Notation 9.23. From now on, we will write F_i (respectively E_i) for the element $F_i \otimes 1$ (respectively $1 \otimes E_i$) in U_q. We also write K_λ for $K_\lambda \otimes 1 = 1 \otimes K_\lambda$ in U_q.

The quantum electric Hopf algebra is very similar to the quantized universal enveloping algebra (of adjoint type in the sense of [Jan96, 4.5]) of $\mathfrak{sl}_\mathbb{Z}$ of type A_∞.

Corollary 9.24. *The quantum electrical Hopf algebra* U_q *has as algebra a presentation with generators* E_i, F_i *for* $i \in \mathbb{Z}$ *and* K_λ *for* $\lambda \in X$ *subject to the relations*

$$K_\lambda K_\mu = K_{\lambda+\mu}, \qquad K_0 = 1, \tag{U-1}$$

$$K_\lambda F_i = q^{-\langle \lambda,\alpha_i^\vee\rangle} F_i K_\lambda, \qquad K_\lambda E_i = q^{\langle \lambda,\alpha_i^\vee\rangle} E_i K_\lambda \tag{U-2}$$

$$[E_i, F_j]_{\beta_{ij}} = \delta_{ij}\frac{K_{\alpha_i} - K_{-\alpha_i}}{q-q^{-1}}, \tag{U-3}$$

$$\begin{aligned}
&F_iF_j = q^{b_{ij}} F_jF_i \text{ if } |i-j| > 1, \\
&q^3F_i^2F_{i+1} - [2]F_iF_{i+1}F_i + q^{-3}F_{i+1}F_i^2 = 0, \\
&q^{-3}F_i^2F_{i-1} - [2]F_iF_{i-1}F_i + q^3F_{i-1}F_i^2 = 0,
\end{aligned} \tag{U-4}$$

$$\text{(U-5)}\qquad \begin{aligned} &E_iE_j = q^{b_{ij}}E_jE_i \quad \textit{if } |i-j|>1,\\ &q^3E_i^2E_{i+1} - [2]E_iE_{i+1}E_i + q^{-3}E_{i+1}E_i^2 = 0,\\ &q^{-3}E_i^2E_{i-1} - [2]E_iE_{i-1}E_i + q^3E_{i-1}E_i^2 = 0, \end{aligned}$$

where $[E_i, F_j]_{\beta_{ij}}$ *denotes the q-commutator* $E_iF_j - q^{\beta_{ij}}F_jE_i$ *and* $\beta_{ij} = \langle \gamma_i - \alpha_i, \alpha_j^\vee \rangle$. *Spelling out* β_{ij} *explicitly, we have*

$$\text{(9.4)}\qquad \beta_{ij} = \begin{cases} 0 & \textit{if } i=j,\\ 3(j-i) & \textit{if } |i-j|=1,\\ -4\cdot\operatorname{sgn}(j-i)(-1)^{j-i} & \textit{otherwise.} \end{cases}$$

The Hopf algebra structure is given by Lemma 9.13 and Lemma 9.15.

Proof. The relations (U-1)–(U-2) and (U-4)–(U-5) hold by definition of U_q and (U-3) follows from Remark 9.20 noting that $\alpha_i + \beta_i - \gamma_i = -\alpha_i$ by Lemma 9.11 and (9.2). The given relations suffice by comparison with $U_q(\mathfrak{sl}_\mathbb{Z})$: one obtains a PBW-type basis for U_q from compatible PBW-type bases of U_q^+ and U_q^- via Definition 9.22. □

9.3. Realization of the q-ϵlectrical algebra as a coideal

The quantum electrical Hopf algebra allows treating $\mathfrak{el}_q^\epsilon$ in a more conceptual way as coideal in U_q:

Theorem 9.25 (Coideal realisation)**.** *The q-ϵlectric algebra* $\mathfrak{el}_q^\epsilon$ *embeds into the quantum electrical Hopf algebra* U_q *as a right coideal via* $\mathcal{E}_i \mapsto F_i + q^{\epsilon-1}E_{i-1}K_{-\alpha_{i-1}}$.

Proof. In Section 11.3 we show that the assignment provides a well-defined algebra homomorphism j which is moreover injective. It remains to show that its image $C \coloneqq \operatorname{im}(\mathrm{j})$ is in fact a right coideal. We have

$$\begin{aligned} \Delta(\mathrm{j}(\mathcal{E}_{i+1})) &= \Delta(F_{i+1} + q^{\epsilon-1}E_iK_{-\alpha_i})\\ &= F_{i+1}\otimes K_{\beta_{i+1}} + 1\otimes F_{i+1} + q^{\epsilon-1}\otimes E_iK_{-\alpha_i} + q^{\epsilon-1}E_iK_{-\alpha_i}\otimes K_{-\gamma_i}. \end{aligned}$$

Since $\beta_{i+1} = -\gamma_i$, we obtain

$$\Delta(\mathrm{j}(\mathcal{E}_{i+1})) = \mathrm{j}(1)\otimes \mathrm{j}(\mathcal{E}_{i+1}) + \mathrm{j}(\mathcal{E}_{i+1})\otimes K_{\beta_{i+1}} \in C\otimes U_q.$$

This shows that C is a right coideal in U_q and finishes the proof. □

Notation 9.26. From now on we identify $\mathfrak{el}_q^\epsilon$ with its image in U_q and thus view it as coideal subalgebra of U_q with $\Delta(\mathcal{E}_i) = 1\otimes\mathcal{E}_i + \mathcal{E}_i\otimes K_{\beta_i}$.

9.4. (Dual) Natural representation of U_q and their exterior powers

The Hopf algebra U_q is another quantization of the universal enveloping algebra of $\mathfrak{sl}_{\mathbb{Z}}$. In analogy to $\mathfrak{sl}_{\mathbb{Z}}$ we define a natural representation $V = \mathbb{Q}(q)^{\mathbb{Z}}$ of U_q.

Proposition 9.27. *Let $V = \mathbb{Q}(q)^{\mathbb{Z}}$ with basis v_i, $i \in \mathbb{Z}$. Then there exists a well-defined right action of U_q on V given, for $i, j \in \mathbb{Z}$ and $\lambda \in X$, by*

$$v_j F_i = \delta_{ij} v_{j+1}, \quad v_j E_i = \delta_{i+1,j} v_{j-1}, \quad v_j K_\lambda = q^{\langle \lambda, e_j \rangle} v_j.$$

Proof. The relation (U-1) is immediate, and (U-2) follows directly from the definition of $\alpha_j^\vee$. For (U-4) and (U-5) both sides act by 0, hence these relations are satisfied. It remains to check the compatibility with (U-3). If $i \neq j$, then both sides act by 0. Otherwise, we have $v_k[E_i, F_i]_{\beta_{ii}} = v_k E_i F_i - v_k F_i E_i = \delta_{k,i+1} v_{i+1} - \delta_{ki} v_i$, which equals $v_k \frac{K_{\alpha_i} - K_{-\alpha_i}}{q - q^{-1}} = v_k \frac{q^{\langle \alpha_i, e_k \rangle} - q^{-\langle \alpha_i, e_k \rangle}}{q - q^{-1}}$, because $\alpha_i = \varepsilon_{i+1} - \varepsilon_i$. □

Definition 9.28. Let $V^{\circledast}$ be the (restricted) dual vector space of V, i.e. the vector space with basis $v^i \in \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$, where $v^i(v_j) \coloneqq \delta_{ij}$. The right U_q-module structure on V defines a left U_q-module structure on $V^{\circledast}$. In formulas, it is given by

$$F_i v^j = \delta_{i+1,j} v^i, \quad E_i v^j = \delta_{ij} v^{i+1}, \quad v^j K_\lambda = q^{\langle \lambda, e_j \rangle} v^j.$$

Define a q-bilinear pairing $(\_,\_) \colon V^{\circledast} \otimes V \to \mathbb{Q}(q)$ by $(v^i, v_j) = \delta_{ij}$.

Lemma 9.29. *The bilinear pairing satisfies $(wu, v) = (w, v\sigma(u))$ for all $w \in V^{\circledast}$, $v \in V$ and $u \in \mathfrak{cl}^{\epsilon}_{q^{-1}}$ with σ as in Lemma 9.5.*

Proof. It is enough to consider $v = v^l, w = v_k, u = \mathcal{E}_i$ for any l, k, i. We compute $(v^l \mathcal{E}_i, v_k) = (\delta_{(i+2)l} q^{-\epsilon} v^{i+1} + \delta_{il} v^{i+1}, v_k) = \delta_{(i+1)k}(q^{-\epsilon}\delta_{(i+2)l} + \delta_{il})$. On the other hand $(v^l, v_k \sigma(\mathcal{E}_i)) = q^{-\epsilon}(v^l, v_k \mathcal{E}_{i+1}) = q^{-\epsilon}(\delta_{k(i+1)}(v^l, v_{i+2}) + q^{\epsilon}\delta_{k(i+1)}(v^l, v_i))$. Since the latter equals $q^{-\epsilon}\delta_{k(i+1)}\delta_{l(i+2)} + \delta_{k(i+1)}\delta_{li}$ the assertion follows. □

We next define an alternative comultiplication on U_q:

Definition 9.30. Given $\lambda \in X$ let $\lambda' \in X$ such that $\langle \lambda', e_j \rangle = \langle \lambda, e_{j+1} \rangle$. Define the algebra isomorphism

$$\mathrm{shift} \colon U_q \to U_q, \quad F_i \mapsto F_{i+1}, E_i \mapsto E_{i+1}, K_\lambda \mapsto K_{\lambda'}.$$

Let $\Delta' \coloneqq (\mathrm{shift} \otimes \mathrm{shift})\, \Delta\, \mathrm{shift}^{-1}$ be the induced comultiplication, cf. [Jan96, §7.2].

Remark 9.31. We have $\Delta' = (\mathrm{shift}^{-1} \otimes \mathrm{shift}^{-1})\, \Delta\, \mathrm{shift}$ as $\langle \beta_{i-1}, e_j \rangle = \langle \beta_{i+1}, e_{j+1} \rangle$.

Notation 9.32. Definition 9.30 defines a second monoidal structure on the category of U_q-modules, cf. [Jan96, §3.8]. To keep track of the tensor products we use the symbol $\otimes$ for the usual tensor product of vector spaces and $\odot_1$ and $\odot_2$ for the tensor product of U_q-modules with the action given by Δ and Δ' respectively. The notation $M \odot N$ means that $\odot$ can be $\odot_1$ or $\odot_2$.

We will use mixed tensor products involving $\odot_1$ and $\odot_2$.

Definition 9.33. Given a U_q-module M and $\underline{d} = (l_1, \ldots, l_{d-1}) \in \{1,2\}^{d-1}$, we define the corresponding *mixed tensor product of* M as $M^{\odot \underline{d}} := M \odot_{l_1} \cdots \odot_{l_{d-1}} M$.

Warning. *When writing mixed tensor products, we have to be careful with the bracketings, since e.g.* $(M \odot_1 N) \odot_2 P \not\cong M \odot_1 (N \odot_2 P)$ *in general. If in the following we suppress the bracketing in mixed tensor products, we will implicitly always assume that the bracketing is left adjusted, e.g.* $M \odot_1 N \odot_2 P := (M \odot_1 N) \odot_2 P$.

Next, we analyze the U_q-linear endomorphisms of (mixed) tensor powers of V and $V^{\circledast}$. In analogy to $U_q(\mathfrak{sl}_{\mathbb{Z}})$, we expect to find a Hecke algebra action.
Recall the Hecke algebra $\mathcal{H}_d$, which is the $\mathbb{Q}(q)$-algebra generated by $H_1, \ldots, H_{d-1}$ subject to the relations

$$\begin{gathered} H_i^2 = 1 + (q^{-1} - q) H_i \\ H_i H_j = H_j H_i \quad \text{if } |i-j| > 1, \quad H_i H_{i+1} H_i = H_{i+1} H_i H_{i+1}. \end{gathered} \tag{9.5}$$

Given a $\mathbb{Q}(q)$-vector space W and a linear endomorphism ϕ of $W \otimes W$, define the endomorphisms $\phi_i := \mathrm{id}^{\otimes(i-1)} \otimes \phi \otimes \mathrm{id}^{\otimes(d-i-1)}$ of $W^{\otimes d}$ for $i = 1, \ldots, d-1$. Then ϕ *satisfies the Hecke relations* if (9.5) hold with H_i replaced by ϕ_i.

Proposition 9.34. *Consider the natural right U_q-module V. The linear map*

$$H \colon V \odot V \to V \odot V, \quad v_i \odot v_j \mapsto a_{ij} v_j \odot v_i + \delta_{i<j} (q^{-1} - q) v_i \odot v_j,$$

is U_q-linear and satisfies the Hecke relations, where for $\odot = \odot_l$ we set

$$a_{ij} = \begin{cases} q^3 & \text{if } i \geq j,\ i-l \text{ odd},\ j-l \text{ even}, \\ q^{-1} & \text{if } i \geq j,\ \text{otherwise}, \\ q^{-3} & \text{if } i < j,\ i-l \text{ even},\ j-l \text{ odd}, \\ q & \text{if } i < j,\ \text{otherwise}. \end{cases}$$

Proof. Noting that

$$a_{ii} = q^{-1} \quad \text{and} \quad a_{ij} a_{ji} = 1 \quad \text{for any} \quad i \neq j, \tag{9.6}$$

the statement follows by straight-forward calculations, see Section 11.4. ☐

There is no reason to prefer V to $V^{\circledast}$. Analogously to Proposition 9.34 we obtain:

Proposition 9.35. *The linear map*

$$H^{\circledast} \colon V^{\circledast} \odot V^{\circledast} \to V^{\circledast} \odot V^{\circledast}, \quad v^i \odot v^j \mapsto a_{ji} v^j \odot v^i + \delta_{i<j} (q^{-1} - q) v^i \odot v^j,$$

is U_q-linear and satisfies the Hecke relations, with a_{ij} as in Proposition 9.34.

Remark 9.36. As a consequence of Propositions 9.34 and 9.35 we obtain a (left) action of $\mathcal{H}_d$ on any d-fold mixed tensor product $V^{\odot \underline{d}}$ of V by U_q-module homomorphisms commuting with the (right) U_q-action. With our implicit bracketing convention we have for instance $v_i \odot_1 v_j \odot_2 v_k := (v_i \odot_1 v_j) \odot_2 v_k$. Then F_i acts as $F_i \otimes K_{\beta_i} \otimes K_{\beta'_{i-1}} + 1 \otimes F_i \otimes K_{\beta'_{i-1}} + 1 \otimes 1 \otimes F_i$. This commutes with the $\mathcal{H}_d$-action.

Remark 9.37. One can even check that the Hecke algebra centralizes U_q and that we have an isomorphism
$$\mathcal{H}_d \to \operatorname{End}_{U_q}(V^{\otimes d}).$$
This even works for any mixed tensor product $V^{\odot \underline{d}}$.
To see this recall that by quantum Schur–Weyl duality, $\mathcal{H}_d \cong \operatorname{End}_{U_q(\mathfrak{sl}_{\mathbb{Z}})}(V^{\otimes d})$, where H_i acts in $V \otimes V$ by $v_a \otimes v_b \mapsto v_b \otimes v_a + \delta_{a<b}(q^{-1} - q) v_a \otimes v_b$. We now claim that $V \otimes V \cong V \odot V$ as $\mathcal{H}_2$-modules. Indeed, an isomorphism as desired is given by
$$v_i \otimes v_j \mapsto \begin{cases} v_i \odot v_j & \text{if } i \geq j, \\ a_{ji} v_i \odot v_j & \text{if } i < j. \end{cases}$$
This can now easily be extended to arbitrary mixed tensor products.

The Hecke algebra actions from Propositions 9.34 and 9.35 finally allow defining q-wedge products of V and of $V^{\circledast}$.

Definition 9.38. Consider a mixed tensor product $V^{\odot \underline{d}}$. We define the *q-wedge product* $\bigwedge^{\underline{d}} V$ to be the subspace of $V^{\odot \underline{d}}$ spanned by all elements of the form
$$v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_d} := \sum_{w \in \mathfrak{S}_d} (-q)^{\ell(w)} H_w (v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_d}),$$
for $i_1 > i_2 > \cdots > i_d$. If $\underline{d} = (2, 1, 2, 1, \dots)$, we just write $\bigwedge^d V$ for $\bigwedge^{\underline{d}} V$.

The goal of the next section is a definition of a Fock space $\mathscr{F}_\delta$ and its dual $\mathscr{F}_\delta^{\circledast}$ for the electrical Lie algebras $\mathfrak{el}_q^\epsilon$. We use the $\bigwedge^{\underline{d}} V$ with their U_q-actions to define Fock spaces for $\mathfrak{el}_q^\epsilon$ following in principle the standard constructions, [LT96], as a space of semiinfinite wedges. In detail, the construction is however more involved. We have to make sure that the action of the Cartan part in U_q is well-defined. For this the combination of the two monoidal structures $\odot_1, \odot_2$, i.e. the choice of $\underline{d} = (2, 1, 2, 1, \dots)$ in the definition of the q-wedge product will be crucial.

9.5. The electric Fock space representations $\mathscr{F}$ and $\mathscr{F}^{\circledast}$

In the following we will consider V and $V^{\circledast}$ as right $\mathfrak{el}_q^\epsilon$-modules:

Definition 9.39. The *natural $\mathfrak{el}_q^\epsilon$-module* V is the vector space V with the action restricted from U_q to $\mathfrak{el}_q^\epsilon$. In formulas, the action is given by $v_j \mathcal{E}_i = \delta_{ij}(v_{i+1} + q^\epsilon v_{i-1})$. The *dual natural $\mathfrak{el}_q^\epsilon$-module* $V^{\circledast}$ is the vector space $V^{\circledast}$ with the action of $\mathfrak{el}_q^\epsilon$ given by the restriction from U_q to $\mathfrak{el}_q^\epsilon$ twisted by the shift anti-automorphism σ from Lemma 9.5. In formulas, we have $v^j \mathcal{E}_i = \delta_{i+2,j} q^{-\epsilon} v^{i+1} + \delta_{ij} v^{i+1}$.

Indeed, one calculates $v^j \mathcal{E}_i = \sigma(\mathcal{E}_i) v^j = q^{-\epsilon} \mathcal{E}_{i+1} \cdot v^j = \delta_{i+2,j} q^{-\epsilon} v^{i+1} + \delta_{ij} v^{i+1}$.

Definition 9.40. We define the *Fock space* $\mathscr{F}$ as the vector space

$$\mathscr{F} := \varinjlim \bigwedge^d V, \quad \text{using the linear maps } \_ \wedge v_{-d} \colon \bigwedge^d V \to \bigwedge^{d+1} V. \tag{9.7}$$

The Fock space has as basis formal *semiinfinite wedges*

$$v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \wedge \cdots,$$

where $i_j > i_{j+1}$ and $i_j \neq 1-j$ for only finitely many $j \in \mathbb{Z}_{>0}$.

Unfortunately, the action of U_q on q-wedge products extends only partially to $\mathscr{F}$:

Proposition 9.41. *Let U_q^{fsupp} be the subalgebra of U_q generated by E_i, F_i with $i \in \mathbb{Z}$ and K_λ with $\lambda \in X^{\mathrm{fsupp}}$. Then there is a well-defined action of U_q^{fsupp} on $\mathscr{F}$ induced from the U_q-action on q-wedge products.*

Proof. Consider for any d, the map $\_ \wedge v_{-d} \colon \bigwedge^d V \to \bigwedge^{d+1} V$. Our (implicit) choice of $\underline{l}$ implies that in the comultiplication of F_i we obtain a K_{β_i} in even spots and a $K_{\beta'_i}$ in odd spots. However, for $i > -d$ we have $v_{-d} K_{\beta_i} = v_{-d}$ if $-d$ is even and have $v_{-d} K_{\beta'_i} = v_{-d}$ if $-d$ is odd. Hence, the action of F_i is well-defined. Similarly, the action of E_i is well-defined. By our assumption on λ, we have $v_{-d} K_\lambda = v_{-d}$ for $d \gg 0$, hence the action is well-defined as well. $\square$

We finally arrive at a well-defined electric Fock space:

Corollary 9.42. *The action of U_q^{fsupp} on $\mathscr{F}$ restricts to a right action of $\mathfrak{el}_q^\epsilon$.*

Proof. From the formulas for $\mathfrak{el}_q^\epsilon \subseteq U_q$ in Theorem 9.25 we see that $\mathfrak{el}_q^\epsilon \subseteq U_q^{\mathrm{fsupp}}$. $\square$

Definition 9.43. Similarly to $\mathscr{F}$, we can define the *dual Fock space* $\mathscr{F}^\circledast$ using $V^\circledast$ instead of V. This has then a basis given by formal semiinfinite wedges

$$v^{i_1} \wedge v^{i_2} \wedge v^{i_3} \wedge \cdots,$$

where $i_j > i_{j+1}$ and $i_j \neq 1-j$ for only finitely many $j \in \mathbb{Z}_{>0}$. As above, we can identify partitions with the basis vectors of $\mathscr{F}^\circledast$ and write v^λ for the corresponding basis vector. With the same arguments we get an induced (left) action of U_q^{fsupp} on $\mathscr{F}^\circledast$ and thus a right action of $\mathfrak{el}_{q^{-1}}^\epsilon$ via the shift automorphism from Lemma 9.5.

We call $\mathscr{F}$ the *electric Fock space* and $\mathscr{F}^\circledast$ the *dual electric Fock space.*

Corollary 9.44. *Both, the Fock space $\mathscr{F}$ and the dual Fock space $\mathscr{F}^\circledast$, are cyclic $\mathfrak{el}_q^\epsilon$-module generated by the vacuum vector $v_\emptyset$.*

One can consider also Fock spaces $\mathscr{F}_\delta$ depending on a charge $\delta \in \mathbb{R}$. For this let $V_\delta = \mathbb{Q}(q)^{\mathbb{Z}}$ with basis v_i, with $i \in \delta + \mathbb{Z}$ and let $\mathscr{F}_\delta$ be the corresponding Fock space defines as before. Via the identification of vector spaces $V \cong V_\delta$, $v_i \mapsto v_{\delta+i}$ V_δ inherits an action of $\mathfrak{el}_q^\epsilon$. Similarly, we define $\mathscr{F}_\delta^\circledast$, the *dual Fock* space of charge δ. In the special case $\delta = 0$ we have $\mathscr{F}_0 = \mathscr{F}$ and $\mathscr{F}_0^\circledast = \mathscr{F}^\circledast$. The following is straight-forward:

Proposition 9.45. *All results in Section 9.5 hold for $\mathscr{F}_\delta$, $\mathscr{F}_\delta^{\circledast}$ instead of $\mathscr{F}$, $\mathscr{F}^{\circledast}$.*

Lemma 9.46. *The annihilator of $v_\emptyset \in \mathscr{F}_\delta$ and of $v^\emptyset \in \mathscr{F}_\delta^{\circledast}$ is the right ideal generated by $\mathcal{E}_i$ for $i \neq \delta$ and by the two-sided ideal generated by $\mathcal{E}_i^2$ for $i \in \mathbb{Z}$.*

Proof. We compute the annihilator A of $v_\emptyset$. By definition, we have $v_\emptyset \mathcal{E}_i = 0$ if and only if $i \neq \delta$. By Remark 9.47, the element $\mathcal{E}_i^2$ acts by 0 on $\mathscr{F}_\delta$. Thus, $J \subseteq A$. To show $J = A$ let $u = \mathcal{E}_{i_1} \cdots \mathcal{E}_{i_r}$ be a product of generators, that is nonzero in $\mathfrak{el}_q^\epsilon / J$. In particular $i_1 = 0$ and we may assume moreover by (ϵl-2) and (ϵl-3) that u is braid-avoiding, that is there is no subsequence in the sense of Definition 2.2 of the form $(j, j \pm 1, j)$ for some j. It is straightforward to check that $(i_r, \dots, i_1)$ is the residue sequence of a partition λ, see [Neh24, Proposition 2.7] for a similar argument. By definition of the $\mathfrak{el}_q^\epsilon$-action we have $v_\emptyset \cdot u = c_\lambda v_\lambda + \sum_{|\mu|<|\lambda|} c_\mu v_\mu$ with $c_\lambda \neq 0$.
Conversely, any residue sequence $(i_r, \dots, i_1)$ defining λ is braid-avoiding and provides, thanks to (ϵl-1), up to powers of q the same element u in $A \backslash \mathfrak{el}_q^\epsilon$. Therefore, the linear map $J \backslash \mathfrak{el}_q^\epsilon \to \mathscr{F}_\delta$, $u \mapsto v_\emptyset \cdot u$ is surjective and in fact an isomorphism. This shows the claim $J = A$.
The same arguments show that A is also the annihilator of $v^\emptyset \in \mathscr{F}_\delta^{\circledast}$. □

As usual, see e.g. [Ari02], we label, depending on the fixed charge δ, the basis vectors (i.e. the semi-infinite wedges) of $\mathscr{F}$ by partitions. Namely, to a partition λ we assign the basis vector with indices given by $\{\lambda_i + 1 - i + \delta\}$ and also write v_λ for this basis vector. Similarly, for $\mathscr{F}_\delta^{\circledast}$ using $V_\delta^{\circledast}$ with basis vectors v^i, $i \in \delta + \mathbb{Z}$.

Remark 9.47. Note that Definitions 9.38 and 9.39 provide explicit formulas for the $\mathfrak{el}_q^\epsilon$-action on $\mathscr{F}_\delta$ and $\mathscr{F}_\delta^{\circledast}$. Up to powers of q, this action is given in the basis of partitions in familiar terms (cf. e.g. [Ari02]) using (2.1):

- $\mathcal{E}_i$ sends a partition λ to the linear combination of all partitions μ where a box of charged content $\delta + i$ was added or a box of charged content $\delta + i - 1$ was removed. In the language of Section 2.2 this means $\mathrm{res}(\lambda \to \mu) = \delta + i$.
- Similarly, for $\mathscr{F}^{\circledast}$ we have that $\mathcal{E}_i \in \mathfrak{el}_{q^{-1}}^\epsilon$ adds boxes of charged content $\delta + i$ and removes boxes of charged content $\delta + i + 1$ that means $\mathrm{res}^{\circledast}(\lambda \to \mu) = \delta + i$.

Proof. This follows from Corollary 9.42 using Definition 9.43 and Remark 9.47. □

Remark 9.48. Remark 9.47 should justify the notation res and $\mathrm{res}^{\circledast}$ by referring to $\mathscr{F}_\delta$ and $\mathscr{F}_\delta^{\circledast}$. The introduction of these two slightly different functions is necessary because σ not only scales by a power of q, but also shifts the indices.

Lemma 9.49. *There is a unique isomorphism of vector spaces $\tau \colon \mathscr{F}_\delta \to \mathscr{F}_\delta^{\circledast}$ satisfying $\tau(v_\emptyset) = v_\emptyset$ and $\tau(v\mathcal{E}) = \tau(v)\tau(\mathcal{E})$ for $v \in \mathscr{F}_\delta$, $\mathcal{E} \in \mathfrak{el}_q^\epsilon$.*

Remark 9.50. Up to some q-power, τ transposes the partition, i.e. $\tau(v_\lambda) = q^{c(\lambda)} v_{\lambda^t}$.

Proof. This follows directly from Lemma 9.6 and Corollary 9.44, since the annihilator J in Lemma 9.46 is τ-invariant. □

9.5.1. Pairing and Bar involution on Fock spaces

From the definition we expect $\mathscr{F}_\delta^{\circledast}$ to be dual to $\mathscr{F}_\delta$ via the following pairing.

Definition 9.51. Define a q-bilinear pairing $(\_,\_)\colon \mathscr{F}_\delta^{\circledast} \otimes \mathscr{F}_\delta \to \mathbb{Q}(q)$ by

$$(v^\lambda, v_\mu) = \delta_{\lambda\mu}.$$

Lemma 9.52. *The bilinear pairing satisfies (with σ as in Lemma 9.5)*

$$(wu, v) = (w, v\sigma(u))$$

for all $w \in \mathscr{F}_\delta^{\circledast}$, $v \in \mathscr{F}_\delta$ and $u \in \mathfrak{el}^{\epsilon}_{q^{-1}}$.

Proof. This holds by definition recalling the twist by σ in the action. □

Warning. *For readers familiar with categorification the shift σ appearing in Lemma 9.52 should be alarming, since we cannot expect that a functor categorifying $\mathcal{E}_i$ is self-adjoint (even up to grading shifts). One should also observe that we do not define a scalar product on $\mathscr{F}_\delta$, but only a pairing with $\mathscr{F}_\delta^{\circledast}$.*

We next define a bar involution compatible with the bar involution on $\mathfrak{el}^{\epsilon}_q$ and $\mathfrak{el}^{\epsilon}_{q^{-1}}$.

Proposition 9.53. *There exists a* bar involution on $\mathscr{F}_\delta$*, that is a unique q-antilinear isomorphism $\overline{}\colon \mathscr{F}_\delta \to \mathscr{F}_\delta^{\circledast}$ satisfying $\overline{v_\emptyset} := v^\emptyset$ and $\overline{u.v} = \overline{u}.\overline{v}$.*

Proof. By Corollary 9.44, the $\mathfrak{el}^{\epsilon}_q$-module $\mathscr{F}_\delta$ is cyclic with generator $v_\emptyset$. Therefore, the bar involution on $\mathscr{F}_\delta$ is unique if it exists. If $A \subseteq \mathfrak{el}^{\epsilon}_q$ is the annihilator of $v_\emptyset$, then $A\backslash\mathfrak{el}^{\epsilon}_q \to \mathscr{F}_\delta$, $u \mapsto v_\emptyset \cdot u$ is an isomorphism of (right) $\mathfrak{el}^{\epsilon}_q$-modules. Now $A = J$ by Lemma 9.46. Since J is obviously preserved under the bar involutions on $\mathfrak{el}^{\epsilon}_q$ and $\mathfrak{el}^{\epsilon}_{q^{-1}}$, the desired (unique) bar involution maps on $\mathscr{F}_\delta$ and $\mathscr{F}_\delta^{\circledast}$ exist. □

Definition 9.54. For a charge vector $\boldsymbol{\delta}$ and a level ℓ we define the *level ℓ Fock space* $\mathscr{F}_{\boldsymbol{\delta},\ell} = \mathscr{F}_{\delta_1} \otimes \cdots \otimes \mathscr{F}_{\delta_\ell}$ of charge $\boldsymbol{\delta}$. It comes with an obvious $\left(\mathfrak{el}^{\epsilon}_q\right)^{\otimes \ell}$-action.

Remark 9.47 generalizes to higher levels by identifying the standard basis vectors from $\mathscr{F}_{\boldsymbol{\delta},\ell}$ with ℓ-multipartitions and then using the residue functions (2.2).

10. Categorification of the ϵlectrical Lie algebra

10.1. Gradings, free $\mathbb{Z}$-actions and categories of representations

Instead of working with (strict monoidal) $\mathcal{GSV}\mathrm{ec}^\circ$-categories $\mathcal{C}$, we could equivalently work with (strict monoidal) $\mathcal{SV}\mathrm{ec}^\circ$-categories $\mathcal{C}^{\mathbb{Z}}$, but equipped with a free $\mathbb{Z}$-action given by (strict monoidal) isomorphisms $\langle i\rangle$, $i\in\mathbb{Z}$, such that $\langle i\rangle\langle j\rangle=\langle i+j\rangle$.
More precisely we have the following, see [MOS09, (2.1)]:

Lemma 10.1. *There is a correspondence*

$$\begin{array}{rcl} \mathrm{Cat}_{\mathbb{Z}} \coloneqq \{\mathcal{GSV}\mathrm{ec}^\circ\textit{-categories}\} & \leftrightarrow & \{\mathcal{SV}\mathrm{ec}^\circ\textit{-categories with a free } \mathbb{Z}\textit{-action}\} \eqqcolon \mathrm{Cat}^{\mathbb{Z}} \\ \mathcal{C} & \mapsto & \mathcal{C}^{\mathbb{Z}} \\ \mathcal{C}_{\mathbb{Z}} & \leftarrow\!\shortmid & \mathcal{C} \end{array}$$

Here, a $\mathbb{Z}$-*action* means an action by automorphisms $\langle i\rangle, i\in\mathbb{Z}$ such that $\langle i\rangle\langle j\rangle=\langle i+j\rangle$ (and freely means that the stabilizer of every object is trivial).
In $\mathcal{C}^{\mathbb{Z}}$, the objects are $\langle i\rangle c$, with $i\in\mathbb{Z}$, $c\in\mathcal{C}$ and $\mathrm{Hom}_{\mathcal{C}^{\mathbb{Z}}}(\langle i\rangle c,\langle j\rangle c')\coloneqq\mathrm{Hom}_{\mathcal{C}}(c,c')_{i-j}$.
The *orbit category* $\mathcal{C}_{\mathbb{Z}}$ has objects the orbits $[c]$ of objects in $\mathcal{C}$ with a fixed representative $\hat{c}$. The morphisms are $\mathrm{Hom}_{\mathcal{C}_{\mathbb{Z}}}([c],[c'])_i\coloneqq\mathrm{Hom}_{\mathcal{C}_{\mathbb{Z}}}(\langle i\rangle\hat{c},\hat{c})=\mathrm{Hom}_{\mathcal{C}_{\mathbb{Z}}}(\hat{c},\langle -i\rangle\hat{c})$.

Remark 10.2. One could work with any group G and with G-graded vector spaces instead. If we work with $G=\mathbb{Z}/2\mathbb{Z}$ and with Vec instead of $\mathcal{SV}\mathrm{ec}^\circ$ our notion of supercategories turns into the notion of supercategories using free $\mathbb{Z}_2$-actions as defined e.g. in [KKT16].

Concretely, in $(\mathrm{sR}_\epsilon)^{\mathbb{Z}}$, objects are $\langle i\rangle\mathbf{a}$, $i\in\mathbb{Z}$ with $\mathbf{a}\in\mathrm{sR}_\epsilon$ and $\mathrm{Hom}_{\mathrm{sR}_\epsilon^{\mathbb{Z}}}(\langle i\rangle\mathbf{a},\langle j\rangle\mathbf{b})=\mathrm{Hom}_{\mathrm{sR}_\epsilon}(\mathbf{a},\mathbf{b})_{i-j}$, the degree $i-j$ morphisms in sR_ϵ. As monoidal supercategory with $\mathbb{Z}$-action, $\mathrm{sR}_\epsilon^{\mathbb{Z}}$ is generated by objects $a=\langle 0\rangle a$, $a\in\mathbb{R}$, and morphisms $(f\colon\mathbf{a}\to\langle -i\rangle\mathbf{b})\in\mathcal{SV}\mathrm{ec}^\circ$ for any $(f\colon\mathbf{a}\to\mathbf{b})\in\mathcal{GSV}\mathrm{ec}^\circ$ from (3.5) of degree i, subject to (sR-1)-(sR-7) interpreted in the same way.

Remark 10.3. Given $\mathcal{C}\in\mathrm{Cat}_{\mathbb{Z}}$ there is an equivalence $(\mathcal{C}^{\mathrm{op}})^{\mathbb{Z}}\cong(\mathcal{C}^{\mathbb{Z}})^{\mathrm{op}}$ given by $\langle i\rangle c\mapsto\langle -i\rangle c$ noting that the following holds for morphisms $\mathrm{Hom}_{(\mathcal{C}^{\mathrm{op}})^{\mathbb{Z}}}(\langle i\rangle c,\langle j\rangle d)=\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(c,d)_{i-j}=\mathrm{Hom}_{\mathcal{C}}(d,c)_{i-j}=\mathrm{Hom}_{\mathcal{C}^{\mathbb{Z}}}(\langle -j\rangle d,\langle -i\rangle c)=\mathrm{Hom}_{(\mathcal{C}^{\mathbb{Z}})^{\mathrm{op}}}(\langle -i\rangle c,\langle -j\rangle d)$.

Remark 10.4. We can view $\mathrm{Cat}^{\mathbb{Z}}$ and $\mathrm{Cat}_{\mathbb{Z}}$ as categories with morphisms given by functors compatible with the $\mathbb{Z}$-action respectively by $\mathcal{GSV}\mathrm{ec}$-functors[1]. Then the correspondence from Lemma 10.1 extends to a functor

$$\mathrm{Cat}^{\mathbb{Z}}\to\mathrm{Cat}_{\mathbb{Z}}\ \text{ sending a morphism } F\colon\mathcal{C}\to\mathcal{D}\text{ to }F_{\mathbb{Z}}\colon\mathcal{C}_{\mathbb{Z}}\to\mathcal{D}_{\mathbb{Z}},\tag{10.1}$$

with $F_\mathbb{Z}$ defined as follows. On objects $F_\mathbb{Z}([c]) = [F(c)]$ and $f \in \mathrm{Hom}_{\mathcal{C}_\mathbb{Z}}([\hat{c_1}], [\hat{c_2}])_i = \mathrm{Hom}_\mathcal{C}(\langle i\rangle \hat{c_1}, \hat{c_2})$ is sent to $F_\mathbb{Z}(f) = F(f) \in \mathrm{Hom}_{\mathcal{D}_\mathbb{Z}}(F_\mathbb{Z}([\hat{c_1}]), F_\mathbb{Z}([\hat{c_2}]))_{m_1+i-m_2}$, where $m_i \in \mathbb{Z}$ for $i = 1, 2$ such that $F(\hat{c_i}) = \langle m_i\rangle \widehat{F(\hat{c_i})}$. Here we use that the $\mathbb{Z}$-action is free and that $F(f) \in \mathrm{Hom}_\mathcal{D}(F(\langle i\rangle \hat{c_1}), F(\hat{c_2})) = \mathrm{Hom}_\mathcal{D}(\langle i\rangle F(\hat{c_1}), F(\hat{c_2}))$.

Warning. *A $\mathcal{GSV}$ec-functor $F\colon \mathcal{C} \to \mathcal{D}$ might not have a preimage under* (10.1)*. The functors relevant for representation theory however usually have lifts. For instance, the functors Σ and $\mathcal{T}$ have graded lifts which are given on objects by $a \mapsto \langle\epsilon\rangle(a + 1)$ respectively $a \mapsto -\langle\epsilon\rangle a$. For an example of the existence and construction of graded lifts which are less obvious see e.g.* [Str03].

Definition 10.5. A left (resp. right) $\mathcal{C}$-module for $\mathcal{C} \in \mathrm{Cat}_\mathbb{Z}$ is a co(ntra)variant $\mathcal{GSV}\mathrm{ec}^\circ$-functor $M\colon \mathcal{C} \to \mathcal{GSV}\mathrm{ec}$. The categories $\mathcal{C}$- Rep (and Rep -$\mathcal{C}$) of left (resp. right) modules can again be viewed as $\mathcal{GSV}\mathrm{ec}^\circ$-categories as explained in [Kel05]. These are objects in $\mathrm{Cat}^\mathbb{Z}$ with $\mathbb{Z}$-action given by $\langle i\rangle M(c) = \langle i\rangle(M(c))$, where the $\mathbb{Z}$-action on $\mathcal{GSV}$ec is given by $(\langle i\rangle V)_{n+i} = V_n$ for $i, n \in \mathbb{Z}$.
A left (resp. right) $\mathcal{C}$-module for $\mathcal{C} \in \mathrm{Cat}^\mathbb{Z}$ is a co(ntra)variant functor $M\colon \mathcal{C} \to \mathcal{SV}\mathrm{ec}^\circ$ of $\mathcal{SV}\mathrm{ec}^\circ$-categories. We denote by $\mathcal{C}$- Rep (and Rep -$\mathcal{C}$) the corresponding $\mathcal{SV}\mathrm{ec}^\circ$-category of left (resp. right) modules. This is an object in $\mathrm{Cat}^\mathbb{Z}$ with $\mathbb{Z}$-action given by $\langle i\rangle(M)(c) = M(\langle -i\rangle c)$ (resp. $\langle i\rangle(M)(c) = M(\langle i\rangle c)$).

Remark 10.6. We have Rep -$\mathcal{C} \coloneqq \mathcal{C}^{\mathrm{op}}$- Rep using the opposite category, [Kel05, §1.4].

The following are important examples of left and right modules:

Definition 10.7. Let $\mathcal{C} \in \mathrm{Cat}^\mathbb{Z}$ or $\mathcal{C} \in \mathrm{Cat}_\mathbb{Z}$. The corresponding *projective modules* are $P_c \coloneqq \mathrm{Hom}_\mathcal{C}(c, \_) \in \mathcal{C}$- Rep and ${}_cP \coloneqq \mathrm{Hom}_\mathcal{C}(\_, c) \in$ Rep -$\mathcal{C}$. The *regular $\mathcal{C}$-modules $\mathcal{C}$* are defined as $\mathcal{C} = \bigoplus_c P_c \in \mathcal{C}$- Rep and $\mathcal{C} = \bigoplus_c {}_cP \in$ Rep -$\mathcal{C}$.

For readers who refer less categorical notions the following remark is important:

Remark 10.8. The data of a module $M \in \mathrm{sR}_\epsilon$- Rep or $M \in$ Rep -sR_ϵ is, by taking $\bigoplus_{\boldsymbol{i}} M(\boldsymbol{i})$, equivalent to the data of an ordinary (locally unital) left, respectively right, module for the electric KLR superalgebra from Definition 3.8. The notion of projective and regular modules then boils down to the usual notion of projective modules for a (locally unital) superalgebra.

Lemma 10.9. *Let $\mathcal{C} \in \mathrm{Cat}^\mathbb{Z}$ and $\mathcal{D} \in \mathrm{Cat}_\mathbb{Z}$. Then $\mathcal{D}$-* Rep *can be viewed as object in* $\mathrm{Cat}^\mathbb{Z}$ *by setting $(\langle a\rangle M)(d) := \langle a\rangle(M(d))$ and there are isomorphisms in* $\mathrm{Cat}^\mathbb{Z}$*:*

$$\begin{array}{ccc} \mathcal{C}\text{- Rep} \cong \mathcal{C}_\mathbb{Z}\text{- Rep} & & \mathcal{D}\text{- Rep} \cong \mathcal{D}^\mathbb{Z}\text{- Rep} \\ M \mapsto M_\mathbb{Z} & \textit{and} & M \mapsto M^\mathbb{Z} \\ M^\mathbb{Z} \leftarrow M & & M_\mathbb{Z} \leftarrow M \end{array} \tag{10.2}$$

Proof. In the first case let $M_\mathbb{Z}([c]) = \oplus_{m\in\mathbb{Z}} M(\langle -m\rangle \hat{c})$ and $M^\mathbb{Z}(\langle m\rangle \hat{c}) = M([c])_{-m}$. Any $f \in \mathrm{Hom}_{C_\mathbb{Z}}([c], [c'])_k$ defines an element in $\mathrm{Hom}_\mathcal{C}(\langle k - a\rangle \hat{c}, \langle -a\rangle \hat{c'})$ for any $a \in \mathbb{Z}$ and

then in $\mathrm{Hom}_{\mathcal{SV}\mathrm{ec}^\circ}(M(\langle k-a\rangle\hat{c}), M(\langle -a\rangle\hat{c'})) = \mathrm{Hom}_{\mathcal{SV}\mathrm{ec}^\circ}(M_{\mathbb{Z}}([c])_{a-k}, M_{\mathbb{Z}}([c'])_a)$. These maps, for $a \in \mathbb{Z}$, are the components of $M_{\mathbb{Z}}([c])(f)$.
Conversely, if $f \in \mathrm{Hom}_{\mathcal{C}}(\langle m\rangle\hat{c}, \langle n\rangle\hat{c'}) = \mathrm{Hom}_{\mathcal{C}_{\mathbb{Z}}}([c],[c']))_{m-n}$ we get $M^{\mathbb{Z}}(f) = M(f) \in \mathrm{Hom}_{\mathcal{SV}\mathrm{ec}^\circ}(M([c])_{-m}, M(c')_{-n}) = \mathrm{Hom}_{\mathcal{SV}\mathrm{ec}^\circ}(M^{\mathbb{Z}}(\langle m\rangle\hat{c}), M^{\mathbb{Z}}(\langle n\rangle\hat{c'}))$. We omit checking that these define the isomorphisms. The second case is analogous. □

Remark 10.10. Consider the case $\mathcal{D} = \mathrm{sR}_\epsilon$ or $\mathcal{D} = \mathrm{sR}^\ell_\epsilon$. Under the second isomorphism (10.2) the projective module $\langle m\rangle P^{(\ell)}_{\boldsymbol{i}}$ correspond to $P^{(\ell)}_{\langle m\rangle\boldsymbol{i}}$ for $m \in \mathbb{Z}$.

Notation 10.11. Let $\mathcal{C} \in \mathrm{Cat}^{\mathbb{Z}}$. Given an additive subcategory $\mathcal{A}$ of $\mathcal{C}$-Rep closed under the $\mathbb{Z}$-action, we denote by $K'_0(\mathcal{A})$ the usual additive Grothendieck group. This is a $\mathbb{Z}[q,q^{-1}]$-module by identifying q with $\langle 1\rangle$ in case $\mathcal{A}$ is invariant under the $\mathbb{Z}$-action. We write then $K_0(\mathcal{A}) := \mathbb{Q}(q) \otimes_{\mathbb{Z}[q,q^{-1}]} K'_0(\mathcal{A})$.

This definition applies in particular to the following categories:

Definition 10.12. For $\mathcal{C} \in \mathrm{Cat}^{\mathbb{Z}}$ let $\mathcal{C}$-proj be the idempotent closed additive subcategory of $\mathcal{C}$-Rep generated by the projectives P_c, $c \in \mathcal{C}$. Given $\mathcal{C} \in \mathrm{Cat}_{\mathbb{Z}}$ we write by abuse of language $\mathcal{C}$-proj for the category $\mathcal{D}$-proj where $\mathcal{D} = \mathcal{C}^{\mathbb{Z}}$.

Remark 10.13. The identity on objects and morphisms defines a *contravariant* functor $\mathrm{id}\colon \mathrm{sR}_\epsilon \to \mathrm{sR}_\epsilon^{\mathrm{op}}$ which induces via Remark 10.3 a contravariant functor $\mathrm{sR}^{\mathbb{Z}}_\epsilon \to (\mathrm{sR}^{\mathbb{Z}}_\epsilon)^{\mathrm{op}}$. It induces a *$q$-antilinear* map on K_0 of the representation categories.

10.2. Projective modules for the (cyclotomic) electric KLR algebras

In this section we study the category of projective modules for sR_ϵ and $\mathrm{sR}^\ell_\epsilon$ with their Grothendieck groups. We start with some definitions.

Definition 10.14. Let sR_ϵ-proj be the idempotent closed additive subcategory of sR_ϵ-Rep generated by the projectives $P_{\boldsymbol{i}}$. Similarly, we define $\mathrm{sR}^\ell_\epsilon$-proj for $\mathrm{sR}^\ell_\epsilon$ and denote here the projective module associated with $\boldsymbol{i}$ as $P^\ell_{\boldsymbol{i}}$ to indicate the dependence on ℓ. Let proj-sR_ϵ, proj-$\mathrm{sR}^\ell_\epsilon$ be the analogues for right modules.

Similarly, let sR_ϵ-$\mathrm{proj}^{\mathbb{Z}}$ be the idempotent closed additive subcategory of $\mathrm{sR}^{\mathbb{Z}}_\epsilon$-*Rep* generated by the projectives $P_{\langle m\rangle\boldsymbol{i}}$, $m \in \mathbb{Z}$.

Notation 10.15. Given $\mathcal{SV}\mathrm{ec}^\circ$-categories $\mathcal{C}$ and $\mathcal{D}$, we denote by $\mathcal{C}\boxtimes\mathcal{D}$ the *Deligne–Kelly tensor product* of $\mathcal{C}$ and $\mathcal{D}$. Given $M \in \mathcal{C}$-Rep, $N \in \mathcal{D}$-Rep, we have the *outer tensor product* $M \boxtimes N \in \mathcal{C}\boxtimes\mathcal{D}$-Rep given by $M \boxtimes N(c,d) := M(c) \otimes N(d)$.

Remark 10.16. More precisely, objects of $\mathcal{C}\boxtimes\mathcal{D}$ are pairs (c,d) with $c \in \mathcal{C}$, $d \in \mathcal{D}$ and $\mathrm{Hom}_{\mathcal{C}\boxtimes\mathcal{D}}((c,d),(c',d')) = \mathrm{Hom}_{\mathcal{C}}(c,c') \otimes \mathrm{Hom}_{\mathcal{D}}(d,d')$. The tensor product is in $\mathcal{GSV}\mathrm{ec}$ or $\mathcal{GSV}\mathrm{ec}^\circ$ if the original categories were enriched in these. For details on the abstract definition see [Kel05, §6.5]. Note that this construction is compatible with Lemma 10.1 in the sense that $(\mathcal{C}\boxtimes\mathcal{D})_{\mathbb{Z}} \cong \mathcal{C}_{\mathbb{Z}} \boxtimes \mathcal{D}_{\mathbb{Z}}$ and $(\mathcal{C}\boxtimes\mathcal{D})^{\mathbb{Z}} \cong \mathcal{C}^{\mathbb{Z}} \boxtimes \mathcal{D}^{\mathbb{Z}}$.

10.2.1. Tensor products of projective modules for sR_ϵ and $\mathrm{sR}_\epsilon^\ell$

Using horizontal stacking of diagrams in sR_ϵ we have a canonical map $\mathrm{sR}_\epsilon \boxtimes \mathrm{sR}_\epsilon \to \mathrm{sR}_\epsilon$ which allows us to view the regular module sR_ϵ as a $(\mathrm{sR}_\epsilon, \mathrm{sR}_\epsilon \boxtimes \mathrm{sR}_\epsilon)$-bimodule. As in [KL09, §2.6] this provides induction and restriction functors and the following definition:

Definition 10.17. For $M, N \in \mathrm{sR}_\epsilon\text{-}\mathrm{Rep}$ define their *tensor product*

$$M \cdot N := \mathrm{ind}_{\mathrm{sR}_\epsilon \boxtimes \mathrm{sR}_\epsilon}^{\mathrm{sR}_\epsilon} M \boxtimes N \in \mathrm{sR}_\epsilon\text{-}\mathrm{Rep}\,. \tag{10.3}$$

The tensor product $M \cdot N$ of two right sR_ϵ-modules is defined similarly.

The following statements about sR_ϵ-proj and proj-sR_ϵ are clear from the definitions:

Lemma 10.18. *We have $P_{\boldsymbol{i}} \cdot P_{\boldsymbol{j}} \cong P_{\boldsymbol{ij}}$ and ${}_{\boldsymbol{i}}P \cdot {}_{\boldsymbol{j}}P \cong {}_{\boldsymbol{ij}}P$. In particular, $K_0(\mathrm{sR}_\epsilon\text{-proj})$ and $K_0(\text{proj-}\mathrm{sR}_\epsilon)$ are $\mathbb{Q}(q)$-algebras with multiplication given by tensor product.*

Remark 10.19. The tensor product $\cdot$ provides a monoidal structure on sR_ϵ-proj with unit object $\mathbf{1} = P_\emptyset$. Moreover, $\mathrm{sR}_\epsilon^\ell\text{-}\mathrm{Rep}$ is a right module category over sR_ϵ-proj, see Lemma 10.21 below. The same holds for proj-sR_ϵ with $\mathbf{1} = {}_\emptyset P$ and $\mathrm{Rep}\text{-}\mathrm{sR}_\epsilon^\ell$.

Notation 10.20. For any object $\boldsymbol{i}$ in sR_ϵ let $P_{\boldsymbol{i}}^\ell \in \mathrm{sR}_\epsilon^\ell\text{-}\mathrm{Rep}$ and ${}_{\boldsymbol{i}}P^\ell \in \mathrm{Rep}\text{-}\mathrm{sR}_\epsilon^\ell$ be the corresponding projective module (in contrast to $P_{\boldsymbol{i}} \in \mathrm{sR}_\epsilon\text{-}\mathrm{Rep}$ and ${}_{\boldsymbol{i}}P \in \mathrm{Rep}\text{-}\mathrm{sR}_\epsilon$).

Horizontal stacking of diagrams gives a morphism $\mathrm{sR}_\epsilon^\ell \boxtimes \mathrm{sR}_\epsilon \to \mathrm{sR}_\epsilon^\ell$. Thus, given $M \in \mathrm{sR}_\epsilon^\ell\text{-}\mathrm{Rep}$ and $N \in \mathrm{sR}_\epsilon\text{-}\mathrm{Rep}$ we obtain $M \cdot N := \mathrm{ind}_{\mathrm{sR}_\epsilon^\ell \boxtimes \mathrm{sR}_\epsilon}^{\mathrm{sR}_\epsilon^\ell} M \boxtimes N \in \mathrm{sR}_\epsilon^\ell\text{-}\mathrm{Rep}$. Similarly, for right modules. The following is immediate from the definitions.

Lemma 10.21. *We have $P_{\boldsymbol{i}}^\ell \cdot P_{\boldsymbol{j}} \cong P_{\boldsymbol{ij}}^\ell$ and ${}_{\boldsymbol{i}}P^\ell \cdot {}_{\boldsymbol{j}}P \cong {}_{\boldsymbol{ij}}P^\ell$. In particular, the tensor product turns $K_0(\mathrm{sR}_\epsilon^\ell\text{-proj})$ into a right module for $K_0(\mathrm{sR}_\epsilon\text{-proj})$ and $K_0(\text{proj-}\mathrm{sR}_\epsilon^\ell)$ into a right module for $K_0(\text{proj-}\mathrm{sR}_\epsilon)$.*

Definition 10.22. For $\lambda \in \mathrm{Par}^\ell$ let $P_\lambda^\ell := P_{\boldsymbol{i}_{t\lambda}}^\ell$ and ${}^\lambda P^\ell := {}_{\boldsymbol{i}_{t\lambda}^\circledast}P$. From Theorem 3.25, we also get the *left standard module* $\Delta_\lambda \in \mathrm{sR}_\epsilon^\ell\text{-}\mathrm{Rep}$ and the *right standard* $\mathrm{sR}_\epsilon^\ell$*-module* ${}^\lambda\Delta \in \mathrm{Rep}\text{-}\mathrm{sR}_\epsilon^\ell$ defined as the respective quotients by all morphism which factor through some P_μ with $\mu < \lambda$.

Lemma 10.21 does in fact not require the level to be generic. If it is however generic, then $P_{\boldsymbol{i}}^\ell = 0$ or we find $\lambda \in \mathrm{Par}^\ell$ such that $P_{\boldsymbol{i}}^\ell \cong P_\lambda^\ell$, see Corollary 3.37, similarly for right modules. This observation should motivate the following:

Lemma 10.23. *The following sets each from a $\mathbb{Z}$-basis:*

$$\begin{aligned} &\text{for } K_0'(\mathrm{sR}_\epsilon^\ell\text{-proj}): && \{[\langle i\rangle P_\lambda^\ell] \mid \lambda \in \mathrm{Par}, i \in \mathbb{Z}\}, && \{[\langle i\rangle \Delta_\lambda] \mid \lambda \in \mathrm{Par}, i \in \mathbb{Z}\},\\ &\text{for } K_0'(\text{proj-}\mathrm{sR}_\epsilon^\ell): && \{[\langle i\rangle {}^\lambda P^\ell] \mid \lambda \in \mathrm{Par}, i \in \mathbb{Z}\}, && \{[\langle i\rangle {}^\lambda\Delta] \mid \lambda \in \mathrm{Par}, i \in \mathbb{Z}\},\end{aligned}$$

and the sets $\{[P_\lambda^\ell]\}$, $\{[\Delta_\lambda]\}$, $\{[{}^\lambda P^\ell]\}$, $\{[{}^\lambda\Delta]\}$ with $\lambda \in \mathrm{Par}$ form $\mathbb{Q}(q)$-bases of K_0.

Proof. The statements for $[P_\lambda^\ell]$ follow directly from Theorem 3.25. By definition of Δ_λ and the upper-finite labelling set, the "base change" matrix is upper triangular with 1's on the diagonal and only finitely many non-zero entries in each row. Therefore, it is invertible and the $[\Delta_\lambda]$ form a basis as well. Alternatively, one could apply [Bru25, Theorem 8.3]. The same arguments work for right modules. □

10.3. Bar involutions and pairings

We define $\mathrm{HOM}_{\mathcal{C}\text{-}\mathrm{Rep}}(M,N) \coloneqq \bigoplus \mathrm{Hom}_{\mathcal{C}\text{-}\mathrm{Rep}}(M\langle i\rangle, N) \in \mathcal{GSV}\mathrm{ec}$ for any $\mathcal{GSV}\mathrm{ec}$-category $\mathcal{C}$ and $M, N \in \mathcal{C}\text{-}\mathrm{Rep}$, which is the space of morphisms when $\mathcal{C}\text{-}\mathrm{Rep}$ is viewed as a $\mathcal{GSV}\mathrm{ec}$-category.
For a graded (super)vector space $V = \oplus_{n\in\mathbb{Z}} V_n$ with $V_n = 0$ for $n \ll 0$ we let $\mathrm{gdim}(V) = \sum \dim V_n q^n \in \mathbb{N}[q^{-1}][[q]]$ be its graded dimension.

Definition 10.24. Define the grading-reversing contravariant functor

$$\overline{}\colon \mathrm{sR}_\epsilon\text{-proj} \to \text{proj-}\mathrm{sR}_\epsilon, \quad \overline{P} \coloneqq \mathrm{HOM}_{\mathrm{sR}_\epsilon}(P, \mathrm{sR}_\epsilon).$$

It satisfies $\overline{P_{\boldsymbol{i}}\langle a\rangle} = {}_{\boldsymbol{i}}P\langle -a\rangle$. It descends to a functor $\overline{}\colon \mathrm{sR}^\ell_\epsilon\text{-proj} \to \text{proj-}\mathrm{sR}^\ell_\epsilon$ which satisfies the analogous property on projectives P^ℓ.
We also define $\overline{}\colon \text{proj-}\mathrm{sR}^{(\ell)}_\epsilon \to \mathrm{sR}_\epsilon\text{-proj}^{(\ell)}$ by the same formula. It satisfies $\overline{{}_{\boldsymbol{i}}P\langle a\rangle} = P_{\boldsymbol{i}}\langle -a\rangle$ and descends again to the cyclotomic quotients.
In particular, we have $\overline{\overline{P}} = P$ for any left or right sR_ϵ or $\mathrm{sR}^\ell_\epsilon$-module P. This is why we also call this functor *Bar involution.*

The following is immediate from the monoidality of $\overline{}$.

Lemma 10.25. *The Bar involutions on* $\mathrm{sR}^\ell_\epsilon$-proj *and* sR_ϵ-proj *are compatible with the right module structure, that is* $\overline{M \cdot N} \cong \overline{M} \cdot \overline{N}$ *for* $M \in \mathrm{sR}^{(\ell)}_\epsilon$-proj, $N \in \mathrm{sR}_\epsilon$-proj. *The same holds true for the right* sR_ϵ-proj-*module structure on* proj-$\mathrm{sR}^\ell_\epsilon$.

Definition 10.26. We define a q-bilinear pairing

$$(\_,\_)\colon K_0(\text{proj-}\mathrm{sR}^\ell_\epsilon) \otimes K_0(\mathrm{sR}^\ell_\epsilon\text{-proj}) \to \mathbb{Q}(q),$$
$$[P'] \otimes [P] \mapsto \mathrm{gdim}(P' \otimes_{\mathrm{sR}^\ell_\epsilon} P).$$

Remark 10.27. The pairing $(\_,\_)$ is related to the HOM pairing as follows. Given P and $Q \in \mathrm{sR}^\ell_\epsilon$-proj we have

$$\mathrm{gdim}\,\mathrm{HOM}_{\mathrm{sR}^\ell_\epsilon}(P,Q) = (\overline{P}, Q).$$

For two P' and $Q' \in$ proj-$\mathrm{sR}^\ell_\epsilon$ we have

$$\mathrm{gdim}\,\mathrm{HOM}_{\mathrm{sR}^\ell_\epsilon}(P',Q') = (Q', \overline{P'}). \tag{10.4}$$

The next lemma essentially follows from (sR-4), but we prove it to make sure that all the grading shifts agree.

Lemma 10.28. *The bilinear form satisfies*

$$([P']\cdot[Q],[P]) = ([P'],[P]\cdot[\Sigma(Q)])$$

Proof. We may assume that $P' = {}_{\boldsymbol{j}}P$, $P = P_{\boldsymbol{i}}$ and $Q = {}_kP$. Then we have $P'\cdot Q = {}_{\boldsymbol{j}k}P$ and $P\cdot\Sigma(Q) = P\cdot P_{k+1}\langle -\epsilon\rangle = P_{\boldsymbol{i}k+1}\langle -\epsilon\rangle$. And thus,

$${}_{\boldsymbol{j}k}P \otimes_{\mathrm{sR}^\ell_\epsilon} P_{\boldsymbol{i}} = \mathrm{HOM}_{\mathrm{sR}^\ell_\epsilon}(\boldsymbol{i}, \boldsymbol{j}k) = \mathrm{HOM}_{\mathrm{sR}^\ell_\epsilon}(\boldsymbol{i}k+1, \boldsymbol{j})\langle -\epsilon\rangle = {}_{\boldsymbol{j}}P \otimes_{\mathrm{sR}^\ell_\epsilon} P_{\boldsymbol{i}k+1}\langle -\epsilon\rangle. \qquad \square$$

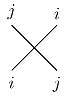
10.4. Relations in Grothendieck groups

As preparation for the categorification results in the next section we calculate some crucial relations in $K_0(\mathrm{sR}_\epsilon\text{-proj})$ and $K_0(\text{proj-}\mathrm{sR}_\epsilon)$. For this we extend the parameters b_{ij} from Definition 9.1 to $i, j \in \mathbb{R}$:

Definition 10.29. For $i, j \in \mathbb{R}$ let $b_{ij} = -2$ if $j = i, i+1$, let $b_{ij} = 0$ if $|i-j| \notin \mathbb{Z}$, and set $b_{ij} = 4 \cdot \mathrm{sgn}(j-i)(-1)^{j-i}$ otherwise.

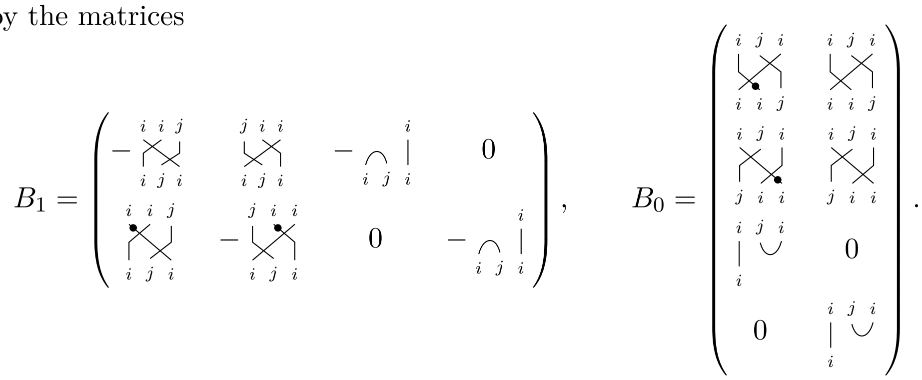
Proposition 10.30. *In* sR_ϵ-proj *and* proj-sR_ϵ *we have for any* $i \neq j \in \mathbb{R}$*:*

$$\begin{aligned}
P_{iji}\langle 1\rangle \oplus P_{iji}\langle -1\rangle &\cong P_{iij}\langle 3\rangle \oplus P_{jii}\langle -3\rangle \oplus P_i\langle \epsilon+1\rangle \oplus P_i\langle \epsilon-1\rangle && \textit{if } j = i+1,\\
P_{iji}\langle 1\rangle \oplus P_{iji}\langle -1\rangle &\cong P_{iij}\langle -3\rangle \oplus P_{jii}\langle 3\rangle \oplus P_i\langle \epsilon+1\rangle \oplus P_iP\langle \epsilon-1\rangle && \textit{if } j = i-1,\\
{}_{iji}P\langle 1\rangle \oplus {}_{iji}P\langle -1\rangle &\cong {}_{iij}P\langle -3\rangle \oplus {}_{jii}P\langle 3\rangle \oplus {}_iP\langle \epsilon-1\rangle \oplus {}_iP\langle -\epsilon-1\rangle && \textit{if } j = i+1,\\
{}_{iji}P\langle 1\rangle \oplus {}_{iji}P\langle -1\rangle &\cong {}_{iij}P\langle 3\rangle \oplus {}_{jii}P\langle -3\rangle \oplus {}_iP\langle 1-\epsilon\rangle \oplus {}_iP\langle -1-\epsilon\rangle && \textit{if } j = i-1,\\
P_{ij} \cong P_{ji}\langle b_{ij}\rangle \ &\textit{and } {}_{ij}P \cong {}_{ji}P\langle -b_{ij}\rangle && \textit{otherwise.}
\end{aligned}$$

Proof. The morphism  has degree $b_{ij} = -b_{ji}$. It defines homogeneous degree 0 maps $P_{ji} \to P_{ij}P\langle b_{ji}\rangle$ and ${}_{ij}P \to {}_{ji}P\langle -b_{ij}\rangle$. Since both are isomorphisms by (sR-6) the last two claims follow.
Of the remaining relations we will only prove the first one as they are all similar. For this let $j = i+1$ and consider

$$\begin{aligned}
B_1 &\colon P_{iij}\langle 3\rangle \oplus P_{jii}\langle -3\rangle \oplus P_i\langle 1+\epsilon\rangle \oplus P_i\langle 1+\epsilon\rangle \to P_{iji}\langle 1\rangle \oplus P_{iji}\langle -1\rangle\\
B_0 &\colon P_{iji}\langle 1\rangle \oplus P_{iji}\langle -1\rangle \to P_{iij}\langle 3\rangle \oplus P_{jii}\langle -3\rangle \oplus P_i\langle 1+\epsilon\rangle \oplus P_i\langle 1+\epsilon\rangle
\end{aligned}$$

given by the matrices



Note that all the entries are homogeneous and provide two degree zero maps. They are mutually inverses by Lemma 11.1 in Section 11.5. □

10.5. Categorification Theorems

In this section we finally apply our results to deduce some categorification results.

10.5.1. A categorification of the q-ϵlectrical algebra

The first result is the following *q-ϵlectric Categorification Theorem* analogous to [KL09], [Rou08]:

Theorem 10.31. *There are $\mathbb{Q}(q)$-linear algebra isomorphisms*

$$\begin{aligned} \Phi_q \colon \mathfrak{el}_q^\epsilon &\to K_0(\mathrm{sR}_\epsilon(\mathbb{Z})\text{-proj}) & \qquad \Phi_{q^{-1}} \colon \mathfrak{el}_{q^{-1}}^\epsilon &\to K_0(\text{proj-}\mathrm{sR}_\epsilon(\mathbb{Z})), \\ \mathcal{E}_i &\mapsto [P_i], & \mathcal{E}_i &\mapsto [{}_iP]. \end{aligned}$$

Proof. By Proposition 10.30 the assignments extend to a well-defined algebra homomorphism. Recall from Lemma 9.7 that the algebra $\mathfrak{el}_q^\epsilon$ is a filtered with $\mathcal{E}_{i_1} \cdots \mathcal{E}_{i_k}$ in filtration degree k. On the other hand $\mathrm{sR}_\epsilon(\mathbb{Z})$-proj is a filtered category in the sense of [FLP23, §4.3], where $P_{i_1 \dots i_k}$ sits in filtration degree k. This induces a filtration on $K_0(\mathrm{sR}_\epsilon(\mathbb{Z})\text{-proj})$ so that Φ_q is actually a morphism of filtered algebras. We obtain a commutative diagram in vector spaces with vertical isomorphisms:

$$\begin{tikzcd} \mathfrak{el}_q^\epsilon \arrow[r, "\Phi_q"] \arrow[d, "\mathrm{gr}"] & K_0(\mathrm{sR}_\epsilon(\mathbb{Z})\text{-proj}) \arrow[d, "\mathrm{gr}"] \\ \mathrm{gr}\, \mathfrak{el}_q^\epsilon \arrow[r, "\mathrm{gr}\, \Phi_q"] & \mathrm{gr}\, K_0(\mathrm{sR}_\epsilon(\mathbb{Z})\text{-proj}) \end{tikzcd}$$

Thus, it suffices to show that $\mathrm{gr}\, \Phi_q$ is an isomorphism. Now by [FLP23, Theorem 4.19] we know that $\mathrm{gr}\, K_0(\mathrm{sR}_\epsilon(\mathbb{Z})\text{-proj}) \cong K_0(\mathrm{gr}\, \mathrm{sR}_\epsilon(\mathbb{Z})\text{-proj})$. The category $\mathrm{gr}\, \mathrm{sR}_\epsilon(\mathbb{Z})$-proj arises by quotienting out everything that factors through a lower filtration degree. In our case this means that we kill every cup and cap. From the defining relations (sR-1)-(sR-7) we see that $\mathrm{gr}(\mathrm{sR}_\epsilon(\mathbb{Z})\text{-proj})$ is equivalent to $R-$proj from [KL09] if we ignore the $\mathbb{Z}$-grading. On the other hand, the algebra $\mathrm{gr}\, \mathfrak{el}_q^\epsilon$ is by Lemma 9.7 up to a different q-shifts exactly the algebra $\mathbf{f}$ from [KL09]. One quickly checks that the q-shifts match the different grading. Then, the statement follows from [KL09, Theorem 1.1]. $\square$

10.5.2. Categorified involutions

Theorem 10.32 (Compatibilities Theorem)**.** *The following diagrams commute:*

$$\begin{tikzcd} \mathfrak{el}_q^\epsilon \arrow[r, "\Phi_q"] \arrow[d, "-"] & K_0(\mathrm{sR}_\epsilon(\mathbb{Z})\text{-proj}) \arrow[d, "-"] \\ \mathfrak{el}_{q^{-1}}^\epsilon \arrow[r, "\Phi_{q^{-1}}"] & K_0(\text{proj-}\mathrm{sR}_\epsilon(\mathbb{Z})) \end{tikzcd}, \qquad \begin{tikzcd} \mathfrak{el}_q^\epsilon \arrow[r, "\Phi_q"] \arrow[d, "\sigma"] & K_0(\mathrm{sR}_\epsilon(\mathbb{Z})\text{-proj}) \arrow[d, "\Sigma"] \\ \mathfrak{el}_{q^{-1}}^\epsilon \arrow[r, "\Phi_{q^{-1}}"] & K_0(\text{proj-}\mathrm{sR}_\epsilon(\mathbb{Z})) \end{tikzcd}.$$

Proof. For the commutative diagrams, it suffices to check the claim for $\mathcal{E}_i$. We have $\overline{P_i} = {}_iP$, whence the left diagram commutes. Similarly, $\Sigma(P_i) = {}_{i+1}P\langle -\epsilon \rangle$. $\square$

The classes $[P_i]$ and $[{}_iP]$ provide a *canonical basis* of $\mathfrak{el}_q^\epsilon$ respectively $\mathfrak{el}_{q^{-1}}^\epsilon$.

Remark 10.33. Alternatively one could work directly with the additive closure of the Karoubian closure of $\mathrm{sR}_\epsilon(\mathbb{Z})$ and take its K_0. Then the bar involution is categorified via the functor in Remark 10.13, whereas σ and τ from Lemmas 9.5 and 9.6 are categorified by Σ and $\mathcal{T}$ respectively from Lemma 3.9.

10.5.3. Categorification of the q-ϵlectrical Fock spaces

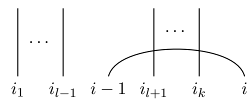
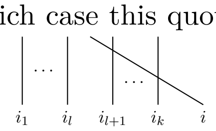
We next categorify the $\ell = 1$ (dual) Fock space of charge zero. We show that the right $\mathrm{sR}_\epsilon(\mathbb{Z})$-proj-module structure on $\mathrm{sR}^\ell_\epsilon(\mathbb{Z})$-proj categorifies the right action of $\mathfrak{el}^\epsilon_q$ on $\mathscr{F}$, similarly for the right proj-$\mathrm{sR}_\epsilon(\mathbb{Z})$-module structure on proj-$\mathrm{sR}^\ell_\epsilon(\mathbb{Z})$ and the action of $\mathfrak{el}^\epsilon_{q^{-1}}$ on $\mathscr{F}^\circledast$.

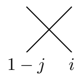
Theorem 10.34 (Fock space categorification)**.** *The q-linear map*

$$\Psi\colon \mathscr{F} \to K_0(\mathrm{sR}^\ell_\epsilon(\mathbb{Z})\text{-proj}),\ v_\lambda \mapsto [\Delta_\lambda]$$

is an isomorphism and the following diagram is commutative:

$$\begin{tikzcd}
\mathscr{F}_\delta \otimes \mathfrak{el}^\epsilon_q \arrow[rr] \arrow[d, "\Psi\otimes\Phi_\delta"] & & \mathscr{F}_\delta \arrow[d, "\Psi"] \\
K_0(\mathrm{sR}^\ell_\epsilon(\mathbb{Z})\text{-proj}) \otimes K_0(\mathrm{sR}_\epsilon(\mathbb{Z})\text{-proj}) \arrow[rr] & & K_0(\mathrm{sR}^\ell_\epsilon(\mathbb{Z})\text{-proj})
\end{tikzcd}$$

Proof. The first part is obvious from Lemma 10.23. For the second part we need to compute $[\Delta_\lambda][P_i]$. The module Δ_λ has a basis given by $\mathtt{B}^{\mathfrak{s}}_{\mathfrak{t}^\lambda}$. The module P_λ has a basis given by all $\mathtt{B}^{\mathfrak{s}}_{\mathfrak{t}}$, where $\mathrm{res}^l\,\mathfrak{t} = \boldsymbol{i}_\lambda$. Additionally, $P_\lambda.P_i$ has a basis given by all $\mathtt{B}^{\mathfrak{s}}_{\mathfrak{t}}$ where $\mathrm{res}^l\,\mathfrak{t} = \boldsymbol{i}_\lambda i$. Therefore, $\Delta_\lambda.P_i$ has a basis given by $\mathtt{B}^{\mathfrak{s}}_{\mathfrak{t}}$ where $\mathrm{res}^l\,\mathfrak{t} = \boldsymbol{i}_\lambda i$ but only in the last step of $\mathfrak{t}$ a box might be removed. Now all such $\mathtt{B}^{\mathfrak{s}}_{\mathfrak{t}}$ where the last step removes a box form a submodule of $\Delta_\lambda.P_i$. This is nonzero if and only if λ has a removable box $\square$ of content $i-1$ in which case it is isomorphic to $\Delta_{\lambda\ominus\square}\langle d\rangle$, where d is the degree of the diagram  and if $\square$ was in row r, then $l = \lambda_1 + \cdots + \lambda_r$. The quotient of $\Delta_\lambda.P_i$ by this submodule is nonzero if and only if λ admits an addable box β of residue i, in which case this quotient is isomorphic to $\Delta_{\lambda\oplus\beta}\langle d'\rangle$, where d' is the degree of the diagram  and if β is added in row r, then $l = \lambda_1 + \cdots + \lambda_r$.

It remains to check that d and d' give the same degree shifts as the K appearing in the comultiplication of $\mathcal{E}_i$. Observe that we act (by our implicit choice of tensor product) on v_{λ_j+1-j} by K_{β_i} if $1-j$ is even and by $K_{\beta'_i}$ if $1-j$ is odd. This means that we get q^0 contribution for every even λ_j, a $q^{(-1)^i 4}$ contribution for even $1-j$ and odd λ_j and $q^{(-1)^{i+1}4}$ contribution for odd $1-j$ and odd λ_j. In other words every even λ_j gives a q^0 contribution and every odd λ_j gives a $q^{(-1)^{i+1-j}4}$ contribution. On the other hand, observe that the crossings swap i with rows λ_k, λ_{k-1} until λ_{r+1} (if λ has k rows). Now if λ_j is even, swapping with this row gives degree 0. If λ_j is odd, as in the even case, consecutive pairs of crossings cancel in their degree, and we are left with the degree of . This has exactly degree $4(-1)^{i+1-j}$ as $i > 1-j$ and the diagram in the theorem commutes. $\square$

Theorem 10.35 (Dual Fock space categorification)**.** *The q-linear map*

$$\Psi' \colon \mathscr{F}^{\circledast} \to K_0(\text{proj-sR}^{\ell}_{\epsilon}(\mathbb{Z})),\ v^{\lambda} \mapsto [{}^{\lambda}\Delta]$$

is an isomorphism and the following diagram is commutative:

$$\begin{tikzcd} \mathscr{F}^{\circledast} \otimes \mathfrak{el}^{\epsilon}_{q^{-1}} \arrow[rr] \arrow[d, "\Psi' \otimes \Phi_{q^{-1}}"] & & \mathscr{F}^{\circledast} \arrow[d, "\Psi'"] \\ K_0(\text{proj-sR}^{\ell}_{\epsilon}(\mathbb{Z})) \otimes K_0(\text{proj-sR}_{\epsilon}(\mathbb{Z})) \arrow[rr] & & K_0(\text{proj-sR}^{\ell}_{\epsilon}(\mathbb{Z})). \end{tikzcd}$$

Proof. This is similar to the left module version from Theorem 10.34. $\square$

Proposition 10.36 (Compatibility with bar involution)**.** *The following diagram commutes.*

$$\begin{tikzcd} \mathscr{F}_{\delta} \arrow[r, "-"] \arrow[d, "\Psi'"] & \mathscr{F}^{\circledast}_{\delta} \arrow[d, "\Psi"] \\ K_0(\text{sR}^{\ell}_{\epsilon}(\mathbb{Z})\text{-proj}) \arrow[r, "-"] & K_0(\text{proj-sR}^{\ell}_{\epsilon}(\mathbb{Z})). \end{tikzcd}$$

Proof. The vector space $\mathscr{F}_{\delta}$ is generated by $v_{\emptyset}$ as an $\mathfrak{el}^{\epsilon}_{q}$-module. $\square$

Remark 10.37. The canonical basis of $\mathfrak{el}^{\epsilon}_{q^{-1}}$ respectively $\mathfrak{el}^{\epsilon}_{q^{-1}}$ induces a canonical basis of $\mathscr{F}_{\delta}$ and $\mathscr{F}^{\circledast}_{\delta}$. They correspond to the classes $[P^{\ell}_{\boldsymbol{i}}]$ and $[{}_{\boldsymbol{i}}P^{\ell}]$ respectively.

Proposition 10.38 (Compatibility with pairing)**.** *We have $(w, v) = (\Phi^{\ell}_{q}{}'(w), \Phi^{\ell}_{q}(v))$ for all $w \in \mathscr{F}^{\circledast}_{\delta}$, $v \in \mathscr{F}_{\delta}$.*

Proof. It suffices to check that $([{}^{\lambda}\Delta], [\Delta_{\mu}]) = \delta_{\lambda\mu}$. But this is immediate from Theorem 3.25 and (10.4) using that projective $\text{sR}^{\ell}_{\epsilon}$-modules have Δ-flags and the Ext-vanishing, see [BS24, Theorem 3.14], between Δ's and ∇'s. $\square$

10.5.4. Universal categorification and higher level Fock spaces

To incorporate $\boldsymbol{\delta}$ and higher level Fock spaces we work now with $\text{sR}_{\epsilon}(\mathbb{R})$ instead of $\text{sR}_{\epsilon}(\mathbb{Z})$.

Definition 10.39. The *universal electrical algebra* $\mathbb{Q}(q)$-algebra $\mathfrak{el}^{\epsilon}_{q}(\mathbb{R})$ is generated by $\mathcal{E}_i$, $i \in \mathbb{R}$, with relations (ϵl-1), (ϵl-2), (ϵl-3) using Definition 10.29.

For a fixed level ℓ and a generic charge vector $\boldsymbol{\delta}$, see Notation 2.8, set $\mathbb{R}(\ell, \boldsymbol{\delta}) = \bigcup_{j=1}^{\ell}(\delta_j + \mathbb{Z}1) \subset \mathbb{R}$. We consider the full monoidal supersubcategory $\mathrm{sR}_\epsilon(\ell, \boldsymbol{\delta}) = \mathrm{sR}_\epsilon(\mathbb{R}(\ell, \boldsymbol{\delta}))$ of $\mathrm{sR}_\epsilon(\mathbb{R})$ with objects sequences of elements in $\mathbb{R}(\ell, \boldsymbol{\delta})$. We also let $\mathrm{sR}_\epsilon^{\ell}(\boldsymbol{\delta})$ be the associated level ℓ cyclotomic quotient.
Denote by $\mathfrak{el}_q^{\epsilon}(\ell, \boldsymbol{\delta})$ the $\mathbb{Q}(q)$-algebra generated by the $\mathcal{E}_i$ for $i \in \mathbb{R}(\ell, \boldsymbol{\delta})$. In particular, $\mathfrak{el}_q^{\epsilon}(1, \boldsymbol{\delta}) = \mathfrak{el}_q^{\epsilon}$ if $\delta_1 \in \mathbb{Z}$ and $\mathfrak{el}_q^{\epsilon}(\ell, \boldsymbol{\delta}) \cong \mathfrak{el}_q^{\epsilon} \otimes \cdots \otimes \mathfrak{el}_q^{\epsilon}$, the ℓ-fold tensor product of $\mathfrak{el}_q^{\epsilon}$, since $\boldsymbol{\delta}$ is generic. Similarly define $\mathfrak{el}_{q^{-1}}^{\epsilon}(\ell, \boldsymbol{\delta})$ with $\mathfrak{el}_{q^{-1}}^{\epsilon}(\ell, \boldsymbol{\delta}) \cong \mathfrak{el}_{q^{-1}}^{\epsilon} \otimes \cdots \otimes \mathfrak{el}_{q^{-1}}^{\epsilon}$.
With these definitions we obtain as in Theorem 10.31 directly the following:

Theorem 10.40 (Universal categorification)**.** *There are algebra isomorphisms*

$$\begin{aligned} \Phi_q \colon \mathfrak{el}_q^{\epsilon}(\ell, \boldsymbol{\delta}) &\to K_0(\mathrm{sR}_\epsilon(\ell, \boldsymbol{\delta})\text{-proj}) & \Phi_{q^{-1}} \colon \mathfrak{el}_{q^{-1}}^{\epsilon}(\ell, \boldsymbol{\delta}) &\to K_0(\text{proj-}\mathrm{sR}_\epsilon(\ell, \boldsymbol{\delta})), \\ \mathcal{E}_i &\mapsto [P_i], & \mathcal{E}_i &\mapsto [{}_iP]. \end{aligned}$$

Recall from Definition 9.54 the higher level Fock space $\mathscr{F}_{\boldsymbol{\delta},\ell}$.

Theorem 10.41 (Higher level Fock space categorification)**.** *The q-linear map*

$$\Psi_\ell \colon \mathscr{F}_{\boldsymbol{\delta},\ell} \to K_0(\mathrm{sR}_\epsilon^{\ell}(\boldsymbol{\delta})),\ v_\lambda \mapsto [\Delta_\lambda]$$

is an isomorphism and the following diagram is commutative:

$$\begin{tikzcd} \mathscr{F}_{\boldsymbol{\delta},\ell} \otimes \mathfrak{el}_q^{\epsilon}(\ell, \boldsymbol{\delta}) \arrow[r] \arrow[d, "\Psi_\ell \otimes \Phi_q"] & \mathscr{F}_{\boldsymbol{\delta},\ell} \arrow[d, "\Psi_\ell"] \\ K_0(\mathrm{sR}_\epsilon^{\ell}(\boldsymbol{\delta})\text{-proj}) \otimes K_0(\mathrm{sR}_\epsilon(\ell, \boldsymbol{\delta})\text{-proj}) \arrow[r] & K_0(\mathrm{sR}_\epsilon^{\ell}(\boldsymbol{\delta})\text{-proj}). \end{tikzcd}$$

Proof. The proof of Theorem 10.34 can be just copied. □

Similar to Definition 9.54 there is the higher level dual Fock space

$$\mathscr{F}_{\boldsymbol{\delta},\ell}^{\circledast} = \mathscr{F}_{\delta_1}^{\circledast} \otimes \cdots \otimes \mathscr{F}_{\delta_\ell}^{\circledast}$$

of level ℓ and charge $\boldsymbol{\delta}$. Theorem 10.35 directly generalizes to the following

Theorem 10.42 (Higher level dual Fock space categorification)**.** *The q-linear map*

$$\Psi_\ell' \colon \mathscr{F}_{\delta_1}^{\circledast} \otimes \cdots \otimes \mathscr{F}_{\delta_\ell}^{\circledast} \to K_0(\text{proj-}\mathrm{sR}_\epsilon^{\ell}(\boldsymbol{\delta})),\ v^\lambda \mapsto [{}^\lambda\Delta]$$

is an isomorphism and the following diagram is commutative:

$$\begin{tikzcd} \mathscr{F}^{\circledast} \otimes \mathfrak{el}_{q^{-1}}^{\epsilon}(\ell, \boldsymbol{\delta}) \arrow[r] \arrow[d, "\Psi_\ell' \otimes \Phi_{q^{-1}}"] & \mathscr{F}^{\circledast} \arrow[d, "\Psi_\ell'"] \\ K_0(\text{proj-}\mathrm{sR}_\epsilon^{\ell}(\boldsymbol{\delta})) \otimes K_0(\text{proj-}\mathrm{sR}_\epsilon) \arrow[r] & K_0(\text{proj-}\mathrm{sR}_\epsilon^{\ell}(\boldsymbol{\delta})). \end{tikzcd}$$

Proof. The proof of Theorem 10.35 can be just copied. □

Remark 10.43. We consider in this article only generic charge vectors, see Notation 2.8. This allows to distinguish the components of a multi-partition. In fact, the charge uniquely determines the corresponding component and the combinatorics of different components do not interact with each other. Correspondingly, the factors in the $\mathscr{F}_{\delta_1} \otimes \cdots \otimes \mathscr{F}_{\delta_\ell}$ and $\mathscr{F}^{\circledast}_{\delta_1} \otimes \cdots \otimes \mathscr{F}^{\circledast}_{\delta_\ell}$ are independent in the sense that $\mathfrak{el}^{\epsilon}_{q}(\ell, \boldsymbol{\delta})$ respectively $\mathfrak{el}^{\epsilon}_{q^{-1}}(\ell, \boldsymbol{\delta})$ act componentwise.

Remark 10.44. Via Theorem 3.17 we could alternatively use modules over the super Brauer algebras for the categorification of Fock spaces and thus also categories of representations of the periplectic Lie superalgebras. The categories of finite dimensional representations of $\mathfrak{p}(n)$ are equivalent to a subquotient category of the categories categorifying the Fock spaces, see Section 3.5 and Theorem 7.6. Each $\mathfrak{p}(n)$ corresponds to a layer in a filtration on $\mathscr{F}_\delta$. Alternatively, one could invoke [Cou18b, Corollary 7.3.2].

Remark 10.45. One might want to define and study more involved arbitrary higher level Fock spaces generalizing work of Uglov, [Ugl00], to the electrical Lie algebra setting. We expect that these can be categorified using parabolic category $\mathcal{O}$ for the periplectic Lie superalgebras. For this parabolic category $\mathcal{O}$ needs to be revisited and studied in more detail first extending e.g. the works [CC20], [CP24].

11. Proofs of Chapter 9 and Proposition 10.30

In this section we collect some technical proofs of statements from Chapter 9.

11.1. Proof of Lemma 9.13

Proof. All the maps clearly satisfy the Hopf algebra conditions if we show that they are well-defined, i.e. compatible with the relations. For ε, this is a straight-forward calculation which is omitted. For Δ, the compatible with (1^-) and (2^-) is obvious. For (3^-), we calculate

$$\begin{aligned}\Delta(K_\lambda)\Delta(F_i) &= K_\lambda \otimes K_\lambda F_i + K_\lambda F_i \otimes K_\lambda K_{\beta_i} \\ &= q^{\langle \lambda, \alpha_i^\vee \rangle}(K_\lambda \otimes F_i K_\lambda + F_i K_\lambda \otimes K_{\beta_i} K_\lambda) = q^{\langle \lambda, \alpha_i^\vee \rangle}\Delta(F_i)\Delta(K_\lambda).\end{aligned}$$

For (4^-), we assume $|i-j|>1$ and compute

$$\begin{aligned}&\Delta(F_i)\Delta(F_j) = 1 \otimes F_i F_j + F_j \otimes F_i K_{\beta_j} + F_i \otimes K_{\beta_i} F_j + F_i F_j \otimes K_{\beta_i+\beta_j} \\ &= q^{b_{ij}} 1 \otimes F_j F_i + q^{\langle \beta_j, \alpha_i^\vee \rangle} F_j \otimes K_{\beta_j} F_i + q^{-\langle \beta_i, \alpha_j^\vee \rangle} F_i \otimes F_j K_{\beta_i} + q^{b_{ij}} F_j F_i \otimes K_{\beta_j+\beta_i} \\ &= q^{b_{ij}} \Delta(F_j)\Delta(F_i).\end{aligned}$$

Here we used that $b_{ij} = -b_{ji}$ if $|i-j|>1$, see Remark 9.2.
Of the remaining Serre relations (5^-) and (6^-) we only consider one, since the arguments are similar. We calculate the parts:

$$\begin{aligned}\Delta(F_i^2 F_{i+1}) &= 1 \otimes F_i^2 F_{i+1} + F_{i+1} \otimes F_i^2 K_{\beta_{i+1}} + F_i \otimes F_i K_{\beta_i} F_{i+1} + F_i \otimes K_{\beta_i} F_i F_{i+1} \\ &\quad + F_i F_{i+1} \otimes F_i K_{\beta_i} K_{\beta_{i+1}} + F_i F_{i+1} \otimes K_{\beta_i} F_i K_{\beta_{i+1}} + F_i^2 \otimes K_{2\beta_i} F_{i+1} \\ &\quad + F_i^2 F_{i+1} \otimes K_{2\beta_i+\beta_{i+1}} \\ &= 1 \otimes F_i^2 F_{i+1} + F_{i+1} \otimes F_i^2 K_{\beta_{i+1}} + (q^{-4}+q^{-2}) F_i \otimes F_i F_{i+1} K_{\beta_i} \\ &\quad + (1+q^2) F_i F_{i+1} \otimes F_i K_{\beta_i} K_{\beta_{i+1}} + q^{-8} F_i^2 \otimes F_{i+1} K_{2\beta_i} \\ &\quad + F_i^2 F_{i+1} \otimes K_{2\beta_i+\beta_{i+1}}. \\ \Delta(F_i F_{i+1} F_i) &= 1 \otimes F_i F_{i+1} F_i + F_i \otimes F_i F_{i+1} K_{\beta_i} + F_{i+1} \otimes F_i K_{\beta_{i+1}} F_i \\ &\quad + F_i \otimes K_{\beta_i} F_{i+1} F_i + F_{i+1} F_i \otimes F_i K_{\beta_{i+1}} K_{\beta_i} + F_i^2 \otimes K_{\beta_i} F_{i+1} K_{\beta_i} \\ &\quad + F_i F_{i+1} \otimes K_{\beta_i} K_{\beta_{i+1}} F_i + F_i^2 F_{i+1} \otimes K_{2\beta_i+\beta_{i+1}} \\ &= 1 \otimes F_i F_{i+1} F_i + F_i \otimes F_i F_{i+1} K_{\beta_i} + q^2 F_{i+1} \otimes F_i^2 K_{\beta_{i+1}}\end{aligned}$$

$$
\begin{aligned}
&+ q^{-2}F_i \otimes F_{i+1}F_iK_{\beta_i} + F_{i+1}F_i \otimes F_iK_{\beta_{i+1}}K_{\beta_i} + q^{-4}F_i^2 \otimes F_{i+1}K_{2\beta_i} \\
&+ q^4F_iF_{i+1} \otimes F_iK_{\beta_i}K_{\beta_{i+1}} + F_i^2F_{i+1} \otimes K_{2\beta_i+\beta_{i+1}}. \\
\Delta(F_{i+1}F_i^2) &= 1 \otimes F_{i+1}F_i^2 + F_i \otimes F_{i+1}F_iK_{\beta_i} + F_i \otimes F_{i+1}K_{\beta_i}F_i + F_{i+1} \otimes K_{\beta_{i+1}}F_i^2 \\
&+ F_i^2 \otimes F_{i+1}K_{2\beta_i} + F_{i+1}F_i \otimes K_{\beta_{i+1}}F_iK_{\beta_i} \\
&+ F_{i+1}F_i \otimes K_{\beta_{i+1}}K_{\beta_i}F_i + F_{i+1}F_i^2 \otimes K_{2\beta_i+\beta_{i+1}} \\
&= 1 \otimes F_{i+1}F_i^2 + (1+q^2)F_i \otimes F_{i+1}F_iK_{\beta_i} + q^4F_{i+1} \otimes K_{\beta_{i+1}}F_i^2 \\
&+ F_i^2 \otimes F_{i+1}K_{2\beta_i} + (q^2+q^4)F_{i+1}F_i \otimes F_iK_{\beta_i+\beta_{i+1}} \\
&+ F_{i+1}F_i^2 \otimes K_{2\beta_i+\beta_{i+1}}
\end{aligned}
$$

Now the first terms from each term give zero thanks to the Serre relation in the second tensor factor. The same for the last term thanks to the Serre relation in the first tenor factor. But then also all other terms cancel (remember to multiply the three cases by q^3, $-[2]$ and q^{-3} respectively!). This shows that Δ is well-defined. For S we compute

$$
\begin{aligned}
S(K_\lambda F_i) = S(F_i)S(K_\lambda) = -F_iK_{-\beta_i}K_{-\lambda} = -q^{-\langle\lambda,\alpha_i^\vee\rangle}K_{-\lambda}F_iK_{\beta_i} = S(q^{-\langle\lambda,\alpha_i^\vee\rangle}F_iK_\lambda), \\
S(F_iF_j) = F_jK_{-\beta_j}F_iK_{-\beta_i} = q^{b_{ij}+b_{ji}-b_{ji}}F_iK_{-\beta_i}F_jK_{-\beta_j} = S(q^{b_{ij}}F_jF_i),
\end{aligned}
$$

For the Serre relations (5^-), and similarly for (6^-), we calculate

$$
\begin{aligned}
&S(q^3F_i^2F_{i+1} - [2]F_iF_{i-1}F_i + q^{-3}F_{i+1}F_i^2) \\
&= -q^3F_{i+1}K_{\beta_{i+1}}(F_iK_{\beta_i})^2 + [2]F_iK_{\beta_i}F_{i+1}K_{\beta_{i+1}}F_iK_{\beta_i} - q^{-3}(F_iK_{\beta_i})^2F_{i+1}K_{\beta_{i+1}} \\
&= -q^{3-6}F_{i+1}F_i^2K_{2\beta_i+\beta_{i+1}} + [2]F_iF_{i+1}F_iK_{2\beta_i+\beta_{i+1}} - q^{-3+6}F_i^2F_{i+1}K_{2\beta_i+\beta_{i+1}} = 0.
\end{aligned}
$$

And therefore, S is also well-defined and Lemma 9.13 is proven. □

11.2. Proof of Proposition 9.21

Proof. The statement is clear for Δ and ε. For S, it suffices to show that

$$S(aK_\lambda \otimes b) = S(a \otimes K_\lambda b)$$

holds in $U_q^- \otimes_{U^0} U_q^+$ for any $a \in U_q^-$, $b \in U_q^+$, $\lambda \in X$.
By definition of S in Corollary 9.19 we get that

$$S(aK_\lambda \otimes b) = (1 \otimes S(b))(S(aK_\lambda) \otimes 1) = (1 \otimes S(b))(K_{-\lambda} \otimes 1)(S(a) \otimes 1) \tag{11.1}$$

since S is an antipode on the factors and U_q^- and U_q^+ are subalgebras. Similarly,

$$S(a \otimes K_\lambda b) = (1 \otimes S(K_\lambda b))(S(a) \otimes 1) = (1 \otimes S(b))(1 \otimes K_{-\lambda})(S(a) \otimes 1). \tag{11.2}$$

Since (11.1)=(11.2) in $U_q^- \otimes_{U^0} U_q^+$, we showed that S is U^0-balanced.
It remains to consider the multiplication. It is U^0-balanced if the equalities

$$(1 \otimes K_\lambda)(a \otimes 1) = (K_\lambda a \otimes 1) \qquad \text{and} \qquad (1 \otimes b)(K_\lambda \otimes 1) = (1 \otimes bK_\lambda) \tag{11.3}$$

hold in $U_q^- \otimes_{U^0} U_q^+$ for any $a \in U_q^-$, $b \in U_q^+$, $\lambda \in X$. By linearity, it suffices to assume $a = K\overline{a}$ for some $K \in U^0$ and some monomial $\overline{a} = F_{i_1} \cdots F_{i_r}$ in the F_is. Note that then the term $\langle a'_{(3)}, b_{(3)} \rangle$ in (9.3) can only get nonzero contributions from monomial summands in $a'_{(3)}$ which are contained in U^0, i.e. contain no F_is. Similarly, for $a'_{(1)}$ using the term $\langle S^{-1}(a'_{(1)}), b_{(1)} \rangle$. By the definition of Δ in Lemma 9.13 this implies that only $a'_{(1)} = K$, $a'_{(2)} = K\overline{a}$, $a'_{(3)} = K \prod_{j=1}^r K_{\beta_{i_j}}$ contributes. Thus,

$$(1 \otimes K_\lambda)(a \otimes 1) = \langle K^{-1}, K_\lambda \rangle K\overline{a} \otimes K_\lambda \langle K \prod_{j=1}^r K_{\beta_{i_j}}, K_\lambda \rangle = \langle \prod_{j=1}^r K_{\beta_{i_j}}, K_\lambda \rangle a \otimes K_\lambda.$$

This simplifies in $U_q^- \otimes_{U^0} U_q^+$ to $q^c a \otimes K_\lambda = q^c a K_\lambda \otimes 1 = q^c K\overline{a} K_\lambda \otimes 1$ with $c = \sum_{j=1}^r (\beta_{i_j}, \lambda)$. But $K\overline{a}K_\lambda$ is by (1^-) and (3^-) equal to $q^d K_\lambda a \otimes 1$ where $d = \sum_{j=1}^r \langle \lambda, \alpha_{i_j} \rangle = -\sum_{j=1}^r (\beta_{i_j}, \lambda) = -c$. Thus, the first equality in (11.3) holds. The second can be shown analogously using E_is instead of F_is. Therefore, the multiplication is U^0-balanced, and $U_q^- \otimes_{U^0} U_q^+$ is a Hopf algebra. □

11.3. Proof of Theorem 9.25

Proof. To prove Theorem 9.25 we need to show that we get a well-defined injective algebra homomorphism. We first check consistency with the relations of $\mathfrak{el}_q^\epsilon$.
For (ϵl-1) we have

$$\begin{aligned}
&\mathrm{j}(\mathcal{E}_i)\,\mathrm{j}(\mathcal{E}_j) = (F_i + q^{\epsilon-1}E_{i-1}K_{-\alpha_{i-1}})(F_j + q^{\epsilon-1}E_{j-1}K_{-\alpha_j}) \\
= \;& F_iF_j + q^{\epsilon-1}E_{i-1}K_{-\alpha_{i-1}}F_j + q^{\epsilon-1}F_iE_{j-1}K_{-\alpha_j} + q^{2\epsilon+2}E_{i-1}K_{-\alpha_{i-1}}E_{j-1}K_{-\alpha_j} \\
= \;& q^{b_{ij}}F_jF_i + q^{\epsilon-1-b_{i-1,j}}F_jE_{i-1}K_{-\alpha_{i-1}} + q^{\epsilon-1+b_{j-1,i}}E_{j-1}K_{-\alpha_j}F_i \\
&\qquad + q^{2\epsilon+2+b_{i-1,j-1}}E_{j-1}K_{-\alpha_j}E_{i-1}K_{-\alpha_{i-1}} \\
= \;& q^{b_{ij}}F_jF_i + q^{\epsilon-1+b_{ij}}F_jE_{i-1}K_{-\alpha_{i-1}} + q^{\epsilon-1+b_{ij}}E_{j-1}K_{-\alpha_j}F_i \\
&\qquad + q^{2\epsilon-2+b_{i,j}}E_{j-1}K_{-\alpha_j}E_{i-1}K_{-\alpha_{i-1}} \\
= \;& q^{b_{ij}}(F_j + q^{\epsilon-1}E_{j-1}K_{-\alpha_j})(F_i + q^{\epsilon-1}E_{i-1}K_{-\alpha_{i-1}}) = \mathrm{j}(\mathcal{E}_j)\,\mathrm{j}(\mathcal{E}_i).
\end{aligned}$$

Here we used that $b_{i+k,j+k} = b_{ij}$ and $b_{k-1,l} = b_{l,k}$, see Remark 9.2.
For (ϵl-2), we first compute the result R of j applied to the left-hand side of (ϵl-2). We break the task into pieces. *Piece 1:* First, let us look at the sum of those summands in R that contain three F's. Together with (U-4) we get

$$q^3F_iF_iF_{i+1} - [2]F_iF_{i+1}F_i + q^{-3}F_{i+1}F_iF_i = 0.$$

Piece 2: The sum of the summands in R that contain three E's is (up to the common $q^{3\epsilon+3}$ factor and calculated for $i+1$ instead)

$$\begin{aligned}
q^3E_iK_{-\alpha_i}E_iK_{-\alpha_i}E_{i+1}K_{-\alpha_{i+1}} - [2]E_iK_{-\alpha_i}E_{i+1}K_{-\alpha_{i+1}}E_iK_{-\alpha_i} \\
+ q^{-3}E_{i+1}K_{-\alpha_{i+1}}E_iK_{-\alpha_i}E_iK_{-\alpha_i}
\end{aligned}$$

$$=(q^3E_iE_iE_{i+1}-[2]E_iE_{i+1}E_i+q^{-3}E_{i+1}E_iE_i)K_{-2\alpha_i-\alpha_{i+1}}\overset{\text{(U-5)}}{=}0.$$

Piece 3: Next we have those terms that contain two F's. We split this case into three subcases, whether we have two F_i or F_{i+1} before F_i or F_{i+1} after F_i.
In case of two F_i, we get (ignoring the common factor $q^{\epsilon-1}$)

$$\begin{aligned}
&(q^3F_iF_iE_iK_{-\alpha_i}-[2]F_iE_iK_{-\alpha_i}F_i+q^{-3}E_iK_{-\alpha_i}F_iF_i)\\
=&(q^3F_iF_iE_iK_{-\alpha_i}-(q^3+q)F_iE_iF_iK_{-\alpha_i}+qE_iF_iF_iK_{-\alpha_i})\\
=&(q^3F_i[F_i,E_i]K_{-\alpha_i}+q[E_i,F_i]F_iK_{-\alpha_i})+q[2]F_i\\
=&(\frac{-q^3}{q-q^{-1}}F_i(K_{\alpha_i}-K_{-\alpha_i})K_{-\alpha_i}+\frac{q}{q-q^{-1}}(K_{\alpha_i}-K_{-\alpha_i})F_iK_{-\alpha_i})\\
=&(\frac{-q^3}{q-q^{-1}}F_i(1-K_{-2\alpha_i})+\frac{q^{-1}}{q-q^{-1}}F_i+\frac{q^3}{q-q^{-1}}F_iK_{-2\alpha_i})=-q[2]F_i=:(*).
\end{aligned}$$

Next assume F_{i+1} appears before a unique F_i. Ignoring a factor $q^{\epsilon-1}$ we get

$$\begin{aligned}
&-[2]E_{i-1}K_{-\alpha_{i-1}}F_{i+1}F_i+q^{-3}F_{i+1}E_{i-1}K_{-\alpha_{i-1}}F_i+q^{-3}F_{i+1}F_iE_{i-1}K_{-\alpha_{i-1}}\\
=&-(1+q^{-2})E_{i-1}F_{i+1}F_iK_{-\alpha_{i-1}}+q^{-4-\beta_{i-1,i+1}}E_{i-1}F_{i+1}F_iK_{-\alpha_{i-1}}\\
&\qquad+q^{-3-\beta_{i-1,i+1}-\beta_{i-1,i}}E_{i-1}F_{i+1}F_iK_{-\alpha_{i-1}}=0.
\end{aligned}$$

The remaining case for two F's is when F_{i+1} appears after a unique F_i. Then,

$$\begin{aligned}
&q^3E_{i-1}K_{-\alpha_{i-1}}F_iF_{i+1}+q^3F_iE_{i-1}K_{-\alpha_{i-1}}F_{i+1}-[2]F_iF_{i+1}E_{i-1}K_{-\alpha_{i-1}}\\
=&q^2E_{i-1}F_iF_{i+1}K_{-\alpha_{i-1}}+q^{3-\beta_{i-1,i}}E_{i-1}F_iF_{i+1}K_{-\alpha_{i-1}}\\
&\qquad-[2]q^{-\beta_{i-1,i}-\beta_{i-1,i+1}}E_{i-1}F_iF_{i+1}K_{-\alpha_{i-1}}=0.
\end{aligned}$$

Piece 5: Now it remains to look at the case, where we have only one F. Similar to before, we split this case into three subcases. Namely, we look at the cases where two E_{i-1} appear, E_i appears before E_{i-1} respectively E_i appears after E_{i-1}.
If we have two E_{i-1}, we get (we calculate again for $i+1$)

$$\begin{aligned}
&q^3E_iK_{-\alpha_i}E_iK_{-\alpha_i}F_{i+2}-[2]E_iK_{-\alpha_i}F_{i+2}E_iK_{-\alpha_i}+q^{-3}F_{i+2}E_iK_{-\alpha_i}E_iK_{-\alpha_i}\\
=&(q^{3+2\beta_{i,i+2}}F_{i+2}E_iK_{-\alpha_i}E_i-[2]q^{\beta_{i,i+2}}F_{i+2}E_iK_{-\alpha_i}E_i+q^{-3}F_{i+2}E_iK_{-\alpha_i}E_i)K_{-\alpha_i}\\
\overset{\text{(U-5)}}{=}&0.
\end{aligned}$$

The next case is when E_i appears before E_{i-1}. We calculate in U_q the following

$$\begin{aligned}
&-[2]F_iE_iK_{-\alpha_i}E_{i-1}K_{-\alpha_{i-1}}+q^{-3}E_iK_{-\alpha_i}F_iE_{i-1}K_{-\alpha_{i-1}}+q^{-3}E_iK_{-\alpha_i}E_{i-1}K_{-\alpha_{i-1}}F_i\\
=&-[2](F_iE_iK_{-\alpha_i}E_{i-1}+q^{-1}E_iF_iK_{-\alpha_i}E_{i-1}+q^{-2+\beta_{i-1,i}}E_iF_iK_{-\alpha_i}E_{i-1})K_{-\alpha_{i-1}}\\
\overset{(9.4)}{=}&[2]\left([E_i,F_i]K_{-\alpha_i}E_{i-1}\right)K_{-\alpha_{i-1}}=[2]\left(\frac{1-K_{-2\alpha_i}}{q-q^{-1}}E_{i-1}\right)K_{-\alpha_{i-1}}\\
=&\frac{q+q^{-1}}{q-q^{-1}}E_{i-1}K_{-\alpha_{i-1}}-\frac{q+q^3}{q-q^{-1}}E_{i-1}K_{-2\alpha_i-\alpha_{i-1}}=:(**).
\end{aligned}$$

Finally, we look at the case where E_i appears after E_{i-1}. Then,

$$\begin{aligned}
&q^3F_iE_{i-1}K_{-\alpha_{i-1}}E_iK_{-\alpha_i}+q^3E_{i-1}K_{-\alpha_{i-1}}F_iE_iK_{-\alpha_i}-[2]E_{i-1}K_{-\alpha_{i-1}}E_iK_{-\alpha_i}F_i\\
&=\left(q^{4-\beta_{i-1,i}}E_{i-1}K_{-\alpha_{i-1}}F_iE_i+q^3E_{i-1}K_{-\alpha_{i-1}}F_iE_i-(q+q^3)E_{i-1}K_{-\alpha_{i-1}}E_iF_i\right)K_{-\alpha_i}\\
&=-(q+q^3)E_{i-1}K_{-\alpha_{i-1}}[E_i,F_i]K_{-\alpha_i}=-\frac{q+q^3}{q-q^{-1}}E_{i-1}K_{-\alpha_{i-1}}(1-K_{-2\alpha_i})=:(***).
\end{aligned}$$

Adding up the nonzero intermediate results (which only appear in Pieces 3-4) and recalling the q^x scaling factors, we find that j maps $q^3\mathcal{E}_i^2\mathcal{E}_j-[2]\mathcal{E}_i\mathcal{E}_j\mathcal{E}_i+q^{-3}\mathcal{E}_j\mathcal{E}_i^2$ to

$$\begin{aligned}
(*)+(**)+(***)=&\ -q^\epsilon[2]F_i+q^{2\epsilon-2}\frac{q^{-1}-q^3}{q-q^{-1}}E_{i-1}K_{-\alpha_{i-1}}\\
=&\ -q^\epsilon[2]F_i-q^{2\epsilon-1}[2]E_{i-1}K_{-\alpha_{i-1}}=-q^\epsilon[2](F_i+q^{\epsilon-1}E_{i-1}K_{-\alpha_{i-1}})=\mathrm{j}(-q^\epsilon[2]\mathcal{E}_i).
\end{aligned}$$

Thus, (ϵl-2) is satisfied. A similar calculation shows the compatibility with (ϵl-3).
This proves that $\mathrm{j}\colon \mathfrak{el}_q^\epsilon\to U_q$ is well-defined. It remains to show injectivity. By definition of the map, the image of a word in the generators $\mathcal{E}_i$ of $\mathfrak{el}_q^\epsilon$ has a unique summand that contains only F_i's (and it is moreover the same word in these F_i's). Now the statement follows from Lemma 9.7, since the algebra $\mathfrak{el}_q^\epsilon$ is filtered with associated graded isomorphic to the subalgebra of U_q generated by the F_is. ☐

11.4. Proof of Proposition 9.34

Proof. We show the first statement and the most complicated braid relations in the general mixed cases as claimed in Remark 9.36. The remaining cases are then straight-forward adaptions of the easy checks. We start by showing that H is U_q-linear.
It clearly commutes with K_λ, so we consider E_i and F_i. The computation for E_i is very similar to the one for F_i, so we only treat the F_i. Recall that $v_jF_i=\delta_{ij}v_{i+1}$. Let $\eta_i:=\beta_i$ if $\odot=\odot_1$ and $\eta_i:=\beta'_{i-1}$ if $\odot=\odot_2$. Then

$$\begin{tikzcd}
v_i\odot v_j \arrow[r, "F_a"] \arrow[dd, "H"] & \delta_{ia}v_{i+1}\odot v_jK_{\eta_i}+\delta_{ja}v_i\odot v_{j+1} \arrow[d, "H"] \\
& \begin{matrix}\delta_{ia}a_{i+1,j}v_jK_{\eta_i}\odot v_{i+1}\\ +\delta_{ia}\delta_{i+1<j}(q^{-1}-q)v_{i+1}\odot v_jK_{\eta_i}\\ +\delta_{ja}a_{i,j+1}v_{j+1}\odot v_i+\delta_{ja}\delta_{i<j+1}(q^{-1}-q)v_i\odot v_{j+1}\end{matrix} \\
\begin{matrix}a_{ij}v_j\odot v_i+\\ \delta_{i<j}(q^{-1}-q)v_i\odot v_j\end{matrix} \arrow[r, "F_a"] & \begin{matrix}\delta_{ia}a_{ij}v_j\odot v_{i+1}+\delta_{ia}\delta_{i<j}(q^{-1}-q)v_{i+1}\odot v_jK_{\eta_i}\\ +\delta_{ja}a_{ij}v_{j+1}\odot v_iK_{\eta_j}+\delta_{ja}\delta_{i<j}(q^{-1}-q)v_i\odot v_{j+1}\end{matrix}
\end{tikzcd}$$

With the abbreviation $c_{ji} := \langle \eta_i, e_j \rangle$, we therefore have to show the following

$$
\begin{aligned}
a_{i+1,j}c_{ji} &= a_{ij} && \text{for } j \notin \{i, i+1\},\\
a_{i+1,j}c_{ji} &= a_{ij} + (q^{-1}-q)c_{ji} && \text{for } j = i+1,\\
a_{i+1,j}c_{ji} + q^{-1} - q &= a_{ij} && \text{for } j = i,\\
a_{i,j+1} &= a_{ij}c_{ij} && \text{for any } i, j.
\end{aligned}
\tag{11.4}
$$

The following table shows all the different possibilities for i and j.

	parity of $i-l$	parity of $j-l$	$a_{i+1,j}$	c_{ji}	a_{ij}	c_{ij}	$a_{i,j+1}$
$j > i+1$	even	even	q^{-3}	q^4	q	1	q
	even	odd	q	1	q	1	q
	odd	even	q	q^{-4}	q^{-3}	q^4	q
	odd	odd	q	1	q	q^{-4}	q^{-3}
$j < i$	even	even	q^{-1}	1	q^{-1}	q^4	q^3
	even	odd	q^{-1}	q^4	q^3	q^{-4}	q^{-1}
	odd	even	q^{-1}	1	q^{-1}	1	q^{-1}
	odd	odd	q^3	q^{-4}	q^{-1}	1	q^{-1}
$j = i$	even	even	q^{-1}	q^2	q^{-1}	q^2	q
	odd	odd	q^3	q^{-2}	q^{-1}	q^{-2}	q^{-3}
$j = i+1$	even	odd	q^{-1}	1	q	1	q
	odd	even	q^{-1}	q^{-4}	q^{-3}	q^4	q

Now it is easy to see that (11.4) is satisfied. This shows that H is U_q-linear.
Next we show that H satisfies the Hecke relations.
We begin by computing $H^2(v_i \odot v_j)$. In case $i \neq j$ we have

$$
\begin{aligned}
H^2(v_i \odot v_j) &= H(a_{ij}v_j \odot v_i + \delta_{i<j}(q^{-1}-q)v_i \odot v_j)\\
&= a_{ij}a_{ji}v_i \odot v_j + a_{ij}\delta_{j<i}(q^{-1}-q)v_j \odot v_i\\
&\qquad + \delta_{i<j}(q^{-1}-q)a_{ij}v_j \odot v_i + \delta_{i<j}(q^{-1}-q)^2 v_i \odot v_j\\
&= a_{ij}a_{ji}v_i \odot v_j + (q^{-1}-q)(a_{ij}v_j \odot v_i + \delta_{i<j}(q^{-1}-q)v_i \odot v_j)\\
&= v_i \odot v_j + (q^{-1}-q)H(v_i \odot v_j).
\end{aligned}
$$

In case $i = j$, we have

$$
H^2(v_i \odot v_i) = H(a_{ii}v_i \odot v_i) = q^{-2}v_i \odot v_i = v_i \odot v_i + (q^{-1}-q)H(v_i \odot v_i),
$$

and thus the first Hecke relation is satisfied. The second Hecke relation is obvious.
It thus remains to show the braid relations for $V \odot V \odot V$, where each of the $\odot$ can be chosen from $\{\odot_1, \odot_2\}$. So let $V \odot_l V \odot_r V$ with $l, r \in \{1, 2\}$. To simplify notation

abbreviate $v_{ijk} := v_i \odot v_j \odot v_k$. We will also write a^l_{ij} and a^r_{ij} to emphasize the dependence on l and r in the definition. We compute

$$\begin{aligned}
&H_1H_2H_1(v_{ijk}) = H_1H_2(a^l_{ij}v_{jik} + \delta_{i<j}(q^{-1}-q)v_{ijk}) \\
&= H_1(a^l_{ij}a^r_{ik}v_{jki} + \delta_{i<k}a^l_{ij}(q^{-1}-q)v_{jik} + \delta_{i<j}(q^{-1}-q)a^r_{jk}v_{ikj} + \delta_{i<j<k}(q^{-1}-q)^2v_{ijk}) \\
&= \underbrace{a^l_{ij}a^r_{ik}a^l_{jk}v_{kji}}_{①} + \underbrace{\delta_{j<k}(q^{-1}-q)a^l_{ij}a^r_{ik}v_{jki}}_{②} + \underbrace{\delta_{i<k}a^l_{ji}a^l_{ij}(q^{-1}-q)v_{ijk}}_{③} \\
&\quad + \underbrace{\delta_{j<i<k}a^l_{ij}(q^{-1}-q)^2v_{jik}}_{④} + \underbrace{\delta_{i<j}(q^{-1}-q)a^r_{jk}a^l_{ik}v_{kij}}_{⑤} + \underbrace{\delta_{k>i<j}(q^{-1}-q)^2a^r_{jk}v_{ikj}}_{⑥} \\
&\qquad + \underbrace{\delta_{i<j<k}a^l_{ij}(q^{-1}-q)^2v_{jik}}_{④} + \underbrace{\delta_{i<j<k}(q^{-1}-q)^3v_{ijk}}_{⑦},
\end{aligned}$$

which we need to compare with

$$\begin{aligned}
&H_2H_1H_2(v_{ijk}) = H_2H_1(a^r_{jk}v_{ikj} + \delta_{j<k}(q^{-1}-q)v_{ijk}) \\
&= H_2(a^r_{ik}a^l_{jk}v_{kij} + \delta_{i<k}a^r_{jk}(q^{-1}-q)v_{ikj} + \delta_{j<k}(q^{-1}-q)a^l_{ij}v_{jik} + \delta_{i<j<k}(q^{-1}-q)^2v_{ijk}) \\
&= \underbrace{a^r_{ij}a^l_{ik}a^r_{jk}v_{kji}}_{①} + \underbrace{\delta_{i<j}(q^{-1}-q)a^l_{ik}a^r_{jk}v_{kij}}_{⑤} + \underbrace{\delta_{i<k}a^r_{jk}a^r_{kj}(q^{-1}-q)v_{ijk}}_{③} \\
&\quad + \underbrace{\delta_{i<k<j}a^r_{jk}(q^{-1}-q)^2v_{ikj}}_{⑥} + \underbrace{\delta_{j<k}(q^{-1}-q)a^r_{ij}a^r_{ik}v_{jki}}_{②} + \underbrace{\delta_{j<k>i}(q^{-1}-q)^2a^l_{ij}v_{jik}}_{④} \\
&\qquad + \underbrace{\delta_{i<j<k}a^r_{jk}(q^{-1}-q)^2v_{ikj}}_{⑥} + \underbrace{\delta_{i<j<k}(q^{-1}-q)^3v_{ijk}}_{⑦}.
\end{aligned}$$

The parts ②, ⑤, ⑦ agree in the two expressions. Let us consider now ③, ④, ⑥.
If $i = j = k$, then the terms for ③, ④, ⑥ match since $a^l_{tt} = q^{-1} = a^r_{tt}$ for any t.
Next assume i, j, k are pairwise distinct. Then the parts ③ agree if $a^l_{ji}a^l_{ij} = a^r_{jk}a^r_{kj}$ which holds by (9.6). The parts ④ agree if $\delta_{j<i<k}a^l_{ij} + \delta_{i<j<k}a^l_{ij} = \delta_{j<k>i}a^l_{ij}$ which obviously holds. Similarly, for ⑥. Assume now $i = j \neq k$. Then the respective sums ③+④+⑥ are $\delta_{i<k}a^l_{ii}a^l_{ii}(q^{-1}-q)v_{iik}$ and $(\delta_{i<k}a^r_{ik}a^r_{ki} + \delta_{j<k>i}a^l_{ii})(q^{-1}-q)^2v_{iik}$. They agree, since $a^l_{ii}a^l_{ii} = q^{-2} = 1 + q^{-1}(q^{-1}-q) = a^r_{ik}a^r_{ki} + a^l_{ii}(q^{-1}-q)$ by (9.6).
Assume next $i \neq j = k$. Then the sums are $\delta_{i<k}a^l_{ji}a^l_{ij}(q^{-1}-q) + \delta_{k>i<j}(q^{-1}-q)^2a^r_{jj}$ and $\delta_{k>i<j}(q^{-1}-q)^2a^r_{jj}$. They agree, since $a^l_{ji}a^l_{ij} = 1 + q^{-2} = q^{-1}(q^{-1}-q)$ by (9.6).
Assume finally $i = k \neq j$. Then both sums ③+④+⑥ vanish.
It remains to compare the two parts labelled ①.
If $i = j = k$ then they agree since $a^l_{tt} = q^{-1} = a^r_{tt}$ for any t. If $i = j \neq k$ then we ask if $a^r_{ik}a^l_{ik} = a^l_{ik}a^r_{ik}$ which is obviously true. If $i = k \neq j$ then we ask if $a^l_{ij}a^l_{ji} = a^r_{ij}a^r_{ji}$ which holds by (9.6). If $j = k \neq i$ then we ask if $a^l_{ij}a^r_{ji} = a^r_{ij}a^j_{il}$ which is clearly true.
Therefore, the ①-parts agree if at least two of i, j, and k are equal, and it remains to consider ① in the case where i, j, k are distinct and $r \neq l$. Using (9.6) we can reduce to the case $i < j < k$. We compute the values depending on whether $i - l$, $j - l$ and $k - l$

parity of $i-l$	parity of $j-l$	parity of $k-l$	$a^l_{ij}a^r_{ik}a^l_{jk}$	$a^r_{ij}a^l_{ik}a^r_{jk}$
even	even	even	q^3	q^3
		odd	q^{-1}	q^{-1}
	odd	even	q^{-1}	q^{-1}
		odd	q^{-1}	q^{-1}
odd	even	even	q^{-1}	q^{-1}
		odd	q^{-1}	q^{-1}
	odd	even	q^{-1}	q^{-1}
		odd	q^3	q^3

Figure 11.1.: Comparison of values

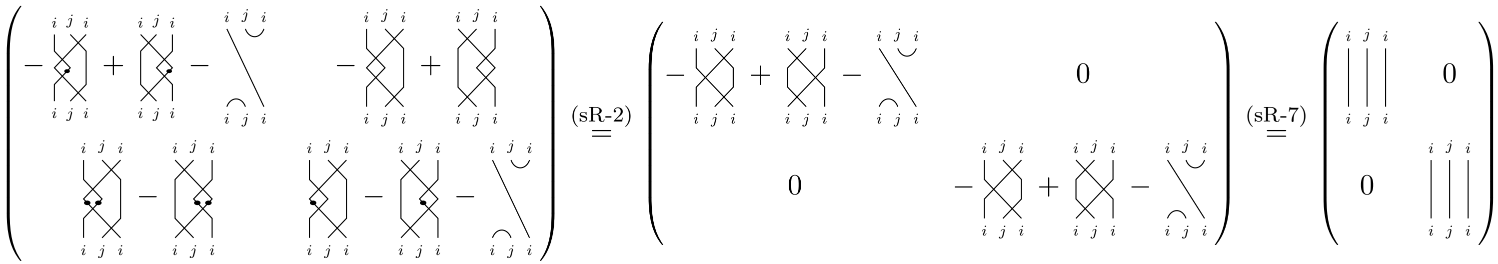
are even or odd in Section 11.4. Note that r has the opposite parity of l since $r \neq l$. We see that the 1-parts agree as well. This finishes the proof. $\square$

11.5. Proof of Proposition 10.30

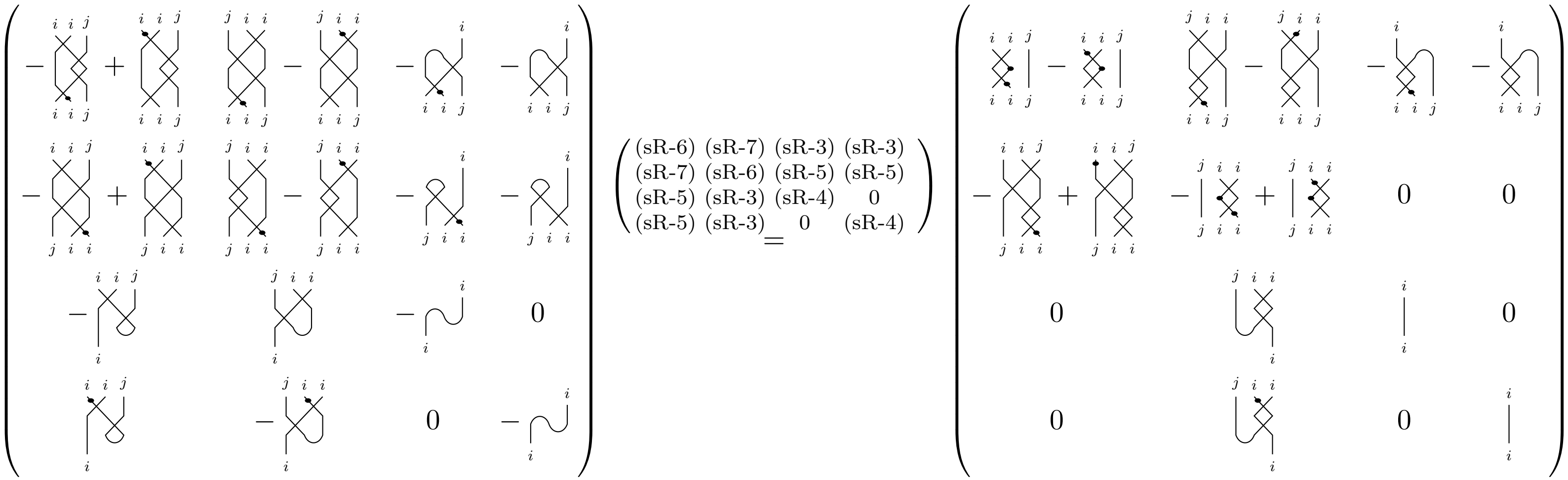
Lemma 11.1. *The matrices B_0 and B_1 from Proposition 10.30 are mutually inverse.*

Proof.

$B_1B_0 =$



$B_0B_1 =$



Rewriting this using (sR-2), (3.4) and (sR-6) gives the identity matrix. □

Part II.

The isomeric Lie superalgebra

12. Introduction

In this part of the thesis we will consider the representation theory of the isomeric Lie superalgebra $\mathfrak{q}(n)$. For this consider a vector superspace V of dimension $(n|n)$ with an odd endomorphism J satisfying $J^2 = -1$.
The *isomeric Lie superalgebra* $\mathfrak{q}(n)$ is then the centralizer of J in $\mathfrak{gl}(V)$. Fixing a basis $\{v_1, \dots, v_n, v_1', \dots, v_1'\}$ for V, we may assume that $Jv_i = v_i'$ and $Jv_i' = -v_i$. We have $\mathfrak{q}(n)_0 \cong \mathfrak{gl}(n)$, and $\mathfrak{q}(n)_1$ is the adjoint representation of $\mathfrak{gl}(n)$.
The most striking difference of $\mathfrak{q}(n)$ to other Lie (super)algebras is that the classical version of Schur's lemma does not hold for $\mathfrak{q}(n)$. In particular, there exist irreducible representations whose endomorphism ring is not $\mathbb{C}$. If that is the case, the endomorphism ring has dimension $(1|1)$ and the odd part is generated by a morphism that squares to -1. These irreducibles are said to be of *type* $\mathtt{Q}$ and the others are of *type* $\mathtt{M}$. For instance, the natural representation V is by definition of type $\mathtt{Q}$.
Our goal is to describe the category of finite dimensional representations of $\mathfrak{q}(n)$ via explicitly describing the endomorphism ring of a projective generator. We approach this via Schur–Weyl–Sergeev duality. By construction, the *natural representation* V of $\mathfrak{q}(n)$ is faithful. In particular, every indecomposable projective representation appears as a direct summand of $V^{\otimes d} \otimes (V^*)^{\otimes d'}$ for some d and d', see e.g. [CH17].
So if we have an explicit description of homomorphisms between tensor products of V and V^* there is hope to extract information on morphisms between projective modules. For this we will use the *oriented Brauer–Clifford supercategory* $\mathcal{OBC}$. This is a cyclotomic quotient of $\mathcal{AOBC}$, the *degenerate affine oriented Brauer–Clifford supercategory.* Both supercategories, $\mathcal{OBC}$ and $\mathcal{AOBC}$, were introduced in [BCK19], extending the works of [Ser84, Naz97, JK14].
Both are symmetric monoidal supercategories generated by two objects $\wedge$ and $\vee$ (the dual of $\wedge$) and morphisms generated by the braiding, evaluation and coevaluation morphisms, as well as one additional generator, which is odd and corresponds to the odd endomorphism J of V. The supercategory $\mathcal{AOBC}$ admits one further even generator, which should be seen as a diagrammatic interpretation of (a rescaling of) the odd Casimir for $\mathfrak{q}(n)$, see Section 14.2 for a precise definition.
By results of [Ser84], [JK14], [HKS11] and [BCK19], there exists a full monoidal superfunctor

$$\mathrm{SWS}_n \colon \mathcal{OBC} \to \mathrm{rep}(\mathfrak{q}(n))$$

of $\mathcal{AOBC}$-module categories. This is also called *(mixed) Schur–Weyl–Sergeev duality.*
To be able to pick out the projective representations in $\mathrm{rep}(\mathfrak{q}(n))$, we would like to understand idempotents in $\mathcal{OBC}$. This can be achieved by replacing $\mathcal{AOBC}$ by a certain 2-supercategory $\mathfrak{U}(B_{0|\infty})$, the super Kac–Moody 2-category of type $B_{0|\infty}$, which was

introduced in [BE17b]. This construction is based on work by [KKT16], [KKO13] and [KKO14], who introduced an idempotent version of the degenerate affine Sergeev algebra, that coincides with the full monoidal subcategory of $\mathcal{AOBC}$ generated by $\wedge$. Disregarding all technicalities, $\mathfrak{U}(B_{0|\infty})$ arises from $\mathcal{AOBC}$ by decomposing the endofunctors $\wedge \otimes \_$ and $\vee \otimes \_$ into generalized eigenspaces for the additional generator of $\mathcal{AOBC}$. The direct summands are then named E_i and F_i and define the 1-morphisms for $\mathfrak{U}(B_{0|\infty})$.
The definition of $\mathfrak{U}(B_{0|\infty})$ arose in the study of categorification of Kac–Moody superalgebras. Generally speaking, representations of type $\mathtt{Q}$ are not very suitable for categorification purposes, one prefers much more the type $\mathtt{M}$ representations. Usually, to obtain some kind of uniqueness result, one requires e.g. your category corresponding to the highest weight space to be Vec (i.e. type $\mathtt{M}$), see [Rou08, LW15, BLW17]. Another phenomenon appears in [Bru04], where many factors of 2 appear while relating the representation theory of $\mathfrak{q}(n)$ to exterior powers of type B. These factors of 2 are mainly due to type $\mathtt{Q}$ representations, and would not appear for type $\mathtt{M}$ representations, see Theorem 19.18.
In [KKT16], a so-called Clifford-Twist was defined, which in broad terms swaps between type $\mathtt{M}$ and type $\mathtt{Q}$ representations. Using this twist, we obtain a supercategory $\operatorname{rep}'(\mathfrak{q}(n))$, which is weakly Morita superequivalent to $\operatorname{rep}(\mathfrak{q}(n))$, but contains only type $\mathtt{M}$ representations. By results of [Bru25], the functor SWS_n upgrades to a full morphism of $\mathfrak{U}(B_{0|\infty})$-2-representations

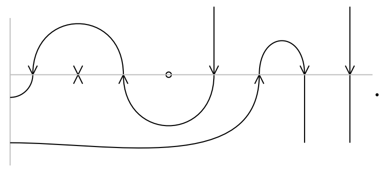
$$\mathcal{F}_n \colon \mathfrak{U}(B_{0|\infty})^{\Lambda} \to \operatorname{rep}'(\mathfrak{q}(n)).$$

The 2-representation $\mathfrak{U}(B_{0|\infty})^{\Lambda}$ is a cyclotomic quotient of $\mathfrak{U}(B_{0|\infty})$ and arises by the "same" cyclotomic condition as $\mathcal{OBC}$ from $\mathcal{AOBC}$.
The main advantage of $\mathfrak{U}(B_{0|\infty})^{\Lambda}$ over $\mathcal{OBC}$ is that the idempotents which we want to understand are part of the definition of $\mathfrak{U}(B_{0|\infty})^{\Lambda}$. However, from the definition it is not clear, which idempotents are actually non-trivial. We then go about proving a basis theorem for $\mathfrak{U}(B_{0|\infty})^{\Lambda}$. This will give insight into the non-trivial idempotents and also describes the morphism spaces of $\mathfrak{U}(B_{0|\infty})^{\Lambda}$, yielding information on morphisms between projective representations of $\mathfrak{q}(n)$.

Theorem A. *The* 2*-representation* $\mathfrak{U}(B_{0|\infty})^{\Lambda}$ *has basis indexed by pairs of up-down-tableaux of strict bipartitions and this endows* $\mathfrak{U}(B_{0|\infty})^{\Lambda}$ *with the structure of an upper finite based quasi-hereditary algebra in the sense of* [BS24].

Its proof comprises two parts. After defining the basis elements in Definition 15.6, we show in Theorem 15.23 that these elements span the morphism spaces of $\mathfrak{U}(B_{0|\infty})^{\Lambda}$.
To show that these are linearly independent, we construct a faithful representation of $\mathfrak{U}(B_{0|\infty})^{\Lambda}$ in Chapter 16. To construct the faithful representation, we introduce a new *Khovanov superalgebra* $\mathbb{K}^Q$ *of type* Q as it should be seen as an analog of the Khovanov algebras of types A, B and P from [BS11a] and [ES16] and Chapter 4. Before we circle back to the linear independence, we will first give a brief overview of the Khovanov algebra of type Q, as this will later also play an important role for our main goal, the

explicit description of the endomorphism ring of a projective generator for $\mathfrak{q}(n)$. The Khovanov algebras (of types A, B, P and Q) are equipped with a distinguished basis and an explicit multiplication rule, which arises from the application of certain surgery procedures. In our case (i.e. type Q), the distinguished basis elements look like



To keep it simple here, we will only define the multiplication rule over $\mathbb{F}_2$. In general, signs will appear in the multiplication rule in a subtle way, similar to the signs appearing for $\mathfrak{osp}(r|2n)$, see [ES16]. We show:

Theorem B. *The graded superalgebra $\mathbb{K}^Q$ is an upper-finite based quasi-hereditary superalgebra in the sense of* [BS24]*.*

As in [BS10] and [HNS24], we then introduce geometric bimodules for $\mathbb{K}^Q$. The results there carry over to our case more or less verbatim. In particular, we can e.g. compute their effect on projective representations of $\mathbb{K}^Q$ and obtain an adjunction theorem.
We define then special geometric bimodules called E_i and F_i for $i \in \mathbb{N}_0$ and obtain the linear independence needed for the basis theorem:

Theorem C. *There is an isomorphism of* 2*-representations* $G\colon \mathfrak{U}(B_{0|\infty})^\Lambda \to 2\mathbb{K}^Q_{\iota_0}$.

Here, $2\mathbb{K}^Q_{\iota_0}$ is the category obtained by repeatedly applying the functor E_i and F_i to a particular projective representation of $\mathbb{K}^Q$. This statement should be taken with a grain of salt, as we only have defined the multiplication rule for $\mathbb{K}^Q$ over $\mathbb{F}_2$. In particular, we can check well-definedness of the 2-representation structure on $\mathbb{K}^Q$ only over $\mathbb{F}_2$. We will indicate, however, which parts of the argument carry over to the general case and where extra care is needed.
The rest of the proof (i.e. obtaining the morphism of representations and showing that this is an isomorphism) does not use the explicit signs in the multiplication rule.
The well-definedness of the 2-representation structure on $\mathbb{K}^Q$ is the sole point where the explicit signs are actually needed. All the other results do not depend on the explicit signs.
With the isomorphism from Theorem C, we can translate the results for $\mathfrak{U}(B_{0|\infty})^\Lambda$ to $\mathbb{K}^Q$ and obtain for free a full functor

$$\mathcal{F}_n\colon \mathbb{K}^Q\text{-proj} \to \operatorname{rep}'(\mathfrak{q}(n)).$$

The image is given by direct summands of mixed tensor powers $V^{\otimes d} \otimes (V^*)^{\otimes d'}$. In particular, the image contains all projective representations of $\mathfrak{q}(n)$. Unfortunately, this functor is not faithful. To remedy this issue, we then introduce certain sub-2-representations of $2\mathbb{K}^Q_{\iota_0}$, called $2\mathbb{I}_n$. From Schur–Weyl–Sergeev duality in conjunction

with easy considerations regarding central characters, we see that $\mathcal{F}_n$ maps $2\mathbb{I}_{n+1}$ to 0 and $2\mathbb{I}_n$ to $\mathrm{proj}'(\mathfrak{q}(n))$. We can give another description of the quotient representation $2\mathbb{I}_n/2\mathbb{I}_{n+1}$ as $\mathbb{K}_n^Q$-proj, where $\mathbb{K}_n^Q$ is a subquotient of $\mathbb{K}^Q$.
With all that we achieved so far, the following theorem is almost immediate.

Theorem D. *We have a superequivalence of* $\mathfrak{U}(B_{0|\infty})$*-2-representations*

$$\hat{\mathcal{F}}_n \colon \mathbb{K}_n^Q\text{-proj} \to \mathrm{proj}'(\mathfrak{q}(n)).$$

In particular, we obtain a weak superequivalence between $\mathbb{K}_n^Q$-mod *and* $\mathrm{rep}(\mathfrak{q}(n))$.

After proving this main theorem, we take a step back and consider again the cyclotomic quotient $\mathfrak{U}(B_{0|\infty})^\Lambda$, but now from a categorification perspective. In this context quantum covering groups as in [CHW13] appear. These quantum covering groups come equipped with another parameter π satisfying $\pi^2 = 1$. This additional parameter can be specialized to 1 or -1. In the former case, giving a classical quantum group and in the latter case, giving a quantum supergroup (of anisotropic type). The original motivation for this twofold definition came from the categorification of the Jones polynomial. Namely, there exist two different categorifications of the Jones polynomial, Khovanov homology as in [Kho00] and *odd* Khovanov homology as introduced by [ORS13]. These two categorifications are not equivalent, as can be seen from results of [Shu11].
As the Jones polynomial bears close connections to the representation theory of $U_q(\mathfrak{sl}_2)$, see e.g. [RT90], it is natural to expect close connections between the corresponding categorifications. With this guiding principle at hand, there should be another (non-equivalent) categorification of quantum groups in addition to the one from [KL10] (or equivalently [Rou08]). In [EKL14], a second categorification of the positive half of $U_q(\mathfrak{sl}_2)$ was introduced, based on the theory of odd symmetric functions and the odd nil-Hecke algebra.
In [KKT16], so-called *quiver Hecke superalgebras* were defined which comprise a generalization of KLR algebras by incorporating the odd nil-Hecke algebra. Following up on this, it was shown in [KKO13] that the respective cyclotomic quotients provide a *supercategorification* of Kac–Moody algebras and their integrable highest weight modules. However, in this categorification setup supercategories are used. Instead of supercategories, one could equivalently work with categories endowed with a parity shift functor Π. See e.g. [MOS09] for a similar consideration in the graded context. Furthermore, refer to [BE17a] for more details in the case of 2-supercategories.
This parity shift functor gives rise to the above-mentioned second parameter π. These quantum covering groups have been defined by [CHW13]. The main difference of this definition compared to other definitions of quantum supergroups is the existence of another set of generators $\mathcal{J}$, which gives a more unified approach to the representation theory. Loosely speaking, the generators $\mathcal{K}$ scale weight spaces by q-powers and $\mathcal{J}$ by π-powers. Without these additional generators, these algebras were already considered in [BKM98].
Following up on [KKO13], the same authors proved in [KKO14] that the cyclotomic quotient of the quiver Hecke superalgebras, now with parity shift Π, provide a categorification of these quantum covering groups. This parallels the classical results from

[KK12] and [Web17], which proved that cyclotomic KLR algebras together with induction and restriction functors provide a categorification of highest weight representations of Kac–Moody algebras.
Another approach to categorify the full quantum group was taken in [Rou08] and [KL10]. In both cases, a certain 2-category was constructed. We can think of KLR algebras as 2-categories with horizontal and vertical composition given via horizontal and vertical stacking of diagrams. The 2-categories in question arise then via adding adjoints F_i to the generating 1-morphisms E_i and imposing one additional relation originating in the Mackey type relations for induction and restriction. This additional relation corresponds decategorified to the commutation relation between E_i and F_j.
For the quantum covering groups, the analogous 2-supercategories have been defined in [BE17b].
A very important step in the categorification process is the proof of a basis theorem. In the classical setup, the case $\mathfrak{sl}_2$ was proven in [Lau10] via iterated flag varieties. This was extended to $\mathfrak{sl}_n$ in [KL10] using similar methods. The result in full generality was proven in [Web24] by means of deformation theory. The basis theorem for the 2-supercategories is not known in full generality. Intermediate results can be extracted from [KKT16], where the authors related (degenerate) affine Sergeev algebras to certain quiver Hecke superalgebras. In particular, type $B_{0|\infty}$ appears among these, providing in turn a basis theorem for $\mathfrak{U}(B_{0|\infty})$.
Therefore, $\mathfrak{U}(B_{0|\infty})$ categorifies $U(B_{0|\infty})$ and we obtain a $U(B_{0|\infty})$-module structure on $K_0(\mathfrak{U}(B_{0|\infty})^\Lambda)$. From the upper-finite highest weight structure on $\mathfrak{U}(B_{0|\infty})^\Lambda$, we obtain two natural bases on $K_0(\mathfrak{U}(B_{0|\infty})^\Lambda)$. One is given by the isomorphism classes of the standard modules $\Delta([\lambda,\mu])$ and the other by the isomorphism classes of the projective modules $P([\lambda,\mu])$. We will show

Theorem E. *There is an isomorphism of $U(B_{0|\infty})$-modules*

$$\begin{aligned}\Phi\colon L(\omega_f)\otimes L^w(-\omega_f) &\to K_0(\mathfrak{U}(B_{0|\infty})^\Lambda),\\ v_\mu\otimes v^\lambda &\mapsto [\Delta([\lambda,\mu])].\end{aligned}$$

Under this isomorphism, $[P([\lambda,\mu])]$ corresponds to the canonical basis element b_μ^λ of $L(\omega_f)\otimes L^w(-\omega_f)$.

Here, $L(\omega_f)$ denotes the integrable $U(B_{0|\infty})$-module with highest weight ω_f and $L^w(-\omega_f)$ the one with lowest weight $-\omega_f$.
Canonical bases for covering quantum supergroups are a straightforward generalization of the classical theory as in [Lus10], see e.g. [CHW13] for this setup. We use however a different comultiplication, resembling the one from [Kas93] in the construction of crystal bases.
Disregarding the statement concerning the canonical basis, a similar result was proven in [GRS24] in the context of cyclotomic oriented Brauer–Clifford algebras.
Note that $L(\omega_f)\otimes L^w(-\omega_f)$ is not irreducible anymore. It rather has a descending Jordan–Hölder filtration

$$\cdots\subseteq J_2\subseteq J_1\subseteq J_0 = L(\omega_f)\otimes L^w(-\omega_f).$$

We will also categorify this filtration.

Theorem F. *There is a filtration of* $\mathfrak{U}(B_{0|\infty})^{\Lambda}$ *by* 2*-subrepresentations*

$$\cdots \subseteq \mathfrak{U}(B_{0|\infty})_2^{\Lambda} \subseteq \mathfrak{U}(B_{0|\infty})_1^{\Lambda} \subseteq \mathfrak{U}(B_{0|\infty})_0^{\Lambda} = \mathfrak{U}(B_{0|\infty})^{\Lambda}$$

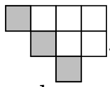
with $\mathfrak{U}(B_{0|\infty})_n^{\Lambda}/\mathfrak{U}(B_{0|\infty})_{n-1}^{\Lambda} \cong \mathrm{rep}'(\mathfrak{q}(n))$*. The induced filtration on* $K_0(\mathfrak{U}(B_{0|\infty})^{\Lambda})$ *turns* Φ *into a filtered isomorphism of filtered* $U(B_{0|\infty})$*-modules (with the filtration from Section 19.3 on* $L(\omega_f) \otimes L(-\omega_f)$*).*

Finally, we will identify the subquotients appearing in the Jordan–Hölder filtration. We will show that these are isomorphic to exterior powers $\mathscr{F}^n$ of the natural representation of $U(B_{0|\infty})$. In particular, we obtain that $\mathrm{rep}(\mathfrak{q}(n))$ categorifies $\mathscr{F}^n$, giving a more conceptual proof for the results in [Bru04].

13. Preliminaries

In this chapter we will introduce some basic definitions. We will recall the combinatorics of strict bipartitions and also recall the definition of graded (2-)supercategories.

13.1. Combinatorics of strict bipartitions

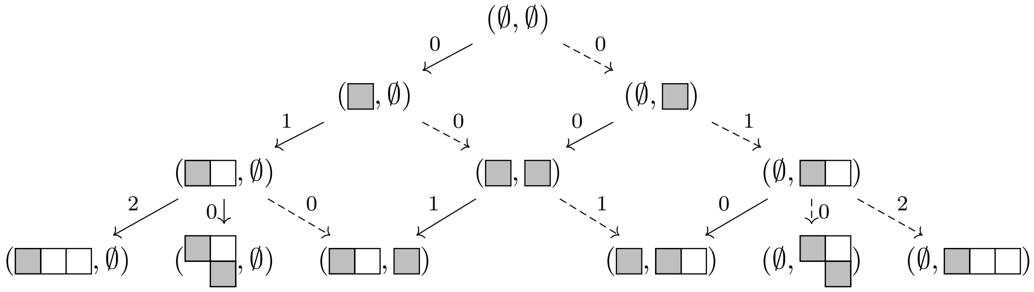
Definition 13.1. A partition $(\lambda_1, \cdots, \lambda_r)$ is called *strict* if $\lambda_1 > \lambda_2 > \cdots > \lambda_r$. We write $\ell(\lambda) = r$ for the length of this sequence and $|\lambda| \coloneqq \sum_{i=1}^{r} \lambda_i$ for its size. We will identify a strict partition with its (shifted) Young diagrams. Namely, we will draw the Young diagram with λ_i boxes in row i but shifted by $i-1$ to the right. To emphasize this slightly unusual convention, we will draw the box on the diagonal in gray. For instance, the Young diagram of $(4,3,1)$ is . We define the residue $\operatorname{res}(\alpha)$ of a box α in the Young diagram to be the difference between the column and the row. In particular, every gray box has residue 0.

Remark 13.2. There are two reasons for this convention. First, the author finds it easier to detect addable and removable boxes in this way. Namely, in this convention all boxes to the right that "look" addable or removable are actually addable or removable. Second, we can also identify strict partitions with symmetric partitions, i.e. partitions λ with $\lambda^t = \lambda$. This is done by reflecting the (shifted) Young diagram at the main diagonal. These strict partitions will later appear again in the context of type B combinatorics. Strict partitions encode then also symmetric partitions, since it is enough to remember half of the diagram.

A strict bipartition $[\lambda, \mu]$ is a pair, i.e. $[\lambda, \mu] = (\lambda, \mu)$, of strict partitions. We denote the set of strict bipartitions by $\Lambda\!\!\!\Lambda$ and also identify these with their pairs of Young diagrams. The residue of a box α for a strict bipartition is the same as the residue for a bipartition. We define a partial order on the set of bipartitions by $[\lambda, \mu] > [\lambda', \mu']$ if $|\lambda| < |\lambda'|$ and $|\mu| < |\mu'|$.
We define a graph (also called $\Lambda\!\!\!\Lambda$) with vertices $\Lambda\!\!\!\Lambda$ and two kinds of edges. There is

- an edge $[\lambda, \mu] \to [\lambda', \mu']$ if λ' can be obtained from λ by adding a box and $\mu = \mu'$,
- an edge $[\lambda, \mu] \dashrightarrow [\lambda', \mu']$ if $\lambda = \lambda'$ and μ' can be obtained from μ by adding a box.

We label every edge by the residue of the box that is added.
The following depicts a small part of this labelled graph.



An up-down-bitableau $\mathfrak{t}$ of shape $\mathrm{Shape}(\mathfrak{t}) = [\lambda, \mu]$ is a walk in the underlying undirected graph of $\mathbb{\Lambda}$ starting at $(\emptyset, \emptyset)$ and ending at $[\lambda, \mu]$. We write $l(\mathfrak{t})$ for the length of this walk. We denote the set of up-down-bitableaux of shape $[\lambda, \mu]$ by $\mathcal{T}^{\mathrm{ud}}([\lambda, \mu])$, and we define $\mathcal{T}_l^{\mathrm{ud}}([\lambda, \mu]) \coloneqq \{\mathfrak{t} \in \mathcal{T}^{\mathrm{ud}}([\lambda, \mu]) \mid l(\mathfrak{t}) = l\}$. Furthermore, we introduce the notation $\mathcal{T}^{\mathrm{ud}}$ for the set of all up-down-bitableaux.

Given an up-down-bitableau $\mathfrak{t}$ of shape $[\lambda, \mu]$ and suppose that $[\lambda, \mu] \xrightarrow{i} [\lambda', \mu]$, then write $\mathfrak{t} \xrightarrow{i} [\lambda', \mu]$ for the up-down-bitableau obtained by concatenation of the walks. Similarly, we use this notation for $\dashrightarrow$ as well as backwards edges.

For every bipartition $[\lambda, \mu]$ there exists a special up-down-bitableau $\mathfrak{t}^{\lambda,\mu}$, which is the walk in the graph by first constructing the Young diagram of λ by adding boxes row by row and afterward constructing the Young diagram of μ in the same way.

Example 13.3. For $[\lambda, \mu] = \left(\ydiagram{2,1+1}, \ydiagram{2}\right)$, the walk $\mathfrak{t}^{\lambda,\mu}$ would be given as follows

$$(\emptyset, \emptyset) \xrightarrow{0} (\ydiagram{1}, \emptyset) \xrightarrow{1} (\ydiagram{2}, \emptyset) \xrightarrow{0} \left(\ydiagram{2,1+1}, \emptyset\right) \overset{0}{\dashrightarrow} \left(\ydiagram{2,1+1}, \ydiagram{1}\right) \overset{1}{\dashrightarrow} \left(\ydiagram{2,1+1}, \ydiagram{2}\right).$$

We define

$$\mathrm{Add}_{E_i}([\lambda, \mu]) \coloneqq \{[\lambda', \mu] \mid [\lambda, \mu] \xrightarrow{i} [\lambda', \mu]\} \text{ and } \mathrm{Add}_{F_i}([\lambda, \mu]) \coloneqq \{[\lambda, \mu'] \mid [\lambda, \mu] \overset{i}{\dashrightarrow} [\lambda, \mu']\}.$$

13.2. Basics on 2-supercategories

This section introduces the main definitions and results on 2-supercategories that we will need. See [BE17a] for a more thorough introduction.

Definition 13.4. Denote by $\mathcal{GSV}\mathrm{ec}$ the symmetric monoidal category of graded (i.e. $\mathbb{Z}$-graded) vector superspaces with linear maps of all degrees and parities. Write $\mathcal{GSV}\mathrm{ec}^{\circ}$ for the subcategory of $\mathcal{GSV}\mathrm{ec}$ with only even linear maps of degree 0. Given a graded vector superspace V, we write $|v|$ for the parity of $v \in V$ and $\deg(v)$ for the degree of v (given that $v \in V$ is homogeneous). The braiding on $\mathcal{GSV}\mathrm{ec}$ is given by $v \otimes w \mapsto (-1)^{|v||w|} w \otimes v$, i.e. the grading plays no role for the braiding.

Given a graded vector superspace $V = \bigoplus_{i \in \mathbb{Z}} V_i$, we define its *shift* $Q^m V$ by $(Q^m V)_i = V_{i-m}$ for $m \in \mathbb{Z}$. We also have the parity shift functor Π swapping the parity.

Definition 13.5. A *graded supercategory* is a category enriched over $\mathcal{GSV}\mathrm{ec}^\circ$. This means that every morphism space is a graded vector superspace and composition induces an even linear map of degree 0. Given a graded supercategory $\mathcal{C}$, we write $\mathcal{C}^\circ$ for the *underlying category*, i.e. the category with the same objects but only even morphisms of degree 0. Observe that $\mathcal{GSV}\mathrm{ec}^\circ$ is the underlying category of $\mathcal{GSV}\mathrm{ec}$.
A *graded superfunctor* $F\colon \mathcal{C} \to \mathcal{D}$ between two graded supercategories is a $\mathcal{GSV}\mathrm{ec}^\circ$-enriched functor, i.e. $\mathrm{Hom}_{\mathcal{C}}(x,y) \to \mathrm{Hom}_{\mathcal{D}}(Fx,Fy)$, $f \mapsto Ff$ is an even linear map of degree 0 for all x, $y \in \mathcal{C}$.
A *degree m supernatural transformation* $\eta\colon F \Rightarrow G$ between graded superfunctors F, $G\colon \mathcal{C} \to \mathcal{D}$ is a family of degree m morphisms $\eta_x = \eta_{x,0} + \eta_{x,1}\colon Fx \to Gx$ such that $|\eta_{x,p}| = p$ and $Gf \circ \eta_{x,p} = (-1)^{p|f|}\eta_{y,p} \circ Ff$ for $f \in \mathrm{Hom}_{\mathcal{C}}(x,y)$. Denote by $\mathrm{Hom}(F,G)_m$ the superspace of all homogeneous degree m supernatural transformations. A *graded supernatural transformation* is then an element in $\bigoplus_{m\in\mathbb{Z}} \mathrm{Hom}(F,G)_m$. An even supernatural transformation of degree 0 is the same as a $\mathcal{GSV}\mathrm{ec}^\circ$-enriched natural transformation.

Classically (i.e. no super), there is a correspondence between graded categories and categories with a free $\mathbb{Z}$-action. Between these two notions, there are graded categories with a $\mathbb{Z}$-action. The next definition introduces these for graded supercategories.

Definition 13.6. A graded Q-Π-*supercategory* is a graded supercategory $\mathcal{C}$ plus the extra data of graded superfunctors Q, Q^{-1}, $\Pi\colon \mathcal{C} \to \mathcal{C}$, a degree 0 odd supernatural isomorphism $\zeta\colon \Pi \Rightarrow I$, and even supernatural isomorphisms $\sigma\colon Q \Rightarrow I$ and $\overline{\sigma}\colon Q^{-1} \Rightarrow I$ of degrees -1 and 1 respectively.
The Q-Π-*envelope* of a graded supercategory $\mathcal{A}$ is the graded Q-Π-supercategory $\mathcal{C}_{q,\pi}$ with objects $\{Q^m\Pi^a\kappa \mid \kappa \in \mathrm{Ob}(\mathcal{A}), m \in \mathbb{Z}, a \in \mathbb{Z}/2\mathbb{Z}\}$ and

$$\mathrm{Hom}_{\mathcal{C}_{q,\pi}}(Q^m\Pi^a\kappa, Q^n\Pi^b\nu) = Q^{n-m}\Pi^{a+b}\,\mathrm{Hom}_{\mathcal{A}}(\kappa,\nu),$$

where on the right-hand side Q and Π denote the grading and parity shift functors on $\mathcal{GSV}\mathrm{ec}^\circ$.

Remark 13.7. If we leave out the graded everywhere in the previous definition and replace $\mathcal{GSV}\mathrm{ec}^\circ$ by $\mathcal{SV}\mathrm{ec}^\circ$, we obtain the notion of supercategories, superfunctors, supernatural transformations etc.

The next definition was introduced in [KKT16] for categories with a $\mathbb{Z}/2\mathbb{Z}$-action. We will use the adaptation from [Bru] to supercategories.

Definition 13.8. Let $\mathcal{C}$ be a supercategory. We define its *Clifford twist* $\mathcal{C}^{CT}$ to be the following supercategory. The objects of $\mathcal{C}^{CT}$ are pairs (X,ϕ), where $X \in \mathrm{Ob}(\mathcal{C})$ and $\phi \in \mathrm{End}(X)$ such that $|\phi| = 1$ and $\phi^2 = -1$. Morphisms $f\colon (X,\phi) \to (Y,\psi)$ are morphisms $f\colon X \to Y$ in $\mathcal{C}$ such that

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d, "f"'] & X \arrow[d, "f"] \\ Y \arrow[r, "\psi"'] & Y \end{tikzcd}$$

supercommutes, i.e. $\psi \circ f = (-1)^{|f|} f \circ \phi$.

As $\mathcal{GSV}\mathrm{ec}^\circ$ is symmetric monoidal, we get a tensor product on $\mathcal{GSV}\mathrm{ec}^\circ$-enriched categories, see [Kel05, §1.4]. Given two $\mathcal{GSV}\mathrm{ec}^\circ$-categories $\mathcal{A}$ and $\mathcal{B}$, their tensor product $\mathcal{A} \boxtimes \mathcal{B}$ is the $\mathcal{GSV}\mathrm{ec}^\circ$-category with objects $\mathrm{Ob}(\mathcal{A}) \times \mathrm{Ob}(\mathcal{B})$ and morphisms

$$\mathrm{Hom}_{\mathcal{A}\boxtimes\mathcal{B}}((\kappa,\nu),(\sigma,\tau)) = \mathrm{Hom}_{\mathcal{A}}(\kappa,\sigma) \otimes \mathrm{Hom}_{\mathcal{B}}(\nu,\tau)$$

for $\kappa, \sigma \in \mathrm{Ob}(\mathcal{A})$ and $\nu, \tau \in \mathrm{Ob}(\mathcal{B})$. The composition of morphisms uses the symmetric braiding on $\mathcal{GSV}\mathrm{ec}^\circ$.

In particular, we can consider categories enriched over the monoidal category of graded supercategories with graded superfunctors. This notion is, however, a bit too restrictive for our purposes, and we recall the weaker version of a graded 2-supercategory (we use the term 2-category for what sometimes is also called bicategory). Roughly speaking, this is a 2-category but the level of 2-morhisms is enriched over $\mathcal{GSV}\mathrm{ec}^\circ$. This is also a special case of [GS16], where the authors consider 2-categories enriched in monoidal 2-categories. In our case, the monoidal 2-category is the (strict) 2-category of graded supercategories, graded superfunctors and *even, degree* 0 supernatural transformations (i.e. $\mathcal{GSV}\mathrm{ec}^\circ$-enriched natural transformations).

We will only list the needed data, for the precise coherence conditions we refer to [BE17a] (or [GS16] for the general setup). In particular, these are the analogs of the coherence conditions for 2-categories.

Definition 13.9.

(i) A *graded* 2*-supercategory* $\mathfrak{A}$ is the data of

  (a) A set of objects $\mathrm{Ob}(\mathfrak{A})$.

  (b) A graded supercategory $\mathcal{H}om_{\mathfrak{A}}(\kappa,\nu)$ for every $\kappa,\ \nu \in \mathrm{Ob}(\mathfrak{A})$, whose objects and morphisms are called 1-morphisms and 2-morphisms, respectively. The composition of 2-morphisms is also referred to as *vertical composition.*

  (c) A family of 1-morphisms $\mathbb{1}_\kappa \colon \kappa \to \kappa$ for every $\kappa \in \mathrm{Ob}(\mathfrak{A})$.

  (d) Graded superfunctors $T_{\kappa,\nu,\iota} \colon \mathcal{H}om_{\mathfrak{A}}(\nu,\kappa) \boxtimes \mathcal{H}om_{\mathfrak{A}}(\iota,\nu) \to \mathcal{H}om_{\mathfrak{A}}(\iota,\kappa)$ for all $\kappa,\ \nu,\ \iota \in \mathrm{Ob}(\mathfrak{A})$. We will usually abbreviate $T_{\kappa,\nu,\iota}$ by $--$ and refer to it as *horizontal composition.*

  (e) Even degree 0 supernatural isomorphisms $a \colon (--)- \Rightarrow -(--)$, $l \colon \mathbb{1}_\kappa - \to -$, and $r \colon -\mathbb{1}_\kappa \to -$ (whenever this makes sense).

  If a, l and r are the identity, we call $\mathfrak{A}$ a *strict graded* 2*-supercategory.*

(ii) A *graded* 2*-superfunctor* $\mathbb{R} \colon \mathfrak{A} \to \mathfrak{D}$ between two graded 2-supercategories is the data of

  (a) A function $\mathbb{R} \colon \mathrm{Ob}(\mathfrak{A}) \to \mathrm{Ob}(\mathfrak{D})$.

  (b) Graded superfunctors $\mathbb{R} \colon \mathcal{H}om_{\mathfrak{A}}(\nu,\kappa) \to \mathcal{H}om_{\mathfrak{D}}(\mathbb{R}\nu,\mathbb{R}\kappa)$ for all $\kappa,\ \nu \in \mathrm{Ob}(\mathfrak{A})$.

  (c) Even supernatural isomorphisms $c \colon (\mathbb{R}-)(\mathbb{R}-) \Rightarrow \mathbb{R}(--)$ that are of degree 0.

(d) Degree 0 even 2-isomorphisms $i\colon \mathbb{1}_{\mathbb{R}\kappa} \Rightarrow \mathbb{R}\mathbb{1}_\kappa$ for all $\kappa \in \mathrm{Ob}(\mathfrak{A})$.

(iii) A *graded 2-supernatural transformation* $(X, x)\colon \mathbb{R} \Rightarrow \mathbb{S}$ is the data of

(a) 1-morphisms $X_\kappa\colon \mathbb{R}\kappa \to \mathbb{S}\kappa$ for all $\kappa \in \mathrm{Ob}(\mathfrak{A})$.

(b) Even degree 0 supernatural transformations $x_{\kappa,\nu}\colon X_\kappa(\mathbb{R}-) \Rightarrow (\mathbb{S}-)X_\nu$ for all κ, $\nu \in \mathrm{Ob}(\mathfrak{A})$. (Both sides are graded superfunctors $\mathcal{H}om_{\mathfrak{A}}(\nu,\kappa) \to \mathcal{H}om_{\mathfrak{D}}(\mathbb{R}\nu, \mathbb{S}\kappa)$, respectively).

A graded 2-supernatural transformation is called *strong* if all $X_{\kappa,\nu}$ are isomorphisms.

Remark 13.10. A strict graded 2-supercategory is the same as a category enriched over the category of graded supercategories with graded superfunctors.
Again there is an obvious notion of 2-supercategories, 2-superfunctors and 2-supernatural transformations by leaving out the grading everywhere.
Note that the forgetful functor $\mathcal{GSV}\mathrm{ec}^\circ \to \mathcal{SV}\mathrm{ec}^\circ$ is symmetric monoidal, in particular any graded 2-supercategory is a 2-supercategory. However, not every (graded) 2-supercategory is a 2-category (but the converse holds).

Example 13.11. The prototypical example of a strict graded 2-supercategory is the graded 2-supercategory $\mathfrak{GSCat}$ consisting of graded supercategories, graded superfunctors and graded supernatural transformations. We also consider a particular sub-2-supercategory $\mathfrak{GSCat}^\oplus$ with Karoubian graded supercategories (by this we mean that the underlying category is Karoubian), graded superfunctors, and graded supernatural transformations.
An example for a non-strict 2-supercategory is the 2-supercategory of superalgebras, superbimodules and superbimodule morphisms.

Definition 13.12. A *graded 2-representation of a graded 2-supercategory* $\mathfrak{A}$ is a graded 2-superfunctor $\mathfrak{A} \to \mathfrak{GSCat}$. A *morphism* of graded 2-representations is a strong graded 2-supernatural transformation. A graded 2-representation $\mathbb{R}'$ is called a *sub-2-representation* if there is a fully faithful morphism $\mathbb{R}' \to \mathbb{R}$.

Remark 13.13. The following is a particular example of a sub-2-representation. Let $\mathbb{R}$ be a graded 2-representation of the graded 2-supercategory $\mathfrak{A}$. Suppose, we are given full subcategories $\mathbb{R}\lambda' \subseteq \mathbb{R}\lambda$ such that the graded superfunctors $\mathbb{R}$ restrict to these subcategories. Then, we obtain a sub-2-representation $\mathbb{R}'$ of $\mathbb{R}$.

The following gives another construction of graded 2-representations as quotients by invariant ideals, see e.g. [BD17, §4.2].

Definition 13.14. Let $\mathbb{R}$ be a graded 2-representation of a graded 2-supercategory $\mathfrak{A}$. An *invariant ideal* $\mathbb{I}$ *of* $\mathbb{R}$ is a family $(\mathbb{I}_\kappa)_\kappa$ of homogeneous (categorical) ideals $\mathbb{I}_\kappa \subseteq \mathbb{R}\kappa$ such that for every 1-morphism $X\colon \kappa \to \nu$ and $g\colon x \to y$ in $\mathbb{I}_\kappa$ we have $Xg \in \mathbb{I}_\nu$. Categorical ideal means that if $g\colon x \to y \in \mathbb{I}_\kappa$ and $f\colon y \to z, h\colon z \to x \in \mathbb{R}\kappa$, then $fg, gh \in \mathbb{I}_\kappa$.
Given an invariant ideal $\mathbb{I}$ of $\mathbb{R}$ we can define $\mathbb{R}/\mathbb{I}$ to be the graded 2-representation of $\mathfrak{A}$ with $(\mathbb{R}/\mathbb{I})\kappa := \mathbb{R}\kappa/\mathbb{I}_\kappa$ (the usual quotient of a category by an ideal).

Remark 13.15. Note that an invariant ideal does not necessarily define a sub-2-representation (it is not necessarily fully faithful) but every sub-2-representation defines an invariant ideal.

The following definition will be later needed for the categorification results.

Definition 13.16. A *graded Q-Π-2-supercategory* is a graded 2-supercategory $\mathfrak{A}$ together with families of 1-morphisms

$$q = (q_\kappa \colon \kappa \to \kappa), \qquad q^{-1} = (q_\kappa^{-1} \colon \kappa \to \kappa), \qquad \pi = (\pi_\kappa \colon \kappa \to \kappa), \qquad \pi^{-1} = (\pi_\kappa^{-1} \colon \kappa \to \kappa)$$

and families of (homogeneous) 2-morphisms

$$\begin{array}{cccc} & \sigma = (\sigma_\kappa \colon q_\kappa \Rightarrow \mathbb{1}_\kappa), & \overline{\sigma} = (\overline{\sigma}_\kappa \colon q_\kappa^{-1} \Rightarrow \mathbb{1}_\kappa), & \zeta = (\zeta_\kappa \colon \pi_\kappa \Rightarrow \mathbb{1}_\kappa). \\ \text{parity} & \text{even} & \text{even} & \text{odd} \\ \text{degree} & -1 & 1 & 0 \end{array}$$

Definition 13.17. Given a graded 2-supercategory $\mathfrak{A}$, we define its Q-Π-*envelope* $\mathfrak{A}_{\pi,q}$ to be the graded Q-Π-2-supercategory with the same objects as $\mathfrak{U}$ and morphism supercategories $\mathcal{H}om_{\mathfrak{A}_{q,\pi}}(\kappa,\nu) \coloneqq \mathcal{H}om_{\mathfrak{c}}(\kappa,\nu)_{q,\pi}$. In other words, we add formal grading and parity shift to every 1-morphism.
The space of 2-morphisms stays the same but is shifted in degree and parity, according to the grading and parity shift of the 1-morphisms.
Observe that for the horizontal composition of 2-morphisms signs are involved (for details see [BE17a]).

14. Isomeric Lie superalgebras and oriented Brauer–Clifford algebras

In this section we collect the basic definitions on isomeric Lie superalgebras and their representation theory.

14.1. Isomeric Lie superalgebras and their representation theory

Consider a vector superspace V with an odd endomorphism J satisfying $J^2 = -1$. This necessarily means that V is of dimension $(n|n)$ for some $n \in \mathbb{N}$.
The *isomeric Lie superalgebra* $\mathfrak{q}(n)$ is the centralizer of J in $\mathfrak{gl}(n|n)$. Fixing a basis $\{v_1, \dots, v_n, v_1', \dots, v_n'\}$ for V, we may assume that $Jv_i = v_i'$ and $Jv_i' = -v_i$. In terms of explicit matrices, $\mathfrak{q}(n)$ is then given by

$$\mathfrak{q}(n) = \{\left(\begin{smallmatrix} A & B \\ B & A \end{smallmatrix}\right)\}$$

such that $\mathfrak{q}(n)_0 \cong \mathfrak{gl}(n)$, and $\mathfrak{q}(n)_1$ is the adjoint representation of $\mathfrak{gl}(n)$.
Denote by $\mathrm{rep}(\mathfrak{q}(n))$ the supercategory of finite dimensional representations of $\mathfrak{q}(n)$ that are integrable with respect to $\mathrm{GL}(n)$, the algebraic group corresponding to $\mathfrak{gl}(n) = \mathfrak{q}(n)_0$. The morphisms in $\mathrm{rep}(\mathfrak{q}(n))$ can be even or odd, and we denote by $\mathrm{rep}^\circ(\mathfrak{q}(n))$ the restriction to the even morphisms. Both of these are rigid symmetric monoidal categories, with braiding $v \otimes w \mapsto (-1)^{|v||w|} w \otimes v$.
The abelian category $\mathrm{rep}(\mathfrak{q}(n))^\circ$ has enough projectives and injectives (and these agree), see e.g. [BKN11, Proposition 2.2.2]. Denote by $\mathrm{proj}(\mathfrak{q}(n))$ be the full subcategory of $\mathrm{rep}(\mathfrak{q}(n))$ generated by the projective objects (in $\mathrm{rep}\,\mathfrak{q}(n)^\circ$).
Two very important representations of $\mathfrak{q}(n)$ are the *natural representation* V and its dual V^*. We denote the full Karoubian monoidal subcategory of $\mathrm{rep}^\circ(\mathfrak{q}(n))$ generated by V and V^* by $\mathrm{Fund}^\circ(\mathfrak{q}(n))$. Objects of $\mathrm{Fund}(\mathfrak{q}(n))$ are direct sums of direct summands of mixed tensor powers $V^{\otimes d} \otimes (V^*)^{\otimes d'}$.
We use the notation $\mathrm{Fund}(\mathfrak{q}(n))$ for the full subcategory of $\mathrm{rep}(\mathfrak{q}(n))$ generated by the objects of $\mathrm{Fund}^\circ(\mathfrak{q}(n))$.

Remark 14.1. Note that $\mathrm{Fund}(\mathfrak{q}(n))$ is not Karoubian. There might exist non-homogeneous idempotents in $\mathrm{Fund}(\mathfrak{q}(n))$ that will not split. However, every homogeneous idempotent (which is then necessarily even) splits.

The natural representation V is a faithful representation and thus a tensor generator for $\mathrm{rep}\,\mathfrak{q}(n)$. The proof is as in the classical case, but an explicit argument for the super case can e.g. be found in [CH17] (the paragraph after Remark 7.4). In particular, we have the following important consequence.

Corollary 14.2. *Every indecomposable projective representation appears as a direct summand of $V^{\otimes d} \otimes (V^*)^{\otimes d'}$ for some d and d'.*

Hence, we have the following inclusions of full subcategories of $\mathrm{rep}(\mathfrak{q}(n))$:

$$\mathrm{proj}(\mathfrak{q}(n)) \subseteq \mathrm{Fund}(\mathfrak{q}(n)) \subseteq \mathrm{rep}(\mathfrak{q}(n)).$$

Irreducible finite dimensional representations of $\mathfrak{q}(n)$ have been classified by [Pen86, Theorem 4] and are parametrized by their highest weight. Denote by Λ_Q the set $\{(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_i \geq \lambda_{i+1}$ and $\lambda_i = \lambda_{i+1}$ only if $\lambda_i = 0\}$. Given $\lambda \in \Lambda_Q$, there is a corresponding irreducible highest weight representation $L_Q(\lambda)$ of $\mathfrak{q}(n)$ and

$$\{L_Q(\lambda) \mid \lambda \in \Lambda_Q\} \tag{14.1}$$

form a complete set of non-isomorphic irreducible representations of $\mathfrak{q}(n)$ (up to parity shift). Some of these are of type $\mathtt{Q}$, i.e. $L_Q(\lambda) \cong \Pi L_Q(\lambda)$, and some are of type $\mathtt{M}$, i.e. $L_Q(\lambda) \ncong \Pi L_Q(\lambda)$. To give an easy criterion to determine whether a representation is of type $\mathtt{Q}$ or $\mathtt{M}$, we introduce some combinatorics.

Definition 14.3. The *combinatorial weight diagram* of $\lambda \in \Lambda_Q$ is a sequence $\lambda^\dagger = (\lambda_i^\dagger)_{i \in \mathbb{N}}$ with $\lambda_i^\dagger \in \{\circ, \times, \wedge, \vee\}$. We draw this as a sequence of symbols next to one another and put a *wall* $|$ to the left of the first entry. The entries are defined as follows: Let $I_\lambda^+ = \{\lambda_i \mid \lambda_i > 0\}$ and $I_\lambda^- = \{-\lambda_i \mid \lambda_i < 0\}$. Then,

$$\lambda_i^\dagger = \begin{cases} \vee & \text{if } i \in I_\lambda^- \cap I_\lambda^+, \\ \times & \text{if } i \in I_\lambda^+ \setminus I_\lambda^-, \\ \circ & \text{if } i \in I_\lambda^- \setminus I_\lambda^+, \\ \wedge & \text{if } i \notin I_\lambda^+ \cup I_\lambda^-. \end{cases}$$

Note that any combinatorial weight diagram has finitely many $\circ$, $\times$ and $\vee$ entries but infinitely many $\wedge$. For a sequence $\lambda^\dagger$ we write $?(\lambda^\dagger)$ for the number of $?$ entries in $\lambda^\dagger$, where $? \in \{\circ, \times, \wedge, \vee\}$. The core $\mathrm{core}(\lambda)$ is the core diagram of $\lambda^\dagger$ which is obtained by replacing all $\wedge$ and $\vee$ with $\bullet$.

Example 14.4. To the weight $\lambda = (7, 5, 2, 0, 0, -1, -2, -3, -7)$ for $\mathfrak{q}(9)$ we have the following combinatorial weight diagram:

$$\lambda^\dagger = |\circ \vee \circ \wedge \times \wedge \vee \wedge \cdots$$

Its core is $\mathrm{core}(\lambda) = |\circ \bullet \circ \bullet \times \bullet \bullet \bullet \cdots$. Note that the combinatorial weight diagram of $(7, 5, 2, -1, -2, -3, -7)$ for $\mathfrak{q}(7)$ is the same as the one for $\mathfrak{q}(9)$. In particular, the combinatorial weight diagram only determines λ if we fix the n for $\mathfrak{q}(n)$.

Lemma 14.5. *The irreducible module $L_Q(\lambda)$ is of type $\mathtt{Q}$ if and only if $\times(\lambda^\dagger) + \circ(\lambda^\dagger)$ is odd, and otherwise of type $\mathtt{M}$.*

Proof. By [Pen86, Proposition 1], we have that $L_Q(\lambda)$ is of type $\mathtt{Q}$ if and only if $|\{\lambda_i \mid \lambda_i \neq 0\}|$ is odd. This number is exactly $\times(\lambda^\dagger) + \circ(\lambda^\dagger) + 2 \vee (\lambda^\dagger)$ and the statement follows. $\square$

The category $\mathrm{rep}(\mathfrak{q}(n))$ decomposes into a direct sum of blocks, i.e.

$$\mathrm{rep}(\mathfrak{q}(n)) = \bigoplus_\kappa \mathrm{rep}(\mathfrak{q}(n))_\kappa,$$

given by the central characters κ.

Definition 14.6. We write $X_{\leq 1}(n)$ for the set of central characters of $\mathfrak{q}(n)$.

The following is a direct consequence of [Ser83, Theorem 2].

Lemma 14.7. *Two finite dimensional irreducible representations $L_Q(\lambda)$ and $L_Q(\mu)$ have the same central character if and only if $\mathrm{core}(\lambda) = \mathrm{core}(\mu)$.*

Next we will introduce the notion of atypicality.
The definition we give here goes back to [Bru04, (2.13)], and it is a slight refinement of the usual notion of atypicality, as defined e.g. in [CW12a, Definition 2.29].

Definition 14.8. We define the *atypicality* of a weight $\lambda \in \Lambda_Q$ to be

$$\mathrm{atyp}(\lambda) := n - \times(\lambda^\dagger) + \circ(\lambda^\dagger).$$

If $\mathrm{atyp}(\lambda) \leq 1$, we call λ *typical* and *strongly typical* if $\mathrm{atyp}(\lambda) = 0$. Observe, that the atypicality only depends on the core of λ, and thus, by Lemma 14.7, we can also define the atypicality of a block κ.
Furthermore, note that $L_Q(\lambda)$ is of type $\mathtt{M}$ if and only if $n - \mathrm{atyp}(\lambda)$ is even.

The usual notion of atypicality can be recovered from Definition 14.8 as $\lfloor \frac{\mathrm{atyp}(\lambda)}{2} \rfloor$. However, the differentiation between typical and strongly typical weights (both of which would yield "usual" typical weights) was e.g. also made in [FM09] and [GS20]. The next result justifies the notion of typical and strongly typical and was shown in [GS20, Theorem 4.1] and the discussion thereafter.

Lemma 14.9. *Let $\lambda \in \Lambda_Q$.*

(i) *If* $\mathrm{atyp}(\lambda) = 0$, *then $L_Q(\lambda) = P_Q(\lambda) = I_Q(\lambda)$.*
Here, $L_Q(\lambda)$ is of type $\mathtt{Q}$ if and only if n is odd.

(ii) *If* $\mathrm{atyp}(\lambda) = 1$, *then there is a short exact sequence*

$$0 \to \Pi L_Q(\lambda) \to P_Q(\lambda) \to L_Q(\lambda) \to 0.$$

Here, $L_Q(\lambda)$ is of type $\mathtt{Q}$ if and only if n is even.

Remark 14.10. Lemma 14.9 shows that the (strongly) typical blocks of $\mathrm{rep}(\mathfrak{q}(n))$ contain only one irreducible module (up to parity shift). The strongly typical blocks are exactly those blocks that are simple.

Working with representations of type $\mathtt{Q}$ is often not very convenient. One reason for this is that their endomorphism ring is not 1-dimensional. In particular, Jordan–Hölder multiplicities cannot be computed via $\dim \mathrm{Hom}_{\mathfrak{q}(n)}(P_Q(\lambda), \_)$. In the following definition, we use the Clifford twist from Definition 13.8 to remedy this.

Definition 14.11. Consider the decomposition $\mathrm{rep}(\mathfrak{q}(n)) = \bigoplus_{\kappa \in X_{\leq 1}(n)} \mathrm{rep}(\mathfrak{q}(n))_\kappa$ (recall the set $X_{\leq 1}(n)$ from Definition 14.6).
Define $\mathrm{rep}'(\mathfrak{q}(n))_\kappa$ to be $\mathrm{rep}(\mathfrak{q}(n))_\kappa$ if it is of type $\mathtt{M}$ and $\mathrm{rep}(\mathfrak{q}(n))_\kappa^{CT}$ if it is of type $\mathtt{Q}$.
By Definition 14.8, we add a Clifford twist if and only if $n - \mathrm{atyp}(\kappa)$ is odd.
Similarly, we define $\mathrm{Fund}'(\mathfrak{q}(n))$ and $\mathrm{proj}'(\mathfrak{q}(n))$.

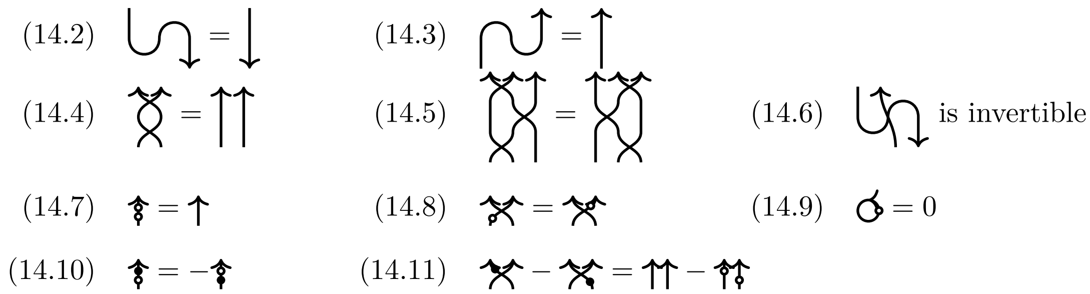
14.2. Schur–Weyl–Sergeev duality and oriented Brauer–Clifford algebras

In this section, we are going to recall Sergeev duality from [Ser84, Theorem 3]. This will be the main ingredient in our construction of an equivalence of abelian categories, as described in the introduction.
We begin by recalling the *degenerate affine oriented Brauer–Clifford supercategory* from [BCK19].

Definition 14.12. The *degenerate affine oriented Brauer–Clifford supercategory* $\mathcal{AOBC}$ is the $\mathbb{C}$-linear strict monoidal supercategory generated by two objects $\vee$ and $\wedge$ and morphisms

$: \wedge\vee \to \mathbb{1}$, $: \mathbb{1} \to \vee\wedge$, $: \wedge\wedge \to \wedge\wedge$, $: \wedge \to \wedge$, $: \wedge \to \wedge$,

where the first four are even and the last one is odd, subject to the following relations:



Next, we introduce an important cyclotomic quotient of $\mathcal{AOBC}$.

Definition 14.13. The oriented Brauer–Clifford supercategory $\mathcal{OBC}$ is the quotient of $\mathcal{AOBC}$ by the left tensor ideal generated by

.

Write $\mathcal{HC}_n$ for the endomorphism algebra of $\vee^{\otimes n}$ in $\mathcal{OBC}$.

Remark 14.14. The above defined supercategory $\mathcal{OBC}$ can alternatively be defined as the monoidal subsupercategory of $\mathcal{AOBC}$ generated by the same objects and morphisms as $\mathcal{AOBC}$, excluding .

The main reason for introducing $\mathcal{AOBC}$ is the following result.

Theorem 14.15 (Mixed Schur–Weyl–Sergeev duality)**.** *There is a full functor*

$$\mathrm{SWS}_n \colon \mathcal{OBC} \to \mathrm{rep}(\mathfrak{q}(n))$$

of left $\mathcal{AOBC}$-module categories.

The fullness for the full monoidal subcategory of $\mathcal{OBC}$ generated by $\vee$ goes back to [Ser84, Theorem 3]. Using adjunctions, it can easily be extended to all of $\mathcal{OBC}$, see e.g. [BCK19, Theorem 4.1]. In [BCK19, Theorem 4.4], it was shown that $\mathrm{rep}(\mathfrak{q}(n))$ is a left $\mathcal{AOBC}$-module category, see also [HKS11] for a similar statement for the endomorphism algebras of $\vee \cdots \vee$. From the definition there, it is clear that this is compatible with SWS_n.
Under the functor SWS_n, the object $\vee$ corresponds to the natural representation V and $\wedge$ to its dual V^*. The cap, cup and crossing of $\mathcal{OBC}$ are sent to the evaluation, coevaluation and braiding of $\mathrm{rep}(\mathfrak{q}(n))$, respectively. The morphisms is sent to the rescaling of J by $\sqrt{-1}$. The final generator is sent to a rescaling of the odd Casimir in $U(\mathfrak{q}(n)) \otimes U(\mathfrak{q}(n))$ by $-J \otimes 1$.
The more classical formulation in [Ser84] is as follows. In this setup everything is semisimple and more can be said about decomposing tensor products into irreducibles. The following summarizes [Ser84, Theorem 3 and 4].

Theorem 14.16. *The actions of $\mathcal{HC}_n$ and $\mathfrak{q}(n)$ on $V^{\otimes d}$ centralize each other. Moreover, $V^{\otimes d}$ is semisimple as $\mathfrak{q}(n)$-representation and decomposes as*

$$V^{\otimes d} = \bigoplus_{\lambda} 2^{-\delta(\lambda)} L_Q(\lambda) \boxtimes D^\lambda.$$

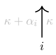
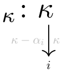
Here, λ runs through all $[\lambda, \emptyset] \in \Lambda\!\!\Lambda$ with $\ell(\lambda) \leq n$, D^λ denotes the irreducible $\mathcal{HC}_n$-module associated to λ and $\delta(\lambda) = \ell(\lambda) \mod 2$. We can easily interpret λ as the integral dominant weight $(\lambda_1 > \cdots > \lambda_{\ell(\lambda)}, 0, \cdots, 0)$ of $\mathfrak{q}(n)$ and $L_Q(\lambda)$ denotes the irreducible module with this highest weight. If $\delta(\lambda) = 1$, both irreducible representation are of type `Q` *and their tensor product decomposes into a direct sum of two irreducible representations of type* `M`*. Then, $2^{-\delta(\lambda)} L_Q(\lambda) \boxtimes D^\lambda$ denotes one summand of this.*

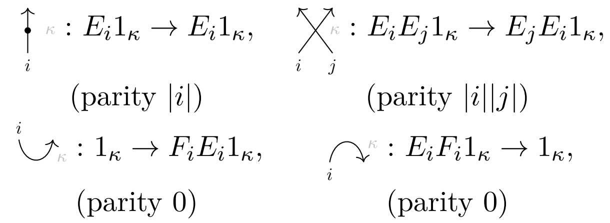
15. Quiver Hecke superalgebras

In this section we are going to recall the definition of quiver Hecke superalgebras from [KKT16] and [BE17b]. These are defined for any super Cartan datum, but we will only be interested in the case of type $B_{0|\infty}$: For this let $I = \mathbb{N}_0$ with parity function $I \to \mathbb{Z}/2\mathbb{Z}$, $i \mapsto \delta_{i,0}$. Let $(-d_{ij})$ be the generalized Cartan matrix of odd type B_∞, i.e. we have

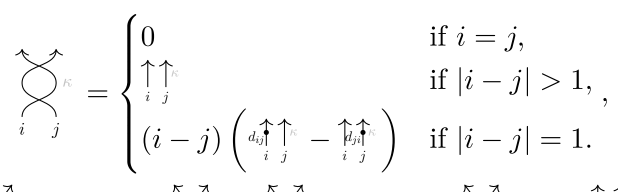
$$d_{ij} = \begin{cases} -2 & \text{if } i = j, \\ 2 & \text{if } i = 0 = j-1, \\ 1 & \text{if } i \neq 0 \text{ and } j = i \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

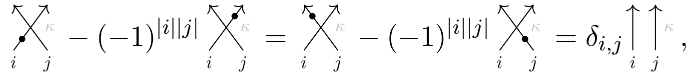
Let $\mathfrak{h}$ be the complex vector space with basis e_i for $i \in \mathbb{N}$. We define $h_i \in \mathfrak{h}$ as $h_0 = -2e_1$ and $h_i = e_i - e_{i+1}$ for $i \in \mathbb{N}$. Furthermore, we define $\alpha_i \in \mathfrak{h}^*$ as $\alpha_0 = -\varepsilon_1$ and $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i \in \mathbb{N}$. In particular, we have $\langle h_i, \alpha_j \rangle = -d_{ij}$. Denote by $X \coloneqq \{\kappa \in \mathfrak{h}^* \mid \langle h_i, \kappa \rangle \in \mathbb{Z}\}$ the weight lattice.
The following definition is taken from [BE17b] and based on [KKT16].

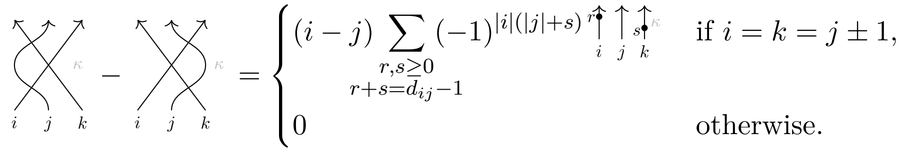
Definition 15.1. The 2-supercategory $\mathfrak{U}(B_{0|\infty})$ is the 2-supercategory with objects X, generating 1-morphisms $E_i 1_\kappa \colon \kappa \to \kappa + \alpha_i$ and $F_i 1_\kappa \colon \kappa \to \kappa - \alpha_i$ for $i \in I$ and $\kappa \in X$, which we depict as  and , and the following generating 2-morphisms

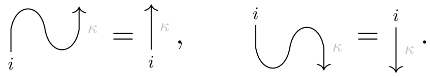


subject to the following relations:

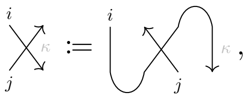
(15.1) 

(15.2) 

(15.3) 

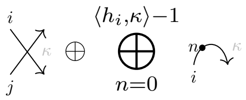
(15.4) 

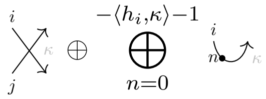
Finally, introducing the shorthand



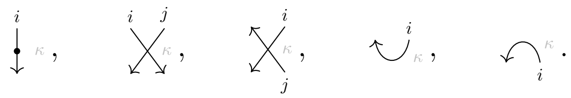
we require the following (not necessarily homogeneous) morphisms to be isomorphisms:

(15.5)  $: E_jF_i1_\kappa \xrightarrow{\sim} F_iE_j1_\kappa$ if $i \neq j$,

(15.6)  $: E_iF_i1_\kappa \xrightarrow{\sim} F_iE_i1_\kappa \oplus 1_\kappa^{\oplus\langle h_i,\kappa\rangle}$ if $\langle h_i,\kappa\rangle \geq 0$,

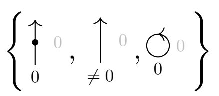
(15.7)  $: E_iF_i1_\kappa \oplus 1_\kappa^{\oplus-\langle h_i,\kappa\rangle} \xrightarrow{\sim} F_iE_i1_\kappa$ if $\langle h_i,\kappa\rangle \leq 0$.

In [BE17b], the authors calculated a spanning set for $\mathfrak{U}(B_{0|\infty})$, and they calculated many additional relations that hold among these generators. We do not repeat all of these relations here, but will frequently refer to them. We advise the reader to consult [BE17b] while reading Chapter 15 and Section 15.1.
Rotating the generating morphisms via (15.4) and using (15.5)–(15.7), we obtain the following important elements



Observe that, by using the defining relations, $\mathfrak{U}(B_{0|\infty})$ can be turned into a *graded* 2-supercategory. Figure 15.1 shows the degree and parity of all the generators.
Next, we introduce a certain cyclotomic quotient $\mathfrak{U}(B_{0|\infty})^\Lambda$ of $\mathfrak{U}(B_{0|\infty})$, i.e. a certain 2-representation of $\mathfrak{U}(B_{0|\infty})$.

Definition 15.2. Consider the universal 2-representation $(\mathcal{R}_\kappa)_{\kappa\in X}$ associated with the weight 0, i.e. $\mathcal{R}_\kappa \coloneqq \mathcal{H}om_{\mathfrak{U}(B_{0|\infty})}(0,\kappa)$, where the 1- and 2-morphisms in $\mathfrak{U}(B_{0|\infty})$ act on the left in the obvious way.
Define $(\mathcal{I}(\alpha_0 \mid \alpha_0)_\kappa)_{\kappa\in X}$ to be the invariant ideal of $\mathcal{R}$ generated by

(15.8) 

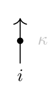
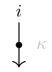
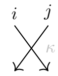
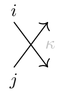
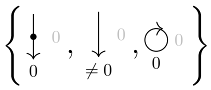
Generator	Degree	Parity	Generator	Degree	Parity
	$2d_i$	$\lvert i\rvert$		$2d_i$	$\lvert i\rvert$
	$d_i d_{ij}$	$\lvert i\rvert\lvert j\rvert$		$d_i d_{ij}$	$\lvert i\rvert\lvert j\rvert$
	0	$\lvert i\rvert\lvert j\rvert$		0	$\lvert i\rvert\lvert j\rvert$
	$d_i(1-\langle h_i,\kappa\rangle)$	0		$d_i(1+\langle h_i,\kappa\rangle)$	$\lvert i\rvert$
	$d_i(1+\langle h_i,\kappa\rangle)$	0		$d_i(1-\langle h_i,\kappa\rangle)$	$\lvert i\rvert$

Figure 15.1.: Degree and parity of the generators of $\mathfrak{U}(B_{0|\infty})$

Remark 15.3. By [BE17b, 2.18, Corollary 5.4], $(\mathcal{I}(\alpha_0 \mid \alpha_0)_\kappa)_{\kappa\in X}$ is also generated by

(15.9)  .

Definition 15.4. We define the cyclotomic level 1 quotient $\mathfrak{U}(B_{0|\infty})^\Lambda$ to be the 2-representation $\mathfrak{U}(B_{0|\infty})/\mathcal{I}(\alpha_0 \mid \alpha_0)$ using Definition 13.14.

Theorem 15.5. *There is an essentially surjective, full functor of 2-representations of* $\mathfrak{U}(B_{0|\infty})$

$$\mathcal{F}_n \colon \mathfrak{U}(B_{0|\infty})^\Lambda \to \mathrm{Fund}'(\mathfrak{q}(n)).$$

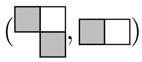
Before we give the proof, note that we use $\mathrm{Fund}'(\mathfrak{q}(n))$ and not $\mathrm{Fund}(\mathfrak{q}(n))$ here. The reason for this is that $\mathfrak{U}(B_{0|\infty})^\Lambda$ is solely of type $\mathtt{M}$, so to have any chance of a full functor, we need to consider $\mathrm{Fund}'(\mathfrak{q}(n))$.

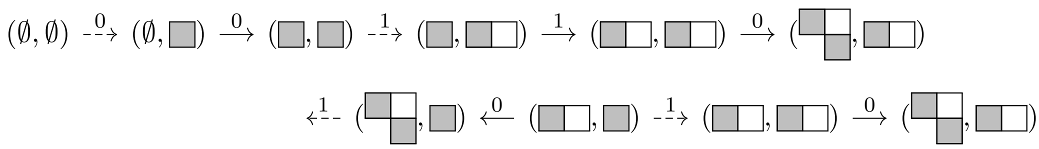
Proof. Theorem 14.15 gives an action of $\mathcal{AOBC}$ on $\mathrm{Fund}(\mathfrak{q}(n))$. By [Bru], this upgrades to a 2-representation of $\mathfrak{U}(B_{0|\infty})$ on $\mathrm{Fund}'(\mathfrak{q}(n))$. The trivial representation $\mathbb{C}$ clearly satisfies the cyclotomic relations (15.8), and thus we get a map of 2-representations $\mathfrak{U}(B_{0|\infty})^\Lambda \to \mathrm{Fund}'(\mathfrak{q}(n))$. Fullness follows from Theorem 14.15. $\square$

We conclude this section by formulating a basis theorem.
The basis will be indexed by pairs of up-down-bitableaux of the same shape.

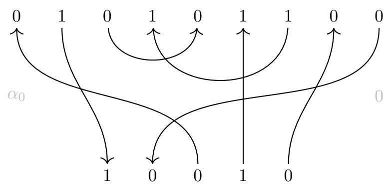
Definition 15.6. Fix an up-down-bitableau $\mathfrak{t}$ of shape $[\lambda,\mu]$. To this, we will associate an upper half basis element, which we denote by $\Psi^{\mathfrak{t}}$.
Let $\kappa = \mathrm{wt}([\lambda,\mu]) \coloneqq \mathrm{wt}(\mu) - \mathrm{wt}(\lambda)$, where $\mathrm{wt}(\lambda) = \sum_{i=1}^{r} \varepsilon_{\lambda_i}$ if $\lambda = (\lambda_1 > \cdots > \lambda_r)$.
Note that $\mathrm{wt}(\lambda) = \sum_{i\in\mathbb{N}_0} -l_i\alpha_i$, where l_i is the number of boxes of λ with residue i.
Then, $\Psi^{\mathfrak{t}}$ lives in the Hom-category $\mathfrak{U}(B_{0|\infty})^\Lambda_\kappa$, which is a quotient of $\mathcal{H}om_{\mathfrak{U}(B_{0|\infty})^\Lambda}(0,\kappa)$.

To $\mathfrak{t}$ we associate the following sequence of 1-morphisms E_i and F_i. Namely, we start with the empty sequence and build it via traversing the walk in $\bigwedge\!\!\!\bigwedge$ from $(\emptyset, \emptyset)$ to $[\lambda, \mu]$. For every edge $\xrightarrow{i}$ we add E_i and for every edge $\overset{i}{\dashrightarrow}$ we add F_i to the front. For edges being traversed backwards, i.e. $\xleftarrow{i}$ and $\overset{i}{\dashleftarrow}$ respectively, we add F_i and E_i respectively to the front. Denote the so obtained sequence by $\operatorname{res}\mathfrak{t}$. Similarly, we define $\operatorname{res}\mathfrak{t}^{\lambda,\mu}$ for the special up-down-bitableau. Then, $\Psi^{\mathfrak{t}}$ is a 2-morphism from $\operatorname{res}\mathfrak{t}^{\lambda,\mu}$ to $\operatorname{res}\mathfrak{t}$ in $\mathcal{H}om_{\mathfrak{U}(B_{0|\infty})^\Lambda}(0, \kappa)$. This two-morphism is obtained as follows. For every edge in the walk for $\mathfrak{t}$ that removes a box, we connect (using a cup) the corresponding E_i (or F_i) to the corresponding F_i (or E_i) where this box was added. After finishing all this, we connect the remaining E_i (or F_i) to the corresponding E_i (or F_i) in $\operatorname{res}\mathfrak{t}^{\lambda,\mu}$ (this is also the unique way with the least number of crossings).

Example 15.7. Consider the up-down-bitableau $\mathfrak{t}$ of shape $[\lambda, \mu] =$ :



We have $\mathrm{wt}(\lambda) = \varepsilon_2 + \varepsilon_1 = -(2\alpha_0 + \alpha_1)$ and $\mathrm{wt}(\mu) = \varepsilon_2 = -(\alpha_0 + \alpha_1)$. In particular, $\kappa = \mathrm{wt}(\mu) - \mathrm{wt}(\lambda) = -\varepsilon_1 = \alpha_0$ and the sequence $\operatorname{res}\mathfrak{t}$ would be $E_0F_1F_0E_1E_0E_1F_1E_0F_0$, while $\operatorname{res}\mathfrak{t}^{\lambda,\mu} = F_1F_0E_0E_1E_0$. The 2-morphism $\Psi^{\mathfrak{t}}$ would be given by the following diagram:



Similarly, we can define a lower half basis element $\Psi_{\mathfrak{t}}$ for every up-down-bitableau $\mathfrak{t}$ of shape $[\lambda, \mu]$. This lives again in $\mathfrak{U}(B_{0|\infty})^\Lambda_\kappa$ but describes now a 2-morphism from $\operatorname{res}\mathfrak{t}$ to $\operatorname{res}\mathfrak{t}^{\lambda,\mu}$. The construction is exactly the same, we just replace cups by caps.
Given now a pair $(\mathfrak{t}, \mathfrak{s})$ of up-down-bitableaux of the same shape, we can define an element $\Psi^{\mathfrak{s}}_{\mathfrak{t}} := \Psi^{\mathfrak{s}} \circ \Psi_{\mathfrak{t}}$ of $\mathfrak{U}(B_{0|\infty})^\Lambda$. In the beginning of Section 15.1, we will argue why these elements are actually well-defined.

Theorem 15.8 (Basis Theorem)**.** *The set*

$$\bigcup_{[\lambda,\mu]\in \bigwedge\!\!\!\bigwedge} \{\Psi^{\mathfrak{s}}_{\mathfrak{t}} \mid \mathfrak{t}, \mathfrak{s} \in \mathcal{T}^{\mathrm{ud}}([\lambda, \mu])\}$$

is a basis of $\mathfrak{U}(B_{0|\infty})^\Lambda$.

In Theorem 15.23, we will show that this set spans $\mathfrak{U}(B_{0|\infty})^\Lambda$. Linear independence will follow from Theorem 17.8. We conclude this section with a direct consequence of the basis theorem.
For this, recall the partial order on $\mathbb{\Lambda}$ from Section 13.1:

$$[\lambda,\mu] > [\lambda',\mu'] \iff |\lambda| < |\lambda'| \text{ and } |\mu| < |\mu'|. \tag{15.10}$$

Let I be the set of finite sequences with entries in $\{E_i, F_i \mid i \in \mathbb{N}_0\}$. We view $\mathbb{\Lambda}$ as a subset of I via $[\lambda,\mu] \mapsto \operatorname{res} \mathfrak{t}^{\lambda,\mu}$. Let

$$\begin{aligned} Y(\boldsymbol{i},[\lambda,\mu]) &:= \{\Psi_{\mathfrak{t}} \mid \mathfrak{t} \in \mathcal{T}^{\mathrm{ud}}([\lambda,\mu]) \text{ and } \operatorname{res}\mathfrak{t} = \boldsymbol{i}\},\\ X([\lambda,\mu],\boldsymbol{i}) &:= \{\Psi^{\mathfrak{t}} \mid \mathfrak{t} \in \mathcal{T}^{\mathrm{ud}}([\lambda,\mu]) \text{ and } \operatorname{res}\mathfrak{t} = \boldsymbol{i}\}.\end{aligned}$$

Theorem 15.9. *The data $\mathbb{\Lambda} \subseteq I$ together with $Y(\boldsymbol{i},[\lambda,\mu])$ and $X([\lambda,\mu],\boldsymbol{i})$ endow $\mathfrak{U}(B_{0|\infty})^\Lambda$ with the structure of an upper-finite based quasi-hereditary algebra in the sense of* [BS24].

Proof. The partial ordering on $\mathbb{\Lambda}$ is clearly upper finite. Furthermore, by Theorem 15.8, the set of all $\Psi_{\mathfrak{t}} \cdot \Psi^{\mathfrak{s}} = \Psi_{\mathfrak{t}}^{\mathfrak{s}}$ is a basis of $\mathfrak{U}(B_{0|\infty})^\Lambda$. Clearly, $Y([\lambda',\mu'],[\lambda,\mu]) = \emptyset$ unless $[\lambda,\mu] = [\lambda',\mu']$ or $|\lambda| < |\lambda'|$ and $|\mu| < |\mu'|$. And finally, $Y([\lambda,\mu],[\lambda,\mu]) = X([\lambda,\mu],[\lambda,\mu]) = \{\Psi_{\mathfrak{t}^{\lambda,\mu}}^{\mathfrak{t}^{\lambda,\mu}}\} = \{1_{\operatorname{res}\mathfrak{t}^{\lambda,\mu}}\}$ and the result follows. $\square$

15.1. Spanning set of $\mathfrak{U}(B_{0|\infty})^\Lambda$

In this section, we are going to show the first half of Theorem 15.8, namely that the designated basis spans.
We start with some remarks concerning the definition of the basis elements.

Remark 15.10. A priori, it is not clear that the elements $\Psi_{\mathfrak{t}}^{\mathfrak{s}}$ are well-defined. Namely, there are several ways to draw the cups and caps in the definitions of $\Psi^{\mathfrak{t}}$ and $\Psi_{\mathfrak{t}}$. By [BE17b, Proposition 7.2] all these possibilities only differ by a sign. In particular, this means that they span the same linear subspace. To be completely precise, one should rather pick for every up-down-bitableau a preferred way to draw cups and caps, but this is not necessary for our purposes.
Furthermore, there might also be a slight ambiguity in the crossings needed after all cups and caps have been constructed. Namely, there might be multiple reduced expressions for the needed permutation diagram. By Matsumoto's theorem, all these reduced expressions differ only by braid moves. The distant swaps all hold on the nose (again up to sign). Additionally, one observes that all upward or downward crossings can only appear for i and j with $|i-j| > 1$, hence the braid relation will always be satisfied by (15.3) and [BE17b, Lemma 3.3, Proposition 7.6].

Remark 15.11. Just by the construction of $\Psi_{\mathfrak{t}}^{\mathfrak{s}}$, the crossings in the middle of the corresponding diagram in general do not give a reduced expression.

However, double-crossings between two E_i and E_j (resp. F_i and F_j) can only occur for $|i-j|>1$ (see Remark 15.10) and thus can be removed (up to sign) by (15.1).
Any double-crossing between E_i and F_j with $i \neq j$ can be removed via (15.5). Finally, if there appears a double-crossing between E_i and F_i, this means that the corresponding weight κ in the region corresponds to a $[\lambda,\mu] \in \Lambda\!\!\Lambda$ with addable boxes of residue i for λ and μ. It is then an easy check that $\langle h_i, \kappa \rangle = 0$ and thus (15.6) allows us to remove this double-crossing. Summarizing, we can always remove all double-crossings in the middle of the diagram, at least up to sign.

We will prove the spanning set property via an induction on $|\lambda|+|\mu|$. This will consist of two main steps. First, we will show that our basis is closed under horizontal composition, i.e. we will show $E_i(\Psi^{\mathfrak{s}}_{\mathfrak{t}})$ and $F_i(\Psi^{\mathfrak{s}}_{\mathfrak{t}})$ are again linear combinations of basis elements, at least up to lower order terms; this will imply, that the identity morphism of every object in $\mathfrak{U}(B_{0|\infty})^{\Lambda}$ is contained in our spanning set. Secondly, we will show that the basis is closed under composition with the generating 2-morphisms of $\mathfrak{U}(B_{0|\infty})$. These together will imply the spanning set property.
We begin by introducing the necessary notation. Let $\mathfrak{U}(B_{0|\infty})^{\Lambda}_{\leq b}$ be the collection of all $((\mathfrak{U}(B_{0|\infty})^{\Lambda}_{\leq b})_{\kappa})_{\kappa \in X}$, where $(\mathfrak{U}(B_{0|\infty})^{\Lambda}_{\leq b})_{\kappa}$ is the collection of all morphisms in $\mathfrak{U}(B_{0|\infty})^{\Lambda}_{\kappa}$ factoring through an object of length $\leq b$. By definition, we have

$$E_i(\mathfrak{U}(B_{0|\infty})^{\Lambda}_{\leq b}) \subseteq \mathfrak{U}(B_{0|\infty})^{\Lambda}_{\leq b+1}, \quad F_i(\mathfrak{U}(B_{0|\infty})^{\Lambda}_{\leq b}) \subseteq \mathfrak{U}(B_{0|\infty})^{\Lambda}_{\leq b+1}$$

Furthermore, given $[\lambda,\mu] \in \Lambda\!\!\Lambda$, we write $C^{[\lambda,\mu]}$ for the span of all the morphisms $\Psi^{\mathfrak{t}}_{\mathfrak{t}^{\lambda,\mu}}$ for all $\mathfrak{t} \in \mathcal{T}^{\mathrm{ud}}([\lambda,\mu])$.
We also write $R_{\leq b}$ for the span of all $\Psi^{\mathfrak{s}}_{\mathfrak{t}}$ for $\mathfrak{t}, \mathfrak{s} \in \mathcal{T}^{\mathrm{ud}}([\lambda,\mu])$ for all $[\lambda,\mu]$ with $|\lambda|+|\mu| \leq b$.
Furthermore, we introduce the notations $R_{\leq b}|_{\leq l}$ and $\mathfrak{U}(B_{0|\infty})^{\Lambda}_{\leq b}\Big|_{\leq l}$ for the restriction to all morphisms between objects of length at most l.
Observe that we have $C^{[\lambda,\mu]} \subseteq R_{\leq |\lambda|+|\mu|} \subseteq \mathfrak{U}(B_{0|\infty})^{\Lambda}_{\leq |\lambda|+|\mu|}$.

15.1.1. Closure under horizontal composition

The 2-representation $\mathfrak{U}(B_{0|\infty})^{\Lambda}$ is equipped with functors $E_i \colon \mathfrak{U}(B_{0|\infty})^{\Lambda}_{\kappa} \to \mathfrak{U}(B_{0|\infty})^{\Lambda}_{\kappa+\alpha_i}$ and $F_i \colon \mathfrak{U}(B_{0|\infty})^{\Lambda}_{\kappa} \to \mathfrak{U}(B_{0|\infty})^{\Lambda}_{\kappa+\alpha_i}$.
Furthermore, we also have natural transformations $x_i \colon E_i \to E_i$, which are induced by $\underset{i}{\uparrow}{\scriptstyle \kappa}$. We write $E_i^1(\Psi^{\mathfrak{s}}_{\mathfrak{t}})$ for the composition $(x_i)_{i\,\mathrm{res}(\mathfrak{s})} \circ E_i(\Psi^{\mathfrak{s}}_{\mathfrak{t}})$. Diagrammatically this is just given by juxtaposition with $\underset{i}{\uparrow}{\scriptstyle \kappa}$. Similarly, we also define F_i^1.
In this section we will show

$$\begin{aligned} E_i(R_{\leq b}) \subseteq R_{\leq b+1} + \mathfrak{U}(B_{0|\infty})^{\Lambda}_{<b}, \quad F_i(R_{\leq b}) \subseteq R_{\leq b+1} + \mathfrak{U}(B_{0|\infty})^{\Lambda}_{<b}, \\ E_i^1(R_{\leq b}) \subseteq \mathfrak{U}(B_{0|\infty})^{\Lambda}_{<b+1}, \quad F_i^1(R_{\leq b}) \subseteq \mathfrak{U}(B_{0|\infty})^{\Lambda}_{<b+1}. \end{aligned}$$

Lemma 15.12. *Let $\mathfrak{t}$ and $\mathfrak{s} \in \mathcal{T}^{\mathrm{ud}}([\lambda,\mu])$. If $[\lambda,\mu] \xrightarrow{i} [\lambda',\mu']$, then $E_i(\Psi^{\mathfrak{s}}_{\mathfrak{t}}) = \Psi^{\mathfrak{s}'}_{\mathfrak{t}'}$, where $\mathfrak{t}' := \mathfrak{t} \xrightarrow{i} [\lambda',\mu']$ and $\mathfrak{s}' := \mathfrak{s} \xrightarrow{i} [\lambda',\mu']$.*
Similarly, if $[\lambda,\mu] \overset{i}{\dashrightarrow} [\lambda',\mu']$, then $F_i(\Psi^{\mathfrak{s}}_{\mathfrak{t}}) = \Psi^{\mathfrak{s}'}_{\mathfrak{t}'}$ for the corresponding $\mathfrak{t}'$ and $\mathfrak{s}'$.

Proof. This follows directly from the construction and Remark 15.11, which allows removing all double-crossings in the middle. □

Proposition 15.13. *Let* $\mathfrak{t}$ *and* $\mathfrak{s}$ *be up-down-bitableaux of shape* $[\lambda, \mu]$.

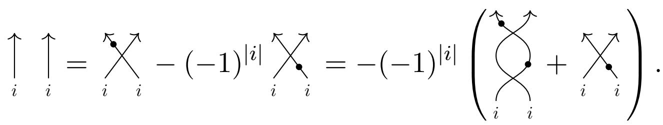
(i) *If* $\mathrm{Add}_{E_i}([\lambda,\mu]) = \emptyset$*, then* $E_i(\Psi_{\mathfrak{t}}^{\mathfrak{s}}) \in \mathfrak{U}(B_{0|\infty})^\Lambda_{\leq |\lambda|+|\mu|}$.

(ii) $E_i^1(\Psi_{\mathfrak{t}}^{\mathfrak{s}}) \in \mathfrak{U}(B_{0|\infty})^\Lambda_{\leq |\lambda|+|\mu|}$.

The same statements hold also for F_i *and* F_i^1.

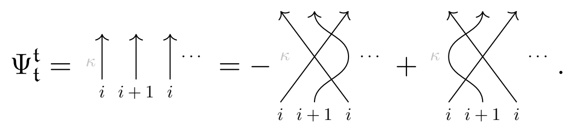
Proof. The proof for F_i is the same as for the E_i case, so we will only show the E_i case. By definition of $\mathfrak{U}(B_{0|\infty})^\Lambda_{\leq|\lambda|+|\mu|}$, it suffices to show the statement for $\mathfrak{t} = \mathfrak{s} = \mathfrak{t}^{\lambda,\mu}$. This means that $\Psi_{\mathfrak{t}}^{\mathfrak{s}} = 1_{\mathrm{res}\,\mathfrak{t}^{\lambda,\mu}}$.
We will prove the statement via downwards induction on $[\lambda,\mu]$. There is a unique maximal element $[\lambda,\mu] = [\emptyset,\emptyset]$. In this case, $\mathfrak{t} = \mathfrak{s} = \mathfrak{t}^{\emptyset,\emptyset}$ and both claims follow from (15.8).
Now assume that the claims hold for all $[\lambda',\mu'] > [\lambda,\mu]$. We make a case distinction depending on the first entry X in $\mathrm{res}\,\mathfrak{t}^{\lambda,\mu}$.

(a) If $X = E_i$, we have by (15.1) and (15.2)



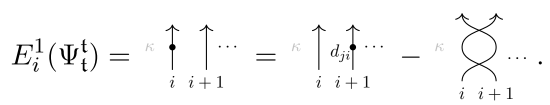
Then, (i) and (ii) follow from the induction hypothesis.

(b) Suppose that $X = E_{i+1}$ and $i \neq 0$. For (i), observe that the second entry in $\mathrm{res}\,\mathfrak{t}^{\lambda,\mu}$ must be E_i by construction of $\mathfrak{t}^{\lambda,\mu}$ and the assumption that $\mathrm{Add}_{E_i}([\lambda,\mu]) = \emptyset$. In particular, we get by (15.3)



In both diagrams on the right-hand side, we can apply the induction hypothesis. In the first one, we can not add $i+1$ before i and in the second one we cannot add twice i (all in the first component). This shows (i).

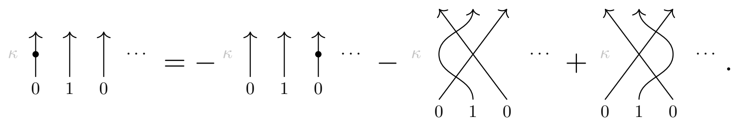
For (ii) we may assume that $\mathrm{Add}_{E_i}([\lambda,\mu]) \neq \emptyset$ as otherwise we can apply the above argument. Then, we have by (15.1)



The first diagram on the right-hand side is in $\mathfrak{U}(B_{0|\infty})^\Lambda_{\leq|\lambda|+|\mu|}$ by the induction hypothesis. For the second diagram note that by assumption there cannot be an addable box of residue i before we add the box with residue $i+1$, hence we can apply the induction hypothesis.

(c) Next consider the case $X = E_{i+1}$ with $i = 0$. Then, $\mathrm{Add}_{E_0}([\lambda, \mu]) \neq \emptyset$, so we only have to show (ii). For this, note that the second step in $\operatorname{res} \mathfrak{t}^{\lambda,\mu}$ must be E_0 by construction. Hence, by (15.3)

$$E_i^1(\Psi^{\mathfrak{t}^{\lambda,\mu}}_{\mathfrak{t}^{\lambda,\mu}}) =$$

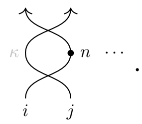


The first diagram is in $\mathfrak{U}(B_{0|\infty})^{\Lambda}_{\leq|\lambda|+|\mu|}$ by induction for (ii). The other two follow from the induction hypothesis for (i) (the middle strand in the (vertical) middle of each diagram comprises a not addable box).

(d) If $X = E_{i-1}$, this is similar to the case $X = E_{i+1}$ for $i \neq 0$, but we must use (15.1) for distant entries to move the first i-entry to the second spot.

(e) If $X = E_j$ with $|i - j| > 1$, (15.1) and (15.2) imply for $n \in \{0, 1\}$

$$E_i^1(\Psi^{\mathfrak{t}^{\lambda,\mu}}_{\mathfrak{t}^{\lambda,\mu}}) =$$



In case $n = 1$, the statement follows from the induction hypothesis for (ii). For $n = 0$, let $[\lambda', \mu']$ denote the second to last bipartition in the walk for $\mathfrak{t}^{\lambda,\mu}$. As $|i - j| > 1$, we have $\mathrm{Add}_{E_i}([\lambda', \mu']) = \emptyset$. Hence, we can apply the induction hypothesis for (i).

(f) If $X = F_j$ with $i \neq j$, using (15.5) and [BE17b, Lemma 3.1], the same argument as for $X = E_j$ with $|i - j| > 1$ shows the claims.

(g) If $X = F_i$, we can use almost the exact same argument as for $X = F_j$ using (15.6) or (15.7) and [BE17b, Lemma 3.1]. The only difference is that the double-crossing might not be equal to the identity, but rather yields some error terms given by some cups and caps. However, the cap and cap terms are again in $\mathfrak{U}(B_{0|\infty})^{\Lambda}_{<|\lambda|+|\mu|}$. $\square$

15.1.2. Closure under vertical composition

We say that a diagram (or morphism) $a \in \mathfrak{U}(B_{0|\infty})^{\Lambda}$ is (2)-*extensive*, if the diagram contains at most one more cup than cap.

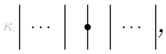
Given a morphism $a \in \mathfrak{U}(B_{0|\infty})^{\Lambda}$ and an up-down-bitableau $\mathfrak{t}$, we want to show the following cellular-like property

$$\text{(CP)} \qquad a \cdot \Psi^{\mathfrak{t}}_{\mathfrak{t}^{\lambda,\mu}} \in C^{[\lambda,\mu]} \mod \mathfrak{U}(B_{0|\infty})^{\Lambda}_{<|\lambda|+|\mu|}.$$

An up-down-bitableau $\mathfrak{t}$ is called (2)-*good* if (CP) holds for all (2)-extensive morphisms a. If we can show that all up-down-bitableaux are (2)-good, the spanning set property follows immediately from this and the Chevalley involution.

The base case

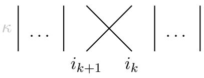
Let $[\lambda, \mu] \in \Lambda\!\!\!\Lambda$ and let $l := |\lambda| + |\mu|$. In this section we are going to prove that any $\mathfrak{t} \in \mathcal{T}_l^{\mathrm{ud}}([\lambda, \mu])$ is (2)-good. So throughout this section, fix $\mathfrak{t} \in \mathcal{T}_l^{\mathrm{ud}}([\lambda, \mu])$. In all of this section, the orientation and label of any strand has to match with $\Psi^{\mathfrak{t}}_{\mathfrak{t}^{\lambda,\mu}}$, so we will not write them down explicitly.

Lemma 15.14. *For $d =$* *, we have $d \cdot \Psi^{\mathfrak{t}}_{\mathfrak{t}^{\lambda,\mu}} \in \mathfrak{U}(B_{0|\infty})^\Lambda_{<b}$.*

Proof. This immediately follows from Proposition 15.13. $\square$

The next lemma is obvious from the definition of $\mathfrak{U}(B_{0|\infty})^\Lambda_{<b}$.

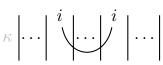
Lemma 15.15. *Let $y =$* *. Then, $y \cdot \Psi^{\mathfrak{t}}_{\mathfrak{t}^{\lambda,\mu}} \in \mathfrak{U}(B_{0|\infty})^\Lambda_{<b}$.*

Lemma 15.16. *Let $z =$*  *with any orientation. Then,*

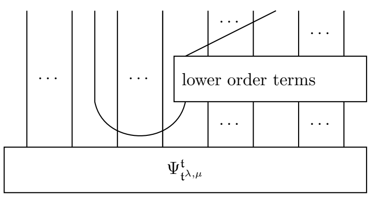
$$\begin{cases} z \cdot \Psi^{\mathfrak{t}}_{\mathfrak{t}^{\lambda,\mu}} = \Psi^{\mathfrak{s}}_{\mathfrak{t}^{\lambda,\mu}} & \text{if } \mathfrak{s} = \mathfrak{t} \cdot s_k \text{ exists,} \\ z \cdot \Psi^{\mathfrak{t}}_{\mathfrak{t}^{\lambda,\mu}} \in \mathfrak{U}(B_{0|\infty})^\Lambda_{<b} & \text{otherwise.} \end{cases}$$

Here, $\mathfrak{t} \cdot s_k$ is the same walk as $\mathfrak{t}$ except we swap the k-th and $(k+1)$-th arrow. Note that this might not exist.

Proof. If $\mathfrak{t} \cdot s_k$ exists, the statement follows from the definition of $\Psi^{\mathfrak{t}\cdot s_k}_{\mathfrak{t}^{\lambda,\mu}}$ and Remarks 15.10 and 15.11. Otherwise, the last box added in $\mathfrak{t}$ cannot be added before. This means we can apply Proposition 15.13 to the sequence $(\operatorname{res}\mathfrak{t}_1, \cdots, \operatorname{res}\mathfrak{t}_{k-2}, \operatorname{res}\mathfrak{t}_k, \operatorname{res}\mathfrak{t}_{k-1})$ to obtain the desired result. $\square$

Lemma 15.17. *Let $x_i =$*  *with either orientation present on the cup. Then, $x_i \cdot \Psi^{\mathfrak{t}}_{\mathfrak{t}^{\lambda,\mu}}$ satisfies* (CP).

Proof. Without loss of generality, we may assume that the cup is oriented rightwards. If inserting an arrow $\overset{i}{\leftarrow\!\!-\,-}$ and $\overset{i}{\rightarrow}$ in the middle of $\mathfrak{t}$ (determined by the endpoints of the cup) gives rise to a valid up-down-bitableau $\mathfrak{v}$, then $x_i \cdot \Psi^{\mathfrak{t}}_{\mathfrak{t}^{\lambda,\mu}} = \Psi^{\mathfrak{v}}_{\mathfrak{t}^{\lambda,\mu}}$.
Otherwise, we find a sequence $(i_1, \cdots, i_k)$ (with $k < l+2$), where we can apply Proposition 15.13. Meaning we get a sum of elements of the following form:



All the lower terms contain at least one cap. If there exists a cap that does not connect to the present cup, this summand is in $\mathfrak{U}(B_{0|\infty})^\Lambda_{<l}$ by Lemma 15.15. If there is a unique

cap that also connects to this cup, we can apply (15.4) to straighten the cup and the cap. This leaves some crossings to which we can apply Lemma 15.16 and one cup where the left endpoint appears later.
For this we can repeat the argument above to eventually arrive at a cup being present at the rightmost position. If this does not give rise to a valid up-down-bitableau, this diagram is already 0 by (15.8). $\square$

Lemma 15.18. *Any* $\mathfrak{t} \in \mathcal{T}_l^{\mathrm{ud}}([\lambda,\mu])$ *for* $l = |\lambda| + |\mu|$ *is* (2)*-good.*

Proof. This is a direct consequence of Lemmas 15.14–15.17. $\square$

The induction step

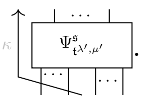
Now we are going to prove some kind of induction step.
The following statement is the main reduction step.

Lemma 15.19. *Let* $\mathfrak{t} \in \mathcal{T}^{\mathrm{ud}}([\lambda,\mu])$ *such that the walk ends with* $[\lambda',\mu'] \xrightarrow{i} [\lambda,\mu]$ *or* $[\lambda',\mu'] \overset{i}{\dashleftarrow} [\lambda,\mu]$*. Define* $\mathfrak{u}$ *to be the* $\mathfrak{t}^{\lambda',\mu'}$ *followed by the last edge of* $\mathfrak{t}$ *and* $\mathfrak{s} := \mathfrak{t}|_{l(\mathfrak{t})-1}$*. Then,*

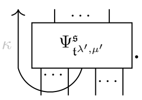
$$\Psi^{\mathfrak{t}}_{\mathfrak{t}^{\lambda,\mu}} = E_i(\Psi^{\mathfrak{s}}_{\mathfrak{t}^{\lambda',\mu'}}) \cdot \Psi^{\mathfrak{u}}_{\mathfrak{t}^{\lambda,\mu}}.$$

If $\mathfrak{t}$ *ends with* $[\lambda',\mu'] \overset{i}{\dashrightarrow} [\lambda,\mu]$ *or* $[\lambda',\mu'] \xleftarrow{i} [\lambda,\mu]$*, the statement holds with* F_i *instead of* E_i*.*

Proof. This is just a fancy formulation of the following easy observation. If $\mathfrak{t}$ ends with $[\lambda',\mu'] \xrightarrow{i} [\lambda,\mu]$ (i.e. a box is added in the last step), then

$\Psi^{\mathfrak{t}}_{\mathfrak{t}^{\lambda,\mu}} =$  .

The bottom part of the diagram is exactly $\Psi^{\mathfrak{u}}_{\mathfrak{t}^{\lambda,\mu}}$.
On the other hand, if $\mathfrak{t}$ ends with $[\lambda',\mu'] \overset{i}{\dashleftarrow} [\lambda,\mu]$, we have

$\Psi^{\mathfrak{t}}_{\mathfrak{t}^{\lambda,\mu}} =$  .

And again the bottom part is exactly $\Psi^{\mathfrak{u}}_{\mathfrak{t}^{\lambda,\mu}}$.
The same argument holds for the other two cases, there the last strand is oriented downwards. $\square$

Essentially, this allows us to reduce to up-down-bitableaux of smaller length. To make this precise, we need the following assumption.

$$(\mathrm{Ass}_{<l}) \qquad R_{<l}|_{<l} = \mathfrak{U}(B_{0|\infty})^{\Lambda}_{<l}\Big|_{<l} \quad \text{and any } \mathfrak{v} \in \mathcal{T}^{\mathrm{ud}}_{<l} \text{ is (2)-good.}$$

This is made precise in the following proposition (using the notation from Lemma 15.19).
For simplicity, we will only consider $\mathfrak{t}$ ending with $[\lambda',\mu'] \xrightarrow{i} [\lambda,\mu]$ and $[\lambda',\mu'] \overset{i}{\dashleftarrow} [\lambda,\mu]$.

Proposition 15.20. *Assume* (Ass$_{<l}$) *for* $l = l(\mathfrak{t})$. *Then,* $\mathfrak{t}$ *has* (CP) *for any* (2)*-extensive diagram of the form* $a = E_i(a')$.

Before we prove this, we need first a technical result.

Lemma 15.21. *Let* $[\lambda,\mu] \in \Lambda\!\!\!\Lambda$ *and* $[\lambda,\mu] \overset{i}{\dashrightarrow} [\lambda',\mu']$ *and* $l \coloneqq |\lambda| + |\mu|$. *Denote by* $\mathfrak{u} \in \mathcal{T}^{\mathrm{ud}}([\lambda,\mu])$ *the up-down-bitableau* $\mathfrak{t}^{\lambda',\mu'} \overset{i}{\dashleftarrow} [\lambda,\mu]$. *Let* z *be the morphism starting at* $\operatorname{res}(\mathfrak{u})$*, which is given by connecting any two strings by a cap. Then,*

$$z \cdot \Psi^{\mathfrak{u}}_{\mathfrak{t}^{\lambda,\mu}} = \sum_{\mathfrak{v} \in \mathcal{T}^{\mathrm{ud}}_l([\lambda,\mu])} c_{\mathfrak{v}} \Psi^{\mathfrak{v}}_{\mathfrak{t}^{\lambda,\mu}} \left(\mod \mathfrak{U}(B_{0|\infty})^\Lambda_{<|\lambda|+|\mu|}\right).$$

Proof. We will only give a sketch here. The result is obtained by first removing any double-crossings with (15.1) and (15.5)–(15.7) (maybe we need to use (15.3) before). This might create more cups, caps and dots. Then, apply any possible snake to straighten some cups and caps (one might need to use [BE17b, (2.4), (2.5), (7.4), (7.5))] for this). If any small circle is created one can move this to the rightmost position to get rid of it by (15.8). In this process one might create some more dots.
If there are still some caps left, one can move these to the bottom and apply Lemma 15.15. Otherwise, there are only crossings and dots left, hence we can apply Lemmas 15.14 and 15.16. ☐

Proof of Proposition 15.20. We use the notation from Lemma 15.19.
First assume that $\mathfrak{t}$ ends with $[\lambda',\mu'] \xrightarrow{i} [\lambda,\mu]$. We have by Lemma 15.19

$$a \cdot \Psi^{\mathfrak{t}}_{\mathfrak{t}^{\lambda,\mu}} = E_i(a' \cdot \Psi^{\mathfrak{s}}_{\mathfrak{t}^{\lambda',\mu'}}) \cdot \Psi^{\mathfrak{u}}_{\mathfrak{t}^{\lambda,\mu}} = E_i \left(\sum_{\mathfrak{v}' \in \mathcal{T}^{\mathrm{ud}}([\lambda',\mu'])} c_{\mathfrak{v}'} \Psi^{\mathfrak{v}'}_{\mathfrak{t}^{\lambda',\mu'}} \right) \cdot \Psi^{\mathfrak{u}}_{\mathfrak{t}^{\lambda,\mu}} = \sum_{\mathfrak{v}} c_{\mathfrak{v}} \Psi^{\mathfrak{v}}_{\mathfrak{t}^{\lambda,\mu}},$$

where we used (Ass$_{<l}$) for the second equality (as $l(\mathfrak{s}) < l$) and the reverse of Lemma 15.19 for the last equality. In particular, $c_{\mathfrak{v}} \neq 0$ only if $\mathfrak{v}$ ends with $[\lambda',\mu'] \xrightarrow{i} [\lambda,\mu]$.
Now we consider the case that $\mathfrak{t}$ ends with $[\lambda',\mu'] \overset{i}{\dashleftarrow} [\lambda,\mu]$. We have

$$a \cdot \Psi^{\mathfrak{t}}_{\mathfrak{t}^{\lambda,\mu}} = E_i(a' \cdot \Psi^{\mathfrak{t}}_{\mathfrak{t}^{\lambda',\mu'}}) \cdot \Psi^{\mathfrak{u}}_{\mathfrak{t}^{\lambda,\mu}} = E_i \left(\sum_{\mathfrak{v}' \in \mathcal{T}^{\mathrm{ud}}([\lambda',\mu'])} c_{\mathfrak{v}'} \Psi^{\mathfrak{v}'}_{\mathfrak{t}^{\lambda',\mu'}} + \sum_{\mathfrak{x}',\mathfrak{y}'} c_{\mathfrak{x}'\mathfrak{y}'} \Psi^{\mathfrak{y}'}_{\mathfrak{x}'} \right) \cdot \Psi^{\mathfrak{u}}_{\mathfrak{t}^{\lambda,\mu}}.$$

In the last step, we used (Ass$_{<l}$) to explicitly write the rest in terms of $\Psi^{\mathfrak{y}'}_{\mathfrak{x}'}$. The second sum runs through all $\sigma \in \Lambda\!\!\!\Lambda$ and $\mathfrak{x}, \mathfrak{y} \in \mathcal{T}^{\mathrm{ud}}(\sigma)$. Note that the coefficient $c_{\mathfrak{x}'\mathfrak{y}'}$ can only be nonzero, if $\operatorname{res}(\mathfrak{x}') = \operatorname{res}(\mathfrak{t}^{\lambda',\mu'})$. By applying the reverse of Lemma 15.19 to the first sum and Lemma 15.12 and Proposition 15.13 to the second, we obtain

$$a \cdot \Psi^{\mathfrak{t}}_{\mathfrak{t}^{\lambda,\mu}} \equiv \sum_{\mathfrak{v} \in \mathcal{T}^{\mathrm{ud}}([\lambda,\mu])} c_{\mathfrak{v}} \cdot \Psi^{\mathfrak{v}}_{\mathfrak{t}^{\lambda,\mu}} + \sum_{\mathfrak{x},\mathfrak{y}} c_{\mathfrak{x}\mathfrak{y}} \Psi^{\mathfrak{y}}_{\mathfrak{x}} \cdot \Psi^{\mathfrak{u}}_{\mathfrak{t}^{\lambda,\mu}} \mod \mathfrak{U}(B_{0|\infty})^\Lambda_{<|\lambda|+|\mu|},$$

where the sum runs through $\mathfrak{x}$, $\mathfrak{y} \in \mathcal{T}^{\mathrm{ud}}([\sigma_1, \sigma_2])$ for all $[\sigma_1, \sigma_2] \in \Lambda\!\!\!\Lambda$ with $|\sigma_1| + |\sigma_2| = |\lambda| + |\mu|$. Furthermore, we must have $\mathrm{res}(\mathfrak{x}) = \mathrm{res}(\mathfrak{u})$ and that $\mathfrak{x}$ and $\mathfrak{y}$ both add a box of residue i in the last step.
There is a unique cap in every $\Psi_{\mathfrak{x}}^{\mathfrak{y}}$ (if $c_{\mathfrak{x}\mathfrak{y}} \neq 0$) (otherwise it lies already in $\mathfrak{U}(B_{0|\infty})^{\Lambda}_{<|\lambda|+|\mu|}$). Therefore, we can apply Lemma 15.21 and obtain

$$a \cdot \Psi^{\mathfrak{t}}_{\mathfrak{t}^{\lambda,\mu}} = \sum_{\mathfrak{v} \in \mathcal{T}^{\mathrm{ud}}([\lambda,\mu])} c_{\mathfrak{v}} \Psi^{\mathfrak{v}}_{\mathfrak{t}^{\lambda,\mu}} + \sum_{\mathfrak{z} \in \mathcal{T}^{\mathrm{ud}}_{|\lambda|+|\mu|}([\lambda,\mu])} x_{\mathfrak{z}} \cdot \Psi^{\mathfrak{z}}_{\mathfrak{t}^{\lambda,\mu}} \mod \mathfrak{U}(B_{0|\infty})^{\Lambda}_{<|\lambda|+|\mu|},$$

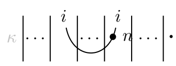
where $x_{\mathfrak{z}}$ is (2)-extensive and of the form $E_i(y_{\mathfrak{z}})$ for some $y_{\mathfrak{z}}$ (as $\mathfrak{x}$ and $\mathfrak{y}$ add boxes in the last step).
Now, observe that any appearing $\mathfrak{z}$ ends with $[\lambda', \mu'] \xrightarrow{i} [\lambda, \mu]$.
Note that $y_{\mathfrak{z}}$ is (k)-extensive for $k = 2 + l(\mathfrak{t}) - |\lambda| - |\mu|$. In particular, we can repeatedly apply $(\mathrm{Ass}_{<l})$ to $\Psi^{\mathfrak{z}}_{\mathfrak{t}^{\lambda,\mu}}$ to obtain a linear combination of $x'_{\mathfrak{z}'} \cdot \Psi^{\mathfrak{z}'}_{\mathfrak{t}^{\lambda,\mu}}$, where $\mathfrak{z}' \in \mathcal{T}^{\mathrm{ud}}_{l(\mathfrak{t})}([\lambda,\mu])$ and $x'_{\mathfrak{z}'}$ is (2)-extensive and of the form $E_i(y'_{\mathfrak{z}})$ for some $y'_{\mathfrak{z}}$.
In particular, we can apply the first part of the proof to $\mathfrak{z}'$ and obtain the desired result. □

The spanning set

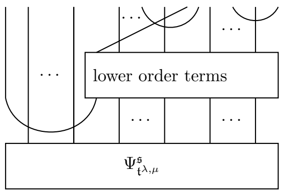
Now we are ready to prove the main result of this section.

Proposition 15.22. *Let* $\mathfrak{t} \in \mathcal{T}^{\mathrm{ud}}_l([\lambda,\mu])$ *and assume* $(\mathrm{Ass}_{<l})$. *Then,* $\mathfrak{t}$ *is* (2)*-good.*

Proof. If $\mathfrak{t} = \mathfrak{t}^{\lambda,\mu}$, the statement follows from Lemma 15.18. Otherwise, it can be obtained from $\mathfrak{x}$ of length $l(\mathfrak{x}) = l - 2$ by composing with a cup. As $\mathfrak{x}$ is (2)-good by $(\mathrm{Ass}_{<l})$, $\mathfrak{t}$ is (0)-good (i.e. closed under composition with diagrams containing as many cups as caps). Therefore, $\mathfrak{t}$ is (2)-good, if we can show (CP) for any diagram of the form

$x =$ 

If there is at least one strand to the left of the cup, we can apply Proposition 15.20 and obtain the claim.
Otherwise, the argument that we will give is very similar to the one in Lemma 15.17. We might have the situation that the resulting diagram is already of the form $\Psi^{\mathfrak{u}}_{\mathfrak{t}^{\lambda,\mu}}$ for some $\mathfrak{u}$ (only if $n = 0$).
If this is not the case, we find a spot, where we can apply Proposition 15.13. If we denote by $\mathfrak{s}$ the up-down-bitableau of length $|\lambda| + |\mu|$, where we removed all the edges corresponding to removing boxes (or adding boxes that will later be removed), the situation looks as follows:



All lower order terms contain at least one cap. If there exists a cap, that does not connect to the present cup, this summand is in $\mathfrak{U}(B_{0|\infty})^\Lambda_{<|\lambda|+|\mu|}$ by Lemma 15.15. Otherwise, there is a unique cap that also connects to this cup, and we can apply (15.4) to straighten the cup and the cap. This leaves some crossings for which we can use Lemma 15.16 and some cups.
Note that there are $k = \frac{l-|\lambda|-|\mu|}{2}+1$ cups in the diagram. Therefore, we can split this part into k terms all of the form $E_i(a')$ for (2)-extensive diagrams a'. As $2\cdot k'+|\lambda|+|\mu|-1 < l$ for all $k' < k$, we can apply Proposition 15.20 to all these terms. Hence, $\mathfrak{t}$ is (2)-good. □

Now we have all the ingredients to show that our proposed basis indeed spans $\mathfrak{U}(B_{0|\infty})^\Lambda$.

Theorem 15.23 (Spanning set)**.** *The set* $\{\Psi^{\mathfrak{t}}_{\mathfrak{s}} \mid \mathfrak{t},\mathfrak{s} \in \mathcal{T}^{\mathrm{ud}}([\lambda,\mu])\}$ *spans* $\mathfrak{U}(B_{0|\infty})^\Lambda$.

Proof. We claim that it suffices to show that every $\mathfrak{t}$ is (2)-good. Applying this inductively, shows that $\Psi^{\mathfrak{t}}_{\mathfrak{s}}$ is closed under left multiplication with all 2-morphisms. Mimicking the arguments for the right multiplication, we obtain that $\Psi^{\mathfrak{t}}_{\mathfrak{s}}$ is closed under right multiplication with all 2-morphisms. Alternatively, one can apply the Chevalley involution from [BE17b, Proposition 3.5]. Then, Lemma 15.12 and Proposition 15.13 imply that the set of all $\Psi^{\mathfrak{t}}_{\mathfrak{s}}$ is closed under horizontal composition. In particular, every identity morphism is in the span of the set.
Therefore, it suffices to show that any $\mathfrak{t}$ is (2)-good. We proceed by induction on the length of $\mathfrak{t}$. If $l(\mathfrak{t}) = 0$, the statement follows from Lemma 15.18.
Now, let $\mathfrak{t}$ be of length $l > 0$. By the induction hypothesis, ($\mathrm{Ass}_{<l}$) holds. By Proposition 15.22, $\mathfrak{t}$ is (2)-good. □

It remains to show that the $\Psi^{\mathfrak{s}}_{\mathfrak{t}}$ are linearly independent. This will be done in Chapter 17 by creating a faithful representation of $\mathfrak{U}(B_{0|\infty})^\Lambda$ on a Khovanov algebra of type Q, which will be defined next.

16. Khovanov algebra of type Q

In this section we are going to define a Khovanov algebra of type Q. This will be backbone to show linear independence for the basis elements from Theorem 15.8.
This algebra will be defined with an explicit basis and multiplication rule using diagrammatics resembling the one from [BS11a]. We will also introduce geometric bimodules and study these.
We begin by introducing the combinatorics.

16.1. Weight, cup and cap diagrams

In this section we are going to define weight diagrams, cup and cap diagrams in our context. The definitions and combinatorics are inspired by the ones in [BS11a] and very similar to the combinatorics of $\mathfrak{osp}(r|2n)$ as in [ES17] Compare these definitions also to the combinatorial weight diagrams that we introduced in Definition 14.3.

Definition 16.1. Let $r \in \mathbb{N} \cup \{\infty\}$. A *weight diagram of length* r is a sequence of symbols $\omega = (\omega_i)_{1 \leq i \leq r}$ with entries $\omega_i \in \{\vee, \wedge, \circ, \times\}$ such that $\omega_i = \vee$ for almost all i (if the sequence is finite, this condition is vacuous).
We usually draw this as a sequence of symbols next to one another and put a $|$ (called *wall*) at the left of this sequence. Write Λ_r for the set of weight diagrams of length $r \in \mathbb{N} \cup \{\infty\}$.

We are mainly interested in the case $r = \infty$. The finite case is only used to simplify some computations for the definition of the multiplication.
Next, we will need a partial order on these weight diagrams.

Definition 16.2. The *Bruhat order* on Λ_r is the partial order generated by $\vee\wedge < \wedge\vee$ and $\wedge < \vee$.

Example 16.3. The following is the Bruhat graph for the block $|\bullet\bullet\circ\bullet$ (the weights are increasing from left to right):

$$\begin{tikzcd}[arrows=dash] & & & \mid\vee\vee\circ\wedge \arrow[dr] & & & \\ \mid\wedge\wedge\circ\wedge \arrow[r] & \mid\vee\wedge\circ\wedge \arrow[r] & \mid\wedge\vee\circ\wedge \arrow[ur] \arrow[dr] & & \mid\vee\wedge\circ\vee \arrow[r] & \mid\wedge\vee\circ\vee \arrow[r] & \mid\vee\vee\circ\vee \\ & & & \mid\wedge\wedge\circ\vee \arrow[ur] & & & \end{tikzcd}$$

Another important thing is the core of a diagram.

Definition 16.4. Given a weight diagram $\omega \in \Lambda_r$, we define $\mathrm{core}(\omega)$ as the weight diagram ω obtained by replacing all $\vee$'s and $\wedge$'s with $\bullet$. In other words, we only remember the position of $\circ$ and $\times$ for the core. Write Cores_r for the set of cores of weight diagrams of length r.
Observe that for any core diagram, there is a unique maximal weight diagram with this given core. It is obtained by replacing all $\bullet$'s with $\vee$'s.

Definition 16.5. To each weight diagram $\omega \in \Lambda_r$, we associate a *cup diagram* as follows:

(i) We connect neighboring pairs of $\vee\wedge$ with a cup. Here, neighboring means that there is no $\vee$ or $\wedge$ in between that is not yet connected to a cup.

(ii) We join every remaining $\wedge$ with a half cup (called *inner cup*) to the wall (such that nothing intersects).

(iii) We attach to each remaining $\vee$ a ray to infinity.

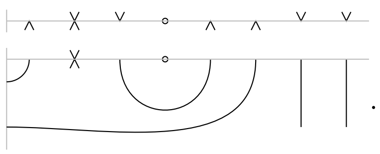
(iv) Finally, we remove all $\vee$'s and $\wedge$'s from the original weight diagram.

We call this cup diagram $\underline{\omega}$.

Example 16.6. Consider the weight diagram $\omega = |\wedge \times \vee \circ \wedge \wedge \vee \vee$. To this we associate the cup diagram

$\omega =$

$\underline{\omega} =$



Definition 16.7. A cap diagram is the horizontal mirror image of a cup diagram. We write a^* for the cap (resp. cup) diagram given by the horizontal mirror image of the cup (resp. cap) diagram a. We also abbreviate $\overline{\omega} := (\underline{\omega})^*$ for any weight diagram ω.
A cup or a cap diagram is called *closed* if it does not contain any lines to infinity (i.e. only cups/caps and inner cups/caps connecting to the wall). Note that a cup or cap diagram can only be closed if $r < \infty$. The cup diagram from Example 16.6 is obviously not closed.

Definition 16.8. Let a be a cup diagram (associated to some weight diagram). We can glue another weight diagram ω onto a, which we call $a\omega$. It is called *oriented* if the positions of $\circ$ and $\times$ agree in ω and a and if the endpoints of any outer cup are $\vee\wedge$ or $\wedge\vee$. Similarly, we define oriented cap diagrams.
Given an oriented cup diagram $a\omega$ and an oriented cap diagram ωb (with the same orientation), we can glue these in the middle to obtain an *oriented circle diagram.*

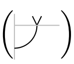
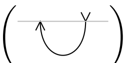
Definition 16.9. Any oriented circle diagram consists of several components. We call those which do not connect to the wall *outer* and the other ones *inner*. There are outer lines and outer circles, and inner components with one endpoint on the wall (which we call *inner lines*) and inner components with both endpoints on the wall (which we call *inner circles*). A circle diagram is called *closed* if its cup and cap diagram are closed. Any circle diagram (i.e. a cup and a cap diagram without a weight diagram in between) admits $2^{\#\text{circles}}$ orientations. If the rightmost vertex of a circle is oriented $\wedge$, the circle is called anticlockwise, and it is called clockwise if the rightmost vertex is oriented $\vee$.

Lemma 16.10. *If $\underline{\eta}\omega$ is an oriented cup diagram, then $\omega \geq \eta$ in the Bruhat order.*

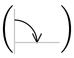
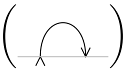
Proof. Consider $\underline{\eta}\eta$. Any inner cup is oriented $\wedge$ and any outer cup $\vee\wedge$. Now any other orientation of $\underline{\eta}$ is obtained by swapping the orientation on any cap. This means that either $\wedge$ is turned into a $\vee$ or $\vee\wedge$ gets replaced by $\wedge\vee$. Both of these operations increase the weight in the Bruhat order. ☐

Next we will associate a degree to every diagram.

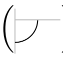
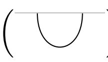
Definition 16.11. Given an oriented cup diagram $a\omega$, we define its *degree*

(16.1) $\deg(a\omega) := \#\Big($  $\Big) + 2\cdot\#\Big($  $\Big).$

In other words, it is the number of clockwise inner cups plus twice the number of clockwise outer cups. Similarly, for an oriented cap diagram ωb, we define its degree as

(16.2) $\deg(\omega b) := \#\Big($  $\Big) + 2\cdot\#\Big($  $\Big).$

Given an oriented circle diagram $a\omega b$, we define its degree as $\deg(a\omega b) := \deg(a\omega) + \deg(\omega b)$.
We also introduce the shorthand

$\mathrm{cups}(a) := \#\Big($  $\Big) + 2\cdot\#\Big($  $\Big)$

for a cup diagram a, and similarly $\mathrm{caps}(b)$ for a cap diagram b. We will also use this notation for circle diagrams and more general diagrams.

Remark 16.12. The oriented cup diagram $\underline{\omega}\omega$ has always degree 0. In particular, for every weight diagram ω there is exactly one oriented cup diagram $\underline{\omega}\eta\overline{\omega}$ of degree 0, namely $\underline{\omega}\omega\overline{\omega}$.

The next result is the analog of [BS11a, Lemma 2.1].

Lemma 16.13. *The degree of an anticlockwise circle C in an oriented anticlockwise circle diagram is $\deg(C) = \mathrm{caps}(C) - 1 - \epsilon$ and $\mathrm{caps}(C) + 1 + \epsilon$ for a clockwise circle, where $\epsilon = 1$ if C is outer and 0 if C is inner. The same statement is true, when replacing $\mathrm{caps}(C)$ with $\mathrm{cups}(C)$.*

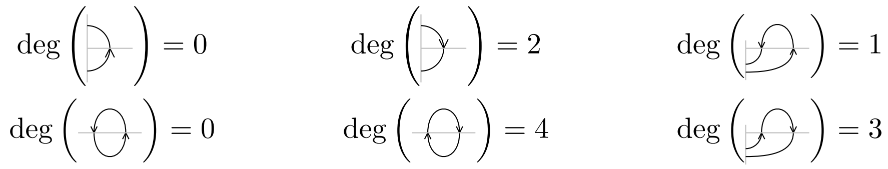


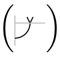
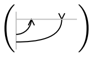
Figure 16.1.: Induction beginning for degree of a circle

Proof. First, observe that caps(C) = cups(C) for any circle C. Hence, we only have to prove one of the statements.
We proceed by induction on caps(C). The cases for caps(C) $= 1$ and caps(C) $= 2$ are shown in Figure 16.1. For instance, for the first picture, we have $\deg(C) = 1 - 1 - 0 = 0$. If we have a circle with caps(C) > 2, then there must be one of the following two subpictures present:

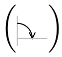
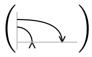
Observe that this contributes degree 2 independent of the orientation of this piece. Replacing this piece by a straight line, we reduce the degree by 2 and also reduce caps(C) by 2 (recall that outer caps count 2). Thus, the result follows from the induction hypothesis. □

We define yet another grading on these diagrams, namely a supergrading.

Definition 16.14. Given an oriented cup diagram $a\omega$, we define its *parity* $|a\omega| \in \mathbb{Z}/2\mathbb{Z}$ as follows:

$|a\omega| := \#\Big($  $\Big) \cdot (\#(\wedge) + \#(\times)) + \#\Big($  $\Big) \mod 2.$

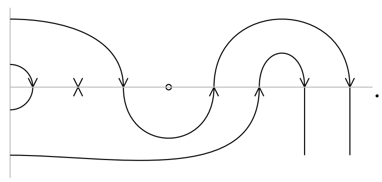
Very similarly, we define the parity of an oriented cap diagram ωb via

$|\omega b| := \#\Big($  $\Big) \cdot (\#(\wedge) + \#(\times) + 1) + \#\Big($  $\Big) \mod 2.$

Observe that there is an extra $+1$ for oriented cap diagrams!
Finally, we define the parity of an oriented circle diagram $a\omega b$ as $|a\omega b| := |a\omega| + |\omega b|$.

Definition 16.15. For every weight diagram ω, we define a special oriented circle diagram $e_\omega := \underline{\omega}\omega\overline{\omega}$. This is always even of degree 0.

Example 16.16. Consider the oriented circle diagram $a\omega b$ given by



Then,

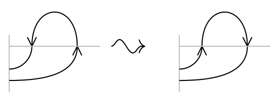
$$
\begin{aligned}
|a\omega| &= 1 \cdot (2+1) + 0 \equiv 1 \mod 2, \\
|\omega b| &= 2 \cdot (2+1+1) + 0 \equiv 0 \mod 2.
\end{aligned}
$$

In particular $a\omega b$ is odd. However, if we change the $\circ$ to a $\times$, we would get an even diagram.

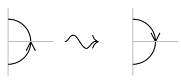
As the previous example demonstrated, the parity is very cumbersome to compute and not very intuitive. However, the following result should give some intuition for it.

Proposition 16.17. *Changing the orientation of an outer circle does not change its parity. However, changing the orientation of an inner circle changes its parity.*

Proof. Changing the orientation of an outer circle only changes the orientation of some outer cups, which do not change the parity, as the number of $\wedge$ stays the same.
For the second part, observe that there are three different ways, how an inner circle might connect to the wall. Namely, either via two inner cups, two inner caps or one inner cup and one inner cap. The first two cases are handled similarly, hence we only show it for the former of these two.
Then our situation is the following:



The number of inner cups oriented downwards is the same in both cases and the same holds for $\#(\wedge) + \#(\times)$. The only difference lies in the pairs of inner cups and for this it obviously suffices to compare those which involve one of the inner cups in the inner circle. Namely, every inner cup inside this circle gives rise to a pair on the right (but not on the left). As this happens inside this circle, there must be an even number of inner cups inside, hence these do not change the parity. Finally, both inner cups on the right also contribute one pair, which is not on the left-hand side. Hence, we get an overall parity change.
Now, assume that we have one inner cup and one inner cap for this circle. Then the situation is as follows:



Note that, apart from these two inner cups/caps, $\#(\wedge) + \#(\times)$ is the same. In particular, the upwards oriented inner cup on the left contributes 1 for every downward inner cup inside the circle. The same holds true for the inner cap on the left. On the right we have in total a contribution given by all upwards oriented inner cups and caps inside the circle (+1 for the downward inner cap). The number of inner cup and inner caps inside the circle must in total be even, hence we get an overall parity change. $\square$

16.2. The algebra $\mathbb{H}_\kappa^Q$

In this section we will define the algebra $\mathbb{H}_\kappa^Q$. It has a distinguished basis given by closed oriented circle diagrams with a fixed core κ. This will be a finite dimensional algebra and should be seen as an analog of the algebra H_Λ from [BS11a].

Definition 16.18. Let $\kappa \in \mathrm{Cores}_r$. We define $\mathbb{H}_\kappa^Q$ to be the graded vector superspace with basis

$$\{a\omega b \mid a\omega b \text{ closed oriented circle diagram with } \mathrm{core}(\omega) = \kappa\}.$$

We also define $\mathbb{H}^Q := \bigoplus_{\kappa\in\mathrm{Cores}_r} \mathbb{H}_\kappa^Q$. Note that this is a finite dimensional vector superspace. Furthermore, observe that $\mathbb{H}_\kappa^Q = \{0\}$ if $r = \infty$.

We will define a multiplication on $\mathbb{H}_\kappa^Q$. For this we make use of surgery procedures, similar to those in [BS11a]´and [ES16].
Given $a\omega b$ and $c\eta d$ in $\mathbb{H}_\kappa^Q$, we compute their product as follows. If $b^* \neq c$, we set $(a\omega b) \cdot (c\eta d) = 0$. Otherwise, we draw $(a\omega b)$ underneath $(c\eta d)$ and connect the rays of b and c.
As $b^* = c$, we have a symmetric middle section consisting of cup/cap pairs (either inner or outer) and we replace these successively by straight line(s) applying a surgery procedure. There are three kinds of surgery procedures, depending on whether the number of components decreases (*Merge*), increases (*Split*) or stays the same (*Reconnect*). After applying all surgery procedures, we obtained a linear combination of diagrams with middle section only consisting of rays. This we then collapse and declare this to be the result of the multiplication.
From the definition, it is neither clear that this is well-defined nor associative. But before we dive into these questions, we will give an explicit description of the surgery procedures.

16.2.1. Merge

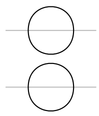
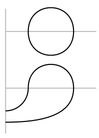
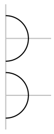
In this situation the number of components decreases. This means, that two components are involved in the surgery. If we denote anticlockwise outer circles by 1, clockwise outer ones by x and use the same symbols with a bar for inner circles, we have listed the possibilities in Figure 16.2a.

16.2.2. Split

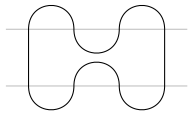
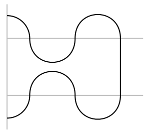
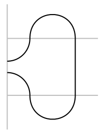
In this situation the number of components increases. This means, that one component is turned into two. Using the same notation as for *Merge*, the various possibilities are listed in Figure 16.2b.

16.2.3. Reconnect

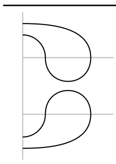
In this situation the number of components stays the same. This can only occur for an outer surgery procedure that involves two inner circles. The possibilities are listed in Figure 16.2c.

Surgery	Orientation	Result
	$1 \otimes 1$	1
	$1 \otimes x$	x
	$x \otimes 1$	x
	$x \otimes x$	0
	$1 \otimes \bar{1}$	$\bar{1}$
	$1 \otimes \bar{x}$	$\bar{x}$
	$x \otimes \bar{1}$	0
	$x \otimes \bar{x}$	0
	$\bar{1} \otimes \bar{1}$	$\bar{1}$
	$\bar{1} \otimes \bar{x}$	$\bar{x}$
	$\bar{x} \otimes \bar{1}$	$\bar{x}$
	$\bar{x} \otimes \bar{x}$	0

(a) Merge

Surgery	Ori.	Result
	1	$1 \otimes x + x \otimes 1$
	x	$x \otimes x$
	$\bar{1}$	$\bar{1} \otimes x$
	$\bar{x}$	$\bar{x} \otimes x$
	$\bar{1}$	0
	$\bar{x}$	x

(b) Split

Surgery	Orientation	Result
	$\bar{1} \otimes \bar{1}$	$\bar{1} \otimes \bar{x} + \bar{x} \otimes \bar{1}$
	$\bar{1} \otimes \bar{x}$	$\bar{x} \otimes \bar{x}$
	$\bar{x} \otimes \bar{1}$	$\bar{x} \otimes \bar{x}$
	$\bar{x} \otimes \bar{x}$	0

(c) Reconnect

Figure 16.2.: Surgery procedures for $\mathbb{H}^Q$

Warning. *To keep it simple here, we work over $\mathbb{F}_2$. Over $\mathbb{F}_2$, the multiplication is exactly given as in Figure 16.2. However, in the general case, signs come into play in a very subtle way, similar to the signs appearing for $\mathfrak{osp}(r|2n)$ case, see* [ES16].

Theorem 16.19. *Over $\mathbb{F}_2$, the multiplication on $\mathbb{H}^Q_\kappa$ is well-defined and associative.*

Before we prove this theorem, we prove the following lemma to give the reader a better intuition for the surgery procedures.

Lemma 16.20. *The multiplication on $\mathbb{H}^Q_\kappa$ is homogeneous of degree* 0.

Proof. We must show that the number of clockwise cups and caps is preserved under the surgery procedures. By means of Lemma 16.13, this is equivalent to showing that

$$\#(\text{caps}) - \#(\text{anti-clockwise circles}) + \#(\text{clockwise circles})$$

stays constant under the surgery procedures. Here, we count with multiplicities, i.e. an outer cap contributes 2 whereas an inner cap contributes 1. The same holds for outer circles in contrast to inner ones.

In case of a Merge, the number of clockwise circles stays the same (with multiplicities!). The number of anticlockwise circles decreases by 1 for an inner Merge and 2 for an outer Merge. The number of caps decreases by 1 for an inner Merge and 2 for an outer Merge. Hence, the sum stays constant.
In case of an outer Split, the number of anticlockwise circles stays the same, whereas the number of clockwise circles increases by 2 (we add an outer clockwise circle). For an inner Split, we change a clockwise inner circle into a clockwise outer circle and remove an inner cap. So, the sum is preserved.
Finally, for a Reconnect, we reduce the number of caps by 2. We also change an inner anticlockwise circle into an inner clockwise one. Thus, we preserve the above sum. □

Proof of Theorem 16.19. In order to show both claims at once it suffices to show that any two surgery procedures commute. Horizontal commutativity implies well-definedness, since any ordering gives the same result. Vertical commutativity will give associativity. We have three different cases, depending on whether 0, 1 or 2 of the surgeries are inner. All base cases are shown in Figure 16.3.

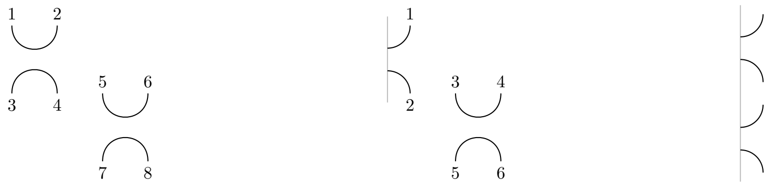
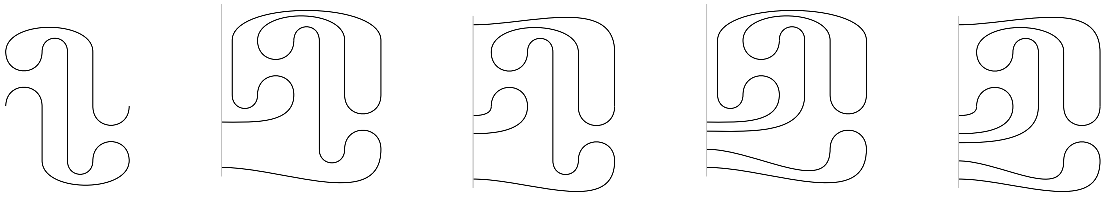


Figure 16.3.: Commutativity of surgery procedures

We will frequently use the term *directly connected* in the following. Two numbers are directly connected if they are connected by a line segment not involving any of the cups or caps depicted.
Let us begin by examining the case of two outer surgeries. If these two do not interact, then they clearly commute. Suppose that 1 and 2 lie on an outer circle, that does not interact with the other cups and caps. Then the surgery involved here is a Merge. The other surgery has the same type (independent of whether we first perform this Merge or not). In particular both commute. The same argument also holds if 3 and 4, 5 and 6, or 7 and 8 lie on a circle that does not interact with the other surgery.
Similarly, if 1 and 3 are directly connected the here involved surgery is a Split. Again the other surgery has the same type and this Split also produces no signs. From the description of the surgeries, we see that they commute in this case. The same argument also holds if 2 and 4, 5 and 7, or 6 and 8 are directly connected.
As we assumed that the surgeries interact, there exists $i \in \{1, 2, 3, 4\}$ that is directly connected to $j \in \{5, 6, 7, 8\}$. Without loss of generality, we assume that $i = 1$. Also by symmetry, we may assume that $j = 5$ or $j = 6$. Consider the case $j = 5$. We make another case distinction on where 8 is connected to. We may exclude the cases where 8 is directly connected to 7 or 6 because these cases have been treated already. We can also exclude the connection to 2 and 3 as these do not produce a valid diagram (every number



(a) (b) (c) (d) (e)

Figure 16.4.: The case that 1 is directly connected to 5

must either connect to another number or to the wall). This leaves 8 being connected to 4 or to the wall. By similar considerations, we finally deduce that there are five different cases to consider, all of them are shown in Figure 16.4.

For (a), note that the first surgery will always be a Merge and the second one always a Split (in both cases). In particular, they commute.

For (b), the first surgery is always a Split and the second one always a Merge (in both cases). In particular, they commute.

For (c), note that the right surgery splits off a clockwise outer component which is then merged with an inner component. In particular, the result is 0. The other order gives two Reconnects. If any of the components is oriented clockwise, we will get 0. However, if both are oriented anticlockwise, we get $\bar{x} \otimes \bar{x} + \bar{x} \otimes \bar{x} = 0$.

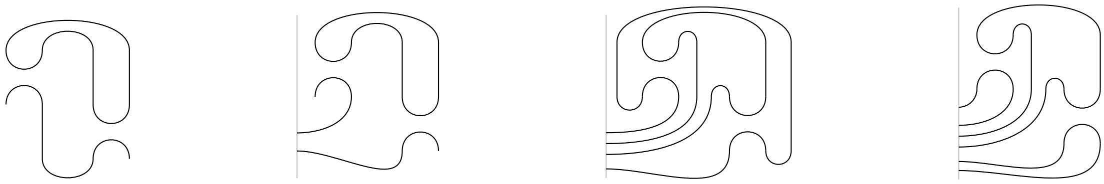
The case (d) is similar to the (c), here the left surgery splits off the clockwise outer component.

Finally, for (e), all involved surgeries are Reconnects. If at least two of the inner circles are oriented clockwise, both outcomes will be 0. If exactly one is oriented clockwise, we will get $\bar{x} \otimes \bar{x} \otimes \bar{x}$ for both sides. And if all are anticlockwise, both orderings will give $\bar{1} \otimes \bar{x} \otimes \bar{x} + \bar{x} \otimes \bar{1} \otimes \bar{x} + \bar{x} \otimes \bar{x} \otimes \bar{1}$. This finishes the case of 1 and 5 being directly connected.

Next, assume that 1 and 6 are directly connected. We make a case distinction on how 7 is connected. As before, we can exclude the cases where 7 is directly connected to 2, 3 as these cannot be part of a valid diagram. We can also exclude the case that 7 is directly connected to 5 or 8, as these cases have been treated already. So it is either connected to 4 or to the wall. With similar considerations for the other vertices, we are left with four different cases, which are shown in Figure 16.5. For (a) and (b), the first surgery is always Merge and the type of the second stays the same. In particular, they commute (use the same argument as before).

Next we consider (c). In both cases, the first surgery is a Reconnect and the second one a Split. In particular, the Split will always split off a clockwise outer component. Hence, they commute.

For the final case (d), all involved surgeries are reconnects. This is similar to (e) for the case 1 and 5 being directly connected. This finishes the argument for 1 and 6 being directly connected and thus also the case of two outer surgeries.



(a) (b) (c) (d)

Figure 16.5.: The case that 1 is directly connected to 6.

Next, consider the case of one inner surgery. Begin by considering the case that 1 is directly connected to the wall. Then, this surgery is an inner Merge and the type of the other is the same throughout. In particular, they commute (this is the same argument when considering 1 and 2 being directly connected).
Hence, we may assume that neither 1 nor 2 are directly connected to the wall. We also may assume that the surgeries are interconnected as the claim is otherwise obvious. With further excluding the cases where 3 is directly connected to 4 or 5 and 6 being directly connected to 4 or 5 (as all these have been treated before), this leaves us with the two cases shown in Figure 16.6. In the first case, one direction is given by first an inner Merge and then an outer Split, resulting in the following assignment:

$$\begin{aligned}
\bar{1} \otimes \bar{1} &\mapsto \bar{1} \otimes x \\
\bar{1} \otimes \bar{x} &\mapsto \bar{x} \otimes x \\
\bar{x} \otimes \bar{1} &\mapsto \bar{x} \otimes x \\
\bar{x} \otimes \bar{x} &\mapsto 0
\end{aligned}$$

The other direction is given by first a Reconnect and then an inner Split, which gives

$$\begin{aligned}
\bar{1} \otimes \bar{1} &\mapsto \bar{1} \otimes x \\
\bar{1} \otimes \bar{x} &\mapsto \bar{x} \otimes x \\
\bar{x} \otimes \bar{1} &\mapsto \bar{x} \otimes x \\
\bar{x} \otimes \bar{x} &\mapsto 0
\end{aligned}$$

For the second one, observe that doing first the inner surgery and then the outer one produces 0. The other order gives a Reconnect and then an inner Merge. In particular, this

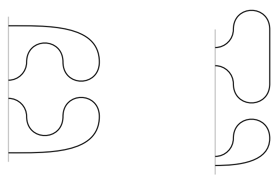


Figure 16.6.: One inner surgery

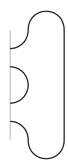


Figure 16.7.: Two inner surgeries

is 0 if any of the involved circles is oriented clockwise. If both are oriented anticlockwise, we get $\bar{x} + \bar{x} = 0$. Hence, the two surgeries commute.
Finally, we consider the case where both surgeries are inner. Again, we may exclude that any of the four endpoints is directly connected to the wall. This leaves exactly one case, shown in Figure 16.7. In this case, the first surgery is always an inner Merge and the second one an inner Split. In particular, they commute. $\square$

Proposition 16.21. *The multiplication on $\mathbb{H}_\kappa^Q$ is parity preserving.*

Proof. We begin by outlining our strategy. By Theorem 16.19, we know that the surgery procedures commute, and thus we can choose any particular order of surgeries. We will first apply inner surgeries and then the outer ones. For the inner ones, we will make sure that after applying any of them, we still have a diagram that is glued from two oriented circle diagram (this allows us to apply Proposition 16.17). For each of these two we know how to compute the parity and this will allow us to track the parity of the result. After all inner surgeries have been applied, the middle section will give no contribution to the parity (as there are no inner cups or caps anymore). Then, we are left to apply the outer surgeries and for these we only need to track orientation changes of inner circles.
As outlined above, we may begin by applying inner surgeries. Figure 16.8 shows a list of all the possible types of surgeries that we can apply.

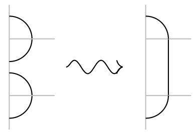
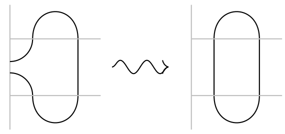
(1)  (2) 

Figure 16.8.: All possible inner surgeries

(1) This is the most difficult case of the four. There are several players that impact the situation here. Namely, we have the orientations of the two circles and the particular orientation of the inner cup and inner cap in the middle. We want to cut the inner cup/cap pair in the middle and keep track of the parity change. Note that if both inner cups/caps are oriented downwards, the parity changes exactly by 1, when replacing this by a straight line. This is because the number of outer cups and caps as well as the number of $\times$ agrees, and there is no $\vee\wedge$ pair of inner cups/caps involved with the ones for the surgery. Thus, the change comes exactly from the removal of the downward oriented inner cup. Now, the idea is to change the orientation of the circles so that both inner cup and cap are oriented downward, cut this pair and reorient to what we actually want to have. Let C_1 (resp. C_2) denote the orientation of a rightmost vertex of the upper (resp. lower) circle. If $C_1 = \vee$ the upper circle is oriented clockwise and anticlockwise if $C_1 = \wedge$. The same also holds true for C_2 and the lower circle. We also use C_3 to encode the orientation of the resulting circle after the reorientation and the cutting (not the proper surgery).

	C_1		C_2	Reorientations before	C_3	actual result	Reorientations after
$\vee$	$\vee$	$\vee$	$\wedge$	0	$\wedge$	$\vee$	1
$\vee$	$\vee$	$\wedge$	$\wedge$	1	$\vee$	$\vee$	0
$\vee$	$\wedge$	$\vee$	$\vee$	0	$\wedge$	$\vee$	1
$\vee$	$\wedge$	$\wedge$	$\wedge$	1	$\wedge$	$\wedge$	0
$\wedge$	$\vee$	$\wedge$	$\wedge$	2	$\wedge$	$\vee$	1
$\wedge$	$\wedge$	$\vee$	$\vee$	1	$\vee$	$\vee$	0
$\wedge$	$\wedge$	$\vee$	$\wedge$	1	$\wedge$	$\wedge$	0
$\wedge$	$\wedge$	$\wedge$	$\vee$	2	$\wedge$	$\vee$	1
$\wedge$	$\wedge$	$\wedge$	$\wedge$	2	$\vee$	$\wedge$	1

Figure 16.9.: The Merge of two inner circles

Note that if both circles are oriented clockwise (i.e. $C_1 = C_2 = \vee$), the result is 0, so we may assume that at least one is oriented anticlockwise. Furthermore, observe that if the first two columns (resp. third and fourth columns) do not agree, this means that the corresponding inner circle wraps around the other inner circle. In particular, if the first two entries are different, then the third and fourth have to agree (and vice versa). Figure 16.9 lists all the remaining possibilities. Now note that the replacement of the inner cup/cap pair by a straight line changes the parity by 1. Also, observe that the total number of reorientations is odd in each case. In total this gives an even parity change.

Finally, we only have to consider, whether cutting the resulting diagrams gives two oriented circle diagrams. This is the case if and only if the "actual result"–column is $\vee$ (as it has to be glued from two inner lines). If it is oriented $\wedge$, there are two possibilities. It is either part of another inner cup/cap surgery or not. If it is, then this surgery has to be an inner Split, which produces 0. Otherwise, we may reorient this inner circle (which changes the parity). This diagram will then be glued from two oriented circle diagrams, and we can continue the argument for the other inner cup/cap surgeries. After all these have been completed, we can reorient this inner circle again to revert the parity change.

(2) Note that the result is 0 if the inner circle is oriented anticlockwise. Hence, we may assume the inner circle to be oriented clockwise and Figure 16.10 shows the precise situation. Observe that the surgery depicted on the right gives the same result.

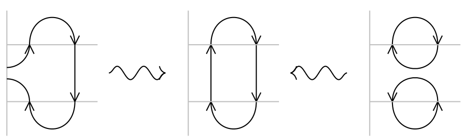


Figure 16.10.: The situation for the second surgery

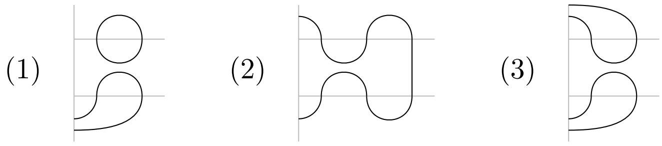


Figure 16.11.: The possibilities for the outer surgery procedures

Both surgeries change the parity by the number of downward inner cups/caps in the middle. In particular, the diagrams on the right and the left have the same parity. And if we replace the diagram by the right-hand side, we will get the same multiplication result, but the diagram has one less inner cup.

Also note, that for both pictures, if we cut them in the middle, we obtain two oriented circle diagrams.

Now we are in a situation, where we have applied all the inner cup/cap surgeries, keeping the same parity. But now the middle section does not contribute anything to the parity anymore. Now, any surgery procedure now that does not involve an inner circle will not change the parity. So we only need to look at outer surgery procedures involving an inner circle. In Figure 16.11, we list all the possibilities.

(1) If the outer circle is clockwise, the result is 0. If the outer circle is anticlockwise, the orientation of the inner circle does not change. All in all, the parity is preserved.

(2) Observe that the orientation of the left end of the cup and cap in the surgery agrees and is $\vee$ if the circle is clockwise and $\wedge$ if it is anticlockwise. In any case, the surgery to be performed splits off a clockwise outer circle and keeps the orientation of the inner circle. In particular, with the above observation no inner cup/cap gets reoriented, hence the parity is preserved.

(3) In Figure 16.12, one fixed orientation for the possible cases is shown.

Note that our diagram is not necessarily glued from two oriented circle diagrams anymore. But as we first applied all inner surgeries, the middle section does not contribute anything to the parity anymore. Furthermore, as our middle section is symmetric and only contains outer cups and caps, the number of $\times$ and $\wedge$ is the same for both numberlines. This means that the arguments in Proposition 16.17 carry over verbatim to this case.

Furthermore, observe that in this case two anticlockwise inner circles produce a sum of two diagrams, namely both orientations, which assure that of the two resulting

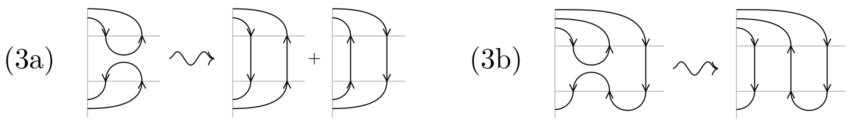


Figure 16.12.: The possibilities for the third surgery procedure

inner circles one is clockwise and the other anticlockwise. If we start with one anticlockwise and one clockwise inner circle, the result gives two clockwise inner circles. And two clockwise circles produce 0.

In particular, via reorienting inner circles before and after the surgery procedure and via symmetry, it suffices to check the statement for the two diagram shown in Figure 16.12. For (3a), note that both summands have the same parity by Proposition 16.17. As we do not need to reorient anything for the first summand, means that the parity is preserved.

For (3b), we also do not reorient anything, hence the result has the same parity.

All together, we found a sequence of surgeries that do not change the parity in total. Hence, by independence of the order, the multiplication preserves the parity. □

The previous results culminate in the following corollary.

Corollary 16.22. *The multiplication procedure turns $\mathbb{H}_\kappa^Q$ into an associative, graded superalgebra.*

16.2.4. Cellular structure of $\mathbb{H}_\kappa^Q$

Next, we will show a cellular property for $\mathbb{H}_\kappa^Q$. The statement and the proof are parallel to [BS11a, Theorem 3.1].

Theorem 16.23. *Let $a\omega b$ and $c\eta d$ be basis vectors of $\mathbb{H}_\kappa^Q$. Then,*

$$(a\omega b)(c\eta d) = \begin{cases} 0 & \text{if } b \neq c^*, \\ s_{a\omega b, c\eta d} \cdot a\eta d + (\dagger) & \text{if } b = c^* \text{ and } a\eta \text{ is oriented,} \\ (\dagger) & \text{otherwise,} \end{cases}$$

where

(i) $(\dagger)$ *is a linear combination of $a\gamma d$ for $\gamma > \eta$ in the Bruhat order;*

(ii) $s_{a\omega b, c\eta d} \in \{-1, 0, 1\}$*;*

(iii) *if $\overline{\omega} = b = c^* = \overline{\eta}$, then $s_{a\omega b, c\eta d} \neq 0$.*

Proof. The statement for $b \neq c^*$ is immediate from the definition, so we may assume from now on that $b = c^*$.
Consider a single iteration of the surgery procedure for $(a\omega b)(c\eta d)$. Let γ denote the top weight at the beginning of the surgery procedure. We claim that

(i) the top weight after every surgery procedure is $\geq \gamma$;

(ii) the total number of diagrams with top weight γ is either 1 or 0;

(iii) if the cup and cap to be cut are anticlockwise, then exactly one diagram with top weight γ is produced.

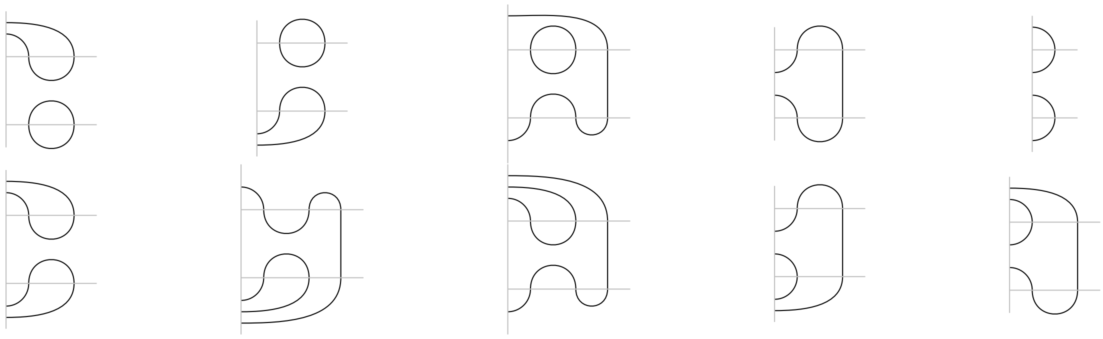


Figure 16.13.: The possibilities for the surgery procedures involving a clockwise cap

If we can show these claims, the statement follows inductively.
If the surgery procedure only involves outer components, we can invoke the proof of [BS11a, Theorem 3.1] to show these claims (note that our claims are significantly weaker than theirs). Hence, it suffices to show the claims for the surgery procedures involving at least one inner component.
Observe that whenever we change the orientation of a circle from anticlockwise to clockwise, we increase all the involved weights. We will make use of this observation many times in the following.
All possibilities for the surgery are listed in Figure 16.13. The proof of the claims is then very similar to the proof of Proposition 16.21 and we do not give the details here. The general strategy is to first reorient components so that the cup and cap to be cut have the same orientation. Then, we can just cut and reorient components again, according to the orientation we actually need to have.
Checking all the cases, one sees that this can always be obtained by reorienting anticlockwise circles. With the above observation, this means that the top weight (and also the bottom one) cannot decrease.
On the other hand, if cup and cap are anticlockwise, then there needs to be no reorientation before the cutting. It is then a quick check to see that this cutting appears as a term in the actual surgery procedure. This implies the second claim. ☐

16.3. The algebra $\mathbb{K}_\kappa^Q$

In the previous section we have defined a multiplication on $\mathbb{H}_\kappa^Q$, i.e. a product of any two closed oriented circle diagrams. In this section we are going to define a larger algebra $\mathbb{K}_\kappa^Q$, extending the multiplication of $\mathbb{H}_\kappa^Q$ to all circle diagrams.

Definition 16.24. Let $\kappa \in \mathrm{Cores}_r$. We define $\mathbb{K}_\kappa^Q$ to be the graded vector superspace with basis

$$\{a\omega b \mid a\omega b \text{ oriented circle diagram with } \operatorname{core}(\omega) = \kappa\},$$

and $\mathbb{K}^Q := \bigoplus_{\kappa \in \mathrm{Cores}_r} \mathbb{K}_\kappa^Q$.

We want to define a multiplication on $\mathbb{K}_\kappa^Q$.
First assume that $\kappa \in \mathrm{Cores}_r$ for $r < \infty$. Let $r' = r+1$ if r is odd and $r' = r+2$ otherwise. Consider ω with $\mathrm{core}(\omega) = \kappa$. Define the *closure* $\mathrm{cl}(\omega)$ *of* ω to be the weight diagram obtained from ω by adding r' new vertices labelled $\wedge$ to the right of the rightmost vertex of ω. Note that $\mathrm{cl}(\omega)$ is a closed weight diagram. All weight diagrams ω with $\mathrm{core}(\omega) = \kappa$ still have the same core after the closure. Denote by ν this block.
This gives rise to an injective map of vector spaces

$$\begin{aligned} \mathrm{cl}\colon \mathbb{K}_\kappa^Q &\to \mathbb{H}_\nu^Q \\ a\omega b &\mapsto \mathrm{cl}(a)\,\mathrm{cl}(\omega)\,\mathrm{cl}(b). \end{aligned}$$

Note that this map is homogeneous of degree 0 and parity preserving (as r' is even). Denote by $\mathrm{cl}(\kappa)$ the set of all $\mathrm{cl}(\gamma)$ for γ with $\mathrm{core}(\gamma) = \kappa$. Furthermore, observe that if $\eta < \mathrm{cl}(\omega)$ in the Bruhat order, then $\eta = \mathrm{cl}(\gamma)$ for some $\gamma \in \Lambda_r$, i.e. $\mathrm{cl}(\kappa)$ is a lower set. By Theorem 16.23, the vector superspace $I_\kappa \coloneqq \{a\eta b \in \mathbb{H}_\nu^Q \mid \eta \notin \mathrm{cl}(\kappa)\}$ is a two-sided ideal in $\mathbb{H}_\nu^Q$. In particular, we can define the quotient superalgebra $\mathbb{H}_\kappa^Q/I_\kappa$.

Lemma 16.25. *The map* cl *induces an isomorphism of graded vector superspaces*

$$\mathrm{cl}\colon \mathbb{K}_\kappa^Q \to \mathbb{H}_\nu^Q/I_\kappa.$$

Proof. Observe that this map is injective by construction. We have to show that cl is surjective. Note that $\mathbb{H}_\nu^Q/I_\kappa$ has a basis given by $a\,\mathrm{cl}(\omega)b + I\kappa$ for all closed oriented circle diagram $a\,\mathrm{cl}(\omega)b$ with $\mathrm{core}(\omega) = \kappa$. By definition of $\mathrm{cl}(\omega)$, the rightmost r' vertices are labelled $\wedge$. In particular, all these vertices must be right endpoints of inner or outer cups. Therefore, there is an obvious way to "open" the diagram. This is done by removing the rightmost r' vertices and the inner cups/caps that are attached to them. We replace the outer cups/caps that are attached to these vertices by rays. This gives a (non-closed) circle diagram of the form $a'\omega b'$ with $\mathrm{core}(\omega) = \kappa$. Hence, cl is surjective. □

Using this isomorphism we define the multiplication on $\mathbb{K}_\kappa^Q$.

Remark 16.26. If we replace r' by $r'+2$, we can do the same construction and obtain a potentially different algebra structure on $\mathbb{K}_\kappa^Q$. Observe that in this case, every diagram for r' is just enclosed in two anticlockwise inner circles to obtain one for $r'+2$. Any surgery involving these, merges two of these anticlockwise inner circles. Hence, the multiplication on $\mathbb{K}_\kappa^Q$ is the same.

Now consider $\kappa \in \mathrm{Cores}_\infty$. In this case, we define the multiplication on $\mathbb{K}_\kappa^Q$ by taking a certain colimit of $\mathbb{K}_\nu^Q$ for $\nu \in \mathrm{Cores}_r$ with $r < \infty$. More precisely, consider the following diagram

$$\mathbb{K}_{\kappa_k}^Q \to \mathbb{K}_{\kappa_{k+1}}^Q \to \mathbb{K}_{\kappa_{k+2}}^Q \to \cdots,$$

where κ_l is the restriction of κ to the first $l+1$ vertices and k some integer such that all $\circ$ and $\times$ appear among the first $k+1$ vertices. The maps are given by adding a ray to the right of any circle diagram (or $\vee$ to any weight).

It is easy to see that the colimit (in $\mathcal{GSV}$ec) of this diagram is $\mathbb{K}_\kappa^Q$. If we can show that all the maps are (non-unital) algebra homomorphisms, we get the desired multiplication on $\mathbb{K}_\kappa^Q$.
Note that, first adding the ray and then closing the diagram is the same as first closing the diagram and then adding an anticlockwise circle in the middle of the diagram (between the original and the new vertices). In particular, the surgery procedures are exactly the same, except that we have to do another Merge of anticlockwise circles for the diagram with the added ray. Hence, first adding a ray and then multiplying is the same as first multiplying and then adding the ray.
Therefore, we get a well-defined algebra structure on $\mathbb{K}_\kappa^Q$.
In Appendix A, we list all the surgery procedures that can occur in the multiplication (ignoring any signs). In contrast to $\mathbb{H}_\kappa^Q$, now also non-closed components can appear.

16.4. Properties of $\mathbb{K}_\kappa^Q$

Recall the element e_ω from Definition 16.15.

Lemma 16.27. *We have*

$$e_\omega \cdot a\eta b = \begin{cases} a\eta b & \text{if } \underline{\omega} = a, \\ 0 & \text{otherwise}, \end{cases} \qquad a\eta b \cdot e_\omega = \begin{cases} a\eta b & \text{if } \overline{\omega} = b, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. We only prove the first statement, the second one is similar. If $\underline{\omega} \neq a$, the result is 0 by definition. So we may assume that $\underline{\omega} = a$. In this case, every surgery involved in the multiplication is a Merge with an anticlockwise (inner) circle (which is part of e_ω). In particular, no surgery changes the orientation of any component of $a\eta b$ and the statement follows. □

In the language of [BS24], we have the following corollary.

Corollary 16.28. *The graded superalgebra $\mathbb{K}_\kappa^Q$ is a locally unital locally finite dimensional superalgebra with idempotents e_ω for $\omega \in \mathbb{\Lambda}$ with* $\operatorname{core}(\omega) = \kappa$.

Proof. By Lemma 16.27, we have

$$\mathbb{K}_\kappa^Q = \bigoplus_{\omega,\eta} e_\omega \mathbb{K}_\kappa^Q e_\eta,$$

where the sum runs through all ω, $\eta \in \mathbb{\Lambda}$ of weight κ. Each of the direct summands on the right-hand side has a basis given by all oriented circle diagrams of the form $\underline{\omega}\gamma\overline{\eta}$. There exist only finitely many such γ, hence each summand is finite dimensional. □

Definition 16.29. Given a graded, locally unital, locally finite dimensional superalgebra $A = \bigoplus_{i,j \in I} e_i A e_j$, we write $A\text{-mod}_{lfd}$ for the category of all locally finite dimensional graded left A-modules M such that $M_i = \bigoplus_{j \in I} e_j M_i$ with $\dim e_j M < \infty$. Given a right A-module N, we write N^* for the left A-module $\bigoplus_{i \in I}(Ne_i)^*$.
We denote by A-proj the full subcategory of A-mod of projective modules.

Theorem 16.30. *The graded superalgebra $\mathbb{K}^Q_\kappa = \bigoplus_{\omega,\gamma} e_\omega \mathbb{K}^Q_\kappa e_\gamma$ is an upper-finite based quasi-hereditary superalgebra in the sense of* [BS24] *with the following data*

(i) *all idempotents are special,*

(ii) *the partial order is given by Definition 16.2,*

(iii) $Y(\omega,\gamma) := \{\underline{\omega}\gamma\overline{\gamma} \mid$ *if oriented*$\}$ *and* $X(\gamma,\omega) := \{\underline{\gamma}\gamma\overline{\omega} \mid$ *if oriented*$\}$.

Proof. Suppose that $Y(\omega,\gamma) \neq \emptyset$, i.e. that $\underline{\omega}\gamma$ is oriented. By Lemma 16.10, we get $\gamma \geq \omega$. Hence, $Y(\omega,\gamma) = \emptyset$ unless $\omega \leq \gamma$, the same holds true for $X(\gamma,\omega)$. We clearly also have $Y(\omega,\omega) = \{e_\omega\} = X(\omega,\omega)$.
Thus, it remains to show that the elements $(a\gamma\overline{\gamma}) \cdot (\underline{\gamma}\gamma b)$ form a basis of $\mathbb{K}^Q_\kappa$. By Theorem 16.23, we get $(a\gamma\overline{\gamma}) \cdot (\underline{\gamma}\gamma b) = \pm(a\gamma b)$ up to lower order terms. In particular, these form a basis of $\mathbb{K}^Q_\kappa$. $\square$

16.5. Geometric bimodules

In this section we are going to introduce geometric bimodules for $\mathbb{K}^Q_\kappa$. Many things are similar to [BS10], though there is one major difference. In our case, we have two different kinds of circles, namely inner and outer circles. It does not make affect any arguments, it just involved more bookkeeping as these behave differently.

Definition 16.31. A *crossingless matching* t is a diagram obtained from drawing a cap diagram underneath a cup diagram, connecting the rays in the unique non-crossing way. We furthermore, require that almost all rays are straight lines, i.e. connect positions i and i.
If the cap diagram has weight κ and the cup diagram has weight ν, we call this a $\kappa\nu$-matching. An orientation of a $\kappa\nu$-matching t are two weight diagrams ω, η such that $\omega t \eta$ is consistently oriented.
Given a sequence of blocks $\boldsymbol{\kappa} = \kappa_k \cdots \kappa_0$, a $\boldsymbol{\kappa}$*-matching* is a sequence of $\kappa_i\kappa_{i-1}$-matchings t_i for $1 \leq i \leq k$. An orientation of a $\boldsymbol{\kappa}$-matching $\boldsymbol{t}$ is a sequence $\boldsymbol{\omega} = \omega_k \cdots \omega_0$ of consistent orientations of the t_i, and we write $\boldsymbol{t}[\boldsymbol{\kappa}]$ for this oriented $\boldsymbol{\kappa}$-matching. Given cup and cap diagrams we can glue these onto (oriented) $\boldsymbol{\kappa}$-matchings to obtain (oriented) $\boldsymbol{\kappa}$-circle diagrams.
The degree of an oriented $\boldsymbol{\kappa}$-circle diagram is computed in the same way as of an oriented circle diagram, namely, by adding the number of clockwise (half) cups and caps.
Its *parity* $|\omega t \eta|$ is defined as

$$
\begin{aligned}
|\omega t \eta| := \#\left(\lrcorner\right) \cdot (\#(\wedge_\eta) + \#(\times_\eta)) + \#\left(\smile\right) \\
+ \#\left(\urcorner\right) \cdot (\#(\wedge_\omega) + \#(\times_\omega) + 1) + \#\left(\frown\right) \\
+ \left(\#(\times_\eta) + \#\left(\cup\right) - \#(\times_\omega) - \#\left(\cap\right)\right) \\
\cdot \left(\#\left(\updownarrow\right) + \frac{1}{2}\left(\#(\times_\eta) + \#\left(\cup\right) + \#(\times_\omega) + \#\left(\cap\right) + 1\right)\right) \quad \bmod 2.
\end{aligned}
$$

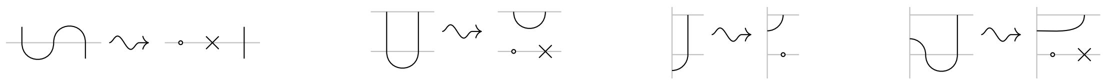


Figure 16.14.: Reduction steps for geometric bimodules

Note that the first two lines look very similar to the definition of parity for oriented cup and cap diagrams.
Given an oriented $\boldsymbol{\kappa}$-circle diagram $a\boldsymbol{t}[\boldsymbol{\omega}]b$, we define the *parity* of $a\boldsymbol{t}[\boldsymbol{\omega}]b$ as

$$|a\boldsymbol{t}[\boldsymbol{\omega}]b| := |a\omega_k| + |\omega_0 b| + \sum_{i=1}^{k} |\omega_i t_i \omega_{i-1}|.$$

For us, the important consequence of this definition is the following analog of Proposition 16.17.

Proposition 16.32. *Switching the orientation of an inner circle changes the parity by* 1*, whereas the orientation of an outer circle does not impact the parity.*

Proof. We will only give a sketch of the proof. The statement is clearly true for outer circles. For inner circles, there are four different reductions shown in Figure 16.14. One easily checks that both orientations of any reduction step give rise to the same parity change. Henceforth, we can reduce to inner circles as for usual circle diagrams and apply the same arguments as in the proof of Proposition 16.17. □

Definition 16.33. For every $\boldsymbol{\kappa}$-matching $\boldsymbol{t}$ we define a graded vector superspace $\mathbb{K}^{Q,\boldsymbol{t}}_{\boldsymbol{\kappa}}$ with basis all oriented $\boldsymbol{\kappa}$-circle diagrams $a\boldsymbol{t}[\boldsymbol{\omega}]b$. Every $a\boldsymbol{t}[\boldsymbol{\omega}]b$ is a homogeneous element, where the degree and parity are defined in Definition 16.31.

Suppose we are given two sequences of blocks $\boldsymbol{\kappa} = \kappa_k \cdots \kappa_0$ and $\boldsymbol{\nu} = \nu_l \cdots \nu_0$ such that $\kappa_0 = \nu_l$. We introduce the notation $\boldsymbol{\kappa} \wr \boldsymbol{\nu} = \kappa_k \cdots \kappa_1 \nu_l \cdots \nu_1$. Given a $\boldsymbol{\kappa}$-matching $\boldsymbol{t}$ and a $\boldsymbol{\nu}$-matching $\boldsymbol{u}$, we can define a multiplication map

$$m\colon \mathbb{K}^{Q,\boldsymbol{t}}_{\boldsymbol{\kappa}} \otimes \mathbb{K}^{Q,\boldsymbol{u}}_{\boldsymbol{\nu}} \to \mathbb{K}^{Q,\boldsymbol{tu}}_{\boldsymbol{\kappa}\wr\boldsymbol{\nu}},$$

which is defined as the multiplication of $\mathbb{K}^Q$. Namely, $m(a\boldsymbol{t}[\boldsymbol{\omega}]b \otimes c\boldsymbol{u}[\boldsymbol{\eta}]d)$ is defined to be 0 if $b \neq c^*$, and otherwise we draw the first underneath the second diagram and iterate the surgery procedures. Via the same arguments, using in particular Proposition 16.32, we obtain a well-defined multiplication (i.e. independent of the order of surgeries), which is homogeneous of degree 0 and parity preserving. Furthermore, it is associative, i.e. the following diagram commutes:

$$\begin{tikzcd}
\mathbb{K}^{Q,\boldsymbol{s}}_{\boldsymbol{\iota}} \otimes \mathbb{K}^{Q,\boldsymbol{t}}_{\boldsymbol{\kappa}} \otimes \mathbb{K}^{Q,\boldsymbol{u}}_{\boldsymbol{\nu}} \arrow[r, "m\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes m"] & \mathbb{K}^{Q,\boldsymbol{st}}_{\boldsymbol{\iota}\wr\boldsymbol{\kappa}} \otimes \mathbb{K}^{Q,\boldsymbol{u}}_{\boldsymbol{\nu}} \arrow[d, "m"] \\
\mathbb{K}^{Q,\boldsymbol{s}}_{\boldsymbol{\iota}} \otimes \mathbb{K}^{Q,\boldsymbol{tu}}_{\boldsymbol{\kappa}\wr\boldsymbol{\nu}} \arrow[r, "m"] & \mathbb{K}^{Q,\boldsymbol{stu}}_{\boldsymbol{\iota}\wr\boldsymbol{\kappa}\wr\boldsymbol{\nu}}
\end{tikzcd} \tag{16.3}$$

In particular, we obtain a $(\mathbb{K}^Q_{\kappa_k}, \mathbb{K}^Q_{\kappa_0})$-bimodule structure on $\mathbb{K}^{Q,\boldsymbol{t}}_{\boldsymbol{\kappa}}$ (use m with the special case of the empty matching). We can define (upper or lower) reduction as in [BS10, §2] and the results on the degree directly carry over to our setting. Furthermore, we can also interpret the $(\mathbb{K}^Q_{\kappa_k}, \mathbb{K}^Q_{\kappa_0})$-bimodule $\mathbb{K}^{Q,\boldsymbol{t}}_{\boldsymbol{\kappa}}$ as a $(\mathbb{K}^Q, \mathbb{K}^Q)$-bimodule by lettings all the other summands act by 0.

Theorem 16.34. *We have $\mathbb{K}^{Q,\boldsymbol{t}}_{\boldsymbol{\kappa}} \otimes_{\mathbb{K}^Q_{\kappa_0}} \mathbb{K}^{Q,\boldsymbol{u}}_{\boldsymbol{\nu}} \cong \mathbb{K}^{Q,\boldsymbol{tu}}_{\boldsymbol{\kappa\wr\nu}}$ as $(\mathbb{K}^Q_{\kappa_k}, \mathbb{K}^Q_{\nu_0})$-bimodules.*

Proof. The proof is exactly the same as in [BS10, Theorem 3.5]. □

Next we obtain a similar reduction result as in [BS10, Theorem 3.6]. For this let $R = \mathbb{C}[X]/(X^2)$ be the graded vector space with $\deg x = 2$ and $\deg 1 = -2$. We write R' for the same vector space but with halved grading, i.e. $\deg x = 1$ and $\deg 1 = -1$. We also put a supergrading on R' via $|X| = 1$ and $|1| = 0$.

Theorem 16.35 (Reduction)**.** *Let $\boldsymbol{t}$ be a generalized crossingless matching and u be its reduction. Denote by n the number of inner circles that got removed in the reduction process and by m the number of outer circles. Then there exists $l \in \{0,1\}$ (only depending on t) such that the following is an even isomorphism of $(\mathbb{K}^Q_{\kappa_k}, \mathbb{K}^Q_{\kappa_0})$-bimodules*

$$\mathbb{K}^{\boldsymbol{t}} \cong \Pi^l \mathbb{K}^u \otimes R^{\otimes m} \otimes R'^{\otimes n} \langle \operatorname{caps}(t_1) + \cdots + \operatorname{caps}(t_k) - \operatorname{caps}(u) \rangle.$$

Proof. We have two different kinds of circles, inner and outer circles. Both of these admit two orientations, namely clockwise and anticlockwise. Lemma 16.13 and Proposition 16.32 show that reorienting an anticlockwise inner circle changes the parity and the degree by 2. Furthermore, reorienting an anticlockwise outer circle does not change the parity and changes the degree by 4. These exactly explain the difference between R and R'. With this in mind, the proof is exactly the same as [BS10, Theorem 3.6]. The statement about the parity follows from Proposition 16.32. □

Theorem 16.35 shows that it suffices to consider crossingless matchings t instead of the generalized crossingless matchings $\boldsymbol{t}$. We can now define the *geometric bimodule* $K^t_{\kappa\nu}$ as follows.

Definition 16.36. Given a $\kappa\nu$-matching t, tensoring with $\mathbb{K}^{Q,t}_{\kappa\nu}$ defines a functor

$$G^{Q,t}_{\kappa\nu} := \mathbb{K}^{Q,t}_{\kappa\nu}\langle -\operatorname{caps}(t)\rangle \otimes_{\mathbb{K}^Q_\nu} \_\colon \mathbb{K}^Q_\nu\text{-mod} \to \mathbb{K}^Q_\kappa\text{-mod}.$$

We will call any functor that is isomorphic to a finite direct sum of (a grading/parity shift of) such functors a *projective functor*. Theorem 16.34 and Theorem 16.35 imply that projective functors are closed under composition.

Using this definition, the proofs and statements of [BS10, Lemma 4.1–Theorem 4.9] carry over verbatim to our setting. At some places the same adjustments as in the proof of Theorem 16.35 are necessary. To gather all these in one place, we state the adapted versions of these results. Throughout all of these assume that t is a $\kappa\nu$-matching.

Lemma 16.37. *If t does not contain any cups or caps, the functor $G^{Q,t}_{\kappa\nu}$ is an equivalence of supercategories.*

Proof. As in [BS10, Lemma 4.1]. □

Theorem 16.38. *Let γ be a weight diagram with* $\operatorname{core}(\gamma) = \nu$. *Then,*

(i) $G^{Q,t}_{\kappa\nu} P(\gamma) \cong K^t_{\kappa\nu} e_\gamma \langle -\operatorname{caps}(t)\rangle$ *as left $\mathbb{K}^Q_\kappa$-modules,*

(ii) *the module $G^{Q,t}_{\kappa\nu} P(\gamma)$ is nonzero if and only if the rays in each upper line in $t\gamma\overline{\gamma}$ are oriented such that one is $\vee$ and one is $\wedge$, and*

(iii) *moreover, in this case we have for some $l \in \{0,1\}$, we have*

$$G^{Q,t}_{\kappa\nu} P(\gamma) \cong \Pi^l P(\omega) \otimes R'^{\otimes n} \otimes R^{\otimes m} \langle \operatorname{cups}(t) - \operatorname{caps}(t)\rangle$$

as graded left $\mathbb{K}^Q_\kappa$-modules ($\mathbb{K}^Q_\kappa$ acts again on the right-hand side only on the first factor), where $\overline{\omega}$ is the upper reduction of $t\overline{\gamma}$ and n (resp. m) denotes the number of inner (resp. outer) circles removed in the reduction process.

Proof. The proof is exactly the same as in [BS10, Theorem 4.2]. □

The next corollaries are proven as [BS10, Corollary 4.3 and 4.4].

Corollary 16.39. *The module $K^t_{\kappa\nu}$ is sweet, i.e. projective as a left $\mathbb{K}^Q_\kappa$-module as well as projective as a right $\mathbb{K}^Q_\nu$-module.*

Corollary 16.40. *Projective functors are exact and preserve the property of being finitely generated.*

Next, we want to show that the projective functors $G^{Q,t}_{\kappa\nu}$ and $G^{t^*}_{\nu\kappa}$ form up to degree shift an adjoint pair. For this we define a linear map

$$\phi\colon K^{t^*}_{\nu\kappa} \otimes K^t_{\kappa\nu} \to \mathbb{K}^Q_\nu \tag{16.4}$$

as follows. Given basis vectors $(a\omega t^*\nu d) \in K^{t^*}_{\nu\kappa}$ and $(d'\kappa t\gamma b) \in K^t_{\kappa\nu}$, we denote by c the upper reduction of t^*d. Then if $d' = d^*$ and all mirror image pairs of upper respectively lower circles in t^*d respectively d^*t are oriented in *opposite* ways in the corresponding basis vectors, we set

$$\phi((a\omega t^*\nu d) \otimes (d'\kappa t\gamma b)) := (a\omega c)(c^*\gamma b), \tag{16.5}$$

and otherwise we set $\phi((a\omega t^*\nu d) \otimes (d'\kappa t\gamma b)) := 0$.

Lemma 16.41. *The map $\phi\colon K^{t^*}_{\nu\kappa} \otimes K^t_{\kappa\nu} \to \mathbb{K}^Q_\nu$ is a homogeneous $(\mathbb{K}^Q_\nu, \mathbb{K}^Q_\nu)$-bimodule homomorphism of degree $-2\operatorname{caps}(t)$ changing the parity by the number of inner caps in t. Moreover, it is $\mathbb{K}^Q_\kappa$-balanced and thus induces a map $\overline{\phi}\colon K^{t^*}_{\nu\kappa} \otimes_{\mathbb{K}^Q_\kappa} K^t_{\kappa\nu} \to \mathbb{K}^Q_\nu$.*

Proof. This is proven as in [BS10, Lemma 4.6], but recall that $\operatorname{caps}(t)$ encodes two different kinds of caps (i.e. all outer caps are counted twice). □

Theorem 16.42. *Let l be the number of inner caps in t. There is an even, graded $(\mathbb{K}^Q_\kappa, \mathbb{K}^Q_\nu)$-bimodule isomorphism*

$$\hat{\phi}\colon \Pi^l K^t_{\kappa\nu}\langle -2\operatorname{caps}(t)\rangle \xrightarrow{\sim} \operatorname{Hom}_{\mathbb{K}^Q_\nu}(K^{t^*}_{\nu\kappa}, \mathbb{K}^Q_\nu)$$

given by sending $y \in K^t_{\kappa\nu}$ to $\hat{\phi}(y)\colon K^{t^*}_{\nu\kappa} \to \mathbb{K}^Q_\nu, x \mapsto \phi(x \otimes y)$.

Proof. This is now the same argument as in [BS10, Theorem 4.7]. □

As in [BS10], we deduce the following two corollaries.

Corollary 16.43. *There is a canonical even isomorphism*

$$\operatorname{Hom}_{\mathbb{K}^Q_\nu}(K^{t^*}_{\nu\kappa}, \_) \cong \Pi^l K^t_{\kappa\nu}\langle -2\operatorname{caps}(t)\rangle \otimes_{\mathbb{K}^Q_\nu} \_$$

of functors from $\mathbb{K}^Q_\nu$-mod to $\mathbb{K}^Q_\kappa$-mod, where l denotes the number of inner caps in t.

Corollary 16.44. *We have an adjoint pair of functors*

$$(G^{t^*}_{\nu\kappa}\langle \operatorname{cups}(t) - \operatorname{caps}(t)\rangle, \Pi^l G^{Q,t}_{\kappa\nu})$$

giving rise to an even, degree 0 adjunction between $\mathbb{K}^Q_\nu$-mod and $\mathbb{K}^Q_\kappa$-mod.

17. Another 2-representation of $\mathfrak{U}(B_{0|\infty})$

In this section we will finish the proof of Theorem 15.8 by showing linear independence. This will be done by constructing another 2-representation of $\mathfrak{U}(B_{0|\infty})$, which is related to $\mathbb{K}^Q$. We begin by observing some combinatorial connections.

17.1. Combinatorial connections between $\mathfrak{U}(B_{0|\infty})$ and $\mathbb{K}^Q$

Lemma 17.1. *There is a bijection between Λ_∞ and $\mathbb{\Lambda}$.*

Proof. To $[\lambda, \mu] \in \mathbb{\Lambda}$, we associate the following two sets

$$A := \{\lambda_i \mid 1 \leq i \leq \ell(\lambda)\} \quad \text{and} \quad B := \{\mu_i \mid 1 \leq i \leq \ell(\mu)\}.$$

To this we associate a weight diagram $\omega = (\omega_i)_{i \in \mathbb{N}}$ via

$$\omega_i = \begin{cases} \times & \text{if } i \in B \setminus A, \\ \circ & \text{if } i \in A \setminus B, \\ \wedge & \text{if } i \in A \cap B, \\ \vee & \text{if } i \notin A \cup B. \end{cases}$$

Observe that we can reconstruct A and B from the weight diagram and then in turn also $[\lambda, \mu]$. $\square$

Additionally, we define its *weight* $\mathrm{wt}(\omega) \in X$ as $\sum_{i \in \mathbb{N}} a_i \varepsilon_i$, where

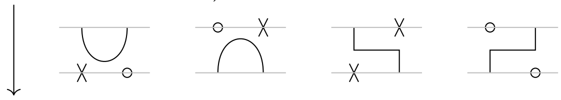
$$a_i = \begin{cases} -1 & \text{if } \omega_i = \circ, \\ 1 & \text{if } \omega_i = \times, \\ 0 & \text{if } \omega_i = \vee \text{ or } \omega_i = \wedge. \end{cases}$$

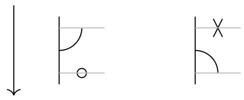
Clearly, this only depends on the core of ω. In turn, we get a bijection between $\kappa = \sum_{i \in \mathbb{N}} a_i \varepsilon_i \in X$ with $|a_i| \leq 1$ and cores of weight diagrams for Λ_∞. We also refer to these κ as blocks and write $X_{\leq 1} \subseteq X$ for the set of these. Note that in the beginning of Definition 15.6 we associated to each $[\lambda, \mu] \in \mathbb{\Lambda}$ a weight $\mathrm{wt}([\lambda, \mu]) \in X$. Now, we associated to each weight diagram another weight $\mathrm{wt}(\omega) \in X$. The next lemma is immediate from the definitions.

Lemma 17.2. *Let $[\lambda, \mu] \in \mathbb{\Lambda}$ and $\omega \in \Lambda_\infty$ be the weight diagram associated to $[\lambda, \mu]$. Then,* $\mathrm{wt}([\lambda, \mu]) = \mathrm{wt}(\omega)$.

We begin by defining some important functors E_i and F_i.
From now on, we will use the convention $\mathbb{K}^Q_\kappa := \{0\}$ for $\kappa \notin X_{\leq 1}$.

Definition 17.3. Fix a block $\kappa \in X_{\leq 1}$ such that also $\kappa + \alpha_i \in X_{\leq 1}$. This means that the core corresponding to κ matches exactly one of the tops of the following diagrams (we depicted positions i and $i+1$):

for $i > 0$: E_i  $t_i^e(\kappa)$

E_0  $t_0^e(\kappa)$

By swapping all $\circ$'s and $\times$'s, we define $t_i^f(\kappa)$ in the same way for a block κ such that $\kappa - \alpha_i \in X_{\leq 1}$.

Definition 17.4. Let $\kappa \in X$ and $\nu := \kappa - \alpha_i$. We define the functor

$$E_i := G_{\nu\kappa}^{t_i^e(\kappa)} \otimes \_ \colon \mathbb{K}^Q_\kappa\text{-mod} \to \mathbb{K}^Q_{\kappa+\alpha_i}\text{-mod},$$

whenever κ and $\nu \in X_{\leq 1}$, otherwise we define E_i to be 0.
Furthermore, we define F_i to be the right adjoint of E_i (given by Corollary 16.44), i.e. we have

$$F_i = \Pi^l K_{\nu\kappa}^{t_i^f(\kappa+\alpha_i)} \langle -\operatorname{cups}(t_i^f(\kappa+\alpha_i))\rangle \otimes \_ \colon \mathbb{K}^Q_{\kappa+\alpha_i}\text{-mod} \to \mathbb{K}^Q_\kappa\text{-mod},$$

where l denotes the number of inner caps in $t_i^f(\kappa)$.

Next, we assemble these functors into a 2-category.

Definition 17.5. We define the 2-supercategory $2\mathbb{K}^Q$ as the 2-subsupercategory of the 2-supercategory of k-linear supercategories with objects $\mathbb{K}^Q_\kappa$-mod for $\kappa \in X$, 1-morphisms generated by E_i and F_i and 2-morphisms given by natural transformations (resp. bimodule maps).

Denote by ω_0 the weight diagram $| \vee \vee \cdots$ and write κ_0 for the corresponding block. The projective $\mathbb{K}^Q_{\kappa_0}$-module $P(\iota_0)$ induces a 2-representation $\Phi\colon 2\mathbb{K}^Q \to \mathfrak{GSCat}$ of $2\mathbb{K}^Q$, denoted by $2\mathbb{K}^Q_{\iota_0}$. Explicitly, on objects, we map the object $\mathbb{K}^Q_\kappa$-mod to the full additive subcategory of $\mathbb{K}^Q_\kappa$-mod given by all objects that are obtained from $P(\iota_0)$ by applying E_i's and F_i's; while on 1- and 2-morphisms, we just act in the obvious way.

Lemma 17.6. *We have* $(2\mathbb{K}^Q_{\iota_0})_\kappa \simeq \mathbb{K}^Q_\kappa\text{-proj}$ *and thus* $\bigoplus_{\kappa\in X}(2\mathbb{K}^Q_{\iota_0})_\kappa \simeq \mathbb{K}^Q\text{-proj}$.

Proof. This follows easily from Theorem 16.38 using the defining diagrams of E_i and F_i. $\square$

Theorem 17.7. *We have a graded 2-superfunctor* $\mathfrak{U}(B_{0|\infty}) \to 2\mathbb{K}^Q$*, given on objects by* $\kappa \mapsto \mathbb{K}^Q_\kappa$-mod *and on 1-morphisms by sending F_i and E_i to the corresponding functors for* $\mathbb{K}^Q_\kappa$.

Warning. *We prove this theorem over $\mathbb{F}_2$, as we did for the multiplication in Theorem 16.19. In addition to the signs appearing in a general multiplication on $\mathbb{K}^Q$, we also have to consider odd maps of superbimodules.*
However, we indicate in the proof, where extra care is needed in the general case.

Proof. We still have to define the 2-superfunctor on 2-morphisms, and then it amounts to checking the defining relations of $\mathfrak{U}(B_{0|\infty})$. For all the 2-morphisms not involving E_0 or F_0, we use the same morphism as [BS11b]. Also recall that we did not introduce any signs in the multiplication, whenever an outer component is involved in an outer surgery. This together with Theorem 16.19 implies that all the relations that do not involve E_0 or F_0 are satisfied.
For the remaining generators, we first map $\underset{0}{\overset{\bullet}{\uparrow}}\ {}_\kappa : E_0 1_\kappa \to E_0 1_\kappa$ to the supernatural transformation, which is given by multiplying the inner cup/cap in the diagram for E_0 with $)$, which is odd and has degree 2. Next, if E_0E_0 is not zero on $\mathbb{K}^Q_\kappa$ we necessarily find a small inner circle in the diagram for E_0E_0. Then, $\underset{0\ 0}{\times}\ {}_\kappa : E_0E_0 1_\kappa \to E_0E_0 1_\kappa$ is given sending an anticlockwise circle to 0 and reorienting a clockwise one to be anticlockwise. This is then also odd and has degree -2.

> Note that further care needs to be taken to make sure that these are superbimodule maps. As these are odd, we must ensure $f(avb) = (-1)^{|a|} a f(v) b$.

For crossings involving 0 and $j \geq 2$, there is an obvious isomorphism given by a height move. This is even and of degree 0.
For the remaining crossings

$$\underset{0\ 1}{\times}\ {}_\kappa : E_0E_1 1_\kappa \to E_1E_0 1_\kappa \quad \text{and} \quad \underset{1\ 0}{\times}\ {}_\kappa : E_1E_0 1_\kappa \to E_0E_1 1_\kappa,$$

we have to apply a surgery procedure to cut the components and glue them back together the other way.
This is done by creating a cup/cap pair (which gives degree 2) and then applying a surgery procedure. This gives an even degree 2 supernatural transformation, see Figure 17.1 for details.
Finally, we map $\overset{i}{\cup}\ {}_\kappa : 1_\kappa \to E_iF_i 1_\kappa$, and $\underset{i}{\cap}\ {}_\kappa : E_iF_i 1_\kappa \to 1_\kappa$, to the corresponding (co)unit of the adjunction Corollary 16.44. In particular, (15.4) is automatically satisfied.
To check the relations, first observe that applying $\underset{0\ 0}{\times}\ {}_\kappa$ or $\underset{i}{\overset{\bullet}{\uparrow}}\ {}_\kappa$ twice, gives 0. Therefore, for (15.1) we only have to show

$$\underset{0\ 1}{\between}\ {}_\kappa = \underset{0}{\uparrow}\ \underset{1}{\overset{\bullet}{\uparrow}}\ {}_\kappa, \tag{17.1}$$

as well as the mirror (i.e. first 0, then 1), which is proven in exactly the same way. There are four possible number lines (at positions 1 and 2) such that E_1 is applicable: $\bullet\bullet$, $\bullet\times$,

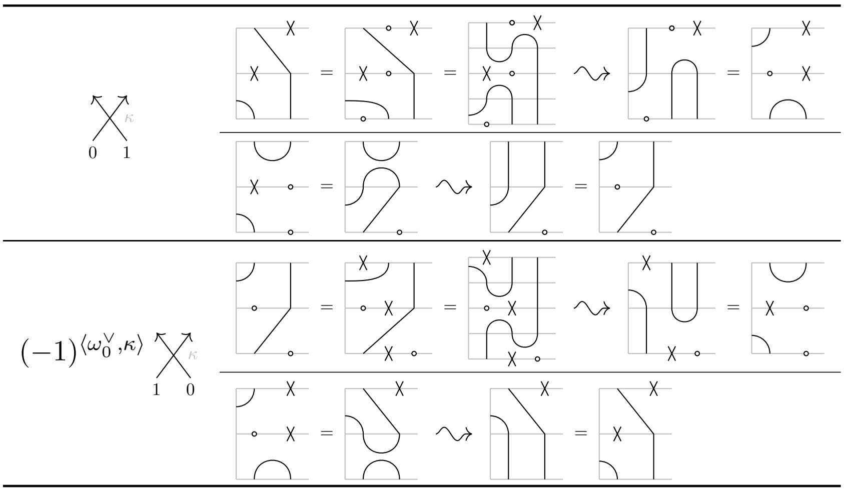


Figure 17.1.: Explicit description for the image of the crossings for 0 and 1

$\circ\bullet$ and $\circ\times$. Note that the last two are exactly those where, E_0 is not applicable and also those where the strands involved in the geometric bimodules for E_0 and E_1 belong to the same components. Hence, for the last two both sides produce 0. The other two cases have exactly the same argument and we will only give the one for $\bullet\times$. First note that both sides are given by surgery procedures plus some signs. Note that these signs agree on both sides, so it suffices to check that the surgeries on both sides give the same result. The left-hand side in terms of geometric bimodules is shown in Figure 17.2. The squiggly arrows denote the application of surgery procedures. Now, as surgery procedures commute by Theorem 16.19, the top composition is the same as the bottom one. For the bottom one, we can first split off the outer circle and then merge this with the inner circle. The Merge of a clockwise outer circle with an inner circle is 0, hence the composition is nonzero, only if we split off an anticlockwise circle. This happens, if and only if this component is part of an anticlockwise circle. The result is then the reorientation of this anticlockwise circle, which agrees with the right-hand side of (17.1) by definition.

For (15.2), note that if the labels do not agree, the crossing and the dot are given by surgery procedures (or height moves). Hence, the relations hold by Theorem 16.19. So

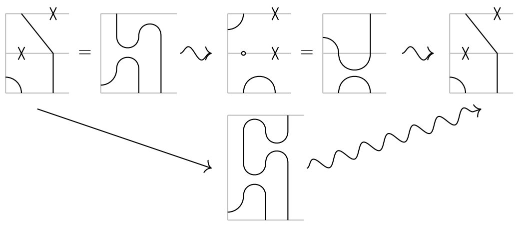


Figure 17.2.: Surgeries for (15.1)

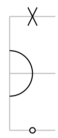
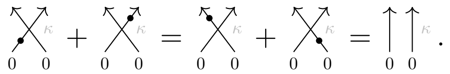


Figure 17.3.: Diagram for (15.2)

we may assume that both labels are 0. Note that if applying E_0 twice is nonzero, then the diagram has to contain a small inner circle (see Figure 17.3). We have to show



Observe that the crossing reorients the small circle to be anticlockwise (and kills the anticlockwise orientation), whereas the dot reorients the small circle to be clockwise (and kills the clockwise circle). In any case, one of the summands kills it and the other gives the identity. Thus, (15.2) is satisfied.

If the dot lives on a strand labelled 0, we would need to check the additional signs.

Next, we consider (15.3). There are three labels involved, i, j and k, at least one of them being 0. If any of the three is distant to the other two, the braid relation is given by some height moves plus one additional crossing. As this crossing commutes with the height moves, the relation is satisfied. Furthermore, if all three labels are pairwise different, the involved crossings are either height moves or surgeries and thus commute. Thus, we can assume that the labels are either 0 or 1. Also, applying E_0 three times is 0, so we may exclude this. We also need to have that $E_iE_jE_k$ as well as $E_kE_jE_i$ are nonzero (as otherwise the relation trivially holds). This leaves exactly two cases to check $i = k = 1 = j + 1$ and $i = k = 0 = j - 1$.
We begin by examining $i = k = 1 = j + 1$. There are exactly two possible top number lines, such that this sequence is nonzero: $\circ\times$ and $\bullet\times$. They are just upside down mirrors of each other and we will only consider the first one. Observe that for this the second summand of the left-hand side of (15.3) is zero. The involved surgeries for the right-hand side are shown in Figure 17.4. In the first step, we split off a clockwise circle. This gets then reoriented and merged again, which results in the identity.
Next, we consider $i = k = 0 = j - 1$.

As before, we show this relation over $\mathbb{F}_2$. We will just show that the resulting basis vectors on the left and right-hand side match.

In this case, there exist 4 possible number lines, such that the sequence is nonzero: $\times\times$, $\times\bullet$, $\bullet\bullet$ and $\bullet\times$. The latter two are the horizontal mirrors of the first two and we will only consider the first two. Note that the second summand of the left-hand side of (15.3) is zero. The involved surgeries for the left-hand side are shown in Figure 17.5. The two

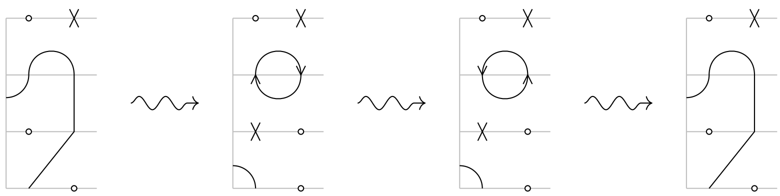


Figure 17.4.: Braid relation for $i = k = 1 = j + 1$

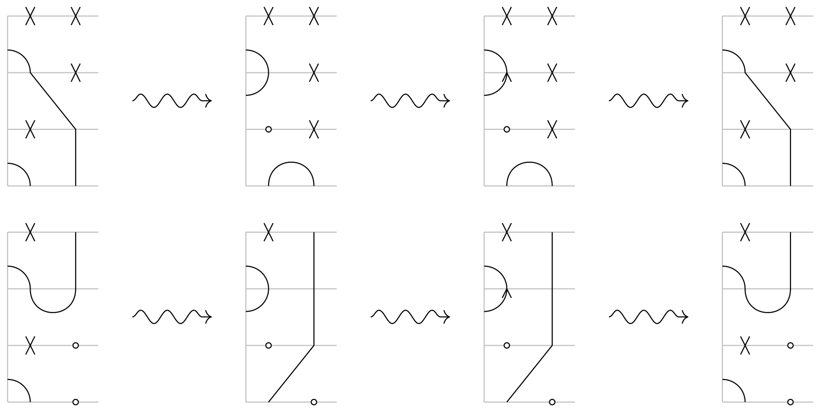
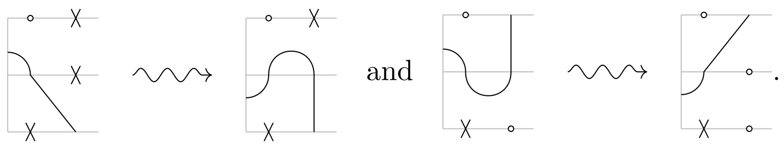


Figure 17.5.: The two possibilities for the braid relation for $i = k = 0 = j - 1$

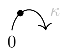
cases, further divide into two subcases each. Namely, whether the two drawn components are connected and form an inner circle or not. If they form an inner circle, the first surgery splits off a clockwise outer circle, which merges in the last step with an inner component, thus the left-hand side is 0. On the right-hand side, each summand reorients this circle and hence, they cancel out.

If the two components are not connected, the first surgery is a Reconnect, then we reorient the small inner circle and do another Reconnect. In case both circles are clockwise, both sides are 0. If exactly one circle is oriented anticlockwise, the first Reconnect produces two clockwise circles. Overall, the result will contain two clockwise circles. One summand on the right-hand side gives 0 (applying the dot to the clockwise circle) and the other summand gives two clockwise circles. If both circles are anticlockwise, after the second step, we will still have two anticlockwise circles. The final Reconnect, then gives the difference of two diagrams, each containing exactly one clockwise circle. This is exactly the effect that the right-hand side has. Thus, the relation is satisfied.

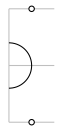
Finally, we consider (15.5). If i and j are distant, this is just a height move, which is clearly an isomorphism. This leaves $i = 0 = j - 1$ and $i = 1 = j + 1$. Both of these are similar, so we only show the first case. There are exactly two number lines, such that the E_1F_0 is nonzero: $\circ\times$ and $\circ\bullet$. These are also exactly those two, such that F_0E_1 is nonzero. The involved crossings are the obvious isomorphisms



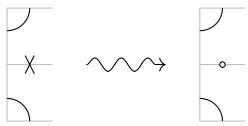
It remains to look at (15.6) and (15.7). Both are shown similarly, so we concentrate on (15.6) for $i = 0$. We are in a situation where $\langle h_0, \kappa\rangle \geq 0$. This leaves two possibilities for position 1 of the top number line: $\circ$ and $\bullet$. In case of $\circ$, we have $\langle h_0, \kappa\rangle = 2$ and for $\bullet$ we have $\langle h_0, \kappa\rangle = 0$. For the former, note that the crossing is 0 as E_0 is not applicable. So we have to show that

 $\oplus$  $: E_0F_0 1_\kappa \to 1_\kappa \oplus 1_\kappa$

is an isomorphism. The diagram for the left-hand side is given by



Observe that the counit of the adjunction is given by projecting onto the clockwise oriented component. Hence, the first summand kills the anticlockwise component and projects onto the clockwise one. Similarly, the second summand kills the clockwise component (because of the dot) and projects onto the anticlockwise one. This is also exactly the isomorphism given in Theorem 16.35.
If $\langle h_0, \kappa\rangle = 0$, the crossing is the obvious isomorphism



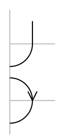
Note that for this argument the signs do not matter. Independent of their appearance, the maps will always be isomorphisms.

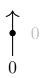
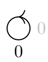
Hence, all relations are satisfied, and we obtain the desired 2-functor. $\square$

Using this theorem, $2\mathbb{K}^Q_{\iota 0}$ exhibits the structure of a 2-representation of $\mathfrak{U}(B_{0|\infty})$ by pulling back the action along the 2-functor.

Theorem 17.8. *There is an isomorphism of* 2*-representations* $G\colon \mathfrak{U}(B_{0|\infty})^\Lambda \to 2\mathbb{K}^Q_{\iota 0}$.

Proof. We being by constructing the morphism G. Observe that $\mathfrak{U}(B_{0|\infty})^\Lambda$ is generated by the object 0 and $2\mathbb{K}^Q_{\iota 0}$ by the projective module $P_{\iota 0}$. If we can show that $P_{\iota 0}$ satisfies (15.8), we get the desired morphism from Theorem 17.7.
Observe that the definition and Theorem 16.38 immediately imply that $E_i P(\iota_0) = 0$ unless $i = 0$. Furthermore, we have

 $= 0$

and thus  is satisfied. Finally, observe that  defines an odd endomorphism of $P_{\iota 0}$, which thus must be trivial.
Therefore, we obtain the desired morphism of 2-representations $G\colon \mathfrak{U}(B_{0|\infty})^\Lambda \to 2\mathbb{K}^Q_{\iota 0}$.
Let t be one of the diagrams that are used in Definition 17.3 to define E_i. Now, observe

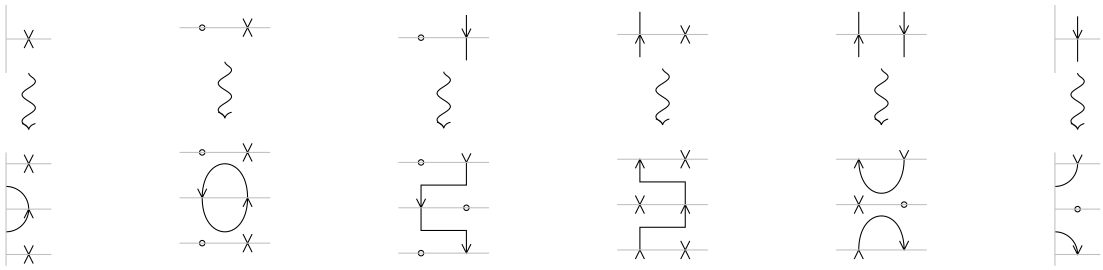


Figure 17.6.: The possibilities for a cup

that $\omega t \eta$ is oriented if and only if (under the identification with $\mathbb{\Lambda}$) either $\eta \xrightarrow{i} \omega$ or $\eta \overset{i}{\dashleftarrow} \omega$. A similar statement is true for F_i (by reversing the orientation of the arrows). This has the following consequence. Let $X_{\boldsymbol{i}}$ and $Y_{\boldsymbol{j}}$ denote two sequences of E_i and F_i and let κ be the weight associated to this. On the one hand, we can consider $\mathcal{H}om_{\mathfrak{U}(B_{0|\infty})^\Lambda}(\kappa, 0)(X_{\boldsymbol{i}}, Y_{\boldsymbol{j}})$. On the other hand, we have $\mathrm{Hom}_{\mathbb{K}^Q_\kappa}(X_{\boldsymbol{i}}P(\iota_0), Y_{\boldsymbol{j}}P(\iota_0))$. Note that G exactly maps the former to the latter. Now the above observation shows that

$$\dim \mathcal{H}om_{\mathfrak{U}(B_{0|\infty})^\Lambda}(\kappa, 0)(X_{\boldsymbol{i}}, Y_{\boldsymbol{j}}) \leq \dim \mathrm{Hom}_{\mathbb{K}^Q_\kappa}(X_{\boldsymbol{i}}P(\iota_0), Y_{\boldsymbol{j}})P(\iota_0).$$

Hence, it suffices to show that G is full. This in turn also proves the linear independence of the spanning set of $\mathfrak{U}(B_{0|\infty})^\Lambda$.

Let $[\lambda, \mu] \in \mathbb{\Lambda}$ and $\mathfrak{s} \in \mathcal{T}^{\mathrm{ud}}([\lambda, \mu])$. Let $X_{\boldsymbol{i}}$ be the sequence of projective functors that is induced by $\operatorname{res} \mathfrak{s}$ and let $\boldsymbol{t} = t_n \cdots t_1$ be the associated sequence of crossingless matchings. A basis of $X_{\boldsymbol{i}}P(\iota_0)$ is given by all compatible orientations $\underline{\omega'}\eta_n t_n \eta_{n-1} \cdots t_1 \iota_0 \overline{\iota_0}$. If we write ω for the weight associated to $[\lambda, \mu]$, the set of all these orientations with $\omega = \omega' = \eta_n$ is in bijection with the subset of $\mathcal{T}^{\mathrm{ud}}([\lambda, \mu])$ with residue sequence $\operatorname{res} \mathfrak{s}$.

Now, $\Psi^{\mathfrak{s}}$ is given as the composition of crossings and cups. Note that considering the diagram from the bottom up, this still defines a valid up-down-bitableau of shape $[\lambda, \mu]$. Thus, we can associate to each of these slices also a basis vector in the corresponding projective module. If we consider two consecutive slices, these differ only by a basic crossing or cup. We will show that in this situation the dedicated basis vector is mapped to the other dedicated basis vector (up to lower order terms).

By the dual argument we obtain the same result for $\Psi_{\mathfrak{t}}$. In the end, we will have $\Psi^{\mathfrak{s}}$ giving rise to a diagram of the form $\underline{\omega}\omega \cdots$ and $\Psi_{\mathfrak{t}}$ gives $\cdots \omega\overline{\omega}$. By Theorem 16.30, we have that $(\cdots \omega\overline{\omega}) \cdot (\underline{\omega}\omega \cdots) = \pm \cdots \omega \cdots$ up to lower order terms. This shows that G is full and thus an isomorphism of 2-representations.

We only have to show the above claim. For this let $\mathfrak{t}$ and $\mathfrak{s}$ two consecutive slices. We look at different cases depending on the elementary diagram connecting these two. First suppose that the elementary diagram is a rightward cup labelled i. This means that at this position a box of residue i can be added to the first component (which then gets removed by the other end of the cup). Figure 17.6 describes all the local possibilities including the orientations of what basis vector needs to be mapped to which other one. For most of these cases it is immediate from the definition, that the basis vector is

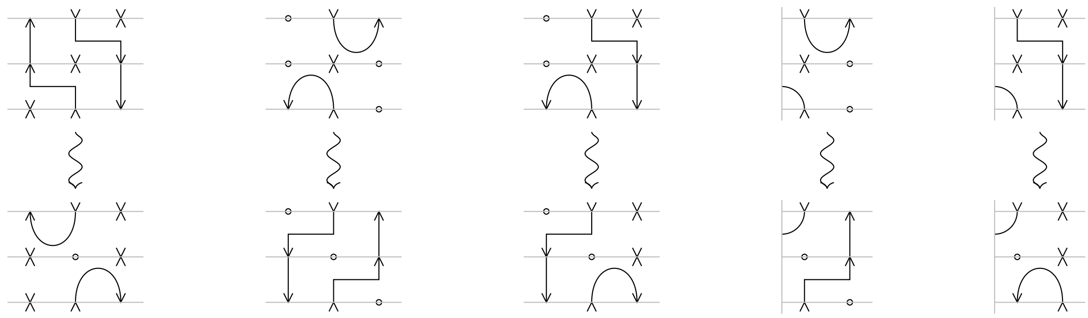
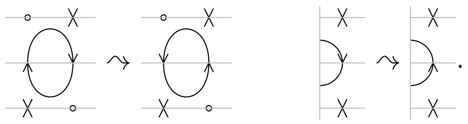
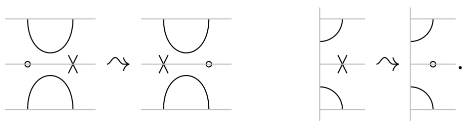


Figure 17.7.: The possibilities for a neighbored crossing

mapped to the other one. The only exception are the last two, where a line is cut into two. Both of these can be rephrased in terms of usual surgeries and then these amount to cutting anticlockwise cups and caps, hence the orientation is preserved by Theorem 16.30. Replacing $\circ$ by $\times$ and vice versa, we obtain the same result for leftward cups.
Next, we move on to crossings. We begin by examining the possibilities for E_iE_i. In our situation, a crossing for these two can only appear, when the first a box is removed and afterward added. The situation looks necessarily like



By definition of the crossing, this exactly reorients the small circle as desired.
Next, we consider E_iE_{i+1}. This can also only appear if E_{i+1} removes a box and E_i adds a box. The possible cases are depicted in Figure 17.7. Recall, that neighbored crossing are defined using one surgery procedure. In all the cases from Figure 17.7, the surgery is given by cutting an anticlockwise cup and cap, hence (up to possible sign) everything is given by reconnecting (without changing any weights). If we reorient all lines and change the orientation of the squiggly arrow, we obtain the diagrams for $E_{i+1}E_i$.
For E_i and E_j with $|i-j|>1$, the crossing is just a height move, in the diagram. In this case, it is easy to see that the two basis vectors are mapped to each other.
If we replace in the above arguments all $\circ$ by $\times$, we obtain the diagrams for crossings involving F_i's.
For crossings involving E_i and F_j with $i \neq j$, the statement is very similar to the distant case for E_i and E_j.
Finally, we are left with F_iE_i. This situation can only appear, if either both add boxes or both remove boxes. In any case, the crossing is the obvious isomorphism, given by the following diagrams



It is easy to see that also in this case the basis vectors are mapped to each other. $\square$

18. Khovanov algebra of type Q_n

In this section, we will foster the relationship between $\mathbb{K}^Q$ and $\mathfrak{q}(n)$. By combining Theorem 15.5 with Theorem 17.8 and Theorem 16.35, we obtain a relationship between $\mathbb{K}^Q$-modules and representations of $\mathfrak{q}(n)$. This is the following corollary.

Corollary 18.1. *We have an essentially surjective, full functor*

$$\mathcal{F}_n \colon \mathbb{K}^Q\text{-proj} \to \mathrm{Fund}'(\mathfrak{q}(n)).$$

Unfortunately, this functor is not faithful. To overcome this issue, we begin by introducing certain subrepresentations of $2\mathbb{K}^Q_{\iota_0}$.

Definition 18.2. Let ι_n be the weight $|\circ\circ\cdots\circ\vee\vee\cdots$ where the number of $\circ$'s is exactly n. (Note that this agrees with the previous definition of ι_0). Denote by $2\mathbb{I}_n$ the 2-subrepresentation of $2\mathbb{K}^Q_{\iota_0}$ generated by P_{ι_n}. This is the full subcategory of $2\mathbb{K}^Q_{\iota_0}$ where objects are given by direct summands of $\mathbb{K}^{Q,\boldsymbol{t}} \otimes_{\mathbb{K}^Q} P_{\iota_n}$, where $\boldsymbol{t}$ is a composite matching build from the elementary E_i and F_i diagrams.
From Theorem 16.38 it is easy to see that $2\mathbb{I}_0 \supseteq 2\mathbb{I}_1 \supseteq 2\mathbb{I}_2 \supseteq \cdots$.

Next, we will introduce certain subsets of Λ, which will be an important tool for understanding $2\mathbb{I}_n$.

Definition 18.3. Denote by $\Lambda_n \subseteq \Lambda$ the set of all weight diagrams ω such that $\underline{\omega}$ satisfies

$$\#(\times) + \#(\circ) + \#\Big(\quad\Big) + 2\cdot\#\Big(\qquad\Big) = n.$$

We also introduce $\Lambda_{\geq n} := \bigcup_{k\geq n} \Lambda_k$ (equivalently we can replace the $=$ by $\geq$ in the definition of Λ_n).

Lemma 18.4. *Let $\omega \in \Lambda$ a weight diagram. We have $P(\omega) \in 2\mathbb{I}_n$ if and only if $\omega \in \Lambda_{\geq n}$. Additionally, any $P(\omega)$ with $\omega \in \Lambda_n$ also generates $2\mathbb{I}_n$.*

Proof. First, observe that any of the elementary diagrams for E_i and F_i never decrease the left-hand side of the inequality and clearly ι_n satisfies the inequality. So we only have to show that any $P(\omega)$ satisfying the inequality is in $2\mathbb{I}_n$. Observe that applying E_iE_i turns $\circ\times$ into $\times\circ$ (if $i > 0$) and $\times$ into $\circ$ (if $i = 0$). Similarly, F_iF_i does exactly the reverse. Furthermore, E_i and F_i can also move $\circ$ and $\times$ around. Applying F_0 and E_0 to $\bullet$, we can create $\circ$ or $\times$. This means, that we can obtain any $\underline{\omega}$ with at $\#(\circ) + \#(\times) \geq n$ that does not contain any cups. Finally, applying E_i to $\circ\times$ (or to $\times$ if $i = 0$) we can create a cup. This shows that any $\underline{\omega} \in \Lambda_{\geq n}$ can be obtained from ι_n.

For the second part, we have to show that $P(\iota_n)$ can be obtained (as a direct summand) from $P(\omega)$ with $\omega \in \mathbb{\Lambda}_n$ via applying E_i's and F_i's. This can be proven by mimicking the same arguments as above. Any cup in the cup diagram can be removed by applying E_i, which turns this into $\times\circ$. Then, move all the $\times$ to the left and apply E_0E_0 to turn them into $\circ$. Finally, move all the $\circ$'s to the left to obtain ι_n. Note that we really need $\omega \in \mathbb{\Lambda}_n$ to ensure that the number of resulting $\circ$ is exactly n. $\square$

The following lemma gives a representation theoretic interpretation of the sets $\mathbb{\Lambda}_n$ in terms of the representation theory of $\mathfrak{q}(n)$.

Lemma 18.5. *We have $\mathcal{F}_n(2\mathbb{I}_{n+1}) = 0$ and $\mathcal{F}_n(2\mathbb{I}_n) \subseteq \mathrm{proj}'(\mathfrak{q}(n))$. Furthermore, $\mathcal{F}_n(P(\omega)) \neq 0$ for all $\omega \in \mathbb{\Lambda}_n$.*

Proof. By Theorem 14.16, we have $\mathcal{F}_n(P(\iota_{n+1})) = 0$ and $\mathcal{F}_n(P(\iota_n)) = L_Q(n, n-1, \dots, 1)$. Now, $L_Q(n, n-1, \dots, 1) = P_Q(n, n-1, \dots, 1)$ by Lemma 14.9. As E_i and F_i are biadjoint functors (on $\mathrm{rep}'(\mathfrak{q}(n))$), they preserve projective objects. In conjunction with $\mathcal{F}_n$ being a map of 2-representations, the first part follows.
For the second part, suppose that $\mathcal{F}_n(P(\omega)) = 0$ for some $\omega \in \mathbb{\Lambda}_n$. As $P(\omega)$ generates $2\mathbb{I}_n$, we get in particular $\mathcal{F}_n(P(\iota_n)) = 0$, contradicting the first part. $\square$

In particular, we obtain an induced map of 2-representations $\mathcal{F}_n\colon 2\mathbb{I}_n/2\mathbb{I}_{n+1} \to \mathrm{proj}'(\mathfrak{q}(n))$, where $2\mathbb{I}_n/2\mathbb{I}_{n+1}$ denotes the quotient 2-representation. Our next goal is to give a better description of this quotient. For this we introduce a subquotient of $\mathbb{K}^Q$.

Definition 18.6. Let $e_n = \sum_{\omega \in \mathbb{\Lambda}_n} e_\omega$. This is not an honest element in $\mathbb{K}^Q$; nevertheless, we can consider $e_n\mathbb{K}^Q e_k = \bigoplus_{\omega \in \mathbb{\Lambda}_n, \eta \in \mathbb{\Lambda}_k} e_\omega \mathbb{K}^Q e_\eta$.
We define $\mathbb{K}_n^Q$ to be the quotient of $e_n\mathbb{K}^Q e_n$ by the two-sided ideal generated by e_{n+1}.

The following is immediate from $\mathbb{K}_n^Q$ and Lemma 18.4.

Corollary 18.7. *The quotient 2-representation $2\mathbb{I}_n/2\mathbb{I}_{n+1}$ is isomorphic to $\mathbb{K}_n^Q$-proj.*

The next proposition provides a basis for $\mathbb{K}_n^Q$.

Proposition 18.8. *The algebra $\mathbb{K}_n^Q$ has a basis given by all circle diagrams $\underline{\gamma}\omega\overline{\eta}$ with $\gamma, \eta \in \mathbb{\Lambda}_n$ that do not contain an inner line.*

Proof. It is clear that $e_n\mathbb{K}^Q e_n$ has a basis given by $\underline{\gamma}\omega\overline{\eta}$ with $\gamma, \eta \in \mathbb{\Lambda}_n$.
First, note that any oriented circle diagram in $e_n\mathbb{K}^Q e_{n+1}$ (respectively $e_{n+1}\mathbb{K}^Q e_n$) has an inner line. This is because the cup and the cap diagram have the same number of $\circ$ and $\times$. Furthermore, each outer line and each circle contribute the same number to the sum in the inequality of Lemma 18.4. Hence, there must be an inner line ending at the bottom for $e_n\mathbb{K}^Q e_{n+1}$ and at the top for $e_{n+1}\mathbb{K}^Q e_n$. From the explicit description of the surgery procedures in Appendix A, we see that this inner line is preserved by the multiplication (or mapped to 0).
On the other hand, we have to show that any circle diagram with an inner line factors through e_{n+1}. So consider $\underline{\gamma}\omega\overline{\eta}$ with γ, $\eta \in \mathbb{\Lambda}_n$ containing an inner line. Consider the

first (from the left) inner line L ending at the top (the number of inner lines ending at the bottom and top is the same, but the same argument also works the other way around). Let $\overline{\eta'}$ be the same diagram as $\overline{\eta}$ but we replace the line segment of L by an inner cap. Then, $\underline{\gamma}\omega\overline{\eta'}$ is also an oriented circle diagram, where the L is now replaced by a clockwise inner circle. Let ω' be the same as ω except that we orient this inner circle anticlockwise. Then, $\underline{\gamma}\omega'\overline{\eta'}$ is an oriented circle diagram in $e_n\mathbb{K}^Q e_{n+1}$. Furthermore, $\underline{\eta'}\eta\overline{\eta}$ is an oriented circle diagram in $e_{n+1}\mathbb{K}^Q e_n$. Observe that $\underline{\eta'}\eta\overline{\eta}$ contains only anticlockwise circles and one inner line. Hence, all surgeries involved in the multiplication process are Merges with anticlockwise circles. All except for the last one come from the anticlockwise circles in $\underline{\eta'}\eta\overline{\eta}$. The last one merges the created anticlockwise circle of $\underline{\gamma}\omega'\overline{\eta'}$ with the inner circle of $\underline{\eta'}\eta\overline{\eta}$. This shows that $\underline{\gamma}\omega\overline{\eta}$ factors through e_{n+1}. □

Theorem 18.9. *We have a superequivalence of* $\mathfrak{U}(B_{0|\infty})$*-2-representations*

$$\hat{\mathcal{F}}_n \colon \mathbb{K}_n^Q\text{-proj} \to \operatorname{proj}'(\mathfrak{q}(n)).$$

In particular, we obtain a weak superequivalence of abelian categories between $\mathbb{K}_n^Q$-mod *and* $\operatorname{rep}(\mathfrak{q}(n))$.

Proof. Consider $\operatorname{Hom}_{\mathbb{K}_n^Q}(P(\gamma), P(\eta))$ for γ and $\eta \in \Lambda\!\!\!\Lambda_n$. By Theorem 16.38 (see also the proof of Lemma 18.4), there exists $\omega \in \Lambda\!\!\!\Lambda_n$ such that $\underline{\omega}$ contains no cups and a sequence $X_{\boldsymbol{i}}$ of E_i's and F_i's such that $X_{\boldsymbol{i}}P(\omega) = P(\gamma)$. Denote by $X_{\boldsymbol{i}}^*$ the adjoint sequence (i.e. reverse the order and swap E's and F's) and consider the following diagram

$$\begin{array}{ccc}
\operatorname{Hom}_{\mathbb{K}_n^Q}(P(\gamma), P(\eta)) & \xrightarrow{\hat{\mathcal{F}}_n} & \operatorname{Hom}_{\operatorname{rep}'(\mathfrak{q}(n))}\Big(\hat{\mathcal{F}}_n(P(\gamma)), \hat{\mathcal{F}}_n(P(\eta))\Big) \\
\wr\| & & \wr\| \\
\operatorname{Hom}_{\mathbb{K}_n^Q}(X_{\boldsymbol{i}}P(\omega), P(\eta)) & \xrightarrow{\hat{\mathcal{F}}_n} & \operatorname{Hom}_{\operatorname{rep}'(\mathfrak{q}(n))}\Big(\hat{\mathcal{F}}_n(X_{\boldsymbol{i}}P(\omega)), \hat{\mathcal{F}}_n(P(\eta))\Big) \\
\wr\| & & \wr\| \\
\operatorname{Hom}_{\mathbb{K}_n^Q}(P(\omega), X_{\boldsymbol{i}}^*P(\eta)) & \xrightarrow{\hat{\mathcal{F}}_n} & \operatorname{Hom}_{\operatorname{rep}'(\mathfrak{q}(n))}\Big(\hat{\mathcal{F}}_n(P(\omega)), \hat{\mathcal{F}}_n(X_{\boldsymbol{i}}^*P(\eta))\Big) \\
\wr\| & & \wr\| \\
\operatorname{Hom}_{\mathbb{K}_n^Q}\Big(P(\omega), P(\omega)^{\oplus k}\Big) & \xrightarrow{\hat{\mathcal{F}}_n} & \operatorname{Hom}_{\operatorname{rep}'(\mathfrak{q}(n))}\Big(\hat{\mathcal{F}}_n(P(\omega)), \hat{\mathcal{F}}_n(P(\omega))^{\oplus k}\Big)
\end{array}.$$

The first vertical isomorphisms come from the definition of $X_{\boldsymbol{i}}$ and ω and the upper square commutes by definition. The second vertical isomorphism follow from adjunction and the square commutes as $\hat{\mathcal{F}}_n$ is a map of 2-representations. Note that ω is strongly typical by construction and thus Theorem 16.38 implies that there exists some $k \in \mathbb{N}_0$ such that the third vertical isomorphism holds (which might be 0). The lower square also trivially commutes.

By Lemma 18.4, we know that $\hat{\mathcal{F}}_n(P(\omega)) \neq 0$ and as $P(\omega)$ is irreducible so is the image. In particular, the dimension of the two homomorphism spaces in the bottom row is the same, namely k. As $\hat{\mathcal{F}}_n$ is full, the lower horizontal map must be an isomorphism and thus also the top horizontal map. Thus, $\hat{\mathcal{F}}_n$ is fully faithful and essentially surjective and hence a superequivalence of supercategories.

As $\mathrm{rep}'(\mathfrak{q}(n))$ and $\mathrm{rep}(\mathfrak{q}(n))$ are weakly superequivalent (by definition of $\mathrm{rep}'(\mathfrak{q}(n))$), we obtain the desired result. $\square$

Remark 18.10. The locally unital algebra $\mathbb{K}_n^Q$ decomposes into blocks $(\mathbb{K}_n^Q)_\kappa$. Let $(\mathbb{K}_n^{Q'})_\kappa$ be $(\mathbb{K}_n^Q)_\kappa$ if $\#(\circ) + \#(\times)$ is even and $(\mathbb{K}_n^Q)_\kappa \otimes \mathfrak{C}_1$ if $\#(\circ) + \#(\times)$ is odd. Here, $\mathfrak{C}_1$ is the Clifford algebra with one generator, i.e. $\mathbb{C}[C]/(C^2 - 1)$ with C odd.
Write $\mathbb{K}_n^{Q'} = \bigoplus_\kappa (\mathbb{K}_n^{Q'})_\kappa$. Then, we have a superequivalence of abelian categories between $(\mathbb{K}_n^{Q'})_\kappa$-mod and $\mathrm{rep}(\mathfrak{q}(n))$ by Theorem 18.9 and [KKT16, Lemma 2.3, Lemma 2.7].

Remark 18.11. In [Fri07], it was shown that integral typical blocks of $\mathcal{O}(\mathfrak{q}(n))$ are fibered highest weight categories. This also holds true (more or less trivially) for the typical blocks of finite dimensional representations of $\mathfrak{q}(n)$ as these are semisimple by Lemma 14.9.
One might hope that this extends to all integral blocks of (finite dimensional) representations of $\mathfrak{q}(n)$. However, this is not the case. From the explicit description of $\mathbb{K}_n^Q$, it is easy to check that already for $n = 2$ the atypical blocks are not fibered highest weight categories anymore.

19. Categorification

In this section, we will show that the 2-representation $\mathfrak{U}(B_{0|\infty})^\Lambda$ categorifies the $U(B_{0|\infty})$-representation $L^w(-\omega_f) \otimes L(\omega_f)$. Here, $L(\omega_f)$ denotes the integrable highest weight representation with highest weight ω_f and $L^w(-\omega_f)$ the integrable lowest weight representation with lowest weight $-\omega_f$.
Throughout this section, let $\mathbb{Q}(q)^\pi$ be the algebra $\mathbb{Q}(q)[\pi]/(\pi^2-1)$, with q and π indeterminates over $\mathbb{Q}$. We also use the notation $\mathbb{Z}^\pi$ for $\mathbb{Z}[\pi]/(\pi^2-1)$.
We replace the usual braiding on $\mathbb{Q}(q)^\pi$-vector superspaces by $v \otimes w \mapsto \pi^{|v||w|} v \otimes w$. Note that setting $\pi = -1$ gives the usual braiding of vector superspaces and $\pi = 1$ gives the usual braiding of vector spaces.

19.1. The Grothendieck groups of $\mathfrak{U}(B_{0|\infty})$ and $\mathfrak{U}(B_{0|\infty})^\Lambda$

We begin with some generalities for the Grothendieck groups of graded 2-supercategories and its 2-representations.

Definition 19.1. Let $\mathfrak{A}$ be a graded 2-supercategory. Then, we define its Grothendieck group

$$K_0'(\mathfrak{A}) \coloneqq \bigoplus_{\kappa,\nu \in \mathrm{Ob}(\mathfrak{A})} K_0(\mathcal{H}om_{\mathfrak{A}}(\kappa,\nu)^\circ_{q,\pi}).$$

This is a locally unital $\mathbb{Z}^\pi[q,q^{-1}]$-algebra with mutually orthogonal idempotents 1_κ, $\kappa \in X$. Here, q, q^{-1} and π are induced by the automorphisms Q, Q^{-1} and Π respectively. The algebra structure is induced by the horizontal composition.
We write $K_0(\mathfrak{A}) \coloneqq K_0'(\mathfrak{A}) \otimes_{\mathbb{Z}^\pi[q,q^{-1}]} \mathbb{Q}^\pi(q)$ for the scalar extension.
Given a 2-representation $\mathbb{R}$ of $\mathfrak{A}$, we define its Grothendieck group as

$$K_0'(\mathbb{R}) \coloneqq \bigoplus_{\kappa\, \mathrm{Ob}(\mathfrak{A})} K_0(\mathbb{R}\kappa^\circ_{q,\pi}).$$

As above, this is a $\mathbb{Z}^\pi[q,q^{-1}]$-module. The 2-representation structure induces a left $K_0'(\mathfrak{A})$-module structure on $K_0'(\mathbb{R})$. We write $K_0(\mathbb{R}) \coloneqq K_0'(\mathbb{R}) \otimes_{\mathbb{Z}^\pi[q,q^{-1}]} \mathbb{Q}^\pi(q)$ for the scalar extension, which is then also a left $K_0(\mathfrak{A})$-module.

Definition 19.2. Given a Q-Π-category $\mathcal{C}$ (i.e. the underlying category of a graded Q-Π-supercategory), we denote by $\mathrm{rep}(\mathcal{C})$ the functor category $\mathrm{Fun}(\mathcal{C}, \mathrm{Vec})$. This is a Q-Π-category where the additional data is given by precomposing with Q, Q^{-1} and Π. Given $c \in \mathrm{Ob}(\mathcal{C})$, we have the *projective module* $P_c \coloneqq \mathrm{Hom}_{\mathcal{C}}(c, \_)$. We denote by $\mathrm{proj}(\mathcal{C})$ the full additive subcategory generated by finite direct sums of P_c.

The following lemma is obvious from the definitions.

Lemma 19.3. *Let $\mathcal{C}$ be a Q-Π-category. Then, there is a $\mathbb{Q}(q)^\pi$-linear isomorphism*

$$K_0(\mathcal{C}) \to K_0(\mathrm{proj}(\mathcal{C})), \quad [c] \mapsto [P_c].$$

This gives us the following bases for $K_0(\mathfrak{U}(B_{0|\infty})^\Lambda)$.

Lemma 19.4. *The $\mathbb{Q}(q)^\pi$-module $K_0(\mathfrak{U}(B_{0|\infty})^\Lambda)$ has the following bases over $\mathbb{Q}(q)^\pi$:*

$$\left\{[P([\lambda,\mu])] \mid [\lambda,\mu] \in \mathbb{\Lambda}\right\} \quad \textit{and} \quad \left\{[\Delta([\lambda,\mu])] \mid [\lambda,\mu] \in \mathbb{\Lambda}\right\}$$

Proof. By Lemma 19.3, we see that the indecomposable projective modules form a basis of $K_0(\mathfrak{U}(B_{0|\infty})^\Lambda)$. Theorem 15.9 identifies the indecomposable projective modules with $P([\lambda,\mu])$. The $\Delta([\lambda,\mu])$ all have finite projective dimension by [BS24, Lemma 3.43], i.e. we can consider them as elements in $K_0(\mathfrak{U}(B_{0|\infty})^\Lambda)$. Furthermore, any projective has a finite filtration by (grading/parity shifts of) $\Delta([\lambda,\mu])$, so they form a basis. □

We will also need a supergrading on $K_0(\mathfrak{U}(B_{0|\infty}))$ and $K_0(\mathfrak{U}(B_{0|\infty})^\Lambda)$.

Definition 19.5. We endow $K_0(\mathfrak{U}(B_{0|\infty}))$ with the structure of a superalgebra by declaring $|[\mathcal{E}_i]1_\kappa| = |[\mathcal{F}_i]1_\kappa| = |i|$. There also exists a supergrading on $K_0(\mathfrak{U}(B_{0|\infty})^\Lambda)$ by declaring that the vector space $1_\kappa K_0(\mathfrak{U}(B_{0|\infty})^\Lambda)$ is pure of parity $\langle \sum_{r=1}^{\infty} e_r, \kappa\rangle \mod 2$. With this definition, $K_0(\mathfrak{U}(B_{0|\infty})^\Lambda)$ is a left $K_0(\mathfrak{U}^\circ_{q,\pi}(B_{0|\infty}))$-module.

19.2. Categorification of $\mathfrak{U}(B_{0|\infty})^\Lambda$

In this section, we state the results from [BE17b] for the categorification of $U(B_{0|\infty})$.
We begin by recalling the definition of the quantum covering group $U(B_{0|\infty})$ from [CFLW14].
Let $q_i = q^{d_i}$ and $\pi_i = \pi^{|i|}$ for $i \in I$.
For $n \in \mathbb{Z}$, we define

$$[n]_{q,\pi} = \frac{q^n - (\pi q)^{-n}}{q - \pi q^{-1}} = \begin{cases} q^{n-1} + \pi q^{n-3} + \cdots + \pi^{n-1} q^{1-n} & \text{if } n \geq 0, \\ -\pi^n (q^{-n-1} + \pi q^{-n-3} + \cdots + \pi^{-n-1} q^{1+n}) & \text{if } n \leq 0, \end{cases}$$

Note that this slightly differs from [CFLW14] but agrees with [BE17b].

Definition 19.6. The covering quantum group $U(B_{0|\infty})$ is the $\mathbb{Q}(q)^\pi$-superalgebra with generators $\mathcal{J}$, $\mathcal{E}_i$, $\mathcal{F}_i$, $\mathcal{K}_i$ for all $i \in \mathbb{N}_0$ subject to the following relations

$$\mathcal{J} \text{ is central,}$$
$$\mathcal{K}_j\mathcal{K}_i = \mathcal{K}_i\mathcal{K}_j, \qquad \mathcal{J}^2 = 1,$$
$$\mathcal{K}_i\mathcal{E}_j = q^{\langle h_i,\alpha_j\rangle}\mathcal{E}_j\mathcal{K}_i, \qquad \mathcal{K}_i\mathcal{F}_j = q^{-\langle h_i,\alpha_j\rangle}\mathcal{F}_j\mathcal{K}_i,$$

$$
\begin{gathered}
\mathcal{E}_i\mathcal{F}_j - \pi^{|i||j|}\mathcal{E}_j\mathcal{F}_i = \delta_{ij}\frac{\mathcal{J}^{|i|}\mathcal{K}_i^{d_i} - \mathcal{K}_i^{-d_i}}{\pi^{|i|}q_i - q_i^{-1}}, \\
\mathcal{E}_i\mathcal{E}_j = \mathcal{E}_j\mathcal{E}_i, \qquad \mathcal{F}_i\mathcal{F}_j = \mathcal{F}_j\mathcal{F}_i \qquad \text{for } |i-j| > 1, \\
\mathcal{E}_i^2\mathcal{E}_{i\pm1} - [2]_{q^2,1}\mathcal{E}_i\mathcal{E}_{i\pm1}\mathcal{E}_i + \mathcal{E}_{i\pm1}\mathcal{E}_i^2 = 0 \qquad \text{for } i \neq 0, \\
\mathcal{F}_i^2\mathcal{F}_{i\pm1} - [2]_{q^2,1}\mathcal{F}_i\mathcal{F}_{i\pm1}\mathcal{F}_i + \mathcal{F}_{i\pm1}\mathcal{F}_i^2 = 0 \qquad \text{for } i \neq 0, \\
\mathcal{E}_0^3\mathcal{E}_1 - [3]_{q,\pi}\mathcal{E}_0^2\mathcal{E}_1\mathcal{E}_0 + \pi[3]_{q,\pi}\mathcal{E}_0\mathcal{E}_1\mathcal{E}_0^2 - \pi\mathcal{E}_1\mathcal{E}_0^3 = 0, \\
\mathcal{F}_0^3\mathcal{F}_1 - [3]_{q,\pi}\mathcal{F}_0^2\mathcal{F}_1\mathcal{F}_0 + \pi[3]_{q,\pi}\mathcal{F}_0\mathcal{F}_1\mathcal{F}_0^2 - \pi\mathcal{F}_1\mathcal{F}_0^3 = 0.
\end{gathered}
$$

Here, $|\mathcal{K}_i| = |\mathcal{J}| = 0$ and $|\mathcal{E}_i| = |\mathcal{F}_i| = |i| = \delta_{i,0}$ for all $i \in \mathbb{N}_0$.

This is a Hopf algebra with comultiplication $\Delta(\mathcal{E}_i) = \mathcal{E}_i \otimes \mathcal{K}_i^{-d_i} + \mathcal{J}^{|i|} \otimes \mathcal{E}_i$, $\Delta(\mathcal{F}_i) = \mathcal{F}_i \otimes 1 + \mathcal{K}_i^{d_i} \otimes \mathcal{F}_i$, $\Delta(\mathcal{K}) = \mathcal{K} \otimes \mathcal{K}$ and $\Delta(\mathcal{J}) = \mathcal{J} \otimes \mathcal{J}$. The chosen comultiplication is the one from [CHW14] and is more similar to the comultiplication of [Kas91] than [Lus10].

Remark 19.7. Observe that [CFLW14] define $U(B_{0|\infty})$ where $\mathcal{K}$ and $\mathcal{J}$ are indexed by general coroots. We restricted to coroots associated to the simple roots. Also note that their $\mathcal{J}_{\alpha_i^\vee} = 1$ for $i \neq 0$ and our $\mathcal{J}$ is their $\mathcal{J}_{\alpha_0^\vee}$.

The following theorem was proven in [BE17b] for all types (not only $B_{0|\infty}$) assuming a basis theorem for $\mathfrak{U}(B_{0|\infty})$. In our specific setup this basis theorem can be deduced from [KKT16, Theorem 3.13, Theorem 5.4], see also [KKO13, KKO14].

Theorem 19.8. *There is an isomorphism of $\mathbb{Q}(q)^\pi$-superalgebras*

$$\Psi\colon \dot{U}(B_{0|\infty}) \to K_0(\mathfrak{U}(B_{0|\infty})),$$

where $\dot{U}(B_{0|\infty})$ denotes the modified covering quantum group from [CFLW14].

19.3. Categorifying $L(\omega_f) \otimes L^w(-\omega_f)$

From Theorem 19.8, we obtain a $U(B_{0|\infty})$-module structure on $K_0(\mathfrak{U}(B_{0|\infty})^\Lambda)$. We will show that this categorifies a tensor product of type B spin representations.
Let us begin by introducing the representations we want to categorify.

Lemma 19.9. *The following gives a well-defined representation of $U(B_{0|\infty})$ on the vector superspace $L(\omega_f)$ with basis v_μ for $[\emptyset, \mu] \in \mathbb{A}\!\!\!\mathbb{A}$, where $|v_\mu| = 1 + \ell(\mu)$.*

$$
\mathcal{K}_i v_\mu = q^{\delta_{i,0} + \langle h_i, \mathrm{wt}(\mu)\rangle} v_\mu, \qquad \mathcal{J} v_\mu = \pi v_\mu,
$$

$$
\mathcal{F}_i v_\mu = \begin{cases} v_{\mu'} & \textit{if } [\emptyset,\mu] \xrightarrow{i} [\emptyset,\mu'], \\ 0 & \textit{otherwise}, \end{cases} \qquad
\mathcal{E}_i v_\mu = \begin{cases} v_{\mu'} & \textit{if } [\emptyset,\mu] \xleftarrow{i} [\emptyset,\mu'], \\ 0 & \textit{otherwise}. \end{cases}
$$

In particular, $L(\omega_f)$ is an integrable highest weight module with highest weight ω_f, where ω_f is the first fundamental weight, i.e. $\langle h_i, \omega_f\rangle = \delta_{i0}$.

Proof. We need to check all the defining relations of $U(B_{0|\infty})$. First, observe that the action is parity preserving, as $\mathcal{F}_0$ and $\mathcal{E}_0$ change $\ell(\mu)$ by 1.
Recall that, if $\mu = (\mu_1 > \cdots > \mu_r)$, then $\mathrm{wt}(\mu) = \sum_{k=1}^r \varepsilon_{\mu_k}$. The element $\mathcal{J}$ operates by a scalar and hence is central. Observe that by definition of $\mathrm{wt}(\mu)$, the commutation relations between $\mathcal{K}$ and $\mathcal{E}$ (respectively $\mathcal{F}$) are satisfied.
For the commutation relation between $\mathcal{E}_i$ and $\mathcal{F}_j$ assume first that $i \neq j$. Note that if $|i-j| = 1$, then both summands on the left-hand side are 0, as a box of residue i cannot be removed after a box of residue j has been added (and vice versa). If $|i-j| > 1$, then the commutation relation is trivially satisfied, as these do not interact. Next if $i = j \neq 0$, note that $\mathcal{F}_i v_\mu$ is non-zero if and only if there is a row with i boxes in μ and no row with $i+1$ boxes. Similarly, $\mathcal{E}_i$ can remove a box of residue i if there is a row with $i+1$ boxes and no row with i boxes. In particular, the commutation relation is satisfied if $\mathcal{E}_i$ and $\mathcal{F}_i$ act by 0 on v_μ (as then $\mathcal{K}_i v_\mu = v_\mu$). If $\mathcal{F}_i v_\mu \neq 0$, then $\mathcal{K}_i v_\mu = q v_\mu$, and if $\mathcal{E}_i v_{\mu'} \neq 0$, then $\mathcal{K}_i v_{\mu'} = q^{-1} v_{\mu'}$. Hence,

$$(\mathcal{E}_i\mathcal{F}_i - \mathcal{F}_i\mathcal{E}_i)v_\mu = v_\mu = \frac{\mathcal{K}_i^2 - \mathcal{K}_i^{-2}}{q^2 - q^{-2}} v_\mu, \quad (\mathcal{E}_i\mathcal{F}_i - \mathcal{F}_i\mathcal{E}_i)v_{\mu'} = -v_{\mu'} = \frac{\mathcal{K}_i^2 - \mathcal{K}_i^{-2}}{q^2 - q^{-2}} v_{\mu'}.$$

For $i = j = 0$, suppose that $[\emptyset, \mu] \overset{0}{\dashrightarrow} [\emptyset, \mu']$, this means that $\langle h_0, \mathrm{wt}(\mu)\rangle = 0$. Then $\mathcal{E}_0 v_\mu = 0$, and we calculate

$$\mathcal{E}_0\mathcal{F}_0 v_\mu = v_\mu = \frac{\mathcal{J}\mathcal{K}_0 - \mathcal{K}_0^{-1}}{\pi q - q^{-1}} v_\mu.$$

If $[\emptyset, \mu] \overset{0}{\dashleftarrow} [\emptyset, \mu']$, then $\mathcal{F}_0 v_\mu = 0$ and $\langle h_0, \mathrm{wt}(\mu)\rangle = -2$. We obtain

$$-\pi\mathcal{F}_0\mathcal{E}_0 v_\mu = -\pi v_\mu = \frac{\mathcal{J}\mathcal{K}_0 - \mathcal{K}_0^{-1}}{\pi q - q^{-1}} v_\mu.$$

The Serre relation for $|i-j| > 1$ trivially holds for F and follows from $\langle h_i, \alpha_j\rangle = 0$ for E. Finally, all the other Serre relations trivially hold as every individual summand operates by 0.
From the definition it is clear, that $L(\omega_f)$ is an integrable module (we even have $\mathcal{F}_i^2 v_\mu = \mathcal{E}_i^2 v_\mu = 0$). The highest weight vector is $v_\emptyset$ and this has highest weight ω_f, as $\mathcal{K}_i v_\emptyset = q^{\delta_{i0}} v_\emptyset$. ☐

We have the following automorphism on $U(B_{0|\infty})$, see [CHW13, §2.2].

Lemma 19.10. *There is a unique $\mathbb{Q}(q)^\pi$-linear involution w on $U(B_{0|\infty})$ given by $w(\mathcal{E}_i) = \pi_i J^{|i|}\mathcal{F}_i$, $w(\mathcal{F}_i) = \mathcal{E}_i$, $w(\mathcal{K}_i) = \mathcal{K}_i^{-1}$ and $w(\mathcal{J}) = \mathcal{J}$.*

With this involution, we can define the following lowest weight representation.

Definition 19.11. Denote by $L^w(-\omega_f)$ the $U(B_{0|\infty})$-module obtained by twisting $L(\omega_f)$ with w from Lemma 19.10. Explicitly, we have the following action (writing now v^λ for $\lambda \in [\lambda, \emptyset]$)

$$\mathcal{K}_j v^\lambda = q^{-\delta_{i,0} - \langle h_i, \mathrm{wt}(\lambda)\rangle} v^\lambda, \qquad \mathcal{J} v^\lambda = \pi v^\lambda,$$

$$\mathcal{F}_i v^\lambda = \begin{cases} v^{\lambda'} & \text{if } [\lambda, \emptyset] \xleftarrow{i} [\lambda', \emptyset], \\ 0 & \text{otherwise,} \end{cases} \qquad \mathcal{E}_i v^\lambda = \begin{cases} v^{\lambda'} & \text{if } [\lambda, \emptyset] \xrightarrow{i} [\lambda', \emptyset], \\ 0 & \text{otherwise.} \end{cases}$$

Recall that $\mathrm{wt}(\lambda) = \sum_{k=1}^{r} \varepsilon_{\lambda_k}$. In particular, $L^w(-\omega_f)$ is an integrable lowest weight module with lowest weight $-\omega_f$.

We want to introduce a bar involution on $L(\omega_f)$ and $L^w(-\omega_f)$. For this, recall the bar involution on $U(B_{0|\infty})$ from [CHW13, §2.2].

Lemma 19.12. *There is a unique $\mathbb{Q}$-algebra automorphism* $\overline{}\colon U(B_{0|\infty}) \to U(B_{0|\infty})$ *such that*

$$\overline{q} = \pi q^{-1}, \quad \overline{\pi} = \pi, \quad \overline{\mathcal{E}_i} = \mathcal{E}_i, \quad \overline{\mathcal{F}_i} = \mathcal{F}_i, \quad \overline{\mathcal{K}_i} = \mathcal{J}^{|i|}\mathcal{K}_i^{-1}, \quad \overline{\mathcal{J}} = \mathcal{J}.$$

We call this automorphism the bar involution*.*

Proposition 19.13. *There is a unique bar involution* $\overline{}$ *on $L(\omega_f)$ (respectively $L^w(-\omega_f)$) such that $\overline{u \cdot v} = \overline{u} \cdot \overline{v}$ and $\overline{v_\emptyset} = v_\emptyset$ (respectively $\overline{v^\emptyset} = v^\emptyset$) for all $u \in U(B_{0|\infty})$, and $v \in L(\omega_f)$ (respectively $v \in L^w(-\omega_f)$).*

Proof. Note that the involution w commutes with the bar involution on $U(B_{0|\infty})$, thus the statement for $L^w(-\omega_f)$ follows from the one for $L(\omega_f)$.
The uniqueness is clear, as $v_\emptyset$ generates $L(\omega_f)$. We only have to show that the bar involution is well-defined.
Let I be the left ideal of $U(B_{0|\infty})$ generated by $\mathcal{E}_i$ for all i and $\mathcal{F}_i$ for $i \neq 0$ as well as $\mathcal{F}_0^2$, $\mathcal{J} - \pi$ and $\mathcal{K}_i - q^{\delta_{i,0}}$ for all i. Consider the quotient $U(B_{0|\infty})/I$. This is an integrable highest weight module of highest weight ω_f, thus we must have $L(\omega_f) \cong U(B_{0|\infty})/I$. It is easy to see that $\overline{I} = I$. Hence, the bar involution on $L(\omega_f)$ is well-defined. □

Using the quasi R-matrix from [CHW13, Theorem 3.11], we can extend these bar involutions to $L(\omega_f) \otimes L^w(-\omega_f)$.

Proposition 19.14. *There is a unique bar involution* $\overline{}$ *on $L(\omega_f) \otimes L^w(-\omega_f)$ such that $\overline{u \cdot v} = \overline{u} \cdot \overline{v}$ and $\overline{v_\emptyset \otimes v^\emptyset} = v_\emptyset \otimes v^\emptyset$ for all $u \in U(B_{0|\infty})$, and $v \in L(\omega_f) \otimes L^w(-\omega_f)$.*

Proof. We follow the arguments from [Lus10, §27.3.1]. The uniqueness is still clear, as $v_\emptyset \otimes v^\emptyset$ generates $L(\omega_f) \otimes L^w(-\omega_f)$. We only have to show that the bar involution is well-defined and an involution.
Recall the quasi R-matrix $\Theta \in (U(B_{0|\infty}) \otimes U(B_{0|\infty}))^\wedge$ from [CHW13, Theorem 3.11]. This is an element $\Theta = \sum_\nu \Theta_\nu$ with ν a positive linear combination of simple roots and $\Theta_\nu \in U(B_{0|\infty})^-_\nu \otimes U(B_{0|\infty})^+_\nu$. It satisfies $\Delta'(u)\Theta = \Theta\overline{\Delta'}(u)$ for all $u \in U(B_{0|\infty})$. Here, Δ' is another comultiplication of $U(B_{0|\infty})$ and $\overline{\Delta'}$ is given by precomposing Δ' with $\overline{}$ and postcomposing with $\overline{} \otimes \overline{}$.
To translate their result to our comultiplication, consider the $\mathbb{Q}$-superalgebra automorphism ι, which is defined by $\iota(q) = q^{-1}$, $\iota(\pi) = \pi$, $\iota(\mathcal{E}_i) = \mathcal{J}^{|i|}\mathcal{F}_i$, $\iota(\mathcal{F}_i) = \mathcal{E}_i$, $\iota(\mathcal{K}_i) = \mathcal{J}^{|i|}\mathcal{K}_i$ and $\iota(\mathcal{J}) = \mathcal{J}$. Then, $\Delta' \circ \iota = \iota \otimes \iota \circ \Delta$. It is also clear that ι commutes with the bar

involution. Hence, we obtain an element $\Theta' = \sum_\nu \Theta'_\nu := \iota \otimes \iota(\Theta) \in (U(B_{0|\infty}) \otimes U(B_{0|\infty}))^\wedge$ with $\Theta'_\nu \in U(B_{0|\infty})^+_\nu \otimes U(B_{0|\infty})^-_\nu$. This satisfies $\Delta(u)\Theta' = \Theta'\overline{\Delta}(u)$ for all $u \in U(B_{0|\infty})$. Now, define $\overline{v \otimes w} := \Theta'(\overline{v} \otimes \overline{w})$ for $v \in L(-\omega_f)$ and $w \in L^w(-\omega_f)$. The quasi-commutativity of Θ' with the comultiplication implies $\overline{u} \cdot \overline{v \otimes w} = \overline{u \cdot (v \otimes w)}$. By [CHW13, Corollary 3.1.3], we have $\Theta'\overline{\Theta'} = \overline{\Theta'}\Theta' = 1 \otimes 1$. So, we get $\overline{\overline{v \otimes w}} = \overline{\Theta'(\overline{v} \otimes \overline{w})} = \Theta'\overline{\Theta'}(\overline{\overline{v}} \otimes \overline{\overline{w}}) = v \otimes w$. □

Remark 19.15. In [CHW13] it was assumed that the set of simple roots is finite. However, looking carefully at the needed statements in the proof of [CHW13, Theorem 3.11], one sees that this is not necessary. Every calculation and claim takes place in a subalgebra generated by $\mathcal{J}$, $\mathcal{E}_i$, $\mathcal{F}_i$ and $\mathcal{K}_i$ for $i \leq N$ for some $N \gg 0$.

The next result is a standard argument.

Lemma 19.16. *There is a unique basis $\{b_\mu^\lambda \mid [\lambda, \mu] \in \mathbb{\Lambda}\}$ (called* canonical basis*) of $L(\omega_f) \otimes L^w(-\omega_f)$ such that*

(i) $\overline{b_\mu^\lambda} = b_\mu^\lambda$,

(ii) $b_\mu^\lambda = v_\mu \otimes v^\lambda + \sum_{[\lambda',\mu'] > [\lambda,\mu]} a_{\lambda,\mu}^{\lambda',\mu'} v_{\mu'} \otimes v^{\lambda'}$ *with* $a_{\lambda,\mu}^{\lambda',\mu'} \in q\mathbb{Z}^\pi[q]$.

Corollary 19.17. *Let $[\lambda, \mu] \in \mathbb{\Lambda}$ such that $\lambda_i \neq \mu_j$ for all i and j (whenever $\lambda_i \neq 0$ and $\mu_j \neq 0$). Then, $b_\mu^\lambda = v_\mu \otimes v^\lambda$.*

Proof. Note that $v_\mu \otimes v^\lambda$ is the minimal element in its weight space. Hence, it must be bar invariant and thus $b_\mu^\lambda = v_\mu \otimes v^\lambda$. □

The next theorem shows that $\mathfrak{U}(B_{0|\infty})^\Lambda$ categorifies $L(\omega_f) \otimes L^w(-\omega_f)$.

Theorem 19.18. *We have an isomorphism of $U(B_{0|\infty})$-modules*

$$\begin{aligned} \Phi\colon L(\omega_f) \otimes L^w(-\omega_f) &\to K_0(\mathfrak{U}(B_{0|\infty})^\Lambda), \\ v_\mu \otimes v^\lambda &\mapsto [\Delta([\lambda, \mu])]. \end{aligned}$$

Under this isomorphism, $[P([\lambda, \mu])]$ corresponds to the canonical basis element b_μ^λ of $L(\omega_f) \otimes L^w(-\omega_f)$.

Remark 19.19. This theorem should by all means fit in the framework of *tensor product categorification* as e.g. in [BLW17, LW15]. All the axioms of [BLW17, Definition 2.10] are satisfied, except for the ordering on the poset. Looking at the proof of Theorem 15.9, we can actually refine the partial ordering on $\mathbb{\Lambda}$. Namely, $\mathfrak{U}(B_{0|\infty})^\Lambda$ is also an upper-finite highest weight category, if we replace the partial ordering by $[\lambda, \mu] \leq [\lambda', \mu']$ if and only if

$$\mathrm{wt}([\lambda, \mu]) = \mathrm{wt}([\lambda', \mu']) \text{ and } \mathrm{wt}(\mu') - \mathrm{wt}(\mu) = \sum_{i=1}^{\infty} l_i \alpha_i \text{ with } l_i \geq 0.$$

The second condition is exactly the opposite of the ordering on $\mathbb{\Lambda}$ that is required in [BLW17, Definition 2.10].

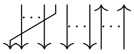
We believe that this is due to the fact that we used a comultiplication which is more in line with Kashiwara's convention [Kas91] than Lusztig's comultiplication [Lus10] (the latter being considered in [BLW17]).

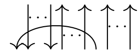
Proof of Theorem 19.18. The map is an isomorphism by Theorem 15.9, so it suffices to check that this respects the $U(B_{0|\infty})$-module structure. Also, note that this action preserves the weight spaces, so we only need to check the action of E and F.
From Theorem 15.8, one easily sees that $E_i\Delta([\lambda,\mu])$ has a filtration by $Q^m\Pi^a\Delta([\lambda',\mu])$ and $Q^n\Pi^b\Delta([\lambda,\mu'])$, where $[\lambda,\mu] \xrightarrow{i} [\lambda',\mu]$ and $[\lambda,\mu] \xleftarrow{i} [\lambda,\mu']$, respectively. The shifts are given by the degree and parity of the following diagrams.

The first diagram determines m and a and the second n and b. Ignoring the non-straight lines, this is the identity on $\mathrm{res}(\mathfrak{t}^{\lambda,\mu})$. From right to left, we iterate along first the rows of λ (upwards lines) and then the rows of μ (downward lines).
The first diagram corresponds to adding a box to λ of content i. In particular, all labels on the upward strands that cross the diagonal one must be distant to i. If λ has an addable box of content i, then every row after has at most $i-1$ boxes. In particular, any of these boxes has residue at most $i-2$. This means that every crossing has degree 0 and thus $m = 0$. The parity is determined by the number of crossings where both strands are labelled 0, so $a = 0$ if $i \neq 0$. If $i = 0$, then there are no crossing between upward strands, and we get $a = \ell(\mu)$ (every row in μ contributes one $\mathcal{F}_0$).
The second diagram corresponds to removing a box of content i from μ. The degree of the second diagram is $d_i(1 - \langle h_i, \kappa\rangle)$, where κ is the weight in the region directly to the right of the cap. Explicitly, $\kappa = \varepsilon_i + \sum_{r=1}^{s} \varepsilon_{\mu_r} - \sum_{r=1}^{\infty} \varepsilon_{\lambda_r}$, if a box in row $s+1$ of μ is removed (if $i = 0$, the ε_0 summand should be considered 0). As we have a removable box of content i, we must have $\langle h_i, \varepsilon_{\mu_r}\rangle = 0$ for all $r \leq s$. Also, $\langle h_i, \varepsilon_i\rangle = 1$ if $i \neq 0$ and 0 otherwise. So the degree is given by $d_i\langle h_i, \omega_f + \mathrm{wt}(\lambda)\rangle$. The parity of the diagram will always be even. So we get $n = d_i\langle h_i, \omega_f + \mathrm{wt}(\lambda)\rangle$ and $b = 0$.
On the other hand,

$$\begin{aligned}
\mathcal{E}_i(v_\mu \otimes v^\lambda) &= (\mathcal{E}_i \otimes \mathcal{K}_i^{-d_i} + J^{|i|} \otimes \mathcal{E}_i)(v_\mu \otimes v^\lambda)\\
&= \mathcal{E}_i v_\mu \otimes \mathcal{K}_i^{-d_i} v^\lambda + \pi^{|i||v_\mu|} J^{|i|} v_\mu \otimes \mathcal{E}_i v^\lambda\\
&= q^{d_i\langle h_i, \omega_f + \mathrm{wt}(\lambda)\rangle} v_{\mu'} \otimes v^\lambda + \pi^{|i|(|v_\mu|+1)} v_\mu \otimes v^{\lambda'}.
\end{aligned}$$

Now, recall that $|v_\mu| = 1 + \ell(\mu)$, and so both computation agree under the proposed isomorphism.
For F, a similar computation shows the compatibility of the action. Since it might be difficult to keep track of the grading and parity, we nevertheless include the argument here.
We have that $F_i\Delta([\lambda,\mu])$ has a filtration by $Q^m\Pi^a\Delta([\lambda',\mu])$ and $Q^n\Pi^b\Delta([\lambda,\mu'])$, where the bipartitions satisfy $[\lambda,\mu] \xleftarrow{i} [\lambda',\mu]$ respectively $[\lambda,\mu] \xrightarrow{i} [\lambda,\mu']$. The shifts are given by the degree and parity of the following diagrams.

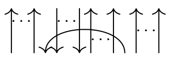




With the same argument as before, we obtain $m = 0$. We also have $a = 0$, as the parity can only possibly occur for $i = 0$, but in this case there are no crossings involved (a box of residue 0 will always be added in the last row).
For the second diagram note that the degree is given by $d_i(1+\langle h_i, \kappa\rangle + \langle h_i, \mathrm{wt}(\mu)\rangle)$. Again κ denotes the region directly to the right of the cap. The first summand corresponds to the degree of the cap, whereas the second summand describes the degree of the crossings. As before, we obtain $\kappa = -\varepsilon_i - \sum_{r=1}^{s} \varepsilon_{\lambda_r}$, if we remove a box in row $s+1$ of λ. Again we obtain $\langle h_i, \varepsilon_{\lambda_r}\rangle = 0$ for all $r \leq s$ and $\langle h_i, \varepsilon_i\rangle = 1$ if $i \neq 0$ and 0 otherwise. So in total we obtain $n = d_i(\langle h_i, \omega_f + \mathrm{wt}(\mu)\rangle)$. The diagram is even if $i \neq 0$ and for $i = 0$ the degree is given by $1 + \ell(\mu)$ (the leftward cap is odd and $\ell(\mu)$ describes the number of F_0).
Now we compare this again with $\mathcal{F}_i(v_\mu \otimes v^\lambda)$. We get

$$\begin{aligned}
\mathcal{F}_i(v_\mu \otimes v^\lambda) &= (\mathcal{F}_i \otimes 1 + \mathcal{K}_i^{d_i} \otimes \mathcal{F}_i)(v_\mu \otimes v^\lambda) \\
&= \mathcal{F}_i v_\mu \otimes v^\lambda + \pi^{|i||v_\mu|}\mathcal{K}_i^{d_i} v_\mu \otimes \mathcal{F}_i v^\lambda \\
&= v_{\mu'} \otimes v^\lambda + \pi^{|i|(1+\ell(\mu))} q^{d_i\langle h_i, \omega_f + \mathrm{wt}(\mu)\rangle} v_\mu \otimes v^{\lambda'}
\end{aligned}$$

and the two computations agree.
For the claim regarding the canonical basis, note first that $P([\emptyset, \emptyset]) = \Delta([\emptyset, \emptyset])$ and $b_\emptyset^\emptyset = v_\emptyset \otimes v^\emptyset$. Any other $P([\lambda, \mu])$ is obtained by applying E_i or F_i to $P([\lambda', \mu'])$ for some $[\lambda', \mu'] < [\lambda, \mu]$. From the definition it is thus immediate that $[P([\lambda, \mu])]$ is bar invariant. By Theorem 15.9, we know that $P([\lambda, \mu])$ has a filtration with section $Q^m \Pi^a \Delta([\lambda', \mu'])$ for some $[\lambda', \mu'] \leq [\lambda, \mu]$, $m \in \mathbb{Z}$ and $a \in \mathbb{Z}/2\mathbb{Z}$. The definition of the upper finite based quasi-hereditary structure implies that every $\Delta([\lambda', \mu'])$ appears at most once (for some m and a). If we can show that $\Delta([\lambda, \mu])$ appears without parity and grading shift and any other standard module only with positive grading shift, the statement follows from the uniqueness of the canonical basis.
The filtration by $\Delta([\lambda', \mu'])$ is obtained by applying caps to $P([\lambda, \mu])$. So to assert the positivity, we need to argue that the following diagram has positive degree:



We calculated above that the degree of this diagram is $d_i(\langle h_i, \omega_f + \nu\rangle)$, where ν denotes the weight in the region below the left end of the cap. As the bottom is the residue sequence for $[\lambda, \mu]$, we know that there is an F-addable box of content i right before the cap. But this implies that $\langle h_i, \omega_f + \nu\rangle \geq 1$. Clearly, $\Delta([\lambda, \mu])$ occurs exactly once with no grading shift (and no parity shift). So the statement follows. $\square$

Observe that $L(\omega_f) \otimes L^w(-\omega_f)$ is indecomposable but not irreducible. The next definition introduces certain subquotients.

Definition 19.20. Let $[\lambda_n, \emptyset] \in \Lambda\!\!\Lambda$ with $\lambda_n = (n, n-1, n-2, \cdots, 1)$ (this is the partition associated to ι_n from Definition 18.2). Let J_n be the subrepresentation of $L(\omega_f) \otimes L^w(-\omega_f)$ generated by $b_\emptyset^{\lambda_n}$.

The subrepresentations J_n give rise to a filtration of $L(\omega_f) \otimes L^w(-\omega_f)$

$$\cdots \subseteq J_2 \subseteq J_1 \subseteq J_0 = L(\omega_f) \otimes L^w(-\omega_f).$$

Theorem 19.21. *There is a filtration of $\mathfrak{U}(B_{0|\infty})^\Lambda$ by 2-subrepresentations*

$$\cdots \subseteq \mathfrak{U}(B_{0|\infty})_2^\Lambda \subseteq \mathfrak{U}(B_{0|\infty})_1^\Lambda \subseteq \mathfrak{U}(B_{0|\infty})_0^\Lambda = \mathfrak{U}(B_{0|\infty})^\Lambda$$

with $\mathfrak{U}(B_{0|\infty})_n^\Lambda / \mathfrak{U}(B_{0|\infty})_{n-1}^\Lambda \cong \operatorname{rep}'(\mathfrak{q}(n))$. The induced filtration on $K_0(\mathfrak{U}(B_{0|\infty})^\Lambda)$ turns Φ into a filtered isomorphism of filtered $U(B_{0|\infty})$-modules (with the filtration from Section 19.3 on $L(\omega_f) \otimes L(-\omega_f)$).

Proof. Using Theorem 17.8, we can translate all the statements to $2\mathbb{K}^Q$. The filtration is then given in Definition 18.2 and the identification of the quotients is given by Theorem 18.9. $\square$

In the following we will give a more explicit description of the subquotient J_n/J_{n+1}.

19.3.1. The natural representation and its exterior powers

For this part, we will mostly follow [Bru04]. However, we will adapt the notation to accommodate the covering quantum group. Specializing $q = 1$, we obtain his results (observe that his $\mathcal{K}_i$ is our $\mathcal{K}_i^{d_i}$). Observe that all proofs carry over verbatim to our setting.

Definition 19.22. The natural representation V of $U(B_{0|\infty})$ is the $\mathbb{Q}(q)^\pi$-superspace with basis $\{v_i \mid i \in \mathbb{Z}\}$ with $|v_i| = \delta_{i,0}$ and action given by

$$\begin{aligned} \mathcal{K}_i v_a &= q^{\langle h_i, \varepsilon_a \rangle} v_a & \mathcal{J} v_a &= v^\lambda, \\ \mathcal{F}_i v_a &= \delta_{a,i} v_{i+1} + \delta_{a,-i-1} v_{-i} & \mathcal{E}_i v_a &= \delta_{a,i+1} v_i + \delta_{a,-i} v_{-i-1} \\ \mathcal{F}_0 v_a &= \delta_{a,0} [2]_{q,\pi} v_1 + \delta_{a,-1} v_0 & \mathcal{E}_0 v_a &= \delta_{a,0} \pi [2]_{q,\pi} v_{-1} + \delta_{a,1} v_0, \end{aligned}$$

with the convention that $\varepsilon_0 = 0$ and $\varepsilon_{-k} = -\varepsilon_k$.

The following is clear from the definition.

Lemma 19.23. *The linear map $\overline{}$ defined by $\overline{v_a} = v_a$ is a bar involution on V, i.e. $\overline{u \cdot v} = \overline{u} \cdot \overline{v}$ for all $u \in U(B_{0|\infty})$, and $v \in V$.*

Definition 19.24. Write $\mathscr{T}$ for the tensor algebra of V and $\mathscr{T}^n$ for $V^{\otimes n}$. Let $\mathscr{F}$ be the quotient of $\mathscr{T}$ by the two-sided ideal generated by

$$\begin{gathered} v_a \otimes v_a \quad (a \neq 0) \\ v_a \otimes v_b + q^2 v_b \otimes v_a \quad (a > b,\ a \neq -b) \\ v_a \otimes v_{-a} + q^2 (v_{a-1} \otimes v_{1-a} + v_{1-a} \otimes v_{a-1}) + q^4 v_{-a} \otimes v_a \quad (a \geq 2) \\ v_1 \otimes v_{-1} + q v_0 \otimes v_0 + q^4 v_{-1} \otimes v_1, \end{gathered}$$

for all admissible $a, b \in \mathbb{Z}$. This is a homogeneous ideal and thus $\mathscr{F} = \bigoplus_{n \geq 0} \mathscr{F}^n$, with $\mathscr{F}^n$ a quotient of $V^{\otimes n}$. Moreover, the ideal is a $U(B_{0|\infty})$-submodule, and we obtain a $U(B_{0|\infty})$-module structure on $\mathscr{F}^n$.

$$\times\wedge \xrightarrow{\tilde{F}_i} \wedge\times \qquad \wedge\circ \xrightarrow{\tilde{F}_i} \circ\wedge \qquad \times\circ \xrightarrow{\tilde{F}_i} \vee\wedge \xrightarrow{\tilde{F}_i} \circ\times$$
$$\times\vee \xrightarrow{\tilde{F}_i} \vee\times \qquad \vee\circ \xrightarrow{\tilde{F}_i} \circ\vee \qquad |\circ \xrightarrow{\tilde{F}_0} |\wedge \xrightarrow{\tilde{F}_0} |\times$$

Figure 19.1.: The crystal strings of length > 0 for $\mathscr{F}$ (at positions i and $i+1$)

The elements $F_\lambda \coloneqq v_{\lambda_n} \wedge \cdots \wedge v_{\lambda_1}$ for $\lambda \in \Lambda_Q$ form a basis of $\mathscr{F}^n$.
Write $\hat{\mathscr{T}}^n$ for the completion of $\mathscr{T}^n$ with respect to the descending filtration $(\mathscr{T}^n_d)_{d\in\mathbb{Z}}$, where $\mathscr{T}^n_d$ is spanned by $v_{a_1} \otimes \cdots \otimes v_{a_n}$ with $\sum_{r=1}^n ra_r \geq d$. We denote the induced completion of $\mathscr{F}^n$ by $\hat{\mathscr{F}}^n$. The next two results are [Bru04, Lemma 3.4 and Theorem 3.5].

Lemma 19.25. *There is a continuous q-antilinear involution $\overline{}$ on $\hat{\mathscr{F}}^n$ satisfying*

(i) $\overline{uv} = \overline{u} \cdot \overline{v}$ *for all* $u \in U(B_{0|\infty})$ *and* $v \in \hat{\mathscr{F}}^n$,

(ii) $\overline{F_\lambda} \in F_\lambda + \sum_{\mu>\lambda} \mathbb{Z}^\pi[q, q^{-1}]F_\mu$ *for all* $\lambda \in \Lambda_Q$.

Theorem 19.26. *There exists a unique topological basis $\{U_\lambda \mid \lambda \in \Lambda_Q\}$ of $\hat{\mathscr{F}}^n$ such that $\overline{U_\lambda} = U_\lambda$ and $U_\lambda \in F_\lambda + \sum_{\mu>\lambda} q\mathbb{Z}^\pi[q]F_\mu$ for all $\lambda \in \Lambda_Q$.*

In [Bru04], the actions of $\mathcal{E}_i$ and $\mathcal{F}_i$ on U_λ were computed in some cases using the crystal structure obtained from the canonical basis.
We give the crystal structure in terms of the combinatorial weight diagrams from Definition 14.3, see Figure 19.1 for a list of all crystal strings of length > 0.
The following result is a general property of canonical basis, see e.g. [Kas93, Proposition 5.3.1].

Lemma 19.27. *Let $\lambda \in \Lambda_Q$ and $i \geq 0$.*

(i) $\mathcal{E}_i U_\lambda = [\varphi_i(\lambda) + 1]_{i,\pi} U_{\tilde{E}_i\lambda} + \sum_{\mu\in\Lambda_Q} y^i_{\mu,\lambda} U_\mu$, *where* $y^i_{\mu,\lambda} \in qq_i^{1-\varphi_i(\mu)}\mathbb{Z}^\pi[q]$ *is zero unless* $\varepsilon_j(\mu) \geq \varepsilon_j(\lambda)$ *for all* $j \geq 0$.

(ii) $\mathcal{F}_i U_\lambda = [\varepsilon_i(\lambda)+1]_{i,\pi} U_{\tilde{F}_i\lambda} + \sum_{\mu\in\Lambda_Q} z^i_{\mu,\lambda} U_\mu$, *where* $z^i_{\mu,\lambda} \in qq_i^{1-\varepsilon_i(\mu)}\mathbb{Z}^\pi[q]$ *is zero unless* $\varphi_j(\mu) \geq \varphi_j(\lambda)$ *for all* $j \geq 0$.

As every crystal string has length at most 2, we obtain the following (see also [Bru04, Corollary 3.25]).

Corollary 19.28. *Let $\lambda \in \Lambda_Q$ and $i \geq 0$.*

(i) *If* $\varepsilon_i(\lambda) > 0$, *then* $\mathcal{E}_i U_\lambda = [\varphi_i(\lambda) + 1]_{i,\pi} U_{\tilde{E}_i\lambda}$.

(ii) *If* $\varphi_i(\lambda) > 0$, *then* $\mathcal{F}_i U_\lambda = [\varepsilon_i(\lambda) + 1]_{i,\pi} U_{\tilde{F}_i\lambda}$.

Theorem 19.29. *We have an isomorphism of $U(B_{0|\infty})$-modules*

$$\Xi\colon J_n/J_{n+1} \to \mathscr{F}^n_\circ, \qquad b^\lambda_\mu \mapsto U_{\omega^\lambda_\mu}.$$

If ω is the weight diagram associated to $[\lambda, \mu]$, then ω_μ^λ is the combinatorial weight diagram obtained from ω by replacing every $\vee$ corresponding to a ray in $\underline{\omega}$ by $\wedge$.
Here, $\mathscr{F}_\circ^n$ is the subrepresentation of $\hat{\mathscr{F}}^n$ with basis U_λ for $\lambda \in \Lambda_Q$ (i.e. not topological).

Proof. The proposed map is clearly a linear isomorphism, so we only have to check that it is compatible with the action of $U(B_{0|\infty})$. The arguments for $\mathcal{F}_i$ are similar to the ones for $\mathcal{E}_i$, so we only give the proof for $\mathcal{E}_i$. By Theorem 16.38, we know how b_μ^λ behaves under $\mathcal{E}_i$. Abbreviate ω_μ^λ by ω. From the explicit crystal strings in Figure 19.1 and Corollary 19.28, we see that $\mathcal{E}_i U_\omega = \Xi(\mathcal{E}_i b_\mu^\lambda)$ whenever $\varepsilon_i(\omega) > 0$.
The case $\varepsilon_i(\omega) = 0$ remains to check. We also may assume that the weight space of $\mathcal{E}_i U_\omega$ is non-zero, as the statement is otherwise trivial. This leaves the following cases for positions i and $i+1$ of the combinatorial weight diagram ω. We either have $\vee\vee$, $\wedge\vee$ or $\wedge\wedge$ if $i > 0$ and $|\vee$ if $i = 0$. We check these case by case.
Suppose it is $\vee\vee$. Looking at the cup diagram associated to ω, we see that the $\vee$'s are left endpoints of cups. Consider the subsequence $\vee\vee\wedge\wedge$ that is given by the $\vee$'s and the right endpoints of the corresponding cups. We claim that $\mathcal{E}_i U_\omega = U_{\omega'}$, where ω' is obtained by replacing this subsequence by $\times \circ \vee\wedge$. By applying distant $\mathcal{E}_j$'s and $\mathcal{F}_j$'s, we may assume that these are actually neighbored.
Let ω_1 be the weight diagram obtained by replacing the (now neighbored) subsequence $\vee\vee\wedge\wedge$ by $\times \vee \circ\wedge$. Then, $\mathcal{F}_{i+1}\mathcal{F}_i U_{\omega_1} = U_\omega$ by Corollary 19.28. We compute using Corollary 19.28 that

$$\mathcal{E}_i U_\omega = \mathcal{E}_i \mathcal{F}_{i+1} \mathcal{F}_i \mathcal{E}_{i+1} U_{\omega_1} = \mathcal{F}_{i+1} \mathcal{E}_i \mathcal{F}_i U_{\omega_1} = \mathcal{F}_{i+1} U_{\omega_1} = U_{\omega'}.$$

For the case $\wedge\vee$, note that $\vee$ is always the right endpoints of a cup. With the same reduction argument as above, we may assume that the next entry is $\wedge$. Let ω_1 be obtained from ω by replacing $\wedge \vee \wedge$ by $\wedge \times \circ$. Then, $\mathcal{F}_{i+1} U_{\omega_1} = U_\omega$. We compute

$$\mathcal{E}_i U_\omega = \mathcal{E}_i \mathcal{F}_{i+1} U_{\omega_1} = \mathcal{F}_{i+1} \mathcal{E}_i U_{\omega_1} = U_{\omega'},$$

where ω' is obtained from ω by replacing $\wedge \vee \wedge$ with $\times \circ \wedge$.
For $\wedge\wedge$, we have two cases. Either the first $\wedge$ is a right endpoint of a cap or not. Suppose that it is not the endpoint of a cap. We may assume that the combinatorial weight diagram contains no $\vee$ (otherwise use $\mathcal{E}_j$ and $\mathcal{F}_j$ distant to i to remove them). Then, the U_ω has weight $\sum_{r=1}^\infty a_r \varepsilon_r$, where exactly n entries a_r are non-zero but a_i and a_{i+1} are zero. In particular, we see that $\mathcal{E}_i U_\omega = 0$ as the corresponding weight space is zero. So assume now that the first $\wedge$ is the right endpoints of a cap. Again we may assume that we have the subsequence $\vee \wedge \wedge$. Let ω_1 and ω' be obtained by replacing $\vee \wedge \wedge$ with $\times \circ \wedge$ and $\wedge \times \circ$, respectively. Then, $\mathcal{F}_{i-1} U_{\omega_1} = U_\omega$ and we compute

$$\mathcal{E}_i U_\omega = \mathcal{E}_i \mathcal{F}_{i-1} U_{\omega_1} = \mathcal{F}_{i-1} \mathcal{E}_i U_{\omega_1} = U_{\omega'}.$$

Finally, it remains to check the case $i = 0$. We must have $|\vee$, and we may assume that the next entry is $\wedge$. Let ω_1 and ω' be obtained by replacing $| \vee \wedge$ by $| \times \circ$ and $| \circ \wedge$ respectively. Then, $\mathcal{F}_1 U_{\omega_1} = U_\omega$ and we compute

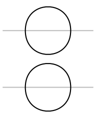
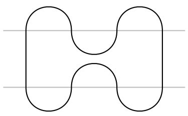
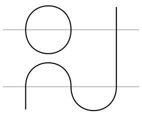
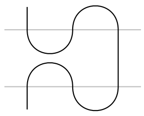
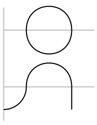
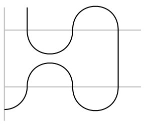
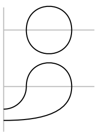
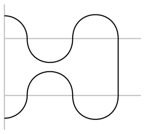
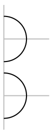
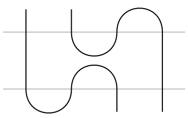
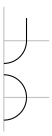
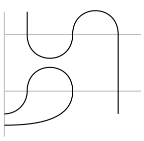
$$\mathcal{E}0 U_\omega = \mathcal{E}0 \mathcal{F}_1 U_{\omega_1} = \mathcal{F}_1 \mathcal{E}0 U_{\omega_1} = U_{\omega'}.$$

Note that in all the computed cases this is exactly the same as the action of $\mathcal{E}_i$ on b_μ^λ. Thus, Ξ is an isomorphism of $U(B_{0|\infty})$-modules. $\square$

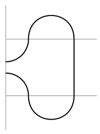
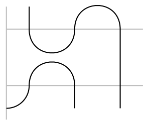
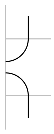
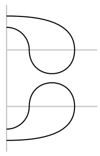
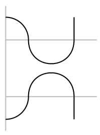
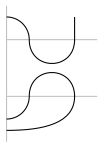
Remark 19.30. Combining Theorem 19.21 with Theorem 19.29, we obtain a new proof of the connection between $K_0(\mathrm{rep}(\mathfrak{q}(n))$ and $\mathscr{F}^n$ that was already observed in [Bru04]. In [Bru04], the standard basis of $\hat{\mathscr{F}}^n$ corresponds to certain classes in the Grothendieck group and a completion was needed to write the classes of projective modules in terms of the standard basis. In Remark 18.11, we have seen that $\mathfrak{q}(n)$ do not admit any kind of nice highest weight structure (except for the typical blocks). Hence, there should exist no "nice" standard basis for $K_0(\mathrm{rep}(\mathfrak{q}(n)))$. We remedy this problem by considering a bigger category $\mathfrak{U}(B_{0|\infty})^\Lambda$. This is an upper-finite highest weight category and thus admits a nice standard basis. Then, $\mathrm{rep}(\mathfrak{q}(n))$ is obtained via a relatively simple subquotient.

A. Explicit surgery procedures

Here, we describe all surgery procedures (over $\mathbb{F}_2$). This allows computing the multiplication without closing the diagrams first. We use the notation 1, x, y for anticlockwise, clockwise circles and lines, respectively. We put a bar on top if these correspond to inner components and put no decoration for outer ones.

Surgery	Orientation	Result	Surgery	Orientation	Result
	$1 \otimes 1$ $1 \otimes x$ $x \otimes 1$ $1 \otimes x$	1 x x 0		1 x	$1 \otimes x + x \otimes 1$ $x \otimes x$
	$1 \otimes y$ $x \otimes y$	y 0		y	$y \otimes x$
	$1 \otimes \bar{y}$ $x \otimes \bar{y}$	$\bar{y}$ 0		$\bar{y}$	$\bar{y} \otimes x$
	$1 \otimes \bar{1}$ $1 \otimes \bar{x}$ $x \otimes \bar{1}$ $x \otimes \bar{x}$	$\bar{1}$ x 0 0		$\bar{1}$ $\bar{x}$	$\bar{1} \otimes x$ $\bar{x} \otimes x$
	$\bar{1} \otimes \bar{1}$ $\bar{1} \otimes \bar{x}$ $\bar{x} \otimes \bar{1}$ $\bar{x} \otimes \bar{x}$	$\bar{1}$ $\bar{x}$ $\bar{x}$ 0		$y \otimes y$	0
	$\bar{y} \otimes \bar{1}$ $\bar{y} \otimes \bar{x}$	$\bar{y}$ 0		$y \otimes \bar{1}$ $y \otimes \bar{x}$	$\bar{y} \otimes \bar{y}$ 0

Continued on next page

Surgery	Orientation	Result	Surgery	Orientation	Result
	$\bar{1}$ $\bar{x}$	0 x		$y \otimes \bar{y}$	0
	$\bar{y} \otimes \bar{y}$	0		$\bar{1} \otimes \bar{1}$ $\bar{1} \otimes \bar{x}$ $\bar{x} \otimes \bar{1}$ $\bar{x} \otimes \bar{x}$	$\bar{1} \otimes \bar{x} + \bar{x} \otimes \bar{1}$ $\bar{x} \otimes \bar{x}$ $\bar{x} \otimes \bar{x}$ 0
	$y \otimes \bar{y}$	0		$\bar{y} \otimes \bar{1}$ $\bar{y} \otimes \bar{x}$	$\bar{y} \otimes \bar{x}$ 0

Bibliography

[AMR06] S. Ariki, A. Mathas, and H. Rui, *Cyclotomic Nazarov–Wenzl Algebras*, Nagoya Math. J. **182** (2006), 47–134.

[Ari02] S. Ariki, *Representations of quantum algebras and combinatorics of Young tableaux*, University Lecture Series, vol. 26, AMS, Providence, RI, 2002.

[BCK19] J. Brundan, J. Comes, and J. R. Kujawa, *A basis theorem for the degenerate affine oriented Brauer–Clifford supercategory*, Canad. J. Math. **71** (2019), no. 5, 1061–1101.

[BD17] J. Brundan and N. Davidson, *Categorical actions and crystals*, Categorification and higher representation theory, 2017, pp. 105–147.

[BDEA$^+$19] M. Balagović, Z. Daugherty, I. Entova-Aizenbud, I. Halacheva, J. Hennig, M. S. Im, G. Letzter, E. Norton, V. Serganova, and Catharina, *Translation functors and decomposition numbers for the periplectic Lie superalgebra* $\mathfrak{p}(n)$, Math. Res. Lett. **26** (2019), no. 3, 643–710.

[BDEA$^+$20] ———, *The affine VW supercategory*, Selecta Math. (N.S.) **26** (2020), no. 2.

[BE17a] J. Brundan and A. P. Ellis, *Monoidal supercategories*, Comm. Math. Phys. **351** (2017), no. 3, 1045–1089.

[BE17b] ———, *Super Kac–Moody 2-categories*, Proc. Lond. Math. Soc. (3) **115** (2017), no. 5, 925–973.

[BGG24] A. Berenstein, A. Gainutdinov, and V. Gorbounov, *Generalised electrical Lie algebras* (2024), available at arXiv:2405.02956.

[BGS96] A. Beilinson, V. Ginzburg, and W. Soergel, *Koszul Duality Patterns in Representation Theory*, J. Amer. Math. Soc. **9** (1996), no. 2, 473–527.

[BK09] J. Brundan and A. Kleshchev, *Blocks of cyclotomic Hecke algebras and Khovanov–Lauda algebras*, Invent. Math. **178** (2009), no. 3, 451–484.

[BKM98] G. Benkart, S.-J. Kang, and D. Melville, *Quantized enveloping algebras for Borcherds superalgebras*, Trans. Amer. Math. Soc. **350** (1998), no. 8, 3297–3319.

[BKN11] B. D. Boe, J. R. Kujawa, and D. K. Nakano, *Complexity and Module Varieties for Classical Lie Superalgebras*, Int. Math. Res. Not. IMRN **3** (2011), 696–724.

[BL80] I. N. Bernšteĭn and D. A. Leĭtes, *A formula for the characters of the irreducible finite-dimensional representations of Lie superalgebras of series* Gl *and* SL, C. R. Acad. Bulgare Sci. **33** (1980), no. 8, 1049–1051.

[BLW17] J. Brundan, I. Losev, and B. Webster, *Tensor product categorifications and the super Kazhdan–Lusztig conjecture*, Int. Math. Res. Not. IMRN **20** (2017), 6329–6410.

[BR87] A. Berele and A. Regev, *Hook Young diagrams with applications to combinatorics and to representations of Lie superalgebras*, Adv. in Math. **64** (1987), no. 2, 118–175.

[Bra37] R. Brauer, *On algebras which are connected with the semisimple continuous groups*, Ann. of Math. (2) **38** (1937), no. 4, 857–872.

[Bru03] J. Brundan, *Kazhdan–Lusztig polynomials and character formulae for the Lie superalgebra* $\mathfrak{gl}(m|n)$, J. Amer. Math. Soc. **16** (2003), no. 1, 185–231.

[Bru04] ———, *Kazhdan–Lusztig polynomials and character formulae for the Lie superalgebra* $\mathfrak{q}(n)$, Adv. Math. **182** (2004), no. 1, 28–77.

[Bru25] ———, *Graded Triangular Bases*, Algebr. Represent. Theory **28** (2025), no. 1, 239–280.

[Bru] J. Brundan, *Isomeric Heisenberg and Kac–Moody categorification*, Representations of Supergroups (Bonn, Germany, July 5, 2024).

[BS10] J. Brundan and C. Stroppel, *Highest Weight Categories Arising from Khovanov's Diagram Algebra. II. Koszulity*, Transform. Groups **15** (2010), no. 1, 1–45.

[BS11a] ———, *Highest Weight Categories Arising from Khovanov's Diagram Algebra I: Cellularity*, Mosc. Math. J. **11** (2011), no. 4, 685–722.

[BS11b] ———, *Highest Weight Categories Arising from Khovanov's Diagram Algebra III: Category* $\mathcal{O}$, Represent. Theory **15** (2011), no. 5, 170–243.

[BS12a] ———, *Gradings on walled Brauer algebras and Khovanov's arc algebra*, Adv. Math. **231** (2012), no. 2, 709–773.

[BS12b] ———, *Highest Weight Categories Arising from Khovanov's Diagram Algebra IV: the General Linear Supergroup*, J. Eur. Math. Soc. (JEMS) **14** (2012), no. 2, 373–419.

[BS24] ———, *Semi-infinite highest weight categories*, Mem. Amer. Math. Soc. **293** (2024), no. 1459, vii+152.

[CC20] C.-W. Chen and K. Coulembier, *The primitive spectrum and category* $\mathcal{O}$ *for the periplectic Lie superalgebra*, Canad. J. Math. **72** (2020), no. 3, 625–655.

[CE18] K. Coulembier and M. Ehrig, *The periplectic Brauer algebra II: Decomposition multiplicities*, J. Comb. Algebra **2** (2018), no. 1, 19–46.

[CE21] ———, *The periplectic Brauer algebra III: The Deligne category*, Algebr. Represent. Theory **24** (2021), no. 4, 993–1027.

[CFLW14] S. Clark, Z. Fan, Y. Li, and W. Wang, *Quantum supergroups III. Twistors*, Comm. Math. Phys. **332** (2014), no. 1, 415–436.

[CH17] J. Comes and T. Heidersdorf, *Thick ideals in Deligne's category* $\underline{\mathrm{Rep}}(O_\delta)$, J. Algebra **480** (2017), 237–265.

[Che15] C.-W. Chen, *Finite-dimensional representations of periplectic Lie superalgebras*, J. Algebra **443** (2015), 99–125.

[CHW13] S. Clark, D. Hill, and W. Wang, *Quantum supergroups I. Foundations*, Transform. Groups **18** (2013), no. 4, 1019–1053.

[CHW14] ———, *Quantum supergroups II. Canonical basis*, Represent. Theory **18** (2014), 278–309.

[CIM98] E. B. Curtis, D. Ingerman, and J. A. Morrow, *Circular planar graphs and resistor networks*, Linear Algebra Appl. **283** (1998), no. 1-3, 115–150.

[CLW11] S.-J. Cheng, N. Lam, and W. Wang, *Super duality and irreducible characters of ortho-symplectic Lie superalgebras*, Invent. Math. **183** (2011), no. 1, 189–224.

[CNS75] L. Corwin, Y. Ne'eman, and S. Sternberg, *Graded Lie algebras in mathematics and physics (Bose-Fermi symmetry)*, Rev. Modern Phys. **47** (1975), 573–603.

[Cou18a] K. Coulembier, *The periplectic Brauer algebra*, Proc. Lond. Math. Soc. (3) **117** (2018), no. 3, 441–482.

[Cou18b] ———, *Tensor ideals, Deligne categories and invariant theory*, Selecta Math. (N.S.) **24** (2018), no. 5, 4659–4710.

[CP24] C.-W. Chen and Y.-N. Peng, *Parabolic category for periplectic Lie superalgebras*, Transform. Groups **29** (2024), no. 1, 17–45.

[CW08] S.-J. Cheng and W. Wang, *Brundan–Kazhdan–Lusztig and super duality conjectures*, Publ. Res. Inst. Math. Sci. **44** (2008), no. 4, 1219–1272.

[CW12a] ———, *Dualities and Representations of Lie Superalgebras*, Graduate Studies in Mathematics, vol. 144, American Mathematical Society, Providence, RI, 2012.

[CW12b] J. Comes and B. Wilson, *Deligne's category* $\underline{\mathrm{Rep}}(GL_\delta)$ *and representations of general linear supergroups*, Represent. Theory **16** (2012), 568–609.

[Del07] P. Deligne, *La catégorie des représentations du groupe symétrique S_t, lorsque t n'est pas un entier naturel*, Algebraic groups and homogeneous spaces, 2007, pp. 209–273.

[Djo76] D. Ž. Djoković, *Classification of some* 2*-graded Lie algebras*, J. Pure Appl. Algebra **7** (1976), no. 2, 217–230.

[DLZ18] P. Deligne, G. I. Lehrer, and R. B. Zhang, *The first fundamental theorem of invariant theory for the orthosymplectic super group*, Adv. Math. **327** (2018), 4–24.

[DM99] P. Deligne and J. W. Morgan, *Notes on supersymmetry (following Joseph Bernstein)*, Quantum fields and strings: a course for mathematicians, Vol. 1, 2 (Princeton, NJ, 1996/1997), 1999, pp. 41–97.

[EAS21] I. Entova-Aizenbud and V. Serganova, *Deligne categories and the periplectic Lie superalgebra*, Mosc. Math. J. **21** (2021), no. 3, 507–565.

[EKL14] A. P. Ellis, M. Khovanov, and A. D. Lauda, *The odd nilHecke algebra and its diagrammatics*, Int. Math. Res. Not. IMRN **4** (2014), 991–1062.

[ES16] M. Ehrig and C. Stroppel, *Diagrammatic description for the categories of perverse sheaves on isotropic Grassmannians*, Selecta Math. (N.S.) **22** (2016), no. 3, 1455–1536.

[ES17] ———, *On the category of finite-dimensional representations of* $\mathrm{OSp}(r|2n)$*: Part I*, Representation theory—current trends and perspectives, 2017, pp. 109–170.

[ES21] ———, *Deligne categories and representations of* $\mathrm{OSp}(r|2n)$ (2021), `http://www.math.uni-bonn.de/ag/stroppel/OSPII.pdf`.

[FK76] P. G. O. Freund and I. Kaplansky, *Simple supersymmetries*, J. Mathematical Phys. **17** (1976), no. 2, 228–231.

[FLP23] J. Flake, R. Laugwitz, and S. Posur, *Indecomposable objects in Khovanov–Sazdanovic's generalizations of Deligne's interpolation categories*, Adv. Math. **415** (2023), Paper No. 108892, 70.

[FM09] A. Frisk and V. Mazorchuk, *Regular strongly typical blocks of $\mathcal{O}^{\mathfrak{q}}$*, Comm. Math. Phys. **291** (2009), no. 2, 533–542.

[Fri07] A. Frisk, *Typical blocks of the category $\mathcal{O}$ for the queer Lie superalgebra*, J. Algebra Appl. **6** (2007), no. 5, 731–778.

[Geo24] T. George, *The twist for electrical networks and the inverse problem*, Int. Math. Res. Not. IMRN **8** (2024), 7001–7031.

[Ger63] M. Gerstenhaber, *The cohomology structure of an associative ring*, Ann. of Math. (2) **78** (1963), 267–288.

[Ger98] J. Germoni, *Indecomposable Representations of Special Linear Lie Superalgebras*, J. Algebra **209** (1998), no. 2, 367–401.

[Gin05] V. Ginzburg, *Lectures on noncommutative geometry* (2005), available at arXiv:math/0506603.

[GQS07] G. Götz, T. Quella, and V. Schomerus, *Representation theory of $\mathfrak{sl}(2|1)$*, J. Algebra **312** (2007), no. 2, 829–848.

[GRS24] M. Gao, H. Rui, and L. Song, *Representations of the cyclotomic oriented Brauer–Clifford supercategory*, J. Pure Appl. Algebra **228** (2024), no. 12, Paper No. 107767, 37.

[GS10] C. Gruson and V. Serganova, *Cohomology of generalized supergrassmannians and character formulae for basic classical Lie superalgebras*, Proc. Lond. Math. Soc. (3) **101** (2010), no. 3, 852–892.

[GS16] R. Garner and M. Shulman, *Enriched categories as a free cocompletion*, Adv. Math. **289** (2016), 1–94.

[GS20] N. Grantcharov and V. Serganova, *Extension quiver for Lie superalgebra $\mathfrak{q}(3)$*, SIGMA Symmetry Integrability Geom. Methods Appl. **16** (2020), Paper No. 141, 32.

[Hei17] T. Heidersdorf, *Mixed tensors of the general linear supergroup*, J. Algebra **491** (2017), 402–446.

[HKS11] D. Hill, J. R. Kujawa, and J. Sussan, *Degenerate affine Hecke–Clifford algebras and type Q Lie superalgebras*, Math. Z. **268** (2011), no. 3-4, 1091–1158.

[HNS24] T. Heidersdorf, J. Nehme, and C. Stroppel, *Khovanov algebras of type B and tensor powers of the natural* OSp*-representation* (2024), available at arXiv:2405.11981.

[Jan96] J.-C. Jantzen, *Lectures on quantum groups*, Graduate Studies in Mathematics, vol. 6, American Mathematical Society, Providence, RI, 1996.

[JK14] J. H. Jung and S.-J. Kang, *Mixed Schur–Weyl–Sergeev duality for queer Lie superalgebras*, J. Algebra **399** (2014), 516–545.

[Jos95] A. Joseph, *Quantum groups and their primitive ideals*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 29, Springer-Verlag, Berlin, 1995.

[Kac77a] V. G. Kac, *Characters of typical representations of classical Lie superalgebras*, Comm. Algebra **5** (1977), no. 8, 889–897.

[Kac77b] V. G. Kac, *Lie superalgebras*, Adv. Math. **26** (1977), no. 1, 8–96.

[Kas91] M. Kashiwara, *On crystal bases of the Q-analogue of universal enveloping algebras*, Duke Math. J. **63** (1991), no. 2, 465–516.

[Kas93] M. Kashiwara, *Global crystal bases of quantum groups*, Duke Math. J. **69** (1993), no. 2, 455–485.

[Kas95] C. Kassel, *Quantum groups*, Graduate Texts in Mathematics, vol. 155, Springer-Verlag, New York, 1995.

[Kel05] G. M. Kelly, *Basic concepts of enriched category theory*, Repr. Theory Appl. Categ. **10** (2005), vi+137.

[Ken99] A. E. Kennelly, *The equivalence of triangles and three-pointed stars in conducting networks*, Electrical World and Engineer **34** (1899), no. 21, 413–414.

[Kho00] M. Khovanov, *A categorification of the Jones polynomial*, Duke Math. J. **101** (2000), no. 3, 359–426.

[Kho02] ———, *A functor-valued invariant of tangles*, Algebr. Geom. Topol. **2** (2002), 665–741.

[KK12] S.-J. Kang and M. Kashiwara, *Categorification of highest weight modules via Khovanov–Lauda–Rouquier algebras*, Invent. Math. **190** (2012), no. 3, 699–742.

[KKO13] S.-J. Kang, M. Kashiwara, and S.-j. Oh, *Supercategorification of quantum Kac–Moody algebras*, Adv. Math. **242** (2013), 116–162.

[KKO14] ———, *Supercategorification of quantum Kac–Moody algebras II*, Adv. Math. **265** (2014), 169–240.

[KKT16] S.-J. Kang, M. Kashiwara, and S. Tsuchioka, *Quiver Hecke superalgebras*, J. Reine Angew. Math. **711** (2016), 1–54.

[KL09] M. Khovanov and A. Lauda, *A diagrammatic approach to categorification of quantum groups1409.1195 I*, Represent. Theory **13** (2009), no. 14, 309–347.

[KL10] M. Khovanov and A. D. Lauda, *A categorification of quantum* $\mathrm{sl}(n)$, Quantum Topol. **1** (2010), no. 1, 1–92.

[Koi89] K. Koike, *On the decomposition of tensor products of the representations of the classical groups: by means of the universal characters*, Adv. Math. **74** (1989), no. 1, 57–86.

[KT17] J. R. Kujawa and B. C. Tharp, *The marked Brauer category*, J. Lond. Math. Soc. (2) **95** (2017), no. 2, 393–413.

[Lam24] T. Lam, *An invitation to positive geometries*, Open problems in algebraic combinatorics, 2024, pp. 159–179.

[Lau10] A. D. Lauda, *A categorification of quantum* $\mathrm{sl}(2)$, Adv. Math. **225** (2010), no. 6, 3327–3424.

[Li14] G. Li, *A KLR Grading of the Brauer Algebras* (2014), available at arXiv:1409.1195.

[LP15] T. Lam and P. Pylyavskyy, *Electrical networks and Lie theory*, Algebra Number Theory **9** (2015), no. 6, 1401–1418.

[LT96] B. Leclerc and J.-Y. Thibon, *Canonical bases of q-deformed Fock spaces*, Internat. Math. Res. Notices **9** (1996), 447–456.

[Lus10] G. Lusztig, *Introduction to quantum groups*, Modern Birkhäuser Classics, Birkhäuser/Springer, New York, 2010. Reprint of the 1994 edition.

[LW15] I. Losev and B. Webster, *On uniqueness of tensor products of irreducible categorifications*, Selecta Math. (N.S.) **21** (2015), no. 2, 345–377.

[Mat15] A. Mathas, *Cyclotomic quiver Hecke algebras of type A*, Modular representation theory of finite and p-adic groups, 2015, pp. 165–266.

[MM12] V. Mazorchuk and V. Miemietz, *Serre functors for Lie algebras and superalgebras*, Ann. Inst. Fourier (Grenoble) **62** (2012), no. 1, 47–75.

[Moo03] D. Moon, *Tensor product representations of the Lie superalgebra $\mathfrak{p}(n)$ and their centralizers*, Communications in Algebra **31** (2003), no. 5, 2095–2140.

[MOS09] V. Mazorchuk, S. Ovsienko, and C. Stroppel, *Quadratic duals, Koszul dual functors, and applications*, Trans. Amer. Math. Soc. **361** (2009), no. 3, 1129–1172.

[Mus12] I. Musson, *Lie Superalgebras and Enveloping Algebras*, Graduate Studies in Mathematics, vol. 131, American Mathematical Society, Providence, RI, 2012.

[Naz97] M. Nazarov, *Young's symmetrizers for projective representations of the symmetric group*, Adv. Math. **127** (1997), no. 2, 190–257.

[Neh24] J. Nehme, *Khovanov algebras for the periplectic lie superalgebras*, Int. Math. Res. Notices **2024** (2024), no. 22, 14008–14060.

[NRS76] W. Nahm, V. Rittenberg, and M. Scheunert, *The classification of graded Lie algebras*, Phys. Lett. B **61** (1976), no. 4, 383–384.

[NS25] J. Nehme and C. Stroppel, *The quantum ϵlectrical Hopf algebra and categorification of Fock space* (2025), available at arXiv:2504.17952.

[ORS13] P. S. Ozsváth, J. Rasmussen, and Z. Szabó, *Odd Khovanov homology*, Algebr. Geom. Topol. **13** (2013), no. 3, 1465–1488.

[Pen86] I. B. Penkov, *Characters of typical irreducible finite-dimensional $\mathfrak{q}(n)$-modules*, Funktsional. Anal. i Prilozhen. **20** (1986), no. 1, 37–45, 96.

[PR75] A. Pais and V. Rittenberg, *Semisimple graded Lie algebras*, J. Mathematical Phys. **16** (1975), no. 10, 2062–2073.

[PS97] I. Penkov and V. Serganova, *Characters of irreducible G-modules and cohomology of G/P for the Lie supergroup $G = Q(N)$*, 1997, pp. 1382–1412. Algebraic geometry, 7.

[Rei01] M. Reineke, *Generic extensions and multiplicative bases of quantum groups at $q = 0$*, Represent. Theory **5** (2001), 147–163.

[Rin96] C. M. Ringel, *PBW-bases of quantum groups*, J. Reine Angew. Math. **470** (1996), 51–88.

[Rou08] R. Rouquier, *2-Kac-Moody algebras* (2008), available at arXiv:0812.5023.

[RS82] V. Rittenberg and M. Scheunert, *A remarkable connection between the representations of the Lie superalgebras* $\mathrm{osp}(1, 2n)$ *and the Lie algebras* $\mathrm{o}(2n+1)$, Comm. Math. Phys. **83** (1982), no. 1, 1–9.

[RT90] N. Yu. Reshetikhin and V. G. Turaev, *Ribbon graphs and their invariants derived from quantum groups*, Comm. Math. Phys. **127** (1990), no. 1, 1–26.

[Ser17] V. Serganova, *Representations of Lie Superalgebras*, Perspectives in Lie Theory, 2017, pp. 125–177.

[Ser83] A. N. Sergeev, *The centre of enveloping algebra for Lie superalgebra* $Q(n, \mathbf{C})$, Lett. Math. Phys. **7** (1983), no. 3, 177–179.

[Ser84] ______, *Tensor algebra of the identity representation as a module over the Lie superalgebras* $\mathrm{Gl}(n, m)$ *and* $Q(n)$, Mat. Sb. (N.S.) **123(165)** (1984), no. 3, 422–430.

[Ser96] V. Serganova, *Kazhdan–Lusztig polynomials and character formula for the Lie superalgebra* $\mathfrak{gl}(m|n)$, Selecta Math. (N.S.) **2** (1996), no. 4, 607–651.

[Shu11] A. N. Shumakovitch, *Patterns in odd Khovanov homology*, J. Knot Theory Ramifications **20** (2011), no. 1, 203–222.

[SNR76] M. Scheunert, W. Nahm, and V. Rittenberg, *Classification of all simple graded Lie algebras whose Lie algebra is reductive. I*, J. Mathematical Phys. **17** (1976), no. 9, 1626–1639.

[Str03] C. Stroppel, *Category* $\mathcal{O}$*: gradings and translation functors*, J. Algebra **268** (2003), no. 1, 301–326.

[Su14] Y. Su, *Electrical Lie Algebra of Classical Types* (2014), available at arXiv:1410.1188.

[SW19] C. Stroppel and A. Wilbert, *Two-block Springer fibers of types C and D: a diagrammatic approach to Springer theory*, Math. Z. **292** (2019), no. 3-4, 1387–1430.

[Tur89] V. G. Turaev, *Operator invariants of tangles, and R-matrices*, Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), no. 5, 1073–1107, 1135.

[Ugl00] D. Uglov, *Canonical bases of higher-level q-deformed Fock spaces and Kazhdan–Lusztig polynomials*, Physical combinatorics (Kyoto, 1999), 2000, pp. 249–299.

[VdJ91] J. Van der Jeugt, *Character formulae for the Lie superalgebra* $C(n)$, Comm. Algebra **19** (1991), no. 1, 199–222.

[VHKTM90] J. Van der Jeugt, J. W. B. Hughes, R. C1. King, and J. Thierry-Mieg, *A character formula for singly atypical modules of the Lie superalgebra* $\mathrm{sl}(m/n)$, Comm. Algebra **18** (1990), no. 10, 3453–3480.

[Web17] B. Webster, *Knot invariants and higher representation theory*, Mem. Amer. Math. Soc. **250** (2017), no. 1191, v+141.

[Web24] ______, *Unfurling Khovanov–Lauda–Rouquier algebras* (2024), available at arXiv:1603.06311.

[Xia97] J. Xiao, *Drinfeld double and Ringel–Green theory of Hall algebras*, J. Algebra **190** (1997), no. 1, 100–144.