Differentiation of higher groupoid objects in tangent categories

Dissertation

zur

Erlangung des Doktorgrades (Dr. rer. nat.)

der

Mathematisch-Naturwissenschaftlichen Fakultät

der

Rheinischen Friedrich-Wilhelms-Universität Bonn

vorgelegt von

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Bonn, November 2024

Angefertigt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät
der Rheinischen Friedrich-Wilhelms-Universität Bonn

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Tag der Promotion: 26.05.2025

Erscheinungsjahr: 2025

For Arek

$$\mathrm{Lie}(G) := \int_{[m]\in\mathbf{\Delta}^{\mathrm{op}}_{\leq n+1}} T^{m+1}G_m = \int^{[m]\in\mathbf{\Delta}_{\leq n+1}} T^{m+1}G_m\,.$$

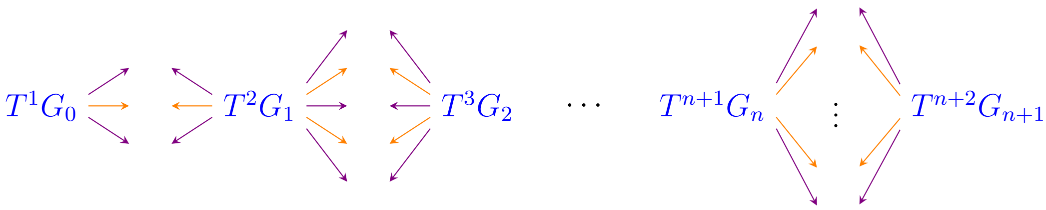

Figure 1: This is the coend of the thesis.

Abstract

The infinitesimal counterpart of a Lie group is its Lie algebra: the space of right-invariant vector fields with their Lie bracket. As a vector space, it is isomorphic to the tangent space to the Lie group at the unit. A Lie group*oid*, which is a many-unit generalization of a Lie group, can be similarly differentiated to its Lie algebr*oid*. As a vector bundle, it is the source-vertical tangent bundle restricted to the units. The Lie bracket on the space of sections of this vector bundle is obtained analogously by its identification with the space of right-invariant vector fields on the manifold of arrows.

The main goal of this thesis is to give satisfactory answers to the following questions: Given a (higher) groupoid object G in a category $\mathcal{C}$, what are the structures of $\mathcal{C}$ and the properties of G needed for its differentiation? How does the differentiation procedure work? What are the generalized infinitesimal objects?

The first part of the answer is to identify the categorical structures needed in the differentiation process. I show that the ambient category has to be equipped with an abstract tangent structure so that there is a Lie bracket of vector fields together with some additional properties. For that, we use the categorical generalization of the tangent functor on smooth manifolds developed by Rosický in the 1980s. An abstract tangent structure on a category $\mathcal{C}$ consists of an endofunctor $T : \mathcal{C} \to \mathcal{C}$, together with the natural transformations of the bundle projection, the zero section, the fiberwise addition, the vertical lift, and the symmetric structure. Rosický's axioms of a tangent structure are the minimal axioms needed to define the Lie bracket of vector fields on an object of $\mathcal{C}$.

For the second part of the answer, I introduce differentiable groupoid objects as the analogue of Lie groupoids, as well as abstract Lie algebroids as the analogue of Lie algebroids in the setting of tangent categories. One of the main results of this thesis is the construction of the abstract Lie algebroid of a differentiable groupoid object in a cartesian tangent category with a scalar multiplication by a ring object. Examples include the differentiation of infinite-dimensional Lie groups and elastic diffeological groupoids.

In the last part of the thesis, I propose a method of differentiation of differentiable higher groupoid objects in a tangent category equipped with a compatible Grothendieck pretopology. In 2006, Ševera has argued that the L_∞-algebroid of a higher Lie groupoid is given by the enriched hom in the category of simplicial supermanifolds from the nerve of the pair groupoid of $\mathbb{R}^{0|1}$ to the higher Lie groupoid. I show that this idea can be rigorously implemented by a universal construction given by a categorical end, that works in any tangent category. Dually, the higher generalized Lie algebroid cohomology is given by a coend in differential complexes.

Acknowledgements

First and foremost, I would like to express my profound gratitude to my doctoral supervisor Christian Blohmann for introducing me to the beautiful world of higher structures in differential geometry and for his guidance in the various stages of this thesis. Christian, your constant support, insightful comments, and mathematical expertise are invaluable. Not only have you guided me throughout my Masters and PhD studies, but you have also shown me how enjoyable mathematical discussions and collaborations can be.

I would also like to extend my sincere gratitude to Catharina Stroppel and Peter Teichner for their support and for being the official referees of this thesis. I thank Margherita Disertori and Albrecht Klemm for being the chair and external member of my PhD committee. Special thanks go to Anna Siffert and Arunima Ray, who have been my mentors during my doctoral studies. I appreciate your precious advice.

I feel deeply indebted to the Bonn International Graduate School (BIGS) of Mathematics and the Max Planck Institute for Mathematics (MPIM) for funding my PhD studies, for providing me one of the most inspiring research environments to write this thesis, as well as for the child-friendly atmosphere at work. My heartfelt appreciation goes to Karen Bingel who accompanied me in my first steps in Bonn. I would also like to thank the staff of MPIM for always being ready to help and cooperate in the most friendly and efficient way. Finally, I thank the beautiful city of Bonn, the Beethovenstadt, for being such a wonderful place to live and study.

I would like to thank Madeleine Jotz, Claudia Scheimbauer and Marco Zambon for their research invitations and the fruitful discussions. I would like to acknowledge the inspirational role the following female professors have played in my life during my doctoral studies. Thank you Madeleine Jotz, Chenchang Zhu, Claudia Scheimbauer and Alice Barbara Tumpach for the stimulating conversations we have had and the understanding and encouragement you have shown to me.

I am grateful for all the interesting mathematical discussions with João Nuno Mestre on Lie groupoid matters, David Miyamoto on interesting examples of diffeological spaces, Madeleine Jotz on the cubic notation and multiple vector bundles, Kalin Krishna on Kan conditions, Chenchang Zhu on coskeletality of simplicial objects, Leonid Ryvkin on the tangent complex of higher Lie groupoids, and David Carchedi on enriched homs.

To my fellow PhD students Janina Bernardy, Annika Tarnowsky, and David Aretz, not only have you become competent colleagues, but you have also become wonderful friends of mine. Your presence in offices 215 and B27, our fruitful mathematical discussions and your friendship in all aspects of life made the past years of my doctoral studies truly enjoyable.

Last but not least, I would like to express my deepest gratitude to my family and friends for their continuous emotional and practical support. Thank you Elly and Shushan for the many inspiring conversations and the positive energy you always give me. Thank you Marine for reminding me that there is nothing impossible when there is determination, hard work and hope. Thank you Janina for being there for me in moments of joy and sadness, thank you for your special role in Arek's life. Thank you Mr. and Mrs. Leon and Waltraud Torossians for your support, encouragement and optimism you have constantly shown to me. Thank you Christian and Andrea for being a wonderful role model of an academic, generous, kind and fun couple. Thank you Madeleine for giving me strength and confidence both in my mathematical and in my personal life.

Thank you my uncle David for making me fall in love with mathematics in my high school years. Thank you Luuk for sharing with me so many simple but unforgettable moments, thank you for showing Arek and me how beautiful and adventurous life can be. Thank you my sisters Arpa and Hasmig for your genuine trust in me as your older sister, I also learn a lot from you. Finally, thank you my dear parents Sossy and Hagop, for your endless love, for your constant support, and for believing in me from the moment that I decided to travel abroad for my graduate studies, until the very last moments of submitting my PhD thesis.

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Introduction

A Lie group is a group object in the category of smooth finite-dimensional manifolds. To every Lie group we can associate its Lie algebra, given by the right-invariant vector fields. In this sense, every group object in manifolds is differentiable. This is not true for every group*oid* object in manifolds. We have to assume that the source (and therefore target) map is a submersion, so that we can construct its Lie algebroid. The groupoid objects that are differentiable in this sense are the Lie groupoids.

The category of smooth manifolds is inconvenient. Other than coproducts and finite products, it does not have many good categorical properties, which puts it at odds with many modern developments of mathematics that are structured and guided by category theoretic concepts. This has motivated the development of a plethora of convenient settings for differential geometry, which can be vaguely defined as categories that contain smooth manifolds as a full subcategory, but have better properties, such as having all small limits, colimits, exponential objects, etc. The price to be paid for this convenience is that such a category is usually too large as to allow for strong geometric results that hold for all its objects. The task is then to identify the structure of a category and the properties of its objects that are needed to generalize a certain differential geometric construction, in our case: the differentiation of Lie groupoids. This can be done for a particular category, e.g. the category of diffeological spaces, Chen spaces, Fröhlicher spaces, differentiable stacks, polyfolds, bornological spaces, etc. [IZ13, BX11, Sta11]. Better still, we can try to identify the structure any category must have and the properties any object must have for the construction to work. In this thesis, we ask and answer the following:

Questions. Given a (higher) groupoid object G in a category $\mathcal{C}$, what are the structures of $\mathcal{C}$ and the properties of G needed for its differentiation? What are the infinitesimal objects in $\mathcal{C}$ that generalize (higher) Lie algebroids?

Lie groupoids and Lie algebroids

A **groupoid** is a small category in which every morphism is invertible. Explicitly, a groupoid is composed of a set of objects G_0 and a set of arrows G_1 together with the source $s : G_1 \to G_0$, target $t : G_1 \to G_0$, multiplication $m : G_1 \times^{s,t}_{G_0} G_1 \to G_1$, unit $1 : G_0 \to G_1$, and inverse $i : G_1 \to G_1$, satisfying the usual axioms of a category.

A **Lie groupoid** is a groupoid internal to the category of smooth manifolds such that the source and target maps are submersions. Lie groupoids are many-unit generalizations of Lie groups. They describe local and global symmetries of geometric structures in more general cases [Wei96]. A large class of geometric structures can be described via Lie groupoids, such as representations, equivalence relations, group actions, foliations, orbifolds, differentiable stacks, etc. [Mac05, CdSW99, MM03].

The theory of Lie groupoids comes together with its infinitesimal counterpart: Lie algebroids. Being a generalization of the notion of Lie algebras, a **Lie algebroid** is a vector bundle $A \to M$ together with a vector bundle map $\rho : A \to TM$, called the **anchor**, and a Lie bracket $[\cdot\,,\cdot]$ on its space of smooth sections $\Gamma(M, A)$, such that the Leibniz rule

$$[a, fb] = f[a, b] + \big((\rho \circ a) \cdot f\big)b$$

holds for all sections $a, b \in \Gamma(M, A)$ and smooth functions $f \in C^\infty(M)$.

Differentiation of Lie groupoids

Lie groupoids can be differentiated to their Lie algebroids. We will now analyze the differentiation procedure from a categorical point of view. Let G be a Lie groupoid. The assumption that the source map s is a submersion is sufficient for the existence of the domain of the groupoid multiplication

$$m : G_1 \times^{s,t}_{G_0} G_1 \longrightarrow G_1\,,$$

as well as for the existence of the iterated k-fold pullback

$$\underbrace{G_1 \times_{G_0} \cdots \times_{G_0} G_1}_{k \text{ factors}} \tag{1}$$

for all $k \geq 0$. While the assumption that the source is a submersion is sufficient for the existence of these pullbacks, it is not necessary.

In the first step of the differentiation of G, we take the derivative of the multiplication by applying the tangent functor,

$$Tm : T(G_1 \times_{G_0} G_1) \longrightarrow TG_1\,.$$

To obtain the multiplication of the tangent groupoid, we need an isomorphism

$$\nu : T(G_1 \times_{G_0} G_1) \xrightarrow{\cong} TG_1 \times_{TG_0} TG_1\,,$$

that is, we need the pullback of the codomain to exist and the natural map ν to have an inverse. This is the step where the assumption that s and t are submersions is necessary. The multiplication of the tangent groupoid is then defined by the composition

$$m_{TG} : TG_1 \times_{TG_0} TG_1 \xrightarrow[\cong]{\nu^{-1}} T(G_1 \times_{G_0} G_1) \xrightarrow{Tm} TG_1\,.$$

In the second step, we restrict the tangent multiplication to a right G-action on the s-vertical tangent bundle,

$$VG_1 := TG_1 \times^{Ts, 0_{G_0}}_{TG_0} G_0\,,$$

where $0_{G_0} : G_0 \to TG_0$ is the zero section. Since s is a submersion, so is Ts, which ensures the existence of the pullback. The right G-action

$$VG_1 \times^{\mathrm{pr}_2, t}_{G_0} G_1 \longrightarrow VG_1$$

is the restriction of m_{TG} to the s-vertical tangent bundle. Since t is a submersion, the pullback of the domain exists.

In the third step, we restrict the vertical tangent bundle $VG_1 \to G_1$ to the identity bisection of the groupoid,

$$A := G_0 \times_{G_1} VG_1 .$$

Since the tangent bundle is a vector bundle, so is the vertical tangent bundle. This implies that $VG_1 \to G_1$ is a submersion, so that the pullback always exists.

The Lie algebroid is constructed as the vector bundle $A \to G_0$ with anchor the restriction of Tt to A. The sections of A can be shown to be in bijection with the right-invariant vector fields on G_1, that is, the right G-equivariant sections of the vertical tangent bundle. The Lie bracket of the sections of A is the Lie bracket of vector fields.

The upshot is the following: For the differentiation of a Lie groupoid to its Lie algebroid, a number of pullbacks must exist and the tangent functor must commute with the fiber product of k composable arrows given by (1). In smooth manifolds, this is the case if the source map is a submersion.

Abstract tangent structures

Any category $\mathcal{C}$ in which we want to differentiate a groupoid must have a tangent functor T and some additional structure that allows us to define a Lie bracket of vector fields. This construction is particularly difficult to generalize, because the map $X \mapsto \Gamma(X, TX)$ that sends a manifold to its space of vector fields is not even a functor.

Fortunately, this issue has been resolved by Rosický in his seminal paper [Ros84]. He introduces the concept of **abstract tangent functor** as an endofunctor $T : \mathcal{C} \to \mathcal{C}$ together with the natural transformations of the bundle projection $\pi : T \to 1$, zero section $0 : 1 \to T$, fiberwise addition of the tangent bundle $+ : T \times_1 T \to T$, the exchange of the order of differentiation $\tau : T^2 \to T^2$, and the vertical lift $\lambda : T \to T^2$, satisfying a number of natural axioms (Definition 3.2.4). A category with an abstract tangent functor is called a **tangent category**. Rosický's axioms are the minimal axioms needed to define the Lie bracket of vector fields on an object of $\mathcal{C}$.

In the category of smooth manifolds, vector fields can be identified with the derivations on the ring of smooth real-valued functions on the manifold. With this identification, the Lie bracket of vector fields is defined as the commutator. However, in other tangent categories, vector fields, which are defined as sections of the tangent bundle, cannot be identified with derivations on the ring of functions in general (Remark 3.3.6). Thus, we need the geometric approach for the definition of the Lie bracket that only uses the tangent structure on smooth manifolds. This was Rosický's observation.

The Lie bracket of two vector fields $v : M \to TM$, $x \mapsto v_x$ and $w : M \to TM$, $x \mapsto w_x$ on a smooth manifold M can be defined by

$$[v, w]_x := Tw(v_x) - \tau_M\big(Tv(w_x)\big) \tag{2}$$

for all $x \in M$. The fiberwise differentials of v and w are given by $Tv : T_xM \to T_{v_x}(TM)$ and $Tw : T_xM \to T_{w_x}(TM)$ respectively. Since $Tw(v_x)$ and $Tv(w_x)$ do not lie in the same fiber, the canonical flip τ_M is applied on the second factor in (2). Moreover, the right hand side of Equation (2) is a vertical vector in $T_{w_x}(TM)$, that is, it lies in the kernel of $T\pi_M$. Identifying it with a tangent vector in T_xM via the vertical lift, we get the Lie bracket. This construction is the cornerstone of the geometric definition of the Lie bracket in tangent categories and will be explored in Section 3.3.

A conceptual understanding of tangent structures can be acquired by the local coordinate description of the tangent functor T on Euclidean spaces. Let $\mathcal{E}\mathrm{ucl}$ denote the category with objects open subsets $U \subset \mathbb{R}^n$, for $n \geq 0$, and morphisms smooth maps. Then, $TU = U \times \mathbb{R}^n$ and $T^2U = U \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$. This yields the following expressions of the tangent structure in local coordinates:

$$\begin{aligned}
\pi_U &: TU \longrightarrow U & (u, u_0) &\longmapsto u \\
0_U &: U \longrightarrow TU & u &\longmapsto (u, 0) \\
+_U &: TU \times_U TU \longrightarrow TU & \big((u, u_0), (u, v_0)\big) &\longmapsto (u, u_0 + v_0) \\
\lambda_U &: TU \longrightarrow T^2U & (u, u_0) &\longmapsto (u, 0, 0, u_0) \\
\tau_U &: T^2U \longrightarrow T^2U & (u, u_0, u_1, u_{01}) &\longmapsto (u, u_1, u_0, u_{01}),
\end{aligned}$$

where the subscripts refer to the differentiation with respect to the different variables. These expressions as well as the description of the Lie bracket in local coordinates are explained in detail in Section 3.4.

Bundles

In the category of smooth manifolds, a *fiber* bundle is a smooth surjective map $p : A \to X$ which is locally the projection of a product manifold to one of its factors. This concept of bundles with local trivializations requires a generalization of open covers, which is provided by the notion of Grothendieck pretopology. However, already for our guiding example of diffeological spaces requiring local trivializations is too strong. Instead, we start with the usual categorical notion of bundle as a morphism without further conditions, in other words, an object in the overcategory $\mathcal{C}{\downarrow}X$. Then we add, step by step, the algebraic structure we need, using the unified Terminology 3.1.1:

Terminology. Let "Wibble" be an algebraic theory. Let X be an object in a category $\mathcal{C}$ such that the overcategory $\mathcal{C}{\downarrow}X$ has all finite products (i.e. pullbacks over X). An object in $\mathcal{C}{\downarrow}X$ will be called a **bundle over** X. A Wibble object in $\mathcal{C}{\downarrow}X$ will be called a **bundle of Wibbles over** X.

In this thesis, "Wibble" will be one of: group, abelian group, and R-module (for $R \in \mathcal{C}$ a ring object). Let $W \to X$ be a bundle of Wibbles. If $\mathcal{C}$ has finite products, then the pullback $W_x := * \times_X W$ over a point $x : * \to X$ is a Wibble object in $\mathcal{C}$. In other words, every fiber of a bundle of Wibbles is a Wibble, which justifies the terminology. We emphasize once more, that the notion of bundle of Wibbles does not make any assumptions on local trivializations, whatsoever. Note that in the axioms of a tangent structure, the tangent bundle $\pi_X : TX \to X$ is required to be a bundle of abelian groups for any object $X \in \mathcal{C}$.

Submersions and differentiability

As we mentioned above, submersions are a class of morphisms in the category of smooth manifolds which play an important role in the definition of a Lie groupoid and in the differentiation to its Lie algebroid. Let us take a moment to delve deeper into their properties. Recall that a smooth map $f : M \to N$ of manifolds is called a **submersion** if:

(i) The tangent map $Tf|_x : T_xM \to T_{f(x)}N$ is surjective for every point $x \in M$.

Submersions have a number of good properties, such as:

(ii) Every point $x \in M$ lies in the image of a smooth local section of f.

(iii) For every smooth map $M' \to N$, the pullback $M \times_N M'$ exists.

(iv) Every pullback via f commutes with the tangent functor, that is, the natural map $T(M \times_N M') \to TM \times_{TN} TM'$ is an isomorphism.

(v) The tangent map $Tf : TM \to TN$ is a submersion.

Note that Properties (i), (ii), and (iv) are equivalent. It is tempting to state the conditions for a groupoid object in a tangent category to be *differentiable* by requiring the source map to have the properties of a surjective submersion. However, Properties (ii)–(v) make use of the fact that a submersion is locally the projection to one of the factors of a product manifold. This is proved with the implicit function theorem, a genuinely analytic method that cannot be easily generalized. In a general category with a tangent structure, none of the Properties (i)–(iv) implies any other of the properties. So which properties should be the basis of our generalization?

For the existence of the pullbacks (1), we need Property (iii), for the first step of the differentiation procedure we need Property (iv), and for the second step we need (v) and (iii). So we would need morphisms that have the Properties (iii), (iv), and (v). We are not aware of any interesting tangent category other than the category of smooth manifolds, in which there is a known class of such morphisms.

Fortunately, the properties of a submersion are unnecessarily strong. We do not need the pullbacks via submersions to exist universally over *all* other morphisms, but only the pullbacks that appear in our construction. Neither do we need the tangent functor to preserve all pullbacks, but only the pullback (1) describing the k composable arrows. For all these reasons, in the Definition 4.1.6 of differentiable groupoid objects in tangent categories we will identify and explicitly require the existence of the pullbacks and their properties needed in the differentiation process.

Higher set theoretic groupoids

Higher groupoid objects can be defined succinctly with simplicial methods. Let $\mathbf{\Delta}$ be the **simplex category** which has non-empty finite ordinals [0], [1], [2], etc., as objects, and order-preserving maps as morphisms. A **simplicial object** in a category $\mathcal{C}$ is a contravariant functor $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$. Equivalently, it can be described by a family of objects in $\mathcal{C}$ together with certain face and degeneracy morphisms satisfying the simplicial identities. A simplicial structure can be depicted by

$$X_0 \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow} X_1 \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow} X_2 \quad \cdots$$

The **standard simplicial m-simplex** is defined by $\Delta^m = y[m] = \mathbf{\Delta}\big(\_ , [m]\big)$, where y is the Yoneda embedding. Its i^{th} **horn** is the subsimplicial set Λ^m_i which is obtained by removing the i^{th} face from it as well as its unique non-degenerate m-simplex. To every groupoid G there is an associated simplicial set $G_\bullet$, called its **nerve**. Its k-simplices are given by the pullback (1) of the strings of k composable arrows, its face maps by composition by arrows, and the degeneracy maps by inserting identities. This simplicial set satisfies horn filling conditions in all degrees and unique horn filling conditions for degrees > 1. In fact, one uniquely recovers a groupoid from a given simplicial set with these horn filling conditions (Remark 2.2.22). This suggests the following definition, originally due to [Dus75, Gle82]. An **n-groupoid** is a simplicial set $G : \mathbf{\Delta}^{\text{op}} \to \mathcal{S}\text{et}$ such that the natural horn projection

$$p_{m,i} : G_m \longrightarrow G(\Lambda^m_i) \tag{3}$$

is a surjection for all $m \geq 1$ and a bijection for all $m > n$ and $0 \leq i \leq m$. Here, $G_m \cong \mathcal{S}\text{et}^{\mathbf{\Delta}^{\text{op}}}(\Delta^m, G)$ by the Yoneda lemma and $G(\Lambda^m_i) := \mathcal{S}\text{et}^{\mathbf{\Delta}^{\text{op}}}(\Lambda^m_i, G)$.

Higher groupoids in a category with a Grothendieck pretopology

In an attempt to define higher groupoid objects in any category, we need to make sense of the horn filling conditions. For that, the need of a class of morphisms that play the role of surjective submersions in smooth manifolds becomes necessary. Following the approach of Henriques, Zhu and Meyer, these will be given by the covers of a Grothendieck pretopology on $\mathcal{C}$ [Hen08, Zhu09, MZ15]. The covers satisfy the following axioms:

(i) Every isomorphism is a cover.

(ii) The composition of two covers is a cover.

(iii) If $U \to X$ is a cover and $Y \to X$ is a morphism in $\mathcal{C}$, then the pullback $Y \times_X U$ exists and the projection $Y \times_X U \to Y$ is a cover.

We will equip $\mathcal{C}$ with a Grothendieck pretopology satisfying Assumption 5.1.3:

Assumption.

(i) The category $\mathcal{C}$ has a terminal object $*$ and the unique morphism $U \to *$ is a cover for every object $U \in \mathcal{C}$ which is not the initial object (if it exists).

(ii) The pretopology is subcanonical, i.e. every cover is a regular epimorphism.

Note that for the horn projection (3) to be well-defined as a morphism in $\mathcal{C}$, the object of horns $G(\Lambda_i^m)$, which is given by a certain limit, must exist in $\mathcal{C}$. This was proved in [Hen08] using the fact that Λ_i^m is a collapsible simplicial set (Corollary 5.1.10). In this setting, an **n-groupoid object** in $\mathcal{C}$ is a simplicial object $G : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$ such that the horn projection (3) is a cover for all $m \geq 1$ and an isomorphism for all $m > n$ and $0 \leq i \leq m$ (Definition 5.1.13). The n-groupoid objects in the category of smooth manifolds with surjective submersions as the covers are called **Lie n-groupoids**.

Differentiation of higher groupoid objects

In 2006, Ševera [Šev06] proposed a method of constructing the L_∞-algebroid (given by a differential non-negatively graded manifold) of a Lie n-groupoid G. He uses that the inner hom in graded manifolds from the odd line $\mathbb{R}^{0|1}$ to a manifold M represents the odd tangent bundle ΠTM of M. Ševera's main idea was that for higher groupoids we should replace $\mathbb{R}^{0|1}$ by the nerve P of its pair groupoid. He then argues that the enriched hom $\underline{\mathrm{Hom}}(P, G)$ is represented by a differential non-negatively graded manifold [Šev06, Prop. 9.2]. The gaps of this statement and its proof have been worked out in detail in [LRWZ23], where the authors prove that the representing object of $\underline{\mathrm{Hom}}(P, G)$ is the tangent complex of G.

In this thesis, we give a categorical generalization of Ševera's idea using the language of categorical ends. It starts with the observation that ends provide a general method to enrich the homs in functor categories over the target category. Using the isomorphism $\underline{\mathrm{Hom}}(\mathbb{R}^{0|1}, M) \cong \Pi TM$, the enriched hom can be entirely expressed in terms of the tangent functor on smooth manifolds. The advantages of our approach can be summarized as follows:

- Ends are universal constructions for which there is a powerful calculus [Lor21].
- Being special limits, ends can be computed in terms of equalizers and products.
- Our end formula works in any tangent category, such as the category of elastic diffeological spaces [Blo24a] and the bicategory of differentiable stacks (Remark 3.2.12).

This is explained in detail in Section 5.2.

Main result 1: differentiation of groupoid objects

The main result of this thesis is to give satisfactory answers to the Questions on page 11. The first part of the answers is to identify the categorical structures needed in the differentiation process. As already explained, the ambient category $\mathcal{C}$ has to be equipped with an abstract tangent structure (Definition 3.2.4), so that there is a Lie bracket of vector fields. We also have to assume that the tangent functor preserves finite products. Tangent categories with this property are called **cartesian** (Definition 3.2.13). Finally, we need a generalization of the scalar multiplication of tangent vectors by real numbers. For this, the tangent bundles $TX \to X$ have to be equipped with the fiberwise structure of an R-module for a ring object R in

$\mathcal{C}$, satisfying some compatibility with the tangent structure. Such a module structure is called a **scalar R-multiplication** (Definition 3.2.19). All these categorical structures have been worked out in [AB].

The second part of the answers to the Questions on page 11 is to identify the properties of a groupoid object in $\mathcal{C}$ needed for its differentiation. This leads to the first main concept of this thesis, introduced in Definition 4.1.6:

Definition. A groupoid object G in a category $\mathcal{C}$ with a tangent structure will be called **differentiable** if the pullbacks in the diagrams

$$\begin{tikzcd} T^nG_1 \times_{T^nG_0} G_k \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & G_k \arrow[d, "0^{[n]}_{G_k}"] \\ T^nG_1 \times_{T^nG_0} T^nG_k \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & T^nG_k \arrow[d, "T^nt_k"] \\ T^nG_1 \arrow[r, "T^ns"'] & T^nG_0 \end{tikzcd} \qquad \begin{tikzcd} T_mG_1 \times_{T_mG_0} G_k \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & G_k \arrow[d, "0_{m,G_0}\circ t_k"] \\ T_mG_1 \arrow[r, "T_ms"'] & T_mG_0 \end{tikzcd} \tag{4}$$

$$\begin{tikzcd} G_0 \times_{G_1} TG_1 \times_{TG_0} G_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & TG_1 \times_{TG_0} G_0 \arrow[d, "\pi_{G_1}\circ \mathrm{pr}_1"] \\ G_0 \arrow[r, "1"'] & G_1 \end{tikzcd} \tag{5}$$

exist, and if the natural morphism

$$T^n(G_1 \times^{s,t_k}_{G_0} G_k) \longrightarrow T^nG_1 \times^{T^ns,T^nt_k}_{T^nG_0} T^nG_k \tag{6}$$

is an isomorphism for all $n \geq 1$, $m \geq 2$ and $k \geq 0$.

Here, $0^{[n]}_{G_0} = (T^{n-1}0 \circ \ldots \circ T0 \circ 0)_{G_0}$ is the iterated zero section, $0_{m,G_0} := (0_{G_0}, \ldots, 0_{G_0})$ the diagonal zero section, and $t_k : G_k \to G_0$ the morphism to the target of the rightmost arrow of the nerve of the groupoid.

The notion of differentiability generalizes to any bundle $E \to G_0$ with a right G-action (Definition 4.2.11). On differentiable groupoid bundles there is a natural notion of invariant vector fields (Definition 4.4.1). We show in Theorem 4.4.6 that the set $\mathcal{X}(E)^G$ of invariant vector fields on E is a Lie subalgebra of $\mathfrak{X}(E) = \Gamma(E, TE)$. The second main concept of this thesis is introduced in Definition 4.5.1:

Definition. Let $\mathcal{C}$ be a cartesian tangent category with scalar R-multiplication. An **abstract Lie algebroid** in $\mathcal{C}$ consists of a bundle of R-modules $A \to X$, a morphism $\rho : A \to TX$ of bundles of R-modules, called the **anchor**, and a Lie bracket on the abelian group $\Gamma(X, A)$, such that

$$\begin{aligned} [a, fb] &= f[a, b] + \big((\rho \circ a) \cdot f\big)b \\ \rho \circ [a, b] &= [\rho \circ a, \rho \circ b] \end{aligned}$$

for all sections a, b of A and all morphisms $f : X \to R$ in $\mathcal{C}$.

The main generalization is that $A \to G_0$ is no longer required to be a vector bundle, but only a bundle of R-modules (Terminology on page 14), so that the sheaf of sections of A is not a locally free $\mathcal{C}(G_0, R)$-module. Therefore, the Leibniz rule does not imply that the composition with the anchor is a homomorphism of Lie algebras, which has to be required separately. Our main results are Theorems 4.5.7 and 4.5.8, whose essence can be stated as follows:

Theorem A. *Let G be a differentiable groupoid object in a cartesian tangent category $\mathcal{C}$ with scalar R-multiplication. Then there is an abstract Lie algebroid $A \to G_0$ with the Lie bracket of invariant vector fields on $\Gamma(G_0, A) \cong \mathfrak{X}(G_1)^G$.*

These results will appear in [AB25].

Main result 2: differentiation of higher groupoid objects

The second main result of this thesis is an answer to the Questions on page 11 in the case of higher groupoid objects. The ambient category $\mathcal{C}$ in this setting has a tangent structure and a compatible Grothendieck pretopology satisfying the Assumption on page 16. A pretopology and a tangent structure are compatible if the tangent functor maps covers to covers. In Definition 5.1.29, we identify the axioms needed for a higher groupoid object in $\mathcal{C}$ to differentiate to its infinitesimal counterpart. The following definition is one of the main constructions in this thesis, which is motivated in Section 5.2.1 and introduced in Definition 5.2.6:

Definition. Let G be a differentiable n-groupoid object in $\mathcal{C}$ for some $n \geq 0$. The **abstract Lie n-algebroid** $\mathrm{Lie}(G)$ of G is defined by

$$\mathrm{Lie}(G) := \int_{[m] \in \mathbf{\Delta}^{\mathrm{op}}_{\leq n+1}} T^{m+1} G_m \, . \tag{7}$$

The end is well-defined since the iterated tangent bundle $T^{\bullet+1}$ has a natural cosimplicial structure, which we prove in Proposition 3.2.25. The end is over the $(n+1)$-truncated simplex category, which reflects the $(n+1)$-coskeletality of n-groupoid objects (Propositions 2.3.12 and 5.1.18). Our main results are Theorems 5.2.8 and 5.2.17, which can be summarized as follows:

Theorem B. *Let $\mathcal{C}$ be a category with a tangent structure and a compatible Grothendieck pretopology satisfying the Assumption on page 16. Let G be a differentiable n-groupoid object in $\mathcal{C}$ for some $n \geq 0$. Fix $k \in \{0, \dots, n+1\}$. Then, the end $\mathrm{Lie}(G)$ is isomorphic to the following fiber products*

$$\mathrm{Lie}(G) \cong TG_0 \times_{Q_1} T^2 G_1 \times_{Q_2} \cdots \times_{Q_n} T^{n+1} G_n \times_{Q_{n+1}} T^{n+2} G_{n+1} \tag{8}$$

$$\cong TG_0 \times_{\widetilde{Q}_1} T^2 G_1 \times_{\widetilde{Q}_2} \cdots \times_{\widetilde{Q}_n} T^{n+1} G_n \times_{\widetilde{Q}_{n+1}} T^{n+2} G(\Lambda^{n+1}_k) \, , \tag{9}$$

where for all $1 \leq m \leq n+1$,

$$Q_m := (T^m G_m)^m \times (T^{m+1} G_{m-1})^{m+1} \, ,$$

and

$$\widetilde{Q}_m := \begin{cases} (T^m G_m)^m & \text{if } 1 \leq m \leq n\,, \\ (T^{n+2} G_n)^{n+2} & \text{if } m = n+1\,. \end{cases}$$

The fiber products are with respect to the (co)face and (co)degeneracy morphisms of $T^{\bullet+1}$ and $G_\bullet$. The fiber product (8) is the result of computing the limit corresponding to the end (7). To achieve the isomorphism from (8) to (9), we show in Section 5.2.4 that half of the (co)face and (co)degeneracy relations are redundant.

Motivation and applications

Broad categorical generalizations of geometric concepts like Lie groupoids and Lie algebroids risk being solutions in search of a problem. That is, while the generalization of a concept may be of interest in its own right, its usefulness in the originating area, differential geometry in our case, may be limited. The concept of differentiable groupoid objects and their abstract Lie algebroids, however, was motivated by and provides answers to existing research questions.

In [BFW13] the symmetry structure of the initial value problem of general relativity was studied. The main result was the construction of a diffeological groupoid whose diffeological Lie algebroid has the same bracket as the somewhat mysterious Poisson bracket of the Gauß-Codazzi constraints for Ricci flat metrics. Without a theory of diffeological groupoids available, the differentiation of the diffeological groupoid of [BFW13] was carried out in an ad-hoc manner. The results of the present thesis can be viewed as a rigorous justification of this construction. In this sense, this application of Theorem A existed before the actual theorem. The groupoid in the general relativity example generalizes to the diffeological groupoids that arise from the gauge and diffeomorphism symmetries of classical field theory.

The construction of the groupoid symmetry of general relativity from [BFW13] is a particular case of a more general construction: reductions of action Lie groupoids by (not necessarily normal) subgroups. Blohmann and Weinstein have computed the Lie algebroid of this quotient groupoid in [BW24, Sec. 8.5] in the setting of smooth finite-dimensional manifolds. With our results, this reduction can now be carried out for diffeomorphism groups acting on sections of natural bundles.

Diffeological groupoids arise in a number of other areas, such as diffeological integration of Lie algebroids [Vil25], the holonomy groupoid of a singular foliation [AS09], and more generally, of a singular subalgebroid [Zam22]. In [AZ23], Androulidakis and Zambon explain a differentiation procedure from holonomy-like diffeological groupoids to singular subalgebroids, where the original goal of the authors is to establish an integration method for singular subalgebroids via diffeological groupoids.

One of the original motivations of this thesis was the application of diffeological groupoids to geometric deformation theory. The idea is that many geometric structures, such as Riemannian metrics or complex structures, are equipped with a natural diffeology of smooth homotopies, so that their moduli space of structures modulo isomorphism are presented by diffeological groupoids. Since we are now working in the bicategory of diffeological stacks, higher categorical considerations come into play. In this framework, deformations can be conceptualized by smooth paths in a diffeological stack, higher deformations by paths of paths, etc. This suggests that the natural object to consider is the path ∞-groupoid of the diffeological moduli stack. We posit that the infinitesimal object of these groupoids is closely related to the differential graded Lie algebras or the L_∞-algebras describing the deformation theory of these moduli. While this ambitious goal was not achieved within the constraints of this thesis, all the relevant objects and constructions are now rigorously defined.

Structure of the thesis

The thesis is composed of five chapters and an appendix. Here is a brief summary:

- In Chapter 1, we describe the notions of Lie groupoids and Lie algebroids as the many-unit generalizations of Lie groups and Lie algebras. In particular, we explain how a Lie groupoid can be differentiated to its Lie algebroid.

- The main subject of Chapter 2 is simplicial objects in categories. It starts with recalling the simplex category with its coface and codegeneracy maps, which models simplicial and cosimplicial objects. The examples of the standard simplicial n-simplex and its horns and boundary are described in detail. We explain the categorical approach to define the horns and boundaries of a simplicial object in a category using right Kan extensions. We then recall the set theoretic horn filling conditions and higher groupoids. The chapter concludes with a discussion on simplicial skeleta and coskeleta.

- In Chapter 3, we review Rosický's abstract tangent structures [Ros84], which generalize the tangent functor on the category of smooth manifolds. We meticulously study bundles whose fibers have the structure of an abelian group or an R-module, where R is a ring object in the ambient category. Extending the ideas in [Blo24a], we introduce cartesian tangent structures with scalar R-multiplication.

  A fundamental part of this chapter is devoted to the construction of the Lie bracket of vector fields in tangent categories. Following an idea of Rosický, we explain the Lie bracket construction in more detail and state the Leibniz rule. We then prove several naturality results for the Lie bracket. Furthermore, we show that the iterated tangent functor, by virtue of being a monad, has an augmented cosimplicial structure. Lastly, we describe the namesake example of tangent categories: the category of Euclidean spaces. Using a notation on iterated local coordinates which we introduce, we revisit our proof of the cosimplicial identities in Euclidean spaces.

- Chapter 4 is the core of this thesis, where we establish a differentiation procedure of groupoid objects to their infinitesimal counterparts in a cartesian tangent category with scalar R-multiplication. To this end, we introduce the notion of differentiable groupoid objects, which are groupoids that satisfy additional conditions needed for their differentiation. On the infinitesimal level, we define abstract Lie algebroids in tangent categories.

To achieve the main construction, we develop the language of differentiable G-bundles. This allows us to consider the restriction of the source-vertical tangent bundle to the identity bisection. One of our main results states that the sections of this bundle can be identified with invariant vector fields. Using an elegant technical lemma, we show that the tangent structure of the ambient category has a source-vertical restriction. It follows that invariant vector fields are closed under the Lie bracket of vector fields. The main theorem states that our differentiation procedure applied to a differentiable groupoid object yields an abstract Lie algebroid. As a conclusion, a list of possible applications and examples is outlined.

- Chapter 5 proposes a method of differentiation of higher groupoid objects in a tangent category equipped with a compatible Grothendieck pretopology. Mainly, we provide a categorical generalization of Ševera's idea [Šev06], which states that the infinitesimal counterpart of a higher Lie groupoid is given by the enriched hom from the nerve of the pair groupoid of $\mathbb{R}^{0|1}$ to the higher Lie groupoid. We show that this idea can be rigorously implemented by a universal construction given by a categorical end.

  Computing the corresponding limit, we show that the end is given by an iterated fiber product with respect to the simplicial structure of the higher groupoid, and the cosimplicial structure of the iterated tangent bundle. Using the naturality of the tangent structure and the (co)simplicial identities, we further show that half of the relations are redundant. This construction recovers the differentiation of Lie groupoids to their Lie algebroids. Finally, we conjecture that the higher generalized Lie algebroid cohomology is given by a coend in differential complexes.

- The primary goal of Appendix A is to provide a brief exposition of category theoretic concepts that are used in this thesis. We recall the Yoneda lemma and its consequences. Moreover, we explain the notions of categorical ends and Kan extensions, which are major universal constructions in category theory. We conclude the appendix with useful categorical machinery and algebraic objects internal to a category. The appendix also serves to fix some notation.

Conventions and Notation

Categories will be denoted by curly letters, such as $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, etc. Given two categories $\mathcal{C}$ and $\mathcal{D}$, we will denote the **functor category** with objects functors $F : \mathcal{C} \to \mathcal{D}$ and morphisms natural transformations by the exponential notation $\mathcal{D}^{\mathcal{C}}$. The categories we consider in this thesis are all **locally small**, that is, the morphisms between two objects form a set. Given objects A, B in a category $\mathcal{C}$, we will denote the set of morphisms from A to B by $\mathcal{C}(A, B)$. We will explicitly state when a category is also **small** (for example groupoids), that is, their objects also form a set.

If $\mathcal{C}$ is a category, $\mathcal{C}^{\mathrm{op}}$ denotes its **opposite category**, whose objects are those of $\mathcal{C}$, and morphisms are those of $\mathcal{C}$ that have the domain and target swapped. By a **concrete structure** on a category $\mathcal{C}$ we mean a faithful functor $\mathcal{C} \to \mathcal{S}\mathrm{et}$. A

concrete category is a category with a concrete structure. A **point** in an object X of a (not necessarily concrete) category $\mathcal{C}$ is a morphism $x : * \to X$ in $\mathcal{C}$, where $*$ is the terminal object of $\mathcal{C}$, if it exists.

The following (non exhaustive) list provides the notation for specific categories that appear in the thesis:

category	objects	morphisms
$\mathcal{S}\mathrm{et}$	sets	functions
$\mathcal{T}\mathrm{op}$	topological spaces	continuous maps
$\mathcal{M}\mathrm{fld}$	smooth manifolds	smooth maps
$\mathcal{E}\mathrm{ucl}$	open subsets of $\mathbb{R}^n$ $(n \geq 0)$	smooth maps
$\mathcal{D}\mathrm{flg}$	diffeological spaces	smooth maps
$\boldsymbol{\Delta}$	non-empty finite ordinals	order-preserving maps
$\mathrm{s}\mathcal{S}\mathrm{et}$	simplicial sets	natural transformations
$\mathrm{s}\mathcal{M}\mathrm{fld}$	simplicial manifolds	natural transformations
$\mathcal{B}\mathrm{un}_G$	G-bundles (G a groupoid object)	G-equivariant bundle morphisms

All manifolds are finite-dimensional smooth manifolds (e.g. [Lee13, Ch. 1]). They are Hausdorff (unless otherwise stated) and second countable. The ring of $\mathbb{R}$-valued smooth maps on M will be denoted by $C^\infty(M) := \mathcal{M}\mathrm{fld}(M, \mathbb{R})$. Given a vector bundle $p : E \to M$, $\Gamma(M, E) = \{a : M \to E \mid p \circ a = \mathrm{id}_M\}$ denotes the space of smooth sections of E. It is a $C^\infty(M)$-module. Given a smooth manifold M, its space of vector fields is denoted by $\mathfrak{X}(M) := \Gamma(M, TM)$. Given a tuple $(x_1, \ldots, \widehat{x}_k, \ldots, x_n)$, the notation $\widehat{x}_k$ means that x_k is omitted.

Chapter 1

Lie groupoids and Lie algebroids

A groupoid is a generalization of the notion of a group, that has many units. It is a category in which every morphism is invertible. Set theoretically, groupoids were first introduced in 1927 by Brandt [Bra27] in his attempt to solve composition problems involving quadratic forms. In the late 1950's, Ehresmann introduced internal groupoids to the topological and smooth settings [Ehr59, Ehr63], which were later refined by Pradines [Pra66]. Lie groupoids are groupoids which are internal to the category of smooth manifolds with the additional property that the source (and therefore the target) map is a smooth submersion. For a more detailed historical note on (Lie) groupoids, see [Bro87] or [Wei96].

Being generalizations of Lie groups, Lie groupoids describe local and global symmetries of geometric structures in more general cases, where one has internal symmetries with respect to its objects and external symmetries via its actions or representations [Wei96]. As such, Lie groupoids model a large class of geometric structures, among which are representations, equivalence relations, group actions, foliations, orbifolds, differentiable stacks, etc. For a rigorous treatment of the subject, the reader may refer to [Mac05, CdSW99, MM03].

The theory of Lie groupoids comes together with its infinitesimal counterpart: Lie algebroids. Being a generalization of the notion of Lie algebras, Lie algebroids are *anchored* vector bundles together with a Lie algebra structure on their space of smooth sections satisfying some Leibniz identity. They were originally introduced by Pradines [Pra67] and arise naturally in Poisson geometry, foliation theory, equivariant geometry and Lie theory. For the general theory of Lie algebroids and its relation to these fields, the reader may refer to [Mac05, CdSW99, MM03, Fer03].

In Section 1.1, we will define the category of Lie groupoids, state their main properties and provide fundamental examples. Lie groupoids will be the prototypical example of *differentiable* groupoids in tangent categories introduced in Section 4.1. We then recall the notion of Lie algebroids in Section 1.2 and discuss the subtlety about their morphisms. Lie algebroids will serve as the namesake example of *abstract* Lie algebroids in tangent categories introduced in Section 4.5.1. We conclude by describing the Lie algebroid of a Lie groupoid in Section 1.3. The constructions therein will provide the preliminaries for a generalized differentiation procedure as discussed in Section 4.5.2.

1.1 Lie groupoids

Definition 1.1.1. A **groupoid** is a small category in which all morphisms are isomorphisms.

Spelled out, a groupoid is composed of two sets G_0 and G_1, called the set of objects and the set of arrows, together with the following structure maps:

- the **source** $s : G_1 \to G_0,\ \ g \mapsto s(g) = s_g$,
- the **target** $t : G_1 \to G_0,\ \ g \mapsto t(g) = t_g$,
- the **multiplication** $m : G_1 \times^{s,t}_{G_0} G_1 \to G_1,\ \ (g,h) \mapsto m(g,h) = gh$,
- the **unit** $1 : G_0 \to G_1,\ \ x \mapsto 1(x) = 1_x$,
- the **inverse** $i : G_1 \to G_1,\ \ g \mapsto i(g) = g^{-1}$,

such that the following diagrams commute:

(i) Conditions on s and t:

$$
\begin{tikzcd}
G_1 \times^{s,t}_{G_0} G_1 \arrow[r, "m"] \arrow[d, "\mathrm{pr}_2"'] & G_1 \arrow[d, "s"] \\
G_1 \arrow[r, "s"'] & G_0
\end{tikzcd}
\qquad
\begin{tikzcd}
G_1 \times^{s,t}_{G_0} G_1 \arrow[r, "m"] \arrow[d, "\mathrm{pr}_1"'] & G_1 \arrow[d, "t"] \\
G_1 \arrow[r, "t"'] & G_0
\end{tikzcd}
\tag{1.1}
$$

$$
\begin{tikzcd}
G_0 \arrow[r, "1"] \arrow[dr, "\mathrm{id}_{G_0}"'] & G_1 \arrow[d, "s"] \\
& G_0
\end{tikzcd}
\quad
\begin{tikzcd}
G_0 \arrow[r, "1"] \arrow[dr, "\mathrm{id}_{G_0}"'] & G_1 \arrow[d, "t"] \\
& G_0
\end{tikzcd}
\quad
\begin{tikzcd}
G_1 \arrow[r, "i"] \arrow[dr, "t"'] & G_1 \arrow[d, "s"] \\
& G_0
\end{tikzcd}
\quad
\begin{tikzcd}
G_1 \arrow[r, "i"] \arrow[dr, "s"'] & G_1 \arrow[d, "t"] \\
& G_0
\end{tikzcd}
\tag{1.2}
$$

(ii) Associativity:

$$
\begin{tikzcd}
G_1 \times^{s,t}_{G_0} G_1 \times^{s,t}_{G_0} G_1 \arrow[r, "m \times_{G_0} \mathrm{id}_{G_1}"] \arrow[d, "\mathrm{id}_{G_1} \times_{G_0} m"'] & G_1 \times^{s,t}_{G_0} G_1 \arrow[d, "m"] \\
G_1 \times^{s,t}_{G_0} G_1 \arrow[r, "m"'] & G_1
\end{tikzcd}
\tag{1.3}
$$

(iii) Unitality:

$$
\begin{tikzcd}
G_1 \arrow[r, "(1\circ t, \mathrm{id}_{G_1})"] \arrow[dr, "\mathrm{id}_{G_1}"'] & G_1 \times^{s,t}_{G_0} G_1 \arrow[d, "m"] & G_1 \arrow[l, "(\mathrm{id}_{G_1}, 1\circ s)"'] \arrow[dl, "\mathrm{id}_{G_1}"] \\
& G_1 &
\end{tikzcd}
\tag{1.4}
$$

(iv) Invertibility:

$$\begin{tikzcd} G_1 \arrow[r, "(\mathrm{id}_{G_1}, i)"] \arrow[d, "t"'] & G_1 \times^{s,t}_{G_0} G_1 \arrow[d, "m"'] & G_1 \arrow[l, "(i, \mathrm{id}_{G_1})"'] \arrow[d, "s"] \\ G_0 \arrow[r, "1"'] & G_1 & G_0 \arrow[l, "1"] \end{tikzcd} \tag{1.5}$$

Remark 1.1.2. The commutativity of the second and fourth diagrams in (1.2) is implied by the first and third diagrams in (1.2) and the invertibility axiom. We keep the diagrams in the axioms for later reference.

A groupoid G is usually depicted by $G_1 \overset{s}{\underset{t}{\rightrightarrows}} G_0$, an arrow $g \in G_1$ by

$$\begin{tikzcd} s_g \arrow[r, bend left, "g"] & t_g \end{tikzcd}$$

and the multiplication of a pair of composable arrows $(g, h) \in G_1 \times^{s,t}_{G_0} G_1$ by

$$\begin{tikzcd} s_{gh}=s_h \arrow[r, bend left, "h"] \arrow[rr, bend right, "gh"'] & t_h=s_g \arrow[r, bend left, "g"] & t_g=t_{gh} \end{tikzcd}$$

Definition 1.1.3. A **Lie groupoid** is a groupoid internal to the category $\mathcal{M}\mathrm{fld}$ of smooth manifolds, where the source (and therefore the target) map is a smooth submersion.

Remark 1.1.4. In order not to exclude interesting examples from the theory of (singular) foliations (e.g. [MM03, Ch. 5], [AS09]), the manifold G_1 of arrows is not required to be Hausdorff, whereas the manifold G_0 of objects will be Hausdorff.

Remark 1.1.5. The requirement that the source (and hence the target) map is a submersion in the definition of a Lie groupoid ensures that the domain $G_1 \times^{s,t}_{G_0} G_1$ of the multiplication is a smooth manifold. There is another significant reason for this assumption, which concerns the construction of the Lie algebroid of a Lie groupoid (Sec. 1.3). Mainly, for the differentiation of a Lie groupoid to its Lie algebroid:

(1) a number of pullbacks must exist, for which the assumption that the source map is a submersion is sufficient;

(2) the tangent functor must commute with the *nerve* of the groupoid (Ex. 2.1.6), for which this assumption is necessary.

In Definition 4.1.6, we will introduce *differentiable* groupoid objects in any category, where (1) and (2) are imposed as axioms to enable their differentiation.

Definition 1.1.6. A **Lie groupoid morphism** is a smooth functor.

Spelled out, a morphism between two Lie groupoids $G_1 \rightrightarrows G_0$ and $H_1 \rightrightarrows H_0$ is a pair of smooth maps $F_1 : G_1 \to H_1$ and $F_0 : G_0 \to H_0$, such that the following diagrams

$$\begin{tikzcd} G_1 \arrow[r, "F_1"] \arrow[d, shift right, "s_G"'] \arrow[d, shift left, "t_G"] & H_1 \arrow[d, shift right, "s_H"'] \arrow[d, shift left, "t_H"] \\ G_0 \arrow[r, "F_0"'] & H_0 \end{tikzcd} \qquad \begin{tikzcd} G_1 \times_{G_0} G_1 \arrow[r, "F_1 \times_{F_0} F_1"] \arrow[d, "m_G"'] & H_1 \times_{H_0} H_1 \arrow[d, "m_H"] \\ G_1 \arrow[r, "F_1"'] & H_1 \end{tikzcd}$$

commute.[1]

Remark 1.1.7. It can be easily checked that a morphism between two Lie groupoids intertwines the unit and inverse maps (e.g. [Ain17, Rem. 1.2.7]).

Lie groupoids together with morphisms between them form a category $\mathcal{L}\mathrm{ie}\mathcal{G}\mathrm{rpd}$. Given a Lie groupoid $G_1 \rightrightarrows G_0$ and an element $x \in G_0$, the source fiber $s^{-1}(x)$ and the target fiber $t^{-1}(x)$ are closed embedded submanifolds of G_1 since s and t are submersions [Lee13, Cor. 5.13]. Moreover, the inverse $i : G_1 \to G_1$ is a diffeomorphism which exchanges the source and target fibers [Mac05, Prop. 1.1.5]. Additionally, the unit $1 : G_0 \to G_1$ has a retract (being a section of both the source and target maps), so that G_0 can be viewed as a closed embedded submanifold of G_1. Figure 1.1, which is inspired from [CdSW99, p. 86], depicts a Lie groupoid.

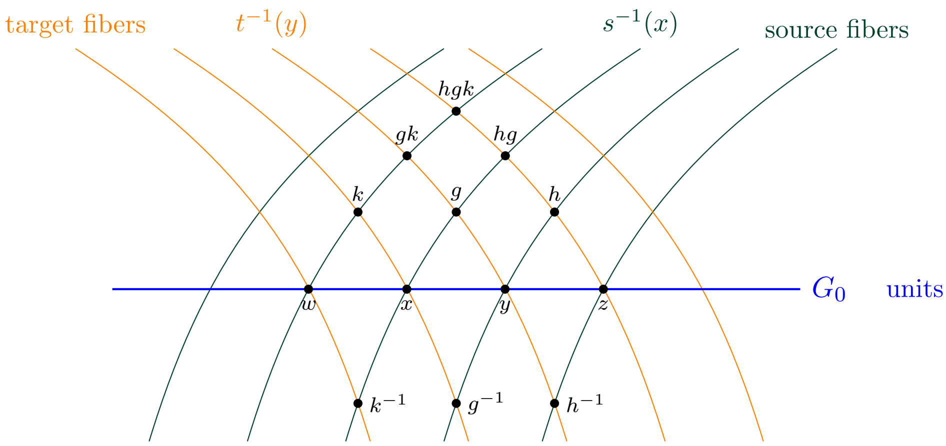


Figure 1.1: A Lie groupoid $G_1 \rightrightarrows G_0$

Example 1.1.8. Here are some of the main examples of Lie groupoids:

(i) A **Lie group** can be viewed as a Lie groupoid over a point $G \rightrightarrows *$.

(ii) Given a smooth manifold M, there is the associated **pair groupoid** $M \times M \rightrightarrows M$. The structure maps are given by $s(x, y) = x$, $t(x, y) = y$, $(y, z)(x, y) = (x, z)$, $1_x = (x, x)$ and $(x, y)^{-1} = (y, x)$ for all $x, y, z \in M$.

[1] The subscripts G and H refer to the structure maps of G and H respectively. The first diagram is composed of two commutative squares, i.e. $s_H \circ F_1 = F_0 \circ s_G$ and $t_H \circ F_1 = F_0 \circ t_G$.

(iii) For a Lie groupoid G, there is the induced **tangent groupoid** $TG_1 \rightrightarrows TG_0$. The structure maps are given by the differentials of the structure maps of G.

(iv) A smooth action of a Lie group G on a smooth manifold M induces a Lie groupoid structure on $G \times M$ over M. For all $g, h \in G$ and $x \in M$, $s(g,x) = x$, $t(g,x) = g \cdot x$, $(g, h \cdot x)(h,x) = (gh, x)$, $1_x = (e,x)$ and $(g,x)^{-1} = (g^{-1}, g \cdot x)$. The resulting Lie groupoid is called the **action groupoid** and is denoted by $G \ltimes M$.

Remark 1.1.9. Two Lie groupoids G and H are isomorphic if there are smooth functors $F : G \to H$ and $F' : H \to G$ such that $FF' = \mathrm{id}_H$ and $F'F = \mathrm{id}_G$. An isomorphism of Lie groupoids is often too strong. The notion of *Morita equivalence* between Lie groupoids provides a notion of weak equivalence, which is crucial for many applications. It is given by a *principal G-H bibundle* [MM03, Sec. 5.4].

A such, Lie groupoids form a bicategory with 1-morphisms given by right principal bibundles and 2-morphisms given by morphisms of bibundles, i.e. biequivariant maps. It is well-known that the bicategory of Lie groupoids is equivalent to the bicategory of *differentiable stacks*. In this way, Lie groupoids present differentiable stacks up to Morita equivalence. As we will not use this concept in this thesis, we refer the reader to [Blo08, BX11, dH13, Gin13].

1.2 Lie algebroids

Definition 1.2.1. A **Lie algebroid** over a smooth manifold M consists of a vector bundle $A \to M$, a vector bundle map $\rho : A \to TM$, called the **anchor**, and a Lie bracket $[\cdot\,,\cdot]$ on its space of smooth sections $\Gamma(M, A)$, such that

$$[a, fb] = f[a,b] + \big((\rho \circ a) \cdot f\big)b \tag{1.6}$$

for all $a, b \in \Gamma(M, A)$ and $f \in C^\infty(M)$.

Remark 1.2.2. Equation (1.6) is called the **Leibniz rule** for Lie algebroids. Recall that the $C^\infty(M)$-module structure of $\Gamma(M, A)$ is given by

$$(fa)(p) = f(p)a(p)$$

for all $f \in C^\infty(M)$, $a \in \Gamma(M, A)$ and $p \in M$. The notation $(\rho \circ a) \cdot f$ refers to the action of the vector field $\rho \circ a$ on the smooth function f.

Remark 1.2.3. The anchor induces a Lie algebra morphism between the space of sections, also denoted by

$$\rho : \Gamma(M, A) \to \mathfrak{X}(M)\,,$$

where $\mathfrak{X}(M) := \Gamma(M, TM)$ is the space of vector fields on M. This was set as an axiom in the original definition of a Lie algebroid (e.g. [Wei96, p. 749]). Later, it was shown that the Jacobi identity, bilinearity and antisymmetry of $[\cdot\,,\cdot]$, and the Leibniz rule imply that ρ preserves the Lie brackets. In Section 4.5.1, we generalize the notion of Lie algebroids in the setting of tangent categories, where it becomes crucial to impose this as a separate axiom again (see Def. 4.5.1 and Rem. 4.5.3).

Definition 1.2.4. A **morphism between two Lie algebroids** A and B over the same smooth manifold M is a vector bundle map $\varphi : A \to B$ over the identity id_M such that the following equations hold:

(i) $\rho_B \circ \varphi = \rho_A$,

(ii) $\varphi([a, a']_A) = [\varphi(a), \varphi(a')]_B$,

for all sections $a, a' \in \Gamma(M, A)$.

Remark 1.2.5. Defining morphisms between Lie algebroids over different bases is more subtle since there is no directly induced map from the sections of A to that of B. In [HM90, Def. 1.3], one encounters a possible way of phrasing the compatibility condition of the Lie brackets. However, this approach utilizes choices of decompositions of sections (where one also shows that the compatibility is independent of the choices involved), and thus is very complicated to work with.

A more succinct way of defining general Lie algebroid morphisms starts with the observation that Lie algebroid structures on a vector bundle $A \to M$ are in one to one correspondence with degree 1 derivations d on the graded algebra $\Gamma(M, \Lambda^\bullet A^*)$ such that $d^2 = 0$ [Vai97]. Here, $\Lambda^\bullet A^* \to M$ denotes the exterior bundle of the dual bundle of A. A Lie algebroid morphism is then defined to be a vector bundle map $\varphi : A \to B$ such that $\varphi^* : \Gamma(M, \Lambda^\bullet B^*) \to \Gamma(M, \Lambda^\bullet A^*)$ is a cochain map (see [Kla17, Sec. 2.3] for a detailed account on this, see also [Jot18, Sec. 2.1]).

Lie algebroids together with morphisms between them form a category $\mathcal{L}\mathrm{ie}\mathcal{A}\mathrm{lgd}$.

Example 1.2.6. Some of the examples of Lie algebroids are as follows:

(i) A **Lie algebra** can be viewed as a Lie algebroid over a point $\mathfrak{g} \to *$.

(ii) The tangent bundle TM of a smooth manifold M is a Lie algebroid over M, called the **tangent Lie algebroid**, together with the usual Lie bracket on the space $\mathfrak{X}(M)$ of vector fields. The anchor is the identity on TM.

(iii) Given an action of a Lie algebra $\mathfrak{g}$ on a manifold M, i.e. a Lie algebra map $\varphi : \mathfrak{g} \to \mathfrak{X}(M)$, one obtains a Lie algebroid $\mathfrak{g} \ltimes M$, called the **action Lie algebroid**. The underlying vector bundle is the trivial bundle $\mathfrak{g} \times M \to M$. The anchor is given by $\rho(X, p) = \varphi(X)_p$ for all $X \in \mathfrak{g}$ and $p \in M$. The Lie bracket is given by

$$[v, w]_p = [v_p, w_p] + \big(\varphi(v_p) \cdot w\big)|_p + \big(\varphi(w_p) \cdot v\big)|_p$$

for all $v, w \in \Gamma(M, \mathfrak{g} \times M) \cong \mathcal{M}\mathrm{fld}(M, \mathfrak{g})$ and $p \in M$.

Examples (i) and (ii) are considered to be the two extreme cases of Lie algebroids. In many algebraic constructions on Lie algebroids, one often recovers known instances in these two cases. For instance, the *Lie algebroid cohomology* of $TM \to M$ recovers the de Rham cohomology of M and that of $\mathfrak{g} \to *$ recovers the Chevalley-Eilenberg cohomology of $\mathfrak{g}$ (see [Fer03] for a concise description of Lie algebroid cohomology, see [Mac87, Ch.IX] for details).

1.3 The Lie algebroid of a Lie groupoid

To each Lie groupoid one can associate a Lie algebroid, which generalizes the differentiation of Lie groups to Lie algebras. Recall that the Lie algebra of a Lie group G is the tangent space to G at the identity e as a vector space. Under the identification of T_eG with the space of right-invariant vector fields on G, and using the fact that the space of right-invariant vector fields is closed under the usual Lie bracket of vector fields, one equips this space with a Lie bracket.

The Lie algebroid of a Lie groupoid will be defined similarly, where there are two subtleties. The first one concerns the existence of many units in a groupoid, which will lead to a vector bundle $A \to G_0$ instead of a vector space. The second one concerns the construction of the Lie bracket on the space $\Gamma(G_0, A)$ of sections. Similar to the case of Lie groups, the Lie bracket will be defined by identifying $\Gamma(G_0, A)$ with the right-invariant vector fields on G_1. However, we restrict the vector fields to be tangent to the source fibers, where the right translation is well-defined.

In this thesis, we do not address the question of integrability of a Lie algebroid to a Lie groupoid. The interested reader is invited to read [CF03] for a detailed exposition on the failure of Lie's third theorem in the Lie groupoid case and obstructions to integrability.

Let G be a Lie groupoid with structure maps $s, t, m, 1, i$. We consider the subbundle of the tangent bundle $TG_1 \to G_1$ consisting of vectors tangent to the source fibers. This is obtained by pulling back the differential $Ts : TG_1 \to TG_0$ of the source map via the zero section:

$$\begin{tikzcd} \ker Ts := TG_1 \times_{TG_0} G_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & TG_1 \arrow[d, "Ts"] \\ G_0 \arrow[r, "0_{G_0}"'] & TG_0 \end{tikzcd} \tag{1.7}$$

Since s is a submersion, it follows that $\ker Ts$ exists in the category of smooth manifolds, and that it is a smooth subbundle of TG_1:

$$\begin{tikzcd} \ker Ts \arrow[r] \arrow[d] & TG_1 \arrow[dl] \\ G_1 & \end{tikzcd}$$

The fiber of $\ker Ts$ at an element $g : x \to y$ in G_1 is given by $\ker Ts|_g = T_g\big(s^{-1}(x)\big)$.

As a second step, we consider the restriction of $\ker Ts$ to the identity bisection of the groupoid:

$$\begin{tikzcd} A := G_0 \times_{G_1} \ker Ts \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \ker Ts \arrow[d] \\ G_0 \arrow[r, "1"'] & G_1 \end{tikzcd} \tag{1.8}$$

The fibers of A are given by $A_x = T_{1_x}\big(s^{-1}(x)\big)$ for all $x \in G_0$, as depicted in Figure 1.2. The anchor is defined by the restriction of the differential of the target map to A:

$$\rho : A \longrightarrow \ker Ts \longrightarrow TG_1 \xrightarrow{Tt} TG_0 \,. \tag{1.9}$$

We now proceed by defining the Lie algebra structure on $\Gamma(G_0, A)$.

Remark 1.3.1. Let $g : x \to y$ be an element of G_1. Then, the right translation by g,

$$\begin{aligned} R_g : s^{-1}(y) &\longrightarrow s^{-1}(x) \\ h &\longmapsto hg\,, \end{aligned}$$

is a diffeomorphism only of the source fibers since the multiplication map of G is defined only on composable pairs. Similarly, the left translation by g,

$$\begin{aligned} L_g : t^{-1}(x) &\longrightarrow t^{-1}(y) \\ h &\longmapsto gh\,, \end{aligned}$$

is a diffeomorphism of the target fibers only.

Definition 1.3.2. A vector field $v : G_1 \to TG_1$ is called s**-vertical** if it is tangent to the source fibers, that is, it takes values in $\ker Ts$. The vector field v is called **right-invariant** if it is s-vertical and if it is invariant under right translations, that is $(TR_h)_g\, v_g = v_{gh}$ for all $(g, h) \in G_1 \times_{G_0} G_1$.

Left-invariant vector fields can be defined similarly, by being tangent to the target fibers and invariant under left translations. Denote the space of right-invariant vector fields on G by $\mathfrak{X}(G_1)^G$. We equip it with a $\mathcal{M}\mathrm{fld}(G_0, \mathbb{R})$-module structure via the assignment

$$(fv)_g := f(t_g)v_g$$

for all $v \in \mathfrak{X}(G_1)^G$ and $f \in \mathcal{M}\mathrm{fld}(G_0, \mathbb{R})$.

Remark 1.3.3. The assignment to each section $a \in \Gamma(G_0, A)$, a right-invariant vector field $\vec{a}$ defined by

$$g : x \to y \quad \longmapsto \quad \vec{a}_g := (TR_g)_{1_y} a_y \tag{1.10}$$

defines an isomorphism of $\mathcal{M}\mathrm{fld}(G_0, \mathbb{R})$-modules

$$\Gamma(G_0, A) \cong \mathfrak{X}(G_1)^G\,.$$

The space of right-invariant vector fields is closed under the usual Lie bracket of vector fields on G_1 and hence is a Lie subalgebra of $\mathfrak{X}(G_1)$. Under the isomorphism in (1.10), there is an induced Lie bracket on $\Gamma(G_0, A)$, uniquely determined by

$$\overrightarrow{[a, b]} = [\vec{a}, \vec{b}\,] \tag{1.11}$$

for all $a, b \in \Gamma(G_0, A)$.

Theorem 1.3.4. *Let G be a Lie groupoid. Then, the vector bundle $A \to G_0$* (1.8) *together with the anchor $\rho : A \to TG_0$* (1.9) *and the Lie bracket* (1.11) *on* $\Gamma(G_0, A)$ *is a Lie algebroid.*

Proof. For a detailed proof of the Leibniz rule (1.6) for Lie algebroids, the reader may refer to [CF11, Prop. 1.24]. □

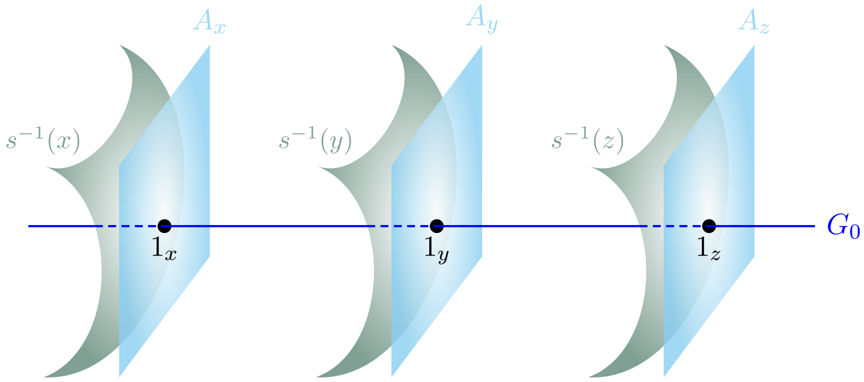


Figure 1.2: The Lie algebroid of a Lie groupoid

We will now discuss the *Lie functor* at the level of morphisms. Let H be another Lie groupoid and let

$$\begin{tikzcd} G_1 \arrow[r, "F_1"] \arrow[d, shift right, "s_G"'] \arrow[d, shift left, "t_G"] & H_1 \arrow[d, shift right, "s_H"'] \arrow[d, shift left, "t_H"] \\ G_0 \arrow[r, "F_0"'] & H_0 \end{tikzcd}$$

be a Lie groupoid morphism. Then, $TF_1 : TG_1 \to TH_1$ restricts to a vector bundle map

$$\begin{tikzcd} \ker Ts_G \arrow[r, "TF_1 \times_{TF_0} F_0"] \arrow[d] & \ker Ts_H \arrow[d] \\ G_1 \arrow[r, "F_1"'] & H_1 \end{tikzcd}$$

where we have used the definition $\ker Ts_G = TG_1 \times_{TG_0} G_0$ of the kernel. Pulling back along the identity bisections of G and H, we obtain the following commutative diagram:

$$\begin{tikzcd} & \ker Ts_G \arrow[rr, "TF_1 \times_{TF_0} F_0"] \arrow[dd] & & \ker Ts_H \arrow[dd] \\ A \arrow[ur] \arrow[rr, "\Phi"] \arrow[dd] & & B \arrow[ur] \arrow[dd] & \\ & G_1 \arrow[rr, "F_1"] & & H_1 \\ G_0 \arrow[ur, "1_G"'] \arrow[rr, "F_0"'] & & H_0 \arrow[ur, "1_H"'] & \end{tikzcd}$$

The left and right squares of this diagram are the Lie algebroids $A = G_0 \times_{G_1} \ker Ts_G$ and $B = H_0 \times_{H_1} \ker Ts_H$ of G and H respectively. The map

$$\Phi := F_0 \times_{F_1} TF_1 \times_{TF_0} F_0$$

is obtained via the universal property of pullbacks. In elements, it is given by

$$\Phi : (x, v) \longmapsto \big(F_0(x), (TF_1)_{1_x} v\big) ,$$

for all $x \in G_0$ and $v \in \ker Ts_G$.

The map Φ is a Lie algebroid morphism. As discussed in Remark 1.2.5, defining morphisms of Lie algebroids is subtle and so we will not prove this statement in this thesis (see [HM90, Thm. 1.7] for a proof using the approach of decompositions of sections). The upshot is that there is a functor

$$\mathrm{Lie} : \mathcal{L}\mathrm{ie}\mathcal{G}\mathrm{rpd} \longrightarrow \mathcal{L}\mathrm{ie}\mathcal{A}\mathrm{lgd}\,,$$

called the **Lie functor** which associates to each Lie groupoid its Lie algebroid, and to each Lie groupoid morphism a Lie algebroid morphism as described above.

Example 1.3.5. Here are some of the basic examples of differentiation of Lie groupoids into their Lie algebroids:

(i) Let G be a Lie group viewed as a Lie groupoid over a point $G \rightrightarrows *$. Then, its Lie algebroid is the Lie algebra $\mathfrak{g}$ of G, viewed as a Lie algebroid over a point $\mathfrak{g} \rightrightarrows *$.

(ii) The Lie algebroid of the pair groupoid $M \times M \rightrightarrows M$ is the tangent Lie algebroid $TM \to M$.

(iii) Given an action of a Lie group G on a smooth manifold M, the Lie algebroid of the action groupoid $G \ltimes M$ is the action Lie algebroid $\mathfrak{g} \ltimes M$ with respect to the induced infinitesimal action, where $\mathfrak{g}$ is the Lie algebra of G.

Remark 1.3.6. The choice of right-invariant vector fields in the construction of the Lie algebroid of a Lie groupoid is arbitrary. One could have as well chosen left-invariant vector fields and proceeded similarly. The symmetry of the construction is given by the diffeomorphism $Ti : TG_1 \to TG_1$.

Remark 1.3.7. The multiplication map on the tangent groupoid has the following explicit formula

$$Tm|_{(g,h)}(v_g, v_h) = (TR_h)_g\, v_g + (TL_g)_h\, v_h \tag{1.12}$$

for all composable pairs $(g,h) \in G_1 \times_{G_0} G_1$, $v_g \in \ker Ts|_g$, and $v_h \in \ker Tt|_h$ (e.g. [Mac05, p. 6]).

Chapter 2

Simplicial objects

Simplicial objects are combinatorial models that can serve as the building blocks of higher structures, such as higher categories. Intuitively, a simplicial object is composed of n-simplices for each $n \geq 0$, where 0-simplices are visualized as points, 1-simplices as line segments, 2-simplices as triangles, 3-simplices as tetrahedra and more generally n-simplices as abstract n-dimensional tetrahedra. As such, simplicial objects generalize directed graphs, simplicial complexes from algebraic topology, as well as categories and groupoids to the higher setting. They were originally introduced by Eilenberg and Zilber [EZ50]. Some references for a detailed account on the subject and its applications are [May92, GJ99, Rie11].

One of the key examples of simplicial sets, that is simplicial objects in the category of sets with functions, is the nerve of a groupoid (or more generally, a small category). Its 0-simplices are given by the objects of the groupoid, its 1-simplices by the arrows, its 2-simplices by pairs of composable arrows, and its n-simplices by strings of n-composable arrows. Since the composition of arrows in a groupoid is unique, the nerve of a groupoid further satisfies (unique) horn filling conditions, called *Kan conditions*. In fact, the nerve construction allows us to identify groupoids with simplicial sets satisfying these Kan conditions. This observation provides a simplicial approach to the notion of (higher) groupoid objects in any category. The reader may refer to [Hen08] and [Zhu09] as the original sources for higher groupoid objects in categories with a *Grothendieck pretopology* (in [MZ15] the case of groupoid objects is elaborated). These will be recalled in Section 5.1.1.

In this chapter we will provide the reader with the foundations of simplicial objects and simplicial constructions for the purposes of this thesis. In Section 2.1, we will define the categories of (co)simplicial objects. The goal of Section 2.2 is twofold. Firstly, we will state the most fundamental and geometric examples of simplicial sets: the n-simplex as well as its horns and boundary; and generalize them as objects in any category using right Kan extensions. Secondly, we will describe higher set theoretic groupoids using Kan conditions. Lastly, in Section 2.3 we will elaborate on the finite data description of simplicial sets via their skeleton and coskeleton functors, and we will discuss a possible generalization to simplicial objects in any category.

2.1 Simplicial and cosimplicial objects

In Section 2.1.1, we will recall the simplex category, which is the model encoding simplicial objects through its coface and codegeneracy maps. This will lead to the functorial as well as the geometric definition of (co)simplicial objects in Sections 2.1.2 and 2.1.3.

2.1.1 The simplex category

Throughout this section and the rest of the thesis, let $\mathbf{\Delta}$ denote the category of non-empty finite ordinals

$$[0] = \{0\}, \quad [1] = \{0 < 1\}, \quad \ldots, \quad [n] = \{0 < \ldots < n\}, \quad \ldots$$

with order-preserving maps as morphisms, called the **simplex category**. This category has two special types of morphisms. Mainly, for all $n \geq 1$ and $0 \leq i \leq n$, let

$$\delta^{n,i} : [n-1] \longrightarrow [n]$$

be the unique injective morphism forgetting i, called the **coface maps**, and for all $n \geq 0$ and $0 \leq i \leq n$, let

$$\sigma^{n,i} : [n+1] \longrightarrow [n]$$

be the unique surjective morphism hitting i twice, called the **codegeneracy maps**. These morphisms satisfy certain obvious relations, called the **cosimplicial identities**, given as follows:

$$\begin{aligned}
\delta^{n+1,i} \circ \delta^{n,j} &= \delta^{n+1,j+1} \circ \delta^{n,i} && \text{if } i \leq j && (2.1)\\
\sigma^{n,j} \circ \sigma^{n+1,i} &= \sigma^{n,i} \circ \sigma^{n+1,j+1} && \text{if } i \leq j && (2.2)\\
\sigma^{n,j} \circ \delta^{n+1,i} &= \delta^{n,i} \circ \sigma^{n-1,j-1} && \text{if } i < j && (2.3)\\
\sigma^{n,j} \circ \delta^{n+1,i} &= \mathrm{id} && \text{if } i = j, j+1 && (2.4)\\
\sigma^{n,j} \circ \delta^{n+1,i} &= \delta^{n,i-1} \circ \sigma^{n-1,j} && \text{if } i > j+1\,, && (2.5)
\end{aligned}$$

whenever they are defined.

Remark 2.1.1. Any morphism of $\mathbf{\Delta}$ is finitely generated by these two kinds, i.e. it can be expressed as a composition of finitely many coface and codegeneracy maps.

On the other hand, we will denote by $\mathbf{\Delta}_+$ the category of *all* finite ordinals with order-preserving maps as morphisms, called the **augmented simplex category**. In particular, $\mathbf{\Delta}_+ = \mathbf{\Delta} \cup [-1]$, where $[-1] = \varnothing$ denotes the empty set. As such, $\mathbf{\Delta}$ is a full subcategory of $\mathbf{\Delta}_+$, where one has an extra coface map, namely the unique map $\delta^{0,0} : [-1] \to [0]$, which satisfies the cosimplicial identity (2.1) for $n = i = j = 0$, equalizing $\delta^{1,0}$ and $\delta^{1,1}$.

2.1.2 Simplicial objects

Definition 2.1.2. A **simplicial object** in a category $\mathcal{C}$ is a $\mathcal{C}$-valued presheaf on $\mathbf{\Delta}$. An **augmented simplicial object** in $\mathcal{C}$ is a $\mathcal{C}$-valued presheaf on $\mathbf{\Delta}_+$.

In other words, a simplicial object in $\mathcal{C}$ is a contravariant functor

$$\begin{aligned} X : \mathbf{\Delta}^{\mathrm{op}} &\longrightarrow \mathcal{C} \\ [n] &\longmapsto X[n] =: X_n \,. \end{aligned}$$

Geometrically, it is composed of a family $\{X_n\}_{n\geq 0}$ of objects in $\mathcal{C}$, where X_n is called the **object of n-simplices**. Furthermore, the coface and codegeneracy maps of $\mathbf{\Delta}$ yield morphisms

$$d_{n,i} := X(\delta^{n,i}) : X_n \longrightarrow X_{n-1}$$

for all $n \geq 1$ and $0 \leq i \leq n$, called the **face morphisms**, and

$$s_{n,i} := X(\sigma^{n,i}) : X_n \longrightarrow X_{n+1}$$

for all $n \geq 0$ and $0 \leq i \leq n$, called the **degeneracy morphisms**. By the functoriality of X, the face and degeneracy morphisms satisfy the relations dual to that of $\mathbf{\Delta}$, called the **simplicial identities**:

$$\begin{aligned} d_{n,j} \circ d_{n+1,i} &= d_{n,i} \circ d_{n+1,j+1} && \text{if } i \leq j && (2.6)\\ s_{n+1,i} \circ s_{n,j} &= s_{n+1,j+1} \circ s_{n,i} && \text{if } i \leq j && (2.7)\\ d_{n+1,i} \circ s_{n,j} &= s_{n-1,j-1} \circ d_{n,i} && \text{if } i < j && (2.8)\\ d_{n+1,i} \circ s_{n,j} &= \mathrm{id} && \text{if } i = j, j+1 && (2.9)\\ d_{n+1,i} \circ s_{n,j} &= s_{n-1,j} \circ d_{n,i-1} && \text{if } i > j+1\,, && (2.10) \end{aligned}$$

whenever defined. Conversely, every simplicial object is determined uniquely in this way by Remark 2.1.1. In many applications, it is easier to define a simplicial object by explicitly describing the family of objects as well as the face and degeneracy morphisms (where one needs to check the simplicial identities). For simplicity, the first index of the face and degeneracy morphisms will be sometimes omitted and we will write d_i and s_i whenever the context is clear. A simplicial object is usually depicted by the following diagram:

$$X_0 \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}\; X_1 \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow}\; X_2 \quad \cdots$$

Definition 2.1.3. A **morphism of simplicial objects** in $\mathcal{C}$ is a natural transformation of functors.

Explicitly, given two simplicial objects $X, Y : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$, a morphism $f : X \to Y$ is composed of a family of morphisms $\{f_n : X_n \to Y_n\}_{n\geq 0}$ in $\mathcal{C}$ which commute with the face and degeneracy morphisms. Pictorially, f is given by the following commutative diagram[1]:

[1]Here, commutative means serially commutative. That is, the squares corresponding to a given face or degeneracy morphism commute. As such, $d_{n,i} \circ f_n = f_{n-1} \circ d_{n,i}$ and $f_{n+1} \circ s_{n,i} = s_{n,i} \circ f_n$.

$$
\begin{tikzcd}
\vdots & \vdots \\
X_2 \arrow[r, "f_2"] \arrow[d, shift right=6] \arrow[d] \arrow[d, shift left=6] & Y_2 \arrow[d, shift right=6] \arrow[d] \arrow[d, shift left=6] \\
X_1 \arrow[r, "f_1"] \arrow[u, shift right=3] \arrow[u, shift left=3] \arrow[d, shift right=3] \arrow[d, shift left=3] & Y_1 \arrow[u, shift right=3] \arrow[u, shift left=3] \arrow[d, shift right=3] \arrow[d, shift left=3] \\
X_0 \arrow[r, "f_0"] \arrow[u] & Y_0 \arrow[u]
\end{tikzcd}
$$

Simplicial objects in $\mathcal{C}$ together with morphisms between them form a category. Being a functor category, it will be denoted by $\mathcal{C}^{\mathbf{\Delta}^{\mathrm{op}}}$.

Remark 2.1.4. The set of morphisms between two simplicial objects X and Y in $\mathcal{C}$ is in natural bijection with the following end[2]

$$\mathcal{C}^{\mathbf{\Delta}^{\mathrm{op}}}(X,Y) \cong \int_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \mathcal{C}(X_n, Y_n)\,.$$

Terminology 2.1.5. A simplicial object in the category $\mathcal{S}\mathrm{et}$ of sets will be called a **simplicial set**; a simplicial object in the category $\mathcal{M}\mathrm{fld}$ of smooth manifolds a **simplicial manifold**, etc. We will sometimes denote the category of simplicial sets by $\mathrm{s}\mathcal{S}\mathrm{et} := \mathcal{S}\mathrm{et}^{\mathbf{\Delta}^{\mathrm{op}}}$ and the category of simplicial manifolds by $\mathrm{s}\mathcal{M}\mathrm{fld} := \mathcal{M}\mathrm{fld}^{\mathbf{\Delta}^{\mathrm{op}}}$.

We will now give an example of a simplicial manifold coming from the nerve construction of Lie groupoids.

Example 2.1.6. Let G be a Lie groupoid. Its nerve NG is a simplicial manifold, where $(NG)_0 := G_0$ and $(NG)_1 := G_1$ are the manifolds of objects and arrows respectively, and for $n \geq 2$,

$$(NG)_n := \underbrace{G_1 \times_{G_0} G_1 \times_{G_0} \cdots \times_{G_0} G_1}_{n\text{-times}}$$

is the manifold of strings of n composable arrows. The face maps $d_{n,i}$ are given by composing the i^{th} and $(i+1)^{\mathrm{st}}$ arrows if $0 < i < n$, and by forgetting the first or last arrow if $i = 0$ or $i = n$ respectively. The degeneracy maps $s_{n,i}$ insert the identity arrow at the i^{th} spot. It follows from the groupoid axioms that the simplicial identities hold.

By Example 2.1.6 we can visualize arrows and strings of composable arrows in a Lie groupoid as line segments, triangles and (generalized) tetrahedra as opposed to curved arrows depicted in Section 1.1. Figure 2.1 demonstrates this.

Remark 2.1.7. Example 2.1.6 can also be worked out for usual small categories or set theoretic groupoids, where the nerve construction gives a simplicial set.

[2]See Section A.2 for a recall on categorical ends; see Remark A.2.3 for the formula of the set of natural transformations in terms of ends.

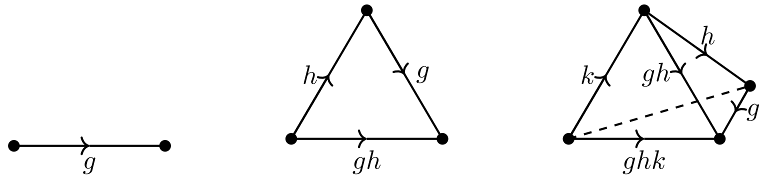


Figure 2.1: Nerve of a Lie groupoid

Definition 2.1.8. A **subsimplicial object** of a simplicial object X in $\mathcal{C}$ is a subpresheaf of X.

Spelling out Definition A.1.6, a subsimplicial set of a simplicial set X is a simplicial set Y such that $Y_n \subseteq X_n$ is a subset, $d^X_{n,i}|_{Y_n} = d^Y_{n,i}$ and $s^X_{n,i}|_{Y_n} = s^Y_{n,i}$ whenever defined. Here, $d^X_{n,i}$ and $d^Y_{n,i}$ are the face maps of X and Y respectively and $s^X_{n,i}$ and $s^Y_{n,i}$ are the degeneracy maps.

Definition 2.1.9. Let X be a simplicial set and $n \geq 1$. An n-simplex $x \in X_n$ is called **degenerate** if it is the image of an $(n-1)$-simplex under a degeneracy map, that is $x = s_{n-1,i}(y)$ for some $y \in X_{n-1}$ and $0 \leq i \leq n-1$. Otherwise, it is called **non-degenerate**.

2.1.3 Cosimplicial objects

Definition 2.1.10. A **cosimplicial object** in a category $\mathcal{C}$ is a $\mathcal{C}$-valued copresheaf on $\mathbf{\Delta}$. An **augmented cosimplicial object** in $\mathcal{C}$ is a $\mathcal{C}$-valued copresheaf on $\mathbf{\Delta}_+$.

Similar to the simplicial setting, a cosimplicial object in $\mathcal{C}$ is given by a covariant functor $X : \mathbf{\Delta} \to \mathcal{C}$ and is equivalently determined by a family $\{X_n\}_{n\geq 0}$ of objects in $\mathcal{C}$, together with **coface morphisms**

$$d^{n,i} := X(\delta^{n,i}) : X_{n-1} \longrightarrow X_n$$

for all $n \geq 1$, $0 \leq i \leq n$ and **codegeneracy morphisms**

$$s^{n,i} = X(\sigma^{n,i}) : X_{n+1} \longrightarrow X_n$$

for all $n \geq 0$, $0 \leq i \leq n$, satisfying the cosimplicial identities (2.1)–(2.5). A cosimplicial object is illustrated by

$$\begin{tikzcd} X_0 \arrow[r, shift left=2] \arrow[r, shift right=2] & X_1 \arrow[l] \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & X_2 \arrow[l, shift left=2] \arrow[l, shift right=2] & \cdots \end{tikzcd}$$

A morphism of two cosimplicial objects in $\mathcal{C}$ is a natural transformation of functors. The category of cosimplicial objects in $\mathcal{C}$ will be denoted by $\mathcal{C}^{\mathbf{\Delta}}$.

Remark 2.1.11. The set of morphisms between two cosimplicial objects X and Y in $\mathcal{C}$ is in natural bijection with the following end

$$\mathcal{C}^{\mathbf{\Delta}}(X,Y) \cong \int_{[n]\in\mathbf{\Delta}} \mathcal{C}(X_n, Y_n)\,,$$

using Remark A.2.3 from the Appendix.

Example 2.1.12. In Section 3.2.5, we will show that given an abstract tangent functor T on a category $\mathcal{C}$ (Def. 3.2.4), the assignment

$$\mathbf{\Delta}_+ \longrightarrow \mathrm{End}(\mathcal{C})\,, \quad [n] \longmapsto T^{n+1}$$

is an augmented cosimplicial object in the category of endofunctors on $\mathcal{C}$.

Remark 2.1.13. The cosimplicial structure of the iterated tangent bundle will be crucial for Definition 5.2.6, where we propose a definition of the *higher Lie algebroid* of a *higher Lie groupoid*.

2.2 The n-simplex, horns, and boundaries

To get a better geometric intuition of simplicial objects, we will consider some of the most basic yet significant examples of simplicial sets: the standard simplicial n-simplex for $n \geq 0$, its outer and inner horns, and its boundary. This is the subject of Section 2.2.1. Using right Kan extensions along the yoneda embedding we will then describe the objects of horns and boundaries of any simplicial object in a category in Section 2.2.2. Lastly, the aim of Section 2.2.3 is to recall the notions of Kan simplicial sets and higher groupoids in the set theoretic sense using horn filling conditions.

2.2.1 The standard simplicial n-simplex

Example 2.2.1. Let $y : \mathbf{\Delta} \to \mathrm{s}\mathcal{S}\mathrm{et}$ be the Yoneda embedding (see Sec. A.1.2).

(i) For all $n \geq 0$, the representable presheaf

$$\Delta^n := y[n] = \mathbf{\Delta}\big(\_\,, [n]\big)$$

is a simplicial set called the **standard simplicial n-simplex**. Its set of m-simplices is given by $\Delta^n_m = \mathbf{\Delta}\big([m], [n]\big)$, and the unique non-degenerate n-simplex by $\mathrm{id}_{[n]} \in \Delta^n_n = \mathbf{\Delta}\big([n], [n]\big)$. Moreover, for all $m > n$, the m-simplices are degenerate.

(ii) For all $n \geq 1$ and $0 \leq i \leq n$, Λ^n_i is the subsimplicial set of Δ^n, called its i^{th} **horn**, which is obtained by removing the i^{th} face from it as well as its unique non-degenerate n-simplex. Namely, its set of non-degenerate m-simplices is given by

$$(\Lambda^n_i)_m = \begin{cases} \Delta^n_m & \text{if } 0 \leq m < n-1 \\ \Delta^n_m \setminus d_{n,i}\big(\,\mathrm{id}_{[n]}\,\big) & \text{if } m = n-1 \\ \varnothing & \text{if } m \geq n\,. \end{cases}$$

The horn Λ^n_i is called an **outer horn** if $i \in \{0, n\}$, and an **inner horn** if $0 < i < n$.

(iii) For all $n \geq 0$, $\partial\Delta^n$ is the subsimplicial set of Δ^n, called its **boundary**, which is obtained by removing its unique non-degenerate n-simplex. Namely, its set of non-degenerate m-simplices is given by

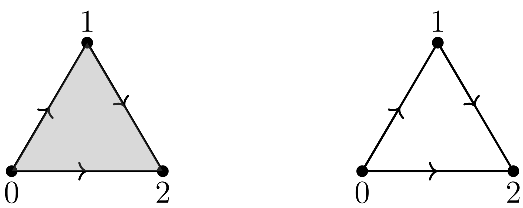
$$(\partial\Delta^n)_m = \begin{cases} \Delta^n_m & \text{if } 0 \leq m < n \\ \varnothing & \text{if } m \geq n\,. \end{cases}$$

Being subsimplicial sets, there are natural monomorphisms of simplicial sets $\Lambda^n_i \to \Delta^n$ and $\partial\Delta^n \to \Delta^n$. At the level of m-simplices, they are given by subsets.

Figures 2.2–2.7 display the standard simplicial n-simplex, its horns and boundary for small values of n. In Figure 2.3, the standard simplicial 2-simplex Δ^2 is colored to highlight the fact that the unique non-degenerate 2-simplex belongs to Δ^2 and not to its boundary. The same argument holds for the standard simplicial 3-simplex Δ^3 in Figure 2.6. However, throughout the thesis these will not be colored in general and it will be clear from the context whether it refers to the whole simplex or only to the boundary. Furthermore, Figure 2.5 depicts the face maps of Δ^2 in the non-degenerate cases.

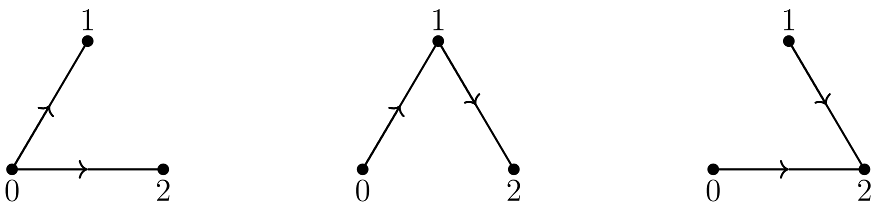
(a) Δ^1 (b) $\partial\Delta^1$

Figure 2.2: The simplicial 1-simplex and its boundary



(a) Δ^2 (b) $\partial\Delta^2$

Figure 2.3: The simplicial 2-simplex and its boundary



(a) Λ^2_0 (b) Λ^2_1 (c) Λ^2_2

Figure 2.4: The horns of Δ^2

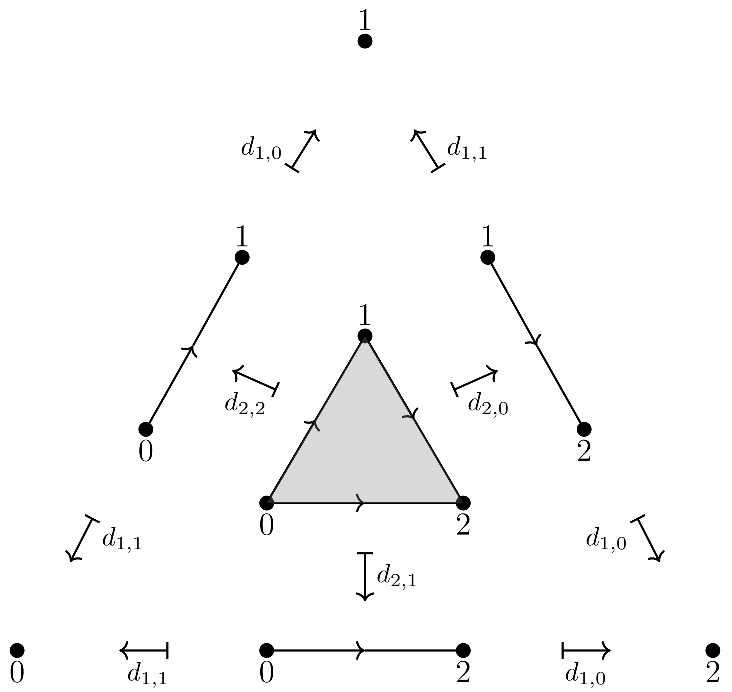
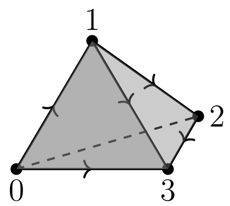
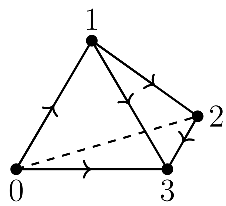


Figure 2.5: The face maps of Δ^2

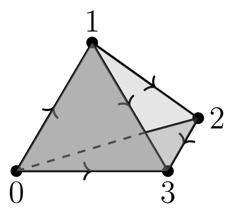


(a) Δ^3

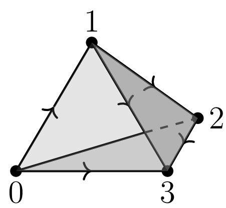


(b) $\partial\Delta^3$

Figure 2.6: The simplicial 3-simplex and its boundary



(a) Λ^3_0



(b) Λ^3_2

Figure 2.7: Some horns of Δ^3

Remark 2.2.2. There is also the notion of the *standard topological n-simplex* Δ^n_{top} which we do not cover in this thesis. Let $\mathcal{T}\mathrm{op}$ be the category of topological spaces with continuous maps. The simplicial and topological simplices are related by a functor, called the *geometric realization functor* $|\_\,| : \mathrm{s}\mathcal{S}\mathrm{et} \to \mathcal{T}\mathrm{op}$, where $|\Delta^n|$ is homeomorphic Δ^n_{top}, which is in turn homeomorphic to the closed n-ball, and $|\partial\Delta^n|$

is homeomorphic to the $(n-1)$-sphere. This has vast applications in simplicial and singular homology theory, and simplicial homotopy theory as a combinatorial model for the standard homotopy theory of topological spaces (see [Hat02], [GJ99] as standard sources).

2.2.2 Horns and boundaries of simplicial objects

It follows from the Yoneda lemma (Lem. A.1.18) that the set of n-simplices of a simplicial set X is in natural bijection with the set of morphisms $\Delta^n \to X$ of simplicial sets, that is, there is a natural bijection of sets

$$\mathrm{s}\mathcal{S}\mathrm{et}(\Delta^n, X) \cong X_n\,. \tag{2.11}$$

This suggests defining the set of (n,i)-horns of X as the set of all morphisms $\Lambda^n_i \to X$ of simplicial sets. Since Λ^n_i is a simplicial object in the category of sets, the definition of horns of simplicial objects in other categories is more subtle. For a rigorous definition, we will use Kan extensions along the Yoneda embedding[3].

Remark 2.2.3. By the density theorem (Prop. A.1.24), there is a natural isomorphism

$$X \cong \operatorname*{colim}_{\Delta^n \to X} \Delta^n\,, \tag{2.12}$$

for every simplicial set X. The index category is the comma category[4] $y \downarrow X$, whose objects are natural transformations $\Delta^n \to X$ and morphisms are commutative triangles.

Right Kan extensions along the Yoneda embedding

Let $\mathcal{C}$ be a category. In what follows, we will sometimes talk about limits of diagrams in $\mathcal{C}$ without knowing about their actual existence. In that case, we consider the corresponding limit in the category $\mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}}$ of presheaves on $\mathcal{C}$ via the Yoneda embedding, as explained in Remark A.1.20. We address the question of representability whenever needed.

Definition 2.2.4. Given a simplicial object X in $\mathcal{C}$, define $\widetilde{X}$ to be the right Kan extension of X along the opposite y^{op} of the Yoneda embedding.

This can be depicted by the following diagram

$$\begin{tikzcd}
\mathbf{\Delta}^{\mathrm{op}} \arrow[r, "X"] \arrow[d, "y^{\mathrm{op}}"'] & \mathcal{C} \\
\mathrm{s}\mathcal{S}\mathrm{et}^{\mathrm{op}} \arrow[ur, dashed, "\widetilde{X} := \mathrm{Ran}_{y^{\mathrm{op}}} X"']
\end{tikzcd}$$

[3] See Section A.3 for a review of the notion of Kan extensions.

[4] For a recap on comma categories, see Definition A.1.21 and Remark A.1.23 in the Appendix.

Remark 2.2.5. Let S be a simplicial set. By Theorem A.3.3, $\widetilde{X}(S)$ is the limit of the diagram

$$\begin{array}{ccccc} S \downarrow y^{\mathrm{op}} & \xrightarrow{q_{\mathbf{\Delta}^{\mathrm{op}}}} & \mathbf{\Delta}^{\mathrm{op}} & \xrightarrow{X} & \mathcal{C} \\ (S \to \Delta^n) & \longmapsto & [n] & \longmapsto & X_n\,. \end{array} \tag{2.13}$$

Observe that $S \to \Delta^n$ in the index category is a morphism in $\mathrm{s}\mathcal{S}\mathrm{et}^{\mathrm{op}}$. We will often write

$$\widetilde{X}(S) = \lim_{\Delta^n \to S} X_n\,, \tag{2.14}$$

where $\Delta^n \to S$ is understood to run through the index category $(y \downarrow S)^{\mathrm{op}} \cong S \downarrow y^{\mathrm{op}}$ (see Remark A.1.25 for the isomorphism). Being an object of $(y \downarrow S)^{\mathrm{op}}$, $\Delta^n \to S$ is a morphism in $\mathrm{s}\mathcal{S}\mathrm{et}$.

In the next lemma we show that if $\mathcal{C} = \mathcal{S}\mathrm{et}$, then the functor $\widetilde{X}$ is representable by X and hence recovers the usual hom functor.

Lemma 2.2.6. *Let X be a simplicial set. Then, $\widetilde{X}$ is naturally isomorphic to yX.*

Proof. For every simplicial set S,

$$\begin{aligned} \widetilde{X}(S) &= \lim_{\Delta^n \to S} X_n \\ &\cong \lim_{\Delta^n \to S} \mathrm{s}\mathcal{S}\mathrm{et}(\Delta^n, X) \qquad (2.15) \\ &\cong \mathrm{s}\mathcal{S}\mathrm{et}(\operatorname*{colim}_{\Delta^n \to S} \Delta^n, X) \qquad (2.16) \\ &\cong \mathrm{s}\mathcal{S}\mathrm{et}(S, X)\,, \end{aligned}$$

where in the first step we have used the Yoneda lemma, spelled out in Equation (2.11), in the second step the fact that the hom functor preserves limits[5], and in the last step the density theorem, spelled out in Equation (2.12). All the isomorphisms are natural in S and thus $\widetilde{X}$ is naturally isomorphic to $yX = \mathrm{s}\mathcal{S}\mathrm{et}(\_, X)$. □

Caution 2.2.7. The limit in (2.15) is indexed by the category $S \downarrow y^{\mathrm{op}} \cong (y \downarrow S)^{\mathrm{op}}$, whereas the colimit in (2.16) is indexed by $y \downarrow S$. It will usually be clear from the context what exactly the index category is.

In particular, if X is a simplicial set and $S = \Delta^n$, it follows from Lemma 2.2.6 that

$$\widetilde{X}(\Delta^n) \cong \mathrm{s}\mathcal{S}\mathrm{et}(\Delta^n, X) \cong X_n$$

for all $n \geq 0$. It turns out that this observation is not only true for simplicial sets, but also for simplicial objects.

Lemma 2.2.8. *Let X be a simplicial object in $\mathcal{C}$. Then, $\widetilde{X}(\Delta^n) \cong X_n$ for all $n \geq 0$.*

[5] The hom functor preserves limits in both of its arguments. In particular, being contravariant in the first argument, and since limits in the opposite category are colimits in the actual category, the hom functor takes colimits in the first argument to limits.

Proof. By Corollary A.1.19, the Yoneda embedding y is fully faithful. Thus, it follows from Corollary A.3.7 that

$$\widetilde{X}(\Delta^n) = \widetilde{X}y[n] \cong X[n] = X_n$$

for all $n \geq 0$. In the second step we have also used that $y[n] = y^{\mathrm{op}}[n]$. □

Remark 2.2.9. If $\mathcal{C} = \mathcal{S}\mathrm{et}$, spelling out the limit of Diagram (2.13) in terms of equalizers and products using [ML98, Thm. V.2.2], one obtains that $\widetilde{X}(S) \cong \mathrm{s}\mathcal{S}\mathrm{et}(S, X)$ is the equalizer of

$$\prod_{n\geq 0} \mathcal{S}\mathrm{et}(S_n, X_n) \rightrightarrows \prod_{[m]\to[n]} \mathcal{S}\mathrm{et}(S_n, X_m)\,,$$

where the second product is over all morphisms $[m] \to [n]$ in $\boldsymbol{\Delta}$. In a similar manner, one can compute the limit of (2.13) in an arbitrary category $\mathcal{C}$. If the limit does not exist in $\mathcal{C}$ we view it as a presheaf on $\mathcal{C}$ as mentioned earlier. This approach is sometimes used as the definition of $\widetilde{X}$ in the literature (see for instance [Hen08, Sec. 2]).

Notation 2.2.10. By the previous observations, we will often denote $\widetilde{X}$ by X itself if the context is clear. This is a slight abuse of notation. However, knowing that the object of n-simplices is usually written as X_n with a subscript, this convention is convincing. Hence, we have that $X(\Delta^n) \cong X_n$.

The objects of horns and boundaries

Let Λ^n_i be the i^{th} horn of Δ^n and $\partial\Delta^n$ its boundary (Example 2.2.1). Given a simplicial object $X : \boldsymbol{\Delta}^{\mathrm{op}} \to \mathcal{C}$, we have

$$X(\Lambda^n_i) = \lim_{\Delta^k\to\Lambda^n_i} X_k \quad \text{and} \quad X(\partial\Delta^n) = \lim_{\Delta^k\to\partial\Delta^n} X_k$$

by using Equation (2.14). Let $\Lambda^n_i \to \Delta^n$ and $\partial\Delta^n \to \Delta^n$ be the natural monomorphisms of simplicial sets.

Definition 2.2.11. Let X be a simplicial object in $\mathcal{C}$. Then, $X(\Lambda^n_i)$ is called the **object of (n,i)-horns** in X and the morphism

$$p_{n,i} := X(\Lambda^n_i \to \Delta^n) : X_n \longrightarrow X(\Lambda^n_i) \tag{2.17}$$

is called the **(n,i)-horn projection** for $n \geq 1$ and $0 \leq i \leq n$.

Definition 2.2.12. Let X be a simplicial object in $\mathcal{C}$. Then, $X(\partial\Delta^n)$ is called the **object of n-boundaries** in X and the morphism

$$q_n := X(\partial\Delta^n \to \Delta^n) : X_n \longrightarrow X(\partial\Delta^n) \tag{2.18}$$

is called the **n-boundary projection** for $n \geq 1$.

We will usually drop the prefix (n, i) or n and just write horns, horn projections, boundaries and boundary projections. The horn projections naturally factor through the boundary projections as follows

$$\begin{tikzcd} X_n \arrow[rr, "p_{n,i}"] \arrow[dr, "q_n"'] & & X(\Lambda_i^n) \\ & X(\partial\Delta^n) \arrow[ur, "r_{n,i}"'] & \end{tikzcd} \tag{2.19}$$

where $r_{n,i}$ is induced by the monomorphism $\Lambda_i^n \to \partial\Delta^n$ of simplicial sets.

Remark 2.2.13. Let X be a simplicial set. By Lemma 2.2.6, a horn in X is a morphism $\Lambda_i^n \to X$ of simplicial sets, and a boundary in X is a morphism $\partial\Delta^n \to X$ of simplicial sets. Using the observation that $X(\Lambda_i^n) \cong \mathrm{s}\mathbb{S}\mathrm{et}(\Lambda_i^n, X)$ is in bijection with

$$\left\{(x_0, \ldots, \widehat{x_i}, \ldots, x_n) \,|\, x_j \in X_{n-1};\ d_j(x_k) = d_{k-1}(x_j);\ j < k;\ j, k \neq i\right\}, \tag{2.20}$$

which is a subset of $X_{n-1} \times \ldots \times X_{n-1}$ (n factors), the horn projection can be written as

$$p_{n,i} = (d_{n,0}, \ldots, \widehat{d_{n,i}}, \ldots, d_{n,n})$$

for all $n \geq 1$ and $0 \leq i \leq n$. It follows from the first simplicial identity, given by Equation (2.6), that the horn projection indeed takes values in the set (2.20).

In a similar fashion, observing that $X(\partial\Delta^n) \cong \mathrm{s}\mathbb{S}\mathrm{et}(\partial\Delta^n, X)$ is in bijection with

$$\left\{(x_0, \ldots, x_n) \,|\, x_j \in X_{n-1};\ d_j(x_k) = d_{k-1}(x_j);\ j < k\right\},$$

which is a subset of $X_{n-1} \times \ldots \times X_{n-1}$ ($n+1$ factors), the boundary projection can be expressed as

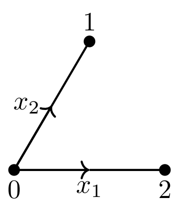
$$q_n = (d_{n,0}, \ldots, d_{n,n})$$

for all $n \geq 1$.

Example 2.2.14. Let X be a simplicial set. The set of $(2, 0)$-horns of X is in bijection with

$$\left\{(x_1, x_2) \,|\, x_j \in X_1;\ d_{1,1}(x_2) = d_{1,1}(x_1)\right\} \subset X_1 \times X_1 .$$

The horn projection $p_{2,0} : X_2 \to X(\Lambda_0^2)$ maps an element $x \in X_2$ to $\big(d_{2,1}(x), d_{2,2}(x)\big)$. Letting $x_1 = d_{2,1}(x)$ and $x_2 = d_{2,2}(x)$, it follows from the first simplicial identity (2.6) that $d_{1,1}(x_2) = d_{1,1}(x_1)$. This can be depicted by the following diagram:



Remark 2.2.15. The expressions of the horn and boundary projections via the face maps of the simplicial set X provide us with a geometric intuition and a tool for explicit computations. This can also be adapted to simplicial objects in a category with a concrete structure. For a detailed exposition on the above element-wise descriptions of the set of horns and boundaries in a simplicial set, the reader may refer to Section 2.1 in [Dus02], and to Sections I.2 and I.3 in [GJ99].

2.2.3 The Kan condition and higher groupoids

Definition 2.2.16. A simplicial set X is called **Kan** if the horn projection $p_{m,i} : X_m \to X(\Lambda_i^m)$ is a surjection for all $m \geq 1$ and $0 \leq i \leq m$.

Since $X_m \cong \mathrm{s}\mathcal{S}\mathrm{et}(\Delta^m, X)$ and $X(\Lambda_i^m) \cong \mathrm{s}\mathcal{S}\mathrm{et}(\Lambda_i^m, X)$, this means that in a Kan simplicial set X, every horn can be extended to a simplex. The horn filling condition can be depicted by the following diagram:

$$\begin{tikzcd} \Lambda_i^m \arrow[r, "\forall"] \arrow[d] & X \\ \Delta^m \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

Remark 2.2.17. In the literature, Kan simplicial sets are sometimes called *Kan complexes* (e.g. [May92, Conventions 1.6]) or *fibrant* simplicial sets with respect to the standard model structure (e.g. [GJ99, p. 11]).

Example 2.2.18. The nerve of a groupoid is a simplicial set (Remark 2.1.7), such that every horn has a filler. This follows from the existence of compositions and inverses of arrows in a groupoid. Thus, the nerve of a groupoid is a Kan simplicial set. For a detailed proof, see [GJ99, Lemma I.3.5]. In addition, its horns of degree greater than 1 can be filled uniquely. That is, $p_{m,i}$ is a bijection for all $m > 1$ and $0 \leq i \leq m$.

This suggests the following simplicial approach to defining set theoretic higher groupoids.

Definition 2.2.19 ([Zhu09, Def. 1.1])**.** Let $n \in \mathbb{N} \cup \{0\} \cup \{\infty\}$. An n**-groupoid** is a Kan simplicial set G such that $p_{m,i} : G_m \to G(\Lambda_i^m)$ is a bijection for all $m > n$ and $0 \leq i \leq m$. If G_0 is a point, then G is called an n**-group**.

Remark 2.2.20. Higher groupoids were originally called *hypergroupoids* by Duskin [Dus79] and Glenn [Gle82]. However, in these original definitions, the lower horns are not required to have fillers at all.

Definition 2.2.21. A **morphism of** n**-groupoids** is a morphism of simplicial sets, i.e. a natural transformation.

For $n \in \mathbb{N} \cup \{0\} \cup \{\infty\}$, n-groupoids together with morphisms between them form a category.

Remark 2.2.22. The category of 1-groupoids is equivalent to the category of groupoids via the nerve construction. We have already discussed in Example 2.2.18 that the nerve of a groupoid is a 1-groupoid. Conversely, a 1-groupoid $G : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ gives rise to a groupoid as follows. The set of objects is the set G_0 of 0-simplices and the set of arrows is the set G_1 of 1-simplices. The face maps $d_{1,0}, d_{1,1} : G_1 \to G_0$ are the source $s \equiv d_{1,1}$ and the target $t \equiv d_{1,0}$. The degeneracy map $s_{0,0} : G_0 \to G_1$ is the identity bisection $1 \equiv s_{0,0}$. The set of $(2,1)$-horns of G is given by

$$G(\Lambda_1^2) = \lim_{\Delta^k \to \Lambda_1^2} G_k \cong G_1 \times_{G_0}^{s,t} G_1 \,.$$

Similarly, the sets of $(2,0)$-horns and $(2,2)$-horns of G are given by

$$G(\Lambda^2_0) \cong G_1 \times^{s,s}_{G_0} G_1 \quad \text{and} \quad G(\Lambda^2_2) \cong G_1 \times^{t,t}_{G_0} G_1 \,.$$

The groupoid multiplication is given by the composition of the inverse of the horn projection and the inner face map

$$m : G_1 \times^{s,t}_{G_0} G_1 \xrightarrow[\cong]{p_{2,1}^{-1}} G_2 \xrightarrow{d_{2,1}} G_1 \,. \tag{2.21}$$

The outer face maps are the first and second projections on the pullback $G_1 \times^{s,t}_{G_0} G_1$. That is, $\mathrm{pr}_1 = d_{2,0} \circ p_{2,1}^{-1}$ and $\mathrm{pr}_2 = d_{2,2} \circ p_{2,1}^{-1}$. The inverse is given by the following composition

$$i : G_1 \xrightarrow{(\mathrm{id}_{G_1}, 1 \circ s)} G_1 \times^{s,s}_{G_0} G_1 \xrightarrow[\cong]{p_{2,0}^{-1}} G_2 \xrightarrow{d_{2,0}} G_1 \,.$$

The associativity of the groupoid multiplication can be proved using the (unique) horn filling conditions of degree 2 and 3 (see a nice geometric proof in [Zhu09, p. 5]). The other axioms of a groupoid follow from the simplicial identities. By induction, we can prove that

$$G_k \cong \underbrace{G_1 \times^{s,t}_{G_0} \ldots \times^{s,t}_{G_0} G_1}_{k \text{ factors}} \,,$$

which shows that the simplicial set we started with is the nerve of the groupoid.

Remark 2.2.23. For $n = 2$ and G_0 a point, there are inequivalent definitions of 2-groups in the literature with various adjectives. In [BL04], the authors introduce the notion of *weak* and *coherent* 2-groups. Henriques [Hen08, Sec. 9] provides a detailed explanation of the relation between these coherent 2-groups and the 2-groups from Definition 2.2.19. The comparison is originally done for the case of Lie 2-groups. The arguments hold for the non-smooth case in a similar fashion.

Strict 2-groups are equivalently defined as strict monoidal categories such that every morphism is invertible and every object has a strict inverse, as group objects in the category of groupoids, as internal categories in the category of groups, and as *crossed modules* (see [For02] for a review on these equivalent definitions). It was shown in [BS76] that the category of strict 2-groups and crossed modules are equivalent.

Defining (higher) groupoid objects in any category $\mathcal{C}$ is more subtle. Given a simplicial object $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$, the main issue is that its horns

$$X(\Lambda^m_i) = \lim_{\Delta^k \to \Lambda^m_i} X_k$$

might not exist in $\mathcal{C}$. Viewing them as presheaves on $\mathcal{C}$ is not sufficient. We need representability, that is, the existence of the horns in $\mathcal{C}$. In Section 4.1.1 we define a groupoid object in any category by directly asking the existence of the horns needed to define the horn filling conditions. Moreover, in Section 5.1.1 we adapt the approach of Henriques, Zhu and Meyer and define higher groupoid objects in categories equipped with a *Grothendieck pretopology* [Hen08, Zhu09, MZ15].

2.3 Simplicial skeleta and coskeleta

In some instances, the information of a simplicial object is encoded in its simplices up to some fixed degree. This phenomenon can be best studied via the *skeleton* and *coskeleton* functors as recalled in Section 2.3.1 for the case of simplicial sets. We then describe the notion of coskeletality of simplicial sets and show that n-groupoids are $(n+1)$-coskeletal in Section 2.3.2. We conclude by commenting on the possible approaches of skeleta and coskeleta in the case of simplicial objects in any category in Section 2.3.3.

2.3.1 The skeleton and coskeleton functors

Fix $n \geq 0$ throughout the rest of this section. Let $\mathbf{\Delta}_{\leq n}$ be the full subcategory of $\mathbf{\Delta}$ with objects $[0], [1], \ldots, [n]$, called the n**-truncated simplex category**. Let $\iota_n : \mathbf{\Delta}_{\leq n} \to \mathbf{\Delta}$ be the natural full and faithful functor. The n**-truncation** functor is the precomposition functor (see Remark A.3.2)

$$\begin{aligned} \mathrm{tr}_n := (\iota_n^{\mathrm{op}})^* : \mathcal{S}\mathrm{et}^{\mathbf{\Delta}^{\mathrm{op}}} &\longrightarrow \mathcal{S}\mathrm{et}^{\mathbf{\Delta}^{\mathrm{op}}_{\leq n}} \\ X &\longmapsto X\iota_n^{\mathrm{op}} , \end{aligned}$$

which restricts a simplicial set to its simplices of degree up to n and forgets its simplices of degree greater than n. Since $\mathcal{S}\mathrm{et}$ is complete and cocomplete, and since $\mathbf{\Delta}_{\leq n}$ is small, it follows from Corollary A.3.5 and Remark A.3.6 that tr_n has a left adjoint, given by the left Kan extension

$$\mathrm{sk}'_n := \mathrm{Lan}_{\iota_n^{\mathrm{op}}} : \mathcal{S}\mathrm{et}^{\mathbf{\Delta}^{\mathrm{op}}_{\leq n}} \longrightarrow \mathcal{S}\mathrm{et}^{\mathbf{\Delta}^{\mathrm{op}}} ,$$

as well as a right adjoint, given by the right Kan extension

$$\mathrm{cosk}'_n := \mathrm{Ran}_{\iota_n^{\mathrm{op}}} : \mathcal{S}\mathrm{et}^{\mathbf{\Delta}^{\mathrm{op}}_{\leq n}} \longrightarrow \mathcal{S}\mathrm{et}^{\mathbf{\Delta}^{\mathrm{op}}} .$$

Being left adjoint to a forgetful functor, sk'_n is a free functor. That is, for $X \in \mathcal{S}\mathrm{et}^{\mathbf{\Delta}^{\mathrm{op}}_{\leq n}}$, the simplicial set $\mathrm{sk}'_n X$ has the same simplices as X up to degree n, and is freely filled with degenerate simplices (coming from X_n) above degree n. Moreover, the set of m-simplices of $\mathrm{cosk}'_n X$ is in natural bijection with

$$(\mathrm{cosk}'_n X)_m \cong \mathrm{s}\mathcal{S}\mathrm{et}(\Delta^m, \mathrm{cosk}'_n X) \cong \mathcal{S}\mathrm{et}^{\mathbf{\Delta}^{\mathrm{op}}_{\leq n}}(\mathrm{tr}_n \Delta^m, X) ,$$

using the Yoneda lemma and the fact that cosk'_n is right adjoint to tr_n.

Definition 2.3.1. The n**-skeleton** and the n**-coskeleton** are the composite endofunctors

$$\begin{aligned} \mathrm{sk}_n &:= \mathrm{sk}'_n \, \mathrm{tr}_n : \mathrm{s}\mathcal{S}\mathrm{et} \longrightarrow \mathrm{s}\mathcal{S}\mathrm{et} \\ \mathrm{cosk}_n &:= \mathrm{cosk}'_n \, \mathrm{tr}_n : \mathrm{s}\mathcal{S}\mathrm{et} \longrightarrow \mathrm{s}\mathcal{S}\mathrm{et} . \end{aligned}$$

Remark 2.3.2. Section 2.2 of [Dus02] provides a more detailed and geometric approach to these constructions by using the perspective of *simplicial kernels* and element-wise descriptions, in the same spirit as Remark 2.2.13. The ideas originally appeared in [Dus75, Sec. 0.8], where the n-coskeleton of X is defined by successively iterating simplicial kernels. In [Gle82] a notation of *simplicial matrices* is introduced, which is a very useful technique for carrying out computations involving skeleta and coskeleta (among other simplicial constructions). For a standard reference on skeleta and coskeleta of simplicial sets, see Section IV.3.2 in [GJ99].

Remark 2.3.3. The following relations hold:

(i) $\mathrm{sk}_n \Delta^m = \Delta^m$ for all $n \geq m$.

(ii) $\mathrm{sk}_{n-1} \Delta^n = \partial\Delta^n$ for all $n \geq 1$.

(iii) $\mathrm{sk}_n \Lambda^m_i = \mathrm{sk}_n \Delta^m$ for all $n < m-1$ and $0 \leq i \leq m$.

(iv) $\mathrm{tr}_n \Delta^m = \mathrm{tr}_n \partial\Delta^m$ for all $n < m$.

(v) $\mathrm{sk}_n \mathrm{sk}_m \Delta^k = \mathrm{sk}_n \Delta^k$ for all $n \leq m$ and $k \geq 0$.

Lemma 2.3.4. *The n-skeleton and the n-coskeleton functors form an adjunction*

$$\mathrm{sk}_n : s\mathcal{S}\mathrm{et} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} s\mathcal{S}\mathrm{et} : \mathrm{cosk}_n \,.$$

Proof. Let X and Y be simplicial sets. Then, there are natural bijections

$$\begin{aligned} s\mathcal{S}\mathrm{et}(\mathrm{sk}_n X, Y) &= s\mathcal{S}\mathrm{et}(\mathrm{sk}'_n \, \mathrm{tr}_n X, Y) \\ &\cong \mathcal{S}\mathrm{et}^{\mathbf{\Delta}^{\mathrm{op}}_{\leq n}}(\mathrm{tr}_n X, \mathrm{tr}_n Y) \\ &\cong s\mathcal{S}\mathrm{et}(X, \mathrm{cosk}'_n \, \mathrm{tr}_n Y) \\ &\cong s\mathcal{S}\mathrm{et}(X, \mathrm{cosk}_n Y)\,, \end{aligned}$$

where we have used the fact that sk'_n is left adjoint to tr_n and cosk'_n is right adjoint to tr_n. □

In particular, for $m \leq n$ and $X = \Delta^m$, we get

$$\begin{aligned} (\mathrm{cosk}_n Y)_m &\cong s\mathcal{S}\mathrm{et}(\Delta^m, \mathrm{cosk}_n Y) \\ &\cong s\mathcal{S}\mathrm{et}(\mathrm{sk}_n \Delta^m, Y) \\ &= s\mathcal{S}\mathrm{et}(\Delta^m, Y) \\ &\cong Y_m\,. \end{aligned} \tag{2.22}$$

In the third step, we have used Relation (i) of Remark 2.3.3.

Lemma 2.3.5. *Let X and Y be simplicial sets such that* $\mathrm{tr}_n X \cong \mathrm{tr}_n Y$. *Then, there is a natural bijection of sets*

$$s\mathcal{S}\mathrm{et}(\mathrm{sk}_n A, X) \cong s\mathcal{S}\mathrm{et}(\mathrm{sk}_n A, Y)$$

for all simplicial sets A.

Proof. Since $\mathrm{tr}_n X \cong \mathrm{tr}_n Y$, we have that $\mathrm{cosk}_n X \cong \mathrm{cosk}_n Y$. Using the fact that sk_n is left adjoint to cosk_n (Lemma 2.3.4), we get natural bijections

$$\begin{aligned} s\mathcal{S}\mathrm{et}(\mathrm{sk}_n A, X) &\cong s\mathcal{S}\mathrm{et}(A, \mathrm{cosk}_n X) \\ &\cong s\mathcal{S}\mathrm{et}(A, \mathrm{cosk}_n Y) \\ &\cong s\mathcal{S}\mathrm{et}(\mathrm{sk}_n A, Y) \end{aligned} \tag{2.23}$$

for all $A \in s\mathcal{S}\mathrm{et}$. □

2.3.2 Coskeletality

Definition 2.3.6. A simplicial set X is called n**-coskeletal** if $\eta_X : X \xrightarrow{\cong} \operatorname{cosk}_n X$ is an isomorphism, where η is the unit of the adjunction $\operatorname{tr}_n \dashv \operatorname{cosk}'_n$.

By Equation (2.22), note that the first non-identity component of η_X is given by its degree $n+1$ map $X_{n+1} \to (\operatorname{cosk}_n X)_{n+1}$.

Lemma 2.3.7. *Let X be a simplicial set. Then, $\operatorname{cosk}_n X$ is m-coskeletal for all $m \geq n$.*

Proof. We need to show that $\eta_{\operatorname{cosk}_n X} : \operatorname{cosk}_n X \to \operatorname{cosk}_m \operatorname{cosk}_n X$ is an isomorphism for all $m \geq n$. By Equation (2.22), we have that

$$(\operatorname{cosk}_n X)_k \cong (\operatorname{cosk}_m \operatorname{cosk}_n X)_k$$

for all $k \leq m$. For $k > m$, we have

$$\begin{aligned} \mathrm{s}\mathcal{S}\mathrm{et}(\Delta^k, \operatorname{cosk}_m \operatorname{cosk}_n X) &\cong \mathrm{s}\mathcal{S}\mathrm{et}(\operatorname{sk}_m \Delta^k, \operatorname{cosk}_n X) \\ &\cong \mathrm{s}\mathcal{S}\mathrm{et}(\operatorname{sk}_n \operatorname{sk}_m \Delta^k, X) \\ &\cong \mathrm{s}\mathcal{S}\mathrm{et}(\operatorname{sk}_n \Delta^k, X) \\ &\cong \mathrm{s}\mathcal{S}\mathrm{et}(\Delta^k, \operatorname{cosk}_n X)\,, \end{aligned}$$

where we have used the adjunction $\operatorname{sk}_n \dashv \operatorname{cosk}_n$ (Lemma 2.3.4) and Relation (v) of Remark 2.3.3. The desired isomorphism follows from the Yoneda lemma. □

Remark 2.3.8. Similarly, when $m < n$, we get that $\operatorname{cosk}_m \operatorname{cosk}_n X \cong \operatorname{cosk}_m X$ for all simplicial sets X.

The next proposition shows that n-coskeletal simplicial sets are characterized by the fact that morphisms into them are uniquely determined by their components up to degree n, and equivalently by the unique existence of boundary fillers above degree n.

Proposition 2.3.9. *Let X be a simplicial set. Then, the following are equivalent:*

(i) X is n-coskeletal;

(ii) the natural map

$$\mathrm{s}\mathcal{S}\mathrm{et}(A, X) \xrightarrow{\cong} \mathcal{S}\mathrm{et}^{\mathbf{\Delta}^{\mathrm{op}}_{\leq n}}(\operatorname{tr}_n A, \operatorname{tr}_n X)$$

is a bijection for all simplicial sets A;

(iii) the natural map

$$\mathrm{s}\mathcal{S}\mathrm{et}(\Delta^m, X) \xrightarrow{\cong} \mathcal{S}\mathrm{et}^{\mathbf{\Delta}^{\mathrm{op}}_{\leq n}}(\operatorname{tr}_n \Delta^m, \operatorname{tr}_n X)$$

is a bijection for all $m > n$;

(iv) the boundary projection

$$q_m : \mathrm{s}\mathcal{S}\mathrm{et}(\Delta^m, X) \xrightarrow{\cong} \mathrm{s}\mathcal{S}\mathrm{et}(\partial\Delta^m, X)$$

is a bijection for all $m > n$.

Proof. Since cosk'_n is right adjoint to tr_n, there is a natural bijection

$$\mathrm{s}\mathcal{S}\mathrm{et}(A, \mathrm{cosk}'_n B) \cong \mathcal{S}\mathrm{et}^{\mathbf{\Delta}^{\mathrm{op}}_{\leq n}}(\mathrm{tr}_n A, B) \tag{2.24}$$

for all $A \in \mathrm{s}\mathcal{S}\mathrm{et}$ and $B \in \mathcal{S}\mathrm{et}^{\mathbf{\Delta}^{\mathrm{op}}_{\leq n}}$. We will first show that parts (i), (ii) and (iii) are equivalent. Lastly, we will show that parts (i) and (iv) are equivalent.

For (i) $\implies$ (ii), assume X is n-coskeletal, that is $X \cong \mathrm{cosk}_n X = \mathrm{cosk}'_n \mathrm{tr}_n X$. Then, substituting B by $\mathrm{tr}_n X$ in (2.24) yields the desired bijection of (ii).

The direction (ii) $\implies$ (iii) trivially follows by letting $A = \Delta^m$. The bijection holds for all $m \geq 0$. In particular, it holds for all $m > n$.

Now, assume (iii) holds. By Equation (2.22), we know that $X_m \cong (\mathrm{cosk}_n X)_m$ for all $m \leq n$. For $m > n$, we get that

$$\begin{aligned} X_m \cong \mathrm{s}\mathcal{S}\mathrm{et}(\Delta^m, X) &\cong \mathcal{S}\mathrm{et}^{\mathbf{\Delta}^{\mathrm{op}}_{\leq n}}(\mathrm{tr}_n \Delta^m, \mathrm{tr}_n X) \\ &\cong \mathrm{s}\mathcal{S}\mathrm{et}(\Delta^m, \mathrm{cosk}_n X) \\ &\cong (\mathrm{cosk}_n X)_m\,, \end{aligned}$$

where we have used the Yoneda lemma, the assumption and (2.24). We conclude that X is n-coskeletal. This shows part (i).

For the direction (i) $\implies$ (iv), assume that X is n-coskeletal. Then,

$$\begin{aligned} \mathrm{s}\mathcal{S}\mathrm{et}(\Delta^m, X) &\cong \mathrm{s}\mathcal{S}\mathrm{et}(\Delta^m, \mathrm{cosk}'_n \mathrm{tr}_n X) \\ &\cong \mathcal{S}\mathrm{et}^{\mathbf{\Delta}^{\mathrm{op}}_{\leq n}}(\mathrm{tr}_n \Delta^m, \mathrm{tr}_n X) \\ &\cong \mathcal{S}\mathrm{et}^{\mathbf{\Delta}^{\mathrm{op}}_{\leq n}}(\mathrm{tr}_n \partial\Delta^m, \mathrm{tr}_n X) \\ &\cong \mathrm{s}\mathcal{S}\mathrm{et}(\partial\Delta^m, \mathrm{cosk}'_n \mathrm{tr}_n X) \\ &\cong \mathrm{s}\mathcal{S}\mathrm{et}(\partial\Delta^m, X)\,, \end{aligned}$$

where we have used (2.24) and that $\mathrm{tr}_n \Delta^m = \mathrm{tr}_n \partial\Delta^m$ for all $m > n$ (Relation (iv) from Remark 2.3.3).

Finally, assume (iv) holds. To show that X is n-coskeletal, we will perform induction on the simplices of $\mathrm{cosk}_n X$. As mentioned above, $X_m \cong (\mathrm{cosk}_n X)_m$ for all $m \leq n$. For $m = n + 1$, we get that

$$\begin{aligned} X_{n+1} &\cong \mathrm{s}\mathcal{S}\mathrm{et}(\Delta^{n+1}, X) \\ &\cong \mathrm{s}\mathcal{S}\mathrm{et}(\partial\Delta^{n+1}, X) \\ &\cong \mathrm{s}\mathcal{S}\mathrm{et}(\mathrm{sk}_n \Delta^{n+1}, X) \\ &\cong \mathrm{s}\mathcal{S}\mathrm{et}(\Delta^{n+1}, \mathrm{cosk}_n X) \\ &\cong (\mathrm{cosk}_n X)_{n+1}\,, \end{aligned}$$

by the assumption of unique boundary fillers, the fact that $\partial\Delta^m = \mathrm{sk}_{m-1} \Delta^m$ from Relation (ii) of Remark 2.3.3, and the adjunction $\mathrm{sk}_n \dashv \mathrm{cosk}_n$ (Lemma 2.3.4).

In particular, this implies that $\operatorname{tr}_{n+1} X \cong \operatorname{tr}_{n+1} \operatorname{cosk}_n X$. It then follows from Lemma 2.3.5 for $A = \Delta^{n+2}$ and $Y = \operatorname{cosk}_n X$ that there is a natural bijection

$$\mathrm{s}\mathcal{S}\mathrm{et}(\operatorname{sk}_{n+1} \Delta^{n+2}, X) \cong \mathrm{s}\mathcal{S}\mathrm{et}(\operatorname{sk}_{n+1} \Delta^{n+2}, \operatorname{cosk}_n X)\,. \tag{2.25}$$

For $m = n+2$, we get by analogous arguments

$$\begin{aligned}
\mathrm{s}\mathcal{S}\mathrm{et}(\Delta^{n+2}, X) &\cong \mathrm{s}\mathcal{S}\mathrm{et}(\partial\Delta^{n+2}, X)\\
&\cong \mathrm{s}\mathcal{S}\mathrm{et}(\operatorname{sk}_{n+1} \Delta^{n+2}, X)\\
&\cong \mathrm{s}\mathcal{S}\mathrm{et}(\operatorname{sk}_{n+1} \Delta^{n+2}, \operatorname{cosk}_n X)\\
&\cong \mathrm{s}\mathcal{S}\mathrm{et}(\Delta^{n+2}, \operatorname{cosk}_{n+1} \operatorname{cosk}_n X)\\
&\cong \mathrm{s}\mathcal{S}\mathrm{et}(\Delta^{n+2}, \operatorname{cosk}_n X)
\end{aligned}$$

where in the third step we have used (2.25) and in the last step Lemma 2.3.7.

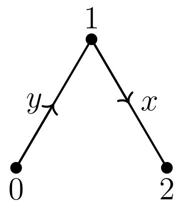
Proceeding inductively, we conclude that

$$\mathrm{s}\mathcal{S}\mathrm{et}(\Delta^{m}, X) \cong \mathrm{s}\mathcal{S}\mathrm{et}(\Delta^{m}, \operatorname{cosk}_n X)$$

for all $m > n$. This shows that $X \cong \operatorname{cosk}_n X$ and hence part (i) holds. □

Example 2.3.10. The nerve of a small category (Remark 2.1.7) is 2-coskeletal. This is essentially true by construction. An explanation of this fact may be found in [Gle82, Sec. 2.1], where the author uses the approach of *simplicial matrices*. Moreover, a rigorous proof using the notion of the *fundamental category* of simplicial sets may be found in [Joy08, Cor. 1.2]. In particular, the nerve of a groupoid is 2-coskeletal.

Remark 2.3.11. Not every simplicial set which is 2-coskeletal comes from the nerve of a category. For instance, the inner horn Λ^2_1 is 2-coskeletal but not the nerve of a category since it lacks the composition xy, as depicted by the following diagram:



Proposition 2.3.12. *Every n-groupoid is $(n+1)$-coskeletal.*

Proof. Let G be an n-groupoid. To show that it is $(n+1)$-coskeletal, we will use the equivalent statement (iv) of Proposition 2.3.9. Let $m > n+1$ and $0 \leq i \leq m$. Consider the following commutative diagram (it is Diagram (2.19)):

$$\begin{tikzcd}
G_m \arrow[rr, "p_{m,i}", "\cong"'] \arrow[dr, "q_m"'] & & G(\Lambda^m_i) \\
& G(\partial\Delta^m) \arrow[ur, "r_{m,i}"'] &
\end{tikzcd}$$

Since G is an n-groupoid, $p_{m,i}$ is a bijection. Hence, $r_{m,i}$ is surjective and q_m is injective. The aim is to show that q_m is also surjective. It is sufficient to show that $r_{m,i}$ has an inverse.

Observe that $q_m \circ p_{m,i}^{-1}$ is a right inverse of $r_{m,i}$ since $r_{m,i} \circ q_m \circ p_{m,i}^{-1} = \mathrm{id}_{G(\Lambda_i^m)}$. We will show that it is also a left inverse of $r_{m,i}$. Let $\alpha \in G(\partial\Delta^m)$. Using the notation in Remark 2.2.13, we can write

$$\alpha = (g_0, \dots, g_m) \quad \text{with} \quad g_j \in G_{m-1},\ d_j(g_k) = d_{k-1}(g_j) \tag{2.26}$$

for $j < k$. Then,

$$\begin{aligned} q_m \circ p_{m,i}^{-1} \circ r_{m,i}(\alpha) &= q_m \circ p_{m,i}^{-1} \circ r_{m,i}(g_0, \dots, g_m) \\ &= q_m \circ p_{m,i}^{-1}(g_0, \dots, \widehat{g_i}, \dots, g_m) \\ &= q_m(g) \end{aligned}$$

for some $g \in G_m$ such that $d_j(g) = g_j$ for all $j \neq i$. Let $x_i := d_i(g)$. Then,

$$q_m(g) = (g_0, \dots, x_i, \dots, g_m)\,.$$

As a last step, we will show that $x_i = g_i$. Using the first simplicial identity, given by Equation (2.6), and the boundary condition in (2.26), we have that

$$\begin{aligned} d_p(x_i) &= d_p \circ d_i(g) \\ &= d_i \circ d_{p+1}(g) \\ &= d_i(g_{p+1}) \\ &= d_p(g_i) \end{aligned}$$

for all $i \leq p$. Similarly,

$$\begin{aligned} d_p(x_i) &= d_p \circ d_i(g) \\ &= d_{i-1} \circ d_p(g) \\ &= d_{i-1}(g_p) \\ &= d_p(g_i) \end{aligned}$$

for all $i > p$. This shows that $d_p(x_i) = d_p(g_i)$ for all $0 \leq p \leq m-1$. Since G is an n-groupoid and $m > n+1$, the horn projection $p_{m-1,0}$ is a bijection, so that x_i and g_i are uniquely determined by a lower horn. In other words, the equalities

$$\begin{aligned} p_{m-1,0}(x_i) &= \big(d_1(x_i), \dots, d_{m-1}(x_i)\big) \\ &= \big(d_1(g_i), \dots, d_{m-1}(g_i)\big) \\ &= p_{m-1,0}(g_i) \end{aligned}$$

imply that $x_i = g_i$. As a conclusion, we get that

$$\alpha = q_m \circ p_{m,i}^{-1} \circ r_{m,i}(\alpha)\,.$$

We showed that $r_{m,i}$ is a bijection with inverse $q_m \circ p_{m,i}^{-1}$. This implies that q_m is a bijection too. Since the above argument holds for all $m > n+1$, we conclude that G is $(n+1)$-coskeletal by using the equivalence (i) $\Leftrightarrow$ (iv) of Proposition 2.3.9. $\square$

Remark 2.3.13. It follows from Proposition 2.3.12 and the equivalence (i) $\Leftrightarrow$ (iv) of Proposition 2.3.9 that n-groupoids are of homotopy n-type. That is, their homotopy groups of degree $> n$ are trivial. As we do not talk about homotopy groups in this thesis, we refer the reader to standard sources, such as [GJ99].

2.3.3 Generalization to simplicial objects

In this section, we comment on a possible generalization of the skeleta and coskeleta of simplicial sets to simplicial objects in any category $\mathcal{C}$. As Goerss and Jardine state: "*Skeleta are most precisely described as Kan extensions of truncated simplicial sets*" [GJ99, Ch. V, p. 251]. However, having no assumptions of completeness and cocompleteness on $\mathcal{C}$, the n-truncation functor

$$\begin{aligned} \mathrm{tr}_n^{\mathcal{C}} : \mathcal{C}^{\mathbf{\Delta}^{\mathrm{op}}} &\longrightarrow \mathcal{C}^{\mathbf{\Delta}_{\leq n}^{\mathrm{op}}} \\ X &\longmapsto X\iota_n^{\mathrm{op}} \end{aligned}$$

might not admit a right or left adjoint (which would be the respective right and left Kan extensions of $\mathrm{tr}_n^{\mathcal{C}}$).

One approach is to explicitly specify what the m-simplices of the n-skeleton of a simplicial object are. We define the **n-skeleton** to be the endofunctor

$$\mathrm{sk}_n^{\mathcal{C}} : \mathcal{C}^{\mathbf{\Delta}^{\mathrm{op}}} \longrightarrow \mathcal{C}^{\mathbf{\Delta}^{\mathrm{op}}}$$

which assigns to each simplicial object X in $\mathcal{C}$, the simplicial object $\mathrm{sk}_n^{\mathcal{C}} X$ which has the same m-simplices as X for all $m \leq n$ and whose simplices of degree $> n$ are all degenerate (coming from X_n).

As mentioned, the subtlety here lies in the fact that $\mathrm{sk}_n^{\mathcal{C}}$ might not admit a right adjoint since for instance $\mathcal{C}$ is not assumed to be complete. One approach to define the n-coskeleton functor is to use the universal property of adjoint functors. In $\mathrm{s}\mathcal{S}\mathrm{et}$, we have that

$$\mathrm{s}\mathcal{S}\mathrm{et}(\Delta^m, \mathrm{cosk}_n A) \cong \mathrm{s}\mathcal{S}\mathrm{et}(\mathrm{sk}_n \Delta^m, A)$$

for all $n, m \geq 0$, $A \in \mathrm{s}\mathcal{S}\mathrm{et}$, using Lemma 2.3.4. We define the **n-coskeleton** to be the endofunctor

$$\mathrm{cosk}_n^{\mathcal{C}} : \mathcal{C}^{\mathbf{\Delta}^{\mathrm{op}}} \longrightarrow \mathcal{C}^{\mathbf{\Delta}^{\mathrm{op}}}$$

which assigns to each simplicial object X in $\mathcal{C}$, the simplicial object $\mathrm{cosk}_n^{\mathcal{C}} X$ whose m-simplices are defined by

$$\left(\mathrm{cosk}_n^{\mathcal{C}} X\right)_m := X(\mathrm{sk}_n \Delta^m) = \lim_{\Delta^k \to \mathrm{sk}_n \Delta^m} X_k \,,$$

for all $m \geq 0$. Here, we have used Definition 2.2.4 of Kan extensions of simplicial objects along the Yoneda embedding, the Expression (2.14) as a limit, and Notation 2.2.10. These objects are a priori only presheaves on $\mathcal{C}$, whose representability is not guaranteed. If $m \leq n$, we have $\mathrm{sk}_n \Delta^m = \Delta^m$ by Relation (i) of Remark 2.3.3. In this case, $\left(\mathrm{cosk}_n^{\mathcal{C}} X\right)_m = X(\Delta^m) \cong X_m \in \mathcal{C}$ by Lemma 2.2.8.

Remark 2.3.14. In [Zhu09], the author overcomes this subtlety by defining the skeleton and coskeleton functors as endofunctors on the category of simplicial presheaves on $\mathcal{C}$. We use this approach in the proof of Proposition 5.1.18.

For the purposes of this thesis, we give the following working definition of n-coskeletality of simplicial objects using the equivalence (i) $\Leftrightarrow$ (ii) from Proposition 2.3.9. This allows us to avoid the question of representability and to have a concise definition without referring to the coskeleton functor.

Definition 2.3.15. A simplicial object $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$ is called n**-coskeletal** if the natural map

$$\mathcal{C}^{\mathbf{\Delta}^{\mathrm{op}}}(A, X) \xrightarrow{\cong} \mathcal{C}^{\mathbf{\Delta}^{\mathrm{op}}_{\leq n}}(\mathrm{tr}_n^{\mathcal{C}} A, \mathrm{tr}_n^{\mathcal{C}} X)$$

is a bijection for all simplicial objects A of $\mathcal{C}$.

Using this definition, the $(n+1)$-coskeletality of n-groupoids also holds for n-groupoid objects in any category. This will be explained in Section 5.1.2.

Remark 2.3.16. If the category $\mathcal{C}$ is complete and cocomplete, the n-truncation functor admits a right and left adjoint, and the skeleton and coskeleton endofunctors are defined in a similar manner as in simplicial sets. In this case, all the results in Sections 2.3.1 and 2.3.2 can be formulated for simplicial objects in $\mathcal{C}$. In particular, the equivalent statements in Proposition 2.3.9 hold, where the set $\mathrm{s}\mathcal{S}\mathrm{et}(\Delta^m, X)$ is replaced by the object $X(\Delta^m) \cong X_m$ of $\mathcal{C}$ and the set $\mathrm{s}\mathcal{S}\mathrm{et}(\partial\Delta^m, X)$ by the object $X(\partial\Delta^m) = \lim_{\Delta^k \to \partial\Delta^m} X_k$ of $\mathcal{C}$.

Remark 2.3.17. In Sections V.1 and VII.1 of [GJ99, Sec. V.1], the authors impose a (co)completeness assumption on the category $\mathcal{C}$. Since one of the primary purposes of the authors there is defining Reedy model structures on $\mathcal{C}$, it is natural to ask for the existence of small limits and colimits. In our situation, the main example of $\mathcal{C}$ will be the category $\mathcal{M}\mathrm{fld}$ of smooth manifolds, which is neither complete nor cocomplete.

Chapter 3

Tangent categories

One of the core concepts in differential geometry is the tangent bundle of a smooth manifold. It is the collection of all tangent spaces at all points of the manifold. It has vast applications in the theory of manifolds and enables one to do linear approximations and to make sense of calculus on manifolds in general. Moreover, it has been used to define other geometric objects, such as vector fields, Riemannian metrics and tensor fields in general.

In the 1980s, Rosický has identified the categorical nature of the tangent functor on the category of smooth manifolds and its key properties [Ros84]. He has defined an abstract tangent functor on a category $\mathcal{C}$ as an endofunctor $T : \mathcal{C} \to \mathcal{C}$ together with certain natural structure maps, such as the bundle projection $\pi_X : TX \to X$, the zero section $0_X : X \to TX$, the fiberwise addition $+_X : TX \times_X TX \to TX$, etc., for each object X in $\mathcal{C}$. Rosický's axioms of a tangent structure are the minimal axioms needed to define the Lie bracket of two vector fields on X.

Tangent structures have been rediscovered and further extended by Cockett and Cruttwell [CC14], where one of the primary interests of the authors lies in the relationship between representable tangent structures and synthetic differential geometry. On the other hand, tangent structures have an important application in diffeological spaces, a convenient setting for differential geometry. In his recent paper [Blo24a], Blohmann shows that the left Kan extension of the tangent structure on Euclidean spaces defines a tangent structure on so-called *elastic* diffeological spaces. Examples of elastic diffeological spaces include manifolds with corners and cusps, diffeological groups, mapping spaces between manifolds and spaces of sections of a fiber bundle[1].

Our primary motivation to thoroughly explore tangent categories lies in its significance to (higher) Lie theory. In Chapters 4 and 5, we show that the structure needed on a category $\mathcal{C}$ to differentiate (higher) groupoid objects in $\mathcal{C}$ is precisely a cartesian tangent structure with scalar R-multiplication, where $R \in \mathcal{C}$ is a ring object. In fact, such tangent structures provide us with the minimal categorical setting where one has a Cartan calculus [AB] and hence can tackle differential geometric problems.

[1] In fact, the original motivation of Blohmann to consider tangent structures was understanding the tangent bundle of the space of fields, that is space of sections of a smooth fiber bundle in Lagrangian field theory [Blo24b, BFW13]

In the following sections, we provide the reader with the necessary background on tangent structures needed for our purposes. In Section 3.1, we give a categorical approach to bundles, whose fibers have the structure of an algebraic theory, yet there is no assumption on local triviality. The goal of Section 3.2 is to discuss in detail abstract tangent functors in the sense of Rosický and to extend the framework to cartesian tangent structures and scalar multiplications. We then study the monad structure of the tangent functor which induces a cosimplicial structure on its powers.

Furthermore, in Section 3.3 we explain the main construction of the Lie bracket of vector fields in a tangent category, using the ideas of Rosický. We then state the Leibniz rule and prove several naturality results for the Lie bracket. Lastly, the goal of Section 3.4 is to provide an intuition of tangent structures by considering the namesake example of a tangent category: the category of Euclidean spaces. The various computations carried out in local coordinates serve as a local model of tangent structures.

3.1 Bundles with algebraic structure

The goal of this section is to provide a unified approach to bundles, whose fibers have a certain algebraic structure. We will start with a general categorical definition using the notion of an overcategory. For a recap on overcategories as a special case of comma categories, the reader may refer to Remark A.1.23 (i) in the Appendix.

Terminology 3.1.1. Let "Wibble" be an algebraic theory. Let X be an object in a category $\mathcal{C}$ such that the overcategory $\mathcal{C}\downarrow X$ has all finite products (that is pullbacks over X). An object in $\mathcal{C}\downarrow X$ will be called a **bundle over** X. A Wibble object in $\mathcal{C}\downarrow X$ will be called a **bundle of Wibbles over** X.

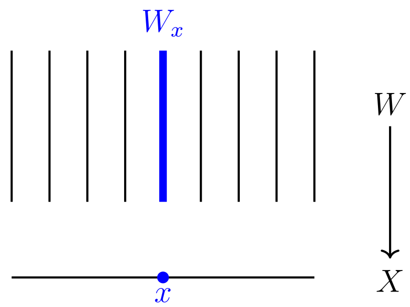
In this thesis, Wibble will mainly refer to a group, an abelian group, or an R-module (for $R \in \mathcal{C}$ a ring object). The assumption that $\mathcal{C} \downarrow X$ has all finite products equips $\mathcal{C} \downarrow X$ with a symmetric monoidal structure[2] with monoidal unit given by the terminal object $\mathrm{id}_X : X \to X$. Hence, it makes sense to talk about Wibble objects in $\mathcal{C} \downarrow X$.

Remark 3.1.2. Assume that $\mathcal{C}$ has all finite products and let $*$ be its terminal object (product of the empty diagram). Let $W \to X$ be a bundle of Wibbles. If $x : * \to X$ is a point in X, then the pullback $W_x := * \times_X W$ is called the **fiber of** W **over** x. The fiber W_x is a Wibble object in $\mathcal{C}$, where the structure maps are induced by the ones on the bundle $W \to X$. In other words, every fiber of a bundle of Wibbles is a Wibble. This justifies Terminology 3.1.1.

Caution 3.1.3. A bundle of vector spaces over a manifold M is more general than a vector bundle over M. This is due to the fact that the notion of bundle of Wibbles does not make any assumptions on local trivializations.

The fibers of a bundle of Wibbles can be visualized by the following diagram:

[2] These categories are often called cartesian monoidal categories.



Notation 3.1.4. The set of sections of a bundle $p : A \to X$ will be denoted by

$$\Gamma(X, A) := \{a : X \to A \mid p \circ a = \mathrm{id}_X\}\,.$$

Definition 3.1.5. An endofunctor $F : \mathcal{C} \to \mathcal{C}$ **preserves the fiber products** of a bundle $p : A \to X$ if the natural morphism of bundles over FX,

$$\nu_{k,X} : F(\underbrace{A \times_X^{p,p} \cdots \times_X^{p,p} A}_{k\text{-times}}) \longrightarrow \underbrace{FA \times_{FX}^{Fp,Fp} \cdots \times_{FX}^{Fp,Fp} FA}_{k\text{-times}}, \tag{3.1}$$

is an isomorphism for all $k \geq 1$.

In Sections 3.1.1 and 3.1.2, we will elaborate Terminology 3.1.1 for the case of bundles of abelian groups and bundles of R-modules, and spell out some of their properties. Throughout the rest of the section, let $\mathcal{C}$ be a category with a terminal object $*$. Assume that the overcategory $\mathcal{C}\!\downarrow\! X$ has all finite products for all objects $X \in \mathcal{C}$. In particular, this implies that $\mathcal{C} \cong \mathcal{C}\!\downarrow\! *$ has all finite products.

3.1.1 Bundles of abelian groups

Let $p : A \to X$ be a bundle of abelian groups, that is, an abelian group object in the overcategory $\mathcal{C} \downarrow X$. Spelled out, the group structure consists of the morphisms

$$\begin{tikzcd} A \times_X A \arrow[rr, "+"] \arrow[dr, "p \circ \mathrm{pr}_1 = p \circ \mathrm{pr}_2"'] & & A \arrow[dl, "p"] \\ & X & \end{tikzcd} \quad \begin{tikzcd} X \arrow[rr, "0"] \arrow[dr, "\mathrm{id}_X"'] & & A \arrow[dl, "p"] \\ & X & \end{tikzcd} \quad \begin{tikzcd} A \arrow[rr, "\iota"] \arrow[dr, "p"'] & & A \arrow[dl, "p"] \\ & X & \end{tikzcd} \tag{3.2}$$

of the addition, the zero, and the inverse, such that the usual axioms of associativity, unitality, invertibility and commutativity hold. The commutative diagrams corresponding to these axioms are spelled out in Definition A.4.15. The terminal object in $\mathcal{C}\!\downarrow\! X$ is the identity morphism $\mathrm{id}_X : X \to X$.

Remark 3.1.6. The subtraction is defined by the commutative triangle

$$\begin{tikzcd} A \times_X A \arrow[rr, "-"] \arrow[dr, "\mathrm{id}_A \times_X \iota"'] & & A \\ & A \times_X A \arrow[ur, "+"'] & \end{tikzcd} \tag{3.3}$$

as in any group object.

Even though our notion of bundles does not assume any local triviality, the bundles behave in the usual way under base changes.

Definition 3.1.7. Let $p : A \to X$ and $p' : A' \to X'$ be bundles of abelian groups with additions $+$ and $+'$ respectively. A **morphism of bundles** is a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "p"'] & A' \arrow[d, "p'"] \\ X \arrow[r, "f"'] & X' \end{tikzcd} \tag{3.4}$$

in the category $\mathcal{C}$. It is called a **morphism of bundles of abelian groups** if furthermore the diagram

$$\begin{tikzcd} A \times_X A \arrow[r, "\varphi \times_f \varphi"] \arrow[d, "+"'] & A' \times_{X'} A' \arrow[d, "+'"] \\ A \arrow[r, "\varphi"'] & A' \end{tikzcd} \tag{3.5}$$

commutes.

As for ordinary groups, the definition implies that the zeros are intertwined, that is, $\varphi \circ 0 = 0' \circ f$, where $0' : X' \to A'$ is the zero of $p' : A' \to X'$. Composing this equation with p' on the left, we obtain $p' \circ \varphi \circ 0 = f$, which shows that f is uniquely determined by φ. Therefore, we can denote a morphism (3.4) of bundles of abelian groups by φ, without loss of generality.

A morphism (3.4) of bundles over different base objects can be identified with the morphism

$$\begin{tikzcd} A \arrow[rr, "{(p,\varphi)}"] \arrow[dr, "p"'] & & X \times_{X'} A' \arrow[dl, "\mathrm{pr}_1"] \\ & X & \end{tikzcd} \tag{3.6}$$

of objects over the same base.

Proposition 3.1.8. *Let $A \to X$ be a bundle of abelian groups and $Y \to X$ a morphism in $\mathcal{C}$. Then the pullback $\mathrm{pr}_1 : Y \times_X A \to Y$ is a bundle of abelian groups and $\mathrm{pr}_2 : Y \times_X A \to A$ is a morphism of bundles of abelian groups.*

Proof. Let the bundle projection be denoted by $p : A \to X$ and the morphism by $f : Y \to X$. Since limits commute with limits, the functor

$$\begin{aligned} Y \times_X^{f,-} - : \quad \mathcal{C}{\downarrow}X &\longrightarrow \mathcal{C}{\downarrow}Y \\ (B \xrightarrow{q} X) &\longmapsto (Y \times_X^{f,q} B \xrightarrow{\mathrm{pr}_1} Y) \end{aligned} \tag{3.7}$$

preserves products. This implies that the functor (3.7) maps abelian groups in $\mathcal{C}{\downarrow}X$ to abelian groups in $\mathcal{C}{\downarrow}Y$ (Remark A.4.25). This shows that the pullback bundle

$f^*A := Y \times_X A \to Y$ is a bundle of abelian groups. The group structure is given as follows. Consider the following commutative diagram:

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "\mathrm{id}_Y"'] & X \arrow[d, "\mathrm{id}_X"] & A \times_X A \arrow[l, "p \circ \mathrm{pr}_1"'] \arrow[d, "+"] \\ Y \arrow[r, "f"'] & X & A \arrow[l, "p"] \end{tikzcd} \tag{3.8}$$

By the commutativity of pullbacks with pullbacks, we have the following isomorphism for the limit of the top row,

$$\begin{aligned} \alpha : Y \times_X (A \times_X A) &\xrightarrow{\cong} (Y \times_Y Y) \times_X (A \times_X A) \\ &\xrightarrow{\cong} (Y \times_X A) \times_Y (Y \times_X A) \,. \end{aligned}$$

Diagram (3.8) induces a unique morphism from the limit of the top row to the limit of the bottom row

$$+_{f^*A} := \big(\mathrm{id}_Y \times_X +\big) \circ \alpha^{-1} : (Y \times_X A) \times_Y (Y \times_X A) \longrightarrow Y \times_X A \,,$$

such that the diagram

$$\begin{tikzcd} (Y \times_X A) \times_Y (Y \times_X A) \arrow[r, "\mathrm{pr}_2 \times_f \mathrm{pr}_2"] \arrow[d, "+_{f^*A}"'] & A \times_X A \arrow[d, "+"] \\ Y \times_X A \arrow[r, "\mathrm{pr}_2"'] & A \end{tikzcd} \tag{3.9}$$

commutes. We conclude that $\mathrm{pr}_2 : Y \times_X A \to A$ is a morphism of bundles of abelian groups. □

Example 3.1.9. Assume that $\mathcal{C}$ has a terminal object $*$ and let $x : * \to X$ be a point in X. Then, the fiber $A_x := * \times_X A$ is an abelian group object in $\mathcal{C} \cong \mathcal{C}{\downarrow}*$. This justifies Remark 3.1.2, where Wibble stands for an abelian group.

Proposition 3.1.10. *A morphism of bundles given by* (3.4) *is a morphism of bundles of abelian groups if and only if the corresponding morphism given by* (3.6) *is a morphism of group objects in* $\mathcal{C}{\downarrow}X$.

Proof. First, we observe that the statement is true if $X = X'$ and $f = \mathrm{id}$. Consider the following commutative diagram:

$$\begin{tikzcd} A \arrow[rr, "\varphi"] \arrow[dr, "(p,\varphi)"'] & & A' \\ & X \times_{X'} A' \arrow[ur, "\mathrm{pr}_2"'] & \end{tikzcd}$$

By Proposition 3.1.8, the map $\mathrm{pr}_2 : X \times_{X'} A' \to A'$ is a morphism of bundles of abelian groups over f.

Assume that the morphism (p, φ) is a morphism of group objects in $\mathcal{C}{\downarrow}X$, in other words, a morphism of bundles of abelian groups over X. It follows that the composition $\varphi = \mathrm{pr}_2 \circ (p,\varphi)$ is a morphism of bundles of abelian groups over $f \circ \mathrm{id}_X = f$.

Conversely, assume that φ is a morphism of bundles of abelian groups over f. This implies that the left square of the following diagram

$$\begin{tikzcd}
A \times_X A \arrow[r, "(p\circ \mathrm{pr}_1, \varphi\times_f\varphi)"] \arrow[d, "+"'] & X \times_{X'} (A' \times_{X'} A') \arrow[r, "\cong"] \arrow[d, "\mathrm{id}_X \times_{X'} +'"'] & (X \times_{X'} A') \times_X (X \times_{X'} A') \arrow[d, "+_{f^*A'}"] \\
A \arrow[r, "(p,\varphi)"'] & X \times_{X'} A' \arrow[r, "\mathrm{id}"'] & X \times_{X'} A'
\end{tikzcd}$$

is commutative. The right square is the definition of $+_{f^*A'}$. It follows that the outer rectangle commutes, which shows that (p,φ) is a morphism of group objects in $\mathcal{C}{\downarrow}X$. $\square$

The kernel of a morphism of bundles of abelian groups

Definition 3.1.11. Let $A \to X$ and $A' \to X'$ be bundles of abelian groups. The **kernel** of a morphism $\varphi : A \to A'$ of bundles of abelian groups, if it exists, is the pullback

$$\begin{tikzcd}
\ker\varphi := X' \times_{A'} A \arrow[r, "i_{\ker\varphi}"] \arrow[d, "p_{\ker\varphi}"'] \arrow[dr, phantom, "\lrcorner"] & A \arrow[d, "\varphi"] \\
X' \arrow[r, "0'"'] & A'
\end{tikzcd}$$

where $0'$ is the zero of $A' \to X'$.

Remark 3.1.12. If A and A' are bundles of abelian groups over the same base X, the definition of the kernel of φ is the same as the kernel of a morphism of group objects (Definition A.4.22).

Proposition 3.1.13. *Let $\varphi : A \to A'$ be a morphism of the bundles of abelian groups $A \to X$ and $A' \to X'$. Then:*

(i) The composition $\ker\varphi \xrightarrow{i_{\ker\varphi}} A \longrightarrow X$ equips the kernel of φ with the structure of a bundle of abelian groups over X.

(ii) The morphism $i_{\ker\varphi} : \ker\varphi \to A$ is a regular monomorphism[3] *of bundles of abelian groups over X.*

Proof. Let the bundle projections be denoted by $p : A \to X$ and $p' : A' \to X'$. Let $f : X \to X'$ be the base morphism of φ. Consider the following commutative diagram:

$$\begin{tikzcd}
\ker\varphi \arrow[r] \arrow[d] & A \arrow[d, "(p,\varphi)"] & \\
X \arrow[r, "(\mathrm{id}_X, 0'\circ f)"'] \arrow[d, "f"'] & X \times_{X'} A' \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
X' \arrow[r, "0'"'] & A' \arrow[r, "p'"'] & X'
\end{tikzcd}$$

[3] See Definition A.4.1 for the different notions of monomorphisms.

The lower right square is a pullback. The composition of the horizontal arrows at the bottom is the identity. The composition of the horizontal arrows in the middle is also the identity. This shows that the horizontal rectangle at the bottom is a pullback. It follows from the pasting lemma (Lemma A.4.13) that the lower left square is a pullback. The vertical rectangle on the left is a pullback by definition of the kernel. It follows from the pasting lemma that the upper left square is a pullback:

$$\begin{tikzcd} \ker\varphi \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "{(p,\varphi)}"] \\ X \arrow[r, "{(\mathrm{id}_X, 0'\circ f)}"'] & X \times_{X'} A' \end{tikzcd} \tag{3.10}$$

By Proposition 3.1.8, the morphism $X \times_{X'} A' \to X$ is a bundle of abelian groups with the zero given by the bottom horizontal arrow of Diagram (3.10). Since φ is a morphism of bundles of abelian groups, the right vertical arrow (p, φ) is a morphism of group objects in $\mathcal{C}\downarrow X$ by Proposition 3.1.10. We conclude that the kernel of φ is the kernel of this morphism. As is the case for any morphism of group objects in any category, the kernel is a subgroup (Lemma A.4.23). That is, the addition $+ : A \times_X A \to A$ restricts to a morphism $\ker\varphi \times_X \ker\varphi \to \ker\varphi$ and the zero $0 : X \to A$ factors through $\ker\varphi$, so that $\ker\varphi \to X$ inherits the structure of a bundle of abelian groups. This completes the proof of part (i).

Observe that the zero $(\mathrm{id}_X, 0' \circ f)$ of the bundle of abelian groups $X \times_{X'} A' \to X$ is a split, hence a regular monomorphism. Since regular monomorphisms are stable under pullback, the top horizontal arrow of Diagram (3.10) is a regular monomorphism (Rem. A.4.2). By definition, it is a morphism of group objects in $\mathcal{C}\downarrow X$, in other words, a morphism of bundles of abelian groups over X. This shows part (ii). $\square$

Sections of bundles of abelian groups

Recall from Notation 3.1.4 that given a bundle $p : A \to X$, its set of sections is denoted by $\Gamma(X, A) := \{a : X \to A \mid p \circ a = \mathrm{id}_X\}$.

Remark 3.1.14. The functor of sections

$$\begin{aligned} \Gamma : \mathcal{C}\downarrow X &\longrightarrow \mathcal{S}\mathrm{et} \\ (A \to X) &\longmapsto \Gamma(X, A) \end{aligned} \tag{3.11}$$

preserves finite products. As is the case for any functor that preserves products, it maps abelian groups in $\mathcal{C}\downarrow X$ to abelian groups in $\mathcal{S}\mathrm{et}$ (Remark A.4.25). In other words, the set of sections of a bundle of abelian groups has the structure of an abelian group. Spelled out, the sum of two sections $a, b : X \to A$ of a bundle of abelian groups $A \to X$ is defined by

$$a + b := + \circ (a, b)\,.$$

Furthermore, (3.11) takes morphisms of group objects in $\mathcal{C}\downarrow X$ to group homomorphisms. That is, if $\varphi : A \to A'$ is a morphism of bundles of abelian groups over

X, then the map

$$\begin{aligned}\varphi_* : \Gamma(X, A) &\longrightarrow \Gamma(X, A') \\ a &\longmapsto \varphi \circ a\end{aligned} \tag{3.12}$$

is a group homomorphism. If φ is a monomorphism, then (3.12) is injective.

3.1.2 Bundles of R-modules

Let R be a commutative ring object in the category $\mathcal{C}$ with addition $\hat{+} : R \times R \to R$, zero $\hat{0} : * \to R$, multiplication $\hat{m} : R \times R \to R$, and unit $\hat{1} : * \to R$. Let $X \in \mathcal{C}$ be an object. Since the functor

$$\begin{aligned}\mathcal{C} &\longrightarrow \mathcal{C}{\downarrow}X \\ C &\longmapsto (X \times C \xrightarrow{\mathrm{pr}_1} X)\end{aligned}$$

preserves finite products, the ring structure on R induces a ring structure on $\mathrm{pr}_1 : X \times R \to X$ in $\mathcal{C}{\downarrow}X$. An $(X \times R \to X)$-module object in $\mathcal{C}{\downarrow}X$ will be called, for short, a bundle of R-modules over X (Terminology 3.1.1).

Spelled out, an R-module structure on a bundle of abelian groups $A \to X$ consists of a morphism

$$\begin{tikzcd} R \times A \arrow[rr, "\kappa"] \arrow[dr, "p \circ \mathrm{pr}_2"'] & & A \arrow[dl, "p"] \\ & X & \end{tikzcd} \tag{3.13}$$

satisfying the usual conditions of a left action and linearity in the first and second argument. The axioms are spelled out in Definition A.4.30 for any module object over a ring in a category. In Section 3.2.3, we write the diagrams explicitly for the case of the tangent bundle $TX \to X$ in a tangent category.

Definition 3.1.15. Let $p : A \to X$ and $p' : A' \to X'$ be bundles of R-modules with module structures κ and κ' respectively. A **morphism of bundles of R-modules** is a morphism $\varphi : A \to A'$ of bundles of abelian groups such that the diagram

$$\begin{tikzcd} R \times A \arrow[r, "\mathrm{id}_R \times \varphi"] \arrow[d, "\kappa"'] & R \times A' \arrow[d, "\kappa'"] \\ A \arrow[r, "\varphi"'] & A' \end{tikzcd} \tag{3.14}$$

commutes.

Proposition 3.1.16. *Let $A \to X$ be a bundle of R-modules and $Y \to X$ a morphism in $\mathcal{C}$. Then, the pullback $\mathrm{pr}_1 : Y \times_X A \to Y$ is a bundle of R-modules and $\mathrm{pr}_2 : Y \times_X A \to A$ is a morphism of bundles of R-modules.*

Proof. Let the bundle projection be denoted by $p : A \to X$, its module structure by κ, and the morphism by $f : Y \to X$. It follows from Proposition 3.1.8 that $\mathrm{pr}_1 : Y \times_X A \to Y$ is a bundle of abelian groups and $\mathrm{pr}_2 : Y \times_X A \to A$ is a morphism of bundles of abelian groups. To describe the R-module structure on the pullback $f^*A := Y \times_X A$, we consider the following diagram:

$$\begin{tikzcd} R \times Y \arrow[r, "\mathrm{id}_R \times f"] \arrow[d, "\mathrm{pr}_2"'] & R \times X \arrow[d, "\mathrm{pr}_2"] & R \times A \arrow[l, "\mathrm{id}_R \times p"'] \arrow[d, "\kappa"] \\ Y \arrow[r, "f"'] & X & A \arrow[l, "p"] \end{tikzcd} \tag{3.15}$$

The right square commutes by the commutative Triangle (3.13). The left square commutes trivially. By the commutativity of pullbacks and products, we have the following isomorphism for the limit of the top row,

$$\begin{aligned} \alpha : (R \times Y) \times_{R \times X} (R \times A) &\xrightarrow{\cong} (R \times_R R) \times (Y \times_X A) \\ &\xrightarrow{\cong} R \times (Y \times_X A) . \end{aligned}$$

Diagram (3.15) induces a unique morphism from the limit of the top row to the limit of the bottom row

$$\kappa_{f^*A} := \big(\mathrm{pr}_2 \times_{\mathrm{pr}_2} \kappa\big) \circ \alpha^{-1} : R \times (Y \times_X A) \longrightarrow Y \times_X A ,$$

such that the diagram

$$\begin{tikzcd} R \times (Y \times_X A) \arrow[r, "\mathrm{id}_R \times \mathrm{pr}_2"] \arrow[d, "\kappa_{f^*A}"'] & R \times A \arrow[d, "\kappa"] \\ Y \times_X A \arrow[r, "\mathrm{pr}_2"'] & A \end{tikzcd}$$

commutes. We conclude that $\mathrm{pr}_2 : Y \times_X A \to A$ is a morphism of bundles of R-modules. □

The kernel of a morphism of bundles of R-modules

Proposition 3.1.17. *Let $\varphi : A \to A'$ be a morphism of the bundles of R-modules $A \to X$ and $A' \to X'$. Then:*

(i) The composition $\ker\varphi \xrightarrow{i_{\ker\varphi}} A \longrightarrow X$ equips the kernel of φ with the structure of a bundle of R-modules over X.

(ii) The morphism $i_{\ker\varphi} : \ker\varphi \to A$ is a regular monomorphism of bundles of R-modules over X.

Proof. It follows from Proposition 3.1.13(i) that $\ker\varphi \to X$ is a bundle of abelian groups. Denote by κ and κ' the R-module structures of $A \to X$ and $A' \to X'$ respectively. To describe the R-module structure on $\ker\varphi$, consider the following diagram:

$$\begin{tikzcd} R \times X' \arrow[r, "\mathrm{id}_R \times 0'"] \arrow[d, "\mathrm{pr}_2"'] & R \times A' \arrow[d, "\kappa'"] & R \times A \arrow[l, "\mathrm{id}_R \times \varphi"'] \arrow[d, "\kappa"] \\ X' \arrow[r, "0'"'] & A' & A \arrow[l, "\varphi"] \end{tikzcd} \tag{3.16}$$

The right square is the commutative Diagram (3.14), since φ is, by assumption, a morphism of bundles of R-modules. The left square commutes since κ', being an R-module structure, is linear in A'. This implies that it maps the zero to the zero. By the commutativity of pullbacks and products, we have the following isomorphism for the limit of the top row,

$$\begin{aligned}\alpha : (R \times X') \times_{R\times A'} (R \times A) &\xrightarrow{\cong} (R \times_R R) \times (X' \times_{A'} A)\\ &\xrightarrow{\cong} R \times \ker \varphi \,.\end{aligned}$$

Diagram (3.16) induces a unique morphism from the limit of the top row to the limit of the bottom row

$$\kappa_{\ker\varphi} := \big(\mathrm{pr}_2 \times_{\kappa'} \kappa\big) \circ \alpha^{-1} : R \times \ker\varphi \longrightarrow \ker\varphi \,,$$

such that the diagram

$$\begin{tikzcd} R \times \ker\varphi \arrow[r, "\mathrm{id}_R \times i_{\ker\varphi}"] \arrow[d, "\kappa_{\ker\varphi}"'] & R \times A \arrow[d, "\kappa"] \\ \ker\varphi \arrow[r, "i_{\ker\varphi}"'] & A \end{tikzcd} \tag{3.17}$$

commutes. It follows from Proposition 3.1.13(ii) that $i_{\ker\varphi} : \ker\varphi \to A$ is a regular monomorphism of bundles of abelian groups over X. By using the commutativity of Diagram (3.17), we conclude that $i_{\ker\varphi}$ is a morphism of bundles of R-modules. □

Sections of bundles of R-modules

Remark 3.1.18. The ring object $\mathrm{pr}_1 : X \times R \to X$ in $\mathcal{C}{\downarrow}X$ is mapped by the functor of sections (3.11) to a ring, which is isomorphic by

$$\begin{aligned}(\mathrm{pr}_2)_* : \Gamma(X, X \times R) &\xrightarrow{\cong} \mathcal{C}(X, R)\\ a &\longmapsto \mathrm{pr}_2 \circ a\end{aligned} \tag{3.18}$$

to the ring of R-valued morphisms with addition and multiplication

$$f + g := \hat{+} \circ (f, g) \,, \quad fg := \hat{m} \circ (f, g) \,,$$

for all $f, g \in \mathcal{C}(X, R)$, with zero $X \to * \xrightarrow{\hat{0}} R$, and unit $X \to * \xrightarrow{\hat{1}} R$.

Remark 3.1.19. Let $A \to X$ be a bundle of R-modules with module structure $\kappa : R \times A \to A$. Using the isomorphism (3.18), we see that $\Gamma(X, A)$ has a $\mathcal{C}(X, R)$-module structure given by

$$fa := \kappa \circ (f, a) \,, \tag{3.19}$$

for all $f \in \mathcal{C}(X, R)$ and $a \in \Gamma(X, A)$. Moreover, if $\varphi : A \to A'$ is a morphism of bundles of R-modules over X, then the map (3.12) is a morphism of $\mathcal{C}(X, R)$-modules.

3.2 Abstract tangent functors

This section provides the reader with the main categorical setting which sets the basis of the rest of the constructions in this thesis. After recalling symmetric structures on endofunctors in Section 3.2.1, we will explore the concept of an abstract tangent functor T according to Rosický and his axiomatization in Section 3.2.2. Moreover, based on the work [AB] in progress, we extend the notion of tangent structures to encompass cartesian tangent structures and scalar multiplications in Section 3.2.3. The goal of Section 3.2.4 is to provide a concise proof of the monad structure of the abstract tangent functor, as first observed by [Jub12] in the category of smooth manifolds and [CC14] in general tangent categories. As an application, we observe and prove that the monad structure on T induces an augmented cosimplicial structure on $T^{\bullet+1}$ in Section 3.2.5.

3.2.1 Symmetric structures

We recall from Section A.3.1 that natural transformations can be composed vertically, denoted by the usual composition symbol $\circ$, and horizontally, denoted by juxtaposition. The usual composition of functors is also denoted by juxtaposition. Let $\mathcal{C}$ be a category. The n-fold composition of an endofunctor $F : \mathcal{C} \to \mathcal{C}$ with itself will be denoted by

$$F^n := \underbrace{F \dots F}_{n\text{-times}}$$

for all $n \geq 1$. The two horizontal compositions of a natural transformation $\tau : F^2 \to F^2$ and the identity natural transformation $F \equiv \mathrm{id}_F : F \to F$ are given by $\tau F : F^3 \to F^3$ and $F\tau : F^3 \to F^3$. In components,

$$(\tau F)_C = \tau_{FC} \quad \text{and} \quad (F\tau)_C = F(\tau_C)$$

for all $C \in \mathcal{C}$.

Definition 3.2.1. A **braiding** on a functor $F : \mathcal{C} \to \mathcal{C}$ is a natural transformation $\tau : F^2 \to F^2$, such that

$$\tau F \circ F\tau \circ \tau F = F\tau \circ \tau F \circ F\tau \,. \tag{3.20}$$

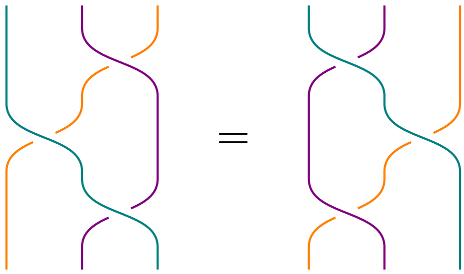
A braiding τ on F is called a **symmetric structure** on F if it further satisfies $\tau \circ \tau = F^2$.

Remark 3.2.2. Using Equation (A.6) for any natural transformation $\alpha = \tau : F^2 \to F^2$ (not necessarily a braiding or a symmetric structure) and $\beta = F^{n-2}\tau : F^n \to F^n$, we get the following equation

$$\tau F^n \circ F^n\tau = F^n\tau \circ \tau F^n \tag{3.21}$$

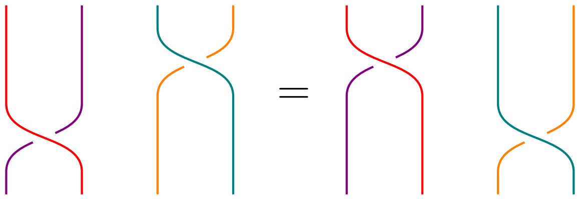
for all $n \geq 2$.

Equations (3.20) and (3.21) are usually called the **braid relations**. We illustrate them by Figures 3.1 and 3.2. The equation $\tau \circ \tau = F^2$ means that τ is an **involution**, that is a natural isomorphism with inverse itself. This is depicted in Figure 3.3.



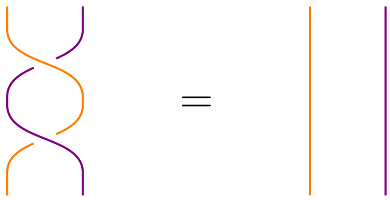
$\tau F \circ F\tau \circ \tau F$ $\qquad$ $F\tau \circ \tau F \circ F\tau$

Figure 3.1: The braid relations, part 1



$\tau F^2 \circ F^2\tau$ $\qquad$ $F^2\tau \circ \tau F^2$

Figure 3.2: The braid relations, part 2



$\tau \circ \tau$ $\qquad$ F^2

Figure 3.3: Involution

Remark 3.2.3. A symmetric structure τ on a functor $F : \mathcal{C} \to \mathcal{C}$ defines an action of the symmetric group S_n on F^n. Spelled out, given an adjacent transposition $\sigma_i = (i, i+1)$ for some $1 \leq i < n$, we define a natural isomorphism

$$F^{i-1}\tau F^{n-i-1} : F^n \longrightarrow F^n . \tag{3.22}$$

Using the fact that the symmetric group on n elements is generated by adjacent transpositions, subject to the braid relations and involutivity, we get that (3.22) induces a group homomorphism $S_n \to \mathrm{Aut}(F^n)$, where $\mathrm{Aut}(F^n)$ denotes the automorphism group consisting of natural isomorphisms $F^n \to F^n$.

3.2.2 Rosický's axioms

In [Ros84], Rosický introduced the notion of *abstract tangent functor*, which captures the natural categorical structure of the tangent functor of manifolds that is needed to define the Lie bracket of vector fields. Recall that the identity endofunctor on a category $\mathcal{C}$ will be denoted by $1 : \mathcal{C} \to \mathcal{C}$.

Definition 3.2.4 (Sec. 2 in [Ros84], Def. 2.3 in [CC14])**.** A **tangent structure** on a category $\mathcal{C}$ is composed of an endofunctor $T : \mathcal{C} \to \mathcal{C}$, called **abstract tangent functor**, together with the following natural transformations:

- the **bundle projection** $\pi : T \to 1$,
- the **zero section** $0 : 1 \to T$,
- the **fiberwise addition** $+ : T \times_1^{\pi,\pi} T \to T$,
- the **vertical lift** $\lambda : T \to T^2$,
- the **symmetric structure** $\tau : T^2 \to T^2$,

satisfying the following axioms:

(i) **Fiber products**: The pullbacks

$$T_k := \underbrace{T \times_1 T \times_1 \ldots \times_1 T}_{k\text{-times}}$$

over $\pi : T \to 1$ exist for all $k \geq 1$, are pointwise, and preserved by T (Def. 3.1.5).

(ii) **Bundle of abelian groups**: The fiberwise addition $+$ and the zero section 0 equip $\pi : T \to 1$ with the structure of a bundle of abelian groups over 1.

(iii) **Symmetric structure**: τ is a symmetric structure on T (Def. 3.2.1). Moreover, τ is a morphism of bundles of abelian groups. That is, the diagrams

$$\begin{tikzcd} T^2 \arrow[rr, "\tau"] \arrow[dr, "T\pi"'] & & T^2 \arrow[dl, "\pi T"] \\ & T & \end{tikzcd} \tag{3.23}$$

and

$$\begin{tikzcd} T^2 \times_T^{T\pi,T\pi} T^2 \arrow[r, "\tau\times_T\tau"] \arrow[d, "\nu_2^{-1}"', "\cong"] & T_2T \arrow[dd, "+T"] \\ TT_2 \arrow[d, "T+"'] & \\ T^2 \arrow[r, "\tau"'] & T^2 \end{tikzcd} \tag{3.24}$$

commute, where ν_2 is the morphism (3.1) for $A = TX \xrightarrow{\pi_X} X$, $F = T$, and $k = 2$.

(iv) **Vertical lift**: $\lambda : T \to T^2$ is a morphism of bundles of abelian groups over $0 : 1 \to T$. That is, the diagrams

$$\begin{tikzcd} T \arrow[r, "\lambda"] \arrow[d, "\pi"'] & T^2 \arrow[d, "\pi T"] \\ 1 \arrow[r, "0"'] & T \end{tikzcd} \qquad \begin{tikzcd} T_2 \arrow[r, "\lambda\times_0\lambda"] \arrow[d, "+"'] & T_2T \arrow[d, "+T"] \\ T \arrow[r, "\lambda"'] & T^2 \end{tikzcd} \tag{3.25}$$

commute. Furthermore, the diagram

$$\begin{tikzcd} T \arrow[r, "\lambda"] \arrow[d, "\lambda"'] & T^2 \arrow[d, "\lambda T"] \\ T^2 \arrow[r, "T\lambda"'] & T^3 \end{tikzcd} \tag{3.26}$$

commutes.

(v) **Compatibility of vertical lift and symmetric structure**: The diagrams

$$\begin{tikzcd} & T \arrow[dl, "\lambda"'] \arrow[dr, "\lambda"] & \\ T^2 \arrow[rr, "\tau"'] & & T^2 \end{tikzcd} \qquad \begin{tikzcd} T^2 \arrow[r, "T\lambda"] \arrow[d, "\tau"'] & T^3 \arrow[r, "\tau T"] & T^3 \arrow[d, "T\tau"] \\ T^2 \arrow[rr, "\lambda T"'] & & T^3 \end{tikzcd} \tag{3.27}$$

commute.

(vi) **The vertical lift is a kernel**: The diagram

$$\begin{tikzcd} T \arrow[r, "\lambda"] \arrow[d, "\pi"'] & T^2 \arrow[d, "(\pi T, T\pi)"] \\ 1 \arrow[r, "(0,0)"'] & T_2 \end{tikzcd} \tag{3.28}$$

is a pointwise pullback.

Terminology 3.2.5. A category with a tangent structure will be called a **tangent category**.

We will often denote a tangent category by a pair $(\mathcal{C}, T)$ or say that $\mathcal{C}$ has an abstract tangent functor T. The rest of the structure natural transformations will be implicit and will be denoted by the same letters as in Definition 3.2.4 throughout the thesis, unless otherwise specified.

Remark 3.2.6. In [CC14], the authors relax the second axiom and only require that $T \to 1$ is a bundle of abelian monoids, having certain applications in computer science and combinatorics in mind. In this thesis, we will use the original axiom of Rosický and require that $T \to 1$ is a bundle of abelian groups. The reason for our choice is that the negatives will be crucial in the construction of the Lie bracket of two vector fields.

Remark 3.2.7. Recall that the tangent space of a finite-dimensional vector space at a point is canonically isomorphic to the vector space itself [Lee13, Prop. 3.13]. Thus, the tangent spaces of the fibers of a smooth vector bundle can be identified with the fibers. This key concept has been generalized in [CC18], where the authors develop the notion of differential bundles and fibrations in tangent categories. A differential bundle is a bundle of abelian groups together with a compatible vertical lift [CC18, Def. 2.3]. The vertical lift in a tangent structure and Axiom (vi) turn the tangent bundle $TX \to X$ into a differential bundle in this sense.

However, for our purposes we need an extra structure of scalar R-multiplication on $TX \to X$, for $R \in \mathcal{C}$ a ring object. Since differential bundles have in general no scalar multiplication and since the only differential bundle we will be working with is the tangent bundle, we will not employ this concept in this thesis.

Notation 3.2.8. The subtraction of the bundle of abelian groups $T \to 1$ will be denoted by $- : T_2 \to T$ (Diagram (3.3)).

Remark 3.2.9. Using Equation (A.6), we get the following identities of the horizontal compositions of various combinations of the natural transformations $\pi : T \to 1$ and $0 : 1 \to T$:

$$\pi\pi = \pi \circ T\pi = \pi \circ \pi T \tag{3.29}$$
$$00 = 0T \circ 0 = T0 \circ 0 \tag{3.30}$$
$$0\pi = T\pi \circ 0T = 0 \circ \pi \tag{3.31}$$
$$\pi 0 = \pi T \circ T0 = 0 \circ \pi\,. \tag{3.32}$$

Using Equation (3.29), we further get:

$$T\pi\pi = T\pi \circ T^2\pi = T\pi \circ T\pi T \tag{3.33}$$
$$\pi T\pi = \pi T \circ T^2\pi = T\pi \circ \pi T^2 \tag{3.34}$$
$$\pi\pi T = \pi T \circ T\pi T = \pi T \circ \pi T^2\,. \tag{3.35}$$

Remark 3.2.10. The vertical lift can be extended by the additive bundle structure to the morphism

$$\lambda_2 : T_2 \xrightarrow{T0 \times_0 \lambda} T_2 T \xrightarrow{+T} T^2 \xrightarrow{\tau} T^2\,.$$

In components,

$$\lambda_{2,X} = \tau_X \circ +_{TX} \circ (T0_X \times_{0_X} \lambda_X)\,, \tag{3.36}$$

for all $X \in \mathcal{C}$. It was shown in [CC14, Lem. 3.10], assuming all other axioms of a tangent structure (with negatives), that Axiom (vi) is satisfied if and only if

$$\begin{tikzcd} T_2 \arrow[r, "\lambda_2"] \arrow[d, "\pi \circ \mathrm{pr}_1"'] & T^2 \arrow[d, "T\pi"] \\ 1 \arrow[r, "0"'] & T \end{tikzcd} \tag{3.37}$$

is a pointwise pullback.

Example 3.2.11. The following are some examples of tangent categories:

(i) The prototypical example for tangent categories is the category of smooth finite-dimensional manifolds with the usual tangent functor [Lee13, Ch. 3]. The local model for smooth manifolds and its tangent functor is the category of Euclidean spaces, which is studied in detail in Section 3.4.

(ii) Diffeological spaces are concrete sheaves on the site of Euclidean spaces with the usual open covers [Blo24a, Def. 3.4]. Elastic diffeological spaces are diffeological spaces with certain *elasticity* axioms [Blo24a, Def. 4.1]. These axioms ensure that the left Kan extension of the tangent structure on Euclidean spaces defines a tangent structure on elastic diffeological spaces [Blo24a, Thm. 4.2].

(iii) Let G be a Lie groupoid. A smooth right G-bundle is a smooth submersion $r_E : E \to G_0$ with a right G-action $E \times_{G_0}^{r_E,t} G_1 \to E$, as studied in detail in Section 4.2. The map that sends E to the vertical tangent bundle

$$VE := TE \times_{TG_0}^{Tr_E, 0_{G_0}} G_0$$

with its right G-action as defined by (4.27) is an abstract tangent functor on the category of smooth right G-bundles with G-equivariant morphisms.

(iv) The category of abelian groups has an abstract tangent functor given by

$$TA := A \times A$$

on objects and $Tf := f \times f$ on morphisms $f : A \to B$. The fiberwise addition, zero section, symmetric structure, and vertical lift are defined exactly as for Euclidean spaces (Section 3.4).

(v) A symmetric algebraic operad is a monoid object in the monoidal category of symmetric modules [LV12, Sec. 5.2.1]. The category of algebras over a symmetric algebraic operad has a tangent structure [ILL24, Thm. 4.3.3], given in terms of semi-direct products.

Remark 3.2.12. Recall from Remark 1.1.9 that the bicategory of Lie groupoids is equivalent to the bicategory of differentiable stacks. In other words, Lie groupoids present differentiable stacks up to Morita equivalence. A natural question is if the tangent structure on smooth manifolds induces a tangent structure on differentiable stacks.

Given a differentiable stack $\mathcal{X}$ presented by a Lie groupoid $G_1 \rightrightarrows G_0$, the tangent stack $\mathcal{T}\mathcal{X}$ is the differentiable stack presented by the tangent groupoid $TG_1 \rightrightarrows TG_0$. This is independent of the choice of the presenting groupoid $G_1 \rightrightarrows G_0$ up to isomorphism. It follows from the naturality of the bundle projection $\pi : T \to 1$ in smooth manifolds that the pair (π_{G_1}, π_{G_0}) forms a Lie groupoid morphism. This induces a morphism of stacks $\mathcal{T}\mathcal{X} \to \mathcal{X}$. The other structure natural transformations are obtained similarly from those on smooth manifolds.

However differentiable stacks form a bicategory. Hence, for a more refined approach, we have to consider the higher categorical aspects of the tangent structure. For instance, we need to consider the question of coherence relations between the associator and unit constraints of the bicategory on one side and the tangent structure on the other side. Generalizing the notion of abstract tangent structures to higher categories is an interesting goal for future research.

3.2.3 Cartesian tangent categories with scalar multiplication

The fibers of the tangent bundle $TM \to M$ of a smooth manifold M have a scalar $\mathbb{R}$-multiplication

$$\mathbb{R} \times T_pM \longrightarrow T_pM\,,$$

for all $p \in M$. In this section we generalize this to arbitrary tangent categories, where the field $\mathbb{R}$ of real numbers is replaced by a ring object in the tangent category.

Let $\mathcal{C}$ be a category with finite products.

Definition 3.2.13. A tangent structure on $\mathcal{C}$ is called **cartesian** if the tangent functor preserves finite products, that is, if the natural morphism

$$\chi_{X,Y} : T(X \times Y) \longrightarrow TX \times TY \tag{3.38}$$

has an inverse for all $X, Y \in \mathcal{C}$ and if $T* \cong *$ for the terminal object $*$ of $\mathcal{C}$.

Definition 3.2.14 ([CC14, Def. 2.9])**.** Let $\mathcal{C}$ be a cartesian tangent category. The **partial tangent morphisms** of a morphism $f : X \times Y \to Z$ in $\mathcal{C}$ are defined by

$$T_{(1)}f : TX \times Y \xrightarrow{\mathrm{id}_{TX} \times 0_Y} TX \times TY \xrightarrow{\chi^{-1}_{X,Y}} T(X \times Y) \xrightarrow{Tf} TZ$$
$$T_{(2)}f : X \times TY \xrightarrow{0_X \times \mathrm{id}_{TY}} TX \times TY \xrightarrow{\chi^{-1}_{X,Y}} T(X \times Y) \xrightarrow{Tf} TZ\,,$$

where the index refers to the factor in the product.

R-module structure

Let R be a commutative ring object in $\mathcal{C}$. Then, $R : \mathcal{C} \to \mathcal{C}$, $C \mapsto R$ can be viewed as the constant endofunctor. The ring structure of R equips $R : \mathcal{C} \to \mathcal{C}$ with the structure of a ring in the category $\mathrm{End}(\mathcal{C})$ of endofunctors on $\mathcal{C}$. As explained in

Section 3.1.2 (where now the ambient category is $\mathrm{End}(\mathcal{C})$ and the object X is the identity endofunctor 1), the functor

$$\begin{aligned} \mathrm{End}(\mathcal{C}) &\longrightarrow \mathrm{End}(\mathcal{C})\downarrow 1 \\ F &\longmapsto (1 \times F \xrightarrow{\mathrm{pr}_1} 1) \end{aligned}$$

preserves finite products[4]. This implies that the projection $\mathrm{pr}_1 : 1 \times R \to 1$ has a ring structure in $\mathrm{End}(\mathcal{C})\downarrow 1$, inherited from that of $R : \mathcal{C} \to \mathcal{C}$. A $(1 \times R \to 1)$-module object in $\mathrm{End}(\mathcal{C})\downarrow 1$ will be called, for short, a bundle of R-modules over 1 (Terminology 3.1.1). In other words, a $(1 \times R \to 1)$-module structure on a bundle $S \to 1$ of abelian groups in $\mathrm{End}(\mathcal{C})$ will be called, for short, an R-module structure on $S \to 1$.

Terminology 3.2.15. A tangent category is called a **tangent category with an R-module structure** if the tangent bundle $T \to 1$ is equipped with an R-module structure.

Explicitly, an R-module structure on $T \to 1$ consists of a natural morphism

$$\kappa_X : R \times TX \longrightarrow TX$$

of bundles over X, such that the following diagrams commute[5] for all $X \in \mathcal{C}$:

(i) Associativity:

$$\begin{tikzcd} R \times R \times TX \arrow[r, "\mathrm{id}_R \times \kappa_X"] \arrow[d, "\hat{m} \times \mathrm{id}_{TX}"'] & R \times TX \arrow[d, "\kappa_X"] \\ R \times TX \arrow[r, "\kappa_X"'] & TX \end{tikzcd}$$

(ii) Unitality:

$$\begin{tikzcd} * \times TX \arrow[r, "\hat{1} \times \mathrm{id}_{TX}"] \arrow[dr, "\cong"'] & R \times TX \arrow[d, "\kappa_X"] \\ & TX \end{tikzcd}$$

(iii) Linearity in R:

$$\begin{tikzcd} R \times R \times TX \arrow[r, "\hat{+} \times \mathrm{id}_{TX}"] \arrow[d, "\left(\kappa_X \circ (\mathrm{pr}_1, \mathrm{pr}_3), \kappa_X \circ (\mathrm{pr}_2, \mathrm{pr}_3)\right)"'] & R \times TX \arrow[d, "\kappa_X"] \\ TX \times_X TX \arrow[r, "+_X"'] & TX \end{tikzcd}$$

[4]The product of endofunctors $F, G : \mathcal{C} \to \mathcal{C}$ is given pointwise, that is, $(F \times G)(X) = FX \times GX$ for all $X \in \mathcal{C}$.

[5]These diagrams are spelled out in Definition A.4.30 for any module over a ring in a category.

(iv) Linearity in TX:

$$\begin{tikzcd}
R \times TX \times_X TX \arrow[r, "\mathrm{id}_R \times +_X"] \arrow[d, "\left(\kappa_X \circ (\mathrm{pr}_1, \mathrm{pr}_2),\, \kappa_X \circ (\mathrm{pr}_1, \mathrm{pr}_3)\right)"'] & R \times TX \arrow[d, "\kappa_X"] \\
TX \times_X TX \arrow[r, "+_X"'] & M
\end{tikzcd}$$

Tangent-stable R-module objects

Given a point $x : * \to X$, the fiber $T_x X := * \times_X TX$ has the structure of an R-module (Remark 3.1.2). Recall that the tangent space of a finite-dimensional vector space at a point is canonically isomorphic to the vector space itself [Lee13, Prop. 3.13]. In tangent categories, we give a special name to objects with this property:

Definition 3.2.16. Let $\mathcal{C}$ be a cartesian tangent category with an R-module structure. An R-module object $A \in \mathcal{C}$ will be called **tangent-stable** if there is an isomorphism

$$T_{\hat{0}} A \cong A$$

of R-modules, where $\hat{0} : * \to A$ is the zero of A.

Proposition 3.2.17. *Let $\mathcal{C}$ be a cartesian tangent category with an R-module structure. An R-module object A is tangent-stable if and only if its tangent bundle has a trivialization*

$$TA \cong A \times A\,. \tag{3.39}$$

Proof. It follows from a result in [AB] that the tangent bundle of any group object (G, m, e) in $\mathcal{C}$ has a natural trivialization $TG \cong G \times \mathfrak{g}$, where

$$\mathfrak{g} := T_e G = * \times_G^{e, \pi_G} TG\,.$$

Applying this to the underlying group structure of A, we get that

$$\begin{aligned} TA &\cong A \times T_{\hat{0}} A \\ &\cong A \times A\,. \end{aligned}$$

In the last step, we have used the property that A is tangent-stable. □

Let A be a tangent-stable R-module in a cartesian tangent category. The projection onto the first factor of $TA \cong A \times A$ is the bundle projection onto the base. The projection onto the second factor, the fiber of the bundle, will be denoted by

$$\eta_A : TA \longrightarrow A\,.$$

Hence, the isomorphism (3.39) can be written as

$$(\pi_A, \eta_A) : TA \xrightarrow{\cong} A \times A\,.$$

Definition 3.2.18. A commutative ring object in a cartesian tangent category will be called **tangent-stable** if it is tangent-stable as module over itself.

Scalar R-multiplication

In a tangent category, we have to require an R-module structure to be compatible with the rest of the tangent structure in order to obtain the usual relations of a Cartan calculus [AB].

Definition 3.2.19. Let R be a commutative ring object in a cartesian tangent category $\mathcal{C}$. An R-module structure $\kappa : R \times T \to T$ on the tangent bundle $\pi : T \to 1$ will be called a **scalar R-multiplication** if R is tangent-stable and if the following diagrams commute for all $X \in \mathcal{C}$:

$$\begin{tikzcd} R \times TX \arrow[r, "\mathrm{id}_R \times \lambda_X"] \arrow[d, "\kappa_X"'] & R \times T^2X \arrow[d, "\kappa_{TX}"] \\ TX \arrow[r, "\lambda_X"'] & T^2X \end{tikzcd} \tag{3.40}$$

$$\begin{tikzcd} TR \times TX \arrow[r, "T_{(1)}\kappa_X"] \arrow[d, "(\pi_R, \eta_R) \times \mathrm{id}_{TX}"', "\cong"] & T^2X \\ R \times R \times TX \arrow[r, "\left(\kappa_X \circ (\mathrm{pr}_1, \mathrm{pr}_3),\, \kappa_X \circ (\mathrm{pr}_2, \mathrm{pr}_3)\right)"'] & T_2X \arrow[u, "\lambda_{2,X}"'] \end{tikzcd} \tag{3.41}$$

$$\begin{tikzcd} R \times T^2X \arrow[r, "T_{(2)}\kappa_X"] \arrow[d, "\mathrm{id}_R \times \tau_X"'] & T^2X \\ R \times T^2X \arrow[r, "\kappa_{TX}"'] & T^2X \arrow[u, "\tau_X"'] \end{tikzcd} \tag{3.42}$$

Terminology 3.2.20. A cartesian tangent category with an R-module structure is called a **cartesian tangent category with scalar R-multiplication** if the R-module structure is a scalar R-multiplication.

Example 3.2.21. Consider the tangent structures in Example 3.2.11.

(i) The category of smooth manifolds is a cartesian tangent category with scalar $\mathbb{R}$-multiplication.

(ii) The category of elastic diffeological spaces is a cartesian tangent category. The left Kan extension of the scalar $\mathbb{R}$-multiplication on Euclidean spaces is a scalar $\mathbb{R}$-multiplication on elastic diffeological spaces [Blo24a, Thm. 4.2].

(iii) Let G be a Lie groupoid. The product of two smooth right G-bundles E and E' is given by the fiber product $E \times_{G_0} E'$ with the diagonal right G-action (Section 4.2.2). The vertical tangent bundle satisfies $V(E \times_{G_0} E') \cong VE \times_{G_0} VE'$, which shows that the tangent category of G-bundles is cartesian.

Every smooth manifold M can be viewed as a G-bundle $\mathrm{pr}_2 : M \times G_0 \to G_0$ with right G-action $(m, t(g)) \cdot g = (m, s(g))$. A smooth map $f : M \to N$ gives rise to the morphism $f \times \mathrm{id}_{G_0} : M \times G_0 \to N \times G_0$ of G-bundles. This defines a functor from the category of smooth manifolds to the category of smooth right G-bundles. The functor preserves finite products, which implies that the ring object $\mathbb{R}$ in manifolds is mapped to a ring object $R = (\mathbb{R} \times G_0 \to G_0)$ in G-bundles. By identifying $(\mathbb{R} \times G_0) \times_{G_0} E \cong \mathbb{R} \times E$, the $\mathbb{R}$-multiplication of vertical tangent vectors can be viewed as a scalar multiplication by the ring object R.

(iv) By construction, the tangent category of abelian groups is cartesian. One might guess that there is a scalar $\mathbb{Z}$-multiplication. However, $\mathbb{Z}$ is not a ring object in the category of abelian groups with the categorical product. (The multiplication $\mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$ is bilinear, not linear.) To fix this, we have to use the tensor product of $\mathbb{Z}$-modules, which would require substantial generalization of the concepts used in this thesis.

3.2.4 The abstract tangent functor as a monad

A **monad** in a category $\mathcal{C}$ is a monoid object in the strict monoidal category $(\mathrm{End}(\mathcal{C}), \circ, 1)$ of endofunctors on $\mathcal{C}$. In [Jub12, Theorem 2.1.1] it was proved that the tangent functor

$$T : \mathcal{M}\mathrm{fld} \longrightarrow \mathcal{M}\mathrm{fld}$$

on the category of smooth manifolds has a monad structure with multiplication

$$T^2 M \xrightarrow{(\pi_{TM}, T\pi_M)} TM \times_M TM \xrightarrow{+_M} TM$$

and the zero section

$$0_M : M \longrightarrow TM\,,$$

expressed componentwise for all smooth manifolds M. It was further shown in [Jub12, Proposition 2.1.1] that this is the unique monad structure on T. The monad structure on the tangent functor of smooth manifolds has vast applications in the foliation theory of manifolds, as studied in [Jub12].

In [CC14, Proposition 3.4], this result was generalized to abstract tangent functors in any category. The proof provided there makes extensive use of the symmetric structure τ on T. In this section, we provide a shorter and more direct proof, where we only use the abelian group structure on $\pi : T \to 1$.

Proposition 3.2.22 ([CC14, Proposition 3.4]). *Let $(\mathcal{C}, T)$ be a tangent category. Then, the abstract tangent functor $T : \mathcal{C} \to \mathcal{C}$ together with the natural transformation $\mu := + \circ (\pi T, T\pi) : T^2 \to T$ and the zero section $0 : 1 \to T$ is a monad in $\mathcal{C}$.*

Proof. We need to check the associativity of μ and the unitality of 0, that is, we need to show that the following diagrams of natural transformations

$$\begin{tikzcd} T^3 \arrow[r, "\mu T"] \arrow[d, "T\mu"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "0T"] \arrow[dr, "T"'] & T^2 \arrow[d, "\mu"] & T \arrow[l, "T0"'] \arrow[dl, "T"] \\ & T & \end{tikzcd} \tag{3.43}$$

commute. For the unitality,

$$\begin{aligned}\mu \circ 0T &= + \circ (\pi T, T\pi) \circ 0T \\ &= + \circ (\pi T \circ 0T, T\pi \circ 0T) \\ &= + \circ (T, 0 \circ \pi) \\ &= T\,,\end{aligned}$$

where we have used $\pi \circ 0 = 1$, Equation (3.31) and the unitality axiom of the zero section in the bundle of abelian groups $\pi : T \to 1$. Using the same arguments and Equation (3.32) instead, we get that

$$\begin{aligned}\mu \circ T0 &= + \circ (\pi T, T\pi) \circ T0 \\ &= + \circ (\pi T \circ T0, T\pi \circ T0) \\ &= + \circ (0 \circ \pi, T) \\ &= T\,.\end{aligned}$$

For the associativity, first of all note that

$$\begin{aligned}\pi \circ \mu &= \pi \circ + \circ (\pi T, T\pi) \\ &= \pi \circ \mathrm{pr}_1 \circ (\pi T, T\pi) \\ &= \pi \circ \pi T \\ &= \pi\pi\,,\end{aligned} \tag{3.44}$$

using the first commutative diagram in (3.2) (for the bundle $T \xrightarrow{\pi} 1$) and Equation (3.29). Now, we calculate

$$\begin{aligned}\mu \circ \mu T &= + \circ (\pi T, T\pi) \circ \mu T \\ &= + \circ (\pi T \circ \mu T,\ T\pi \circ \mu T) \\ &= + \circ \big((\pi \circ \mu)T,\ \mu \circ T^2\pi\big) \\ &= + \circ \big(\pi\pi T,\ + \circ (\pi T, T\pi) \circ T^2\pi\big) \\ &= + \circ \big(\pi\pi T,\ + \circ (\pi T \circ T^2\pi, T\pi \circ T^2\pi)\big) \\ &= + \circ \big(\pi\pi T,\ + \circ (\pi T\pi, T\pi\pi)\big) \\ &= + \circ (T\ \times_1\ +) \circ (\pi\pi T,\ \pi T\pi,\ T\pi\pi)\end{aligned}$$

by using the naturality of μ, Equation (3.44), and Equations (3.34) and (3.33). Note that in the expression $T \times_1 +$ in the last line, T refers to the identity natural transformation $T \to T$. On the other hand,

$$\begin{aligned}\mu \circ T\mu &= + \circ (\pi T, T\pi) \circ T\mu \\ &= + \circ (\pi T \circ T\mu,\ T\pi \circ T\mu) \\ &= + \circ \big(\mu \circ \pi T^2,\ T(\pi \circ \mu)\big) \\ &= + \circ \big(+ \circ (\pi T, T\pi) \circ \pi T^2,\ T\pi\pi\big) \\ &= + \circ \big(+ \circ (\pi T \circ \pi T^2, T\pi \circ \pi T^2),\ T\pi\pi\big) \\ &= + \circ \big(+ \circ (\pi\pi T, \pi T\pi),\ T\pi\pi\big) \\ &= + \circ (+\ \times_1\ T) \circ (\pi\pi T, \pi T\pi, T\pi\pi)\end{aligned}$$

by using the naturality of π, Equation (3.44), and Equations (3.34) and (3.35). Lastly, using the associativity of $+$, we get that $\mu \circ \mu T = \mu \circ T\mu$. □

Remark 3.2.23. In [Man12, Theorem 3.2.6], the result is proved for the tangent bundle $TX = X \times X$ in cartesian differential categories, a notion introduced in [BCS09].

3.2.5 The cosimplicial structure of $T^{\bullet+1}$

The goal of this section is to show that the iterated tangent bundle $T^{\bullet+1}$ of a tangent category $\mathcal{C}$ has an augmented cosimplicial structure (Def. 2.1.10). In other words, we will prove that the assignment

$$\boldsymbol{\Delta}_+ \longrightarrow \mathrm{End}(\mathcal{C}), \quad [n] \longmapsto T^{n+1}$$

is an augmented cosimplicial object in the category $\mathrm{End}(\mathcal{C})$ of endofunctors on $\mathcal{C}$ (Prop. 3.2.25). For $n = -1$, the functor $T^{-1+1} = T^0 = 1$ is the identity endofunctor on $\mathcal{C}$.

As $\mathrm{End}(\mathcal{C})$ is a functor category, the coface and codegeneracy maps will be given by natural transformations of functors. Showing that the cosimplicial identities hold will amount to an interplay between various vertical and horizontal compositions of natural transformations. In Section 3.4.4, we will revisit the proof in the category of Euclidean spaces by using local coordinates.

The following lemma is a general result about natural transformations between specified powers of an endofunctor. The relations therein depend only on one index k, whereas the cosimplicial identities of $T^{\bullet+1}$ will depend on three indices n, i and j. Using this lemma, the main proof of the cosimplicial identities will be vastly simplified.

Lemma 3.2.24. *Let $T : \mathcal{C} \to \mathcal{C}$ be an endofunctor and let $\eta : 1 \to T$ and $\mu : T^2 \to T$ be natural transformations. Then, the following identities hold:*

$$\begin{aligned}
\eta T^{k+1} \circ T^k\eta &= T^{k+1}\eta \circ \eta T^k && \text{for } k \geq 0 && (3.45)\\
\mu T^k \circ T^{k+1}\mu &= T^k\mu \circ \mu T^{k+1} && \text{for } k > 0 && (3.46)\\
\mu T^k \circ T^{k+1}\eta &= T^k\eta \circ \mu T^{k-1} && \text{for } k > 0 && (3.47)\\
\eta T^{k-1} \circ T^{k-2}\mu &= T^{k-1}\mu \circ \eta T^k && \text{for } k > 1\,. && (3.48)
\end{aligned}$$

Proof. The identities will be obtained by the horizontal composition of two cleverly chosen natural transformations, by using Formula (A.6) from the Appendix.

For $k \geq 0$, the horizontal composition of

$$\alpha := \eta : 1 \longrightarrow T \quad \text{and} \quad \beta := T^k\eta : T^k \longrightarrow T^{k+1}$$

is given by

$$\alpha\beta = \eta T^{k+1} \circ T^k\eta = T^{k+1}\eta \circ \eta T^k\,.$$

This shows that Equation (3.45) holds. It can be summarized by the following diagram:

$$\begin{tikzcd}[column sep=huge]
\mathcal{C} \arrow[r, bend left=50, "T^k"] \arrow[r, bend right=50, "T^{k+1}"'] \arrow[r, phantom, "\beta = \Downarrow T^k\eta"] \arrow[rr, bend left=80, "T^k"] \arrow[rr, bend right=80, "T^{k+2}"'] & \mathcal{C} \arrow[from=u, phantom] \arrow[r, bend left=50, "1"] \arrow[r, bend right=50, "T"'] \arrow[r, phantom, "\alpha = \Downarrow \eta"] & \mathcal{C}
\end{tikzcd}$$
$$\Downarrow \alpha\beta$$

Similarly, for $k > 0$, the horizontal composition of

$$\alpha := \mu : T^2 \longrightarrow T \quad \text{and} \quad \beta := T^{k-1}\mu : T^{k+1} \longrightarrow T^k$$

is given by

$$\alpha\beta = \mu T^k \circ T^{k+1}\mu = T^k\mu \circ \mu T^{k+1} ,$$

which proves Equation (3.46). Furthermore, letting

$$\alpha := \mu : T^2 \longrightarrow T \quad \text{and} \quad \beta := T^{k-1}\eta : T^{k-1} \longrightarrow T^k$$

yields the equation

$$\alpha\beta = \mu T^k \circ T^{k+1}\eta = T^k\eta \circ \mu T^{k-1} .$$

This shows that Equation (3.47) holds. Finally, for $k > 1$, the horizontal composition of

$$\alpha := \eta : 1 \longrightarrow T \quad \text{and} \quad \beta := T^{k-2}\mu : T^k \longrightarrow T^{k-1}$$

is given by

$$\alpha\beta = \eta T^{k-1} \circ T^{k-2}\mu = T^{k-1}\mu \circ \eta T^k ,$$

which proves Equation (3.48). □

Let $\mathcal{C}$ be a category equipped with an abstract tangent functor T. Recall from Proposition 3.2.22 that $\mu := + \circ (\pi T, T\pi) : T^2 \to T$ and the zero section $0 : 1 \to T$ equip T with a monad structure. Taking powers of T applied to μ and 0, we get the following natural transformations:

$$\begin{tikzcd}
1 \arrow[r, "0"] & T \arrow[r, shift left, "T0"] \arrow[r, shift right, "0T"'] & T^2 \arrow[r, shift left=3, "T^2 0"] \arrow[r, "T0T"] \arrow[r, shift right=3, "0T^2"'] & T^3 & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
T & T^2 \arrow[l, "\mu"'] & T^3 \arrow[l, shift right, "T\mu"'] \arrow[l, shift left, "\mu T"] & T^4 \arrow[l, shift right=3, "T^2\mu"'] \arrow[l, "T\mu T"'] \arrow[l, shift left=3, "\mu T^2"] & \cdots
\end{tikzcd}$$

The next proposition shows that these transformations equip the iterated tangent bundle with an augmented cosimplicial structure.

Proposition 3.2.25. *Let $(\mathcal{C}, T)$ be a tangent category. Then, the family of endofunctors $\{T^{n+1}\}_{n\geq -1}$ together with the coface and codegeneracy transformations*

$$\begin{aligned} d^{n,i} &:= T^{n-i}0T^i : T^n \longrightarrow T^{n+1}, \\ s^{n,i} &:= T^{n-i}\mu T^i : T^{n+2} \longrightarrow T^{n+1}, \end{aligned}$$

for $n \geq 0$ and $0 \leq i \leq n$ is an augmented cosimplicial object in $\mathrm{End}(\mathcal{C})$.

Proof. We start by proving the first cosimplicial identity, given by Equation (2.1). By the functoriality of T, an explicit calculation shows that

$$\begin{aligned} d^{n+1,i} \circ d^{n,j} &= T^{n-i+1}0T^i \circ T^{n-j}0T^j \\ &= T^{n-j}(T^{j-i+1}0T^i \circ 0T^j) \\ &= T^{n-j}(T^{j-i+1}0 \circ 0T^{j-i})T^i, \end{aligned} \tag{3.49}$$

and

$$\begin{aligned} d^{n+1,j+1} \circ d^{n,i} &= T^{n-j}0T^{j+1} \circ T^{n-i}0T^i \\ &= T^{n-j}(0T^{j+1} \circ T^{j-i}0T^i) \\ &= T^{n-j}(0T^{j-i+1} \circ T^{j-i}0)T^i, \end{aligned} \tag{3.50}$$

for all $n \geq 0$, $i \leq j$. Comparing Equations (3.49) and (3.50), and using Equation (3.45) for the monad multiplication μ, the zero section $\eta = 0$ and $k = j - i$, we get Identity (2.1).

Similarly, we calculate that

$$\begin{aligned} s^{n,j} \circ s^{n+1,i} &= T^{n-j}\mu T^j \circ T^{n-i+1}\mu T^i \\ &= T^{n-j}(\mu T^j \circ T^{j-i+1}\mu T^i) \\ &= T^{n-j}(\mu T^{j-i} \circ T^{j-i+1}\mu)T^i, \end{aligned} \tag{3.51}$$

and

$$\begin{aligned} s^{n,i} \circ s^{n+1,j+1} &= T^{n-i}\mu T^i \circ T^{n-j}\mu T^{j+1} \\ &= T^{n-j}(T^{j-i}\mu T^i \circ \mu T^{j+1}) \\ &= T^{n-j}(T^{j-i}\mu \circ \mu T^{j-i+1})T^i, \end{aligned} \tag{3.52}$$

for all $n \geq 0$, $i < j$. Comparing Equations (3.51) and (3.52), and using Equation (3.46) for $k = j - i$, we get Identity (2.2) for all $i < j$. When $i = j$, we have

$$\begin{aligned} T^{n-i}\mu T^i \circ T^{n-i+1}\mu T^i &= T^{n-i}(\mu \circ T\mu)T^i \\ &= T^{n-i}(\mu \circ \mu T)T^i \\ &= T^{n-i}\mu T^i \circ T^{n-i}\mu T^{i+1}, \end{aligned}$$

where we have used the associativity of μ, as expressed by the left diagram in (3.43). This completes the proof of Identity (2.2).

For the mixed identities between the coface and codegeneracy maps, we have

$$\begin{aligned} s^{n,j} \circ d^{n+1,i} &= T^{n-j}\mu T^j \circ T^{n-i+1}0T^i \\ &= T^{n-j}(\mu T^j \circ T^{j-i+1}0T^i) \\ &= T^{n-j}(\mu T^{j-i} \circ T^{j-i+1}0)T^i \,, \end{aligned} \tag{3.53}$$

and

$$\begin{aligned} d^{n,i} \circ s^{n-1,j-1} &= T^{n-i}0T^i \circ T^{n-j}\mu T^{j-1} \\ &= T^{n-j}(T^{j-i}0T^i \circ \mu T^{j-1}) \\ &= T^{n-j}(T^{j-i}0 \circ \mu T^{j-i-1})T^i \,, \end{aligned} \tag{3.54}$$

for all $n \geq 0$, $i < j$. Comparing Equations (3.53) and (3.54), and using Equation (3.47) for $k = j - i$, we get Identity (2.3).

On the other hand,

$$\begin{aligned} s^{n,j} \circ d^{n+1,i} &= T^{n-j}\mu T^j \circ T^{n-i+1}0T^i \\ &= T^{n-i+1}(T^{i-j-1}\mu T^j \circ 0T^i) \\ &= T^{n-i+1}(T^{i-j-1}\mu \circ 0T^{i-j})T^j \,, \end{aligned} \tag{3.55}$$

and

$$\begin{aligned} d^{n,i-1} \circ s^{n-1,j} &= T^{n-i+1}0T^{i-1} \circ T^{n-j-1}\mu T^j \\ &= T^{n-i+1}(0T^{i-1} \circ T^{i-j-2}\mu T^j) \\ &= T^{n-i+1}(0T^{i-j-1} \circ T^{i-j-2}\mu)T^j \,, \end{aligned} \tag{3.56}$$

for all $n \geq 0$, $i > j + 1$. Comparing Equations (3.55) and (3.56), and using Equation (3.48) for $k = i - j$, we get Identity (2.5).

Lastly, it remains to show Identity (2.4). Using the unitality of the zero section, as expressed by the right diagram in (3.43), we have that

$$\begin{aligned} s^{n,i} \circ d^{n+1,i} &= T^{n-i}\mu T^i \circ T^{n-i+1}0T^i \\ &= T^{n-i}(\mu \circ T0)T^i \\ &= T^{n+1} \end{aligned}$$

and

$$\begin{aligned} s^{n,i} \circ d^{n+1,i+1} &= T^{n-i}\mu T^i \circ T^{n-i}0T^{i+1} \\ &= T^{n-i}(\mu \circ 0T)T^i \\ &= T^{n+1} \end{aligned}$$

for all $n \geq 0$ and $0 \leq i \leq n$. This shows Equation (2.4).

We conclude that the assignment $[n] \mapsto T^{n+1}$ is an augmented cosimplicial object in $\mathrm{End}(\mathcal{C})$. □

Remark 3.2.26. The proof of Proposition 3.2.25 only uses the monad structure of T and the relations from Lemma 3.2.24. Thus, we have shown a much more general result: Any monad on a category is equipped with an augmented cosimplicial structure. In fact, one can similarly show that any monoid object in a (strict) monoidal category has an augmented cosimplicial structure. In the case of non-strict monoidal categories, the role of the associator and the right and left unitors is essential.

Remark 3.2.27. It was shown in [CC14, Proposition 3.7] that T^2 also has the structure of a monad. This uses the symmetric structure τ on T.

3.3 The Lie bracket of vector fields

A vector field on a smooth manifold M is a section of the tangent bundle $TM \to M$. It can be identified with a derivation of the ring $C^\infty(M) = \mathcal{M}\mathrm{fld}(M, \mathbb{R})$ of real-valued smooth functions on M [Lee13, Prop. 8.15]. Since the space of derivations on $\mathcal{M}\mathrm{fld}(M, \mathbb{R})$ is closed under the commutator bracket of the ambient ring of endomorphisms on $\mathcal{M}\mathrm{fld}(M, \mathbb{R})$, one can equip the space of vector fields on M with a Lie bracket.

However, in general cartesian tangent categories with scalar R-multiplication, vector fields on an object X may not be always identified with derivations on the ring of R-valued morphisms on X (Remark 3.3.6). Hence, in order to generalize the construction of the Lie bracket to the setting of tangent categories, we should consider the less used formula of the Lie bracket: the one which only uses the tangent structure on smooth manifolds. This was first observed by Rosický [Ros84].

The Lie bracket of two vector fields $v : M \to TM$, $p \mapsto v_p$ and $w : M \to TM$, $p \mapsto w_p$ is defined by

$$[v, w]_p := Tw(v_p) - \tau_M\big(Tv(w_p)\big) \tag{3.57}$$

for all $p \in M$. The fiberwise differentials of v and w are given by $Tv : T_pM \to T_{v_p}(TM)$ and $Tw : T_pM \to T_{w_p}(TM)$ respectively. Since $Tw(v_p)$ and $Tv(w_p)$ do not lie in the same fiber, the canonical flip map τ_M is applied on the second factor in (3.57). Moreover, the right hand side of Equation (3.57) is a vertical vector in $T_{w_p}(TM)$, that is, it lies in the kernel of $T\pi_M$. Identifying it with a tangent vector in T_pM, we get the Lie bracket. This definition is equivalent to the commutator definition [AMR88, Exercise 4.2K on p. 297].

In Section 3.3.1, we explain the generalized construction of the Lie bracket of vector fields on objects in a tangent category, using the ideas of Rosický. In Section 3.3.2, we state the Leibniz rule for the Lie bracket of vector fields in the setting of cartesian tangent categories with scalar R-multiplication. In Section 3.3.3, we prove several naturality results for the Lie bracket. Sections 3.3.2 and 3.3.3 are based on the work [AB] in progress.

3.3.1 The main construction

Definition 3.3.1. Let $\mathcal{C}$ be a category with a tangent structure. A **vector field** on $X \in \mathcal{C}$ is a section of $\pi_X : TX \to X$.

The bracket of two vector fields $v, w : X \to TX$ is defined as follows. The composition of v and $Tw : TX \to T^2X$ satisfies

$$\begin{aligned}\pi_{TX} \circ Tw \circ v = w \circ \pi_X \circ v = w \circ \mathrm{id}_X \\ = w\,,\end{aligned} \tag{3.58}$$

by the naturality of π. When we exchange v and w, we have $\pi_{TX} \circ Tv \circ w = v$. In order to be able to subtract the two terms in the fiber product $T^2X \times_{TX} T^2X$, we have to apply the symmetric structure on T^2, so that by the commutativity of Diagram (3.23) we obtain

$$\begin{aligned}\pi_{TX} \circ \tau_X \circ Tv \circ w = T\pi_X \circ Tv \circ w = T\,\mathrm{id}_X \circ w = \mathrm{id}_{TX} \circ w \\ = w\,.\end{aligned} \tag{3.59}$$

This shows that $Tw \circ v$ and $\tau_X \circ Tv \circ w$ project to the same fiber of $\pi_{TX} : T^2X \to TX$, and hence there is a unique map

$$(Tw \circ v, \tau_X \circ Tv \circ w) : X \longrightarrow T^2X \times_{TX}^{\pi_{TX},\pi_{TX}} T^2X\,.$$

We denote the subtraction of the two components by

$$\delta(v, w) := -_{TX} \circ (Tw \circ v, \tau_X \circ Tv \circ w) : X \longrightarrow T^2X\,, \tag{3.60}$$

where the minus $-_{TX}$ denotes the difference in the bundle of abelian groups $\pi_{TX} : T^2X \to TX$ (Notation 3.2.8). Similarly, observe that

$$T\pi_X \circ Tw \circ v = T(\pi_X \circ w) \circ v = \mathrm{id}_{TX} \circ v = v\,, \tag{3.61}$$

and

$$T\pi_X \circ \tau_X \circ Tv \circ w = \pi_{TX} \circ Tv \circ w = v\,. \tag{3.62}$$

It follows from Equations (3.61) and (3.62) that the left square in the following diagram

$$\begin{tikzcd} X \arrow[r, "(Tw \circ v, \tau_X \circ Tv \circ w)"] \arrow[d, "\mathrm{id}_X"'] & T^2X \times_{TX}^{\pi_{TX},\pi_{TX}} T^2X \arrow[r, "-_{TX}"] \arrow[d, "T\pi_X \times_{\pi_X} T\pi_X"] & T^2X \arrow[d, "T\pi_X"] \\ X \arrow[r, "(v,v)"'] & TX \times_X^{\pi_X,\pi_X} TX \arrow[r, "-_X"'] & TX \end{tikzcd} \tag{3.63}$$

commutes. The commutativity of the right square follows from the naturality of the subtraction $-$. We conclude that the outer rectangle of Diagram (3.63) commutes. Observe that the composition of the upper horizontal maps is $\delta(v, w)$. The composition of the lower horizontal maps is 0_X by the axiom of the inverse of a group structure. Therefore, we have

$$T\pi_X \circ \delta(v, w) = 0_X\,. \tag{3.64}$$

In other words, the map $\delta(v, w)$ takes values in the kernel of $T\pi_X$, which, by Remark 3.2.10, is isomorphic to $TX \times_X TX$. By projecting to the second factor we thus obtain the vector field $[v, w] : X \to TX$. It is the unique vector field satisfying

$$\delta(v, w) = \lambda_{2,X} \circ \big(w, [v, w]\big)\,,$$

as shown in Lemma 3.3.2. The entire construction can be summarized by the following commutative diagram:

$$
\begin{tikzcd}
TX & X \arrow[l, "{[v,w]}"'] \arrow[r, "{(v,w)}"] \arrow[dl, dashed, "\exists!"'] \arrow[d, "{\delta(v,w)}"] \arrow[dr, dashed, "{\exists!\ (Tw\circ v,\, \tau_X\circ Tv\circ w)}"] & TX \times TX \arrow[r, "Tw\times Tv"] & T^2X \times T^2X \arrow[d, "\mathrm{id}_{T^2X}\times\tau_X"] \\
TX \times_X TX \arrow[u, "\mathrm{pr}_2"] \arrow[r, "\lambda_{2,X}"] \arrow[d, "\pi_X\circ\mathrm{pr}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & T^2X \arrow[d, "T\pi_X"] & T^2X \times_{TX} T^2X \arrow[l, "-_{TX}"'] \arrow[r, "i"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & T^2X \times T^2X \arrow[d, "\pi_{TX}\times\pi_{TX}"] \\
X \arrow[r, "0_X"'] & TX & TX \arrow[r, "\Delta_{TX}"'] & TX \times TX
\end{tikzcd}
\tag{3.65}
$$

We see that all ingredients of the tangent structure are needed. It was announced in [Ros84] and proved in [CC15] with the input of Rosický that $[v, w]$ satisfies the Jacobi relation[6].

By Remark 3.1.14, $\Gamma(X, TX)$ is equipped with the structure of an abelian group, that is a $\mathbb{Z}$-module, where the addition is given by

$$
v + w = +_X \circ (v, w)
$$

for all $v, w \in \Gamma(X, TX)$, and the zero is given by the zero section $0_X : X \to TX$. From the associativity of the addition it follows that the subtraction $-_{TX}$ is linear in each argument, so that $\delta(v, w)$ and, therefore, the Lie bracket $[v, w]$ is bilinear. We conclude that $\Gamma(X, TX)$ is a $\mathbb{Z}$-Lie algebra.

Lemma 3.3.2. *The following relation*

$$
\delta(v, w) = \lambda_{2,X} \circ \big(w, [v, w]\big) \tag{3.66}
$$

holds for all vector fields $v, w : X \to TX$.

Proof. Using Equations (3.58) and (3.59) and that $\pi \circ - = \pi \circ \mathrm{pr}_1 = \pi \circ \mathrm{pr}_2$, we have

$$
\pi_{TX} \circ \delta(v, w) = w\,. \tag{3.67}
$$

Moreover, we calculate that

$$
\begin{aligned}
\pi_{TX} \circ \lambda_{2,X} &= \pi_{TX} \circ \tau_X \circ +_{TX} \circ (T0_X \times_{0_X} \lambda_X) \\
&= T\pi_X \circ +_{TX} \circ (T0_X \times_{0_X} \lambda_X) \\
&= +_X \circ (T\pi_X \times_{\pi_X} T\pi_X) \circ (T0_X \times_{0_X} \lambda_X) \\
&= +_X \circ \big((T\pi_X \circ T0_X) \times_X (T\pi_X \circ \lambda_X)\big) \\
&= +_X \circ \big(\mathrm{id}_{TX} \times_X (0_X \circ \pi_X)\big) \\
&= \mathrm{pr}_1\,,
\end{aligned}
\tag{3.68}
$$

[6] A proof for the Jacobi identity of the Lie bracket using this definition in the category of smooth manifolds can be found in [Mac13], where the author uses double and triple vector bundles.

where we have used Equation (3.36), Diagram (3.23), the naturality of $+$, functoriality, that $\pi \circ 0 = 1$, the commutativity of Diagram (3.28) and the unitality in a group object.

Now, let us denote the unique morphism representing the left dotted arrow in Diagram (3.65) by
$$(\alpha_1, \alpha_2) : X \longrightarrow TX \times_X TX\,.$$
It satisfies
$$\delta(v, w) = \lambda_{2,X} \circ (\alpha_1, \alpha_2)\,. \tag{3.69}$$
By definition, the Lie bracket is given by $[v, w] = \mathrm{pr}_2 \circ (\alpha_1, \alpha_2) = \alpha_2$. Applying π_{TX} on both sides of Equation (3.69) and using Equations (3.67) and (3.68), we obtain $w = \mathrm{pr}_1 \circ (\alpha_1, \alpha_2) = \alpha_1$. Replacing $\alpha_1 = w$ and $\alpha_2 = [v, w]$ in Equation (3.69) we get Equation (3.66). □

Example 3.3.3. Consider the tangent structures in Example 3.2.11.

(i) The Lie bracket of two vector fields on a smooth manifold can be defined by Equation (3.57). As explained in the introduction of this chapter, this equation is the core of the generalization of the bracket in the setting of tangent categories.

(ii) In [Blo24a], Blohmann uses the ingredients of a tangent structure to define the Lie bracket on an elastic diffeological space.

(iii) Let G be a Lie groupoid. In the tangent category of smooth G-bundles, a vector field is a G-equivariant section $v : E \to VE$ of the vertical tangent bundle $VE \to E$, that is, an invariant vector field on E (Def. 4.4.1). The Lie bracket is the bracket of invariant vector fields (Theorem 4.4.6).

(iv) In the tangent category of abelian groups, a vector field is given by a morphism of abelian groups $v : A \to A$. The bracket is given by the commutator $[v, w] = w \circ v - v \circ w$. In other words, the Lie algebra of vector fields on A is given by $\Gamma(A, TA) = \mathrm{gl}(A)$, the Lie algebra of $\mathbb{Z}$-linear maps.

3.3.2 The Leibniz rule

The Lie bracket of vector fields on a smooth manifold M satisfies the usual Leibniz rule
$$[v, fw] = (v \cdot f)w + f[v, w]$$
for all smooth maps $f : M \to \mathbb{R}$ and vector fields v, w on M [Lee13, Prop. 8.28 (iv)]. In this section, we state that this identity holds in the setting of a cartesian tangent category with scalar multiplication over a ring. For that, we first have to describe the action of a vector field on a ring-valued morphism on X.

Let $\mathcal{C}$ be a tangent category and $R \in \mathcal{C}$ a ring object with addition $\hat{+}$, zero $\hat{0}$, multiplication $\hat{m}$, and unit $\hat{1}$. Recall from Remark 3.1.18 that the set $\mathcal{C}(X, R)$ of R-valued morphisms is equipped with a ring structure, where the addition and multiplication are given by
$$f + g := \hat{+} \circ (f, g)\,, \qquad fg := \hat{m} \circ (f, g)$$

for all morphisms $f, g \in \mathcal{C}(X, R)$. The zero is given by $X \to * \xrightarrow{\hat{0}} R$, and the unit by $X \to * \xrightarrow{\hat{1}} R$. Assume that R is tangent-stable (Def. 3.2.18). Then we can define an action of vector fields on R-valued morphisms by

$$v \cdot f : X \xrightarrow{v} TX \xrightarrow{Tf} TR \xrightarrow{\eta_R} R \tag{3.70}$$

for all $v \in \Gamma(X, TX)$ and $f \in \mathcal{C}(X, R)$.

Proposition 3.3.4. *In a cartesian tangent category $\mathcal{C}$ with scalar R-multiplication, the action* (3.70) *is a representation of the Lie algebra of vector fields by derivations on the ring of R-valued morphisms. That is,*

$$\begin{aligned} v \cdot (f+g) &= v \cdot f + v \cdot g \\ (v+w) \cdot f &= v \cdot f + w \cdot f \\ [v,w] \cdot f &= v \cdot (w \cdot f) - w \cdot (v \cdot f) \\ v \cdot (fg) &= (v \cdot f)g + f(v \cdot g)\,. \end{aligned}$$

for all $v, w \in \Gamma(X, TX)$ and $f, g \in \mathcal{C}(X, R)$.

Proof. The detailed proof of the equations can be found in [AB]. □

Remark 3.3.5. Some of the key ingredients of the proof of Proposition 3.3.4 are the Definition (3.70) of the action, the functoriality of T, the naturality of the tangent structure, Relation (3.66) between $\lambda_{2,X}$ and $\delta(v, w)$, as well as compatibility relations between the abelian group structures of the ring R and the bundle $\pi_R : TR \to R$, the projection $\eta_R : TR \to R$ onto the fiber, and the natural isomorphism $\chi_{R,R}$ given in (3.38).

Remark 3.3.6. Proposition 3.3.4 shows that there is a homomorphism from the Lie algebra of vector fields on X to the Lie algebra of derivations on the ring $\mathcal{C}(X, R)$, where the Lie bracket is given by the commutator. However, this homomorphism is generally neither injective nor surjective, so that we cannot identify vector fields on X with derivations on the structure ring of X.

Assume now that the tangent structure has a scalar R-multiplication κ. In particular, this means that $TX \to X$ is an $(X \times R \to X)$-module in $\mathcal{C}{\downarrow}X$. Applying the functor of sections, we see that $\Gamma(X, TX)$ is equipped with the structure of a module over $\Gamma(X, X \times R) \cong \mathcal{C}(X, R)$, given by

$$fv := \kappa_X \circ (f, v)\,,$$

for all $f \in \mathcal{C}(X, R)$ and $v \in \Gamma(X, TX)$, as explained in Remark 3.1.19.

Proposition 3.3.7. *In a cartesian tangent category $\mathcal{C}$ with scalar R-multiplication, the* ***Leibniz rule***

$$[v, fw] = (v \cdot f)w + f[v, w] \tag{3.71}$$

holds for all vector fields $v, w \in \Gamma(X, TX)$ and all morphisms $f \in \mathcal{C}(X, R)$.

Proof. The detailed proof of the Leibniz rule can be found in [AB]. □

Remark 3.3.8. Some of the key ingredients of the proof of Proposition 3.3.7 are the Definition (3.70) of the action, the Definition (3.60) of $\delta(v, w)$, the axioms of the scalar multiplication (Def. 3.2.19), the naturality of the tangent structure and the scalar multiplication, the expression[7] of Tf in terms of the partial tangent morphisms $T_{(1)}f$ and $T_{(2)}f$ (Def. 3.2.14), the linearity of $T_{(2)}f$ in the second argument, the associativity of $+$ and $-$, the linearity of κ_{TX} in T^2X, the Definition (3.36) of $\lambda_{2,X}$, that it is linear in the second argument, Relation (3.66) between $\lambda_{2,X}$ and $\delta(v, w)$ and that $\lambda_{2,X}$ is a monomorphism.

Remark 3.3.8 highlights that all the structures and properties we have are used in the proof of the Leibniz identity.

As is the case for any pullback, the map

$$\begin{aligned} \mathcal{C}(*, R) &\longrightarrow \mathcal{C}(X, R) \\ (* \to R) &\longmapsto (X \xrightarrow{!_X} * \to R) \end{aligned}$$

is a ring homomorphism (Lemma A.4.35), where $!_X : X \to *$ is the unique morphism to the terminal object. Its image is the constant R-valued morphisms on X. By precomposing the $\mathcal{C}(X, R)$-module structure of $\Gamma(X, TX)$ with this ring homomorphism, we equip $\Gamma(X, TX)$ with the structure of a $\mathcal{C}(*, R)$-module (Lemma A.4.33). Spelled out, the module structure is given by

$$rv := \kappa_X \circ (r \circ !_X, v)\,, \tag{3.72}$$

for all $r \in \mathcal{C}(*, R)$ and $v \in \Gamma(X, TX)$.

Corollary 3.3.9. *Let X be an object in a cartesian tangent category $\mathcal{C}$ with scalar R-multiplication. Then the Lie bracket on $\Gamma(X, TX)$ is $\mathcal{C}(*, R)$-bilinear.*

Proof. It follows from the trivialization

$$(\pi_A, \eta_A) : TA \xrightarrow{\cong} A \times A$$

in Proposition 3.2.17, that the following diagram

$$\begin{tikzcd} R \arrow[r, "0_R"] \arrow[d, "!_R"'] & TR \arrow[d, "\eta_R"] \\ * \arrow[r, "\hat{0}"'] & R \end{tikzcd} \tag{3.73}$$

commutes. Since the tangent structure is cartesian, the tangent functor maps the terminal object to the terminal object, so that the bundle projection and the zero section of $T* = *$ are the identity. This implies that the tangent morphism of a point $r : * \to R$ satisfies $Tr = Tr \circ 0_* = 0_R \circ r$. For the action of a vector field v on

[7] This expression can be found in [CC14, Prop. 2.10], where the authors label the partial tangent morphisms by the objects themselves.

a constant function, we obtain

$$\begin{aligned} v \cdot (r \circ !_X) &= \eta_R \circ Tr \circ T!_X \circ v \\ &= \eta_R \circ 0_R \circ r \circ T!_X \circ v \\ &= \hat{0} \circ !_R \circ r \circ T!_X \circ v \\ &= \hat{0} \circ !_X \\ &= 0\,, \end{aligned}$$

where we have used the Definition (3.70) of the action and Diagram (3.73). It follows from Proposition 3.3.7 that $[v, fw] = f[v, w]$ for all constant functions f. □

Corollary 3.3.9 tell us that $\Gamma(X, TX)$ has the structure of a Lie algebra over the ring $\mathcal{C}(*, R)$.

3.3.3 Naturality of the Lie bracket

In this section, we prove that the naturality of the Lie bracket of vector fields on a smooth manifold (c.f. [Lee13, Prop. 8.30]) can be generalized to the setting of tangent categories. The proofs of Propositions 8.16 and 8.30 in [Lee13] rely on the identification of vector fields with derivations on the ring of functions on the manifold. As explained in Remark 3.3.6 we do not have this identification in general tangent categories. We provide a proof which only uses the definition of the Lie bracket as summarized in Diagram (3.65).

Definition 3.3.10. Let $\varphi : X \to Y$ be a morphism in a tangent category. A vector field v on X is called **φ-related** to a vector field v' on Y if the diagram

$$\begin{tikzcd} X \arrow[r, "v"] \arrow[d, "\varphi"'] & TX \arrow[d, "T\varphi"] \\ Y \arrow[r, "v'"'] & TY \end{tikzcd} \tag{3.74}$$

commutes.

Proposition 3.3.11. *Let $\varphi : X \to Y$ be a morphism in a tangent category. If two vector fields v and w on X are φ-related to vector fields v' and w' on Y, respectively, then:*

(i) $v + w$ is φ-related to $v' + w'$;

(ii) $[v, w]$ is φ-related to $[v', w']$.

Definition 3.3.12. Let $\varphi : X \to Y$ be a morphism in some category. A morphism $f : X \to R$ is called **φ-related** to a morphism $f' : Y \to R$ if the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi"'] & R \\ Y \arrow[ur, "f'"'] & \end{tikzcd} \tag{3.75}$$

commutes.

Proposition 3.3.13. *Let $\varphi : X \to Y$ be a morphism in a cartesian tangent category with scalar R-multiplication. If $v \in \Gamma(X, TX)$ is φ-related to $v' \in \Gamma(Y, TY)$ and $f : X \to R$ is φ-related to $f' : Y \to R$, then:*

(i) fv is φ-related to $f'v'$;

(ii) $v \cdot f$ is φ-related to $v' \cdot f'$.

Remark 3.3.14. The statement of part (ii) of Proposition 3.3.13 is only one direction of the statement of Proposition 8.16 in [Lee13]. For the other direction, the identification of vector fields with derivations on the ring of functions is necessary.

Corollary 3.3.15. *Let $\varphi : X \to Y$ be a morphism in a cartesian tangent category with scalar R-multiplication. Let $r : * \to R$ be a point. If $v \in \Gamma(X, TX)$ is φ-related to $v' \in \Gamma(Y, TY)$, then rv is φ-related to rv'.*

Proof of Proposition 3.3.11. For part (i), consider the following diagram:

$$\begin{tikzcd}
X \arrow[r, "{(v,w)}"] \arrow[d, "\varphi"'] & TX \times_X TX \arrow[r, "+_X"] \arrow[d, "T\varphi \times_\varphi T\varphi"] & TX \arrow[d, "T\varphi"] \\
Y \arrow[r, "{(v',w')}"'] & TY \times_Y TY \arrow[r, "+_Y"'] & TY
\end{tikzcd}$$

The left square commutes by (3.74). The right square commutes by the naturality of $+$. The composition of the upper horizontal arrows is $v + w$ and that of the lower horizontal arrows is $v' + w'$. The commutativity of the outer rectangle implies that $v + w$ is φ-related to $v' + w'$.

For part (ii), consider the following diagram:

$$\begin{tikzcd}
X \arrow[r, "{(v,w)}"] \arrow[d, "\varphi"'] & TX \times TX \arrow[r, "Tw \times Tv"] \arrow[d, "T\varphi \times T\varphi"] & T^2X \times T^2X \arrow[r, "\mathrm{id}_{T^2X} \times \tau_X"] \arrow[d, "T^2\varphi \times T^2\varphi"] & T^2X \times T^2X \arrow[d, "T^2\varphi \times T^2\varphi"] \\
Y \arrow[r, "{(v',w')}"'] & TY \times TY \arrow[r, "Tw' \times Tv'"'] & T^2Y \times T^2Y \arrow[r, "\mathrm{id}_{T^2Y} \times \tau_Y"'] & T^2Y \times T^2Y
\end{tikzcd} \tag{3.76}$$

The left square and the center square commute by (3.74) and the functoriality of T. The right square commutes by the naturality of τ. It follows that the outer rectangle commutes.

Now, consider the following diagram:

$$\begin{tikzcd}
X \arrow[rrr, dashed, "{\exists!\ (Tw \circ v, \tau_X \circ Tv \circ w)}"] \arrow[dr, "\mathrm{id}_X"'] \arrow[ddd, "\varphi"'] & & & T^2X \times_{TX} T^2X \arrow[dl, "i"] \arrow[ddd, "T^2\varphi \times_{T\varphi} T^2\varphi"] \\
& X \arrow[r] \arrow[d, "\varphi"'] & T^2X \times T^2X \arrow[d, "T^2\varphi \times T^2\varphi"] & \\
& Y \arrow[r] & T^2Y \times T^2Y & \\
Y \arrow[ur, "\mathrm{id}_Y"] \arrow[rrr, dashed, "{\exists!\ (Tw' \circ v', \tau_Y \circ Tv' \circ w')}"'] & & & T^2Y \times_{TY} T^2Y \arrow[ul, "i'"']
\end{tikzcd} \tag{3.77}$$

The inner square is the commutative outer rectangle of Diagram (3.76). The left trapezoid commutes trivially. The top and bottom trapezoids commute by the universal property of pullbacks and products. The right trapezoid commutes by the naturality of the inclusions i and i'. The map i' is a monomorphism by Lemma A.4.6. It follows from Lemma A.4.14 (i) that the outer square of Diagram (3.77) commutes.

Composing the horizontal morphisms of the outer commutative square of Diagram (3.77) with the differences $-_{TX}$ and $-_{TY}$, we get the following diagram:

$$\begin{tikzcd}
X \arrow[r, "{(Tw \circ v,\, \tau_X \circ Tv \circ w)}"] \arrow[d, "\varphi"'] & T^2X \times_{TX} T^2X \arrow[r, "-_{TX}"] \arrow[d, "T^2\varphi \times_{T\varphi} T^2\varphi"] & T^2X \arrow[d, "T^2\varphi"] \\
Y \arrow[r, "{(Tw' \circ v',\, \tau_X \circ Tv' \circ w')}"'] & T^2Y \times_{TY} T^2Y \arrow[r, "-_{TY}"'] & T^2Y
\end{tikzcd} \tag{3.78}$$

The right square commutes by the naturality of $-$. Thus, the outer rectangle of Diagram (3.78) commutes. The composition of the top horizontal morphisms is $\delta(v,w)$ and that of the bottom horizontal morphisms is $\delta(v',w')$.

Now, consider the following diagram:

$$\begin{tikzcd}
X \arrow[rrr, dashed, "\exists!"] \arrow[dr, "\mathrm{id}_X"] \arrow[ddd, "\varphi"'] & & & TX \times_X TX \arrow[dl, "\lambda_{2,X}"'] \arrow[ddd, "T\varphi \times_\varphi T\varphi"] \\
& X \arrow[r, "\delta(v,w)"] \arrow[d, "\varphi"'] & T^2X \arrow[d, "T^2\varphi"] & \\
& Y \arrow[r, "\delta(v',w')"'] & T^2Y & \\
Y \arrow[ur, "\mathrm{id}_Y"] \arrow[rrr, dashed, "\exists!"'] & & & TY \times_Y TY \arrow[ul, "\lambda_{2,Y}"]
\end{tikzcd} \tag{3.79}$$

The inner square is the commutative outer rectangle of (3.78). The outer right trapezoid commutes by the naturality of λ_2. The commutativity of the upper and lower trapezoids is explained in the construction of the bracket as summarized in Diagram (3.65). Since monomorphisms are stable under pullbacks and the zero section is a monomorphism, it follows from Diagram (3.37) that the morphism $\lambda_{2,Y}$ is a monomorphism. Thus, the outer square of Diagram (3.79) commutes by Lemma A.4.14 (i).

Lastly, by projecting to the second factor, we get that $T\varphi \circ [v,w] = [v',w'] \circ \varphi$. We conclude that the bracket $[v,w]$ is φ-related to $[v',w']$. □

Proof of Proposition 3.3.13. For (i) we consider the following diagram:

$$\begin{tikzcd}
X \arrow[r, "{(f,v)}"] \arrow[d, "\varphi"'] & R \times TX \arrow[r, "\kappa_X"] \arrow[d, "\mathrm{id}_R \times T\varphi"'] & TX \arrow[d, "T\varphi"] \\
Y \arrow[r, "{(f',v')}"'] & R \times TY \arrow[r, "\kappa_Y"'] & TY
\end{tikzcd}$$

where κ denotes the scalar R-multiplication. The square on the left commutes since f and v are φ-related to f' and v', respectively. The square on the right commutes by the naturality of the scalar multiplication. It follows that the outer rectangle commutes. The composition of the upper horizontal arrows is fv and that of the lower horizontal arrows is $f'v'$. We conclude that fv is φ-related to $f'v'$.

For (ii) we consider the following diagram:

$$\begin{tikzcd} X \arrow[r, "v"] \arrow[d, "\varphi"'] & TX \arrow[r, "Tf"] \arrow[d, "T\varphi"'] & TR \arrow[r, "\eta_R"] \arrow[d, "\mathrm{id}_{TR}"] & R \arrow[d, "\mathrm{id}_R"] \\ Y \arrow[r, "v'"'] & TY \arrow[r, "Tf'"'] & TR \arrow[r, "\eta_R"'] & R \end{tikzcd}$$

The square on the left commutes since v is φ-related to v'. The square in the middle commutes since f is φ-related to f' and T is a functor. The square on the right commutes trivially. Thus, the outer rectangle commutes. The composition of the upper horizontal arrows is $v \cdot f$ and that of the lower horizontal arrows is $v' \cdot f'$. We conclude that $v \cdot f$ is φ-related to $v' \cdot f'$. □

Proof of Corollary 3.3.15. Let $!_X : X \to *$ and $!_Y : Y \to *$ be the unique morphisms to the terminal object. Then, the morphism $f : X \xrightarrow{!_X} * \xrightarrow{r} R$ is φ-related to $f' : Y \xrightarrow{!_Y} * \xrightarrow{r} R$. It follows from Proposition 3.3.13 (i) that fv is φ-related to $f'v'$. That is,

$$T\varphi \circ (r \circ !_X)v = (r \circ !_Y)v' \circ \varphi \,,$$

which is equivalent to

$$T\varphi \circ \kappa_X \circ (r \circ !_X, v) = \kappa_Y \circ (r \circ !_Y, v') \circ \varphi \,.$$

Using the $\mathcal{C}(*, R)$-module structure given by Equation (3.72), we get that $T\varphi \circ rv = rv' \circ \varphi$. This shows that rv is φ-related to rv'. □

Remark 3.3.16. Let $\mathcal{C}$ be a cartesian tangent category with scalar R-multiplication and let $\mathcal{M}\mathrm{od}_{\mathcal{C}(*,R)}$ be the category of modules over $\mathcal{C}(*, R)$ with module homomorphisms. Let $\mathcal{C}^{\cong}$ be the maximal subgroupoid of $\mathcal{C}$, that is the subcategory of $\mathcal{C}$ with objects the objects of $\mathcal{C}$ and morphisms only the isomorphisms of $\mathcal{C}$. In general, the assignment $X \mapsto \Gamma(X, TX)$ is not a functor. However, restricting it to the maximal subgroupoid of $\mathcal{C}$ does yield a functor

$$\begin{aligned} F : \mathcal{C}^{\cong} &\longrightarrow \mathcal{M}\mathrm{od}_{\mathcal{C}(*,R)} \\ X &\longmapsto \Gamma(X, TX) \,. \end{aligned}$$

It sends an isomorphism $\varphi : X \to Y$ of $\mathcal{C}$ to the function $F\varphi : \Gamma(X, TX) \to \Gamma(Y, TY)$, that assigns to each vector field v on X the unique vector field v' on Y such that v and v' are φ-related. It follows from Proposition 3.3.11(i) and Corollary 3.3.15 that $F\varphi$ is a $\mathcal{C}(*, R)$-module homomorphism.

On the other hand, the product functor is given by

$$\begin{aligned} F \times F : \mathcal{C}^{\cong} &\longrightarrow \mathcal{M}\mathrm{od}_{\mathcal{C}(*,R)} \\ X &\longmapsto \Gamma(X, TX) \times \Gamma(X, TX) \,. \end{aligned}$$

It sends an isomorphism $\varphi : X \to Y$ of $\mathcal{C}$ to the $\mathcal{C}(*, R)$-module homomorphism $\Gamma(X, TX) \times \Gamma(X, TX) \to \Gamma(Y, TY) \times \Gamma(Y, TY)$, that assigns to each pair of vector fields v, w on X, the unique vector fields v', w' on Y, such that the pairs v and v', and w and w', are φ-related.

It follows from Proposition 3.3.11 (ii) that the Lie bracket $[\cdot,\cdot] : F \times F \to F$, with components $[\cdot,\cdot]_X : \Gamma(X,TX) \times \Gamma(X,TX) \to \Gamma(X,TX)$, is a natural transformation. This justifies the terminology of the title of this section.

3.4 The tangent structure on Euclidean spaces

The prototypical example of tangent structures is the tangent functor of open subsets of real finite-dimensional vector spaces. This is the local model for the tangent functor on the category of smooth manifolds. The goal of this section is to describe this tangent structure in local coordinates, which will provide us with a deeper understanding of tangent structures in general. Mainly, local coordinate expressions will be given to the tangent structure transformations in Section 3.4.2 and to the Lie bracket in Section 3.4.3. Lastly, by developing a notation and calculus for iterated local coordinates, we will revisit the proof of the augmented cosimplicial structure of the iterated tangent bundle $T^{\bullet+1}$ in local coordinates in Section 3.4.4.

3.4.1 Notation on local coordinates

Let $\mathcal{E}\mathrm{ucl}$ be the **category of Euclidean spaces**, whose objects are open subsets of $\mathbb{R}^n$ for $n \geq 0$ and the morphisms are smooth maps. Let

$$T : \mathcal{E}\mathrm{ucl} \longrightarrow \mathcal{E}\mathrm{ucl}$$

be its tangent functor, $U \subset \mathbb{R}^n$ an open subset and $k \geq 1$. Then,

$$\begin{aligned}
TU &= U \times \mathbb{R}^n \\
T^2U &= U \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \\
T^kU &= U \times (\mathbb{R}^n)^{2^k-1} \\
T_2U &= TU \times_U TU = U \times \mathbb{R}^n \times \mathbb{R}^n \\
T_kU &= \underbrace{TU \times_U TU \times_U \cdots \times_U TU}_{k\text{-times}} = U \times (\mathbb{R}^n)^k \,.
\end{aligned}$$

In local coordinates, we will write

$$\begin{aligned}
TU &= \{(u, u_0^i) \,|\, u \in U, u_0^i \in \mathbb{R} \ \text{ for } 1 \leq i \leq n\} = \{(u, u_0^i)\} \\
T^2U &= \{(u, u_0^i, u_1^i, u_{01}^i)\} \\
T^3U &= \{(u, u_0^i, u_1^i, u_{01}^i, u_2^i, u_{02}^i, u_{12}^i, u_{012}^i)\} \,,
\end{aligned} \tag{3.80}$$

etc., where we will often omit the counting index i for simplicity. In Section 3.4.4, we introduce a useful notation for iterated local coordinates on T^kU for any $k \geq 1$ using a simplicial approach.

From a geometric perspective, tangent vectors at a point $u \in U$ can be viewed as equivalence classes of paths, where two paths passing through u are equivalent if they have the same derivative at u in a coordinate chart. Let $Y \in T(TU)$ be a tangent vector to TU represented by a smooth path

$$X : \mathbb{R} \longrightarrow TU\,, \quad s \longmapsto X_s\,.$$

For every $s \in \mathbb{R}$, $X_s \in TU$ is a tangent vector to U represented by some smooth path

$$\gamma_s : \mathbb{R} \longrightarrow U\,, \quad t \longmapsto \gamma_s(t) =: \gamma(t,s)\,.$$

This implies that Y can be expressed as

$$Y = \frac{d}{ds}\bigg|_{s=0} X_s = \frac{d}{ds}\bigg|_{s=0} \left(\frac{d}{dt}\bigg|_{t=0} \gamma(t,s) \right)\,.$$

In other words, Y can be represented by a path $\gamma : \mathbb{R} \times \mathbb{R} \to U$, $(t,s) \mapsto \gamma(t,s)$ with two parameters. Writing $Y = (u, u_0, u_1, u_{01})$, the subscripts 0 and 1 denote differentiation with respect to the parameters t and s respectively.

Now, let $V \subset \mathbb{R}^m$ be another open subset and $f : U \to V$ be a smooth map. Let $(x^1, \ldots, x^n)$ be a system of local coordinates on U. That is, $x^i : U \to \mathbb{R}$ is a smooth map such that the i^{th} component of an element $u \in U$ is given by $u^i = x^i(u)$ for each $i = 1, \ldots, n$. The tangent maps of f and Tf are given by

$$\begin{aligned} Tf : (u, u_0^i) &\longmapsto \left(f(u),\, \frac{\partial f^a}{\partial x^i} u_0^i \right) \\ T^2 f : (u, u_0^i, u_1^i, u_{01}^i) &\longmapsto \left(f(u),\, \frac{\partial f^a}{\partial x^i} u_0^i,\, \frac{\partial f^a}{\partial x^i} u_1^i,\, \frac{\partial f^a}{\partial x^i} u_{01}^i + \frac{\partial^2 f^a}{\partial x^i \partial x^j} u_0^i u_1^j \right)\,, \end{aligned} \tag{3.81}$$

where $a \in \{1, \ldots, m\}$ and we have used the Einstein summation convention (c.f. [Lee13, p.18]). Moreover, the fiber product $T_2 f := Tf \times_f Tf$ is given by

$$T_2 f : (u, u_0^i, v_0^i) \longmapsto \left(f(u),\, \frac{\partial f^a}{\partial x^i} u_0^i,\, \frac{\partial f^a}{\partial x^i} v_0^i \right)\,.$$

These notations are extensively used in [Jub12].

3.4.2 The structure maps in local coordinates

Let $U \subset \mathbb{R}^n$ be an open subset. The natural transformations from Definition 3.2.4 in the tangent structure on $\mathcal{E}$ucl can be expressed in local coordinates as follows:

$$\begin{aligned}
\pi_U : TU &\longrightarrow U & (u, u_0) &\longmapsto u \\
0_U : U &\longrightarrow TU & u &\longmapsto (u, 0) \\
+_U : T_2 U &\longrightarrow TU & (u, u_0, v_0) &\longmapsto (u, u_0 + v_0) \\
\lambda_U : TU &\longrightarrow T^2 U & (u, u_0) &\longmapsto (u, 0, 0, u_0) \\
\tau_U : T^2 U &\longrightarrow T^2 U & (u, u_0, u_1, u_{01}) &\longmapsto (u, u_1, u_0, u_{01})\,.
\end{aligned}$$

The two natural extensions of the bundle projection to T^2 are given by

$$\begin{aligned}
(\pi T)_U &= \pi_{TU} : T^2U \longrightarrow TU & (u,u_0,u_1,u_{01}) &\longmapsto (u,u_0)\\
(T\pi)_U &= T\pi_U : T^2U \longrightarrow TU & (u,u_0,u_1,u_{01}) &\longmapsto (u,u_1)\,.
\end{aligned}$$

The multiplication map of the monad T (Proposition 3.2.22) can be expressed by

$$\begin{aligned}
\mu_U : T^2U &\xrightarrow{(\pi_{TU},T\pi_U)} T_2U \xrightarrow{\quad +_U \quad} TU\\
(u,u_0,u_1,u_{01}) &\longmapsto (u,u_0,u_1) \longmapsto (u,u_0+u_1)\,.
\end{aligned} \tag{3.82}$$

The commutativity of T^2 and T is given by the following isomorphism

$$\begin{aligned}
T(T_2U) &\xrightarrow{\cong} T_2(TU)\\
\big((u,u_0,v_0),(u_1,u_{01},v_{01})\big) &\longmapsto \big((u,u_1),(u_0,u_{01}),(v_0,v_{01})\big)\,.
\end{aligned}$$

On the other hand, T preserves the fiber products T_kU. This can be expressed by Isomorphism (3.1) for $k=2$:

$$\begin{aligned}
\nu_{2,U} : T(T_2U) &\xrightarrow{\cong} T^2U \times_{TU}^{T\pi_U,T\pi_U} T^2U\\
\big((u,u_0,v_0),(u_1,u_{01},v_{01})\big) &\longmapsto \big((u,u_0,u_1,u_{01}),(u,v_0,u_1,v_{01})\big)\,.
\end{aligned}$$

The extension (3.36) of the vertical lift is given by

$$\begin{aligned}
\lambda_{2,U} : T_2U &\xrightarrow{T0_U\times_{0_U}\lambda_U} T_2(TU) \xrightarrow{\quad +_{TU} \quad} T^2U \xrightarrow{\quad \tau_U \quad} T^2U\\
(u,u_0,u_1) &\longmapsto \big((u,0),(u_0,0),(0,u_1)\big) \longmapsto (u,0,u_0,u_1) \longmapsto (u,u_0,0,u_1)\,.
\end{aligned}$$

The expressions of the natural transformations $0T$, $T0$, $+T$, $T+$, λT, $T\lambda$, τT and $T\tau$ in local coordinates can be spelled out in a similar fashion (these will appear in the proof of Proposition 3.4.1).

Moreover, we have the natural fiberwise multiplication by real numbers

$$\begin{aligned}
\kappa_U : \mathbb{R}\times TU &\longrightarrow TU\\
\big(r,(u,u_0)\big) &\longmapsto (u,ru_0)\,,
\end{aligned}$$

whose tangent map is given by

$$\begin{aligned}
T\kappa_U : T(\mathbb{R}\times TU) &\longrightarrow T^2U\\
\big((r,u,u_0),(r_1,u_1,u_{01})\big) &\longmapsto (u,ru_0,u_1,r_1u_0+ru_{01})\,.
\end{aligned}$$

Composing it with the natural isomorphism (3.38) for $X=\mathbb{R}$ and $Y=TU$, we get

$$\begin{aligned}
T\mathbb{R}\times T^2U &\xrightarrow{\chi^{-1}_{\mathbb{R},TU}} T(\mathbb{R}\times TU) \xrightarrow{\quad T\kappa_U \quad} T^2U\\
\big((r,r_1),(u,u_0,u_1,u_{01})\big) &\longmapsto \big((r,u,u_0),(r_1,u_1,u_{01})\big) \longmapsto (u,ru_0,u_1,r_1u_0+ru_{01})\,.
\end{aligned}$$

The partial tangent morphisms (Def. 3.2.14) of κ_U are given by

$$\begin{aligned} T_{(1)}\kappa_U : T\mathbb{R}\times TU &\xrightarrow{\mathrm{id}_{T\mathbb{R}}\times 0_{TU}} T\mathbb{R}\times T^2U \xrightarrow{T\kappa_U\circ\chi^{-1}_{\mathbb{R},TU}} T^2U\\ \big((r,r_1),(u,u_0)\big) &\longmapsto \big((r,r_1),(u,u_0,0,0)\big) \longmapsto (u,ru_0,0,r_1u_0)\,,\end{aligned}$$

and

$$\begin{aligned} T_{(2)}\kappa_U : \mathbb{R}\times T^2U &\xrightarrow{0_{\mathbb{R}}\times\mathrm{id}_{T^2U}} T\mathbb{R}\times T^2U \xrightarrow{T\kappa_U\circ\chi^{-1}_{\mathbb{R},TU}} T^2U\\ \big(r,(u,u_0,u_1,u_{01})\big) &\longmapsto \big((r,0),(u,u_0,u_1,u_{01})\big) \longmapsto (u,ru_0,u_1,ru_{01})\,.\end{aligned}$$

Proposition 3.4.1. *The tangent functor $T:\mathcal{E}\mathrm{ucl}\to\mathcal{E}\mathrm{ucl}$ with the natural transformations π, 0, $+$, λ, τ, and κ is a cartesian tangent structure with scalar $\mathbb{R}$-multiplication.*

Proof. The axioms of a tangent structure can be shown by direct computations in local coordinates. The existence of the fiber products T_kU, that they are pointwise and preserved by T are trivial. So is the fact that $\pi_U : TU \to U$ is a bundle of abelian groups with respect to $+_U$ and 0_U.

Now, we verify that τ is a symmetric structure. First of all, the following diagram

$$\begin{tikzcd} & {(u,u_0,u_1,u_{01},\; u_2,u_{02},u_{12},u_{012})} \arrow[dl, mapsto, "\tau_{TU}"'] \arrow[dr, mapsto, "T\tau_U"] & \\ {(u,u_0,u_2,u_{02},\; u_1,u_{01},u_{12},u_{012})} \arrow[d, mapsto, "T\tau_U"'] & & {(u,u_1,u_0,u_{01},\; u_2,u_{12},u_{02},u_{012})} \arrow[d, mapsto, "\tau_{TU}"] \\ {(u,u_2,u_0,u_{02},\; u_1,u_{12},u_{01},u_{012})} \arrow[dr, mapsto, "\tau_{TU}"'] & & {(u,u_1,u_2,u_{12},\; u_0,u_{01},u_{02},u_{012})} \arrow[dl, mapsto, "T\tau_U"] \\ & {(u,u_2,u_1,u_{12},\; u_0,u_{02},u_{01},u_{012})} & \end{tikzcd}$$

commutes. This shows that τ_U satisfies the braid relations, given by Equation (3.20). Moreover, we have that

$$\tau_U\circ\tau_U : (u,u_0,u_1,u_{01}) \longmapsto (u,u_1,u_0,u_{01}) \longmapsto (u,u_0,u_1,u_{01})$$

is the identity on T^2U. We conclude that τ is a symmetric structure on T. Furthermore, the diagrams

$$\begin{tikzcd} {(u,u_0,u_1,u_{01})} \arrow[rr, mapsto, "\tau_U"] \arrow[dr, mapsto, "T\pi_U"'] & & {(u,u_1,u_0,u_{01})} \arrow[dl, mapsto, "\pi_{TU}"] \\ & {(u,u_1)} & \end{tikzcd}$$

and

$$\begin{tikzcd}
\big((u,u_0,v_0),(u_1,u_{01},v_{01})\big) \arrow[r, mapsto, "(\tau_U\times_{TU}\tau_U)\circ\nu_{2,U}"] \arrow[d, mapsto, "T+_U"'] & \big((u,u_1),(u_0,u_{01}),(v_0,v_{01})\big) \arrow[d, mapsto, "+_{TU}"] \\
(u,u_0+v_0,u_1,u_{01}+v_{01}) \arrow[r, mapsto, "\tau_U"'] & (u,u_1,u_0+v_0,u_{01}+v_{01})
\end{tikzcd}$$

commute. This proves the commutativity of Diagrams (3.23) and (3.24).

Moreover, the vertical lift fits into the following commutative diagrams:

$$\begin{tikzcd}
(u,u_0) \arrow[r, mapsto, "\lambda_U"] \arrow[d, mapsto, "\pi_U"'] & (u,0,0,u_0) \arrow[d, mapsto, "\pi_{TU}"] \\
u \arrow[r, mapsto, "0_U"'] & (u,0)
\end{tikzcd}$$

$$\begin{tikzcd}
(u,u_0,v_0) \arrow[r, mapsto, "\lambda_U\times_{0_U}\lambda_U"] \arrow[d, mapsto, "+_U"'] & ((u,0),(0,u_0),(0,v_0)) \arrow[d, mapsto, "+_{TU}"] \\
(u,u_0+v_0) \arrow[r, mapsto, "\lambda_U"'] & (u,0,0,u_0+v_0)
\end{tikzcd}$$

$$\begin{tikzcd}
(u,u_0) \arrow[r, mapsto, "\lambda_U"] \arrow[d, mapsto, "\lambda_U"'] & (u,0,0,u_0) \arrow[d, mapsto, "\lambda_{TU}"] \\
(u,0,0,u_0) \arrow[r, mapsto, "T\lambda_U"'] & (u,0,0,0,0,0,0,u_0)
\end{tikzcd}$$

This proves the commutativity of Diagrams (3.25) and (3.26).

Furthermore, the compatibility of the vertical lift and the symmetric structure, given by Diagram (3.27), follow from the commutativity of the following diagrams:

$$\begin{tikzcd}
& (u,u_0) \arrow[dl, mapsto, "\lambda_U"'] \arrow[dr, mapsto, "\lambda_U"] & \\
(u,0,0,u_0) \arrow[rr, mapsto, "\tau_U"'] & & (u,0,0,u_0)
\end{tikzcd}$$

$$\begin{tikzcd}
(u,u_0,u_1,u_{01}) \arrow[r, mapsto, "T\lambda_U"] \arrow[d, mapsto, "\tau_U"'] & (u,0,0,u_0,u_1,0,0,u_{01}) \arrow[r, mapsto, "\tau_{TU}"] & (u,0,u_1,0,0,u_0,0,u_{01}) \arrow[d, mapsto, "T\tau_U"] \\
(u,u_1,u_0,u_{01}) \arrow[rr, mapsto, "\lambda_{TU}"'] & & (u,u_1,0,0,0,0,u_0,u_{01})
\end{tikzcd}$$

To show that the vertical lift is a kernel, first of all note that the following diagram

$$\begin{tikzcd}
(u,u_0) \arrow[r, mapsto, "\lambda_U"] \arrow[d, mapsto, "\pi_U"'] & (u,0,0,u_0) \arrow[d, mapsto, "(\pi_{TU},T\pi_U)"] \\
u \arrow[r, mapsto, "(0_U,0_U)"'] & (u,0,0)
\end{tikzcd}$$

commutes. The unique map

$$
\begin{aligned}
TU &\xrightarrow{(\pi_U,\lambda_U)} U \times_{T_2U} T^2U \\
(u,u_0) &\longmapsto \big(u,(u,0,0,u_0)\big)
\end{aligned}
$$

is an isomorphism. We conclude that Diagram (3.28) is a pointwise pullback.

Lastly, it remains to check that κ is a scalar $\mathbb{R}$-multiplication. The tangent functor T preserves finite products. Also, $\mathbb{R}$ is tangent-stable since $T\mathbb{R} \cong \mathbb{R}\times\mathbb{R}$. The associativity, unitality and linearity of κ in $\mathbb{R}$ and TU can be verified easily, so that κ is an $\mathbb{R}$-module structure. The commutativity of the following diagrams

$$
\begin{tikzcd}
\big(r,(u,u_0)\big) \arrow[r, mapsto, "\mathrm{id}_{\mathbb{R}}\times\lambda_U"] \arrow[d, mapsto, "\kappa_U"'] & \big(r,(u,0,0,u_0)\big) \arrow[d, mapsto, "\kappa_{TU}"] \\
(u,ru_0) \arrow[r, mapsto, "\lambda_U"'] & (u,0,0,ru_0)
\end{tikzcd}
$$

$$
\begin{tikzcd}
\big((r,r_1),(u,u_0)\big) \arrow[r, mapsto, "T_{(1)}\kappa_U"] \arrow[d, mapsto, "(\pi_{\mathbb{R}},\eta_{\mathbb{R}})\times\mathrm{id}_{TU}"', "\cong"] & (u,ru_0,0,r_1u_0) \\
\big((r,r_1),(u,u_0)\big) \arrow[r, mapsto, "\big(\kappa_U\circ(\mathrm{pr}_1,\mathrm{pr}_3),\,\kappa_U\circ(\mathrm{pr}_2,\mathrm{pr}_3)\big)"'] & (u,ru_0,u,r_1u_0) \arrow[u, mapsto, "\lambda_{2,U}"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
\big(r,(u,u_0,u_1,u_{01})\big) \arrow[r, mapsto, "T_{(2)}\kappa_U"] \arrow[d, mapsto, "\mathrm{id}_R\times\tau_U"'] & (u,ru_0,u_1,ru_{01}) \\
\big(r,(u,u_1,u_0,u_{01})\big) \arrow[r, mapsto, "\kappa_{TU}"'] & (u,u_1,ru_0,ru_{01}) \arrow[u, mapsto, "\tau_U"']
\end{tikzcd}
$$

prove that Diagrams (3.40), (3.41) and (3.42) commute.

This concludes the proof that $(\mathcal{E}\mathrm{ucl},T)$ is a cartesian tangent category with scalar $\mathbb{R}$-multiplication. □

3.4.3 The Lie bracket in local coordinates

Let $U \subset \mathbb{R}^n$ be an open subset and let $(x^1,\dots,x^n)$ be a system of local coordinates on U. That is, $x^i : U \to \mathbb{R}$ is a smooth map such that the i^{th} component of an element $u \in U$ is given by $u^i = x^i(u)$ for each $i = 1,\dots,n$. In order to have a better insight of the definition of the Lie bracket of two vector fields (Section 3.3.1), we will consider its expression in local coordinates. Let

$$v : U \longrightarrow TU\,,\quad u \longmapsto \big(u, v^i(u)\big) = \big(x^i(u), v^i(u)\big)$$

and

$$w : U \longrightarrow TU\,,\quad u \longmapsto \big(u, w^i(u)\big) = \big(x^i(u), w^i(u)\big)$$

be two vector fields on U. The smooth maps $v^i : U \to \mathbb{R}$ and $w^i : U \to \mathbb{R}$ are the respective component functions of v and w. The tangent map of w can be expressed by

$$
\begin{aligned}
Tw : (u, u_0^i) &\longmapsto \left(x^i(u),\, w^i(u),\, \frac{\partial x^i}{\partial x^j}(u)u_0^j,\, \frac{\partial w^i}{\partial x^j}(u)u_0^j\right)\\
&= \left(x^i(u),\, w^i(u),\, u_0^i,\, \frac{\partial w^i}{\partial x^j}(u)u_0^j\right),
\end{aligned}
$$

and its composition with v by

$$
Tw \circ v : u \longmapsto \left(x^i(u), v^i(u)\right) \longmapsto \left(x^i(u),\, w^i(u),\, v^i(u),\, \frac{\partial w^i}{\partial x^j}(u)v^j(u)\right).
$$

In short, we will write

$$
\begin{aligned}
Tw \circ v &= \left(x^i,\, w^i,\, v^i,\, v^j\frac{\partial w^i}{\partial x^j}\right)\\
Tv \circ w &= \left(x^i,\, v^i,\, w^i,\, w^j\frac{\partial v^i}{\partial x^j}\right).
\end{aligned}
$$

We cannot subtract the terms $Tw\circ v$ and $Tv\circ w$, since their basepoints are different. We first have to exchange the order of differentiation of the second term, which yields

$$
\tau_U \circ Tv \circ w = \left(x^i,\, w^i,\, v^i,\, w^j\frac{\partial v^i}{\partial x^j}\right).
$$

Now we can apply the fiberwise subtraction $-_{TU} : T^2U \times_{TU} T^2U \to T^2U$ and obtain

$$
\begin{aligned}
\delta(v, w) &= -_{TU} \circ (Tw \circ v, \tau_U \circ Tv \circ w)\\
&= \left(x^i,\, w^i,\, 0,\, v^j\frac{\partial w^i}{\partial x^j} - w^j\frac{\partial v^i}{\partial x^j}\right).
\end{aligned}
$$

The map $\delta(v, w) : U \to T^2U$ lies in the image of $\lambda_{2,U} : T_2U \to T^2U$. In other words, it takes values in the kernel of $T\pi_U$. By projecting to the second factor we obtain the Lie bracket

$$
[v, w] = \left(x^i,\, v^j\frac{\partial w^i}{\partial x^j} - w^j\frac{\partial v^i}{\partial x^j}\right).
$$

More precisely, this provides a local coordinate proof of Equation (3.66):

$$
\delta(v, w) = \lambda_{2,U} \circ (w, [v, w]).
$$

This computation aligns with the coordinate formula for the Lie bracket of v and w (e.g. [Lee13, Prop. 8.26]), when seen as derivations on the ring of functions on U.

3.4.4 The cosimplicial structure of $T^{\bullet+1}$ in local coordinates

Recall from Proposition 3.2.25 that the iterated tangent bundle $T^{\bullet+1}$ has an augmented cosimplicial structure. The aim of this section is to revisit the main proof from a local coordinate perspective in the category of Euclidean spaces.

The key point is to perform induction on the powers of T applied to the multiplication $\mu = + \circ (\pi T, T\pi)$ of the monad T and the zero section 0 using local coordinates. The proof will be some sort of *proof by notation.*

Notation 3.4.2. In order to avoid confusion between the zero section $0 : 1 \to T$ as a natural transformation and 0 as a number appearing in a local coordinate system, we will denote the zero section by η throughout this section.

Notation on iterated local coordinates

Let $U \subset \mathbb{R}^n$ be an open subset. Let $\mathbf{\Delta}_+ \downarrow\!\!\!\!\int [k]$ denote the full subcategory of the overcategory $\mathbf{\Delta}_+ \downarrow [k]$ with objects the order-preserving injective maps $[l] \hookrightarrow [k]$. Then, for $k \geq -1$, the $(k+1)$-iterated tangent bundle of U has local coordinates

$$T^{k+1}U = \left\{(u_\sigma) \mid \sigma \in \mathbf{\Delta}_+ \downarrow\!\!\!\!\int [k]\right\},$$

where for $\sigma : [l] \hookrightarrow [k]$,

$$u_\sigma = \begin{cases} u, & \text{if} \quad l = -1 \\ u_{\sigma(0)\cdots\sigma(l)}, & \text{if} \quad 0 \leq l \leq k\,. \end{cases}$$

Remark 3.4.3. The categories $\mathbf{\Delta}_+ \downarrow\!\!\!\!\int [k]$ and $(\square^{k+1})^{\mathrm{op}}$ are isomorphic, where $\square^{k+1}$ denotes the *standard* $(k+1)$*-cube category* with objects subsets of $\{0, \dots, k\}$ and morphisms $I \to J$ if and only if $J \subseteq I$. The injective map $[-1] \hookrightarrow [k]$ is then trivially identified with the empty set. The cube category is for instance used to model multiple vector bundles in [JH20].

Under the light of Remark 3.4.3, we will often view injective maps $\sigma : [l] \hookrightarrow [k]$ as subsets of $\{0, \dots, k\}$ of cardinality $|\sigma| = l + 1$. This is convenient since one can employ the usual set theoretic operations, such as unions or intersections. We will often write σ_k instead of σ to emphasize that it is a subset of $\{0, \dots, k\}$.

Example 3.4.4. For small values of k, we recover the notation introduced in Section 3.4.1:

(i) for $k = -1$, $U = \{(u_{\sigma_{-1}})\} = \{(u)\}$;

(ii) for $k = 0$, $TU = \{(u_{\sigma_0})\} = \{(u, u_0)\}$;

(iii) for $k = 1$, $T^2U = \{(u_{\sigma_1})\} = \{(u, u_0, u_1, u_{01})\}$.

Notation 3.4.5. Let $\sigma_k \subseteq \{0, \dots, k\}$ be a subset and let $k < i_1 < \dots < i_p$. Then, we denote the union of σ_k and $\{i_1, \dots, i_p\}$ by

$$\sigma_k, i_1, \dots, i_p := \sigma_k \cup \{i_1, \dots, i_p\}\,.$$

The coordinate tuples will be expressed by

$$\left(u_{\sigma_k}\right)_{i_1,\dots,i_p} := \left(u_{\sigma_k,i_1,\dots,i_p}\right) = \left(u_{\sigma_k \cup \{i_1,\dots,i_p\}}\right).$$

Example 3.4.6. We get the following expressions in tuples:

(i) for $k = 0$, $i_1 = 1$, $(u_{\sigma_0})_1 = (u_{\sigma_0,1}) = (u_1, u_{01})$;

(ii) for $k = 1$, $i_1 = 2$, $(u_{\sigma_1})_2 = (u_{\sigma_1,2}) = (u_2, u_{02}, u_{12}, u_{012})$;

(iii) for $k = 1$, $i_1 = 2$, $i_2 = 3$, $(u_{\sigma_1})_{2,3} = (u_{\sigma_1,2,3}) = (u_{23}, u_{023}, u_{123}, u_{0123})$.

Remark 3.4.7. If $\{i\}$ is a singleton and $k = i - 1$, the notation σ_k, i provides an emphasis on the $(i+1)^{\mathrm{st}}$ order of differentiation. For instance, $(u_{\sigma_0,1})$ highlights the 2^{nd} order differentiation of u. In other words, we have that

$$(u_{\sigma_k}) = \big((u_{\sigma_{k-1}}), (u_{\sigma_{k-1},k})\big) \tag{3.83}$$

for all $k \geq 0$. Proceeding iteratively, we get

$$(u_{\sigma_k}) = \big((u_{\sigma_0}), (u_{\sigma_0,1}), (u_{\sigma_1,2}), \ldots, (u_{\sigma_{k-1},k})\big)\,. \tag{3.84}$$

From now on, the order of the local coordinates of $T^{k+1}U$ will always be given by Equations (3.83) and (3.84), where the extra parentheses will be removed for simplicity. This is quite natural, since the local coordinates of $T^{k+1}U$ depict the different orders of differentiation. For instance, when $k = 2$,

$$\begin{aligned}(u_{\sigma_2}) = (u_{\sigma_1}, u_{\sigma_1,2}) &= \big((u, u_0, u_1, u_{01}), (u_2, u_{02}, u_{12}, u_{012})\big) \\ &= (u, u_0, u_1, u_{01}, u_2, u_{02}, u_{12}, u_{012})\,.\end{aligned}$$

Notation 3.4.8. Let $i, j \geq 0$. We will consider smooth maps of the form

$$\begin{aligned}\alpha : T^iU &\longrightarrow T^jU \\ (u_{\sigma_{i-1}}) &\longmapsto \alpha(u_{\sigma_{i-1}})\,,\end{aligned}$$

such that the differential of α can be expressed combinatorially by

$$\begin{aligned}T\alpha : T^{i+1}U &\longrightarrow T^{j+1}U \\ (u_{\sigma_i}) &\longmapsto \big(\alpha(u_{\sigma_{i-1}}),\, \alpha(u_{\sigma_{i-1}})_i\big)\,.\end{aligned}$$

Note that the image of α is itself of the form

$$\alpha(u_{\sigma_{i-1}}) = (v_{\xi_{j-1}})\,,$$

where the index $\xi_{j-1} \in \mathbf{\Delta}_+ \downarrow [j-1]$ represents a subset of $\{0, \ldots, j-1\}$ (Remark 3.4.3).

For our purposes, α will be either the bundle projection $\pi : T \to 1$, or the monad multiplication $\mu = + \circ (\pi T, T\pi) : T^2 \to T$, or the zero section $\eta : 1 \to T$. Recall from Notation 3.4.2 that in this section the zero section will be denoted by η. These maps satisfy the assumption in Notation 3.4.8.

Example 3.4.9. The differential of the zero section $\eta_U \equiv \eta : U \to TU$, $u \mapsto (u, 0)$ can be expressed by

$$\begin{aligned}T\eta_U : TU &\longrightarrow T^2U \\ (u, u_0) &\longmapsto (\underbrace{u, 0}_{\eta(u)}, \underbrace{u_0, 0}_{\eta(u)_0})\,,\end{aligned}$$

and its second order differential by

$$\begin{aligned}T^2\eta_U : T^2U &\longrightarrow T^3U \\ (u, u_0, u_1, u_{01}) &\longmapsto (\underbrace{u, 0}_{\eta(u)}, \underbrace{u_0, 0}_{\eta(u)_0}, \underbrace{u_1, 0}_{\eta(u)_1}, \underbrace{u_{01}, 0}_{\eta(u)_{01}})\,.\end{aligned}$$

Example 3.4.10. More generally, for $i \geq 0$, consider the zero section

$$\begin{aligned} \eta_i := \eta_{T^iU} : T^iU &\longrightarrow T^{i+1}U \\ (u_{\sigma_{i-1}}) &\longmapsto (u_{\sigma_{i-1}}, 0) \end{aligned} \tag{3.85}$$

of the iterated tangent bundle T^iU. Note that in the tuple $(u_{\sigma_{i-1}}, 0)$, there are 2^i copies of zero. Then, its differential can be expressed by

$$\begin{aligned} T\eta_i : T^{i+1}U &\longrightarrow T^{i+2}U \\ (u_{\sigma_i}) &\longmapsto (\underbrace{u_{\sigma_{i-1}}, 0}_{\eta_i\left(u_{\sigma_{i-1}}\right)}, \underbrace{u_{\sigma_{i-1},i}, 0}_{\eta_i\left(u_{\sigma_{i-1}}\right)_i}), \end{aligned}$$

and its second order differential by

$$\begin{aligned} T^2\eta_i : T^{i+2}U &\longrightarrow T^{i+3}U \\ (u_{\sigma_{i+1}}) &\longmapsto (\underbrace{u_{\sigma_{i-1}}, 0}_{\eta_i\left(u_{\sigma_{i-1}}\right)}, \underbrace{u_{\sigma_{i-1},i}, 0}_{\eta_i\left(u_{\sigma_{i-1}}\right)_i}, \underbrace{u_{\sigma_{i-1},i+1}, 0}_{\eta_i\left(u_{\sigma_{i-1}}\right)_{i+1}}, \underbrace{u_{\sigma_{i-1},i,i+1}, 0}_{\eta_i\left(u_{\sigma_{i-1}}\right)_{i,i+1}}). \end{aligned}$$

When $i = 1$, we obtain:

$$\begin{aligned} \eta_{TU} : (u, u_0) &\longmapsto (u, u_0, 0, 0) \\ T\eta_{TU} : (u, u_0, u_1, u_{01}) &\longmapsto (u, u_0, 0, 0, u_1, u_{01}, 0, 0), \end{aligned}$$

and a similar computation for $T^2\eta_{TU}$.

Example 3.4.11. The differential of the bundle projection $\pi_U \equiv \pi : TU \to U$, $(u, u_0) \mapsto u$ can be expressed by

$$\begin{aligned} T\pi_U : T^2U &\longrightarrow TU \\ (u, u_0, u_1, u_{01}) &\longmapsto (\underbrace{u}_{\pi(u,u_0)}, \underbrace{u_1}_{\pi(u,u_0)_1}), \end{aligned}$$

and its second order differential by

$$\begin{aligned} T^2\pi_U : T^3U &\longrightarrow T^2U \\ (u, u_0, u_1, u_{01}, u_2, u_{02}, u_{12}, u_{012}) &\longmapsto (\underbrace{u}_{\pi(u,u_0)}, \underbrace{u_1}_{\pi(u,u_0)_1}, \underbrace{u_2}_{\pi(u,u_0)_2}, \underbrace{u_{12}}_{\pi(u,u_0)_{12}}). \end{aligned}$$

Example 3.4.12. More generally, for $i \geq 0$, the differential of the bundle projection

$$\begin{aligned} \pi_i := \pi_{T^iU} : T^{i+1}U &\longrightarrow T^iU \\ (u_{\sigma_i}) &\longmapsto (u_{\sigma_{i-1}}) \end{aligned}$$

can be expressed by

$$\begin{aligned} T\pi_i : T^{i+2}U &\longrightarrow T^{i+1}U \\ (u_{\sigma_{i+1}}) &\longmapsto (\underbrace{u_{\sigma_{i-1}}}_{\pi_i\left(u_{\sigma_i}\right)}, \underbrace{u_{\sigma_{i-1},i+1}}_{\pi_i\left(u_{\sigma_i}\right)_{i+1}}), \end{aligned}$$

and its second order differential by

$$\begin{aligned} T^2\pi_i : T^{i+3}U &\longrightarrow T^{i+2}U \\ (u_{\sigma_{i+2}}) &\longmapsto (\underbrace{u_{\sigma_{i-1}}}_{\pi_i(u_{\sigma_i})}, \underbrace{u_{\sigma_{i-1},i+1}}_{\pi_i(u_{\sigma_i})_{i+1}}, \underbrace{u_{\sigma_{i-1},i+2}}_{\pi_i(u_{\sigma_i})_{i+2}}, \underbrace{u_{\sigma_{i-1},i+1,i+2}}_{\pi_i(u_{\sigma_i})_{i+1,i+2}}). \end{aligned}$$

When $i = 1$, we obtain:

$$\begin{aligned} \pi_{TU} : (u, u_0, u_1, u_{01}) &\longmapsto (u, u_0) \\ T\pi_{TU} : (u, u_0, u_1, u_{01}, u_2, u_{02}, u_{12}, u_{012}) &\longmapsto (u, u_0, u_2, u_{02}), \end{aligned}$$

and a similar computation for $T^2\pi_{TU}$.

Notation 3.4.13. Given a subset $\sigma_k = \{\sigma^0, \dots, \sigma^l\} \subseteq \{0, \dots, k\}$ and $i \geq 0$, we define the set

$$\sigma_k[i] := \{\sigma^0 + i, \dots, \sigma^l + i\},$$

where the elements of σ_k are shifted by i.

Example 3.4.14. We get the following expressions in tuples:

(i) for $k = 0$, $\left(u_{\sigma_0[i]}\right) = (u, u_i)$;

(ii) for $k = 1$, $\left(u_{\sigma_1[i]}\right) = (u, u_i, u_{i+1}, u_{i,i+1})$.

Example 3.4.15. The differentials in Examples 3.4.10 and 3.4.12 can be expressed by:

$$\begin{aligned} T\eta_i(u_{\sigma_i}) &= \left(\eta_i(u_{\sigma_{i-1}})_{\xi_0[i]}\right) \\ T^2\eta_i(u_{\sigma_{i+1}}) &= \left(\eta_i(u_{\sigma_{i-1}})_{\xi_1[i]}\right) \\ T\pi_i(u_{\sigma_{i+1}}) &= \left(\pi_i(u_{\sigma_i})_{\xi_0[i+1]}\right) \\ T^2\pi_i(u_{\sigma_{i+2}}) &= \left(\pi_i(u_{\sigma_i})_{\xi_1[i+1]}\right) . \end{aligned}$$

Remark 3.4.16. Many of the previous notations carry on naturally to the shifted setting. For instance, Equation (3.83) can be written as

$$\left(u_{\sigma_k[i]}\right) = \left(u_{\sigma_{k-1}[i]},\, u_{\sigma_{k-1}[i],k+i}\right) \tag{3.86}$$

for all $k, i \geq 0$. When $k = 1$ and $i = 2$, we get

$$\left(u_{\sigma_1[2]}\right) = (u, u_2, u_3, u_{23}) = \left(u_{\sigma_0[2]},\, u_{\sigma_0[2],3}\right).$$

Lemma 3.4.17. *Let $i, j \geq 0$ and let $\alpha : T^iU \to T^jU$ be a smooth map satisfying the assumption in Notation 3.4.8. Then, the k^{th} order differential of α can be expressed by*

$$T^k\alpha(u_{\sigma_{k+i-1}}) = \left(\alpha(u_{\sigma_{i-1}})_{\xi_{k-1}[i]}\right) ,$$

for all $k \geq 0$.

Proof. We proceed by induction on k. For $k = 0$, the index ξ_{-1} is identified with the empty set, so that the equation holds trivially. For $k = 1$, the result follows from (3.4.8), by assumption. Assume that the formula holds for k. Then,

$$\begin{aligned} T^{k+1}\alpha(u_{\sigma_{k+i}}) &= \big(T^k\alpha(u_{\sigma_{k+i-1}}),\, T^k\alpha(u_{\sigma_{k+i-1}})_{k+i}\big) \\ &= \big(\alpha(u_{\sigma_{i-1}})_{\xi_{k-1}[i]}, \alpha(u_{\sigma_{i-1}})_{\xi_{k-1}[i],k+i}\big) \\ &= \big(\alpha(u_{\sigma_{i-1}})_{\xi_k[i]}\big)\,, \end{aligned}$$

using (3.4.8), the assumption for k and Equation (3.86). □

Lemma 3.4.18. *Let $k > 0$ and let $\alpha : U \to TU$ be a smooth map satisfying the assumption in Notation 3.4.8. Then, the components of the image $(v_{\xi_{k+1}})$ of*

$$T^{k+1}\alpha : T^{k+1}U \longrightarrow T^{k+2}U$$

are given by

$$\begin{aligned} (v_{\xi_{k-1}}) &= T^{k-1}\alpha(u_{\sigma_{k-2}}) \\ (v_{\xi_{k-1},k}) &= T^{k-1}\alpha(u_{\sigma_{k-2}})_{k-1} \\ (v_{\xi_{k-1},k+1}) &= T^{k-1}\alpha(u_{\sigma_{k-2}})_{k} \\ (v_{\xi_{k-1},k,k+1}) &= T^{k-1}\alpha(u_{\sigma_{k-2}})_{k-1,k}\,. \end{aligned}$$

Proof. First of all, the iterated local coordinates of $T^{k+2}U$ can be expressed by

$$\begin{aligned} (v_{\xi_{k+1}}) &= \big(v_{\xi_k},\, v_{\xi_k,k+1}\big) \\ &= \big(v_{\xi_{k-1}},\, v_{\xi_{k-1},k},\, v_{\xi_{k-1},k+1},\, v_{\xi_{k-1},k,k+1}\big)\,, \end{aligned}$$

by Equation (3.83). Now, consider the differential

$$\begin{aligned} T^{k-1}\alpha : T^{k-1}U &\longrightarrow T^kU \\ (u_{\sigma_{k-2}}) &\longmapsto T^{k-1}\alpha(u_{\sigma_{k-2}})\,. \end{aligned}$$

Applying T, we get that $T^{k-1}\alpha(u_{\sigma_{k-2}})$ is mapped to

$$\big(T^{k-1}\alpha(u_{\sigma_{k-2}}),\, T^{k-1}\alpha(u_{\sigma_{k-2}})_{k-1}\big) \in T^{k+1}U\,, \tag{3.87}$$

be Lemma 3.4.17. Applying T once more, we get that (3.87) is mapped to

$$\begin{aligned} &\big(T^{k-1}\alpha(u_{\sigma_{k-2}}),\, T^{k-1}\alpha(u_{\sigma_{k-2}})_{k-1}, \\ &\;T^{k-1}\alpha(u_{\sigma_{k-2}})_{k},\, T^{k-1}\alpha(u_{\sigma_{k-2}})_{k-1,k}\big) \in T^{k+2}U\,, \end{aligned} \tag{3.88}$$

by Lemma 3.4.17. This finishes the proof. □

Lemma 3.4.19. *Let $k > 0$ and let $\alpha : T^2U \to TU$ be a smooth map satisfying the assumption in Notation 3.4.8. Then, the components of the image $(v_{\xi_{k+1}})$ of*

$$T^{k+1}\alpha : T^{k+3}U \longrightarrow T^{k+2}U$$

are given by

$$\begin{aligned} (v_{\xi_{k-1}}) &= T^{k-1}\alpha(u_{\sigma_k}) \\ (v_{\xi_{k-1},k}) &= T^{k-1}\alpha(u_{\sigma_k})_{k+1} \\ (v_{\xi_{k-1},k+1}) &= T^{k-1}\alpha(u_{\sigma_k})_{k+2} \\ (v_{\xi_{k-1},k,k+1}) &= T^{k-1}\alpha(u_{\sigma_k})_{k+1,k+2}\,. \end{aligned}$$

Proof. The proof is completely analogous to that of Lemma 3.4.18. □

Useful formulas

We start by a brief summary of the main formulas we need for the proof in local coordinates. Let $i \geq 0$. For simplicity, we will write

$$\begin{aligned} \eta_i &:= \eta_{T^iU} : T^iU \to T^{i+1}U & \eta_0 &:= \eta_U \equiv \eta \\ \pi_i &:= \pi_{T^iU} : T^{i+1}U \to T^iU & \pi_0 &:= \pi_U \equiv \pi \\ \mu_i &:= \mu_{T^iU} : T^{i+2}U \to T^{i+1}U & \mu_0 &:= \mu_U \equiv \mu \end{aligned}$$

for the zero section, the bundle projection and the monad multiplication, respectively.

Formula 3.4.20. For all $k \geq 0$, the k^{th} order differential of η_i is given by

$$T^k\eta_i(u_{\sigma_{k+i-1}}) = \big(\eta_i(u_{\sigma_{i-1}})_{\xi_{k-1}[i]}\big) .$$

Proof. The equation follows from Lemma 3.4.17 for $\alpha = \eta_i$. □

Formula 3.4.21. The multiplication μ_i is given by

$$\mu_i(u_{\sigma_{i+1}}) = (u_{\sigma_{i-1}}, u_{\sigma_{i-1},i} + u_{\sigma_{i-1},i+1}) .$$

Proof. By definition, $\mu_i = \pi_{i+1} + T\pi_i : T^{i+2}U \to T^{i+1}U$, where

$$\pi_{i+1} : (u_{\sigma_{i+1}}) \longmapsto (u_{\sigma_i}) = (u_{\sigma_{i-1}}, u_{\sigma_{i-1},i})$$

and

$$T\pi_i : (u_{\sigma_{i+1}}) \longmapsto (u_{\sigma_{i-1}}, u_{\sigma_{i-1},i+1}) ,$$

by Example 3.4.12. Adding the two terms yields the result, where the base point $(u_{\sigma_{i-1}})$ of T^iU stays fixed and addition is performed elementwise. □

Example 3.4.22. For small values of i, the monad multiplication can be expressed by:

(i) for $i = 0$, $\mu_U = \pi_{TU} + T\pi_U$:

$$\begin{aligned} (u, u_0, u_1, u_{01}) \longmapsto \quad & (u, u_0) + (u, u_1) \\ = \quad & (u, u_0 + u_1) \\ = \quad & (u_{\sigma_{-1}}, u_{\sigma_{-1},0} + u_{\sigma_{-1},1}) ; \end{aligned}$$

(ii) for $i = 1$, $\mu_{TU} = \pi_{T^2U} + T\pi_{TU}$:

$$\begin{aligned} (u, u_0, u_1, u_{01}, u_2, u_{02}, u_{12}, u_{012}) \quad \longmapsto \quad & (u, u_0, u_1, u_{01}) + (u, u_0, u_2, u_{02}) \\ = \quad & (u, u_0, u_1 + u_2, u_{01} + u_{02}) \\ = \quad & (u_{\sigma_0}, (u_1, u_{01}) + (u_2, u_{02})) \\ = \quad & (u_{\sigma_0}, u_{\sigma_0,1} + u_{\sigma_0,2}) . \end{aligned}$$

Formula 3.4.23. For all $k \geq 0$, the k^{th} order differential of μ_i is given by

$$T^k\mu_i(u_{\sigma_{k+i+1}}) = \big(\mu_i(u_{\sigma_{i+1}})_{\xi_{k-1}[i+2]}\big) .$$

Proof. The equation follows from Lemma 3.4.17 for $\alpha = \mu_i$. □

Lemma 3.4.24. *Let $k > 0$ and let $(u_{\sigma_k}) \in T^{k+1}U$ and $(w_{\zeta_{k-1}}) \in T^k U$ be such that $(w_{\zeta_{k-2}}) = (u_{\sigma_{k-2}})$ and $(w_{\zeta_{k-2},k-1}) = (u_{\sigma_{k-2},k-1} + u_{\sigma_{k-2},k})$. Then,*

$$\eta(w)_{k-1} = \eta(u)_{k-1} + \eta(u)_k \,.$$

Proof. By straightforward calculation, we get that

$$\begin{aligned}\eta(w)_{k-1} &= (w,\, 0)_{k-1}\\ &= (w_{k-1},\, 0)\\ &= (u_{k-1} + u_k,\, 0)\\ &= (u_{k-1},\, 0) + (u_k,\, 0)\\ &= \eta(u)_{k-1} + \eta(u)_k\,,\end{aligned}$$

where in the third step we have used the assumption, and in the fourth step that addition is elementwise. □

Corollary 3.4.25. *Let the assumptions of Lemma 3.4.24 hold. Then, the following identity holds:*

$$T^{k-1}\eta(w_{\zeta_{k-2}})_{k-1} = T^{k-1}\eta(u_{\sigma_{k-2}})_{k-1} + T^{k-1}\eta(u_{\sigma_{k-2}})_k \,.$$

Proof. By Formula 3.4.20, we have that

$$T^{k-1}\eta(w_{\zeta_{k-2}}) = \big(\eta(w)_{\xi_{k-2}}\big) \,.$$

Hence, by Lemma 3.4.24, we get

$$\begin{aligned}T^{k-1}\eta(w_{\zeta_{k-2}})_{k-1} &= \big(\eta(w)_{\xi_{k-2}}\big)_{k-1}\\ &= \big(\eta(u)_{\xi_{k-2}}\big)_{k-1} + \big(\eta(u)_{\xi_{k-2}}\big)_k\\ &= T^{k-1}\eta(u_{\sigma_{k-2}})_{k-1} + T^{k-1}\eta(u_{\sigma_{k-2}})_k\,,\end{aligned}$$

where in the last step we have used Formula 3.4.20 again. □

Lemma 3.4.26. *Let $k > 0$ and let $(u_{\sigma_{k+2}}) \in T^{k+3}U$ and $(w_{\zeta_{k+1}}) \in T^{k+2}U$ be such that $(w_{\zeta_k}) = (u_{\sigma_k})$ and $(w_{\zeta_k,k+1}) = (u_{\sigma_k,k+1} + u_{\sigma_k,k+2})$. Then,*

$$\mu(w_{\sigma_1})_{k+1} = \mu(u_{\sigma_1})_{k+1} + \mu(u_{\sigma_1})_{k+2} \,.$$

Proof. By straightforward calculation, we get that

$$\begin{aligned}\mu(w_{\sigma_1})_{k+1} &= (w,\, w_0 + w_1)_{k+1}\\ &= (w_{k+1},\, w_{0,k+1} + w_{1,k+1})\\ &= (u_{k+1} + u_{k+2},\, u_{0,k+1} + u_{0,k+2}\\ &\qquad + u_{1,k+1} + u_{1,k+2})\\ &= (u_{k+1},\, u_{0,k+1} + u_{1,k+1})\\ &\qquad + (u_{k+2},\, u_{0,k+2} + u_{1,k+2})\\ &= \mu(u_{\sigma_1})_{k+1} + \mu(u_{\sigma_1})_{k+2}\,,\end{aligned}$$

where in the third step we have used the assumption, and in the fourth step that addition is elementwise. □

Corollary 3.4.27. *Let the assumptions of Lemma 3.4.26 hold. Then, the following identity holds:*

$$T^{k-1}\mu(w_{\zeta_k})_{k+1} = T^{k-1}\mu(u_{\sigma_k})_{k+1} + T^{k-1}\mu(u_{\sigma_k})_{k+2}\,.$$

Proof. By Formula 3.4.23, we have that

$$T^{k-1}\mu(w_{\zeta_k}) = \big(\mu(w_1)_{\xi_{k-2}[2]}\big)\;.$$

Hence, by Lemma 3.4.26, we get

$$\begin{aligned} T^{k-1}\mu(w_{\zeta_k})_{k+1} &= \big(\mu(w_1)_{\xi_{k-2}[2]}\big)_{k+1} \\ &= \big(\mu(u_1)_{\xi_{k-2}[2]}\big)_{k+1} + \big(\mu(u_1)_{\xi_{k-2}[2]}\big)_{k+2} \\ &= T^{k-1}\mu(u_{\sigma_k})_{k+1} + T^{k-1}\mu(u_{\sigma_k})_{k+2}\,, \end{aligned}$$

where in the last step we have used Formula 3.4.23 again. □

Proof of Proposition 3.2.25 in local coordinates

By the argument presented in the global proof of Proposition 3.2.25, it suffices to show that Equations (3.45)- (3.48) hold.

For Equation (3.45), let $(u_\sigma) = (u_{\sigma_{k-1}}) \in T^kU$ for $k \geq 0$. Consider the following composition:

$$(u_\sigma) \xmapsto{T^k\eta} T^k\eta(u_\sigma) \xmapsto{\eta_{k+1}} \big(T^k\eta(u_\sigma),\, 0\big)\,, \tag{3.89}$$

where we have used the expression (3.85) of the zero section. On the other hand, using the notation given by (3.4.8), we get the following composition:

$$(u_\sigma) \xmapsto{\eta_k} (u_\sigma,\, 0) \xmapsto{T^{k+1}\eta} \big(T^k\eta(u_\sigma),\, T^k\eta(u_\sigma)_k\big)\,. \tag{3.90}$$

Observe that the second component vanishes since it is composed of elements of the form $u_{\sigma,k}$ which are zero in the preimage. We conclude that the compositions (3.89) and (3.90) are equal. This proves Equation (3.45).

For Equation (3.46), let $(u_\sigma) = (u_{\sigma_{k+2}}) \in T^{k+3}U$ and $k > 0$. Consider the following composition:

$$(u_\sigma) \xmapsto{T^{k+1}\mu} T^{k+1}\mu(u_\sigma) \xmapsto{\mu_k} (v_{\xi_{k-1}},\, v_{\xi_{k-1},k} + v_{\xi_{k-1},k+1})\,, \tag{3.91}$$

where we denote the image in the middle by

$$(v_{\xi_{k+1}}) := T^{k+1}\mu(u_\sigma)\,,$$

and use Formula 3.4.21 for the second map. By Lemma 3.4.19, we have that

$$(v_{\xi_{k-1}}) = T^{k-1}\mu(u_{\sigma_k}) \tag{3.92}$$
$$(v_{\xi_{k-1},k}) = T^{k-1}\mu(u_{\sigma_k})_{k+1} \tag{3.93}$$
$$(v_{\xi_{k-1},k+1}) = T^{k-1}\mu(u_{\sigma_k})_{k+2}\,. \tag{3.94}$$

On the other hand, using Formula 3.4.21, we get the following composition:

$$(u_\sigma) \xmapsto{\mu_{k+1}} (u_{\sigma_k},\, u_{\sigma_k,k+1} + u_{\sigma_k,k+2}) \xmapsto{T^k\mu} \big(T^{k-1}\mu(w_{\zeta_k}),\, T^{k-1}\mu(w_{\zeta_k})_{k+1}\big)\ , \tag{3.95}$$

where we denote the image in the middle by

$$(w_{\zeta_{k+1}}) := (u_{\sigma_k},\, u_{\sigma_k,k+1} + u_{\sigma_k,k+2})\,.$$

We use the notation given by (3.4.8) for the second map. The first component in the image of (3.95) is equal to (3.92) since $(w_{\zeta_k}) = (u_{\sigma_k})$. Moreover, the second component is equal to the sum of (3.93) and (3.94) by Corollary 3.4.27. Thus, the compositions (3.91) and (3.95) are equal. This proves Equation (3.46).

For Equation (3.47), let $(u_\sigma) = (u_{\sigma_k}) \in T^{k+1}U$ for $k > 0$. Consider the following composition:

$$(u_\sigma) \xmapsto{T^{k+1}\eta} T^{k+1}\eta(u_\sigma) \xmapsto{\mu_k} (v_{\xi_{k-1}},\, v_{\xi_{k-1},k} + v_{\xi_{k-1},k+1})\,, \tag{3.96}$$

where we denote the image in the middle by

$$(v_{\xi_{k+1}}) := T^{k+1}\eta(u_\sigma)\,,$$

and use Formula 3.4.21 for the second map. By Lemma 3.4.18, we have that

$$(v_{\xi_{k-1}}) = T^{k-1}\eta(u_{\sigma_{k-2}}) \tag{3.97}$$

$$(v_{\xi_{k-1},k}) = T^{k-1}\eta(u_{\sigma_{k-2}})_{k-1} \tag{3.98}$$

$$(v_{\xi_{k-1},k+1}) = T^{k-1}\eta(u_{\sigma_{k-2}})_k\,. \tag{3.99}$$

On the other hand, using Formula 3.4.21, we get the following composition:

$$(u_\sigma) \xmapsto{\mu_{k-1}} (u_{\sigma_{k-2}}, u_{\sigma_{k-2},k-1} + u_{\sigma_{k-2},k}) \xmapsto{T^k\eta} \big(T^{k-1}\eta(w_{\zeta_{k-2}}), T^{k-1}\eta(w_{\zeta_{k-2}})_{k-1}\big)\,, \tag{3.100}$$

where we denote the image in the middle by

$$(w_{\zeta_{k-1}}) := (u_{\sigma_{k-2}},\, u_{\sigma_{k-2},k-1} + u_{\sigma_{k-2},k})\,.$$

For the second map, we again use the notation given by (3.4.8). The first component in the image of (3.100) is equal to (3.97) since $(w_{\zeta_{k-2}}) = (u_{\sigma_{k-2}})$. Moreover, the second component is equal to the sum of (3.98) and (3.99) by Corollary 3.4.25. Hence, the compositions (3.96) and (3.100) are equal. This proves Equation (3.47).

For Equation (3.48), let $(u_\sigma) = (u_{\sigma_{k-1}}) \in T^kU$ for $k > 1$. Consider the following composition:

$$(u_\sigma) \xmapsto{T^{k-2}\mu} T^{k-2}\mu(u_\sigma) \xmapsto{\eta_{k-1}} \big(T^{k-2}\mu(u_\sigma),\, 0\big)\,, \tag{3.101}$$

where we use the expression (3.85) of the zero section for the second map. On the other hand, using the notation given by (3.4.8), we get the following composition:

$$(u_\sigma) \xmapsto{\eta_k} (u_\sigma,\, 0) \xmapsto{T^{k-1}\mu} \big(T^{k-2}\mu(u_\sigma),\, T^{k-2}\mu(u_\sigma)_k\big)\,. \tag{3.102}$$

The second component of the image of (3.102) vanishes since it is composed of (sums of) elements of the form $u_{\sigma,k}$ which are zero in the preimage. We conclude that the compositions (3.101) and (3.102) are equal, and thus Equation (3.48) holds. □

Chapter 4

Differentiable groupoid objects and their abstract Lie algebroids

A Lie groupoid can be differentiated to its Lie algebroid. As a vector bundle, it is given by the source-vertical tangent bundle restricted to the identity bisection. The Lie bracket on its space of sections is defined by its identification with the right-invariant vector fields on the manifold of arrows. This procedure generalizes the well-known differentiation of Lie groups to their Lie algebras, and is recalled in Section 1.3.

However, many interesting examples of geometric groupoids lie outside of the category of finite-dimensional smooth manifolds. The reason is that the category of smooth manifolds is inconvenient. Other than coproducts and finite products, it does not have many good categorical properties. In particular, it does not have all pullbacks. For Lie groupoids, the assumption that the source map (and therefore the target map) is a submersion solves all technical problems. For instance, the pullback of a submersion along any other smooth map exists. For the tangent functor of smooth manifolds to commute with the nerve of the Lie groupoid, the assumption that the source map is a submersion is also necessary (Rem. 1.1.5).

A common approach to tackle this problem is to embed the category of smooth manifolds in diffeological spaces, which is a convenient setting for differential geometry. Being a quasi-topos, it has small limits, colimits and many desirable categorical properties [BH11, Thm. 5.25], [Blo24a, Thm. 3.5, Prop. 3.6]. More importantly, the category of diffeological spaces encompasses a rich class of examples of diffeological group(oid)s. This includes infinite-dimensional Lie groups, such as the diffeomorphism groups of (not necessarily compact) smooth manifolds and the group-valued mapping spaces of local gauge transformations [Blo24b, Sec. 9.3], as well as the groupoid symmetry in general relativity [BFW13, BSW23], reduction of action Lie groupoids by (not necessarily normal) subgroups [BW24, Ch. 8], the holonomy groupoid of a singular foliation [AS09, GV21, AZ23], etc.

This motivates us to develop a differentiation procedure for a class of diffeological groupoids having additional good properties. Indeed, Blohmann has shown that the full subcategory of *elastic* diffeological spaces admits a tangent structure in the sense of Rosický [Blo24a]. Moreover, the axioms of a tangent structure are the minimal axioms needed to define a Lie bracket of vector fields, as explained in Section 3.3.

In Chapter 3, we extend the framework of Rosický's tangent categories to cartesian tangent structures with scalar R-multiplication, for R a ring object in the ambient category.

These observations pave the way to the ultimate goal of this chapter: generalizing the differentiation procedure of Lie groupoids to the setting of cartesian tangent categories with scalar R-multiplication. Such a construction can then be directly applied to elastic diffeological group(oid)s and to any groupoid object in a category with a tangent structure. In this thesis, we do not address the question of integrability. A recent work of Villatoro provides one approach in that direction [Vil25].

The first step in the differentiation process is to identify the properties of a groupoid object internal to a category needed for its differentiation. This is accomplished in Section 4.1, where we introduce the notion of *differentiable* groupoids. This leads to the construction of the source-vertical tangent bundle of the groupoid object. By restricting it to the identity bisection, we obtain a bundle A of R-modules, depicted by the pullback (4.74).

One of the key results of this chapter is the definition of the Lie bracket on the sections of A. It consists of two steps:

1) We introduce *invariant* vector fields in Section 4.4, and show that they are closed under the Lie bracket of vector fields (Theorem 4.4.6). The proof of this result uses the language of (differentiable) right groupoid bundles and higher vertical tangent bundles, which we develop in Sections 4.2 and 4.3.

2) We prove that invariant vector fields can be identified with the sections of A (Theorem 4.5.7).

Finally, in Section 4.5 we introduce *abstract* Lie algebroids in tangent categories, which are generalizations of Lie algebroids in smooth manifolds. The main theorem states that our differentiation procedure of a differentiable groupoid object yields an abstract Lie algebroid (Theorem 4.5.8). We conclude by providing the reader with an outlook of applications and examples in Section 4.6. This chapter is based on the paper [AB25].

4.1 Differentiable groupoids

The first step towards defining differentiable groupoid objects is to state precisely what we mean by a groupoid internal to a category $\mathcal{C}$. This is the subject of Section 4.1.1. We then proceed with identifying the minimal axioms on a groupoid object needed for its differentiation. In Section 4.1.2, we motivate and define the notion of differentiability of groupoid objects in $\mathcal{C}$ with a tangent structure.

4.1.1 Groupoid objects

In Section 2.2.3, we explained the simplicial approach to (higher) set theoretic groupoids. In particular, the nerve of a groupoid is a simplicial set such that every horn has a filler, and every horn of degree > 1 has a unique filler (Example 2.2.18). More generally, a simplicial set $G : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ is called Kan if the horn projection

$$p_{n,i} : G_n \longrightarrow G(\Lambda^n_i) \tag{4.1}$$

is a surjection for all $n \geq 1$ and $0 \leq i \leq n$ (Def. 2.2.16). It is called a 1-groupoid if, in addition, $p_{n,i}$ is a bijection for all $n > 1$ (Def. 2.2.19). By the Yoneda lemma, there is a natural bijection $G_n \cong \mathrm{s}\mathcal{S}\mathrm{et}(\Delta^n, G)$ of sets. In (4.1), $G(\Lambda^n_i) = \mathrm{s}\mathcal{S}\mathrm{et}(\Lambda^n_i, G)$ is the set of (n, i)-horns of G and $p_{n,i}$ is the natural morphism induced by the monomorphism $\Lambda^n_i \to \Delta^n$. Using this terminology, the nerve of a groupoid is a 1-groupoid. In fact, the nerve construction gives an equivalence of categories between the category of 1-groupoids and the category of groupoids (Remark 2.2.22). In particular, every 1-groupoid gives rise to a unique groupoid up to isomorphism. We will use this observation for the definition of groupoid objects in any category.

Before we proceed with the definition, let us recall one subtlety. As explained in Section 2.2.2, horns can be defined in any simplicial object in a category $\mathcal{C}$ by Kan extensions along the Yoneda embedding. More precisely, we define the object of (n, i)-horns of a simplicial object $G : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$ by

$$G(\Lambda^n_i) := \big(\mathrm{Ran}_{y^{\mathrm{op}}} G\big)(\Lambda^n_i) \cong \lim_{\Delta^k \to \Lambda^n_i} G_k \tag{4.2}$$

(Def. 2.2.11). Without any completeness assumptions on $\mathcal{C}$, these objects are a priori only presheaves on $\mathcal{C}$. In our definition of a groupoid object, we explicitly ask for the existence of these horns.

Definition 4.1.1. Let $\mathcal{C}$ be a category. A **groupoid object** in $\mathcal{C}$ is a simplicial object $G : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$ such that the horns $G(\Lambda^n_i)$ exist in $\mathcal{C}$ and the horn projections $p_{n,i} : G_n \to G(\Lambda^n_i)$ are isomorphisms for all $n > 1$ and $0 \leq i \leq n$.

Definition 4.1.2. A **morphism between two groupoid objects** in $\mathcal{C}$ is a morphism of simplicial objects, i.e. a natural transformation of functors.

Groupoid objects in $\mathcal{C}$ together with morphisms between them form a category. Let us recall the description of a groupoid object G in $\mathcal{C}$ in terms of its structure morphisms (see Remark 2.2.22 for the set theoretic case). G_0 is the object of points (objects) and G_1 the object of arrows. The face morphisms $d_{1,0}, d_{1,1} : G_1 \to G_0$ are the source $s \equiv d_{1,1}$ and the target $t \equiv d_{1,0}$. The degeneracy morphism $s_{0,0} : G_0 \to G_1$ is the identity bisection $1 \equiv s_{0,0}$. The objects of $(2, 0)$-horns, $(2, 1)$-horns and $(2, 2)$-horns of G are given respectively by

$$\begin{aligned} G(\Lambda^2_0) &\cong G_1 \times^{s,s}_{G_0} G_1 \\ G(\Lambda^2_1) &\cong G_1 \times^{s,t}_{G_0} G_1 \\ G(\Lambda^2_2) &\cong G_1 \times^{t,t}_{G_0} G_1 \,. \end{aligned} \tag{4.3}$$

The groupoid multiplication is given by the composition

$$m : G_1 \times^{s,t}_{G_0} G_1 \xrightarrow[\cong]{p^{-1}_{2,1}} G_2 \xrightarrow{d_{2,1}} G_1 \,.$$

The inverse is given by the composition

$$i : G_1 \xrightarrow{(\mathrm{id}_{G_1}, 1\circ s)} G_1 \times^{s,s}_{G_0} G_1 \xrightarrow[\cong]{p_{2,0}^{-1}} G_2 \xrightarrow{d_{2,0}} G_1 \,.$$

By the (unique) horn filling conditions and the simplicial identities, the usual axioms of a groupoid hold. These are depicted by the commutativity of Diagrams (1.1)–(1.5). It follows by induction from the isomorphism $G_2 \cong G_1 \times^{s,t}_{G_0} G_1$ that

$$G_k \cong \underbrace{G_1 \times^{s,t}_{G_0} \ldots \times^{s,t}_{G_0} G_1}_{k \text{ factors}}$$

for all $k \geq 2$.

Remark 4.1.3. Let $G : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$ be a simplicial object. For $n = 1$, the objects of $(1,0)$-horns and $(1,1)$-horns always exist and are isomorphic to the object of 0-simplices. That is,

$$G(\Lambda^1_0) \cong G_0 \cong G(\Lambda^1_1)\,.$$

Now, let G be a groupoid object in $\mathcal{C}$. Under these isomorphisms, the $(1,0)$-horn projection is given by the source morphism $p_{1,0} \equiv d_{1,1} \equiv s$ and the $(1,1)$-horn projection by the target morphism $p_{1,1} \equiv d_{1,0} \equiv t$. Since $t \circ 1 = \mathrm{id}_{G_0}$ and $s \circ 1 = \mathrm{id}_{G_0}$, we conclude that the horn projections $p_{1,0}$ and $p_{1,1}$ are split epimorphisms, and hence regular epimorphisms[1].

Remark 4.1.4. In [Hen08], [Zhu09] and [MZ15], the authors solve the subtlety of the existence of the horns by equipping the category with a *Grothendieck pretopology* (Def. A.1.8) with additional good properties. A pretopology defines a class of morphisms called covers that play the role of surjective submersions in the category of smooth manifolds. In that case, the horn $X(\Lambda^n_i)$ exists as an object in $\mathcal{C}$, using the fact that Λ^n_i is a *collapsible* simplicial set (Cor. 5.1.10). This makes it possible to talk about a geometric analog of the horn filling conditions of simplicial sets.

Our definition of groupoid objects works in any category without any extra structure of a pretopology. The reader may refer to Remark 5.1.15 for a comparison between our definition of groupoid objects and that of [Hen08], [Zhu09] and [MZ15].

Remark 4.1.5. In [Lur09, Sec. 6.1.2], Lurie defines groupoid objects in $\mathcal{C}$ using a similar simplicial approach. However, $\mathcal{C}$ is assumed to have finite limits, a condition which we do not assume for reasons explained above. In his definition of groupoid objects in the context of higher categories, the author requires the existence of pullback squares, which encode both the existence of the horns and the horn filling conditions in one step [Lur09, Def. 6.1.2.7, Prop. 6.1.2.6 (4”)]. For instance, G_2 is required to be isomorphic to each of the fiber products in (4.3). We do not employ this approach since for us it is important to emphasize how these isomorphisms are obtained via the horn projections.

[1] See Definition A.4.3 for the different notions of epimorphisms.

4.1.2 Differentiability

One of the main goals of this chapter is to identify the minimal axioms needed on a groupoid object in order to differentiate it to its (abstract) Lie algebroid. Let us first fix some notation.

Notation on the iterated and diagonal zero sections

Let $\mathcal{C}$ be a category with a tangent structure. For an object $X \in \mathcal{C}$, let

$$0_X^{[n]} : X \longrightarrow T^n X \tag{4.4}$$

denote the natural morphism given by iterating the zero section for $n \geq 1$. For example, we have

$$0_X^{[n]} = T^{n-1}0_X \circ \ldots \circ T0_X \circ 0_X .$$

Since $T0_X \circ 0_X = 0_{TX} \circ 0_X$ (Remark 3.2.9), there are many different ways of expressing this morphism. In the trivial case $n = 0$, we define $0_X^{[0]} := \mathrm{id}_X$. Furthermore, let

$$0_{m,X} := \underbrace{(0_X, \ldots, 0_X)}_{m \text{ factors}} : X \longrightarrow T_m X$$

denote the diagonal zero section for all $m \geq 1$.

Let $G : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$ be a groupoid object. Denote by

$$t_k := t \circ \mathrm{pr}_1 : G_k \longrightarrow G_0 \tag{4.5}$$

the morphism that sends the k-simplices in G_k to their terminal vertex for $k \geq 1$. In the trivial case $k = 0$, we define $t_0 := \mathrm{id}_{G_0}$. Using the naturality of the (iterated and diagonal) zero section, we obtain the relations

$$T^n t_k \circ 0_{G_k}^{[n]} = 0_{G_0}^{[n]} \circ t_k \tag{4.6}$$

$$T_m t_k \circ 0_{m,G_k} = 0_{m,G_0} \circ t_k . \tag{4.7}$$

The main definition, its motivation and consequences

Definition 4.1.6. A groupoid object G in a category $\mathcal{C}$ with a tangent structure will be called **differentiable** if the pullbacks in the diagrams

$$\begin{tikzcd} T^nG_1 \times_{T^nG_0} G_k \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & G_k \arrow[d, "0_{G_k}^{[n]}"] \\ T^nG_1 \times_{T^nG_0} T^nG_k \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & T^nG_k \arrow[d, "T^nt_k"] \\ T^nG_1 \arrow[r, "T^ns"'] & T^nG_0 \end{tikzcd} \qquad \begin{tikzcd} T_mG_1 \times_{T_mG_0} G_k \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & G_k \arrow[d, "0_{m,G_0}\circ t_k"] \\ T_mG_1 \arrow[r, "T_ms"'] & T_mG_0 \end{tikzcd} \tag{4.8}$$

$$\begin{tikzcd} G_0 \times_{G_1} TG_1 \times_{TG_0} G_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & TG_1 \times_{TG_0} G_0 \arrow[d, "\pi_{G_1}\circ \mathrm{pr}_1"] \\ G_0 \arrow[r, "1"'] & G_1 \end{tikzcd} \tag{4.9}$$

exist, and if the natural morphism

$$T^n(G_1 \times^{s,t_k}_{G_0} G_k) \longrightarrow T^n G_1 \times^{T^n s, T^n t_k}_{T^n G_0} T^n G_k \tag{4.10}$$

is an isomorphism for all $n \geq 1$, $m \geq 2$ and $k \geq 0$.

Example 4.1.7. Every Lie groupoid is differentiable. In fact, every differentiable groupoid object in the category of finite-dimensional smooth manifolds is a Lie groupoid (see [AB25] for a proof).

Let us now motivate Definition 4.1.6. Let $G : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$ be a differentiable groupoid. For $n = 1$, the outer pullback on the left of (4.8) is the pullback of Ts along the zero section (4.6). The existence of this pullback makes it possible to restrict the tangent bundle TG_1 *vertically* with respect to the source fibers. Similarly, for $n \geq 2$ we get higher vertical restrictions of the tangent bundle.

For $m \geq 2$, the pullback on the right of (4.8) is the pullback of $T_m s$ along the diagonal zero section (4.7). The existence of this pullback enables us to talk about fiber products of the vertical tangent bundle. This will be necessary when we vertically restrict the fiberwise addition of the tangent structure. Higher vertical tangent bundles and vertical restrictions of tangent structures are thoroughly studied in Section 4.3 in the context of right groupoid bundles.

The pullback in (4.9) is needed for the definition of the abstract Lie algebroid of G, where we consider the pullback (4.74) of the vertical tangent bundle along the identity bisection.

To understand the importance of the existence of the pullback and the isomorphism in (4.10), we consider the case for $n = k = 1$, which states that the natural morphism

$$T(G_1 \times^{s,t}_{G_0} G_1) \longrightarrow TG_1 \times^{Ts,Tt}_{TG_0} TG_1$$

is an isomorphism. This implies that for the simplicial object $TG : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$, given by $(TG)_k = TG_k$, the object of $(2,1)$-horns

$$(TG)(\Lambda^2_1) \cong TG_1 \times^{Ts,Tt}_{TG_0} TG_1$$

exists in $\mathcal{C}$. Moreover, the horn projection

$$TG_2 \xrightarrow[Tp_{2,1}]{\cong} T(G_1 \times^{s,t}_{G_0} G_1) \xrightarrow{\cong} TG_1 \times^{Ts,Tt}_{TG_0} TG_1 \,,$$

being the composition of isomorphisms, is an isomorphism. Using the isomorphism $Ti : TG_1 \to TG_1$ and the existence of $G_1 \times^{s,s}_{G_0} G_1$ and $G_1 \times^{t,t}_{G_0} G_1$ in $\mathcal{C}$, we conclude that the objects of $(2,0)$-horns and $(2,2)$-horns of TG, given respectively by

$$\begin{aligned}(TG)(\Lambda^2_0) &\cong TG_1 \times^{Ts,Ts}_{TG_0} TG_1 \\ (TG)(\Lambda^2_2) &\cong TG_1 \times^{Tt,Tt}_{TG_0} TG_1 \,,\end{aligned}$$

exist in $\mathcal{C}$ too. Analogously, the respective horn projections are isomorphisms. Proceeding similarly, we conclude that TG is a groupoid object in $\mathcal{C}$.

The structure maps of TG are inherited from those on the groupoid G. Explicitly, its source, target, unit and inverse morphisms are the tangent morphisms of those of G. The multiplication of TG is given by

$$TG_1 \times_{TG_0}^{Ts,Tt} TG_1 \xleftarrow{\cong} T(G_1 \times_{G_0}^{s,t} G_1) \xrightarrow{Tm} TG_1 .$$

If $k = 2$ and by applying the isomorphism (4.10) twice we get that the natural morphism

$$TG_3 \longrightarrow TG_1 \times_{TG_0}^{Ts,Tt} TG_1 \times_{TG_0}^{Ts,Tt} TG_1$$

is an isomorphism. This allows us to talk about associativity of the multiplication of TG.

For general $n \geq 1$ and $k \geq 0$, it follows by induction from the isomorphism (4.10), that the natural morphism

$$T^nG_k \longrightarrow \underbrace{T^nG_1 \times_{T^nG_0}^{T^ns,T^nt} \ldots \times_{T^nG_0}^{T^ns,T^nt} T^nG_1}_{k \text{ factors}} \tag{4.11}$$

is an isomorphism. By similar arguments as for the case when $n = 1$, this implies that $T^nG : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$ given by $(T^nG)_k = T^nG_k$ is a groupoid object in $\mathcal{C}$. Analogously, we get that $T_mG : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$, given by $(T_mG)_k = T_mG_k$ is a groupoid object in $\mathcal{C}$ for all $m \geq 2$.

Remark 4.1.8. For the construction of the abstract Lie algebroid of a differentiable groupoid G, we only need the existence of the pullbacks in Definition 4.1.6 for $n = 1, 2$, $m \geq 2$ and $k \geq 0$. The reason that we ask for the existence of these pullbacks for all $n \geq 1$ can be summarized as follows:

(1) The concepts of the higher vertical tangent bundle and the vertical restriction of the tangent structure that we develop in Section 4.3 make sense for all $n \geq 0$ (with $n = 0$ being the trivial case). This enables us to talk about the functor $V^{[n]}$, which behaves like the powers of the vertical restriction functor V (see Remarks 4.3.6 and 4.3.7).

(2) It is the first step to getting a vertical restriction endofunctor on differentiable groupoids, and more generally an endofunctor on differentiable right groupoid bundles (Def. 4.2.11). This makes it possible to obtain an honest tangent structure on differentiable right groupoid bundles (and not just a vertical restriction of the tangent structure).

(3) It seems unnatural to only consider the cases for $n = 1$ and $n = 2$.

4.2 Differentiable groupoid bundles

Given a groupoid object G, its source morphism $s : G_1 \to G_0$ is a G-bundle together with the right groupoid multiplication. If G is differentiable, the tangent morphism $Ts : TG_1 \to TG_0$ is a TG-bundle. Using the axioms of differentiability, the pasting lemma and universal constructions, we will show that the restriction of the tangent bundle *vertically* with respect to the source fibers yields a G-bundle

$$VG_1 := TG_1 \times_{TG_0}^{Ts, 0_{G_0}} G_0 \longrightarrow G_0 \,.$$

In this section, we develop the language of G-bundles, which will be the cornerstone of proving that the Lie bracket of (right) invariant vector fields is invariant in the setting of tangent categories (Section 4.4). This will be crucial for the definition of the Lie bracket on the sections of the abstract Lie algebroid of G via its identification to the invariant vector fields (Theorem 4.5.7).

In Section 4.2.1, we define the category of right G-bundles with G-equivariant bundle morphisms. We then discuss the diagonal action of a groupoid object on pullbacks of groupoid bundles in Section 4.2.2. Lastly, we generalize the notion of differentiability of groupoids to groupoid bundles in a category with a tangent structure in Section 4.2.3. This generalization allows us to make sense of (higher) vertical tangent bundles and their fiber products.

4.2.1 Groupoid bundles and equivariant morphisms

Definition 4.2.1. Let G be a groupoid object in a category $\mathcal{C}$. A **right groupoid bundle** is a morphism $r : E \to G_0$ in $\mathcal{C}$ such that the pullbacks

$$E \times_{G_0}^{r,t_k} G_k \cong E \times_{G_0}^{r,t} \underbrace{G_1 \times_{G_0}^{s,t} \ldots \times_{G_0}^{s,t} G_1}_{k \text{ factors}} \tag{4.12}$$

exist for all $k \geq 1$, together with a morphism

$$\beta : E \times_{G_0}^{r,t} G_1 \longrightarrow E \,,$$

called the **right action**, such that the following diagrams commute:

(i) Condition on fibers:

$$\begin{tikzcd} E \times_{G_0}^{r,t} G_1 \arrow[r, "\beta"] \arrow[d, "\mathrm{pr}_2"'] & E \arrow[d, "r"] \\ G_1 \arrow[r, "s"'] & G_0 \end{tikzcd}$$

(ii) Unitality:

$$\begin{tikzcd} E \arrow[r, "(\mathrm{id}_E, 1\circ r)"] \arrow[dr, "\mathrm{id}_E"'] & E \times_{G_0}^{r,t} G_1 \arrow[d, "\beta"] \\ & E \end{tikzcd}$$

(iii) Associativity:

$$\begin{tikzcd} E \times_{G_0}^{r,t_2} G_2 \arrow[r, "\mathrm{id}_E \times_{G_0} m"] \arrow[d, "\beta \times_{G_0} \mathrm{id}_{G_1}"'] & E \times_{G_0}^{r,t} G_1 \arrow[d, "\beta"] \\ E \times_{G_0}^{r,t} G_1 \arrow[r, "\beta"'] & E \end{tikzcd}$$

Definition 4.2.2. Let $r_E : E \to G_0$ and $r_F : F \to G_0$ be right G-bundles. A morphism $\Phi : E \to F$ in $\mathcal{C}$ is called a **morphism of bundles** if the diagram

$$\begin{tikzcd} E \arrow[rr, "\Phi"] \arrow[dr, "r_E"'] & & F \arrow[dl, "r_F"] \\ & G_0 & \end{tikzcd}$$

commutes. It is called G**-equivariant** if the diagram

$$\begin{tikzcd} E \times_{G_0}^{r_E,t} G_1 \arrow[r, "\Phi \times_{G_0} \mathrm{id}_{G_1}"] \arrow[d, "\beta_E"'] & F \times_{G_0}^{r_F,t} G_1 \arrow[d, "\beta_F"] \\ E \arrow[r, "\Phi"'] & F \end{tikzcd}$$

commutes, where the vertical arrows are the G-actions.

Terminology 4.2.3. In this thesis we will only consider right groupoid bundles. Hence, we will often refer to right groupoid bundles by groupoid bundles or G-bundles, dropping the adjective *right*.

Given a groupoid object G in $\mathcal{C}$, G-bundles together with G-equivariant bundle morphisms form a category, which will be denoted by $\mathcal{B}\mathrm{un}_G$.

Remark 4.2.4. The pullbacks (4.12) required to exist are the levels of the action groupoid $E \rtimes G \rightrightarrows E$. Such a condition is generally needed if the ambient category does not have finite limits. There is a fully simplicial version of Definition 4.2.1, which uses bisimplicial objects or higher correspondences [BKZ].

Example 4.2.5. The identity map $E := G_0 \xrightarrow{\mathrm{id}} G_0$ with the right G-action $\beta := s \circ \mathrm{pr}_2 : G_0 \times_{G_0}^{\mathrm{id},t} G_1 \to G_0$ is the terminal object in $\mathcal{B}\mathrm{un}_G$.

Example 4.2.6. The source map $G_1 \xrightarrow{s} G_0$ and the right groupoid multiplication $\beta := m : G_1 \times_{G_0}^{s,t} G_1 \to G_1$ equip $E := G_1$ with the structure of a G-bundle.

Restriction of a groupoid bundle

The following lemma shows that if we restrict a groupoid bundle F to a subbundle $E \to F$ and pull back the action along a morphism of groupoids which is an inclusion at the level of 0-simplices, the restricted action on E takes the target fibers to the source fibers, and inherits unitality and associativity from the ambient bundle F.

Lemma 4.2.7. *Let G and H be groupoids; let $r_E : E \to G_0$ be a bundle together with a morphism $\beta_E : E \times_{G_0}^{r_E,t_G} G_1 \to E$; let $r_F : F \to H_0$ be an H-bundle with action β_F; let $\phi : G \to H$ be a morphism of groupoids with simplicial components $\phi_k : G_k \to H_k$ for all $k \geq 0$; let $i : E \to F$ be a morphism such that the diagrams*

$$\begin{tikzcd} E \arrow[r, "i"] \arrow[d, "r_E"'] & F \arrow[d, "r_F"] \\ G_0 \arrow[r, "\phi_0"'] & H_0 \end{tikzcd} \qquad \begin{tikzcd} E \times_{G_0} G_1 \arrow[r, "i \times_{\phi_0} \phi_1"] \arrow[d, "\beta_E"'] & F \times_{H_0} H_1 \arrow[d, "\beta_F"] \\ E \arrow[r, "i"'] & F \end{tikzcd} \tag{4.13}$$

commute. If i and ϕ_0 are monomorphisms, then β_E is a right G-action on E.

Proof. For the unitality, we consider the following diagram:

$$
\begin{tikzcd}
E \arrow[rrr, "(\mathrm{id}_E, 1_G \circ r_E)"] \arrow[dr, "i"] \arrow[ddd, "\mathrm{id}_E"'] & & & E \times_{G_0} G_1 \arrow[dl, "i \times_{\phi_0} \phi_1"'] \arrow[ddd, "\beta_E"] \\
& F \arrow[r, "(\mathrm{id}_F, 1_H \circ r_F)"] \arrow[d, "\mathrm{id}_F"'] & F \times_{H_0} H_1 \arrow[d, "\beta_F"] & \\
& F \arrow[r, "\mathrm{id}_F"'] & F & \\
E \arrow[ur, "i"'] \arrow[rrr, "\mathrm{id}_E"'] & & & E \arrow[ul, "i"]
\end{tikzcd}
$$

The inner square commutes since by assumption F is a groupoid bundle, so that the H-action on F is unital. The right trapezoid is the commutative diagram on the right of (4.13). The upper trapezoid commutes since ϕ is a morphism of groupoids and by using the commutativity of the left diagram of (4.13). Spelled out,

$$
\begin{aligned}
(i \times_{\phi_0} \phi_1) \circ (\mathrm{id}_E, 1_G \circ r_E) &= (i \circ \mathrm{id}_E, \phi_1 \circ 1_G \circ r_E) \\
&= (\mathrm{id}_F \circ i, 1_H \circ \phi_0 \circ r_E) \\
&= (\mathrm{id}_F \circ i, 1_H \circ r_F \circ i) \\
&= (\mathrm{id}_F, 1_H \circ r_F) \circ i \,.
\end{aligned}
$$

The left and bottom trapezoids commute trivially. Since i is by assumption a monomorphism, it follows from Lemma A.4.14 (i) that the outer square commutes. This shows that β_E is unital.

For the associativity, we consider the following diagram:

$$
\begin{tikzcd}
E \times_{G_0} G_2 \arrow[rrr, "\mathrm{id}_E \times_{G_0} m_G"] \arrow[dr, "i \times_{\phi_0} \phi_2"] \arrow[ddd, "\beta_E \times_{G_0} \mathrm{id}_{G_1}"'] & & & E \times_{G_0} G_1 \arrow[dl, "i \times_{\phi_0} \phi_1"'] \arrow[ddd, "\beta_E"] \\
& F \times_{H_0} H_2 \arrow[r, "\mathrm{id}_F \times_{H_0} m_H"] \arrow[d, "\beta_F \times_{H_0} \mathrm{id}_{H_1}"'] & F \times_{H_0} H_1 \arrow[d, "\beta_F"] & \\
& F \times_{H_0} H_1 \arrow[r, "\beta_F"'] & F & \\
E \times_{G_0} G_1 \arrow[ur, "i \times_{\phi_0} \phi_1"'] \arrow[rrr, "\beta_E"'] & & & E \arrow[ul, "i"]
\end{tikzcd}
$$

The inner square commutes since F is by assumption a groupoid bundle, so that the H-action on F is associative. The right and bottom trapezoids are both the commutative diagram on the right of (4.13). The upper trapezoid commutes since ϕ is a morphism of groupoids and by functoriality. Spelled out,

$$
\begin{aligned}
(i \times_{\phi_0} \phi_1) \circ (\mathrm{id}_E \times_{G_0} m_G) &= (i \circ \mathrm{id}_E) \times_{\phi_0} (\phi_1 \circ m_G) \\
&= (\mathrm{id}_F \circ i) \times_{\phi_0} (m_H \circ \phi_2) \\
&= (\mathrm{id}_F \times_{H_0} m_H) \circ (i \times_{\phi_0} \phi_2) \,.
\end{aligned}
$$

The commutativity of the left trapezoid follows from the commutativity of the right diagram of (4.13), observing that $G_2 \cong G_1 \times_{G_0} G_1$ and $\phi_2 = \phi_1 \times_{\phi_0} \phi_1$. Since i is by assumption a monomorphism, it follows from Lemma A.4.14 (i) that the outer square commutes. This shows that β_E is associative.

To show that β_E takes the target fibers to the source fibers of G, we consider the following diagram:

$$\begin{tikzcd}
E \times_{G_0} G_1 \arrow[rrr, "\beta_E"] \arrow[dr, "i \times_{\phi_0} \phi_1"] \arrow[ddd, "\mathrm{pr}_2"'] & & & E \arrow[dl, "i"'] \arrow[ddd, "r_E"] \\
& F \times_{H_0} H_1 \arrow[r, "\beta_F"] \arrow[d, "\mathrm{pr}_2"'] & F \arrow[d, "r_F"] & \\
& H_1 \arrow[r, "s_H"'] & H_0 & \\
G_1 \arrow[ur, "\phi_1"'] \arrow[rrr, "s_G"'] & & & G_0 \arrow[ul, "\phi_0"]
\end{tikzcd}$$

The inner square commutes since by assumption F is a groupoid bundle, so that the H-action on F takes the target fibers to the source fibers of H. The right and upper trapezoids are the commutative diagrams in (4.13). The lower trapezoid commutes since ϕ is a morphism of groupoids. The left trapezoid commutes trivially. Since ϕ_0 is by assumption a monomorphism, it follows from Lemma A.4.14 (i) that the outer square commutes. Hence, β_E takes the target fibers to the source fibers of G.

We conclude that β_E satisfies the commutative diagrams in Definition 4.2.1 and thus is a right G-action on E. $\square$

4.2.2 Pullbacks and the diagonal action

Let $r_A : A \to G_0$, $r_B : B \to G_0$ and $r_C : C \to G_0$ be G-bundles with right actions β_A, β_B, and β_C. Let $\phi : A \to B$ and $\psi : C \to B$ be G-equivariant bundle morphisms. Assume that the pullbacks

$$(A \times_B C) \times_{G_0}^{r_C \circ \mathrm{pr}_2, t_k} G_k \tag{4.14}$$

exist in $\mathcal{C}$ for all $k \geq 0$. The pullback $A \times_B C$ is equipped with the natural morphism

$$r_{A \times_B C} : A \times_B C \longrightarrow G_0 \,,$$

which is given explicitly by $r_{A \times_B C} = r_A \circ \mathrm{pr}_1 = r_C \circ \mathrm{pr}_2$. The condition that ϕ and ψ are G-equivariant is equivalent to the commutativity of the following diagram:

$$\begin{tikzcd}
A \times_{G_0} G_1 \arrow[r, "\phi \times_{G_0} \mathrm{id}"] \arrow[d, "\beta_A"'] & B \times_{G_0} G_1 \arrow[d, "\beta_B"] & C \times_{G_0} G_1 \arrow[l, "\psi \times_{G_0} \mathrm{id}"'] \arrow[d, "\beta_C"] \\
A \arrow[r, "\phi"'] & B & C \arrow[l, "\psi"]
\end{tikzcd}$$

This induces a unique morphism from the limit of the top row to the limit of the bottom row,

$$\beta_A \times_{\beta_B} \beta_C : (A \times_{G_0} G_1) \times_{B \times_{G_0} G_1} (C \times_{G_0} G_1) \longrightarrow A \times_B C\,. \tag{4.15}$$

The domain is isomorphic to

$$\begin{aligned}(A \times_{G_0} G_1) \times_{B \times_{G_0} G_1} (C \times_{G_0} G_1) &\cong (A \times_B C) \times_{G_0 \times_{G_0} G_0} (G_1 \times_{G_1} G_1)\\ &\cong (A \times_B C) \times_{G_0} G_1\,,\end{aligned}$$

where we have used that pullbacks commute with pullbacks and the cancellation of pullbacks over the identity. By composing the isomorphism with (4.15), we obtain the **diagonal action**

$$\beta_{A \times_B C} : A \times_B C \times_{G_0} G_1 \longrightarrow A \times_B C\,. \tag{4.16}$$

Explicitly, we can express the diagonal action as

$$\beta_{A \times_B C} = \big(\beta_A \circ (\mathrm{pr}_1, \mathrm{pr}_3), \beta_C \circ (\mathrm{pr}_2, \mathrm{pr}_3)\big)\,. \tag{4.17}$$

Lemma 4.2.8. *The morphism* (4.16) *is a right G-action on $A \times_B C$.*

Proof. To prove unitality, we write $\mathrm{id}_{A \times_B C} = (\mathrm{pr}_1, \mathrm{pr}_2)$ and calculate

$$\begin{aligned}&\beta_{A \times_B C} \circ (\mathrm{id}_{A \times_B C}, 1 \circ r_{A \times_B C})\\ &= \beta_{A \times_B C} \circ (\mathrm{pr}_1, \mathrm{pr}_2, 1 \circ r_{A \times_B C})\\ &= \big(\beta_A \circ (\mathrm{pr}_1, \mathrm{pr}_3), \beta_C \circ (\mathrm{pr}_2, \mathrm{pr}_3)\big) \circ (\mathrm{pr}_1, \mathrm{pr}_2, 1 \circ r_{A \times_B C})\\ &= \big(\beta_A \circ (\mathrm{pr}_1, 1 \circ r_{A \times_B C}), \beta_C \circ (\mathrm{pr}_2, 1 \circ r_{A \times_B C})\big)\\ &= \big(\beta_A \circ (\mathrm{pr}_1, 1 \circ r_A \circ \mathrm{pr}_1), \beta_C \circ (\mathrm{pr}_2, 1 \circ r_C \circ \mathrm{pr}_2)\big)\\ &= \big(\beta_A \circ (\mathrm{id}_A, 1 \circ r_A) \circ \mathrm{pr}_1, \beta_C \circ (\mathrm{id}_C, 1 \circ r_C) \circ \mathrm{pr}_2\big)\\ &= (\mathrm{id}_A \circ \mathrm{pr}_1, \mathrm{id}_C \circ \mathrm{pr}_2)\\ &= \ \mathrm{id}_{A \times_B C}\,,\end{aligned}$$

where we have used the explicit formulation (4.17) of the diagonal action, and the unitality of the actions β_A and β_C.

For associativity, we have

$$\begin{aligned}\beta_{A \times_B C} \circ (\mathrm{id}_{A \times_B C} \times_{G_0} m) &= \beta_{A \times_B C} \circ (\mathrm{pr}_1, \mathrm{pr}_2, m \circ \mathrm{pr}_3)\\ &= \big(\beta_A \circ (\mathrm{pr}_1, \mathrm{pr}_3), \beta_C \circ (\mathrm{pr}_2, \mathrm{pr}_3)\big) \circ (\mathrm{pr}_1, \mathrm{pr}_2, m \circ \mathrm{pr}_3)\\ &= \big(\beta_A \circ (\mathrm{pr}_1, m \circ \mathrm{pr}_3), \beta_C \circ (\mathrm{pr}_2, m \circ \mathrm{pr}_3)\big)\\ &= \big(\beta_A \circ (\beta_A \times_{G_0} \mathrm{id}_{G_1}), \beta_C \circ (\beta_C \times_{G_0} \mathrm{id}_{G_1})\big)\,,\end{aligned} \tag{4.18}$$

where we have used the diagonal action explicitly, and the associativity of β_A and β_C. On the other hand, writing $G_2 \cong G_1 \times_{G_0} G_1$, we calculate

$$\begin{aligned}&\beta_{A \times_B C} \circ (\beta_{A \times_B C} \times_{G_0} \mathrm{id}_{G_1})\\ &= \big(\beta_A \circ (\mathrm{pr}_1, \mathrm{pr}_3), \beta_C \circ (\mathrm{pr}_2, \mathrm{pr}_3)\big) \circ \big(\beta_A \circ (\mathrm{pr}_1, \mathrm{pr}_3), \beta_C \circ (\mathrm{pr}_2, \mathrm{pr}_3), \mathrm{pr}_4\big)\\ &= \big(\beta_A \circ (\beta_A \circ (\mathrm{pr}_1, \mathrm{pr}_3), \mathrm{pr}_4), \beta_C \circ (\beta_C \circ (\mathrm{pr}_2, \mathrm{pr}_3), \mathrm{pr}_4)\big)\\ &= \big(\beta_A \circ (\beta_A \times_{G_0} \mathrm{id}_{G_1}), \beta_C \circ (\beta_C \times_{G_0} \mathrm{id}_{G_1})\big)\,.\end{aligned} \tag{4.19}$$

Comparing Equations (4.18) and (4.19), we get that

$$\beta_{A\times_B C}\circ(\mathrm{id}_{A\times_B C}\times_{G_0} m)=\beta_{A\times_B C}\circ(\beta_{A\times_B C}\times_{G_0}\mathrm{id}_{G_1})\,.$$

This shows that $\beta_{A\times_B C}$ is associative.

To show that the target fibers are taken to the source fibers, we have that

$$\begin{aligned}r_{A\times_B C}\circ\beta_{A\times_B C}&=r_A\circ\mathrm{pr}_1\circ\big(\beta_A\circ(\mathrm{pr}_1,\mathrm{pr}_3),\beta_C\circ(\mathrm{pr}_2,\mathrm{pr}_3)\big)\\&=r_A\circ\beta_A\circ(\mathrm{pr}_1,\mathrm{pr}_3)\\&=s\circ\mathrm{pr}_2\circ(\mathrm{pr}_1,\mathrm{pr}_3)\\&=s\circ\mathrm{pr}_3\,,\end{aligned}$$

where we use that β_A is a groupoid action and so takes the target fibers to the source fibers of G.

We conclude that $\beta_{A\times_B C}$ is a right G-action.

□

Proposition 4.2.9. *Let $A\to B\leftarrow C$ be G-equivariant bundle morphisms such that the pullbacks* (4.14) *exist for all $k\geq 0$. Then the pullback $A\times_B C$ in $\mathcal{C}$ with the diagonal G-action is the pullback in the category $\mathcal{B}\mathrm{un}_G$ of G-bundles and equivariant bundle morphisms.*

Proof. Let the G-equivariant bundle morphisms be denoted by $\phi:A\to B$, and $\psi:C\to B$. Consider the pullback diagram

$$\begin{tikzcd} A\times_B C \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & C \arrow[d, "\psi"] \\ A \arrow[r, "\phi"'] & B \end{tikzcd} \tag{4.20}$$

in $\mathcal{C}$. The diagonal action (4.17) satisfies

$$\begin{aligned}\mathrm{pr}_1\circ\beta_{A\times_B C}&=\mathrm{pr}_1\circ\big(\beta_A\circ(\mathrm{pr}_1,\mathrm{pr}_3),\beta_C\circ(\mathrm{pr}_2,\mathrm{pr}_3)\big)\\&=\beta_A\circ(\mathrm{pr}_1,\mathrm{pr}_3)\\&=\beta_A\circ(\mathrm{pr}_1\times_{G_0}\mathrm{id}_{G_1})\,,\end{aligned}$$

which shows that pr_1 is G-equivariant. In an analogous way, we can show that pr_2 is G-equivariant. This shows that the pullback square (4.20) is a commutative square in $\mathcal{B}\mathrm{un}_G$.

Let $r_E:E\to G_0$ be another G-bundle with right groupoid action β_E; let $f:E\to A$ and $g:E\to C$ be G-equivariant bundle morphisms such that $\phi\circ f=\psi\circ g$; let $(f,g):E\to A\times_B C$ be the unique morphism in $\mathcal{C}$ given by the universal property of the pullback in $\mathcal{C}$. It remains to show that (f,g) is a G-equivariant bundle morphism. The bundle projections satisfy

$$\begin{aligned}r_{A\times_B C}\circ(f,g)=r_A\circ\mathrm{pr}_1\circ(f,g)&=r_A\circ f\\&=r_E\,,\end{aligned}$$

where in the last step we have used that f is a morphism of bundles. This shows that (f,g) is a morphism of bundles. The actions satisfy

$$\begin{aligned}\beta_{A\times_B C}\circ\big((f,g)\times_{G_0}\mathrm{id}_{G_1}\big) &= \big(\beta_A\circ(\mathrm{pr}_1,\mathrm{pr}_3),\beta_C\circ(\mathrm{pr}_2,\mathrm{pr}_3)\big)\circ(f\circ\mathrm{pr}_1,g\circ\mathrm{pr}_1,\mathrm{pr}_2)\\ &= \big(\beta_A\circ(f\circ\mathrm{pr}_1,\mathrm{pr}_2),\beta_C\circ(g\circ\mathrm{pr}_1,\mathrm{pr}_2)\big)\\ &= (f\circ\beta_E,g\circ\beta_E)\\ &= (f,g)\circ\beta_E\,,\end{aligned}$$

where we have used the explicit form (4.17) of the diagonal action and that f and g are G-equivariant. This shows that (f,g) is G-equivariant. $\square$

Example 4.2.10. Let A and C be G-bundles and B the terminal G-bundle $G_0\xrightarrow{\mathrm{id}}G_0$ from Example 4.2.5. Since G_0 is the terminal object in $\mathcal{B}\mathrm{un}_G$ we have unique morphisms of G-bundles $A\to G_0\leftarrow C$. The pullback is the fiber product $A\times_{G_0}C$ equipped with the diagonal action.

4.2.3 Differentiability

Definition 4.2.11. Let G be a groupoid in a category $\mathcal{C}$ with a tangent structure. A G-bundle $r:E\to G_0$ in $\mathcal{C}$ will be called **differentiable** if G is differentiable, if the pullbacks in the diagrams

$$\begin{tikzcd} T^nE\times_{T^nG_0}G_k \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & G_k \arrow[d, "0^{[n]}_{G_k}"] \\ T^nE\times_{T^nG_0}T^nG_k \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & T^nG_k \arrow[d, "T^nt_k"] \\ T^nE \arrow[r, "T^nr"'] & T^nG_0 \end{tikzcd} \qquad \begin{tikzcd} T_mE\times_{T_mG_0}G_k \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & G_k \arrow[d, "0_{m,G_0}\circ t_k"] \\ T_mE \arrow[r, "T_mr"'] & T_mG_0 \end{tikzcd} \tag{4.21}$$

exist, and if the natural morphism

$$\nu_{n,k}:T^n(E\times_{G_0}G_k)\longrightarrow T^nE\times_{T^nG_0}T^nG_k \tag{4.22}$$

is an isomorphism for all $n\geq 1$, $m\geq 2$ and $k\geq 0$.

Example 4.2.12. For a differentiable groupoid G, the G-bundle $E=G_1$ of Example 4.2.6 is differentiable.

The rest of this section is devoted to motivating the axioms in Definition 4.2.11 and stating their consequences.

The higher vertical tangent bundle

The condition that (4.22) is defined and is an isomorphism means that T^n applied to the nerve of the action groupoid $E\rtimes G\rightrightarrows E$ (Remark 4.2.4) is the nerve of an action groupoid $T^nE\rtimes T^nG\rightrightarrows T^nE$. More explicitly, $T^nr:T^nE\to T^nG_0$ with the action

$$\beta_{T^nE}:T^nE\times_{T^nG_0}T^nG_1\xrightarrow[\cong]{\nu_{n,1}^{-1}}T^n(E\times_{G_0}G_1)\xrightarrow{T^n\beta_E}T^nE \tag{4.23}$$

is a T^nG-bundle. This action can be restricted along the zero section $0^{[n]}_{G_1} : G_1 \to T^nG_1$ to a G-action as follows.

Applying the pasting lemma to the left diagram of (4.21), using Equation (4.6), and using the pasting lemma again, we obtain the natural isomorphisms

$$\begin{aligned}(T^nE \times_{T^nG_0} T^nG_k) \times^{\mathrm{pr}_2, 0^{[n]}_{G_k}}_{T^nG_k} G_k &\cong T^nE \times^{T^nr, T^nt_k \circ 0^{[n]}_{G_k}}_{T^nG_0} G_k \\ &= T^nE \times^{T^nr, 0^{[n]}_{G_0} \circ t_k}_{T^nG_0} G_k \\ &\cong (T^nE \times^{T^nr, 0^{[n]}_{G_0}}_{T^nG_0} G_0) \times^{\mathrm{pr}_2, t_k}_{G_0} G_k \\ &= V^{[n]}E \times^{\mathrm{pr}_2, t_k}_{G_0} G_k \,,\end{aligned} \tag{4.24}$$

where

$$V^{[n]}E := T^nE \times^{T^nr, 0^{[n]}_{G_0}}_{T^nG_0} G_0 = \ker T^nr \tag{4.25}$$

is the higher **vertical tangent bundle** for $0 \le n \le 2$. For $n = 0$, $V^{[0]}E \cong E$. For $n = 1$, we will write $VE \equiv V^{[1]}E$. Consider the following diagram:

$$\begin{tikzcd} T^nE \times^{T^nr, T^nt}_{T^nG_0} T^nG_1 \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\beta_{T^nE}"'] & T^nG_1 \arrow[d, "T^ns"] & G_1 \arrow[l, "0^{[n]}_{G_1}"'] \arrow[d, "s"] \\ T^nE \arrow[r, "T^nr"'] & T^nG_0 & G_0 \arrow[l, "0^{[n]}_{G_0}"] \end{tikzcd} \tag{4.26}$$

The left square commutes because β_{T^nE} is a groupoid action. The right square commutes due to the naturality of the iterated zero section. The diagram induces a morphism from the limit of the top row to the limit of the bottom row. By precomposing it with the isomorphism (4.24), we obtain a morphism

$$\beta_{V^{[n]}E} : V^{[n]}E \times_{G_0} G_1 \longrightarrow V^{[n]}E \tag{4.27}$$

that equips $V^{[n]}E \to G_0$ with a right G-action (Proposition 4.3.1).

Fiber products of the vertical tangent bundle

Since the tangent functor commutes with the nerve of the action groupoid and since limits commute with limits, so do its fiber products,

$$T_m(E \times_{G_0} G_k) \cong T_mE \times_{T_mG_0} T_mG_k \,.$$

This implies that $T_mr : T_mE \to T_mG_0$ is a T_mG-bundle with the action

$$\beta_{T_mE} : T_mE \times_{T_mG_0} T_mG_1 \xrightarrow{\cong} T_m(E \times_{G_0} G_1) \xrightarrow{T_m\beta_E} T_mE \,. \tag{4.28}$$

The T_mG-action can be restricted along the diagonal zero section

$$0_{m,G_1} : G_1 \longrightarrow T_mG_1$$

to a G-action as follows.

The pullback on the right side of (4.21) for $m = 2$ is naturally isomorphic to

$$\begin{aligned} T_2E \times_{T_2G_0} G_k &\cong T_2E \times_{T_2G_0}^{T_2r,\, 0_{2,G_0} \circ t_k} G_k \\ &\cong T_2E \times_{T_2G_0}^{T_2r,\, 0_{2,G_0}} G_0 \times_{G_0}^{\mathrm{id},\, t_k} G_k \\ &\cong (TE \times_E TE) \times_{TG_0 \times_{G_0} TG_0} (G_0 \times_{G_0} G_0) \times_{G_0} G_k \\ &\cong (TE \times_{TG_0} G_0) \times_{E \times_{G_0} G_0} (TE \times_{TG_0} G_0) \times_{G_0} G_k \\ &\cong (VE \times_E VE) \times_{G_0} G_k \\ &\cong V_2E \times_{G_0} G_k\,, \end{aligned} \tag{4.29}$$

where we have used the pasting lemma, the definition of T_2, that pullbacks commute with pullbacks, the definition of the vertical tangent bundle, and finally the notation

$$V_mE := \underbrace{VE \times_E \ldots \times_E VE}_{m \text{ factors}} \tag{4.30}$$

for the fiber products of the vertical tangent bundle. Here, the morphism $VE \to E$ is given by the composition

$$VE = TE \times_{TG_0} G_0 \xrightarrow{\mathrm{pr}_1} TE \xrightarrow{\pi_E} E\,. \tag{4.31}$$

It is straightforward to generalize the isomorphism (4.29) to an isomorphism

$$T_mE \times_{T_mG_0} G_k \cong V_mE \times_{G_0} G_k \tag{4.32}$$

for all $m > 2$ and $k \geq 0$. Consider the diagram:

$$\begin{tikzcd} T_mE \times_{T_mG_0}^{T_mr, T_mt} T_mG_1 \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\beta_{T_mE}"'] & T_mG_1 \arrow[d, "T_ms"] & G_1 \arrow[l, "0_{m,G_1}"'] \arrow[d, "s"] \\ T_mE \arrow[r, "T_mr"'] & T_mG_0 & G_0 \arrow[l, "0_{m,G_0}"] \end{tikzcd} \tag{4.33}$$

where the left square commutes because β_{T_mE} is a groupoid action and the right square commutes due to the naturality of the diagonal zero section. The diagram induces a morphism from the limit of the top row to the limit of the bottom row. By the pasting lemma and the isomorphism (4.32) for $k = 1$, the limit of the top row is isomorphic to

$$T_mE \times_{T_mG_0} T_mG_1 \times_{T_mG_1} G_1 \cong V_mE \times_{G_0} G_1\,.$$

On the other hand, by the isomorphism (4.32) for $k = 0$, the limit of the bottom row is isomorphic to

$$T_mE \times_{T_mG_0} G_0 \cong V_mE \times_{G_0} G_0 \cong V_mE\,.$$

Thus, we obtain a morphism

$$\beta_{V_mE} : V_mE \times_{G_0} G_1 \longrightarrow V_mE \tag{4.34}$$

that equips the bundle $V_mE \to G_0$ with a right G-action (Proposition 4.3.2).

Summary of the purpose of the axioms of Definition 4.2.11

1. The isomorphism (4.22) is needed so that by differentiating the action of a G-bundle $E \to G_0$ we obtain a T^nG-action on $T^nE \to T^nG_0$ and a T_mG-action on $T_mE \to T_mG_0$.

2. The existence of the pullback $T^nE \times_{T^nG_0} G_k$ is needed so that the T^nG-action on T^nE restricts to a G-action on the higher vertical tangent bundle $V^{[n]}E \to G_0$.

3. The existence of the pullback $T_mE \times_{T_mG_0} G_k$ is needed so that the T_mG-action on T_mE restricts to a G-action on the fiber product $V_mE \to G_0$ of the vertical tangent bundle.

4.3 Higher vertical tangent functors

In order to achieve our ultimate goal of proving that the Lie bracket of invariant vector fields is invariant (Section 4.4), we will prove that each step of the construction of the Lie bracket of vector fields is invariant. As explained in Section 3.3.1, this involves all the structure natural transformations of the tangent category.

We will start by showing that the assignment to a differentiable G-bundle its higher vertical tangent bundle (and the fiber products of its vertical tangent bundle) is functorial. This is explained in Section 4.3.1. Then, we prove a fundamental technical lemma, which states that natural transformations between different powers and fiber products of T can be vertically restricted, under the assumption that the zero section is preserved. This is accomplished in Section 4.3.2.

Using this powerful tool, we prove that the tangent structure and its R-module structure admit vertical restrictions in Sections 4.3.4 and 4.3.5 respectively. Moreover, Section 4.3.3 provides a method of vertical prolongation of G-equivariant bundles morphisms. As a consequence, we show that the vertically restricted vertical lift is a pointwise pullback. These results will play a crucial role in showing that each step of the Lie bracket construction is invariant (Theorem 4.4.6).

4.3.1 Functoriality

The goal of this section is to show that the assignments $E \mapsto V^{[n]}E$ and $E \mapsto V_mE$ define functors from differentiable G-bundles to G-bundles.

Proposition 4.3.1. *If a G-bundle $E \to G_0$ is differentiable, then the higher vertical tangent bundle $V^{[n]}E \to G_0$ defined in* (4.25) *together with the action* (4.27) *is a G-bundle.*

Proof. Consider the pullback diagram that defines the higher vertical tangent bundle,

$$\begin{tikzcd} V^{[n]}E = T^nE \times_{T^nG_0} G_0 \arrow[r] \arrow[d] & T^nE \arrow[d, "T^nr"] \\ G_0 \arrow[r, "0^{[n]}_{G_0}"'] & T^nG_0 \end{tikzcd} \tag{4.35}$$

Let us denote the top horizontal arrow, which is the projection onto the first factor, by

$$i_{V^{[n]}E} : V^{[n]}E \longrightarrow T^n E\,.$$

Since $0^{[n]}_{G_0}$ is a split monomorphism, it is a fortiori a strong monomorphism. Since strong monomorphisms are preserved by pullbacks, $i_{V^{[n]}E}$ is a strong monomorphism (Rem. A.4.2). Moreover, the iterated zero sections $0^{[n]}_{G_k} : G_k \to T^n G_k$ define a morphism $G \to T^n G$ of groupoids.

By definition of the groupoid actions on $T^n E$ and $V^{[n]}E$, the diagram

$$\begin{tikzcd} V^{[n]}E \times_{G_0} G_1 \arrow[r, "i_{V^{[n]}E} \times_{0^{[n]}_{G_0}} 0^{[n]}_{G_1}"] \arrow[d, "\beta_{V^{[n]}E}"'] & T^n E \times_{T^n G_0} T^n G_1 \arrow[d, "\beta_{T^n E}"] \\ V^{[n]}E \arrow[r, "i_{V^{[n]}E}"'] & T^n E \end{tikzcd} \tag{4.36}$$

commutes. It now follows from Lemma 4.2.7, that $\beta_{V^{[n]}E}$ is a right G-action. □

Proposition 4.3.2. *If a G-bundle $E \to G_0$ is differentiable, then the fiber products of the vertical tangent bundle $V_m E \to G_0$ defined in* (4.30) *together with the action* (4.34) *is a G-bundle.*

Proof. The proof is analogous to the proof of Proposition 4.3.1. □

Remark 4.3.3. The right G-action (4.34) is precisely the diagonal action (4.16).

Proposition 4.3.4. *Let $E \to G_0$ and $F \to G_0$ be G-bundles and $\Phi : E \to F$ a G-equivariant bundle morphism. If E and F are differentiable, then the morphism*

$$V^{[n]}E = T^n E \times_{T^n G_0} G_0 \xrightarrow{T^n\Phi \times_{T^n G_0} \mathrm{id}_{G_0}} T^n F \times_{T^n G_0} G_0 = V^{[n]}F\,, \tag{4.37}$$

which we will denote by $V^{[n]}\Phi$, is a G-equivariant bundle morphism.

Proof. Since by definition $V^{[n]}\Phi$ is the identity on G_0, it is a morphism of bundles over G_0. For the equivariance, we consider the following diagram:

$$\begin{tikzcd} V^{[n]}E \times_{G_0} G_1 \arrow[rrr, "V^{[n]}\Phi \times_{G_0} \mathrm{id}"] \arrow[dr, "i_{V^{[n]}E} \times_{0^{[n]}_{G_0}} 0^{[n]}_{G_1}"] \arrow[ddd, "\beta_{V^{[n]}E}"'] & & & V^{[n]}F \times_{G_0} G_1 \arrow[dl, "i_{V^{[n]}F} \times_{0^{[n]}_{G_0}} 0^{[n]}_{G_1}"'] \arrow[ddd, "\beta_{V^{[n]}F}"] \\ & T^n E \times_{T^n G_0} T^n G_1 \arrow[r, "T^n\Phi \times_{T^n G_0} \mathrm{id}"] \arrow[d, "\beta_{T^n E}"] & T^n F \times_{T^n G_0} T^n G_1 \arrow[d, "\beta_{T^n F}"] & \\ & T^n E \arrow[r, "T^n\Phi"'] & T^n F & \\ V^{[n]}E \arrow[rrr, "V^{[n]}\Phi"'] \arrow[ur, "i_{V^{[n]}E}"] & & & V^{[n]}F \arrow[ul, "i_{V^{[n]}F}"'] \end{tikzcd} \tag{4.38}$$

Spelling out the actions β_{T^nE} and β_{T^nF} as given in (4.23), we get that the inner square is the outer rectangle of the following diagram

$$\begin{tikzcd}
T^nE \times_{T^nG_0} T^nG_1 \arrow[r, "T^n\Phi\times_{T^nG_0}\mathrm{id}"] \arrow[d, "\nu_{n,1}^{-1}"', "\cong"] & T^nF \times_{T^nG_0} T^nG_1 \arrow[d, "\cong"', "\nu_{n,1}^{-1}"] \\
T^n(E \times_{G_0} G_1) \arrow[r, "T^n(\Phi\times_{G_0}\mathrm{id})"] \arrow[d, "T^n\beta_E"'] & T^n(F \times_{G_0} G_1) \arrow[d, "T^n\beta_F"] \\
T^nE \arrow[r, "T^n\Phi"'] & T^nF
\end{tikzcd}$$

which commutes since Φ is G-equivariant and T^n is a functor.

The left and right trapezoids are both the commutative diagram (4.36). The bottom trapezoid commutes since $i_{V^{[n]}E}$ and $i_{V^{[n]}F}$ are projections onto the first factor and since $V^{[n]}\Phi$ is, by definition, $T^n\Phi$ on the first factor. The commutativity of the top trapezoid follows from the commutativity of the bottom trapezoid and the functoriality of the pullback along the zero sections. Explicitly,

$$\begin{aligned}
\big(T^n\Phi \times_{T^nG_0} \mathrm{id}\big) \circ \big(i_{V^{[n]}E} \times_{0^{[n]}_{G_0}} 0^{[n]}_{G_1}\big) &= \big(T^n\Phi \circ i_{V^{[n]}E}\big) \times_{0^{[n]}_{G_0}} \big(\mathrm{id} \circ 0^{[n]}_{G_1}\big) \\
&= \big(i_{V^{[n]}F} \circ V^{[n]}\Phi\big) \times_{0^{[n]}_{G_0}} \big(0^{[n]}_{G_1} \circ \mathrm{id}\big) \qquad (4.39)\\
&= \big(i_{V^{[n]}F} \times_{0^{[n]}_{G_0}} 0^{[n]}_{G_1}\big) \circ \big(V^{[n]}\Phi \times_{G_0} \mathrm{id}\big)\,.
\end{aligned}$$

The morphism $i_{V^{[n]}F}$ is a monomorphism, as was shown in the proof of Proposition 4.3.1. It follows from Lemma A.4.14 (i) that the outer square commutes, which is the condition of equivariance. □

Proposition 4.3.5. *Let $E \to G_0$ and $F \to G_0$ be G-bundles and $\Phi : E \to F$ a G-equivariant bundle morphism. If E and F are differentiable, then the morphism*

$$V_m\Phi := \underbrace{V\Phi \times_\Phi \ldots \times_\Phi V\Phi}_{m \text{ factors}} : V_mE \longrightarrow V_mF\,, \qquad (4.40)$$

is a G-equivariant bundle morphism, where we write $V\Phi \equiv V^{[1]}\Phi$.

Proof. Firstly, consider the following diagram:

$$\begin{tikzcd}
VE \arrow[r, "i_{VE}"] \arrow[d, "V\Phi"'] & TE \arrow[r, "\pi_E"] \arrow[d, "T\Phi"] & E \arrow[d, "\Phi"] \\
VF \arrow[r, "i_{VF}"'] & TF \arrow[r, "\pi_F"'] & F
\end{tikzcd}$$

The right square commutes by the naturality of π. The left square is the commutative bottom trapezoid of Diagram (4.38) with $i_{VE} \equiv i_{V^{[1]}E}$. Using (4.31), we conclude that the map (4.40) is well-defined.

The rest of the proof is analogous to the proof of Proposition 4.3.4. □

Remark 4.3.6. Let $\mathcal{B}\mathrm{un}_G^{\mathrm{diff}}$ denote the full subcategory of differentiable G-bundles in $\mathcal{B}\mathrm{un}_G$. Propositions 4.3.4 and 4.3.5 show that the higher vertical tangent bundle and the fiber product of the vertical tangent bundle are functors

$$V^{[n]}, V_m : \mathcal{B}\mathrm{un}_G^{\mathrm{diff}} \longrightarrow \mathcal{B}\mathrm{un}_G\,.$$

For $n = 0$, the functor $V^{[0]}$, which forgets the differentiability axioms, will be denoted by 1.

Remark 4.3.7. If we apply the vertical tangent functor twice and if the category $\mathcal{C}$ has all pullbacks, we obtain an object

$$V^2E \cong V(TE \times_{TG_0} G_0) \cong T(TE \times_{TG_0} G_0) \times_{TG_0} G_0\,.$$

In general, the limit on the right hand side does not exist. This means that VE is generally not differentiable. Only if we assume the existence of this limit and that the tangent functor commutes with the pullback defining VE, we obtain the isomorphism

$$V^2E \cong T^2E \times_{T^2G_0} \times TG_0 \times_{TG_0} G_0 \cong V^{[2]}E\,.$$

However, we will *not* make the assumptions for this isomorphism to exist. As it turns out, it is not needed for the main results of this thesis. In all relevant aspects, the operators $V^{[n]}$ behave like the powers of the vertical tangent functor, as we will show in the following sections.

4.3.2 Vertical restriction of natural transformations

In this section, we present a powerful technical tool which will be essential in showing that the tangent structure can be vertically restricted.

Lemma 4.3.8. *Let $\alpha : T^n \to T^m$ be a natural transformation for some $n, m \geq 0$; let G be a differentiable groupoid. If*

$$\alpha_X \circ 0_X^{[n]} = 0_X^{[m]}$$

for $X = G_0$ and $X = G_1$, then there is a unique natural transformation $\alpha' : V^{[n]} \to V^{[m]}$ of functors $\mathcal{B}\mathrm{un}_G^{\mathrm{diff}} \to \mathcal{B}\mathrm{un}_G$ such that

$$\begin{tikzcd} V^{[n]}E \arrow[r, "\alpha'_E"] \arrow[d, "i_{V^{[n]}E}"'] & V^{[m]}E \arrow[d, "i_{V^{[m]}E}"] \\ T^nE \arrow[r, "\alpha_E"'] & T^mE \end{tikzcd} \tag{4.41}$$

commutes for all differentiable G-bundles E.

Proof. Let $r : E \to G_0$ be a differentiable G-bundle. Consider the following diagram:

$$\begin{tikzcd} T^nE \arrow[r, "T^n r"] \arrow[d, "\alpha_E"'] & T^nG_0 \arrow[d, "\alpha_{G_0}"] & G_0 \arrow[l, "0_{G_0}^{[n]}"'] \arrow[d, "\mathrm{id}_{G_0}"] \\ T^mE \arrow[r, "T^m r"'] & T^mG_0 & G_0 \arrow[l, "0_{G_0}^{[m]}"] \end{tikzcd}$$

The left square commutes by the naturality of α. The right square commutes by assumption. Hence, the diagram induces a unique map from the limit of the top row to the limit of the bottom row

$$\alpha'_E := \alpha_E \times_{\alpha_{G_0}} \mathrm{id}_{G_0} : V^{[n]}E \longrightarrow V^{[m]}E\,,$$

such that Diagram (4.41) and the diagram

$$\begin{tikzcd}
V^{[n]}E \arrow[rr, "\alpha'_E"] \arrow[dr] & & V^{[m]}E \arrow[dl] \\
& G_0 &
\end{tikzcd}$$

commute. This shows that α'_E is a bundle morphism. To show that α'_E is G-equivariant, we consider the following diagram:

$$\begin{tikzcd}
V^{[n]}E \times_{G_0} G_1 \arrow[rrr, "\alpha'_E \times_{G_0} \mathrm{id}"] \arrow[dr, "i_{V^{[n]}E} \times_{0^{[n]}_{G_0}} 0^{[n]}_{G_1}"] \arrow[ddd, "\beta_{V^{[n]}E}"'] & & & V^{[m]}E \times_{G_0} G_1 \arrow[dl, "i_{V^{[m]}E} \times_{0^{[m]}_{G_0}} 0^{[m]}_{G_1}"'] \arrow[ddd, "\beta_{V^{[m]}E}"] \\
& T^nE \times_{T^nG_0} T^nG_1 \arrow[r, "\alpha_E \times_{\alpha_{G_0}} \alpha_{G_1}"] \arrow[d, "\beta_{T^nE}"'] & T^mE \times_{T^mG_0} T^mG_1 \arrow[d, "\beta_{T^mE}"] & \\
& T^nE \arrow[r, "\alpha_E"'] & T^mE & \\
V^{[n]}E \arrow[ur, "i_{V^{[n]}E}"'] \arrow[rrr, "\alpha'_E"'] & & & V^{[m]}E \arrow[ul, "i_{V^{[m]}E}"]
\end{tikzcd} \tag{4.42}$$

The inner square is the outer rectangle of the following diagram:

$$\begin{tikzcd}
T^nE \times_{T^nG_0} T^nG_1 \arrow[r, "\alpha_E \times_{\alpha_{G_0}} \alpha_{G_1}"] \arrow[d, "\nu_{n,1}^{-1}"', "\cong"] & T^mE \times_{T^mG_0} T^mG_1 \arrow[d, "\cong"', "\nu_{m,1}^{-1}"] \\
T^n(E \times_{G_0} G_1) \arrow[r, "\alpha_{E \times_{G_0} G_1}"'] \arrow[d, "T^n\beta_E"'] & T^m(E \times_{G_0} G_1) \arrow[d, "T^m\beta_E"] \\
T^nE \arrow[r, "\alpha_E"'] & T^mE
\end{tikzcd}$$

The bottom square commutes by the naturality of α, the upper square by Lemma A.4.11. We conclude that the outer rectangle, and thus, the inner square of Diagram (4.42) is commutative.

The right and left trapezoids of (4.42) are both the commutative diagram (4.36). The lower trapezoid is Diagram (4.41), which we have already shown to commute. The commutativity of the top trapezoid follows from the commutativity of the bottom trapezoid, the assumption that $\alpha_X \circ 0^{[n]}_X = 0^{[m]}_X$ for both $X = G_0$ and $X = G_1$, and from the functoriality of the pullback along the zero sections. The explicit calculation is analogous to that in (4.39).

The morphism $i_{V^{[m]}E}$ is a monomorphism, as was shown in the proof of Proposition 4.3.1. It follows from Lemma A.4.14 (i) that the outer square commutes, which is the condition of equivariance.

Since α is a natural transformation and pullbacks are natural, $\alpha'_E := \alpha_E \times_{\alpha_{G_0}} \mathrm{id}_{G_0}$ is natural in $E \in \mathcal{B}\mathrm{un}_G^{\mathrm{diff}}$. $\square$

Lemma 4.3.9. *Let $\alpha : T_n \to T_m$, $\beta : T_n \to T^m$, $\gamma : T^n \to T_m$ be natural transformations for some $n, m \geq 0$; let G be a differentiable groupoid. If*

$$\alpha_X \circ 0_{n,X} = 0_{m,X}\,, \quad \beta_X \circ 0_{n,X} = 0_X^{[m]}\,, \quad \gamma_X \circ 0_X^{[n]} = 0_{m,X}\,,$$

for $X = G_0$ and $X = G_1$, then there are unique natural transformations $\alpha' : V_n \to V_m$, $\beta' : V_n \to V^{[m]}$, $\gamma : V^{[n]} \to V_m$ of functors $\mathcal{B}\mathrm{un}_G^{\mathrm{diff}} \to \mathcal{B}\mathrm{un}_G$ such that

$$\begin{array}{ccc}
\begin{tikzcd} V_nE \arrow[r, "\alpha'_E"] \arrow[d, "i_{V_nE}"'] & V_mE \arrow[d, "i_{V_mE}"] \\ T_nE \arrow[r, "\alpha_E"'] & T_mE \end{tikzcd}
&
\begin{tikzcd} V_nE \arrow[r, "\beta'_E"] \arrow[d, "i_{V_nE}"'] & V^{[m]}E \arrow[d, "i_{V^{[m]}E}"] \\ T_nE \arrow[r, "\beta_E"'] & T^mE \end{tikzcd}
&
\begin{tikzcd} V^{[n]}E \arrow[r, "\gamma'_E"] \arrow[d, "i_{V^{[n]}E}"'] & V_mE \arrow[d, "i_{V_mE}"] \\ T^nE \arrow[r, "\gamma_E"'] & T_mE \end{tikzcd}
\end{array} \tag{4.43}$$

commute for all differentiable G-bundles E.

Proof. The proof is analogous to the proof of Lemma 4.3.8. $\square$

Lemma 4.3.10. *Let $\alpha : T^n \to T^m$ and $\beta : T^m \to T^l$ be natural transformations for some $n, m, l \geq 0$ and $\gamma := \beta \circ \alpha : T^n \to T^l$ their composition; let G be a differentiable groupoid. If*

$$\alpha_X \circ 0_X^{[n]} = 0_X^{[m]}\,, \quad \beta_X \circ 0_X^{[m]} = 0_X^{[l]}$$

for $X = G_0$ and $X = G_1$, then $\gamma_X \circ 0_X^{[n]} = 0_X^{[l]}$ for $X = G_0$ and $X = G_1$, and the vertical restrictions $\alpha' : V^{[n]} \to V^{[m]}$, $\beta' : V^{[m]} \to V^{[l]}$, $\gamma' : V^{[n]} \to V^{[l]}$ from Lemma 4.3.8 satisfy $\gamma' = \beta' \circ \alpha'$.

The statement also holds, mutatis mutandis, for natural transformations of type $T_n \to T_m$, $T_n \to T^m$, and $T^n \to T_m$.

Proof. We have

$$\gamma_X \circ 0_X^{[n]} = \beta_X \circ \alpha_X \circ 0_X^{[n]} = \beta_X \circ 0_X^{[m]} = 0_X^{[l]}\,,$$

which implies that γ has a vertical restriction. By Lemma 4.3.8, we have the commutative diagram

$$\begin{tikzcd}
V^{[n]}E \arrow[r, "\alpha'_E"] \arrow[d, "i_{V^{[n]}E}"'] & V^{[m]}E \arrow[r, "\beta'_E"] \arrow[d, "i_{V^{[m]}E}"] & V^{[l]}E \arrow[d, "i_{V^{[l]}E}"] \\
T^nE \arrow[r, "\alpha_E"'] \arrow[rr, bend right, "\gamma_E"'] & T^mE \arrow[r, "\beta_E"'] & T^lE
\end{tikzcd}$$

This shows that $\beta'_E \circ \alpha'_E$ is the vertical restriction of γ_E. Since the vertical restriction is unique, it follows that $\beta'_E \circ \alpha'_E = \gamma'_E$ for all $E \in \mathcal{B}\mathrm{un}_G^{\mathrm{diff}}$.

The proof of the statements for natural transformations of type $T_n \to T_m$, $T_n \to T^m$, and $T^n \to T_m$ is analogous. $\square$

4.3.3 Vertical prolongation of equivariant bundle morphisms

Recall that $V^{[k]} : \mathcal{B}\mathrm{un}_G^{\mathrm{diff}} \to \mathcal{B}\mathrm{un}_G$ is a functor (Rem. 4.3.6). Given differentiable G-bundles E and F, a G-equivariant bundle morphism $\Phi : E \to V^{[n]}F$ can in general not be extended to $V^{[k]}\Phi : V^{[k]}E \to V^{[n+k]}F$. The reason is that $V^{[n]}F$ might not be differentiable (Rem. 4.3.7), and so we cannot simply apply the functor $V^{[k]}$ to the morphism Φ. The following proposition solves this subtlety by introducing the notion by vertical prolongation using a universal construction.

Proposition 4.3.11. *Let E and F be differentiable G-bundles and $\Phi : E \to V^{[n]}F$ a G-equivariant bundle morphism. Then for every $k \geq 1$ there is a unique G-equivariant bundle morphism $\Phi^{[k]} : V^{[k]}E \to V^{[n+k]}F$, called the k-th* ***vertical prolongation*** *of Φ, such that*

$$\begin{tikzcd} V^{[k]}E \arrow[rr, "\Phi^{[k]}"] \arrow[d, "i_{V^{[k]}E}"'] & & V^{[n+k]}F \arrow[d, "i_{V^{[n+k]}F}"] \\ T^kE \arrow[r, "T^k\Phi"'] & T^kV^{[n]}F \arrow[r, "T^k i_{V^{[n]}F}"'] & T^{n+k}F \end{tikzcd} \tag{4.44}$$

commutes.

Proof. Let

$$\Psi := T^k i_{V^{[n]}F} \circ T^k\Phi \circ i_{V^{[k]}E}$$

denote the composition of the counterclockwise arrows of Diagram (4.44). Let $r_E : E \to G_0$, $r_F : F \to G_0$, $r_{V^{[k]}E} : V^{[k]}E \to G_0$ and $r_{V^{[k]}F} : V^{[k]}F \to G_0$ denote the bundle projections. Then

$$\begin{aligned} T^{n+k}r_F \circ \Psi &= T^{n+k}r_F \circ T^k i_{V^{[n]}F} \circ T^k\Phi \circ i_{V^{[k]}E} \\ &= T^k(T^n r_F \circ i_{V^{[n]}F} \circ \Phi) \circ i_{V^{[k]}E} \\ &= T^k(0_{G_0}^{[n]} \circ r_{V^{[n]}F} \circ \Phi) \circ i_{V^{[k]}E} \\ &= T^k(0_{G_0}^{[n]} \circ r_E) \circ i_{V^{[k]}E} \\ &= T^k 0_{G_0}^{[n]} \circ T^k r_E \circ i_{V^{[k]}E} \\ &= T^k 0_{G_0}^{[n]} \circ 0_{G_0}^{[k]} \circ r_{V^{[k]}E} \\ &= 0_{G_0}^{[n+k]} \circ r_{V^{[k]}E} \,, \end{aligned}$$

where we have used the pullback squares defining $V^{[n]}F$ and $V^{[k]}E$ and that Φ is a bundle morphism. It follows from the universal property of the pullback defining $V^{[n+k]}F$, that there is a unique morphism $\Phi^{[k]}$, such that

$$\begin{tikzcd} V^{[k]}E \arrow[dr, dashed, "\exists!\,\Phi^{[k]}"] \arrow[drr, bend left, "r_{V^{[k]}E}"] \arrow[ddr, bend right, "\Psi"'] & & \\ & V^{[n+k]}F \arrow[r] \arrow[d, "i_{V^{[n+k]}F}"] & G_0 \arrow[d, "0_{G_0}^{[n+k]}"] \\ & T^{n+k}F \arrow[r, "T^{n+k}r_F"'] & T^{n+k}G_0 \end{tikzcd}$$

commutes. In particular, we have that $\Phi^{[k]}$ is a bundle morphism. It remains to show that it is G-equivariant.

Let $\psi := i_{V^{[n]}F} \circ \Phi$, so that $i_{V^{[n+k]}F} \circ \Phi^{[k]} = T^k\psi \circ i_{V^{[k]}E}$. Consider the following diagram:

$$\begin{tikzcd}
E \times_{G_0} G_1 \arrow[r, "\Phi \times_{G_0} \mathrm{id}_{G_1}"] \arrow[d, "\beta_E"'] & V^{[n]}F \times_{G_0} G_1 \arrow[r, "i_{V^{[n]}F} \times_{0^{[n]}_{G_0}} 0^{[n]}_{G_1}"] \arrow[d, "\beta_{V^{[n]}F}"] & T^nF \times_{T^nG_0} T^nG_1 \arrow[d, "\beta_{T^nF}"] \\
E \arrow[r, "\Phi"'] & V^{[n]}F \arrow[r, "i_{V^{[n]}F}"'] & T^nF
\end{tikzcd} \tag{4.45}$$

The left square commutes since Φ is, by assumption, G-equivariant. The right square is the commutative diagram (4.36). The composition of the lower horizontal arrows is ψ and that of the upper horizontal arrows is

$$\begin{aligned}
\left(i_{V^{[n]}F} \times_{0^{[n]}_{G_0}} 0^{[n]}_{G_1}\right) \circ \left(\Phi \times_{G_0} \mathrm{id}_{G_1}\right) &= \left(i_{V^{[n]}F} \circ \Phi\right) \times_{0^{[n]}_{G_0}} 0^{[n]}_{G_1} \\
&= \psi \times_{0^{[n]}_{G_0}} 0^{[n]}_{G_1} ,
\end{aligned}$$

by functoriality.

Applying the functor T^k to Diagram (4.45) and stacking it with the commutative square obtained via the natural universal morphisms to the pullbacks, we get the following commutative diagram:

$$\begin{tikzcd}
T^kE \times_{T^kG_0} T^kG_1 \arrow[r, "T^k\psi \times_{T^k 0^{[n]}_{G_0}} T^k 0^{[n]}_{G_1}"] \arrow[d, "\nu_{k,1}^{-1}"', "\cong"] & T^{n+k}F \times_{T^{n+k}G_0} T^{n+k}G_1 \arrow[d, "\cong"', "\nu_{k,1}^{-1}"] \\
T^k(E \times_{G_0} G_1) \arrow[r, "T^k\left(\psi \times_{0^{[n]}_{G_0}} 0^{[n]}_{G_1}\right)"] \arrow[d, "T^k\beta_E"'] & T^k(T^nF \times_{T^nG_0} T^nG_1) \arrow[d, "T^k\beta_{T^nF}"] \\
T^kE \arrow[r, "T^k\psi"'] & T^{n+k}F
\end{tikzcd} \tag{4.46}$$

By definition, the composition of the left vertical arrows is the T^kG-action β_{T^kE} and that of the right vertical arrows is the $T^{n+k}G$-action $\beta_{T^{n+k}E}$. Thus, the outer square of Diagram (4.46) is the inner square in the following diagram:

$$
\begin{tikzcd}
V^{[k]}E \times_{G_0} G_1 \arrow[rrr, "\Phi^{[k]} \times_{G_0} \mathrm{id}"] \arrow[dr, "i_{V^{[k]}E} \times_{0^{[k]}_{G_0}} 0^{[k]}_{G_1}"] \arrow[ddd, "\beta_{V^{[k]}E}"'] & & & V^{[n+k]}F \times_{G_0} G_1 \arrow[dl, "i_{V^{[n+k]}F} \times_{0^{[n+k]}_{G_0}} 0^{[n+k]}_{G_1}"'] \arrow[ddd, "\beta_{V^{[n+k]}F}"] \\
& T^kE \times_{T^kG_0} T^kG_1 \arrow[r, "T^k\psi \times_{T^k 0^{[n]}_{G_0}} T^k 0^{[n]}_{G_1}"] \arrow[d, "\beta_{T^kE}"'] & T^{n+k}F \times_{T^{n+k}G_0} T^{n+k}G_1 \arrow[d, "\beta_{T^{n+k}F}"] & \\
& T^kE \arrow[r, "T^k\psi"'] & T^{n+k}F & \\
V^{[k]}E \arrow[ur, "i_{V^{[k]}E}"'] \arrow[rrr, "\Phi^{[k]}"'] & & & V^{[n+k]}F \arrow[ul, "i_{V^{[n+k]}F}"]
\end{tikzcd}
$$

The bottom and top trapezoids commute by the definition of $\Phi^{[k]}$ and the functoriality of pullbacks. The left and right trapezoids commute by the definition of the G-actions on the vertical bundles (Diagram (4.36)). Since $i_{V^{[n+k]}F}$ is a monomorphism, it follows from Lemma A.4.14 (i) that the outer square commutes, which shows that $\Phi^{[k]}$ is G-equivariant. $\square$

It is a consequence of Lemmas 4.3.8 and 4.3.9 that vertical prolongations also exist for vertically restricted natural transformations.

Lemma 4.3.12. *Let $\alpha : T^n \to T^m$ be a natural transformation for some $n, m \geq 0$; let G be a differentiable groupoid. Assume that*

$$\alpha_X \circ 0^{[n]}_X = 0^{[m]}_X$$

for $X = G_0$ and $X = G_1$, and let $\alpha' : V^{[n]} \to V^{[m]}$ be the unique natural transformation from Lemma 4.3.8. Then, there exists a unique natural transformation

$$\alpha'^{[k]} : V^{[n+k]} \longrightarrow V^{[m+k]},$$

called the k-th ***vertical prolongation*** *of α', such that*

$$
\begin{tikzcd}
V^{[n+k]}E \arrow[r, "\alpha'^{[k]}_E"] \arrow[d, "i_{V^{[n+k]}E}"'] & V^{[m+k]}E \arrow[d, "i_{V^{[m+k]}E}"] \\
T^{n+k}E \arrow[r, "T^k\alpha_E"'] & T^{m+k}E
\end{tikzcd}
\tag{4.47}
$$

commutes for all differentiable G-bundles E.

Proof. The natural transformation $T^k\alpha : T^{n+k} \to T^{m+k}$ is given componentwise by

$(T^k\alpha)_X = T^k\alpha_X$ for all $X \in \mathcal{C}$. It satisfies

$$\begin{aligned}(T^k\alpha)_X \circ 0_X^{[n+k]} &= T^k\alpha_X \circ 0_X^{[n+k]}\\ &= T^k\alpha_X \circ T^k 0_X^{[n]}\\ &= T^k(\alpha_X \circ 0_X^{[n]})\\ &= T^k 0_X^{[m]}\\ &= 0_X^{[m+k]}\end{aligned}$$

for $X = G_0$ and $X = G_1$. It then follows from Lemma 4.3.8 that there is a unique natural transformation $\alpha'^{[k]} := (T^k\alpha)' : V^{[n+k]} \longrightarrow V^{[m+k]}$ such that Diagram (4.47) commutes. $\square$

4.3.4 Vertical restriction of the tangent structure

The next proposition is one of the main results of this thesis, where we show that the natural transformations of the tangent structure on $\mathcal{C}$ can be restricted to the (higher) vertical tangent bundles and their fiber products. It is a consequence of Lemma 4.3.8.

Proposition 4.3.13. *Let G be a differentiable groupoid in a tangent category $\mathcal{C}$. Let $1 : \mathcal{B}\mathrm{un}_G^{\mathrm{diff}} \to \mathcal{B}\mathrm{un}_G$ denote the inclusion. There are natural transformations $\pi' : V \to 1$, $0' : 1 \to V$, $+' : V_2 \to V$, $\tau' : V^{[2]} \to V^{[2]}$, $\lambda' : V \to V^{[2]}$ and $\lambda'_2 : V_2 \to V^{[2]}$ of functors $\mathcal{B}\mathrm{un}_G^{\mathrm{diff}} \to \mathcal{B}\mathrm{un}_G$ such that the following diagrams*

$$\begin{tikzcd} VE \arrow[rr, "\pi'_E"] \arrow[dr, "i_{VE}"'] & & E & E \arrow[rr, "0'_E"] \arrow[dr, "0_E"'] & & VE \arrow[dl, "i_{VE}"] & V_2E \arrow[r, "+'_E"] \arrow[d, "i_{V_2E}"'] & VE \arrow[d, "i_{VE}"] \\ & TE \arrow[ur, "\pi_E"'] & & & TE & & T_2E \arrow[r, "+_E"'] & TE \end{tikzcd}$$

$$\begin{tikzcd} V^{[2]}E \arrow[r, "\tau'_E"] \arrow[d, "i_{V^{[2]}E}"'] & V^{[2]}E \arrow[d, "i_{V^{[2]}E}"] & VE \arrow[r, "\lambda'_E"] \arrow[d, "i_{VE}"'] & V^{[2]}E \arrow[d, "i_{V^{[2]}E}"] & V_2E \arrow[r, "\lambda'_{2,E}"] \arrow[d, "i_{V_2E}"'] & V^{[2]}E \arrow[d, "i_{V^{[2]}E}"] \\ T^2E \arrow[r, "\tau_E"'] & T^2E & TE \arrow[r, "\lambda_E"'] & T^2E & T_2E \arrow[r, "\lambda_{2,E}"'] & T^2E \end{tikzcd} \tag{4.48}$$

commute in $\mathcal{C}$ for all differentiable G-bundles E.

Proof. We will show that the assumption $\alpha_X \circ 0_X^{[n]} = 0_X^{[m]}$ of Lemma 4.3.8 is satisfied for the natural transformations $\alpha = \pi, 0, \tau, \lambda$ (with the right choice of n and m) of the tangent structure for all $X \in \mathcal{C}$, so in particular for $X = G_0$ and $X = G_1$. Similarly, we will show that the assumptions $\alpha_X \circ 0_{2,X} = 0_X$ and $\beta_X \circ 0_{2,X} = 0_X^{[2]}$ of Lemma 4.3.9 hold for $\alpha = + : T_2 \to T$ and $\beta = \lambda_2 : T_2 \to T^2$.

Since 0 is a section of π, we have that $\pi_X \circ 0_X = \mathrm{id}_X = 0_X^{[0]}$. For the zero section, we have

$$0_X \circ 0_X^{[0]} = 0_X \circ \mathrm{id}_X = 0_X\,.$$

By Definition 3.2.4 of tangent structures, τ is a morphism of bundles of abelian groups, so that it maps the zero section to the zero section,

$$\tau_X \circ 0_X^{[2]} = \tau_X \circ T0_X \circ 0_X = 0_{TX} \circ 0_X = 0_X^{[2]} . \tag{4.49}$$

Similarly, λ is a morphism of bundles of abelian groups over $0 : 1 \to T$, so that it maps the zero section to the zero section,

$$\lambda_X \circ 0_X = 0_{TX} \circ 0_X = 0_X^{[2]} . \tag{4.50}$$

The natural transformation $\alpha = +$ is an abelian group structure, so that

$$+_X \circ (0_X, 0_X) = 0_X .$$

By Definition (3.36) of the vertical lift, Equation (4.50), that the addition of zero and zero is zero, and Equation (4.49), we get that

$$\begin{aligned}
\lambda_{2,X} \circ (0_X, 0_X) &= \tau_X \circ +_{TX} \circ (T0_X \times_{0_X} \lambda_X) \circ (0_X, 0_X) \\
&= \tau_X \circ +_{TX} \circ (T0_X \circ 0_X, \lambda_X \circ 0_X) \\
&= \tau_X \circ +_{TX} \circ (0_X^{[2]}, 0_X^{[2]}) \\
&= \tau_X \circ 0_X^{[2]} \\
&= 0_X^{[2]} .
\end{aligned}$$

The proof now follows from Lemma 4.3.8 for $\alpha = \pi, 0, \tau, \lambda$ and from Lemma 4.3.9 for $\alpha = +$ and $\beta = \lambda_2$. □

Remark 4.3.14. By Definition (3.36), the extension λ_2 of the vertical lift is given by the composition $\lambda_2 := \tau \circ (+T) \circ (T0 \times_0 \lambda)$. It is natural to ask why its vertical restriction λ_2' is not defined similarly by composing the corresponding vertical restrictions. The reason is that VE is generally not differentiable as described in Remark 4.3.7. Thus, the pullback $V_2VE = V^2E \times_{VE} V^2E$ does not generally exist and so the addition $+'_{VE} : V_2VE \to V^2E$ is not well-defined. Proposition 4.3.13 shows that this issue can be completely overcome by providing us with a direct and universal way of defining λ_2'.

The next proposition shows that λ_2' is still a vertical lift, where the notion of vertical prolongation from Proposition 4.3.11 becomes crucial.

Proposition 4.3.15. *The diagram*

$$\begin{tikzcd}
V_2 \arrow[r, "\lambda_2'"] \arrow[d, "\pi' \circ \mathrm{pr}_1"'] & V^{[2]} \arrow[d, "\pi'^{[1]}"] \\
1 \arrow[r, "0'"'] & V
\end{tikzcd} \tag{4.51}$$

is a pointwise pullback in $\mathcal{B}\mathrm{un}_G$*, where* $\pi'^{[1]}$ *is the 1st vertical prolongation of* π'*.*

Proof. Let $r_E : E \to G_0$ be a differentiable G-bundle. Consider the following diagram:

$$\begin{tikzcd}
V_2E \arrow[rrr, "\lambda'_{2,E}"] \arrow[dr, "i_{V_2E}"] \arrow[ddd, "\pi'_E \circ \mathrm{pr}_1"'] & & & V^{[2]}E \arrow[dl, "i_{V^{[2]}E}"'] \arrow[ddd, "\pi'^{[1]}_E"] \\
& T_2E \arrow[r, "\lambda_{2,E}"] \arrow[d, "\pi_E \circ \mathrm{pr}_1"'] & T^2E \arrow[d, "T\pi_E"] & \\
& E \arrow[r, "0_E"'] & TE & \\
E \arrow[ur, "\mathrm{id}"'] \arrow[rrr, "0'_E"'] & & & VE \arrow[ul, "i_{VE}"]
\end{tikzcd} \tag{4.52}$$

The commutativity of the upper, lower and left trapezoids follows from the corresponding commutative diagrams in (4.48). The right trapezoid is the commutative diagram (4.47) with $\alpha = \pi$ and $k = 1$. The inner square is the commutative square (3.37) evaluated at $E \in \mathcal{C}$. Since i_{VE} is a monomorphism, we get, by Lemma A.4.14 (i), that the outer square is a commutative square in $\mathcal{B}\mathrm{un}_G$.

Now, consider the following diagram:

$$\begin{tikzcd}
E \arrow[r, "0_E"] \arrow[d, "r_E"'] & TE \arrow[d, "Tr_E"] & T^2E \arrow[l, "T\pi_E"'] \arrow[d, "T^2r_E"] \\
G_0 \arrow[r, "0_{G_0}"] & TG_0 & T^2G_0 \arrow[l, "T\pi_{G_0}"'] \\
G_0 \arrow[r, "\mathrm{id}"'] \arrow[u, "\mathrm{id}"] & G_0 \arrow[u, "0_{G_0}"'] & G_0 \arrow[l, "\mathrm{id}"] \arrow[u, "0^{[2]}_{G_0}"']
\end{tikzcd}$$

The upper left square commutes by the naturality of 0 and the upper right square commutes by the naturality of π and the functoriality of T. The lower right square commutes since $T\pi \circ T0 \circ 0 = T(\pi \circ 0) \circ 0 = T1 \circ 0 = 0$. The commutativity of the lower left square is trivial. It follows from the commutativity of pullbacks that

$$(E \times_{TE} T^2E) \times_{G_0 \times_{TG_0} T^2G_0} (G_0 \times_{G_0} G_0) \;\cong\; (E \times_{G_0} G_0) \times_{TE \times_{TG_0} G_0} (T^2E \times_{T^2G_0} G_0)\,. \tag{4.53}$$

Using the fact that Diagram (3.37) is a pointwise pullback, we know that

$$E \times_{TE} T^2E \cong T_2E \quad \text{and} \quad G_0 \times_{TG_0} T^2G_0 \cong T_2G_0\,.$$

We conclude that the left hand side of (4.53) is isomorphic to

$$\begin{aligned} T_2E \times_{T_2G_0} G_0 &\cong V_2E \times_{G_0} G_0 \\ &\cong V_2E\,, \end{aligned}$$

where we have used the isomorphism in (4.29) for $k = 0$ and cancellation of pullbacks along the identity.

On the other hand, the right hand side of (4.53) is isomorphic to $E \times_{VE} V^{[2]}E$. We conclude that (4.53) induces the isomorphism

$$V_2E \cong E \times_{VE} V^{[2]}E$$

in $\mathcal{C}$. Lastly, it follows from Proposition 4.2.9 that this is the pullback in the category $\mathcal{B}\mathrm{un}_G$. $\square$

4.3.5 Vertical restriction of the module structure

Let E be a differentiable G-bundle and R a ring object of the tangent category $\mathcal{C}$. Then,

$$r_{VE} \circ \mathrm{pr}_2 : R \times VE \to G_0$$

is a G-bundle with right G-action given by

$$\mathrm{id}_R \times \beta_{VE} : R \times VE \times_{G_0} G_1 \longrightarrow R \times VE\,.$$

Let $\kappa : R \times T \to T$ be an R-module structure on $\pi : T \to 1$. It is a natural transformation with components $\kappa_X : R \times TX \to TX$ for all $X \in \mathcal{C}$ (Section 3.2.3). We will show that κ restricts to a G-equivariant R-module structure on $\pi' : V \to 1$.

Proposition 4.3.16. *Let G be a differentiable groupoid in a tangent category $\mathcal{C}$ with an R-module structure $\kappa : R \times T \to T$. Then there is a natural transformation $\kappa' : R \times V \to V$ of functors $\mathcal{B}\mathrm{un}_G^{\mathrm{diff}} \to \mathcal{B}\mathrm{un}_G$, such that*

$$\begin{tikzcd} R \times VE \arrow[r, "\kappa'_E"] \arrow[d, "\mathrm{id}_R \times i_{VE}"'] & VE \arrow[d, "i_{VE}"] \\ R \times TE \arrow[r, "\kappa_E"'] & TE \end{tikzcd} \tag{4.54}$$

commutes in $\mathcal{C}$ for all differentiable G-bundles E.

Proof. Let $r : E \to G_0$ be a differentiable G-bundle. Consider the following diagram:

$$\begin{tikzcd} R \times TE \arrow[r, "\mathrm{id}_R \times Tr"] \arrow[d, "\kappa_E"'] & R \times TG_0 \arrow[d, "\kappa_{G_0}"] & R \times G_0 \arrow[l, "\mathrm{id}_R \times 0_{G_0}"'] \arrow[d, "\mathrm{pr}_2"] \\ TE \arrow[r, "Tr"'] & TG_0 & G_0 \arrow[l, "0_{G_0}"] \end{tikzcd} \tag{4.55}$$

The left square commutes by the naturality of κ. The right square commutes since the scalar multiplication is linear in T, which implies that it maps the zero section to the zero section. By the commutativity of pullbacks and products, we have the following isomorphism for the limit of the top row,

$$\begin{aligned} \phi : (R \times TE) \times_{R \times TG_0} (R \times G_0) &\xrightarrow{\cong} (R \times_R R) \times (TE \times_{TG_0} G_0) \\ &\xrightarrow{\cong} R \times VE\,. \end{aligned}$$

Diagram (4.55) induces a unique map

$$\kappa'_E := \big(\kappa_E \times_{\kappa_{G_0}} \mathrm{pr}_2\big) \circ \phi^{-1} : R \times VE \longrightarrow VE\,, \tag{4.56}$$

such that Diagram (4.54) and the diagram

$$\begin{tikzcd} R \times VE \arrow[r, "\kappa'_E"] \arrow[d, "\mathrm{id}_R \times r_{VE}"'] & VE \arrow[d, "r_{VE}"] \\ R \times G_0 \arrow[r, "\mathrm{pr}_2"'] & G_0 \end{tikzcd}$$

commute. This shows that κ'_E is a morphism of bundles over G_0.

Next, we show that κ is right TG-equivariant. From Lemma A.4.11 with $F(\_) = R \times T(\_)$ and $G = T$ we obtain the following commutative square:

$$\begin{tikzcd}
R \times T(E \times_{G_0} G_1) \arrow[r, "\kappa_{E\times_{G_0}G_1}"] \arrow[d] & T(E \times_{G_0} G_1) \arrow[d] \\
(R \times TE) \times_{R\times TG_0} (R \times TG_1) \arrow[r, "\kappa_E \times_{\kappa_{G_0}} \kappa_{G_1}"'] & TE \times_{TG_0} TG_1
\end{tikzcd}$$

For the domain of the bottom horizontal arrow, we have the isomorphism

$$\begin{aligned}
\psi : (R \times TE) \times_{R\times TG_0} (R \times TG_1) &\xrightarrow{\cong} (R \times_R R) \times (TE \times_{TG_0} TG_1) \\
&\xrightarrow{\cong} R \times TE \times_{TG_0} TG_1 \,,
\end{aligned}$$

so that we obtain the commutative square

$$\begin{tikzcd}
R \times T(E \times_{G_0} G_1) \arrow[r, "\kappa_{E\times_{G_0}G_1}"] \arrow[d, "\mathrm{id}_R \times \nu_{1,1}"'] & T(E \times_{G_0} G_1) \arrow[d, "\nu_{1,1}"] \\
R \times TE \times_{TG_0} TG_1 \arrow[r, "(\kappa_E \times_{\kappa_{G_0}} \kappa_{G_1}) \circ \psi^{-1}"'] & TE \times_{TG_0} TG_1
\end{tikzcd} \tag{4.57}$$

Expressing ψ explicitly in terms of projections, we can write the morphism represented by the bottom arrow as

$$\begin{aligned}
\tilde{\kappa} &:= (\kappa_E \times_{\kappa_{G_0}} \kappa_{G_1}) \circ \psi^{-1} \\
&= \big(\kappa_E \circ (\mathrm{pr}_1, \mathrm{pr}_2), \kappa_{G_1} \circ (\mathrm{pr}_1, \mathrm{pr}_3)\big) \,.
\end{aligned}$$

It can be viewed as the diagonal R-module structure or as the R-module structure of the pullback of R-modules. By the assumption that E is differentiable, the vertical arrows of (4.57) are isomorphisms, so that we obtain the commutative diagram

$$\begin{tikzcd}
R \times TE \times_{TG_0} TG_1 \arrow[r, "\tilde{\kappa}"] \arrow[d, "\mathrm{id}_R \times \nu_{1,1}^{-1}"] \arrow[dd, bend right=60, "\mathrm{id}_R \times \beta_{TE}"'] & TE \times_{TG_0} TG_1 \arrow[d, "\nu_{1,1}^{-1}"'] \arrow[dd, bend left=60, "\beta_{TE}"] \\
R \times T(E \times_{G_0} G_1) \arrow[r, "\kappa_{E\times_{G_0}G_1}"'] \arrow[d, "\mathrm{id}_R \times T\beta_E"] & T(E \times_{G_0} G_1) \arrow[d, "T\beta_E"'] \\
R \times TE \arrow[r, "\kappa_E"'] & TE
\end{tikzcd}$$

which shows that the R-module structure and the TG-action on TE commute. This diagram is the inner square of the following diagram:

$$\begin{tikzcd}
R \times VE \times_{G_0} G_1 \arrow[rrr, "\kappa'_E \times_{G_0} \mathrm{id}_{G_1}"] \arrow[dr, "\mathrm{id}_R \times i_{VE} \times_{0_{G_0}} 0_{G_1}"] \arrow[ddd, "\mathrm{id}_R \times \beta_{VE}"'] & & & VE \times_{G_0} G_1 \arrow[dl, "i_{VE} \times_{0_{G_0}} 0_{G_1}"'] \arrow[ddd, "\beta_{VE}"] \\
& R \times TE \times_{TG_0} TG_1 \arrow[r, "\tilde{\kappa}"] \arrow[d, "\mathrm{id}_R \times \beta_{TE}"'] & TE \times_{TG_0} TG_1 \arrow[d, "\beta_{TE}"] & \\
& R \times TE \arrow[r, "\kappa_E"'] & TE & \\
R \times VE \arrow[rrr, "\kappa'_E"'] \arrow[ur, "\mathrm{id}_R \times i_{VE}"'] & & & VE \arrow[ul, "i_{VE}"]
\end{tikzcd} \tag{4.58}$$

The right and left trapezoids of (4.58) commute by the commutativity of Diagram (4.36). The lower trapezoid is Diagram (4.54), which we have already shown to commute. The top trapezoid commutes since

$$\begin{aligned}
&\tilde{\kappa} \circ (\mathrm{id}_R \times i_{VE} \times_{0_{G_0}} 0_{G_1}) \\
&= \big(\kappa_E \circ (\mathrm{pr}_1, \mathrm{pr}_2),\ \kappa_{G_1} \circ (\mathrm{pr}_1, \mathrm{pr}_3)\big) \circ (\mathrm{id}_R \times i_{VE} \times_{0_{G_0}} 0_{G_1}) \\
&= \big(\kappa_E \circ (\mathrm{id}_R \times i_{VE}) \circ (\mathrm{pr}_1, \mathrm{pr}_2), \kappa_{G_1} \circ (\mathrm{id}_R \times 0_{G_1}) \circ (\mathrm{pr}_1, \mathrm{pr}_3)\big) \\
&= \big((i_{VE} \circ \kappa'_E) \circ (\mathrm{pr}_1, \mathrm{pr}_2), (0_{G_1} \circ \mathrm{pr}_2) \circ (\mathrm{pr}_1, \mathrm{pr}_3)\big) \\
&= \big((i_{VE} \circ \kappa'_E) \circ (\mathrm{pr}_1, \mathrm{pr}_2), (0_{G_1} \circ \mathrm{pr}_3)\big) \\
&= (i_{VE} \circ \kappa'_E) \times_{0_{G_0}} 0_{G_1} \\
&= \big(i_{VE} \times_{0_{G_0}} 0_{G_1}\big) \circ \big(\kappa'_E \times_{G_0} \mathrm{id}_{G_1}\big)\,,
\end{aligned}$$

where we have used the commutativity of the bottom trapezoid, that κ preserves the zero section, and the functoriality of pullbacks. Since the morphism i_{VE} is a monomorphism, it follows from Lemma A.4.14 (i) that the outer square commutes, which shows that κ' is G-equivariant. Moreover, since κ is a natural transformation and pullbacks are natural, $\kappa'_E := \big(\kappa_E \times_{\kappa_{G_0}} \mathrm{pr}_2\big) \circ \phi^{-1}$ is natural in $E \in \mathcal{B}\mathrm{un}_G^{\mathrm{diff}}$. $\square$

Remark 4.3.17. The R-module structure κ'_E of $VE \to E$ defined in (4.56) is the same module structure described in the proof of Proposition 3.1.17 for $\phi = Tr$ and $\ker \phi = VE$.

Corollary 4.3.18. *The natural transformation $\pi' : V \to 1$ with neutral element $0' : 1 \to V$ and addition $+' : V_2 \to V$ is a bundle of abelian groups over 1. If the tangent category $\mathcal{C}$ has an R-module structure, then $\pi' : V \to 1$ is a bundle of R-modules.*

Proof. Let $r : E \to G_0$ be a differentiable G-bundle. Then, $Tr : TE \to TG_0$ is a morphism of bundles of abelian groups by the naturality of the tangent structure. It follows from Proposition 3.1.13(i) that its kernel $VE = TE \times_{TG_0}^{Tr, 0_{G_0}} G_0$, together with the vertical restriction of the bundle projection

$$\pi'_E : VE \xrightarrow{i_{VE}} TE \xrightarrow{\pi_E} E\,,$$

has the structure of a bundle of abelian groups. As explained in the proof of Proposition 3.1.13, the group structure is given by the vertical restrictions $+'_E$ of the addition $+_E$ and $0'_E$ of the zero section 0_E.

The R-module structure is given by the unique map κ'_E, as defined in Equation (4.56) in the proof of Proposition 4.3.16. □

Remark 4.3.19. The commutative diagrams of associativity, unitality, and linearity of the R-module structure κ restrict to commutative diagrams at the level of the vertical tangent bundle. This can be shown by similar arguments using Lemma A.4.14 (i). It follows that the natural morphism $\kappa'_E : R \times VE \to VE$ is an R-module structure on $\pi'_E : VE \to E$ for all differentiable G-bundles E.

4.4 The Lie bracket of invariant vector fields

The goal of this section is to state and prove one of the main results of this thesis: invariant vector fields are closed under the Lie bracket. We will first introduce vertical and invariant vector fields on differentiable G-bundles in the context of tangent categories in Section 4.4.1. After having all the ingredients ready, we state and prove the main theorem in Section 4.4.2.

4.4.1 Invariant vector fields on groupoid bundles

Definition 4.4.1. Let E be a differentiable G-bundle in a category $\mathcal{C}$ with a tangent structure. A vector field $v : E \to TE$ is called **vertical** if it factors through $i_{VE} : VE \to TE$, that is, if there is a (necessarily unique) morphism $v' : E \to VE$ such that

$$\begin{tikzcd} & VE \arrow[d, "i_{VE}"] \\ E \arrow[ur, "v'"] \arrow[r, "v"'] & TE \end{tikzcd} \tag{4.59}$$

commutes. The vector field v is called **invariant** if it is vertical and if v' is G-equivariant.

Remark 4.4.2. Since the groupoid bundles that we consider in this thesis are all right groupoid bundles (Terminology 4.2.3), we do not include the adjective *right* in the definition of invariant vector fields. In the category of smooth manifolds, these are precisely the usual right-invariant vector fields (vector fields that are invariant under all right translations). As a standard reference, see for instance [Lee13, Equation (8.12)] for the identity satisfied by left invariant vector fields.

Lemma 4.4.3. *The vertical lift v' of a vertical vector field v is a section of $\pi'_E : VE \to E$.*

Proof. Using the first commutative triangle in (4.48), we have that

$$\pi'_E \circ v' = \pi_E \circ i_{VE} \circ v' = \pi_E \circ v = \mathrm{id}_E \ .$$

Moreover, v' is a bundle morphism since

$$\begin{aligned} r_E = r_E \circ \mathrm{id}_E &= r_E \circ \pi_E \circ v \\ &= r_E \circ \pi_E \circ i_{VE} \circ v' \\ &= \pi_{G_0} \circ Tr_E \circ i_{VE} \circ v' \\ &= \pi_{G_0} \circ 0_{G_0} \circ r_{VE} \circ v' \\ &= \mathrm{id}_{G_0} \circ r_{VE} \circ v' \\ &= r_{VE} \circ v' , \end{aligned}$$

where $r_E : E \to G_0$ and $r_{VE} : VE \to G_0$ are the bundle projections. In the calculation, we have used the naturality of π, the commutative square defining the pullback VE and that $\pi \circ 0 = 1$. □

Definition 4.4.4. Let E be a G-bundle with right G-action β_E in a category $\mathcal{C}$; let R be an object of $\mathcal{C}$. A morphism $f : E \to R$ in $\mathcal{C}$ is called **invariant** if the diagram

$$\begin{tikzcd} E \times_{G_0} G_1 \arrow[r, "\beta_E"] \arrow[d, "\mathrm{pr}_1"'] & E \arrow[d, "f"] \\ E \arrow[r, "f"'] & R \end{tikzcd}$$

commutes.

We will denote by

$$\mathfrak{X}(E)^G := \{v \in \Gamma(E, TE) \mid v \text{ is invariant}\} \tag{4.60}$$

the set of invariant vector fields on E and by

$$\mathcal{C}(E, R)^G := \{f \in \mathcal{C}(E, R) \mid f \text{ is invariant}\}$$

the set of invariant morphisms from E to R. If R has a ring structure, then $\mathcal{C}(E, R)^G \subset \mathcal{C}(E, R)$ is a subring.

Proposition 4.4.5. *Let E be a differentiable G-bundle in a tangent category $\mathcal{C}$ with an R-module structure. Then, the set $\mathfrak{X}(E)^G$ of invariant vector fields on E is a $\mathcal{C}(E, R)^G$-submodule of $\Gamma(E, TE)$.*

Proof. Let $v, w : E \to TE$ be invariant vector fields with vertical lifts $v', w' : E \to VE$, that is, $v = i_{VE} \circ v'$ and $w = i_{VE} \circ w'$. The addition of the vector fields and the addition of their lifts are defined by

$$\begin{aligned} v + w &= +_E \circ (v, w) \\ v' + w' &= +'_E \circ (v', w') . \end{aligned}$$

Consider the following diagram:

$$\begin{tikzcd} & V_2E \arrow[r, "+'_E"] \arrow[d, "i_{V_2E}"] & VE \arrow[d, "i_{VE}"] \\ E \arrow[ur, "(v',w')"] \arrow[r, "(v,w)"'] & T_2E \arrow[r, "+_E"'] & TE \end{tikzcd}$$

The left triangle commutes by Diagram (4.59). The right square is the third commutative diagram of (4.48). The commutativity of the outer pentagon shows that $v' + w'$ is the vertical lift of $v + w$. The morphisms v' and w' are G-equivariant by assumption. The sum $+'_E$ is G-equivariant by Proposition 4.3.13. We conclude that the composition $v' + w' = +'_E \circ (v', w')$ is G-equivariant, which shows that $v + w$ is invariant.

Composing a vector field with the inverse $\iota_E : TE \to TE$ of the addition $+_E$ yields the inverse with respect to the addition of vector fields, that is,

$$\begin{aligned} v + (\iota_E \circ v) &= +_E \circ (v, \iota_E \circ v) \\ &= +_E \circ (\mathrm{id}_{TE} \times_E \iota_E) \circ \Delta_{TE} \circ v \\ &= 0_E \circ \pi_E \circ v \\ &= 0_E \,, \end{aligned}$$

where $\Delta_{TE} : TE \to TE \times_E TE$ is the diagonal morphism in $\mathcal{C}{\downarrow}E$. Since $\iota_X \circ 0_X = 0_X$ for all objects X in $\mathcal{C}$, so in particular for $X = G_0$ and $X = G_1$, it follows from Lemma 4.3.8 that the inverse has a G-equivariant vertical restriction $\iota'_E : VE \to VE$ such that the diagram

$$\begin{tikzcd} VE \arrow[r, "\iota'_E"] \arrow[d, "i_{VE}"'] & VE \arrow[d, "i_{VE}"] \\ TE \arrow[r, "\iota_E"'] & TE \end{tikzcd}$$

commutes. This implies that $\iota'_E \circ v'$ is a G-equivariant vertical lift of $\iota_E \circ v$, so that $\iota_E \circ v$ is invariant. By Proposition 4.3.13, the zero vector field $0_E : E \to TE$ is invariant with vertical lift $0'_E$. We conclude that $\mathcal{X}(E)^G$ is an abelian subgroup of $\Gamma(E, TE)$.

Now, let $f : E \to R$ be an invariant morphism in $\mathcal{C}$. The $\mathcal{C}(E, R)^G$-module structure on $\Gamma(E, TE)$ is given by

$$fv = \kappa_E \circ (f, v) \,.$$

Consider the following diagram:

$$\begin{tikzcd} & R \times VE \arrow[r, "\kappa'_E"] \arrow[d, "\mathrm{id}_R \times i_{VE}"] & VE \arrow[d, "i_{VE}"] \\ E \arrow[ur, "{(f,v')}"] \arrow[r, "{(f,v)}"'] & R \times TE \arrow[r, "\kappa_E"'] & TE \end{tikzcd}$$

where κ'_E is the natural map from Proposition 4.3.16. The left triangle commutes since v is, by assumption, vertical. The right square is the commutative diagram (4.54). The commutativity of the outer pentagon shows that

$$fv' := \kappa'_E \circ (f, v')$$

is the vertical lift of fv. The morphism κ'_E is G-equivariant by Proposition 4.3.16. By assumption, the morphism f is invariant and the morphism v' is G-equivariant.

This implies that (f, v') is G-equivariant. We conclude that $fv' = \kappa'_E \circ (f, v')$ is G-equivariant, which shows that fv is invariant.

We have shown that $\mathcal{X}(E)^G \subset \Gamma(E, TE)$ is an abelian subgroup that is closed under the $\mathcal{C}(E, R)^G$-module structure. □

4.4.2 Invariant vector fields are closed under the Lie bracket

Theorem 4.4.6. *Let E be a differentiable G-bundle in a tangent category $\mathcal{C}$. Then, the set $\mathcal{X}(E)^G$ of invariant vector fields on E is a Lie subalgebra of $\Gamma(E, TE)$.*

In the proof of Proposition 4.4.5 we have shown that $\mathcal{X}(E)^G \subset \Gamma(E, TE)$ is an abelian subgroup. The rest of this section is devoted to the proof of Theorem 4.4.6, where we show that (vertical) invariant vector fields are closed under the Lie bracket.

Proof that the Lie bracket of vertical vector fields is vertical

Let v and w be vertical, but not necessarily invariant vector fields with vertical lifts $v', w' : E \to VE$. Let $\delta(v, w) : E \to T^2E$ be the morphism defined in Equation (3.60). In a first step, we will show that $\delta(v, w)$ factors through $i_{V^{[2]}E} : V^{[2]}E \to T^2E$. That is, we want to show that there is a morphism $\delta'(v', w')$, such that

$$\begin{tikzcd} & V^{[2]}E \arrow[d, "i_{V^{[2]}E}"] \\ E \arrow[ur, "\delta'(v',w')"] \arrow[r, "\delta(v,w)"'] & T^2E \end{tikzcd} \tag{4.61}$$

commutes.

Let $r_E : E \to G_0$ and $r_{VE} : VE \to G_0$ denote the bundle projections. Firstly, we have that

$$\begin{aligned} Tr_E \circ v &= Tr_E \circ i_{VE} \circ v' \\ &= 0_{G_0} \circ r_{VE} \circ v' \\ &= 0_{G_0} \circ r_E \end{aligned} \tag{4.62}$$

by the commutative square defining the pullback VE and since v' is a bundle map (Lemma 4.4.3). For $\delta(v, w)$ we have the relation

$$\begin{aligned} T^2r_E \circ \delta(v, w) &= T^2r_E \circ -_{TE} \circ (Tw \circ v, \tau_E \circ Tv \circ w) \\ &= -_{TG_0} \circ (T^2r_E \times_{Tr_E} T^2r_E) \circ (Tw \circ v, \tau_E \circ Tv \circ w) \\ &= -_{TG_0} \circ (T^2r_E \circ Tw \circ v, T^2r_E \circ \tau_E \circ Tv \circ w) \\ &= -_{TG_0} \circ (T^2r_E \circ Tw \circ v, \tau_{G_0} \circ T^2r_E \circ Tv \circ w) \\ &= -_{TG_0} \circ (T0_{G_0} \circ Tr_E \circ v, \tau_{G_0} \circ T0_{G_0} \circ Tr_E \circ w) \\ &= -_{TG_0} \circ (T0_{G_0} \circ 0_{G_0} \circ r_E, \tau_{G_0} \circ T0_{G_0} \circ 0_{G_0} \circ r_E) \\ &= -_{TG_0} \circ (0^{[2]}_{G_0} \circ r_E, 0^{[2]}_{G_0} \circ r_E) \\ &= 0^{[2]}_{G_0} \circ r_E\,, \end{aligned}$$

where we have used the Definition (3.60) of $\delta(v,w)$, the naturality of the subtraction $-_X$, the naturality of τ_X, Equation (4.62) twice, Equation (4.49), and finally that the subtraction of zero by zero is zero, as is the case for any group object.

Using this relation, we conclude from the universal property of the pullback defining $V^{[2]}E$, that there is a unique morphism $\delta'(v',w')$, such that

$$\begin{tikzcd}
E \arrow[dr, dashed, "\exists!\,\delta'(v',w')"] \arrow[drr, bend left, "r_E"] \arrow[ddr, bend right, "\delta(v,w)"'] & & \\
& V^{[2]}E \arrow[r, "r_{V^{[2]}E}"] \arrow[d, "i_{V^{[2]}E}"] & G_0 \arrow[d, "0^{[2]}_{G_0}"] \\
& T^2E \arrow[r, "T^2 r_E"'] & T^2G_0
\end{tikzcd}$$

commutes. Consider the relation

$$\begin{aligned}
i_{VE} \circ \pi'^{[1]}_E \circ \delta'(v',w') &= T\pi_E \circ i_{V^{[2]}E} \circ \delta'(v',w') \\
&= T\pi_E \circ \delta(v,w) \\
&= 0_E \\
&= i_{VE} \circ 0'_E \,,
\end{aligned}$$

where we have used the relation defining the vertical prolongation of π'_E, the Relation (4.61) defining $\delta'(v',w')$, and Equation (3.64). Since i_{VE} is a monomorphism, it follows that

$$\pi'^{[1]}_E \circ \delta'(v',w') = 0'_E \,.$$

We can summarize the situation in the following commutative diagram:

$$\begin{tikzcd}
E \arrow[r, "{[v',w']}"] \arrow[dr, dashed, "\exists!\,\phi"'] \arrow[drr, bend left, "\delta'(v',w')"] \arrow[ddr, bend right, "\mathrm{id}_E"'] & VE & \\
& V_2E \arrow[u, "\mathrm{pr}_2"'] \arrow[r, "\lambda'_{2,E}"] \arrow[d, "\pi'_E \circ \mathrm{pr}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & V^{[2]}E \arrow[d, "\pi'^{[1]}_E"] \\
& E \arrow[r, "0'_E"'] & VE
\end{tikzcd} \tag{4.63}$$

We have shown in Proposition 4.3.15 that the bottom right square is a pullback. The unique dashed arrow exists due to the universal property of the pullback. It remains to show that its projection onto the second factor of V_2E, which we will denote by $[v',w']$, is the vertical lift of $[v,w]$. For this, we consider the following diagram:

$$
\begin{tikzcd}
E \arrow[rrr, dashed, "\exists!"] \arrow[rrrr, bend left, "{[v',w']}"] \arrow[dr, "\mathrm{id}"] \arrow[ddd, "\mathrm{id}"'] & & & V_2E \arrow[r, "\mathrm{pr}_2"'] \arrow[dl, "\lambda'_{2,E}"] \arrow[ddd, "i_{V_2E}"] & VE \arrow[ddd, "i_{VE}"] \\
& E \arrow[r, "\delta'(v',w')"] \arrow[d, "\mathrm{id}"'] & V^{[2]}E \arrow[d, "i_{V^{[2]}E}"] & & \\
& E \arrow[r, "\delta(v,w)"'] & T^2E & & \\
E \arrow[ur, "\mathrm{id}"] \arrow[rrr, dashed, "\exists!"'] \arrow[rrrr, bend right, "{[v,w]}"'] & & & T_2E \arrow[r, "\mathrm{pr}_2"] \arrow[ul, "\lambda_{2,E}"'] & TE
\end{tikzcd}
\tag{4.64}
$$

The inner square is the commutative triangle (4.61). The right trapezoid is the last commutative square in (4.48). The commutativity of the upper and lower trapezoids is explained in the construction of the bracket as summarized in Diagrams (3.65) and (4.63). The left trapezoid commutes trivially. Since monomorphisms are stable under pullbacks and the zero section is a monomorphism, it follows from Diagram (3.37) that the map $\lambda_{2,E}$ is a monomorphism. Thus, the left outer square commutes by Lemma A.4.14 (i). The right square commutes by the naturality of the projection. From the commutativity of the outer rectangle it follows that

$$
i_{VE} \circ [v', w'] = [v, w] = [i_{VE} \circ v', i_{VE} \circ w'] \,. \tag{4.65}
$$

This shows that $[v, w]$ is vertical.

Proof that the Lie bracket of invariant vector fields is invariant

Assume now that the vertical lifts $v', w' : E \to VE$ are G-equivariant. Consider the following diagram:

$$
\begin{tikzcd}
E \times_{G_0} G_1 \arrow[r, "v' \times_{G_0} \mathrm{id}_{G_1}"] \arrow[d, "\beta_E"'] & VE \times_{G_0} G_1 \arrow[r, "w'^{[1]} \times_{G_0} \mathrm{id}_{G_1}"] \arrow[d, "\beta_{VE}"] & V^{[2]}E \times_{G_0} G_1 \arrow[d, "\beta_{V^{[2]}E}"] \\
E \arrow[r, "v'"'] \arrow[d, "\mathrm{id}"'] & VE \arrow[r, "w'^{[1]}"'] \arrow[d, "i_{VE}"] & V^{[2]}E \arrow[d, "i_{V^{[2]}E}"] \\
E \arrow[r, "v"'] & TE \arrow[r, "Tw"'] & T^2E
\end{tikzcd}
\tag{4.66}
$$

The upper left square commutes since v' is G-equivariant. The lower left square commutes because v' is the vertical lift of v. The upper and lower right squares commute by Proposition 4.3.11, where $w'^{[1]} : VE \to V^{[2]}E$ is the first vertical prolongation of

w'. We conclude that the outer square commutes, so that we obtain

$$\begin{aligned} Tw \circ v \circ \beta_E &= i_{V^{[2]}E} \circ \beta_{V^{[2]}E} \circ \big((w'^{[1]} \circ v') \times_{G_0} \mathrm{id}_{G_1}\big) \\ &= \beta_{T^2E} \circ \big(i_{V^{[2]}E} \times_{0^{[2]}_{G_0}} 0^{[2]}_{G_1}\big) \circ \big((w'^{[1]} \circ v') \times_{G_0} \mathrm{id}_{G_1}\big) \\ &= \beta_{T^2E} \circ \big((i_{V^{[2]}E} \circ w'^{[1]} \circ v') \times_{0^{[2]}_{G_0}} 0^{[2]}_{G_1}\big) \\ &= \beta_{T^2E} \circ \big((Tw \circ v) \times_{0^{[2]}_{G_0}} 0^{[2]}_{G_1}\big)\,, \end{aligned} \tag{4.67}$$

where we have used the Diagram (4.36) defining the vertical G-action, the functoriality of pullbacks, and the bottom rectangle of Diagram (4.66). The diagram

$$\begin{tikzcd} T^2E \times_{T^2G_0} T^2G_1 \arrow[r, "\tau_E \times_{\tau_{G_0}} \tau_{G_1}"] \arrow[d, "\beta_{T^2E}"'] & T^2E \times_{T^2G_0} T^2G_1 \arrow[d, "\beta_{T^2E}"] \\ T^2E \arrow[r, "\tau_E"'] & T^2E \end{tikzcd} \tag{4.68}$$

is the inner commutative square of (4.42) for $\alpha = \tau$. Similarly, for the subtraction $\alpha = - : T_2 \to T$, we obtain the commutative diagram

$$\begin{tikzcd} (T^2E \times_{T^2G_0} T^2G_1) \times_{TE \times_{TG_0} TG_1} (T^2E \times_{T^2G_0} T^2G_1) \arrow[r, "\cong"] \arrow[dd, "\beta_{T^2E} \times_{\beta_{TE}} \beta_{T^2E}"'] & T_2TE \times_{T_2TG_0} T_2TG_1 \arrow[d, "-_{TE} \times_{-_{TG_0}} -_{TG_1}"] \\ & T^2E \times_{T^2G_0} T^2G_1 \arrow[d, "\beta_{T^2E}"] \\ T_2TE \arrow[r, "-_{TE}"'] & T^2E \end{tikzcd} \tag{4.69}$$

where the isomorphism is a simple reordering of the factors since pullbacks commute with pullbacks. Using (4.67) with v and w swapped and (4.68), we get

$$\begin{aligned} \tau_E \circ Tv \circ w \circ \beta_E &= \tau_E \circ \beta_{T^2E} \circ \big((Tv \circ w) \times_{0^{[2]}_{G_0}} 0^{[2]}_{G_1}\big) \\ &= \beta_{T^2E} \circ \big(\tau_E \times_{\tau_{G_0}} \tau_{G_1}\big) \circ \big((Tv \circ w) \times_{0^{[2]}_{G_0}} 0^{[2]}_{G_1}\big) \\ &= \beta_{T^2E} \circ \big((\tau_E \circ Tv \circ w) \times_{0^{[2]}_{G_0}} 0^{[2]}_{G_1}\big)\,, \end{aligned} \tag{4.70}$$

where in the last step we have used that τ takes the zero section to the zero section as expressed by Equation (4.49).

Now we can subtract (4.67) and (4.70), which yields

$$
\begin{aligned}
&\delta(v,w)\circ\beta_E \\
&= -_{TE}\circ(Tw\circ v,\ \tau_E\circ Tv\circ w)\circ\beta_E \\
&= -_{TE}\circ(Tw\circ v\circ\beta_E,\ \tau_E\circ Tv\circ w\circ\beta_E) \\
&= -_{TE}\circ\big\{\beta_{T^2E}\circ\big((Tw\circ v)\times_{0^{[2]}_{G_0}}0^{[2]}_{G_1}\big),\ \beta_{T^2E}\circ\big((\tau_E\circ Tv\circ w)\times_{0^{[2]}_{G_0}}0^{[2]}_{G_1}\big)\big\} \\
&= -_{TE}\circ(\beta_{T^2E}\times_{\beta_{TE}}\beta_{T^2E})\circ\big((Tw\circ v)\times_{0^{[2]}_{G_0}}0^{[2]}_{G_1},\ (\tau_E\circ Tv\circ w)\times_{0^{[2]}_{G_0}}0^{[2]}_{G_1}\big) \\
&= \beta_{T^2E}\circ(-_{TE}\times_{-_{TG_0}}-_{TG_1})\circ\big((Tw\circ v,\ \tau_E\circ Tv\circ w)\times_{\left(0^{[2]}_{G_0},0^{[2]}_{G_0}\right)}(0^{[2]}_{G_1},0^{[2]}_{G_1})\big) \\
&= \beta_{T^2E}\circ\big(\delta(v,w)\times_{0^{[2]}_{G_0}}0^{[2]}_{G_1}\big)\,,
\end{aligned}
$$

where we have used Equations (3.60), (4.67), (4.70), Diagram (4.69), and that the subtraction of zero by zero is zero. This relation means that the inner square of the following diagram commutes:

$$
\begin{tikzcd}
E\times_{G_0}G_1 \arrow[rrr, "\delta'(v',w')\times_{G_0}\mathrm{id}"] \arrow[ddd, "\beta_E"'] \arrow[dr, "\mathrm{id}"] & & & V^{[2]}E\times_{G_0}G_1 \arrow[dl, "i_{V^{[2]}E}\times_{0^{[2]}_{G_0}}0^{[2]}_{G_1}"'] \arrow[ddd, "\beta_{V^{[2]}E}"] \\
& E\times_{G_0}G_1 \arrow[r, "\delta(v,w)\times_{0^{[2]}_{G_0}}0^{[2]}_{G_1}"] \arrow[d, "\beta_E"'] & T^2E\times_{T^2G_0}T^2G_1 \arrow[d, "\beta_{T^2E}"] & \\
& E \arrow[r, "\delta(v,w)"'] & T^2E & \\
E \arrow[ur, "\mathrm{id}"] \arrow[rrr, "\delta'(v',w')"'] & & & V^{[2]}E \arrow[ul, "i_{V^{[2]}E}"']
\end{tikzcd}
$$

The left trapezoid commutes trivially. The bottom trapezoid commutes by (4.61). The top trapezoid commutes by (4.61) and the functoriality of pullbacks. The right trapezoid commutes by (4.36). Since $i_{V^{[2]}E}$ is a monomorphism, it follows from Lemma A.4.14 (i) that the outer square commutes, which shows that $\delta'(v',w')$ is G-equivariant.

By Proposition 4.3.15, the pullback square in Diagram (4.63) is a pullback in the category $\mathcal{B}un_G$ of G-bundles and G-equivariant bundle morphisms. Since $\delta'(v',w')$ is a G-equivariant bundle morphism, it follows by the universal property of the pullback that the morphism $\phi : E\to V_2E$ depicted by the dashed arrow in (4.63) is G-equivariant. Since $\mathrm{pr}_2 : V_2E\to VE$ is also G-equivariant, so is the composition $[v',w'] = \mathrm{pr}_2\circ\phi$. We conclude that $[v',w']$ is G-equivariant, which shows that $[v,w]$ is invariant.

We have shown that the abelian subgroup $\mathfrak{X}(E)^G\subset\Gamma(E,TE)$ is closed under the Lie bracket, which finishes the proof of Theorem 4.4.6. $\square$

4.5 The abstract Lie algebroid of a differentiable groupoid

This section is the core of this chapter, where we describe the construction of the infinitesimal counterpart of a differentiable groupoid object. The first step is to generalize the notion of Lie algebroids in the category of smooth manifolds to the setting of tangent categories. This is the goal of Section 4.5.1. In Section 4.5.2, we show the steps of the differentiation procedure, and finally state and prove the main theorems of this chapter.

4.5.1 Abstract Lie algebroids in a tangent category

Definition 4.5.1. Let $\mathcal{C}$ be a cartesian tangent category with scalar R-multiplication. An **abstract Lie algebroid** in $\mathcal{C}$ consists of a bundle of R-modules $A \to X$, a morphism $\rho : A \to TX$ of bundles of R-modules, called the **anchor**, and a Lie bracket $[\cdot\,,\cdot]$ on the abelian group $\Gamma(X, A)$, such that

$$[a, fb] = f[a, b] + \big((\rho \circ a) \cdot f\big)b \tag{4.71}$$

$$\rho \circ [a, b] = [\rho \circ a, \rho \circ b] \tag{4.72}$$

for all sections a, b of A and all morphisms $f : X \to R$ in $\mathcal{C}$.

Remark 4.5.2. Recall that the $\mathcal{C}(X, R)$-module structure fa is defined by (3.19). Moreover, the action of vector fields on R-valued morphisms is defined by (3.70). Equation (4.71) is the Leibniz rule for Lie algebroids. By an analogous argument as in Corollary 3.3.9, we have that the Lie bracket on $\Gamma(X, A)$ is $\mathcal{C}(*, R)$-linear. Observe that $\mathcal{C}(*, R)$ is a ring in sets that can generally not be identified with the ring object R in $\mathcal{C}$. In the category of manifolds where $R = \mathbb{R}$, we retrieve the usual $\mathbb{R}$-linearity of the Lie bracket.

Remark 4.5.3. In the definition of a usual Lie algebroid (Def. 1.2.1), the condition (4.72) that ρ is a morphism of Lie algebras is redundant, since it can be proved using the Jacobi identity, bilinearity and antisymmetry of the Lie bracket, and the Leibniz rule (Rem. 1.2.3). However, this proof relies on the identification of vector fields as derivations of the ring of functions on X and the definition of the Lie bracket as the commutator[2]. In our case, vector fields are sections of the tangent bundle and the Lie bracket is defined by Diagram (3.65). In the smooth manifold setting, these coincide.

Example 4.5.4. Lie algebroids are abstract Lie algebroids in the category of smooth manifolds.

Remark 4.5.5. In [BM19], Burke and MacAdam introduce the notion of *involution algebroids* as generalizations of Lie algebroids in tangent categories. The underlying bundles of the involution algebroids are differential bundles (see Remark 3.2.7). This presents a crucial difference to the abstract Lie algebroids that we introduce in

[2] A detailed proof can be found in [Ain17, Remark 1.3.2].

Definition 4.5.1, where the underlying bundle is a bundle of R-modules. Another key difference is that in an involution algebroid, the Lie bracket of sections is replaced with an involution instead. As our goal is to generalize the differentiation of Lie groupoids to their Lie algebroids by the identification of the sections with invariant vector fields, we need a Lie algebra structure on the sections of the abstract Lie algebroids.

4.5.2 The differentiation procedure

Let G be a groupoid in a tangent category $\mathcal{C}$. The source bundle $s : G_1 \to G_0$ is equipped with the right G-action given by groupoid multiplication. If G is differentiable in the sense of Definition 4.1.6, then this bundle is differentiable in the sense of Definition 4.2.11. Its vertical tangent bundle will be denoted by

$$VG_1 = TG_1 \times_{TG_0}^{Ts, 0_{G_0}} G_0 \xrightarrow{r_{VG_1}} G_0 \,.$$

It follows from Proposition 4.3.1 that VG_1, equipped with the right G-action defined in (4.27) for $n = 1$, is a G-bundle. Recall from Definition 4.4.1 that a vector field $v : G_1 \to TG_1$ is called invariant if it factors through a morphism $v' : G_1 \to VG_1$ that is G-equivariant with respect to the right G-actions.

Proposition 4.5.6. *Let G be a differentiable groupoid in a cartesian tangent category $\mathcal{C}$ with scalar R-multiplication $\kappa : R \times T \to T$. Then, the set $\mathfrak{X}(G_1)^G$ of invariant vector fields is naturally a $\mathcal{C}(G_0, R)$-module with the addition of vector fields and the module structure*

$$fv := (t^*f)v = \kappa_{G_1} \circ (t^*f, v) \tag{4.73}$$

for all $v \in \mathfrak{X}(G_1)^G$ and $f \in \mathcal{C}(G_0, R)$.

Proof. It follows from Proposition 4.4.5 that $\mathfrak{X}(G_1)^G$ is a $\mathcal{C}(G_1, R)^G$-submodule of $\Gamma(G_1, TG_1)$. The pullback

$$\begin{aligned} t^* : \mathcal{C}(G_0, R) &\longrightarrow \mathcal{C}(G_1, R) \\ f &\longmapsto f \circ t \,, \end{aligned}$$

is a ring homomorphism by Lemma A.4.35. Its image is the ring $\mathcal{C}(G_1, R)^G$ of invariant morphisms by Axiom (1.1) of groupoids. Then, the assignment $fv = (t^*f)v$ equips $\mathfrak{X}(G_1)^G$ with a $\mathcal{C}(G_0, R)$-module structure by Lemma A.4.33. $\square$

In the Definition 4.1.6 of differentiable groupoids, we have assumed that the pullback (4.9) exists, which can be written as the restriction of the vertical bundle to the identity bisection:

$$\begin{tikzcd} A := G_0 \times_{G_1} VG_1 \arrow[r, "i_A"] \arrow[d, "p_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & VG_1 \arrow[d, "\pi'_{G_1}"] \\ G_0 \arrow[r, "1"'] & G_1 \end{tikzcd} \tag{4.74}$$

The vertical tangent bundle $\pi'_{G_1} : VG_1 \to G_1$ is a bundle of R-modules by Corollary 4.3.18. It follows from Proposition 3.1.16 that $p_A : A \to G_0$ is a bundle of R-modules. The composition

$$\rho : A \xrightarrow{i_A} VG_1 \xrightarrow{i_{VG_1}} TG_1 \xrightarrow{Tt} TG_0\,, \tag{4.75}$$

is a morphism of bundles of R-modules, since all three arrows are. Indeed, i_A and i_{VG_1} are morphisms of bundles of R-modules by Propositions 3.1.16 and 3.1.17(ii) respectively. So is the map Tt by the naturality of the tangent structure.

Theorem 4.5.7. *Let G be a differentiable groupoid in a cartesian tangent category $\mathcal{C}$ with scalar R-multiplication. Let $A \to G_0$ be as defined in* (4.74)*. Then there is a natural isomorphism of $\mathcal{C}(G_0, R)$-modules*

$$\phi : \Gamma(G_0, A) \xrightarrow{\cong} \mathfrak{X}(G_1)^G\,.$$

Theorem 4.5.8. *Let G be a differentiable groupoid in a cartesian tangent category $\mathcal{C}$ with scalar R-multiplication. Then the bundle $A \to G_0$ defined in* (4.74)*, with the anchor $\rho : A \to TG_0$ defined in* (4.75) *and the Lie bracket of invariant vector fields on $\mathfrak{X}(G_1)^G \cong \Gamma(G_0, A)$ is an abstract Lie algebroid.*

The rest of this section is devoted to the proofs of Theorems 4.5.7 and 4.5.8.

Proof of Theorem 4.5.7

Let $\Gamma(G_1, VG_1)^G$ denote the set of G-equivariant sections of $\pi'_{G_1} : VG_1 \to G_1$. By the definition of invariant vector fields, the map

$$\begin{aligned}(i_{VG_1})_* : \Gamma(G_1, VG_1)^G &\longrightarrow \mathfrak{X}(G_1)^G \\ v' &\longmapsto i_{VG_1} \circ v'\end{aligned} \tag{4.76}$$

is surjective. Since i_{VG_1} is a monomorphism, the map (4.76) is also injective. By Proposition 4.5.6, $\mathfrak{X}(G_1)^G$ has a $\mathcal{C}(G_0, R)$-module structure given by (4.73). Similarly, $\Gamma(G_1, VG_1)^G$ has a $\mathcal{C}(G_0, R)$-module structure as follows. Let $f : G_0 \to R$ be a morphism in $\mathcal{C}$ and let $v' : G_1 \to VG_1$ be a G-equivariant section. Consider the composition

$$fv' : G_1 \xrightarrow{(t^*f,\, v')} R \times VG_1 \xrightarrow{\kappa'_{G_1}} VG_1\,.$$

The morphism κ'_{G_1} is G-equivariant by Proposition 4.3.16. By assumption, the section v' is also G-equivariant. Moreover, the morphism t^*f is invariant by the commutative diagram on the right of (1.1). This implies that $fv' = \kappa'_{G_1} \circ (t^*f, v')$ is G-equivariant. It follows from the R-linearity of i_{VG_1}, as expressed by Diagram (4.54), that

$$\begin{aligned}(i_{VG_1})_*(fv') &= i_{VG_1} \circ \kappa'_{G_1} \circ (t^*f, v') \\ &= \kappa_{G_1} \circ (\mathrm{id}_R \times i_{VG_1}) \circ (t^*f, v') \\ &= \kappa_{G_1} \circ (t^*f, i_{VG_1} \circ v') \\ &= f\big((i_{VG_1})_*(v')\big)\end{aligned}$$

We conclude that (4.76) is an isomorphism of $\mathcal{C}(G_0, R)$-modules. It remains to show that there is an isomorphism of $\mathcal{C}(G_0, R)$-modules between $\Gamma(G_0, A)$ and $\Gamma(G_1, VG_1)^G$.

Let a be a section of A. Consider the following morphism:

$$\phi'(a) : G_1 \xrightarrow{(a \circ t, \mathrm{id}_{G_1})} A \times_{G_0}^{p_A, t} G_1 \xrightarrow{i_A \times_{G_0} \mathrm{id}_{G_1}} VG_1 \times_{G_0}^{r_{VG_1}, t} G_1 \xrightarrow{\beta_{VG_1}} VG_1 .$$

In formulas,

$$\phi'(a) = \beta_{VG_1} \circ (i_A \circ a \circ t, \mathrm{id}_{G_1}) . \tag{4.77}$$

First, we show that $\phi'(a)$ is a section of the vertical tangent bundle by a straightforward calculation,

$$\begin{aligned}
\pi'_{G_1} \circ \phi'(a) &= \pi'_{G_1} \circ \beta_{VG_1} \circ (i_A \circ a \circ t, \mathrm{id}_{G_1}) \\
&= \pi_{G_1} \circ i_{VG_1} \circ \beta_{VG_1} \circ (i_A \circ a \circ t, \mathrm{id}_{G_1}) \\
&= \pi_{G_1} \circ \beta_{TG_1} \circ (i_{VG_1} \times_{0_{G_0}} 0_{G_1}) \circ (i_A \circ a \circ t, \mathrm{id}_{G_1}) \\
&= m \circ (\pi_{G_1} \times_{\pi_{G_0}} \pi_{G_1}) \circ (i_{VG_1} \times_{0_{G_0}} 0_{G_1}) \circ (i_A \circ a \circ t, \mathrm{id}_{G_1}) \\
&= m \circ ((\pi_{G_1} \circ i_{VG_1}) \times_{G_0} \mathrm{id}_{G_1}) \circ (i_A \circ a \circ t, \mathrm{id}_{G_1}) \\
&= m \circ (\pi'_{G_1} \circ i_A \circ a \circ t, \mathrm{id}_{G_1}) \\
&= m \circ (1 \circ p_A \circ a \circ t, \mathrm{id}_{G_1}) \\
&= m \circ (1 \circ t, \mathrm{id}_{G_1}) \\
&= \mathrm{id}_{G_1} ,
\end{aligned}$$

where we have used Equation (4.77), the first diagram in (4.48), Diagram (4.36), the inner square of (4.42) (for β_{G_1} given by the group multiplication m and $\alpha = \pi$), the functoriality of pullbacks, that $\pi \circ 0 = 1$, Diagram (4.74), that a is a section of $p_A : A \to G_0$, and the unitality axiom (1.4).

To show the G-equivariance of $\phi'(a)$, we consider the following diagram:

$$\begin{tikzcd}
G_1 \times_{G_0} G_1 \arrow[r, "m"] \arrow[d, "(t,\mathrm{id}_{G_1}) \times_{G_0} \mathrm{id}_{G_1}"', "\cong"] & G_1 \arrow[d, "(t,\mathrm{id}_{G_1})", "\cong"'] \\
G_0 \times_{G_0}^{\mathrm{id},t} G_1 \times_{G_0} G_1 \arrow[r, "\mathrm{id}_{G_0} \times_{G_0} m"] \arrow[d, "a \times_{G_0} \mathrm{id}_{G_2}"'] & G_0 \times_{G_0}^{\mathrm{id},t} G_1 \arrow[d, "a \times_{G_0} \mathrm{id}_{G_1}"] \\
A \times_{G_0} G_1 \times_{G_0} G_1 \arrow[r, "\mathrm{id}_A \times_{G_0} m"] \arrow[d, "i_A \times_{G_0} \mathrm{id}_{G_2}"'] & A \times_{G_0} G_1 \arrow[d, "i_A \times_{G_0} \mathrm{id}_{G_1}"] \\
VG_1 \times_{G_0} G_1 \times_{G_0} G_1 \arrow[r, "\mathrm{id}_{VG_1} \times_{G_0} m"] \arrow[d, "\beta_{VG_1} \times_{G_0} \mathrm{id}_{G_1}"'] & VG_1 \times_{G_0} G_1 \arrow[d, "\beta_{VG_1}"] \\
VG_1 \times_{G_0} G_1 \arrow[r, "\beta_{VG_1}"'] & VG_1
\end{tikzcd} \tag{4.78}$$

The commutativity of the top square follows from the pasting law of pullbacks and by the right diagram in (1.1). The second square commutes due to the functoriality of the fiber product,

$$\begin{aligned}(a \times_{G_0} \mathrm{id}_{G_1}) \circ (\mathrm{id}_{G_0} \times_{G_0} m) &= (a \circ \mathrm{id}_{G_0}) \times_{G_0} (\mathrm{id}_{G_1} \circ m) \\ &= (\mathrm{id}_A \circ a) \times_{G_0} (m \circ \mathrm{id}_{G_2}) \\ &= (\mathrm{id}_A \times_{G_0} m) \circ (a \times_{G_0} \mathrm{id}_{G_2}) \,.\end{aligned}$$

The commutativity of the third square follows from an analogous argument, where we replace a with i_A. The commutativity of the bottom square expresses the associativity of the right action β_{VG_1}. It follows that the outer rectangle of Diagram (4.78) commutes. The composition of the right vertical arrows is $\phi'(a)$. The composition of the left vertical arrows is $\phi'(a) \times_{G_0} \mathrm{id}_{G_1}$. This shows that $\phi'(a)$ is G-equivariant.

So far, we have constructed a map

$$\phi' : \Gamma(G_0, A) \longrightarrow \Gamma(G_1, VG_1)^G \,.$$

Next, we construct a map in the opposite direction. Given a section $v' \in \Gamma(G_1, VG_1)$, we get a section of A by restriction to the identity bisection. More precisely, there is a unique morphism $\psi'(v') : G_0 \to A$, such that

$$i_A \circ \psi'(v') = v' \circ 1 \,. \tag{4.79}$$

It follows from this equation that

$$\begin{aligned}p_A \circ \psi'(v') &= s \circ 1 \circ p_A \circ \psi'(v') \\ &= s \circ \pi'_{G_1} \circ i_A \circ \psi'(v') \\ &= s \circ \pi'_{G_1} \circ v' \circ 1 \\ &= s \circ 1 \\ &= \mathrm{id}_{G_0} \,,\end{aligned}$$

where we have used that $s \circ 1 = \mathrm{id}_{G_0}$, Diagram (4.74), and that v' is a section of $\pi'_{G_1} : VG_1 \to G_1$. It follows that $\psi'(v')$ is a section of $p_A : A \to G_0$. By restricting to the G-equivariant sections, we obtain a map

$$\psi' : \Gamma(G_1, VG_1)^G \longrightarrow \Gamma(G_0, A) \,.$$

In the next step, we will show that ϕ' and ψ' are mutually inverse. We have

$$\begin{aligned}\phi'(a) \circ 1 &= \beta_{VG_1} \circ (i_A \circ a \circ t, \mathrm{id}_{G_1}) \circ 1 \\ &= \beta_{VG_1} \circ (i_A \circ a, 1) \\ &= i_A \circ a \,,\end{aligned}$$

where we have used the defining Equation (4.77) for $\phi'(a)$, that $t \circ 1 = \mathrm{id}_{G_0}$, and that the right action is unital. Comparing this equation with (4.79) for $v' = \phi'(a)$, we conclude that $a = \psi'\big(\phi'(a)\big)$. Conversely,

$$\begin{aligned}\phi'\big(\psi'(v')\big) &= \beta_{VG_1} \circ \big(i_A \circ \psi'(v') \circ t, \mathrm{id}_{G_1}\big) \\ &= \beta_{VG_1} \circ \big(v' \circ 1 \circ t, \mathrm{id}_{G_1}\big) \\ &= \beta_{VG_1} \circ (v' \times_{G_0} \mathrm{id}_{G_1}) \circ (1 \circ t, \mathrm{id}_{G_1}) \\ &= v' \circ m \circ (1 \circ t, \mathrm{id}_{G_1}) \\ &= v' \,,\end{aligned}$$

where we have used (4.77), (4.79), that v' is G-equivariant and Diagram (1.4). We conclude that ϕ' and ψ' are inverse to each other.

Finally, we show that the map ψ' is $\mathcal{C}(G_0, R)$-linear. Denote by $\kappa_A : R \times A \to A$ the R-module structure on A. For $f \in \mathcal{C}(G_0, R)$ and $v' \in \Gamma(G_1, VG_1)^G$ we have that

$$\begin{aligned} i_A \circ f\psi'(v') &= i_A \circ \kappa_A \circ \big(f, \psi'(v')\big) \\ &= \kappa'_{G_1} \circ (\mathrm{id}_R \times i_A) \circ \big(f, \psi'(v')\big) \\ &= \kappa'_{G_1} \circ \big(f, i_A \circ \psi'(v')\big) \\ &= \kappa'_{G_1} \circ \big(f, v' \circ 1\big) \\ &= \kappa'_{G_1} \circ \big(f \circ t, v'\big) \circ 1 \\ &= \kappa'_{G_1} \circ \big(t^* f, v'\big) \circ 1 \\ &= (t^* f) v' \circ 1 \,, \end{aligned}$$

where we have used the definition of the module structure on $\Gamma(G_0, A)$, the R-linearity of the inclusion i_A, Equation (4.79), that $t \circ 1 = \mathrm{id}_{G_0}$ and the module structure (4.73) on $\Gamma(G_1, VG_1)^G$. Comparing this equation with (4.79), we conclude that

$$\psi'\big((t^* f) v'\big) = f\psi'(v') \,,$$

which shows that ψ' is $\mathcal{C}(G_0, R)$-linear. It follows from Lemma A.4.36 that its inverse ϕ' is also $\mathcal{C}(G_0, R)$-linear.

Composing ϕ' with Isomorphism (4.76), we obtain the isomorphism of $\mathcal{C}(G_0, R)$-modules

$$\phi = (i_{VG_1})_* \circ \phi' : \Gamma(G_0, A) \xrightarrow{\cong} \mathfrak{X}(G_1)^G \,, \tag{4.80}$$

which concludes the proof. $\square$

Proof of Theorem 4.5.8

It follows from Theorem 4.4.6 that the Lie bracket of vector fields on G_1 restricts to a Lie bracket on the abelian group $\mathfrak{X}(G_1)^G$ of invariant vector fields. Let a and b be sections of $A \to G_0$; let $\phi(a)$ and $\phi(b)$ be the corresponding invariant vector fields given by Isomorphism (4.80). The Lie bracket $[a, b]$ is defined by

$$\phi\big([a, b]\big) = [\phi(a), \phi(b)] \,. \tag{4.81}$$

The target map t of the groupoid is invariant under the right G-multiplication, as given in the right of Diagram (1.1). This implies that Tt is invariant under the right TG-action, so that the bottom square of the following diagram

$$\begin{tikzcd} VG_1 \times_{G_0} G_1 \arrow[r, "\beta_{VG_1}"] \arrow[d, "i_{VG_1} \times_{0_{G_0}} 0_{G_1}"] \arrow[dd, bend right=60, "i_{VG_1} \circ \mathrm{pr}_1"'] & VG_1 \arrow[d, "i_{VG_1}"] \\ TG_1 \times_{TG_0} TG_1 \arrow[r, "\beta_{TG_1}"'] \arrow[d, "\mathrm{pr}_1"] & TG_1 \arrow[d, "Tt"] \\ TG_1 \arrow[r, "Tt"'] & TG_0 \end{tikzcd}$$

commutes. The top square commutes by the definition of the G-action on the vertical tangent bundle (Diagram 4.36 for $n = 1$ and $E = G_1$). It follows that the outer square is equivalent, so that we obtain the relation

$$Tt \circ i_{VG_1} \circ \beta_{VG_1} = Tt \circ i_{VG_1} \circ \mathrm{pr}_1 \,. \tag{4.82}$$

To prove Equation (4.72), we first calculate that

$$\begin{aligned} Tt \circ \phi(a) &= Tt \circ i_{VG_1} \circ \phi'(a) \\ &= Tt \circ i_{VG_1} \circ \beta_{VG_1} \circ (i_A \circ a \circ t, \mathrm{id}_{G_1}) \\ &= Tt \circ i_{VG_1} \circ \mathrm{pr}_1 \circ (i_A \circ a \circ t, \mathrm{id}_{G_1}) \\ &= Tt \circ i_{VG_1} \circ i_A \circ a \circ t \\ &= (\rho \circ a) \circ t \,, \end{aligned} \tag{4.83}$$

where we have used the Isomorphism (4.80), Equations (4.77) and (4.82), and the defining Equation (4.75) of the anchor. This shows that the vector field $\phi(a)$ on G_1 is t-related to the vector field $\rho \circ a$ on G_0. It follows from Proposition 3.3.11 that $[\phi(a), \phi(b)]$ is t-related to $[\rho \circ a, \rho \circ b]$, so that

$$\begin{aligned} \rho \circ [a, b] \circ t = Tt \circ \phi([a, b]) &= Tt \circ [\phi(a), \phi(b)] \\ &= [\rho \circ a, \rho \circ b] \circ t \,, \end{aligned}$$

where we have used (4.83) for the section $[a, b]$ and Equation (4.81). Since t is an epimorphism, we can cancel t on both sides of this equation, which yields Equation (4.72).

Let $f \in \mathcal{C}(G_0, R)$. By definition, the function $t^*f : G_1 \to R$ is t-related to $f : G_0 \to R$. From (4.83), we know that the vector field $\phi(a)$ on G_1 is t-related to the vector field $\rho \circ a$ on G_0. It follows from Proposition 3.3.13 (ii) that the function $\phi(a) \cdot (t^*f)$ is t-related to $(\rho \circ a) \cdot f$, that is,

$$\begin{aligned} \phi(a) \cdot (t^*f) &= \big((\rho \circ a) \cdot f\big) \circ t \\ &= t^*\big((\rho \circ a) \cdot f\big) \,. \end{aligned} \tag{4.84}$$

To prove the Leibniz rule, we calculate

$$\begin{aligned} \phi\big([a, fb]\big) &= [\phi(a), \phi(fb)] \\ &= [\phi(a), (t^*f)\phi(b)] \\ &= \big(\phi(a) \cdot (t^*f)\big)\phi(b) + (t^*f)[\phi(a), \phi(b)] \\ &= \big(t^*((\rho \circ a) \cdot f)\big)\phi(b) + (t^*f)[\phi(a), \phi(b)] \\ &= \phi\big(((\rho \circ a) \cdot f)b\big) + (t^*f)\phi([a, b]) \\ &= \phi\big(((\rho \circ a) \cdot f)b + f[a, b]\big) \,, \end{aligned}$$

where we have used Equation (4.81), the $\mathcal{C}(G_0, R)$-linearity of ϕ, Proposition 3.3.7, Equation (4.84), and the $\mathcal{C}(G_0, R)$-linearity of ϕ again. By Theorem 4.5.7, ϕ is an isomorphism, so that we can cancel ϕ on both sides of the equation, which yields Equation (4.71). □

4.6 Outlook: applications and examples

The goal of this section is to highlight possible applications and to state classes of examples of the differentiation procedure developed in this chapter. The computation of explicit examples consists of three steps:

(1) Identify a cartesian tangent category $\mathcal{C}$ with scalar R-multiplication, where $R \in \mathcal{C}$ is a ring object.

(2) Identify a differentiable groupoid object in $\mathcal{C}$.

(3) Apply the differentiation procedure of Section 4.5.2 step by step.

By Theorem 4.5.8, this yields an abstract Lie algebroid in the tangent category.

Lie groupoids

The prototypical cartesian tangent category is the category of finite-dimensional smooth manifolds with the usual scalar $\mathbb{R}$-multiplication (Ex. 3.2.21). The differentiable groupoid objects in the category of smooth manifolds coincide with Lie groupoids (Ex. 4.1.7). Our generalized differentiation procedure recovers the usual construction of the Lie algebroid of a Lie groupoid (Sec. 1.3).

Elastic diffeological groups

The category of elastic diffeological spaces is a cartesian tangent category with scalar $\mathbb{R}$-multiplication given by the left Kan extension of the scalar $\mathbb{R}$-multiplication on Euclidean spaces (Ex. 3.2.21). We will first consider examples of elastic diffeological groups, where the base space is a point.

- **Diffeomorphism groups:** Given a smooth (not necessarily compact) manifold, its group $\mathrm{Diff}(M)$ of diffeomorphisms is not a finite-dimensional Lie group. It is commonly viewed as an infinite-dimensional Lie group, in the sense that it is locally modeled on compactly supported vector fields [KM97]. $\mathrm{Diff}(M)$ can alternatively be viewed as a diffeological space equipped with the functional diffeology [Blo24a, Ex. 3.8(e)]. Since $\mathrm{Diff}(M)$ is an elastic diffeological group [Blo24a, Ex. 5.4], we can differentiate it using our method.

  It is well-known that the Lie algebra of the diffeomorphism group of a compact smooth manifold viewed as a Fréchet Lie group is given by its space of vector fields with the (opposite) Lie bracket. While our construction recovers this result, it has the following advantages:

  1) Our approach is based entirely on universal categorical constructions, sidestepping all technicalities of functional analysis and Fréchet manifolds.
  2) As a consequence of 1), our construction applies without modification to the diffeomorphism group of a non-compact manifold. We posit that its Lie algebra is the Lie algebra of *all* vector fields, without conditions on their support or behavior at infinity [BM].

The reader may refer to [KM97] for the applications of differential calculus in infinite-dimensional manifolds, which are modeled by c^∞-open subsets of *convenient* vector spaces, to infinite-dimensional Lie theory. For a survey on Lie theory of Lie groups modeled on *locally convex* spaces, as well as for a detailed historical note on infinite-dimensional Lie theory, see [Nee06].

- **Gauge transformations:** Given a smooth manifold M and a Lie group G with Lie algebra $\mathfrak{g}$, the diffeological mapping space $\underline{\mathfrak{Dflg}}(M,G)$ is elastic with tangent space

$$\begin{aligned} T\underline{\mathfrak{Dflg}}(M,G) &\cong \underline{\mathfrak{Dflg}}(M,TG) \\ &\cong \underline{\mathfrak{Dflg}}(M,G\times\mathfrak{g})\,, \end{aligned}$$

where we have used [Blo24a, Cor. 5.12]. For a recap on diffeological mapping spaces, see [Blo24a, Ex. 3.8(e)]. This has numerous applications in the setting of gauge theory [Blo24b, Sec. 9.3], where the fields are connections on a principal G-bundle $P\to M$. Here, $\underline{\mathfrak{Dflg}}(M,G)$ is the symmetry group of local gauge transformations. Its Lie algebra is $\underline{\mathfrak{Dflg}}(M,\mathfrak{g})$.

- **Bisection groups of Lie groups:** The last two classes of examples belong to a larger class: bisection groups of Lie groupoids. A **bisection** of a Lie groupoid $G_1 \rightrightarrows G_0$ is a section $\sigma: G_0 \to G_1$ of the source map s such that $t\circ\sigma: G_0\to G_0$ is a diffeomorphism. The set of bisections on a Lie groupoid has a natural group structure, and is denoted by $\mathrm{Bis}(G)$ (e.g. [Mac05, Prop. 1.4.2]).

  Given a smooth manifold M, its diffeomorphism group is the bisection group of the pair groupoid $M\times M\rightrightarrows M$ (Ex. 1.1.8 (ii)). For a Lie group G, the symmetry group $\underline{\mathfrak{Dflg}}(M,G)$ of global sections of the trivial principal G-bundle over M is the bisection group of the trivial Lie group bundle[3] $G\times M\to M$.

  It is known that the Lie algebra of the bisection group $\mathrm{Bis}(G)$ of a Lie groupoid G with compact base is isomorphic to the Lie algebra of sections of the Lie algebroid of G. In [SW15], the authors compute the Lie algebra of $\mathrm{Bis}(G)$ by using sophisticated machinery from functional analysis [Mil84, Nee06], where $\mathrm{Bis}(G)$ has a natural locally convex Lie group structure. Another approach is given in [CdSW99, Sec. 15], where the authors differentiate $\mathrm{Bis}(G)$ using paths, choosing a splitting of the tangent bundle over G_0. This can, for instance, be done with a Riemannian metric. The heuristic approach of [CdSW99] is in essence diffeological.

  As explained for the case of diffeomorphism groups, the advantage of our approach is that it does not involve choices of norms or metrics, but only universal constructions, so that it encompasses the case when M is not compact. We conjecture that bisection groups of Lie groupoids are elastic and that their Lie algebras are the Lie algebras of sections of the Lie algebroid.

[3] A **Lie group bundle** is a Lie groupoid where the source and target maps coincide. In this case, the source (or target) fibers have the structure of a Lie group. As such, a Lie group bundle can be viewed as a family of Lie groups smoothly parametrized by the objects of the Lie groupoid.

Elastic diffeological groupoids

We now present classes of possible applications of elastic diffeological groupoids, where the base is not necessarily a point. We start with action groupoids induced by actions of elastic diffeological groups on elastic diffeological spaces. Concrete examples of such action groupoids appear naturally in classical field theory.

- **Actions of diffeomorphism groups:** Consider the natural action of the diffeomorphism group $\mathrm{Diff}(M)$ of a smooth manifold M on the sections of a natural fiber bundle[4] $p : F \to M$. As we have seen above, $\mathrm{Diff}(M)$ is elastic. Moreover, $\Gamma(M, F)$ is elastic [Blo24a, Cor. 5.12] with tangent space

  $$T\Gamma(M,F) \cong \Gamma(M, VF)\,,$$

  where $VF = \ker Tp = TF \times_{TM}^{Tp,0_M} M$ is the vertical tangent bundle. The induced action diffeological groupoid

  $$\mathrm{Diff}(M) \times \Gamma(M,F) \rightrightarrows \Gamma(M,F)$$

  is elastic, which follows from the fact that the finite product of elastic spaces is elastic [Blo24a, Prop. 4.9]. This observation has applications in Lagrangian field theory [Blo24b].

- **Actions of gauge transformations:** Similarly, given a principal G-bundle $P \to M$ and an open subset $U \subset M$, the diffeological mapping space $\underline{\mathfrak{Dflg}}(U, G)$ acts on the diffeological space of local sections $\Gamma(U, P)$. This yields a sheaf of elastic diffeological groupoids

  $$\underline{\mathfrak{Dflg}}(U,G) \times \Gamma(U,P) \rightrightarrows \Gamma(U,P)\,.$$

  By differentiation, we obtain a sheaf of abstract Lie algebroids in diffeological spaces.

- **Elastic diffeological action groupoids:** The last two classes of examples belong to a larger class: diffeological action groupoids. An action of an elastic diffeological group G on an elastic diffeological space M induces an action diffeological groupoid $G \times M \rightrightarrows M$ (similar to Example 1.1.8 (iv) in the diffeological setting). As mentioned above, since the finite product of elastic spaces is elastic [Blo24a, Prop. 4.9], this action groupoid is elastic.

- **Groupoid symmetry of general relativity:**

  In [BFW13], the authors embarked on a project to understand the symmetry structure of the initial value problem of general relativity [BW24,BSW23]. The main result was the construction of a diffeological groupoid whose diffeological Lie algebroid has the same bracket as the somewhat mysterious Poisson

[4] In a natural bundle, the diffeomorphisms between open subsets of the base lift functorially to diffeomorphisms between local sections (e.g. [Blo24b, Rem. 8.3.13]). Examples of natural bundles include the (co)tangent bundle of a manifold, tensor bundles, and finite order jet bundles. For an exhaustive characterization see [ET79]. Taking sections of these bundles we obtain vector fields, differential forms, Riemannian metrics, etc.

bracket of the Gauß-Codazzi constraints for Ricci flat metrics. Without a theory of diffeological groupoids available, the differentiation of the diffeological groupoid of [BFW13] was carried out in an ad-hoc manner. The reason why this worked is that the authors had constructed a differentiable groupoid object in the tangent category of elastic diffeological spaces.

- **Reduction of action Lie groupoids by subgroups:**

  The construction of the groupoid symmetry of general relativity from [BFW13] is a particular case of a more general construction: reductions of action Lie groupoids by subgroups.

  Let $\varphi : G \to \mathrm{Diff}(M)$ be a smooth Lie group action on a smooth manifold. Let $H \subseteq G$ be a closed (not necessarily normal) Lie subgroup, such that $\varphi|_H$ is free and proper. There is a unique Lie groupoid structure on $H \setminus G \times_H M$ over M/H, where $H \setminus G \times_H M = (G \times M)/(H \times H)$ with respect to the action given by $(h_1, h_2) \cdot (g, m) = \big(h_1 g h_2^{-1}, \varphi_{h_2}(m)\big)$. For the notation and statement of the main proposition, see [BW24, Sec. 8.1]. In particular, when H is not normal, this reduction is not an action Lie groupoid. Blohmann and Weinstein have computed the Lie algebroid of this quotient groupoid in [BW24, Sec. 8.5] in the setting of smooth finite-dimensional manifolds.

  To see how this construction applies to the example in general relativity, the reader may refer to [BW24, Ex. 8.13].

Diffeological integration of Lie algebroids

It is known that Lie's third theorem fails in the setting of Lie groupoids and Lie algebroids [CF03]. In [Vil25], Villatoro addresses the question of integrability of Lie algebroids via diffeological groupoids. In his construction, he introduces a class of diffeological spaces, called *quasi-étale* diffeological spaces (QUED), which are locally quotients of smooth manifolds by a well-behaved equivalence relation. He then defines a Lie functor from *singular* Lie groupoids (groupoid objects in QUED with source and target QUED-*submersions*, and the space of objects given by a smooth manifold) to Lie algebroids.

The crucial part of the integration procedure in [Vil25] is to show that the Weinstein groupoid $\Pi_1(A) \rightrightarrows M$ of a Lie algebroid $A \to M$ (e.g. [CF03, Sec. 2]) is a singular Lie groupoid. In an attempt to compare Villatoro's construction to ours, there is one subtlety. The Weinstein groupoid (at least of a non-transitive Lie algebroid) does not seem to be elastic and so its differentiation by our methods seems to be more involved.

Holonomy groupoids of singular foliations

The question of diffeological integrability has been also considered in the context of *holonomy groupoids*. Associated to a singular foliation, there is a diffeological groupoid, called its holonomy groupoid [AS09, Def. 3.5]. More generally, associated to a singular subalgebroid of an integrable Lie algebroid, there is a holonomy groupoid [Zam22]. (The relation between the two is that the singular subalgebroids of $A = TM$ are precisely the singular foliations on M [Zam22, Ex. 1.2]).

In [AZ23], Androulidakis and Zambon propose an integration method for singular subalgebroids via diffeological groupoids. In the first step of establishing the differentiation procedure, the authors consider a certain class of diffeological groupoids, which behave like the holonomy groupoid of a singular subalgebroid. These diffeological spaces have a smooth manifold as their space of objects, and come together with a morphism to a Lie groupoid, satisfying additional properties. Using families of global bisections, the authors differentiate such a holonomy-like diffeological groupoid to a singular subalgebroid [AZ23, Thm. 3.2].

Under the assumptions on the diffeological groupoids in [AZ23], the tangent structure on smooth manifolds is sufficient and the question of a rigorous definition of tangent spaces to diffeological spaces can be circumvented. In an attempt to compare their construction to our differentiation procedure in the tangent category of elastic diffeological spaces, the following questions arise: are holonomy groupoids of singular foliations (or more generally of singular subalgebroids) elastic? What are the minimal conditions making them elastic? Providing satisfactory answers to these questions would be an interesting future work.

Moduli spaces and geometric deformation theory

The core of deformation theory lies in the study of moduli spaces of structures of a certain type, such as the multiplication map of an algebra, a complex structure, or a Riemannian metric, modulo isomorphisms. Deformations of a structure X are *smooth* paths $\varepsilon \mapsto X_\varepsilon$ in the corresponding moduli space $\mathcal{M}$ through $X = X_0$. Differentiating it at $\varepsilon = 0$ yields an infinitesimal deformation

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} X_\varepsilon \,.$$

However, owing to the fact that moduli spaces might admit singularities and do not come with a natural smooth structure, the differentiation often requires a derived approach in which the tangent space is replaced with a differential complex.

It is a well-known conjecture that deformation problems are governed by differential graded Lie algebras (dgLa) or L_∞-algebras via Maurer-Cartan elements. Prototypical examples are the deformation theory of associative algebras (Hochschild cohomology) [Ger64], Lie algebras (Chevalley-Eilenberg cohomology) [NR67], and complex structures (Kodaira-Spencer cohomology) [Kod05]. This conjecture is rigorously proved by Lurie [Lur11] and independently Pridham [Pri10] using the language of higher category theory. However, as promising as this sounds, constructing a computable model of the dgLa controlling a specific deformation problem turns out to be very difficult. As Kontsevich states: "*It is an 'art' to discover these objects for a general deformation theory*" [Kon94, p. 6].

For instance, it is surprising that, while there is consensus that the deformation complex of a Lie group is given by its group cohomology with values in the adjoint representation on its Lie algebra, the Lie brackets of a dgLa- or L_∞-structure are not known. More generally, Crainic, Mestre and Struchiner have constructed a deformation complex controlling deformations of Lie groupoids [CMS20]. It is still an open question what the compatible Lie bracket on this complex is.

One approach to tackle the issue of singularities in the moduli space $\mathcal{M}$ is by equipping it with a diffeology, where one can talk about smooth parametrizations. In fact, the moduli spaces of the structures mentioned above admit a natural functional diffeology. Geometrically, $\mathcal{M}$ can then be viewed as a stack presented by a diffeological groupoid $\mathcal{G}_{\mathcal{M}}$. The infinitesimal deformation theory of X should then be given by the fiber of $\mathcal{T}\mathcal{M}$ (Rem. 3.2.12) at X, presented by the abstract Lie algebroid of $\mathcal{G}_{\mathcal{M}}$. To make these heuristic ideas rigorous, many questions arise, such as:

1) How to generalize tangent structures in higher categorical settings, so that the bicategory of diffeological stacks is equipped with a tangent structure (see Remark 3.2.12)?

2) Are the diffeological groupoids $\mathcal{G}_{\mathcal{M}}$ corresponding to a moduli space $\mathcal{M}$, viewed as a stack, elastic?

My ultimate goal is to use this methodology to describe the deformation theory of geometric structures, such as Lie groupoids.

Chapter 5

Differentiation of higher groupoid objects

Higher groupoids are simplicial sets together with certain horn filling conditions, called *Kan conditions*. Historically, the notion of higher set theoretic groupoids dates back to Duskin [Dus79, Dus02], Glenn [Gle82], and Getzler [Get09]. These notions were recalled in Section 2.2.3. This simplicial approach has been further employed by Henriques [Hen08] and Zhu [Zhu09] who introduced higher Lie groupoids as simplicial manifolds that satisfy a geometric analog of the Kan conditions of simplicial sets. More generally, the authors in [Hen08] and [Zhu09] have introduced higher groupoid objects in any category equipped with a Grothendieck pretopology.

In 2006, Ševera has proposed a method of differentiation of higher Lie groupoids [Šev06]. In his construction, he has extensively used the odd line $\mathbb{R}^{0|1}$ as an infinitesimal model. Ševera has argued that the L_∞-algebroid of a higher Lie groupoid is given by the inner hom in the category of simplicial supermanifolds from the nerve of the pair groupoid of $\mathbb{R}^{0|1}$ to the higher Lie groupoid. The resulting L_∞-algebroid is a priori only a presheaf in graded manifolds. In his PhD thesis, Li has proposed a combinatorial solution to the representability problem [Li14]. In a recent work by Li, Ryvkin, Wessel and Zhu, Ševera's construction has been rigorously formulated and proved [LRWZ23], filling in the gaps of the proof in [Li14]. The authors have shown that the presheaf describing the infinitesimal counterpart of the higher Lie groupoid is representable by its tangent complex.

In this chapter, we give a categorical generalization of Ševera's approach using the language of ends. The first advantage of our construction is that it (only) makes use of the tangent structure of the category of smooth manifolds. Hence, not only it generalizes the differentiation process of usual Lie groupoids to the higher case, but also it gives a method of differentiating higher groupoid objects to their infinitesimal counterparts in any tangent category in the sense of Rosický (Def. 3.2.4). The second advantage is the fact that categorical ends are universal constructions, being special kinds of limits. Moreover, they support a rich calculus of computational and deduction rules [Lor21]. Thus, our construction gives both conceptual understanding of the differentiation procedure and allows us to perform explicit computations.

To achieve our goal of differentiating higher groupoid objects, the first step is to precisely state what we mean by a higher groupoid object, as well as identify the conditions needed for its differentiation. This will be covered in Section 5.1. The differentiation procedure via categorical ends and the computation of the corresponding limit will be described in Section 5.2. We then comment on how to recover the well-known differentiation of Lie groupoids in Section 5.3. Lastly, we comment on a possible generalization of the notion of higher Lie algebroids in the category of smooth manifolds to the setting of tangent categories in Section 5.4. We also make a conjecture about the higher Lie algebroid structure of the categorical end formulation of the infinitesimal counterpart of the higher groupoid object.

5.1 Differentiable higher groupoids

The main goal of this section is to describe the global geometric objects that we will differentiate: differentiable higher groupoids. We first define an n-groupoid object in a category equipped with a Grothendieck pretopology, using the approach of Henriques [Hen08] and Zhu [Zhu09]. These notions are recalled in Section 5.1.1. In Section 5.1.2, we illustrate the $(n+1)$-coskeletality of an n-groupoid object, and describe its higher multiplications and degeneracies in the $n+1$ level under concreteness assumptions. This is a generalization of the descriptions in [Dus02] and [Zhu09] for the cases of $n = 0, 1$ and 2. Lastly, in Section 5.1.3 we identify the axioms on an n-groupoid object needed for its differentiation in a tangent category equipped with a compatible Grothendieck pretopology.

5.1.1 Higher groupoids in categories with a pretopology

Higher set theoretic groupoids are simplicial sets that satisfy certain horn filling conditions. More precisely, an n-groupoid is a simplicial set $G : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ such that the horn projection

$$p_{m,i} : G_m \longrightarrow G(\Lambda^m_i) \tag{5.1}$$

is a surjection for all $m \geq 1$ and a bijection for all $m > n$ and $0 \leq i \leq m$ (Def. 2.2.19). By the Yoneda lemma, there is a natural bijection $G_m \cong \mathrm{s}\mathcal{S}\mathrm{et}(\Delta^m, G)$ of sets. In (5.1), $G(\Lambda^m_i) = \mathrm{s}\mathcal{S}\mathrm{et}(\Lambda^m_i, G)$ is the set of (m,i)-horns of G and $p_{m,i}$ is the natural morphism induced by the monomorphism $\Lambda^n_i \to \Delta^n$. These notions are recalled in Section 2.2. For the case $n = 1$, the nerve construction induces an equivalence of categories between the category of 1-groupoids and the category of groupoids (Remark 2.2.22).

Singleton Grothendieck pretopology

In an attempt to define higher groupoid objects in any category, there are two subtleties:

(1) As described in Section 2.2.2, horns can be defined in any simplicial object in a category $\mathcal{C}$ by Kan extensions along the Yoneda embedding. Explicitly, the object of (m,i)-horns of a simplicial object $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$ is defined by

$$X(\Lambda^m_i) := \big(\mathrm{Ran}_{y^{\mathrm{op}}} X\big)(\Lambda^m_i) \cong \lim_{\Delta^k \to \Lambda^m_i} X_k \tag{5.2}$$

(Def. 2.2.11). Without any completeness assumptions on $\mathcal{C}$, these objects are a priori only presheaves on $\mathcal{C}$. In Definition 4.1.1 of groupoid objects in $\mathcal{C}$, we explicitly require the existence of these horns in $\mathcal{C}$.

(2) In order to make sense of the non-unique horn filling conditions, we need a class of morphisms that play the role of surjective submersions in the category of smooth manifolds. Following the approach of Henriques and Zhu, these will be given by the covers of a Grothendieck pretopology on $\mathcal{C}$ [Hen08,Zhu09]. The properties of the pretopology will then guarantee the existence of the horns in $\mathcal{C}$ (Cor. 5.1.10).

A **singleton Grothendieck pretopology** on $\mathcal{C}$ is given by a class of morphisms, called **covers**, subject to the following axioms (Def. A.1.9):

(i) Every isomorphism is a cover.

(ii) The composition of two covers is a cover.

(iii) If $U \to X$ is a cover and $Y \to X$ is a morphism in $\mathcal{C}$, then the pullback $Y \times_X U$ exists and the projection $Y \times_X U \to Y$ is a cover.

Remark 5.1.1. If $\mathcal{C}$ has coproducts, a singleton Grothendieck pretopology coincides with a Grothendieck pretopology (see Def. A.1.8 and Rem. A.1.10). Since we will only need singletons as covers, we do not make the assumption that $\mathcal{C}$ has coproducts.

Terminology 5.1.2. In the rest of this chapter, we will just write a Grothendieck pretopology or a pretopology to mean a singleton Grothendieck pretopology.

A pretopology on $\mathcal{C}$ is **subcanonical** if every cover is a regular epimorphism (Def. A.1.16), that is, if every cover is a coequalizer of some parallel pair of morphisms.

We will equip $\mathcal{C}$ with a Grothendieck pretopology satisfying the following assumption.

Assumption 5.1.3.

(i) The category $\mathcal{C}$ has a terminal object $*$ and the unique morphism $U \to *$ is a cover for every object $U \in \mathcal{C}$ which is not the initial object (if it exists).

(ii) The pretopology is subcanonical.

These are Assumptions 2.1 from [Zhu09], except that we exclude the initial object in part (i). The reason is that in many situations the unique morphism $\varnothing \to *$ is not a cover, such as in $\mathcal{S}et$ with any subcanonical pretopology as explained in [MZ15, p. 1913].

Remark 5.1.4. A category equipped with a Grothendieck pretopology satisfying Assumption 5.1.3 has all finite products. This follows from Axiom (iii) of the pretopology and the assumption that the unique morphism to the terminal object is a cover.

Example 5.1.5. Consider the category $\mathcal{M}\mathrm{fld}$ of smooth manifolds with smooth maps. The covers given by surjective submersions equip $\mathcal{M}\mathrm{fld}$ with a Grothendieck pretopology satisfying Assumption 5.1.3.

Example 5.1.6. Diffeological spaces are concrete sheaves on the site of Euclidean spaces with the usual open covers [Blo24a, Def. 3.4]. They form a category $\mathcal{D}\mathrm{flg}$ with morphisms of the underlying presheaves. Subductions are the strong epimorphisms (Def. A.4.3) in $\mathcal{D}\mathrm{flg}$ [Blo24a, Term. 3.7]. The covers given by subductions equip $\mathcal{D}\mathrm{flg}$ with a Grothendieck pretopology satisfying Assumption 5.1.3. This follows from the many convenient properties of diffeological spaces [BH11, Thm. 5.25], [Blo24a, Thm. 3.5, Prop. 3.6].

Collapsible simplicial sets and existence of the horns

Definition 5.1.7. A **filtration** of a simplicial set S is a sequence of monomorphisms

$$\Delta^0 = S^0 \to S^1 \to \ldots \to S^k = S \tag{5.3}$$

of simplicial sets for some $k \geq 1$.

Definition 5.1.8. A simplicial set S is called **collapsible** if it admits a filtration (5.3) of simplicial sets, where for each $1 \leq i \leq k$, there exist some $n_i \geq 1$, $0 \leq j_i \leq n_i$, and a morphism $f_i : \Lambda^{n_i}_{j_i} \to S^{i-1}$ of simplicial sets, such that S^i is the pushout in $\mathrm{s}\mathcal{S}\mathrm{et}$ of the diagram

$$\begin{tikzcd} S^i \cong S^{i-1} \amalg_{\Lambda^{n_i}_{j_i}} \Delta^{n_i} & \Delta^{n_i} \arrow[l] \\ S^{i-1} \arrow[u] \arrow[ur, phantom, "\lrcorner"] & \Lambda^{n_i}_{j_i} \arrow[l, "f_i"] \arrow[u] \end{tikzcd} \tag{5.4}$$

Geometrically, the pushout (5.4) means that each S^i comes from the preceding one by filling a horn. Note that (5.4) is computed objectwise, being a colimit in a functor category.

Recall that the object of (m, i)-horns of a simplicial object $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$ is given by Equation (5.2). This definition applies to any simplicial set S (see Def. 2.2.4 and Rem. 2.2.4). Explicitly,

$$X(S) := \big(\mathrm{Ran}_{y^{\mathrm{op}}} X\big)(S) \cong \lim_{\Delta^k \to S} X_k\,.$$

Using the properties of collapsible simplicial sets, the next lemma gives a sufficient condition for the existence of these limits in $\mathcal{C}$. It is another formulation of [Hen08, Lem. 2.4].

Lemma 5.1.9. *Let S be a collapsible subsimplicial set of Δ^n for some $n \geq 1$. Let X be a simplicial object in a category $\mathcal{C}$ equipped with a pretopology, such that the object $X(\Lambda^m_j)$ exists in $\mathcal{C}$ and the horn projection*

$$p_{m,j} : X_m \longrightarrow X(\Lambda^m_j)$$

is a cover for all $0 < m < n$ and $0 \leq j \leq m$. Then, $X(S)$ exists in $\mathcal{C}$.

Proof. Since S is a collapsible subsimplicial set of Δ^n, it admits a filtration

$$\Delta^0 = S^0 \to S^1 \to \ldots \to S^k = S$$

such that for all $1 \leq i \leq k$, there exist $1 \leq n_i < n$, $0 \leq j_i \leq n_i$, and a morphism $f_i : \Lambda^{n_i}_{j_i} \to S^{i-1}$ making Diagram (5.4) a pushout square. We will proceed by induction on i and show that $X(S^i)$ is an object of $\mathcal{C}$ for all $0 \leq i \leq k$.

For $i = 0$, it follows from Lemma 2.2.8 that $X(S^0) = X(\Delta^0) \cong X_0 \in \mathcal{C}$. Assume now that $X(S^{i-1}) \in \mathcal{C}$ for some $0 < i \leq k$. Consider the following diagram:

$$\begin{tikzcd} & X(\Delta^{n_i}) \arrow[d, "p_{n_i,j_i}"] \\ X(S^{i-1}) \arrow[r, "X(f_i)"'] & X(\Lambda^{n_i}_{j_i}) \end{tikzcd} \tag{5.5}$$

By assumption and since $n_i < n$, we have that $X(\Lambda^{n_i}_{j_i}) \in \mathcal{C}$ and p_{n_i,j_i} is a cover. Using Axiom (iii) of a Grothendieck pretopology, we get that the pullback of Diagram (5.5) exists. By the universal property of pullbacks and by the pushout square (5.4), we conclude that the pullback of Diagram (5.5) is isomorphic to $X(S^i)$. Thus, $X(S^i)$ exists in $\mathcal{C}$. In particular, for $i = k$, we get that $X(S^k) = X(S)$ exists in $\mathcal{C}$. □

Corollary 5.1.10. *Let $n \geq 1$. Let X be a simplicial object in a category $\mathcal{C}$ equipped with a pretopology, such that the object $X(\Lambda^m_j)$ exists in $\mathcal{C}$ and the horn projection*

$$p_{m,j} : X_m \longrightarrow X(\Lambda^m_j)$$

is a cover for all $0 < m < n$ and $0 \leq j \leq m$. Then, $X(\Lambda^n_i)$ exists in $\mathcal{C}$ for all $0 \leq i \leq n$.

Proof. For $n = 1$, the objects of $(1,0)$-horns and $(1,1)$-horns always exist and are isomorphic to the object of 0-simplices. That is,

$$X(\Lambda^1_0) \cong X_0 \cong X(\Lambda^1_1)\,.$$

For $n \geq 2$, since the horns Λ^n_i are collapsible subsimplicial sets of Δ^n, the result follows from Lemma 5.1.9. □

The Kan conditions and n-groupoids

We will now generalize the notion of Kan simplicial sets from Definition 2.2.16 to the setting of categories with a pretopology. It gives a geometric analog of the horn filling conditions. The following definition is another formulation of [Hen08, Def. 2.3].

Definition 5.1.11. Let $\mathcal{C}$ be a category equipped with a pretopology satisfying Assumption 5.1.3. A simplicial object X in $\mathcal{C}$ is called **Kan** if the horn projection $p_{m,i} : X_m \to X(\Lambda^m_i)$ is a cover for all $m \geq 1$ and $0 \leq i \leq m$.

Remark 5.1.12. The nerve of a Lie groupoid (see Example 2.1.6) is a Kan simplicial manifold since the horn projections are surjective submersions. In fact, more is true: All horns of degree > 1 can be filled uniquely, that is $p_{m,i}$ is a diffeomorphism for all $m > 1$ and $0 \leq i \leq m$ (similar to Example 2.2.18). This observation motivates the next definition, which generalizes Definition 2.2.19 of higher set theoretic groupoids.

Definition 5.1.13 ([Zhu09, Def. 1.3]). Let $\mathcal{C}$ be a category equipped with a pretopology satisfying Assumption 5.1.3 and let $n \in \mathbb{N} \cup \{0\} \cup \{\infty\}$. An ***n*-groupoid object** in $\mathcal{C}$ is a Kan simplicial object G in $\mathcal{C}$ such that $p_{m,i} : G_m \to G(\Lambda_i^m)$ is an isomorphism for all $m > n$ and $0 \leq i \leq m$. If G_0 is the terminal object $*$, then G is called an ***n*-group object**.

Definition 5.1.14. A **morphism of *n*-groupoid objects** in $\mathcal{C}$ is a morphism of simplicial objects, i.e. a natural transformation.

For $n \in \mathbb{N} \cup \{0\} \cup \{\infty\}$, n-groupoid objects together with morphisms between them form a category.

Remark 5.1.15. Let $\mathcal{C}$ be a category equipped with a pretopology satisfying Assumption 5.1.3. A 1-groupoid object in $\mathcal{C}$ is a groupoid object in $\mathcal{C}$ in the sense of Definition 4.1.1 since the existence of the horns follows from Corollary 5.1.10. The converse only holds if we add the assumption that the horn projections $p_{1,0}$ and $p_{1,1}$ are covers. Note that if G is a groupoid object in $\mathcal{C}$ in the sense of Definition 4.1.1, we have that $p_{1,0}$ and $p_{1,1}$ are regular epimorphisms (Remark 4.1.3). However, this does not in general imply that they are covers.

Terminology 5.1.16. Lie *n*-group(oid)s are n-group(oid) objects in the category of smooth manifolds with the pretopology given by surjective submersions. The comparison between the various notions of 2-groups in Remark 2.2.23 holds for the case of Lie 2-groups. For the equivalence of the category of strict Lie 2-groups and the category of Lie crossed modules, see [BL04, Sec. 8.4].

5.1.2 Coskeletality, higher multiplications and degeneracies

In this section we will show that the data of an n-groupoid object is given by its first $n+1$ levels. We will assume that the functor of points

$$\begin{aligned} |\_| : \mathcal{C} &\longrightarrow \mathcal{S}\mathrm{et} \\ C &\longmapsto |C| := \mathcal{C}(*, C) \end{aligned} \tag{5.6}$$

is faithful, so that it equips $\mathcal{C}$ with a concrete structure. In the last part, we will give explicit formulas for the face and degeneracy morphisms in the $(n+1)$-level.

Coskeletality

Every set theoretic n-groupoid G is $(n+1)$-coskeletal (Prop. 2.3.12). In this section, we show that this also holds for n-groupoid objects in categories with a pretopology. To prove the statement, we will use Definition 2.3.15, which states that a simplicial object $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$ is ***n*-coskeletal** if the natural map

$$\mathcal{C}^{\mathbf{\Delta}^{\mathrm{op}}}(A, X) \xrightarrow{\cong} \mathcal{C}^{\mathbf{\Delta}^{\mathrm{op}}_{\leq n}}(\mathrm{tr}_n^{\mathcal{C}} A, \mathrm{tr}_n^{\mathcal{C}} X)$$

is a bijection for all simplicial objects A of $\mathcal{C}$. Here, $\mathrm{tr}_n^{\mathcal{C}} : \mathcal{C}^{\mathbf{\Delta}^{\mathrm{op}}} \to \mathcal{C}^{\mathbf{\Delta}^{\mathrm{op}}_{\leq n}}$ is the n-truncation functor, which restricts a simplicial object to its simplices of degree up to n and forgets its simplices of degree greater than n (see Section 2.3.1 for details).

Remark 5.1.17. The reason why we use Definition 2.3.15 of n-coskeletality is explained in Section 2.3.3. Under no assumptions of completeness in the ambient category, the coskeleton functor is defined only in presheaves. In this case, Definition 2.3.6 that $X \cong \operatorname{cosk}_n X$ does not make sense to the nose, but only in presheaves. The equivalence of (i) and (ii) in Proposition 2.3.9 justifies this definition, which is sufficient for our purposes.

Proposition 5.1.18. *Let $\mathcal{C}$ be a category equipped with a pretopology satisfying Assumption 5.1.3. Every n-groupoid object in $\mathcal{C}$ is $(n+1)$-coskeletal.*

Proof. Let $G : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$ be an n-groupoid object and $A : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$ a simplicial object. Denote by $\hat{\mathcal{C}} := \mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}}$ the category of presheaves on $\mathcal{C}$ and let $y : \mathcal{C} \to \hat{\mathcal{C}}$ be the Yoneda embedding. Consider the following simplicial presheaves:

$$\hat{G} : \mathbf{\Delta}^{\mathrm{op}} \xrightarrow{G} \mathcal{C} \xrightarrow{y} \hat{\mathcal{C}}$$
$$\hat{A} : \mathbf{\Delta}^{\mathrm{op}} \xrightarrow{A} \mathcal{C} \xrightarrow{y} \hat{\mathcal{C}}\,.$$

Since y is full and faithful, we have the following natural bijections

$$\begin{aligned} \mathcal{C}^{\mathbf{\Delta}^{\mathrm{op}}}(A, G) &\cong \hat{\mathcal{C}}^{\mathbf{\Delta}^{\mathrm{op}}}(\hat{A}, \hat{G}) \\ \mathcal{C}^{\mathbf{\Delta}^{\mathrm{op}}_{\leq n+1}}(\operatorname{tr}^{\mathcal{C}}_{n+1} A,\, \operatorname{tr}^{\mathcal{C}}_{n+1} G) &\cong \hat{\mathcal{C}}^{\mathbf{\Delta}^{\mathrm{op}}_{\leq n+1}}(\operatorname{tr}^{\hat{\mathcal{C}}}_{n+1} \hat{A},\, \operatorname{tr}^{\hat{\mathcal{C}}}_{n+1} \hat{G})\,. \end{aligned} \tag{5.7}$$

Furthermore, since $\mathcal{C}$ has a concrete structure and since $\hat{\mathcal{C}}$ is complete and cocomplete, it follows from the proof of Proposition 2.3.12 that

$$\hat{G} \cong \operatorname{cosk}^{\hat{\mathcal{C}}}_{n+1} \hat{G}\,,$$

where $\operatorname{cosk}^{\hat{\mathcal{C}}}_{n+1} : \hat{\mathcal{C}}^{\mathbf{\Delta}^{\mathrm{op}}} \to \hat{\mathcal{C}}^{\mathbf{\Delta}^{\mathrm{op}}}$ is the coskeleton endofunctor (see also Rem. 2.3.14 and [Zhu09, Prop. 2.15]). By the equivalence of (i) and (ii) in Proposition 2.3.9, we have a natural bijection

$$\hat{\mathcal{C}}^{\mathbf{\Delta}^{\mathrm{op}}}(\hat{A}, \hat{G}) \cong \hat{\mathcal{C}}^{\mathbf{\Delta}^{\mathrm{op}}_{\leq n+1}}(\operatorname{tr}^{\hat{\mathcal{C}}}_{n+1} \hat{A},\, \operatorname{tr}^{\hat{\mathcal{C}}}_{n+1} \hat{G}) \tag{5.8}$$

(see Remark 2.3.16). Using (5.7) and (5.8), we conclude that there is a natural bijection

$$\mathcal{C}^{\mathbf{\Delta}^{\mathrm{op}}}(A, G) \cong \mathcal{C}^{\mathbf{\Delta}^{\mathrm{op}}_{\leq n+1}}(\operatorname{tr}^{\mathcal{C}}_{n+1} A,\, \operatorname{tr}^{\mathcal{C}}_{n+1} G)\,,$$

which shows that G is $(n+1)$-coskeletal. □

Remark 5.1.19. The geometric interpretation of the previous proposition is that n-groupoid objects can be completely described by the first $n+1$ levels together with some additional data, which are given by the Kan conditions. When $n = 1$, the extra data is given by the inverse and the associative multiplication of the groupoid. The case for $n = 2$ is rigorously treated in [Zhu09, Sec. 2.3], where Zhu proves that a 2-truncated simplicial object together with so-called *associative 3-multiplications* is equivalent to the data of a 2-groupoid object via a generalized nerve construction. The reader may refer to [Dus02] for a detailed exposition in the set theoretic setting for $n = 0, 1$ and 2.

Let G be an n-groupoid object in $\mathcal{C}$ for some $n \geq 0$. We will now describe the $(n+1)$-level of G explicitly. We shall denote the face and degeneracy morphisms of G by $d_{m,i}$ for $m \geq 1$, $0 \leq i \leq m$ and $s_{m,i}$ for $m \geq 0$, $0 \leq i \leq m$. As a simplicial object, G is depicted by

$$\begin{tikzcd} G_0 \arrow[r, shift left=2, leftarrow, "d_{1,0}"] \arrow[r] \arrow[r, shift right=2, leftarrow, "d_{1,1}"'] & G_1 \arrow[r, shift left=3, leftarrow, "d_{2,0}"] \arrow[r, shift left=1] \arrow[r, shift right=1, leftarrow] \arrow[r, shift right=3] \arrow[r, shift right=5, leftarrow, "d_{2,2}"'] & G_2 & \cdots \end{tikzcd}$$

Let $k \in \{0, \ldots, n+1\}$ be fixed.

Remark 5.1.20. Since we assume that $\mathcal{C}$ has a concrete structure, the horn projections of G can be expressed in components as described in Remark 2.2.13. For instance, the $(n+1,k)$-horn projection can be written as

$$\begin{aligned} p_{n+1,k} : G_{n+1} &\longrightarrow G(\Lambda_k^{n+1}) \\ g &\longmapsto (g_0, \ldots, \widehat{g}_k, \ldots, g_{n+1}), \end{aligned}$$

where $g_i := d_{n+1,i}(g)$ for all $0 \leq i \leq n+1$. Since G is an n-groupoid object, $p_{n+1,k}$ is an isomorphism. Hence, we will usually identify g with its horn projection and write $g \equiv (g_0, \ldots, \widehat{g}_k, \ldots, g_{n+1})$.

The higher multiplications

For all $0 \leq i \leq n+1$, consider the composition

$$\widetilde{d}_{n+1,i} : G(\Lambda_k^{n+1}) \xrightarrow[\cong]{p_{n+1,k}^{-1}} G_{n+1} \xrightarrow{d_{n+1,i}} G_n\,,$$

which we also call the face morphisms. It is easy to observe that

$$\widetilde{d}_{n+1,i} = \begin{cases} \mathrm{pr}_{i+1} & \text{if } i < k \\ \mathrm{pr}_i & \text{if } i > k\,. \end{cases}$$

Example 5.1.21. Let $n = 1$ and $k = 1$. If $i = 1$, $\widetilde{d}_{2,1} := d_{2,1} \circ p_{2,1}^{-1}$ is the groupoid multiplication

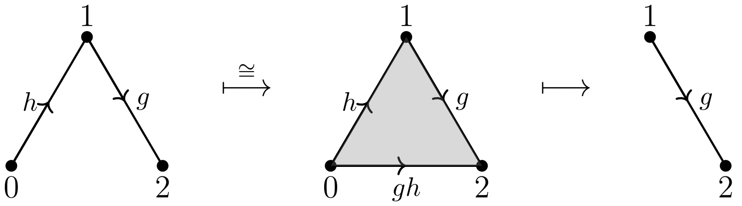
$$\begin{aligned} m:\ G(\Lambda_1^2) \ \cong G_1 \times_{G_0}^{s,t} G_1 &\xrightarrow[\cong]{p_{2,1}^{-1}} G_2 \xrightarrow{d_{2,1}} G_1 \\ (g,h) &\longmapsto gh\,. \end{aligned} \tag{5.9}$$

Here, $s \equiv d_{1,1}$ is the source and $t \equiv d_{1,0}$ is the target of the groupoid. If $i = 0$, the morphism $\widetilde{d}_{2,0} = \mathrm{pr}_1$ is the projection onto the first factor:

$$\begin{aligned} \mathrm{pr}_1 : G_1 \times_{G_0}^{s,t} G_1 &\xrightarrow[\cong]{p_{2,1}^{-1}} G_2 \xrightarrow{d_{2,0}} G_1 \\ (g,h) &\longmapsto g\,. \end{aligned}$$

It can be depicted by:

$\widetilde{d}_{2,0}$:



Similarly, if $i = 2$, we get that $\widetilde{d}_{2,2} = \mathrm{pr}_2$ is the projection onto the second factor. These constructions are recalled from Remark 2.2.22, where we explain the equivalence of groupoids and 1-groupoids through the nerve construction.

The fact that the inner face morphism $\widetilde{d}_{2,1}$ is the multiplication of the groupoid suggests the following definition.

Definition 5.1.22. The $(n+1, k)$**-multiplication** of the n-groupoid object G in $\mathcal{C}$ is the morphism

$$m_{n+1,k} := \widetilde{d}_{n+1,k} : G(\Lambda_k^{n+1}) \longrightarrow G_n\,.$$

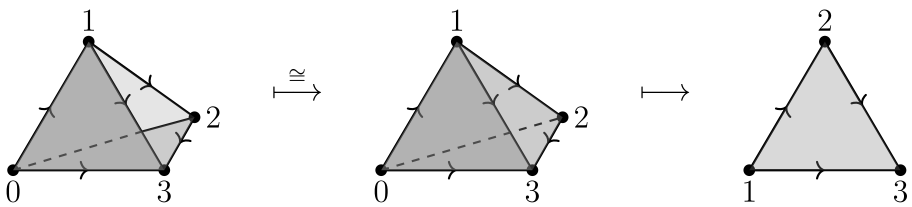
Remark 5.1.23. Often we will just write $m \equiv m_{n+1} \equiv m_{n+1,k}$ and call it *the* multiplication of G if it is clear from the context what k is. Note however that there are $n+2$ choices of such multiplication morphisms for an n-groupoid. When $n = 1$, G has a unique degree 2 inner horn $G(\Lambda_1^2)$ and hence the multiplication $m = m_{2,1}$, as given in (5.9), is canonical. In this case, the other two non-standard multiplications are given by $m_{2,0} : (g,h) \mapsto gh^{-1}$ and $m_{2,2} : (g,h) \mapsto g^{-1}h$. For higher groupoid objects however, there is no natural choice of an inner horn.

Example 5.1.24. Let $n = 2$ and $k = 0$. Then, the $(3,0)$-multiplication of the 2-groupoid object G is given by

$$\begin{aligned} m_{3,0} : G(\Lambda_0^3) &\xrightarrow[\cong]{p_{3,0}^{-1}} G_3 \xrightarrow{d_{3,0}} G_2 \\ (g_1, g_2, g_3) &\longmapsto g_1 g_2 g_3\,, \end{aligned}$$

where $g \equiv (g_1, g_2, g_3)$ is such that $g_i := d_{3,i}(g) \in G_2$ for $i = 1,2,3$. The multiplication can be depicted by:

$m_{3,0}$:



Remark 5.1.25. The (unique) Kan conditions of degree $n+1$ and $n+2$ allow us to rigorously formulate and prove that the $(n+1,k)$-multiplication of the n-groupoid G is associative. This is spelled out for the case $n = 2$ in [Zhu09, Prop.-Def. 2.16] for the multiplications $m_{3,0}$, $m_{3,1}$, $m_{3,2}$ and $m_{3,3}$.

The degeneracy morphisms

For all $0 \leq i \leq n$, consider the composition

$$\widetilde{s}_{n,i} : G_n \xrightarrow{s_{n,i}} G_{n+1} \xrightarrow[\cong]{p_{n+1,k}} G(\Lambda_k^{n+1}) \,,$$

which we also call the degeneracy morphisms. The next lemma gives a useful description of these morphisms in terms of the components of the horn. For simplicity, the composition symbol as well as the first index in the face and degeneracy morphisms are omitted.

Lemma 5.1.26. *The following equations hold:*

$$\widetilde{s}_i = \begin{cases} (s_{i-1}d_0,\, \dots,\, s_{i-1}d_{i-1},\, \mathrm{id},\, \mathrm{id},\, s_i d_{i+1},\, \dots,\, \widehat{s_i d_{k-1}},\, \dots,\, s_i d_n) & \textit{if } i < k \,, \\ (s_{k-1}d_0,\, \dots,\, s_{k-1}d_{k-1},\, \mathrm{id},\, s_k d_{k+1},\, \dots,\, s_k d_n) & \textit{if } i = k \,, \\ (s_{i-1}d_0,\, \dots,\widehat{s_{i-1}d_k},\, \dots,\, s_{i-1}d_{i-1},\, \mathrm{id},\, \mathrm{id},\, s_i d_{i+1},\, \dots,\, s_i d_n) & \textit{if } i > k \,, \end{cases} \tag{5.10}$$

whenever defined.

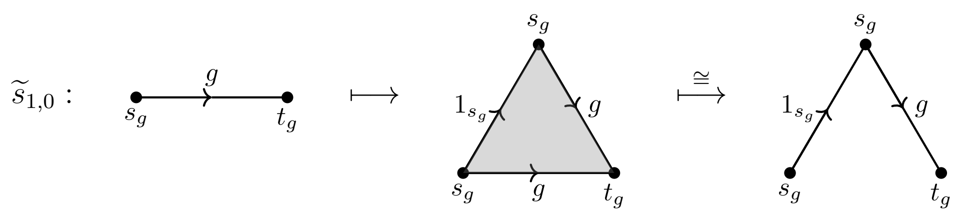
Proof. The result follows by the fact that $p_{n+1,k} = (d_{n+1,0}, \dots, \widehat{d}_{n+1,k}, \dots, d_{n+1,n+1})$ (Remark 2.2.13) and the simplicial identities. □

Remark 5.1.27. Let us analyze Equation (5.10). If $i < k$, then the $(i+1)$ and $(i+2)$ slots of $\widetilde{s}_i$ are the identity morphisms. Moreover, the $(k+1)$-component is omitted. This implies that if $i = k-1$, the $(k+1)$-component coincides with the second identity map, which is omitted. In this case, only one identity component appears in the equation. The situation for $i > k$ is analogous.

Example 5.1.28. Let $n = 1$ and $k = 1$. If $i = 0$, the morphism $\widetilde{s}_{1,0}$ is given by the composition

$$\begin{aligned} \widetilde{s}_{1,0} : G_1 \xrightarrow{s_{1,0}} G_2 \xrightarrow[\cong]{p_{2,1}} G_1 \times^{s,t}_{G_0} G_1 &\cong\; G(\Lambda_1^2) \\ g \longmapsto &\; \big(g, 1_{s(g)}\big) \,. \end{aligned}$$

It can be depicted by:



Since $s \equiv d_{1,1}$ is the source and $1 \equiv s_{0,0}$ is the unit of the groupoid, we have

$$\widetilde{s}_{1,0} = (\mathrm{id}_{G_1},\, s_{0,0} \circ d_{1,1})$$

as expected. Similarly, if $i = 2$, we get that

$$\widetilde{s}_{1,1} = (s_{0,0} \circ d_{1,0},\, \mathrm{id}_{G_1}) \,.$$

For $n = 2$, we recover the explicit formulations of the degeneracy morphisms $s_{2,0}$, $s_{2,1}$ and $s_{2,2}$ as described in [Zhu09, Sec. 2.3].

5.1.3 Differentiability

In this section, we identify the axioms needed for a higher groupoid object to differentiate to its infinitesimal counterpart. We will give a tentative definition of the notion of differentiability and explain the motives of our axioms.

The setting is a category $\mathcal{C}$ with a tangent structure (Def. 3.2.4) and a Grothendieck pretopology satisfying Assumption 5.1.3. We say that the tangent structure and the Grothendieck pretopology are **compatible** if the abstract tangent functor T maps covers to covers.

Definition 5.1.29. Let $n \in \mathbb{N} \cup \{0\} \cup \{\infty\}$. Let $\mathcal{C}$ be a category equipped with a tangent structure and a compatible Grothendieck pretopology satisfying Assumption 5.1.3. An n-groupoid object G in $\mathcal{C}$ will be called **differentiable** if:

(i) for all $p \geq 1$, the natural morphism

$$T^p\big(G(\Lambda_i^m)\big) = T^p\Big(\lim_{\Delta^l \to \Lambda_i^m} G_l\Big) \longrightarrow \lim_{\Delta^l \to \Lambda_i^m} T^p G_l = (T^p G)(\Lambda_i^m) \tag{5.11}$$

is a cover for all $m \geq 1$ and an isomorphism for all $m > n$ and $0 \leq i \leq m$;

(ii) the end

$$\int_{[m] \in \mathbf{\Delta}^{\mathrm{op}}_{\leq n+1}} T^{m+1} G_m \,. \tag{5.12}$$

exists in $\mathcal{C}$.

Let us motivate this definition. It is a consequence of condition (i) that the simplicial object $(T^pG)[l] = T^p G_l$ is an n-groupoid object in $\mathcal{C}$ for all $p \geq 1$. Let us explain this for $p = 1$. The horn projections[1] $p^{TG}_{m,i} := (TG)(\Lambda_i^m \to \Delta^m)$ of TG factor through the tangent morphism of the horn projections $p_{m,i}$ of G as follows:

$$\begin{tikzcd} TG_m \arrow[rr, "p^{TG}_{m,i}"] \arrow[dr, "Tp_{m,i}"'] & & (TG)(\Lambda_i^m) \\ & T\big(G(\Lambda_i^m)\big) \arrow[ur] & \end{tikzcd} \tag{5.13}$$

Since G is an n-groupoid object, the horn projection $p_{m,i}$ is a cover for all $m \geq 1$ and an isomorphism for all $m > n$. Since T maps covers to covers, we get that $Tp_{m,i}$ is a cover for all $m \geq 1$. Moreover, by the functoriality of T, we have that $Tp_{m,i}$ is an isomorphism for all $m > n$. Using the condition that (5.11) is a cover for all $m \geq 1$ and the axioms of the pretopology, we obtain that the horn projection $p^{TG}_{m,i}$, being a composition of covers, is a cover for all $m \geq 1$. Lastly, using the condition that (5.11) is an isomorphism for all $m > n$, we get that $p^{TG}_{m,i}$, being a composition of isomorphisms, is an isomorphism for all $m > n$. This concludes that TG is an n-groupoid object in $\mathcal{C}$.

Remark 5.1.30. The existence of the horns $(T^pG)(\Lambda_i^m)$ follows from the fact that the horns Λ_i^m are collapsible (Cor. 5.1.10).

[1] For a recall on the objects of horns and the horn projections, see Definition 2.2.11.

Notation 5.1.31. If G is a differentiable n-groupoid object, we will often omit the extra parenthesis and write $T^pG(\Lambda_i^m) \equiv T^p\big(G(\Lambda_i^m)\big) \cong (T^pG)(\Lambda_i^m)$ for all $p \geq 1$, $m > n$, and $0 \leq i \leq m$. This is justified by condition (i) of Definition 5.1.29. This will appear in the component expression of the horn projection in Remark 5.2.12 and in the last box (5.28) of the computation of the end.

The motivation for condition (ii) of Definition 5.1.29 will become clear in the coming sections. For now, it may look like a very strong unexpected condition. In Section 5.2.2, we will generalize an idea of Ševera and define the infinitesimal counterpart of a differentiable n-groupoid by the categorical end (5.12) (see Def. 5.2.6), where we also use the $(n+1)$-coskeletality of G (Prop. 5.1.18). The existence of the end (5.12) turns out to be equivalent to the requirement that certain fiber products of the cosimplicial iterated tangent functor applied to the simplicial structure of G exist (Theorems 5.2.8 and 5.2.17).

Remark 5.1.32. Let G be a 1-groupoid object in a category $\mathcal{C}$ equipped with a Grothendieck pretopology satisfying Assumption 5.1.3. Then G is a groupoid object in $\mathcal{C}$ in the sense of Definition 4.1.1. If $\mathcal{C}$ has a tangent structure compatible with the pretopology, the differentiability of G in the sense of Definition 5.1.29 implies its differentiability in the sense of Definition 4.1.6.

Note that in Definition 4.1.6, making no assumption whatsoever of a pretopology on the ambient category, we have identified and explicitly spelled out all the pullbacks that we require to exist for the construction of the abstract Lie algebroid of a differentiable groupoid. If $\mathcal{C}$ has a pretopology, the existence of some of the pullbacks in Definition 4.1.6 already follows from the axioms of the pretopology. For instance, the vertical tangent bundle $VG_1 := TG_1 \times_{TG_0}^{Ts,0_{G_0}} G_0$ exists since s is a cover and T maps covers to covers.

The converse is more subtle already at the level of groupoid objects without the differentiability conditions. This is explained in Remark 5.1.15.

Remark 5.1.33. Every Lie n-groupoid is a differentiable n-groupoid object in the category of smooth finite-dimensional manifolds, with covers the surjective submersions and with the usual tangent structure. Note that in this case, the pretopology and the tangent structure are compatible since T maps surjective submersions to surjective submersions.

5.2 The differentiation procedure

This section is the core of this chapter, where we describe a method of differentiation of higher groupoid objects to their infinitesimal counterparts. In Section 5.2.1, we will explain the ideas of Ševera that he used to construct the L_∞-algebroid of a higher Lie groupoid [Šev06]. Generalizing these ideas, we will give a categorical formulation of the abstract higher Lie algebroid of a differentiable higher groupoid object in a category with a tangent structure and a Grothendieck pretopology. In Section 5.2.2 we state the main definition in terms of categorical ends, and in Section 5.2.3 we compute its corresponding limit. Using the simplicial identities, the naturality of the tangent structure, and concreteness assumptions, we show in Section 5.2.4 that half of the relations encoded by this limit are redundant.

5.2.1 Motivation: Ševera's idea

In 2006, Ševera has proposed a method of differentiating higher Lie groupoids, given by Kan simplicial manifolds, to higher Lie algebroids, given by differential graded manifolds [Šev06]. In order to explain Ševera's ideas, we need the notion of the *odd line* $\mathbb{R}^{0|1}$ as an infinitesimal model. In the literature, this object is sometimes called a *superpoint* (e.g. [HKST11]), or a *fat point* (e.g. [LRWZ23]).

The odd line, the odd tangent bundle, and the de Rham complex

In the same way smooth manifolds locally look like a finite-dimensional vector space, graded manifolds locally look like a graded vector space. If the grading is given by the group $\mathbb{Z}_2$, we get the concept of supermanifolds. The main idea lies in the fact that supermanifolds are endowed with both commuting and anti-commuting coordinate functions, called *even* and *odd* coordinates respectively.

Explicitly, a **(p,q)-supermanifold** $\mathcal{M}$ consists of a p-dimensional smooth manifold M, called its **core**, and a structure sheaf $\mathcal{O}_{\mathcal{M}}$ of $\mathbb{Z}_2$-graded commutative algebras on M, where for all open subsets $U \subseteq M$, $\mathcal{O}_{\mathcal{M}}(U) \cong \mathcal{M}\mathrm{fld}(U,\mathbb{R}) \otimes S(\theta^1,\dots,\theta^q)$. Here, $S(\theta^1,\dots,\theta^q)$ denotes the free graded symmetric algebra generated by odd coordinates θ^i. It follows that $\theta^i\theta^j = -\theta^j\theta^i$ and thus $(\theta^i)^2 = 0$ for all $1 \leq i,j \leq q$. Note that a smooth manifold can be viewed as a supermanifold with only even coordinates.

Remark 5.2.1. In general, a **sign convention** on an abelian group G is given by a group homomorphism $G \to \mathbb{Z}_2$. Elements of G whose image is 0 are called **even**, and those with image 1 are called **odd**. In graded geometry, G is commonly taken to be $\mathbb{Z}$, $\mathbb{Z}_2$ or $\mathbb{Z} \times \mathbb{Z}_2$. In the case when $G = \mathbb{Z}_2$, the sign convention is naturally given by the identity map.

A **morphism** $\varphi : \mathcal{M} \to \mathcal{N}$ **of supermanifolds** is given by a smooth map $\widetilde{\varphi} : M \to N$ on the cores and a natural transformation $\varphi^* : \mathcal{O}_{\mathcal{N}} \to \widetilde{\varphi}_*\mathcal{O}_{\mathcal{M}}$ of sheaves on N, where $\widetilde{\varphi}_*\mathcal{O}_{\mathcal{M}}$ is the direct image sheaf that assigns to each open subset $V \subseteq N$, $\widetilde{\varphi}_*\mathcal{O}_{\mathcal{M}}(V) := \mathcal{O}_{\mathcal{M}}\big(\widetilde{\varphi}^{-1}(V)\big)$. We denote the category of supermanifolds by $\mathcal{SM}\mathrm{fld}$.

Remark 5.2.2. Supergeometry was originally motivated by the discovery of supersymmetry in physics in the early 1970s [WZ74], which plays an essential role in the study of elementary particles, *bosons* and *fermions*. For a detailed and rigorous treatment of the subject, the reader may refer to [Lei80, Ber87, Man97] as the classical sources, as well as [Var04, CCF11] as more recent ones.

For us, the first interesting example is the **odd line** $\mathbb{R}^{0|1}$. It is the supermanifold with core the point space $\mathbb{R}^0 = *$ and the ring of functions $\mathcal{O}_{\mathbb{R}^{0|1}}$ the algebra of dual numbers. That is, its structure sheaf is given by

$$\mathcal{O}_{\mathbb{R}^{0|1}}(*) \cong S(\theta) \cong \mathbb{R} \oplus \theta\mathbb{R}\,,$$

where θ denotes the odd generator of $\mathbb{R}^{0|1}$.

The second interesting example for our purposes is the **odd tangent bundle** of a smooth manifold M. It is the supermanifold ΠTM with core M, where the structure sheaf $\mathcal{O}_{\Pi TM}$ is given by the sheaf Ω of differential forms on M. Spelled out, for all open subsets $U \subseteq M$,

$$\mathcal{O}_{\Pi TM}(U) = \Gamma(U, \Lambda T^*M|_U) = \Omega(U)\,.$$

Remark 5.2.3. The odd tangent bundle is a particular case of a more general construction. To every smooth vector bundle $E \to M$ there is an associated supermanifold ΠE with core M and structure sheaf given by $\mathcal{O}_{\Pi E}(U) := \Gamma(U, \Lambda E^*|_U)$ for each $U \subseteq M$ open. The notation comes from the parity reversing endofunctor Π on $\mathbb{Z}_2$-graded vector spaces. Moreover, Batchelor's theorem states that for any supermanifold $\mathcal{M}$ with core M, there exists a vector bundle E over M such that there is a (non-canonical) isomorphism $\Pi E \cong \mathcal{M}$ [Bat79, Bat85]. However, there are many more morphisms in supermanifolds than the ones coming from morphisms of vector bundles, so that we do not obtain an equivalence of categories. One may refer to [Var04, Sec. 4.2] for a detailed explanation or [Lei80, Ex. 2.1.3.(c)] for an example.

Given a (super)manifold M, it is folklore in the history of supermanifolds that the inner hom from $\mathbb{R}^{0|1}$ to M is represented by the odd tangent bundle of M [Kon03, ŠK03, Šev06], that is,

$$\underline{\mathrm{Hom}}(\mathbb{R}^{0|1}, M) \cong \Pi TM\,.$$

By the universal property of the inner hom, this means that there is a natural bijection

$$\mathrm{S}\mathcal{M}\mathrm{fld}(X \times \mathbb{R}^{0|1}, M) \cong \mathrm{S}\mathcal{M}\mathrm{fld}(X, \Pi TM)$$

for all $X \in \mathrm{S}\mathcal{M}\mathrm{fld}$ (see [HKST11, Prop. 3.1] for a rigorous proof). By induction, we get that for all $k \in \mathbb{N}$,

$$\underline{\mathrm{Hom}}\left((\mathbb{R}^{0|1})^k, M\right) \cong (\Pi T)^k M\,. \tag{5.14}$$

Furthermore, there is a natural right action of the graded ring $\underline{\mathrm{End}}(\mathbb{R}^{0|1}) := \underline{\mathrm{Hom}}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1})$ on $\underline{\mathrm{Hom}}(\mathbb{R}^{0|1}, M)$. This action is equivalent to the structure of a differential complex on the sheaf $\Omega(M)$ of differential forms on M, that is, to its $\mathbb{Z}$-grading and to the de Rham differential. Further details can be found in [Kon03, ŠK03, Šev06, HKST11].

Remark 5.2.4. In the smooth setting, a common approach to model higher Lie algebroids are differential (non-negatively) graded manifolds [SZ17] (also see Remark 1.2.5 for the case of Lie algebroids). Thus, the above mentioned right action of $\underline{\mathrm{End}}(\mathbb{R}^{0|1})$ on $\underline{\mathrm{Hom}}(\mathbb{R}^{0|1}, M)$ is crucial for the description of the L_∞-structure of the infinitesimal counterpart of the higher Lie groupoid.

The pair groupoid of the odd line

Ševera's main idea suggests replacing $\mathbb{R}^{0|1}$ by the nerve P of the pair groupoid[2] associated to $\mathbb{R}^{0|1}$. Spelled out, the supermanifold of m-simplices is given by

$$P_m = \underbrace{\mathbb{R}^{0|1} \times \cdots \times \mathbb{R}^{0|1}}_{(m+1)\text{-times}} = (\mathbb{R}^{0|1})^{m+1}$$

[2]See part (ii) of Example 1.1.8 for the notion of the pair groupoid in the category of smooth manifolds, and Example 2.1.6 for the nerve construction.

for all $m \geq 0$. The simplicial supermanifold P is then depicted by

$$\begin{tikzcd} \mathbb{R}^{0|1} \arrow[r, shift left=2, leftarrow] \arrow[r] \arrow[r, shift right=2, leftarrow] & \mathbb{R}^{0|1} \times \mathbb{R}^{0|1} \arrow[r, shift left=4, leftarrow] \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] \arrow[r, shift right=4, leftarrow] & \mathbb{R}^{0|1} \times \mathbb{R}^{0|1} \times \mathbb{R}^{0|1} & \cdots \end{tikzcd}$$

where the face morphisms are given by

$$\begin{aligned} d_{m,i} : (\mathbb{R}^{0|1})^{m+1} &\longrightarrow (\mathbb{R}^{0|1})^{m} \\ (\theta_0, \ldots, \theta_m) &\longmapsto (\theta_0, \ldots, \hat{\theta}_i, \ldots, \theta_m) \end{aligned}$$

for all $m \geq 1$, $0 \leq i \leq m$, and the degeneracy morphisms by

$$\begin{aligned} s_{m,i} : (\mathbb{R}^{0|1})^{m+1} &\longrightarrow (\mathbb{R}^{0|1})^{m+2} \\ (\theta_0, \ldots, \theta_m) &\longmapsto (\theta_0, \ldots, \theta_i, \theta_i, \ldots, \theta_m) \end{aligned}$$

for all $m \geq 0$, $0 \leq i \leq m$. Here, θ_i denotes different copies of the odd variable of $\mathbb{R}^{0|1}$.

From inner homs to categorical ends

Ševera has argued that the infinitesimal counterpart of a higher Lie groupoid $G : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{M}\mathrm{fld}$ is given by the *enriched* hom from P to G, where G is viewed as a simplicial supermanifold. An element of $\mathcal{SM}\mathrm{fld}^{\mathbf{\Delta}^{\mathrm{op}}}(P, G)$ is depicted by a commutative diagram[3] of the form

$$\begin{tikzcd} \vdots & \vdots \\ \mathbb{R}^{0|1} \times \mathbb{R}^{0|1} \times \mathbb{R}^{0|1} \arrow[r] \arrow[d, shift left=3] \arrow[d, shift left=1, leftarrow] \arrow[d, shift right=1] \arrow[d, shift right=3, leftarrow] \arrow[d, shift right=5] & G_2 \arrow[d, shift left=3] \arrow[d, shift left=1, leftarrow] \arrow[d, shift right=1] \arrow[d, shift right=3, leftarrow] \arrow[d, shift right=5] \\ \mathbb{R}^{0|1} \times \mathbb{R}^{0|1} \arrow[r] \arrow[d, shift left=2] \arrow[d, leftarrow] \arrow[d, shift right=2] & G_1 \arrow[d, shift left=2] \arrow[d, leftarrow] \arrow[d, shift right=2] \\ \mathbb{R}^{0|1} \arrow[r] & G_0 \end{tikzcd}$$

Using the language of functor of points, he defines the *1-jet* of G by the enriched hom from P to G, given by the presheaf

$$\underline{\mathrm{Hom}}(P, G) : N \longmapsto \mathcal{SM}\mathrm{fld}^{\mathbf{\Delta}^{\mathrm{op}}}(N \times P, G)$$

on the category of supermanifolds. In fact, this definition does not use the Kan conditions on G and makes sense for any simplicial manifold. Ševera's representability theorem [Šev06, Prop. 9.2] claims that under some truncation conditions, the 1-jet of the higher Lie groupoid G is representable and the representing object is a differential non-negatively graded manifold.

[3]Here, commutative means serially commutative. That is, the squares corresponding to a given face or degeneracy morphism commute.

Remark 5.2.5. The proof provided in [Šev06] lacks many details and has serious gaps. In his PhD thesis, Li has tried to overcome many of them [Li14]. However, Lemma 8.34 in [Li14] contains a mistake. Filling the gaps of [Šev06] and [Li14], Ševera's representability theorem has been rigorously formulated and proved in [LRWZ23]. The authors have shown that the 1-jet of the higher Lie groupoid G is represented by its tangent complex [LRWZ23, Thm. 3.3].

In the forthcoming section, we shall give a categorical generalization of Ševera's idea using the language of ends. It starts with the observation that ends provide a general framework to concisely formulate (enriched) homs in functor categories (see Remark A.2.4). As such, if G is a Lie n-groupoid, there is a natural bijection

$$\mathcal{SM}\mathrm{fld}^{\mathbf{\Delta}^{\mathrm{op}}}(P, G) \cong \int_{[m]\in \mathbf{\Delta}^{\mathrm{op}}_{\leq n+1}} \mathcal{SM}\mathrm{fld}\big((\mathbb{R}^{0|1})^{m+1}, G_m\big)\,,$$

using Remark 2.1.4 and the fact that G is $(n+1)$-coskeletal (Proposition 5.1.18). Thus, the enriched hom from P to G may be defined as

$$\begin{aligned}\underline{\mathrm{Hom}}(P, G) &:= \int_{[m]\in \mathbf{\Delta}^{\mathrm{op}}_{\leq n+1}} \underline{\mathrm{Hom}}\left((\mathbb{R}^{0|1})^{m+1}, G_m\right)\\ &\cong \int_{[m]\in \mathbf{\Delta}^{\mathrm{op}}_{\leq n+1}} (\Pi T)^{m+1} G_m\,, \end{aligned} \tag{5.15}$$

where the isomorphism follows from (5.14). The end (5.15) is independent of the infinitesimal model $\mathbb{R}^{0|1}$ and only contains the tangent functor T in the category of smooth manifolds.

The motivation described above can be summarized by the following diagram:

$$\begin{tikzcd}
\Pi TM & \cong & \underline{\mathrm{Hom}} \arrow[d, rightsquigarrow, "\text{[Kel05]}", "\substack{\text{enrichment of}\\ \text{functor}\\ \text{categories}}"'] & (\mathbb{R}^{0|1} \arrow[d, rightsquigarrow, "\text{[Šev06]}"] & , & M) \arrow[d, rightsquigarrow, "\substack{\text{simplicial}\\ \text{manifold}}"] \\
\mathrm{Lie}(G) & := & \int_{[m]\in\Delta^{\mathrm{op}}} \underline{\mathrm{Hom}} & \big((\mathbb{R}^{0|1})^{m+1} & , & G_m\big)
\end{tikzcd}$$

5.2.2 The higher Lie algebroid as a categorical end

The goal of this section is to give a categorical generalization of Ševera's ideas from the previous section. Using the language of ends, we will explain a differentiation method of higher groupoid objects. The reader may refer to Section A.2 for details on categorical ends and their main properties needed for our purposes. Let $\mathcal{C}$ be a category with a tangent structure and a compatible Grothendieck pretopology satisfying Assumption 5.1.3.

Definition 5.2.6. Let G be a differentiable n-groupoid object in $\mathcal{C}$ for some $n \geq 0$. The **abstract Lie n-algebroid** $\mathrm{Lie}(G)$ of G is defined by

$$\mathrm{Lie}(G) := \int_{[m]\in \mathbf{\Delta}^{\mathrm{op}}_{\leq n+1}} T^{m+1}G_m \,. \tag{5.16}$$

Let us spell out Definition 5.2.6 in more detail. Recall that $T^{\bullet+1}$ has the following (augmented) cosimplicial structure (Prop. 3.2.25): The coface and codegeneracy transformations are given by

$$\begin{aligned} d^{m,i} &:= T^{m-i}0T^i : T^m \longrightarrow T^{m+1}\,, \\ s^{m,i} &:= T^{m-i}\mu T^i : T^{m+2} \longrightarrow T^{m+1}\,, \end{aligned} \tag{5.17}$$

for $m \geq 0$ and $0 \leq i \leq m$. Here, $0 : 1 \to T$ is the zero section of the tangent structure and $\mu := + \circ (\pi T, T\pi) : T^2 \to T$ is the monad multiplication (Prop. 3.2.22).

Let G be a differentiable n-groupoid object in $\mathcal{C}$ for some $n \geq 0$. Denote its face and degeneracy morphisms by $d_{m,i}$ and $s_{m,i}$. Also, let $\delta^{m,i}$ and $\sigma^{m,i}$ denote the coface and codegeneracy maps of the simplex category $\mathbf{\Delta}$, and $\delta_{m,i}$ and $\sigma_{m,i}$ the corresponding maps in $\mathbf{\Delta}^{\mathrm{op}}$ (see Sec. 2.1.1). The end (5.16) is the end of the functor

$$\begin{aligned} T^{\bullet+1}G_\bullet : \big(\mathbf{\Delta}^{\mathrm{op}}_{\leq n+1}\big)^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}_{\leq n+1} &\longrightarrow \mathcal{C} \\ \big([m],[l]\big) &\longmapsto T^{m+1}G_l\,. \end{aligned} \tag{5.18}$$

It maps the morphism

$$(\delta^{m,i},\, \delta_{l,j}) : \big([m-1],[l]\big) \longrightarrow \big([m],[l-1]\big)$$

in the domain category to the morphism $T^mG_l \to T^{m+1}G_{l-1}$ in $\mathcal{C}$ given by the diagonal of

$$\begin{tikzcd} T^mG_l \arrow[r, "T^m d_{l,j}"] \arrow[d, "(T^{m-i}0T^i)_{G_l}"'] \arrow[dr] & T^mG_{l-1} \arrow[d, "(T^{m-i}0T^i)_{G_{l-1}}"] \\ T^{m+1}G_l \arrow[r, "T^{m+1}d_{l,j}"'] & T^{m+1}G_{l-1} \end{tikzcd} \tag{5.19}$$

for all $1 \leq m,l \leq n+1$, $0 \leq i \leq m$ and $0 \leq j \leq l$. Similarly, the functor (5.18) maps the morphism

$$(\sigma^{m,i},\, \sigma_{l,j}) : \big([m+1],[l]\big) \longrightarrow \big([m],[l+1]\big)$$

in the domain category to the morphism $T^{m+2}G_l \to T^{m+1}G_{l+1}$ in $\mathcal{C}$ given by the diagonal of

$$\begin{tikzcd} T^{m+2}G_l \arrow[r, "T^{m+2} s_{l,j}"] \arrow[d, "(T^{m-i}\mu T^i)_{G_l}"'] \arrow[dr] & T^{m+2}G_{l+1} \arrow[d, "(T^{m-i}\mu T^i)_{G_{l+1}}"] \\ T^{m+1}G_l \arrow[r, "T^{m+1}s_{l,j}"'] & T^{m+1}G_{l+1} \end{tikzcd} \tag{5.20}$$

The commutativity of Diagram (5.19) follows from the naturality of 0, that of Diagram (5.20) from the naturality of μ. It is easy to spell out the other commutative diagrams which are obtained as the images of different combinations of the (co)face and (co)degeneracy maps. This is sufficient to describe the functor (5.18) since any morphism of $\mathbf{\Delta}$ is finitely generated by its coface and codegeneracy maps (Rem. 2.1.1).

Remark 5.2.7. Note that we have not yet defined what general abstract Lie n-algebroids in tangent categories are. In Definition 5.2.6, we define the abstract Lie n-algebroid of a differentiable n-groupoid as an end, which a priori is only an object of $\mathcal{C}$. In Section 5.4 we will make a conjecture about the higher Lie algebroid structure on this object.

5.2.3 Computing the limit

Ends are universal constructions in category theory. They can be expressed as the limit of a particular diagram indexed by the *subdivision category.* This is explained in detail in Section A.2.2. In the case of the end (5.16), the limit can be greatly simplified.

Theorem 5.2.8. *Let $\mathcal{C}$ be a category with a tangent structure and a compatible Grothendieck pretopology satisfying Assumption 5.1.3. Let G be a differentiable n-groupoid object in $\mathcal{C}$ for some $n \geq 0$. Then, the end* $\mathrm{Lie}(G)$ *is isomorphic to the fiber product*

$$\mathrm{Lie}(G) \cong TG_0 \times_{Q_1} T^2G_1 \times_{Q_2} \cdots \times_{Q_{n+1}} T^{n+2}G_{n+1}\,, \tag{5.21}$$

where for all $1 \leq m \leq n+1$,

$$Q_m := (T^m G_m)^m \times (T^{m+1}G_{m-1})^{m+1}\,,$$

and where the fiber product is with respect to the (co)face and (co)degeneracy morphisms of $T^{\bullet+1}$ and $G_\bullet$:

$$T^m G_{m-1} \xrightarrow{\left((T^m s_{m-1,j})_{0\leq j\leq m-1},\ (T^{m-i}0T^i)_{G_{m-1},\,0\leq i\leq m}\right)} Q_m$$
$$T^{m+1}G_m \xrightarrow{\left((T^{m-j-1}\mu T^j)_{G_m,\,0\leq j\leq m-1},\ (T^{m+1}d_{m,i})_{0\leq i\leq m}\right)} Q_m\,.$$

Remark 5.2.9. Before we proceed with the proof of this theorem, let us analyze the fiber product (5.21) for $n = 1$. It is the limit of the following diagram:

$$
\begin{tikzcd}
 & & & T^3G_1 & \\
 & T^2G_0 & & T^2G_2 & \\
TG_0 \arrow[ur, "(T0)_{G_0}"] \arrow[r, "Ts_{0,0}"] \arrow[dr, "(0T)_{G_0}"'] & TG_1 & T^2G_1 \arrow[ul, "T^2d_{1,0}"'] \arrow[l, "\mu_{G_1}"'] \arrow[dl, "T^2d_{1,1}"] \arrow[uur, "(T^20)_{G_1}"] \arrow[ur, "T^2s_{1,0}"] \arrow[r, "(T0T)_{G_1}"] \arrow[dr, "T^2s_{1,1}"] \arrow[ddr, "(0T^2)_{G_1}"'] & T^3G_1 & T^3G_2 \arrow[uul, "T^3d_{2,0}"'] \arrow[ul, "(T\mu)_{G_2}"'] \arrow[l, "T^3d_{2,1}"'] \arrow[dl, "(\mu T)_{G_2}"] \arrow[ddl, "T^3d_{2,2}"] \\
 & T^2G_0 & & T^2G_2 & \\
 & & & T^3G_1 &
\end{tikzcd}
\tag{5.22}
$$

In other words, it is the fiber product of the objects TG_0, T^2G_1 and T^3G_2 in blue over the objects

$$
\begin{aligned}
Q_1 &= TG_1 \times (T^2G_0)^2 \\
Q_2 &= (T^2G_2)^2 \times (T^3G_1)^3
\end{aligned}
$$

in black. Moreover, the fiber product is with respect to the (co)face morphisms depicted in purple and the (co)degeneracy morphisms depicted in orange (see Equation (5.17) for the cosimplicial structure of $T^{\bullet+1}$).

Proof of Theorem 5.2.8. By definition, $\mathrm{Lie}(G)$ is the end of the functor

$$
S := T^{\bullet+1}G_\bullet : \big(\mathbf{\Delta}^{\mathrm{op}}_{\leq n+1}\big)^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}_{\leq n+1} \longrightarrow \mathcal{C}\,,
$$

as explained in (5.18). We will use Proposition A.2.5 to compute the limit corresponding to the end of this functor. For that, we first need to describe the subdivision category $\mathcal{I}^\S$ of $\mathcal{I} := \mathbf{\Delta}^{\mathrm{op}}_{\leq n+1}$. Its objects are composed of the following set of symbols

$$
\big\{[m]^\S \mid 0 \leq m \leq n+1\big\} \cup \big\{\delta^\S_{m,i}\,,\, \sigma^\S_{m-1,j} \mid 1 \leq m \leq n+1\big\}
$$

and its non-identity morphisms are given by

$$
\big\{[m]^\S \to \delta^\S_{m,i} \leftarrow [m-1]^\S\big\} \cup \big\{[m-1]^\S \to \sigma^\S_{m-1,j} \leftarrow [m]^\S\big\}\,.
$$

As explained in Remark A.2.7, it is sufficient to consider the generating morphisms of the simplex category to compute the limit of the associated functor $S^\S : \mathcal{I}^\S \to \mathcal{C}$. At the level of objects, $S^\S$ maps

$$
\begin{array}{rcl}
[m]^\S & \longmapsto & S\big([m],[m]\big) = T^{m+1}G_m \\
\delta^\S_{m,i} & \longmapsto & S\big([m],[m-1]\big) = T^{m+1}G_{m-1} \\
\sigma^\S_{m-1,j} & \longmapsto & S\big([m-1],[m]\big) = T^{m}G_{m}\,.
\end{array}
$$

It maps the morphisms $[m]^\S \to \delta^\S_{m,i} \leftarrow [m-1]^\S$ and $[m-1]^\S \to \sigma^\S_{m-1,j} \leftarrow [m]^\S$ of $\mathcal{J}^\S$ to the morphisms

$$\begin{tikzcd} T^{m+1}G_m \arrow[r, "S\left([m],\delta_{m,i}\right)"] & T^{m+1}G_{m-1} & T^m G_{m-1} \arrow[l, "S\left(\delta^{m,i},[m-1]\right)"'] \\ T^m G_{m-1} \arrow[r, "S\left([m-1],\sigma_{m-1,j}\right)"] & T^m G_m & T^{m+1}G_m \arrow[l, "S\left(\sigma^{m-1,j},[m]\right)"'] \end{tikzcd} \tag{5.23}$$

of $\mathcal{C}$ respectively, where

$$\begin{aligned} S\big([m],\delta_{m,i}\big) &= T^{m+1}d_{m,i} \\ S\big(\delta^{m,i},[m-1]\big) &= (T^{m-i}0T^i)_{G_{m-1}} \\ S\big([m-1],\sigma_{m-1,j}\big) &= T^m s_{m-1,j} \\ S\big(\sigma^{m-1,j},[m]\big) &= (T^{m-j-1}\mu T^j)_{G_m}\,, \end{aligned}$$

for all $1 \le m \le n+1$, $0 \le i \le m$ and $0 \le j \le m-1$. Using Proposition A.2.5, we have that

$$\mathrm{Lie}(G) \cong \lim(S^\S : \mathcal{J}^\S \to \mathcal{C})\,,$$

which is the limit of successive diagrams of the form

$$\begin{tikzcd} & T^{m+1}G_{m-1} & \\ & T^m G_m & \\ T^m G_{m-1} \arrow[uur, "(T^m 0)_{G_{m-1}}"] \arrow[ur, "T^m s_{m-1,0}"'] \arrow[dr, "T^m s_{m-1,m-1}"] \arrow[ddr, "(0T^m)_{G_{m-1}}"'] & \vdots & T^{m+1}G_m \arrow[uul, "T^{m+1}d_{m,0}"'] \arrow[ul, "(T^{m-1}\mu)_{G_m}"] \arrow[dl, "(\mu T^{m-1})_{G_m}"'] \arrow[ddl, "T^{m+1}d_{m,m}"] \\ & T^m G_m & \\ & T^{m+1}G_{m-1} & \end{tikzcd} \tag{5.24}$$

for all $1 \le m \le n+1$. This follows from the morphisms (5.23) in the image of $S^\S$. We conclude that $\mathrm{Lie}(G)$ is the fiber product expressed in (5.21). $\square$

The relations in components

For the rest of this chapter, we assume that $\mathcal{C}$ has a concrete structure given by the functor of points (5.6). Theorem 5.2.8 and its proof suggest that the elements of $\mathrm{Lie}(G)$ are given by tuples

$$(x^1,\ldots,x^{n+2}) \quad \in \quad TG_0 \times \cdots \times T^{n+2}G_{n+1}$$

satisfying certain relations. We will denote the **(co)face relations**, that is the relations obtained from the limit of

$$S^{\S}\big([m]^{\S} \to \delta^{\S}_{m,i} \leftarrow [m-1]^{\S}\big)$$

by $R^d_{m,i}$ for all $1 \leq m \leq n+1$ and $0 \leq i \leq m$. These are depicted in purple in Diagram (5.24). The **(co)degeneracy relations**, that is the ones obtained from the limit of

$$S^{\S}\big([m-1]^{\S} \to \sigma^{\S}_{m-1,j} \leftarrow [m]^{\S}\big)$$

will be denoted by $R^s_{m-1,j}$ for all $1 \leq m \leq n+1$ and $0 \leq j \leq m-1$. These are depicted in orange in Diagram (5.24).

Terminology 5.2.10. Let $m \in \{1, \ldots, n+1\}$. The set of relations $R^d_{m,i}$ and $R^s_{m-1,j}$ will be called the **box** m of the fiber product (5.21). It is depicted by Diagram (5.24).

Spelled out, the relations in box m are given by

$$\begin{aligned} R^d_{m,i} : (T^{m-i}0T^i)_{G_{m-1}}(x^m) &= T^{m+1}d_{m,i}(x^{m+1}) \\ R^s_{m-1,j} : T^m s_{m-1,j}(x^m) &= (T^{m-j-1}\mu T^j)_{G_m}(x^{m+1}) \end{aligned} \tag{5.25}$$

for all $x^m \in T^m G_{m-1}$ and $x^{m+1} \in T^{m+1}G_m$.

Remark 5.2.11. In box m, there are $m+1$ (co)face relations and m (co)degeneracy relations.

The higher multiplication in the last box

Now, let $k \in \{0, \ldots, n+1\}$ be fixed. Since we assume that $\mathcal{C}$ is a concrete category, we will employ the component notation from Remark 2.2.13. Also, we will use the notations introduced in Section 5.1.2.

Remark 5.2.12. Since G is differentiable, $T^m G$ is an n-groupoid object in $\mathcal{C}$ for all $m \geq 1$ (see Definition 5.1.29 and its consequences). Its $(n+1,k)$-horn projection factors as follows:

$$\begin{tikzcd} T^m G_{n+1} \arrow[rr, "\cong"] \arrow[dr, "\cong", "T^m p_{n+1,k}"'] & & (T^m G)(\Lambda^{n+1}_k) \\ & T^m\big(G(\Lambda^{n+1}_k)\big) \arrow[ur, "\cong"] & \end{tikzcd} \tag{5.26}$$

where $p_{n+1,k} : G_{n+1} \xrightarrow{\cong} G(\Lambda^{n+1}_k)$ is the horn projection of G. In components, the horn projection can be expressed as

$$\begin{aligned} T^m G_{n+1} &\xrightarrow{\cong} T^m G(\Lambda^{n+1}_k) \\ u &\longmapsto (u_0, \ldots, \widehat{u}_k, \ldots, u_{n+1}), \end{aligned} \tag{5.27}$$

where $u_i := T^m d_{n+1,i}(u)$ for all $0 \leq i \leq n+1$. Here, we have used the component description of the object $(T^m G)(\Lambda^{n+1}_k)$ of horns and Notation 5.1.31 to omit the extra parenthesis. We shall similarly identify u with its horn projection and write $u \equiv (u_0, \ldots, \widehat{u}_k, \ldots, u_{n+1})$, as done for the n-groupoid G in Remark 5.1.20.

Recall from Section 5.1.2 that $\widetilde{d}_{n+1,i} := d_{n+1,i} \circ p^{-1}_{n+1,k} : G(\Lambda^{n+1}_k) \to G_n$ is the $(i+1)$-projection if $i < k$ and the i-projection if $i > k$. Let

$$m := \widetilde{d}_{n+1,k} : G(\Lambda^{n+1}_k) \longrightarrow G_n$$

be the $(n+1, k)$-multiplication of G (Def. 5.1.22). Moreover, the degeneracy morphisms $\widetilde{s}_{n,j} := p_{n+1,k} \circ s_{n,j} : G_n \to G(\Lambda^{n+1}_k)$ are described by Lemma 5.1.26.

Using Isomorphism (5.27), the face morphisms $\widetilde{d}_{n+1,i}$, and the degeneracy morphisms $\widetilde{s}_{n,j}$, we can replace box $n+1$ (this is Diagram (5.24) for $m = n+1$) by the following new diagram

$$\begin{tikzcd}
 & T^{n+2}G_n & \\
 & T^{n+1}G(\Lambda^{n+1}_k) & \\
 & \vdots & \\
T^{n+1}G_n \arrow[uuur, "(T^{n+1}0)_{G_n}"] \arrow[uur, "T^{n+1}\widetilde{s}_{n,0}"'] \arrow[r, "(T^{n-k+1}0T^k)_{G_n}"'] \arrow[ddr, "T^{n+1}\widetilde{s}_{n,n}"] \arrow[dddr, "(0T^{n+1})_{G_n}"'] & T^{n+2}G_n & T^{n+2}G(\Lambda^{n+1}_k) \arrow[uuul, "\mathrm{pr}_1"'] \arrow[uul, "(T^n\mu)_{G(\Lambda^{n+1}_k)}"] \arrow[l, "T^{n+2}m"] \arrow[ddl, "(\mu T^n)_{G(\Lambda^{n+1}_k)}"'] \arrow[dddl, "\mathrm{pr}_{n+1}"] \\
 & \vdots & \\
 & T^{n+1}G(\Lambda^{n+1}_k) & \\
 & T^{n+2}G_n &
\end{tikzcd} \tag{5.28}$$

Explicitly, let $x^{n+1} \in T^{n+1}G_n$ and

$$x^{n+2} = (x^{n+2}_0, \ldots, \widehat{x^{n+2}_k}, \ldots, x^{n+2}_{n+1}) \quad \in \quad T^{n+2}G_{n+1} \cong T^{n+2}G(\Lambda^{n+1}_k)$$

with $x^{n+2}_i := T^{n+2}d_{n+1,i}\,(x^{n+2})$ for all $0 \le i \le n+1$ (see Rem. 5.2.12). Using these component expressions, the (co)face relations $R^d_{n+1,i}$ in box $n+1$ are equivalent to the relations

$$\widetilde{R}^d_{n+1,i}:\ (T^{n-i+1}0T^i)_{G_n}(x^{n+1}) \ = \ \begin{cases} \mathrm{pr}_{i+1}(x^{n+2}) = x_i^{n+2} & \text{if } i<k\,, \\ T^{n+2}m(x^{n+2}) & \text{if } i=k\,, \\ \mathrm{pr}_i(x^{n+2}) = x_i^{n+2} & \text{if } i>k\,. \end{cases} \tag{5.29}$$

Similarly, the (co)degeneracy relations $R^s_{n,j}$ in box $n+1$ are equivalent to the relations

$$\widetilde{R}^s_{n,j}:\ T^{n+1}\widetilde{s}_{n,j}(x^{n+1}) \ = \ (T^{n-j}\mu T^j)_{G(\Lambda_k^{n+1})}(x^{n+2})\,, \tag{5.30}$$

where the right hand side has the following component expression

$$\Big((T^{n-j}\mu T^j)_{G_n}(x_0^{n+2}),\ \ldots,\ \widehat{(T^{n-j}\mu T^j)_{G_n}}(x_k^{n+2}),\ \ldots,\ (T^{n-j}\mu T^j)_{G_n}(x_{n+1}^{n+2})\Big)\ , \tag{5.31}$$

by using Remark 5.2.12 and the following lemma.

Lemma 5.2.13. *For all $u \in T^2G_{n+1}$, the following equation*

$$\mu_{G(\Lambda_k^{n+1})}(u_0,\ldots,\widehat{u}_k,\ldots,u_{n+1}) = \big(\mu_{G_n}(u_0),\ldots,\widehat{\mu_{G_n}(u_k)},\ldots,\mu_{G_n}(u_{n+1})\big)$$

holds, where $u_i := T^2d_{n+1,i}(u)$ for all $0 \le i \le n+1$.

Proof. Using Isomorphism (5.27) for $m=2$, we let $u = (u_0,\ldots,\widehat{u}_k,\ldots,u_{n+1}) \in T^2G_{n+1} \cong T^2G(\Lambda_k^{n+1})$, where $u_i := T^2d_{n+1,i}(u)$. Let $v := \mu_{G_{n+1}}(u)$, By the naturality of μ, the following diagram

$$\begin{tikzcd} T^2G_{n+1} \arrow[r, "\mu_{G_{n+1}}"] \arrow[d, "T^2p_{n+1,k}"', "\cong"] & TG_{n+1} \arrow[d, "\cong"', "Tp_{n+1,k}"] \\ T^2G(\Lambda_k^{n+1}) \arrow[r, "\mu_{G(\Lambda_k^{n+1})}"] & TG(\Lambda_k^{n+1}) \end{tikzcd}$$

commutes, which implies that

$$\mu_{G(\Lambda_k^{n+1})}(u_0,\ldots,\widehat{u}_k,\ldots,u_{n+1}) = (v_0,\ldots,\widehat{v}_k,\ldots,v_{n+1})\,.$$

We have for all $0 \le i \le n+1$,

$$\begin{aligned} v_i &:= Td_{n+1,i}(v) \\ &= Td_{n+1,i}\circ\mu_{G_{n+1}}(u) \\ &= \mu_{G_n}\circ T^2d_{n+1,i}(u) \\ &= \mu_{G_n}(u_i) \end{aligned}$$

by using the naturality of μ once more. $\square$

5.2.4 Redundancy of relations

The aim of this section is to prove that half of the relations given in (5.25) are redundant. The next lemma shows that the (co)face relations in box m and the (co)degeneracy relations in box $m-1$ imply the (co)face relations in box $m-1$.

Lemma 5.2.14. *Let $m \in \{2, \dots, n+1\}$. Assume that Relations $R^d_{m,i}$ and $R^s_{m-2,j}$ hold for all $0 \le i \le m$ and $0 \le j \le m-2$. Then, Relation $R^d_{m-1,p}$ holds for all $0 \le p \le m-1$.*

Proof. Since we assume that $\mathcal{C}$ has a concrete structure, we will prove the result with elements. Let

$$\begin{aligned} x^{m-1} &\in T^{m-1}G_{m-2} \\ x^m &\in T^m G_{m-1} \\ x^{m+1} &\in T^{m+1}G_m \,. \end{aligned}$$

As a first step, we will show that the Relations $R^d_{m,i}$ and $R^s_{m-2,j}$ induce some equations, which will then be used to prove that $R^d_{m-1,p}$ holds.

For all $0 \le j \le m-2$, applying $T^{m-1}d_{m-1,j}$ to Relation $R^s_{m-2,j}$ yields

$$T^{m-1}d_{m-1,j} \circ T^{m-1}s_{m-2,j}\,(x^{m-1}) = T^{m-1}d_{m-1,j} \circ (T^{m-j-2}\mu T^j)_{G_{m-1}}\,(x^m)\,.$$

The left hand side of this equation is the identity by using the simplicial identity (2.9) and the functoriality of T. Thus, we get that

$$x^{m-1} = T^{m-1}d_{m-1,j} \circ (T^{m-j-2}\mu T^j)_{G_{m-1}}\,(x^m) \tag{5.32}$$

for all $0 \le j \le m-2$. Similarly, by applying $T^{m-1}d_{m-1,m-1}$ to Relation $R^s_{m-2,m-2}$, we get that

$$x^{m-1} = T^{m-1}d_{m-1,m-1} \circ (\mu T^{m-2})_{G_{m-1}}\,(x^m)\,. \tag{5.33}$$

Moreover, for all $0 \le p \le m-1$, by applying $T^{m+1}d_{m-1,p}$ to Relation $R^d_{m,p}$, we have

$$\begin{aligned} T^{m+1}d_{m-1,p} \circ (T^{m-p}0T^p)_{G_{m-1}}\,(x^m) &= T^{m+1}d_{m-1,p} \circ T^{m+1}d_{m,p}\,(x^{m+1}) \\ &= T^{m+1}d_{m-1,p} \circ T^{m+1}d_{m,p+1}\,(x^{m+1}) \\ &= T^{m+1}d_{m-1,p} \circ (T^{m-p-1}0T^{p+1})_{G_{m-1}}\,(x^m)\,, \end{aligned} \tag{5.34}$$

where in the second step we have used the simplicial identity (2.6) and the functoriality of T, and in the last step Relation $R^d_{m,p+1}$. Similarly, applying $T^{m+1}d_{m-1,m-1}$ to Relation $R^d_{m,m}$ yields

$$T^{m+1}d_{m-1,m-1} \circ (0T^m)_{G_{m-1}}\,(x^m) = T^{m+1}d_{m-1,m-1} \circ (T0T^{m-1})_{G_{m-1}}\,(x^m)\,, \tag{5.35}$$

where we have used $R^d_{m,m-1}$.

As a second step, we show that Relation $R^d_{m-1,p}$ holds. For all $0 \leq p \leq m-2$,

$$\begin{aligned}
&(T^{m-p-1}0T^p)_{G_{m-2}}(x^{m-1}) \\
&= (T^{m-p-1}0T^p)_{G_{m-2}} \circ T^{m-1}d_{m-1,p} \circ (T^{m-p-2}\mu T^p)_{G_{m-1}}(x^m) \\
&= T^m d_{m-1,p} \circ (T^{m-p-1}0T^p)_{G_{m-1}} \circ (T^{m-p-2}\mu T^p)_{G_{m-1}}(x^m) \\
&= T^m d_{m-1,p} \circ (T^{m-p-2}\mu T^{p+1})_{G_{m-1}} \circ (T^{m-p}0T^p)_{G_{m-1}}(x^m) \\
&= (T^{m-p-2}\mu T^{p+1})_{G_{m-2}} \circ T^{m+1}d_{m-1,p} \circ (T^{m-p}0T^p)_{G_{m-1}}(x^m) \\
&= (T^{m-p-2}\mu T^{p+1})_{G_{m-2}} \circ T^{m+1}d_{m-1,p} \circ (T^{m-p-1}0T^{p+1})_{G_{m-1}}(x^m) \\
&= T^m d_{m-1,p} \circ (T^{m-p-2}\mu T^{p+1})_{G_{m-1}} \circ (T^{m-p-1}0T^{p+1})_{G_{m-1}}(x^m) \\
&= T^m d_{m-1,p}(x^m),
\end{aligned}$$

where we have consecutively used Equation (5.32) for $j = p$, the naturality of 0, the cosimplicial identity (2.3) for $T^{\bullet+1}$ (this is spelled out by Equations (3.53) and (3.54) for $n = m-1$, $i = p$ and $j = p+1$), the naturality of μ, Equation (5.34), the naturality of μ once more, and finally the unitality of 0 with respect to μ, as expressed by the right of Diagram (3.43). This proves $R^d_{m-1,p}$ for all $0 \leq p \leq m-2$.

By using a similar sequence of arguments, and Equations (5.33) and (5.35), it can be shown that Relation $R^d_{m-1,m-1}$ holds too. □

Corollary 5.2.15. *Assume that Relations $\widetilde{R}^d_{n+1,i}$ and $R^s_{n-1,j}$ hold for all $0 \leq i \leq n+1$ and $0 \leq j \leq n-1$. Then, Relation $R^d_{n,p}$ holds for all $0 \leq p \leq n$.*

Proof. With the component expression of elements in $T^{n+2}G_{n+1} \cong T^{n+2}G(\Lambda^{n+1}_k)$, the (co)face relations $R^d_{n+1,i}$ are equivalent to $\widetilde{R}^d_{n+1,i}$, as described in (5.29). The result then follows from Lemma 5.2.14. □

We now show that the (co)face relations in box $n+1$ and the (co)degeneracy relations in box n imply the (co)degeneracy relations in box $n+1$.

Lemma 5.2.16. *Assume that Relations $\widetilde{R}^d_{n+1,i}$ and $R^s_{n-1,j}$ hold for all $0 \leq i \leq n+1$ and $0 \leq j \leq n-1$. Then, Relation $\widetilde{R}^s_{n,p}$ holds for all $0 \leq p \leq n$.*

Proof. Since we assume that $\mathcal{C}$ has a concrete structure, we will prove the result with elements. Let

$$\begin{aligned}
x^n &\in T^n G_{n-1} \\
x^{n+1} &\in T^{n+1} G_n \\
x^{n+2} = (x^{n+2}_0, \ldots, \widehat{x^{n+2}_k}, \ldots, x^{n+2}_{n+1}) &\in T^{n+2}G_{n+1} \cong T^{n+2}G(\Lambda^{n+1}_k),
\end{aligned}$$

where $x^{n+2}_i := T^{n+2}d_{n+1,i}(x^{n+2})$ for all $0 \leq i \leq n+1$ (see Rem. 5.2.12).

Our aim is to show that Relation $\widetilde{R}^s_{n,p}$, as described in (5.30), holds, by using the formula of the degeneracy morphisms from Lemma 5.1.26. Explicitly, we will show that the components of the right hand side and the left hand side of Relation $\widetilde{R}^s_{n,p}$ are equal.

Let $p < k$. Then, for all $0 \leq i \leq p-1$, the i^{th} component of $T^{n+1}\widetilde{s}_{n,p}\,(x^{n+1})$ is given by

$$\begin{aligned} T^{n+1}(s_{n-1,p-1} \circ d_{n,i})\,(x^{n+1}) &= T^{n+1}s_{n-1,p-1} \circ T^{n+1}d_{n,i}\,(x^{n+1}) \\ &= T^{n+1}s_{n-1,p-1} \circ (T^{n-i}0T^i)_{G_{n-1}}\,(x^n) \\ &= (T^{n-i}0T^i)_{G_n} \circ T^n s_{n-1,p-1}\,(x^n) \\ &= (T^{n-i}0T^i)_{G_n} \circ (T^{n-p}\mu T^{p-1})_{G_n}\,(x^{n+1}) \\ &= (T^{n-p}\mu T^p)_{G_n} \circ (T^{n-i+1}0T^i)_{G_n}\,(x^{n+1})\,, \end{aligned}$$

where we have used Lemma 5.1.26, the functoriality of T, Relation $R^d_{n,i}$ (which holds by Corollary 5.2.15), the naturality of 0, Relation $R^s_{n-1,p-1}$ from the assumptions, and finally the cosimplicial identity (2.3) for $T^{\bullet+1}$ (this is spelled out by Equations (3.53) and (3.54) for $j = p$).

On the other hand, the i^{th} component of $(T^{n-p}\mu T^p)_{G(\Lambda^{n+1}_k)}\,(x^{n+2})$ is given by

$$\begin{aligned} (T^{n-p}\mu T^p)_{G_n}\,(x^{n+2}_i) &= (T^{n-p}\mu T^p)_{G_n} \circ \mathrm{pr}_{i+1}\,(x^{n+2}) \\ &= (T^{n-p}\mu T^p)_{G_n} \circ (T^{n-i+1}0T^i)_{G_n}\,(x^{n+1}) \end{aligned}$$

for all $0 \leq i \leq p-1$, where we have used Expression (5.31) and Relation $\widetilde{R}^d_{n+1,i}$ given in (5.29).

We conclude that the i^{th} components of the right hand side and the left hand side of Relation $\widetilde{R}^s_{n,p}$ are equal for all $0 \leq i \leq p-1$.

It follows from Lemma 5.1.26 that the $(p+1)$ and $(p+2)$ slots of $T^{n+1}\widetilde{s}_{n,p}\,(x^{n+1})$ are the identity morphisms, where we assume without loss of generality that $p \neq k-1$ (see Remark 5.1.27). By a similar argument as above, the $(p+1)$ component of $(T^{n-p}\mu T^p)_{G(\Lambda^{n+1}_k)}\,(x^{n+2})$ is given by

$$(T^{n-p}\mu T^p)_{G_n}\,(x^{n+2}_p) = (T^{n-p}\mu T^p)_{G_n} \circ (T^{n-p+1}0T^p)_{G_n}\,(x^{n+1}) = x^{n+1}\,,$$

and its $(p+2)$ component by

$$(T^{n-p}\mu T^p)_{G_n}\,(x^{n+2}_{p+1}) = (T^{n-p}\mu T^p)_{G_n} \circ (T^{n-p}0T^{p+1})_{G_n}\,(x^{n+1}) = x^{n+1}\,,$$

where we have used Expression (5.31), Relation $\widetilde{R}^d_{n+1,i}$ and the unitality of 0 with respect to μ, as expressed by the right of Diagram (3.43).

This shows that the $(p+1)$ and $(p+2)$ components of the right hand side and the left hand side of Relation $\widetilde{R}^s_{n,p}$ are equal too.

Lastly, the remaining components of $T^{n+1}\widetilde{s}_{n,p}\,(x^{n+1})$, namely the $(p+3)$ component up to the $(n+1)$ component, are given by

$$\begin{aligned} T^{n+1}(s_{n-1,p}\circ d_{n,i})\,(x^{n+1}) &= T^{n+1}s_{n-1,p}\circ T^{n+1}d_{n,i}\,(x^{n+1}) \\ &= T^{n+1}s_{n-1,p}\circ (T^{n-i}0T^{i})_{G_{n-1}}\,(x^{n}) \\ &= (T^{n-i}0T^{i})_{G_{n}}\circ T^{n}s_{n-1,p}\,(x^{n}) \\ &= (T^{n-i}0T^{i})_{G_{n}}\circ (T^{n-p-1}\mu T^{p})_{G_{n}}\,(x^{n+1}) \\ &= (T^{n-p}\mu T^{p})_{G_{n}}\circ (T^{n-i}0T^{i+1})_{G_{n}}\,(x^{n+1}) \end{aligned}$$

for all $i\in\{p+1,\dots,\widehat{k-1},\dots,n\}$, by using Lemma 5.1.26, the functoriality of T, Relation $R^{d}_{n,i}$ (which holds by Corollary 5.2.15), the naturality of 0, Relation $R^{s}_{n-1,p}$ from the assumptions, and finally the cosimplicial identity (2.5) for $T^{\bullet+1}$ (this is spelled out by Equations (3.55) and (3.56) for $j=p$ and i replaced by $i+1$).

On the other hand, the remaining components of $(T^{n-p}\mu T^{p})_{G(\Lambda^{n+1}_{k})}\,(x^{n+2})$, mainly the $(p+3)$ component up to the $(n+1)$ component, are given by

$$(T^{n-p}\mu T^{p})_{G_{n}}\,(x^{n+2}_{j}) = (T^{n-p}\mu T^{p})_{G_{n}}\circ (T^{n-j+1}0T^{j})_{G_{n}}\,(x^{n+1})$$

for all $j\in\{p+2,\dots,\widehat{k},\dots,n+1\}$. Letting $j=i+1$ we get the desired equality.

As a conclusion, we have shown that all the components of the right hand side and the left hand side of Relation $\widetilde{R}^{s}_{n,p}$ are equal. This was done for the case $p<k$. The proof for $k\le p\le n$ is analogous. $\square$

Lemmas 5.2.14 and 5.2.16 imply that the (co)face relations in box $n+1$ and the (co)degeneracy relations in boxes 1 to n yield all the other relations. This is summarized by the following theorem, which is a reduced form of Theorem 5.2.8.

Theorem 5.2.17. *Let $\mathcal{C}$ be a category with a tangent structure and a compatible Grothendieck pretopology satisfying Assumption 5.1.3. Assume that the functor of points equip $\mathcal{C}$ with a concrete structure. Let G be a differentiable n-groupoid object in $\mathcal{C}$ for some $n\ge 0$. Then, the end* $\mathrm{Lie}(G)$ *is isomorphic to the fiber product*

$$\mathrm{Lie}(G)\cong TG_0\times_{\widetilde{Q}_1}T^2G_1\times_{\widetilde{Q}_2}\cdots\times_{\widetilde{Q}_n}T^{n+1}G_n\times_{\widetilde{Q}_{n+1}}T^{n+2}G(\Lambda^{n+1}_{k})\,,$$

where

$$\widetilde{Q}_m := \begin{cases} (T^mG_m)^m & \text{if } 1\le m\le n\,,\\ (T^{n+2}G_n)^{n+2} & \text{if } m=n+1\,,\end{cases}$$

and where the fiber product is with respect to

$$T^mG_{m-1}\xrightarrow{\left((T^m s_{m-1,j})_{0\le j\le m-1}\right)}\widetilde{Q}_m$$

$$T^{m+1}G_m\xrightarrow{\left((T^{m-j-1}\mu T^{j})_{G_m,\,0\le j\le m-1}\right)}\widetilde{Q}_m$$

for all $1 \leq m \leq n$*, and*

$$\begin{aligned} T^{n+1}G_n &\xrightarrow{\big((T^{n-i+1}0T^i)_{G_n,\,0\leq i\leq n+1}\big)} \widetilde{Q}_{n+1} \\ T^{n+2}G(\Lambda^{n+1}_k) &\xrightarrow{\big((\mathrm{pr}_{i+1})_{0\leq i<k},\,T^{n+2}m,\,(\mathrm{pr}_i)_{k<i\leq n+1}\big)} \widetilde{Q}_{n+1}\,. \end{aligned}$$

Proof. The result follows from Theorem 5.2.8, Diagram (5.28), Lemma 5.2.14, Corollary 5.2.15 and Lemma 5.2.16. $\square$

Example 5.2.18. Let $n = 1$ and $k = 1$. Then, $\mathrm{Lie}(G)$ is the fiber product

$$TG_0 \times_{TG_1} T^2G_1 \times_{(T^3G_1)^3} T^3G(\Lambda^2_1)\,,$$

which is the limit of the following diagram:

$$\begin{tikzcd}
 & & & T^3G_1 & \\
TG_0 \arrow[r, "Ts_{0,0}"] & TG_1 & T^2G_1 \arrow[l, "\mu_{G_1}"'] \arrow[ur, "(T^20)_{G_1}"] \arrow[r, "(T0T)_{G_1}"] \arrow[dr, "(0T^2)_{G_1}"'] & T^3G_1 & T^3G(\Lambda^2_1) \arrow[l, "T^3m"'] \arrow[ul, "\mathrm{pr}_1"'] \arrow[dl, "\mathrm{pr}_2"] \\
 & & & T^3G_1 &
\end{tikzcd}$$

where m is the groupoid multiplication. This shows that half of the arrows in Diagram (5.22) are redundant. However, this does not mean that the redundant relations can be fully ignored. As it will be clear in Section 5.3.2, the (co)face relations $R^d_{1,0}$ and $R^d_{1,1}$ will be crucial in the description of the symmetric version of the Lie algebroid of the Lie groupoid G.

5.3 Example: Differentiation of Lie groupoids

In this section, we show that in the case of a Lie groupoid, the end construction yields the symmetric version of its usual Lie algebroid. For the end construction, which is (co)simplicial in nature, a Lie groupoid will be given by its nerve, which is a Lie 1-groupoid (Terminology 5.1.16). Recall that the category of Lie groupoids and the category of Lie 1-groupoids are equivalent (Remark 2.2.22). Also recall that every Lie 1-groupoid is a differentiable 1-groupoid object in the category $\mathcal{M}\mathrm{fld}$ of smooth finite-dimensional manifolds with the usual tangent structure and the pretopology given by surjective submersions (Remark 5.1.33). Hence, our differentiation method is applicable.

In Section 5.3.1, we will spell out the end formula, as well as the (co)face and (co)degeneracy relations it induces for Lie groupoids. We proceed by computations in local coordinates in Section 5.3.2, where the Lie algebroid of a Lie groupoid is recovered.

5.3.1 The end construction

Let $G : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{M}\mathrm{fld}$, $[m] \mapsto G_m$ be (the nerve of) a Lie groupoid. The face maps of G will be denoted by

$$d_{m,i} : G_m \longrightarrow G_{m-1}$$

for $m \geq 1$, $0 \leq i \leq m$, and the degeneracy maps by

$$s_{m,i} : G_m \longrightarrow G_{m+1}$$

for $m \geq 0$, $0 \leq i \leq m$. Let $p_{2,j} : G_2 \xrightarrow{\cong} G(\Lambda^2_j)$ be the horn projections for $j = 0, 1, 2$. Let us recall how we obtain the structure maps of the Lie groupoid, as explained in detail in Remark 2.2.22. The source map is given by $s = d_{1,1}$, the target map by $t = d_{1,0}$, the unit by $1 = s_{0,0}$, and the groupoid multiplication by

$$m = \widetilde{d}_{2,1} : G_1 \times^{s,t}_{G_0} G_1 \xrightarrow[\cong]{p_{2,1}^{-1}} G_2 \xrightarrow{d_{2,1}} G_1 \,.$$

The outer face maps are the first and second projections of the pullback $G_1 \times^{s,t}_{G_0} G_1 \cong G(\Lambda^2_1)$. That is, $\widetilde{d}_{2,0} = d_{2,0} \circ p_{2,1}^{-1} = \mathrm{pr}_1$ and $\widetilde{d}_{2,2} = d_{2,2} \circ p_{2,1}^{-1} = \mathrm{pr}_2$. These maps are spelled out in Example 5.1.21. Finally, the inverse is given by

$$i : G_1 \xrightarrow{(\mathrm{id}_{G_1}, 1 \circ s)} G_1 \times^{s,s}_{G_0} G_1 \xrightarrow[\cong]{p_{2,0}^{-1}} G_2 \xrightarrow{d_{2,0}} G_1 \,.$$

Recall that

$$G_m \cong \underbrace{G_1 \times_{G_0} \cdots \times_{G_0} G_1}_{m \text{ factors}}$$

is the string of m-composable arrows for all $m \geq 2$.

Our goal is to spell out the end from Definition 5.2.6 for the case of the Lie groupoid G. Let

$$\mathrm{Lie}(G) = \int_{[m] \in \mathbf{\Delta}^{\mathrm{op}}_{\leq 2}} T^{m+1} G_m \tag{5.36}$$

be the end of the functor

$$\begin{aligned} T^{\bullet+1} G_\bullet : \big(\mathbf{\Delta}^{\mathrm{op}}_{\leq 2}\big)^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}_{\leq 2} &\longrightarrow \mathcal{M}\mathrm{fld} \\ \big([m], [l]\big) &\longmapsto T^{m+1} G_l \,. \end{aligned} \tag{5.37}$$

This is the functor (5.18) for $n = 1$ and $\mathcal{C} = \mathcal{M}\mathrm{fld}$. As explained in Section 5.2.2, it is well-defined because of the cosimplicial structure of the iterated tangent bundle $T^{\bullet+1}$ (Prop. 3.2.25). The functor (5.37) is illustrated by the following commutative[4]

[4]Commutativity here means serial commutativity, that is, the squares corresponding to a particular face or degeneracy map commute.

diagram:

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\textcolor{blue}{T^3G_2}
  \arrow[r, shift left=4, "T^3d_{2,0}"]
  \arrow[r]
  \arrow[r, shift right=4, "T^3d_{2,2}"']
  \arrow[d, shift left=2]
  \arrow[d, shift right=2]
& T^3G_1
  \arrow[l, shift left=2]
  \arrow[l, shift right=2]
  \arrow[r, shift left=2, "T^3d_{1,0}"]
  \arrow[r, shift right=2, "T^3d_{1,1}"']
  \arrow[d, shift left=2]
  \arrow[d, shift right=2]
& T^3G_0
  \arrow[l]
  \arrow[d, shift left=2]
  \arrow[d, shift right=2]
\\
T^2G_2
  \arrow[u, shift left=4]
  \arrow[u]
  \arrow[u, shift right=4, "(T^20)_{G_2}"]
  \arrow[r, shift left=4, "T^2d_{2,0}"]
  \arrow[r]
  \arrow[r, shift right=4, "T^2d_{2,2}"']
  \arrow[d]
& \textcolor{blue}{T^2G_1}
  \arrow[u, shift left=4]
  \arrow[u]
  \arrow[u, shift right=4]
  \arrow[l, shift left=2]
  \arrow[l, shift right=2]
  \arrow[r, shift left=2, "T^2d_{1,0}"]
  \arrow[r, shift right=2, "T^2d_{1,1}"']
  \arrow[d]
& T^2G_0
  \arrow[u, shift left=4, "(0T^2)_{G_0}"']
  \arrow[u]
  \arrow[u, shift right=4]
  \arrow[l]
  \arrow[d]
\\
TG_2
  \arrow[u, shift left=2]
  \arrow[u, shift right=2, "(T0)_{G_2}"]
  \arrow[r, shift left=4, "Td_{2,0}"]
  \arrow[r]
  \arrow[r, shift right=4, "Td_{2,2}"']
& TG_1
  \arrow[u, shift left=2]
  \arrow[u, shift right=2]
  \arrow[l, shift left=2]
  \arrow[l, shift right=2]
  \arrow[r, shift left=2, "Td_{1,0}"]
  \arrow[r, shift right=2, "Td_{1,1}"']
& \textcolor{blue}{TG_0}
  \arrow[u, shift left=2, "(0T)_{G_0}"']
  \arrow[u, shift right=2]
  \arrow[l]
\end{tikzcd}
\tag{5.38}
$$

The horizontal arrows depict the simplicial structure of G, the vertical arrows the cosimplicial structure of the iterated tangent bundle. By Theorem 5.2.8, the end of Diagram (5.38) is given by the fiber product of the blue objects TG_0, T^2G_1 and T^3G_2 with respect to the (co)face maps, depicted in purple, and the (co)degeneracy maps, depicted in orange. Explicitly,

$$
\mathrm{Lie}(G) \cong TG_0 \times_{Q_1} T^2G_1 \times_{Q_2} T^3G_2\,, \tag{5.39}
$$

where

$$
\begin{aligned}
Q_1 &:= TG_1 \times (T^2G_0)^2 \\
Q_2 &:= (T^2G_2)^2 \times (T^3G_1)^3\,.
\end{aligned}
$$

The fiber product (5.39) is the limit of Diagram (5.22). As explained in the last part of Section 5.2.3, box 2 of this diagram can be replaced by Diagram (5.28) for $n = 1$, by using the corresponding arrows obtained by the (pre)composition with the horn projection $p_{2,1} : G_2 \xrightarrow{\cong} G(\Lambda^2_1)$. As such, the end of Diagram (5.38) or, equivalently, the fiber product (5.39) is the limit of the following diagram:

$$
\begin{tikzcd}
 & & & T^3G_1 & \\
 & T^2G_0 & & T^2G(\Lambda^2_1) & \\
TG_0 \arrow[ur, "(T0)_{G_0}"] \arrow[r, "T1"] \arrow[dr, "(0T)_{G_0}"'] & TG_1 & T^2G_1 \arrow[ul, "T^2t"'] \arrow[l, "\mu_{G_1}"'] \arrow[dl, "T^2s"] \arrow[uur, "(T^20)_{G_1}"] \arrow[ur, "T^2\widetilde{s}_{1,0}"] \arrow[r, "(T0T)_{G_1}"] \arrow[dr, "T^2\widetilde{s}_{1,1}"'] \arrow[ddr, "(0T^2)_{G_1}"'] & T^3G_1 & T^3G(\Lambda^2_1) \arrow[uul, "\mathrm{pr}_1"'] \arrow[ul, "(T\mu)_{G(\Lambda^2_1)}"'] \arrow[l, "T^3m"'] \arrow[dl, "(\mu T)_{G(\Lambda^2_1)}"] \arrow[ddl, "\mathrm{pr}_2"] \\
 & T^2G_0 & & T^2G(\Lambda^2_1) & \\
 & & & T^3G_1 &
\end{tikzcd}
\tag{5.40}
$$

where $T^3G(\Lambda^2_1)$ is given by any of the isomorphic objects $T^3\big(G(\Lambda^2_1)\big) \cong (T^3G)(\Lambda^2_1) \cong T^3G_1 \times^{T^3s,T^3t}_{T^3G_0} T^3G_1$.

The relations in components

Following the notation introduced in Section 5.2.3, we will now express the relations induced by the fiber product (5.39) in components. Let

$$
\begin{aligned}
x &\in TG_0 \\
y &\in T^2G_1 \\
z = (z_0, \widehat{z}_1, z_2) &\in T^3G_2 \cong T^3G(\Lambda^2_1)\,,
\end{aligned}
$$

where $z_i = T^3d_{2,i}(z)$ for $i = 0,1,2$ (see Rem. 5.2.12). The relations are given by

$$
\begin{array}{llll}
 & & \widetilde{R}^d_{2,0}: & (T^20)_{G_1}(y) = \mathrm{pr}_1(z) = z_0 \\
R^d_{1,0}: & (T0)_{G_0}(x) = T^2t(y) & \widetilde{R}^s_{1,0}: & T^2\widetilde{s}_{1,0}(y) = (T\mu)_{G(\Lambda^2_1)}(z) \\
R^s_{0,0}: & T1(x) = \mu_{G_1}(y) & \widetilde{R}^d_{2,1}: & (T0T)_{G_1}(y) = T^3m(z) \\
R^d_{1,1}: & (0T)_{G_0}(x) = T^2s(y) & \widetilde{R}^s_{1,1}: & T^2\widetilde{s}_{1,1}(y) = (\mu T)_{G(\Lambda^2_1)}(z) \\
 & & \widetilde{R}^d_{2,2}: & (0T^2)_{G_1}(y) = \mathrm{pr}_2(z) = z_2\,.
\end{array}
\tag{5.41}
$$

The relations on the left column depict box 1 and those on the right column depict box 2. Recall that the superscript d denotes the (co)face relations and the superscript s the (co)degeneracy ones. These are Relations (5.25), (5.29) and (5.30).

Remark 5.3.1. It follows from Example 5.1.28 and Expression (5.31) that Relations $\widetilde{R}^s_{1,0}$ and $\widetilde{R}^s_{1,1}$ can be rewritten as

$$\begin{aligned}\big(y,\, T^2 s_{0,0} \circ T^2 d_{1,1}\,(y)\big) &= \big((T\mu)_{G_1}\,(z_0),\, (T\mu)_{G_1}\,(z_2)\big)\\ \big(T^2 s_{0,0} \circ T^2 d_{1,0}\,(y), y\big) &= \big((\mu T)_{G_1}\,(z_0),\, (\mu T)_{G_1}\,(z_2)\big)\end{aligned}$$

respectively.

5.3.2 Computations in local coordinates

The tangent structure on the category of smooth finite-dimensional manifolds is locally modeled by the tangent structure on Euclidean spaces. The aim of this section is to compute the relations in box 1 and 2 in local coordinates, where we use the descriptions from Section 3.4.

Notation 5.3.2. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open subsets. Let $f : U \to V$ be a smooth map. Consider the local coordinate expressions of the tangent maps

$$\begin{aligned}Tf : (u, u^i_0) &\longmapsto \left(f(u),\, \frac{\partial f^a}{\partial x^i} u^i_0\right)\\ T^2 f : (u, u^i_0, u^i_1, u^i_{01}) &\longmapsto \left(f(u),\, \frac{\partial f^a}{\partial x^i} u^i_0,\, \frac{\partial f^a}{\partial x^i} u^i_1,\, \frac{\partial f^a}{\partial x^i} u^i_{01} + \frac{\partial^2 f^a}{\partial x^i \partial x^j} u^i_0 u^j_1\right) ,\end{aligned}$$

as described in (3.81). For conciseness, we will use the following notation:

$$\begin{aligned}Df(u_0) &\equiv \frac{\partial f^a}{\partial x^i} u^i_0\\ Df(u_1) &\equiv \frac{\partial f^a}{\partial x^i} u^i_1\\ D^2 f(u_{01}) &\equiv \frac{\partial f^a}{\partial x^i} u^i_{01} + \frac{\partial^2 f^a}{\partial x^i \partial x^j} u^i_0 u^j_1 \,.\end{aligned}$$

Put together, we get the expression

$$T^2 f : (u, u_0, u_1, u_{01}) \longmapsto \big(f(u), Df(u_0), Df(u_1), D^2 f(u_{01})\big) ,$$

where we omit the index i for simplicity. This is in the same spirit as Notation 3.4.8. Higher order tangent maps are expressed combinatorially in the same manner.

Remark 5.3.3. Corollary 5.2.15 and Lemma 5.2.16 state that if Relations $\widetilde{R}^d_{2,0}$, $\widetilde{R}^d_{2,1}$, $\widetilde{R}^d_{2,2}$ and $R^s_{0,0}$ hold, we automatically obtain Relations $R^d_{1,0}$, $R^d_{1,1}$, $\widetilde{R}^s_{1,0}$ and $\widetilde{R}^s_{1,1}$. This can be summarized by Theorem 5.2.17, which states that there is an isomorphism

$$\mathrm{Lie}(G) \cong TG_0 \times_{TG_1} T^2 G_1 \times_{(T^3 G_1)^3} T^3 G(\Lambda^2_1)\,.$$

This is the limit of the diagram illustrated in Example 5.2.18. Under this observation, the first step is to spell out Relations $R^s_{0,0}$ and $\widetilde{R}^d_{2,i}$ for $i = 0, 1, 2$. It will afterwards become clear that the redundant relations $R^d_{1,0}$ and $R^d_{1,1}$ are crucial in the description of the symmetric version of the Lie algebroid of G.

Using the local coordinate expressions in (3.80), we let

$$\begin{aligned} x = (u, u_0) &\in TG_0 \\ y = (v, v_0, v_1, v_{01}) &\in T^2 G_1 \\ z = (z_0, \widehat{z}_1, z_2) &\in T^3 G_2\,, \end{aligned}$$

where we have used the identification $T^3G_2 \cong T^3G(\Lambda^2_1) \cong T^3G_1 \times^{T^3s,T^3t}_{T^3G_0} T^3G_1$, so that $\mathrm{pr}_1(z) = z_0$ and $\mathrm{pr}_2(z) = z_2$. Here, $z_i = T^3 d_{2,i}(z)$ for $i = 0, 1, 2$. We will write

$$\begin{aligned} z_0 = (w, w_0, w_1, w_{01}, w_2, w_{02}, w_{12}, w_{012}) &\in T^3 G_1 \\ z_2 = (w', w'_0, w'_1, w'_{01}, w'_2, w'_{02}, w'_{12}, w'_{012}) &\in T^3 G_1\,. \end{aligned}$$

By Notation 5.3.2, the tangent map of the identity bisection is given by

$$T1 : (u, u_0) \longmapsto \big(1_u, D1(u_0)\big)\,.$$

Moreover, the monad multiplication is given by

$$\mu_{G_1} : (v, v_0, v_1, v_{01}) \longmapsto (v, v_0 + v_1)\,,$$

as explained in (3.82). Thus, Relation $R^s_{0,0}$ is equivalent to

$$v = 1_u \tag{5.42}$$

$$v_0 + v_1 = D1(u_0)\,. \tag{5.43}$$

Similarly, the third order differential of the groupoid multiplication is given in local coordinates by

$$\begin{aligned} T^3m : (z_0, z_2) \longmapsto \big(&m(w, w'), Dm(w_0, w'_0), Dm(w_1, w'_1), D^2m(w_{01}, w'_{01}), \\ &Dm(w_2, w'_2), D^2m(w_{02}, w'_{02}), D^2m(w_{12}, w'_{12}), D^3m(w_{012}, w'_{012})\big)\,. \end{aligned}$$

Furthermore, the different combinations of the zero section and twice the tangent functor can be expressed by

$$\begin{aligned} (T^2 0)_{G_1} &: (v, v_0, v_1, v_{01}) \longmapsto (v, 0, v_0, 0, v_1, 0, v_{01}, 0) \\ (T0T)_{G_1} &: (v, v_0, v_1, v_{01}) \longmapsto (v, v_0, 0, 0, v_1, v_{01}, 0, 0) \\ (0T^2)_{G_1} &: (v, v_0, v_1, v_{01}) \longmapsto (v, v_0, v_1, v_{01}, 0, 0, 0, 0)\,. \end{aligned}$$

Hence, Relation $\widetilde{R}^d_{2,0}$ is equivalent to

$$\begin{aligned} w &= v & w_2 &= v_1 \\ w_0 &= 0 & w_{02} &= 0 \\ w_1 &= v_0 & w_{12} &= v_{01} \\ w_{01} &= 0 & w_{012} &= 0\,, \end{aligned}$$

and Relation $\widetilde{R}^d_{2,2}$ to

$$\begin{aligned} w' &= v & w'_2 &= 0 \\ w'_0 &= v_0 & w'_{02} &= 0 \\ w'_1 &= v_1 & w'_{12} &= 0 \\ w'_{01} &= v_{01} & w'_{012} &= 0\,. \end{aligned}$$

Since half of the coordinates of z_0 and z_2 are zero, the coordinate relations resulting from Relation $\widetilde{R}^d_{2,1}$ are greatly simplified. They are given by

$$\begin{align} m(v,v) &= v \tag{5.44}\\ Dm(0,v_0) &= v_0 \tag{5.45}\\ Dm(v_0,v_1) &= 0 \tag{5.46}\\ D^2m(0,v_{01}) &= 0 \tag{5.47}\\ Dm(v_1,0) &= v_1 \tag{5.48}\\ D^2m(0,0) &= v_{01} \tag{5.49}\\ D^2m(v_{01},0) &= 0 \tag{5.50}\\ D^3m(0,0) &= 0\,. \tag{5.51}\end{align}$$

Let us now discuss these relations. By plugging Equation (5.42) into (5.44), we get that $m(1_u,1_u)=1_u$, which is redundant. Equation (5.49) implies that $v_{01}=0$. It follows that Equations (5.47) and (5.50) hold trivially. Similarly, Equation (5.51) is redundant.

Remark 5.3.4. Even though Relations $R^d_{1,0}$ and $R^d_{1,1}$ do not contribute in the computation of the end (Rem. 5.3.3), the relations they induce show that Equations (5.45) and (5.48) are redundant. Let us spell out these relations. We have the following expressions in local coordinates

$$\begin{aligned} T^2t : (v,v_0,v_1,v_{01}) &\longmapsto \big(t_v, Dt(v_0), Dt(v_1), D^2t(v_{01})\big)\\ T^2s : (v,v_0,v_1,v_{01}) &\longmapsto \big(s_v, Ds(v_0), Ds(v_1), D^2s(v_{01})\big)\\ (T0)_{G_0} : (u,u_0) &\longmapsto (u,0,u_0,0)\\ (0T)_{G_0} : (u,u_0) &\longmapsto (u,u_0,0,0)\,.\end{aligned}$$

Hence, Relations $R^d_{1,0}$ and $R^d_{1,1}$ are equivalent to

$$\begin{align} t_v &= u & s_v &= u \notag\\ Dt(v_0) &= 0 & Ds(v_0) &= u_0 \tag{5.52}\\ Dt(v_1) &= u_0 & Ds(v_1) &= 0 \tag{5.53}\\ D^2t(v_{01}) &= 0 & D^2s(v_{01}) &= 0\,. \notag\end{align}$$

Now, using Expression (1.12) of the multiplication of the tangent groupoid in terms of right and left translations, we get that

$$\begin{aligned} Dm(0,v_0) &= DR_{1_u}(0) + DL_{1_u}(v_0)\\ &= 0 + v_0\\ &= v_0\,.\end{aligned}$$

This shows that Equation (5.45) is redundant. An analogous calculation shows that Equation (5.48) is redundant.

As a conclusion, the interesting equations induced by Relation $\widetilde{R}^d_{2,1}$ which carry new information are given by

$$\begin{aligned} v_{01} &= 0\\ Dm(v_0,v_1) &= 0\,.\end{aligned}$$

Remark 5.3.5. The Equation $Dm(v_0, v_1) = 0$ implies that

$$Ds(v_1) = Ds \circ Dm(v_0, v_1) = Ds(0) = 0$$

and

$$Dt(v_0) = Dt \circ Dm(v_0, v_1) = Dt(0) = 0\,.$$

These equalities also appear in the relations (5.52) and (5.53) induced by $R^d_{1,0}$ and $R^d_{1,1}$.

Remark 5.3.6. The following equations are equivalent:

(i) $Dm(v_0, v_1) = 0$;

(ii) $v_0 = Di(v_1)$.

This can be best understood geometrically. Let

$$\begin{aligned} (g,h) : \mathbb{R} &\longrightarrow G_1 \times_{G_0} G_1 \\ t &\longmapsto (g_t, h_t) \end{aligned}$$

be a smooth path representing the tangent vector $(v_0, v_1) \in T_{(1_u,1_u)}(G_1 \times_{G_0} G_1) \cong T_{1_u}G_1 \times_{T_uG_0} T_{1_u}G_1$. Here, $(g_0, h_0) = (1_u, 1_u)$ and

$$\left.\frac{d}{dt}\right|_{t=0} (g_t, h_t) = (v_0, v_1)\,.$$

Then, $t \mapsto m(g_t, h_t)$ is a path in G_1 through $m(g_0, h_0) = m(1_u, 1_u) = 1_u$ and where

$$\left.\frac{d}{dt}\right|_{t=0} m(g_t, h_t) = Dm\left(\left.\frac{d}{dt}\right|_{t=0} g_t,\, \left.\frac{d}{dt}\right|_{t=0} h_t\right) = Dm(v_0, v_1)\,.$$

This implies that the equation $Dm(v_0, v_1) = 0$ is equivalent to the path $m(g_t, h_t)$ being constantly equal to $m(g_0, h_0) = 1_u$, which is equivalent to g_t and h_t being inverse to each other, i.e. $g_t = i(h_t)$. Differentiating at $t = 0$, we get

$$v_0 = \left.\frac{d}{dt}\right|_{t=0} g_t = \left.\frac{d}{dt}\right|_{t=0} i(h_t) = Di\left(\left.\frac{d}{dt}\right|_{t=0} h_t\right) = Di(v_1)\,.$$

Remark 5.3.7. Plugging in $Dt(v_1) = Ds(v_0) = u_0$ from (5.52) and (5.53) in Equation (5.43), we get that

$$v_0 + v_1 = D1(u_0) = D1 \circ Ds(v_0) = D1 \circ Dt(v_1)\,. \tag{5.54}$$

Also note that if (one of) the equivalent statements in Remark 5.3.6 hold, then $v_0 + v_1 = D1 \circ Ds(v_0)$ holds. This implies that Equation (5.43) is redundant.

Relationship to the usual Lie algebroid of G

Let us summarize the minimal coordinate relations that describe the end in the case of the Lie groupoid G. The base point is given by $u \in G_0$ or, equivalently, by $1_u \in G_1$. The main relation that yields the data of the Lie algebroid is

$$Dm(v_0, v_1) = 0\,,$$

where v_0 and v_1 are tangent vectors to G_1 at the unit 1_u. By Remark 5.3.5, it implies that v_0 is tangent to the target fiber at 1_u, and v_1 tangent to the source fiber at 1_u. It follows from Theorem 5.2.17 that the end is given in local coordinates by

$$\begin{aligned}\mathrm{Lie}(G) &\cong \{(x,y,z) \in TG_0 \times T^2G_1 \times T^3G_2 \mid R^s_{0,0} \text{ and } \widetilde{R}^d_{2,i} \text{ hold for } i = 0,1,2\}\\ &\cong \{(u, v_0, v_1) \in G_0 \times T_{1_u}G_1 \times T_{1_u}G_1 \mid Dm(v_0,v_1) = 0\}\,.\end{aligned}$$

Having made no choice between the source-vertical and the target-vertical tangent directions, we can view this expression as the source-target symmetric version of the Lie algebroid of G. This can be visualized by the following picture:

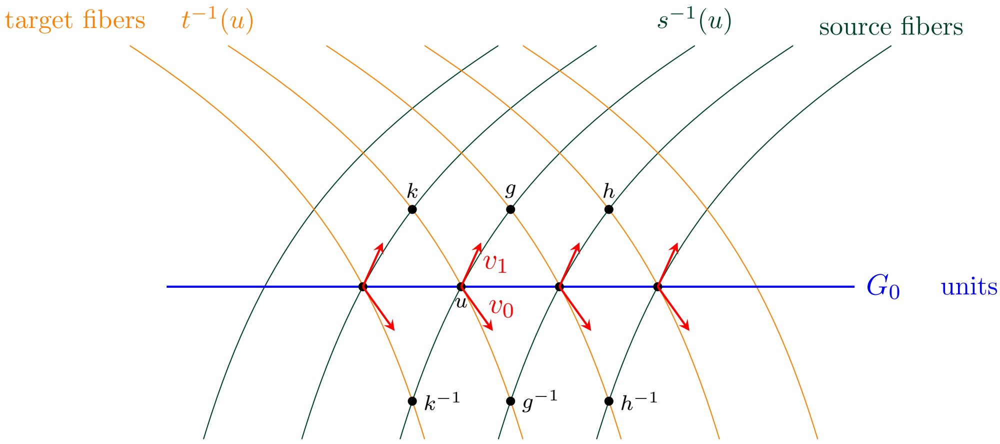


Figure 5.1: The symmetric version of the Lie algebroid of G

The symmetry is described by the inverse map, that is, by the equivalent condition $v_0 = Di(v_1)$, as explained in Remark 5.3.6. Making the choice of the source-vertical tangent direction, we recover the non-symmetric version of the Lie algebroid

$$A \cong \{(u, v_1) \in G_0 \times T_{1_u}G_1 \mid Ds(v_1) = 0\} \cong \ker Ts \times_{G_1} G_0\,,$$

with anchor $\rho : A \to TG_0$, $(u, v_1) \mapsto Dt(v_1)$. The relationship between the symmetric and non-symmetric versions follows from Remarks 5.3.5 and 5.3.6: The map

$$\begin{aligned}\mathrm{Lie}(G) &\longrightarrow A\\ (u, v_0, v_1) &\longmapsto (u, v_1)\end{aligned}$$

is an isomorphism of vector bundles with inverse $(u, v_1) \mapsto (u, Di(v_1), v_1)$. By Equation (5.54), the assignment $\mathrm{Lie}(G) \to TG_1$, $(u, v_0, v_1) \mapsto v_0 + v_1$ makes the following diagram

$$
\begin{tikzcd}
\mathrm{Lie}(G) \arrow[r, "\cong"] \arrow[d] & A \arrow[d, "\rho"] \\
TG_1 & TG_0 \arrow[l, "T1"]
\end{tikzcd}
$$

commute. By identifying TG_0 with its image under $T1$, we obtain an isomorphism of anchored vector bundles.

Our next goal in this project is to carefully examine the Lie bracket or the differential of the Lie algebroid cohomology on A in terms of the end formula. The observations in the Lie groupoid case will serve as a guideline to derive the higher Lie brackets or the differential describing the higher Lie algebroid cohomology for $n > 1$. This is outlined in the next section.

5.4 Outlook: the higher Lie algebroid structure of $\mathrm{Lie}(G)$

Vaintrob [Vai97] has shown that the Lie algebroid structures on a given vector bundle $A \to M$ are in one-to-one correspondence with degree 1 differentials on the graded algebra $\Gamma(M, \Lambda A^*)$ (see Remark 1.2.5). This observation can be formulated succinctly in the language of graded geometry, which has led to the description of higher Lie algebroids in terms of differential (non-negatively) graded manifolds [SZ17, BP13, Vor10, Šev06].

Recall that an **$\mathbb{N}$-graded manifold** $\mathcal{M}$ consists of a smooth manifold M, called its **core**, and a structure sheaf $\mathcal{O}_\mathcal{M}$ of $\mathbb{N}$-graded commutative $C^\infty(M)$-algebras, locally freely generated by elements of strictly positive degree [Meh06, Fai17, Jot18]. A **differential $\mathbb{N}$-graded manifold** is an $\mathbb{N}$-graded manifold $\mathcal{M}$ equipped with a degree 1 derivation Q on the sheaf $\mathcal{O}_\mathcal{M}$ such that $[Q, Q] = 0$ (the bracket is given by the graded commutator), also called a **homological vector field** on $\mathcal{M}$. Differential $\mathbb{N}$-graded manifolds are often called **$\mathbb{N}Q$-manifolds** in the literature. In the graded geometry language, the Lie algebroid structures on a vector bundle $A \to M$ are in one-to-one correspondence with homological vector fields on the degree one graded manifold $A[1]$, which has core M and sheaf of functions $\mathcal{O}_{A[1]}(U) = \Gamma(U, \Lambda A^*)$ for all $U \subset M$ open.

From a geometric perspective, Sheng and Zhu have introduced the notion of **split Lie n-algebroids**, which are graded vector bundles equipped with an anchor and higher brackets satisfying higher Jacobi relations and the Leibniz identity [SZ17, Def. 2.1]. Bonavolontà and Poncin have shown that the category of split Lie n-algebroids and the category of split $\mathbb{N}Q$-manifolds of degree n (graded manifolds coming from a graded vector bundle) are equivalent [BP13, Theorems 2 and 3].

In Section 5.2.2, we have defined the abstract Lie n-algebroid of a differentiable n-groupoid object G in a category $\mathcal{C}$ with a tangent structure and a compatible pretopology. Definition 5.2.6 in terms of an end is a priori only an object of $\mathcal{C}$. In order to fully capture the higher abstract Lie algebroid structure of $\mathrm{Lie}(G)$, the following questions arise:

Questions 5.4.1.

(i) What are the structures needed on the category $\mathcal{C}$ so that there is a well-defined extension of the notion of (split) Lie n-algebroids in $\mathcal{C}$? How are (split) Lie n-algebroids precisely defined in $\mathcal{C}$?

- Algebraic approach: develop differential graded geometry in $\mathcal{C}$. In this case, the minimal structure $\mathcal{C}$ should have is a Grothendieck pretopology so that it makes sense to talk about open covers and sheaves.
- Geometric approach: generalize graded vector bundles to graded bundles of R-modules, define the anchor and investigate the higher Lie brackets. In this case, the minimal structure on $\mathcal{C}$ is a cartesian tangent structure with scalar R-multiplication, where $R \in \mathcal{C}$ is a ring object.

(ii) Given the necessary structure on $\mathcal{C}$ and a well-defined definition of a higher abstract Lie algebroid in $\mathcal{C}$, how can we describe the abstract Lie n-algebroid structure of the end Lie(G)?

(iii) Is the end construction homotopy invariant? In order to tackle this question, we need a model structure on the ambient category. Since our objects and the end construction are (co)simplicial in nature, one approach is to employ Reedy model structures, which are model structures commonly used for functor categories [GJ99].

Geometrization of graded manifolds

In [JH24], Heuer and Jotz propose a geometrization of $\mathbb{N}$-graded manifolds. Generalizing the results in [Jot18] for the degree 2 case, the authors show that the category of $\mathbb{N}$-graded manifolds of degree n is equivalent to the category of (signed) symmetric n-fold vector bundles. For the $n = 2$ case, Jotz has further shown that Li-Bland's correspondence of Lie 2-algebroids and VB-Courant algebroids is an equivalence of categories [LB12], [Jot19]. This can be viewed as a geometrization of the homological vector fields on an $\mathbb{N}$-graded manifold of degree 2 in terms of linear Courant algebroid structures.

A possible geometrization of the homological vector fields on an $\mathbb{N}$-graded manifold of degree n in terms of structures on the corresponding symmetric n-fold vector bundle opens a new perspective of future work. The presence of a (signed) symmetry in the equivalence of categories in [JH24] seems tightly connected to the relations induced by our categorical end formula. Lastly, we conjecture that there is a close relationship between Lie(G) and the core of the multiple vector bundle [JH20, Sec. 2.4] corresponding to the $\mathbb{N}$-graded manifold describing Lie(G). This conjecture builds upon the observation that the core of the LA-groupoid

$$\begin{tikzcd} TG_1 \arrow[r, "\pi_{G_1}"] \arrow[d, shift right, "Ts"'] \arrow[d, shift left, "Tt"] & G_1 \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\ TG_0 \arrow[r, "\pi_{G_0}"'] & G_0 \end{tikzcd}$$

for a Lie groupoid G is given by the Lie algebroid of G. A natural question we are interested in is a possible generalization of this approach to categories with a tangent structure.

The tangent complex

Using the ideas of Ševera [Šev06], the authors in [LRWZ23] have proved by explicit combinatorial computation that the Lie n-algebroid of a Lie n-groupoid is represented by the underlying graded vector bundle of its tangent complex [LRWZ23, Thm. 3.3]

$$\ker Tp_{1,0}|_{G_0}[1] \oplus \ldots \oplus \ker Tp_{n,0}|_{G_0}[n]\,, \tag{5.55}$$

where for all $1 \leq j \leq n$, $\ker Tp_{j,0}|_{G_0} := \ker Tp_{j,0} \times_{G_j} G_0$ is the restriction of the kernel of the tangent map of the horn projection to G_0 via (compositions of) the degeneracy maps. The notation $[j]$ indicates the degrees of the graded vector bundle. This is the model of the tangent complex given in [CZ23, Def. 2.8]. The tangent complex is invariant up to weak equivalence under Morita equivalence of higher Lie groupoids described in terms of hypercovers [CZ23, Cor. 2.28], [Zhu09, Sec. 2].

Note that for $n = 1$, the tangent complex recovers the usual Lie algebroid of the Lie groupoid, where the non-trivial differential of the complex is the anchor map [CZ23, Ex. 2.12]. As such, the differential of the tangent complex is different than the differential of the Lie algebroid cohomology. For the description of the differential of the (higher) Lie algebroid cohomology, the key observation is that there is a natural right action of the graded ring $\underline{\mathrm{End}}(\mathbb{R}^{0|1}) := \underline{\mathrm{Hom}}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1})$ on $\Pi TM \cong \underline{\mathrm{Hom}}(\mathbb{R}^{0|1}, M)$, and that this action is equivalent to the structure of a differential complex on the sheaf $\Omega(M)$ of differential forms on the manifold M, that is, to its $\mathbb{Z}$-grading and to the de Rham differential [Kon03, ŠK03, Šev06, HKST11]. This is recalled in Section 5.2.1. In future work, the authors of [LRWZ23] aim to compute the higher Lie brackets using this approach, where a choice of a connection and splitting is necessary.

In the geometric approach to Questions 5.4.1 (i), the goal would be to first describe $\mathrm{Lie}(G)$ as a graded bundle of R-modules. We observe that the (co)face and (co)degeneracy relations $R^d_{m,i}$ and $R^s_{m-1,j}$ our end formula induces (see (5.25)) look very similar to the description of the tangent complex as a limit in [LRWZ23, Sec. 3.1]. Thus, we conjecture that in the category of smooth manifolds, our construction recovers the tangent complex of the Lie n-groupoid G. Moreover, the definition of the tangent complex naturally generalizes to a general category $\mathcal{C}$ with a tangent structure and a compatible pretopology. In this case, we aim to similarly show that $\mathrm{Lie}(G)$ is the (symmetric version) of the tangent complex (5.55) of G.

The higher Lie algebroid as a coend in differential complexes

In the algebraic approach to Questions 5.4.1 (i), the goal would be to understand the sheaf of functions on $\mathrm{Lie}(G)$ and the homological vector field. Using the fact that $\mathcal{O}(\Pi TM) \equiv \mathcal{O}_{\Pi TM}(M) = \Omega(M)$ is the cochain complex of differential forms on the manifold M, we propose to consider the differential complex

$$\mathcal{O}\big(\mathrm{Lie}(G)\big) := \int^{[m]\in\Delta^{\mathrm{op}}_{\leq n+1}} \mathcal{O}\big((\Pi T)^{m+1}G_m\big)\,.$$

The idea behind this definition is to describe the differential forms on a multiple vector bundle V as graded symmetric maps from fiber products of the tangent functor applied to V into a ring object R. The formulation in terms of the coend is motivated by the fact that mapping out of an object is contravariant. By construction, $\mathcal{O}\big(\mathrm{Lie}(G)\big)$ is a differential graded algebra. We conjecture the following statements:

(i) If G is a Lie 1-groupoid, $\mathcal{O}\big(\mathrm{Lie}(G)\big)$ is the complex of its Lie algebroid cohomology, that is,
$$\mathcal{O}\big(\mathrm{Lie}(G)\big) \cong \Gamma(G_0, \Lambda A^*)$$
where A is the Lie algebroid of G.

(ii) The assignment $G \mapsto \mathcal{O}\big(\mathrm{Lie}(G)\big)$ is homotopy invariant, that is, a hypercover $G \to H$ of Kan simplicial manifolds induces a quasi-isomorphism $\mathcal{O}\big(\mathrm{Lie}(H)\big) \to \mathcal{O}\big(\mathrm{Lie}(G)\big)$.

In future work, we aim to make these heuristic ideas rigorous, provide satisfactory answers to Questions 5.4.1 and prove the conjectures mentioned in this section.

Appendix A

Category theoretic notions

The goal of this appendix is to recall some of the concepts and universal constructions in category theory, that will be essential to this thesis; and to fix some notation. Mainly, we will discuss the Yoneda embedding in Section A.1, categorical ends in Section A.2, and Kan extensions in Section A.3. Lastly, in Section A.4, we will explain useful categorical tools involving pullbacks and monomorphisms, as well as recall the notions of group, ring and module objects in any category.

The reader is assumed to have some background in basic category theory, such as categories, functors, natural transformations, limits and colimits. The categories that we consider in this thesis are all locally small. It will be clear from the context if they are also small. A comprehensive exposition of the subject can be found in [ML98] and [Rie16]. Throughout the coming sections, let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ be categories.

A.1 The Yoneda embedding

A *presheaf* is called representable by an object $C \in \mathcal{C}$ if it is naturally isomorphic to the contravariant hom functor into the object C. The *Yoneda embedding* assigns to each object in $\mathcal{C}$ the associated representable presheaf. In this way, any category can be embedded fully and faithfully into its category of presheaves. This is a consequence of the well-known *Yoneda lemma*, due to Nobuo Yoneda [ML98, p. 76]. Together with its applications, it is considered to be one of the most fundamental results in category theory.

Presheaves which behave well with respect to *covers* are called sheaves. In Section A.1.1, we will recall the notion of presheaves and sheaves, and in Section A.1.2 we will state the Yoneda lemma and highlight one of its important consequences: the density theorem.

A.1.1 Presheaves and sheaves

Presheaves and representability

Definition A.1.1. A $\mathcal{D}$-valued **presheaf** on the category $\mathcal{C}$ is a contravariant functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$.

Whenever the codomain category of a presheaf is clear from the context, we will drop the prefix $\mathcal{D}$-valued.

Definition A.1.2. A **morphism between two $\mathcal{D}$-valued presheaves** F and G on $\mathcal{C}$ is a natural transformation of functors.

$\mathcal{D}$-valued presheaves on $\mathcal{C}$ together with morphisms between them form a category. Being a functor category, it will be denoted by $\mathcal{D}^{\mathcal{C}^{\mathrm{op}}}$.

Example A.1.3. Some of the main examples of presheaves are as follows:

(i) Let X be a topological space and denote by $\mathcal{O}\mathrm{pen}(X)$ the category of open subsets of X whose morphisms are given by inclusions. Then, a presheaf on $\mathcal{O}\mathrm{pen}(X)$ coincides with the usual notion of a presheaf on X. An example of this is the assignment to each open subset $U \subseteq X$, the ring $\mathcal{T}\mathrm{op}(U, \mathbb{R})$ of continuous real-valued functions on U. Here, $\mathcal{T}\mathrm{op}$ denotes the category of topological spaces with continuous maps.

(ii) Let M be a smooth manifold. Then, assigning to each open subset $U \subseteq M$, the ring $C^\infty(U) = \mathcal{M}\mathrm{fld}(U, \mathbb{R})$ of real-valued smooth functions on U, and the ring $\Omega(U) = \Gamma(U, \Lambda T^* M)$ of differential forms on U, are presheaves on $\mathcal{O}\mathrm{pen}(M)$. Here, $\mathcal{M}\mathrm{fld}$ denotes the category of smooth manifolds with smooth maps.

(iii) Let $E \to M$ be a smooth vector bundle. Then, $\Gamma(\_, E)$, which assigns to each open subset U of M the $C^\infty(U)$-module $\Gamma(U, E)$ of smooth sections, is a presheaf on $\mathcal{O}\mathrm{pen}(M)$.

Example A.1.4. Let C be an object of $\mathcal{C}$ and denote by $\mathcal{S}\mathrm{et}$ the category of sets with functions. The hom functor

$$\begin{aligned} \mathcal{C}(\_, C) : \mathcal{C}^{\mathrm{op}} &\longrightarrow \mathcal{S}\mathrm{et} \\ A &\longmapsto \mathcal{C}(A, C)\,, \end{aligned}$$

which assigns to each morphism $f : A \to B$ in $\mathcal{C}$ its pullback

$$\begin{aligned} f^* : \mathcal{C}(B, C) &\longrightarrow \mathcal{C}(A, C) \\ g &\longmapsto g \circ f\,, \end{aligned}$$

is a $\mathcal{S}\mathrm{et}$-valued presheaf on $\mathcal{C}$.

Definition A.1.5. A $\mathcal{S}\mathrm{et}$-valued presheaf F on $\mathcal{C}$ is **representable** if it is naturally isomorphic to $\mathcal{C}(\_, C)$ for some **representing object** $C \in \mathcal{C}$.

Definition A.1.6. A **subpresheaf** of a presheaf $F \in \mathcal{D}^{\mathcal{C}^{\mathrm{op}}}$ is a subobject of F in the category $\mathcal{D}^{\mathcal{C}^{\mathrm{op}}}$ of presheaves.

In other words, a subpresheaf is given by a monomorphism (see Definition A.4.1) $i : G \to F$ in $\mathcal{D}^{\mathcal{C}^{\mathrm{op}}}$. Spelled out, G is a $\mathcal{D}$-valued presheaf on $\mathcal{C}$ such that $i_C : GC \to FC$ is a monomorphism, and is natural in $C \in \mathcal{C}$. If $\mathcal{D} = \mathcal{S}\mathrm{et}$, a subpresheaf of F is a presheaf G such that for all $C \in \mathcal{C}$, $GC \subseteq FC$ is a subset, and for each morphism $f : C \to C'$ in $\mathcal{C}$, $Gf = (Ff)|_{GC'}$.

Remark A.1.7. In fact, subobjects of objects in a category are defined as equivalence classes of monomorphisms. The reader may refer to [ML98, Sec. V.7] for more details on subobjects.

Grothendieck pretopology and sheaves

Presheaves which behave well with respect to so-called *covers* or *covering families* are called sheaves. We will first recall the concept of abstract covers in a category, which generalizes the prototypical case of open covers of topological spaces.

Definition A.1.8. A **Grothendieck pretopology** on $\mathcal{C}$ is given by a class of families of morphisms $\{U_i \to U\}$ of $\mathcal{C}$, called **covers**, subject to the following axioms:

(i) Every isomorphism $\{V \xrightarrow{\cong} U\}$ is a cover.

(ii) If $\{U_j \to U\}$ and $\{U_{ij} \to U_j\}$ are covers, then so is $\{U_{ij} \to U_j \to U\}$.

(iii) If $\{U_i \to U\}$ is a cover and $V \to U$ is a morphism of $\mathcal{C}$, then the pullback $V \times_U U_i$ exists and $\{V \times_U U_i \to V\}$ is a cover.

The original definition of a pretopology dates back to Artin, Grothendieck and Verdier [AGV72, p. 221, Déf. 1.3]. Some definitions assume that $\mathcal{C}$ has pullbacks and thus the existence condition in (iii) is eliminated (e.g. [MLM92, Def. III.2.2]). In some definitions in the literature covers are given by singleton sets with the same axioms (e.g. [Hen08, Def. 2.1], [Zhu09, Def. 1.2]). This will be of special importance in the definition of higher groupoid objects in Section 5.1.1.

Definition A.1.9. A **singleton Grothendieck pretopology** on $\mathcal{C}$ is a Grothendieck pretopology, where the covers are given by singleton sets.

Remark A.1.10. If $\mathcal{C}$ has all coproducts, we recover Definition A.1.8 by declaring $\{U_i \to U\}$ to be a cover if $\amalg_i U_i \to U$ is a cover in the sense of Definition A.1.9.

Example A.1.11. The following are basic examples of Grothendieck pretopologies:

(i) Let X be a topological space and $\mathcal{O}\mathrm{pen}(X)$ the category of open subsets of X whose morphisms are given by inclusions. Then, the usual open covers equip $\mathcal{O}\mathrm{pen}(X)$ with a Grothendieck pretopology.

(ii) The category of sets with surjections as the covers.

(iii) The category of topological spaces with covers the surjective local homeomorphisms.

(iv) The category of smooth manifolds with covers the surjective submersions.

Remark A.1.12. The covers in parts (ii), (iii) and (iv) of Example A.1.11 are given by singleton sets.

Definition A.1.13. Let $\mathcal{C}$ be equipped with a Grothendieck pretopology. A presheaf $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ is called a **sheaf** if the diagram

$$F(U) \longrightarrow \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_U U_j) \tag{A.1}$$

is an equalizer for every object $U \in \mathcal{C}$ and cover $\{U_i \to U\}$. A **morphism of sheaves** is a morphism of the underlying presheaves.

Example A.1.14. Parts (i), (ii) and (iii) of Example A.1.3 are all sheaves.

Remark A.1.15. If $\mathcal{C} = \mathcal{O}\mathrm{pen}(X)$ for a topological space X, then the pullback $U_i \times_U U_j = U_i \cap U_j$ is the intersection and one recovers the usual notion of sheaves on X (see [MLM92, Sec. II.1]). In this case, the equalizer diagram (A.1) can be geometrically interpreted as a gluing condition of the local sections, i.e. the elements of $F(U)$, along their restricted values at the intersections $U_i \cap U_j$.

Definition A.1.16. A (singleton) Grothendieck pretopology on $\mathcal{C}$ is **subcanonical** if every cover is a regular epimorphism (Def. A.4.3).

Remark A.1.17. A Grothendieck pretopology is subcanonical if and only if all representable presheaves are sheaves [MZ15, Def. and Lem. 2.2].

All the definitions and constructions for (pre)sheaves can be similarly formulated in the setting of co(pre)sheaves, which we omit in this appendix to avoid repetitions.

A.1.2 The Yoneda lemma and the density theorem

The Yoneda embedding and the Yoneda lemma

The goal of this section is to recall that any category can be embedded into its category of presheaves fully and faithfully. Let

$$y : \mathcal{C} \longrightarrow \mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}}$$

be the functor which assigns to each object $C \in \mathcal{C}$ the representable presheaf

$$yC := \mathcal{C}(\_\,, C)\,,$$

and to each morphism $f : C \to C'$ in $\mathcal{C}$ the natural transformation

$$yf : yC \longrightarrow yC' \quad \text{with components} \quad (yf)_A : g \longmapsto f \circ g$$

for all $A \in \mathcal{C}$.

Lemma A.1.18 (Yoneda lemma, [ML98, p. 61])**.** *Given a presheaf $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ and an object $C \in \mathcal{C}$, there is a natural bijection of sets*

$$\mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}}(yC, F) \cong FC\,.$$

Corollary A.1.19 ([ML98, p. 61])**.** *The functor y is full and faithful.*

The functor y is called the **Yoneda embedding**, a terminology justified by Corollary A.1.19. More details on the Yoneda lemma can be found in [ML98, Sec. III.2].

Remark A.1.20. Since $\mathcal{S}\mathrm{et}$ is (co)complete, and (co)limits in functor categories are computed objectwise, the category $\mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}}$ is also (co)complete. Thus, the Yoneda embedding is a useful tool to consider (co)limits in any category without even knowing their existence, by considering the corresponding (co)limits in its category of presheaves. However, this does not tackle the problem of representability.

Comma categories and the density theorem

As a consequence of the Yoneda lemma, we get that any presheaf is a colimit of representable presheaves. In order to rigorously state that, we first need to describe the index category. For that, we need the notion of comma categories.

Definition A.1.21. Let $F : \mathcal{A} \to \mathcal{C}$ and $G : \mathcal{B} \to \mathcal{C}$ be functors. The **comma category** $F \downarrow G$ is the category with objects triples (A, B, h), where A is an object of $\mathcal{A}$, B is an object of $\mathcal{B}$, and $h : FA \to GB$ is a morphism in $\mathcal{C}$. The morphisms $(A, B, h) \to (A', B', h')$ are given by morphisms $\alpha : A \to A'$ in $\mathcal{A}$ and $\beta : B \to B'$ in $\mathcal{B}$, such that the diagram

$$\begin{tikzcd} FA \arrow[r, "F\alpha"] \arrow[d, "h"'] & FA' \arrow[d, "h'"] \\ GB \arrow[r, "G\beta"'] & GB' \end{tikzcd}$$

commutes.

Remark A.1.22. There are canonical forgetful functors $q_{\mathcal{A}} : F \downarrow G \to \mathcal{A}$ and $q_{\mathcal{B}} : F \downarrow G \to \mathcal{B}$.

Remark A.1.23. A number of well-known concepts are special cases of comma categories. The following cases will be important in this thesis:

(i) Let C be an object of $\mathcal{C}$. The **overcategory** $\mathcal{C} \downarrow C$ is the comma category $\mathrm{id}_{\mathcal{C}} \downarrow C$, where C is viewed as the constant functor $* \to \mathcal{C}$ that sends the unique object $*$ to C. Here, $*$ refers to the category with one object and the identity morphism. Explicitly, the objects of $\mathcal{C} \downarrow C$ are given by arrows $A \to C$ in $\mathcal{C}$, and the morphisms are commutative triangles

$$\begin{tikzcd} A \arrow[rr, "\alpha"] \arrow[dr] & & A' \arrow[dl] \\ & C & \end{tikzcd}$$

Similarly, the **undercategory** $C \downarrow \mathcal{C}$ is the comma category $C \downarrow \mathrm{id}_{\mathcal{C}}$.

(ii) Let $[1]$ be the **interval category** composed of two objects 0 and 1 and one non-identity morphism $0 \to 1$. The **arrow category** is the functor category $\mathcal{C}^{[1]}$, whose objects are morphisms $A_0 \to A_1$ in $\mathcal{C}$, and whose morphisms are commutative squares

$$\begin{tikzcd} A_0 \arrow[r] \arrow[d] & B_0 \arrow[d] \\ A_1 \arrow[r] & B_1 \end{tikzcd}$$

There is an isomorphism of categories $\mathcal{C}^{[1]} \cong \mathrm{id}_{\mathcal{C}} \downarrow \mathrm{id}_{\mathcal{C}}$. Hence, comma categories also generalize arrow categories.

(iii) Let $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ be a presheaf. Its **category of elements**[1] $\mathrm{el}(F)$ is the category whose objects are pairs (C, x), with C an object of $\mathcal{C}$ and x an element of the set FC. A morphism $(C, x) \to (C', x')$ is given by a morphism $\alpha : C \to C'$ in $\mathcal{C}$ such that $(F\alpha)(x') = x$.

On the other hand, consider the comma category $y \downarrow F$, where F is viewed as the constant functor $* \to \mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}}$, described in part (i) of this remark. The objects are pairs (C, h), where C is an object of $\mathcal{C}$ and $h : yC \to F$ is a natural transformation. A morphism $(C, h) \to (C', h')$ is given by a morphism $\alpha : C \to C'$ in $\mathcal{C}$ such that the triangle

$$\begin{tikzcd} yC \arrow[rr, "y\alpha"] \arrow[dr, "h"'] & & yC' \arrow[dl, "h'"] \\ & F & \end{tikzcd}$$

commutes. By the Yoneda lemma, it can be shown that there is an isomorphism of categories $\mathrm{el}(F) \cong y \downarrow F$. We will often denote the objects of $y \downarrow F$ by natural transformations $yC \to F$.

Proposition A.1.24 (Density theorem, [MLM92, Prop. I.5.1])**.** *Any presheaf* $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ *is isomorphic to the colimit of the following functor*

$$\begin{array}{ccccc} y \downarrow F & \xrightarrow{q_{\mathcal{C}}} & \mathcal{C} & \xrightarrow{y} & \mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}} \\ (yC \to F) & \longmapsto & C & \longmapsto & yC \end{array}$$

where $q_{\mathcal{C}}$ *is the forgetful functor from Remark A.1.22.*

The colimit described in Proposition A.1.24 will be often written as

$$F \cong \operatorname*{colim}_{yC \to F} yC\,.$$

For the name *density* and for more details the reader may refer to [ML98, Sec. X.6].

Remark A.1.25. Given two functors $F : \mathcal{A} \to \mathcal{C}$ and $G : \mathcal{B} \to \mathcal{C}$, the categories $(F \downarrow G)^{\mathrm{op}}$ and $G^{\mathrm{op}} \downarrow F^{\mathrm{op}}$ are isomorphic. Here, $F^{\mathrm{op}} : \mathcal{A}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ denotes the canonical opposite functor induced by F.

A.2 Categorical ends

Ends are universal constructions in category theory, which are associated to functors of two variables, contravariant in the first argument and covariant in the second one. They are special cases of so-called *dinatural* transformations, which are extensions to the notion of naturality. Originally, they are due to Yoneda in his study on Ext functors in homological algebra [Yon60]. The dual notion of ends is called coends.

[1]This can be regarded as a special case of a more general construction, called the *Grothendieck construction*, where $\mathcal{S}\mathrm{et}$ is replaced by the category $\mathcal{C}\mathrm{at}$ of small categories with functors (e.g. [Tho79, Def. 1.1], [MLM92, Sec. I.5]).

It turns out that the uses of (co)ends are numerous. On one hand, they characterize many constructions in different fields of mathematics, such as tensor products of modules over a ring, geometric realization of simplicial sets, etc. [ML98, Sec. IX.6]. On the other hand, (co)ends are special kinds of (co)limits and support a calculus provided by a rich set of computational and deduction rules. As such, they provide a computational approach to category theory.

The goal of Section A.2.1 is to introduce the notion of ends and give some important examples. In Section A.2.2, we will explore the computation of ends as limits. For a general introduction to ends, the reader may refer to [ML98, Sec. IX.5]. A comprehensive treatment on (co)end calculus can be found in a recent book by Loregian [Lor21].

A.2.1 Definition and examples

We will sometimes denote the identity morphism on an object C in the category $\mathcal{C}$ by $C \equiv \mathrm{id}_C : C \to C$. For example, in Diagram (A.2) below, $S(B, f) = S(\mathrm{id}_B, f)$ and $S(f, C) = S(f, \mathrm{id}_C)$.

Definition A.2.1. An **end** of a functor $S : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ is an object E of $\mathcal{D}$ together with a family of morphisms $\big\{\omega_C : E \to S(C, C)\big\}_{C\in\mathcal{C}}$ in $\mathcal{D}$ such that for every morphism $f : B \to C$ in $\mathcal{C}$, the following diagram

$$\begin{tikzcd} E \arrow[r, "\omega_B"] \arrow[d, "\omega_C"'] & S(B,B) \arrow[d, "S(B,f)"] \\ S(C,C) \arrow[r, "S(f,C)"'] & S(B,C) \end{tikzcd} \tag{A.2}$$

commutes and such that E is a universal such object.

The universality condition means that if $E' \in \mathcal{D}$ is another object and if $\big\{\omega'_C : E' \to S(C, C)\big\}$ is a family of morphisms in $\mathcal{D}$, such that for all morphisms $f : B \to C$ in $\mathcal{C}$ the equation $S(B, f) \circ \omega'_B = S(f, C) \circ \omega'_C$ holds, then there exists a unique morphism $h : E' \to E$ in $\mathcal{D}$ such that $\omega_C \circ h = \omega'_C$ for all $C \in \mathcal{C}$. Being a universal object, the end E of a functor S as above is unique up to a unique isomorphism. It is usually denoted by[2]

$$E = \int_{C\in\mathcal{C}} S(C, C)\,,$$

where the reference to the category $\mathcal{C}$ may be dropped if it is clear from the context.

Example A.2.2. Consider the arrow category $\mathcal{C}^{[1]}$ as recalled in Remark A.1.23 (ii). Let $A = \{A_0 \xrightarrow{a} A_1\}$ and $B = \{B_0 \xrightarrow{b} B_1\}$ be objects of $\mathcal{C}^{[1]}$. Then, the set of morphisms from A to B is given by the following pullback

$$\mathcal{C}^{[1]}(A, B) = \mathcal{C}(A_0, B_0) \times^{b_*,\, a^*}_{\mathcal{C}(A_0,B_1)} \mathcal{C}(A_1, B_1)\,,$$

[2]The integral notation is due to Yoneda in his introduction of the notion of ends in homological algebra [Yon60].

whose elements are commutative squares

$$\begin{tikzcd} A_0 \arrow[r, "g_0"] \arrow[d, "a"'] & B_0 \arrow[d, "b"] \\ A_1 \arrow[r, "g_1"'] & B_1 \end{tikzcd}$$

Here, the pullback by a and the pushforward by b are given by $a^*g_1 = g_1 \circ a$ and $b_*g_0 = b \circ g_0$. On the other hand, consider the hom functor

$$\begin{aligned} S := \mathcal{C}(A_{\_}, B_{\_}) : [1]^{\mathrm{op}} \times [1] &\longrightarrow \mathcal{S}\mathrm{et} \\ (i,j) &\longmapsto S(i,j) := \mathcal{C}(A_i, B_j)\,, \end{aligned} \tag{A.3}$$

where for the unique morphism $f : 0 \to 1$ in the interval category $[1]$, we have $S(\mathrm{id}_0, f) = b_*$ and $S(f, \mathrm{id}_1) = a^*$. Then, the pullback diagram

$$\begin{tikzcd} \mathcal{C}^{[1]}(A,B) \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"'] & \mathcal{C}(A_0, B_0) \arrow[d, "b_*"] \\ \mathcal{C}(A_1, B_1) \arrow[r, "a^*"'] & \mathcal{C}(A_0, B_1) \end{tikzcd}$$

gives the end of S, where the universality follows from the universality of the pullback. That is,

$$\mathcal{C}^{[1]}(A,B) \cong \int_{i \in [1]} \mathcal{C}(A_i, B_i)\,.$$

The previous example is a special case of the following more general statement, which expresses the set of natural transformations between two functors in terms of an end.

Remark A.2.3 ([Lor21, Thm. 1.4.1])**.** Given two functors $F, G : \mathcal{C} \to \mathcal{D}$, there is a natural bijection of sets

$$\mathcal{D}^{\mathcal{C}}(F,G) \cong \int_{C \in \mathcal{C}} \mathcal{D}(FC, GC)\,.$$

In this thesis, the functor categories we are primarily interested in are of the form $\mathcal{C}^{\mathbf{\Delta}^{\mathrm{op}}}$, where $\mathbf{\Delta}$ is the *simplex category*. The objects in this functor category are called simplicial objects. In Section 2.1, we give an introduction to the simplex category and (co)simplicial objects.

Remark A.2.4. One of the main applications of Remark A.2.3 is to define enriched homs of functor categories. Heuristically, if $\mathcal{D}$ is enriched over a symmetric monoidal category $\mathcal{V}$, then the functor category $\mathcal{D}^{\mathcal{C}}$ is also enriched over $\mathcal{V}$ by defining

$$\underline{\mathcal{D}}^{\mathcal{C}}(F,G) := \int_{C \in \mathcal{C}} \underline{\mathcal{D}}(FC, GC)\,,$$

for all functors $F, G : \mathcal{C} \to \mathcal{D}$. Here, the end on the right hand side might not always exist. In our applications, it will always exist for specific choices of the functors F and G. In Chapter 2 of [Kel05], the reader may find a detailed exposition on this matter.

A.2.2 Ends as limits

The goal of this section is to show that ends are limits of certain diagrams. We will first describe the index category. Given a category $\mathcal{C}$, the **subdivision category** $\mathcal{C}^\S$ of $\mathcal{C}$ has objects the symbols $C^\S$ and $f^\S$ for every object $C \in \mathcal{C}$ and for every morphism f in $\mathcal{C}$. The morphisms of $\mathcal{C}^\S$ are given by the identity morphisms and for each morphism $f : B \to C$ in $\mathcal{C}$, two morphisms $B^\S \to f^\S \leftarrow C^\S$. The only well-defined composition of morphisms in $\mathcal{C}^\S$ is the one with identity morphisms. Hence, $\mathcal{C}^\S$ is a category by construction.

Given a functor $S : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$, we construct the functor $S^\S : \mathcal{C}^\S \to \mathcal{D}$ as follows. For each object $C \in \mathcal{C}$ and morphism $f : B \to C$ in $\mathcal{C}$,

$$\begin{aligned} S^\S : C^\S &\longmapsto S(C,C) \\ f^\S &\longmapsto S(B,C)\,. \end{aligned}$$

Moreover, $S^\S$ maps identity morphisms to identity morphisms and

$$(B^\S \to f^\S \leftarrow C^\S) \;\longmapsto\; \Big(S(B,B) \xrightarrow{S(B,f)} S(B,C) \xleftarrow{S(f,C)} S(C,C)\Big)\,.$$

Proposition A.2.5 ([ML98, Prop. IX.5.1])**.** *Let $S : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ be a functor. Then,*

$$\int_C S(C,C) \cong \lim(S^\S : \mathcal{C}^\S \to \mathcal{D})\,,$$

where $\mathcal{C}^\S$ is the subdivision category of $\mathcal{C}$ and $S^\S$ is the functor described above.

In fact, every limit can also be viewed as an end [ML98, Prop. IX.5.3]. Coends, the dual notion of ends, can be similarly regarded as special colimits. Since ends are special limits and the hom functor preserves limits, we get the following corollary.

Corollary A.2.6 ([Lor21, Cor. 1.2.8])**.** *For every functor $S : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ and object $D \in \mathcal{D}$, there is a natural bijection of sets*

$$\mathcal{D}\left(D, \int_C S(C,C)\right) \cong \int_C \mathcal{D}\big(D, S(C,C)\big)\,.$$

Corollary A.2.6 states one of the many computational rules provided by ends. In [Lor21], the author presents a meticulous study of such rules, which provide conceptual arguments, such as deduction rules, as opposed to element-wise arguments, to prove categorical results.

Remark A.2.7. If $\{f_i\}$ is a finite generating set of the morphisms of a small category $\mathcal{C}$, then we will consider the full subcategory $\widetilde{\mathcal{C}}^\S$ of the subdivision category $\mathcal{C}^\S$ of $\mathcal{C}$, whose objects are given by the symbols $C^\S$ for each object $C \in \mathcal{C}$, and by the symbols $f_i^\S$ corresponding to the generators. In this case, the limits of $S^\S$ with indexing categories $\widetilde{\mathcal{C}}^\S$ and $\mathcal{C}^\S$ are isomorphic. For simplicity, we will also call $\widetilde{\mathcal{C}}^\S$ the subdivision category of $\mathcal{C}$ and denote it by $\mathcal{C}^\S$ too.

For our purposes, this will be particularly important for the simplex category $\boldsymbol{\Delta}$ whose morphisms are finitely generated by its coface and codegeneracy maps (see Section 2.1.1 and the proof of Theorem 5.2.8).

We *end* this section by considering Example A.2.2 again and computing the corresponding limit.

Example A.2.8. The subdivision category of the interval category $[1] = \{0 \xrightarrow{f} 1\}$ has objects $0^\S$, $1^\S$, $f^\S$, $(\mathrm{id}_0)^\S$ and $(\mathrm{id}_1)^\S$. Its non-identity morphisms are given by

$$0^\S \to f^\S \leftarrow 1^\S\,, \quad 0^\S \to (\mathrm{id}_0)^\S \leftarrow 0^\S \quad \text{and} \quad 1^\S \to (\mathrm{id}_1)^\S \leftarrow 1^\S\,.$$

Let $A = \{A_0 \xrightarrow{a} A_1\}$ and $B = \{B_0 \xrightarrow{b} B_1\}$ be objects of $\mathcal{C}^{[1]}$, where $a = A(f)$ and $b = B(f)$. Consider the functor $S := \mathcal{C}(A_-, B_-)$ given in (A.3). We can construct the associated functor $S^\S : [1]^\S \to \mathcal{S}\mathrm{et}$ as follows:

$$\begin{aligned} S^\S : 0^\S &\longmapsto S(0,0) = \mathcal{C}(A_0, B_0) \\ 1^\S &\longmapsto S(1,1) = \mathcal{C}(A_1, B_1) \\ f^\S &\longmapsto S(0,1) = \mathcal{C}(A_0, B_1) \\ (\mathrm{id}_0)^\S &\longmapsto S(0,0) = \mathcal{C}(A_0, B_0) \\ (\mathrm{id}_1)^\S &\longmapsto S(1,1) = \mathcal{C}(A_1, B_1)\,. \end{aligned}$$

On the other hand, $S^\S$ maps all morphisms to identity functions except the following:

$$(0^\S \to f^\S \leftarrow 1^\S) \;\longmapsto\; \Big(\mathcal{C}(A_0, B_0) \xrightarrow{b_*} \mathcal{C}(A_0, B_1) \xleftarrow{a^*} \mathcal{C}(A_1, B_1)\Big)\,,$$

where we have used that $S(\mathrm{id}_0, f) = b_*$ and $S(f, \mathrm{id}_1) = a^*$, as explained in Example A.2.2. By Proposition A.2.5, the end of S is isomorphic to the limit of $S^\S$, that is

$$\begin{aligned} \int_{i \in [1]} \mathcal{C}(A_i, B_i) &\cong \lim(S^\S : [1]^\S \to \mathcal{S}\mathrm{et}) \\ &\cong \mathcal{C}(A_0, B_0) \times^{b_*, a^*}_{\mathcal{C}(A_0, B_1)} \mathcal{C}(A_1, B_1) \\ &= \mathcal{C}^{[1]}(A, B)\,. \end{aligned}$$

A.3 Kan extensions

Kan extensions generalize the idea of extending a function defined on a subset to the whole set. From a categorical perspective, given a subcategory $\mathcal{A}$ of $\mathcal{B}$ and a functor $F : \mathcal{A} \to \mathcal{C}$, they address the question of extending F to a functor $R : \mathcal{B} \to \mathcal{C}$ in a natural and universal way. However, sometimes such extensions do not exist on the nose, but are only conceptualized as an approximation to an extension, given by a universal natural transformation. They are considered to be one of the most significant universal constructions in category theory. As Mac Lane states: "*The notion of Kan extensions subsumes all the other fundamental concepts of category theory*" [ML98, Sec. X.7].

We will start by recalling the horizontal and vertical compositions of natural transformations in Section A.3.1. Then, we will give a brief introduction to right Kan extensions in Section A.3.2. Left Kan extensions are defined similarly. We will conclude with some results about pointwise Kan extensions in Section A.3.3, that are needed in this thesis. The reader may refer to [ML98, Ch. X] or [Rie16, Ch. 6] for more details.

A.3.1 Compositions of natural transformations

Let $F, F' : \mathcal{B} \to \mathcal{C}$ and $G, G' : \mathcal{A} \to \mathcal{B}$ be functors and let $\alpha : F \to F'$ and $\beta : G \to G'$ be natural transformations. The composition of G and F will be denoted by juxtaposition $FG : \mathcal{A} \to \mathcal{C}$. The **horizontal composition** of α and β will also be denoted by juxtaposition $\alpha\beta : FG \to F'G'$. It is the natural transformation with components $(\alpha\beta)_A$ given by the diagonal of the following commutative diagram

$$\begin{tikzcd} FGA \arrow[r, "F(\beta_A)"] \arrow[d, "\alpha_{GA}"'] \arrow[dr, "(\alpha\beta)_A"] & FG'A \arrow[d, "\alpha_{G'A}"] \\ F'GA \arrow[r, "F'(\beta_A)"'] & F'G'A \end{tikzcd} \tag{A.4}$$

for all $A \in \mathcal{A}$. We will use the notation F for the identity natural transformation $\mathrm{id}_F : F \to F$. Hence, we have the identities

$$(F\beta)_A = F(\beta_A) \quad \text{and} \quad (\alpha G)_A = \alpha_{GA}\,. \tag{A.5}$$

Now, let $F'' : \mathcal{B} \to \mathcal{C}$ be another functor and $\alpha' : F' \to F''$ a natural transformation. The **vertical composition** of α and α' is the natural transformation $\alpha' \circ \alpha : F \to F''$ with components $(\alpha' \circ \alpha)_B = \alpha'_B \circ \alpha_B$ for all $B \in \mathcal{B}$. The vertical composition is precisely the composition in the functor category $\mathcal{C}^{\mathcal{B}}$. The strict monoidal category of endofunctors on $\mathcal{C}$ will be denoted by $\mathrm{End}(\mathcal{C})$ and the identity endofunctor by $1 : \mathcal{C} \to \mathcal{C}$. Using Diagram (A.4) and Equation (A.5), the horizontal composition of α and β can be written as the following vertical compositions

$$\alpha\beta = \alpha G' \circ F\beta = F'\beta \circ \alpha G\,. \tag{A.6}$$

Figures A.1 and A.2 illustrate the horizontal and vertical compositions of natural transformations.

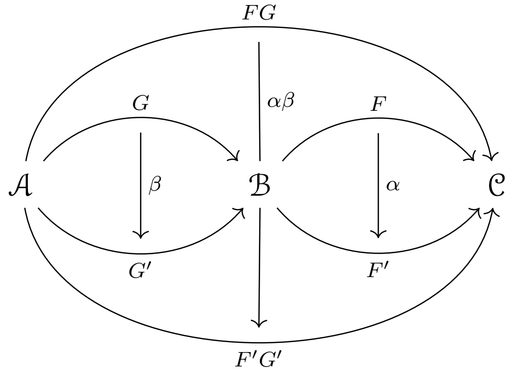


Figure A.1: Horizontal composition of natural transformations

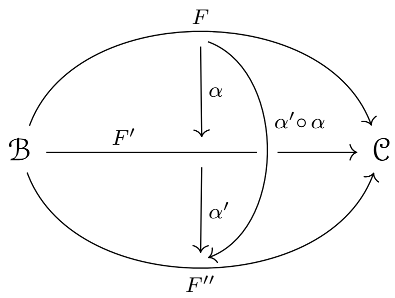


Figure A.2: Vertical composition of natural transformations

A.3.2 Right Kan extensions

Definition A.3.1. Let $K : \mathcal{A} \to \mathcal{B}$ and $F : \mathcal{A} \to \mathcal{C}$ be functors. A **right Kan extension** of F along K is a functor $R : \mathcal{B} \to \mathcal{C}$, together with a universal natural transformation $\alpha : RK \to F$.

Explicitly, being universal means that if $S : \mathcal{B} \to \mathcal{C}$ is a functor and $\beta : SK \to F$ is a natural transformation, then there exists a unique natural transformation $\sigma : S \to R$ such that $\beta = \alpha \circ \sigma K$. Here, σK is the horizontal composition of σ with the identity natural transformation on K, given by Equation (A.5). Being a universal object, the right Kan extension R of F along K as above is unique up to a unique natural isomorphism. It is usually written as $R = \operatorname{Ran}_K F$ and depicted by the following (in general non-commutative) diagram

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "F"] \arrow[d, "K"'] & \mathcal{C} \\
\mathcal{B} \arrow[ur, dashed, "\operatorname{Ran}_K F"'] \arrow[u, phantom, shift left=4, "\nearrow\alpha"] &
\end{tikzcd} \tag{A.7}$$

Under certain conditions this diagram commutes up to natural isomorphism (see Corollary A.3.7).

Remark A.3.2. Given a functor $K : \mathcal{A} \to \mathcal{B}$, consider the precomposition functor

$$K^* : \mathcal{C}^{\mathcal{B}} \longrightarrow \mathcal{C}^{\mathcal{A}} ,$$

which maps a functor $S : \mathcal{B} \to \mathcal{C}$ to the composition $SK : \mathcal{A} \to \mathcal{C}$, and a natural transformation $\alpha : S \to S'$ to the horizontal composition $\alpha K : SK \to S'K$. If $\operatorname{Ran}_K F$ exists for all functors $F : \mathcal{A} \to \mathcal{C}$, then the functor

$$\begin{aligned} \operatorname{Ran}_K : \mathcal{C}^{\mathcal{A}} &\longrightarrow \mathcal{C}^{\mathcal{B}} \\ F &\longmapsto \operatorname{Ran}_K F , \end{aligned}$$

which maps a natural transformation $F \to F'$ in $\mathcal{C}^{\mathcal{A}}$ to the unique natural transformation $\operatorname{Ran}_K F \to \operatorname{Ran}_K F'$ obtained by the universality of $\operatorname{Ran}_K F'$, is right adjoint to K^*.

Left Kan extensions are defined analogously. In that case, one may show that the left Kan extension functor Lan_K along K, if it exists, is left adjoint to K^*.

A.3.3 Pointwise Kan extensions

In what follows, let $K : \mathcal{A} \to \mathcal{B}$ be a functor. In many situations, not all functors from $\mathcal{A}$ to $\mathcal{C}$ have right Kan extensions along K and hence Ran_K as a right adjoint to K^* does not in general exist. However, individual right Kan extensions for given functors may still exist and may be rightfully considered. The following theorem states that if some special limits exist in $\mathcal{C}$, then right Kan extensions can be viewed as pointwise limits.

Theorem A.3.3 ([ML98, Thm. X.3.1])**.** *Let $F : \mathcal{A} \to \mathcal{C}$ be a functor and assume that for all $B \in \mathcal{B}$, the limit RB of the diagram*

$$\begin{aligned} B \downarrow K &\xrightarrow{q_{\mathcal{A}}} \mathcal{A} \xrightarrow{F} \mathcal{C} \\ (B \to KA) &\longmapsto A \longmapsto FA \end{aligned}$$

exists in $\mathcal{C}$ with limiting cone λ. Then,

(i) the right Kan extension of F along K exists and is pointwise, i.e. for all $B \in \mathcal{B}$,

$$(\mathrm{Ran}_K F)(B) = RB = \lim_{B \to KA} FA\,, \tag{A.8}$$

and for all maps $f : B \to B'$ in $\mathcal{B}$, $(\mathrm{Ran}_K F)(f)$ is the unique map

$$Rf : RB \longrightarrow RB'$$

that commutes with the limiting cones;

(ii) for each $A \in \mathcal{A}$ and letting $B := KA$, the assignment

$$\alpha_A := \lambda_{\mathrm{id}_{KA}} : RKA \longrightarrow FA$$

defines the universal natural transformation $\alpha : RK \to F$ of $\mathrm{Ran}_K F$.

In the statement of Theorem A.3.3, $B \downarrow K$ is the comma category where B is viewed as the constant functor $* \to \mathcal{B}$ (see Def. A.1.21 and Rem. A.1.23). The functor $q_{\mathcal{A}}$ is the forgetful functor (see Rem. A.1.22).

Remark A.3.4. In the literature, pointwise Kan extensions have a formal definition in terms of preservation by representable presheaves. It can be proved that the definition is equivalent to computing Kan extensions pointwise as a limit in $\mathcal{C}$, as given by Equation (A.8) in the case of right Kan extensions [ML98, Thm. X.5.3]. A detailed exposition on this is given in [Rie16, Sec. 6.3].

Corollary A.3.5 ([ML98, Cor. X.3.2])**.** *Assume that $\mathcal{A}$ is small and $\mathcal{C}$ is complete. Then, every functor $F : \mathcal{A} \to \mathcal{C}$ has a right Kan extension along K and it is pointwise.*

Remark A.3.6. In particular, it follows from Corollary A.3.5 that if $\mathcal{A}$ is small and $\mathcal{C}$ is complete, the functor K^* has a right adjoint Ran_K given by the right Kan extension along K (see Rem. A.3.2).

Corollary A.3.7 ([ML98, Cor. X.3.3]). *Let the assumptions of Theorem A.3.3 hold and let K be fully faithful. Then, the universal natural transformation $\alpha : RK \to F$ associated to $R = \operatorname{Ran}_K F$ is a natural isomorphism.*

Corollary A.3.7 implies that Diagram (A.7) commutes up to natural isomorphism, and thus right Kan extensions define actual extensions up to natural isomorphism under the given assumptions.

Remark A.3.8. As mentioned earlier, all the above results have dual formulations for left Kan extensions. In this setting, one would for instance encounter pointwise left Kan extensions given by pointwise colimits under suitable assumptions. For additional reading, Chapter 6 of [Rie16] covers both right and left Kan extensions in full detail.

A.4 Useful categorical tools

In this section we summarize the main categorical tools that we extensively use in Chapters 3 and 4. We start by recalling the notions of monomorphisms and epimorphisms with their different adjectives in Section A.4.1. The goal of Section A.4.2 is to fix some notation involving pullbacks and universal morphisms, and to state the well-known pasting lemma, which gives a cancellation law for pullbacks. We proceed with Section A.4.3 where we show an easy yet powerful tool to prove the commutativity of a given diagram that is *included* in another commutative diagram. Finally, in Section A.4.4, we recall the notions of (abelian) group objects, ring objects and module objects over them, as well as prove some useful results for the purposes of this thesis. Throughout the rest of this section, whenever we talk about certain limits in the category $\mathcal{C}$, we assume their existence.

A.4.1 Monomorphisms and epimorphisms

Monomorphisms and epimorphisms are categorical generalizations of injections and surjections respectively. In this section, we state the main definitions and properties of different kinds of monomorphisms and epimorphisms. The reader may refer to [Kel69], [Bor94, Sec. 1.7, 1.8, 4.3] and [ML98, Sec. I.5] for proofs, details and examples.

Definition A.4.1. A morphism $f : A \to B$ in $\mathcal{C}$ is called

(i) a **monomorphism** if it satisfies the left-cancellation rule, that is, for all morphisms $g_1, g_2 : X \to A$, the equality $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$.

(ii) a **split monomorphism** if it has a retract, that is, if there is a morphism $g : B \to A$ in $\mathcal{C}$ such that $g \circ f = \mathrm{id}_A$.

(iii) a **regular monomorphism** if it is the equalizer of some parallel pair of arrows $B \rightrightarrows Y$.

(iv) a **strong monomorphism** if it is a monomorphism that has the right lifting property with respect to epimorphisms (Def. A.4.3), that is, given a commutative square

$$\begin{tikzcd} X \arrow[r] \arrow[d, "g"'] & A \arrow[d, "f"] \\ Y \arrow[r] \arrow[ur, dashed, "\exists!\, h"] & B \end{tikzcd}$$

such that g is an epimorphism, there exists a unique morphism $h : Y \to A$ such that the inner triangles commute.

Remark A.4.2. The following statements hold:

(i) Split and regular monomorphisms are, in particular, monomorphisms.

(ii) Any split monomorphism is a regular monomorphism (it is the equalizer of $f \circ g$ and id_B).

(iii) Any regular monomorphism is a strong monomorphism (it follows from the universal property of equalizers).

(iv) Monomorphisms, regular monomorphisms and strong monomorphisms are stable under pullback[3].

Definition A.4.3. A morphism $g : B \to A$ in $\mathcal{C}$ is called

(i) an **epimorphism** if it satisfies the right-cancellation rule, that is, for all morphisms $f_1, f_2 : A \to Y$, the equality $f_1 \circ g = f_2 \circ g$ implies $f_1 = f_2$.

(ii) a **split epimorphism** if it has a section, that is, if there is a morphism $f : A \to B$ in $\mathcal{C}$ such that $g \circ f = \mathrm{id}_A$.

(iii) a **regular epimorphism** if it is the coequalizer of some parallel pair of arrows $X \rightrightarrows B$.

(iv) a **strong epimorphism** if it is an epimorphism that has the left lifting property with respect to monomorphisms, that is, given a commutative square

$$\begin{tikzcd} B \arrow[r] \arrow[d, "g"'] & X \arrow[d, "f"] \\ A \arrow[r] \arrow[ur, dashed, "\exists!\, h"] & Y \end{tikzcd}$$

such that f is a monomorphism, there exists a unique morphism $h : A \to X$ such that the inner triangles commute.

Remark A.4.4. The following statements hold:

(i) Split and regular epimorphisms are, in particular, epimorphisms.

[3] It is in general true that morphisms with the right lifting property with respect to a class of morphisms are stable under pullback.

(ii) Any split epimorphism is a regular epimorphism (it is the coequalizer of $f \circ g$ and id_B).

(iii) Any regular epimorphism is a strong epimorphism [Kel69, Prop. 3.1].

(iv) Epimorphisms, regular epimorphisms and strong epimorphisms are stable under pushout [Kel69, Prop. 5.2].

Remark A.4.5. The relationship between isomorphisms, monomorphisms and epimorphisms can be described as follows:

(i) Every isomorphism is a monomorphism and an epimorphism.

(ii) A morphism that is both a monomorphism and a split epimorphism is an isomorphism.

(iii) A morphism that is both a split monomorphism and an epimorphism is an isomorphism.

Lemma A.4.6. *Let $f, g : A \to B$ be morphisms in $\mathcal{C}$. Then, the natural morphism $A \times_B^{f,g} A \to A \times A$ is a monomorphism.*

Proof. The pullback of $f \times g$ along the diagonal morphism $\Delta_B : B \to B \times B$ is isomorphic to $A \times_B^{f,g} A$. By definition, Δ_B has a retract $\mathrm{pr}_1 : B \times B \to B$ and hence is a split monomorphism. In particular, Δ_B is a monomorphism. Since monomorphisms are stable under pullbacks, we get that $A \times_B^{f,g} A \longrightarrow A \times A$ is a monomorphism. $\square$

A.4.2 Pullbacks and the pasting lemma

Fiber products of morphisms

Let $A : \mathcal{I} \to \mathcal{C}$, $i \mapsto A_i$ and $B : \mathcal{I} \to \mathcal{C}$, $i \mapsto B_i$ be two diagrams indexed by a small category $\mathcal{I}$. Let $f : A \to B$ be a natural transformation with components $\{f_i : A_i \to B_i\}_{i \in \mathcal{I}}$. Then, by the universal property of limits, we obtain a unique morphism in $\mathcal{C}$ denoted by

$$\lim_{i \in \mathcal{I}} f_i : \lim_{i \in \mathcal{I}} A_i \longrightarrow \lim_{i \in \mathcal{I}} B_i \,,$$

making all diagrams induced by the limiting cones commute.

Notation A.4.7. If $\mathcal{I} = \{0 \xrightarrow{p} 2 \xleftarrow{q} 1\}$ so that diagrams indexed by $\mathcal{I}$ are pullback diagrams, f is given by a commutative diagram

$$\begin{tikzcd}
A_0 \arrow[r, "A(p)"] \arrow[d, "f_0"'] & A_2 \arrow[d, "f_2"'] & A_1 \arrow[l, "A(q)"'] \arrow[d, "f_1"] \\
B_0 \arrow[r, "B(p)"'] & B_2 & B_1 \arrow[l, "B(q)"]
\end{tikzcd}$$

The induced unique morphism between the pullbacks will be denoted by

$$f_0 \times_{f_2} f_1 : A_0 \times_{A_2}^{A(p),\, A(q)} A_1 \longrightarrow B_0 \times_{B_2}^{B(p),\, B(q)} B_1 \,. \tag{A.9}$$

It makes the following squares

$$\begin{tikzcd}
A_0 \arrow[d, "f_0"'] & A_0 \times_{A_2} A_1 \arrow[l, "\mathrm{pr}_1"'] \arrow[r, "\mathrm{pr}_2"] \arrow[d, "f_0 \times_{f_2} f_1"'] & A_1 \arrow[d, "f_1"] \\
B_0 & B_0 \times_{B_2} B_1 \arrow[l, "\mathrm{pr}_1"] \arrow[r, "\mathrm{pr}_2"'] & B_1
\end{tikzcd}$$

commute. If $B_2 = A_2$ and $f_2 = \mathrm{id}_{A_2} : A_2 \to A_2$, the morphism (A.9) will be often written as

$$f_0 \times_{A_2} f_1 : A_0 \times_{A_2} A_1 \longrightarrow B_0 \times_{A_2} B_1 .$$

Commutativity of pullbacks with pullbacks

Let $A : \mathcal{I} \times \mathcal{J} \to \mathcal{C}$, $(i,j) \mapsto A_{ij}$ be a diagram in $\mathcal{C}$ indexed by the product of two small categories $\mathcal{I}$ and $\mathcal{J}$. Then, by the commutativity of limits, we have that

$$\lim_{(i,j)\,\in\,\mathcal{I}\times\mathcal{J}} A_{ij} \cong \lim_{i\in\mathcal{I}} \lim_{j\in\mathcal{J}} A_{ij} \cong \lim_{j\in\mathcal{J}} \lim_{i\in\mathcal{I}} A_{ij} .$$

In particular, if $\mathcal{I} = \mathcal{J} = \{0 \to 2 \leftarrow 1\}$, the functor A is given by the commutative diagram

$$\begin{tikzcd}
A_{00} \arrow[r] \arrow[d] & A_{02} \arrow[d] & A_{01} \arrow[l] \arrow[d] \\
A_{20} \arrow[r] & A_{22} & A_{21} \arrow[l] \\
A_{10} \arrow[r] \arrow[u] & A_{12} \arrow[u] & A_{11} \arrow[l] \arrow[u]
\end{tikzcd}$$

and the commutativity of pullbacks is given by the following isomorphism

$$\begin{aligned}
& (A_{00} \times_{A_{02}} A_{01}) \times_{(A_{20} \times_{A_{22}} A_{21})} (A_{10} \times_{A_{12}} A_{11}) \\
\cong \; & (A_{00} \times_{A_{20}} A_{10}) \times_{(A_{02} \times_{A_{22}} A_{12})} (A_{01} \times_{A_{21}} A_{11}) .
\end{aligned}$$

The parenthesis notation of morphisms

Let A, B and X be objects of $\mathcal{C}$ and let $f : X \to A$ and $g : X \to B$ be morphisms in $\mathcal{C}$. Then, the unique morphism induced by the universal property of the product $A \times B$ will be denoted by $(f,g) : X \to A \times B$. It makes the following diagram

$$\begin{tikzcd}
& X \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, dashed, "\exists!\,(f,g)"] & \\
A & A \times B \arrow[l, "\mathrm{pr}_1"] \arrow[r, "\mathrm{pr}_2"'] & B
\end{tikzcd}$$

commute. By uniqueness, we have the following equality

$$(f,g) = (f \times g) \circ \Delta_X ,$$

where $\Delta_X : X \to X \times X$ is the diagonal morphism in $\mathcal{C}$.

Remark A.4.8. A special case of this is when we consider two morphisms in the overcategory $\mathcal{C}\downarrow C$, given by the commutative diagram:

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[d, "h"] \arrow[dr, "g"] & \\ A \arrow[r, "\alpha"'] & C & B \arrow[l, "\beta"] \end{tikzcd}$$

Then, the unique morphism (f, g) induced by the universal property of the pullback $A \times_C^{\alpha,\beta} B$ makes the following diagram

$$\begin{tikzcd} X \arrow[dr, dashed, "\exists!\,(f,g)"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\ & A \times_C B \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & B \arrow[d, "\beta"] \\ & A \arrow[r, "\alpha"'] & C \end{tikzcd} \tag{A.10}$$

commutative. Note that products in the overcategory $\mathcal{C}\downarrow C$ are pullbacks over C. By uniqueness, we have the following equality

$$(f, g) = (f \times_C g) \circ \Delta_X \,,$$

where $\Delta_X : X \to X \times_C X$ is the diagonal morphism in $\mathcal{C}\downarrow C$.

Caution A.4.9. We use the same notation (f, g) for the unique morphism from X to the product $A \times B$ or to the pullback $A \times_C B$. It will be clear from the context whether we are in the category $\mathcal{C}$ or $\mathcal{C}\downarrow C$.

Lemma A.4.10. *Consider the following commutative diagram:*

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[d, "h"] \arrow[dr, "g"] & \\ A \arrow[r, "\alpha"'] \arrow[d, "\varphi"'] & C \arrow[d, "\mathrm{id}_C"'] & B \arrow[l, "\beta"] \arrow[d, "\psi"] \\ D \arrow[r, "r"'] & C & E \arrow[l, "s"] \end{tikzcd}$$

Let $k : Y \to X$ be a morphism in $\mathcal{C}$ and let $(f, g) : X \to A \times_C B$ be the unique morphism from Diagram (A.10)*. Then, the following equations hold:*

(i) $(f, g) \circ k = (f \circ k, g \circ k)$,

(ii) $(\varphi \times_C \psi) \circ (f, g) = (\varphi \circ f, \psi \circ g)$.

Proof. The desired equalities follow from the universal property of pullbacks. $\square$

Lemma A.4.11. *Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors and $\alpha : F \to G$ a natural transformation. Let $A \to C \leftarrow B$ be a pullback diagram in $\mathcal{C}$ such that $FA \to FC \leftarrow FB$ and $GA \to GC \leftarrow GB$ are pullback diagrams in $\mathcal{D}$. Then, the following diagram*

$$\begin{tikzcd}
F(A \times_C B) \arrow[r, "\alpha_{A\times_C B}"] \arrow[d, "(F\,\mathrm{pr}_1, F\,\mathrm{pr}_2)"'] & G(A \times_C B) \arrow[d, "(G\,\mathrm{pr}_1, G\,\mathrm{pr}_2)"] \\
FA \times_{FC} FB \arrow[r, "\alpha_A \times_{\alpha_C} \alpha_B"'] & GA \times_{GC} GB
\end{tikzcd}$$

commutes, where $\mathrm{pr}_1 : A \times_C B \to A$ and $\mathrm{pr}_2 : A \times_C B \to B$ are the projections onto the first and second factor respectively.

Proof. Using Lemma A.4.10 and the naturality of α, we get that

$$\begin{aligned}
(\alpha_A \times_{\alpha_C} \alpha_B) \circ (F\,\mathrm{pr}_1, F\,\mathrm{pr}_2) &= (\alpha_A \circ F\,\mathrm{pr}_1,\ \alpha_B \circ F\,\mathrm{pr}_2) \\
&= (G\,\mathrm{pr}_1 \circ \alpha_{A\times_C B},\ G\,\mathrm{pr}_2 \circ \alpha_{A\times_C B}) \\
&= (G\,\mathrm{pr}_1,\ G\,\mathrm{pr}_2) \circ \alpha_{A\times_C B}\,.
\end{aligned}$$

□

Notation A.4.12. Using the associativity of pullbacks, we will denote the limit of the diagram

$$A \xrightarrow{\ f\ } B \xleftarrow{\ g\ } C \xrightarrow{\ h\ } D \xleftarrow{\ k\ } E$$

by

$$A \times_B^{f,g} C \times_D^{h,k} E := (A \times_B^{f,g} C) \times_D^{h \circ \mathrm{pr}_2,\, k} E \cong A \times_B^{f,\, g\circ \mathrm{pr}_1} (C \times_D^{h,k} E)\,.$$

The notation for more than two iterated pullbacks will be analogous.

Similarly, any nested parentheses of morphisms can be removed. In other words, given morphisms $f_i : A \to B_i$ in $\mathcal{C}{\downarrow}C$ for $i = 1, \ldots, n$, we will write

$$(f_1, \ldots, f_n) := \Big(\big((f_1, f_2), f_3\big), \ldots, f_n\Big) : A \longrightarrow B_1 \times_C \cdots \times_C B_n\,.$$

The pasting lemma

Pullbacks satisfy the cancellation law with respect to the identity morphism. This is a result of the *pasting lemma*:

Lemma A.4.13 ([ML98, Exercise 8, p. 72])**.** *Consider the following commutative diagram:*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & Y \times_W Z \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, "h"] \\
X \arrow[r, "f"'] & Y \arrow[r, "g"'] & W
\end{tikzcd}$$

where the right square is a pullback. Then, the left square is a pullback if and only if the outer rectangle is a pullback.

Proof. The result follows from the universal property of pullbacks. □

Explicitly, we get that

$$X \times_Y^{f,\,\mathrm{id}_Y} Y \times_W^{g,h} Z = X \times_Y^{f,\,\mathrm{pr}_1} (Y \times_W^{g,h} Z) \cong X \times_W^{g\circ f,\,h} Z\,,$$

and

$$Y \times_W^{g,h} Z \times_Z^{\mathrm{id}_Z,\,k} V = (Y \times_W^{g,h} Z) \times_Z^{\mathrm{pr}_2,\,k} V \cong Y \times_W^{g,\,h\circ k} V\,,$$

where we have used the associativity of pullbacks (Not. A.4.12).

A.4.3 Outer and inner squares

The following lemma provides a simple *outer-inner* technique of proving the commutativity of some diagrams. It is easy to state and to prove, yet it has vast applications. The proof of commutativity for numerous diagrams in Chapter 4 is based on this lemma.

Lemma A.4.14. *Consider the following diagram:*

$$\begin{tikzcd}
A \arrow[rrr, "f"] \arrow[dr, "\alpha"] \arrow[ddd, "g"'] & & & B \arrow[dl, "\beta"'] \arrow[ddd, "h"] \\
& A' \arrow[r, "f'"] \arrow[d, "g'"'] & B' \arrow[d, "h'"] & \\
& C' \arrow[r, "k'"'] & D' & \\
C \arrow[ur, "\gamma"'] \arrow[rrr, "k"'] & & & D \arrow[ul, "\delta"]
\end{tikzcd}$$

Assume that the four trapezoids commute.

(i) If the inner square commutes and δ is a monomorphism, then the outer square commutes.

(ii) If the outer square commutes and α is an epimorphism, then the inner square commutes.

Proof. From the commutativity of the right and the top trapezoids it follows that

$$\begin{aligned} \delta \circ h \circ f &= h' \circ \beta \circ f \\ &= h' \circ f' \circ \alpha\,. \end{aligned} \tag{A.11}$$

Similarly, from the commutativity of the left and the bottom trapezoids it follows that

$$\begin{aligned} \delta \circ k \circ g &= k' \circ \gamma \circ g \\ &= k' \circ g' \circ \alpha\,. \end{aligned} \tag{A.12}$$

We first prove (i). By assumption, the inner square commutes, that is, $h' \circ f' = k' \circ g'$. Comparing Equations (A.11) and (A.12), we get that

$$\delta \circ h \circ f = \delta \circ k \circ g\,.$$

Since, by assumption, δ is a monomorphism, it follows that $h \circ f = k \circ g$. In other words, the outer square commutes.

For part (ii), we assume that the outer square commutes, that is, $h \circ f = k \circ g$. Again, comparing Equations (A.11) and (A.12), we get that

$$h' \circ f' \circ \alpha = k' \circ g' \circ \alpha\,.$$

Since α is assumed to be an epimorphism, it follows that $h' \circ f' = k' \circ g'$, which means that the inner square commutes. $\square$

A.4.4 Group, ring and module objects

Assume that the category $\mathcal{C}$ has finite products. Denote its terminal object by $*$ and the unique morphism from an object $C \in \mathcal{C}$ to the terminal object by $!_C : C \to *$.

Abelian group objects

Definition A.4.15. An **abelian group object** in $\mathcal{C}$ is an object $G \in \mathcal{C}$ together with morphisms $+ : G \times G \to G$, called the addition, $0 : * \to G$, called the zero, and $i : G \to G$, called the inverse, such that the following diagrams commute:

(i) Associativity:

$$\begin{tikzcd} G \times G \times G \arrow[r, "\mathrm{id}_G \times +"] \arrow[d, "+ \times \mathrm{id}_G"'] & G \times G \arrow[d, "+"] \\ G \times G \arrow[r, "+"'] & G \end{tikzcd}$$

(ii) Unitality:

$$\begin{tikzcd} * \times G \arrow[r, "0 \times \mathrm{id}_G"] \arrow[dr, "\mathrm{pr}_2"', "\cong"] & G \times G \arrow[d, "+"] & G \times * \arrow[l, "\mathrm{id}_G \times 0"'] \arrow[dl, "\mathrm{pr}_1", "\cong"'] \\ & G & \end{tikzcd}$$

(iii) Invertibility:

$$\begin{tikzcd} G \arrow[r, "(i, \mathrm{id}_G)"] \arrow[d, "!_G"'] & G \times G \arrow[d, "+"] & G \arrow[l, "(\mathrm{id}_G, i)"'] \arrow[d, "!_G"] \\ * \arrow[r, "0"'] & G & * \arrow[l, "0"] \end{tikzcd}$$

(iv) Commutativity:

$$\begin{tikzcd} G \times G \arrow[rr, "(\mathrm{pr}_2, \mathrm{pr}_1)", "\cong"'] \arrow[dr, "+"'] & & G \times G \arrow[dl, "+"] \\ & G & \end{tikzcd}$$

In the last diagram, $(\mathrm{pr}_2, \mathrm{pr}_1)$ denotes the canonical flip of the product.

Remark A.4.16. In the definition of an abelian group object, we only need to require right unitality and right invertibility. The left unitality and left invertibility are implied by the commutativity of the addition (or the other way around). However, we keep the whole diagrams of unitality and invertibility in the definition for later reference.

Remark A.4.17. The inverse is a property rather than a structure. The existence of the inverse can be formulated in terms of the existence of the pullback of the diagram $* \xrightarrow{0} G \xleftarrow{+} G \times G$. Hence, we will usually only write the addition and zero of a group structure.

Definition A.4.18. A **morphism between two abelian group objects** G and H is a morphism $\varphi : G \to H$ in $\mathcal{C}$ such that the following diagram

$$\begin{tikzcd} G \times G \arrow[r, "\varphi\times\varphi"] \arrow[d, "+_G"'] & H \times H \arrow[d, "+_H"] \\ G \arrow[r, "\varphi"'] & H \end{tikzcd}$$

commutes, where $+_G$ and $+_H$ denote the additions of G and H respectively.

Abelian group objects in $\mathcal{C}$ together with morphisms between them form a category. There is also the notion of non-abelian group objects, where the diagram encoding the commutativity of the addition is dropped in Definition A.4.15. In the rest of the section, group objects will refer to abelian group objects, although most of the statements also hold for non-abelian group objects.

Remark A.4.19. A morphism $\varphi : G \to H$ of abelian group objects intertwines the zeros and the inverses. Explicitly, $\varphi \circ 0_G = 0_H$ and $\varphi \circ i_G = i_H \circ \varphi$, where the subscripts refer to the structure maps of the respective groups.

Example A.4.20. Groups are group objects in the category $\mathcal{S}et$ of sets with functions; topological groups are group objects in the category $\mathcal{T}op$ of topological spaces with continuous maps; Lie groups are group objects in the category $\mathcal{M}fld$ of smooth manifolds with smooth maps.

Definition A.4.21. A **subgroup** of an abelian group object G in $\mathcal{C}$ is a subobject of G in the category of abelian group objects in $\mathcal{C}$.

In other words, it is given by (an equivalence class of) a monomorphism $K \to G$ in the category of abelian group objects in $\mathcal{C}$. Spelled out, a subgroup of $(G, +_G, 0_G)$ is an abelian group object $(K, +_K, 0_K)$ together with a monomorphism $j : K \to G$ in $\mathcal{C}$ such that the following diagram

$$\begin{tikzcd} K \times K \arrow[r, "j\times j"] \arrow[d, "+_K"'] & G \times G \arrow[d, "+_G"] \\ K \arrow[r, "j"'] & G \end{tikzcd}$$

commutes since j is a morphism of group objects. By Remark A.4.19, we get similar diagrams with the zeros and inverses of G and K. In $\mathcal{S}\mathrm{et}$, subgroups are usual subgroups; in $\mathcal{T}\mathrm{op}$, they are topological subgroups; in $\mathcal{M}\mathrm{fld}$, they are Lie subgroups. In these categories (and analogously in categories with a concrete structure), the group structure of K is the restriction of the group structure of G to K.

Definition A.4.22. The **kernel** of a morphism $\varphi : G \to H$ of abelian group objects is the pullback

$$\begin{tikzcd} \ker\varphi := \ast \times_H G \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & G \arrow[d, "\varphi"] \\ \ast \arrow[r, "0_H"'] & H \end{tikzcd} \tag{A.13}$$

in the category $\mathcal{C}$.

Lemma A.4.23. *The kernel of a morphism $\varphi : G \to H$ of abelian group objects is a subgroup of G.*

Proof. Let $K := \ker\varphi = \ast \times_H G$ and denote the second projection of the pullback (A.13) by $j_K : K \to G$. The first projection is given by the unique morphism $!_K : K \to \ast$ to the terminal object. Since 0_H is a monomorphism, and since monomorphisms are stable under pullback (e.g. [ML98, Lemma, p. 122]), we get that j_K is a monomorphism. It remains to show that K has a group structure such that j_K is a morphism of group objects. Consider the following diagram

$$\begin{tikzcd} K \arrow[drr, "j_K"] \arrow[ddd, bend right=60, "!_K"'] & & \\ K \times K \arrow[u, dashed, "\exists!\,+_K"] \arrow[r, "j_K \times j_K"'] \arrow[d, "!_K \times !_K"'] & G \times G \arrow[r, "+_G"'] \arrow[d, "\varphi\times\varphi"] & G \arrow[d, "\varphi"] \\ \ast \times \ast \arrow[r, "0_H \times 0_H"] \arrow[d, "\mathrm{pr}_2"', "\cong"] & H \times H \arrow[r, "+_H"] & H \\ \ast \arrow[urr, "0_H"'] & & \end{tikzcd} \tag{A.14}$$

The right square commutes since φ is a morphism of group objects. The left square commutes by the commutativity of Diagram (A.13) and functoriality. The commutativity of the triangle in the bottom follows from the unitality of 0_H and functoriality. Indeed,

$$\begin{aligned} +_H \circ (0_H \times 0_H) &= +_H \circ (0_H \times \mathrm{id}_H) \circ (\mathrm{id}_\ast \times 0_H) \\ &= \mathrm{pr}_2' \circ (\mathrm{id}_\ast \times 0_H) \\ &= 0_H \circ \mathrm{pr}_2 \,, \end{aligned}$$

where $\mathrm{pr}_2 : \ast \times \ast \xrightarrow{\cong} \ast$ and $\mathrm{pr}_2' : \ast \times H \xrightarrow{\cong} H$ are the projections onto the second factors. Hence, by the universal property of pullbacks, there exists a unique map

$$+_K : K \times K \longrightarrow K \,,$$

such that the upper triangle and the utmost left part of Diagram (A.14) commute. That is,

$$j_K \circ +_K = +_G \circ (j_K \times j_K)\,. \tag{A.15}$$

To show the associativity of $+_K$, consider the following diagram:

$$\begin{tikzcd}
K \times K \times K \arrow[rrr, "\mathrm{id}_K \times +_K"] \arrow[dr, "j_K \times j_K \times j_K"] \arrow[ddd, "+_K \times \mathrm{id}_K"'] & & & K \times K \arrow[dl, "j_K \times j_K"'] \arrow[ddd, "+_K"] \\
& G \times G \times G \arrow[r, "\mathrm{id}_G \times +_G"] \arrow[d, "+_G \times \mathrm{id}_G"'] & G \times G \arrow[d, "+_G"] & \\
& G \times G \arrow[r, "+_G"'] & G & \\
K \times K \arrow[ur, "j_K \times j_K"'] \arrow[rrr, "+_K"'] & & & K \arrow[ul, "j_K"]
\end{tikzcd}$$

The inner square commutes by the associativity of $+_G$. The commutativity of the four outer trapezoids follows from Equation (A.15) and functoriality. Using Lemma A.4.14, we get that the outer square commutes. This shows that $+_K$ is associative.

Next, consider the diagram

$$\begin{tikzcd}
* \arrow[dr, dashed, "\exists!\, 0_K"] \arrow[drr, bend left, "0_G"] \arrow[ddr, bend right, "\mathrm{id}_*"'] & & \\
& K \arrow[r, "j_K"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & G \arrow[d, "\varphi"] \\
& * \arrow[r, "0_H"'] & H
\end{tikzcd} \tag{A.16}$$

Since φ is a morphism of group objects, we have that $\varphi \circ 0_G = 0_H$. Thus, by the universal property of pullbacks, there is a unique map $0_K : * \to K$ such that $j_K \circ 0_K = 0_G$. The unitality of 0_K follows by a similar argument as the associativity of $+_K$.

Similarly, there is an inverse $i_K : K \to K$ satisfying the invertibility condition. As a conclusion, $(K, +_K, 0_K)$ is a group object in $\mathcal{C}$ such that Equation (A.15) holds. This implies that $j_K : K \to G$ is a monomorphism of group objects and so the kernel of φ is a subgroup of G. $\square$

Definition A.4.24. A functor $F : \mathcal{C} \to \mathcal{D}$ **preserves finite products** if the natural morphism

$$\chi_{X,Y} : F(X \times Y) \longrightarrow FX \times FY$$

has an inverse for all $X, Y \in \mathcal{C}$ and if $F* \cong *$.

Remark A.4.25. A finite-product preserving functor $F : \mathcal{C} \to \mathcal{D}$ maps abelian group objects in $\mathcal{C}$ to abelian group objects in $\mathcal{D}$. Spelled out, if $(G, +, 0, i)$ is an abelian group object in $\mathcal{C}$, FG is an abelian group object in $\mathcal{D}$ with addition

$$FG \times FG \xrightarrow[\cong]{\chi^{-1}_{G,G}} F(G \times G) \xrightarrow{F+} FG\,,$$

zero $F0 : * \to FG$, and inverse $Fi : FG \to FG$.

Ring objects

Definition A.4.26. A **ring object** in $\mathcal{C}$ is an abelian group object $(R, +, 0)$ of $\mathcal{C}$ together with morphisms $m : R \times R \to R$, called the multiplication, and $1 : * \to R$, called the unit, such that the following diagrams commute:

(i) Associativity:

$$\begin{tikzcd} R \times R \times R \arrow[r, "\mathrm{id}_R \times m"] \arrow[d, "m \times \mathrm{id}_R"'] & R \times R \arrow[d, "m"] \\ R \times R \arrow[r, "m"'] & R \end{tikzcd}$$

(ii) Unitality:

$$\begin{tikzcd} * \times R \arrow[r, "1 \times \mathrm{id}_R"] \arrow[dr, "\cong", "\mathrm{pr}_2"'] & R \times R \arrow[d, "m"'] & R \times * \arrow[l, "\mathrm{id}_R \times 1"'] \arrow[dl, "\cong"', "\mathrm{pr}_1"] \\ & R & \end{tikzcd}$$

(iii) Right distributivity:

$$\begin{tikzcd} R \times R \times R \arrow[r, "+ \times \mathrm{id}_R"] \arrow[d, "\left(m \circ (\mathrm{pr}_1, \mathrm{pr}_3),\, m \circ (\mathrm{pr}_2, \mathrm{pr}_3)\right)"'] & R \times R \arrow[d, "m"] \\ R \times R \arrow[r, "+"'] & R \end{tikzcd}$$

(iv) Left distributivity:

$$\begin{tikzcd} R \times R \times R \arrow[r, "\mathrm{id}_R \times +"] \arrow[d, "\left(m \circ (\mathrm{pr}_1, \mathrm{pr}_2),\, m \circ (\mathrm{pr}_1, \mathrm{pr}_3)\right)"'] & R \times R \arrow[d, "m"] \\ R \times R \arrow[r, "+"'] & R \end{tikzcd}$$

Remark A.4.27. We define ring objects to have a unit. These are sometimes called *unital* ring objects, since there is also the notion of non-unital ring objects. Since the ring objects we consider in this thesis are all unital, we include the unit in the definition and drop the adjective *unital*.

Definition A.4.28. A **morphism between two ring objects** R and R' is a morphism $\varphi : R \to R'$ of abelian group objects in $\mathcal{C}$, such that the following diagrams

$$\begin{tikzcd} R \times R \arrow[r, "\varphi \times \varphi"] \arrow[d, "m_R"'] & R' \times R' \arrow[d, "m_{R'}"] \\ R \arrow[r, "\varphi"'] & R' \end{tikzcd} \qquad \begin{tikzcd} * \arrow[r, "1"] \arrow[dr, "1'"'] & R \arrow[d, "\varphi"] \\ & R' \end{tikzcd}$$

commute.

Ring objects in $\mathcal{C}$ together with morphisms between them form a category.

Remark A.4.29. A functor that preserves finite products maps ring objects to ring objects. This can be spelled out analogously to Remark A.4.25.

Module objects

Definition A.4.30. Let $(R, \hat{+}, \hat{0}, \hat{m}, \hat{1})$ be a ring object in $\mathcal{C}$. An R**-module object** in $\mathcal{C}$ is an abelian group object $(M, +, 0)$ in $\mathcal{C}$ together with a morphism $\kappa : R\times M \to M$, called the action, such that the following diagrams commute:

(i) Associativity:

$$\begin{tikzcd}
R\times R\times M \arrow[r, "\mathrm{id}_R\times\kappa"] \arrow[d, "\hat{m}\times\mathrm{id}_M"'] & R\times M \arrow[d, "\kappa"] \\
R\times M \arrow[r, "\kappa"'] & M
\end{tikzcd}$$

(ii) Unitality:

$$\begin{tikzcd}
*\times M \arrow[r, "\hat{1}\times\mathrm{id}_M"] \arrow[dr, "\mathrm{pr}_2"', "\cong"] & R\times M \arrow[d, "\kappa"] \\
& M
\end{tikzcd}$$

(iii) Linearity in R:

$$\begin{tikzcd}
R\times R\times M \arrow[r, "\hat{+}\times\mathrm{id}_M"] \arrow[d, "\big(\kappa\circ(\mathrm{pr}_1,\mathrm{pr}_3),\,\kappa\circ(\mathrm{pr}_2,\mathrm{pr}_3)\big)"'] & R\times M \arrow[d, "\kappa"] \\
M\times M \arrow[r, "+"'] & M
\end{tikzcd}$$

(iv) Linearity in M:

$$\begin{tikzcd}
R\times M\times M \arrow[r, "\mathrm{id}_R\times +"] \arrow[d, "\big(\kappa\circ(\mathrm{pr}_1,\mathrm{pr}_2),\,\kappa\circ(\mathrm{pr}_1,\mathrm{pr}_3)\big)"'] & R\times M \arrow[d, "\kappa"] \\
M\times M \arrow[r, "+"'] & M
\end{tikzcd}$$

Remark A.4.31. Linearity in R, expressed by (iii), implies that the action by $\hat{0} : * \to R$ is zero, that is, the diagram

$$\begin{tikzcd}
*\times M \arrow[r, "\hat{0}\times\mathrm{id}_M"] \arrow[d, "\mathrm{pr}_1"'] & R\times M \arrow[d, "\kappa"] \\
* \arrow[r, "0"'] & M
\end{tikzcd}$$

is commutative. Similarly, linearity in M, expressed by (iv), implies that the action sends zero to zero, that is, the diagram

$$\begin{tikzcd}
R\times * \arrow[r, "\mathrm{id}_R\times 0"] \arrow[d, "\mathrm{pr}_2"'] & R\times M \arrow[d, "\kappa"] \\
* \arrow[r, "0"'] & M
\end{tikzcd}$$

commutes.

Definition A.4.32. A **morphism between two R-module objects** M and M' is a morphism $\varphi : M \to M'$ of abelian group objects in $\mathcal{C}$ such that the following diagram

$$\begin{tikzcd}
R \times M \arrow[r, "\mathrm{id}_R \times \varphi"] \arrow[d, "\kappa"'] & R \times M' \arrow[d, "\kappa'"] \\
M \arrow[r, "\varphi"'] & M'
\end{tikzcd}$$

commutes, where κ and κ' are the actions of R on M and M' respectively.

Given a ring object R in $\mathcal{C}$, R-module objects in $\mathcal{C}$ together with morphisms between them form a category.

Lemma A.4.33. *Let $\varphi : R' \to R$ be a morphism of ring objects in $\mathcal{C}$. Let M be an R-module object in $\mathcal{C}$ with action $\kappa : R \times M \to M$. Then, the morphism $\kappa' := \kappa \circ (\varphi \times \mathrm{id}_M) : R' \times M \to M$ equips M with the structure of an R'-module.*

Proof. Denote by $+$ the addition of M; denote the ring structures by $(R, \hat{+}, \hat{0}, \hat{m}, \hat{1})$ and $(R', \hat{+}', \hat{0}', \hat{m}', \hat{1}')$. We need to show that the diagrams in Definition A.4.30 commute for the R'-action κ' on M.

For the associativity, consider the following diagram:

$$\begin{tikzcd}
R' \times R' \times M \arrow[rrr, "\mathrm{id}_{R'} \times \kappa'"] \arrow[dr, "\varphi \times \varphi \times \mathrm{id}_M"] \arrow[ddd, "\hat{m}' \times \mathrm{id}_M"'] & & & R' \times M \arrow[dl, "\varphi \times \mathrm{id}_M"'] \arrow[ddd, "\kappa'"] \\
& R \times R \times M \arrow[r, "\mathrm{id}_R \times \kappa"] \arrow[d, "\hat{m} \times \mathrm{id}_M"'] & R \times M \arrow[d, "\kappa"] & \\
& R \times M \arrow[r, "\kappa"'] & M & \\
R' \times M \arrow[ur, "\varphi \times \mathrm{id}_M"'] \arrow[rrr, "\kappa'"'] & & & M \arrow[ul, "\mathrm{id}_M"]
\end{tikzcd}$$

The inner square commutes by the associativity of the R-action κ on M. The right and bottom trapezoids commute by the definition of κ'. The left trapezoid commutes since φ is a morphism of rings. Moreover,

$$\begin{aligned}
(\varphi \times \mathrm{id}_M) \circ (\mathrm{id}_{R'} \times \kappa') &= (\varphi \circ \mathrm{id}_{R'}) \times (\mathrm{id}_M \circ \kappa') \\
&= (\mathrm{id}_R \circ \varphi) \times \big(\kappa \circ (\varphi \times \mathrm{id}_M)\big) \\
&= (\mathrm{id}_R \times \kappa) \circ (\varphi \times \varphi \times \mathrm{id}_M)\,,
\end{aligned}$$

using functoriality and the definition of κ'. This shows that the upper trapezoid commutes. It follows by Lemma A.4.14 (i) that the outer square commutes. We conclude that κ' is associative.

For the unitality, we calculate

$$\begin{aligned}
\kappa' \circ (\hat{1}' \times \mathrm{id}_M) &= \kappa \circ (\varphi \times \mathrm{id}_M) \circ (\hat{1}' \times \mathrm{id}_M) \\
&= \kappa \circ \big((\varphi \circ \hat{1}') \times (\mathrm{id}_M \circ \mathrm{id}_M)\big) \\
&= \kappa \circ (\hat{1} \times \mathrm{id}_M) \\
&= \mathrm{pr}_2\,,
\end{aligned}$$

by using the definition of κ', functoriality, that φ takes the unit to the unit and the unitality of κ.

For linearity in R' consider the following diagram:

$$
\begin{tikzcd}
R' \times R' \times M \arrow[rrr, "\hat{+}' \times \mathrm{id}_M"] \arrow[dr, "\varphi \times \varphi \times \mathrm{id}_M"] \arrow[ddd, "(\kappa'_{13}, \kappa'_{23})"'] & & & R' \times M \arrow[dl, "\varphi \times \mathrm{id}_M"'] \arrow[ddd, "\kappa'"] \\
& R \times R \times M \arrow[r, "\hat{+} \times \mathrm{id}_M"] \arrow[d, "(\kappa_{13}, \kappa_{23})"'] & R \times M \arrow[d, "\kappa"] & \\
& M \times M \arrow[r, "+"'] & M & \\
M \times M \arrow[rrr, "+"'] \arrow[ur, "\mathrm{id}_M \times \mathrm{id}_M"'] & & & M \arrow[ul, "\mathrm{id}_M"]
\end{tikzcd}
$$

where $\kappa_{ij} = \kappa \circ (\mathrm{pr}_i, \mathrm{pr}_j)$ and $\kappa'_{ij} = \kappa' \circ (\mathrm{pr}_i, \mathrm{pr}_j)$ for $1 \leq i < j \leq 3$. The inner square commutes by the linearity of κ in R. The right and left trapezoids commute by the definition of κ'. The bottom trapezoid commutes trivially. The upper trapezoid commutes since φ, being a morphism of rings, is a priori a morphism of abelian groups. Using Lemma A.4.14 (i), we get that the outer square commutes and hence the action κ' is R'-linear.

Lastly, the proof of linearity in M is analogous. We conclude that κ' equips the R-module object M with the structure of an R'-module. □

Remark A.4.34. Given a ring object $(R, \hat{+}, \hat{0}, \hat{m}, \hat{1})$ in $\mathcal{C}$ and an object A of $\mathcal{C}$, the set $\mathcal{C}(A, R)$ of R-valued morphisms has the structure of a ring with addition and multiplication

$$
f + g := \hat{+} \circ (f, g), \quad fg := \hat{m} \circ (f, g),
$$

for all $f, g \in \mathcal{C}(A, R)$, with zero $A \xrightarrow{!_A} * \xrightarrow{\hat{0}} R$, and unit $A \xrightarrow{!_A} * \xrightarrow{\hat{1}} R$.

Lemma A.4.35. *Let $(R, \hat{+}, \hat{0}, \hat{m}, \hat{1})$ be a ring object in $\mathcal{C}$ and $\alpha : B \to A$ a morphism in $\mathcal{C}$. Then, the pullback*

$$
\begin{aligned}
\alpha^* : \mathcal{C}(A, R) &\longrightarrow \mathcal{C}(B, R) \\
f &\longmapsto f \circ \alpha
\end{aligned}
$$

is a morphism of rings.

Proof. Let $f, g : A \to R$ be morphisms in $\mathcal{C}$. Then,

$$
\begin{aligned}
\alpha^*(f + g) &= (f + g) \circ \alpha \\
&= \hat{+} \circ (f, g) \circ \alpha \\
&= \hat{+} \circ (f \circ \alpha, g \circ \alpha) \\
&= (f \circ \alpha) + (g \circ \alpha) \\
&= \alpha^* f + \alpha^* g,
\end{aligned}
$$

which shows that α^* is a morphism of groups. Similarly, we have that

$$\begin{aligned}\alpha^*(fg) &= (fg) \circ \alpha \\ &= \hat{m} \circ (f, g) \circ \alpha \\ &= \hat{m} \circ (f \circ \alpha, g \circ \alpha) \\ &= (f \circ \alpha)(g \circ \alpha) \\ &= (\alpha^* f)(\alpha^* g) \,.\end{aligned}$$

Lastly, the unit of $\mathcal{C}(A, R)$ is mapped to the unit of $\mathcal{C}(B, R)$ since

$$\alpha^*(\hat{1} \circ !_A) = \hat{1} \circ !_A \circ \alpha = \hat{1} \circ !_B \,.$$

This concludes the proof that α^* is a morphism of rings. □

Lemma A.4.36. *Let $R \in \mathcal{C}$ be a ring object and let $M, N \in \mathcal{C}$ be R-module objects. Let $f : M \to N$ and $g : N \to M$ be inverse morphisms in $\mathcal{C}$, i.e. $f \circ g = \mathrm{id}_N$ and $g \circ f = \mathrm{id}_M$. If f is a morphism of R-modules, then so is g.*

Proof. Denote by $+_M$ and $+_N$ the additions of M and N; denote by κ_M and κ_N the actions of R on M and N respectively. Then,

$$\begin{aligned}f \circ +_M \circ (g \times g) &= +_N \circ (f \times f) \circ (g \times g) \\ &= +_N \circ \big((f \circ g) \times (f \circ g)\big) \\ &= +_N \circ (\mathrm{id}_N \times \mathrm{id}_N) \\ &= +_N \,,\end{aligned}$$

where we have used that f is a morphism of abelian groups, functoriality and the assumption $f \circ g = \mathrm{id}_N$. Composing both sides with g and using the assumption $g \circ f = \mathrm{id}_M$, we get that

$$+_M \circ (g \times g) = g \circ +_N \,.$$

This shows that g is a morphism of abelian groups.

For the equivariance with respect to the R-actions, we similarly get that

$$\begin{aligned}f \circ \kappa_M \circ (\mathrm{id}_R \times g) &= \kappa_N \circ (\mathrm{id}_R \times f) \circ (\mathrm{id}_R \times g) \\ &= \kappa_N \circ \big((\mathrm{id}_R \circ \mathrm{id}_R) \times (f \circ g)\big) \\ &= \kappa_N \circ (\mathrm{id}_R \times \mathrm{id}_N) \\ &= \kappa_N \,,\end{aligned}$$

where we have used that f is a morphism of R-modules, functoriality and the assumption $f \circ g = \mathrm{id}_N$. Composing both sides with g and using the assumption $g \circ f = \mathrm{id}_M$, we get that

$$\kappa_M \circ (\mathrm{id}_R \times g) = g \circ \kappa_N \,.$$

It follows that g is also a morphism of R-modules. □

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