Silting t-Structures, $\mathbb{P}$-Objects, and Weyl Groupoids

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Lukas Bonfert
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Gutachterin/Betreuerin: Prof. Dr. Dr. h.c. Catharina Stroppel
Gutachter: Prof. Dr. Volodymyr Mazorchuk

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Summary

The main topics of this thesis are triangulated categories with $\mathbf{t}$-structures and weight structures, Koszul duality, and certain generalizations of spherical objects known as $\mathbb{P}$-objects. The secondary topic are Weyl groupoids, which are a certain aspect of the structure theory of Lie superalgebras.

Triangulated categories play an important role in representation theory and related areas of mathematics. Interesting examples of triangulated categories with $\mathbf{t}$-structures can be obtained as derived categories of abelian categories such as certain representation categories of Lie algebras, or module categories over finite-dimensional algebras. Further examples arise from complexes of sheaves on stratified varieties. Weight structures, also known as co-$\mathbf{t}$-structures, dualize the notion of $\mathbf{t}$-structures in a certain sense, and are inspired by the theory of weights from complex geometry.

The thesis consists of four rather independent parts. In the first part we study the orthogonality relation of weight structures and $\mathbf{t}$-structures, and the closely related silting $\mathbf{t}$-structures in the sense of Psaroudakis–Vitória. We introduce derived projective covers and relate them to the notion of enough derived projectives introduced by Genovese–Lowen–Van den Bergh. Our main result uses derived projective covers to provide an if-and-only-if criterion for a $\mathbf{t}$-structure with finite-length heart to be a silting $\mathbf{t}$-structure. We also provide equivalent axioms for the ST pairs introduced by Adachi–Mizuno–Yang, and formulate the bijection between simple-minded collections and silting collections due to Koenig–Yang in terms of derived projective covers.

In the second part we show that the non-positive respectively positive dg algebras obtained from silting and simple-minded collections corresponding to orthogonal weight structures and $\mathbf{t}$-structures are dg Koszul dual to each other. This can be seen as a first step towards a tentative Koszul duality of weight structures and $\mathbf{t}$-structures.

In the third part we consider the constructible derived category $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n)$ of complex projective space, equipped with the middle-perverse $\mathbf{t}$-structure. We show that the simple perverse sheaf IC_n is a $\mathbb{P}^n$-object in the sense of Huybrechts–Thomas, and that its associated $\mathbb{P}$-twist is the inverse Serre functor of $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n)$. Moreover, we classify the $\mathbb{P}$-like objects in $\mathbf{Perv}(\mathbb{P}^n)$. This part is joint work with Alessio Cipriani.

In the fourth part we study Weyl groupoids of contragredient Lie superalgebras. Weyl groupoids are an analog of the Weyl group for Lie superalgebras, constructed to also take odd simple roots into account. We provide a convenient graphical formulation of the definitions of Cartan graphs and Weyl groupoids introduced by Heckenberger in the context of Nichols algebras, and apply this to Lie superalgebras following Heckenberger–Yamane. We explicitly describe the Weyl groupoids of $\mathfrak{sl}(m|n)$, $\mathfrak{osp}(2m+1|2n)$ and $\mathfrak{osp}(2m|2n)$ in terms of partitions. Furthermore, we compare this notion of Weyl groupoid to other similar constructions, and in particular to the root groupoid recently introduced by Gorelik–Hinich–Serganova. This part is joint work with Jonas Nehme.

Contents

Chapter 1

Introduction

This thesis consists of four parts, all of which are connected to triangulated categories with **t**-structures, homological algebra, and braid group actions. In particular, we consider the following four questions:

1) Silting **t**-structures are **t**-structures defined by a set of objects. Is there a characterization of silting **t**-structures by homological properties?
2) The definition of weight structures appears dual to that of **t**-structures. Can this duality be formalized, specifically via Koszul duality?
3) The constructible derived category of a partial flag variety is known to admit a Serre functor. Is there a description of the Serre functor that is intrinsic to this category?
4) For a contragredient Lie superalgebra, the Weyl group is replaced by a Weyl groupoid. Is there an explicit combinatorial description of the Weyl groupoids of some Lie superalgebras?

Since their introduction in [Ver96] and [Pup62], triangulated categories have become the natural setup for homological algebra. There are many natural sources of triangulated categories. For instance, given an abelian category $\mathscr{A}$ one can form its derived category $\mathbf{D}(\mathscr{A})$, and given an additive category $\mathscr{A}$ one can form its homotopy category $\mathbf{K}(\mathscr{A})$. Conversely, given a triangulated category $\mathscr{T}$, one may want to recover an abelian or additive category embedded into $\mathscr{T}$. This requires additional structures on $\mathscr{T}$, namely ***t**-structures* and *weight structures*.

The definition of **t**-structures introduced in [BBD82] models the truncation of complexes in the derived category. For the weight structures introduced in [Bon10b; Pau08], there are two different motivations: On the one hand, the definition of weight structures is obtained algebraically from that of **t**-structures by "dualizing" some axioms, and this is why they are also known as co-**t**-structures. This is the perspective primarily used in silting theory, for instance in [Pau08; KY14; KN13], and also in this thesis. On the other hand, weight structures are heavily inspired by geometry, and provide an axiomatic framework for Deligne's theory of weights in the abstract setup of triangulated categories. This is the motivation for the study and applications of weight structures in geometry, for instance in [Bon10b; Bon10c; ES22].

Weight structures and **t**-structures on the same triangulated category are often closely related. In particular, a *silting collection* $\mathcal{P}$ in the sense of [PV18] defines both a **t**-structure and a weight structure, which are orthogonal to each other. The first goal of this thesis is to find a characterization of such silting **t**-structures by homological properties.

Although the definitions of **t**-structures and weight structures appear dual, it is hard to formalize this tentative duality. We provide a first step towards such a formalization via *Koszul duality*, which is an important standard tool in representation theory. The classical version of Koszul duality from [BGS96; MOS09] provides an equivalence between the derived category of a Koszul algebra A and that of its Koszul dual $A^! = \mathrm{Ext}^*_A(A_0, A_0)$, and there are also variants of Koszul duality involving dg algebras [Pri70; Kel94], A_∞-algebras [LH03], and (dg) coalgebras

[Pri70; LH03; Pos11]. Many examples of Koszul duality occur naturally in representation theory, and in particular in the representation theory of Lie algebras. For instance, symmetric and exterior algebras are Koszul-dual to each other, and the algebras describing blocks of (parabolic) category $\mathcal{O}$ are Koszul. For category $\mathcal{O}$, Koszul duality can also be used to interpret the relations between many important functors: in particular, translation functors are Koszul-dual to Zuckerman functors [RH04], and shuffling functors are Koszul-dual to Arkhipov twisting functors [MOS09].

Shuffling functors have many applications. In particular, by [MS08], the shuffling functors can be used to describe the Serre functor of the derived category of the principal block of parabolic category $\mathcal{O}$. These derived categories are equivalent to the constructible derived categories of (partial) flag varieties G/P. Therefore, we want to describe the Serre functor of $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(G/P)$ in the language intrinsic to these categories. For $G/P = \mathbb{P}^1$, such a description is provided by [Woo10] in terms of the *spherical twists* at spherical objects introduced in [ST01]. Extending this description of the Serre functor of $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^1)$ to the constructible derived categories of other flag varieties requires a generalization of spherical twists. The appropriate generalization for $\mathbb{P}^n$ are the *$\mathbb{P}$-twists* introduced in [HT06], which use $\mathbb{P}$-objects instead of spherical objects. Other (partial) flag varieties will presumably require further generalizations of $\mathbb{P}$-twists to "(partial) flag variety twists", but these are yet to be defined.

Spherical twists can be used to construct braid group actions on triangulated categories [ST01], and hence they play an important role in both algebraic geometry and representation theory. In algebraic geometry, spherical twists are a useful tool to describe stability manifolds and automorphism groups of varieties, see for instance [Bri08; Bri09; IU05; BP14], and also [AL17, §1] for an overview of applications and generalizations of spherical twists. In representation theory, in certain special cases the shuffling functors on parabolic category $\mathcal{O}$ can be realized as spherical twists [Len21]. Furthermore, spherical twists are related to tilting of $\mathfrak{t}$-structures, see e.g. [Woo10; Tho18], and can also be used to obtain braid group actions from classical silting collections [MY25]. The applications of $\mathbb{P}$-twists in algebraic geometry are similar to those of spherical twists [Huy06], and like spherical twists they moreover appear in symplectic geometry, see for instance [MW19]. However, $\mathbb{P}$-twists have not been considered in representation-theoretic settings yet, and our results are a first step towards applications of $\mathbb{P}$-twists in this area.

Braid groups are closely related to Weyl groups, whose associated combinatorics controls the structure of semisimple Lie algebras. The analog of simple Lie algebras in the $\mathbb{Z}/2\mathbb{Z}$-graded setting are the classical simple Lie superalgebras such as $\mathfrak{sl}(m|n)$, $\mathfrak{osp}(2m+1|2n)$ and $\mathfrak{osp}(2m|2n)$. However, in this setting the Weyl group is insufficient to describe their structure as it does not see the odd simple roots. One proposed solution is to replace the Weyl group by a *Weyl groupoid*. There are various different constructions of Weyl groupoids, in particular the Weyl groupoid from [SV11] and the root groupoid introduced in [GHS24]. An at first glance different notion of Weyl groupoids plays an important role in the theory of Nichols algebras [Hec06], and their associated combinatorics has been considered in depth, see the survey [AA17]. Weyl groupoids in this sense can also be used as an analog of the Weyl group in the context of Lie superalgebras [HY08; HS20], and turn out be closely related to the root groupoid. Our goal is to explicitly describe the Weyl groupoids of $\mathfrak{sl}(m|n)$, $\mathfrak{osp}(2m+1|2n)$ and $\mathfrak{osp}(2m|2n)$.

In the following we give detailed separate overviews over the main results in the four parts of this thesis.

1.1 Characterization of silting t-structures via derived projectives

Silting theory studies the interaction of **t**-structures, weight structures, simple-minded collections, and silting collections. In particular, one considers **t**-structures that are obtained from silting collections. Our goal is to find a characterization of these silting **t**-structures by homological properties.

By definition [BBD82], a ***t**-structure* $t = (\mathscr{D}^{t\le 0}, \mathscr{D}^{t\ge 0})$ on a triangulated category $\mathscr{D}$ provides a **t**-decomposition triangle $t_{\le 0}X \to X \to t_{>0}X \to t_{\le 0}X[1]$ for any $X \in \mathscr{D}$. Its *heart* $\heartsuit_t = \mathscr{D}^{t\le 0} \cap \mathscr{D}^{t\ge 0}$ is then an abelian category. The prototypical example is the *standard **t**-structure* on the derived category $\mathbf{D}(\mathscr{A})$ of an abelian category $\mathscr{A}$, given by

$$\mathscr{D}^{t\le 0} = \{X \in \mathbf{D}(\mathscr{A}) \mid H^n(X) = 0 \; \forall n > 0\}, \qquad \mathscr{D}^{t\ge 0} = \{X \in \mathbf{D}(\mathscr{A}) \mid H^n(X) = 0 \; \forall n < 0\}.$$

Its heart consists of all those cochain complexes whose cohomology is concentrated in degree 0, and is thus equivalent to $\mathscr{A}$. The **t**-decompositions are given by the "soft truncations" of cochain complexes.

The definition of weight structures from [Bon10b; Pau08] is very similar to that of **t**-structures: a *weight structure* on a triangulated category $\mathscr{C}$ is a pair of subcategories $w = (\mathscr{C}_{w\ge 0}, \mathscr{C}_{w\le 0})$, providing weight decomposition triangles $w_{>0}X \to X \to w_{\le 0}X \to w_{>0}X[1]$ for any $X \in \mathscr{C}$. Its *coheart* $\diamondsuit_w = \mathscr{C}_{w\ge 0} \cap \mathscr{C}_{w\le 0}$ is an additive category. The easiest example of a weight structure is the *standard weight structure* on the homotopy category $\mathscr{C} = \mathbf{K}(\mathscr{A})$ of an additive category $\mathscr{A}$, given by

$$\mathscr{C}_{w\ge 0} = \{X \in \mathscr{C} \mid \exists Y \cong X : Y^n = 0 \; \forall n < 0\}, \quad \mathscr{C}_{w\le 0} = \{X \in \mathscr{C} \mid \exists Y \cong X : Y^n = 0 \; \forall n > 0\}.$$

By definition, its coheart consists of all those complexes isomorphic to complexes concentrated in degree 0 (at least if $\mathscr{A}$ is idempotent-complete), and is thus equivalent to $\mathscr{A}$. The weight decompositions are given by the "brutal truncations" of cochain complexes.

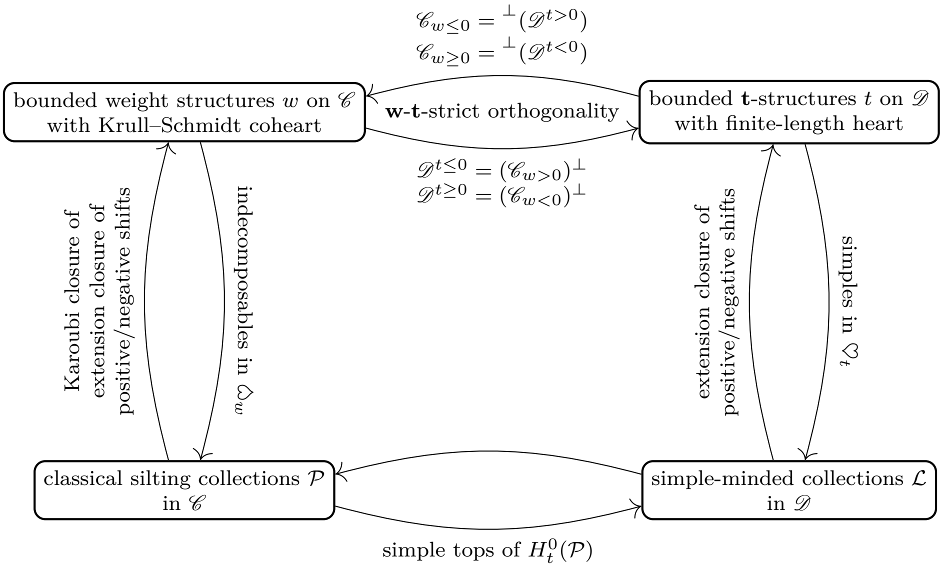
The subtle difference between the definitions of weight structures and **t**-structures lies in the order of the terms in the decomposition triangles, and how these interact with shifts. As a result, **t**-decompositions are unique and functorial, but weight decompositions are unique if and only if $\diamondsuit_w = \{0\}$. Moreover, the heart of a **t**-structure is an abelian category with short exact sequences given by triangles, but this is not true for weight structures.

If t is a bounded **t**-structure (i.e. if $\heartsuit_t$ generates $\mathscr{D}$ as a triangulated category) and $\heartsuit_t$ is finite-length, then t can be reconstructed from the set of simple objects in $\heartsuit_t$. These form a *simple-minded collection*. Similarly, if w is a bounded weight structure and $\diamondsuit_w$ is Krull–Schmidt, then w can be recovered from the indecomposable objects in $\diamondsuit_w$, and these form a *classical silting collection*. The axiomatic definitions of simple-minded collections and classical silting collections are easily obtained from these characterizations.

Now let $\mathscr{T}$ be a triangulated category and $\mathscr{C}$ and $\mathscr{D}$ thick subcategories of $\mathscr{T}$. A weight structure w on $\mathscr{C}$ is *orthogonal* to a **t**-structure t on $\mathscr{D}$ if $\mathscr{C}_{w>0} \perp \mathscr{D}^{t\le 0}$ and $\mathscr{C}_{w<0} \perp \mathscr{D}^{t\ge 0}$. Moreover, such an orthogonality is ***w**-**t**-strict* if these relations define the weight structure and the **t**-structure. For instance, if A is a finite-dimensional algebra, then the standard weight structure on $\mathscr{C} = \mathbf{K}^{\mathrm{b}}(\mathbf{proj}_{\mathrm{fg}}\text{-}A)$ is **w**-**t**-strictly left orthogonal to the standard **t**-structure on $\mathscr{D} = \mathbf{D}^{\mathrm{b}}(\mathbf{mod}_{\mathrm{fd}}\text{-}A)$, with both viewed as subcategories of $\mathscr{T} = \mathbf{D}^{-}(\mathbf{mod}_{\mathrm{fd}}\text{-}A) = \mathbf{K}^{-}(\mathbf{proj}_{\mathrm{fg}}\text{-}A)$.

The pair of subcategories $(\mathbf{K}^{\mathrm{b}}(\mathbf{proj}_{\mathrm{fg}}\text{-}A), \mathbf{D}^{\mathrm{b}}(\mathbf{mod}_{\mathrm{fd}}\text{-}A))$ of $\mathbf{D}^{-}(\mathbf{mod}_{\mathrm{fd}}\text{-}A) = \mathbf{K}^{-}(\mathbf{proj}_{\mathrm{fg}}\text{-}A)$ is the prototypical example of a *WT pair*, introduced in [AMY19] as ST pair. In general, a WT pair in $\mathscr{T}$ is roughly speaking a pair of thick subcategories $(\mathscr{C}, \mathscr{D})$ such that $\mathscr{T}$ admits a bounded above weight structure w and a bounded above **t**-structure t with $\mathscr{T}_{w\leq 0} = \mathscr{T}^{t\leq 0}$, such that $\mathscr{C}$ and $\mathscr{D}$ are the thick subcategories generated by $\rotatebox[origin=c]{180}{$\heartsuit$}_w$ respectively $\heartsuit_t$. In particular, w and t restrict to a bounded weight structure on $\mathscr{C}$ and a bounded **t**-structure on $\mathscr{D}$, respectively. For a WT pair $(\mathscr{C}, \mathscr{D})$, the following result from [Fus24] shows that **w**-**t**-strict orthogonality provides a bijection between weight structures on $\mathscr{C}$ and **t**-structures on $\mathscr{D}$. This unifies and generalizes earlier results for Dynkin quivers [KV88], finite-dimensional algebras [KY14], non-positive dg algebras [BY14], and positive dg algebras [KN13].

Theorem (WT correspondence)**.** *Let $(\mathscr{C}, \mathscr{D})$ be a WT pair in $\mathscr{T}$. Then the following diagram of bijections commutes:*



For the standard example $(\mathbf{D}^{\mathrm{b}}(\mathbf{mod}_{\mathrm{fd}}\text{-}A), \mathbf{K}^{\mathrm{b}}(\mathbf{proj}_{\mathrm{fg}}\text{-}A))$ in $\mathbf{D}^{-}(\mathbf{mod}_{\mathrm{fd}}\text{-}A) = \mathbf{K}^{-}(\mathbf{proj}_{\mathrm{fg}}\text{-}A)$, the bijections identify the standard **t**-structure on $\mathbf{D}^{\mathrm{b}}(\mathbf{mod}_{\mathrm{fd}}\text{-}A)$, the simple-minded collection consisting of the simple A-modules, the standard weight structure on $\mathbf{K}^{\mathrm{b}}(\mathbf{proj}_{\mathrm{fg}}\text{-}A)$, and the classical silting collection consisting of the indecomposable projective A-modules.

The crucial step in the proof of the WT correspondence (see e.g. [KY14, §5.6], [Ric02, §5], or [Fus24, §4] for the general case) is the construction of a classical silting collection $\mathcal{P}$ corresponding to a given simple-minded collection $\mathcal{L}$. This classical silting collection is characterized by the existence of a bijection $\phi\colon \mathcal{P} \to \mathcal{L}$ such that

$$\mathrm{Hom}_{\mathscr{T}}(P, L[m]) \cong \begin{cases} \mathrm{End}_{\mathscr{T}}(L) & \text{if } L = \phi(P),\ m = 0, \\ 0 & \text{otherwise,} \end{cases}$$

but this does not provide a good conceptual interpretation of the relation of $\mathcal{P}$ and $\mathcal{L}$.

However, this characterization looks very similar to the relation of the indecomposable projective objects to the simple objects in an abelian category. Moreover, in the standard example of the WT correspondence the simple-minded collection consists of the simple modules, while the correspnding classical silting collection consists of the indecomposable projective modules. This shows that classical silting collections should play the role of the set of indecomposable projective objects in the triangulated setup, while simple-minded collections should be seen as analogs of the set of simple objects.

The formalization of this observation is the first main result of this thesis. For this, we use *derived projective objects*, which are the analog of projective objects for triangulated categories equipped with **t**-structures. By definition, an object $P \in \mathscr{T}$ is derived projective with respect to t if $P \in \mathscr{T}^{t\leq 0}$ and $P \perp \mathscr{T}^{t<0}$. For instance, if A is a finite-dimensional algebra, then A is derived projective in $\mathbf{D}^{\mathrm{b}}(\mathbf{mod}_{\mathrm{fd}}\text{-}A)$ with respect to the standard **t**-structure. More generally, let $(\mathscr{C}, \mathscr{D})$ be a WT pair and $\mathcal{P}$ a classical silting collection in $\mathscr{C}$ corresponding to a bounded **t**-structure t on $\mathscr{D}$. Then $\mathcal{P}$ does not necessarily lie in $\mathscr{D}$, and thus does not consist of derived projective in $\mathscr{D}$. However, by [AMY19, Prop. 5.2] t extends to a **t**-structure $t_{\mathcal{P}} = (\mathcal{P}^{\perp_{>0}}, \mathcal{P}^{\perp_{<0}})$ on $\mathscr{T}$, and hence any $P \in \mathcal{P}$ is derived projective in $\mathscr{T}$ with respect to $t_{\mathcal{P}}$.

In analogy to the abelian setting, [GLVdB21] introduced the notion of *enough derived projectives.* Therein, it is shown that a pretriangulated dg category $\mathscr{A}$ equipped with a non-degenerate bounded above **t**-structure on $H^0(\mathscr{A})$ with enough derived projectives can be recovered from the dg category consisting of the derived projective objects in $\mathscr{A}$. This yields a correspondence between left homotopically coherent dg categories and pretriangulated dg categories with enough derived projectives [GLVdB21, Thm. 7.12]. These results are applied in [GLVdB22; GLSVdB24] to develop a deformation theory for pretriangulated dg categories with **t**-structures. In [GRG23], the dual notion of enough derived injective objects is moreover used to prove a derived version of the Gabriel–Popescu theorem.

To complete the picture, we introduce *derived projective covers* as analogs of projective covers in the triangulated setting. The following general theorem then in particular implies that the classical silting collection $\mathcal{P}$ corresponding to a simple-minded collection $\mathcal{L}$ under the WT correspondence consists of the derived projective covers of $\mathcal{L}$:

Theorem A (Characterization of silting **t**-structures, Theorem 2.3.16)**.** *Let t be a non-degenerate **t**-structure with finite-length heart on a triangulated category $\mathscr{D}$. Let $\mathcal{L}$ be a full set of isomorphism representatives of the simple objects in $\heartsuit_t$ and $\mathcal{P}$ a full set of isomorphism representatives of the indecomposable derived projectives. Then the following are equivalent:*

*I) t is a silting **t**-structure in the sense of [PV18], i.e. $t = (\mathcal{P}^{\perp_{>0}}, \mathcal{P}^{\perp_{<0}})$.*

II) There is a bijection $\phi\colon \mathcal{P} \to \mathcal{L}$ such that

$$\operatorname{Hom}_{\mathscr{D}}(P, L[m]) \cong \begin{cases} \operatorname{End}_{\mathscr{D}}(L) & \text{if } L = \phi(P) \text{ and } m = 0, \\ 0 & \text{otherwise} \end{cases} \tag{1.1}$$

as $\operatorname{End}_{\mathscr{D}}(L)$-modules.

III) Every $L \in \mathcal{L}$ admits a derived projective cover (and $\mathcal{P}$ is the set of these derived projective covers).

IV) $\mathscr{D}$ has enough derived projectives with respect to t.

Theorem A is closely related to [CSPP22, Thm. 2.4] and [Bon19, Thm. 5.3.1], which provide a similar criterion for the existence of an adjacent weight structure for a given **t**-structure. As an application, in Definition 2.4.1 and Proposition 2.4.3 we provide a definition of WT pairs that is equivalent to that from [AMY19]. Compared to the original definition, our definition uses weight structures instead of classical silting collections, and as a result it is more symmetric. This is also the reason why we prefer the name WT pair over ST pair, to reflect the use of weight structures (instead of classical silting collections) and **t**-structure in the definition.

A different question is whether the bijection between weight structures and **t**-structures provided by the WT correspondence is natural with respect to weight exact functors and **t**-exact functors. Since this bijection is given by **w**-**t**-strict orthogonality, a straightforward generalization of [Bon10b, Prop. 4.4.5] shows that the WT correspondence is natural in the following sense:

Theorem B (Naturality of the WT correspondence, Corollary 2.5.2)**.** *Let $(\mathscr{C}, \mathscr{D})$ be a WT pair in $\mathscr{T}$ and $(\mathscr{C}', \mathscr{D}')$ a WT pair in $\mathscr{T}'$. Let w, w' be bounded weight structures on $\mathscr{C}$ respectively $\mathscr{C}'$, and t, t' the corresponding bounded* **t***-structures on $\mathscr{D}$ respectively $\mathscr{D}'$ under the WT correspondence. Suppose that $F\colon \mathscr{C} \to \mathscr{C}'$ is left pseudo-adjoint to $G\colon \mathscr{D}' \to \mathscr{D}$ in the sense that $\mathrm{Hom}_{\mathscr{T}'}(F(-), -) \cong \mathrm{Hom}_{\mathscr{T}}(-, G(-))$.*

Then F is weight exact with respect to w and w' if and only if G is **t***-exact with respect to t and t'.*

1.2 Koszul duality of simple-minded and silting collections

A *Koszul algebra* is a positively graded $\Bbbk$-algebra A such that the simple A-modules have linear projective resolutions. Its *Koszul dual* $A^! = \mathrm{Ext}^*_A(A_0, A_0)$ is again a Koszul algebra, and we have $(A^!)^! \cong A$. Moreover, the Koszul duality theorem from [BGS96; MOS09] provides a triangulated equivalence

$$\mathrm{RHom}_A(A_0, -)\colon \mathbf{D}^{\downarrow}(A) \cong \mathbf{D}^{\uparrow}(A^!),$$

where $\mathbf{D}^{\downarrow}(A) \subseteq \mathbf{D}(A)$ and $\mathbf{D}^{\uparrow}(A^!) \subseteq \mathbf{D}(A^!)$ are the subcategories consisting of complexes of graded modules that are bounded below and "linearly bounded above", respectively bounded above and "linearly bounded below". In particular, this identifies the simple A-modules with the indecomposable projective $A^!$-modules.

By definition, simple-minded collections are analogs of the set of simple modules, and by Theorem A silting collections can be seen as analogs of the set of indecomposable projective modules. As these are precisely the classes of objects exchanged by the classical Koszul duality, we want to relate simple-minded collections and silting collections by some kind of Koszul duality.

The required notion of Koszul duality is provided by the dg Koszul duality for augmented dg categories introduced in [Kel94]. Augmented dg categories include in particular the non-positive dg algebras and positive dg algebras, viewed as dg categories via a primitive orthogonal collection of idempotents. The dg Koszul dual $A^{!,\mathrm{dg}}$ of a non-positive (respectively positive) dg algebra A is a positive (respectively non-positive) dg algebra by [BY14; KN13]. Under some finiteness assumptions, the double dg Koszul dual of a non-positive or positive dg algebras is the original dg algebra, and there are equivalences between certain subcategories of $\mathbf{D}(A)$ and $\mathbf{D}(A^{!,\mathrm{dg}})$, see [Kel94; Fus25].

The dg Koszul dual is related to the classical Koszul dual $A^!$ considered in [BGS96] as follows. A Koszul algebra A can be viewed as a positive dg algebra with the same grading and trivial differential. If A has finite global dimension, it follows by combining [Sch11, Thm. 39] and [KN13, Lemma 5.2] that

$$H^n(A^{!,\mathrm{dg}}) \cong \begin{cases} A^! & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

To apply dg Koszul duality to silting collections and simple-minded collections, we first have to obtain dg algebras from these. For this, we use a dg enhancement $\widetilde{\mathscr{T}}$ of the ambient triangulated category $\mathscr{T}$. For any object $X \in \mathscr{T}$, the dg enhancement provides a dg algebra $\mathrm{End}_{\widetilde{\mathscr{T}}}(X)$ such that $H^*(\mathrm{End}_{\widetilde{\mathscr{T}}}(X)) \cong \mathrm{End}^*_{\mathscr{T}}(X)$. If $\mathcal{P}$ is a silting collection and $\mathcal{L}$ a simple-minded collection, then as an immediate consequence of the definitions the dg algebra $\mathrm{End}_{\widetilde{\mathscr{T}}}(\bigoplus_{P \in \mathcal{P}} P)$ is non-positive and the dg algebra $\mathrm{End}_{\widetilde{\mathscr{T}}}(\bigoplus_{L \in \mathcal{L}} L)$ is positive. By applying dg Koszul duality to such dg algebras arising from classical silting collections and simple-minded collections related by the orthogonality relation (1.1), we obtain:

Theorem C (Koszul duality of simple-minded/silting, Theorem 3.4.2)**.** *Let $\mathscr{T}$ be a compactly generated dg-enhanced triangulated category. Let $\mathcal{P}$ be a classical silting collection in the subcategory of compact objects of $\mathscr{T}$ and $\mathcal{L}$ a simple-minded collection in a subcategory of $\mathscr{T}$, and suppose there is a bijection $\phi\colon \mathcal{P} \to \mathcal{L}$ satisfying* (1.1)*.*

1) If $\operatorname{End}_{\mathscr{T}}(\bigoplus_{P\in\mathcal{P}} P)$ is finite-dimensional, then the dg algebra $\operatorname{End}_{\widetilde{\mathscr{T}}}(\bigoplus_{L\in\mathcal{L}} L)$ is the dg Koszul dual of $\operatorname{End}_{\widetilde{\mathscr{T}}}(\bigoplus_{P\in\mathcal{P}} P)$.

2) If $\operatorname{Hom}_{\mathscr{T}}(\bigoplus_{P\in\mathcal{P}} P, \bigoplus_{P\in\mathcal{P}} P[n])$ is finite-dimensional for all $n \in \mathbb{Z}$, then $\operatorname{End}_{\widetilde{\mathscr{T}}}(\bigoplus_{P\in\mathcal{P}} P)$ is the dg Koszul dual of $\operatorname{End}_{\widetilde{\mathscr{T}}}(\bigoplus_{L\in\mathcal{L}} L)$.

This result provides a notion of Koszul duality between simple-minded and silting collections, which can be seen as a first step towards understanding the tentative Koszul duality between weight structures and $\mathbf{t}$-structures. More precisely, we ask:

Questions. Does the Koszul duality of simple-minded and silting collections from Theorem C extend to a Koszul duality theorem relating...

1) ...hearts and cohearts of orthogonal $\mathbf{t}$-structures and weight structures?
2) ...orthogonal $\mathbf{t}$-structures and weight structures?
3) ...the machinery of $\mathbf{t}$-structures and weight structures, such as $\mathbf{t}$-decompositions respectively weight decompositions, and the realization functor respectively strong weight complex functor?

1.3 Serre functor and $\mathbb{P}$-objects for perverse sheaves on $\mathbb{P}^n$

The notion of Serre functor introduced in [BK90] generalizes Serre duality from algebraic geometry. Serre functors are an important tool in the theory triangulated categories, and in particular they allow to construct left adjoints of functors that have a right adjoint, and vice versa. Besides algebraic geometry, Serre functors also appear in algebra and representation theory: by [Hap88] the derived Nakayama functor is a Serre functor of the bounded derived category of a finite-dimensional algebra of finite global dimension. This general result abstractly provides Serre functors for many interesting triangulated categories in representation theory, and in particular for the constructible derived categories of partial flag varieties G/P which play a central role in geometric representation theory. However, this does not provide a description of the Serre functor in the language intrinsic to the constructible derived category.

A partial flag variety G/P comes with a natural stratification provided by the double cosets for a Borel subgroup of G, and this stratification can be used to construct the perverse $\mathbf{t}$-structures on the constructible derived category $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(G/P)$ [BBD82]. Of particular importance is the middle-perverse $\mathbf{t}$-structure, and its heart is the category $\mathbf{Perv}(G/P)$ of *middle-perverse sheaves.* By well-known results from [BB81; BK81; BGS96], there are equivalences of triangulated respectively abelian categories given by the diagram

$$\begin{tikzcd}
\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(G/P) \arrow[r, "\cong"] & \mathbf{D}^{\mathrm{b}}(\mathcal{O}_0^{\mathfrak{p}}(\mathfrak{g})) \arrow[r, "\cong"] & \mathbf{D}^{\mathrm{b}}(A^{\mathfrak{p}}(\mathfrak{g})\text{-}\mathbf{mod}_{\mathrm{fd}}) \\
\mathbf{Perv}(G/P) \arrow[r, "\cong"] \arrow[u, hook] & \mathcal{O}_0^{\mathfrak{p}}(\mathfrak{g}) \arrow[r, "\cong"] \arrow[u, hook] & A^{\mathfrak{p}}(\mathfrak{g})\text{-}\mathbf{mod}_{\mathrm{fd}}. \arrow[u, hook]
\end{tikzcd}$$

Here $A^{\mathfrak{p}}(\mathfrak{g})$ is a certain finite-dimensional algebra which in general cannot be described explicitly, see [Str03] for some known cases and an overview of the difficulties. In particular these equivalences identify the middle-perverse $\mathbf{t}$-structure on $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(G/P)$ with the standard $\mathbf{t}$-structures on $\mathbf{D}^{\mathrm{b}}(\mathcal{O}_0^{\mathfrak{p}}(\mathfrak{g}))$ and $\mathbf{D}^{\mathrm{b}}(A^{\mathfrak{p}}(\mathfrak{g})\text{-}\mathbf{mod}_{\mathrm{fd}})$.

Since $A^{\mathfrak{p}}(\mathfrak{g})\text{-}\mathbf{mod}_{\mathrm{fd}}$ is a highest weight category in the sense of [CPS88; BS24], it has finite global dimension, and thus by [Hap88] the left derived Nakayama functor is a Serre functor for $\mathbf{D}^{\mathrm{b}}(A^{\mathfrak{p}}(\mathfrak{g})\text{-}\mathbf{mod}_{\mathrm{fd}})$. From the above equivalences it follows that $\mathbf{D}^{\mathrm{b}}(\mathcal{O}_0^{\mathfrak{p}}(\mathfrak{g}))$ and $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(G/P)$ also admit Serre functors. For $\mathbf{D}^{\mathrm{b}}(\mathcal{O}_0^{\mathfrak{p}}(\mathfrak{g}))$, the Serre functor can be explicitly described in Lie-theoretic language as the shifted derived shuffling functor $\mathrm{Sh}^2_{w_0}[-2\ell(w_0^{\mathfrak{p}})]$ by results from [MS08].

The main result of Chapter 4 provides a description of the Serre functor in the language intrinsic to the constructible derived category for the special case $G = \mathrm{GL}_{n+1}(\Bbbk)$ and the parabolic subgroup $P \subseteq G$ with block sizes $(n, 1)$, i.e. for $G/P = \mathbb{P}^n$. For this we use the $\mathbb{P}$-twists at $\mathbb{P}$-objects introduced in [HT06]. These are autoequivalences that generalize the spherical twists at spherical objects from [ST01], and are defined as follows.

A $\mathbb{P}^k$*-like object* in a triangulated category $\mathscr{D}$ is an object $E \in \mathscr{D}$ which cohomologically looks like a projective space, i.e. such that $\mathrm{End}^*_{\mathscr{D}}(E) \cong \Bbbk[t]/(t^{k+1})$ with $\deg(t) = 2$. It is a $\mathbb{P}^k$-object if it is moreover $2k$-Calabi–Yau, i.e. there is a natural isomorphism $\mathrm{Hom}_{\mathscr{D}}(P, -) \cong \mathrm{Hom}_{\mathscr{D}}(-, P[2k])^\vee$. Slightly more generally, there are also $\mathbb{P}^k[d]$-objects [Kru18; HK19], for which $\mathrm{End}^*_{\mathscr{D}}(P) \cong \Bbbk[t]/(t^{k+1})$ with $\deg(t) = d$. In particular d-spherical objects are the same as $\mathbb{P}^1[d]$-objects, and exceptional objects are the same as $\mathbb{P}^0$-like objects.

The value of the $\mathbb{P}$-twist PT_E associated to a $\mathbb{P}$-like object E at $X \in \mathscr{D}$ is then defined by the double cone construction

$$\begin{tikzcd}
\big(\mathrm{Hom}^*_{\mathscr{D}}(E,X) \otimes E\big)[-2] \arrow[r, "t^* \otimes \mathrm{id} - \mathrm{id} \otimes t"] & \mathrm{Hom}^*_{\mathscr{D}}(E,X) \otimes E \arrow[r] \arrow[d, "\mathrm{ev}"] & \mathrm{cone}(t^* \otimes \mathrm{id} - \mathrm{id} \otimes t) \arrow[dl, dashed, "\exists \overline{\mathrm{ev}}"] \\
& X \arrow[dl] & \\
\mathrm{cone}(\overline{\mathrm{ev}}) = \mathsf{PT}_E(X). & &
\end{tikzcd}$$

where the factorization $\overline{\mathrm{ev}}\colon \mathrm{cone}(t^* \otimes \mathrm{id} - \mathrm{id} \otimes t) \to X$ exists since $\mathrm{ev} \circ (t^* \otimes \mathrm{id} - \mathrm{id} \otimes t) = 0$. The precise definition of the $\mathbb{P}$-twist PT_E as a triangulated functor requires some care and involves a dg enhancement of $\mathscr{D}$. If E is a spherelike object, then the $\mathbb{P}$-twist PT_E is related to the spherical twist ST_E defined in [ST01] by $\mathsf{PT}_E = \mathsf{ST}_E^2$.

In our setting $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n)$, the total endomorphism ring of the simple perverse sheaf $\mathrm{IC}_k = \mathrm{incl}_* \underline{\Bbbk}_{\mathbb{P}^k}[k]$ is the cohomology ring of $\mathbb{P}^k$, and hence IC_k is a $\mathbb{P}^k$-like object almost by definition. Among the simple perverse sheaves, $\mathrm{IC}_n = \underline{\Bbbk}_{\mathbb{P}^n}[n]$ is moreover Calabi–Yau, and thus a $\mathbb{P}^n$-object. Therefore the $\mathbb{P}$-twist $\mathsf{PT}_{\mathrm{IC}_n}$ at IC_n is an autoequivalence of $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n)$, and this yields the desired description of the Serre functor:

Theorem D (Serre functor via $\mathbb{P}$-twists, Theorem 4.3.11)**.** *The $\mathbb{P}$-twist $\mathsf{PT}_{\mathrm{IC}_n}$ is the inverse Serre functor of $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n)$.*

Theorem D in particular recovers the result for $\mathbb{P}^1$ from [Woo10], where it is shown that $\mathsf{ST}^2_{\mathrm{IC}_1}$ is the inverse Serre functor of $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^1)$.

It would be desirable to extend Theorem D to other partial flag varieties. However, in this case the appropriate notion of twist functors is yet to be defined.

Questions. Let G be a reductive algebraic group and $P \subseteq G$ a parabolic subgroup.

1) Is the simple perverse sheaf $\underline{\Bbbk}_{G/P}[\dim G/P]$ a "partial flag variety object" in $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(G/P)$?
2) Can one define a "partial flag variety twist" at a partial flag variety object, such that the partial flag variety twist at $\underline{\Bbbk}_{G/P}[\dim G/P]$ is the inverse Serre functor for $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(G/P)$?

Motivated by Theorem D, one may wonder whether there are further interesting $\mathbb{P}$-objects in $\mathbf{Perv}(\mathbb{P}^n)$. However, it is easy to see that no indecomposable object except IC_n and the indecomposable projective-injectives can be Calabi–Yau, and hence any other indecomposable object can at best be $\mathbb{P}$-like. By using the description of $\mathbf{Perv}(\mathbb{P}^n)$ in terms of finite-dimensional algebras, the indecomposable objects can be listed explicitly: there are the indecomposable projective-injective objects and certain *string objects* $\mathrm{M}^{\pm}_{a,b}$ for $0 \leq b \leq a \leq n$. The indecomposable projective-injective objects are 0-spherical (i.e. $\mathbb{P}^1[0]$-objects), and for the string objects we obtain the following classification result:

Theorem E ($\mathbb{P}$-like string objects, Theorem 4.4.17)**.** *Let $0 \leq b \leq a \leq n$.*

1) If $a-b$ is even, then the string objects $\mathrm{M}^{\pm}_{a,b}$ are $\mathbb{P}^{(a+b)/2}$-like.

2) If $a-b$ is odd, then the string objects $\mathrm{M}^{\pm}_{a,b}$ are $\mathbb{P}^{(a-b-1)/2}$-like.

In other words, all indecomposable perverse sheaves on $\mathbb{P}^n$ are $\mathbb{P}$-like. Theorem E also yields a classification of the spherelike objects (i.e. $\mathbb{P}^1$-like objects) in $\mathbf{Perv}(\mathbb{P}^n)$, and moreover recovers the classification of the exceptional objects (i.e. $\mathbb{P}^0$-like objects) from [PW20].

1.4 The Weyl groupoids of $\mathfrak{sl}(m|n)$ and $\mathfrak{osp}(r|2n)$

Weyl groups play a central role in the structure theory of complex semisimple Lie algebras. In particular, it is well-known that the systems of simple roots of a complex simple Lie algebra are conjugate under the Weyl group. Classical simple Lie superalgebras are natural analogs of simple Lie algebras in the $\mathbb{Z}/2\mathbb{Z}$-graded setting, and like their ungraded counterparts they have systems of simple roots corresponding to Borel subalgebras. However, in contrast to the ungraded situation, not all systems of simple roots are conjugate under the action of the Weyl group of the even part. One way to remedy this issue is to use Weyl groupoids instead.

We consider the notion of Weyl groupoids introduced in [Hec06] in the theory of Nichols algebras. Our main goal is to explicitly describe the Weyl groupoids of the classical simple Lie superalgebras $\mathfrak{sl}(m|n)$, $\mathfrak{osp}(2m+1|2n)$ and $\mathfrak{osp}(2m|2n)$.

Weyl groupoids are constructed from *(semi-)Cartan graphs*, i.e. edge-colored graphs together with an assignment of a generalized Cartan matrix called *Serre matrix* to each vertex [HY08; HS20]. This combinatorial data can be used to classify Weyl groupoids and Nichols algebras [AA17], similarly to how Dynkin diagrams are used to classify Weyl groups and Lie algebras. The Weyl groupoid is obtained from a Cartan graph by purely combinatorial means: an edge $x \overset{i}{\longleftrightarrow} y$ gives rise to simple reflections $(s_i)_x \colon x \to y$ and $(s_i)_y \colon y \to x$ defined in terms of the Serre matrices, and these simple reflections generate the Weyl groupoid. In the theory of Nichols algebras, the edges correspond to reflections of Yetter–Drinfeld modules, and for Lie superalgebras the edges correspond to reflections at simple roots.

Weyl groupoids share many properties of Weyl groups. In particular they are Coxeter groupoids, i.e. the simple reflections are subject (only) to the Coxeter relations $(s_i s_j)^{m_{ij}} = \mathrm{id}$ for certain $m_{ij} \in \mathbb{Z}$ with $m_{ii} = 2$. Like Coxeter groups, Coxeter groupoids have an associated root system with powerful combinatorics, and in particular there are positive and negative roots which can be used to determine the length of a morphism. There are also analogs of the exchange condition and Matsumoto's theorem [HY08].

We define the Cartan graph and the Weyl groupoid of a contragredient Lie superalgebras as follows:

Definition F (Weyl groupoids of Lie superalgebras, Definition 5.2.10)**.** The *Cartan graph* $\mathcal{G}_{\mathfrak{g}}$ of a regular symmetrizable contragredient Lie superalgebra $\mathfrak{g}$ has

- vertices: a labelling set X for the ordered root bases of $\mathfrak{g}$.
- edges according to the rules:
    - for each odd isotropic root α_i^x in an ordered root basis $\Pi(x)$ and $\Pi(x')$ obtained from $\Pi(x)$ by an odd reflection at α_i^x in the sense of [PS89], there is an edge $x \overset{i}{\longleftrightarrow} x'$ of color i.
    - for each root $\alpha_i^x \in \Pi(x)$ that is not odd isotropic, there is an edge $x \overset{i}{\longleftrightarrow} x$ of color i.
- the Serre matrices $A(x)$, which are the matrices defining the Serre relations among the generators of $\mathfrak{g}$ corresponding to the ordered root basis $\Pi(x)$.

The *Weyl groupoid* $\mathcal{W}_{\mathfrak{g}}$ is the Weyl groupoid of $\mathcal{G}_{\mathfrak{g}}$.

Cartan graphs and Weyl groupoids of finite-dimensional Lie superalgebras were first considered in [HY08], and the notion has been extended to contragredient Lie superalgebras in [HS20]. Our definition is equivalent to the construction in [HS20, §11.2], but uses a language that is more convenient if one wants to understand the structure of a Lie superalgebra.

The following theorem justifies the terminology in Definition F.

Theorem G (Cartan Graph Theorem, Corollary 5.2.14)**.** *The graph $\mathcal{G}_{\mathfrak{g}}$ is a Cartan graph in the sense of [HS20].*

In the structure theory of Lie superalgebras, there are several other notions of "Weyl groupoid". These are related to the Weyl groupoid $\mathcal{W}_{\mathfrak{g}}$ from Definition F as follows:

1) $\mathcal{W}_{\mathfrak{g}}$ is obtained from a component of the Weyl groupoid introduced in [Ser11] by forgetting all morphisms corresponding to rescaling rows of the Cartan matrices B.
2) A connected component of the skeleton of the root groupoid introduced in [GHS24] is the simply connected cover of $\mathcal{W}_{\mathfrak{g}}$ in the sense of [HS20, Def. 9.1.10 and 10.1.1].

   Conversely, the subgroupoid $\mathcal{W}'_{\mathfrak{g}}$ of $\mathcal{W}_{\mathfrak{g}}$ generated by all isotropic reflections is isomorphic to a connected component of the spine of the root groupoid.
3) There is no connection to the Weyl groupoid in the sense of [SV11].

The Weyl groupoids of the classical simple Lie superalgebras $\mathfrak{sl}(m|n)$, $\mathfrak{osp}(2m+1|2n)$ and $\mathfrak{osp}(2m|2n)$ can be explicitly described as follows. For these Lie superalgebras, the systems of simple roots have been classified in [Kac77] and can be written down explicitly in terms of (m,n)-shuffles, i.e. permutations of $\{1,\dots,m+n\}$ that do not swap the relative order of the elements in $\{1,\dots,m\}$ and $\{m+1,\dots,m+n\}$. These shuffles can moreover be identified with partitions fitting into a rectangle of size $m \times n$, which is more convenient if one wants to write down the corresponding Borel subalgebras. In this graphical language, odd reflections correspond to adding or removing boxes to (respectively from) partitions, and the edge coloring is then determined by numbering the boxes of the partition as follows:

1	2	3	4
2	3	4	5
3	4	5	6

We use the slightly unusual convention that the longest row of the partition is at the bottom, which makes it easier to write down the corresponding Borel subalgebras. For instance, the shaded boxes in the above picture are the partition $(4, 2, 1)$. To this, the boxes numbered 2 and 4 can be added, and the boxes numbered 1, 3 and 6 can be removed.

The description of the ordered root bases and odd reflections in terms of partitions makes it very easy to describe the Cartan graphs and Weyl groupoids of $\mathfrak{sl}(m|n)$, $\mathfrak{osp}(2m+1|2n)$ and $\mathfrak{osp}(2m|2n)$, see Propositions 5.4.1, 5.4.2 and 5.4.4. In the following we describe the smallest non-trivial examples, namely those determined by partitions fitting in a 1×2-rectangle.

For $\mathfrak{sl}(1|2)$, the underlying graph of the Cartan graph is

$$\begin{tikzcd} \emptyset \arrow[loop left, "2"] \arrow[r, leftrightarrow, "1"] & \square \arrow[r, leftrightarrow, "2"] & \square\square \arrow[loop right, "1"] \end{tikzcd}$$

and the Serre matrix is $A_2 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ at every vertex. The Weyl groupoid is then generated by the simple reflections $(s_1)_x$ and $(s_2)_x$ corresponding to the edges, where x is any partition fitting in the 1×2-rectangle indicating the source of the reflection. For instance, the edge $\emptyset \xleftrightarrow{1} \square$ gives rise to $(s_1)_\emptyset \colon \emptyset \to \square$ and $(s_1)_\square \colon \square \to \emptyset$. These are subject to the usual type A Coxeter relations, i.e. $s_i^2 = \mathrm{id}$ and $s_1 s_2 s_1 = s_2 s_1 s_2$ for any composition that makes sense. The observations from this small example generalize directly to $\mathfrak{sl}(m|n)$.

For $\mathfrak{osp}(3|4)$, the Cartan graph has the shape

$$\begin{tikzcd} \emptyset \arrow[loop above, "2"] \arrow[loop below, "3"] \arrow[r, leftrightarrow, "1"] & \square \arrow[loop above, "3"] \arrow[r, leftrightarrow, "2"] & \square\square \arrow[loop above, "1"] \arrow[loop below, "3"] \end{tikzcd}$$

and the Serre matrix is $B_3 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}$ at every vertex. It follows that the simple reflections $(s_i)_x$ (for $1 \le i \le 3$ and any partition x) are subject to the type B Coxeter relations $s_i^2 = \mathrm{id}$, $s_1 s_2 s_1 = s_2 s_1 s_2$, $s_1 s_3 = s_3 s_1$ and $s_2 s_3 s_2 s_3 = s_3 s_2 s_3 s_2$. As for $\mathfrak{sl}(m|n)$, the observations from this example generalize easily to $\mathfrak{osp}(2m+1|2n)$.

For $\mathfrak{osp}(2m|2n)$ the situation is more complicated. In this case, the Borel subalgebras are labelled by partitions and an additional sign $\varepsilon \in \{+, -, \pm\}$. For instance, the Cartan graph of $\mathfrak{osp}(2|4)$ has the shape

$$\begin{tikzcd} (\emptyset,+) \arrow[loop above, "2"] \arrow[loop below, "3"] \arrow[r, leftrightarrow, "1"] & (\square,+) \arrow[loop above, "3"] \arrow[r, leftrightarrow, "2"] & (\square\square,\pm) \arrow[loop above, "1"] \arrow[r, leftrightarrow, "3"] & (\square,-) \arrow[loop above, "2"] \arrow[r, leftrightarrow, "1"] & (\emptyset,-) \arrow[loop above, "2"] \arrow[loop below, "3"] \end{tikzcd}$$

From left to right, the Serre matrices are $C_3 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$, $A_3 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$, the matrix $\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$, $A_3' = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$ and $C_3' = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -2 \\ -1 & -1 & 2 \end{pmatrix}$. The simple reflections of the Weyl groupoid are subject to the Coxeter relations $(s_i s_j)^{m_{ij}^x} = \mathrm{id}_x$, where m_{ij}^x is determined from the Serre matrix $A(x)$ corresponding to the source x by the usual rules. In particular, in this example the relations depend on the vertices. The general case $\mathfrak{osp}(2m|2n)$ is similar to this, but in addition to the Serre matrices of the above types there will also be Serre matrices of type D_{m+n} appearing.

1.5 Structure of the thesis

The thesis consists of four parts, corresponding to the four main questions. These can be read mostly independently of each other, except that Chapter 3 relies on some definitions from Chapter 2.

In Chapter 2 we consider the relation of weight structures and $\mathbf{t}$-structures by orthogonality. We recall the definitions and some important examples of weight structures, $\mathbf{t}$-structures, simple-minded collections and silting collections, as well as the definition of orthogonality and adjacency, in Section 2.2. In Sections 2.3.1 and 2.3.2 we discuss derived projective objects and derived projective covers in general, and in Section 2.3.3 we show how these notions are related to silting $\mathbf{t}$-structures. These results are applied to the WT correspondence in Section 2.4, and in Section 2.5 we show that the WT correspondence is natural. This chapter is an expanded version of §2, §3 and §5 of [Bon25].

The Koszul duality of simple-minded collections and silting collections is studied in Chapter 3. The definition of dg Koszul duality and its relation to the classical Koszul duality can be found in Section 3.3. In Section 3.4 we prove the Koszul duality theorem for simple-minded collections and silting collections, and in Section 3.5 we provide three small examples of the Koszul duality. The chapter is an expanded version of §4 of [Bon25].

In Chapter 4 we describe the Serre functor of the constructible derived category $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n)$ and classify the $\mathbb{P}$-like objects in $\mathbf{Perv}(\mathbb{P}^n)$. In Section 4.2 we provide the necessary technical background about $\mathbb{P}$-twists and the definition of Serre functors. We also recall the definition of $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n)$ and $\mathbf{Perv}(\mathbb{P}^n)$ and describe the simple, standard and costandard objects in $\mathbf{Perv}(\mathbb{P}^n)$, and also the construction of the indecomposable projective and injective objects. We summarize the relation to parabolic category $\mathcal{O}$ and the description in terms of finite-dimensional algebras in Section 4.2.5. The description of the Serre functor is obtained in Section 4.3, and we compare the different descriptions of the Serre functor in Section 4.3.6. In Section 4.4 we construct the string objects, and compute the morphisms between them to show that they are $\mathbb{P}$-like. This chapter is joint work with Alessio Cipriani [BC25].

Chapter 5 is about Weyl groupoids of Lie superalgebras. Section 5.2 contains the definition of Cartan graphs and Weyl groupoids, both in general and for contragredient Lie superalgebras. In Section 5.2.5 we compare the various notions of Weyl groupoids. The Weyl groupoids of $\mathfrak{sl}(m|n)$, $\mathfrak{osp}(2m+1|2n)$ and $\mathfrak{osp}(2m|2n)$ are described in Section 5.4. For this we recall the required descriptions of the ordered root bases and Borel subalgebras in Section 5.3, and compute the required Cartan data in Section 5.A. This chapter is joint work with Jonas Nehme [BN24].

[BC25] L. Bonfert and A. Cipriani. *Serre functor and $\mathbb{P}$-objects for perverse sheaves on $\mathbb{P}^n$*. Preprint. 2025. arXiv:`2506.06051v1 [math.RT]`.

[BN24] L. Bonfert and J. Nehme. "The Weyl groupoids of $\mathfrak{sl}(m|n)$ and $\mathfrak{osp}(r|2n)$". *J. Algebra* 641 (2024).

[Bon25] L. Bonfert. "Derived projective covers and Koszul duality of simple-minded and silting collections". To appear in *Appl. Categ. Struct.* (2025). arXiv:`2309.00554v3 [math.RT]`.

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Chapter 2

Characterization of silting t-structures via derived projectives

In this chapter we use derived projective objects to provide a characterization of silting **t**-structures. We begin by recalling the notions of **t**-structures, weight structures, simple-minded collections and silting collections, and how these are related to each other. In particular, we consider variants of the adjacency and orthogonality of weight structures and **t**-structures introduced in [Bon10a].

To characterize silting **t**-structures, we study derived projective objects with respect to a **t**-structure and the notion of enough derived projectives from [GLVdB21]. We show in Theorem 2.3.7 that this definition agrees with the notion of enough Ext-projectives from [CSPP22]. By [CSPP22, Thm. 2.4] enough derived projectives provide a criterion for the existence of an adjacent weight structure for a given **t**-structure, and we slightly refine this result in Corollary 2.3.9.

We then introduce derived projective covers and provide several equivalent definitions for special cases of the definition, in analogy to the equivalent definitions of projective covers in abelian categories. The main result of this chapter (Theorem 2.3.16) characterizes silting **t**-structures with finite-length heart as those non-degenerate **t**-structures with respect to which the triangulated category has enough derived projectives. A further equivalent criterion is that every simple object of the heart admits a derived projective cover.

In Section 2.4 we apply Theorem 2.3.16 to the WT correspondence from [KY14; Fus24]. We provide an equivalent definition for WT pairs (introduced as ST pairs in [AMY19]), and show that the bijection between weight structures and **t**-structures provided by the WT correspondence is given by **w**-**t**-strict orthogonality. From this it follows that the silting collection corresponding to a simple-minded collection under the WT correspondence consists of its derived projective covers. In Theorem 2.5.1 we show that orthogonality of weight structures and **t**-structures, and in particular the WT correspondence, is natural with respect to weight exact functors and **t**-exact functors.

The chapter is based on [Bon25, §1–3 and §5].

[Bon25] L. Bonfert. "Derived projective covers and Koszul duality of simple-minded and silting collections". To appear in *Appl. Categ. Struct.* (2025). arXiv:`2309.00554v3 [math.RT]`.

2.1 Motivation and overview of results

In any triangulated category $\mathscr{C}$ there is a bijection between simple-minded collections in $\mathscr{C}$ and bounded **t**-structures with finite-length heart on $\mathscr{C}$, sending a **t**-structure t to the set of simple objects in its heart $\heartsuit_t$. Similarly, there is a bijection between (classical) silting collections in $\mathscr{C}$ and bounded weight structures (also known as co-**t**-structures) with Krull–Schmidt coheart, sending a weight structure w to the set of indecomposable objects in its coheart $\diamondsuit_w$.

Weight structures and **t**-structures on (not necessarily the same) triangulated categories are closely related by the notion of *orthogonality* introduced in [Bon10a]. For instance, for a finite-dimensional algebra A [KY14] establishes a bijection between bounded weight structures on $\mathbf{K}^{\mathrm{b}}(\mathbf{proj}_{\mathrm{fg}}\text{-}A)$ and bounded **t**-structures on $\mathbf{D}^{\mathrm{b}}(\mathbf{mod}_{\mathrm{fd}}\text{-}A)$ with finite-length heart. This *WT correspondence* can be formulated in terms of **w**-**t**-strict left orthogonality, where we say that a weight structure w is **w**-**t**-strictly left orthogonal to a **t**-structure t if $(\mathscr{C}_{w\leq 0})^{\perp} = \mathscr{D}^{t>0}$ and ${}^{\perp}(\mathscr{D}^{t>0}) = \mathscr{C}_{w\leq 0}$, and similarly for $\mathscr{C}_{w\geq 0}$ and $\mathscr{D}^{t<0}$. This is slightly stronger than the strict orthogonality considered in [Bon19], and there are several other variants. However, in many cases at least some of them coincide, see Section 2.2.5.

In terms of the corresponding silting collection $\mathcal{P}$ and simple-minded collection $\mathcal{L}$, the bijection from [KY14] is characterized by the existence of a bijection $\phi\colon \mathcal{P} \to \mathcal{L}$ such that

$$\operatorname{Hom}(P, L[m]) \cong \begin{cases} \operatorname{End}(L) & \text{if } L = \phi(P),\ m = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

This bijection sends $P \in \mathcal{P}$ to the simple top of $t_{\geq 0}P \in \heartsuit_t$. Recently, [Fus24] proved such bijections in the setup of *WT pairs*, which were introduced as ST pairs in [AMY19]. This provides a common generalization of the results from [KY14] as well as the analogous results for non-positive dg algebras with finite-dimensional total cohomology from [BY14] and homologically smooth non-positive dg algebras from [KN11]. For positive dg algebras, the results in [KN13] provide further examples of WT pairs. Here we call a dg algebra A *(cohomologically) non-positive* if $H^n(A) = 0$ for $n > 0$, and *(cohomologically) positive* if $H^n(A) = 0$ for $n < 0$ and $H^0(A)$ is semisimple.

The main result of this chapter relates silting collections to derived projective objects (also known as Ext-projective objects) which are an analog of projective objects in the triangulated setting. Analogously to the setting of abelian categories, [GLVdB21] introduced the term *enough derived projectives*. In Theorem 2.3.7 we show that this definition agrees with the notion of *enough Ext-projectives* from [CSPP22]. Enough derived projectives are used in [CSPP22, Thm. 2.4] to provide a criterion for the existence of a left adjacent weight structure, see Corollary 2.3.9 for a slightly refined version.

For a **t**-structure $t_{\mathcal{P}} = (\mathcal{P}^{\perp_{>0}}, \mathcal{P}^{\perp_{<0}})$ obtained from a silting collection $\mathcal{P}$ in the sense of [PV18], an easy but important observation shows that $\mathcal{P}$ consist of derived projective objects with respect to $t_{\mathcal{P}}$. As is evident from (2.1), the relation of silting collections to simple-minded collections is somewhat similar to the relation of indecomposable projective objects to simple objects in finite-length abelian categories. To formalize this, in Definition 2.3.11 we introduce *derived projective covers*, and we show the following result:

Theorem 2.1.1 (Theorem 2.3.16)**.** *Let t be a non-degenerate **t**-structure on $\mathscr{D}$ with finite-length heart. Let $\mathcal{L}$ be a full set of isomorphism representatives of the simple objects in $\heartsuit_t$ and $\mathcal{P}$ a full set of isomorphism representatives of the indecomposable derived projectives. Then the following are equivalent:*

I) t is silting (and $\mathcal{P}$ is the silting collection).

II) There is a bijection $\phi\colon \mathcal{P} \to \mathcal{L}$ satisfying (2.1)*.*

III) Every $L \in \mathcal{L}$ admits a derived projective cover (and $\mathcal{P}$ is the set of these derived projective covers).

IV) $\mathscr{D}$ has enough derived projectives with respect to t.

This result is somewhat analogous to [CSPP22, Thm. 2.4]. As an application, in Theorem 2.3.18 we show that for a **t**-structure obtained from a simple-minded collection $\mathcal{L}$ and a weight structure obtained from a silting collection $\mathcal{P}$, the relations (2.1) characterize orthogonality, and moreover that these relations are equivalent to $\mathcal{P}$ consisting of the derived projective covers of $\mathcal{L}$. This is not very surprising, as it is similar to the results contained in [KY14].

Theorem 2.1.1 also allows us to study the bijections between weight structures and **t**-structures in more detail. For this, in Definition 2.4.1 we define WT pairs, and in Proposition 2.4.3 we show that this definition is equivalent to the axioms for the ST pairs from [AMY19]. Compared to the definition in [AMY19], our definition uses weight structures instead of silting collections, which makes the definition more symmetric. In Theorem 2.4.4 we show that at the level of weight structures and **t**-structures the bijection from [Fus24] is given by **w**-**t**-strict orthogonality. From this it follows by Theorem 2.1.1 that the silting collection corresponding to a simple-minded collection under the WT correspondence consists of its derived projective covers.

A related question is whether **w**-**t**-strict orthogonality between weight structures and **t**-structures is natural with respect to weight exact functors and **t**-exact functors that are (in a certain sense) adjoint to each other. The setup of the main result Theorem 2.5.1 is somewhat technical, although the proof is straightforward and essentially the same as [Bon19, Prop. 4.4.5]. In particular, it follows from this that the bijection between weight structures and **t**-structures provided by the WT correspondence is natural:

Theorem 2.1.2 (Corollary 2.5.2)**.** *Let $(\mathscr{C}, \mathscr{D})$ be a WT pair in $\mathscr{T}$ and $(\mathscr{C}', \mathscr{D}')$ a WT pair in $\mathscr{T}'$. Let w, w' be bounded weight structures on $\mathscr{C}$ resp. $\mathscr{C}'$, and t, t' the corresponding bounded **t**-structures on $\mathscr{D}$ resp. $\mathscr{D}'$ under the WT correspondence. Suppose that $F\colon \mathscr{C} \to \mathscr{C}'$ is left pseudo-adjoint to $G\colon \mathscr{D}' \to \mathscr{D}$ in the sense that $\operatorname{Hom}_{\mathscr{T}'}(F(-), -) \cong \operatorname{Hom}_{\mathscr{T}}(-, G(-))$.*

*Then F is weight exact with respect to w and w' if and only if G is **t**-exact with respect to t and t'.*

2.2 Definitions

We begin by recalling the definitions of **t**-structures, weight structures, simple-minded collections and silting collections. For silting collections, we also compare two slightly different definitions. Finally we recall the notion of orthogonality between weight structures and **t**-structures, and compare the various strictness levels in special cases.

Unless explicitly mentioned, all categories will be linear over some (fixed) field $\Bbbk$. (Dg) modules over a (dg) algebra will be right modules, unless stated otherwise. For subcategories $\mathscr{A}, \mathscr{B} \subseteq \mathscr{C}$ we write $\mathscr{A} \perp \mathscr{B}$ if $\operatorname{Hom}_{\mathscr{C}}(A, B) = 0$ for all $A \in \mathscr{A}$, $B \in \mathscr{B}$. Moreover we write $\mathscr{A}^{\perp} = \{C \in \mathscr{C} \mid \operatorname{Hom}_{\mathscr{C}}(A, C) = 0\ \forall A \in \mathscr{A}\}$ and ${}^{\perp}\mathscr{A} = \{C \in \mathscr{C} \mid \operatorname{Hom}_{\mathscr{C}}(C, A) = 0\ \forall A \in \mathscr{A}\}$. For a triangulated category $\mathscr{D}$ and $\mathcal{X} \subseteq \mathscr{D}$, the full subcategory whose objects are the direct summands of finite coproducts of objects in $\mathcal{X}$ is denoted by $\operatorname{Kar}_{\mathscr{D}}(\mathcal{X})$, and the closure of $\mathcal{X}$ under extensions is denoted by $\operatorname{extclos}_{\mathscr{D}}(\mathcal{X})$.

2.2.1 t-structures

The notion of **t**-structures on triangulated categories was introduced in [BBD82].

Definition 2.2.1. A ***t**-structure* on a triangulated category $\mathscr{D}$ is a pair $t = (\mathscr{D}^{t\leq 0}, \mathscr{D}^{t\geq 0})$ of strict full subcategories such that

- $\mathscr{D}^{t\leq 0}[1] \subseteq \mathscr{D}^{t\leq 0}$ and $\mathscr{D}^{t\geq 0}[-1] \subseteq \mathscr{D}^{t\geq 0}$,
- $\mathscr{D}^{t\leq 0} \perp \mathscr{D}^{t\geq 0}[-1]$,

- for all $X \in \mathscr{D}$ there is a triangle (called ***t**-decomposition* of X)

$$t_{\leq 0}X \to X \to t_{>0}X \to t_{\leq 0}X[1]$$

  with $t_{\leq 0}X \in \mathscr{D}^{t\leq 0}$ and $t_{>0}X \in \mathscr{D}^{t\geq 0}[-1]$.

The full subcategory $\heartsuit_t = \mathscr{D}^{t\leq 0} \cap \mathscr{D}^{t\geq 0}$ is called the *heart* of t.

We also write $\mathscr{D}^{t>0} = \mathscr{D}^{t\geq 0}[-1]$ and $\mathscr{D}^{t<0} = \mathscr{D}^{t\leq 0}[1]$, and also $\mathscr{D}^{t\geq n} = \mathscr{D}^{t\geq 0}[-n]$ for $n \in \mathbb{Z}$ (and analogously $\mathscr{D}^{t\leq n}$).

A **t**-structure t is called *non-degenerate* if $\bigcap_{n\in\mathbb{Z}} \mathscr{D}^{t\leq n} = \{0\} = \bigcap_{n\in\mathbb{Z}} \mathscr{D}^{t\geq n}$. It is *bounded above* if $\mathscr{D} = \bigcup_{n\in\mathbb{Z}} \mathscr{D}^{t\leq n}$, *bounded below* if $\mathscr{D} = \bigcup_{n\in\mathbb{Z}} \mathscr{D}^{t\geq n}$, and *bounded* if $\mathscr{D} = \operatorname{tria}_{\mathscr{D}}(\heartsuit_t)$. Here $\operatorname{tria}_{\mathscr{D}}(\heartsuit_t)$ denotes the triangulated subcategory of $\mathscr{D}$ generated by $\heartsuit_t$. Note that t is bounded if and only if it is bounded above and bounded below.

Recall that **t**-decompositions are unique up to isomorphism, and furthermore $t_{\geq 0}\colon \mathscr{D} \to \mathscr{D}^{t\geq 0}$ and $t_{\leq 0}\colon \mathscr{D} \to \mathscr{D}^{t\leq 0}$ define functors that are left (resp. right) adjoint to the respective inclusions [BBD82, Prop. 1.3.3]. Also recall that $\mathscr{D}^{t\leq 0} = {}^{\perp}(\mathscr{D}^{t>0})$ and $\mathscr{D}^{t\geq 0} = (\mathscr{D}^{t<0})^{\perp}$.

Example 2.2.2. The following are some examples of **t**-structures:

1) Let $\mathscr{A}$ be an abelian category. The *standard **t**-structure* on the derived category $\mathbf{D}(\mathscr{A})$ is given by

$$\mathscr{D}^{t\leq 0} = \{X \in \mathbf{D}(\mathscr{A}) \mid H^n(X) = 0\ \forall n > 0\}, \quad \mathscr{D}^{t\geq 0} = \{X \in \mathbf{D}(\mathscr{A}) \mid H^n(X) = 0\ \forall n < 0\}.$$

   It restricts to a bounded (resp. bounded above, resp. bounded below) **t**-structure on $\mathbf{D}^{\mathrm{b}}(\mathscr{A})$ (resp. $\mathbf{D}^{-}(\mathscr{A})$, resp. $\mathbf{D}^{+}(\mathscr{A})$). Its heart is equivalent to $\mathscr{A}$.

2) Let A be a non-positive dg algebra, i.e. a dg algebra such that $H^n(A) = 0$ for $n > 0$. By [HKM02, Thm. 1.3], there is a *standard **t**-structure* on the derived category $\mathbf{D}(A)$ defined by

$$\mathscr{D}^{t\leq 0} = \{X \in \mathbf{D}(A) \mid H^n(X) = 0\ \forall n > 0\}, \quad \mathscr{D}^{t\geq 0} = \{X \in \mathbf{D}(A) \mid H^n(X) = 0\ \forall n < 0\},$$

   and its heart is equivalent to $\mathbf{Mod}\text{-}H^0(A)$. It restricts to a bounded **t**-structure on $\mathbf{D}_{\mathrm{fd}}(A)$, with heart equivalent to $\mathbf{mod}_{\mathrm{fd}}\text{-}H^0(A)$.

3) Let A be a locally finite-dimensional positive dg algebra, i.e. a dg algebra such that $H^n(A) = 0$ for $n < 0$, $H^0(A)$ is a semisimple algebra, and each $H^n(A)$ is finite-dimensional. By [KN13, Thm. 7.1] there is a **t**-structure on $\mathbf{perf}(A)$, which is defined by

$$\mathscr{D}^{t\leq 0} = \operatorname{Kar}_{\mathbf{D}(A)} \operatorname{extclos}_{\mathbf{D}(A)}\{A[n] \mid n \geq 0\}, \quad \mathscr{D}^{t\geq 0} = \operatorname{Kar}_{\mathbf{D}(A)} \operatorname{extclos}_{\mathbf{D}(A)}\{A[n] \mid n \leq 0\}.$$

4) The *Postnikov **t**-structure* (also called *standard **t**-structure*) on the stable homotopy category $\mathbf{SH}$ is given by

$$\mathscr{D}^{t\leq 0} = \{X \in \mathbf{SH} \mid \pi_i(X) = 0\ \forall i < 0\}, \qquad \mathscr{D}^{t\geq 0} = \{X \in \mathbf{SH} \mid \pi_i(X) = 0\ \forall i > 0\}.$$

   Its heart is equivalent to $\mathbf{Mod}\text{-}\mathbb{Z}$.

Further examples include the *perverse **t**-structures* from [BBD82] and the *Koszul **t**-structure* from [BGS96; MOS09].

2.2.2 Weight structures

Weight structures (also known as co-**t**-structures) were originally defined in [Bon10b] and [Pau08].

Definition 2.2.3. A *weight structure* on a triangulated category $\mathscr{C}$ is a pair $w = (\mathscr{C}_{w\leq 0}, \mathscr{C}_{w\geq 0})$ of Karoubi-closed full subcategories such that

- $\mathscr{C}_{w\leq 0}[1] \subseteq \mathscr{C}_{w\leq 0}$ and $\mathscr{C}_{w\geq 0}[-1] \subseteq \mathscr{C}_{w\geq 0}$,
- $\mathscr{C}_{w\geq 0}[-1] \perp \mathscr{C}_{w\leq 0}$,
- for all $X \in \mathscr{C}$ there is a triangle (called *weight decomposition* of X)
$$w_{>0}X \to X \to w_{\leq 0}X \to w_{>0}X[1]$$
with $w_{>0}X \in \mathscr{C}_{w\geq 0}[-1]$ and $w_{\leq 0}X \in \mathscr{C}_{w\leq 0}$.

The full subcategory $\heartsuit_w = \mathscr{C}_{w\leq 0} \cap \mathscr{C}_{w\geq 0}$ is called the *coheart* of w.

As for **t**-structures we write $\mathscr{C}_{w>0} = \mathscr{C}_{w\geq 0}[-1]$, $\mathscr{C}_{w\geq n} = \mathscr{C}_{w\geq 0}[-n]$, and so on.

A weight structure w is called *non-degenerate* if $\bigcap_{n\in\mathbb{Z}} \mathscr{C}_{w\leq n} = \{0\} = \bigcap_{n\in\mathbb{Z}} \mathscr{C}_{w\geq n}$. It is *bounded above* if $\mathscr{C} = \bigcup_{n\in\mathbb{Z}} \mathscr{C}_{w\leq n}$, *bounded below* if $\mathscr{C} = \bigcup_{n\in\mathbb{Z}} \mathscr{C}_{w\geq n}$, and *bounded* if $\mathscr{C} = \operatorname{thick}_{\mathscr{C}}(\heartsuit_w)$. Here $\operatorname{thick}_{\mathscr{C}}(\heartsuit_w)$ denotes the thick subcategory of $\mathscr{C}$ generated by $\heartsuit_w$. Note that w is bounded if and only if it is bounded above and bounded below.

Analogously to the situation for **t**-structures we have $\mathscr{C}_{w\leq 0} = (\mathscr{C}_{w>0})^{\perp}$ and $\mathscr{C}_{w\geq 0} = {}^{\perp}(\mathscr{C}_{w<0})$, see [Bon10b, Prop. 1.3.3]. In contrast to **t**-decompositions, by [Bon10b, Rem. 1.2.2] weight decompositions are usually not unique. In particular $w_{\leq 0}$ and $w_{>0}$ do not define functors.

Example 2.2.4. The following are some examples of weight structures:

1) Let $\mathscr{A}$ be an additive category. The *standard weight structure* on the homotopy category $\mathbf{K}(\mathscr{A})$ is given by
$$\mathscr{C}_{w\geq 0} = \{X \in \mathbf{K}(\mathscr{A}) \mid \exists Y \cong X : Y^n = 0\ \forall n < 0\},$$
$$\mathscr{C}_{w\leq 0} = \{X \in \mathbf{K}(\mathscr{A}) \mid \exists Y \cong X : Y^n = 0\ \forall n > 0\}.$$
Its coheart is $\operatorname{Kar}_{\mathbf{K}(\mathscr{A})}(\mathscr{A})$, where $\mathscr{A}$ is embedded into $\mathbf{K}(\mathscr{A})$ as the complexes concentrated in degree 0. In particular, if $\mathscr{A}$ is idempotent-complete, then $\heartsuit_w$ consists of the complexes concentrated in degree 0, and thus is equivalent to $\mathscr{A}$.

2) Let A be a non-positive dg algebra. By [BY14, Thm. A.1] (see also [Bon10b, Prop. 6.2.1]), there is a *standard weight structure* on $\mathbf{perf}(A)$ is given by
$$\mathscr{C}_{w\geq 0} = \operatorname{Kar}_{\mathbf{D}(A)} \operatorname{extclos}_{\mathbf{D}(A)}\{A[n] \mid n \leq 0\}, \quad \mathscr{C}_{w\leq 0} = \operatorname{Kar}_{\mathbf{D}(A)} \operatorname{extclos}_{\mathbf{D}(A)}\{A[n] \mid n \geq 0\}.$$

3) Let A be a positive dg algebra. By [KN13, Cor. 4.1] there is a weight structure on $\mathbf{D}(A)$ defined by
$$\mathscr{C}_{w\geq 0} = \{X \in \mathbf{D}(\mathscr{A}) \mid H^n(X) = 0\ \forall n < 0\}, \quad \mathscr{C}_{w\leq 0} = \{X \in \mathbf{D}(\mathscr{A}) \mid H^n(X) = 0\ \forall n > 0\}.$$

4) The *spherical weight structure* from [Bon10b, §4.6] on the subcategory $\mathbf{SH}_{\text{fin}} \subseteq \mathbf{SH}$ of finite spectra is given by
$$\mathscr{C}_{w\geq 0} = \operatorname{Kar}_{\mathbf{SH}} \operatorname{extclos}_{\mathbf{SH}}\{\mathbb{S}[n] \mid n \leq 0\}, \qquad \mathscr{C}_{w\leq 0} = \operatorname{Kar}_{\mathbf{SH}} \operatorname{extclos}_{\mathbf{SH}}\{\mathbb{S}[n] \mid n \geq 0\},$$
where $\mathbb{S}$ denotes the sphere spectrum.

Further examples include the weight structures obtained from Koszul duality and Ringel duality, see [ES22, §2.4–2.5], the weight structure on the derived category of mixed Hodge modules from [Bon10c, Prop. 2.3.9], and the *Chow weight structure* on Voevodsky's category of effective geometric motives from [Bon10b, §6–7].

2.2.3 Simple-minded collections

The following definition is from [AN09] and [KY14], and the axioms already appeared in [Ric02].

Definition 2.2.5. A *simple-minded collection* in a triangulated category $\mathscr{D}$ is a (not necessarily finite) set $\mathcal{L}$ of objects of $\mathscr{D}$ such that

- $\mathrm{Hom}_{\mathscr{D}}(L, L'[m]) = 0$ for all $L, L' \in \mathcal{L}$, $m < 0$,
- $\mathrm{Hom}_{\mathscr{D}}(L, L') = 0$ for $L, L' \in \mathcal{L}$, $L \neq L'$,
- $\mathrm{End}_{\mathscr{D}}(L)$ is a division algebra for all $L \in \mathcal{L}$,
- $\mathrm{tria}_{\mathscr{D}}(\mathcal{L}) = \mathscr{D}$.

A simple-minded collection is *finite* if it consists of finitely many objects.

Remark 2.2.6. Note that in contrast to most of the existing literature we do not assume simple-minded collections to be finite, see also [Sch20] where infinite simple-minded collections are also studied. However, if a triangulated category $\mathscr{D}$ admits a finite simple-minded collection, then automatically any simple-minded collection in $\mathscr{D}$ is finite, since it follows from Proposition 2.2.7 that simple-minded collections form bases of the Grothendieck group of $\mathscr{D}$.

An abelian category is *finite-length* (or a *length category*) if all of its objects have finite length. The definition of simple-minded collections is based on properties of the simple objects in the heart of a bounded **t**-structure with finite-length heart, and in fact specifying a simple-minded collection is equivalent to specifying such a **t**-structure. This is already mentioned in [BBD82, Rem. 1.3.14], and explicitly spelled out in [AN09].

Proposition 2.2.7.

1) *Let $\mathcal{L}$ be a simple-minded collection in $\mathscr{D}$. Then $t = (\mathscr{D}^{t\leq 0}, \mathscr{D}^{t\geq 0})$, with*
$$\mathscr{D}^{t\leq 0} = \mathrm{extclos}_{\mathscr{D}}\{L[m] \mid L \in \mathcal{L}, m \geq 0\}, \qquad \mathscr{D}^{t\geq 0} = \mathrm{extclos}_{\mathscr{D}}\{L[m] \mid L \in \mathcal{L}, m \leq 0\},$$
*is a bounded **t**-structure with finite-length heart, and $\mathcal{L}$ is a full set of isomorphism representatives of the simple objects in $\heartsuit_t$.*

2) *Let t be a bounded **t**-structure on $\mathscr{D}$ such that $\heartsuit_t$ is finite-length, and let $\mathcal{L}$ be a full set of isomorphism representatives of the simple objects in $\heartsuit_t$. Then $\mathcal{L}$ is a simple-minded collection in $\mathscr{D}$.*

Proof. See [AN09, Prop. 2 and Prop. 4]. Although the propositions there are formulated only for the bounded derived category of a self-injective algebra, the proofs work in a general triangulated category without modifications. □

The following examples are standard examples of simple-minded collections.

Example 2.2.8.

1) Let $\mathscr{A}$ be a finite-length abelian category and $\mathcal{L}$ a full set of isomorphism representatives of the simple objects in $\mathscr{A}$. Then $\mathcal{L}$ is a simple-minded collection in $\mathbf{D}^{\mathrm{b}}(\mathscr{A})$, corresponding to the standard **t**-structure.
2) Let A be a non-positive dg algebra. A finite-dimensional simple $H^0(A)$-module can be viewed as a dg A-module via the quasi-isomorphism $t_{\leq 0}A \to A$ (where $t_{\leq 0}$ denotes the **t**-truncation with respect to the standard **t**-structure) and the quotient map $t_{\leq 0}A \twoheadrightarrow H^0(A)$. Then a full set $\mathcal{L}$ of isomorphism representatives of the simple $H^0(A)$-modules is a simple-minded collection in $\mathbf{D}_{\mathrm{fd}}(A)$.
3) Let A be a locally finite-dimensional positive dg algebra. Then the indecomposable summands of A form a simple-minded collection in $\mathbf{perf}(A)$.

2.2.4 Silting collections

For a collection of objects $\mathcal{X}$ in a triangulated category $\mathscr{D}$ we write $\mathcal{X}^{\perp_{>0}} = \{D \in \mathscr{D} \mid \operatorname{Hom}_{\mathscr{D}}(X, D[m]) = 0 \ \forall m > 0, X \in \mathcal{X}\}$, and analogously define $\mathcal{X}^{\perp_{<0}}$, $\mathcal{X}^{\perp_{\geq 0}}$, etc.

The following definition is based on [PV18, Def. 4.1]. There are other definitions of silting, see [PV18, Ex. 4.2] for an overview and comparison of different definitions.

Definition 2.2.9. A *silting collection* in a triangulated category $\mathscr{D}$ is a (not necessarily finite) set $\mathcal{P}$ of objects of $\mathscr{D}$ such that

- $\operatorname{Kar}_{\mathscr{D}}(\mathcal{P})$ is Krull–Schmidt,
- objects in $\mathcal{P}$ are indecomposable and pairwise non-isomorphic,
- $t_{\mathcal{P}} = (\mathcal{P}^{\perp_{>0}}, \mathcal{P}^{\perp_{<0}})$ is a $\mathbf{t}$-structure on $\mathscr{D}$, called the *silting* $\boldsymbol{t}$*-structure associated with* $\mathcal{P}$.

We say that $\mathcal{P}$ is *finite* if it consists of finitely many objects. A silting collection consisting of compact objects (in a triangulated category with small coproducts) is called *compact*.

Remark 2.2.10. In [PV18, Def. 4.1] it is moreover required that $\operatorname{Hom}_{\mathscr{D}}(P, P'[m]) = 0$ for all $P, P' \in \mathcal{P}$ and $m > 0$. However, as mentioned in [AHLSV22, Prop. 2.5], this assumption is automatic: for $P \in \mathcal{P}$, take the $\mathbf{t}$-decomposition $t_{\leq 0}P \to P \to t_{>0}P \to t_{\leq 0}P[1]$. Then $t_{>0}P \in \mathcal{P}^{\perp_{\leq 0}}$, so $t_{\leq 0}P[1] \cong P[1] \oplus t_{>0}P$ and thus $P \cong t_{\leq 0}P \in \mathcal{P}^{\perp_{>0}}$.

In the literature usually silting objects (rather than silting collections) are used, see e.g. [KY14; PV18]. However these provide exactly the same data, at least in the finite case: given a silting object P, (isomorphism representatives of) its indecomposable summands form a silting collection. Conversely, if $\mathcal{P}$ is a silting collection, then $\coprod_{P \in \mathcal{P}} P$ is a silting object (assuming the coproduct exists). We prefer to use silting collections rather than silting objects since we are mostly interested in the indecomposable summands, but will nevertheless occasionally use the word silting object if it is more convenient.

Remark 2.2.11. It is important to specify the ambient triangulated category $\mathscr{D}$ for a silting collection $\mathcal{P}$. Note that if $\mathcal{P}$ is a silting collection in $\mathscr{D}$, then not necessarily $\mathscr{D} = \operatorname{thick}_{\mathscr{D}}(\mathcal{P})$. In particular, $\mathcal{P}$ is in general not a silting collection in $\operatorname{thick}_{\mathscr{D}}(\mathcal{P})$, since the associated silting $\mathbf{t}$-structure need not restrict to a $\mathbf{t}$-structure on $\operatorname{thick}_{\mathscr{D}}(\mathcal{P})$.

Example 2.2.12. The following examples are standard examples of silting objects. In each case, silting collections can be obtained by taking their indecomposable direct summands.

1) For an algebra A and $X \in \mathbf{D}(\mathbf{Mod}\text{-}A)$ we have $H^n(X) \cong \operatorname{Hom}_{\mathbf{D}(\mathbf{Mod}\text{-}A)}(A, X[n])$, and thus A is a silting object in $\mathbf{D}(\mathbf{Mod}\text{-}A)$, defining the standard $\mathbf{t}$-structure. If A is finite-dimensional, it is also a silting object in $\mathbf{D}^{\mathrm{b}}(\mathbf{mod}_{\mathrm{fd}}\text{-}A)$ and $\mathbf{D}^{-}(\mathbf{mod}_{\mathrm{fd}}\text{-}A)$.
2) A non-positive dg algebra A is a silting object in $\mathbf{D}(A)$ by [HKM02, Thm. 1.3] (see also [BY14, Thm. A.1]), and its associated silting $\mathbf{t}$-structure is the standard $\mathbf{t}$-structure. If $H^*(A)$ is finite-dimensional, then A is also a silting object in $\mathbf{D}_{\mathrm{fd}}(A)$.
3) The sphere spectrum $\mathbb{S}$ is a silting object in the stable homotopy category $\mathbf{SH}$, since $\pi_i(X) = \operatorname{Hom}_{\mathbf{SH}}(\mathbb{S}, X[-i])$. Its associated silting $\mathbf{t}$-structure is the Postnikov $\mathbf{t}$-structure.

Using the terminology from [Stacks, Tag 09SJ], we say that a set of objects $\mathcal{X}$ *weakly generates* $\mathscr{D}$ if $\operatorname{Hom}_{\mathscr{D}}(X, Y[n]) = 0$ for all $n \in \mathbb{Z}$ and $X \in \mathcal{X}$ implies $Y = 0$. The following result is also stated in [PV18, Prop. 4.3], however we were unable to verify their proof.

Lemma 2.2.13. *A silting collection $\mathcal{P}$ in $\mathscr{D}$ weakly generates $\mathscr{D}$. In particular, the associated silting $\boldsymbol{t}$-structure $t_{\mathcal{P}}$ is non-degenerate.*

Proof. Let $X \in \mathscr{D}$ such that $\mathrm{Hom}_{\mathscr{D}}(P, X[n]) = 0$ for all $P \in \mathcal{P}$ and $n \in \mathbb{Z}$. Then in particular $X \in \mathcal{P}^{\perp_{>0}}$ and $X \in \mathcal{P}^{\perp_{\leq 0}}$, and thus both $X \to X \to 0 \to X[1]$ and $0 \to X \to X \to 0$ are **t**-decompositions of X with respect to $t_{\mathcal{P}}$. But since **t**-decompositions are unique it follows that $X = 0$.

That $t_{\mathcal{P}}$ is non-degenerate is equivalent to $\mathcal{P}$ weakly generating $\mathscr{D}$ since $\mathscr{D}^{t_{\mathcal{P}} \leq n} = \mathcal{P}^{\perp_{>n}}$ for all $n \in \mathbb{Z}$, and analogously for the positive part. ☐

For us the main class of examples will be silting collections according to the following "classical" definition going back to [KV88] and [AI12, Def. 2.1].

Definition 2.2.14. A *classical silting collection* in a triangulated category $\mathscr{C}$ is a set $\mathcal{P}$ of pairwise non-isomorphic objects of $\mathscr{C}$ such that

- $\mathrm{Kar}_{\mathscr{C}}(\mathcal{P})$ is Krull–Schmidt,
- objects in $\mathcal{P}$ are indecomposable,
- $\mathrm{Hom}_{\mathscr{C}}(P, P'[m]) = 0$ for all $P, P' \in \mathcal{P}$, $m > 0$,
- $\mathscr{C} = \mathrm{thick}_{\mathscr{C}}(\mathcal{P})$.

The difference to Definition 2.2.9 is that silting collections by definition provide **t**-structures, while classical silting collections have to generate $\mathscr{C}$ as thick subcategory. By Remark 2.2.10 a silting collection $\mathcal{P}$ in $\mathscr{D}$ is a classical silting collection in $\mathrm{thick}_{\mathscr{D}}(\mathcal{P})$. In particular, the examples listed in Example 2.2.12 can also be seen as classical silting collections:

Example 2.2.15.

1) Let A be an algebra. Then a full set of isomorphism representatives of the indecomposable projective A-modules form a classical silting object in $\mathbf{K}^{\mathrm{b}}(\mathbf{proj}_{\mathrm{fg}}\text{-}A)$.
2) Let A be a non-positive dg algebra. The direct summands of A (up to isomorphism) as a dg A-module form a classical silting object in $\mathbf{perf}(A)$.
3) The sphere spectrum $\mathbb{S}$ forms a classical silting collection in the triangulated category $\mathbf{SH}_{\mathrm{fin}}$ of finite spectra.

The following lemma describes the relation between silting collections and classical silting collections in compactly generated triangulated categories. We write $\mathscr{D}^{\mathrm{c}}$ for the full subcategory of compact objects of a triangulated category $\mathscr{D}$ with small coproducts.

Lemma 2.2.16. *Let $\mathscr{D}$ be a compactly generated triangulated category. Then a set of objects $\mathcal{P}$ is a compact silting collection in $\mathscr{D}$ if and only if $\mathcal{P}$ is a classical silting collection in $\mathscr{D}^{\mathrm{c}}$.*

Proof. By [AI12, Cor. 4.7] a classical silting collection $\mathcal{P}$ in $\mathscr{D}^{\mathrm{c}}$ provides a **t**-structure $t_{\mathcal{P}} = (\mathcal{P}^{\perp_{>0}}, \mathcal{P}^{\perp_{<0}})$ on $\mathscr{D}$ and hence is a silting collection in $\mathscr{D}$.

Conversely, if $\mathcal{P}$ is a compact silting collection in $\mathscr{D}$, then $\mathcal{P}$ weakly generates $\mathscr{D}$ by Lemma 2.2.13, and it follows from general facts (see e.g. [Kra21, Prop. 3.4.15]) that $\mathrm{thick}_{\mathscr{D}}(\mathcal{P}) = \mathscr{D}^{\mathrm{c}}$. Thus $\mathcal{P}$ is a classical silting collection in $\mathscr{D}^{\mathrm{c}}$. ☐

One often considers classical silting collections in $\mathbf{K}^{\mathrm{b}}(\mathbf{proj}_{\mathrm{fg}}\text{-}A)$ for a finite-dimensional algebra A. For instance [KY14] describes the relation of classical silting collections in $\mathbf{K}^{\mathrm{b}}(\mathbf{proj}_{\mathrm{fg}}\text{-}A)$ to **t**-structures on $\mathbf{D}^{\mathrm{b}}(\mathbf{mod}_{\mathrm{fd}}\text{-}A)$. We would like to rephrase these results using silting collections instead of classical silting collections. However, we can't apply Lemma 2.2.16 directly, as $\mathbf{D}^{\mathrm{b}}(\mathbf{mod}_{\mathrm{fd}}\text{-}A)$ is not compactly generated since it does not have small coproducts.

Proposition 2.2.17. *Let A be a finite-dimensional algebra and $\mathcal{P}$ be a set of objects of $\mathscr{D} = \mathbf{D}(\mathbf{Mod}\text{-}A)$. Then the following are equivalent:*

I) $\mathcal{P}$ is a classical silting collection in $\mathbf{K}^{\mathrm{b}}(\mathbf{proj}_{\mathrm{fg}}\text{-}A)$.

II) $\mathcal{P}$ is a compact silting collection in $\mathbf{D}(\mathbf{Mod}\text{-}A)$.

III) $\mathcal{P}$ is a silting collection in $\mathbf{D}^{\mathrm{b}}(\mathbf{mod}_{\mathrm{fd}}\text{-}A)$ and $t_{\mathcal{P}}$ is a bounded $\mathbf{t}$-structure on $\mathbf{D}^{\mathrm{b}}(\mathbf{mod}_{\mathrm{fd}}\text{-}A)$.

IV) $\mathcal{P}$ is a silting collection in $\mathbf{D}^{\mathrm{b}}(\mathbf{mod}_{\mathrm{fd}}\text{-}A)$ and $\operatorname{thick}_{\mathscr{D}}(\mathcal{P}) = \mathbf{K}^{\mathrm{b}}(\mathbf{proj}_{\mathrm{fg}}\text{-}A)$.

Proof. I) $\iff$ II): It is well-known that $\mathscr{D} = \mathbf{D}(\mathbf{Mod}\text{-}A)$ is compactly generated, and $\mathscr{D}^{\mathrm{c}} = \mathbf{K}^{\mathrm{b}}(\mathbf{proj}_{\mathrm{fg}}\text{-}A)$. Thus by Lemma 2.2.16 classical silting collections in $\mathbf{K}^{\mathrm{b}}(\mathbf{proj}_{\mathrm{fg}}\text{-}A)$ are the same as compact silting collections in $\mathbf{D}(\mathbf{Mod}\text{-}A)$.

I) $\Longrightarrow$ III): It follows from (the proof of) [KY14, Lemma 5.3] that classical silting collections $\mathcal{P}$ in $\mathbf{K}^{\mathrm{b}}(\mathbf{proj}_{\mathrm{fg}}\text{-}A)$ are silting collections $\mathcal{P}$ in $\mathbf{D}^{\mathrm{b}}(\mathbf{mod}_{\mathrm{fd}}\text{-}A)$, and that $t_{\mathcal{P}}$ is bounded.

III) $\Longrightarrow$ IV): Let L be a simple A-module. If $t_{\mathcal{P}}$ is bounded, then for $P \in \mathcal{P}$ we have $\operatorname{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathbf{mod}_{\mathrm{fd}}\text{-}A)}(P, L[m]) = 0$ for $m \gg 0$ or $m \ll 0$, which implies $\mathcal{P} \subseteq \mathbf{K}^{\mathrm{b}}(\mathbf{proj}_{\mathrm{fg}}\text{-}A)$. It then follows from the proof of [AMY19, Cor. 6.9] that $\operatorname{thick}_{\mathscr{D}}(\mathcal{P}) = \mathbf{K}^{\mathrm{b}}(\mathbf{proj}_{\mathrm{fg}}\text{-}A)$.

IV) $\Longrightarrow$ I): This is immediate from Remark 2.2.10. $\square$

Remark 2.2.18. It seems very likely that every silting $\mathbf{t}$-structure on $\mathbf{D}^{\mathrm{b}}(\mathbf{mod}_{\mathrm{fd}}\text{-}A)$ is bounded, or that (equivalently) every silting collection of $\mathbf{D}^{\mathrm{b}}(\mathbf{mod}_{\mathrm{fd}}\text{-}A)$ lies in $\mathbf{K}^{\mathrm{b}}(\mathbf{proj}_{\mathrm{fg}}\text{-}A)$. If this is the case, then both III) and IV) in Proposition 2.2.17 reduce to $\mathcal{P}$ being a silting collection in $\mathbf{D}^{\mathrm{b}}(\mathbf{mod}_{\mathrm{fd}}\text{-}A)$.

The definition of classical silting collections is reminiscent of the properties of indecomposable objects in the coheart of a weight structure. Indeed, this is not a coincidence. Using silting collections instead of classical silting collections, we obtain:

Proposition 2.2.19. *Let $\mathscr{C} \subseteq \mathscr{D}$ be a thick subcategory of a triangulated category.*

1) Let $\mathcal{P}$ be a silting collection in $\mathscr{D}$ such that $\operatorname{thick}_{\mathscr{D}}(\mathcal{P}) = \mathscr{C}$. Then $w = (\mathscr{C}_{w\leq 0}, \mathscr{C}_{w\geq 0})$ with

$$\mathscr{C}_{w\leq 0} = \operatorname{Kar}_{\mathscr{C}} \operatorname{extclos}_{\mathscr{C}}\{P[m] \mid P \in \mathcal{P}, m \geq 0\},$$
$$\mathscr{C}_{w\geq 0} = \operatorname{Kar}_{\mathscr{C}} \operatorname{extclos}_{\mathscr{C}}\{P[m] \mid P \in \mathcal{P}, m \leq 0\}$$

is a bounded weight structure on $\mathscr{C}$, and $\mathcal{P}$ is a full set of isomorphism representatives of the indecomposable objects in $\heartsuit_w$.

2) Let w be a bounded weight structure on $\mathscr{C}$ such that $(\heartsuit_w^{\perp_{>0}}, \heartsuit_w^{\perp_{<0}})$ is a $\mathbf{t}$-structure on $\mathscr{D}$ and $\heartsuit_w$ is Krull–Schmidt. Then a full set $\mathcal{P}$ of isomorphism representatives of the indecomposable objects in $\heartsuit_w$ is a silting collection in $\mathscr{D}$.

Proof. The first part is [Bon10b, Thm. 4.3.2]. For the second part, note that since $\heartsuit_w$ is Krull–Schmidt, $(\mathcal{P}^{\perp_{>0}}, \mathcal{P}^{\perp_{<0}}) = (\heartsuit_w^{\perp_{>0}}, \heartsuit_w^{\perp_{>0}})$ is a $\mathbf{t}$-structure on $\mathscr{D}$, and the remaining axioms from Definition 2.2.9 are clear. $\square$

Remark 2.2.20. Under the bijection from Proposition 2.2.19, finite silting collections correspond to weight structures such that the coheart contains finitely many indecomposable objects (up to isomorphism). Moreover, Proposition 2.2.19 remains valid if one uses classical silting collections instead of silting collections and leaves out the assumption that the coheart defines a $\mathbf{t}$-structure. This version is commonly used, for instance it occurs in [KY14].

In the setup of Proposition 2.2.19 we would like to know when the coheart of a bounded weight structure on a thick subcategory $\mathscr{C} \subseteq \mathscr{D}$ weakly generates $\mathscr{D}$. The following criterion is proved analogously to Lemma 2.2.13.

Corollary 2.2.21. *Let $\mathscr{C} \subseteq \mathscr{D}$ be a thick subcategory of a triangulated category and w be a bounded weight structure on $\mathscr{C}$. If $(\rotatebox[origin=c]{180}{$\heartsuit$}_w^{\perp_{>0}}, \rotatebox[origin=c]{180}{$\heartsuit$}_w^{\perp_{<0}})$ defines a $\boldsymbol{t}$-structure on $\mathscr{D}$, then $\rotatebox[origin=c]{180}{$\heartsuit$}_w$ weakly generates $\mathscr{D}$.*

2.2.5 Adjacency and orthogonality

By definition, a silting collection $\mathcal{P}$ in a triangulated category $\mathscr{D}$ defines a $\mathbf{t}$-structure $t = (\mathcal{P}^{\perp_{>0}}, \mathcal{P}^{\perp_{<0}})$ on $\mathscr{D}$. On the other hand, by Proposition 2.2.19 $\mathcal{P}$ also defines a weight structure w on $\mathscr{C} = \mathrm{thick}_{\mathscr{D}}(\mathcal{P})$. From the definition of t it is clear that $\mathscr{D}^{t\leq 0} = (\mathscr{C}_{w>0})^{\perp}$ and $\mathscr{D}^{t\geq 0} = (\mathscr{C}_{w<0})^{\perp}$. If moreover $\mathscr{C} = \mathscr{D}$, then even $\mathscr{D}^{t\leq 0} = \mathscr{C}_{w\leq 0}$. These relations are described, and generalized by, the notions of orthogonality and adjacency between weight structures and **t**-structures.

Let $\mathscr{C}$ and $\mathscr{D}$ be triangulated categories and $\mathscr{A}$ an abelian category. Following [Bon19, Def. 5.2.1], by *duality* we mean a biadditive bifunctor $\Phi\colon \mathscr{C} \times \mathscr{D} \to \mathscr{A}$ which is contravariant and cohomological in the first argument, covariant and homological in the second argument, and comes with a natural isomorphism $\Phi(-,-) \cong \Phi(-[1], -[1])$.

Most of the time both $\mathscr{C}$ and $\mathscr{D}$ will be subcategories of a triangulated category $\mathscr{T}$ and $\Phi = \mathrm{Hom}_{\mathscr{T}}(-,-)\colon \mathscr{C} \times \mathscr{D} \to \mathbf{Mod}\text{-}\Bbbk$. For sets of objects $\mathcal{X} \subseteq \mathscr{C}$ and $\mathcal{Y} \subseteq \mathscr{D}$ we write $\mathcal{X} \perp_{\Phi} \mathcal{Y}$ if $\Phi(X, Y) = 0$ for all $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$, and we define

$$\mathcal{X}^{\perp_\Phi} = \{Y \in \mathscr{D} \mid \Phi(X, Y) = 0 \ \forall X \in \mathcal{X}\}, \qquad {}^{\perp_\Phi}\mathcal{Y} = \{X \in \mathscr{C} \mid \Phi(X, Y) = 0 \ \forall Y \in \mathcal{Y}\}.$$

The following definition is based on [Bon19, Def. 5.2.1].

Definition 2.2.22. Let w be a weight structure on $\mathscr{C}$ and t a **t**-structure on $\mathscr{D}$.

- w is *left orthogonal (with respect to Φ)* to t if $\mathscr{C}_{w\geq 0} \perp_{\Phi} \mathscr{D}^{t<0}$ and $\mathscr{C}_{w\leq 0} \perp_{\Phi} \mathscr{D}^{t>0}$.
- The orthogonality is *$\boldsymbol{w}$-strict* if $\mathscr{C}_{w\geq 0} = {}^{\perp_\Phi}(\mathscr{D}^{t<0})$ and $\mathscr{C}_{w\leq 0} = {}^{\perp_\Phi}(\mathscr{D}^{t>0})$.
- The orthogonality is *$\boldsymbol{t}$-strict* if $\mathscr{D}^{t<0} = (\mathscr{C}_{w\geq 0})^{\perp_\Phi}$ and $\mathscr{D}^{t>0} = (\mathscr{C}_{w\leq 0})^{\perp_\Phi}$.
- The orthogonality is *$\boldsymbol{w}$-$\boldsymbol{t}$-strict* if it is both **w**-strict and **t**-strict.

If both $\mathscr{C}$ and $\mathscr{D}$ are subcategories of a triangulated category $\mathscr{T}$, then any orthogonality will be with respect to $\Phi = \mathrm{Hom}_{\mathscr{T}}(-,-)$ unless explicitly mentioned. If moreover $\mathscr{C} = \mathscr{D}$, then left orthogonality is also called *left adjacency*.

In [Bon19] only orthogonality and **t**-strict orthogonality are considered, and there **t**-strict orthogonality is just called strict orthogonality.

Remark 2.2.23. Note that Proposition 2.2.19 establishes a bijection between silting collections and bounded weight structures that are **t**-strictly left orthogonal to a **t**-structure.

If $\mathscr{C} \subseteq \mathscr{D}$, then it is possible to characterize left orthogonality in terms of the negative and positive part, and moreover orthogonality and **w**-strict orthogonality coincide. The non-obvious implication I) $\Longrightarrow$ III) of the following statement is already shown in [Bon19, Prop. 5.2.3].

Lemma 2.2.24. *Let $\mathscr{C} \subseteq \mathscr{D}$ be a thick subcategory, w a weight structure on $\mathscr{C}$ and t a $\boldsymbol{t}$-structure on $\mathscr{D}$. Then the following are equivalent:*

I) w is left orthogonal to t.

*II) w is **w**-strictly left orthogonal to t.*

III) $\mathscr{C}_{w\le 0} = \mathscr{D}^{t\le 0} \cap \mathscr{C}$ and $\mathscr{C}_{w\ge 0} = {}^{\perp}(\mathscr{D}^{t<0}) \cap \mathscr{C}$.

Proof. II) $\implies$ I) is trivial.

I) $\implies$ III): From $\mathscr{C}_{w\le 0} \perp \mathscr{D}^{t>0}$ it is clear that $\mathscr{C}_{w\le 0} \subseteq {}^{\perp}(\mathscr{D}^{t>0}) \cap \mathscr{C} = \mathscr{D}^{t\le 0} \cap \mathscr{C}$. The converse inclusion follows from $\mathscr{C}_{w>0} \perp \mathscr{D}^{t\le 0}$ and $\mathscr{C}_{w\le 0} = (\mathscr{C}_{w>0})^{\perp}$.

For $\mathscr{C}_{w\ge 0}$, we have by assumption $\mathscr{C}_{w\ge 0} \subseteq {}^{\perp}(\mathscr{D}^{t<0}) \cap \mathscr{C}$, and from $\mathscr{C}_{w\le 0} = \mathscr{D}^{t\le 0} \cap \mathscr{C}$ we get ${}^{\perp}(\mathscr{D}^{t<0}) \cap \mathscr{C} \subseteq {}^{\perp}(\mathscr{D}^{t<0} \cap \mathscr{C}) = {}^{\perp}(\mathscr{C}_{w<0}) = \mathscr{C}_{w\ge 0}$.

III) $\implies$ II): This is obvious from $\mathscr{D}^{t\le 0} = {}^{\perp}(\mathscr{D}^{t>0})$ and the assumptions. □

Corollary 2.2.25. *Let $\mathcal{P}$ be a silting collection in $\mathscr{D}$, t its associated silting **t**-structure and w the induced weight structure on* $\operatorname{thick}_{\mathscr{D}}(\mathcal{P})$. *Then w is **w**-**t**-strictly left orthogonal to t.*

Proof. From the construction of t it is clear that w is **t**-strictly left orthogonal to t, and the orthogonality is **w**-strict by Lemma 2.2.24. □

The following lemma shows that in the case of adjacent weight structures and **t**-structures we do not need to distinguish between the various levels of strictness of orthogonality at all. The equivalence I) $\iff$ II), which recovers the original definition [Bon10b, Def. 4.4.1] of adjacency, is also shown in [Bon19, Prop. 1.3.3].

Lemma 2.2.26. *Let t be a **t**-structure and w a weight structure on $\mathscr{C}$. Then the following are equivalent:*

I) $\mathscr{C}^{t\le 0} = \mathscr{C}_{w\le 0}$,

II) w is left orthogonal to t,

*III) w is **w**-strictly left orthogonal to t,*

*IV) w is **t**-strictly left orthogonal to t,*

*V) w is **w**-**t**-strictly left orthogonal to t.*

Proof. II) $\implies$ I) follows from Lemma 2.2.24, and the implications V) $\implies$ IV), V) $\implies$ III), III) $\implies$ II) and IV) $\implies$ II) are obvious from the definitions. For the remaining implication I) $\implies$ V) observe that

$$\begin{aligned} \mathscr{C}_{w\ge 0} &= {}^{\perp}(\mathscr{C}_{w<0}) = {}^{\perp}(\mathscr{C}^{t<0}), & \mathscr{C}_{w\le 0} &= \mathscr{C}^{t\le 0} = {}^{\perp}(\mathscr{C}^{t>0}), \\ \mathscr{C}^{t\le 0} &= \mathscr{C}_{w\le 0} = (\mathscr{C}_{w>0})^{\perp}, & \mathscr{C}^{t\ge 0} &= (\mathscr{C}^{t<0})^{\perp} = (\mathscr{C}_{w<0})^{\perp}. \end{aligned}$$ □

We end this section by listing some standard examples of left orthogonal weight structures and **t**-structures.

Example 2.2.27.

1) Let A be a finite-dimensional algebra. Then $\mathbf{K}^{-}(\mathbf{proj}_{\mathrm{fg}}\text{-}A) \cong \mathbf{D}^{-}(\mathbf{mod}_{\mathrm{fd}}\text{-}A)$, and the standard weight structure is left adjacent to the standard **t**-structure. Moreover, the standard weight structure on $\mathbf{K}^{\mathrm{b}}(\mathbf{proj}_{\mathrm{fg}}\text{-}A)$ is **w**-**t**-strictly left orthogonal to the standard **t**-structure on $\mathbf{D}^{\mathrm{b}}(\mathbf{mod}_{\mathrm{fd}}\text{-}A)$.
2) Let A be a non-positive dg algebra such that $H^n(A)$ is finite-dimensional for all $n \in \mathbb{Z}$. The standard weight structure on $\mathbf{perf}(A)$ is **w**-**t**-strictly left orthogonal to the standard **t**-structure on $\mathbf{D}_{\mathrm{fd}}(A)$.

3) Let A be a positive dg algebra such that $\mathbf{D}_{\mathrm{fd}}(A) = \mathrm{thick}_{\mathbf{D}(A)}(A^\vee)$, where $(-)^\vee = \mathrm{Hom}_{\mathbf{dgMod}\text{-}\Bbbk}(-, \Bbbk)$ is the $\Bbbk$-linear duality functor. Then [KN13, Cor. 4.1 and Thm. 7.1] provide a weight structure on $\mathbf{D}_{\mathrm{fd}}(A)$ and (via the equivalence provided by the Nakayama functor) a $\mathbf{t}$-structure on $\mathrm{thick}_{\mathbf{D}(A)}(A^\vee)$. By taking K-injective resolutions, one sees that these are left adjacent to each other.
4) By [Bon10b, §4.6 and Thm. 4.3.2], the spherical weight structure on the category of finite spectra $\mathbf{SH}_{\mathrm{fin}}$ extends to a weight structure on $\mathbf{SH}^- = \{X \in \mathbf{SH} \mid \pi_n(X) = 0\ \forall n \ll 0\}$, and this extension is left adjacent to the Postnikov $\mathbf{t}$-structure.

2.3 Silting collections and derived projectives

It is well-known that silting collections behave very similar to projective objects. To make this precise, in this section we introduce derived projective covers, and show that under some assumptions the derived projective covers of simple objects of the heart are the same as a silting collection. As an application, we use derived projective covers to formulate criteria for orthogonality.

2.3.1 Derived projective objects

We begin by showing some basic facts about derived projective objects. Let $\mathscr{D}$ be a Krull–Schmidt triangulated category and t a $\mathbf{t}$-structure on $\mathscr{D}$.

Definition 2.3.1. An object $P \in \mathscr{D}$ is *derived projective* (with respect to t) if $P \in \mathscr{D}^{t\leq 0}$ and $\mathrm{Hom}_{\mathscr{D}}(P, X[1]) = 0$ for all $X \in \mathscr{D}^{t\leq 0}$. We write $\mathbf{DProj}_t(\mathscr{D})$ for the full subcategory of derived projective objects with respect to t.

[GLVdB21, Def. 6.1] gives a different definition of derived projective objects, which is equivalent to the above by [GRG23, Prop. 2.3.5]. Derived projective objects are also known as *Ext-projectives* or just *projectives*, see for instance [CSPP22], [Lur17, §7.2.2] and [Lur18b, §C.5.7].

The definition of derived projectives is motivated by the well-known fact that an object P of an abelian category $\mathscr{A}$ is projective if and only if $\mathrm{Ext}^1_{\mathscr{A}}(P, X) = 0$ for all $X \in \mathscr{A}$. From this point of view, the following lemma is an analog of the statement that $\mathrm{Hom}_{\mathscr{A}}(P, -)$ is right exact if $P \in \mathscr{A}$ is projective.

Lemma 2.3.2. *Let $P \in \mathbf{DProj}_t(\mathscr{D})$ and $X, Y \in \mathscr{D}$.*

1) *For $f\colon X \to Y$ with $\mathrm{cone}(f) \in \mathscr{D}^{t<0}$, the map $\mathrm{Hom}_{\mathscr{D}}(P, f)\colon \mathrm{Hom}_{\mathscr{D}}(P, X) \to \mathrm{Hom}_{\mathscr{D}}(P, Y)$ is surjective.*
2) *$t_{\geq 0}$ and $t_{\leq 0}$ induce isomorphisms*

$$\mathrm{Hom}_{\mathscr{D}}(P, X) \cong \mathrm{Hom}_{\mathscr{D}^{t\geq 0}}(t_{\geq 0}P, t_{\geq 0}X) \cong \mathrm{Hom}_{\heartsuit_t}(H^0_t(P), H^0_t(X)).$$

3) *P is indecomposable if and only if $H^0_t(P)$ is.*

Proof.

1) This is immediate from the long exact sequence obtained by applying $\mathrm{Hom}_{\mathscr{D}}(P, -)$ to the triangle $X \to Y \to \mathrm{cone}(f) \to X[1]$.

2) From the long exact sequence obtained by applying $\mathrm{Hom}_{\mathscr{D}}(P,-)$ to the triangle $t_{<0}X \to X \to t_{\geq 0}X \to t_{<0}X[1]$, and derived projectivity of P, we get

$$\mathrm{Hom}_{\mathscr{D}}(P,X) \cong \mathrm{Hom}_{\mathscr{D}}(P,t_{\geq 0}X) \cong \mathrm{Hom}_{\mathscr{D}^{t\geq 0}}(t_{\geq 0}P,t_{\geq 0}X).$$

This isomorphism is given by the functor $t_{\geq 0}$. The second isomorphism follows since $t_{\leq 0}$ is right adjoint to $\mathscr{D}^{t\leq 0} \hookrightarrow \mathscr{D}$, using that $t_{\geq 0}P = H_t^0(P)$.

3) By 2) we have $\mathrm{End}_{\mathscr{D}}(P) \cong \mathrm{End}_{\heartsuit_t}(H_t^0(P))$. Since $\mathscr{D}$ is Krull–Schmidt, P is indecomposable if and only if $\mathrm{End}_{\mathscr{D}}(P)$ is local (and analogously for $H_t^0(P)$). ☐

An easy but important observation is that silting collections consist of derived projectives with respect to their associated silting **t**-structures, see Lemma 2.3.15 below. With this in mind, Lemma 2.3.2 as well as the following lemma is contained in [AHLSV22, Prop. 2.5]. Variants of Lemma 2.3.3 have already appeared several times in the literature, see for instance [AN09] or [PV18, Prop. 4.3].

Lemma 2.3.3. *If $P \in \mathbf{DProj}_t(\mathscr{D})$, then $t_{\geq 0}P = H_t^0(P)$ is projective in $\heartsuit_t$.*

Proof. Since $P \in \mathscr{D}^{t\leq 0}$ we obviously have $t_{\geq 0}P \in \heartsuit_t$. It is well-known (see e.g. [Ach21, Prop. A.7.18]) that $\mathrm{Ext}^1_{\heartsuit_t}(t_{\geq 0}P,X) \cong \mathrm{Hom}_{\mathscr{D}}(t_{\geq 0}P,X[1])$ for $X \in \heartsuit_t$, where $\mathrm{Ext}^1_{\heartsuit_t}$ is defined via equivalence classes of short exact sequences (Yoneda ext). From the long exact sequence obtained by applying $\mathrm{Hom}_{\mathscr{D}}(-,X[1])$ to the triangle $t_{<0}P \to P \to t_{\geq 0}P \to t_{<0}P[1]$ and derived projectivity of P it follows that $\mathrm{Hom}_{\mathscr{D}}(t_{\geq 0}P,X[1]) = 0$, and hence $t_{\geq 0}P$ is projective in $\heartsuit_t$. ☐

Corollary 2.3.4. *For $f\colon P \to P'$ with $P,P' \in \mathbf{DProj}_t(\mathscr{D})$ the following are equivalent:*

I) *f is a split epimorphism.*

II) $\mathrm{cone}(f) \in \mathscr{D}^{t<0}$.

III) *$t_{\geq 0}f = H_t^0(f)$ is an epimorphism in $\heartsuit_t$.*

IV) *$t_{\geq 0}f = H_t^0(f)$ is a split epimorphism in $\heartsuit_t$.*

We will often need to assume that all projectives in $\heartsuit_t$ are obtained as truncations of derived projectives. More precisely, we use the following definition from [GLVdB21, Def. 6.1 and Def. 6.6]

Definition 2.3.5. *$\mathscr{D}$ has derived projectives* (with respect to t) if for every projective $P \in \heartsuit_t$ there is $\hat{P} \in \mathbf{DProj}_t(\mathscr{D})$ with $H_t^0(\hat{P}) \cong P$. If moreover $\heartsuit_t$ has enough projectives, we say that *$\mathscr{D}$ has enough derived projectives* (with respect to t).

In Theorem 2.3.16 we will show that if $\heartsuit_t$ is finite-length, then $\mathscr{D}$ has enough derived projectives if and only if t is silting. In general, $\mathscr{D}$ does not necessarily have enough derived projectives, even if $\heartsuit_t$ has enough projectives. For instance, this is the case for the standard **t**-structure on $\mathbf{D}_{\mathrm{fd}}(A)$ if A is a non-positive dg algebra such that $H^n(A)$ is finite-dimensional for all $n \in \mathbb{Z}$, but $H^*(A)$ is not, see Example 2.3.17 below.

Corollary 2.3.6. *If $\mathscr{D}$ has derived projectives with respect to t, then $t_{\geq 0} = H_t^0\colon \mathbf{DProj}_t(\mathscr{D}) \to \mathbf{Proj}(\heartsuit_t)$ is an equivalence of categories.*

Proof. The functor is well-defined by Lemma 2.3.3 and fully faithful by Lemma 2.3.2, and that $\mathscr{D}$ has derived projectives ensures that it is dense. ☐

The following theorem shows that the definition of enough derived projectives given in [CSPP22, Def. 2.2] is equivalent to the one we use.

Theorem 2.3.7. *$\mathscr{D}$ has enough derived projectives with respect to t if and only if $\mathbf{DProj}_t(\mathscr{D})$ is contravariantly finite in $\mathscr{D}^{t\leq 0}$ and $\mathbf{DProj}_t(\mathscr{D})^{\perp} \cap \heartsuit_t = \{0\}$.*

Proof. "$\implies$": For $X \in \mathscr{D}^{t\leq 0}$ we have $H^0_t(X) \in \heartsuit_t$. Since $\mathscr{D}$ has enough derived projectives, there is an epimorphism $\pi \colon P \to H^0_t(X)$ with P projective in $\heartsuit_t$, and moreover $\hat{P} \in \mathbf{DProj}_t(\mathscr{D})$ with $H^0_t(\hat{P}) = t_{\geq 0}\hat{P} \cong P$. By Lemma 2.3.2 there is a unique morphism $\hat{\pi} \colon \hat{P} \to X$ such that $H^0_t(\hat{\pi}) = \pi$. We claim that $\hat{P}$ is a right $\mathbf{DProj}_t(\mathscr{D})$-approximation. Indeed, for $P' \in \mathbf{DProj}_t(\mathscr{D})$ by Lemma 2.3.2 we get a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{D}}(P', \hat{P}) \arrow[r, "\mathrm{Hom}_{\mathscr{D}}(P', \hat{\pi})"] \arrow[d, "H^0_t", "\cong"'] & \mathrm{Hom}_{\mathscr{D}}(P', X) \arrow[d, "\cong", "H^0_t"'] \\
\mathrm{Hom}_{\heartsuit_t}(H^0_t(P'), P) \arrow[r, "\mathrm{Hom}_{\heartsuit_t}(H^0_t(P'), \pi)"] & \mathrm{Hom}_{\heartsuit_t}(H^0_t(P'), H^0_t(X)),
\end{tikzcd}$$

and the bottom map is surjective since $\pi \colon H^0_t(\hat{P}) \to H^0_t(X)$ is an epimorphism and $H^0_t(P') \in \mathbf{Proj}(\heartsuit_t)$ by Lemma 2.3.3.

Since $\heartsuit_t$ has enough projectives, $X \in \heartsuit_t$ is zero if and only if $\mathrm{Hom}_{\heartsuit_t}(P, X) = 0$ for all $P \in \mathbf{Proj}(\heartsuit_t)$. Since $\mathscr{D}$ has derived projectives, for every $P \in \mathbf{Proj}(\heartsuit_t)$ there is $\hat{P} \in \mathbf{DProj}_t(\mathscr{D})$ with $H^0_t(\hat{P}) \cong P$. By Lemma 2.3.2 it follows that $X \in \heartsuit_t$ is zero if and only if $\mathrm{Hom}_{\mathscr{D}}(\hat{P}, X) = 0$ for all $\hat{P} \in \mathbf{DProj}_t(\mathscr{D})$, as required.

"$\impliedby$": We first show that $\heartsuit_t$ has enough projectives. If $X \in \heartsuit_t$, then $X \in \mathscr{D}^{t\leq 0}$. By assumption, there is a right $\mathbf{DProj}_t(\mathscr{D})$-approximation $\pi \colon P \to X$. By Lemma 2.3.3 $H^0_t(P)$ is projective in $\heartsuit_t$, and so it suffices to show that $H^0_t(\pi) \colon H^0_t(P) \to H^0_t(X) = X$ is an epimorphism.

For this, let $P' \in \mathbf{DProj}_t(\mathscr{D})$ and apply $\mathrm{Hom}_{\mathscr{D}}(P', -)$ to the triangle $P \xrightarrow{\pi} X \to \mathrm{cone}(\pi) \to P[1]$. This gives an exact sequence

$$\mathrm{Hom}_{\mathscr{D}}(P', P) \to \mathrm{Hom}_{\mathscr{D}}(P', X) \to \mathrm{Hom}_{\mathscr{D}}(P', \mathrm{cone}(\pi)) \to \mathrm{Hom}_{\mathscr{D}}(P', P[1]).$$

The first map is surjective since $\pi \colon P \to X$ is a right $\mathbf{DProj}_t(\mathscr{D})$-approximation, and the last term vanishes as P' is derived projective and $P[1] \in \mathscr{D}^{t<0}$. Thus $\mathrm{Hom}_{\mathscr{D}}(P', \mathrm{cone}(\pi)) = 0$. As $t_{\geq 0}$ is left adjoint to $\mathscr{D}^{t\geq 0} \hookrightarrow \mathscr{D}$ and $t_{\geq 0}P' = H^0_t(P')$, we get

$$\begin{aligned}
\mathrm{Hom}_{\mathscr{D}}(P', H^0_t(\mathrm{cone}(\pi))) &\cong \mathrm{Hom}_{\heartsuit_t}(H^0_t(P'), H^0_t(\mathrm{cone}(\pi)) \\
&\cong \mathrm{Hom}_{\mathscr{D}}(P', \mathrm{cone}(\pi)) = 0,
\end{aligned}$$

where the last isomorphism is by Lemma 2.3.2. Thus $H^0_t(\mathrm{cone}(\pi)) \in \mathbf{DProj}_t(\mathscr{D})^{\perp} \cap \heartsuit_t = \{0\}$.

To show that $\mathscr{D}$ has derived projectives, let $P \in \heartsuit_t$ be projective and let $\pi \colon \tilde{P} \to P$ be a right $\mathbf{DProj}_t(\mathscr{D})$-approximation. By the previous argument, we have $\mathrm{cone}(\pi) \in \mathscr{D}^{t<0}$, and thus we get an epimorphism $H^0_t(\tilde{P}) \to P$ in $\heartsuit_t$. This splits since P is projective, and thus P is a summand of $H^0_t(\tilde{P})$. Since $H^0_t \colon \mathbf{DProj}_t(\mathscr{D}) \to \mathbf{Proj}(\heartsuit_t)$ is fully faithful, there must be a corresponding summand $\hat{P}$ of $\tilde{P}$ with $H^0_t(\hat{P}) \cong P$. $\square$

Remark 2.3.8. As is explained in [CSPP22, Rem. 2.3], in Theorem 2.3.7 the assumption that $\mathbf{DProj}_t(\mathscr{D})$ is contravariantly finite is unnecessary if $\mathbf{DProj}_t(\mathscr{D})$ contains only finitely many indecomposables.

By combining Theorem 2.3.7 with [CSPP22, Thm. 2.4] (see also [Bon19, Thm. 5.3.1]) we obtain the following criterion for the existence of a weight structure that is left adjacent to a given **t**-structure.

Corollary 2.3.9. *For a bounded above **t**-structure t on a Hom-finite Krull–Schmidt triangulated category $\mathscr{D}$ the following are equivalent:*

I) $\mathbf{DProj}_t(\mathscr{D})$ *is contravariantly finite in* $\mathscr{D}^{t\leq 0}$ *and* $\mathbf{DProj}_t(\mathscr{D})^{\perp} \cap \heartsuit_t = \{0\}$.

II) $\mathscr{D}$ *has enough derived projectives with respect to* t.

III) t *admits a left adjacent weight structure.*

Moreover, if these conditions hold, then $\heartsuit_t$ *is covariantly finite in* $\mathscr{D}$.

Proof. By Theorem 2.3.7, I) is equivalent to II). Moreover II) is equivalent to [CSPP22, Thm. 2.4 (2)] by Corollary 2.3.6, and I) is [CSPP22, Thm. 2.4 (1)] without the assumption that $\heartsuit_t$ is covariantly finite in $\mathscr{D}$. Thus the remaining implications follow from [CSPP22, Thm. 2.4 and Rem. 2.5]. □

Example 2.3.10. In particular, Corollary 2.3.9 shows that for a finite-dimensional algebra A, the standard $\mathbf{t}$-structure on $\mathbf{D}^{\mathrm{b}}(\mathbf{mod}_{\mathrm{fd}}\text{-}A)$ (see [BBD82, Ex. 1.3.2]) admits a left adjacent weight structure. Indeed, in this case II) is obviously satisfied: the projective generator A of $\heartsuit_t \cong \mathbf{mod}_{\mathrm{fd}}\text{-}A$ is derived projective since $\mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathbf{mod}_{\mathrm{fd}}\text{-}A)}(A, X[n]) \cong H^n(X)$ for all $n \in \mathbb{Z}$ (or, in other words, since A is the silting object defining the standard $\mathbf{t}$-structure). An alternative way to obtain this weight structure on $\mathbf{D}^{\mathrm{b}}(\mathbf{mod}_{\mathrm{fd}}\text{-}A)$ is via [AMY19, Lemma 4.10] and Proposition 2.4.3 below.

The adjacent weight structure can also be described explicitly, as follows. Let $\mathscr{D} \subseteq \mathbf{D}^{-}(\mathbf{mod}_{\mathrm{fd}}\text{-}A)$ be the full triangulated subcategory of complexes with finite-dimensional total cohomology, and $\mathscr{C} \subseteq \mathbf{K}^{-}(\mathbf{proj}_{\mathrm{fg}}\text{-}A)$ the full subcategory of complexes with finite-dimensional total cohomology. The obvious inclusions

$$\mathscr{C} \hookrightarrow \mathscr{D} \hookleftarrow \mathbf{D}^{\mathrm{b}}(\mathbf{mod}_{\mathrm{fd}}\text{-}A)$$

are equivalences since any $Y \in \mathscr{D}$ can be t-truncated to an isomorphic object that lies in $\mathbf{D}^{\mathrm{b}}(\mathbf{mod}_{\mathrm{fd}}\text{-}A)$, and $\mathscr{C}$ precisely consists of the projective resolutions of objects in $\mathscr{D}$. Note that (by construction) the equivalence $\mathbf{D}^{\mathrm{b}}(\mathbf{mod}_{\mathrm{fd}}\text{-}A) \to \mathscr{C}$ sends a complex to a projective resolution.

The standard weight structure on $\mathbf{K}^{-}(\mathbf{proj}_{\mathrm{fg}}\text{-}A)$ from [Bon10b, §1.1] restricts to a weight structure w on $\mathscr{C}$, and thus yields a weight structure on $\mathbf{D}^{\mathrm{b}}(\mathbf{mod}_{\mathrm{fd}}\text{-}A)$. For this it suffices to check that if $X \in \mathscr{C}$, then there is a weight decomposition $w_{>0}X \to X \to w_{\leq 0}X \to w_{>0}X[1]$ with $w_{>0}X, w_{\leq 0}X \in \mathscr{C}$. But this is obvious since for the standard weight structure, $w_{>0}X$ and $w_{\leq 0}X$ are given by "brutal truncation" of X (note that X is, by definition, a complex of finitely generated projectives).

The weight structure w is left adjacent to the standard $\mathbf{t}$-structure on $\mathbf{D}^{\mathrm{b}}(\mathbf{mod}_{\mathrm{fd}}\text{-}A)$ since $\mathscr{C}_{w\leq 0}$ precisely consists of the projective resolutions of objects in $\mathbf{D}^{\mathrm{b}}(\mathbf{mod}_{\mathrm{fd}}\text{-}A)^{t\leq 0}$. Note that w is always bounded above, but bounded below if and only if A has finite global dimension.

2.3.2 Derived projective covers

For an object $X \in \mathscr{D}$ we would like to find a minimal derived projective object approximating X. This is made precise by the following definition, which is dual to [Lur18b, Ex. C.5.7.9].

Definition 2.3.11. A *derived projective cover* of $X \in \mathscr{D}$ is a morphism $\pi\colon P \to X$ such that P is derived projective and $H^0_t(\pi)\colon H^0_t(P) \to H^0_t(X)$ is a projective cover of $H^0_t(X)$ in $\heartsuit_t$.

Lemma 2.3.12. *The derived projective cover of* $X \in \mathscr{D}$ *is unique up to isomorphism (if it exists).*

Proof. Let $\pi_1 \colon P_1 \to X$ and $\pi_2 \colon P_2 \to X$ be derived projective covers of X. Then $H_t^0(\pi_1) \colon H_t^0(P_1) \to H_t^0(X)$ and $H_t^0(\pi_2) \colon H_t^0(P_2) \to H_t^0(X)$ are projective covers of $H_t^0(X)$ in $\heartsuit_t$. Since projective covers are unique up to isomorphism, there is an isomorphism $g \colon P_1 \to P_2$ with $H_t^0(\pi_2)g = H_t^0(\pi_1)$, and by Lemma 2.3.2 there is an isomorphism $\hat{g} \colon P_1 \to P_2$ with $\pi_2\hat{g} = \pi_1$. $\square$

Lemma 2.3.13. *Let P be derived projective. Then $\pi \colon P \to X$ is a derived projective cover of X if and only if $t_{\leq 0}\pi \colon P \to t_{\leq 0}X$ is a derived projective cover of $t_{\leq 0}X$.*

Proof. This is obvious since $H_t^0 \circ t_{\leq 0} = H_t^0$. $\square$

Recall that in a Krull–Schmidt abelian category, a morphism $\pi \colon P \to X$ is a projective cover if and only if it satisfies one of the following equivalent conditions:

I) P is projective, π an epimorphism, and for any epimorphism $\pi' \colon P' \to X$ with P' projective there is $g \colon P' \to P$ with $\pi g = \pi'$, and any such g is an epimorphism.

II) $\pi \colon P \to X$ is a minimal right approximation of X by projectives.

If X is simple, then moreover $\pi \colon P \to X$ is a projective cover if and only if it satisfies one of the following equivalent conditions:

I) P is projective and $\pi \neq 0$, and for any non-zero $\pi' \colon P' \to X$ with P' projective there is $g \colon P' \to P$ with $\pi g = \pi'$, and any such g is the projection onto a direct summand.

II) P is projective, indecomposable, and $\pi \neq 0$.

The following lemma provides analogous characterizations of derived projective covers in more specific situations. In general, a good strategy to pass from statements about projective objects to statements about derived projective objects is to replace "f is an epimorphism" by "$\operatorname{cone}(f) \in \mathscr{D}^{t<0}$". This can also be seen in Lemma 2.3.2 above, which is also the main reason behind this phenomenon.

Lemma 2.3.14. *Assume that $\mathscr{D}$ has derived projectives with respect to t.*

1) If $X \in \mathscr{D}^{t\leq 0}$, then for $\pi \colon P \to X$ the following are equivalent:

- *I) $\pi \colon P \to X$ is a derived projective cover of X.*
- *II) $\pi \colon P \to X$ satisfies the following conditions:*
  - *P is derived projective,*
  - $\operatorname{cone}(\pi) \in \mathscr{D}^{t<0}$,
  - *for $\pi' \colon P' \to X$ with P' derived projective and* $\operatorname{cone}(\pi') \in \mathscr{D}^{t<0}$ *there is $g \colon P' \to P$ with $\pi g = \pi'$,*
  - *and* $\operatorname{cone}(g) \in \mathscr{D}^{t<0}$ *for any such g.*
- *III) $\pi \colon P \to X$ is a minimal right approximation of X by derived projective objects.*

2) If $L \in \heartsuit_t$ is simple, then for $\pi \colon P \to L$ the following are equivalent:

- *I) $\pi \colon P \to L$ is a derived projective cover of L.*
- *II) P is indecomposable and $\pi \neq 0$.*
- *III) $\pi \colon P \to L$ satisfies the following conditions:*
  - *P is derived projective,*
  - *$\pi \neq 0$,*

- *for any non-zero $\pi'\colon P' \to L$ with P' derived projective there is $g\colon P' \to P$ such that $\pi' = \pi g$,*
- *and any such g is the projection onto a direct summand.*

Proof.

1) I) $\Longrightarrow$ II): Let $X \in \mathscr{D}^{t\leq 0}$ and $\pi\colon P \to X$ be a derived projective cover. From the triangle $P \to X \to \mathrm{cone}(\pi) \to P[1]$ we get $\mathrm{cone}(\pi) \in \mathscr{D}^{t\leq 0}$, and as $H^0_t(\pi)$ is an epimorphism we have $H^0_t(\mathrm{cone}(\pi)) = 0$, thus $\mathrm{cone}(\pi) \in \mathscr{D}^{t<0}$. Now let $\pi'\colon P' \to X$ with P' derived projective and $\mathrm{cone}(\pi') \in \mathscr{D}^{t<0}$. Applying $\mathrm{Hom}_{\mathscr{D}}(P', -)$ to the triangle $P \to X \to \mathrm{cone}(\pi) \to P[1]$ and using $\mathrm{Hom}_{\mathscr{D}}(P', \mathrm{cone}(\pi)) = 0$ (since $\mathrm{cone}(\pi) \in \mathscr{D}^{t<0}$ and P' is derived projective) shows that π' factors through π. So let $g\colon P' \to P$ be any morphism with $\pi g = \pi'$. Since $P, P' \in \mathscr{D}^{t\leq 0}$ it is clear that $\mathrm{cone}(g) \in \mathscr{D}^{t\leq 0}$. Note that by Lemma 2.3.3 $H^0_t(\pi')\colon H^0_t(P') \to H^0_t(X)$ is an epimorphism from a projective object in $\heartsuit_t$. As $H^0_t(\pi)H^0_t(g) = H^0_t(\pi')$ and $H^0_t(\pi)\colon H^0_t(P) \to H^0_t(X)$ is a projective cover it follows that $H^0_t(g)$ must be an epimorphism. This means $H^0_t(\mathrm{cone}(g)) = 0$, and hence $\mathrm{cone}(g) \in \mathscr{D}^{t<0}$.

   II) $\Longrightarrow$ III): Let $\pi'\colon P' \to X$ be any morphism with P' derived projective. The long exact sequence obtained by applying $\mathrm{Hom}_{\mathscr{D}}(P', -)$ to the triangle $P \xrightarrow{\pi} X \to \mathrm{cone}\,\pi \to P[1]$ shows that $\mathrm{Hom}_{\mathscr{D}}(P', \pi)\colon \mathrm{Hom}_{\mathscr{D}}(P', P) \to \mathrm{Hom}_{\mathscr{D}}(P', X)$ is surjective, since $\mathrm{cone}(\pi) \in \mathscr{D}^{t<0}$ and P' is derived projective. Thus $\pi\colon P \to X$ is a right approximation.

   For minimality, let $g\colon P \to P$ with $\pi g = \pi$. Then by assumption $\mathrm{cone}(g) \in \mathscr{D}^{t<0}$, so by Corollary 2.3.4 g is a split epimorphism. Since $\mathscr{D}$ is Krull–Schmidt, it follows that g is an isomorphism.

   III) $\Longrightarrow$ I): We show that $H^0_t(\pi)\colon H^0_t(P) \to H^0_t(X)$ is a minimal right approximation by projectives. Let $P' \in \heartsuit_t$ be projective and $\pi'\colon P' \to X$. By assumption there is a derived projective $\hat{P}'$ with $H^0_t(\hat{P}') \cong P'$, and we get an induced morphism $\hat{\pi}'\colon \hat{P}' \to t_{\geq 0}\hat{P}' = H^0_t(\hat{P}') \xrightarrow{\pi'} X$ with $H^0_t(\hat{\pi}') = \pi'$. Lemma 2.3.2 implies that H^0_t induces a bijection between morphisms $\hat{g}\colon \hat{P}' \to P$ with $\hat{\pi}' = \pi\hat{g}$ and morphisms $g\colon P' = H^0_t(\hat{P}') \to H^0_t(P)$ with $\pi' = H^0_t(\pi)g$, and the claim follows from this.

2) I) $\Longleftrightarrow$ II): If P is derived projective, then $\pi\colon P \to L$ is a derived projective cover of L iff $H^0_t(\pi)\colon H^0_t(P) \to L$ is a projective cover of L in $\heartsuit_t$ iff $H^0_t(\pi) \neq 0$ and $H^0_t(P)$ is indecomposable projective iff $\pi \neq 0$ and P is indecomposable (by Lemma 2.3.2).

   I) $\Longleftrightarrow$ III): Observe that for $\pi\colon P \to L$ we have $\mathrm{cone}(\pi) \in \mathscr{D}^{t\leq 0}$, and by Lemma 2.3.2 and simplicity of L we get $\pi \neq 0$ iff $H^0_t(\pi) \neq 0$ iff $H^0_t(\pi)$ is an epimorphism iff $\mathrm{cone}(\pi) \in \mathscr{D}^{t<0}$. Similarly $g\colon P \to P'$ satisfies $\mathrm{cone}(g) \in \mathscr{D}^{t\leq 0}$, and by Corollary 2.3.4 g is the projection onto a direct summand if and only if $\mathrm{cone}(g) \in \mathscr{D}^{t<0}$. Therefore the claim follows from 1). ☐

2.3.3 Silting collections as derived projective covers

Silting collections provide an important source of derived projective objects.

Lemma 2.3.15. *Let $\mathcal{P}$ be a silting collection in $\mathscr{D}$ and t its associated silting $\boldsymbol{t}$-structure. Then any $P \in \mathcal{P}$ is derived projective with respect to t.*

Proof. By definition we have $\mathscr{D}^{t<0} = \mathcal{P}^{\perp_{\geq 0}}$, and also $\mathcal{P} \subseteq \mathcal{P}^{\perp_{>0}} = \mathscr{D}^{t\leq 0}$ by Remark 2.2.10, which precisely means that $\mathcal{P}$ consists of derived projective objects. ☐

The following theorem shows that derived projective covers provide a convenient description of the relation between a silting collection and the simple objects in the heart of the associated silting **t**-structure. In particular, this observation can be used to formulate the bijections between simple-minded collections and silting collections from [KY14], see Theorem 2.4.4 below. This result is very similar to [CSPP22, Thm. 2.4] and [Bon19, Thm. 5.3.1 II.].

Theorem 2.3.16. *Let t be a non-degenerate **t**-structure on $\mathscr{D}$ with finite-length heart. Let $\mathcal{L}$ be a full set of isomorphism representatives of the simple objects in $\heartsuit_t$ and $\mathcal{P}$ a full set of isomorphism representatives of the indecomposable derived projectives. Then the following are equivalent:*

I) t is silting (and $\mathcal{P}$ is the silting collection).

II) There is a bijection $\phi\colon \mathcal{P} \to \mathcal{L}$ such that for $P \in \mathcal{P}$, $L \in \mathcal{L}$, $m \in \mathbb{Z}$ we have

$$\operatorname{Hom}_{\mathscr{D}}(P, L[m]) \cong \begin{cases} \operatorname{End}_{\mathscr{D}}(L) & \text{if } L = \phi(P),\ m = 0, \\ 0 & \text{otherwise} \end{cases} \tag{2.2}$$

as left $\operatorname{End}_{\mathscr{D}}(L)$-modules.

III) Every $L \in \mathcal{L}$ admits a derived projective cover (and $\mathcal{P}$ is the set of these derived projective covers).

IV) $\mathscr{D}$ has enough derived projectives with respect to t.

Proof. I) $\implies$ II): Let $\mathcal{P}'$ be a silting collection with $t = (\mathcal{P}'^{\perp_{<0}}, \mathcal{P}'^{\perp_{>0}})$. By Lemma 2.2.13 $\mathcal{P}'$ weakly generates $\mathscr{D}$, and thus for each $L \in \mathcal{L}$ there is some $P \in \mathcal{P}'$ and $m \in \mathbb{Z}$ with $\operatorname{Hom}_{\mathscr{D}}(P, L[m]) \neq 0$. From $L \in \heartsuit_t$ we get $m = 0$, so using $P \in \mathscr{D}^{t\leq 0}$ we get

$$0 \neq \operatorname{Hom}_{\mathscr{D}}(P, L) = \operatorname{Hom}_{\mathscr{D}^{t\geq 0}}(t_{\geq 0}P, L) = \operatorname{Hom}_{\heartsuit_t}(H^0_t(P), L).$$

Since L is simple in $\heartsuit_t$ it follows that there is an epimorphism $H^0_t(P) \to L$. As P is indecomposable, so is $H^0_t(P)$ by Lemma 2.3.2, and thus $H^0_t(P)$ is the projective cover of L in $\heartsuit_t$. From this it follows that $H^0_t(P)$, and (by Lemma 2.3.2 again) also P, is unique up to isomorphism. So we get a bijection $\phi\colon \mathcal{P}' \to \mathcal{L}$ by defining $\phi(P) = L$, and moreover

$$\operatorname{Hom}_{\mathscr{D}}(P, L) \cong \operatorname{Hom}_{\heartsuit_t}(H^0_t(P), L) \cong \operatorname{End}_{\heartsuit_t}(L)$$

as left $\operatorname{End}_{\heartsuit_t}(L)$-module, as desired. Finally, since $H^0_t(\mathcal{P}')$ is a full set of indecomposable projectives in $\heartsuit_t$ and $\mathcal{P}'$ consists of derived projectives by Lemma 2.3.15, it follows from Corollary 2.3.6 that $\mathcal{P}' = \mathcal{P}$ is the set of indecomposable derived projectives.

II) $\implies$ III): Let $L \in \mathcal{L}$, $P = \phi^{-1}(L)$, and $\pi\colon P \to L$ correspond to id_L. Then $\pi \neq 0$, and moreover $H^0_t(P)$ (and thus, by Lemma 2.3.2, also P) must be indecomposable since otherwise it would admit two simple quotients, which is impossible by (2.2). Thus $\pi\colon P \to L$ is the derived projective cover of L by Lemma 2.3.14.

III) $\implies$ IV): Since every $L \in \mathcal{L}$ has a derived projective cover, it by definition has a projective cover in $\heartsuit_t$. As $\heartsuit_t$ is finite-length, it follows that $\heartsuit_t$ has enough projectives. Moreover, the projective covers of the simple objects are a full set of isomorphism representatives of the indecomposable projectives in $\heartsuit_t$. Thus the indecomposable projectives arise as t-truncations of derived projectives, and therefore $\mathscr{D}$ has enough derived projectives.

IV) $\implies$ I): By Corollary 2.3.6, $H^0_t(\mathcal{P})$ is the set of indecomposable projectives of $\heartsuit_t$. We claim that $\mathscr{D}^{t\leq 0} = \mathcal{P}^{\perp_{>0}}$ and $\mathscr{D}^{t\geq 0} = \mathcal{P}^{\perp_{<0}}$. For $X \in \mathscr{D}$, $P \in \mathcal{P}$ and $n \in \mathbb{Z}$ we get from Lemma 2.3.2

$$\operatorname{Hom}_{\mathscr{D}}(P, X[n]) \cong \operatorname{Hom}_{\heartsuit_t}(H^0_t(P), H^0_t(X[n])) = \operatorname{Hom}_{\heartsuit_t}(H^0_t(P), H^n_t(X)).$$

Since $\{H_t^0(P) \mid P \in \mathcal{P}\}$ is a full set of indecomposable projectives in $\heartsuit_t$ and $\heartsuit_t$ is finite-length, we have $H_t^n(X) = 0$ if and only if $\operatorname{Hom}_{\mathscr{D}}(P, X[n]) = 0$ for all $P \in \mathcal{P}$. The claim follows from this since by non-degeneracy of t we know that $X \in \mathscr{D}^{t\leq 0}$ if and only if $H_t^n(X) = 0$ for all $n > 0$, and similarly for $\mathscr{D}^{t\geq 0}$. □

Using Theorem 2.3.16 we can now show that not every triangulated category has derived projectives with respect to every **t**-structure.

Example 2.3.17. Let A be a non-positive dg algebra such that $H^n(A)$ is finite-dimensional for all $n \in \mathbb{Z}$ but $H^*(A)$ is not finite-dimensional. Then A is a silting object in $\mathbf{D}(A)$ (see e.g. [BY14, Appendix A]), and the silting **t**-structure restricts to $\mathbf{D}_{\mathrm{fd}}(A)$. These **t**-structures are the *standard **t**-structures* on $\mathbf{D}(A)$ and $\mathbf{D}_{\mathrm{fd}}(A)$. The heart of the standard **t**-structure on $\mathbf{D}_{\mathrm{fd}}(A)$ is equivalent to $\mathbf{mod}_{\mathrm{fd}}$-$H^0(A)$ and thus has enough projectives. However, this **t**-structure is not silting, and so by Theorem 2.3.16 there is no derived projective that truncates to the projective generator of $\heartsuit_t$.

To see that the standard **t**-structure is not silting, suppose for a contradiction that $P \in \mathbf{D}_{\mathrm{fd}}(A)$ is a silting object defining the standard **t**-structure. Observe that then $t_{\geq n}P \cong t_{\geq n}A$ for all $n \leq 0$, since both these objects represent the functor $H^0(-)\colon \mathbf{D}_{\mathrm{fd}}(A)^{t\leq 0} \cap \mathbf{D}_{\mathrm{fd}}(A)^{t\geq n} \to \heartsuit_t$ (this uses the equivalences $\mathbf{mod}_{\mathrm{fd}}\text{-}A \cong \heartsuit_t \cong \mathbf{mod}_{\mathrm{fd}}\text{-}\operatorname{End}_{\heartsuit_t}(H^0(P))$, and Lemma 2.3.2). As $P \in \mathbf{D}_{\mathrm{fd}}(A)$, there is $N \leq 0$ with $t_{\geq n}P \cong P$ for all $n \leq N$, and thus we also have $t_{\geq n}A \cong t_{\geq n}P \cong P$ for all $n \leq N$. But this implies $H^n(A) \cong H^n(P) = 0$ for $n < N$ and thus $A \in \mathbf{D}_{\mathrm{fd}}(A)$, a contradiction.

As an application, we obtain the following criterion for left orthogonality between weight structures defined from silting collections and **t**-structures defined from simple-minded collections.

Theorem 2.3.18. *Let $\mathscr{D}$ be a triangulated category, $\mathcal{P}$ a silting collection in $\mathscr{D}$ and $\mathcal{L}$ a simple-minded collection in $\mathscr{D}$. Let w be the weight structure on $\mathscr{C} = \operatorname{thick}_{\mathscr{D}}(\mathcal{P})$ defined by $\mathcal{P}$ and t the **t**-structure on $\mathscr{D}$ defined by $\mathcal{L}$. Then the following are equivalent:*

- *I) w is left orthogonal to t.*
- *II) w is **w**-**t**-strictly left orthogonal to t.*
- *III) $t = (\mathcal{P}^{\perp_{>0}}, \mathcal{P}^{\perp_{<0}})$ is the silting **t**-structure associated with $\mathcal{P}$.*
- *IV) There is a bijection $\phi\colon \mathcal{P} \to \mathcal{L}$ such that $P \in \mathcal{P}$ is the derived projective cover of $\phi(\mathcal{P})$.*
- *V) There is a bijection $\phi\colon \mathcal{P} \to \mathcal{L}$ such that for $P \in \mathcal{P}$, $L \in \mathcal{L}$ and $m \in \mathbb{Z}$ we have isomorphisms of left $\operatorname{End}_{\mathscr{D}}(L)$-modules*

$$\operatorname{Hom}_{\mathscr{D}}(P, L[m]) \cong \begin{cases} \operatorname{End}_{\mathscr{D}}(L) & \text{if } L = \phi(P),\ m = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. I) $\implies$ II): **w**-strictness follows from Lemma 2.2.24. For **t**-strictness, note that by Lemma 2.2.24 we have $\mathscr{C}_{w\leq 0} = \mathscr{D}^{t\leq 0} \cap \mathscr{C}$ and $\mathscr{C}_{w\geq 0} = {}^{\perp}(\mathscr{D}^{t<0}) \cap \mathscr{C}$. Therefore

$$\mathscr{D}^{t\geq 0} = (\mathscr{D}^{t<0})^{\perp} \subseteq (\mathscr{C}_{w<0})^{\perp}.$$

and similarly $\mathscr{D}^{t\leq 0} \subseteq (\mathscr{C}_{w>0})^{\perp}$. But by assumption both $((\mathscr{C}_{w>0})^{\perp}, (\mathscr{C}_{w<0})^{\perp}) = (\mathcal{P}^{\perp_{>0}}, \mathcal{P}^{\perp_{<0}})$ and $t = (\mathscr{D}^{t\leq 0}, \mathscr{D}^{t\geq 0})$ are **t**-structures on $\mathscr{D}$, and therefore they must agree.

II) $\implies$ III): This is clear from the construction of w from $\mathcal{P}$ in Proposition 2.2.19.

III) $\implies$ IV): This is part of Theorem 2.3.16.

IV) $\Longrightarrow$ V): As $L \in \heartsuit_t$ and P is derived projective, we have $\mathrm{Hom}_{\mathscr{D}}(P, L[m]) = 0$ for $m \neq 0$. For $m = 0$ we get from Lemma 2.3.2

$$\mathrm{Hom}_{\mathscr{D}}(P, L) \cong \mathrm{Hom}_{\heartsuit_t}(H^0_t(P), L) \cong \begin{cases} \mathrm{End}_{\heartsuit_t}(L) & \text{if } L = \phi(P), \\ 0 & \text{else,} \end{cases}$$

since by definition $H^0_t(P)$ is the projective cover of $\phi(P)$ in $\heartsuit_t$.

V) $\Longrightarrow$ I): From the construction of w and t it follows that left orthogonality is equivalent to $\mathrm{Hom}_{\mathscr{D}}(P[m], L[n]) = 0$ for all $P \in \mathcal{P}$, $L \in \mathcal{L}$, and either $m \leq 0$ and $n > 0$, or $m \geq 0$ and $n < 0$. This condition is obvious from the assumptions. ☐

2.4 The WT correspondence revisited

In many important examples, **w**-**t**-strict orthogonality yields a bijection between weight structures and **t**-structures. A unified setup for this is provided by the following definition.

Definition 2.4.1. Let $\mathscr{T}$ be an idempotent-complete triangulated category and $\mathscr{C}, \mathscr{D} \subseteq \mathscr{T}$ thick subcategories. We call $(\mathscr{C}, \mathscr{D})$ a *WT pair in* $\mathscr{T}$ if there is a weight structure w and a non-degenerate **t**-structure t on $\mathscr{T}$ such that

1) w is left adjacent to t,
2) w and t are bounded above,
3) $\mathscr{C} = \mathrm{thick}_{\mathscr{T}}(\diamondsuit_w)$ and $\mathscr{D} = \mathrm{tria}_{\mathscr{T}}(\heartsuit_t)$,
4) $\diamondsuit_w$ is Krull–Schmidt with finitely many indecomposables, and $\heartsuit_t$ is Hom-finite finite-length with finitely many simples.

In Proposition 2.4.3 below we will show that WT pairs are the same as the *ST pairs* defined in [AMY19, Def. 4.3]. In contrast to that definition, we do not use silting collections and instead define WT pairs via weight structures and **t**-structures. The axioms for WT pairs are similar to the conditions from [AMY19, Prop. 4.17].

Example 2.4.2. We list some known examples of WT pairs.

1) Let A be a finite-dimensional algebra. Then $(\mathbf{K}^{\mathrm{b}}(\mathbf{proj}_{\mathrm{fg}}\text{-}A), \mathbf{D}^{\mathrm{b}}(\mathbf{mod}_{\mathrm{fd}}\text{-}A))$ is a WT pair in $\mathbf{D}^{-}(\mathbf{mod}_{\mathrm{fd}}\text{-}A)$.

   By using the weight structure on $\mathbf{D}^{\mathrm{b}}(\mathbf{mod}_{\mathrm{fd}}\text{-}A)$ described in Example 2.3.10, one sees that $(\mathbf{K}^{\mathrm{b}}(\mathbf{proj}_{\mathrm{fg}}\text{-}A), \mathbf{D}^{\mathrm{b}}(\mathbf{mod}_{\mathrm{fd}}\text{-}A))$ is also a WT pair in $\mathbf{D}^{\mathrm{b}}(\mathbf{mod}_{\mathrm{fd}}\text{-}A)$, cf. [AMY19, Lemma 4.10].
2) Let A be a non-positive dg algebra such that $H^n(A)$ is finite-dimensional for all $n \in \mathbb{Z}$. Then $(\mathbf{perf}(A), \mathbf{D}_{\mathrm{fd}}(A))$ is a WT pair in $\mathbf{D}^{-}_{\mathrm{fd}}(A) = \{X \in \mathbf{D}(A) \mid \sum_{k \geq n} \dim H^k(X) < \infty \; \forall n \in \mathbb{Z}\}$ by [Fus24, Ex. 3.4].
3) Let A be a non-positive dg algebra such that $H^0(A)$ is finite-dimensional and $\mathbf{D}_{\mathrm{fd}}(A) \subseteq \mathbf{perf}(A)$. Then $(\mathbf{perf}(A), \mathbf{D}_{\mathrm{fd}}(A))$ is a WT pair in $\mathbf{perf}(A)$ by [AMY19, Lemma 4.15].
4) Let A be a positive dg algebra such that $\mathbf{D}_{\mathrm{fd}}(A) = \mathrm{thick}_{\mathbf{D}(A)}(A^\vee)$, where $(-)^\vee = \mathrm{Hom}_{\mathbf{dgMod}\text{-}\Bbbk}(-, \Bbbk)$ is the $\Bbbk$-linear duality functor. Then [KN13, Cor. 4.1 and Thm. 7.1] provide a weight structure on $\mathbf{D}_{\mathrm{fd}}(A)$ and (via the equivalence provided by the Nakayama functor) a **t**-structure on $\mathrm{thick}_{\mathbf{D}(A)}(A^\vee)$. By taking K-injective resolutions, one sees that these are left adjacent to each other, and it follows that $(\mathbf{D}_{\mathrm{fd}}(A), \mathrm{thick}_{\mathbf{D}(A)}(A^\vee))$ is a WT pair in $\mathbf{D}_{\mathrm{fd}}(A)$.

Note that for 1)–3) it is very easy to check the axioms from [AMY19, Def. 4.3], but hard to give explicit descriptions of the adjacent weight structure required for Definition 2.4.1.

The first example also shows that the ambient triangulated category $\mathscr{T}$ for a WT pair is in general not unique. In fact, as observed in [AMY19, §6.1], if $(\mathscr{C}, \mathscr{D})$ is a WT pair in $\mathscr{T}$, then it is also a WT pair in any thick subcategory $\mathscr{T}' \subseteq \mathscr{T}$ containing both $\mathscr{C}$ and $\mathscr{D}$.

The following proposition shows that WT pairs are the same as the ST pairs defined in [AMY19, Def. 4.3].

Proposition 2.4.3. *$(\mathscr{C}, \mathscr{D})$ is a WT pair in $\mathscr{T}$ if and only if there is a finite silting collection $\mathcal{P}$ in $\mathscr{T}$ such that*

- $\operatorname{Hom}_{\mathscr{T}}(P, X)$ *is finite-dimensional for all $X \in \mathscr{T}$ and $P \in \mathcal{P}$,*
- $\operatorname{thick}_{\mathscr{T}}(\mathcal{P}) = \mathscr{C}$,
- $\mathscr{T} = \bigcup_{n\in\mathbb{Z}} \mathscr{T}^{t_{\mathcal{P}} \le n}$ *and* $\mathscr{D} = \bigcup_{n\in\mathbb{Z}} \mathscr{T}^{t_{\mathcal{P}} \ge n}$.

Proof. " $\Longrightarrow$ ": Let $\mathcal{P}$ be a set of isomorphism representatives of the indecomposable objects in $\diamondsuit_w$. By Lemma 2.2.24, w is **w-t**-strictly left orthogonal to t on $\mathscr{T}$. Thus

$$\diamondsuit_w = \mathscr{T}_{w\ge 0} \cap \mathscr{T}_{w\le 0} = {}^{\perp}(\mathscr{T}^{t<0}) \cap \mathscr{T}^{t\le 0} = \mathbf{DProj}_t(\mathscr{T}),$$

so $\mathcal{P}$ consists of the indecomposable derived projectives. Moreover, by [Bon19, Thm. 5.3.1] (cf. Corollary 2.3.9 above) $\mathscr{T}$ has enough derived projectives with respect to t. It follows from Theorem 2.3.16 that $\mathcal{P}$ is a (by assumption finite) silting collection, and t the associated silting **t**-structure. For $P \in \mathcal{P}$ and $X \in \mathscr{T}$, Lemma 2.3.2 gives $\operatorname{Hom}_{\mathscr{T}}(P, X) \cong \operatorname{Hom}_{\heartsuit_t}(H^0_t(P), H^0_t(X))$, which is finite-dimensional by assumption.

Since t is bounded above on $\mathscr{T}$, we have (by definition) $\mathscr{T} = \bigcup_{n\in\mathbb{Z}} \mathscr{T}^{t\le n}$. Finally, if $X \in \mathscr{T}^{t\ge m}$ then $X \in \mathscr{T}^{t\ge m} \cap \mathscr{T}^{t\le n}$ for some $n \in \mathbb{Z}$ as t is bounded above, so $X \in \operatorname{tria}_{\mathscr{T}}(\heartsuit_t) = \mathscr{D}$. The converse inclusion is obvious since $\bigcup_{n\in\mathbb{Z}} \mathscr{T}^{t\ge n} \subseteq \mathscr{T}$ is a triangulated subcategory and $\heartsuit_t \subseteq \bigcup_{n\in\mathbb{Z}} \mathscr{T}^{t\ge n}$.

" $\Longleftarrow$ ": Note that $\heartsuit_{t_{\mathcal{P}}}$ is Hom-finite finite-length with finitely many simples by [AMY19, Prop. 4.6]. As $t_{\mathcal{P}}$ is a silting **t**-structure on $\mathscr{T}$, it is non-degenerate. By Theorem 2.3.16 $\mathscr{T}$ has enough derived projectives with respect to $t_{\mathcal{P}}$ and $\mathcal{P}$ is a full set of indecomposable derived projectives in $\mathscr{T}$. Therefore by [CSPP22, Thm. 2.4] (see Corollary 2.3.9 above) there is a weight structure w on $\mathscr{T}$ that is left adjacent to $t_{\mathcal{P}}$. As $t_{\mathcal{P}}$ is bounded above, so is w.

By Lemma 2.2.24, w is **w-t**-strictly left orthogonal to $t_{\mathcal{P}}$. As above we get $\diamondsuit_w = \mathbf{DProj}_t(\mathscr{T})$, and therefore $\diamondsuit_w = \operatorname{Kar}_{\mathscr{T}}(\mathcal{P})$ as $\mathcal{P}$ consists of the indecomposable derived projectives. In particular $\diamondsuit_w$ contains finitely many indecomposables up to isomorphism, and hence $\mathscr{C} = \operatorname{thick}_{\mathscr{T}}(\mathcal{P}) = \operatorname{thick}_{\mathscr{T}}(\diamondsuit_w)$. Finally, we have $\operatorname{tria}_{\mathscr{T}}(\heartsuit_{t_{\mathcal{P}}}) \subseteq \operatorname{tria}_{\mathscr{T}}(\mathscr{T}^{t_{\mathcal{P}}\ge 0}) \subseteq \mathscr{D}$. Conversely, if $X \in \mathscr{D}$ then $X \in \mathscr{T}^{t_{\mathcal{P}}\ge m} \cap \mathscr{T}^{t_{\mathcal{P}}\le n}$ for some $m, n \in \mathbb{Z}$ (as $t_{\mathcal{P}}$ is bounded above), so $X \in \operatorname{tria}_{\mathscr{T}}(\heartsuit_{t_{\mathcal{P}}})$. ☐

In [Fus24] it is shown that for a WT pair $(\mathscr{C}, \mathscr{D})$ in an algebraic triangulated category $\mathscr{T}$ there are bijections between bounded weight structures on $\mathscr{C}$ and bounded **t**-structures $\mathscr{D}$ with finite-length heart. By the examples listed in Example 2.4.2, this unifies several earlier results: for the WT pair $(\mathbf{K}^{\mathrm{b}}(\mathbf{proj}_{\mathrm{fg}}\text{-}A), \mathbf{D}^{\mathrm{b}}(\mathbf{mod}_{\mathrm{fd}}\text{-}A))$ in $\mathbf{D}^{\mathrm{b}}(\mathbf{mod}_{\mathrm{fd}}\text{-}A)$, where A is a finite-dimensional algebra, the theorem is due to Koenig and Yang [KY14]. It seems the version for non-positive dg algebras with finite-dimensional total cohomology stated in [BY14] was originally a folklore result. Recently [SY19] and [Zha23] provided new proofs using Koszul duality for A_∞-algebras and dg algebras, respectively. For homologically smooth non-positive dg algebras the theorem is stated in [KN11].

The following theorem is a slight refinement of the results in [KY14, §5] and [Fus24, Thm. 4.8], making the bijections explicit.

Theorem 2.4.4 (WT correspondence)**.** *Let $(\mathscr{C}, \mathscr{D})$ be a WT pair in an algebraic triangulated category $\mathscr{T}$. Taking derived projective covers gives one of the eight bijections fitting in the following commutative diagram from [KY14] and [Fus24]:*

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\boxed{\begin{array}{c}\text{bounded weight structures } w \text{ on } \mathscr{C}\\ \text{with Krull–Schmidt coheart}\end{array}}
\arrow[r, bend right=15, "\begin{array}{c}\mathscr{D}^{t\le 0} = (\mathscr{C}_{w>0})^{\perp}\\ \mathscr{D}^{t\ge 0} = (\mathscr{C}_{w<0})^{\perp}\end{array}"']
\arrow[d, bend left=15, "\text{indecomposables in } \heartsuit_w"]
\arrow[d, phantom, "\text{Proposition 2.2.19}"]
&
\boxed{\begin{array}{c}\text{bounded } \mathbf{t}\text{-structures } t \text{ on } \mathscr{D}\\ \text{with finite-length heart}\end{array}}
\arrow[l, bend right=15, "\begin{array}{c}\mathscr{C}_{w\le 0} = {}^{\perp}(\mathscr{D}^{t>0})\\ \mathscr{C}_{w\ge 0} = {}^{\perp}(\mathscr{D}^{t<0})\end{array}"']
\arrow[d, bend left=15, "\text{simples in } \heartsuit_t"]
\arrow[d, phantom, "\text{Proposition 2.2.7}"]
\\
\boxed{\begin{array}{c}\text{silting collections } \mathcal{P} \text{ in } \mathscr{T}\\ \text{with } \mathrm{thick}_{\mathscr{T}}(\mathcal{P}) = \mathscr{C}\end{array}}
\arrow[u, bend left=15, "\begin{array}{c}\text{Karoubi closure of}\\ \text{extension closure of}\\ \text{positive/negative shifts}\end{array}"]
\arrow[r, bend right=15, "\text{simple tops of } H^0_t(\mathcal{P})"']
\arrow[r, phantom, "\text{Theorem 2.3.16}"]
&
\boxed{\begin{array}{c}\text{simple-minded collections } \mathcal{L}\\ \text{in } \mathscr{D}\end{array}}
\arrow[l, bend right=15, "\text{derived projective covers}"']
\arrow[u, bend left=15, "\begin{array}{c}\text{extension closure of}\\ \text{positive/negative shifts}\end{array}"]
\end{tikzcd}
$$

*The bijection at the top in particular says that the correspondence between weight structures and **t**-structures is given by **w**-**t**-strict orthogonality.*

Proof. In view of the proof of [Fus24, Thm. 4.8], we essentially only have to show that the bijection between weight structures and **t**-structures is given by **w**-**t**-strict orthogonality. For convenience of the reader, we first briefly describe the other bijections.

It is well-known (cf. Proposition 2.2.7 above) that simple-minded collections in $\mathscr{D}$ are in bijection with bounded **t**-structures on $\mathscr{D}$ with finite-length heart. Similarly (cf. Proposition 2.2.19 and Remark 2.2.20 above), classical silting collections in $\mathscr{C}$ are in bijection with bounded weight structures on $\mathscr{C}$ with Krull–Schmidt coheart. By [AMY19, Prop. 5.2], classical silting collections in $\mathscr{C}$ are the same as silting collections $\mathcal{P}$ in $\mathscr{T}$ with $\mathrm{thick}_{\mathscr{T}}(\mathcal{P}) = \mathscr{C}$. Thus the vertical maps are bijections.

Let $\mathcal{P}$ be a silting collection in $\mathscr{T}$ with $\mathrm{thick}_{\mathscr{T}}(\mathcal{P}) = \mathscr{C}$, and $t = t_{\mathcal{P}}$ the **t**-structure on $\mathscr{T}$ defined by $\mathcal{P}$. By Theorem 2.3.16, $\mathcal{P}$ consists of the derived projective covers of the simple objects in $\heartsuit_t$, which form a simple-minded collection $\mathcal{L}$ in $\mathscr{D}$ by [AMY19, Prop. 5.2 and Prop. 4.6]. From the definition of derived projective covers it is clear that $\mathcal{L}$ consists of the simple tops of $H^0_t(\mathcal{P})$. Sending $\mathcal{P}$ to $\mathcal{L}$ defines a bijection by [Fus24, Prop. 4.6].

It remains to show that the induced bijection between weight structures and **t**-structures is given by **w**-**t**-strict orthogonality. Let w be a bounded weight structure on $\mathscr{C}$. By the above, w is a bounded weight structure obtained from a silting collection $\mathcal{P}$, and the corresponding **t**-structure on $\mathscr{D}$ is the restriction of the associated silting **t**-structure $t = t_{\mathcal{P}}$ on $\mathscr{T}$. Hence **t**-strict left orthogonality is obvious.

For **w**-strictness, let $X \in \mathscr{C} \cap {}^{\perp}(\mathscr{D}^{t>0})$ and let $t_{\le 0}X \to X \to t_{>0}X \to t_{\le 0}X[1]$ be the **t**-decomposition of X with respect to the **t**-structure t on $\mathscr{T}$. As t is bounded above on $\mathscr{T}$, we have $t_{>0}X \in \mathrm{tria}_{\mathscr{T}}(\heartsuit_t) = \mathscr{D}$, so $t_{>0}X \in \mathscr{D}^{t>0}$ and thus $\mathrm{Hom}_{\mathscr{T}}(X, t_{>0}X) = 0$ by assumption. Hence the **t**-decomposition triangle splits and gives $t_{\leq 0}X \cong t_{>0}X[-1] \oplus X$, which implies $X \cong t_{\leq 0}X \in \mathscr{T}^{t\leq 0} \cap \mathscr{C} = \mathscr{T}_{w\leq 0} \cap \mathscr{C} = \mathscr{C}_{w\leq 0}$.

Let $X \in \mathscr{C} \cap {}^{\perp}(\mathscr{D}^{t<0})$. As w is bounded on $\mathscr{C}$, there is $n \in \mathbb{Z}$ with $X \in \mathscr{C}_{w>n}$. Let $Y \in \mathscr{T}^{t<0}$ and consider the **t**-decomposition $t_{\leq n}Y \to Y \to t_{>n}Y \to t_{\leq n}Y[1]$. Since w is left adjacent to t on w, we have $t_{\leq n}Y \in \mathscr{T}_{w\leq n}$, and since t is bounded above on $\mathscr{T}$, we have $t_{>n}Y \in \mathscr{D}^{t<0}$. Since $X \in \mathscr{C}_{w>n}$ and $X \in {}^{\perp}(\mathscr{D}^{t<0})$, applying $\mathrm{Hom}_{\mathscr{T}}(X, -)$ to the **t**-decomposition of Y shows that $\mathrm{Hom}_{\mathscr{T}}(X, Y) = 0$. Hence from Lemma 2.2.26 we get $X \in {}^{\perp}(\mathscr{T}^{t<0}) \cap \mathscr{C} = \mathscr{T}_{w\geq 0} \cap \mathscr{C} = \mathscr{C}_{w\geq 0}$. $\square$

2.5 Naturality of orthogonality

In this section we show that **w**-**t**-strict orthogonality is natural with respect to weight exact functors and **t**-exact functors. This is a slight generalization of the results from [Bon10b, Prop. 4.4.5] for adjacent **t**-structures. The main result in this section (Theorem 2.5.1) is proved in essentially the same way except for the more technical notation required to set up the statements, which simplifies a lot in most interesting cases.

Theorem 2.5.1. *Let $\mathscr{C}$, $\mathscr{C}'$, $\mathscr{D}$, $\mathscr{D}'$ be triangulated categories, w, w' weight structures on $\mathscr{C}$ and $\mathscr{C}'$, and t, t' **t**-structures on $\mathscr{D}$ and $\mathscr{D}'$, respectively. Let $\Phi\colon \mathscr{C} \times \mathscr{D} \to \mathscr{A}$ and $\Phi'\colon \mathscr{C}' \times \mathscr{D}' \to \mathscr{A}$ be dualities and suppose that w (resp. w') is **w**-**t**-strictly left orthogonal to t (resp. t') with respect to Φ (resp. Φ'). Let $F\colon \mathscr{C} \to \mathscr{C}'$ and $G\colon \mathscr{D}' \to \mathscr{D}$ be "Φ-Φ'-adjoint" in the sense that $\Phi'(F(X), Y) \cong \Phi(X, G(Y))$ naturally for $X \in \mathscr{C}$ and $Y \in \mathscr{D}'$. Then*

1) $F(\mathscr{C}_{w>0}) \subseteq \mathscr{C}'_{w>0}$ if and only if $G(\mathscr{D}'^{t'\leq 0}) \subseteq \mathscr{D}^{t\leq 0}$.

2) $F(\mathscr{C}_{w<0}) \subseteq \mathscr{C}'_{w<0}$ if and only if $G(\mathscr{D}'^{t'\geq 0}) \subseteq \mathscr{D}^{t\geq 0}$.

*In particular, F is weight exact if and only if G is **t**-exact.*

Proof. We only show the first part as the argument for the second claim is entirely analogous.

Suppose $F(\mathscr{C}_{w>0}) \subseteq \mathscr{C}'_{w'>0}$ and let $Y \in \mathscr{D}'^{t'\leq 0}$. By assumption we have $\mathscr{D}^{t\leq 0} = (\mathscr{C}_{w>0})^{\perp_\Phi}$, and thus

$$G(Y) \in \mathscr{D}^{t\leq 0} \iff \mathscr{C}_{w>0} \perp_\Phi G(Y) \iff F(\mathscr{C}_{w>0}) \perp_{\Phi'} Y.$$

But this condition is satisfied since by assumption $F(\mathscr{C}_{w>0}) \subseteq \mathscr{C}'_{w'>0}$, and $Y \in \mathscr{D}'^{t'\leq 0} = (\mathscr{C}'_{w'>0})^{\perp_{\Phi'}}$.

For the converse suppose $G(\mathscr{D}'^{t'\leq 0}) \subseteq \mathscr{D}^{t\leq 0}$ and let $X \in \mathscr{C}_{w>0}$. By assumption we have $\mathscr{C}'_{w'>0} = {}^{\perp_{\Phi'}}(\mathscr{D}'^{t'\leq 0})$, and thus

$$F(X) \in \mathscr{C}'_{w'>0} \iff F(X) \perp_{\Phi'} \mathscr{D}'^{t'\leq 0} \iff X \perp_\Phi G(\mathscr{D}'^{t'\leq 0}).$$

But this is satisfied since by assumption $G(\mathscr{D}'^{t'\leq 0}) \subseteq \mathscr{D}^{t\leq 0}$ and $X \in \mathscr{C}_{w>0} = {}^{\perp_\Phi}(\mathscr{D}^{t\leq 0})$. $\square$

Most notably it follows that the bijection between bounded **t**-structures with finite-length heart and bounded weight structures provided by the WT correspondence (see Theorem 2.4.4 above) is natural:

Corollary 2.5.2. *Let $(\mathscr{C}, \mathscr{D})$ be a WT pair in $\mathscr{T}$ and $(\mathscr{C}', \mathscr{D}')$ a WT pair in $\mathscr{T}'$. Let w, w' be bounded weight structures on $\mathscr{C}$ resp. $\mathscr{C}'$, and t, t' the corresponding bounded **t**-structures on $\mathscr{D}$ resp. $\mathscr{D}'$ under the WT correspondence. Suppose that $F\colon \mathscr{C} \to \mathscr{C}'$ is left pseudo-adjoint to $G\colon \mathscr{D}' \to \mathscr{D}$ in the sense that $\mathrm{Hom}_{\mathscr{T}'}(F(-), -) \cong \mathrm{Hom}_{\mathscr{T}}(-, G(-))$.*

*Then F is weight exact with respect to w and w' if and only if G is **t**-exact with respect to t and t'.*

Proof. By the WT correspondence, w and t (respectively w' and t') are **w-t**-strictly left orthogonal to each other, and so the claim follows from Theorem 2.5.1. □

For the standard WT pair $(\mathbf{K}^{\mathrm{b}}(\mathbf{Proj}(\mathscr{A})), \mathbf{D}^{\mathrm{b}}(\mathscr{A}))$ in $\mathbf{D}^{-}(\mathscr{A})$, where $\mathscr{A}$ is a Hom-finite finite-length abelian categories with enough projectives and finitely many simples, the naturality of the WT correspondence can also be formulated as follows. Following [Che21] we write $\mathbb{K}^b$ for the strict 2-category whose objects are the Hom-finite finite-length abelian categories with enough projectives and finitely many simples, with 1-morphisms given by the functors $\mathbf{K}^{\mathrm{b}}(\mathbf{Proj}(\mathscr{A})) \to \mathbf{K}^{\mathrm{b}}(\mathbf{Proj}(\mathscr{B}))$ and 2-morphisms the natural transformations between these. The 2-category $\mathbb{D}^b$ is defined similarly, using functors $\mathbf{D}^{\mathrm{b}}(\mathscr{A}) \to \mathbf{D}^{\mathrm{b}}(\mathscr{B})$ instead. Let $(\mathbb{D}^b)^{\mathrm{coop}}$ denote the bidual of $\mathbb{D}^b$, i.e. the 2-category obtained by reversing all 1-morphisms and 2-morphisms. By [Che21, Thm. 3.2] there is an equivalence $\mathbb{K}^b \to (\mathbb{D}^b)^{\mathrm{coop}}$ that is the identity on objects and sends a 1-morphism (i.e. a functor) $F\colon \mathbf{K}^{\mathrm{b}}(\mathbf{Proj}(\mathscr{B})) \to \mathbf{K}^{\mathrm{b}}(\mathbf{Proj}(\mathscr{A}))$ to its *right pseudo-adjoint* $F^{\vee}\colon \mathbf{D}^{\mathrm{b}}(\mathscr{A}) \to \mathbf{D}^{\mathrm{b}}(\mathscr{B})$, which is defined by natural isomorphisms

$$\mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathscr{A})}(F(X), Y) \cong \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathscr{B})}(X, F^{\vee}(Y))$$

for $X \in \mathbf{K}^{\mathrm{b}}(\mathbf{Proj}(\mathscr{B}))$ and $Y \in \mathbf{D}^{\mathrm{b}}(\mathscr{A})$.

Corollary 2.5.3. *Let $\mathscr{A}$, $\mathscr{B}$ be Hom-finite finite-length abelian categories with enough projectives and finitely many simples. Let t, t' be bounded **t**-structures on $\mathbf{D}^{\mathrm{b}}(\mathscr{A})$ and $\mathbf{D}^{\mathrm{b}}(\mathscr{B})$, respectively, and w, w' the bounded weight structures on $\mathbf{K}^{\mathrm{b}}(\mathbf{Proj}(\mathscr{A}))$ and $\mathbf{K}^{\mathrm{b}}(\mathbf{Proj}(\mathscr{B}))$ corresponding to t and t' under the bijections from Theorem 2.4.4. Assume that under the equivalence from [Che21, Thm. 3.2], $G\colon \mathbf{D}^{\mathrm{b}}(\mathscr{A}) \to \mathbf{D}^{\mathrm{b}}(\mathscr{B})$ corresponds to $F\colon \mathbf{K}^{\mathrm{b}}(\mathbf{Proj}(\mathscr{B})) \to \mathbf{K}^{\mathrm{b}}(\mathbf{Proj}(\mathscr{A}))$. Then G is t-exact if and only if F is weight exact.*

Proof. By Theorem 2.4.4 the weight structure w (resp. w') is **w-t**-strictly left orthogonal to the **t**-structure t (resp. t'). Thus the result follows from Theorem 2.5.1, since the construction of G as a right pseudo-adjoint to F is precisely the required adjunction property. □

Remark 2.5.4. Using [Che21, Prop. 3.4], we also obtain the following consequence of Corollary 2.5.3:

1) Suppose $F\colon \mathbf{K}^{\mathrm{b}}(\mathbf{Proj}(\mathscr{B})) \to \mathbf{K}^{\mathrm{b}}(\mathbf{Proj}(\mathscr{A}))$ is weight exact and admits a right adjoint G. Then G extends to $\tilde{G}\colon \mathbf{D}^{\mathrm{b}}(\mathscr{A}) \to \mathbf{D}^{\mathrm{b}}(\mathscr{B})$, and $\tilde{G}$ is **t**-exact.
2) If $\tilde{G}\colon \mathbf{D}^{\mathrm{b}}(\mathscr{A}) \to \mathbf{D}^{\mathrm{b}}(\mathscr{B})$ is **t**-exact and restricts to $G\colon \mathbf{K}^{\mathrm{b}}(\mathbf{Proj}(\mathscr{A})) \to \mathbf{K}^{\mathrm{b}}(\mathbf{Proj}(\mathscr{B}))$, then G admits a left adjoint F, which is weight exact.

In this setup, there is an alternative proof of 2): By Lemma 2.2.24 we have $\mathscr{C}_{w<0} = \mathscr{D}^{t<0} \cap \mathbf{K}^{\mathrm{b}}(\mathbf{Proj}(\mathscr{A}))$, and analogously for t' and w'. Now let $X \in \mathscr{C}'_{w'\geq 0}$ and $Y \in \mathscr{C}_{w<0} = \mathscr{D}^{t<0} \cap \mathbf{K}^{\mathrm{b}}(\mathbf{Proj}(\mathscr{A}))$. Then

$$G(Y) = \tilde{G}(Y) \in \mathscr{D}'^{t'<0} \cap \mathbf{K}^{\mathrm{b}}(\mathbf{Proj}(\mathscr{B})) = \mathscr{C}'_{w'<0}$$

by **t**-exactness of G, and from the adjunction and $\mathscr{C}'_{w'\geq 0} \perp \mathscr{C}'_{w'<0}$ we get

$$\mathrm{Hom}_{\mathbf{K}^{\mathrm{b}}(\mathbf{Proj}(\mathscr{A}))}(F(X), Y) = \mathrm{Hom}_{\mathbf{K}^{\mathrm{b}}(\mathbf{Proj}(\mathscr{B}))}(X, G(Y)) = 0,$$

so $F(X) \in {}^{\perp}(\mathscr{C}_{w<0}) = \mathscr{C}_{w\geq 0}$.

Similarly for $X \in \mathscr{C}'_{w'\leq 0} = \mathscr{D}'^{t'\leq 0} \cap \mathbf{K}^{\mathrm{b}}(\mathbf{Proj}(\mathscr{B}))$ and $Y \in \mathscr{D}^{t>0}$ we have

$$\mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathscr{A})}(F(X), Y) = \mathrm{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathscr{B})}(X, \tilde{G}(Y)) = 0,$$

and thus $F(X) \in \mathscr{C}_{w\leq 0} = \mathscr{D}^{t\leq 0} \cap \mathbf{K}^{\mathrm{b}}(\mathbf{Proj}(\mathscr{A}))$.

Chapter 3

Koszul duality of simple-minded and silting collections

In this chapter we prove a theorem about Koszul duality between simple-minded and silting collections. We begin by recalling the classical Koszul duality from [BGS96; MOS09] and the Koszul duality for dg categories and dg algebras from [Kel94]. This requires us to work in dg-enhanced triangulated categories. As we want to apply this to silting collections and simple-minded collections, we are particularly interested in the dg Koszul duals of non-positive and positive dg algebras.

The Koszul duality of simple-minded and silting collections is established by the main result of this chapter (Theorem 3.4.2). This is closely related to the construction of silting collections corresponding to simple-minded collections in the proof of the WT correspondence, see [Zha23] and also [Fus24]. We also provide three small examples of the Koszul duality from Theorem 3.4.2, namely for simple-minded and silting collections in the derived category of the A_2-quiver.
The chapter is based on [Bon25, §1 and §4].

[Bon25] L. Bonfert. "Derived projective covers and Koszul duality of simple-minded and silting collections". To appear in *Appl. Categ. Struct.* (2025). arXiv:`2309.00554v3 [math.RT]`.

3.1 Motivation and overview of results

The classical Koszul duality from [BGS96; MOS09] provides an equivalence of derived categories of graded modules that interchanges the simple and indecomposable projective objects. By definition, simple-minded collections are an analog of the set of simple objects, and by Theorem 2.3.16 a silting collection in the sense of Definition 2.2.9 can be seen as an analog of the set of indecomposable projective objects. Therefore, we want to relate simple-minded collections and silting collections via Koszul duality.

The required notion is the dg Koszul duality from [Kel94], which defines the dg Koszul dual $\mathscr{A}^{!,\mathrm{dg}}$ of an augmented dg category $\mathscr{A}$. This construction can in particular be applied to non-positive and positive dg algebras. In our situation, such dg algebras arise as the endomorphism algebras of silting collections respectively silting collections in a dg-enhanced triangulated category. Following a suggestion by Bernhard Keller, we obtain:

Theorem 3.1.1 (Theorem 3.4.2)**.** *Let $\mathscr{T} = H^0(\widetilde{\mathscr{T}})$ be a compactly generated dg-enhanced triangulated category. For a compact silting collection $\mathcal{P}$ in $\mathscr{T}$ such that $\operatorname{End}_{\mathscr{T}}(\bigoplus_{P\in\mathcal{P}} P)$ is finite-dimensional, let $\mathcal{L}$ be the set of simple objects in the heart of the silting $\boldsymbol{t}$-structure associated with $\mathcal{P}$. Note that $\mathcal{L}$ is a simple-minded collection in $\mathscr{D} = \operatorname{tria}_{\mathscr{T}}(\mathcal{L})$.*

1) The dg algebra $\operatorname{End}_{\widetilde{\mathscr{T}}}(\bigoplus_{L\in\mathcal{L}} L)$ is the dg Koszul dual of $\operatorname{End}_{\widetilde{\mathscr{T}}}(\bigoplus_{P\in\mathcal{P}} P)$.

2) *If* $H^n(\mathrm{End}_{\widetilde{\mathscr{T}}}(\bigoplus_{P\in\mathcal{P}} P))$ *is finite-dimensional for all* $n \in \mathbb{Z}$*, then* $\mathrm{End}_{\widetilde{\mathscr{T}}}(\bigoplus_{P\in\mathcal{P}} P)$ *is the dg Koszul dual of* $\mathrm{End}_{\widetilde{\mathscr{T}}}(\bigoplus_{L\in\mathcal{L}} L)$.

This result is a first step towards the tentative Koszul duality of weight structures and **t**-structures, which is supposed to formalize the apparent duality of their definitions. The theorem is inspired by [BY14], see also the revised version [BY23]. In the case of finite-dimensional algebras or non-positive dg algebras with finite-dimensional total cohomology, the second part of Theorem 3.1.1 can also be shown by a construction from [Zha23], which was used there to construct a silting collection corresponding to a simple-minded collection.

3.2 Classical Koszul duality

Before we come to dg Koszul duality we briefly recall the classical Koszul duality from [BGS96; MOS09]. By definition, a *Koszul algebra* is a positively graded algebra $A = \bigoplus_{n\geq 0} A_n$ such that

- $A_n = 0$ for $n < 0$, and each graded piece A_n is a finitely generated A^0-module,
- A_0 is a semisimple algebra,
- A_0 has a *linear projective resolution*, i.e. there is a projective resolution $P \to A_0$ of A_0 as graded A-module such that each P^{-i} is generated in degree i.

The *Koszul dual* of A is $A^! = \mathrm{Ext}^*_A(A_0, A_0)$.

Example 3.2.1.

1) If $\Bbbk$ has characteristic 0, then $\Bbbk[x]$ with $\deg(x) = 1$ and $\Bbbk[y]/(y^2)$ with $\deg(y) = 1$ are Koszul algebras. Moreover, we have $(\Bbbk[x])^! \cong \Bbbk[y]/(y^2)$ and $(\Bbbk[y]/(y^2))^! \cong \Bbbk[x]$.
2) Assume that $\Bbbk$ is algebraically closed of characteristic 0 and let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over $\Bbbk$. Then the finite-dimensional algebras describing the blocks of $\mathcal{O}(\mathfrak{g})$ are Koszul by [BGS96, Thm. 1.1.3].

The main result about Koszul algebras is the following Koszul duality theorem from [BGS96, Thm. 2.10.2 and Thm. 2.12.1], see also [MOS09, Thm. 30].

Theorem 3.2.2 (Koszul duality)**.** *Let A be a Koszul algebra.*

1) $A^!$ *is a Koszul algebra.*
2) $(A^!)^! \cong A$.
3) *Let* $\mathbf{D}^{\downarrow}(A) \subseteq \mathbf{D}(\mathbf{Mod}^{\mathbb{Z}}\text{-}A)$ *be the full subcategory of bounded below cochain complexes* X *such that* $X^i_j = 0$ *for* $i+j \gg 0$*, and* $\mathbf{D}^{\uparrow}(A^!) \subseteq \mathbf{D}(\mathbf{Mod}^{\mathbb{Z}}\text{-}A^!)$ *the full subcategory of bounded above cochain complexes* X *such that* $X^i_j = 0$ *for* $i + j \ll 0$*. Then there is an equivalence of triangulated categories* $\mathrm{RHom}_A(A_0, -)\colon \mathbf{D}^{\downarrow}(A) \cong \mathbf{D}^{\uparrow}(A^!)$.

Note that by construction the Koszul duality functor $\mathrm{RHom}_A(A_0, -)$ maps the simple A-modules to the indecomposable projective $A^!$-modules.

An equivalent formulation of Koszul duality is as follows. A Koszul algebra can be seen as an augmented algebra over A_0 via the quotient map $A \twoheadrightarrow A_0 = A/(A_{>0})$, and every simple A_0-module L lifts to an A-module $\hat{L}$ via this augmentation. It is easy to see that in fact every graded simple A-module is of the form $\hat{L}$ for some simple A_0-module L.

With this point of view, the Koszul dual of A can also be defined as $A^! = \bigoplus_L \mathrm{Ext}^*_A(\hat{L}, \hat{L})$, where the direct sum runs over all simple A_0-modules up to isomorphism. This is Morita-equivalent to the definition $A^! = \mathrm{Ext}^*_A(A_0, A_0)$. Using this definition, the Koszul duality functor is $\bigoplus_L \mathrm{RHom}_A(\hat{L}, -)\colon \mathbf{D}^{\downarrow}(A) \to \mathbf{D}^{\uparrow}(A^!)$.

3.3 dg Koszul duality

To be able to define the dg Koszul dual of dg algebras and dg categories we have to briefly recall dg-enhanced triangulated categories. By definition, a *dg enhancement* (originally called *enhancement* in [BK91]) of a triangulated category $\mathscr{T}$ is a *pretriangulated dg category* $\widetilde{\mathscr{T}}$ in the sense of [BK91, §3, Def. 1] together with an equivalence $\mathscr{T} \cong H^0(\widetilde{\mathscr{T}})$. An object $X \in \mathscr{T}$ yields a dg functor $\mathrm{Hom}_{\widetilde{\mathscr{T}}}(X,-)\colon \widetilde{\mathscr{T}} \to \mathbf{dgMod}\text{-}\mathrm{End}_{\widetilde{\mathscr{T}}}(X)$, which induces a triangulated functor $\mathrm{Hom}_{\widetilde{\mathscr{T}}}(X,-)\colon \mathscr{T} \to \mathbf{D}(\mathrm{End}_{\widetilde{\mathscr{T}}}(X))$.

Note that by [Kra07, §7.5] and the first part of the proof of [Kel94, Thm. 4.3], dg-enhanced triangulated categories are precisely the stable categories of Frobenius categories considered in [Kel94].

Example 3.3.1. A dg enhancement $\widetilde{\mathbf{D}}(\mathscr{A})$ of the derived category $\mathbf{D}(\mathscr{A})$ of a dg category $\mathscr{A}$ is given by the dg category of K-projective dg $\mathscr{A}$-modules, see [Kel94, §4.1] for details. Here a dg $\mathscr{A}$-module P is *K-projective* if $H^0(\mathrm{Hom}_{\mathbf{dgMod}\text{-}\mathscr{A}}(P,N)) = 0$ for all acyclic dg $\mathscr{A}$-modules N. A dg $\mathscr{A}$-module M can be viewed as an object of $\widetilde{\mathbf{D}}(\mathscr{A})$ by replacing it by a K-projective resolution, i.e. a K-projective dg module pM that is quasi-isomorphic to M. Note that by [Kel94, §3.1, p. 70] K-projectivity is equivalent to the *property (P)* considered in [Kel94], and in particular K-projective resolutions are precisely the P-resolutions defined in [Kel94, §3.1].

We abbreviate $\mathrm{Hom}_{\widetilde{\mathbf{D}}(\mathscr{A})}(-,-) = \mathrm{RHom}_{\mathscr{A}}(-,-)$, and by slight abuse of notation we also write $\mathrm{REnd}_{\mathscr{A}}(X) = \mathrm{RHom}_{\mathscr{A}}(X,X)$ for $X \in \mathbf{D}(\mathscr{A})$. Dually one can also use K-injective resolutions instead.

Recall that a non-positive dg algebra is a dg algebra A such that $H^n(A) = 0$ for $n > 0$, and a positive dg algebra is a dg algebra A such that $H^n(A) = 0$ for $n < 0$ and $H^0(A)$ is semisimple. In these cases, the dg Koszul dual is defined as follows, cf. [KN13, §1 and Not. 5.1], [Zha23], and also [Fus25, Def. 4.1]:

Definition 3.3.2. Let A be a non-positive or positive dg algebra such that $H^0(A)$ is finite-dimensional. The *dg Koszul dual* of A is $A^{!,\mathrm{dg}} = \mathrm{REnd}_A(\hat{L})$, where $\hat{L}$ is defined as follows:

1) If A is non-positive, let $\hat{L}$ be the direct sum of the simple $H^0(A)$-modules, viewed as dg A-modules concentrated in degree 0 via the quasi-isomorphism $t_{\leq 0}A \to A$ and the quotient map $t_{\leq 0}A \twoheadrightarrow H^0(A)$.
2) If A is positive, let $\hat{L}$ be the unique dg module such that $H^0(\hat{L})$ is the direct sum of the simple $H^0(A)$-modules and $H^n(\hat{L}) = 0$ for $n \neq 0$ (this exists by [KN13, Cor. 4.7]).

If A is a non-positive dg algebra, then $A^{!,\mathrm{dg}}$ is a positive dg algebra as a consequence of [BY14, Thm. A.1]. Conversely, if A is a positive dg algebra then $A^{!,\mathrm{dg}}$ is non-positive by [KN13, Lemma 5.2]. Note that the dg Koszul dual is well-defined only up to quasi-equivalence. For explicit computations of dg Koszul duals see Section 3.5 below.

By [Fus25, Thm. 4.17] the double dg Koszul dual $(A^{!,\mathrm{dg}})^{!,\mathrm{dg}}$ is quasi-isomorphic to A if A is either locally finite-dimensional non-positive, or locally finite-dimensional pvd-finite positive in the sense of [Fus25, Def. 3.23]. Furthermore, dg Koszul duality provides equivalences between certain subcategories of $\mathbf{D}(A)$ and $\mathbf{D}(A^{!,\mathrm{dg}})$, see [Fus25, Thm. 4.4].

The definition of the dg Koszul dual of a non-positive or positive dg algebra in Definition 3.3.2 is based on the following general definition of Koszul duality for augmented dg categories from [Kel94, §10.2].

Definition 3.3.3.

1) An *augmented dg category* is a dg category $\mathscr{A}$ with pairwise non-isomorphic objects such that for every object $A \in \mathscr{A}$ there is a dg $\mathscr{A}$-module $\overline{A}$ (called *augmenting dg module*) with

$$H^k(\overline{A})(B) \cong \begin{cases} \Bbbk & \text{if } k = 0, A = B, \\ 0 & \text{otherwise.} \end{cases}$$

2) The *dg Koszul dual* of an augmented dg category $\mathscr{A}$ is the dg category $\mathscr{A}^{!,\mathrm{dg}}$ with objects $\{A^! \mid A \in \mathscr{A}\}$ and morphisms

$$\mathscr{A}^{!,\mathrm{dg}}(A^!, A'^!) = \mathrm{RHom}_{\mathscr{A}}(\overline{A}, \overline{A'}) = \mathrm{Hom}_{\mathbf{dgMod}\text{-}\mathscr{A}}(p\overline{A}, p\overline{A'}),$$

where $p\overline{A}$ and $p\overline{A'}$ are K-projective resolutions of $\overline{A}$ and $\overline{A'}$.

Remark 3.3.4.

1) $\mathscr{A}^{!,\mathrm{dg}}$ is well-defined only up to quasi-equivalence, cf. [Kel94, §10.2].
2) In [Kel94] the Koszul dual is defined more abstractly as a lift of the augmenting modules $\{\overline{A} \mid A \in \mathscr{A}\}$. With the notation from Definition 3.3.3 the dg category $\mathscr{A}^{!,\mathrm{dg}}$ and the $\mathscr{A}^{!,\mathrm{dg}}$-$\mathscr{A}$-bimodule $\bigoplus_{B\in\mathscr{A}} p\overline{B}$ provide a lift, and hence the "abstract" definition agrees with the "concrete" definition we use here.
3) By [Kel94, §10.2], $\mathscr{A}^{!,\mathrm{dg}}$ becomes an augmented dg category with augmenting dg modules $\overline{A^!} = \mathrm{RHom}_{\mathscr{A}}(\bigoplus_{B\in\mathscr{A}} p\overline{B}, D\mathscr{A}(A,-))$. Here $D = \mathrm{Hom}_{\mathbf{dgMod}\text{-}\Bbbk}(-,\Bbbk)$ denotes the $\Bbbk$-linear duality functor.
4) If the augmenting dg modules $\overline{A}$ are compact and generate $\mathbf{D}(\mathscr{A})$ as triangulated subcategory closed under arbitrary coproducts (or, equivalently, if $\mathrm{thick}\{\overline{A} \mid A \in \mathscr{A}\} = \mathbf{D}^{\mathrm{c}}(\mathscr{A})$), then by [Kel94, Lemma 10.5 "The finite case"] the $\mathscr{A}^{!,\mathrm{dg}}$-$\mathscr{A}$-dg bimodule $\bigoplus_{B\in\mathscr{A}} p\overline{B}$ provides an equivalence of categories $\mathbf{D}(\mathscr{A}^{!,\mathrm{dg}}) \to \mathbf{D}(\mathscr{A})$, sending $\mathscr{A}^{!,\mathrm{dg}}(-,A^!)$ to $\overline{A}$ for all $A \in \mathscr{A}$.

Under mild assumptions, Definition 3.3.2 can be recovered from Definition 3.3.3 as follows.

Example 3.3.5. Let A be a dg algebra such that $H^0(A)$ is finite-dimensional and all simple $H^0(A)$-modules are 1-dimensional. Take a complete set of primitive orthogonal idempotents of $H^0(A)$ and suppose they lift to idempotents $\{e_i \mid i \in I\}$ in A with $\sum_{i\in I} e_i = 1$. Consider $A = \bigoplus_{i,j\in I} e_iAe_j$ as a dg category with objects I and morphisms $A(j,i) = e_iAe_j$.

1) If A is a non-positive dg algebra, we can consider the simple $H^0(A)$-modules as dg A-modules concentrated in degree 0 as in Definition 3.3.2. These "simple dg A-modules" make A an augmented dg category. It follows that the dg Koszul dual of A in the sense of Definition 3.3.3 (viewed as a dg algebra by taking the direct sum over the finitely many objects) is precisely the dg Koszul dual defined in Definition 3.3.2.
2) If A is a positive dg algebra, the unique dg A-modules L_i corresponding to the simple $H^0(A)$-modules $e_iH^0(A)$ constructed in [KN13, Cor. 4.7] make A an augmented dg category. Again the dg Koszul dual of A in the sense of Definition 3.3.3 is precisely the dg Koszul dual defined in Definition 3.3.2.

The easiest examples of positive dg algebras are Koszul algebras, viewed as positive dg algebras with the same grading and trivial differential. In this case, the definition of the dg Koszul dual recovers the classical Koszul dual by the following result, which is essentially contained in [Sch11, Thm. 39].

Proposition 3.3.6. *Let A be a Koszul algebra of finite global dimension and consider A as a positive dg algebra with the same grading and trivial differential. Then*

$$H^n(A^{!,\mathrm{dg}}) \cong \begin{cases} A^! & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Note that since A has trivial differential, the simple A-modules become dg A-modules when equipped with the trivial differential, and thus these the augmenting dg A-modules constructed in [KN13, Cor. 4.7] (cf. Definition 3.3.2 above). Let L be the direct sum of the simple A-modules and P a linear projective resolution of L. Consider the complex P of graded A-modules as a dg A-module $\mathrm{Tot}(P)$, with $\mathrm{Tot}(P)^n = \bigoplus_{i+j=n} P^i_j$ and the differential consisting of $(-1)^i d\colon P^i_j \to P^{i+1}_j$. Note that $\mathrm{Tot}(P)$ is K-projective since it can be obtained as an iterated cone of the morphisms of dg A-modules $(-1)^i d\colon P^i \to P^{i+1}$ (where P^i and P^{i+1} are equipped with the trivial differential). The quasi-isomorphism $P \to L$ provides a quasi-isomorphism of dg A-modules $\mathrm{Tot}(P) \to L$, and so this is a K-projective resolution of L. Hence by definition $A^{!,\mathrm{dg}} = \mathrm{Hom}_{\mathbf{dgMod}\text{-}A}(\mathrm{Tot}(P), \mathrm{Tot}(P))$.

Since P is a linear projective resolution, each P^i is generated in degree $-i$, and hence the dg A-module $\mathrm{Tot}(P)$ is generated in degree 0, and moreover $\mathrm{Tot}(P)^n = 0$ for $n < 0$. This immediately implies that $(A^{!,\mathrm{dg}})^n = 0$ for $n < 0$. Moreover, by [KN13, Lemma 5.2] $A^{!,\mathrm{dg}}$ is a non-positive dg algebra, i.e. we have $H^n(A^{!,\mathrm{dg}}) = 0$ for $n > 0$. Finally, by [Sch11, Thm. 39] we get $H^0(A^{!,\mathrm{dg}}) \cong A^!$. □

3.4 Koszul duality of simple-minded and silting collections

The following lemma provides a convenient description of certain subcategories of a compactly generated dg-enhanced triangulated category with a compact silting collection. This is a slight generalization of [KY18, Lemma 3.1], although its proof uses essentially the same arguments.

Lemma 3.4.1. *Let $\mathscr{T} = H^0(\widetilde{\mathscr{T}})$ be a compactly generated dg-enhanced triangulated category. For a compact silting collection $\mathcal{P}$ in $\mathscr{T}$ such that $\mathrm{End}_{\mathscr{T}}(\bigoplus_{P\in\mathcal{P}} P)$ is finite-dimensional, let t be its associated silting $\mathbf{t}$-structure and $\mathscr{D} \subseteq \mathscr{T}$ be the triangulated subcategory generated by the simple objects in $\heartsuit_t$. Let $E = \mathrm{End}_{\widetilde{\mathscr{T}}}(\bigoplus_{P\in\mathcal{P}} P)$.*

1) *There is an equivalence $\mathbf{D}(E) \to \mathscr{T}$ that takes the simple E-modules to the simple objects in $\heartsuit_t$. Moreover it identifies $\mathbf{perf}(E)$ with $\mathrm{thick}_{\mathscr{T}}(\mathcal{P})$, and $\mathbf{D}_{\mathrm{fd}}(E)$ with $\mathscr{D}$.*
2) *If $H^*(E)$ is finite-dimensional, then $\mathrm{thick}_{\mathscr{T}}(\mathcal{P}) = \mathscr{T}^{\mathrm{c}} \subseteq \mathscr{D}$.*

Proof.

1) As $\mathcal{P}$ weakly generates $\mathscr{T}$ by Lemma 2.2.13, $\mathrm{Hom}_{\widetilde{\mathscr{T}}}(\bigoplus_{P\in\mathcal{P}} P, -)\colon \mathscr{T} \to \mathbf{D}(E)$ is an equivalence by (the proof of) [Kel94, Thm. 4.3]. Its inverse takes E to $\bigoplus_{P\in\mathcal{P}} P$, and hence identifies $t = (\mathcal{P}^{\perp_{>0}}, \mathcal{P}^{\perp_{<0}})$ with the standard $\mathbf{t}$-structure on $\mathbf{D}(E)$, as this is the silting $\mathbf{t}$-structure associated with the silting object E in $\mathbf{D}(E)$. In particular it also identifies the simple objects in the hearts. The rest is clear since (by definition) $\mathscr{D}$ is the triangulated subcategory generated by the simple objects of $\heartsuit_t$, while on the other side $\mathbf{D}_{\mathrm{fd}}(E)$ is the triangulated subcategory generated by the augmenting dg E-modules (note that these lie in $\mathbf{D}_{\mathrm{fd}}(E)$, since $H^0(E) = \mathrm{End}(\bigoplus_{P\in\mathcal{P}} P)$ is finite-dimensional).
2) The assumption that $H^*(E)$ is finite-dimensional ensures that $\mathbf{perf}(E) \subseteq \mathbf{D}_{\mathrm{fd}}(E)$, and thus we get $\mathrm{thick}_{\mathscr{T}}(\mathcal{P}) \subseteq \mathscr{D}$ from 1). □

The following result was suggested by Bernhard Keller. It establishes a Koszul duality between simple-minded collections and silting collections.

Theorem 3.4.2. *Let $\mathscr{T} = H^0(\widetilde{\mathscr{T}})$ be a compactly generated dg-enhanced triangulated category. For a compact silting collection $\mathcal{P}$ in $\mathscr{T}$ such that $\operatorname{End}_{\mathscr{T}}(\bigoplus_{P\in\mathcal{P}} P)$ is finite-dimensional, let $\mathcal{L}$ be the set of simple objects in the heart of the silting $\mathbf{t}$-structure associated with $\mathcal{P}$.*

1) *The dg algebra $\operatorname{End}_{\widetilde{\mathscr{T}}}(\bigoplus_{L\in\mathcal{L}} L)$ is the dg Koszul dual of $\operatorname{End}_{\widetilde{\mathscr{T}}}(\bigoplus_{P\in\mathcal{P}} P)$.*
2) *If $H^n(\operatorname{End}_{\widetilde{\mathscr{T}}}(\bigoplus_{P\in\mathcal{P}} P))$ is finite-dimensional for all $n \in \mathbb{Z}$, then $\operatorname{End}_{\widetilde{\mathscr{T}}}(\bigoplus_{P\in\mathcal{P}} P)$ is the dg Koszul dual of $\operatorname{End}_{\widetilde{\mathscr{T}}}(\bigoplus_{L\in\mathcal{L}} L)$.*

Proof. For brevity we write $P = \bigoplus_{P'\in\mathcal{P}} P'$ and $L = \bigoplus_{L'\in\mathcal{L}} L'$.

1) By definition, the cohomology of $E = \operatorname{End}_{\widetilde{\mathscr{T}}}(P)$ is given by

$$H^n(\operatorname{End}_{\widetilde{\mathscr{T}}}(P)) = \operatorname{Hom}_{\mathscr{T}}(P, P[n])$$

and therefore is concentrated in non-positive degrees. By Definition 3.3.2 the Koszul dual of the non-positive dg algebra $E = \operatorname{End}_{\widetilde{\mathscr{T}}}(P)$ is given by

$$E^{!,\mathrm{dg}} = \operatorname{REnd}_E(L_E),$$

where L_E is the sum of the simple $H^0(E)$-modules viewed as dg E-modules concentrated in degree 0. To compute this, we use the equivalence $\mathbf{D}(E) \to \mathscr{T}$ from Lemma 3.4.1, which takes L_E to L and therefore provides a quasi-isomorphism

$$\operatorname{End}_{\widetilde{\mathscr{T}}}(P)^{!,\mathrm{dg}} = E^{!,\mathrm{dg}} = \operatorname{REnd}_E(L_E) \simeq \operatorname{End}_{\widetilde{\mathscr{T}}}(L).$$

2) This follows from [Fus25, Thm. 4.17], since $\operatorname{End}_{\widetilde{\mathscr{T}}}(L)$ is the dg Koszul dual of $\operatorname{End}_{\widetilde{\mathscr{T}}}(P)$ by 1). □

In the case of finite-dimensional algebras (and analogously for non-positive dg algebras with finite-dimensional total cohomology) one can prove Theorem 3.4.2.2) more directly. The proof is interesting since it uses an approach that was used in [Zha23] to construct silting collections corresponding to simple-minded collections in $\mathbf{D}_{\mathrm{fd}}(A)$, where A is a non-positive dg algebra with finite-dimensional total cohomology.

Theorem 3.4.3. *Let A be a finite-dimensional algebra. Let $\mathcal{L}$ be a simple-minded collection in $\mathbf{D}^{\mathrm{b}}(\mathbf{mod}_{\mathrm{fd}}\text{-}A)$ and $\mathcal{P}$ be the corresponding classical silting collection in $\mathbf{K}^{\mathrm{b}}(\mathbf{proj}_{\mathrm{fg}}\text{-}(A))$ under the bijection from Theorem 2.4.4. Then $\operatorname{REnd}_A(\bigoplus_{P\in\mathcal{P}} P)$ is the dg Koszul dual of $\operatorname{REnd}_A(\bigoplus_{L\in\mathcal{L}} L)$.*

Proof. For brevity we write $L = \bigoplus_{L'\in\mathcal{L}} L'$. The cohomology of $E^! = \operatorname{REnd}_A(L)$ is given by

$$H^n(E^!) = H^n(\operatorname{REnd}_A(L)) = \operatorname{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathbf{mod}_{\mathrm{fd}}\text{-}A)}(L, L[n])$$

and therefore is concentrated in non-negative degrees, and moreover

$$H^0(E^!) = \operatorname{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathbf{mod}_{\mathrm{fd}}\text{-}A)}(L, L) = \bigoplus_{L'\in\mathcal{L}} \operatorname{End}_{\mathbf{D}^{\mathrm{b}}(\mathbf{mod}_{\mathrm{fd}}\text{-}A)}(L')$$

is semisimple. Hence by definition the Koszul dual of $E^!$ is

$$E = \operatorname{REnd}_{E^!}(H^0(E^!)).$$

Let L^0 be the sum of the simple A-modules and $A^! = \mathrm{REnd}_A(L^0)$ the Koszul dual of A viewed as a dg algebra concentrated in non-positive degrees. As is explained in [Zha23] we obtain a commutative diagram

$$\begin{tikzcd} \mathbf{perf}(E^!) \arrow[r, "\Psi"] & \mathbf{perf}(A^!) \arrow[r, "\Phi"] & \mathbf{D}^{\mathrm{b}}(\mathbf{mod}_{\mathrm{fd}}\text{-}A) & \\ \mathbf{D}_{\mathrm{fd}}(E^!) \arrow[u, hook] \arrow[r, "\Psi"] & \mathbf{D}_{\mathrm{fd}}(A^!) \arrow[u, hook] \arrow[r, "\Phi"] & \mathbf{K}^{\mathrm{b}}(\mathbf{inj}_{\mathrm{fg}}\text{-}A) \arrow[u, hook] \arrow[r, "\nu^{-1}"] & \mathbf{K}^{\mathrm{b}}(\mathbf{proj}_{\mathrm{fg}}\text{-}A) \end{tikzcd} \tag{3.1}$$

where ν^{-1} is the inverse Nakayama functor and the horizontal functors Ψ and Φ are equivalences defined by

$$\Phi = - \otimes^{\mathrm{L}}_{A^!} L^0, \qquad \Psi = - \otimes^{\mathrm{L}}_{E} \Phi^{-1}(L).$$

Note that the definition of Ψ implicitly also uses the equivalence induced by the quasi-isomorphism $E^! \cong \mathrm{REnd}_{A^!}(\Phi^{-1}(L))$ induced by Φ, which we leave out for brevity.

Now observe that by construction of the diagram (3.1) the equivalences in the bottom row map $H^0(\mathrm{REnd}_A(L)) = \mathrm{End}_{\mathbf{D}^{\mathrm{b}}(\mathbf{mod}_{\mathrm{fd}}\text{-}A)}(L)$ to P, and therefore we obtain a quasi-isomorphism

$$\mathrm{REnd}_A(L)^! = E \cong \mathrm{REnd}_A(P). \qquad \square$$

Remark 3.4.4. Koszul duality of $\mathrm{End}_{\widetilde{\mathscr{T}}}(\bigoplus_{P\in\mathcal{P}} P)$ and $\mathrm{End}_{\widetilde{\mathscr{T}}}(\bigoplus_{L\in\mathcal{L}} L)$ does not imply that $\mathcal{P}$ and $\mathcal{L}$ correspond to each other. For a (trivial) counterexample one can simply shift $\mathcal{L}$ or $\mathcal{P}$, and for further non-trivial examples with the same dg algebras occuring see Examples 3.5.1 and 3.5.2.

3.5 Some small examples: the A_2 quiver

We illustrate Theorem 3.4.2 by some examples over the algebra $A = \Bbbk(2 \to 1)$. For a simple-minded collection $\mathcal{L}$ in $\mathbf{D}^{\mathrm{b}}(\mathbf{mod}_{\mathrm{fd}}\text{-}A)$ and a silting collection $\mathcal{P}$ in $\mathbf{K}^{\mathrm{b}}(\mathbf{proj}_{\mathrm{fg}}\text{-}A)$, let $E = \mathrm{REnd}_A(\bigoplus_{P\in\mathcal{P}} P)$ and $E^! = \mathrm{REnd}_A(\bigoplus_{L\in\mathcal{L}} L)$. To compute the dg Koszul duals of E and $E^!$ we use the description of the dg Koszul dual from Definition 3.3.2.

Recall that for a dg algebra B and dg B-modules X and Y, the dg algebra $\mathrm{REnd}_B(X, Y)$ can be computed by replacing both X and Y by K-projective resolutions, i.e. quasi-isomorphic perfect dg modules. Replacing both is convenient to determine the composition of morphisms, as otherwise one would have to use formal inverses to quasi-isomorphisms. The degree n part of $\mathrm{REnd}_B(X, Y)$ consists of all B-linear morphisms $X \to Y[n]$ (not necessarily dg morphisms), and the differential is defined by $d(f) = df - (-1)^{|f|} fd$. Alternatively one can use K-injective resolutions. If B has trivial differential, K-projective resolutions are just projective resolutions.

Example 3.5.1 (The standard example)**.** Consider the standard simple-minded collection $\mathcal{L} = \{1, 2\}$ consisting of the simple A-modules, and the corresponding standard silting collection $\mathcal{P} = \{\begin{smallmatrix}1\\2\end{smallmatrix}, 2\}$ consisting of the indecomposable projective A-modules.

1) We have $E = \mathrm{REnd}_A(A) \cong A$, viewed as a non-positive dg algebra concentrated in degree 0 with trivial differential.
2) To compute $E^! = \mathrm{REnd}_A(1 \oplus 2)$ we replace the non-projective simple by its projective resolution: $1 \cong (2 \to \begin{smallmatrix}1\\2\end{smallmatrix})$. From this it follows that $E^!$ is 7-dimensional, as it is the direct sum of

$$\mathrm{RHom}_A(2,2) = \Bbbk e_2, \qquad \mathrm{RHom}_A(2,1) = \Bbbk f_0 \oplus \Bbbk f_{-1},$$
$$\mathrm{RHom}_A(1,2) = \Bbbk g, \qquad \mathrm{RHom}_A(1,1) = \Bbbk e_{11} \oplus \Bbbk e_{12} \oplus \Bbbk h,$$

with the degrees and the differentials of the basis elements given by

	e_{11}	e_{12}	e_2	f_0	f_{-1}	g	h
deg	0	0	0	0	-1	1	1
d	h	$-h$	0	0	f_0	0	0

The morphisms e_{11}, e_{12} and e_2 are orthogonal idempotents, and the algebra structure of $E^!$ is given by the quiver with relations

$$E^! = \Bbbk \left(\begin{tikzcd} e_{11} \arrow[rr, "h"] \arrow[dr, bend left, "g"] & & e_{12} \\ & e_2 \arrow[ul, bend left, "f_{-1}"] \arrow[ur, "f_0"'] & \end{tikzcd} \right) \Big/ \left(\begin{array}{c} f_0 g = h \\ f_{-1} g = e_{11} \\ h f_{-1} = f_0 \\ g f_{-1} = e_2 \end{array} \right).$$

It follows that the cohomology $H^*(E^!) = \operatorname{Ext}^*_A(1 \oplus 2, 1 \oplus 2)$ has a basis consisting of the classes of $e_1 = e_{11} + e_{12}$, e_2 and g, where g spans the 1-dimensional $\operatorname{Ext}^1_A(1,2)$. It is easy to see that the map $H^*(E^!) \to E^!$ defined by sending this basis of $H^*(E^!)$ to these representatives is a quasi-isomorphism.

3) As $E \cong A$ as dg algebras, there is nothing to do: the dg Koszul dual of E is literally $\operatorname{REnd}_A(1 \oplus 2) = E^!$.

4) To compute the dg Koszul dual of $E^!$ we use the quasi-isomorphism $E^! \cong H^*(E^!) = \Bbbk(e_1 \xrightarrow{g} e_2)$ with $|g| = 1$ and trivial differential. As the differential is trivial and $E^!$ is actually (not just cohomologically) concentrated in positive degrees, the augmenting dg $E^!$-modules are just the simple modules over $H^0(E^!) = (E^!)^0$ with trivial action of $(E^!)^{>0}$. K-projective resolutions of these are given by

$$e_1 \cong \left(\begin{array}{c} 0 \\ \\ \Bbbk \end{array} \right), \qquad e_2 \cong \left(\begin{tikzcd} \Bbbk \arrow[dr, "g"] & 0 \\ \Bbbk \arrow[r, "d"'] & \Bbbk \end{tikzcd} \right)$$

Here the top row indicates the vertex e_2 and the bottom row the vertex e_1, and in both cases the left-most non-zero term is in degree 0. From this it follows that the dg Koszul dual is

$$\operatorname{REnd}_{E^!}(e_1 \oplus e_2) = \Bbbk \left(\begin{tikzcd} E_{21} \arrow[rr, "H"] \arrow[dr, bend left, "G"] & & E_{22} \\ & E_1 \arrow[ul, bend left, "F_0"] \arrow[ur, "F_1"'] & \end{tikzcd} \right) \Big/ \left(\begin{array}{c} F_1 G = H \\ F_0 G = E_{21} \\ H F_0 = F_1 \\ G F_0 = E_1 \end{array} \right)$$

with dg structure

	E_{21}	E_{22}	E_1	F_0	F_1	G	H
deg	0	0	0	0	1	0	1
d	$-H$	H	0	F_1	0	0	0

By similar arguments as in 2), this dg algebra is quasi-isomorphic to its cohomology, which is $\Bbbk(E_2 \xrightarrow{G} E_1) \cong E$ with $E_2 = E_{21} + E_{22}$.

Example 3.5.2 (Non-standard, faithful heart)**.** Consider the simple-minded collection $\mathcal{L} = \{ {}^{1}_{2}, 2[1] \}$ and the corresponding silting collection $\mathcal{P} = \{ {}^{1}_{2}, 1 \}$. This is obtained from the standard example by left mutation at 2.

1) We have $E = \mathrm{REnd}_A(\frac{1}{2} \oplus 1) \cong \Bbbk(*_{\frac{1}{2}} \xrightarrow{x} *_1) \cong A$, with $|x| = 0$ and trivial differential. Explicitly, x is the morphism $x\colon \frac{1}{2} \to 1$.
2) We have $E^! = \mathrm{REnd}_A(\frac{1}{2} \oplus 2[1]) \cong \Bbbk(*_{2[1]} \xrightarrow{y} *_{\frac{1}{2}})$ with $|y| = 1$ and trivial differential, where $y\colon 2[1] \to \frac{1}{2}[1]$.
3) Note that $E \cong A$ and therefore the dg Koszul dual of E is $\mathrm{REnd}_A(1 \oplus 2)$ as described in Example 3.5.1 above. A quasi-isomorphism $E^! \to \mathrm{REnd}_A(1 \oplus 2)$ is given by $*_{\frac{1}{2}} \mapsto e_2$, $*_{2[1]} \mapsto e_{11} + e_{12}$, $y \mapsto g$, as mentioned in Example 3.5.1.
4) We already computed the Koszul dual of $E^!$ in Example 3.5.1, where we saw that it is quasi-isomorphic to E.

Example 3.5.3 (Non-standard, non-faithful heart)**.** Consider the simple-minded collection $\mathcal{L} = \{1, 2[-1]\}$ and the corresponding silting collection $\mathcal{P} = \{\frac{1}{2}, 2[-1]\}$. This is obtained from the standard example by right mutation at 2. The corresponding heart is semisimple.

1) We have $E = \mathrm{REnd}_A(\frac{1}{2} \oplus 2[-1]) \cong \Bbbk(*_{2[-1]} \xrightarrow{x} *_1)$ with $|x| = -1$ and trivial differential. Explicitly, x is the morphism $2[-1] \to \frac{1}{2}[-1]$.
2) As $\mathcal{L}$ is obtained from the standard simple-minded collection by shifting one object, the algebra structure of $E^! = \mathrm{REnd}_A(1 \oplus 2[-1])$ is the same as in Example 3.5.1. However the degrees and differentials are now given by

	e_{11}	e_{12}	e_2	f_0	f_{-1}	g	h
deg	0	0	0	-1	-2	2	1
d	h	$-h$	0	0	$-f_0$	0	0

3) To compute the dg Koszul dual of E, we need to take K-projective resolutions of the two augmenting dg E-modules $*_{2[-1]}$ and $*_1$. These are given by

$$*_{2[-1]} \cong \begin{pmatrix} 0 \\ \Bbbk \end{pmatrix}, \qquad *_1 \cong \left(\begin{tikzcd} 0 & 0 & \Bbbk \arrow[dl, "x"] \\ \Bbbk \arrow[r, "d"'] & \Bbbk & 0 \end{tikzcd}\right)$$

with the right-most terms in degree 0. Here the top row represents the vertex $*_1$ and the bottom row the vertex $*_{2[-1]}$. It follows that

$$\mathrm{REnd}_E(*_{2[-1]} \oplus *_1) \cong \Bbbk \left(\begin{tikzcd} E_{11} \arrow[rr, "H"] \arrow[dr, bend left, "G"] & & E_{12} \\ & E_2 \arrow[ul, bend left, "F_{-1}"] \arrow[ur, "F_0"'] & \end{tikzcd}\right) \Big/ \begin{pmatrix} F_0G=H \\ F_{-1}G=E_{11} \\ HF_{-1}=F_0 \\ GF_{-1}=E_2 \end{pmatrix}.$$

with the degrees and differentials given by

	E_{11}	E_{12}	E_2	F_0	F_{-1}	G	H
deg	0	0	0	-1	-2	2	1
d	H	$-H$	0	0	$-F_0$	0	0

It is obvious that $\mathrm{REnd}_E(*_1 \oplus *_2)$ is (quasi-)isomorphic to $E^!$.

4) Similarly to the previous examples, it follows that $E^!$ is quasi-isomorphic to its cohomology, which is $H^*(E^!) = \Bbbk(e_1 \xrightarrow{g} e_2)$ where $e_1 = e_{11} + e_{12}$ and $|g| = 2$. The dg Koszul dual of $E^!$ is computed similarly to Example 3.5.1: the K-projective resolutions of the augmenting dg $E^!$-modules are given by

$$e_1 \cong \left(\begin{tikzcd} 0 \\ \Bbbk \end{tikzcd}\right), \qquad\qquad e_2 \cong \left(\begin{tikzcd} \Bbbk \arrow[drr, "y"] & 0 & 0 \\ 0 & \Bbbk \arrow[r, "d"'] & \Bbbk \end{tikzcd}\right),$$

and from this we get

$$\mathrm{REnd}_{E^!}(e_1 \oplus e_2) = \Bbbk \left(\begin{tikzcd} E_{21} \arrow[rr, "H"] \arrow[dr, bend left, "G"] & & E_{22} \\ & E_1 \arrow[ul, bend left, "F_1"] \arrow[ur, "F_2"'] & \end{tikzcd}\right) \Big/ \left(\begin{smallmatrix} F_2G=H \\ F_1G=E_{21} \\ HF_1=F_2 \\ GF_1=E_1 \end{smallmatrix}\right)$$

with dg structure

	E_{21}	E_{22}	E_1	F_1	F_2	G	H
deg	0	0	0	1	2	-1	1
d	$-H$	H	0	F_2	0	0	0

Similarly to the previous examples, $\mathrm{REnd}_{E^!}(e_1 \oplus e_2)$ is quasi-isomorphic to its cohomology, which is $\Bbbk(E_2 \xrightarrow{G} E_1) = E$ where $E_2 = E_{21} + E_{22}$.

Chapter 4

Serre functor and $\mathbb{P}$-objects for perverse sheaves on $\mathbb{P}^n$

In this chapter we show that the $\mathbb{P}$-twist at the IC sheaf $\mathrm{IC}_n = \underline{\Bbbk}_{\mathbb{P}^n}[n]$ is the inverse Serre functor of the constructible derived category $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n)$, and furthermore we classify the $\mathbb{P}$(-like) objects in $\mathbf{Perv}(\mathbb{P}^n)$.

We begin by recalling the required definitions. In particular, following [HT06; HK19] we define $\mathbb{P}$(-like) objects and $\mathbb{P}$-twists in the general setup of dg-enhanced triangulated categories, and we recall the notion of Serre functors introduced in [BK90]. After explaining our setup, i.e. the constructible derived category $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n)$ and the (middle-)perverse $\mathbf{t}$-structure, we explicitly describe the simple, standard and projective objects in $\mathbf{Perv}(\mathbb{P}^n)$. We also summarize the well-known equivalences between perverse sheaves, parabolic category $\mathcal{O}$, and a description in terms of finite-dimensional algebras.

To show that the $\mathbb{P}$-twist at IC_n is the inverse Serre functor of $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n)$ we use a criterion adapted from [MS08], see Lemma 4.3.10. To apply this in the proof of the main result Theorem 4.3.11, we first compute the morphisms between the simple, standard, and projective objects of $\mathbf{Perv}(\mathbb{P}^n)$. This also serves as preparation for the classification of $\mathbb{P}$-like objects. We also compare our description of the Serre functor to the descriptions in terms of category $\mathcal{O}$ and finite-dimensional algebras, and also to the description of the Serre functor of $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(G/B)$ from [BBM04].

For the classification of the $\mathbb{P}$-like objects in $\mathbf{Perv}(\mathbb{P}^n)$ we first recursively construct certain string objects from the simple and standard objects. By the classification of indecomposable objects obtained from the description of $\mathbf{Perv}(\mathbb{P}^n)$ in terms of finite-dimensional algebras, these are all the indecomposable objects that are not projective-injective. We compute the Hom spaces between the string objects via their recursive definition, which also yields canonical morphisms spanning these Hom spaces. We then show that all string objects are $\mathbb{P}$-like by determining the composition of these canonical morphisms up to non-zero scalars, see Theorem 4.4.17. However, none of the string objects are $\mathbb{P}$-objects except for IC_n, since only IC_n and the indecomposable projective-injective objects can be Calabi–Yau.
The chapter is joint work with Alessio Cipriani, and has appeared as the preprint [BC25].

[BC25] L. Bonfert and A. Cipriani. *Serre functor and $\mathbb{P}$-objects for perverse sheaves on $\mathbb{P}^n$*. Preprint. 2025. arXiv:`2506.06051v1 [math.RT]`.

4.1 Motivation and overview of results

A Serre functor on a $\Bbbk$-linear triangulated category is an autoequivalence $\mathbb{S}\colon \mathscr{D} \to \mathscr{D}$ such that for any pair of objects $E, F \in \mathscr{D}$ there exists a functorial isomorphism $\mathrm{Hom}_{\mathscr{D}}(E,F) \cong \mathrm{Hom}_{\mathscr{D}}(F, \mathbb{S}(E))^\vee$. Serre functors generalize Serre duality from algebraic geometry, and are an important tool in the theory of triangulated categories. For instance, they can be used to construct left (resp. right) adjoints to functors having a right (resp. left) adjoint.

Another important class of automorphisms of triangulated categories in algebraic geometry and representation theory are the spherical twists associated to spherical objects [ST01]. For example, these can be used to construct braid group actions on triangulated categories, and certain functors from representation theory such as shuffling functors can be realized as spherical twists [Len21]. By definition, an object $E \in \mathscr{D}$ is d-spherical if there is an isomorphism of graded algebras $\mathrm{End}^*_{\mathscr{D}}(E) \cong \Bbbk[t]/(t^2)$ with $\deg(t) = d$ and E is d-Calabi–Yau. The value of the spherical twist ST_E at $X \in \mathscr{D}$ is then defined by the triangle

$$\mathrm{Hom}^*_{\mathscr{D}}(E, X) \otimes E \xrightarrow{\mathrm{ev}} X \to \mathsf{ST}_E(X) \to \mathrm{Hom}^*_{\mathscr{D}}(E, X) \otimes E[1].$$

Consider the constructible derived category $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n)$ of the complex projective space $\mathbb{P}^n$ with the usual Bruhat stratification, whose strata (the Bruhat cells) have complex dimension $0, 1, \ldots, n$. By definition, $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n)$ consists of those complexes of sheaves of $\Bbbk$-vector spaces on $\mathbb{P}^n$ whose cohomology is locally constant on all strata of $\mathbb{P}^n$. By gluing the standard $\mathbf{t}$-structures on the constructible derived category of each stratum (shifted by the dimension of the stratum) one obtains the (middle-)perverse $\mathbf{t}$-structure on $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n)$, and its heart is the category $\mathbf{Perv}(\mathbb{P}^n)$ of (middle-)perverse sheaves [BBD82]. This perverse $\mathbf{t}$-structure plays an important role in representation theory since there is an equivalence $\mathbf{Perv}(\mathbb{P}^n) \cong \mathcal{O}^{\mathfrak{p}}_0(\mathfrak{sl}_{n+1}(\Bbbk))$, where $\mathfrak{p} \subseteq \mathfrak{sl}_{n+1}(\Bbbk)$ denotes the parabolic Lie subalgebra with block sizes $(n, 1)$. As this perverse $\mathbf{t}$-structure has faithful heart, this yields an equivalence $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n) \cong \mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathcal{O}^{\mathfrak{p}}_0(\mathfrak{sl}_{n+1}(\Bbbk)))$ [BGS96].

The category of perverse sheaves on $\mathbb{P}^n$ is moreover equivalent to the category of finite-dimensional modules over an explicit finite-dimensional algebra A_n [KS02]. Since A_n has finite global dimension, it follows from results of Happel [Hap88] and Bondal–Kapranov [BK90] that $\mathbf{D}^{\mathrm{b}}(\mathbf{mod}_{\mathrm{fd}}\text{-}A_n)$ admits a Serre functor, namely the left derived functor of the Nakayama functor $A_n^\vee \otimes_{A_n} -$. However, these results do not provide a description of the Serre functor that is intrinsic to the constructible derived category $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n)$.

In the case of the complex projective line $\mathbb{P}^1 = \mathbf{pt} \dot\cup \mathbb{A}^1$ stratified by a point and its complement, such an intrinsic description is provided in [Woo10]. In this example, the category $\mathbf{Perv}(\mathbb{P}^1)$ has two simple objects corresponding to the two strata. Explicitly, these are the skyscraper sheaf $\mathrm{IC}_0 = \mathrm{incl}_* \underline{\Bbbk}_{\mathbf{pt}}$ and the shifted constant sheaf $\mathrm{IC}_1 = \underline{\Bbbk}_{\mathbb{P}^1}[1]$. The simple perverse sheaf IC_1 is a 2-spherical object in $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^1)$, and the inverse Serre functor for $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n)$ is then given by $\mathbb{S}^{-1} = \mathsf{ST}^2_{\mathrm{IC}_1}$.

The main result of this chapter is a generalization of this description of the Serre functor to $\mathbb{P}^n$. In this case, the simple perverse sheaf $\mathrm{IC}_n = \underline{\Bbbk}_{\mathbb{P}^n}[n]$ corresponding to the open stratum is not a spherical object, but rather a $\mathbb{P}^n$-object in the sense of Huybrechts and Thomas [HT06]. Explicitly this means that there is an isomorphism of graded algebras $\mathrm{End}_{\mathbb{P}^n}(\mathrm{IC}_n) \cong \Bbbk[t]/(t^{n+1})$ with $\deg(t) = 2$, and that IC_n is $2n$-Calabi–Yau. The corresponding generalization of spherical twists is provided by the $\mathbb{P}$-twists from [HT06], which are defined as certain “double cones”, see Definition 4.2.3 below. In our situation, we obtain:

Theorem 4.1.1 (Theorem 4.3.11)**.** *The inverse Serre functor $\mathbb{S}^{-1}$ of $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n)$ is isomorphic to the $\mathbb{P}$-twist $\mathsf{PT}_{\mathrm{IC}_n}$.*

This in particular recovers the result from [Woo10] for $\mathbb{P}^1$, since by definition a $\mathbb{P}^1$-object is the same as a 2-spherical object, and for any $\mathbb{P}^1$-object E we have $\mathsf{PT}_E \cong \mathsf{ST}^2_E$.

The proof of Theorem 4.1.1 relies on a characterization of the Serre functor adapted from [MS08], see Lemma 4.3.10. The main idea is to compare the “candidate inverse Serre functor” to the inverse Serre functor by studying its action on the injective and projective-injective objects.

In [MS08] the dual version of this criterion was used to describe the Serre functor of $\mathbf{D}^{\mathrm{b}}(\mathcal{O}_0(\mathfrak{g}))$ for any finite-dimensional complex semisimple Lie algebra $\mathfrak{g}$. By an entirely formal argument this description also descends to $\mathbf{D}^{\mathrm{b}}(\mathcal{O}_0^{\mathfrak{p}}(\mathfrak{g}))$ for any parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{g}$, and thus also to $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n)$. In Section 4.3.6, we summarize these results and their relation to the description of the Serre functor in terms of finite-dimensional algebras, and also relate our description of the Serre functor of $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n)$ to the description of the Serre functor for the full flag variety obtained in [BBM04].

Motivated by Theorem 4.1.1, one may ask whether there are further $\mathbb{P}$-objects in $\mathbf{Perv}(\mathbb{P}^n)$. However, it is easy to see that no indecomposable object except for IC_n and the projective-injectives can be Calabi–Yau, see Corollary 4.4.2. Hence any other indecomposable object E can at best be $\mathbb{P}^k$-like in the sense that $\mathrm{End}^*_{\mathbb{P}^n}(E) \cong \Bbbk[t]/(t^{k+1})$.

To describe the indecomposable perverse sheaves, in Section 4.4.2 we inductively construct certain string objects $\mathrm{M}^{\pm}_{a,b} \in \mathbf{Perv}(\mathbb{P}^n)$ for $0 \le b \le a \le n$, starting from the simple objects and (co)standard objects. Alternative constructions of these objects can be found in [CL23], where they are used to describe the wall-and-chamber structure of $\mathbf{Perv}(\mathbb{P}^n)$. Since $\mathbf{Perv}(\mathbb{P}^n) \cong A_n\text{-}\mathbf{mod}_{\mathrm{fd}}$ for a special biserial algebra A_n, the classification of indecomposable modules over special biserial algebras from [BR87; WW85] shows that the string objects together with the indecomposable projective-injective objects are all the indecomposable perverse sheaves.

As the indecomposable projective-injective objects are 0-spherical, the second result of this chapter then shows that all indecomposable perverse sheaves are either $\mathbb{P}$-like or 0-spherical:

Theorem 4.1.2 (Theorem 4.4.17). *Let $0 \le b \le a \le n$.*

1) If $a - b$ is even, then the string objects $\mathrm{M}^{\pm}_{a,b}$ are $\mathbb{P}^{(a+b)/2}$-like.

2) If $a - b$ is odd, then the string objects $\mathrm{M}^{\pm}_{a,b}$ are $\mathbb{P}^{(a-b-1)/2}$-like.

As easy consequences of Theorem 4.1.2, one also obtains a classification of the spherical, spherelike and exceptional objects in $\mathbf{Perv}(\mathbb{P}^n)$, see Corollary 4.4.19. In particular, this recovers the classification of the exceptional objects from [PW20].

The proof of Theorem 4.1.2 is rather technical and occupies most of Sections 4.3 and 4.4. The first step is to compute $\mathrm{End}^*_{\mathbb{P}^n}(\mathrm{M}^{\pm}_{a,b})$ by chasing the long exact sequences obtained from the inductive construction of the string objects, see Section 4.4.3. As the base cases of this construction are the simple objects and the (co)standard objects, this requires us to explicitly fix morphisms between these objects and to determine their compositions, see Sections 4.3.1 to 4.3.3. The computation of $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^{\pm}_{a,b}, \mathrm{M}^{\pm}_{a,b}[2i])$ also yields canonical non-zero morphisms $\Phi^{2i}_{a,b} \colon \mathrm{M}^{\pm}_{a,b} \to \mathrm{M}^{\pm}_{a,b}[2i]$, and the final step is then to check that $\Phi^{2i}_{a,b}\Phi^{2}_{a,b} = \Phi^{2i+2}_{a,b}$ up to a non-zero scalar (whenever this is possible by degree reasons), see Section 4.4.3. To quickly check conjectures about Ext spaces we often used Haruhisa Enomoto's FD Applet [Eno].

4.1.1 Notation

In this chapter $\Bbbk$ denotes an algebraically closed field of characteristic 0. The $\Bbbk$-linear duality functor is denoted by $(-)^\vee = \mathrm{Hom}_\Bbbk(-, \Bbbk)$.

For a $\Bbbk$-linear triangulated category $\mathscr{D}$ and $A, B \in \mathscr{D}$ we denote the total Hom space by $\mathrm{Hom}^*_{\mathscr{D}}(A, B) = \bigoplus_{r\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{D}}(A, B[r])$, with the degree r part given by $\mathrm{Hom}_{\mathscr{D}}(A, B[r])$. We do not write shifts of morphisms, i.e. we just write $f \colon A[1] \to B[1]$ for $f \colon A \to B$.

We write $\mathrm{R}F \colon \mathbf{D}^+(\mathscr{A}) \to \mathbf{D}^+(\mathscr{B})$ for the right derived functor of a left exact functor $F \colon \mathscr{A} \to \mathscr{B}$ of abelian categories. As usual, we will however suppress the notation for derived functors for functors arising from geometry, such as the pushforward. The right derived functor in the ∞-categorical sense will be denoted by $\mathbb{R}F \colon \mathbf{D}^+_\infty(\mathscr{A}) \to \mathbf{D}^+_\infty(\mathscr{B})$.

The following table is a list of notation for the morphisms in $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n)$ between the simple perverse sheaves IC_k, standard objects Δ_k (see Section 4.2.4) and string objects $\mathrm{M}^+_{a,b} \in \mathbf{Perv}(\mathbb{P}^n)$ (see Section 4.4.2) which will be used throughout the chapter.

Morphism	Definition	Morphism	Definition
$\epsilon^r_{k,l} \colon \mathrm{IC}_k \to \mathrm{IC}_l[r]$	Section 4.3.1	$\psi_{a-2,b} \colon \mathrm{M}^+_{a-2,b} \to \Delta_a[1]$	Section 4.4.2
$\mu_{k,l} \colon \Delta_k \to \mathrm{IC}_l[l-k]$	Lemma 4.3.5	$m^r_{a,b} \colon \mathrm{M}^+_{a,b} \to \mathrm{IC}_b[r]$	Lemma 4.4.4
$\delta^r_{k,l} \colon \Delta_k \to \Delta_l[r]$	Lemma 4.3.6	$n^r_{a,b} \colon \mathrm{M}^+_{a,b} \to \mathrm{IC}_a[r]$	Remark 4.4.6
$\phi^r_{k,l} \colon \mathrm{IC}_l \to \Delta_k[r]$	Lemma 4.3.7	$\Phi^{2i}_{a,b} \colon \mathrm{M}^+_{a,b} \to \mathrm{M}^+_{a,b}[2i]$	Remark 4.4.14
$\iota_{a,b} \colon \Delta_a \to \mathrm{M}^+_{a,b}$	Section 4.4.2	$\overline{\Phi}^{2i}_{a,b} \colon \mathrm{M}^+_{a-2i,b} \to \mathrm{M}^+_{a,b}[2i]$	Remark 4.4.14
$\pi_{a,b} \colon \mathrm{M}^+_{a,b} \to \mathrm{M}^+_{a-2,b}$	Section 4.4.2	$\zeta^{2i}_{a-2i,b} \colon \Delta_{a-2i} \to \mathrm{M}^+_{a,b}[2i]$	Remark 4.4.14

4.2 Definitions and background

4.2.1 $\mathbb{P}$-objects and $\mathbb{P}$-twists

We begin by recalling the $\mathbb{P}$-twists at $\mathbb{P}$-objects from [HT06], using the nomenclature from [HK19]. Let $\mathscr{D}$ be a $\Bbbk$-linear triangulated category.

Definition 4.2.1. Let $E \in \mathscr{D}$ and $k \in \mathbb{Z}$ with $k \geq 0$.

1) E is $\mathbb{P}^k$*-like* if there is an isomorphism of graded $\Bbbk$-algebras $\mathrm{End}^*_{\mathscr{D}}(E) \cong \Bbbk[t]/(t^{k+1})$ with $\deg(t) = 2$.
2) E is a $\mathbb{P}^k$*-object* if it is $\mathbb{P}^k$-like, $\mathrm{Hom}^*_{\mathscr{D}}(E, X)$ is finite-dimensional for all $X \in \mathscr{D}$, and E is $2k$-Calabi–Yau.
3) A $\mathbb{P}$*-(like) object* is a $\mathbb{P}^k$-(like) object for any k.

Recall that $E \in \mathscr{D}$ is d-Calabi–Yau if there is a natural isomorphism

$$\mathrm{Hom}_{\mathscr{D}}(E, -) \cong \mathrm{Hom}_{\mathscr{D}}(-, E[d])^{\vee}.$$

It is immediate from the definitions that if a $\mathbb{P}^k$-like object E is d-Calabi–Yau, then necessarily $d = 2k$, cf. [HK25, Def. 2.1] and [HT06, Rem. 1.2].

Slightly more generally, [Kru18] and [HK19] also introduced $\mathbb{P}^k[d]$-(like) objects, for which 1) is replaced by $\mathrm{End}^*_{\mathscr{D}}(E) \cong \Bbbk[t]/(t^{k+1})$ with $\deg(t) = d$. Thus $\mathbb{P}^k[2]$(-like) objects are the same as $\mathbb{P}^k$(-like) objects. As well-known special cases, $\mathbb{P}^1[d]$-objects and $\mathbb{P}^1[d]$-like objects are the same as d-spherical objects and d-spherelike objects, respectively, and $\mathbb{P}^0$-like objects are the same as exceptional objects.

The following lemma provides a useful criterion for when $\mathbb{P}^k$-like objects are $\mathbb{P}^k$-objects.

Lemma 4.2.2. *Let $E \in \mathscr{D}$ be a $\mathbb{P}^k$-like object.*

1) *E is $2k$-Calabi–Yau if and only if the composition pairing*

$$\mathrm{Hom}_{\mathscr{D}}(E, X) \otimes \mathrm{Hom}_{\mathscr{D}}(X, E[2k]) \to \mathrm{Hom}_{\mathscr{D}}(E, E[2k]) \cong \Bbbk$$

is non-degenerate for all $X \in \mathscr{D}$.

2) *If $\mathscr{D} = \mathbf{D}^{\mathrm{b}}(\mathscr{A})$ for an abelian category $\mathscr{A}$ of finite global dimension with enough projectives, then E is $2k$-Calabi–Yau if and only if the composition pairing is non-degenerate for all $X = P[r]$ with $P \in \mathbf{Proj}(\mathscr{A})$ and $r \in \mathbb{Z}$.*

Proof.

1) For spherelike objects, this is [ST01, Lemma 2.15]. The same argument works for $\mathbb{P}$-like objects, as it only requires $\operatorname{Hom}_{\mathscr{D}}(E, E[2k]) \cong \Bbbk$

2) For spherelike objects, this is shown in [Len21, Lemma 3.3]. The same argument works for $\mathbb{P}$-like objects, as it only requires $\operatorname{Hom}_{\mathscr{D}}(E, E[2k]) \cong \Bbbk$ and the first part of the lemma. □

For the rest of this subsection we fix a dg enhancement of $\mathscr{D}$. By definition, a dg enhancement consists of a $\Bbbk$-linear pretriangulated dg category $\widetilde{\mathscr{D}}$ and an equivalence of triangulated categories $H^0(\widetilde{\mathscr{D}}) \cong \mathscr{D}$. Note that if $\widetilde{\mathscr{D}}$ is pretriangulated, then $\mathbf{Fun}_{\mathrm{dg}}(\widetilde{\mathscr{D}}, \widetilde{\mathscr{D}})$ is again a pretriangulated dg category by [BK91, §3, Examples, 4.]. In particular the cone of a morphism of dg functors is again a dg functor.

If $\widetilde{\mathscr{D}}$ has all coproducts, then for an object $E \in \widetilde{\mathscr{D}}$ there is the dg functor $- \otimes E \colon \mathbf{dgVect}_{\Bbbk} \to \widetilde{\mathscr{D}}$, which is defined as the right adjoint to $\operatorname{Hom}_{\widetilde{\mathscr{D}}}(E, -) \colon \widetilde{\mathscr{D}} \to \mathbf{dgVect}_{\Bbbk}$. We denote the counit of this adjunction by $\mathrm{ev} \colon \operatorname{Hom}_{\widetilde{\mathscr{D}}}(E, -) \otimes E \to \mathrm{id}_{\widetilde{\mathscr{D}}}$.

A generalization of spherical twists at spherical objects is provided by the $\mathbb{P}$-twists at $\mathbb{P}$-objects from [HT06, §2]. These are defined as follows:

Definition 4.2.3. Let $E \in \mathscr{D}$ be a $\mathbb{P}$-like object. Assume that the tensor product $\operatorname{Hom}_{\widetilde{\mathscr{D}}}(E, X) \otimes E \in \widetilde{\mathscr{D}}$ exists for all $X \in \widetilde{\mathscr{D}}$.

1) Pick a closed morphism $\tilde{t} \in \operatorname{Hom}_{\widetilde{\mathscr{D}}}(E, E[2])$ of degree 0 representing an algebra generator of $\operatorname{End}^*_{\mathscr{D}}(E)$. Define the dg functor $\widetilde{\mathsf{PT}}_E = \operatorname{cone}(\overline{\mathrm{ev}}) \colon \widetilde{\mathscr{D}} \to \widetilde{\mathscr{D}}$ by the following commutative diagram in $\mathbf{Fun}_{\mathrm{dg}}(\widetilde{\mathscr{D}}, \widetilde{\mathscr{D}})$:

$$\begin{tikzcd}
\big(\operatorname{Hom}_{\widetilde{\mathscr{D}}}(E,-) \otimes E\big)[-2] \arrow[r, "\tilde{t}^* \otimes \mathrm{id} - \mathrm{id} \otimes \tilde{t}"] & \operatorname{Hom}_{\widetilde{\mathscr{D}}}(E,-) \otimes E \arrow[r] \arrow[d, "\mathrm{ev}"] & \operatorname{cone}(\tilde{t}^* \otimes \mathrm{id} - \mathrm{id} \otimes \tilde{t}) \arrow[dl, dashed, "\exists \overline{\mathrm{ev}}"] \\
 & \mathrm{id}_{\widetilde{\mathscr{D}}} \arrow[dl] & \\
\operatorname{cone}(\overline{\mathrm{ev}}) & &
\end{tikzcd} \tag{4.1}$$

Here $\overline{\mathrm{ev}} \colon \operatorname{cone}(\tilde{t}^* \otimes \mathrm{id} - \mathrm{id} \otimes \tilde{t}) \to \mathrm{id}_{\widetilde{\mathscr{D}}}$ is the canonical morphism of dg functors induced by ev, which exists since $\mathrm{ev} \circ (\tilde{t}^* \otimes \mathrm{id} - \mathrm{id} \otimes \tilde{t}) = 0$.

2) The $\mathbb{P}$-*twist at* E is the induced triangulated functor $\mathsf{PT}_E = H^0(\widetilde{\mathsf{PT}}_E) \colon \mathscr{D} \to \mathscr{D}$.

Remark 4.2.4.

1) A priori, the functor $\mathsf{PT}_E \colon \mathscr{D} \to \mathscr{D}$ depends on the choices made in the definition. However, as E is a $\mathbb{P}$-like object, the generator $t \colon E \to E[2]$ of $\operatorname{End}^*_{\mathscr{D}}(E)$ is unique up to non-zero scalar, and rescaling t obviously results in naturally isomorphic triangles. Similarly, choosing a different representative for t results in quasi-isomorphic cones. By [AL22, Thm. 3.2], the functor $\mathsf{PT}_E \colon \mathscr{D} \to \mathscr{D}$ is furthermore independent of the choice of cones and factorization $\overline{\mathrm{ev}}$. Thus $\mathsf{PT}_E \colon \mathscr{D} \to \mathscr{D}$ is well-defined up to natural isomorphism.

2) The assumption that $\operatorname{Hom}_{\widetilde{\mathscr{D}}}(E, X) \otimes E$ exists for all $X \in \widetilde{\mathscr{D}}$ is automatically satisfied if $\widetilde{\mathscr{D}}$ has all coproducts, or if $\operatorname{Hom}_{\widetilde{\mathscr{D}}}(E, X)$ is finite-dimensional.

3) If $\mathscr{D} = \mathbf{D}^+(\mathscr{A})$ is the derived category of an abelian category with enough injectives, a dg enhancement of $\mathscr{D}$ is given by the pretriangulated dg category $\widetilde{\mathscr{D}} = \mathbf{Ch}^+(\mathbf{Inj}(\mathscr{A}))$ together with the canonical equivalence $H^0(\widetilde{\mathscr{D}}) = \mathbf{K}^+(\mathbf{Inj}(\mathscr{A})) \to \mathbf{D}^+(\mathscr{A})$. Hence by definition, to compute $\mathrm{Hom}_{\widetilde{\mathscr{D}}}(E, X)$ for a $\mathbb{P}$-like object $E \in \mathscr{A}$ and any $X \in \widetilde{\mathscr{D}}$, one first has to replace E by a (fixed) injective resolution. However, the derived Hom functor $\mathrm{RHom}_{\mathscr{A}}(E, -) = \mathrm{Hom}_{\mathbf{Ch}^+(\mathscr{A})}(E, -)\colon \widetilde{\mathscr{D}} \to \mathbf{dgVect}_{\Bbbk}$ is quasi-isomorphic to $\mathrm{Hom}_{\widetilde{\mathscr{D}}}(E, -)$. Thus, in this situation we can use $\mathrm{RHom}_{\mathscr{A}}(E, -)$ instead of $\mathrm{Hom}_{\widetilde{\mathscr{D}}}(E, -)$ to define the $\mathbb{P}$-twist $\mathsf{PT}_E\colon \mathscr{D} \to \mathscr{D}$. This is easier to compute in practice, since here E does not need to be replaced by an injective resolution.

Remark 4.2.5.

1) Spherical twists at spherical objects can be generalized to spherical twists at spherical functors. Similarly, $\mathbb{P}$-twists at $\mathbb{P}$-objects can be generalized further to $\mathbb{P}$-twists at (split) $\mathbb{P}$-functors, see [Add16; Cau11; AL19]. However, we will not use these constructions.
2) By [Seg18], any autoequivalence of a triangulated category can be realized as a spherical twist at a spherical functor. For $\mathbb{P}$-twists at $\mathbb{P}$-objects this can be carried out explicitly, see [Seg18, §4].

The following main properties of $\mathbb{P}$-twists were proved in [HT06] using Fourier–Mukai transforms. We briefly sketch how the required properties can be shown purely in terms of dg-enhanced triangulated categories. That all statements carry over to the dg setup is presumably well-known to experts, see for instance [HK19, Prop. 2.5].

Proposition 4.2.6 (Huybrechts–Thomas)**.** *Let $E \in \mathscr{D}$ be a $\mathbb{P}^k$-object.*

1) $\mathsf{PT}_E\colon \mathscr{D} \to \mathscr{D}$ *is an equivalence.*
2) $\mathsf{PT}_E(E) \cong E[-2k]$.
3) *If E is spherical (i.e. if $k = 1$), then* $\mathsf{PT}_E \cong \mathsf{ST}_E^2$.

Proof.

1) A computation similar to the proof of [ST01, Lemma 2.8] shows that PT_E has a left adjoint PT'_E, which is defined dually. Moreover, by a similar argument and the Calabi–Yau property it follows that PT'_E is also right adjoint to PT_E. The claim then follows by the arguments from [HT06, Prop. 2.6].
2) This is straightforward, see [HT06, Lemma 2.5] for details.
3) See [HT06, Prop. 2.9]. Alternatively, this can be seen by comparing the diagram defining ST_E^2 to the octahedral axiom diagram for the factorization in the definition of the $\mathbb{P}$-twist. □

4.2.2 Serre functors

We recall the notion of Serre functor from [BK90].

Definition 4.2.7. A *Serre functor* for a $\Bbbk$-linear triangulated category $\mathscr{D}$ is a functor $\mathbb{S}\colon \mathscr{D} \to \mathscr{D}$ such that there are natural isomorphisms

$$\mathrm{Hom}_{\mathscr{D}}(X, Y) \cong \mathrm{Hom}_{\mathscr{D}}(Y, \mathbb{S}(X))^\vee.$$

As an immediate consequence of the Yoneda lemma, there is at most one Serre functor for $\mathscr{D}$ up to natural isomorphism [BK90, Prop. 3.4].

For $\mathscr{D} = \mathbf{D}^{\mathrm{b}}(A\text{-}\mathbf{mod}_{\mathrm{fd}})$ for a finite-dimensional algebra A, the Serre functor has the following description:

Proposition 4.2.8 (Happel)**.** *Let A be a finite-dimensional algebra. Then $\mathbf{D}^{\mathrm{b}}(A\text{-}\mathbf{mod}_{\mathrm{fd}})$ has a Serre functor if and only if* $\operatorname{gldim}(A) < \infty$*. In this case, the Serre functor is the left derived functor of the Nakayama functor $A^\vee \otimes_A -$.*

Proof. If $\operatorname{gldim}(A) < \infty$, then the left derived Nakayama functor is a Serre functor by [Hap88, Prop. I.4.10]. Conversely, it is clear that if $\mathbb{S}$ is a Serre functor, then

$$\operatorname{Ext}^r_A(X, L) \cong \operatorname{Hom}_{\mathbf{D}^{\mathrm{b}}(A\text{-}\mathbf{mod}_{\mathrm{fd}})}(X, L[r]) \cong \operatorname{Hom}_{\mathbf{D}^{\mathrm{b}}(A\text{-}\mathbf{mod}_{\mathrm{fd}})}(L, \mathbb{S}(X)[-r])^\vee$$

for any $X \in A\text{-}\mathbf{mod}_{\mathrm{fd}}$ and any simple A-module L, and the right-hand side vanishes for r large enough such that $\mathbb{S}(X)[-r] \in \mathscr{D}^{>0}$, where $\mathscr{D}^{>0}$ denotes the positive part of the standard $\mathbf{t}$-structure. ☐

4.2.3 The constructible derived category $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n)$

We recall the constructible derived category of the complex projective space $\mathbb{P}^n = \mathbb{P}^n_{\mathbb{C}}$, which will be the focus of the rest of the chapter. The same construction and all of the tools work in much greater generality, see [BBD82], [HTT08] or [Ach21].

Consider $\mathbb{P}^n$ with the usual stratification by Bruhat cells, i.e. by the subspaces

$$S_k = \{[x_0 : \cdots : x_{k-1} : 1 : 0 : \cdots : 0]\} \subseteq \mathbb{P}^n$$

for $0 \leq k \leq n$. We identify $S_k \cong \mathbb{A}^k$ by projection to the first k coordinates. We denote the strata inclusions by $\jmath_k \colon \mathbb{A}^k \to \mathbb{P}^n$, and write $\imath_k \colon \mathbb{P}^k \hookrightarrow \mathbb{P}^n$. By slight abuse of notation, we also write $\jmath_k$ and $\imath_k$ for the inclusions of $\mathbb{A}^k$ and $\mathbb{P}^k$, respectively, into any $\mathbb{P}^l$ with $k \leq l \leq n$.

We denote by $\mathbf{D}^{\mathrm{b}}(\mathrm{Sh}(\mathbb{P}^n))$ the bounded derived category of sheaves of finite-dimensional $\Bbbk$-vector spaces on $\mathbb{P}^n$. The *constructible derived category* is the full triangulated subcategory $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n) \subset \mathbf{D}^{\mathrm{b}}(\mathrm{Sh}(\mathbb{P}^n))$ consisting of the complexes whose cohomologies are (locally) constant on all strata. For brevity we write $\operatorname{Hom}_{\mathbb{P}^n}(-,-) = \operatorname{Hom}_{\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n)}(-,-)$. We emphasize that we always consider the constructible derived category with respect to a fixed stratification, in constrast to e.g. [Ach21].

The constructible derived category has a natural $\mathbf{t}$-structure given by

$$\begin{aligned}
\mathscr{D}^{t\leq 0} &= \{F \in \mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(X) \mid \imath_k^* X \in \mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(S)^{\leq -k}\ \forall S \in \mathcal{S}\},\\
\mathscr{D}^{t\geq 0} &= \{F \in \mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(X) \mid \imath_k^! X \in \mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(S)^{\geq -k}\ \forall S \in \mathcal{S}\}.
\end{aligned}$$

This is obtained by iterated gluing of the shifted (by $-\frac{1}{2}\dim S_k = -k$) standard $\mathbf{t}$-structures on $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(S_k) \cong \mathbf{D}^{\mathrm{b}}(\Bbbk\text{-}\mathbf{mod}_{\mathrm{fd}})$ along the recollements provided by the strata inclusions, see (the proof of) [BBD82, Prop. 2.1.3]. Its heart is the category of *(middle-)perverse sheaves* $\mathbf{Perv}(\mathbb{P}^n)$.

The *Verdier duality functor* on $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n)$ is $\mathbb{D} = \mathrm{R}\mathcal{H}\mathrm{om}_{\mathbf{D}^{\mathrm{b}}(\mathbb{P}^n)}(-, \omega_{\mathbb{P}^n})$, where $\omega_{\mathbb{P}^n} = a^!_{\mathbb{P}^n}\underline{\Bbbk}_{\mathbf{pt}}$ (with $a_{\mathbb{P}^n} \colon \mathbb{P}^n \to \mathbf{pt}$) is the dualizing sheaf. Since $\mathbb{P}^n$ is smooth, we in fact have $\omega_{\mathbb{P}^n} \cong \underline{\Bbbk}_{\mathbb{P}^n}[2n]$. Verdier duality restricts to a (contravariant) involution $\mathbb{D} \colon \mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n) \to \mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n)$. Moreover, from the definition of perverse sheaves it follows that $\mathbb{D}$ preserves $\mathbf{Perv}(\mathbb{P}^n)$.

By variants of Beilinson's theorem (see [BGS96, Cor. 3.3.2] or [BBM04, Prop. 1.5]) the perverse $\mathbf{t}$-structure on $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n)$ has *faithful heart.* This means that there is a *realization functor*, i.e. a triangulated functor $\mathbf{D}^{\mathrm{b}}(\mathbf{Perv}(\mathbb{P}^n)) \to \mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n)$ such that the diagram

$$\begin{tikzcd}
\mathbf{Perv}(\mathbb{P}^n) \arrow[d, hook] \arrow[dr, hook] & \\
\mathbf{D}^{\mathrm{b}}(\mathbf{Perv}(\mathbb{P}^n)) \arrow[r] & \mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n)
\end{tikzcd}$$

commutes, and that the realization functor is an equivalence. In particular, the realization functor provides isomorphisms $\mathrm{Ext}^r_{\mathbf{Perv}(\mathbb{P}^n)}(X, Y) \cong \mathrm{Hom}_{\mathbb{P}^n}(X, Y[r])$ for all $r \geq 0$ (note that for $r \leq 1$ this holds even if the **t**-structure does not have faithful heart). For $r = 1$ this allows to interpret triangles $Y \xrightarrow{f} Z \xrightarrow{g} X \to Y[1]$ with $X, Y, Z \in \mathbf{Perv}(\mathbb{P}^n)$ as short exact sequences $0 \to Y \xrightarrow{f} Z \xrightarrow{g} X \to 0$ in $\mathbf{Perv}(\mathbb{P}^n)$.

4.2.4 Simple, standard and projective objects

We recall the explicit description of the simple perverse sheaves, and also the standard and costandard objects. This also allows one to describe the indecomposable projective and injective perverse sheaves, and derive some important properties of $\mathbf{Perv}(\mathbb{P}^n)$. While all of this is well-known, see for instance [Ach21; BBD82; BGS96; CW22; MV87], the explicit constructions in this subsection are central to the arguments in Sections 4.3 and 4.4.

Simple objects

The simple perverse sheaves are the *IC sheaves* $\mathrm{IC}_k = \jmath_{k,!*}\underline{\Bbbk}_{\mathbb{A}^k}[k]$ for $0 \leq k \leq n$, where $\jmath_{k,!*}$ denotes the intermediate extension functor. As all strata closures are smooth, the IC sheaves are extensions by zero of shifted constant sheaves supported on the strata closures, i.e. we have $\mathrm{IC}_k \cong \imath_{k,*}\underline{\Bbbk}_{\mathbb{P}^k}[k]$. Note that $\mathbb{D}(\mathrm{IC}_k) \cong \mathrm{IC}_k$, and $\imath_{k-1,*}\imath^*_{k-1}\mathrm{IC}_k \cong \mathrm{IC}_{k-1}[1]$ and $\imath_{k-1,*}\imath^!_{k-1}\mathrm{IC}_k \cong \mathrm{IC}_{k-1}[-1]$.

Standard objects

For $0 \leq k \leq n$, the *standard objects* $\Delta_k = \jmath_{k,!}\underline{\Bbbk}_{\mathbb{A}^k}[k]$ are perverse sheaves. We have $\Delta_0 \cong \mathrm{IC}_0$, and for $k \geq 1$ the recollements provide triangles

$$\mathrm{IC}_{k-1} \xrightarrow{\phi^0_{k-1,k}} \Delta_k \xrightarrow{\mu_{k,k}} \mathrm{IC}_k \xrightarrow{\epsilon^1_{k,k-1}} \mathrm{IC}_{k-1}[1], \tag{4.2}$$

where $\epsilon^1_{k,k-1} \colon \mathrm{IC}_k \to \imath_{k-1,*}\imath^*_{k-1}\mathrm{IC}_k \cong \mathrm{IC}_{k-1}[1]$ is the adjunction unit and $\mu_{k,k} \colon \Delta_k \cong \jmath_{k,!}\jmath^*_k\mathrm{IC}_k \to \mathrm{IC}_k$ the adjunction counit. Note that for the applications in Sections 4.3 and 4.4 one has to be careful with the choice of the (co)units, see Section 4.3.1 for details. The notation here is more complicated than necessary at this point, but chosen for consistency with Sections 4.3.2 and 4.3.3 below where we will describe more general morphisms $\phi^r_{k,l} \colon \mathrm{IC}_k \to \Delta_l[r]$ and $\mu_{k,l} \colon \Delta_k \to \mathrm{IC}_l[l-k]$.

Interpreted as short exact sequences in $\mathbf{Perv}(\mathbb{P}^n)$, the triangles (4.2) are the composition series of the standard objects.

Dually, the *costandard objects* $\nabla_k = \jmath_{k,*}\underline{\Bbbk}_{\mathbb{A}^k}[k] \cong \mathbb{D}(\Delta_k)$ are also perverse sheaves. Explicitly, their composition series are given by $\nabla_0 \cong \mathrm{IC}_0$ and the recollement triangles

$$\mathrm{IC}_k \to \nabla_k \to \mathrm{IC}_{k-1} \to \mathrm{IC}_k[1]$$

for $k > 0$.

Projective objects

The category $\mathbf{Perv}(\mathbb{P}^n)$ has enough projectives. The explicit construction of the indecomposable projective objects from the proof of [CW22, Thm. 4.6] shows that $P_n \cong \Delta_n$, while for $k < n$ the object P_k arises from a triangle

$$\Delta_{k+1} \to P_k \to \Delta_k \xrightarrow{\delta_{k,k+1}} \Delta_{k+1}[1]. \tag{4.3}$$

In particular the indecomposable projective objects have Δ-flags. The morphism $\delta_{k,k+1}$ will be described precisely after Lemma 4.3.6.

Dually, $\mathbf{Perv}(\mathbb{P}^n)$ also has enough injectives, and the indecomposable injective objects have ∇-flags given by $I_n = \nabla_n$ and the triangles

$$\nabla_k \to I_k \to \nabla_{k+1} \to \nabla_k[1]$$

for $k < n$.

Highest weight structure

It follows that $\mathbf{Perv}(\mathbb{P}^n)$ is a highest weight category in the sense of [CPS88] (see [BS24] for a modern treatment) with respect to the order $0 < 1 < \cdots < n$, since the (co)standard objects as defined above are indeed the (co)standard objects in the sense of highest weight categories: the triangles defining the indecomposable projective objects and the composition series of the standard objects show that Δ_k is the maximal a quotient of P_k such that all composition factors except the top are IC_l with $l < k$.

By [Don98, Prop. A2.3] or [BS24, Rem. 3.28], the highest weight structure of $\mathbf{Perv}(\mathbb{P}^n)$ gives a bound on the global dimension, namely $\mathrm{gldim}(\mathbf{Perv}(\mathbb{P}^n)) \leq 2n$. Since the perverse $\mathbf{t}$-structure has faithful heart, from Lemma 4.3.1 below we get $\mathrm{Ext}^{2n}_{\mathbf{Perv}(\mathbb{P}^n)}(\mathrm{IC}_n, \mathrm{IC}_n) \cong \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_n, \mathrm{IC}_n[2n]) \cong \Bbbk$, and thus $\mathrm{gldim}(\mathbf{Perv}(\mathbb{P}^n)) = 2n$.

4.2.5 Other descriptions of the category of perverse sheaves

There are two other well-known descriptions of the category $\mathbf{Perv}(\mathbb{P}^n)$, namely in terms of finite-dimensional algebras and via Lie algebras. We will not use these descriptions throughout the chapter, with the exception that the classification results in Corollaries 4.4.19 and 4.4.20 rely on the classification of indecomposable perverse sheaves, which is obtained from the finite-dimensional algebras description.

The Lie-theoretic description of the constructible derived category is given by the equivalence $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n) \cong \mathbf{D}^{\mathrm{b}}(\mathcal{O}_0^{\mathfrak{p}}(\mathfrak{sl}_{n+1}(\Bbbk)))$ from [BGS96, Thm. 3.5.3], see also [Ach21, Rem. 7.3.10] for an overview. Here $\mathfrak{p} \subseteq \mathfrak{sl}_{n+1}(\Bbbk)$ is the maximal parabolic subalgebra with block sizes $(n, 1)$. This equivalence identifies the standard $\mathbf{t}$-structure on $\mathbf{D}^{\mathrm{b}}(\mathcal{O}_0^{\mathfrak{p}}(\mathfrak{sl}_{n+1}(\Bbbk)))$ with the perverse $\mathbf{t}$-structure on $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n)$, and thus yields $\mathbf{Perv}(\mathbb{P}^n) \cong \mathcal{O}_0^{\mathfrak{p}}(\mathfrak{sl}_{n+1}(\Bbbk))$.

Furthermore, there is an equivalence $\mathbf{Perv}(\mathbb{P}^n) \cong A_n\text{-}\mathbf{mod}_{\mathrm{fd}}$, where

$$A_n = \Bbbk\left(\begin{tikzcd} 0 \arrow[r, bend left, "b_1"] & 1 \arrow[l, bend left, "a_1"] \arrow[r, bend left, "b_2"] & \cdots \arrow[l, bend left, "a_2"] \arrow[r, bend left, "b_{n-1}"] & n-1 \arrow[l, bend left, "a_{n-1}"] \arrow[r, bend left, "b_n"] & n \arrow[l, bend left, "a_n"] \end{tikzcd}\right) \Big/ \left(\begin{matrix} a_{i-1}a_i \\ b_{i+1}b_i \\ a_ib_i - b_{i-1}a_{i-1} \\ b_na_n \end{matrix}\right).$$

The algebra A_n is directly linked to the Lie-theoretic description by an equivalence $\mathcal{O}_0^{\mathfrak{p}}(\mathfrak{sl}_{n+1}) \cong A_n\text{-}\mathbf{mod}_{\mathrm{fd}}$, see [KS02, Prop. 2.9] and [Str06, Ex. 1.1].

From the finite-dimensional algebras description, it is very easy to check the properties mentioned in Section 4.2.4: for instance one can easily write down the indecomposable projective objects and the standard objects to see that A_n-$\mathbf{mod}_{\mathrm{fd}}$ is highest weight. One can also explicitly determine projective and injective resolutions of the simple A_n-modules, which in particular shows that $\mathrm{gldim}(A_n) = 2n$.

The algebra A_n is special biserial, and thus [BR87, p. 161, Thm.] and [WW85, Prop. 2.3] provide a combinatorial description of the indecomposable A_n-modules. Explicitly, there are the indecomposable projective-injective objects P_i for $0 \le i \le n-1$, and certain *string modules* $\mathrm{M}^{\pm}_{a,b}$ with $0 \le b \le a \le n$, see [PW20, §2.4] or [CL23, §4] for an explicit list. It follows that A_n has $n + (n+1) + 2\binom{n+1}{2} = n + (n+1)^2$ isomorphism classes of indecomposable modules. In Section 4.4.2 below we provide a construction of the string modules in terms of perverse sheaves, which can also be found in [CL23].

Note that it is a special property of $\mathbb{P}^n$ that the indecomposable perverse sheaves can be classified. For more general (partial) flag varieties, the category of perverse sheaves is usually of wild representation type.

4.3 Description of the Serre functor of $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n)$

In this section we show that the IC sheaf IC_n corresponding to the open stratum is a $\mathbb{P}^n$-object in $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n)$, and that the $\mathbb{P}$-twist at IC_n is the inverse Serre functor of $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n)$. To do this, we first need to understand morphisms between simple perverse sheaves, and morphisms from simple to projective perverse sheaves. These technical results will also be used in Section 4.4.

4.3.1 Morphisms between simples

Recall that for a variety X we have by definition $\mathrm{Hom}_X(\underline{\Bbbk}_X, \underline{\Bbbk}_X[r]) = H^r(X)$, see e.g. [Ach21, Rem. 1.2.5]. This allows to determine the morphisms between the shifted simple perverse sheaves in terms of the stratification. We briefly recall this well-known fact, see for instance [KS02, p. 217, after the Remark].

Lemma 4.3.1. *For $0 \le k, l \le n$ and $r \ge 0$, there is an isomorphism of vector spaces*

$$\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_k, \mathrm{IC}_l[r]) \cong H^{r-|k-l|}(\mathbb{P}^l \cap \mathbb{P}^k)$$

In particular, if $k = l$ there is an isomorphism of graded algebras $\mathrm{End}^*_{\mathbb{P}^n}(\mathrm{IC}_k) = H^*(\mathbb{P}^k) \cong \Bbbk[t]/(t^{k+1})$ with $\deg(t) = 2$.

Proof. This is obvious from $\imath_l^* \mathrm{IC}_k \cong \underline{\Bbbk}_{\mathbb{P}^l}[k]$ for $k \ge l$, respectively $\imath_k^! \mathrm{IC}_l \cong \underline{\Bbbk}_{\mathbb{P}^k}[2k-l]$ for $l \ge k$, and the definition of cohomology. $\square$

We want to fix non-zero morphisms $\epsilon^r_{k,l} \colon \mathrm{IC}_k \to \mathrm{IC}_l[r]$ in a way that is compatible with composition. This requires us to inductively fix the (co)units for the recollement adjunctions, as follows.

Suppose we have already fixed the adjunction (co)units $\eta_l \colon \mathrm{id}_{\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^{l+1})} \to \imath_{l,*}\imath_l^*$ and $\varepsilon_l \colon \imath_{l,*}\imath_l^! \to \mathrm{id}_{\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^{l+1})}$ for the recollements corresponding to the strata S^{l+1} with $l+1 \le k < n$, such that $\eta_l \varepsilon_l = \varepsilon_{l-1}\eta_{l-1} \colon \mathrm{IC}_l \to \mathrm{IC}_l[2]$ and $\mathbb{D}(\varepsilon_l)_X = (\eta_l)_{\mathbb{D}(X)}$ for $X \in \mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^{l+1})$, and that the composition $\mathrm{IC}_k \xrightarrow{\eta_{k-1}} \mathrm{IC}_{k-1}[1] \xrightarrow{\varepsilon_{k-1}} \mathrm{IC}_k[2]$ is non-zero.

The inclusion of the $(k+1)$-dimensional stratum $S_{k+1} \cong \mathbb{A}^{k+1}$ provides the recollement

$$\begin{tikzcd} \mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^k) \arrow[r, "\imath_{k,*}"] & \mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^{k+1}) \arrow[l, bend right, "\imath_k^*"'] \arrow[l, bend left, "\imath_k^!"] \arrow[r, "\jmath_{k+1}^*"] & \mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{A}^{k+1}) \arrow[l, bend right, "\jmath_{k+1,!}"'] \arrow[l, bend left, "\jmath_{k+1,*}"] \end{tikzcd}$$

As part of the recollement data, there are the adjunction unit $\tilde{\eta}_k \colon \mathrm{id}_{\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^{k+1})} \to \imath_{k,*}\imath_k^*$ and the counit $\tilde{\varepsilon}_k \colon \imath_{k,*}\imath_k^! \to \mathrm{id}_{\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^{k+1})}$. These can be chosen to be Verdier-dual to each other in the sense that $\mathbb{D}((\tilde{\varepsilon}_k)_{\mathbb{D}(X)}) = (\tilde{\eta}_k)_X$ for $X \in \mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^{k+1})$.

The proof of the following lemma is based on conversations with Jon Woolf.

Lemma 4.3.2. *Let $0 \le k \le n-1$.*

1) *The composition $\mathrm{IC}_k \xrightarrow{\tilde{\varepsilon}_k} \mathrm{IC}_{k+1}[1] \xrightarrow{\tilde{\eta}_k} \mathrm{IC}_k[2]$ is non-zero for $k > 0$.*
2) *The composition $\mathrm{IC}_{k+1} \xrightarrow{\tilde{\eta}_k} \mathrm{IC}_k[1] \xrightarrow{\tilde{\varepsilon}_k} \mathrm{IC}_{k+1}[2]$ is non-zero.*

Proof. To see that the compositions are non-zero, we apply $a_{\mathbb{P}^{k+1},*}$ to them, where $a_{\mathbb{P}^{k+1}} \colon \mathbb{P}^{k+1} \to \mathbf{pt}$. We have $a_{\mathbb{P}^{k+1},*}\mathrm{IC}_k \cong \bigoplus_{i=0}^k \underline{\Bbbk}_{\mathbf{pt}}[-k+2i]$ since

$$\begin{aligned} \mathrm{Hom}_{\mathbf{pt}}(\underline{\Bbbk}_{\mathbf{pt}}, a_{\mathbb{P}^{k+1},*}\mathrm{IC}_k[r]) &\cong \mathrm{Hom}_{\mathbb{P}^{k+1}}(\underline{\Bbbk}_{\mathbb{P}^{k+1}}, \imath_{k,*}\underline{\Bbbk}_{\mathbb{P}^k}[k+r]) \\ &\cong \begin{cases} \Bbbk & \text{if } 0 \le k+r \le 2k \text{ and } k+r \text{ even,} \\ 0 & \text{else,} \end{cases} \end{aligned}$$

and analogously we get $a_{\mathbb{P}^{k+1},*}\mathrm{IC}_{k+1} \cong \bigoplus_{i=0}^{k+1} \underline{\Bbbk}_{\mathbf{pt}}[-k-1+2i]$. Similar computations show that $a_{\mathbb{P}^{k+1},*}\nabla_{k+1} \cong \underline{\Bbbk}_{\mathbf{pt}}[k+1]$ and $a_{\mathbb{P}^{k+1},*}\Delta_{k+1} \cong \underline{\Bbbk}_{\mathbf{pt}}[-k-1]$ (for the latter, use $a_{\mathbb{P}^{k+1},*} = a_{\mathbb{P}^{k+1},!}$).

Hence applying $a_{\mathbb{P}^{k+1},*}$ to the triangles defining $\nabla_{k+1}[1]$ and $\Delta_{k+1}[2]$ yields triangles

$$\bigoplus_{i=0}^{k} \underline{\Bbbk}_{\mathbf{pt}}[-k+2i] \xrightarrow{a_{\mathbb{P}^{k+1},*}(\tilde{\varepsilon}_k)} \bigoplus_{i=0}^{k+1} \underline{\Bbbk}_{\mathbf{pt}}[-k+2i] \to \underline{\Bbbk}_{\mathbf{pt}}[k+2] \to \bigoplus_{i=0}^{k} \underline{\Bbbk}_{\mathbf{pt}}[-k+2i+1],$$
$$\bigoplus_{i=0}^{k+1} \underline{\Bbbk}_{\mathbf{pt}}[-k+2i] \xrightarrow{a_{\mathbb{P}^{k+1},*}(\tilde{\eta}_k)} \bigoplus_{i=0}^{k} \underline{\Bbbk}_{\mathbf{pt}}[-k+2i+2] \to \underline{\Bbbk}_{\mathbf{pt}}[1-k] \to \bigoplus_{i=0}^{k+1} \underline{\Bbbk}_{\mathbf{pt}}[-k+2i+1].$$

Since $\mathrm{Hom}_{\mathbf{pt}}(\underline{\Bbbk}_{\mathbf{pt}}, \underline{\Bbbk}_{\mathbf{pt}}[r]) = 0$ for $r \ne 0$, the morphisms $a_{\mathbb{P}^{k+1},*}(\tilde{\varepsilon}_k)$ and $a_{\mathbb{P}^{k+1},*}(\tilde{\eta}_k)$ must identify all the summands occuring in both their source and target.

It follows that the composition $a_{\mathbb{P}^{k+1},*}(\mathrm{IC}_k \xrightarrow{\tilde{\varepsilon}_k} \mathrm{IC}_{k+1}[1] \xrightarrow{\tilde{\eta}_k} \mathrm{IC}_k[2])$ identifies all the summands occuring in both $a_{\mathbb{P}^{k+1},*}\mathrm{IC}_k$ and $a_{\mathbb{P}^{k+1},*}\mathrm{IC}_k[2]$, and such a common summand exists if and only if $k > 0$.

For the other composition, one sees similarly that $a_{\mathbb{P}^{k+1},*}(\mathrm{IC}_{k+1} \xrightarrow{\tilde{\eta}_k} \mathrm{IC}_k[1] \xrightarrow{\tilde{\varepsilon}_k} \mathrm{IC}_{k+1}[2])$ identifies all the summands occuring in both $a_{\mathbb{P}^{k+1},*}\mathrm{IC}_{k+1}$ and $a_{\mathbb{P}^{k+1},*}\mathrm{IC}_{k+1}[2]$, and thus is non-zero. ☐

In particular, since $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_k, \mathrm{IC}_k[2])$ is 1-dimensional by Lemma 4.3.1, we have $\tilde{\eta}_k\tilde{\varepsilon}_k = \lambda\varepsilon_{k-1}\eta_{k-1}$ for some scalar $\lambda \ne 0$. Set $\eta_k = \frac{1}{\sqrt{\lambda}}\tilde{\eta}_k$ and $\varepsilon_k = \frac{1}{\sqrt{\lambda}}\tilde{\varepsilon}_k$. These are again adjunction (co)units for the recollement for gluing the $(k+1)$-dimensional stratum, and rescaling both by the same factor ensures that the new (co)units are still Verdier-dual to each other. Note that to ensure that the recollement triangles are isomorphic to those obtained from the original (co)units, one also has to rescale the (co)units for the adjunctions between $\jmath_{k+1,!}$, $\jmath_{k+1}^*$ and $\jmath_{k+1,*}$.

By construction, the square

$$\begin{tikzcd} \mathrm{IC}_k \arrow[r, "\eta_{k-1}"] \arrow[d, "\varepsilon_k"'] & \mathrm{IC}_{k-1}[1] \arrow[d, "\varepsilon_{k-1}"] \\ \mathrm{IC}_{k+1}[1] \arrow[r, "\eta_k"] & \mathrm{IC}_k[2] \end{tikzcd} \tag{4.4}$$

now commutes, as desired. Moreover, the composition $\mathrm{IC}_k \xrightarrow{\varepsilon_k} \mathrm{IC}_{k+1}[1] \xrightarrow{\eta_k} \mathrm{IC}_k[2]$ is still non-zero, which completes the induction.

From now on, we assume the (co)units η_k, ε_k of the recollement adjunctions are fixed for all k, and make the square (4.4) commute. With this data, we can now fix the desired morphisms $\epsilon_{k,l}^r\colon \mathrm{IC}_k \to \mathrm{IC}_l[r]$ spanning $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_k, \mathrm{IC}_l[r]) \neq 0$ (with $|k-l| \leq r \leq |k-l| + 2\min(k,l)$ and $r - |k-l|$ even), as follows.

Set $\epsilon_{k,k-1}^1 = \eta_{k-1}\colon \mathrm{IC}_k \to \mathrm{IC}_{k-1}[1]$ and $\epsilon_{k,k+1}^1 = \varepsilon_k\colon \mathrm{IC}_k \to \mathrm{IC}_{k+1}[1]$. Furthermore, for $k > 0$ let $\epsilon_{k,k}^2 = \epsilon_{k-1,k}^1\epsilon_{k,k-1}^1$, which is a generator of $\mathrm{End}_{\mathbb{P}^n}(\mathrm{IC}_k)$ by Lemmas 4.3.1 and 4.3.2. The proof of Lemma 4.3.1 shows that $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_k, \mathrm{IC}_l[r])$ is spanned by the composition

$$\mathrm{IC}_k \xrightarrow{\epsilon_{k,k-1}^1} \mathrm{IC}_{k-1}[1] \xrightarrow{\epsilon_{k-1,k-2}^1} \dots \xrightarrow{\epsilon_{l+1,l}^1} \mathrm{IC}_l[k-l] \xrightarrow{(\epsilon_{l,l}^2)^{(r-k+l)/2}} \mathrm{IC}_l[r]$$

for $l \leq k$, and by

$$\mathrm{IC}_k \xrightarrow{(\epsilon_{k,k}^2)^{(r+k-l)/2}} \mathrm{IC}_k[r+k-l] \xrightarrow{\epsilon_{k,k+1}^1} \mathrm{IC}_{k+1}[r+k-l+1] \xrightarrow{\epsilon_{k+1,k+2}^1} \dots \xrightarrow{\epsilon_{l-1,l}^1} \mathrm{IC}_l[r]$$

for $l \geq k$. We write $\epsilon_{k,l}^r\colon \mathrm{IC}_k \to \mathrm{IC}_l[r]$ for these compositions. Note that $\mathbb{D}(\epsilon_{k,l}^r) = \epsilon_{l,k}^r$ since $\mathbb{D}(\epsilon_{k,k-1}^1) = \epsilon_{k-1,k}^1$.

As an immediate consequence of the relation (4.4), in fact any non-zero composition of the morphisms $\epsilon_{i,i\pm 1}^1$ between IC_k and $\mathrm{IC}_l[r]$ yields the same morphism. This shows that $\epsilon_{l,m}^s\epsilon_{k,l}^r = \epsilon_{k,m}^{r+s}$ if $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_k, \mathrm{IC}_m[r+s]) \neq 0$.

The above construction also yields the following explicit description of the algebra $\bigoplus_{k,l} \mathrm{Hom}_{\mathbb{P}^n}^*(\mathrm{IC}_k, \mathrm{IC}_l)$, as obtained in [KS02, Proof of Prop. 2.9].

Proposition 4.3.3. *There is an isomorphism of graded algebras* $\bigoplus_{k,l} \mathrm{Hom}_{\mathbb{P}^n}^*(\mathrm{IC}_k, \mathrm{IC}_l) \cong E_n$, *where*

$$E_n = \Bbbk\left(\begin{tikzcd} 0 \arrow[r, bend left, "\epsilon_{0,1}"] & 1 \arrow[l, bend left, "\epsilon_{1,0}"] \arrow[r, bend left, "\epsilon_{1,2}"] & \cdots \arrow[l, bend left, "\epsilon_{2,1}"] \arrow[r, bend left, "\epsilon_{n-2,n-1}"] & n-1 \arrow[l, bend left, "\epsilon_{n-1,n-2}"] \arrow[r, bend left, "\epsilon_{n-1,n}"] & n \arrow[l, bend left, "\epsilon_{n,n-1}"] \end{tikzcd}\right) \Big/ \left(\begin{smallmatrix} \epsilon_{1,0}\epsilon_{0,1} \\ \epsilon_{k+1,k}\epsilon_{k,k+1} - \epsilon_{k-1,k}\epsilon_{k,k-1} \end{smallmatrix}\right)$$

with $\deg(\epsilon_{k,k\pm 1}) = 1$.

Proof. The isomorphism $E_n \to \bigoplus_{k,l} \mathrm{Hom}_{\mathbb{P}^n}^*(\mathrm{IC}_k, \mathrm{IC}_l)$ is given by $\epsilon_{k,k\pm 1} \mapsto \epsilon_{k,k\pm 1}^1$, which is well-defined since $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_0, \mathrm{IC}_0[2]) = 0$ and the square (4.4) commutes for all k. Surjectivity follows from the proof of Lemma 4.3.1 and Lemma 4.3.2. One then easily checks that the graded dimensions of both algebras agree. ☐

Remark 4.3.4. For $0 < k < n$ we moreover have $\epsilon_{k,k}^2 = \imath_k^!(\epsilon_{k+1,k}^1)$. Indeed, naturality of the recollement triangles yields the commutative diagram

$$\begin{tikzcd} \mathrm{IC}_k[-1] \arrow[r, "\epsilon_{k,k+1}^1"] \arrow[d, "\imath_k^!(\epsilon_{k+1,k}^1)"'] & \mathrm{IC}_{k+1} \arrow[r] \arrow[d, "\epsilon_{k+1,k}^1"] & \nabla_{k+1} \arrow[r] \arrow[d] & \mathrm{IC}_k \arrow[d] \\ \mathrm{IC}_k[1] \arrow[r, "\mathrm{id}"] & \mathrm{IC}_k[1] \arrow[r] & 0 \arrow[r] & \mathrm{IC}_k[2]. \end{tikzcd}$$

From this it also follows that $\epsilon^2_{k,k} = \imath_k^!(\epsilon^2_{k+1,k+1})$: we have $\epsilon^2_{k,k} = \epsilon^1_{k+1,k}\epsilon^1_{k,k+1}$ and $\imath_k^!(\epsilon^1_{k,k+1}) = \mathrm{id}$, as $\epsilon^1_{k,k+1}$ is the counit for the adjunction between $\imath_{k,*}$ and $\imath_k^!$.

4.3.2 Morphisms between standards and simples

Morphisms from the standard objects to IC sheaves, and dually from IC sheaves to costandard objects, are easily calculated by the recollement adjunctions:

Lemma 4.3.5. *We have*

$$\mathrm{Hom}_{\mathbb{P}^n}(\Delta_k, \mathrm{IC}_l[r]) \cong \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_l, \nabla_k[r]) \cong \begin{cases} \Bbbk & \text{if } l \geq k \text{ and } r = l-k, \\ 0 & \text{else.} \end{cases}$$

Proof. By Verdier duality it suffices to compute $\mathrm{Hom}_{\mathbb{P}^n}(\Delta_k, \mathrm{IC}_l[r])$. For $l \geq k$ we have

$$\begin{aligned} \mathrm{Hom}_{\mathbb{P}^n}(\Delta_k, \mathrm{IC}_l[r]) &\cong \mathrm{Hom}_{\mathbb{P}^k}(\jmath_{k,!}\underline{\Bbbk}_{\mathbb{A}^k}[k], \underline{\Bbbk}_{\mathbb{P}^k}[2k+r-l]) \\ &\cong \mathrm{Hom}_{\mathbb{A}^k}(\underline{\Bbbk}_{\mathbb{A}^k}, \underline{\Bbbk}_{\mathbb{A}^k}[k+r-l]), \end{aligned}$$

and from this the claim follows. The case $l < k$ is similar but easier, as $\jmath_k^*(\mathrm{IC}_l) = 0$. $\square$

By the proof of Lemma 4.3.5, for $l \geq k$ there is a canonical non-zero morphism $\mu_{k,l} \colon \Delta_k \to \mathrm{IC}_l[l-k]$ corresponding to $\mathrm{id}_{\mathbb{A}^k}$ under the adjunctions. In particular, for $k = l$ these are the (co)units from the recollement triangles (4.2) defining Δ_k. Moreover, the proof also shows that $\mu_{k,l}$ is the unique morphisms making the diagram

$$\begin{tikzcd} \Delta_k \arrow[d, "\mu_{k,k}"'] \arrow[dr, dashed, "\mu_{k,l}"] & \\ \mathrm{IC}_k \arrow[r, "\epsilon^{l-k}_{k,l}"] & \mathrm{IC}_l[l-k] \end{tikzcd}$$

commute.

Lemma 4.3.6. *For $0 \leq k, l \leq n$ we have*

$$\mathrm{Hom}_{\mathbb{P}^n}(\Delta_k, \Delta_l[r]) \cong \mathrm{Hom}_{\mathbb{P}^n}(\nabla_l, \nabla_k[r]) \cong \begin{cases} \Bbbk & \text{if } l > k \text{ and } r \in \{l-k-1, l-k\}, \\ \Bbbk & \text{if } l = k \text{ and } r = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By Verdier duality it suffices to compute $\mathrm{Hom}_{\mathbb{P}^n}(\Delta_k, \Delta_l[r])$. For this, the claim follows from Lemma 4.3.5 and the long exact sequence obtained by applying $\mathrm{Hom}_{\mathbb{P}^n}(\Delta_k, -)$ to the triangle (4.2) defining Δ_l. $\square$

The proof of Lemma 4.3.6 also yields the following descriptions of canonical morphisms $\delta^r_{k,l} \colon \Delta_k \to \Delta_l[r]$ spanning $\mathrm{Hom}_{\mathbb{P}^n}(\Delta_k, \Delta_l[r])$:

- For $r = l-k$, we define $\delta^{l-k}_{k,l} \colon \Delta_k \to \Delta_l[l-k]$ as the unique morphism making the diagram

$$\begin{tikzcd} \Delta_k \arrow[r, dashed, "\delta^{l-k}_{k,l}"] \arrow[dr, "\mu_{k,l}"'] & \Delta_l[l-k] \arrow[d, "\mu_{l,l}"] \\ & \mathrm{IC}_l[l-k] \end{tikzcd}$$

  commute.

- For $r = l-k-1$, we define $\delta^{l-k-1}_{k,l}\colon \Delta_k \to \Delta_l[l-k-1]$ as the composition

$$\Delta_k \xrightarrow{\mu_{k,l-1}} \mathrm{IC}_{l-1}[l-1-k] \xrightarrow{\phi^0_{l-1,l}} \Delta_l[l-1-k],$$

  where $\phi^0_{l-1,l}\colon \mathrm{IC}_{l-1} \to \Delta_l$ is the morphism from the recollement triangle (4.2) defining Δ_l.

In particular, the morphisms $\delta_{k,k+1}\colon \Delta_k \to \Delta_{k+1}[1]$ can be used to define the indecomposable projective objects via the triangles (4.3).

4.3.3 Morphisms between simples and projectives

We start by computing the morphisms from simple perverse sheaves to standard objects.

Lemma 4.3.7. *For $0 \le k, l \le n$ and $r \in \mathbb{Z}$ we have*

$$\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_l, \Delta_k[r]) \cong \mathrm{Hom}_{\mathbb{P}^n}(\nabla_k, \mathrm{IC}_l[r]) \cong \begin{cases} \Bbbk & \text{if } r = k+l, \\ \Bbbk & \text{if } r = k-l-1 \text{ and } l < k, \\ 0 & \text{else.} \end{cases}$$

Proof. By Verdier duality it suffices to compute $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_l, \Delta_k[r])$. Applying the functor $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_l, -)$ to (4.2) yields a long exact sequence

$$\ldots \to \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_l, \mathrm{IC}_{k-1}[r]) \to \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_l, \Delta_k[r]) \to \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_l, \mathrm{IC}_k[r]) \to \ldots$$

in which the left and right-hand side can be computed by Lemma 4.3.1, and these Hom spaces are spanned by $\epsilon^r_{l,k-1}$ and $\epsilon^r_{l,k}$, respectively. Since $\epsilon^{r+1}_{l,k-1} = \epsilon^1_{k,k-1}\epsilon^r_{l,k}$, the connecting morphisms $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_l, \mathrm{IC}_k[r]) \to \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_l, \mathrm{IC}_{k-1}[r+1])$ are isomorphisms unless $r = l+k$, or $l < k$ and $r = k-l-2$. $\square$

The proof of Lemma 4.3.7 shows that $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_l, \Delta_k[k+l])$ is spanned by the unique morphism $\phi^{l+k}_{l,k}\colon \mathrm{IC}_l \to \Delta_k[k+l]$ making the diagram

$$\begin{tikzcd} & \mathrm{IC}_l \arrow[dl, dashed, "\phi^{l+k}_{l,k}"'] \arrow[d, "\epsilon^{k+l}_{k,l}"] \\ \Delta_k[k+l] \arrow[r, "\mu_{k,k}"'] & \mathrm{IC}_k[k+l]. \end{tikzcd}$$

commute.

For $l < k$, the proof shows that $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_l, \Delta_k[k-l-1])$ is spanned by the composition

$$\phi^{k-l-1}_{l,k}\colon \mathrm{IC}_l \xrightarrow{\epsilon^{k-l-1}_{l,k-1}} \mathrm{IC}_{k-1}[k-l-1] \xrightarrow{\phi^0_{k-1,k}} \Delta_k[k-l-1],$$

where $\phi^0_{k-1,k}\colon \mathrm{IC}_{k-1} \to \Delta_k$ is the morphism from the recollement triangle (4.2). The non-zero morphisms $\nabla_k \to \mathrm{IC}_l[r]$ are defined by the dual diagrams.

From Lemma 4.3.7 it also follows that all indecomposable projective perverse sheaves except P_n are projective-injective:

Proposition 4.3.8. *For $0 \le k \le n-1$ and $0 \le l \le n$ we have*

$$\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_l, P_k[r]) \cong \begin{cases} \Bbbk & \text{if } l = k \text{ and } r = 0, \\ 0 & \text{else.} \end{cases}$$

In particular, if $k < n$ then P_k is the injective hull of IC_k in $\mathbf{Perv}(\mathbb{P}^n)$.

Proof. That P_k is the injective hull of IC_k in $\mathbf{Perv}(\mathbb{P}^n)$ is immediate from the first part, using $\mathrm{Ext}^1_{\mathbf{Perv}(\mathbb{P}^n)}(-,-) \cong \mathrm{Hom}_{\mathbb{P}^n}(-,-[1])$.

To compute $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_l, P_k[r])$, we apply the functor $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_l,-)$ to (4.3) to get the long exact sequence

$$\ldots \to \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_l, \Delta_{k+1}[r]) \to \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_l, P_k[r]) \to \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_l, \Delta_k[r]) \to \ldots$$

We claim that $\delta_{k,k+1}\colon \Delta_k \to \Delta_{k+1}[r]$ induces isomorphisms in all degrees.

Case 1: $l > k$. By Lemma 4.3.7, $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_l, \Delta_{k+1}[r+1])$ and $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_l, \Delta_k[r])$ are 1-dimensional for $r = l+k$, and vanish otherwise. By the construction of the morphisms spanning these Hom spaces, we have to show that in the diagram

$$\begin{tikzcd}
\Delta_k[k+l] \arrow[rr, dashed, "\delta_{k,k+1}"] \arrow[dd, "\mu_{k,k}"'] & & \Delta_{k+1}[k+l+1] \arrow[dd, "\mu_{k+1,k+1}"] \\
& \mathrm{IC}_l \arrow[ul, dashed, "\phi^{k+l}_{l,k}"'] \arrow[ur, dashed, "\phi^{k+1+l}_{l,k+1}"] \arrow[dl, "\epsilon^{k+l}_{l,k}"'] \arrow[dr, "\epsilon^{k+l+1}_{l,k+1}"] & \\
\mathrm{IC}_k[k+l] \arrow[rr, "\epsilon^1_{k,k+1}"'] & & \mathrm{IC}_{k+1}[k+l+1].
\end{tikzcd}$$

the triangle consisting of dashed arrows commutes. By construction we have $\epsilon^{k+l+1}_{l,k+1} = \epsilon^1_{k,k+1}\epsilon^{k+l}_{l,k}$, and the outer square commutes by the definition of the morphism $\delta_{k,k+1}\colon \Delta_k \to \Delta_{k+1}[1]$. An easy diagram chase then shows $\phi^{k+1+l}_{l,k+1} = \delta_{k,k+1}\phi^{k+l}_{l,k}$, as required.

Case 2: $l = k$. From Lemma 4.3.7 we know that $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_l, \Delta_k[r])$ is 1-dimensional for $r \in \{0, 2k+1\}$ and $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_l, \Delta_{k+1}[r+1])$ is 1-dimensional for $r = 2k$, and they vanish otherwise. By the same argument as in Case 1, $\delta_{k,k+1}$ induces an isomorphism for $r = 2k$, and it follows from the long exact sequence that $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_k, P_k[r])$ is 1-dimensional for $r = 0$ and vanishes otherwise.

Case 3: $l < k$. From Lemma 4.3.7 we know that $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_l, \Delta_k[r])$ is 1-dimensional for $r \in \{k-l-1, k+l\}$ and $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_l, \Delta_{k+1}[r+1])$ is 1-dimensional for $r \in \{k-l, k+l+1\}$, and they vanish otherwise. For $r = k+l$, that $\delta_{k,k+1}$ induces an isomorphism follows by the same argument as in Case 1.

For $r = k-l-1$, we need to show $\phi^{k-l}_{l,k+1} = \delta_{k,k+1}\phi^{k-l-1}_{l,k}$, which amounts to checking that the diagram

$$\begin{tikzcd}
\mathrm{IC}_{k-1}[k-l-1] \arrow[rr, "\epsilon^1_{k-1,k}"] \arrow[dd, "\phi^0_{k-1,k}"'] & & \mathrm{IC}_k[k-l] \arrow[dd, "\phi^0_{k,k+1}"] \\
& \mathrm{IC}_l \arrow[ul, "\epsilon^{k-l-1}_{l,k-1}"'] \arrow[ur, "\epsilon^{k-l}_{l,k}"] \arrow[dl, dashed, "\phi^{k-l-1}_{l,k}"'] \arrow[dr, dashed, "\phi^{k-l}_{l,k+1}"] & \\
\Delta_k[k-l-1] \arrow[rr, "\delta_{k,k+1}"] & & \Delta_{k+1}[k-l]
\end{tikzcd}$$

commutes. By construction, we have $\epsilon^{k-l}_{l,k} = \epsilon^1_{k-1,k}\epsilon^{k-l-1}_{l,k-1}$, and the claim follows by a straightforward diagram chase provided the outer square commutes.

To see this, observe that $\epsilon^1_{k-1,k}\epsilon^1_{k,k-1} = \epsilon^1_{k+1,k}\epsilon^1_{k,k+1}$, the definition of $\delta_{k,k+1}$, and the axiom

(TR3) give the commutative diagram

$$\begin{tikzcd}
\mathrm{IC}_{k-1} \arrow[r, "\phi^0_{k-1,k}"] \arrow[d, "\epsilon^1_{k-1,k}"] & \Delta_k \arrow[r, "\mu_{k,k}"] \arrow[d, "\delta_{k,k+1}"] & \mathrm{IC}_k \arrow[r, "\epsilon^1_{k,k-1}"] \arrow[d, "\epsilon^1_{k,k+1}"] & \mathrm{IC}_{k-1}[1] \arrow[d, "\epsilon^1_{k-1,k}"] \\
\mathrm{IC}_k[1] \arrow[r, "\phi^0_{k,k+1}"] & \Delta_{k+1}[1] \arrow[r, "\mu_{k+1,k+1}"] & \mathrm{IC}_{k+1}[1] \arrow[r, "\epsilon^1_{k+1,k}"] & \mathrm{IC}_k[2],
\end{tikzcd}$$

in which the left square is the desired commutative square (up to shift). □

4.3.4 $\mathbb{P}$-like simple perverse sheaves

By the explicit description of the simple perverse sheaves as constant sheaves on the stratum closures, it is obvious that they are $\mathbb{P}^k$-like objects. However, only one of them is Calabi–Yau:

Proposition 4.3.9.

1) *For $0 \leq k \leq n$, the simple perverse sheaf IC_k is a $\mathbb{P}^k$-like object in $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n)$.*
2) *The simple perverse sheaf IC_n is a $\mathbb{P}^n$-object in $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n)$.*
3) *IC_k is not Calabi–Yau in $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n)$ if $k < n$.*

Proof.

1) Immediate by Lemma 4.3.1.
2) Since IC_n is $\mathbb{P}^n$-like by the first part and $\mathrm{Hom}^*_{\mathbb{P}^n}(X, Y)$ is finite-dimensional for any $X, Y \in \mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n)$, we only need to check the Calabi–Yau property. As the perverse **t**-structure has faithful heart, by Lemma 4.2.2 it is enough to check that the composition pairing

$$\mathrm{Hom}_{\mathbb{P}^n}(P, \mathrm{IC}_n[r]) \otimes \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_n, P[2n-r]) \to \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_n, \mathrm{IC}_n[2n]) \cong \Bbbk$$

is non-degenerate for any r and any indecomposable projective object $P \in \mathbf{Perv}(\mathbb{P}^n)$.

For $P = P_k$ with $k < n$, we apply $\mathrm{Hom}_{\mathbb{P}^n}(-, \mathrm{IC}_n)$ to (4.3). From Lemmas 4.3.5 and 4.3.6 it follows that all connecting morphisms are isomorphisms, so $\mathrm{Hom}_{\mathbb{P}^n}(P, \mathrm{IC}_n[r]) = 0$ for all r. We also have $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_n, P[2n-r]) = 0$ for all r by Proposition 4.3.8, and thus the only non-trivial case is $P = P_n$. Alternatively, for this one can also use that $P_k = I_k$ is the projective cover and injective hull of IC_k, and that the perverse **t**-structure has faithful heart.

As $P_n = \Delta_n$, we know from Lemma 4.3.5 that $\mathrm{Hom}_{\mathbb{P}^n}(P_n, \mathrm{IC}_n[r])$ is one-dimensional if $r = 0$, and vanishes otherwise. From Lemma 4.3.7 we know that $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_n, P_n[2n-r])$ is one-dimensional for $r = 0$ and vanishes otherwise. Moreover, by construction of $\phi^{2n}_{n,n}$ the composition $\mathrm{IC}_n \xrightarrow{\phi^{2n}_{n,n}} \Delta_n[2n] \xrightarrow{\mu_{n,n}} \mathrm{IC}_n[2n]$ is precisely $\epsilon^{2n}_{n,n} \colon \mathrm{IC}_n \to \mathrm{IC}_n[2n]$, and thus the composition pairing is non-degenerate.
3) For $k < n$, the composition pairing

$$\mathrm{Hom}_{\mathbb{P}^n}(P, \mathrm{IC}_k[r]) \otimes \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_k, P[2k-r]) \to \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_k, \mathrm{IC}_k[2k]) \cong \Bbbk$$

cannot be non-degenerate since for $P = P_k = I_k$ the tensor factors on the left-hand side are non-zero only for $r = 0$ and $r = 2k$, respectively. □

4.3.5 Characterization of the Serre functor

The following characterization of Serre functors is adapted from [MS08, Thm. 3.4]. The main difference is that we would like to start with a triangulated functor which looks like a derived functor, but is not a priori known to arise as a derived functor. Showing that such a functor is indeed a derived functor is hard if one only uses triangulated categories, see e.g. [Ric16]. However, this technical issue can be resolved by using ∞-enhancements.

Lemma 4.3.10. *Let $\mathscr{A}$ be a finite-length abelian category with finitely many simples, enough projectives and enough injectives. Assume that $\mathscr{A}$ is of finite global dimension, all the projective-injective objects in $\mathscr{A}$ have isomorphic top and socle, and that there is a projective generator P of $\mathscr{A}$ admitting a presentation $0 \to P \to X_1 \to X_2$ with X_1, X_2 projective-injective. Let $\mathbf{D}^+_\infty(\mathscr{A})$ be the derived category of $\mathscr{A}$ in the ∞-categorical sense as defined in [Lur17, Variant 1.3.2.8], and let $F\colon \mathbf{D}^+_\infty(\mathscr{A}) \to \mathbf{D}^+_\infty(\mathscr{A})$ be a functor of ∞-categories.*

If the triangulated functor $hF\colon \mathbf{D}^+(\mathscr{A}) \to \mathbf{D}^+(\mathscr{A})$ satisfies the conditions

1) hF restricts to an equivalence $hF\colon \mathbf{D}^b(\mathscr{A}) \to \mathbf{D}^b(\mathscr{A})$,

2) $hF(\mathbf{D}^+(\mathscr{A})^{\geq 0}) \subseteq \mathbf{D}^+(\mathscr{A})^{\geq 0}$, where $\mathbf{D}^+(\mathscr{A})^{\geq 0}$ denotes the non-negative part of the standard $\boldsymbol{t}$-structure,

3) $hF(\mathbf{Inj}(\mathscr{A})) \subseteq \mathbf{Proj}(\mathscr{A})$,

4) $H^0 \circ hF$ preserves the subcategory $\mathbf{ProjInj}(\mathscr{A})$ of projective-injective objects, and restricted to this category is isomorphic to the inverse Nakayama functor ν^{-1},

then $hF\colon \mathbf{D}^{\mathrm{b}}(\mathscr{A}) \to \mathbf{D}^{\mathrm{b}}(\mathscr{A})$ is an inverse Serre functor for $\mathbf{D}^{\mathrm{b}}(\mathscr{A})$.

Proof. The argument essentially follows the proof of [MS08, Thm. 3.4]. We write $\mathbb{R}\nu^{-1}\colon \mathbf{D}^+_\infty(\mathscr{A}) \to \mathbf{D}^+_\infty(\mathscr{A})$ for the right derived functor of the inverse Nakayama functor in the ∞-categorical sense, see [Lur17, Ex. 1.3.3.4] for the definition (actually we use the dual version, obtained by $\mathbf{D}^+_\infty(\mathscr{A}) = \mathbf{D}^-_\infty(\mathscr{A}^{\mathrm{op}})^{\mathrm{op}}$). Then $h\mathbb{R}\nu^{-1} = \mathrm{R}\nu^{-1}\colon \mathbf{D}^+(\mathscr{A}) \to \mathbf{D}^+(\mathscr{A})$ is the usual right derived functor, and its restriction $\mathrm{R}\nu^{-1}\colon \mathbf{D}^{\mathrm{b}}(\mathscr{A}) \to \mathbf{D}^{\mathrm{b}}(\mathscr{A})$ is the inverse Serre functor by Proposition 4.2.8. We show that $\mathbb{R}\nu^{-1} \cong F$, which then implies the claim.

Step 1: On the subcategory $\mathbf{Proj}(\mathscr{A})$, we have $H^0 \circ hF \cong \nu^{-1}$.

Proof: By assumption, the projective generator P of $\mathscr{A}$ admits a presentation $0 \to P \to X_1 \to X_2$ with X_1, X_2 projective-injective. Since $H^0 \circ hF$ and ν^{-1} are left exact, and $H^0 \circ hF \cong \nu^{-1}$ on the subcategory $\mathbf{ProjInj}(\mathscr{A})$, it follows that $H^0(F(P)) \cong \nu^{-1}(P)$. It is easy to see that this isomorphism is functorial in P and compatible with taking direct sums and summands, which proves the claim.

Step 2: $hF\colon \mathbf{D}^{\mathrm{b}}(\mathscr{A}) \to \mathbf{D}^{\mathrm{b}}(\mathscr{A})$ commutes with the (inverse) Serre functor.

Proof: From the Yoneda lemma it follows that the Serre functor commutes with autoequivalences.

Step 3: On the subcategory $\mathbf{Inj}(\mathscr{A})$, we have $H^0 \circ hF \circ \nu^{-1} \cong \nu^{-1} \circ H^0 \circ hF$.

Proof: By assumption we have $hF \cong H^0 \circ hF$ on $\mathbf{Inj}(\mathscr{A})$. With this and Step 2 we get

$$\mathbb{S}^{-1} \circ H^0 \circ hF \cong \mathbb{S}^{-1} \circ hF \cong hF \circ \mathbb{S}^{-1} \cong hF \circ \nu^{-1}.$$

Observe that $H^0 \circ hF$ takes $\mathbf{Inj}(\mathscr{A})$ to $\mathscr{A}$, and therefore taking H^0 on both sides yields

$$\nu^{-1} \circ H^0 \circ hF \cong H^0 \circ \mathbb{S}^{-1} \circ H^0 \circ hF \cong H^0 \circ hF \circ \nu^{-1}.$$

Step 4: $H^0 \circ hF$ is fully faithful on the subcategory $\mathbf{Proj}(\mathscr{A})$.

Proof: By assumption, $H^0 \circ hF$ is isomorphic to ν^{-1} on the full subcategory $\mathbf{ProjInj}(\mathscr{A})$, and $\nu^{-1}\colon \mathbf{ProjInj}(\mathscr{A}) \to \mathbf{ProjInj}(\mathscr{A})$ is an autoequivalence. Let $\mathscr{C} \subseteq \mathscr{A}$ be the full subcategory of objects M admitting a presentation $0 \to M \to X_1 \to X_2$ with X_1, X_2 projective-injective. Then $H^0 \circ hF$ restricts/extends to $H^0 \circ hF\colon \mathscr{C} \to \mathscr{C}$. Moreover, $(H^0 \circ hF)^{-1}$ can be extended to $(H^0 \circ hF)^{-1}\colon \mathscr{C} \to \mathscr{C}$ by setting $(H^0 \circ hF)^{-1}(\ker(\phi\colon X_1 \to X_2)) = \ker((H^0 \circ hF)^{-1}(\phi)\colon (H^0 \circ hF)^{-1}(X_1) \to (H^0 \circ hF)^{-1}(X_2))$. By assumption all projective objects lie in $\mathscr{C}$, and thus $H^0 \circ hF$ is fully faithful on $\mathbf{Proj}(\mathscr{A})$.

Step 5: We have $H^0 \circ hF \cong \nu^{-1}$ as functors $\mathscr{A} \to \mathscr{A}$.

Proof: On $\mathbf{Inj}(\mathscr{A})$ we get

$$(H^0 \circ hF)^2 \cong \nu^{-1} \circ H^0 \circ hF \cong H^0 \circ hF \circ \nu^{-1},$$

where we apply Step 1 using that $H^0 \circ hF$ takes $\mathbf{Inj}(\mathscr{A})$ to $\mathbf{Proj}(\mathscr{A})$, and Step 3. As both ν^{-1} and $H^0 \circ hF$ take $\mathbf{Inj}(\mathscr{A})$ to $\mathbf{Proj}(\mathscr{A})$, and $H^0 \circ hF$ is fully faithful on $\mathbf{Proj}(\mathscr{A})$ (and thus an equivalence to its image) by Step 4, it follows that $H^0 \circ hF \cong \nu^{-1}$ on $\mathbf{Inj}(\mathscr{A})$. Moreover, both functors are left exact, so the claim follows from this by replacing any object by an injective resolution and applying an argument similar to the proof of Step 1.

Step 6: We have $hF \cong \mathrm{R}\nu^{-1}$ as functors $\mathbf{D}^+(\mathscr{A}) \to \mathbf{D}^+(\mathscr{A})$.

Proof: Note that $\mathbf{D}^+_\infty(\mathscr{A})$ with the standard $\mathbf{t}$-structure satisfies the assumptions of [Lur17, Thm. 1.3.3.2] (in particular, it is right complete by [Lur17, Prop. 1.3.3.16]). Therefore $F\colon \mathbf{D}^+_\infty(\mathscr{A}) \to \mathbf{D}^+_\infty(\mathscr{A})$ is up to isomorphism the only functor of ∞-categories restricting to $t_{\leq 0} \circ hF|_{\mathscr{A}} \in N(\mathbf{Fun}_{\mathrm{lex}}(\mathscr{A},\mathscr{A}))$, where $N(\mathbf{Fun}_{\mathrm{lex}}(\mathscr{A},\mathscr{A}))$ denotes the nerve (see [Lur18a, Tag 002M]) of the category of left exact functors $\mathscr{A} \to \mathscr{A}$. On the other hand, by the above we know $H^0 \circ hF|_{\mathscr{A}} \cong \nu^{-1} = t_{\leq 0} \circ h\mathbb{R}\nu^{-1}|_{\mathscr{A}}$ as ordinary functors $\mathscr{A} \to \mathscr{A}$, and thus they are also isomorphic in $N(\mathbf{Fun}_{\mathrm{lex}}(\mathscr{A},\mathscr{A}))$. By [Lur17, Thm. 1.3.3.2] it follows that $F \cong \mathbb{R}\nu^{-1}$ as functors of ∞-categories, and therefore $hF \cong h\mathbb{R}\nu^{-1} = \mathrm{R}\nu^{-1}$ as triangulated functors $\mathbf{D}^+(\mathscr{A}) \to \mathbf{D}^+(\mathscr{A})$.

Step 7: As $\mathscr{A}$ has finite global dimension, $\mathrm{R}\nu^{-1}\colon \mathbf{D}^{\mathrm{b}}(\mathscr{A}) \to \mathbf{D}^{\mathrm{b}}(\mathscr{A})$ is the inverse Serre functor by Proposition 4.2.8, and by Step 6 we have $hF \cong \mathrm{R}\nu^{-1}$. $\square$

Since IC_n is a $\mathbb{P}^n$-object in $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n)$ by Proposition 4.3.9, we can consider the $\mathbb{P}$-twist $\mathsf{PT}_{\mathrm{IC}_n}\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n) \to \mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n)$ as in Definition 4.2.3. By applying Lemma 4.3.10 to $\mathsf{PT}_{\mathrm{IC}_n}$, we obtain:

Theorem 4.3.11. *The $\mathbb{P}$-twist $\mathsf{PT}_{\mathrm{IC}_n}\colon \mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n) \to \mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n)$ is the inverse Serre functor.*

Proof. In order to apply Lemma 4.3.10 we first have to check the technical assumptions.

Recall from Sections 4.2.3 and 4.2.4 that $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n) \cong \mathbf{D}^{\mathrm{b}}(\mathbf{Perv}(\mathbb{P}^n))$, and that the category $\mathbf{Perv}(\mathbb{P}^n)$ has finite global dimension, and enough projectives and enough injectives. By Proposition 4.3.8, all indecomposable projective objects except the projective cover P_n of IC_n are injective, and moreover there is an exact sequence $0 \to P_n \to P_{n-1} \to P_{n-2}$ in $\mathbf{Perv}(\mathbb{P}^n)$ (this can be seen from the Δ-flags).

Since IC_n is a $\mathbb{P}^n$-object in $\mathbf{D}^{\mathrm{b}}(\mathbf{Perv}(\mathbb{P}^n)) \cong \mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n)$, it is also $\mathbb{P}^n$-like in $\mathbf{D}^+(\mathbf{Perv}(\mathbb{P}^n))$. We want to consider the $\mathbb{P}$-twist $\mathsf{PT}_{\mathrm{IC}_n}\colon \mathbf{D}^+(\mathbf{Perv}(\mathbb{P}^n)) \to \mathbf{D}^+(\mathbf{Perv}(\mathbb{P}^n))$. To define this, we use the usual dg enhancement $\widetilde{\mathscr{D}} = \mathbf{Ch}^+(\mathbf{Inj}(\mathbf{Perv}(\mathbb{P}^n)))$ of $\mathbf{D}^+(\mathbf{Perv}(\mathbb{P}^n))$. By Remark 4.2.4, we can use $\mathrm{RHom}_{\mathbf{Perv}(\mathbb{P}^n)}(\mathrm{IC}_n,-)$ instead of $\mathrm{Hom}_{\widetilde{\mathscr{D}}}(\mathrm{IC}_n,-)$ to define the $\mathbb{P}$-twist. Observe that $\mathrm{RHom}_{\mathbf{Perv}(\mathbb{P}^n)}(\mathrm{IC}_n,Y)$ is degreewise finite-dimensional for all $Y \in \mathbf{Ch}^+(\mathbf{Inj}(\mathbf{Perv}(\mathbb{P}^n)))$, and thus the tensor product $\mathrm{RHom}_{\mathscr{D}}(\mathrm{IC}_n, Y) \otimes \mathrm{IC}_n$ exists in $\mathbf{D}^+(\mathbf{Perv}(\mathbb{P}^n))$ for all $Y \in \widetilde{\mathscr{D}}$, as required. Hence $\mathsf{PT}_{\mathrm{IC}_n} \colon \mathbf{D}^+(\mathbf{Perv}(\mathbb{P}^n)) \to \mathbf{D}^+(\mathbf{Perv}(\mathbb{P}^n))$ is well-defined.

By construction, we have $\mathsf{PT}_{\mathrm{IC}_n} = H^0(\widetilde{\mathsf{PT}}_{\mathrm{IC}_n}) \colon \mathbf{D}^+(\mathbf{Perv}(\mathbb{P}^n)) \to \mathbf{D}^+(\mathbf{Perv}(\mathbb{P}^n))$, where $\widetilde{\mathsf{PT}}_{\mathrm{IC}_n} \colon \widetilde{\mathscr{D}} \to \widetilde{\mathscr{D}}$ is a dg functor. By definition (see [Lur17, Variant 1.3.2.8]) we have $\mathbf{D}^+_\infty(\mathbf{Perv}(\mathbb{P}^n)) = N_{\mathrm{dg}}(\mathbf{Ch}^+(\mathbf{Inj}(\mathbf{Perv}(\mathbb{P}^n))))$. Here N_{dg} denotes the dg nerve from [Lur17, Constr. 1.3.1.6], see also [Lur18a, Tag 00PK]. By [Lur17, Prop. 1.3.1.20], $N_{\mathrm{dg}}(\widetilde{\mathsf{PT}}_{\mathrm{IC}_n})$ is a functor of ∞-categories $\mathbf{D}^+_\infty(\mathbf{Perv}(\mathbb{P}^n)) \to \mathbf{D}^+_\infty(\mathbf{Perv}(\mathbb{P}^n))$, and by passing to homotopy categories, by [Lur17, Rem. 1.3.1.11] we recover $hN_{\mathrm{dg}}(\widetilde{\mathsf{PT}}_{\mathrm{IC}_n}) = H^0(\widetilde{\mathsf{PT}}_{\mathrm{IC}_n}) = \mathsf{PT}_{\mathrm{IC}_n} \colon \mathbf{D}^+(\mathbf{Perv}(\mathbb{P}^n)) \to \mathbf{D}^+(\mathbf{Perv}(\mathbb{P}^n))$.

It remains to check the conditions from Lemma 4.3.10:

1) As $\mathbf{D}^+(\mathbf{Perv}(\mathbb{P}^n))^{\geq 0}$ is the extension closure of $\mathbf{D}^+(\mathbf{Perv}(\mathbb{P}^n))^{>0}$ and the IC sheaves, it suffices to show $\mathsf{PT}_{\mathrm{IC}_n}(\mathbf{D}^+(\mathbf{Perv}(\mathbb{P}^n))^{>0}) \subseteq \mathbf{D}^+(\mathbf{Perv}(\mathbb{P}^n))^{\geq 0}$ and $\mathsf{PT}_{\mathrm{IC}_n}(\mathrm{IC}_k) \in \mathbf{D}^+(\mathbf{Perv}(\mathbb{P}^n))^{\geq 0}$ for all $0 \leq k \leq n$.

   For this we use the triangles (4.1) defining the $\mathbb{P}$-twist. First, observe that $\mathsf{PT}_{\mathrm{IC}_n}(\mathrm{IC}_n) \cong \mathrm{IC}_n[-2n] \in \mathbf{D}^+(\mathbf{Perv}(\mathbb{P}^n))^{\geq 0}$. Furthermore, for an object $X \in \mathbf{D}^+(\mathbf{Perv}(\mathbb{P}^n))^{>0}$ or $X = \mathrm{IC}_k$ with $k < n$, $\mathrm{RHom}_{\mathbf{Perv}(\mathbb{P}^n)}(\mathrm{IC}_n, X)$ is cohomologically concentrated in positive degrees. Thus $\mathrm{RHom}_{\mathbf{Perv}(\mathbb{P}^n)}(\mathrm{IC}_n, X) \otimes \mathrm{IC}_n[-2]$ has cohomologies in degrees > 2, and $\mathrm{RHom}_{\mathbf{Perv}(\mathbb{P}^n)}(\mathrm{IC}_n, X) \otimes \mathrm{IC}_n$ has cohomologies in degrees > 0, so $\mathrm{cone}(t^* \otimes \mathrm{id} - \mathrm{id} \otimes t)(X)$ has cohomologies in degrees > 0. As $X \in \mathbf{D}^+(\mathbf{Perv}(\mathbb{P}^n))^{\geq 0}$, it follows that $\mathsf{PT}_{\mathrm{IC}_n}(X) \in \mathbf{D}^+(\mathbf{Perv}(\mathbb{P}^n))^{\geq 0}$.

2) Let $I \in \mathbf{Perv}(\mathbb{P}^n)$ be indecomposable injective. If $I = I_k$ for $k < n$, then $\mathrm{RHom}_{\mathbf{Perv}(\mathbb{P}^n)}(\mathrm{IC}_n, I_k) = 0$ and therefore $\mathsf{PT}_{\mathrm{IC}_n}(I_k) \cong I_k = P_k$.

   For $I = I_n = \nabla_n$ we have $\mathrm{RHom}_{\mathbf{Perv}(\mathbb{P}^n)}(\mathrm{IC}_n, I_n) \cong \Bbbk$ (concentrated in degree 0), and so by evaluating (4.1) at I_n we obtain the diagram

$$\begin{tikzcd}
\mathrm{IC}_n[-2] \arrow[r, "\epsilon^2_{n,n}"] & \mathrm{IC}_n \arrow[r] \arrow[d, "\mathbb{D}(\mu_{n,n})"'] & \mathrm{cone}(\epsilon^2_{n,n}) \arrow[dl, dashed] \\
& I_n \arrow[dl] & \\
\mathsf{PT}_{\mathrm{IC}_n}(I_n) & &
\end{tikzcd}$$

   where $\epsilon^2_{n,n} \colon \mathrm{IC}_n[-2] \to \mathrm{IC}_n$ is the generator of $\mathrm{End}^*_{\mathbb{P}^n}(\mathrm{IC}_n)$.

   From the long exact sequence obtained by applying $\mathrm{Hom}_{\mathbb{P}^n}(-, I_n)$ to the horizontal triangle and Lemma 4.3.5 it follows that $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{cone}(\epsilon^2_{n,n}), I_n)$ is 1-dimensional, i.e. the induced morphism $\mathrm{cone}(\epsilon^2_{n,n}) \to I_n$ is unique up to scalar. Therefore it suffices to find a non-split triangle of the form $\mathrm{cone}(\epsilon^2_{n,n}) \to I_n \to P_n \to \mathrm{cone}(\epsilon^2_{n,n})[1]$. We know that $\mathrm{Hom}_{\mathbb{P}^n}(I_n, P_n) = \mathrm{Hom}_{\mathbb{P}^n}(\nabla_n, \Delta_n)$ is 1-dimensional, spanned by the composition $f \colon I_n \xrightarrow{\mathbb{D}(\phi^0_{n-1,n})} \mathrm{IC}_{n-1} \xrightarrow{\phi^0_{n-1,n}} P_n$. From the octahedral axiom (using the triangle (4.2) defining $P_n = \Delta_n$ as well as its dual) we obtain the diagram

$$\begin{tikzcd}
I_n \arrow[r, "\mathbb{D}(\phi^0_{n-1,n})"] \arrow[d, "="'] & \mathrm{IC}_{n-1} \arrow[r, "\epsilon^1_{n-1,n}"] \arrow[d, "\phi^0_{n-1,n}"] & \mathrm{IC}_n[1] \arrow[r] \arrow[d] & I_n[1] \arrow[d, "="] \\
I_n \arrow[r, "f"] \arrow[d, "\mathbb{D}(\phi^0_{n-1,n})"'] & P_n \arrow[r] \arrow[d, "="] & \mathrm{cone}(f) \arrow[r] \arrow[d] & I_n[1] \arrow[d] \\
\mathrm{IC}_{n-1} \arrow[r, "\phi^0_{n-1,n}"] & P_n \arrow[r] & \mathrm{IC}_n \arrow[r, "\epsilon^1_{n,n-1}"] \arrow[d, "\epsilon^2_{n,n}"] & \mathrm{IC}_{n-1}[1] \arrow[d, "\epsilon^1_{n-1,n}"] \\
 & & \mathrm{IC}_n[2] \arrow[r, "="] & \mathrm{IC}_n[2].
\end{tikzcd}$$

Thus $\mathrm{cone}(f)[-1] \cong \mathrm{cone}(\epsilon^2_{n,n} \colon \mathrm{IC}_n[-2] \to \mathrm{IC}_n)$, and hence the second row is the desired triangle (up to rotation).

3) From the above it also follows that $\mathsf{PT}_{\mathrm{IC}_n}$ is the identity functor on the full subcategory $\mathbf{ProjInj}(\mathbf{Perv}(\mathbb{P}^n))$. Since the endomorphism algebra of the direct sum of the projective-injective objects is symmetric, we also have $\nu^{-1} \cong \mathrm{id}$ on $\mathbf{ProjInj}(\mathbf{Perv}(\mathbb{P}^n))$ by [MS08, Prop. 3.5]. $\square$

Remark 4.3.12. In particular, Theorem 4.3.11 recovers the description of the Serre functor of $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^1)$ from [Woo10, §3.1, p. 680], since $\mathsf{PT}_{\mathrm{IC}_1} \cong \mathsf{ST}^2_{\mathrm{IC}_1}$.

4.3.6 Other descriptions of the Serre functor

Recall the equivalences $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n) \cong \mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathcal{O}^{\mathfrak{p}}_0(\mathfrak{sl}_{n+1}(\Bbbk))) \cong \mathbf{D}^{\mathrm{b}}(A_n\text{-}\mathbf{mod}_{\mathrm{fd}})$ mentioned in Section 4.2.5. The Serre functor of $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n)$ also has explicit descriptions in terms of finite-dimensional algebras and in terms of Lie algebras, and furthermore there is a description of the Serre functor for the constructible category of the full flag variety. We summarize these results and explain how they are related to Theorem 4.3.11.

In terms of finite-dimensional algebras, the Serre functor is given by the derived functor of the Nakayama functor by results of Happel [Hap88, Prop. 4.10]. This actually underlies our argument, as the proof of the criterion Lemma 4.3.10 (which we adapted from [MS08, Thm. 3.4]) compares the candidate Serre functor with the derived functor of the Nakayama functor.

In [MS08] Mazorchuk and Stroppel provide a description of the Serre functor of $\mathbf{D}^{\mathrm{b}}(\mathcal{O}^{\mathfrak{p}}_0(\mathfrak{sl}_{n+1}(\Bbbk)))$ in Lie-theoretic language. For this, the criterion [MS08, Thm. 3.4] is first used to show that the Serre functor of $\mathbf{D}^{\mathrm{b}}(\mathcal{O}_0(\mathfrak{sl}_{n+1}(\Bbbk)))$ is given by the (derived) shuffling functor $\mathrm{Sh}^2_{w_0}$, where $\mathrm{Sh}_{w_0} = \mathrm{Sh}_{s_{i_1}} \dots \mathrm{Sh}_{s_{i_r}}$ for a reduced expression $s_{i_1} \dots s_{i_r}$ of the longest element w_0 of the Weyl group. Alternatively, the Serre functor of $\mathbf{D}^{\mathrm{b}}(\mathcal{O}_0(\mathfrak{sl}_{n+1}(\Bbbk)))$ is also isomorphic to the (derived) Arkhipov twisting functor $\mathrm{Tw}^2_{w_0}$, where $\mathrm{Tw}_{w_0} = \mathrm{Tw}_{s_{i_1}} \dots \mathrm{Tw}_{s_{i_r}}$.

By [MS08, Prop. 4.4], the Serre functor of $\mathbf{D}^{\mathrm{b}}(\mathcal{O}^{\mathfrak{p}}_0(\mathfrak{sl}_{n+1}(\Bbbk)))$ is then $\mathrm{Sh}^2_{w_0}[-2\ell(w^{\mathfrak{p}}_0)]$, where $w^{\mathfrak{p}}_0 \in W_{\mathfrak{p}} \cong S_1 \times S_n$ is the longest element of the parabolic Weyl group. Rather than applying the criterion Lemma 4.3.10, the proof uses the inclusion $\mathbf{D}^{\mathrm{b}}(\mathcal{O}^{\mathfrak{p}}_0(\mathfrak{sl}_{n+1}(\Bbbk))) \hookrightarrow \mathbf{D}^{\mathrm{b}}(\mathcal{O}_0(\mathfrak{sl}_{n+1}(\Bbbk)))$ and its left and right adjoints (i.e. the (derived) Zuckerman functors) to "push down" the description of the Serre functor from $\mathbf{D}^{\mathrm{b}}(\mathcal{O}_0(\mathfrak{sl}_{n+1}(\Bbbk)))$. In particular, note that since the inclusion $\mathbf{D}^{\mathrm{b}}(\mathcal{O}^{\mathfrak{p}}_0(\mathfrak{sl}_{n+1}(\Bbbk))) \hookrightarrow \mathbf{D}^{\mathrm{b}}(\mathcal{O}_0(\mathfrak{sl}_{n+1}(\Bbbk)))$ is not full, the Serre functor of $\mathbf{D}^{\mathrm{b}}(\mathcal{O}^{\mathfrak{p}}_0(\mathfrak{sl}_{n+1}(\Bbbk)))$ is not the restriction of that of $\mathbf{D}^{\mathrm{b}}(\mathcal{O}_0(\mathfrak{sl}_{n+1}(\Bbbk)))$.

In the language of perverse sheaves, in [BBM04] Beilinson, Bezrukavnikov and Mirković provide a description of the Serre functor for the full flag variety G/B, where (for us) $G =$

$\mathrm{GL}_{n+1}(\Bbbk)$ and $B \subseteq G$ is the usual Borel subgroup of upper triangular matrices. By [BBM04, Prop. 2.5], the Serre functor of $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(G/B)$ is given by the Radon transform $(R^*_{w_0})^2$, where $R^*_{w_0} = R^*_{s_{i_1}} \dots R^*_{s_{i_r}}$.

Under the equivalences $\mathcal{O}_0(\mathfrak{sl}_{n+1}(\Bbbk)) \cong \mathbf{Perv}(G/B)$ and $\mathcal{O}_0^{\mathfrak{p}}(\mathfrak{sl}_{n+1}(\Bbbk)) \cong \mathbf{Perv}(G/P) = \mathbf{Perv}(\mathbb{P}^n)$, the inclusion functor corresponds to $\pi^![-d] \cong \pi^*[d]$ and the (dual) Zuckerman functors are $\pi_![d]$ and $\pi_*[-d]$, see e.g. [BGS96, p. 504, Rem. (2)]. Here $P \subseteq G$ is the parabolic subgroup with block sizes $(n, 1)$, $\pi\colon G/B \to G/P$ is the canonical map, and $d = \dim G/B - \dim G/P = \ell(w_0^{\mathfrak{p}})$. Using this, one can apply the purely formal argument from [MS08, Prop. 4.4] to obtain a description of the Serre functor of $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n)$ from the description of the Serre functor of $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(G/B)$.

Combining Theorem 4.3.11 with the above observations yields the following relation between $\mathsf{PT}_{\mathrm{IC}_n}$ and the Radon transform $R^!_{w_0}$, and also a decomposition of $\mathsf{PT}_{\mathrm{IC}_n}$ into a sequence of spherical twists:

Corollary 4.3.13.

1) The square

$$\begin{array}{ccc} \mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(G/B) & \xrightarrow{(R^!_{w_0})^2[2d]} & \mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(G/B) \\ {\scriptstyle \pi^*[d]}\big\uparrow & & \big\downarrow{\scriptstyle \pi_*[-d]} \\ \mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n) & \xrightarrow{\mathsf{PT}_{\mathrm{IC}_n}} & \mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n) \end{array}$$

commutes up to natural isomorphism.

2) For any reduced expression $w_0 = s_{i_1} \dots s_{i_r}$ there is a natural isomorphism

$$\mathsf{PT}_{\mathrm{IC}_n}[2\ell(w_0) - 2\ell(w_0^{\mathfrak{p}})] \cong (\mathsf{ST}_{P_{n-i_1}} \dots \mathsf{ST}_{P_{n-i_r}})^2.$$

Proof.

1) By [MS08, Prop. 4.1 and Prop. 4.4], the inverse Serre functor of $\mathbf{D}^{\mathrm{b}}(\mathcal{O}_0^{\mathfrak{p}}(\mathfrak{sl}_{n+1}(\Bbbk)))$ is $\hat{Z}\mathbb{S}_{\mathfrak{b}}^{-1}\,\mathrm{incl}[2d]$, where $\mathrm{incl}\colon \mathbf{D}^{\mathrm{b}}(\mathcal{O}_0^{\mathfrak{p}}(\mathfrak{sl}_{n+1}(\Bbbk))) \hookrightarrow \mathbf{D}^{\mathrm{b}}(\mathcal{O}_0(\mathfrak{sl}_{n+1}(\Bbbk)))$, $\mathbb{S}_{\mathfrak{b}}^{-1}$ is the inverse Serre functor of $\mathbf{D}^{\mathrm{b}}(\mathcal{O}_0(\mathfrak{sl}_{n+1}(\Bbbk)))$, and $\hat{Z}$ the dual Zuckerman functor. In geometric language, incl is $\pi^*[d]$ and $\hat{Z}$ is $\pi_*[-d]$, and $(R^!_{w_0})^2$ is the inverse Serre functor of $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(G/B)$ by [BBM04, Prop. 2.5 and Fact 2.2]. The claim then follows from Theorem 4.3.11 and uniqueness of the Serre functor.
2) By Theorem 4.3.11 and [MS08, Prop. 4.4] the inverse Serre functor of $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n) \cong \mathbf{D}^{\mathrm{b}}(\mathcal{O}_0^{\mathfrak{p}}(\mathfrak{sl}_{n+1}(\Bbbk)))$ is

$$\mathsf{PT}_{\mathrm{IC}_n} \cong \mathbb{S}^{-1} \cong \mathrm{Sh}_{w_0}^{-2}[2\ell(w_0^{\mathfrak{p}})] = (\mathrm{Sh}_{s_{i_r}}^{-1} \dots \mathrm{Sh}_{s_{i_1}}^{-1})^2[2\ell(w_0^{\mathfrak{p}})].$$

By [Len21, Thm. 4.14] there is a natural isomorphism $\mathrm{Sh}_{s_i}^{-1} \cong \mathsf{ST}_{P_{n-i}}[-1]$, which proves the claim. ☐

For $n = 1$, we in particular get $\mathsf{ST}^2_{\mathrm{IC}_1} \cong \mathsf{PT}_{\mathrm{IC}_1} \cong \mathsf{ST}^2_{P_0}[-2]$, and in fact we even have $\mathsf{ST}_{\mathrm{IC}_1} \cong \mathsf{ST}_{P_0}[-1]$ by [Len21, Thm. 3.6 and Thm. 3.10].

4.4 Classification of $\mathbb{P}$-objects in $\mathbf{Perv}(\mathbb{P}^n)$

In this section we classify the $\mathbb{P}$-objects and $\mathbb{P}$-like objects in $\mathbf{Perv}(\mathbb{P}^n) \subset \mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{P}^n)$. For this we first determine the indecomposable Calabi–Yau objects. After that, we introduce certain string objects, and show that all of them are $\mathbb{P}$-like.

4.4.1 Calabi–Yau objects

The following easy lemma provides obstructions for Calabi–Yau objects in the presence of projective-injective objects.

Lemma 4.4.1. *Let $\mathscr{A}$ be a finite-length Hom-finite Krull–Schmidt abelian category with enough projectives and finite global dimension, and let $P \in \mathscr{A}$ be an indecomposable projective-injective object.*

1) *If P has isomorphic top and socle, then P is 0-Calabi–Yau in $\mathbf{D}^{\mathrm{b}}(\mathscr{A})$.*
2) *If $X \in \mathscr{A}$ involves a composition factor $\operatorname{top}(P)$ or $\operatorname{soc}(P)$, then X cannot be d-Calabi–Yau for $d > 0$.*

Proof.

1) This is clear since the Serre functor, which is the Nakayama functor, fixes such projective-injective objects.
2) Let $P \in \mathscr{A}$ be projective-injective and assume that $X \in \mathscr{A}$ is d-Calabi–Yau with $d > 0$. By definition, this means that the composition pairing

$$\operatorname{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathscr{A})}(P, X[r]) \otimes \operatorname{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathscr{A})}(X, P[d-r]) \to \operatorname{Hom}_{\mathbf{D}^{\mathrm{b}}(\mathscr{A})}(X, X[d])$$

is non-degenerate for all $r \in \mathbb{Z}$. If X involves a simple subquotient $\operatorname{top}(P)$, then for $r = 0$ the first tensor factor is non-zero while the second one is not. If X involves a simple subquotient $\operatorname{soc}(P)$, then for $r = d$ the second tensor factor is non-zero while the first is not. Thus in these cases the pairing cannot be non-degenerate, a contradiction. ☐

As an application, we recover the classification of the indecomposable Calabi–Yau objects in $\mathbf{Perv}(\mathbb{P}^n)$, which was obtained algebraically in [Maz25, §7.4].

Corollary 4.4.2. *An indecomposable object $E \in \mathbf{Perv}(\mathbb{P}^n)$ is Calabi–Yau if and only if $E \in \{\mathrm{IC}_n\} \cup \{P_i \mid 0 \leq i \leq n-1\}$.*

Proof. The simple object IC_n is $2n$-Calabi–Yau by Proposition 4.3.9, while the projective-injective objects $P_i \in \mathbf{Perv}(\mathbb{P}^n)$ for $0 \leq i \leq n-1$ are 0-Calabi–Yau by Lemma 4.4.1.

It is obvious that objects in the heart of a $\mathbf{t}$-structure cannot be d-Calabi–Yau for $d < 0$, and that only projective objects in the heart can be 0-Calabi–Yau: if $E \in \mathbf{Perv}(\mathbb{P}^n)$ is 0-Calabi–Yau, then $\operatorname{Hom}_{\mathbb{P}^n}(E, X[1]) \cong \operatorname{Hom}_{\mathbb{P}^n}(X[1], E)^\vee = 0$ for all $X \in \mathbf{Perv}(\mathbb{P}^n)$, so E has to be projective. Moreover, by Lemma 4.4.1 no object in $\mathbf{Perv}(\mathbb{P}^n)$ involving simple subquotients IC_k for $0 \leq k < n$ can be d-Calabi–Yau with $d > 0$. As $\operatorname{Hom}_{\mathbb{P}^n}(\mathrm{IC}_n, \mathrm{IC}_n[1]) = 0$, the only indecomposable object such that all of its simple subquotients are IC_n is IC_n itself. ☐

Alternatively, in the proof of Corollary 4.4.2 one can also use the classification of indecomposable perverse sheaves (using the language of finite-dimensional algebras), and [PW20, Prop. 2] or Proposition 4.4.13 below, to show that there are no 0-Calabi–Yau objects besides the projective-injective objects.

4.4.2 String objects

For $0 \leq b \leq a \leq n$ we recursively define the *string objects* $\mathrm{M}^+_{a,b}$ as follows. Set $\mathrm{M}^+_{a,a} = \mathrm{IC}_a$ and $\mathrm{M}^+_{a,a-1} = \Delta_a$, and for $a \geq b+2$ define $\mathrm{M}^+_{a,b} = \operatorname{cone}(\psi_{a-2,b})[-1]$, where $\psi_{a-2,b} \colon \mathrm{M}^+_{a-2,b} \to \Delta_a[1]$ is a non-zero morphism that will be fixed recursively. Hence $\mathrm{M}^+_{a,b}$ fits into a triangle

$$\Delta_a \xrightarrow{\iota_{a,b}} \mathrm{M}^+_{a,b} \xrightarrow{\pi_{a,b}} \mathrm{M}^+_{a-2,b} \xrightarrow{\psi_{a-2,b}} \Delta_a[1]. \tag{4.5}$$

We also define $\mathrm{M}^-_{a,b} = \mathbb{D}(\mathrm{M}^+_{a,b})$; alternatively these can be obtained inductively by the dual construction. Note that the string objects $\mathrm{M}^{\pm}_{a,b}$ lie in $\mathbf{Perv}(\mathbb{P}^n)$.

To properly define $\psi_{a-2,b}$, so that $\mathrm{M}^+_{a,b}$ is well-defined, we need:

Lemma 4.4.3. *Let $a \geq b+2$ and assume by induction that $\mathrm{M}^+_{a-2,b}$ is already defined. Then $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a-2,b}, \Delta_a[1])$ is 1-dimensional.*

Proof. For $a = b+2$ and $a = b+3$, the claim follows from Lemma 4.3.7 respectively Lemma 4.3.6.

For $a > b+3$ we apply $\mathrm{Hom}_{\mathbb{P}^n}(-, \Delta_a[1])$ to the triangle (4.5), which by induction is unique up to rescaling of the morphisms. From the construction of $\mathrm{M}^+_{a-4,b}$ and Lemmas 4.3.6 and 4.3.7 it follows that $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a-4,b}, \Delta_a[r]) = 0$ for $r \leq 2$, and thus

$$\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a-2,b}, \Delta_a[1]) \cong \mathrm{Hom}_{\mathbb{P}^n}(\Delta_{a-2}, \Delta_a[1]).$$

By Lemma 4.3.6, this is 1-dimensional, as claimed. □

From the proof we obtain the following explicit definition of a canonical non-zero morphism $\psi_{a-2,b} \colon \mathrm{M}^+_{a-2,b} \to \Delta_a[1]$:

- For $a = b+2$, we take $\psi_{a-2,a-2} = \phi^1_{a-2,a} \colon \mathrm{IC}_{a-2} \to \Delta_a[1]$.
- For $a = b+3$, we define the morphism $\psi_{a-2,a-3}$ as the composition $\Delta_{a-2} \xrightarrow{\mu_{a-2,a-1}} \mathrm{IC}_{a-1}[1] \xrightarrow{\phi^0_{a-1,a}} \Delta_a[1]$.
- For $a > b+3$, the morphism $\psi_{a-2,b} \colon \mathrm{M}^+_{a-2,b} \to \Delta_a[1]$ is uniquely defined by the commutative diagram

$$\begin{tikzcd} \mathrm{M}^+_{a-2,b} \arrow[r, dashed, "\psi_{a-2,b}"] & \Delta_a[1] \\ \Delta_{a-2}, \arrow[u, "\iota_{a-2,b}"] \arrow[ur, "\delta^1_{a-2,a}"'] & \end{tikzcd}$$

  where $\iota_{a-2,b}$ is fixed by the choice of the triangle (4.5) defining $\mathrm{M}^+_{a-2,b}$.

This completes the construction of the string objects $\mathrm{M}^{\pm}_{a,b}$.

In terms of finite-dimensional algebras, the string objects as defined above are precisely the *string modules* mentioned in Section 4.2.5.

4.4.3 String objects are $\mathbb{P}$-like

We show that all the string objects $\mathrm{M}^{\pm}_{a,b}$ are $\mathbb{P}^k$-like, where k depends on a and b by an explicit formula. Note that with the exception of $\mathrm{M}^{\pm}_{n,n} = \mathrm{IC}_n$, the string objects cannot be $\mathbb{P}^k$-objects, since Corollary 4.4.2 obstructs them from having the Calabi–Yau property.

Morphisms between string objects and IC sheaves

We want to understand the total endomorphism spaces $\mathrm{End}^*_{\mathbb{P}^n}(\mathrm{M}^{\pm}_{a,b})$. Due to the inductive definition of the string objects, we first need to compute morphisms between $\mathrm{M}^{\pm}_{a,b}$ and some IC sheaves.

Lemma 4.4.4. *Let $0 \leq b \leq a \leq n$ and $r \in \mathbb{Z}$.*

1) If $a-b$ is even, then

$$\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a,b}, \mathrm{IC}_b[r]) \cong \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_b, \mathrm{M}^-_{a,b}[r]) \cong \begin{cases} \Bbbk & \text{if } 0 \le r \le 2b \text{ and } r \text{ even}, \\ 0 & \text{otherwise.} \end{cases}$$

2) If $a-b$ is odd, then

$$\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a,b}, \mathrm{IC}_b[r]) \cong \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_b, \mathrm{M}^-_{a,b}[r]) \cong 0.$$

Proof. In both cases the first isomorphism is given by Verdier duality, so it suffices to compute $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a,b}, \mathrm{IC}_b[r])$.

We prove both statements by induction on $a-b$. As the argument for the inductive step is the same in both cases, we only give details for the first statement.

1) For the base case $a = b$, we have $\mathrm{M}^+_{a,b} = \mathrm{IC}_b$ and the claim is immediate by Lemma 4.3.1.

   For the inductive step, we apply the functor $\mathrm{Hom}_{\mathbb{P}^n}(-, \mathrm{IC}_b)$ to (4.5) to get the long exact sequence

$$\ldots \to \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a-2,b}, \mathrm{IC}_b[r]) \to \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a,b}, \mathrm{IC}_b[r]) \to \mathrm{Hom}_{\mathbb{P}^n}(\Delta_a, \mathrm{IC}_b[r]) \to \ldots$$

   As $a > b$, we have $\mathrm{Hom}_{\mathbb{P}^n}(\Delta_a, \mathrm{IC}_b[r]) = 0$ for all r by Lemma 4.3.5. Hence

$$\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a,b}, \mathrm{IC}_b[r]) \cong \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a-2,b}, \mathrm{IC}_b[r]),$$

   and so the statement follows from the inductive hypothesis.

2) For the base case $a = b+1$, we have $\mathrm{M}^+_{a,b} = \Delta_a$ and $\mathrm{Hom}_{\mathbb{P}^n}(\Delta_a, \mathrm{IC}_b[r]) = 0$ for any r by Lemma 4.3.5. The claim follows from this by the same arguments as in the first part. □

The proof for $a-b$ even also yields a canonical non-zero morphism $m^r_{a,b} \colon \mathrm{M}^+_{a,b} \to \mathrm{IC}_b[r]$, which is defined as the composition

$$\mathrm{M}^+_{a,b} \xrightarrow{\pi_{a,b}} \mathrm{M}^+_{a-2,b} \xrightarrow{\pi_{a-2,b}} \ldots \xrightarrow{\pi_{b+2,b}} \mathrm{IC}_b \xrightarrow{\epsilon^r_{b,b}} \mathrm{IC}_b[r].$$

Since $\epsilon^{r+2}_{b,b} = \epsilon^2_{b,b}\epsilon^r_{b,b}$, we also obtain the commutative diagram

$$\begin{tikzcd} \mathrm{M}^+_{a,b} \arrow[d, "m^r_{a,b}"'] \arrow[dr, "m^{r+2}_{a,b}"] & \\ \mathrm{IC}_b[r] \arrow[r, "\epsilon^2_{b,b}"] & \mathrm{IC}_b[r+2]. \end{tikzcd} \tag{4.6}$$

Lemma 4.4.5. *Let $0 \le b \le a \le n$ and $r \in \mathbb{Z}$.*

1) If $a-b$ is even, then

$$\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a,b}, \mathrm{IC}_a[r]) \cong \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_a, \mathrm{M}^-_{a,b}[r]) \cong \begin{cases} \Bbbk & \text{if } 0 \le r \le a+b \text{ and } r \text{ even}, \\ 0 & \text{otherwise.} \end{cases}$$

2) If $a-b$ is odd, then

$$\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a,b}, \mathrm{IC}_a[r]) \cong \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_a, \mathrm{M}^-_{a,b}[r]) \cong \begin{cases} \Bbbk & \text{if } 0 \le r \le a-b-1 \text{ and } r \text{ even}, \\ 0 & \text{otherwise.} \end{cases}$$

3) For $0 \leq i \leq n-a$ we have

$$\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a,b}, \mathrm{IC}_{a+i}[r]) \cong \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a,b}, \mathrm{IC}_a[r-i]).$$

Proof. For 1) and 2), by Verdier duality it suffices to compute $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a,b}, \mathrm{IC}_a[r])$. We prove the claim by induction on $a-b$. The base cases are $a=b$ for $a-b$ even, respectively $a=b+1$ for $a-b$ odd, and for these the claim holds by Lemma 4.3.1 and Lemma 4.3.7, respectively.

For the inductive step, we apply $\mathrm{Hom}_{\mathbb{P}^n}(-, \mathrm{IC}_a)$ to (4.5) to get the long exact sequence

$$\cdots \to \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a-2,b}, \mathrm{IC}_a[r]) \to \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a,b}, \mathrm{IC}_a[r]) \to \mathrm{Hom}_{\mathbb{P}^n}(\Delta_a, \mathrm{IC}_a[r]) \to \cdots.$$

By Lemma 4.3.7, $\mathrm{Hom}_{\mathbb{P}^n}(\Delta_a, \mathrm{IC}_a[r])$ is 1-dimensional for $r=0$ and vanishes otherwise. By restriction to $\mathbb{P}^{a-2}$, we have $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a-2,b}, \mathrm{IC}_a[r]) \cong \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a-2,b}, \mathrm{IC}_{a-2}[r-2])$. In particular, this is concentrated in degrees ≥ 2, and hence the claim follows.

The last claim is immediate from $i^!_a \mathrm{IC}_{a+i} \cong \mathrm{IC}_a[-i]$. □

Remark 4.4.6. From the proof of Lemma 4.4.5 we obtain the following explicit description of a morphism $n^r_{a,b}$ spanning $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a,b}, \mathrm{IC}_a[r])$. For $r=0$, we define $n^0_{a,b} \colon \mathrm{M}^+_{a,b} \to \mathrm{IC}_a$ as the unique morphism making the diagram

$$\begin{tikzcd}
\mathrm{M}^+_{a-r,b} \arrow[r, dashed, "n^0_{a-r,b}"] & \mathrm{IC}_{a-r} \\
\Delta_{a-r} \arrow[u, "\iota_{a-r,b}"] \arrow[ur, "\mu_{a-r,a-r}"'] &
\end{tikzcd}$$

commute. For $0 < r \leq a-b$, we define $n^r_{a,b} \colon \mathrm{M}^+_{a,b} \to \mathrm{IC}_a[r]$ as the composition

$$\mathrm{M}^+_{a,b} \xrightarrow{\pi_{a,b}} \mathrm{M}^+_{a-2,b} \xrightarrow{\pi_{a-2,b}} \dots \xrightarrow{\pi_{a-r+2,b}} \mathrm{M}^+_{a-r,b} \xrightarrow{n^0_{a-r,b}} \mathrm{IC}_{a-r} \xrightarrow{\epsilon^r_{a-r,a}} \mathrm{IC}_a[r].$$

For $r > a-b$ (this can only happen if $a-b$ is even), we define $n^r_{a,b}$ as the composition

$$\mathrm{M}^+_{a,b} \xrightarrow{\pi_{a,b}} \mathrm{M}^+_{a-2,b} \xrightarrow{\pi_{a-2,b}} \dots \xrightarrow{\pi_{b+2,b}} \mathrm{IC}_b \xrightarrow{\epsilon^r_{b,a}} \mathrm{IC}_a[r].$$

Moreover, for $0 \leq i \leq n-a$, a canonical morphism $\mathrm{M}^+_{a,b} \to \mathrm{IC}_{a+i}[r]$ is given by the composition

$$\mathrm{M}^+_{a,b} \xrightarrow{n^{r-i}_{a,b}} \mathrm{IC}_a[r-i] \xrightarrow{\epsilon^i_{a,a+i}} \mathrm{IC}_{a+i}[r].$$

Remark 4.4.7. The proof of Lemma 4.4.5 and the octahedral axiom for the composition $n^0_{a,b}\iota_{a,b} = \mu_{a,a}$ also yields triangles

$$\mathrm{M}^-_{a-1,b} \to \mathrm{M}^+_{a,b} \xrightarrow{n^0_{a,b}} \mathrm{IC}_a \to \mathrm{M}^-_{a-1,b}[1]. \tag{4.7}$$

By Verdier duality one also obtains triangles $\mathrm{IC}_a \to \mathrm{M}^-_{a,b} \to \mathrm{M}^+_{a-1,b} \to \mathrm{IC}_a[1]$.

Lemma 4.4.8. *Let $0 \leq b \leq a \leq n$ and $0 \leq r \leq a+b-2$ if $a-b$ is even, respectively $0 \leq r \leq a-b-3$ if $a-b$ is odd. Then the diagram*

$$\begin{tikzcd}
\mathrm{M}^+_{a,b} \arrow[r, "n^r_{a,b}"] \arrow[dr, "n^{r+2}_{a,b}"'] & \mathrm{IC}_a[r] \arrow[d, "\epsilon^2_{a,a}"] \\
& \mathrm{IC}_a[r+2]
\end{tikzcd}$$

commutes up to a non-zero scalar.

Proof. We prove the claim by induction on $a-b$. In the base cases $\mathrm{End}^*_{\mathbb{P}^n}(\mathrm{IC}_a)$ respectively $\mathrm{Hom}^*_{\mathbb{P}^n}(\Delta_a, \mathrm{IC}_a)$ (depending on the parity of $a-b$) there is nothing to show.

For the inductive step, for $r > 0$ Lemma 4.4.5 and its proof yields the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a,b}, \mathrm{IC}_a[r]) \arrow[r, "\cong"] \arrow[d] & \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a-2,b}, \mathrm{IC}_a[r]) \arrow[r, "\cong"] \arrow[d] & \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a-2,b}, \mathrm{IC}_{a-2}[r-2]) \arrow[d] \\
\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a,b}, \mathrm{IC}_a[r+2]) \arrow[r, "\cong"] & \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a-2,b}, \mathrm{IC}_a[r+2]) \arrow[r, "\cong"] & \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a-2,b}, \mathrm{IC}_{a-2}[r]).
\end{tikzcd}$$

Here the vertical morphisms are given by postcomposition with $\epsilon^2_{a,a}$ and $\imath^!_{a-2}(\epsilon^2_{a,a})$, respectively. By Remark 4.3.4 we have $\imath^!_{a-2}(\epsilon^2_{a,a}) = \epsilon^2_{a-2,a-2}$, and therefore by induction it only remains to show the claim for $r = 0$.

For $r = 0$, the square

$$\begin{tikzcd}
\mathrm{M}^+_{a,b} \arrow[r, "\pi_{a,b}"] \arrow[d, "n^0_{a,b}"'] & \mathrm{M}^+_{a-2,b} \arrow[d, "n^2_{a-2,b}"] \\
\mathrm{IC}_a \arrow[r, "\epsilon^2_{a,a}"] & \mathrm{IC}_a[2]
\end{tikzcd}$$

commutes up to a possibly zero scalar, as $\mathrm{cone}(\pi_{a,b}) = \Delta_a[1]$ and $\mathrm{Hom}_{\mathbb{P}^n}(\Delta_a, \mathrm{IC}_a[r]) = 0$ for $r > 0$. By Remark 4.4.6, the composition $n^2_{a,b} = n^2_{a-2,b}\pi_{a,b}$ is non-zero, and thus we have to show that it factors through $\epsilon^2_{a,a} \colon \mathrm{IC}_a \to \mathrm{IC}_a[2]$.

For this we apply $\mathrm{Hom}_{\mathbb{P}^n}(-, \mathrm{IC}_a)$ to the triangle (4.7) to get the long exact sequence

$$\cdots \to \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_a, \mathrm{IC}_a[2]) \to \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a,b}, \mathrm{IC}_a[2]) \to \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^-_{a-1,b}, \mathrm{IC}_a[2]) \to \dots.$$

By Lemmas 4.3.1 and 4.4.5 we know that $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_a, \mathrm{IC}_a[2])$ is spanned by the generator $\epsilon^2_{a,a}$ of $\mathrm{End}^*_{\mathbb{P}^n}(\mathrm{IC}_a)$ and $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a,b}, \mathrm{IC}_a[2])$ is spanned by the composition $n^2_{a-2,b}\pi_{a,b}$, so it suffices to show $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^-_{a-1,b}, \mathrm{IC}_a[2]) = 0$.

By restriction to $\mathbb{P}^{a-1}$, we have $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^-_{a-1,b}, \mathrm{IC}_a[2]) \cong \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^-_{a-1,b}, \mathrm{IC}_{a-1}[1])$, and that this vanishes can be seen by applying $\mathrm{Hom}_{\mathbb{P}^n}(-, \mathrm{IC}_{a-1})$ to the Verdier-dual version of the triangle (4.7) defining $\mathrm{M}^-_{a-1,b}$. ☐

Lemma 4.4.9. *Let $0 \le b \le a \le n$ and $r \in \mathbb{Z}$.*

1) If $a-b$ is even, then

$$\begin{aligned}
\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_b, \mathrm{M}^+_{a,b}[r]) &\cong \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^-_{a,b}, \mathrm{IC}_b[r]) \\
&\cong \begin{cases} \Bbbk & \text{if } a-b \le r \le a+b \text{ and } r \text{ even}, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

2) If $a-b$ is odd, then

$$\begin{aligned}
\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_b, \mathrm{M}^+_{a,b}[r]) &\cong \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^-_{a,b}, \mathrm{IC}_b[r]) \\
&\cong \begin{cases} \Bbbk & \text{if } 0 \le r \le \min(2b, a-b) \text{ and } r \text{ even}, \\ \Bbbk & \text{if } \max(2b, a-b) \le r \le a+b \text{ and } r \text{ odd}, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Proof. By Verdier duality it suffices to compute $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_b, \mathrm{M}^+_{a,b}[r])$.

1) Applying the functor $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_b, -)$ to the triangle (4.7) yields the long exact sequence

$$\ldots \to \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_b, \mathrm{M}^-_{a-1,b}[r]) \to \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_b, \mathrm{M}^+_{a,b}[r]) \to \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_b, \mathrm{IC}_a[r]) \to \ldots.$$

As $a-b$ is even, we have $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_b, \mathrm{M}^-_{a-1,b}[r]) \cong 0$ by Lemma 4.4.4.2). Therefore $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_b, \mathrm{M}^+_{a,b}[r]) \cong \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_b, \mathrm{IC}_a[r])$ and the claim follows by Lemma 4.3.1.

2) We distinguish two cases to avoid having to determine connecting morphisms (note that $a = 3b$ is impossible since $a-b$ is odd).

If $a > 3b$, we apply the functor $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_b, -)$ to the triangle (4.7) to get the long exact sequence

$$\ldots \to \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_b, \mathrm{M}^-_{a-1,b}[r]) \to \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_b, \mathrm{M}^+_{a,b}[r]) \to \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_b, \mathrm{IC}_a[r]) \to \ldots,$$

and the claim follows by using Lemma 4.4.4.1) for the left-hand side and Lemma 4.3.1 for the right-hand side.

If $a < 3b$, we do induction on $a-b$. The base case $a = b+1$ is given by Lemma 4.3.7. For the inductive step, applying the functor $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_b, -)$ to (4.5) yields the long exact sequence

$$\ldots \to \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_b, \Delta_a[r]) \to \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_b, \mathrm{M}^+_{a,b}[r]) \to \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_b, \mathrm{M}^+_{a-2,b}[r]) \to \ldots.$$

The claim follows by using Lemma 4.3.7 for the left-hand side and the inductive hypothesis for the right-hand side. $\square$

Remark 4.4.10. For $a-b$ odd, the case distinction in the proof of Lemma 4.4.9 avoids analysis of the connecting morphisms. Note that to compute $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_b, \mathrm{M}^+_{a,b})$ (i.e. the case $r = 0$) one can always use the argument for $a < 3b$ to obtain

$$\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_b, \mathrm{M}^+_{a,b}) \cong \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_b, \mathrm{M}^+_{a-2,b}) \cong \ldots \cong \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{IC}_b, \Delta_{b+1}).$$

Hence a canonical non-zero morphism $\mathrm{IC}_b \to \mathrm{M}^+_{a,b}$ is defined by the commutative diagram

$$\begin{tikzcd} \mathrm{IC}_b \arrow[d, dashed, "\exists!"'] \arrow[dr, "\phi^0_{b,b+1}"] & \\ \mathrm{M}^+_{a,b} \arrow[r, "\pi_{b+2,b}\cdots\pi_{a,b}"] & \Delta_{b+1} \end{tikzcd}$$

By induction on $a-b$, the octahedral axiom for the composition $\mathrm{IC}_b \to \mathrm{M}^+_{a,b} \xrightarrow{\pi_{a,b}} \mathrm{M}^+_{a-2,b}$ yields triangles

$$\mathrm{IC}_b \to \mathrm{M}^+_{a,b} \to \mathrm{M}^+_{a,b+1} \to \mathrm{IC}_b[1]. \tag{4.8}$$

Dually, there are also triangles $\mathrm{M}^-_{a,b+1} \to \mathrm{M}^-_{a,b} \to \mathrm{IC}_b \to \mathrm{M}^-_{a,b+1}[1]$. These triangles and the ones from Remark 4.4.7 are used in [CL23] to inductively construct the string objects, starting from $\mathrm{M}^\pm_{a,a} = \mathrm{IC}_a$.

Morphisms between standard objects and string objects

Lemma 4.4.11. *For* $0 \le i < \frac{a-b}{2}$ *we have*

$$\mathrm{Hom}_{\mathbb{P}^n}(\Delta_{a-2i}, \mathrm{M}^+_{a,b}[r]) \cong \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^-_{a,b}, \nabla_{a-2i}[r]) \cong \begin{cases} \Bbbk & \text{if } r = 2i, \\ 0 & \text{else.} \end{cases}$$

Proof. By Verdier duality it suffices to compute $\mathrm{Hom}_{\mathbb{P}^n}(\Delta_{a-2i}, \mathrm{M}^+_{a,b}[r])$.

If $a-b$ is odd, we apply $\mathrm{Hom}_{\mathbb{P}^n}(\Delta_{a-2i}, -)$ to the triangle (4.8) and the first triangle in (4.7) defining $\mathrm{M}^+_{a,b+1}$. Since $a-2i > b$, we have $\mathrm{Hom}_{\mathbb{P}^n}(\Delta_{a-2i}, \mathrm{IC}_b[r]) = 0$ for all r by Lemma 4.3.5. Moreover, since the ∇-flag of $\mathrm{M}^-_{a-1,b+1}$ does not involve ∇_{a-2i} it follows that $\mathrm{Hom}_{\mathbb{P}^n}(\Delta_{a-2i}, \mathrm{M}^-_{a-1,b+1}[r]) = 0$ for all r. Together these observations imply

$$\mathrm{Hom}_{\mathbb{P}^n}(\Delta_{a-2i}, \mathrm{M}^+_{a,b}[r]) \cong \mathrm{Hom}_{\mathbb{P}^n}(\Delta_{a-2i}, \mathrm{M}^+_{a,b+1}[r]) \cong \mathrm{Hom}_{\mathbb{P}^n}(\Delta_{a-2i}, \mathrm{IC}_a[r]),$$

which has the claimed form by Lemma 4.3.5.

The argument for $a-b$ even is similar, using the first triangle in (4.7) and the fact that $\mathrm{M}^-_{a-1,b}$ has a ∇-flag not involving ∇_{a-2i}. $\square$

The following lemma determines the morphisms from string modules to some standard modules.

Lemma 4.4.12. *Let $0 \leq b \leq a \leq n$ and $r \in \mathbb{Z}$.*

1) If $a-b$ is even, then

$$\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a,b}, \Delta_a[r]) \cong \mathrm{Hom}_{\mathbb{P}^n}(\nabla_a, \mathrm{M}^-_{a,b}[r]) \cong \begin{cases} \Bbbk & \textit{if } r = a+b, \\ 0 & \textit{otherwise.} \end{cases}$$

2) If $a-b$ is odd, then

$$\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a,b}, \Delta_a[r]) \cong \mathrm{Hom}_{\mathbb{P}^n}(\nabla_a, \mathrm{M}^-_{a,b}[r]) \cong \begin{cases} \Bbbk & \textit{if } r = a-b-1, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. By Verdier duality it suffices to compute $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a,b}, \Delta_a[r])$. For $a=b$ and $a=b+1$, the claim follows from Lemma 4.3.7 and Lemma 4.3.6, respectively.

For $a > b+1$, by applying the functor $\mathrm{Hom}_{\mathbb{P}^n}(-, \Delta_a)$ to the triangle $\Delta_a \to \mathrm{M}^+_{a,b} \to \mathrm{M}^+_{a-2,b} \to \Delta_a[1]$ we get the long exact sequence

$$\ldots \to \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a-2,b}, \Delta_a[r]) \to \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a,b}, \Delta_a[r]) \to \mathrm{Hom}_{\mathbb{P}^n}(\Delta_a, \Delta_a[r]) \to \ldots.$$

By Lemma 4.3.6 the right-hand side is one-dimensional and concentrated in degree 0, and hence $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a,b}, \Delta_a[r]) \cong \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a-2,b}, \Delta_a[r])$ for $r \geq 2$. For $r \leq 1$ we get $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a,b}, \Delta_a[r]) = 0$: as a consequence of Lemma 4.4.3, the connecting morphism $\mathrm{Hom}_{\mathbb{P}^n}(\Delta_a, \Delta_a) \to \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a-2,b}, \Delta_a[1])$ is an isomorphism, and $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a-2,b}, \Delta_a) = 0$ by the inductive construction of $\mathrm{M}^+_{a-2,b}$ and Lemma 4.3.6 (and Lemma 4.3.7 if $a-b$ is even).

To compute $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a-2,b}, \Delta_a[r])$ for $r \geq 2$, we analyze the long exact sequence obtained by applying $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a-2,b}, -)$ to the triangle (4.2) definying Δ_a. By restriction to $\mathbb{P}^{a-2}$ and naturality of the adjunction, we have a commutative square

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a-2,b}, \mathrm{IC}_a[r]) \arrow[r] \arrow[d, "\cong"'] & \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a-2,b}, \mathrm{IC}_{a-1}[r+1]) \arrow[d, "\cong"] \\
\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a-2,b}, \mathrm{IC}_{a-2}[r-2]) \arrow[r] & \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a-2,b}, \mathrm{IC}_{a-2}[r])
\end{tikzcd}$$

where the bottom morphism is given by postcomposition with $\imath^!_{a-2}(\epsilon^1_{a,a-1})$, which by Remark 4.3.4 is $\epsilon^2_{a-2,a-2} \colon \mathrm{IC}_{a-2}[-2] \to \mathrm{IC}_{a-2}$. By Lemmas 4.4.5 and 4.4.8, this is an isomorphism unless $r = a+b$ if $a-b$ is even, respectively unless $r = a-b-1$ if $a-b$ is odd, and the claim follows from this. $\square$

Morphisms between string objects

Now we can determine the morphisms between string objects. Although the proof is mostly the same, we need to distinguish two cases depending on the parity of $a-b$.

Proposition 4.4.13. *Let $0 \leq b \leq a \leq n$, $0 \leq i \leq \frac{a-b}{2}$ and $r \in \mathbb{Z}$.*

1) If $a-b$ is even, then

$$\begin{aligned}\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a-2i,b}, \mathrm{M}^+_{a,b}[r]) &\cong \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^-_{a,b}, \mathrm{M}^-_{a-2i,b}[r]) \\ &\cong \begin{cases} \Bbbk & \text{if } 2i \leq r \leq a+b \text{ and } r \text{ even}, \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

2) If $a-b$ is odd, then

$$\begin{aligned}\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a-2i,b}, \mathrm{M}^+_{a,b}[r]) &\cong \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^-_{a,b}, \mathrm{M}^-_{a-2i,b}[r]) \\ &\cong \begin{cases} \Bbbk & \text{if } 2i \leq r \leq a-b-1 \text{ and } r \text{ even}, \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

Proof.

1) By Verdier duality it suffices to compute $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a-2i,b}, \mathrm{M}^+_{a,b}[r])$. We do this by downward induction on i. For the base case $i = \frac{a-b}{2}$ we have $\mathrm{M}^+_{a-2i,b} = \mathrm{IC}_b$, so the claim follows from Lemma 4.4.9.

   For the inductive step, we apply $\mathrm{Hom}_{\mathbb{P}^n}(-, \mathrm{M}^+_{a,b})$ to the triangle (4.5) defining $\mathrm{M}^+_{a-2i,b}$, and $\mathrm{Hom}_{\mathbb{P}^n}(\Delta_{a-2i}, -)$ to the triangle (4.7) to obtain the diagram

$$\begin{tikzcd} & & \cdots \arrow[d] & \\ & & \mathrm{Hom}_{\mathbb{P}^n}(\Delta_{a-2i}, \mathrm{M}^-_{a-1,b}[r]) \arrow[d] & \\ \cdots \arrow[r] & \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a-2i-2,b}, \mathrm{M}^+_{a,b}[r]) \arrow[r] & \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a-2i,b}, \mathrm{M}^+_{a,b}[r]) \arrow[r] & \mathrm{Hom}_{\mathbb{P}^n}(\Delta_{a-2i}, \mathrm{M}^+_{a,b}[r]) \arrow[r] \arrow[d] & \cdots \\ & & & \mathrm{Hom}_{\mathbb{P}^n}(\Delta_{a-2i}, \mathrm{IC}_a[r]) \arrow[d] & \\ & & & \cdots & \end{tikzcd} \tag{4.9}$$

   By the inductive hypothesis $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a-2i-2,b}, \mathrm{M}^+_{a,b}[r]) = \Bbbk$ for $r \in \{2i+2, 2i+4, \ldots, a+b\}$, and it vanishes otherwise. Since $\mathrm{M}^-_{a-1,b}$ has a ∇-flag not involving ∇_{a-2i} by the dual version of (4.5), the first term in the column of (4.9) vanishes (this follows either by direct calculation, or from e.g. [BS24, Thm. 3.11]). Moreover, since by Lemma 4.3.5 the last term in the column of (4.9) is one dimensional and concentrated in degree $2i$, so is $\mathrm{Hom}_{\mathbb{P}^n}(\Delta_{a-2i}, \mathrm{M}^+_{a,b}[r])$, from which the claim follows.

2) By Verdier duality it suffices to compute $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a-2i,b}, \mathrm{M}^+_{a,b}[r])$. We do this by downward induction on i. In the base case $i = \frac{a-b-1}{2}$ we have $\mathrm{M}^+_{a-2i,b} = \Delta_{b+1}$, so this follows from Lemma 4.4.11.

   For the inductive step, we apply the functor $\mathrm{Hom}_{\mathbb{P}^n}(-, \mathrm{M}^+_{a,b})$ to the triangle (4.5). By induction, $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a-2i-2,b}, \mathrm{M}^+_{a,b}[r]) \cong \Bbbk$ for $2i+2 \leq r \leq a-b-1$ and r even. By

Lemma 4.4.11 we know that $\mathrm{Hom}_{\mathbb{P}^n}(\Delta_{a-2i}, \mathrm{M}^+_{a,b}[r]) \cong \Bbbk$ for $r = 2i$, and that it vanishes otherwise, which implies the claim. $\square$

Remark 4.4.14. From the proof of Proposition 4.4.13 we obtain the following descriptions of canonical non-zero morphisms $\Phi^{2i}_{a,b}\colon \mathrm{M}^+_{a,b} \to \mathrm{M}^+_{a,b}[2i]$, $\overline{\Phi}^{2i}_{a-2i,b}\colon \mathrm{M}^+_{a-2i,b} \to \mathrm{M}^+_{a,b}[2i]$ and $\zeta^{2i}_{a-2i,b}\colon \Delta_{a-2i} \to \mathrm{M}^+_{a,b}[2i]$: for $0 \leq i \leq \frac{a-b}{2}$, they arise from the diagram

$$\begin{tikzcd}
\mathrm{M}^+_{a,b} \arrow[rr, dashed, "\exists!\Phi^{2i}_{a,b}"] \arrow[d, "\pi_{a-2i+2,b}\cdots\pi_{a,b}"'] & & \mathrm{M}^+_{a,b}[2i] \arrow[d, "n^0_{a,b}"] \\
\mathrm{M}^+_{a-2i,b} \arrow[urr, dashed, "\exists!\overline{\Phi}^{2i}_{a-2i,b}"] & & \mathrm{IC}_a[2i] \\
& \Delta_{a-2i} \arrow[ul, "\iota_{a-2i,b}"] \arrow[ur, "\mu_{a-2i,a}"'] \arrow[uur, dashed, "\exists!\zeta^{2i}_{a-2i,b}"] &
\end{tikzcd}$$

Note that this diagram does not depend on whether $a-b$ is even or odd, though in the proof of Proposition 4.4.13 one has to distinguish this since the induction results in different base cases.

Moreover, by Remark 4.4.6 $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a-2i,b}, \mathrm{IC}_a[2i])$ is spanned by the composition $\mathrm{M}^+_{a-2i,b} \xrightarrow{n^0_{a-2i,b}} \mathrm{IC}_{a-2i} \xrightarrow{\epsilon^{2i}_{a-2i,a}} \mathrm{IC}_a[2i]$. Since we have $n^0_{a-2i,b}\iota_{a-2i,b} = \mu_{a-2i,a-2i}$ by construction, it follows that $\epsilon^{2i}_{a-2i,a}n^0_{a-2i,b}\iota_{a-2i,b} = \mu_{a-2i,a}$. An easy diagram chase (using that $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a-2i,b}, \mathrm{IC}_a[2i]) \cong \Bbbk$) then shows that $\epsilon^{2i}_{a-2i,a}n^0_{a-2i,b}\colon \mathrm{M}^+_{a-2i,b} \to \mathrm{IC}_a[2i]$ makes the entire diagram above commute, and therefore the morphism $\Phi^{2i}_{a,b}\colon \mathrm{M}^+_{a,b} \to \mathrm{M}^+_{a,b}[2i]$ can also be defined by the diagram

$$\begin{tikzcd}
\mathrm{M}^+_{a,b} \arrow[r, dashed, "\exists!\Phi^{2i}_{a,b}"] \arrow[d, "\pi_{a-2i+2,b}\cdots\pi_{a,b}"'] & \mathrm{M}^+_{a,b}[2i] \arrow[d, "n^0_{a,b}"] \\
\mathrm{M}^+_{a-2i,b} \arrow[r, "\epsilon^{2i}_{a-2i,a}n^0_{a-2i,b}"] & \mathrm{IC}_a[2i].
\end{tikzcd}$$

For $i > \frac{a-b}{2}$ (this can only happen if $a-b$ is even), the morphism $\Phi^{2i}_{a,b}$ arises from the diagram

$$\begin{tikzcd}
\mathrm{M}^+_{a,b} \arrow[r, dashed, "\exists!\Phi^{2i}_{a,b}"] \arrow[d, "\pi_{b+2,b}\cdots\pi_{a,b}"'] & \mathrm{M}^+_{a,b}[2i] \arrow[d, "n^0_{a,b}"] \\
\mathrm{IC}_b \arrow[r, "\epsilon^{2i}_{b,a}"'] \arrow[ur, dashed, "\exists!"] & \mathrm{IC}_a[2i].
\end{tikzcd}$$

The morphisms $\mathrm{M}^-_{a,b} \to \mathrm{M}^-_{a,b}[2i]$ are described by the dual diagrams.

Determining the composition

To show that the string objects are $\mathbb{P}$-like, we are left to determine the composition of morphisms in $\mathrm{End}^*_{\mathbb{P}^n}(\mathrm{M}^{\pm}_{a,b})$. This requires the following two technical lemmas, which show that the morphisms from Remark 4.4.14 are compatible with the quotient maps between the string objects and the morphisms betwen IC sheaves.

Lemma 4.4.15. *For $0 \le b \le a \le n$ and $1 \le i \le \frac{a-b-2}{2}$ if $a-b$ is even, respectively $1 \le i \le \frac{a-b-3}{2}$ if $a-b$ is odd, the square*

$$\begin{tikzcd} \mathrm{M}^+_{a-2,b} \arrow[r, "\overline{\Phi}^2_{a-2,b}"] \arrow[d, "\pi_{a-2i,b}\cdots\pi_{a-2,b}"'] & \mathrm{M}^+_{a,b}[2] \arrow[d, "\pi_{a-2i+2,b}\cdots\pi_{a,b}"] \\ \mathrm{M}^+_{a-2i-2,b} \arrow[r, "\overline{\Phi}^2_{a-2i-2,b}"] & \mathrm{M}^+_{a-2i,b}[2] \end{tikzcd}$$

commutes up to a non-zero scalar.

Proof. We prove the claim by induction on i. For $i=1$, the composition $\Phi^2_{a-2,b} = \overline{\Phi}^2_{a-4,b}\pi_{a-2,b}$ spans $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a-2,b}, \mathrm{M}^+_{a-2,b}[2])$ by Remark 4.4.14 and Proposition 4.4.13.

We want to show that this morphism factors through $\mathrm{M}^+_{a,b}[2]$. Completing the right column of the square to the triangle $\Delta_a[2] \xrightarrow{\iota_{a,b}} \mathrm{M}^+_{a,b}[2] \xrightarrow{\pi_{a,b}} \mathrm{M}^+_{a-2,b}[2] \xrightarrow{\psi_{a-2,b}} \Delta_a[3]$ and applying $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a-2,b}, -)$ yields the exact sequence

$$\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a-2,b}, \mathrm{M}^+_{a,b}[2]) \to \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a-2,b}, \mathrm{M}^+_{a-2,b}[2]) \to \mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a-2,b}, \Delta_a[3]).$$

By Proposition 4.4.13 the first two terms are 1-dimensional, and by the same argument as in the proof of Lemma 4.4.12 we get $\mathrm{Hom}_{\mathbb{P}^n}(\mathrm{M}^+_{a-2,b}, \Delta_a[3]) = 0$, which yields the required factorization.

For the inductive step, we use the diagram

$$\begin{tikzcd} \mathrm{M}^+_{a-2,b} \arrow[r, "\pi_{a-2i,b}\cdots\pi_{a-2,b}"] \arrow[d, "\overline{\Phi}^2_{a-2,b}"'] & \mathrm{M}^+_{a-2i-2,b} \arrow[r, "\pi_{a-2i-2,b}"] \arrow[d, "\overline{\Phi}^2_{a-2i-2,b}"] & \mathrm{M}^+_{a-2i-4,b} \arrow[d, "\overline{\Phi}^2_{a-2i-4,b}"] \\ \mathrm{M}^+_{a,b}[2] \arrow[r, "\pi_{a-2i+2,b}\cdots\pi_{a,b}"] & \mathrm{M}^+_{a-2i,b}[2] \arrow[r, "\pi_{a-2i,b}"] & \mathrm{M}^+_{a-2i-2,b}[2]. \end{tikzcd}$$

By the base case, the right square commutes, and by induction the left square commutes (in both cases up to a non-zero scalar). Thus the outer rectangle commutes up to a non-zero scalar by an easy diagram chase. ☐

Lemma 4.4.16. *For $0 < b < a$ and $a-b$ even, the square*

$$\begin{tikzcd} \mathrm{M}^+_{a,b} \arrow[r, "\Phi^2_{a,b}"] \arrow[d, "\pi_{b+2,b}\cdots\pi_{a,b}"'] & \mathrm{M}^+_{a,b}[2] \arrow[d, "\pi_{b+2,b}\cdots\pi_{a,b}"] \\ \mathrm{IC}_b \arrow[r, "\epsilon^2_{b,b}"] & \mathrm{IC}_b[2] \end{tikzcd}$$

commutes up to a non-zero scalar.

Proof. The bottom left path is non-zero by (4.6), and hence we have to show that it factors through the vertical morphism on the right. By the triangle $\mathrm{M}^+_{a,b+1} \to \mathrm{M}^+_{a,b} \to \mathrm{IC}_b \to \mathrm{M}^+_{a,b+1}[1]$, it suffices to show $\mathrm{Hom}(\mathrm{M}^+_{a,b}, \mathrm{M}^+_{a,b+1}[3]) = 0$. We prove this by induction on $a-b$. In the base case $a = b+2$ we have $\mathrm{M}^+_{a,b+1} = \Delta_a$, and thus the claim follows from Lemma 4.4.12.

For the inductive step, assume that $a > b+2$. We apply the functor $\mathrm{Hom}(\mathrm{M}^+_{a,b}, -)$ to the triangle $\Delta_a \to \mathrm{M}^+_{a,b+1} \to \mathrm{M}^+_{a-2,b+1} \to \Delta_a[1]$ defining $\mathrm{M}^+_{a,b+1}$ to get the long exact sequence

$$\ldots \to \mathrm{Hom}(\mathrm{M}^+_{a,b}, \Delta_a[r]) \to \mathrm{Hom}(\mathrm{M}^+_{a,b}, \mathrm{M}^+_{a,b+1}[r]) \to \mathrm{Hom}(\mathrm{M}^+_{a,b}, \mathrm{M}^+_{a-2,b+1}[r]) \to \ldots.$$

By Lemma 4.4.12 we have $\mathrm{Hom}(\mathrm{M}^+_{a,b}, \Delta_a[r]) = 0$ unless $r = a+b$, and therefore we have $\mathrm{Hom}(\mathrm{M}^+_{a,b}, \mathrm{M}^+_{a,b+1}[3]) \cong \mathrm{Hom}(\mathrm{M}^+_{a,b}, \mathrm{M}^+_{a-2,b+1}[3])$ (note that $a > b+2$ and $a-b$ even implies $a+b > 4$).

To compute $\mathrm{Hom}(\mathrm{M}^+_{a,b}, \mathrm{M}^+_{a-2,b+1}[3])$, we use the long exact sequence obtained by applying $\mathrm{Hom}(-, \mathrm{M}^+_{a-2,b+1})$ to the triangle $\Delta_a \to \mathrm{M}^+_{a,b} \to \mathrm{M}^+_{a-2,b} \to \Delta_a[1]$. By Lemma 4.3.6 and the inductive construction of $\mathrm{M}^+_{a-2,b+1}$, we have $\mathrm{Hom}(\Delta_a, \mathrm{M}^+_{a-2,b+1}[r]) = 0$ for all r, and therefore in particular

$$\mathrm{Hom}(\mathrm{M}^+_{a,b}, \mathrm{M}^+_{a-2,b+1}[3]) \cong \mathrm{Hom}(\mathrm{M}^+_{a-2,b}, \mathrm{M}^+_{a-2,b+1}[3])$$

which vanishes by induction on $a-b$. $\square$

Theorem 4.4.17. *Let $0 \leq b \leq a \leq n$.*

1) If $a-b$ is even, then $\mathrm{M}^{\pm}_{a,b}$ is $\mathbb{P}^{(a+b)/2}$-like.

2) If $a-b$ is odd, then $\mathrm{M}^{\pm}_{a,b}$ is $\mathbb{P}^{(a-b-1)/2}$-like.

Proof. By Verdier duality it suffices to show that $\mathrm{M}^+_{a,b}$ is $\mathbb{P}$-like. By Proposition 4.4.13 we have isomorphisms of graded $\Bbbk$-vector spaces $\mathrm{End}^*_{\mathbb{P}^n}(\mathrm{M}^+_{a,b}) \cong \Bbbk[t]/(t^{(a+b)/2+1})$ if $a-b$ is even (respectively $\mathrm{End}^*_{\mathbb{P}^n}(\mathrm{M}^+_{a,b}) \cong \Bbbk[t]/(t^{(a-b-1)/2+1})$ if $a-b$ is odd) with $\deg(t) = 2$. Therefore we only have to show that the composition of the canonical morphisms $\mathrm{M}^+_{a,b} \xrightarrow{\Phi^2_{a,b}} \mathrm{M}^+_{a,b}[2] \xrightarrow{\Phi^{2i}_{a,b}} \mathrm{M}^{\pm}_{a,b}[2i+2]$ is non-zero for $0 < i < \frac{a+b}{2}$ if $a-b$ is even (respectively $0 < i < \frac{a-b-1}{2}$ if $a-b$ is odd). We show that up to a non-zero scalar this composition agrees with the canonical morphism $\Phi^{2i+2}_{a,b} \colon \mathrm{M}^{\pm}_{a,b} \to \mathrm{M}^{\pm}_{a,b}[2i+2]$.

First assume that $i < \frac{a-b}{2}$. By Remark 4.4.14, the composition $\Phi^{2i}_{a,b}\Phi^2_{a,b}$ and the morphism $\Phi^{2i+2}_{a,b}$ (which is not drawn) are defined by the non-dashed arrows in the diagram

$$\begin{tikzcd}
\mathrm{M}^+_{a,b} \arrow[rrr, "\Phi^2_{a,b}"] \arrow[dr, "\pi_{a,b}"] \arrow[dd, two heads, "\pi_{a-2i,b}\cdots\pi_{a,b}"'] & & & \mathrm{M}^+_{a,b}[2] \arrow[rrrr, "\Phi^{2i}_{a,b}"] \arrow[dl, "n^0_{a,b}"] \arrow[dr, "\pi_{a-2i+2,b}\cdots\pi_{a,b}"] & & & & \mathrm{M}^+_{a,b}[2i+2] \arrow[dl, "n^0_{a,b}"] \arrow[dd, "n^0_{a,b}"] \\
& \mathrm{M}^+_{a-2,b} \arrow[r, "\epsilon^2_{a-2,a}n^0_{a-2,b}"] \arrow[dl, "\pi_{a-2i,b}\cdots\pi_{a-2,b}"] & \mathrm{IC}_a[2] & & \mathrm{M}^+_{a-2i,b}[2] \arrow[rr, "\epsilon^{2i}_{a-2i,a}n^0_{a-2i,b}"] & & \mathrm{IC}_a[2i+2] \arrow[dr, "="] & \\
\mathrm{M}^+_{a-2i-2,b} \arrow[rrrrrrr, "\epsilon^{2i+2}_{a-2i-2,a}n^0_{a-2i-2,b}"] \arrow[urrrr, dashed, "\overline{\Phi}^2_{a-2i-2,b}"] & & & & & & & \mathrm{IC}_a[2i+2]
\end{tikzcd}$$

We moreover have the canonical morphism $\overline{\Phi}^2_{a-2i-2,b} \colon \mathrm{M}^+_{a-2i-2,b} \to \mathrm{M}^+_{a-2i,b}[2]$ from Remark 4.4.14 (the dashed arrow in the diagram).

Step 1: The square

$$\begin{tikzcd}
\mathrm{M}^+_{a,b} \arrow[r, "\Phi^2_{a,b}"] \arrow[d, "\pi_{a-2i,b}\cdots\pi_{a,b}"'] & \mathrm{M}^+_{a,b}[2] \arrow[d, "\pi_{a-2i+2,b}\cdots\pi_{a,b}"] \\
\mathrm{M}^+_{a-2i-2,b} \arrow[r, dashed, "\overline{\Phi}^2_{a-2i-2,b}"] & \mathrm{M}^+_{a-2i,b}[2]
\end{tikzcd}$$

commutes up to a non-zero scalar.

Proof: From the construction of $\Phi^2_{a,b}$ in Remark 4.4.14 we obtain the diagram

$$\begin{tikzcd}
\mathrm{M}^+_{a,b} \arrow[dr, "\pi_{a,b}"] \arrow[drr, bend left, "\Phi^2_{a,b}"] \arrow[ddr, bend right, "\pi_{a-2i,b}\cdots\pi_{a,b}"'] & & \\
& \mathrm{M}^+_{a-2,b} \arrow[r, "\overline{\Phi}^2_{a-2,b}"] \arrow[d, "\pi_{a-2i,b}\cdots\pi_{a-2,b}"'] & \mathrm{M}^+_{a,b}[2] \arrow[d, "\pi_{a-2i+2,b}\cdots\pi_{a,b}"] \\
& \mathrm{M}^+_{a-2i-2,b} \arrow[r, dashed, "\overline{\Phi}^2_{a-2i-2,b}"] & \mathrm{M}^+_{a-2i,b}[2],
\end{tikzcd}$$

where the triangle at the top commutes by definition of $\Phi^2_{a,b}$ and the triangle at the left commutes obviously. By Lemma 4.4.15 the inner square commutes up to a non-zero scalar, and the claim then follows by an easy diagram chase.

Step 2: The diagram

$$\begin{tikzcd}
\mathrm{M}^+_{a-2i-2,b} \arrow[r, dashed, "\overline{\Phi}^2_{a-2i-2,b}"] \arrow[dr, "\epsilon^{2i+2}_{a-2i-2,a} n^0_{a-2i-2,b}"'] & \mathrm{M}^+_{a-2i,b}[2] \arrow[d, "\epsilon^{2i}_{a-2i,a} n^0_{a-2i,b}"] \\
& \mathrm{IC}_a[2i+2]
\end{tikzcd}$$

commutes.

Proof: Since $\epsilon^{2i+2}_{a-2i-2,a} = \epsilon^{2i}_{a-2i,a}\epsilon^2_{a-2i-2,a-2i}$, it suffices to show that the diagram

$$\begin{tikzcd}
\mathrm{M}^+_{a-2i-2,b} \arrow[r, dashed, "\overline{\Phi}^2_{a-2i-2,b}"] \arrow[d, "n^0_{a-2i-2,b}"'] & \mathrm{M}^+_{a-2i,b}[2] \arrow[d, "n^0_{a-2i,b}"] \\
\mathrm{IC}_{a-2i-2} \arrow[r, "\epsilon^2_{a-2i-2,a-2i}"] & \mathrm{IC}_{a-2i}[2]
\end{tikzcd}$$

commutes. For this, the construction of the morphism $\overline{\Phi}^2_{a-2i-2,b}\colon \mathrm{M}^+_{a-2i-2,b} \to \mathrm{M}^+_{a-2i,b}[2]$ from Remark 4.4.14 gives the diagram

$$\begin{tikzcd}
\Delta_{a-2i-2} \arrow[dr, "\iota_{a-2i-2,b}"] \arrow[drr, bend left, "\zeta^{2i}_{a-2i,b}"] \arrow[ddr, bend right, "\mu_{a-2i-2,a-2i-2}"'] & & \\
& \mathrm{M}^+_{a-2i-2,b} \arrow[r, "\overline{\Phi}^2_{a-2i-2,b}"] \arrow[d, "n^0_{a-2i-2,b}"'] & \mathrm{M}^+_{a-2i,b}[2] \arrow[d, "n^0_{a-2i,b}"] \\
& \mathrm{IC}_{a-2i-2} \arrow[r, "\epsilon^2_{a-2i-2,a-2i}"] & \mathrm{IC}_{a-2i}[2]
\end{tikzcd}$$

where the outer square and the triangle at the top and the left commute by construction. By Remark 4.4.6, we know that $\mathrm{Hom}(\mathrm{M}^+_{a-2i-2,b}, \mathrm{IC}_{a-2i}[2]) \cong \Bbbk$ is spanned by the composition $\epsilon^2_{a-2i-2,a-2i} n^0_{a-2i-2,b}\colon \mathrm{M}^+_{a-2i-2,b} \to \mathrm{IC}_{a-2i}[2]$. Moreover, this morphism is uniquely characterized by its precomposition with $\iota_{a-2i-2,b}\colon \Delta_{a-2i-2} \hookrightarrow \mathrm{M}^+_{a-2i-2,b}$, which by the proof of Lemma 4.4.5 gives $\mu_{a-2i-2,a-2i}\colon \Delta_{a-2i-2} \to \mathrm{IC}_{a-2i}[2]$. From the diagram it follows that

$$n^0_{a-2i,b}\overline{\Phi}^2_{a-2i-2,b}\iota_{a-2i-2,b} = \epsilon^2_{a-2i-2,a-2i}\mu_{a-2i-2,a-2i-2} = \mu_{a-2i-2,a-2i},$$

and therefore $n^0_{a-2i,b}\overline{\Phi}^2_{a-2i-2,b} = \epsilon^2_{a-2i-2,a-2i} n^0_{a-2i-2,b}$, as required.

Step 3: The claim then follows from Steps 1 and 2 by a straightforward diagram chase, using the definition of $\Phi^{2i+2}_{a,b}$ from Remark 4.4.14.

For $i \geq \frac{a-b}{2}$ the proof is very similar. First, if $a = b$, then $\mathrm{M}^+_{a,b} = \mathrm{IC}_b$ and $\Phi^2_{b,b} = \epsilon^2_{b,b}$, so there is nothing to show. For $a > b$, the "big diagram" is almost the same, except that $\mathrm{M}^+_{a-2i,b}$ and $\mathrm{M}^+_{a-2i-2,b}[2]$ have to be replaced by IC_b and $\mathrm{IC}_b[2]$, respectively, and one has to use $\epsilon^2_{b,b}\colon \mathrm{IC}_b \to \mathrm{IC}_b[2]$ as the "dashed morphism". This satisfies (by construction) $\epsilon^{2i+2}_{b,a} = \epsilon^{2i}_{b,a}\epsilon^2_{b,b}$. By Lemma 4.4.16 the square

$$\begin{tikzcd} \mathrm{M}^+_{a,b} \arrow[r, "\Phi^{2i}_{a,b}"] \arrow[d, "\pi_{b+2,b}\cdots\pi_{a,b}"'] & \mathrm{M}^+_{a,b}[2] \arrow[d, "\pi_{b+2,b}\cdots\pi_{a,b}"] \\ \mathrm{IC}_b \arrow[r, "\epsilon^2_{b,b}"] & \mathrm{IC}_b[2] \end{tikzcd}$$

commutes up to a non-zero scalar (note that if $\frac{a-b}{2} \leq i < \frac{a+b}{2}$, then $b > 0$), and the claim follows from this by an easy diagram chase. $\square$

Remark 4.4.18. It would be desirable to show that the squares in Lemmas 4.4.15 and 4.4.16 commute (not only up to non-zero scalar). This cannot be obtained from our argument, which in both cases uses that morphisms to some cone must vanish to get a factorization. If these squares actually commute, then the proof of Theorem 4.4.17 shows that the canonical morphisms $\Phi^{2i}_{a,b}\colon \mathrm{M}^+_{a,b} \to \mathrm{M}^+_{a,b}[2i]$ form a multiplicative basis of $\mathrm{End}^*_{\mathbb{P}^n}(\mathrm{M}^+_{a,b})$.

Recall that $\mathbb{P}^1$-objects and $\mathbb{P}^1$-like objects are also known as spherical and spherelike objects, and that $\mathbb{P}^0$-like objects are also known as exceptional objects. From Theorem 4.4.17 and Corollary 4.4.2 one can easily read off the spherelike, spherical and exceptional string objects. Note that by Proposition 4.4.13 spherelike string objects in $\mathbf{Perv}(\mathbb{P}^n)$ are necessarily 2-spherelike.

Corollary 4.4.19. *For the string objects in* $\mathbf{Perv}(\mathbb{P}^n)$ *we have:*

1) $\mathrm{M}^\pm_{a,b}$ *is* 2*-spherelike if and only if* $a - b = 3$*, or* $a = 2$ *and* $b = 0$*, or* $a = b = 1$*.*

2) The only 2*-spherical string object is* $\mathrm{M}^\pm_{1,1} = \mathrm{IC}_1$ *for* $n = 1$*.*

3) The only string objects that are exceptional are the standard objects and the costandard objects.

Combining this with the classification of indecomposable perverse sheaves mentioned in Section 4.2.5 (which is obtained from the description in terms of finite-dimensional algebras), this in particular recovers the classification of exceptional objects from [PW20, Prop. 3]. Moreover, we also obtain:

Corollary 4.4.20. *All indecomposable objects in* $\mathbf{Perv}(\mathbb{P}^n)$ *are either* $\mathbb{P}$*-like or* 0*-spherical.*

Proof. By the classification of indecomposable objects over a special biserial algebra from [BR87, p. 161, Thm.] and [WW85, Prop. 2.3], the indecomposable objects in $\mathbf{Perv}(\mathbb{P}^n)$ are the string objects $\mathrm{M}^\pm_{a,b}$ for $0 \leq a \leq b \leq n$ and the indecomposable projective-injective objects $P_k = I_k$ for $0 \leq k < n$. The string objects are $\mathbb{P}$-like by Theorem 4.4.17, and that P_k is 0-spherical is obvious from the description in Section 4.2.4 and Lemma 4.4.1. $\square$

Chapter 5

The Weyl groupoids of $\mathfrak{sl}(m|n)$ and $\mathfrak{osp}(r|2n)$

In this chapter we provide an explicit description of the Weyl groupoids of the Lie superalgebras $\mathfrak{sl}(m|n)$, $\mathfrak{osp}(2m+1|2n)$ and $\mathfrak{osp}(2m|2n)$.

For this we first recall the combinatorial definitions of Cartan graphs and Weyl groupoids from [HS20], and we apply these definitions to construct Weyl groupoids of contragredient Lie superalgebras. We also compare the automorphism group of an object of the Weyl groupoid of a contragredient Lie superalgebra to its Weyl group.

The Weyl groupoids of $\mathfrak{sl}(m|n)$, $\mathfrak{osp}(2m+1|2n)$ and $\mathfrak{osp}(2m|2n)$ are described in Section 5.4. For this we first recall the classification of their Borel subalgebras from [Kac77]. To obtain a convenient graphical description of the Weyl groupoids, we reformulate this classification in terms of partitions, and we explicitly compute the corresponding Cartan data. This makes it very easy to explicitly write down the Weyl groupoids in practice.
The chapter is joint work with Jonas Nehme and has been published as [BN24].

[BN24] L. Bonfert and J. Nehme. "The Weyl groupoids of $\mathfrak{sl}(m|n)$ and $\mathfrak{osp}(r|2n)$". *J. Algebra* 641 (2024).

5.1 Motivation and overview of results

Let $\mathfrak{g}$ be a basic classical simple Lie superalgebra and let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Cartan subalgebra. It is well-known that, in contrast to the situation for semisimple Lie algebras, not all Borel subalgebras of $\mathfrak{g}$ containing $\mathfrak{h}$ are conjugate to each other. As a consequence there are several systems of simple roots that are not conjugate under the action of the Weyl group. The number of conjugacy classes is however finite (see e.g. [Mus12, Thm. 3.1.2]), which is equivalent to saying that there are only finitely many Borel subalgebras with fixed even part $\mathfrak{b}_{\bar{0}}$.

In [PS89] Penkov and Serganova introduced *odd reflections* to pass between Borel subalgebras with the same even part. Explicitly, they can be described as follows (see e.g. [CW12, §1.4] or [Mus12, §3.5]). Let $\{\alpha_1, \dots, \alpha_n\}$ be the simple roots corresponding to some Borel subalgebra $\mathfrak{b}$ and suppose the simple root α_i is odd isotropic. Then the simple roots $\alpha'_j = r_i(\alpha_j)$ for the Borel subalgebra $\mathfrak{b}'$ obtained from $\mathfrak{b}$ by odd reflection at α_i are

$$\alpha'_j = r_i(\alpha_j) = \begin{cases} -\alpha_i & \text{if } j = i, \\ \alpha_j & \text{if } j \neq i, \alpha_j(h_i) = 0 \\ \alpha_j + \alpha_i & \text{if } j \neq i, \alpha_j(h_i) \neq 0, \end{cases} \tag{5.1}$$

where $h_i \in \mathfrak{h}$ is the coroot corresponding to α_i. More generally, these definitions also work for contragredient Lie superalgebras. In [Ser11] the odd reflections (and certain other maps) are used to construct a *Weyl groupoid* that acts transitively on the set of Borel subalgebras.

On the other hand, a (seemingly unrelated at first glance) notion of Weyl groupoid was also introduced by Heckenberger and Yamane [HY08] as an analogue of Weyl groups in the theory of

Nichols algebras. Weyl groupoids in this context are constructed from (semi-)Cartan graphs, see [HS20, §9]. A (semi-)Cartan graph is an undirected graph $\mathcal{G}$ with edges labelled by $\{1, \dots, n\}$ and a generalized Cartan matrix $A(x)$ (called the Serre matrix) for every vertex x, subject to certain conditions. In [HY08, Ex. 3] examples of Cartan graphs are obtained from finite-dimensional contragredient Lie superalgebras. Furthermore, [HS20, §11] provides a different combinatorial construction of Cartan graphs for regular contragredient Lie superalgebras, using only the Cartan data. Roughly speaking, the vertices of the Cartan graph are the ordered bases of the roots of $\mathfrak{g}$, the edges correspond to odd reflections, and the Serre matrices define the Serre relations in $\mathfrak{g}$. By results of [HY08] the Weyl groupoid of a Cartan graph is a Coxeter groupoid, i.e. it is generated by *simple reflections* $(s_i)_x \colon x \to r_i(x)$ subject only to Coxeter-type relations $\mathrm{id}_x (s_i s_j)^{m(x)_{ij}} \mathrm{id}_x = \mathrm{id}_x$ (for any possible composition).

The first result of this chapter is a formulation of the general construction of Cartan graphs and Weyl groupoids from [HS20] in more convenient graphical language, see Section 5.2.1. Moreover, in Section 5.2.2 we generalize the construction of Cartan graphs for finite-dimensional contragredient Lie superalgebras from [HY08] to regular symmetrizable contragredient Lie superalgebras, and show that this is equivalent to the combinatorial definition from [HS20]. In particular, this implies that we actually obtained a Cartan graph. In comparison to [HS20] we put the focus on the Borel subalgebras instead of the Cartan data, which is advantageous from the perspective of Lie theory. This point of view allows us to compare this notion of Weyl groupoid to other constructions in the theory of Lie superalgebras, see Section 5.2.5. In Proposition 5.2.15 we show that the automorphism group of an object of the Weyl groupoid $\mathcal{W}$ of a contragredient Lie superalgebra coincides with its Weyl group.

The second result of this chapter is an explicit description of the Cartan graphs and Weyl groupoids for the Lie superalgebras $\mathfrak{sl}(m|n)$, $\mathfrak{osp}(2m+1|2n)$ and $\mathfrak{osp}(2m|2n)$. These Weyl groupoids were previously considered in [AA17]. We provide a detailed, different description in terms of partitions. For this we first recall some standard results about the realizations of $\mathfrak{sl}(m|n)$, $\mathfrak{osp}(2m+1|2n)$ and $\mathfrak{osp}(2m|2n)$ as contragredient Lie superalgebras, which amounts to classifying all Borel subalgebras up to conjugation. Based on [Mus12, §3] we describe the Borel subalgebras (up to conjugation) in terms of partitions λ fitting in an $m \times n$-rectangle. Given such a partition (and an additional sign $\varepsilon \in \{+, -\}$ in the case of $\mathfrak{osp}(2m|2n)$) one can easily construct a Borel subalgebra $\mathfrak{b}(\lambda)$ (resp. $\mathfrak{b}(\lambda, \varepsilon)$), see Sections 5.3.1 to 5.3.3 for details. To determine the Cartan graphs we also need an explicit description of the the Cartan data for all Borel subalgebras, which we compute in Section 5.A.

An observation crucial to determining the Cartan graphs of the Lie superalgebras $\mathfrak{sl}(m|n)$, $\mathfrak{osp}(2m+1|2n)$ and $\mathfrak{osp}(2m|2n)$ is that the combinatorial description of Borel subalgebras in terms of partitions allows for a convenient description of the odd reflections: an odd reflection corresponds to adding or removing a certain box to (resp. from) λ, see Section 5.3.4 for details. This makes it very easy to describe the shape of the Cartan graph in concrete examples, see Proposition 5.4.1. In Proposition 5.4.2 we compute the Serre matrices, which are obtained from the Cartan data computed in Section 5.3. Finally we determine the Coxeter relations in their Weyl groupoids. Somewhat surprisingly, these are as one would expect from the Serre matrices, see Proposition 5.4.4.

5.2 Cartan graphs and Weyl groupoids

5.2.1 Definition and generalities

We begin by reformulating the definitions of Cartan graphs and Weyl groupoids from [HS20, §9]. The notion of Weyl groupoids was first axiomatically introduced in [HY08].

Let I and J be sets with $|I| < \infty$. By an *(I, J)-labelled graph* we mean an (undirected) graph $\mathcal{G}$ with vertices X and edges E together with maps of sets $A\colon X \to J$ and $c\colon E \to I$. For an edge e we call $c(e) \in I$ its *color*. We draw the i-colored edges of an (I, J)-labelled graph as $x \overset{i}{\longleftrightarrow} x'$. The set I will usually be left implicit.

For a finite set I let $\mathrm{GCM}_I(\mathbb{Z})$ be the set of generalized Cartan matrices with entries indexed by I.

Definition 5.2.1. A *semi-Cartan graph* is an $(I, \mathrm{GCM}_I(\mathbb{Z}))$-labelled graph such that

(CG1) for every vertex $x \in X$ and every $i \in I$ there is a unique edge e incident to x with $c(e) = i$,

(CG2) and $A(x)_{ij} = A(y)_{ij}$ for every edge $x \overset{i}{\longleftrightarrow} y$ and all $j \in I$.

The matrices $A(x)$ are called *Serre matrices*.

We call the matrices $A(x)$ Serre matrices since in special cases they describe Serre relations, see Remark 5.2.9.

Note that loops in semi-Cartan graphs are explicitly allowed, and in fact occur very often.

Remark 5.2.2. By (CG1) we obtain involutions $r_i\colon X \to X$, sending a vertex x to its neighbor along the unique i-colored edge at x. This recovers the axioms in [HS20, Def. 9.1.1].

Let $\mathcal{G}$ be a semi-Cartan graph. To each vertex $x \in X$ we associate a $\mathbb{Z}$-lattice $\mathbb{Z}^I_x$ with basis $\{\alpha^x_i \mid i \in I\}$, and for $i \in I$ we define $\mathbb{Z}$-linear maps $s_i = (s_i)_x\colon \mathbb{Z}^I_x \to \mathbb{Z}^I_{r_i(x)}$ by $(s_i)_x(\alpha^x_j) = \alpha^{r_i(x)}_j - A(x)_{ij}\alpha^{r_i(x)}_i$.

Definition 5.2.3. The *Weyl groupoid* $\mathcal{W}$ of $\mathcal{G}$ is the groupoid with set of objects X and the morphisms generated by the $(s_i)_x\colon x \to r_i(x)$ for $i \in I$, $x \in X$. The composition of morphisms is given by the usual composition of $\mathbb{Z}$-linear maps, and two morphisms $x \to y$ are equal if and only if they agree as $\mathbb{Z}$-linear maps $\mathbb{Z}^I_x \to \mathbb{Z}^I_y$.

Observe that from (CG2) we get $(s_i)_{r_i(x)}(s_i)_x = \mathrm{id}_x$ for all $i \in I$ and $x \in X$, and thus $\mathcal{W}$ is indeed a groupoid.

Remark 5.2.4. Usually one thinks of the lattices $\mathbb{Z}^I_x$ as lying inside a fixed ambient $\mathbb{C}$-vector space of dimension $|I|$. Obviously the lattices $\mathbb{Z}^I_x$ are isomorphic as abstract lattices, but they are usually rather different (and in particular depend on the vertex x) as sublattices of the ambient vector space. To emphasize this we will always keep track of the vertex x for the lattice $\mathbb{Z}^I_x$ and its basis vectors α^x_i.

Definition 5.2.5. Let $\mathcal{G}$ be a semi-Cartan graph and $\mathcal{W}$ its Weyl groupoid, and define the following sets of roots at vertex $x \in X$:

- The *real roots* are
$$\Delta^{\mathrm{real}}_x = \{w(\alpha^y_i) \mid y \in X, w \in \mathcal{W}(y, x), i \in I\} \subseteq \mathbb{Z}^I_x,$$
- the *positive* (resp. *negative*) real roots are $\Delta^{\pm,\mathrm{real}}_x = \Delta^{\mathrm{real}}_x \cap \pm \sum_{i\in I} \mathbb{N}_0\alpha^x_i$.

Definition 5.2.6. A semi-Cartan graph $\mathcal{G}$ is a *Cartan graph* if

(CG3) $\Delta_x^{\mathrm{real}} = \Delta_x^{+,\mathrm{real}} \cup \Delta_x^{-,\mathrm{real}}$ for all $x \in X$,

(CG4) and for all $x \in X$ and $i, j \in I$ we have

$$(r_i r_j)^{m(x)_{ij}}(x) = x,$$

where $m(x)_{ij} = |\Delta_x^{\mathrm{real}} \cap (\mathbb{N}_0\alpha_i^x + \mathbb{N}_0\alpha_j^x)|$.

5.2.2 Cartan graphs for contragredient Lie superalgebras

Now we construct a generalized Cartan graph from a contragredient Lie superalgebra. We begin by recalling the construction of contragredient Lie superalgebras from [Mus12, §5]. A *Cartan datum* is a pair (B, τ) consisting of a matrix $B \in \mathbb{C}^{n\times n}$ and a *parity vector* $\tau \in (\mathbb{Z}/2\mathbb{Z})^n$ for some $n \in \mathbb{N}$. We further fix a *minimal realization* of B, i.e. we choose a vector space $\mathfrak{h}$ of dimension $2n - \mathrm{rk}(B)$ with linearly independent roots $\alpha_i \in \mathfrak{h}^*$ and coroots $h_i \in \mathfrak{h}$ for $1 \le i \le n$ such that $\alpha_j(h_i) = a_{ij}$. This data can be used to construct a Lie superalgebra $\tilde{\mathfrak{g}}(B, \tau)$ with *Chevalley generators* e_i and f_i (of parity τ_i) subject to the following relations which are analogous to the defining relations of Kac–Moody Lie algebras:

$$\begin{aligned} [h, h'] &= 0 \quad \text{for all } h, h' \in \mathfrak{h}, \\ [h, e_i] &= \alpha_i(h)e_i \quad \text{for all } h \in \mathfrak{h}, \\ [h, f_i] &= -\alpha_i(h)f_i \quad \text{for all } h \in \mathfrak{h}, \\ [e_i, f_j] &= \delta_{i,j}h_i. \end{aligned}$$

As usual, we call $\mathfrak{h}$ the *Cartan subalgebra*. Note that $\mathfrak{h}$ is concentrated in even degree, and is abelian. Let $\mathfrak{g}(B, \tau) = \tilde{\mathfrak{g}}(B, \tau)/\mathfrak{r}$, where $\mathfrak{r}$ is the maximal ideal of $\tilde{\mathfrak{g}}$ intersecting $\mathfrak{h}$ trivially. From the construction of $\mathfrak{g}(B, \tau)$ it is clear that rescaling the rows of B by non-zero scalars results in isomorphic Lie superalgebras.

In the following we will restrict ourselves to regular symmetrizable Cartan data in the sense of [HS07, Def. 4.8] (up to rescaling of rows). A Cartan datum (B, τ) (with the same notation as in Section 5.3) is called *symmetrizable* if the matrix B is symmetrizable. We call (B, τ) *regular* if

- B has no zero rows and is indecomposable (i.e. does not split into blocks $B = \left(\begin{smallmatrix} B_1 & 0 \\ 0 & B_2 \end{smallmatrix}\right)$),
- $b_{ij} = 0$ if and only if $b_{ji} = 0$,
- if $\tau_i = \bar{0}$ then $b_{ii} \neq 0$ and $\frac{2b_{ij}}{b_{ii}} \in \mathbb{Z}_{\le 0}$ for all j, and
- if $\tau_i = \bar{1}$ and $b_{ii} \neq 0$, then $\frac{2b_{ij}}{b_{ii}} \in 2\mathbb{Z}_{\le 0}$ for all j.

Regularity of (B, τ) implies that $\mathrm{ad}_{e_i} : \mathfrak{g}(B, \tau) \to \mathfrak{g}(B, \tau)$ is locally nilpotent for all i, see [Ser11, §2].

We call the Lie superalgebra $\mathfrak{g}(B, \tau)$ *regular* if any Cartan datum (B', τ') with $\mathfrak{g}(B', \tau') \cong \mathfrak{g}(B, \tau)$ is regular. In particular $\mathfrak{sl}(m|n)$, $\mathfrak{osp}(2m+1|2n)$ and $\mathfrak{osp}(2m|2n)$ are regular, which follows from the computation of all possible Cartan data (up to rescaling of rows) in Section 5.A.

The following definition is [HS20, Def. 11.2.4]. It simplifies slightly since we have restricted ourselves to regular Cartan data.

Definition 5.2.7. For a regular Cartan datum (B,τ), the *Serre matrix* $A^{B,\tau}$ is the $n\times n$-matrix with entries

$$a_{ij}^{B,\tau}=\begin{cases}2 & \text{if } i=j,\\ 0 & \text{if } i\neq j, b_{ij}=0,\\ -1 & \text{if } i\neq j, b_{ij}\neq 0, b_{ii}=0,\\ \frac{2b_{ij}}{b_{ii}} & \text{if } i\neq j, b_{ij}\neq 0, b_{ii}\neq 0.\end{cases}$$

Observe that due to regularity of (B,τ) the Serre matrix $A^{B,\tau}$ is a generalized Cartan matrix. Also note that in particular $A^{B,\tau}$ is invariant under multiplication of rows of B by non-zero scalars.

Remark 5.2.8. The definition of $A^{B,\tau}$ can be seen as a normalization of B, bringing it as close as possible to the form of a generalized Cartan matrix. The rows with non-zero diagonal entries are rescaled so that the diagonal entries become 2, while for odd isotropic roots (those with $b_{ii}=0$) we replace all non-zero off-diagonal entries by -1. In particular if B is a generalized Cartan matrix, then $A^{B,\tau}=B$.

Remark 5.2.9. We call the matrix $A^{B,\tau}$ a Serre matrix since it is used to formulate Serre relations for the contragredient Lie superalgebra $\mathfrak{g}(B,\tau)$. Explicitly we have

$$(\operatorname{ad} e_i)^{1-a_{ij}}(e_j)=0,$$

see for instance [Mus12, Lem. 5.2.13].

For the rest of the section fix a regular symmetrizable contragredient Lie superalgebra $\mathfrak{g}=\mathfrak{g}(B,\tau)$ with $B\in\mathbb{C}^{n\times n}$ and let $I=\{1,\dots,n\}$.

As a direct consequence of the construction, $\mathfrak{g}$ admits a root space decomposition $\mathfrak{g}=\mathfrak{h}\oplus\bigoplus_{\alpha\in\mathfrak{h}^*}\mathfrak{g}_\alpha$. An *ordered root basis* of $\mathfrak{g}$ (simply called *base* in [Ser11, §3]) is a sequence $\Pi'=(\beta_1,\dots,\beta_n)$ of linearly independent roots such that there are $e_i'\in\mathfrak{g}_{\beta_i}$, $f_i'\in\mathfrak{g}_{-\beta_i}$ that together with $\mathfrak{h}$ generate $\mathfrak{g}$ and satisfy $[e_i',f_j']=0$ for $i\neq j$. The β_i are called *simple roots*, and every root can be written as a $\mathbb{Z}$-linear combination of the simple roots such that all the coefficients are either non-negative or non-positive.

A choice of Chevalley generators e_i', f_i' for an ordered root basis determines a Cartan datum (B',τ') by $\tau_i'=|e_i'|$ and $b_{ij}'=\beta_j(h_i')$ with $h_i'=[e_i',f_i']\in\mathfrak{h}$. Observe that the rank of B and B' coincide and thus this gives rise to an isomorphism $\mathfrak{g}\cong\mathfrak{g}(B',\tau')$. Note that the e_i' and f_i' are unique up to scalar, so the Cartan datum (B',τ') is unique up to rescaling of the rows of B'.

Let X be a labelling set for the ordered root bases of $\mathfrak{g}$. For each $x\in X$ fix Chevalley generators corresponding to the simple roots in $\Pi(x)$ and let $(B(x),\tau(x))$ be the Cartan datum obtained from these.

Definition 5.2.10. The *Cartan graph* $\mathcal{G}_\mathfrak{g}$ of $\mathfrak{g}$ is the $(I,\mathrm{GCM}_I(\mathbb{Z}))$-labelled graph consisting of

- the set of vertices X,
- edges according to the rules:
  - For each odd isotropic root $\alpha_i^x\in\Pi(x)$ and $\Pi(x')$ obtained from $\Pi(x)$ by an odd reflection at α_i^x (as defined in (5.1)), there is an edge $x\overset{i}{\longleftrightarrow}x'$ of color $i\in I$.
  - For each root $\alpha_i^x\in\Pi(x)$ that is not odd isotropic there is an edge $x\overset{i}{\longleftrightarrow}x$ of color $i\in I$.
- the Serre matrices $A(x)=A^{B(x),\tau(x)}$.

The *Weyl groupoid* of $\mathfrak{g}$ is the Weyl groupoid of $\mathcal{G}_\mathfrak{g}$.

Remark 5.2.11. A priori the Cartan graph depends on the choice of Chevalley generators. However, as mentioned above, these are unique up to scalar. Rescaling the Chevalley generators corresponds to rescaling the rows of $B(x)$, and this does not affect the Serre matrix $A^{B(x),\tau(x)}$. Thus $\mathcal{G}_\mathfrak{g}$ is well-defined.

It is not obvious that $\mathcal{G}_\mathfrak{g}$ is indeed a Cartan graph (or even a semi-Cartan graph) as the name suggests, this will be checked in Corollary 5.2.14 below. For this we first have to show that the above definition of $\mathcal{G}$ is equivalent to the construction from [HS20, Def. 11.2.6]. The idea to associate a Weyl groupoid to a contragredient Lie superalgebra in this way goes back to [HY08, Ex. 3].

Remark 5.2.12. For the Cartan graph $\mathcal{G}_\mathfrak{g}$ the basis vectors α_i^x from Definition 5.2.3 can be identified with the simple roots in the ordered root basis $\Pi(x) \subseteq \mathfrak{h}^*$, and in general many of these coincide when viewed as elements of $\mathfrak{h}^*$. However, the simple roots in different ordered root bases should always be distinguished. In terms of the Cartan graph, this corresponds to distinguishing the simple roots at different vertices as explained in Remark 5.2.4.

To see that $\mathcal{G}_\mathfrak{g}$ is indeed the same object as the one constructed in [HS20] we need to determine the effect of odd reflections on a symmetric Cartan datum. Similar formulas (without the symmetry assumption) can also be found in [GHS24, §2.2.4] and [HS07, §4] (for further context see also [AA17]).

Proposition 5.2.13. *Let Π be an ordered root basis of $\mathfrak{g}$ and $\alpha_i \in \Pi$ odd isotropic. Let Π' be obtained from Π by odd reflection at α_i. Suppose there are Chevalley generators for Π such that in the resulting Cartan datum (B, τ) the matrix B is symmetric. Then there is a choice of Chevalley generators for Π' such that corresponding Cartan datum (B', τ') is given by*

$$b'_{jk} = \begin{cases} -b_{jk} & \text{if } j = i \text{ or } k = i, \\ b_{jk} & \text{if } j,k \neq i, b_{ji}b_{ik} = 0, \\ b_{jk} + b_{ik} + b_{ji} & \text{if } j,k \neq i, b_{ji}b_{ik} \neq 0, \end{cases} \qquad \tau'_j = \begin{cases} \tau_j & \text{if } b_{ij} = 0, \\ \tau_j + \bar{1} & \text{if } b_{ij} \neq 0. \end{cases}$$

In particular any Cartan datum obtained from a symmetrizable Cartan datum under odd reflections is symmetrizable.

Proof. Recall from (5.1) that the simple roots in Π' are

$$\{-\alpha_i\} \cup \{\alpha_j \mid j \neq i, b_{ij} = 0\} \cup \{\alpha_j + \alpha_i \mid j \neq i, b_{ij} \neq 0\}.$$

One possible choice for the corresponding Chevalley generators is

$$e'_j = \begin{cases} f_i & \text{if } j = i, \\ e_j & \text{if } j \neq i, b_{ij} = 0, \\ [e_i, e_j] & \text{if } j \neq i, b_{ij} \neq 0, \end{cases} \qquad f'_j = \begin{cases} -e_i & \text{if } j = i, \\ f_j & \text{if } j \neq i, b_{ij} = 0, \\ \frac{1}{b_{ij}}(-1)^{\tau_j}[f_i, f_j] & \text{if } j \neq i, b_{ij} \neq 0. \end{cases}$$

Observe that this choice of root vectors is unique up to scalar since the root spaces for simple roots and sums of two simple roots are 1-dimensional, and therefore any other choice of root vectors results in a rescaling of the matrix B'. Our choice of scaling ensures that B' will be symmetric.

From the root vectors we compute (using in particular that e_i and f_i are odd, and that B is symmetric)

$$h'_j = [e'_j, f'_j] = \begin{cases} -h_i & \text{if } j = i, \\ h_j & \text{if } j \neq i, b_{ij} = 0, \\ h_i + h_j & \text{if } j \neq i, b_{ij} \neq 0, \end{cases}$$

and this implies

$$b'_{jk} = \begin{cases} -b_{jk} & \text{if } j = i \text{ or } k = i, \\ b_{jk} & \text{if } j, k \neq i, b_{ji}b_{ik} = 0, \\ b_{jk} + b_{ik} + b_{ji} & \text{if } j, k \neq i, b_{ji}b_{ik} \neq 0 \end{cases}$$

as claimed. Finally, $\tau'_j = |e'_j| = |e_j| = \tau_j$ unless $b_{ij} \neq 0$, in which case $\tau'_j = |e'_j| = |e_j| + |e_i| = |e_j| + \bar{1} = \tau_j + \bar{1}$. ☐

Corollary 5.2.14. *Let (B, τ) be a regular symmetrizable Cartan datum, $\mathfrak{g} = \mathfrak{g}(B, \tau)$, and $\tilde{B}$ a symmetrization of B. Then $\mathcal{G}_\mathfrak{g}$ is precisely the object constructed from $(\tilde{B}, \tau)$ in [HS20, Def. 11.2.6]. In particular $\mathcal{G}_\mathfrak{g}$ is a Cartan graph.*

Proof. In [HS20], the Weyl groupoid is defined using the Lie superalgebra $\mathfrak{g}'(B, \tau) \subseteq \mathfrak{g}(B, \tau)$ generated by the e_i and f_i. The only difference is that the Cartan subalgebra of $\mathfrak{g}'(B, \tau)$ is just spanned by the h_i, and therefore this does not affect the construction of the Weyl groupoid.

The definitions then agree by the observation that the Serre matrix is invariant under rescaling of rows of the matrix B, and that the effect of odd reflections on a symmetric Cartan datum determined in Proposition 5.2.13 agrees with the formulas from [HS20, Lem. 11.2.7]. That $\mathcal{G}_\mathfrak{g}$ is a Cartan graph is then [HS20, Thm. 11.2.10]. ☐

5.2.3 Automorphisms

We would like to compare the automorphism group of an object of the Weyl groupoid $\mathcal{W}$ of a contragredient Lie superalgebra $\mathfrak{g}$ with its Weyl group.

In [GHS24] the Weyl group of a connected component of $\mathcal{W}$ is introduced. It follows from the observations in Section 5.2.5 that the roots in a connected component of the Cartan graph $\mathcal{G}_\mathfrak{g}$ coincide with the real roots in a connected component of the spine of the root groupoid, as defined in [GHS24, Def. 4.1.2]. By [GHS24, Prop. 4.3.12] the Weyl group of a connected component of $\mathcal{W}$ is generated by the reflections at non-isotropic roots that appear as simple roots in some ordered root basis in this connected component.

In [Ser11, §4] the Weyl group of $\mathfrak{g}$ is defined as the subgroup of $\mathrm{GL}(\mathfrak{h}^*)$ generated by all reflections at all principal even roots of $\mathfrak{g}$, where an even root α is *principal* if either α or $\frac{1}{2}\alpha$ appears as a simple root for $\mathfrak{g}$ in some ordered root basis. Note that this definition does not depend on the connected component of $\mathcal{G}_\mathfrak{g}$.

Proposition 5.2.15. *Let $\mathfrak{g}$ be a contragredient Lie superalgebra and $\mathcal{W}$ its Weyl groupoid. For any object $x \in \mathcal{W}$ the group $\mathrm{Aut}_\mathcal{W}(x)$ is isomorphic to the Weyl group W_x of the connected component containing x.*

In particular if for every even root α either α or $\frac{1}{2}\alpha$ appears in an ordered root basis in this connected component, then $\mathrm{Aut}_\mathcal{W}(\mathfrak{b})$ is isomorphic to the Weyl group of $\mathfrak{g}$.

Proof. By definition the Weyl group W_x is a subgroup of $\mathrm{GL}(\mathfrak{h}^*)$. On the other hand, by identifying the $\mathbb{Z}$-lattices in the definition of $\mathcal{W}$ with the $\mathbb{Z}$-lattice in $\mathfrak{h}^*$ spanned by the roots as in Remark 5.2.12 we can also see $\mathrm{Aut}_\mathcal{W}(x)$ as a subgroup of $\mathrm{GL}(\mathfrak{h}^*)$. We claim that the respective generators of these groups act by the same reflections on $\mathfrak{h}^*$.

For $x' \in \mathcal{W}$ and an odd isotropic simple root $\alpha_i^{x'} \in \Pi(x')$ let $r_i(\Pi(x'))$ the ordered root basis obtained from $\Pi(x')$ by odd reflection at $\alpha_i^{x'}$. By construction the corresponding generator $(s_i)_{x'}$ of $\mathcal{W}$ only does an explicit base change between the bases of $\mathfrak{h}^*$ given by the simple roots $\Pi(x')$ and $r_i(\Pi(x'))$, and hence acts as the identity map on $\mathfrak{h}^*$. But this means that $\mathrm{Aut}_\mathcal{W}(x)$ is generated by the reflections at all non-isotropic roots that appear as a simple root in some ordered root basis. As the formulas defining the reflections are the same in both cases (see Definition 5.2.3 and [Ser11, §4], [GHS24, Eq. (7)]) we see that $\mathrm{Aut}_{\mathcal{W}}(x) \subseteq W_x$.

The converse inclusion is clear by definition of W_x. □

Remark 5.2.16. From Proposition 5.2.15 it follows that the real roots of $\mathcal{W}$ in the sense of Definition 5.2.5 are the same as the real roots of $\mathfrak{g}$.

5.2.4 Components of the Cartan graph

In general, the Cartan graph of a contragredient Lie superalgebra will have many connected components, see Remark 5.2.17 below. However in some cases it is enough to consider only one of these.

Remark 5.2.17. An ordered root basis Π' of $\mathfrak{g}$ determines a decomposition $\mathfrak{g} = \mathfrak{n}'^{-} \oplus \mathfrak{h} \oplus \mathfrak{n}'^{+}$ into a positive and negative part with respect to its simple roots. By slight abuse of language we call $\mathfrak{b}' = \mathfrak{h} \oplus \mathfrak{n}'^{+}$ a *Borel subalgebra* of $\mathfrak{g}$.

Since odd reflections do not change the even part of a Borel subalgebra, the Cartan graph of a contragredient Lie superalgebra splits into several components without edges between them. If the Borel subalgebras with the same even part represent all conjugacy classes of Borel subalgebras, then these components all look the same and we restrict our attention to one of these components. This is for instance the case for $\mathfrak{sl}(m|n)$, $\mathfrak{osp}(2m+1|2n)$ and $\mathfrak{osp}(2m|2n)$, see e.g. [Mus12, §3.1].

Remark 5.2.18. From [HS20, Thm. 9.3.5] it follows that it is possible to order the simple roots consistently under odd reflections in the sense that for an ordered root basis $\Pi = (\alpha_1, \dots, \alpha_n)$ and a non-trivial reordering Π' of Π it is impossible to obtain Π' from Π by odd reflections. Therefore the Cartan graph $\mathcal{G}_{\mathfrak{g}}$ decomposes into $n!$ identical (up to renumbering of edges) components without edges between them. However, this uses that $\mathcal{G}_{\mathfrak{g}}$ is a Cartan graph, and therefore we cannot choose a consistent ordering of the simple roots a priori. As far as we know, a consistent ordering of the simple roots cannot be found purely in terms of root combinatorics of Lie superalgebras, although its existence is a purely Lie-theoretical question.

5.2.5 Relation to other notions of Weyl groupoids

There are several constructions called *Weyl groupoid* in the literature. Our definition of the Weyl groupoid $\mathcal{W}$ generalizes the construction from [HY08]. The relation to the other notions is as follows.

In [Ser11] Serganova introduced another notion of *Weyl groupoid* $\mathcal{C}$ whose objects are Cartan data (B, τ) and whose morphisms are isomorphisms $\mathfrak{g}(B, \tau) \to \mathfrak{g}(B', \tau')$ of the associated contragredient Lie superalgebras that preserve the Cartan subalgebra. In virtue of [Ser11, Thm. 4.14] this compares to $\mathcal{W}$ as follows.

Comparison 5.2.19. The Weyl groupoid $\mathcal{W}$ is the subgroupoid of the component of $\mathcal{C}$ containing $\mathfrak{g}(B, \tau)$, obtained by forgetting all morphisms corresponding to rescaling rows of the matrices B.

One could say that this is a restriction to the essentially important information as rescaling rows only amounts to different choices of Chevalley generators.

In a recent paper [GHS24], Gorelik, Hinich and Serganova constructed a different version of a Weyl groupoid called *root groupoid*. For a fixed finite set X, their objects are quadruples $(\mathfrak{h}, a, b, p)$ where $\mathfrak{h}$ denotes a Cartan subalgebra, $a\colon X \to \mathfrak{h}$ a map with image a set of linearly independent coroots, $b\colon X \to \mathfrak{h}^*$ a map with image a set of linearly independent roots and p the corresponding parities. The root groupoid is generated by the following three types of morphisms:

- $(\mathfrak{h}, a, b, p) \to (\mathfrak{h}', \theta \circ a, \theta^{-1} \circ b, p)$ for any isomorphism $\mathfrak{h} \xrightarrow{\cong} \mathfrak{h}'$,
- $(\mathfrak{h}, a, b, p) \to (\mathfrak{h}, a', b, p)$, where $a'(x) = \lambda(x)a(x)$ for some $\lambda \colon X \to \mathbb{C}^*$,
- even and odd reflections.

In [GHS24, §4.2.5] the *skeleton* of the root groupoid is defined as the subgroupoid generated by the even and odd reflections. Furthermore, the *spine* of the root groupoid is defined as the subgroupoid generated by odd reflections only, see [GHS24, §4.2.8].

Given a regular symmetrizable Cartan datum (B, τ) of rank n, we can choose a minimal realization of $\mathfrak{h}$, i.e. a vector space $\mathfrak{h}$ together with linearly independent coroots $a_1, \dots a_n$ and linearly independent roots $b_1, \dots, b_n$ such that the natural pairing satisfies $\langle a_i, b_j \rangle = B_{ij}$. Thus we obtain a quadruple $v = (\mathfrak{h}, a, b, \tau)$. The connected component of this quadruple in the root groupoid is an admissible, fully reflectable component in the sense of [GHS24, Def. 3.2.3, §3.4.1].

Comparison 5.2.20. The connected component of v in the skeleton of the root groupoid is the simply connected cover of $\mathcal{W}$ in the sense of [HS20, Def. 9.1.10 and 10.1.1].

Comparison 5.2.21. The subgroupoid $\mathcal{W}'$ of $\mathcal{W}$ generated by all isotropic reflections is isomorphic to the connected component of v in the spine of the root groupoid.

Yet another notion of Weyl groupoids was suggested by Sergeev and Veselov in [SV11]. However as they remark this construction is completely unrelated to the notion of Weyl groupoid we work with.

5.3 Borel subalgebras of $\mathfrak{sl}(m|n)$, $\mathfrak{osp}(2m+1|2n)$ and $\mathfrak{osp}(2m|2n)$

To give a detailed description of the Weyl groupoids of $\mathfrak{sl}(m|n)$, $\mathfrak{osp}(2m+1|2n)$ and $\mathfrak{osp}(2m|2n)$ we first need some preparation. Recall that the Lie superalgebra $\mathfrak{sl}(m|n)$ is given in matrices by

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

such that $\mathrm{tr}(A) - \mathrm{tr}(D) = 0$ together with the usual supercommutator $[x, y] = x \circ y - (-1)^{|x||y|} y \circ x$.

The orthosymplectic Lie superalgebra $\mathfrak{osp}(2m+1|2n)$ is explicitly given by all matrices of the form

$$\left(\begin{array}{ccc|cc} 0 & -u^t & -v^t & x & x_1 \\ v & a & b & y & y_1 \\ u & c & -a^t & z & z_1 \\ \hline -x_1^t & -z_1^t & -y_1^t & d & e \\ x^t & z^t & y^t & f & -d^t \end{array}\right) \tag{5.2}$$

where a is any $(m \times m)$-matrix; b and c are skew-symmetric $(m \times m)$-matrices; d is any $(n \times n)$-matrix; e and f are symmetric $(n \times n)$-matrices; u and v are $(m \times 1)$-matrices; y, y_1, z and z_1 are $(m \times n)$-matrices; and x as well as x_1 are $(1 \times n)$-matrices. The Lie superalgebra $\mathfrak{osp}(2m|2n)$ is given by the same matrices, except that we have to delete the first row and column. We label the rows and columns by $0, 1, \dots, m, -1, \dots, -m, (m+1), \dots, (m+n), -(m+1), \dots, -(m+n)$ in this order (leaving out 0 for $\mathfrak{osp}(2m|2n)$).

To determine the Weyl groupoids of $\mathfrak{sl}(m|n)$, $\mathfrak{osp}(2m+1|2n)$ and $\mathfrak{osp}(2m|2n)$ we require an explicit description of all the possible realizations as contragredient Lie superalgebras. Hence we need to determine all their ordered root bases, which mostly amounts to determining their Borel subalgebras, and the corresponding Cartan data. These results are standard and are essentially already contained in [Kac77, §2.5.5], though the formulation there is less convenient and not explicit enough for our purposes.

For $\mathfrak{sl}(n|n)$ there is a minor extra difficulty as the simple roots are not linearly independent, and the construction of the contragredient Lie superalgebra from the Cartan data computed from $\mathfrak{sl}(n|n)$ yields $\mathfrak{gl}(n|n)$ rather than $\mathfrak{sl}(n|n)$. Nevertheless we will use $\mathfrak{sl}(n|n)$ in all computations below, as all statements about Borel subalgebras and simple roots carry over to $\mathfrak{gl}(n|n)$.

Let $\mathfrak{g}$ be either $\mathfrak{sl}(m|n)$, $\mathfrak{osp}(2m+1|2n)$ or $\mathfrak{osp}(2m|2n)$. Recall (see e.g. [Mus12, §3.1]) that for a fixed Borel subalgebra $\mathfrak{b}_{\bar{0}} \subseteq \mathfrak{g}_{\bar{0}}$ there are only finitely many Borel subalgebras $\mathfrak{b} \subseteq \mathfrak{g}$ with even part $\mathfrak{b}_{\bar{0}}$. Moreover these Borel subalgebras can be relatively easily described in terms of partitions fitting in an $m \times n$-rectangle, see e.g. [Mus12, Proposition 3.3.8]: For $\mathfrak{sl}(m|n)$ and $\mathfrak{osp}(2m+1|2n)$ the Borel subalgebras with fixed even part are in bijection with the set of partitions λ fitting in an $m \times n$-rectangle. For $\mathfrak{osp}(2m|2n)$ each partition λ fitting in an $m \times n$-rectangle determines two Borel subalgebras $\mathfrak{b}(\lambda, +)$ and $\mathfrak{b}(\lambda, -)$, which coincide if and only if $\lambda_1 = n$.

In Sections 5.3.1 to 5.3.3 below we provide a detailed description of the Borel subalgebras of $\mathfrak{sl}(m|n)$, $\mathfrak{osp}(2m+1|2n)$ and $\mathfrak{osp}(2m|2n)$, based on the description in [Mus12, §3.3–3.4]. We compute the corresponding Cartan data in Section 5.A. For this it is more convenient to work with permutations instead of partitions, and therefore we will frequently use Lemma 5.A.1 to pass between these.

5.3.1 Borel subalgebras for $\mathfrak{sl}(m|n)$

For the Lie superalgebra $\mathfrak{sl}(m|n)$ we fix the $m+n-1$-dimensional Cartan subalgebra $\mathfrak{h}$ given by all the diagonal matrices, and we are interested in Borel subalgebras with even part $\mathfrak{b}_{\bar{0}}$ given by the standard even Borel subalgebra of upper triangular matrices. As usual, we let $\varepsilon_i \in \mathfrak{h}^*$ $(1 \leq i \leq m+n)$ denote the projection to the i-th diagonal entry.

Given a partition $\lambda \in \mathcal{P}_{m \times n}$ we can construct the odd part of a Borel subalgebra $\mathfrak{b}(\lambda)$ with even part $\mathfrak{b}_{\bar{0}}$ as follows: Draw λ in an $m \times n$-rectangle as in (5.4), which we identify with the top right $m \times n$-block. The entries corresponding to the boxes of λ are required to be 0, while the other entries in the top right block can be arbitrary. Similarly drawing the transpose of the complement of λ (taken in the $m \times n$-rectangle) into the lower right $n \times m$-block determines the zeros and arbitrary entries there. For instance the "standard" Borel subalgebra of upper triangular matrices corresponds to $\lambda = \emptyset$, and for another concrete example see Example 5.3.1 below. By [Mus12, Prop. 3.3.8] the $\mathfrak{b}(\lambda)$'s are all the Borel subalgebras of $\mathfrak{sl}(m|n)$ with even part $\mathfrak{b}_{\bar{0}}$.

Example 5.3.1. Let $m = 3$, $n = 4$ and $\lambda = (4, 2, 1)$. Then the corresponding Borel subalgebra of $\mathfrak{sl}(3|4)$ is given by

$$\mathfrak{b}(\lambda) = \left(\begin{array}{ccc|cccc} * & * & * & 0 & * & * & * \\ 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & * & 0 & 0 & 0 & 0 \\ \hline * & * & * & * & * & * & * \\ 0 & * & * & 0 & * & * & * \\ 0 & 0 & * & 0 & 0 & * & * \\ 0 & 0 & * & 0 & 0 & 0 & * \end{array}\right) \tag{5.3}$$

Using Lemma 5.A.1 to pass between shuffles and partitions, we can explicitly describe the ordered root bases and Cartan data for the Borel subalgebras of $\mathfrak{sl}(m|n)$, see Proposition 5.A.3

5.3.2 Borel subalgebras for $\mathfrak{osp}(2m+1|2n)$

To describe all Borel subalgebras of $\mathfrak{osp}(2m+1|2n)$ we fix the Cartan subalgebra $\mathfrak{h}$ consisting of the diagonal matrices and let $\varepsilon_i \in \mathfrak{h}^*$ ($i \in \{\pm 1, \dots, \pm(m+n)\}$) denote the projections to the diagonal entries. We also fix the standard Borel subalgebra $\mathfrak{b}_{\bar{0}}$ of the even part, which is defined by the simple roots $\varepsilon_i - \varepsilon_{i+1}$ for $1 \leq i \leq m-1$ as well as ε_m and $\varepsilon_{m+j} - \varepsilon_{m+j+1}$ for $1 \leq j \leq n-1$ together with $2\varepsilon_{m+n}$.

For a partition $\lambda \in \mathcal{P}_{m\times n}$ we can construct the odd part of a Borel subalgebra $\mathfrak{b}(\lambda)$ with even part $\mathfrak{b}_{\bar{0}}$ as follows: In the notation from (5.2) we demand that $z = 0$ and $x = 0$, while x_1 and y_1 can be chosen arbitrarily. Note that y and z_1 are $m \times n$-matrices, and we identify these with the $m \times n$-rectangle from Lemma 5.A.1. For y, the entries in the boxes corresponding to λ must be zero while the other entries are arbitrary, and for z_1 the rule is exactly the opposite. Again by [Mus12, Prop. 3.3.8] the $\mathfrak{b}(\lambda)$'s are all the Borel subalgebras of $\mathfrak{osp}(2m+1|2n)$ with even part $\mathfrak{b}_{\bar{0}}$.

Example 5.3.2. Let $m = 1$, $n = 2$ and consider the partition $\lambda = (1)$. The corresponding Borel subalgebra of $\mathfrak{osp}(3|4)$ is given by

$$\mathfrak{b}(\lambda) = \left(\begin{array}{ccc|cccc} 0 & 0 & * & 0 & 0 & * & * \\ * & * & 0 & 0 & * & * & * \\ 0 & 0 & * & 0 & 0 & * & 0 \\ \hline * & * & * & * & * & * & * \\ * & 0 & * & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * & 0 \\ 0 & 0 & * & 0 & 0 & * & * \end{array}\right)$$

The corresponding ordered root bases and Cartan data are described in terms of shuffles in Proposition 5.A.4.

5.3.3 Borel subalgebras for $\mathfrak{osp}(2m|2n)$

The case of $\mathfrak{osp}(2m|2n)$ is slightly more involved. Again we fix the Cartan subalgebra $\mathfrak{h}$ consisting of diagonal matrices and let $\varepsilon_i \in \mathfrak{h}^*$ denote the projection to the i-th diagonal entry (with the same ordered basis and index conventions as for $\mathfrak{osp}(2m+1|2n)$). Furthermore, we fix the standard Borel $\mathfrak{b}_{\bar{0}}$ of the even part, which is given by the simple roots $\varepsilon_i - \varepsilon_{i+1}$ for $1 \leq i \leq m-1$ as well as $\varepsilon_{m-1} + \varepsilon_m$ and $\varepsilon_{m+j} - \varepsilon_{m+j+1}$ for $1 \leq j \leq n-1$ together with $2\varepsilon_{m+n}$.

Suppose we are given a partition $\lambda \in \mathcal{P}_{m\times n}$ and $\varepsilon \in \{+, -\}$. From this we construct the odd part of a Borel subalgebra $\mathfrak{b}(\lambda, \varepsilon)$ with even part $\mathfrak{b}_{\bar{0}}$ as follows. If $\varepsilon = +$, the entries are determined by λ by the same rules as in Section 5.3.2. If $\varepsilon = -$, we do the same construction as for $+$ but afterwards we swap the m-th and the $(-m)$-th row of the top right block. The $\mathfrak{b}(\lambda, \varepsilon)$'s are all the Borel subalgebras of $\mathfrak{osp}(2m|2n)$ with even part $\mathfrak{b}_{\bar{0}}$ by [Mus12, Prop. 3.3.8].

Observe that $\mathfrak{b}(\lambda, +) = \mathfrak{b}(\lambda, -)$ if and only if $\lambda_1 = n$, since we need the m-th row of y to be zero and the m-th row of z_1 to be arbitrary. In this case we also denote the resulting Borel subalgebra by $\mathfrak{b}(\lambda, \pm)$.

Example 5.3.3. Let $m = n = 2$. From the partitions $\lambda = (1, 1)$ and $\mu = (2, 1)$ we obtain the following Borel subalgebras $\mathfrak{b}(\lambda, +)$, $\mathfrak{b}(\lambda, -)$ and $\mathfrak{b}(\mu, \pm)$ of $\mathfrak{osp}(4|4)$. Here $\mathfrak{b}(\lambda, -)$ is obtained from $\mathfrak{b}(\lambda, +)$ by swapping the rows and columns as indicated.

$$\left(\begin{array}{cccc|cccc} * & * & 0 & * & 0 & * & * & * \\ 0 & * & * & 0 & 0 & * & * & * \\ 0 & 0 & * & 0 & 0 & 0 & * & 0 \\ 0 & 0 & * & * & 0 & 0 & * & 0 \\ \hline * & * & * & * & * & * & * & * \\ 0 & 0 & * & * & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & 0 \\ 0 & 0 & * & * & 0 & 0 & * & * \end{array}\right) \qquad \left(\begin{array}{cccc|cccc} * & * & 0 & * & 0 & * & * & * \\ 0 & * & * & 0 & 0 & 0 & * & 0 \\ 0 & 0 & * & 0 & 0 & 0 & * & 0 \\ 0 & 0 & * & * & 0 & * & * & * \\ \hline * & * & * & * & * & * & * & * \\ 0 & * & * & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & 0 \\ 0 & * & * & 0 & 0 & 0 & * & * \end{array}\right) \qquad \left(\begin{array}{cccc|cccc} * & * & 0 & * & 0 & * & * & * \\ 0 & * & * & 0 & 0 & 0 & * & * \\ 0 & 0 & * & 0 & 0 & 0 & * & 0 \\ 0 & 0 & * & * & 0 & 0 & * & * \\ \hline * & * & * & * & * & * & * & * \\ 0 & * & * & * & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & 0 \\ 0 & 0 & * & 0 & 0 & 0 & * & * \end{array}\right)$$

$\mathfrak{b}(\lambda,+)$ $\qquad$ $\mathfrak{b}(\lambda,-)$ $\qquad$ $\mathfrak{b}(\mu,\pm)$

We describe the corresponding ordered root bases and Cartan data in Proposition 5.A.5, again in terms of shuffles rather than partitions. To connect this to the above description of the Borel subalgebras, observe that if a shuffle σ corresponds to a partition λ under the bijection from Lemma 5.A.1, then $\sigma(m+n) = m$ if and only if $\lambda_1 = n$.

Remark 5.3.4. As explained in [Mus12, §3.3] the extra difficulties for $\mathfrak{osp}(2m|2n)$ are due to the existence of an outer automorphism of $\mathfrak{o}(2m)$ that on $\mathfrak{h}^*$ swaps ε_m and $\varepsilon_{-m} = -\varepsilon_m$. This corresponds precisely to the swapping of rows in the construction of the Borel subalgebra $\mathfrak{b}(\lambda,-)$ from $\mathfrak{b}(\lambda,+)$.

5.3.4 Odd reflections in terms of partitions

For $\mathfrak{sl}(m|n)$, $\mathfrak{osp}(2m+1|2n)$ and $\mathfrak{osp}(2m|2n)$ we can describe odd reflections in terms of partitions as follows. Consider the $m \times n$-rectangle and number the boxes ascendingly in each row and column, starting from a 1 in the top left as in the following example.

1	2	3	4
2	3	4	5
3	4	5	6

Let $\lambda \in \mathcal{P}_{m\times n}$ and let $\mathfrak{b}(\lambda)$ be the Borel subalgebra of $\mathfrak{sl}(m|n)$ or $\mathfrak{osp}(2m+1|2n)$ constructed from λ. Observe that the numbers in the boxes that can be removed from or added to λ so that the result is still a partition $\lambda' \in \mathcal{P}_{m\times n}$ are precisely the indices of the odd isotropic simple roots for $\mathfrak{b}(\lambda)$. In particular such a box (if it exists) is unique. The Borel subalgebra obtained from $\mathfrak{b}(\lambda)$ by an odd reflection at the simple root α_i is $\mathfrak{b}(\lambda')$, with $\lambda' \in \mathcal{P}_{m\times n}$ obtained from λ by adding or removing a box numbered with i.

Unsurprisingly the description of odd reflections for $\mathfrak{osp}(2m|2m)$ is slightly more complicated due to the different series of Borels. In this case the odd reflection at α_i takes $\mathfrak{b}(\lambda,\varepsilon)$ to $\mathfrak{b}(\lambda',\varepsilon')$ according to the following rules (using the implicit convention $\varepsilon' = \pm$ if $\lambda'_1 = n$):

- If $\lambda_1 < n$ and $\varepsilon = +$, then $\varepsilon' = +$ and λ' is obtained from λ by adding or removing a box numbered i (note that in this case $\alpha_{m+n} = 2\varepsilon_{m+n}$ is even).
- If $\lambda_1 < n$ and $\varepsilon = -$, then $\varepsilon' = -$ and λ' is obtained from λ by adding or removing a box numbered i for $i < m+n-1$, and by adding the box numbered $m+n-1$ for $i = m+n$ (note that $\alpha_{m+n-1} = 2\varepsilon_{m+n}$ is even).
- If $\lambda_1 = n$, then $\varepsilon = \pm$.
    - If $i < m+n-1$, then $\varepsilon' = \pm$ and λ' is obtained from λ by adding or removing a box numbered i.

- If $i = m+n-1$, then $\varepsilon' = +$ and λ' is obtained from λ by removing the box numbered $m+n-1$.
- If $i = m+n$, then $\varepsilon' = -$ and λ' is obtained from λ by removing the box numbered $m+n-1$.

5.4 The Weyl groupoids of $\mathfrak{sl}(m|n)$, $\mathfrak{osp}(2m+1|2n)$ and $\mathfrak{osp}(2m|2n)$

In this section we give a detailed description of the Cartan graphs and the Weyl groupoids of $\mathfrak{sl}(m|n)$, $\mathfrak{osp}(2m+1|2n)$ and $\mathfrak{osp}(2m|2n)$. We begin by describing the underlying graph of the Cartan graph and then list the Serre matrices. Finally we determine the Coxeter-type relations among the generators of the Weyl groupoids.

These Weyl groupoids also appear in [AA17], where they are studied from the perspective of Nichols algebras. However, our combinatorics is based on the the graphical description of Borel subalgebras in terms of partitions. This directly reflects the structural theory of the Lie superalgebra, and therefore makes it very easy to pass between Weyl groupoids and Lie superalgebras. A further advantage of our description is that it is very easy to write down the Weyl groupoids in concrete examples.

5.4.1 Shape of the Cartan graph

In Sections 5.3.1 to 5.3.3 we gave a detailed description of the (finitely many) Borel subalgebras of $\mathfrak{sl}(m|n)$, $\mathfrak{osp}(2m+1|2n)$ and $\mathfrak{osp}(2m|2n)$ with fixed even part. As mentioned above, these represent all conjugacy classes of Borel subalgebras. Therefore by Remark 5.2.17 their Cartan graphs split into several identical subgraphs without edges between each other. Hence we only need to consider one of these subgraphs, namely the one corresponding to the Borel subalgebras described above. The number of these subgraphs is the order of the Weyl group, i.e. $m!n!$ for $\mathfrak{sl}(m|n)$, $2^{m+n}m!n!$ for $\mathfrak{osp}(2m+1|2n)$ and $2^{m+n-1}m!n!$ for $\mathfrak{osp}(2m|2n)$.

Moreover, by Remark 5.2.18 each of the above subgraphs again splits into several identical (up to a permutation of the edge colors) components without edges between them, corresponding to the possible reorderings of the simple roots. Observe that the ordering of the simple roots from Propositions 5.A.3 to 5.A.5 is consistent under odd reflections. Therefore we only describe the component corresponding to this ordering, and for convenience also call it the Cartan graph. Since the number of simple roots is $m+n-1$ for $\mathfrak{sl}(m|n)$ and $m+n$ for $\mathfrak{osp}(2m+1|2n)$ and $\mathfrak{osp}(2m|2n)$ it follows that altogether their Cartan graphs consist of $(m+n-1)!m!n!$ (resp. $2^{m+n}m!n!(m+n)!$, $2^{m+n-1}n!m!(m+n)!$) copies of this component. By [Mus12, Thm. 3.1.3] any two Borel subalgebras with the same even part are connected by a sequence of odd reflections, and therefore these components are moreover connected.

In Sections 5.3.1 to 5.3.3 we used partitions to describe the Borel subalgebras, and the corresponding shuffles to describe the simple roots. Also recall from Section 5.3.4 that in this description odd reflections correspond to adding or removing single boxes. From these observations we obtain:

Proposition 5.4.1. *The Cartan graphs of $\mathfrak{sl}(m|n)$ (resp. $\mathfrak{gl}(n|n)$), $\mathfrak{osp}(2m+1|2n)$ and $\mathfrak{osp}(2m|2n)$ have the following underlying graphs:*

1) *For $\mathfrak{sl}(m|n)$ (resp. $\mathfrak{gl}(n|n)$) the set of vertices is the set $\mathcal{P}_{m\times n}$ of partitions fitting in an $m\times n$-rectangle. The edges are colored by $\{1,\dots,m+n-1\}$. There are edges $\lambda \xleftrightarrow{i} \lambda'$ if*

λ' is obtained from λ by adding a box numbered i (using the numbering from Section 5.3.4), and loops $\lambda \stackrel{i}{\longleftrightarrow} \lambda$ if no box numbered i can be added to λ.

2) *For $\mathfrak{osp}(2m+1|2n)$ the set of vertices is $\mathcal{P}_{m\times n}$. The edges are colored by $\{1,\dots,m+n\}$. There are edges $\lambda \stackrel{i}{\longleftrightarrow} \lambda'$ if λ' is obtained from λ by adding a box numbered i, and loops $\lambda \stackrel{i}{\longleftrightarrow} \lambda$ if no box numbered i can be added to λ. In particular there are loops of color $m+n$ at every vertex.*
3) *For $\mathfrak{osp}(2m|2n)$ the set of vertices is*

$$\{(\lambda,\varepsilon) \mid \lambda \in \mathcal{P}_{m\times n}, \lambda_1 < n, \varepsilon \in \{+,-\}\} \cup \{(\lambda,\pm) \mid \lambda \in \mathcal{P}_{m\times n}, \lambda_1 = n\}.$$

The edges are colored by $\{1,\dots,m+n\}$, and the non-loop edges are as follows:

- *$(\lambda,\varepsilon) \stackrel{i}{\longleftrightarrow} (\lambda',\varepsilon)$ for $\lambda_1 < n$ and λ' obtained from λ by adding a box numbered i, with $1 \le i \le m+n-2$.*
- *$(\lambda,+) \stackrel{m+n-1}{\longleftrightarrow} (\lambda',\pm)$ for $\lambda_1 = n-1$ and λ' obtained from λ by adding the box numbered $m+n-1$.*
- *$(\lambda,-) \stackrel{m+n}{\longleftrightarrow} (\lambda',\pm)$ for $\lambda_1 = n-1$ and λ' obtained from λ by adding the box numbered $m+n-1$.*
- *$(\lambda,\pm) \stackrel{i}{\longleftrightarrow} (\lambda',\pm)$ for $\lambda_1 = n$ and λ' obtained from λ by adding a box numbered i, with $1 \le i \le m+n-2$.*

Hence the connected components of the Cartan graph of $\mathfrak{osp}(2m+1|2n)$ are almost the same as those of $\mathfrak{sl}(m|n)$, with the only difference being the additional loops of color $m+n$ at every vertex for $\mathfrak{osp}(2m+1|2n)$. For some concrete small examples see Section 5.4.5

5.4.2 The Serre matrices

Proposition 5.4.2. *The Serre matrices for $\mathfrak{sl}(m|n)$ (resp. $\mathfrak{gl}(n|n)$), $\mathfrak{osp}(2m+1|2n)$ and $\mathfrak{osp}(2m|2n)$ have the following form:*

1) *For $\mathfrak{sl}(m|n)$ (resp. $\mathfrak{gl}(n|n)$) the Serre matrix is A_{m+n-1} everywhere.*
2) *For $\mathfrak{osp}(2m+1|2n)$ the Serre matrix is B_{m+n} everywhere.*
3) *Let $\lambda \in \mathcal{P}_{m\times n}$ and $\varepsilon \in \{+,-\}$. Then the Serre matrix at the vertex (λ,ε) of the Cartan graph of $\mathfrak{osp}(2m|2n)$ is:*

- *C_{m+n} if $\lambda_1 < n-1$ and $\varepsilon = +$,*
- *C'_{m+n} (obtained by swapping the last two rows and columns of C_{m+n}) if $\lambda_1 < n-1$ and $\varepsilon = -$,*
- *A_{m+n} if $\lambda_1 = n-1$ and $\varepsilon = +$,*
- *A'_{m+n} (obtained by swapping the last two rows and columns of A_{m+n}) if $\lambda_1 = n-1$ and $\varepsilon = -$,*
- *D_{m+n} if $\lambda_2 = n$ and $\varepsilon = \pm$,*
- *the generalized Cartan matrix*

$$\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & \ddots & \ddots & \vdots & \vdots \\ 0 & \ddots & \ddots & -1 & 0 & 0 \\ \vdots & \ddots & -1 & 2 & -1 & -1 \\ 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & -1 & 2 \end{pmatrix}$$

if $\lambda_2 < \lambda_1 = n$ and $\varepsilon = \pm$.

Proof. In all cases the Serre matrix is obtained from the Cartan data determined in Propositions 5.A.3 to 5.A.5 according to the rules from Definition 5.2.7. □

Remark 5.4.3. Let $\mathfrak{g}$ be $\mathfrak{sl}(m|n)$ (resp. $\mathfrak{gl}(n|n)$), $\mathfrak{osp}(2m+1|2n)$ or $\mathfrak{osp}(2m|2n)$ and let $\mathfrak{b} \subseteq \mathfrak{g}$ be the Borel subalgebra corresponding to a vertex x of the Cartan graph $\mathcal{G}_{\mathfrak{g}}$. Observe that the Serre matrix at x is the generalized Cartan matrix corresponding to the Dynkin–Kac diagram for $\mathfrak{b}$ considered as a Dynkin diagram, see [Mus12, §3.4.3] for a list.

5.4.3 The Coxeter relations

By [HS20, Thm. 9.4.8] the Weyl groupoid of a Cartan graph $\mathcal{G} = (I, X, r, A)$ is a *Coxeter groupoid*, i.e. the generators s_i are only subject to relations of the form $\mathrm{id}_x(s_i s_j)^{m(x)_{ij}}\,\mathrm{id}_x = \mathrm{id}_x$ for some symmetric matrices $(m(x)_{ij})$ with $m(x)_{ii} = 1$ (with $i, j \in I$ and $x \in X$, and the implicit assumption that $(r_i r_j)^{m(x)_{ij}}(x) = x$ unless $m(x)_{ij} = \infty$). In fact $m(x)_{ij} = |\Delta_x^{\mathrm{real}} \cap (\mathbb{N}_0\alpha_i^x + \mathbb{N}_0\alpha_j^x)|$. Therefore to obtain a presentation in terms of generators and relations we only have to determine the orders of $s_i s_j$, starting from all vertices of the Cartan graph.

Proposition 5.4.4. *For $\mathfrak{sl}(m|n)$ (resp. $\mathfrak{gl}(n|n)$), $\mathfrak{osp}(2m+1|2n)$ and $\mathfrak{osp}(2m|2n)$, the $m(x)_{ij}$ are determined from the Serre matrices by the same rules as for semisimple Lie algebras. Explicitly,*

$$\begin{aligned} A(x)_{ij}A(x)_{ji} = 0 &\implies m(x)_{ij} = 2, \\ A(x)_{ij}A(x)_{ji} = 1 &\implies m(x)_{ij} = 3, \\ A(x)_{ij}A(x)_{ji} = 2 &\implies m(x)_{ij} = 4. \end{aligned}$$

Proof. By Proposition 5.4.2 we know for $\mathfrak{sl}(m|n)$ (resp. $\mathfrak{gl}(n|n)$) and $\mathfrak{osp}(2m+1|2n)$ that we have the same Serre matrices at every vertex in our Cartan graph. As the Serre matrices are the same at all vertices, the linear maps $s_i \colon \mathfrak{h}^* \to \mathfrak{h}^*$ corresponding to the generators of the Weyl groupoid are independent of the vertex. From this it follows that $(s_i s_j)^{m(x)_{ij}} = \mathrm{id}_{\mathfrak{h}^*}$, and that $m(x)_{ij}$ is the lowest number fulfilling this. Thus we only need to check that the induced path in the Cartan graph ends at the same vertex that we started. But this follows easily from the explicit description in terms of partitions.

For $\mathfrak{osp}(2m|2n)$ the situation is a bit more tedious. We need to compute the intersection of the linear span of two simple roots with the roots of $\mathfrak{osp}(2m|2n)$. Using the explicit description of the simple roots in Proposition 5.A.5 this is rather straightforward. We will only do this for the interesting cases, i.e. when $i, j \in \{n+m-2, n+m-1, n+m\}$, as the remaining cases are similar (and easier).

- For $x = (\lambda, +)$ with $\lambda_1 \leq m-2$, the last three simple roots are $\varepsilon_i - \varepsilon_{m+n-1}$, $\varepsilon_{m+n-1} - \varepsilon_{m+n}$, $2\varepsilon_{m+n}$, where i is either m or $m+n-2$. Therefore we get the claimed $m(x)_{ij}$.
- For $x = (\lambda, +)$ with $\lambda_1 = m-1$, the last three simple roots are $\varepsilon_i - \varepsilon_m$, $\varepsilon_m - \varepsilon_{m+n}$, $2\varepsilon_{m+n}$ where i is either $m-1$ or $m+n-1$. As $2\varepsilon_m$ is not a root of $\mathfrak{osp}(2m|2n)$ we get type A relations.
- For $x = (\lambda, -)$ we can apply the same argument as above since in this case only the last two simple roots are swapped.
- For $x = (\lambda, \pm)$ with $\lambda_1 = m > \lambda_2$, the corresponding last three simple roots are given by $\varepsilon_i - \varepsilon_{m+n}, \varepsilon_{m+n} - \varepsilon_m, \varepsilon_{m+n} + \varepsilon_m$ where i is either $m-1$ or $m+n-1$. As $2\varepsilon_{m+n}$ is a root we directly see that $m(x)_{ij} = 3$.
- Lastly suppose that $x = (\lambda, \pm)$ with $\lambda_1 = \lambda_2 = m$. The simple roots are then given by $\varepsilon_i - \varepsilon_{m-1}$, $\varepsilon_{m-1} - \varepsilon_m$, $\varepsilon_{m-1} + \varepsilon_m$ where i is either $m+n$ or $m-2$. Now $2\varepsilon_{m-1}$ is not a root of $\mathfrak{osp}(2m|2n)$, therefore the linear span of $\varepsilon_{m-1} - \varepsilon_m$ and $\varepsilon_{m-1} + \varepsilon_m$ consists only of 2 roots. Additionally $\varepsilon_i \pm \varepsilon_m$ are indeed roots, so we see the claimed type D phenomenon. □

5.4.4 Automorphisms

For $\mathfrak{sl}(m|n)$ (resp. $\mathfrak{gl}(n|n)$), $\mathfrak{osp}(2m+1|2n)$ and $\mathfrak{osp}(2m|2n)$ Proposition 5.2.15 yields the following explicit description of the automorphism group of an object of the Weyl groupoid.

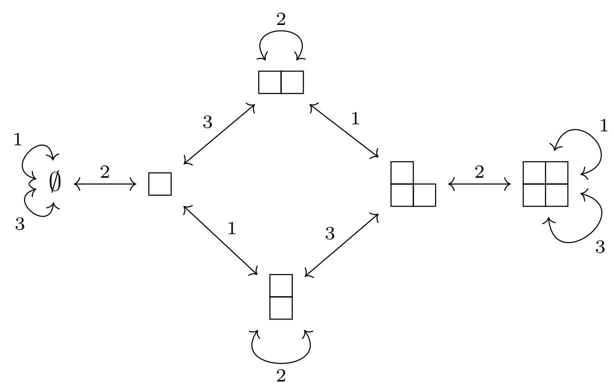
Corollary 5.4.5. *If $\mathfrak{g}$ is $\mathfrak{sl}(m|n)$ (resp. $\mathfrak{gl}(n|n)$), $\mathfrak{osp}(2m+1|2n)$ or $\mathfrak{osp}(2m|2n)$ and x any vertex of the Cartan graph of $\mathfrak{g}$, then $\mathrm{Aut}_{\mathcal{W}}(x)$ is isomorphic to the Weyl group of $\mathfrak{g}_{\bar{0}}$.*

Proof. We only have to show that any even root is principal, which follows easily from the explicit description in Propositions 5.A.3 to 5.A.5. □

5.4.5 Some small examples

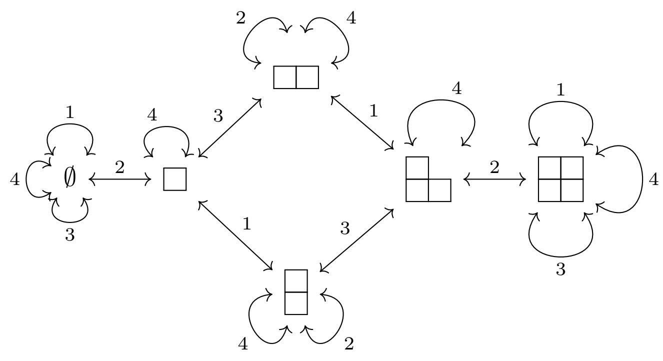
We explicitly describe the Weyl groupoids of $\mathfrak{sl}(m|n)$, $\mathfrak{osp}(2m+1|2n)$ and $\mathfrak{osp}(2m|2n)$ in a few examples.

Example 5.4.6. The contragredient Lie superalgebra $\mathfrak{gl}(2|2)$ has three pairs of Chevalley generators, so the associated Weyl groupoid has three generators s_1, s_2 and s_3. By Section 5.3.1 we can index the Borel subalgebras (with a fixed even part) by partitions fitting into an 2×2-rectangle, and by Proposition 5.4.2 the Serre matrix is $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$ everywhere. From the description of odd reflections in Section 5.3.4 it follows that the generators of $\mathcal{W}$ can be drawn in a diagram



By Proposition 5.4.4 the only relations are the familiar braid relations $s_1s_2s_1 = s_2s_1s_2$, $s_2s_3s_2 = s_3s_2s_3$ and $s_1s_3 = s_3s_1$ (for all vertices of the Cartan graph).

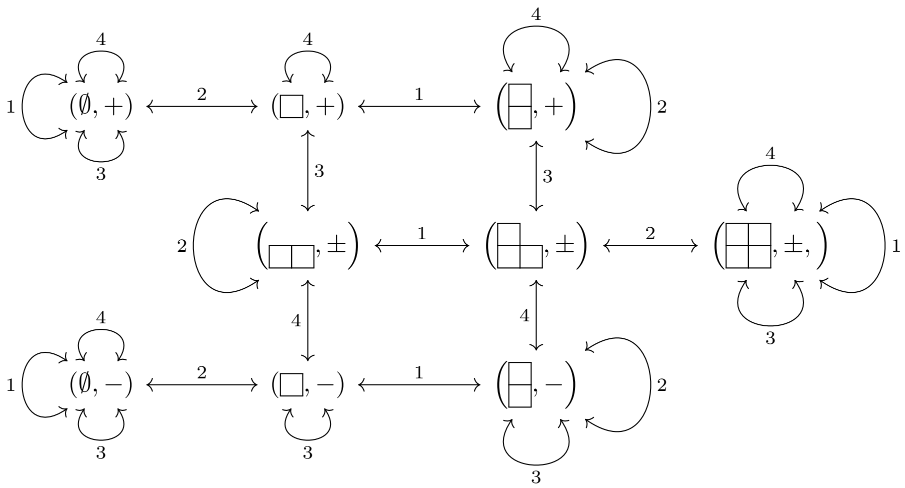
Example 5.4.7. For the Lie superalgebra $\mathfrak{osp}(5|4)$, the underlying graph looks as in Example 5.4.6 but its Cartan subalgebra is 4-dimensional instead of 3-dimensional. So every vertex gets an additional loop with index 4.



In this case the Serre matrix is $B_4 = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & -1 & 2 \end{pmatrix}$ everywhere. The generators are subject to the usual "type B braid relations", which are the relations from Example 5.4.6 as well as $s_3s_4s_3s_4 = s_4s_3s_4s_3$ and $s_is_4 = s_4s_i$ for $i \in \{1, 2\}$.

Example 5.4.6 and Example 5.4.7 had in common that the Serre matrices were all the same at every vertex. This is however not true for $\mathfrak{osp}(2m|2n)$:

Example 5.4.8. The Cartan graph of $\mathfrak{osp}(4|4)$ is



The Serre matrix in the top left corner is of type $C_4 = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -2 & 2 \end{pmatrix}$. The other two Serre matrices in the first row are of type $A_4 = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$. The bottom row has the same Serre matrices as the first row except we swap the third and fourth row and column, i.e. we have $\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & -1 \\ 0 & 0 & 2 & -2 \\ 0 & -1 & -1 & 2 \end{pmatrix}$ in the bottom left corner and $\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & -1 \\ 0 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}$ for the other two. At $\left(\ydiagram{2,2}, \pm\right)$ we have $D_4 = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}$. The remaining two Serre matrices are given by $\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}$, in particular these are not of Dynkin type.

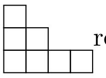
The generators of $\mathcal{W}$ are subject to the braid relations (including relations of type $C = B$) specified by the Serre matrices.

5.A Computation of Cartan data

In this appendix we explicitly describe the simple roots and Cartan data for the Borel subalgebras of $\mathfrak{sl}(m|n)$, $\mathfrak{osp}(2m+1|2n)$ and $\mathfrak{osp}(2m|2n)$. For this it is more convenient to work with permutations instead of partitions, so we first need to set up a bit of combinatorics to pass between the two notions.

5.A.1 Combinatorics: Shuffles and partitions

Let $\mathcal{P}_{m\times n}$ be the set of partitions whose Young diagram fits into an $m \times n$-rectangle. We use the slightly unusual convention that the longest row is at the bottom, so for instance the diagram

 represents the partition $\lambda = (4, 2, 1)$.

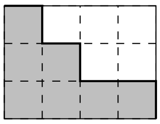
Recall that a permutation $\sigma \in S_{m+n}$ is an (m,n)-*shuffle* if $\sigma^{-1}(i) < \sigma^{-1}(j)$ for all pairs $i < j$ with either $i, j \leq m$ or $i, j > m$. We write $\mathrm{Shff}(m,n)$ for the set of (m,n)-shuffles. Equivalently, $\mathrm{Shff}(m,n)$ can be defined as a set of shortest coset representatives for the parabolic quotient $(S_m \times S_n)\backslash S_{m+n}$. Note that if σ is an (m,n)-shuffle, then either $\sigma(m+n) = m$ or $\sigma(m+n) = m+n$.

We will identify shuffles with partitions as follows:

Lemma 5.A.1. *There is a bijection between* $\mathrm{Shff}(m,n)$ *and the set of Young diagrams (partitions) fitting into an $m \times n$-rectangle, as follows: for an (m,n)-shuffle σ we draw a path in the $m \times n$-rectangle, where in the i-th step we go down if $\sigma(i) \leq m$ and right if $\sigma(i) > m$. Then the partition λ consists of the boxes below the path.*

This bijection is best explained in an example.

Example 5.A.2. Let $m = 3$, $n = 4$ and $\sigma = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 1 & 5 & 2 & 6 & 7 & 3 \end{smallmatrix}\right) \in \mathrm{Shff}(3,4)$. According to Lemma 5.A.1 σ encodes the boundary path $rdrdrrd$ (where r means "right" and d "down"):



(5.4)

The permutation corresponding to σ is $\lambda = (4, 2, 1)$.

5.A.2 Cartan data for $\mathfrak{sl}(m|n)$

Proposition 5.A.3. *Let $\sigma \in \mathrm{Shff}(m,n)$. The simple roots corresponding to the Borel subalgebra $\mathfrak{b}(\sigma)$ of $\mathfrak{sl}(m|n)$ are*

$$\Pi(\sigma) = \{\alpha_i = \varepsilon_{\sigma(i)} - \varepsilon_{\sigma(i+1)} \mid 1 \leq i \leq m+n-1\}.$$

For the corresponding Cartan datum (B, τ) we have $\tau_i = \bar{0}$ if either $\sigma(i), \sigma(i+1) \leq m$ or $\sigma(i), \sigma(i+1) > m$, and $\tau_i = \bar{1}$ otherwise. The i-th row of the matrix B is given by

$$(b_{i,1}, \ldots, b_{i,m+n-1}) = \begin{cases} (0, \ldots, 0, -1, 2, -1, 0, \ldots, 0) & \textit{if } |e_i| = \bar{0}, \\ (0, \ldots, 0, -1, 0, 1, 0, \ldots, 0) & \textit{if } |e_i| = \bar{1}, \end{cases}$$

where the entry 2 (resp. the "middle" 0) is in the i-th spot.

Proof. The simple roots are listed in [Mus12, Lem. 3.4.3]. Since the elementary matrix E_{rs} is of weight $\varepsilon_r - \varepsilon_s$ we can take

$$e_i = E_{\sigma(i),\sigma(i+1)}, \qquad f_i = E_{\sigma(i+1),\sigma(i)}$$

as Chevalley generators. Clearly e_i (and f_i) is even if and only if $\sigma(i)$ and $\sigma(i+1)$ are either both $\leq m$ or both $\geq m+1$. Therefore $h_i = [e_i, f_i] = E_{\sigma(i),\sigma(i)} - (-1)^{|e_i|} E_{\sigma(i+1),\sigma(i+1)}$, and thus

$$\alpha_j(h_i) = \begin{cases} 0 & \text{if } j \neq i, i \pm 1, \\ 2 & \text{if } j = i, |e_i| = \bar{0}, \\ 0 & \text{if } j = i, |e_i| = \bar{1}, \\ -1 & \text{if } j = i-1, \\ -(-1)^{\tau_i} & \text{if } j = i+1. \end{cases}$$

This shows that the matrix B has the claimed form. $\square$

5.A.3 Cartan data for $\mathfrak{osp}(2m+1|2n)$

Proposition 5.A.4. *Let $\sigma \in \mathrm{Shff}(m,n)$. The simple roots for the Borel subalgebra $\mathfrak{b}(\sigma)$ of $\mathfrak{osp}(2m+1|2n)$ are*

$$\Pi(\sigma) = \{\alpha_i = \varepsilon_{\sigma(i)} - \varepsilon_{\sigma(i+1)} \mid 1 \leq i \leq m+n-1\} \cup \{\alpha_{m+n} = \varepsilon_{\sigma(m+n)}\}.$$

The corresponding Cartan datum (B,τ) can be described as follows. For $1 \leq i \leq m+n-1$ we have $\tau_i = \bar{0}$ if either $\sigma(i), \sigma(i+1) \leq m$ or $\sigma(i), \sigma(i+1) > m$, and $\tau_{m+n} = \bar{0}$ if $\sigma(m+n) \leq m$ and $\tau_{m+n} = \bar{1}$ if $\sigma(m+n) > m$. The i-th row of the matrix B is given by

$$(b_{i,1}, \ldots, b_{i,m+n-1}) = \begin{cases} (0,\ldots,0,-1,2,-1,0,\ldots,0) & \text{if } i < m+n, |e_i| = \bar{0}, \\ (0,\ldots,0,-1,0,1,0,\ldots,0) & \text{if } i < m+n, |e_i| = \bar{1}, \\ (0,\ldots,0,-1,1) & \text{if } i = m+n, \end{cases}$$

where the entry 2 (resp. the "middle" 0) is in the i-th spot.

Proof. The simple roots are listed in [Mus12, Lem. 3.4.3]. From the explicit description of the root spaces (see e.g. [Mus12, Exercise 2.7.4]) it follows that one possible choice for the Chevalley generators (for $1 \leq i \leq m+n-1$) is

$$\begin{aligned} e_i &= \begin{cases} E_{\sigma(i),\sigma(i+1)} + E_{-\sigma(i+1),-\sigma(i)} & \text{if } \sigma(i) \leq m, \sigma(i+1) > m, \\ E_{\sigma(i),\sigma(i+1)} - E_{-\sigma(i+1),-\sigma(i)} & \text{otherwise,} \end{cases} \\ e_{m+n} &= E_{\sigma(m+n),0} - E_{0,-\sigma(m+n)} \\ f_i &= \begin{cases} E_{\sigma(i+1),\sigma(i)} + E_{-\sigma(i),-\sigma(i+1)} & \text{if } \sigma(i) > m, \sigma(i+1) \leq m, \\ E_{\sigma(i+1),\sigma(i)} - E_{-\sigma(i),-\sigma(i+1)} & \text{otherwise,} \end{cases} \\ f_{m+n} &= \begin{cases} E_{0,m} - E_{-m,0} & \text{if } \sigma(m+n) = m, \\ E_{0,m+n} + E_{-(m+n),0} & \text{if } \sigma(m+n) = m+n. \end{cases} \end{aligned}$$

Hence e_i (and thus f_i) is even if and only if $\sigma(i)$ and $\sigma(i+1)$ are either both $\leq m$ or both $\geq m+1$. From this it follows that

$$h_i = [e_i, f_i] = E_{\sigma(i),\sigma(i)} - E_{-\sigma(i),-\sigma(i)} - (-1)^{|e_i|}(E_{\sigma(i+1),\sigma(i+1)} - E_{-\sigma(i+1),-\sigma(i+1)})$$

for $1 \leq i \leq m+n-1$, and (independent of the value of $\sigma(m+n)$)

$$h_{m+n} = E_{\sigma(m+n),\sigma(m+n)} - E_{-\sigma(m+n),-\sigma(m+n)}.$$

Therefore the entries of the matrix B are

$$b_{ij} = \alpha_j(h_i) = \begin{cases} 0 & \text{if } j \neq i, i \pm 1, \\ 2 & \text{if } j = i < m+n, \tau_i = \bar{0}, \\ 0 & \text{if } j = i < m+n, \tau_i = \bar{1}, \\ 1 & \text{if } j = i = m+n, \\ -1 & \text{if } j = i-1, \\ -(-1)^{\tau_i} & \text{if } j = i+1, \end{cases}$$

and thus B has the claimed form. □

5.A.4 Cartan data for $\mathfrak{osp}(2m|2n)$

Proposition 5.A.5. *Let $\sigma \in \mathrm{Shff}(m,n)$ and $\varepsilon \in \{+,-\}$. The simple roots and the Cartan data (B,τ) for the Borel subalgebras $\mathfrak{b}(\sigma,\varepsilon)$ of $\mathfrak{osp}(2m|2n)$ can be described as follows:*

1) For $\sigma(m+n) = m+n$ and $\mathfrak{b}(\sigma,+)$ the simple roots are

$$\Pi(\sigma,+) = \{\alpha_i = \varepsilon_{\sigma(i)-\sigma(i+1)} \mid 1 \le i \le m+n-1\} \cup \{\alpha_{m+n} = 2\varepsilon_{m+n}\}.$$

The Chevalley generators e_i and f_i for $1 \le i \le m+n-1$ are even if and only if either $\sigma(i), \sigma(i+1) \le m$ or $\sigma(i), \sigma(i+1) > m$, while e_{m+n} and f_{m+n} are even. The i-th row of B is

$$(b_{i,1},\dots,b_{i,m+n}) = \begin{cases} (0,\dots,0,-1,2,-1,0,\dots,0) & \text{if } i \ne m+n-1, |e_i| = \bar{0}, \\ (0,\dots,0,-1,0,1,0,\dots,0) & \text{if } i \ne m+n-1, |e_i| = \bar{1}, \\ (0,\dots,0,-1,2,-2) & \text{if } i = m+n-1, |e_i| = \bar{0}, \\ (0,\dots,0,-1,0,2) & \text{if } i = m+n-1, |e_i| = \bar{1}. \end{cases}$$

again with the entry 2 (resp. the “middle” 0) in the i-th spot.

2) For $\sigma(m+n) = m+n$ and $\mathfrak{b}(\sigma,-)$ the simple roots $\Pi(\sigma,-)$ are obtained from $\Pi(\sigma,+)$ by replacing every ε_m by $\varepsilon_{-m} = -\varepsilon_m$, and swapping the $m+n-1$-th and the $m+n$-th simple root. The Cartan datum is as above except that the last two rows and columns of the matrix B are swapped.

3) For $\sigma(m+n) = m$ and $\mathfrak{b}(\sigma,\pm)$ the simple roots are

$$\Pi(\sigma,\pm) = \{\alpha_i = \varepsilon_{\sigma(i)-\sigma(i+1)} \mid 1 \le i \le m+n-1\} \cup \{\alpha_{m+n} = \varepsilon_{\sigma(m+n-1)} + \varepsilon_{\sigma(m+n)}\}.$$

The Chevalley generators e_i and f_i for $1 \le i \le m+n-1$ are even if and only if either $\sigma(i), \sigma(i+1) \le m$ or $\sigma(i), \sigma(i+1) > m$, while e_{m+n} and f_{m+n} have the same parity as e_{m+n-1}. The i-th row $(b_{i,1},\dots,b_{i,m+n})$ of the matrix B is given by the following table:

	$\tau_i = \bar{0}$	$\tau_i = \bar{1}$
$i < m+n-2$	$(0,\dots,0,-1,2,-1,0,\dots,0)$	$(0,\dots,0,-1,0,1,0,\dots,0)$
$i = m+n-2$	$(0,\dots,0,-1,2,-1,-1)$	$(0,\dots,0,-1,0,1,1)$
$i = m+n-1$	$(0,\dots,0,-1,2,0)$	$(0,\dots,0,-1,0,2)$
$i = m+n$	$(0,\dots,0,-1,0,2)$	$(0,\dots,0,-1,2,0)$

Proof. The simple roots are listed in [Mus12, Lem. 3.4.3]. A choice for the Chevalley generators e_i, f_i and $h_i = [e_i, f_i]$ associated with the simple roots is:

- for $\alpha_i = \varepsilon_{\sigma(i)} - \varepsilon_{\sigma(i+1)}$,

$$e_i = \begin{cases} E_{\sigma(i),\sigma(i+1)} + E_{-\sigma(i+1),-\sigma(i)} & \text{if } \sigma(i) \le m, \sigma(i+1) > m, \\ E_{\sigma(i),\sigma(i+1)} - E_{-\sigma(i+1),-\sigma(i)} & \text{otherwise,} \end{cases}$$

$$f_i = \begin{cases} E_{\sigma(i+1),\sigma(i)} + E_{-\sigma(i),-\sigma(i+1)} & \text{if } \sigma(i) > m, \sigma(i+1) \le m, \\ E_{\sigma(i+1),\sigma(i)} - E_{-\sigma(i),-\sigma(i+1)} & \text{otherwise,} \end{cases}$$

$$h_i = E_{\sigma(i),\sigma(i)} - E_{-\sigma(i),-\sigma(i)} - (-1)^{|e_i|}(E_{\sigma(i+1),\sigma(i+1)} - E_{-\sigma(i+1),-\sigma(i+1)}).$$

- for $\alpha_i = \varepsilon_{\sigma(i)} + \varepsilon_m$,

$$\begin{aligned} e_i &= E_{\sigma(i),-m} - E_{m,-\sigma(i)}, \\ f_i &= \begin{cases} E_{-m,\sigma(i)} - E_{-\sigma(i),m} & \text{if } \sigma(i) \leq m, \\ E_{-m,\sigma(i)} + E_{-\sigma(i),m} & \text{if } \sigma(i) > m, \end{cases} \\ h_i &= E_{\sigma(i),\sigma(i)} - E_{-\sigma(i),-\sigma(i)} + (-1)^{|e_i|}(E_{m,m} - E_{-m,-m}). \end{aligned}$$

- for $\alpha_i = -\varepsilon_m - \varepsilon_{\sigma(i+1)}$,

$$\begin{aligned} e_i &= \begin{cases} E_{-m,\sigma(i+1)} - E_{-\sigma(i+1),m} & \text{if } \sigma(i+1) \leq m, \\ E_{-m,\sigma(i+1)} + E_{-\sigma(i+1),m} & \text{if } \sigma(i+1) > m, \end{cases} \\ f_i &= E_{\sigma(i+1),-m} - E_{m,-\sigma(i+1)}, \\ h_i &= E_{-m,-m} - E_{m,m} - (-1)^{|e_i|}(E_{\sigma(i+1),\sigma(i+1)} - E_{-\sigma(i+1),-\sigma(i+1)}). \end{aligned}$$

- for $\alpha_i = 2\varepsilon_{m+n}$,

$$\begin{aligned} e_i &= E_{m+n,-(m+n)}, & f_i &= E_{-(m+n),m+n}, \\ h_i = [e_i, f_i] &= E_{m+n,m+n} - E_{-(m+n),-(m+n)}. \end{aligned}$$

Now we plug the h_i into the α_j. For this we have to consider each of the three classes of Borel subalgebras separately.

Case I: $\sigma(m+n) = m+n, +$. Then

$$\alpha_j(h_i) = \begin{cases} 0 & \text{if } j \neq i, i \pm 1, \\ 2 & \text{if } j = i, \tau_i = \bar{0}, \\ 0 & \text{if } j = i, \tau_i = \bar{1}, \\ -1 & \text{if } j = i-1, \\ -(-1)^{\tau_i} & \text{if } j = i+1 < m+n, \\ -2(-1)^{\tau_i} & \text{if } j = i+1 = m+n, \end{cases}$$

and it follows that the matrix B has the claimed form.

Case II: $\sigma(m+n) = m+n, -$. In this case let $i_0 = \sigma^{-1}(m)$. The simple roots are as in the previous case except that $\varepsilon_{\sigma(i_0-1)} - \varepsilon_m$ and $\varepsilon_m - \varepsilon_{\sigma(i_0+1)}$ are replaced by $\varepsilon_{\sigma(i_0-1)} + \varepsilon_m$ and $-\varepsilon_m - \varepsilon_{\sigma(i_0+1)}$, respectively, and the numbering is changed. The values $\alpha_j(h_i)$ are as in the previous case except we have to check the cases involving i_0 separately. Nevertheless it follows that the matrix B is as in Case I except we have to swap the last two rows and columns to account for the renumbering of the simple roots.

Case III: $\sigma(m+n)=m$. Similarly to the previous cases we obtain

$$\alpha_j(h_i)=\begin{cases}0 & \text{if } j\neq i, i\pm 1, i\pm 2,\\ 2 & \text{if } j=i, \tau_i=\bar{0},\\ 0 & \text{if } j=i, \tau_i=\bar{1},\\ -1 & \text{if } j=i-1, i<m+n,\\ 1-(-1)^{\tau_i} & \text{if } j=i-1, i=m+n,\\ 0 & \text{if } j=i-2, i<m+n,\\ -1 & \text{if } j=i-2, i=m+n,\\ -(-1)^{\tau_i} & \text{if } j=i+1<m+n,\\ 1+(-1)^{\tau_i} & \text{if } j=i+1=m+n,\\ 0 & \text{if } j=i+2<m+n,\\ -(-1)^{\tau_i} & \text{if } j=i+2=m+n,\end{cases}$$

as claimed. □

Bibliography

[AA17] N. Andruskiewitsch and I. Angiono. "On finite dimensional Nichols algebras of diagonal type". *Bull. Math. Sci.* 7, no. 3 (2017).

[Ach21] P. N. Achar. *Perverse sheaves and applications to representation theory.* Mathematical Surveys and Monographs 258. American Mathematical Society, Providence, RI, 2021.

[Add16] N. Addington. "New derived symmetries of some hyperkähler varieties". *Algebr. Geom.* 3, no. 2 (2016).

[AHLSV22] L. Angeleri Hügel, R. Laking, J. Šťovíček, and J. Vitória. *Mutation and torsion pairs.* Preprint. 2022. arXiv:`2201.02147v1 [math.RT]`.

[AI12] T. Aihara and O. Iyama. "Silting mutation in triangulated categories". *J. Lond. Math. Soc. (2)* 85, no. 3 (2012).

[AL17] R. Anno and T. Logvinenko. "Spherical DG-functors". *J. Eur. Math. Soc. (JEMS)* 19, no. 9 (2017).

[AL19] R. Anno and T. Logvinenko. *$\mathbb{P}^n$-functors.* Preprint. 2019. arXiv:`1905.05740v2 [math.AG]`.

[AL22] R. Anno and T. Logvinenko. "On uniqueness of $\mathbb{P}$-twists". *Int. Math. Res. Not. IMRN* 2022, no. 14 (2022).

[AMY19] T. Adachi, Y. Mizuno, and D. Yang. "Discreteness of silting objects and t-structures in triangulated categories". *Proc. London Math. Soc.* 118, no. 3 (2019).

[AN09] S. Al-Nofayee. "Simple objects in the heart of a t-structure". *J. Pure Appl. Algebra* 213, no. 1 (2009).

[BB81] A. Beilinson and J. Bernstein. "Localisation de $\mathfrak{g}$-modules". *C. R. Acad. Sci., Paris, Sér. I* 292 (1981).

[BBD82] A. Beilinson, J. Bernstein, and P. Deligne. "Faisceaux pervers". *Astérisque* 100 (1982).

[BBM04] A. Beilinson, R. Bezrukavnikov, and I. Mirković. "Tilting exercises". *Mosc. Math. J.* 4, no. 3 (2004).

[BC25] L. Bonfert and A. Cipriani. *Serre functor and $\mathbb{P}$-objects for perverse sheaves on $\mathbb{P}^n$.* Preprint. 2025. arXiv:`2506.06051v1 [math.RT]`.

[BGS96] A. Beilinson, V. Ginzburg, and W. Soergel. "Koszul duality patterns in representation theory". *J. Amer. Math. Soc.* 9, no. 2 (1996).

[BK81] J.-L. Brylinski and M. Kashiwara. "Kazhdan–Lusztig conjecture and holonomic systems". *Invent. Math.* 64, no. 3 (1981).

[BK90] A. I. Bondal and M. M. Kapranov. "Representable functors, Serre functors, and mutations". *Math. USSR, Izv.* 35, no. 3 (1990).

[BK91] A. I. Bondal and M. M. Kapranov. “Enhanced triangulated categories”. *Math. USSR, Sb.* 70, no. 1 (1991).

[BN24] L. Bonfert and J. Nehme. “The Weyl groupoids of $\mathfrak{sl}(m|n)$ and $\mathfrak{osp}(r|2n)$”. *J. Algebra* 641 (2024).

[Bon10a] M. V. Bondarko. “Motivically functorial coniveau spectral sequences; direct summands of cohomology of function fields”. *Documenta Math.* Extra Vol. (2010).

[Bon10b] M. V. Bondarko. “Weight structures vs. t-structures; weight filtrations, spectral sequences, and complexes (for motives and in general)”. *J. K-Theory* 6 (2010).

[Bon10c] M. V. Bondarko. *Weights and t-structures: in general triangulated categories, for 1-motives, mixed motives, and for mixed Hodge complexes and modules.* Preprint. 2010. arXiv:`1011.3507v4 [math.AG]`.

[Bon19] M. V. Bondarko. *From weight structures to (orthogonal) t-structures and back.* Preprint. 2019. arXiv:`1907.03686v1 [math.KT]`.

[Bon25] L. Bonfert. “Derived projective covers and Koszul duality of simple-minded and silting collections”. To appear in *Appl. Categ. Struct.* (2025). arXiv:`2309.00554v3 [math.RT]`.

[BP14] N. Broomhead and D. Ploog. “Autoequivalences of toric surfaces”. *Proc. Am. Math. Soc.* 142, no. 4 (2014).

[BR87] M. C. R. Butler and C. M. Ringel. “Auslander–Reiten sequences with few middle terms and applications to string algebras”. *Comm. Algebra* 15, no. 1–2 (1987).

[Bri08] T. Bridgeland. “Stability conditions on $K3$ surfaces”. *Duke Math. J.* 141, no. 2 (2008).

[Bri09] T. Bridgeland. “Stability conditions and Kleinian singularities”. *Int. Math. Res. Not. IMRN* 2009, no. 21 (2009).

[BS24] J. Brundan and C. Stroppel. “Semi-infinite highest weight categories”. *Mem. Amer. Math. Soc.* 293, no. 1459 (2024).

[BY14] T. Brüstle and D. Yang. “Ordered exchange graphs”. In: *Advances in representation theory of algebras.* Ed. by D. J. Benson, H. Krause, and A. Skowroński. EMS Series of Congress Reports. European Mathematical Society, 2014.

[BY23] T. Brüstle and D. Yang. *Ordered exchange graphs.* 2023. arXiv:`1302.6045v6 [math.RT]`.

[Cau11] S. Cautis. “Flops and about: a guide”. In: *Derived categories in algebraic geometry.* Ed. by Y. Kawamata. EMS Series of Congress Reports 8. European Mathematical Society, 2011.

[Che21] X.-W. Chen. “Representability and autoequivalence groups”. *Math. Proc. Camb. Phil. Soc.* 171, no. 3 (2021).

[CL23] A. Cipriani and M. Lanini. *Wall and chamber structure for a special biserial algebra coming from perverse sheaves on $\mathbb{P}^n$.* Preprint. 2023. arXiv:`2308.00991 [math.RT]`.

[CPS88] E. Cline, B. Parshall, and L. Scott. “Finite dimensional algebras and highest weight categories”. *J. Reine Angew. Math.* 391 (1988).

[CSPP22] R. Coelho Simões, D. Pauksztello, and D. Ploog. “Functorially finite hearts, simple-minded systems in negative cluster categories, and noncrossing partitions”. *Compositio Math.* 158 (2022).

[CW12] S.-J. Cheng and W. Wang. *Dualities and representations of Lie superalgebras*. Graduate Studies in Mathematics 144. American Mathematical Society, Providence, RI, 2012.

[CW22] A. Cipriani and J. Woolf. "When are there enough projective perverse sheaves?" *Glasg. Math. J.* 64, no. 1 (2022).

[Don98] S. Donkin. *The q-Schur algebra*. London Mathematical Society Lecture Note Series 253. Cambridge University Press, 1998.

[Eno] H. Enomoto. *FD Applet—an applet for finite-dimensional algebras*. URL: `https://haruhisa-enomoto.github.io/fd-applet/`.

[ES22] J. N. Eberhardt and C. Stroppel. "Motivic Springer Theory". *Indag. Math.* 33, no. 1 (2022).

[Fus24] R. Fushimi. *The correspondence between silting objects and t-structures for non-positive dg algebras*. Preprint. 2024. arXiv:`2312.17597v2 [math.RT]`.

[Fus25] R. Fushimi. *Contravariant Koszul duality between non-positive and positive dg algebras*. Preprint. 2025. arXiv:`2409.08842v2 [math.RT]`.

[GHS24] M. Gorelik, V. Hinich, and V. Serganova. "Root groupoid and related Lie superalgebras". *Ann. Represent. Theory* 1, no. 4 (2024).

[GLSVdB24] F. Genovese, W. Lowen, J. Symons, and M. Van den Bergh. *Deformations of triangulated categories with t-structures via derived injectives*. Preprint. 2024. arXiv:`2411.15359v1 [math.CT]`.

[GLVdB21] F. Genovese, W. Lowen, and M. Van den Bergh. "t-structures and twisted complexes on derived injectives". *Adv. Math.* 387 (2021).

[GLVdB22] F. Genovese, W. Lowen, and M. Van den Bergh. *t-structures on dg-categories and derived deformations*. Preprint. 2022. arXiv:`2212.12564v1 [math.CT]`.

[GRG23] F. Genovese and J. Ramos González. "A derived Gabriel-Popescu theorem for t-structures via derived injectives". *Int. Math. Res. Not. IMRN* 2023, no. 6 (2023).

[Hap88] D. Happel. *Triangulated categories in the representation theory of finite dimensional algebras*. London Mathematical Society Lecture Note Series 119. Cambridge University Press, 1988.

[Hec06] I. Heckenberger. "The Weyl groupoid of a Nichols algebra of diagonal type". *Invent. Math.* 164, no. 1 (2006).

[HK19] A. Hochenegger and A. Krug. "Formality of $\mathbb{P}$-objects". *Compos. Math.* 155, no. 5 (2019).

[HK25] A. Hochenegger and A. Krug. "Relations among $\mathbb{P}$-twists". *Int. Math. Res. Not. IMRN* 2025, no. 7 (2025).

[HKM02] M. Hoshino, Y. Kato, and J.-I. Miyachi. "On t-structures and torsion theories induced by compact objects". *J. Pure Appl. Algebra* 167, no. 1 (2002).

[HS07] C. Hoyt and V. Serganova. "Classification of finite-growth general Kac–Moody superalgebras". *Communications in Algebra* 35, no. 3 (2007).

[HS20] I. Heckenberger and H.-J. Schneider. *Hopf Algebras and Root Systems*. Mathematical Surveys and Monographs 247. American Mathematical Society, Providence, RI, 2020.

[HT06] D. Huybrechts and R. Thomas. "$\mathbb{P}$-objects and autoequivalences of derived categories". *Math. Res. Lett.* 13, no. 1 (2006).

[HTT08] R. Hotta, K. Takeuchi, and T. Tanisaki. *D-modules, perverse sheaves, and representation theory*. Progress in Mathematics 236. Birkhäuser, 2008.

[Huy06] D. Huybrechts. *Fourier–Mukai transforms in algebraic geometry*. Oxford Mathematical Monographs. Oxford University Press, 2006.

[HY08] I. Heckenberger and H. Yamane. "A generalization of Coxeter groups, root systems, and Matsumoto's theorem". *Math. Z.* 259, no. 2 (2008).

[IU05] A. Ishii and H. Uehara. "Autoequivalences of derived categories on the minimal resolutions of A_n-singularities on surfaces". *J. Differ. Geom.* 71, no. 3 (2005).

[Kac77] V. G. Kac. "Lie Superalgebras". *Advances in Mathematics* 26 (1977).

[Kel94] B. Keller. "Deriving DG categories". *Ann. Sci. Éc. Norm. Supér. (4)* 27, no. 1 (1994).

[KN11] B. Keller and P. Nicolás. *Notes of the talk "Cluster-hearts and cluster-tilting objects"*. Unpublished note. 2011. URL: `https://pnp.mathematik.uni-stuttgart.de/iaz/iaz1/activities/t-workshop/NicolasNotes.pdf`.

[KN13] B. Keller and P. Nicolás. "Weight structures and simple dg modules for positive dg algebras". *Int. Math. Res. Not. IMRN* 2013, no. 5 (2013).

[Kra07] H. Krause. "Derived categories, resolutions, and Brown representability". In: *Interactions between homotopy theory and algebra*. Ed. by L. L. Avramov, J. D. Christensen, W. G. Dwyer, M. A. Mandell, and B. E. Shipley. Contemporary Mathematics 436. 2007.

[Kra21] H. Krause. *Homological theory of representations*. Cambridge Studies in Advanced Mathematics 195. Cambridge University Press, 2021.

[Kru18] A. Krug. "Varieties with $\mathbb{P}$-units". *Trans. Amer. Math. Soc.* 370, no. 11 (2018).

[KS02] M. Khovanov and P. Seidel. "Quivers, Floer cohomology, and braid group actions". *J. Am. Math. Soc.* 15, no. 1 (2002).

[KV88] B. Keller and D. Vossieck. "Aisles in derived categories". *Bull. Soc. Math. Belg.* 40 (1988).

[KY14] S. Koenig and D. Yang. "Silting objects, simple-minded collections, t-structures and co-t-structures for finite-dimensional algebras". *Documenta Math.* 19 (2014).

[KY18] M. Kalck and D. Yang. *Relative singularity categories II: dg models*. Preprint. 2018. arXiv:`1803.08192v1 [math.AG]`.

[Len21] F. Lenzen. "Shuffling functors and spherical twists on $D^b(\mathcal{O}_0)$". *J. Algebra* 579 (2021).

[LH03] K. Lefèvre-Hasegawa. "Sur les A_∞-catégories". PhD thesis. Université Paris 7—Denis Diderot, 2003. arXiv:`math/0310337v1 [math.CT]`.

[Lur17] J. Lurie. *Higher Algebra*. Unpublished. Sept. 18, 2017. URL: `https://people.math.harvard.edu/~lurie/papers/HA.pdf`.

[Lur18a] J. Lurie. *Kerodon*. `https://kerodon.net`. 2018.

[Lur18b] J. Lurie. *Spectral Algebraic Geometry*. Unpublished. Feb. 3, 2018. URL: `https://www.math.ias.edu/~lurie/papers/SAG-rootfile.pdf`.

[Maz25] V. Mazorchuk. “Some homological properties of category $\mathcal{O}$, VII”. *Adv. Math.* 468 (2025).

[MOS09] V. Mazorchuk, S. Ovsienko, and C. Stroppel. “Quadratic duals, Koszul dual functors, and applications”. *Trans. Amer. Math. Soc.* 361, no. 3 (2009).

[MS08] V. Mazorchuk and C. Stroppel. “Projective-injective modules, Serre functors and symmetric algebras”. *J. reine angew. Math.* 616 (2008).

[Mus12] I. M. Musson. *Lie Superalgebras and Enveloping Algebras.* Graduate Studies in Mathematics 131. American Mathematical Society, Providence, RI, 2012.

[MV87] R. Mirollo and K. Vilonen. “Bernstein–Gelfand–Gelfand reciprocity on perverse sheaves”. *Ann. Sci. Éc. Norm. Supér. (4)* 20, no. 3 (1987).

[MW19] C. Y. Mak and W. Wu. “Dehn twists and Lagrangian spherical manifolds”. *Sel. Math., New Ser.* 25, no. 5 (2019).

[MY25] Y. Mizuno and D. Yang. “Derived preprojective algebras and spherical twist functors”. To appear in *Bull. Lond. Math. Soc.* (2025). arXiv:`2407.02725v3 [math.RT]`.

[Pau08] D. Pauksztello. “Compact corigid objects in triangulated categories and co-t-structures”. *Centr. Eur. J. Math.* 6, no. 1 (2008).

[Pos11] L. Positselski. “Two kinds of derived categories, Koszul duality, and comodule-contramodule correspondence”. *Mem. Am. Math. Soc.* 212, no. 996 (2011).

[Pri70] S. B. Priddy. “Koszul resolutions”. *Trans. Am. Math. Soc.* 152 (1970).

[PS89] I. B. Penkov and V. Serganova. “Cohomology of G/P for classical complex Lie supergroups G and characters of some atypical G-modules”. *Ann. Inst. Fourier* 39, no. 4 (1989).

[Pup62] D. Puppe. “On the formal structure of stable homotopy theory”. In: *Colloquium on Algebraic Topology* (Matematisk Institut, Aarhus Universitet). 1962.

[PV18] C. Psaroudakis and J. Vitória. “Realisation functors in tilting theory”. *Math. Z.* 288 (2018).

[PW20] E. Persson Westin. “Tilting modules and exceptional sequences for leaf quotients of type A zig-zag algebras”. *Beitr. Algebra Geom.* 61, no. 2 (2020).

[RH04] S. Ryom-Hansen. “Koszul duality of translation—and Zuckerman functors”. *J. Lie Theory* 14, no. 1 (2004).

[Ric02] J. Rickard. “Equivalences of derived categories for symmetric algebras”. *J. Algebra* 257, no. 2 (2002).

[Ric16] J. Rickard. *Answer to: “When is a functor a right derived functor?”* MathOverflow. July 19, 2016. URL: `https://mathoverflow.net/a/244590`.

[Sch11] O. M. Schnürer. “Perfect derived categories of positively graded DG algebras”. *Appl. Categ. Struct.* 19, no. 5 (2011).

[Sch20] O. M. Schnürer. *Simple-minded subcategories and t-structures.* Notes. 2020. URL: `https://math.uni-paderborn.de/fileadmin-eim/mathematik/AG-Algebra/Notes/simple-minded-subcategories-and-t-structures.pdf`.

[Seg18] E. Segal. “All autoequivalences are spherical twists”. *Int. Math. Res. Not. IMRN* 2018, no. 10 (2018).

[Ser11] V. Serganova. “Kac–Moody superalgebras and integrability”. In: *Developments and Trends in Infinite-Dimensional Lie Theory*. Ed. by K.-H. Neeb and A. Pianzola. Progress in Mathematics 288. Birkhäuser, 2011, pp. 169–218.

[ST01] P. Seidel and R. Thomas. “Braid group actions on derived categories of coherent sheaves”. *Duke Math. J.* 108, no. 1 (2001).

[Stacks] The Stacks Project Authors. *Stacks Project*. `https://stacks.math.columbia.edu`. 2018.

[Str03] C. Stroppel. “Category $\mathcal{O}$: quivers and endomorphism rings of projectives”. *Represent. Theory* 7 (2003).

[Str06] C. Stroppel. *TQFT with corners and tilting functors in the Kac-Moody case*. Preprint. 2006. arXiv:`math/0605103v1`.

[SV11] A. N. Sergeev and P. Veselov. “Grothendieck rings of basic classical Lie superalgebras”. *Ann. Math.* 173, no. 2 (2011).

[SY19] H. Su and D. Yang. “From simple-minded collections to silting collections via Koszul duality”. *Algebr. Represent. Theor.* 22 (2019).

[Tho18] H. Thomas. “Stability, shards, and preprojective algebras”. In: *Representations of Algebras. 17th Workshop and International Conference on Representation of Algebras* (ICRA 2016). Contemporary Mathematics 705. 2018.

[Ver96] J.-L. Verdier. “Des catégories dérivées des catégories abéliennes”. *Astérisque* 239 (1996).

[Woo10] J. Woolf. “Stability conditions, torsion theories and tilting”. *J. Lond. Math. Soc., II. Ser.* 82, no. 3 (2010).

[WW85] B. Wald and J. Waschbüsch. “Tame biserial algebras”. *J. Algebra* 95 (1985).

[Zha23] H. Zhang. “The ST correspondence for proper non-positive dg algebras”. *Comm. Algebra* 51, no. 11 (2023).