A Twisted Bass-Heller-Swan Decomposition for Localising Invariants and Equivariant Poincaré Duality

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Abstract

This thesis consists of two essentially independent parts.

The first part contains a generalisation of the so called Bass-Heller-Swan decomposition in algebraic K-theory, which relates the algebraic K-theory of a Laurent polynomial ring to the algebraic K-theory of its coefficients. We extend it to a splitting for localising invariants of certain categorical mapping tori. This contains most known generalisations of the Bass-Heller-Swan decomposition as a special case. As an application, we obtain splitting results for Waldhausen's A-theory of mapping tori as well as the K-theory of certain localised tensor algebras.

The second part is concerned with the question of what equivariant Poincaré duality for a compact Lie group G is supposed to be, which is the basic homological property of smooth closed G-manifolds. We first introduce the notion of parametrised Poincaré duality in the setting of parametrised category. Specialising this to the equivariant world, we obtain a robust theory of equivariant Poincaré duality for compact Lie groups and we show that this is compatible with various equivariant constructions and operations. Finally, we apply this theory to give, among other things, a new proof of the rigidity theorem of Atiyah-Bott and Conner-Floyd on group actions on manifolds with a single fixed point.

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And peruse manifold objects, no two alike and every one good, The earth good and the stars good, and their adjuncts all good.

Walt Whitman, "Song of Myself", Verse 7

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Contents

Introduction

The two parts of this thesis, though seemingly independent at first, can both be located in the field of homotopical algebra – the algebra of structures up to coherent homotopy – where the notion of equality is replaced by homotopy. Instead of classical algebraic objects like monoids or rings, one considers spaces with operations which satisfy algebraic relations such as associativity only up to homotopy. Importantly, these homotopies, together with higher coherences witnessing e.g. that multiple ways of rebracketing a fourfold product $x_1 \cdot x_2 \cdot x_3 \cdot x_4$ are equivalent, are part of the structure of such a homotopy coherent algebraic object. On the geometric side, structures up to coherent homotopy naturally arise in various situations. As one of the main examples, consider a fibration $p\colon E \to B$. Classically, the fibre transport describes the action of the fundamental group of B on the fibre F, but is only defined up to homotopy. If one, instead, remembers not only the action of the fundamental group but of the whole homotopy type of B on the space of self homotopy equivalences of F, one recovers p up to homotopy. Thus, by taking a classical question from geometric topology, but considering its fibreed version, one is naturally led to work with such homotopy coherent structures.

The two parts of this thesis both fit into this framework, though in seemingly different instances.

1. Part I is concerned with a generalisation of the fundamental theorem in algebraic K-theory. It is a majorly revised and extended version of the authors work [KK24], joint with Christian Kremer.

2. Part II introduces the notion of equivariant Poincaré duality for compact Lie groups, which captures the homological property of smooth closed G-manifolds. It essentially is a reproduction of the authors work [HKK24b], joint with Kaif Hilman and Christian Kremer, with only minor corrections.

We begin by giving a nontechnical overview of each of those parts and explain how they are connected to aforementioned theme, after which we will give a more technical summary of the results. The individual introductions for each part can be found in Chapters 1 and 5.

Part I: A Twisted Bass-Heller-Swan Decomposition for Localising Invariants

Contextual overview

In algebra one is often faced with classification problems of modules over a ring R. One fundamental invariant in that direction, originally introduced by Grothendieck, is the 0th K-group $K_0(R)$, assembling certain isomorphism classes of R-modules. It should not be surprising that for a space X the group $K_0(\mathbb{Z}[\pi_1(X)])$ is important in algebraic topology, where some sort of universal Euler characteristic of X lives. This universal Euler characteristic for example features in Wall's finiteness obstruction, measuring whether X is homotopy equivalent to a finite CW-complex.

Mathematicians realised that K_0 should just be a shadow of a general "homology theory" K_* of rings. One can similarly define $K_1(R)$ by assembling isomorphism classes of automorphisms of R-modules. And again, the group $K_1(\mathbb{Z}[\pi_1(X)])$ is the home for important variants in geometric topology, let us just mention the Whitehead torsion appearing in the s-cobordism theorem as an example. For a while, it was a big question how one can extend K_0 and K_1 to all positive and negative degrees, as an explicit description through generators and relations was no longer known. In positive degrees, Quillen and Segal realised that if one takes the classical definition of K_0 as group completion of the monoid of isomorphism classes of finitely generated projective R-modules, but now interprets this in a homotopy coherent setting, one obtains a K-theory space whose homotopy groups recover K_0 and K_1. For K-theory to behave as a homology theory, there should be a splitting resembling the formula $H_n(X \times S^1) \simeq H_n(X) \oplus H_{n-1}(X)$. The algebraic analogue of the product $X \times S^1$ is the Laurent polynomial ring $R[t^{\pm 1}]$ over a ring R. The fundamental theorem of algebraic K-theory, sometimes also called the Bass-Heller-Swan decomposition, states that

$$K_n(R[t^{\pm 1}]) \simeq K_n(R) \oplus K_{n-1}(R) \oplus NK_n(R) \oplus NK_n(R). \tag{0.1}$$

This was first proven for regular rings and $n = 1$ by Bass-Heller-Swan [BHS64]. Quillen took this as a reality check that his definition of higher K-theory is the correct one and proved the decomposition for the connective K-theory of regular rings [Gra]. The appearance of the Nil-terms $NK_n(R)$ in (0.1) might be surprising at first, but is rooted in a phenomenon in algebraic geometry called non-$\mathbb{A}^1$-invariance. From the point of view of many invariants of schemes, the affine line $\mathbb{A}^1$ is not contractible. This is measured by the splitting $K_n(R[t]) \simeq K_n(R) \oplus NK_n(R)$ for K-theory. The splitting (0.1) is classically obtained by studying the K-theory of the projective line $\mathbb{P}^1_R$, glued from two copies of $R[t]$ along $R[t^{\pm 1}]$. Bass used (0.1) for the first definitions of negative K-groups [Bas68], even before Quillen's construction of positive K-groups. These two constructions were later combined to a K-theory spectrum, recovering both in positive and negative degrees.

These constructions, however, do not endow $K(R)$ with a similar universal property such as K_0. Quillen and Waldhausen realised that the essential property of K-theory is the additivity theorem, saying that K maps certain cofibre sequences of categories to cofibre sequences of spectra. It was not until much later that this could be turned into a universal property by Blumberg-Gepner-Tabuada [BGT13], who crucially work in the setting of ∞-categories where homotopical constructions on the level of categories become possible.

The main result of Part I is a generalisation of (0.1) to the setting of localising invariants of stable ∞-categories. For the sake of this overview, a localising invariant is a functor $E\colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Sp}$ assigning to each stable ∞-category[1] a spectrum which sends so called bifibre sequences of stable categories to cofibre sequences of spectra. Working in this generality has two advantages: On the one hand, it reduces the proof to the essential property of algebraic K-theory, the localisation property. Thus, it immediately applies to other localising invariants such as topological Hochschild homology and its cousins topological cyclic homology and topological restriction theory. On the other hand, this approach allows more homotopical inputs than just rings, e.g. ring spectra. This generalises and unifies most variants of (0.1) which have appeared in the literature so far.

Let us state the main two special cases of our main result here, while we present the more general statements below.

Theorem A (Corollary 4.1.4)**.** *Consider a ring spectrum $R \in \mathrm{Alg}(\mathrm{Sp})$ together with a (R,R)-bimodule M which admits a left dual $M^\vee$. Then there are natural splittings*

$$E(T_R(M)[M^{-1}]) \simeq E(R)_{h\mathbb{N}} \oplus NE_M(R) \oplus \overline{NE}_M(R)$$

and

$$E(T_R(M)) \simeq E(R) \oplus NE_M(R) \quad \textit{and} \quad E(T_R(M^\vee)) \simeq E(R) \oplus \overline{NE}_M(R).$$

Here, the tensor algebra $T_R(M)$ is the ring spectrum

$$T_R(M) = \bigoplus_{n\geq 0} M^{\otimes_R^n} M$$

and the localised tensor algebra $T_R(M)[M^{-1}]$ is the colimit

$$T_R(M)[M^{-1}] \simeq \operatorname{colim}\left(\bigoplus_{k\geq 0} M^{\otimes_R^k} \xrightarrow{\text{coev}} \bigoplus_{k\geq 0} M^\vee \otimes_R M^{\otimes_R^k} \xrightarrow{\text{coev}} \dots\right),$$

[1] E.g. the perfect derived category $\mathrm{D}^{\mathrm{perf}}(R)$ of a ring.

where the maps are induced by the coevaluation $\mathrm{coev}\colon R \to M^\vee \otimes_R M$. The spectrum $E(R)_{h\mathbb{N}}$ is the mapping torus of the map $E(R) \to E(R)$ induced by M. It fits into a fibre sequence $E(R) \xrightarrow{1-M} E(R) \to E(R)_{h\mathbb{N}}$, from which one obtains a long exact sequence of homotopy groups. Consider the special case where $M = R$ with trivial left R-module structure and right R-module structure given by an automorphism α of R. Then $T_R(M)[M^{-1}] = R_\alpha[t^{\pm 1}]$ is the twisted Laurent polynomial ring, with multiplication defined by $rt^m \cdot st^n = r\alpha^m(s)t^{m+n}$. In this situation, Theorem A reduces to a splitting

$$E(R_\alpha[t^{\pm 1}]) \simeq E(R)_{h\mathbb{N}} \oplus NE_\alpha(R) \oplus \overline{NE}_\alpha(R).$$

Technical summary

Let us now give a detailed summary of the contents of Part I. The reader might also want to consult the introduction Chapter 1 of Part I first.

Chapter 2 sets the foundations for the rest of Part I. Our proof of Theorem A relies on a categorification of the rings $R[t]$ and $R[t^{\pm 1}]$, which we study in §2.1. The first appears in two variants, $\mathrm{End}_\alpha(\mathcal{C})$ and $\overline{\mathrm{End}}_\alpha(\mathcal{C})$, consisting (x, f) of an object $x \in \mathcal{C}$ together with a morphism $f\colon x \to \alpha(x)$ in the first case and a morphism $f\colon \alpha(x) \to x$ in the second case. In the case where α is an equivalence, one has $\overline{\mathrm{End}}_\alpha(\mathcal{C}) \simeq \mathrm{End}_{\alpha^{-1}}(\mathcal{C})$, while these two variants have quite different categorical properties in general. They are special instances of (op)lax colimits, but we will not need this perspective. There is also the category $\mathrm{Aut}_\alpha(\mathcal{C}) \subseteq \mathrm{End}_\alpha(\mathcal{C})$ consisting of pairs (x, f) for which f is an equivalence. We show that, under certain completeness assumptions, the forgetful functor $\mathrm{fgt}_\alpha\colon \overline{\mathrm{End}}_\alpha(\mathcal{C}) \to \mathcal{C}, (x, f) \mapsto x$ admits a left adjoint $\overline{\mathrm{free}}_\alpha$ and the inclusion $\mathrm{Aut}_\alpha(\mathcal{C}) \subseteq \mathrm{End}_\alpha(\mathcal{C})$ admits a left adjoint loc_α, which categorifies induction along the ring homomorphisms $R \to R[t]$ and $R[t] \to R[t^{\pm 1}]$.

§2.2 introduces the category $\mathrm{Nil}_\alpha(\mathcal{C})$ of nilpotent endomorphisms, which will be relevant for the Nil-terms in the splitting. It is a categorification of the t-power torsion $R[t]$-modules. The main result of this section proves two equivalent characterisations of twisted nilpotent endomorphisms:

Theorem B (Theorem 2.2.2)**.** *Let $\alpha\colon \mathcal{C} \to \mathcal{C}$ be an exact endofunctor of a perfect category. Then the following two categories agree:*

1. *The full perfect subcategory of $\overline{\mathrm{End}}_\alpha(\mathcal{C})$ generated by the trivial endomorphisms $(x, 0\colon \alpha(x) \to x)$ for $x \in \mathcal{C}$.*

2. *The full subcategory of $\overline{\mathrm{End}}_\alpha(\mathcal{C})$ consisting of homotopy nilpotent endomorphisms (x, f), for which there is some $n \geq 0$ such that $f^n \simeq 0\colon \alpha^n(x) \to x$.*

In Chapter 3 we prove the two main splitting results of Part I. We begin with a recollection of Land-Tammes work [LT23] on K-theory of pushouts in §3.1. Using that, we can give an explicit model for the assembly map for the K-theory of pushouts. This will be an important ingredient for the proofs of the splitting results. §3.2 contains the proof of the main result of Part I.

Theorem C (Theorem 3.2.5)**.** *Let $\alpha\colon \mathcal{C} \to \mathcal{C}$ be an exact endofunctor of an idempotent complete stable ∞-category and $E\colon \mathrm{Cat}^{\mathrm{perf}} \to \mathcal{E}$ a localising invariant. The assembly map $E(\mathcal{C})_{h\mathbb{N}} \to E(\mathcal{C}_{h\mathbb{N}})$ admits a splitting, natural in $\alpha \in \mathrm{Fun}(B\mathbb{N}, \mathrm{Cat}^{\mathrm{perf}})$, inducing an equivalence*

$$E(\mathcal{C}_{h\mathbb{N}}) \simeq E(\mathcal{C})_{h\mathbb{N}} \oplus NE_\alpha(\mathcal{C}) \oplus \overline{NE}_\alpha(\mathcal{C}).$$

The strategy is to realise $\mathcal{C}_{h\mathbb{N}}$ as the pushout of the span $\mathcal{C} \xleftarrow{\mathrm{id}\oplus\alpha} \mathcal{C}\oplus\mathcal{C} \xrightarrow{\mathrm{id}\oplus\mathrm{id}} \mathcal{C}$ to which one can apply Land-Tamme's machinery from the previous section. From this, one obtains a square

$$\begin{tikzcd}
\mathrm{Im}(\mathcal{C}\oplus\mathcal{C}) \arrow[r, "\mathrm{id}\oplus\mathrm{id}"] \arrow[d, "\mathrm{id}\oplus\alpha"'] & \mathcal{C} \arrow[d] \\
\mathcal{C} \arrow[r] & \mathcal{C}_{h\mathbb{N}}
\end{tikzcd}$$

which becomes a pushout after applying any localising invariant. We show that the Nil-categories $\mathrm{Nil}_\alpha(\mathcal{C})$ and $\overline{\mathrm{Nil}}_\alpha(\mathcal{C})$ form a semiorthogonal decomposition of $\mathrm{Im}(\mathcal{C}\oplus\mathcal{C})$. These contain $\mathcal{C}$ as a retract from which one obtain splittings $E(\mathrm{Nil}_\alpha(\mathcal{C})) \simeq E(\mathcal{C}) \oplus NE_\alpha(\mathcal{C})$ and $E(\overline{\mathrm{Nil}}_\alpha(\mathcal{C})) \simeq E(\mathcal{C}) \oplus \overline{NE}_\alpha(\mathcal{C})$. This explains the appearance of the Nil-terms in the splitting.

§3.3 establishes the second main result of Part I, providing a different splitting featuring the same Nil-terms.

Theorem D (Theorem 3.3.2)**.** *The functor $\overline{\mathrm{free}}_\alpha$ has a splitting*

$$E(\overline{\mathrm{End}}_\alpha(\mathrm{Ind}(\mathcal{C}))^\omega) \simeq E(\mathcal{C}) \oplus NE_\alpha(\mathcal{C}).$$

The strategy again is to realise $\overline{\mathrm{End}}_\alpha(\mathrm{Ind}(\mathcal{C}))^\omega$ as a pushout, this time of the span $\mathcal{C} \leftarrow \mathcal{C}\oplus\mathcal{C} \to \mathcal{C}\otimes\Delta^1$. Here, the replacement $\mathrm{Im}(\mathcal{C}\oplus\mathcal{C})$ admits a semiorthogonal decomposition by the categories $\mathcal{C}$ and $\mathrm{Nil}_\alpha(\mathcal{C})$, which is the reason for the $NE_\alpha(\mathcal{C})$-term in the splitting.

Classically the Nil-terms $NK(R)$ vanish for a regular ring. In §3.4, we first explain how Burklund-Levy's abstract dévissage result [BL23] directly implies vanishing of $NK_\alpha(\mathcal{C})$ if $\mathcal{C}$ admits a bounded t-structure and α is left t-exact. In the rest of this section, we construct under the similar hypotheses the pointwise t-structure on $\overline{\mathrm{End}}_\alpha(\mathcal{C})$ and $\overline{\mathrm{Nil}}_\alpha(\mathcal{C})$. The latter crucially uses the description of $\overline{\mathrm{Nil}}_\alpha(\mathcal{C})$ as homotopy nilpotent endomorphisms.

The final Chapter 4 contains multiple applications and special cases of the main results. We begin by showing that for a ring spectrum R and a (R,R)-bimodule M the category $\mathrm{End}_M(\mathrm{Mod}_R)$ is equivalent to the categories of modules over the tensor algebra $T_R(M)$ in §4.1. This is easy to see if we apply the Schwede-Shipley theorem, though this does not identify the multiplicative structure on $T_R(M)$ with the free (R,R)-algebra on M. Proving this takes a considerable amount of work. Specialising Theorems 3.2.5 and 3.3.2 to that case proves the splitting Theorem A for the K-theory of tensor algebras. In §4.2 we consider a space X with an selfmap α and apply the main result to the colimit $(\operatorname{colim}_X \mathrm{Sp}^{\omega})_{h\mathbb{N}}$ to obtain splitting for Waldhausen's A-theory of the mapping torus $A(X_{h\mathbb{N}})$. This generalises the splitting for $A(X \times S^1)$ from [HKV+01]. As a further special case, we obtain a splitting for the K-theory of group ring spectra for certain generalised HNN-extensions in Corollary 4.2.4. In the final section §4.3 of Part I we describe A-theoretic Nil-terms through free loop spaces. The main ingredients for this result are the Dundas-Goodwillie-McCarthy theorem [DGM13], relating K-theory and TC, together with Bökstedt-Hsian-Madsen's [BHM93] computation of TC of suspension spectra of loops spaces.

Theorem E. *For a connected based space X together with a selfmap $\alpha\colon X \to X$, there is a natural equivalence*

$$N\,\mathrm{TC}^{\mathrm{tot}}_{\alpha}(X) \simeq \Sigma(\Sigma^{\infty}_{+} L(\neq 0)(X_{h\mathbb{N}}))_{hS^1}$$

after p-completion at an arbitrary prime p.

The map $X_{h\mathbb{N}} \to *_{h\mathbb{N}} \simeq S^1$ induces a map $L(X_{h\mathbb{N}}) \to L(S^1) \simeq \mathbb{Z} \times S^1$. We denote by $L(\neq 0)(X_{h\mathbb{N}}) \subseteq L(X_{h\mathbb{N}})$ the components mapping to $\mathbb{Z} \setminus 0$. $N\,\mathrm{TC}^{\mathrm{tot}}_{\alpha}(X)$ is the cofibre of the assembly map $N\,\mathrm{TC}(X)_{n\mathbb{N}} \to N\,\mathrm{TC}(X_{n\mathbb{N}})$. Using the Dundas-Goodwillie-McCarthy theorem together with the A-theory splitting Corollary 4.2.2, we get that, up to π_0, there is an equivalence $N\,\mathrm{TC}^{\mathrm{tot}}_{\alpha}(X) \simeq NA_{\alpha}(X) \oplus \overline{NA}_{\alpha}(X)$.

Let us end this summary of Part I by clarifying the contribution of the author to [KK24]. Originally, the author and the coauthor could prove a version of the splitting Theorem C in the case where α is an automorphism by generalising the methods used in [Sau23]. The author realised that, using Land-Tamme' s machinery on K-theory of pushouts instead, one obtains the more general result where α is only an endomorphism of $\mathcal{C}$. The work was fully collaborative and each coauthor contributed to each section. In the current form, however, most of the results in [KK24] except for §§3.4and 4.2 can be attributed to the author. The material presented in Part I is an extended and revised version of [KK24]. In many places, it is more natural to work with $\overline{\mathrm{End}}_{\alpha}(\mathcal{C})$ instead of $\mathrm{End}_{\alpha}(\mathcal{C})$, which is the perspective we take in this dissertation. Their relation is clarified in Lemma 2.1.5, where we show that $\overline{\mathrm{End}}_{\alpha^L}(\mathcal{C}) \simeq \mathrm{End}_{\alpha}(\mathcal{C})$ if α has a left adjoint α^L. This allows us to remove some unnecessary hypothesis on α in Theorem 2.2.2 and Corollary 3.4.4. Furthermore, the identification of $\overline{\mathrm{End}}_M(\mathrm{Mod}_R)$ with the tensor algebra in Construction 4.1.2 was not part of [KK24].

Part II: Equivariant Poincaré Duality

Contextual overview

Poincaré duality is a concept that appears in many fields of mathematics as a symmetry between homology and cohomology. In geometric topology, it is the fundamental homological property of manifolds. Classically, it says that for a closed connected orientable manifold M, the homology and cohomology of M agree up to a shift by its dimension d. One can get rid of the orientability assumption by working with coefficients in the orientation local system $\mathcal{O}_M$, which remembers the orientation behaviour of loops in M. There is a canonical fundamental class $[M] \in H_d(M; \mathcal{O}_M)$ such that the capping map

$$[M] \cap - \colon H^*(M) \to H_{d-*}(M; \mathcal{O}_M) \tag{0.2}$$

is an equivalence. One fundamental problem in geometric topology is the question whether a given space is homotopy equivalent to a closed manifold. From the previous discussion it is clear that Poincaré duality is the first main obstruction to this problem. This led Wall to introduce Poincaré duality spaces, which satisfy (0.2) for all local coefficient systems. Surgery theory can be used to further decide whether a given Poincaré space is homotopy equivalent to a manifold, and for the classification of such.

Apart from their classification, group actions on manifolds have drawn the attention of topologists for many years. One of the main question is the existence of group actions on manifolds with prescribed fixed points, as well as their classification. The most prominent example is probably the spherical space form problem, asking which finite groups can act freely on a sphere. To attack such problems, one might again hope to have some abstract homological constraints, obstructing the existence of certain group actions. Taking the approach from above, this leads to the following question, which Part II of this dissertation is concerned with:

Question F. What is a good notion of equivariant Poincaré duality, that is also satisfied by smooth closed G-manifolds?

The main difficulty is to decide what homology theories to use in (0.2) when M is a G-space. A naive suggestion would be to require (0.2) for all fixed points M^H for all subgroups $H \leq G$. This, however, does not remember any normal data of the embeddings $M^H \hookrightarrow M^K$ for subgroups $K \leq H$, which certainly is information relevant to a surgery theoretical approach to the classification problem. Costenoble–Waner [CW92; CW16] gave a definition of equivariant Poincaré duality for finite groups and compact Lie groups using complicated equivariant homology theories, graded by representations with an action of an equivariant fundamental groupoid, keeping track of the different dimensions and orientation behaviours of various fixed points.

We take a slightly different approach, using more homotopy theoretic methods. For this, the following alternative perspective to (0.2) is useful, which replaces homology with $\mathbb{Z}$-coefficients by homology with $\mathbb{S}$-coefficients. Similar to $\mathbb{Z}$ being the initial ring, the sphere spectrum $\mathbb{S}$ is the initial "homotopy coherent ring". The orientation system $\mathcal{O}_M$ can be replaced by the local system $\mathbb{S}^{-TM}$ of spectra over M, assigning to each point of M the Thom spectrum of its stable normal bundle, or equivalently the inverse of the Thom spectrum of its Tangent space. Then (0.2) can be reformulated by saying that there is a class $[M] \in H_0(M, \mathbb{S}^{-TM})$ such that for any local system $E \in \mathrm{Sp}^M$ of spectra over M the capping map

$$[M] \cap - \colon H^*(M; E) \to H_{-*}(M; E \otimes \mathbb{S}^{-TM}) \tag{0.3}$$

is an equivalence.[2] The degree shift by the dimension here is conveniently packaged into the orientation system $\mathbb{S}^{-TM}$. A finite space (i.e. finite CW-complex) X now always has a dualising local system $D_X \in \mathrm{Sp}^X$ together with a fundamental class $[X] \in H_0(X, D_X)$ for which capping induces an equivalence between $\mathbb{S}$-cohomology and homology for all local coefficient systems. It is a Poincaré duality space if and only if the local system D_X pointwise is just a shift of the sphere spectrum. This essentially is Klein's [Kle01] charactersiation of Poincaré spaces, which is a reformulation of the characterisation of Poincaré spaces through their Spivak normal fibration. As every cohomology theory can be represented by a spectrum, this says that (0.2) even holds for all homology theories.

If interpreted correctly, all terms in this formulation also make sense equivariantly over a compact Lie group G. This is a bit technical and requires the use of parametrised homotopy category, which we will ignore for the sake of this summary. Let us just explain in the case of smooth G-manifolds which additional structure our definition remembers. For each point $x \in M^H$, the tangent space T_xM carries a H-action, making $\mathbb{S}^{-T_xM}$ into a genuine H-spectrum[3]. These are compatible in the sense that for a subgroup $K \leq H$ we obtain the commutative diagram

$$\begin{tikzcd} M^H \arrow[r, "\mathbb{S}^{-TM}"] \arrow[d] & \mathrm{Sp}_H \arrow[d, "\mathrm{Res}^H_K"] \\ M^K \arrow[r, "\mathbb{S}^{-TM}"] & \mathrm{Sp}_K. \end{tikzcd} \tag{0.4}$$

Our answer to Question F roughly means that there is an invertible local system of G-spectra together with fundamental class in twisted equivariant homology in-

[2] One should really view this as an equivalence of spectra of the limit $\lim_X E$ and the colimit $\mathrm{colim}_X E \otimes \mathbb{S}^{-TM}$, which recovers (0.3) after taking homotopy groups.

[3] A genuine G-spectrum E can be thought of the $\mathbb{S}$-variant of the category $\mathrm{Mack}(G, \mathrm{Ab})$ of Mackey functors. In particular, for each subgroup $H \leq G$ it consists of a spectrum E^H together with restriction maps $\mathrm{Res}^H_K \colon E^K \to E^H$ and transfer maps $\mathrm{tr}^H_K \colon E^H \to E^K$ for subgroups $K \leq H$ satisfying the double coset formula.

ducing an isomorphism between cohomology and twisted homology for all equivariant cohomology theories. An alternative characterisation, analogous to Spivak's characterisation, also works equivariantly. Any finite G-CW-complex X admits an equivariant embedding $X \hookrightarrow V$ into a finite dimensional G-representation with an equivariant regular neighbourhood $X \hookrightarrow N \hookrightarrow V$, that admits an equivariant deformation retraction. X is now a G-Poincaré space if and only if the homotopy fibres of the projection $N \to X$ are generalised homotopy representations[4].

After spending a considerable amount of work on setting up the theory, we show that our notion of equivariant Poincaré duality is compatible with all kinds of equivariant operations. More precisely, if X is a G-Poincaré space and $H \leq G \leq L$ are closed subgroups, then the induction $\mathrm{Ind}_G^L X$ is a L-Poincaré space, the restriction $\mathrm{Res}_H^G X$ is a H-Poincaré space and the fixed points X^H are a Poincaré space (even equivariantly for the residual action of the Weyl group $W_G H$). Conversely, using a construction of Jones [Jon71], we can show that there are examples of G-spaces for which all fixed points are nonequivariant Poincaré spaces, but they are not G-Poincaré spaces themselves. As main examples, we establish that closed smooth G-manifolds as well as tom Dieck's generalised homotopy represenations are examples of G-Poincaré spaces. This already allows for some counterintuitive phenomena. Bredon [Bre72] constructs a generalised C_p-homotopy represention X such that $X^e \simeq X^{C_p} \simeq S^2$ but the inclusion $X^{C_p} \to X^e$ is not a homotopy equivalence.

In the final part, we apply this theory to study rigidity phenomena of group actions. As the main result, we can generalise a result due to Atiyah-Bott and Conner-Floyd on actions of cyclic groups of prime power order with a single fixed point.

Theorem G (Theorem 9.2.2)**.** *Let p be an odd prime and X a compact C_{p^k}-Poincaré space. If X^e is connected, $\mathbb{Z}$-orientable, and of nonzero dimension, then $X^{C_{p^k}} \not\simeq *$.*

Technical summary

Let us now give a detailed summary of the contents of Part II. The reader might also want to consult the introduction Chapter 5 of Part II first.

In §6.1, we recall the necessary background from parametrised category theory over an ∞-topos $\mathcal{B}$, which we will use throughout this part. Parametrised category theory is motivated by Elmendorf's theorem saying that, up to G-homotopy, a G-space X can be recovered from the functor $\mathcal{O}(G)^{\mathrm{op}} \to \mathcal{S}, G/H \to X^H$, where $\mathcal{O}(G)$ is the orbit category of X. A convenient way to encode structures like the equivariant stable normal bundle (0.4) is through G-categories $\mathrm{Cat}_G = \mathrm{Fun}(\mathcal{O}(G)^{\mathrm{op}}, \mathrm{Cat})$.

[4] A generalised homotopy representation is a compact G-space Y such that for all closed subgroups $H \leq G$, the fixed points Y^H are homotopy equivalent to a sphere $S^{n(h)}$ of some dimension $n(H)$. They are the equivariant homotopical analogue of nonequivariant spheres.

One can now play the game to find appropriate analogues of notions from category theory and homotopy theory (e.g. limits, spectra, symmetric monoidal categories, etc.) in the world of G-categories.

We start in §6.1 with a short summary of parametrised category theory as developed by Martini-Wolf [Mar22b; Mar22a; MW22; MW24]. The central notion for the rest of this work is that of presentably symmetric monoidal $\mathcal{B}$-categories, Definition 6.1.29, which will later replace the equivariant (co)homology theories mentioned before. As a main tool, enabling us to later pass between various groups, we study how base change functors interact with presentably symmetric monoidal $\mathcal{B}$-categories in Lemmas 6.1.31 to 6.1.33.

In §6.2 we specialise this to the setting of G-categories, where we develop isotropy separation methods. Classically, given a family $\mathcal{F}$ of subgroups of G and a G-spectrum E, there is a fibre sequence $E \otimes E\mathcal{F}_+ \to E \to E \otimes \widetilde{EF}$ where the fixed points of $E \otimes E\mathcal{F}_+$ are concentrated in $\mathcal{F}$ and the fixed points of $E \otimes \widetilde{EF}$ are concentrated away from $\mathcal{F}$. In the first main result of Part II, Theorem 6.2.26, we extend this to a isotropy separation result for G-categories.

Chapter 7 is a technical section introducing the main concept of this article, parametrised Poincaré duality, in a very general context. In §7.1 we define notion of a Spivak datum on a $\mathcal{B}$-space $\underline{X}$ with coefficients in a symmetric monoidal $\mathcal{B}$-category $\underline{\mathcal{C}}$. It is a pair of a local system $\xi \in \underline{\mathrm{Fun}}(\underline{X}, \underline{\mathcal{C}})$ together with a fundamental class $c\colon \mathbb{1} \to X_!\xi$. We associate to each Spivak datum a transformation $X_* \to X_!(- \otimes \xi)$ of functors $\underline{\mathrm{Fun}}(\underline{X}, \underline{\mathcal{C}}) \to \underline{\mathcal{C}}$, generalising the capping map in ordinary homology and cohomology. In §7.2 we study two special instances of those: Twisted ambidextrous Spivak data, for which the capping map is an equivalence, and Poincaré Spivak data, for which the local system is additionally invertible. The first is a genrealisation of Cnossen's theory of twisted ambidexterity [Cno23] to the case of nonpresentable coefficients. In the case where $\underline{\mathcal{C}}$ is presentably symmetric monoidal, Spivak data are unique if they exist, and it becomes a property of $\underline{X}$ to be Poincaré. In §7.3 we explain how this notion behaves under various constructions, e.g. changing the coefficient category $\underline{\mathcal{C}}$, changing the base topos $\mathcal{B}$ and descent within the topos $\mathcal{B}$. Later, when specialised to the equivariant case, these will be the fundamental tools enabling us to perform isotropy separation manoeuvres. The final §7.4 introduces the notion of the *degree* of a map $f\colon \underline{X} \to \underline{Y}$ with respect to two Spivak data on $\underline{X}$ and $\underline{Y}$. We explain how this relates to the classical degree of a map of closed oriented manifolds. In Lemma 7.4.6 we also establish a version of the Umkehr square, built out of the homological pushforward and the cohomological pullback along f, related through the capping transformations.

After setting up the very abstract notion of $\mathcal{B}$-Poincaré spaces in the previous chapter, we specialise this to the equivariant situation in Chapter 8. The first main result concerning equivariant Poincaré spaces appears in §8.2, where we employ the categorical isotropy separation techniques developed in §6.2 in a crucial way.

Theorem H (Theorem 8.2.9)**.** *Suppose that $X \in \mathcal{S}_G$ is G–Poincaré. Then for any closed subgroup $H \leq G$, the fixed points $X^H \in \mathcal{S}_{W_G H}$ with the residual Weyl group action are a $W_G H$–Poincaré space. In particular, $X^H \in \mathcal{S}$ is a nonequivariant Poincaré space.*

In §8.3 we study the stability of equivariant Poincaré duality under the equivariant constructions induction, restriction, inflation and indexed products. As one of the main results, we also prove the following equivariant generalisation of a result of Klein [Kle01, Cor. F], which he attributes to Quinn.

Theorem I (Theorem 8.3.12)**.** *Let $f\colon X \to Y$ be a map of G–spaces. If Y is G–Poincaré and for every closed subgroup $H \leq G$, the fibres of f over every H–point of Y is H–Poincaré, then X is G-Poincaré.*

In §8.4 we come to the main two examples of equivariant Poincaré spaces: Smooth closed G-manifolds and generalised homotopy representation. In both cases, the strategy is to construct an equivariant orientation system together with an equivariant fundamental class. We then apply the recognition result Theorem 8.2.10 to reduce it to showing that this fundamental class exhibits all fixed points as a nonequivariant Poincaré space. In the manifold case, this equivariant fundamental class can be constructed through an equivariant Pontryagin-Thom collapse map associated to an equivariant embedding into a finite dimensional G-representation. For generalised homotopy representations, similar to spheres being parallelisable, one would expect them to have a trivial equivariant dualising spectrum. This turns out to be correct, and the construction of the equivariant fundamental class crucally uses that the equivarinat suspension spectrum $\Sigma^\infty V$ of a generalised homotopy representation V is invertible.

In the final §8.6 we specialise the abstract degree theory from §7.4 to the equivariant setting, generalising work of Lück [Lüc88]. As the main result of this section, we show the following.

Theorem J (Corollary 8.6.13). *Consider a finite group G and a G-Poincaré space Y such that Y^H is nonempty connected for all $H \leq G$. Given a degree datum $(f\colon X \to Y, \psi)$, the collection $(\deg_{\mathbb{Z}}(f^H, \psi^H))_{(H)}$ lies in the image of the character map*

$$\chi\colon A(G) \to \prod_{(H)} \mathbb{Z}.$$

Here, a degree datum roughly is a choice of compatible orientations of various fixed points. We work over the Burnside ring, which is the equivariant analogue of $\mathbb{Z}$ as coefficients for our equivariant homology theories. The image of the character map is characterised by the so called Burnside congurences. For $G = C^p$, this for example says that the degrees of f^e and f^{C_p} are congurent modulo p.

The final section Chapter 9 applies this theory to rigidity phenomena of group actions on manifolds. The main result is Theorem G on rigidity of single fixed points for C_{p^k}-actions. Our approach uses the equivariant fundamental class of a G-Poincaré X space to extract what we call the *gluing class*, which is a homological shadow of how the fixed points of X are glued into the underlying space. We introduced this gluing class in §8.5. It also appears in Lück's work [Lüc22] as condition (H). Giving a homotopy theoretic description of it was also one of the main motivations our work.

Let us end this summary of Part I by clarifying the authors contribution to [HKK24b]. The work was fully collaborative, and each coauthor contributed to each section and result.

Future work

We hope that the theory of equivariant Poincaré duality, as developed in Part II, has applications to a range of problems in equivariant surgery theory. In the final part of this introduction, let us comment on some future directions of research.

A smooth G-manifold M has the property that M can be glued from an equivariant tubular neighbourhood of the fixed points M^G and its complement, both of which are smooth G-manifolds with boundary. Bredon's example of the exotic generalised homotopy representation shows that an analogue of this won't hold for G-Poincaré spaces in general. But it is interesting to study in which special cases such embeddings exist. The correct homotopical replacement of an embedding of smooth manifolds is Levitt's notion of Poincaré embeddings, which was extensively studied by Klein [Lev68; Kle99a; Kle99b; Kle02; Kle07]. A Poicaré embedding structure on a map $f\colon X \to Y$ of Poincaré spaces is a pushout of the form

$$\begin{tikzcd} \partial C \arrow[r] \arrow[d, "\nu"] & C \arrow[d] \\ X \arrow[r, "f"] & Y, \end{tikzcd}$$

where $(C, \partial C)$ is a Poincaré pair and $\nu\colon \partial C \to X$ a spherical fibration. Levitt shows that such a Poincaré embedding structure always exists if the codimension $\dim Y - \dim X$ is roughly larger than half the dimension of Y. This has an analogue in the equivariant setting, which is studied in work progress of the author joint with C. Kremer [KK]. We can, among other things, show that under certain codimension hypothesis and conditions on the group G, the fixed points X^G of a G-Poincaré space always equivariantly Poincaré embed into X. This result generalises work of Lück's work [Lüc22] on the Nielsen realisation problem, where he studies the question whether the classifying space $E_\Gamma\mathrm{Fin}$ of finite subgroups admits a manifold model. For the relation to the generalised Nielsen realisation problem, asking whether outer automorphisms of the fundamental group of a closed aspherical manifolds can be realised by actions through homeomorphisms, let us refer the reader to [DL24]. Lučk first studies if $E_\Gamma\mathrm{Fin}$ is a Poincaré duality space, and Davis-Lück then use equivariant surgery theory to turn this into a manifold [DL24]. In [HKK24a], the author together with C. Kremer and K. Hilman put this problem into the context of equivariant Poincaré duality. We explain that Lück's result is basically equivalent to the problem whether a certain quotient of $E_\Gamma\mathrm{Fin}$ is a C_p-Poincaré space with discrete fixed points in the sense of Part II. Using this, we can show that a mysterious homological obstruction (H) on the group Γ in Lück's work is automatically satisfied. The hope is that with the theory of equivariant Poincaré embeddings, one can generalise this result to the case where the C_p-fixed points are no longer discrete.

Let us mention an unrelated second direction for future research. A fundamental invarint of manifolds and Poincaré spaces is their signature. Remembering more structure than just the index of the intersection form on middle homology, one obtains the visible symmetric signature $\sigma^{\mathrm{v}}(X) \in L^{\mathrm{v}}(X)$ in the visible L-theory of X as introduced in [CDH+23a, Section 4.4], generalising the Ranicki-Weiss visible symmetric signature. This signature invariant is essential in surgery theory as variants of it define the surgery obstruction in the surgery sequence or Ranicki's total surgery obstruction. The search for equivariant generalisations of these results, and equivariant surgery in general, has a long history. Let us just mention two results in that direction. Costenoble-Waner define a variant of equivariant Poincaré spaces, for which they prove a π-π-theorem [CW92; CW17]. Lück-Madsen establish a geometric equivariant surgery sequence [LM90a; LM90b]. They can show that for groups of odd order and spaces satisfying that strong codimension hypothesis their geometric equivariant L-groups split into algebraic L-groups over various subgroups of G.

In future work, the author joint with K. Hilman and C. Kremer would like to give a new perspective on these results, using our framework of equivariant Poincaré duality. The following is a very rough sketch of the program. So far, everything that follows is just a distant dream and has not been worked out. Starting from Hilman's equivariant L-theory [Hil22], one can associate to each G-Poincaré space X an equivariant visible signature $\sigma^{\mathrm{v}}(X) \in L^{\mathrm{v}}_G(X)$. Using isotropy separation techniques, these equivariant algebraic L-groups should admit a splitting of $L^{\mathrm{v}}_G(X)$ into (nonequivariant) visible L-theory of various fixed points of X, similar to Lück-Madsen's result. There should also be an alternative construction of the equivariant surgery sequence, directly using those algebraic equivariant L-groups. It actually appears in two instances, one isovariant sequence and one equivariant sequence, where an isovariant Poincaré space is a G-Poincaré space X together with the structure of equivariant Poincaré embeddings $X^K \to X^H$ for all subgroups $H \leq K$. The isovariant surgery sequence should always be exact as a consequence of a π-π-theorem, while the equivariant one is not always exact. If X satisfies what is called the strong gap hypothesis $\dim(X^H) \geq 2\dim(X^K) + 2$, it should automatically refine to an isovariant space by the existence result for equivariant Poincaré embeddings mentioned before. This would explain exactness of the equivariant sequence in this situation.

Part I

A Twisted Bass-Heller-Swan Decomposition for Localising Invariants

This part is a majorly revised and extended version of the authors work [KK24], joint with Christian Kremer.

Chapter 1

Introduction

The algebraic K-theory of polynomial- and Laurent extensions has been an object of interest since the very beginnings of the subject. The fundamental theorem of algebraic K-theory, sometimes also called the Bass-Heller-Swan decomposition, states that for a ring R there is a splitting

$$K_n(R[t,t^{-1}]) \simeq K_n(R) \oplus K_{n-1}(R) \oplus NK_n(R) \oplus NK_n(R).$$

This result dates back to the very beginnings of K-theory. It was first established for regular rings and $n = 1$ by Bass-Heller-Swan [BHS64] and Quillen later proved the general version for connective K-theory [Gra]. The numerous applications of algebraic K-theory to geometric topology, such as the s-cobordism theorem or Farrell's fibering theorem [Far72], allowed to extract concrete geometric applications. The splitting also served as source for the first constructions of negative algebraic K-theory.

There have been various generalisations of this result in two directions, and we give an incomplete list here. In one direction, one replaces the ring R by a homotopy coherent version. Hüttemann-Klein-Vogell-Waldhausen-Williams [HKV+01] prove an A-theoretic splitting for products $X \times S^1$, which can be thought of as a splitting for the Laurent ring spectrum $\mathbb{S}[\Omega X][t,t^{-1}]$. Fontes-Ogle [FO20] prove a version for connective $\mathbb{S}$-algebras, Hüttemann [Hüt21] proves a version for $\mathbb{Z}$-graded rings. Most recently, Saunier [Sau23] establishes such a splitting for general localising invariants of stable ∞-categories. In a different direction, generalisations allow for twisted Laurent extensions, where the variable t and the coefficients in R only commute up to an automorphism of R, which naturally appear as the group ring $R[G \rtimes \mathbb{Z}]$ of semidirect products. The first version of that for classical rings was shown by Grayson [Gra88]. Waldhausen [Wal78a] proves a version for his generalised Laurent extensions and Lück-Steimle [LS16a] prove a version for additive categories. These results, combined with the Farrell-Jones conjecture, provide a powerful tool for computational and qualitative results about the algebraic K-theory of group rings [LS16b].

The goal of this work is to provide a common generalisation of most of the previously mentioned results. We prove a general splitting result for the K-theory of certain categorical mapping tori, with the essential ingredient being Land-Tamme's work on the K-theory of pushouts [LT23]. In contrast to most of the previous work, we also allow twists by noninvertible endomorphisms. We work in the setting of localising invariants of stable ∞-categories as pioneered by Blumberg-Gepner-Tabuada [BGT13]. This immediately proves the splitting not only for nonconnective K-theory but also other localising invariants like topological Hochschild homology and its cousins.

Main results

Let $\mathcal{C}$ be an idempotent complete stable ∞-category together with an exact functor $\alpha\colon \mathcal{C} \to \mathcal{C}$. Denote by $\mathcal{C}_{h\mathbb{N}}$ its mapping torus, i.e. the pushout of the span $\mathcal{C} \xleftarrow{\mathrm{id}\oplus\alpha} \mathcal{C}\oplus\mathcal{C} \xrightarrow{\mathrm{id}\oplus\mathrm{id}} \mathcal{C}$ in $\mathrm{Cat}^{\mathrm{perf}}$, the ∞-category of idempotent complete stable ∞-categories. In the case where α is an equivalence, this mapping torus (up to ignoring certain finiteness conditions) consists of pairs (x, f) of an object $x \in \mathcal{C}$ together with an equivalence $f\colon \alpha(x) \xrightarrow{\simeq} x$. We also call $\mathcal{C}_{h\mathbb{N}}$ the category of *twisted automorphisms*. The general description of $\mathcal{C}_{h\mathbb{N}}$ can be found in Definition 2.1.2 and Lemma 2.1.10.

Theorem K (Theorem 3.2.5)**.** *Let $\alpha\colon \mathcal{C} \to \mathcal{C}$ be an exact endofunctor of an idempotent complete stable ∞-category and $E\colon \mathrm{Cat}^{\mathrm{perf}} \to \mathcal{E}$ a localising invariant. The assembly map $E(\mathcal{C})_{h\mathbb{N}} \to E(\mathcal{C}_{h\mathbb{N}})$ admits a splitting, natural in $\alpha \in \mathrm{Fun}(B\mathbb{N}, \mathrm{Cat}^{\mathrm{perf}})$, inducing an equivalence*

$$E(\mathcal{C}_{h\mathbb{N}}) \simeq E(\mathcal{C})_{h\mathbb{N}} \oplus NE_\alpha(\mathcal{C}) \oplus \overline{NE}_\alpha(\mathcal{C}).$$

There is a cofibre sequence $E(\mathcal{C}) \xrightarrow{\mathrm{id}-\alpha} E(\mathcal{C}) \to E(\mathcal{C})_{h\mathbb{N}}$ characterising the mapping torus $E(\mathcal{C})_{h\mathbb{N}}$. If E takes values in spectra, one obtains an associated long exact sequence of homotopy groups relating the homotopy groups of $E(\mathcal{C})$ and $E(\mathcal{C})_{h\mathbb{N}}$.

Let us explain the Nil-terms appearing in this decomposition. There is a category $\overline{\mathrm{End}}_\alpha(\mathcal{C})$ of twisted endomorphisms, which consists of pairs (x, f) of an object $x \in \mathcal{C}$ together with a morphism $f\colon \alpha(x) \to x$. Denote by $\overline{\mathrm{Nil}}_\alpha(\mathcal{C}) \subseteq \overline{\mathrm{End}}_\alpha(\mathcal{C})$ the subcategory of *twisted nilpotent endomorphisms*, generated by the trivial endomorphisms $(x, 0\colon \alpha(x) \to x)$ under finite colimits and retracts. By Theorem 2.2.2, these admit the alternative characterisation as those pairs (x, f) for which the composite

$$\alpha^n(x) \xrightarrow{\alpha^{n-1}(f)} \cdots \to \alpha(x) \xrightarrow{f} x$$

is trivial for large n. The inclusion $\overline{\mathrm{triv}}\colon \mathcal{C} \to \overline{\mathrm{Nil}}_\alpha(\mathcal{C})$ of trivial endomorphisms admits a retraction $(x, f) \mapsto x$ and $\overline{NE}_\alpha(\mathcal{C})$ is defined by the splitting

$$E(\overline{\mathrm{Nil}}_\alpha(\mathcal{C})) \simeq E(\mathcal{C}) \oplus \Omega\overline{NE}_\alpha(\mathcal{C}). \tag{1.1}$$

The description of $NE_\alpha(\mathcal{C})$ is similar, using twisted endomorphisms $x \to \alpha(x)$ instead.

The Nil-terms in (1.1) also arise in a different context. There is the category $\overline{\mathrm{End}}_\alpha(\mathrm{Ind}(\mathcal{C}))^\omega$ defined similar to $\overline{\mathrm{End}}_\alpha(\mathcal{C})$ up to a finiteness condition. To any object $x \in \mathcal{C}$ one can associate the free twisted endomorphism

$$\mathrm{free}_\alpha(x) = \left(\bigoplus_{n\geq 0} \alpha^n(x), \mathrm{shift}\right),$$

where shift denotes the composition $\alpha(\bigoplus_{n\geq 0}\alpha^n(x)) \simeq \bigoplus_{n\geq 1}\alpha^n(x) \to \bigoplus_{n\geq 0}\alpha^n(x)$. The map $\mathrm{free}_\alpha\colon \mathcal{C} \to \overline{\mathrm{End}}_\alpha(\mathrm{Ind}(\mathcal{C}))^\omega$ again admits a retraction. Our second main result identifies its cofibre with the Nil-term from above.

Theorem L (Theorem 3.3.2)**.** *The functor* free_α *induces a splitting*

$$E(\overline{\mathrm{End}}_\alpha(\mathrm{Ind}(\mathcal{C}))^\omega) \simeq E(\mathcal{C}) \oplus NE_\alpha(\mathcal{C}).$$

There is an similar splitting for the $\overline{NE}_\alpha$-term. This result goes back to Waldhausen [Wal78a, Theorem 13.5] in the setting of generalised polynomial extensions of discrete rings. Land-Tamme [LT23, Corollary 4.5, Proposition 4.7] prove a version for tensor algebras. Our result is a generalisation to the case of categories not generated by a single object.

Using a recent dévissage result of Burklund-Levy [BL23], we can prove vanishing of Nil-terms under regularity assumptions, which generalises the classical vanishing result for regular rings.

Corollary M (Corollary 3.4.2)**.** *Suppose that* $\alpha\colon \mathcal{C} \to \mathcal{C}$ *is a left t-exact endofunctor of a stable category with a bounded t-structure. Then the connective* Nil*-term* $\tau_{\geq 0}NK_\alpha(\mathcal{C})$ *vanishes. If* $\mathcal{C}^\heartsuit$ *is Noetherian, then also the nonconnective* Nil*-term* $NK_\alpha(\mathcal{C})$ *vanishes. The same results hold for* $\overline{NK}_\alpha(\mathcal{C})$ *if* α *is right t-exact.*

Unrelated to that, we also construct a t-structure on the categories $\overline{\mathrm{End}}_\alpha(\mathcal{C})$ and $\overline{\mathrm{Nil}}_\alpha(\mathrm{Ind}(\mathcal{C}))$ in Proposition 3.4.3 and Corollary 3.4.4, which might be of independent interest.

Applications

For a ring spectrum R and a (R,R)-bimodule M denote by $T_R(M) = \bigoplus_{n\geq 0} M^{\otimes_R^n}$ its tensor algebra. Suppose that M admits a left dual $M^\vee$. Define the localised tensor algebra by

$$T_R(M)[M^{-1}] \simeq \operatorname{colim}\left(\bigoplus_{n\geq 0} M^{\otimes_R^n} \xrightarrow{\mathrm{coev}} \bigoplus_{n\geq 0} M^\vee \otimes_R M^{\otimes_R^n} \xrightarrow{\mathrm{coev}} \dots\right).$$

The bimodule M induces an endomorphism $M \otimes_R -\colon \mathrm{Mod}_R^\omega \to \mathrm{Mod}_R^\omega$ of the category of perfect left R-modules. It turns out that $\overline{\mathrm{End}}_M(\mathrm{Mod}_R^\omega) \simeq \mathrm{Mod}_{T_R(M)}^\omega$ and $(\mathrm{Mod}_R^\omega)_{h\mathbb{N}} \simeq \mathrm{Mod}_{T_R(M)[M^{-1}]}^\omega$. Theorem K and Theorem L then reduce to the following result.

Theorem N (Corollary 4.1.4)**.** *There are natural splittings*

$$E(T_R(M)[M^{-1}]) \simeq E(R)_{h\mathbb{N}} \oplus NE_M(R) \oplus \overline{NE}_M(R)$$

and

$$E(T_R(M)) \simeq E(R) \oplus NE_M(R) \quad \textit{and} \quad E(T_R(M^\vee)) \simeq E(R) \oplus \overline{NE}_M(R).$$

In the case where R is discrete and M is the bimodule R with trivial left R-module structure and right R-module structure coming from a pure embedding $\alpha\colon R \to R$, the localised tensor algebra $T_R(M)[M^{-1}] \simeq R_\alpha\{t^{\pm 1}\}$ identifies with Waldhausen's generalised Laurent extension [Wal78a]. It is the universal ring containing R and an invertible element t which satisfies $tr = \alpha(r)t$. If α is an isomorphism, this is the classical ring of twisted Laurent polynomials given by $\bigoplus_{n\in\mathbb{Z}} Rt^n$ with multiplication $rt^m \cdot st^n = r\alpha^m(s)t^{m+n}$.

As another application, we can use Theorem K to obtain the following splitting for Waldhausen's (finitely dominated) A-theory of mapping tori.

Theorem O (Corollary 4.2.2)**.** *Let $\alpha\colon X \to X$ be a selfmap of a space. Then there is a splitting*

$$A(X_{h\mathbb{N}}) \simeq \tau_{\geq 0}(\mathbb{A}(X)_{h\mathbb{N}}) \oplus NA_\alpha(X) \oplus \overline{NA}_\alpha(X).$$

We actually prove a version of this for a nonconnective delooping $\mathbb{A}$ of A-theory and obtain Theorem O by passing to connective covers. In particular, on 1-connective covers one has $\tau_{\geq 1}(\mathbb{A}(X)_{h\mathbb{N}}) \simeq \tau_{\geq 1}(A(X)_{h\mathbb{N}})$.

In §4.3 we express A-theoretic Nil-terms through free loops spaces by envoking the work of Bökstedt-Hsiang-Madsen [BHM93] on the topological cyclic homology of spaces.

Outline of the proof

Most of the results in the literature only consider the case where the twist α is an equivalence. The proof usually follows the classical algebraic geometric approach and first establishes a splitting for sheaves over the (twisted) projective line over R glued from (twisted) polynomial algebras $R_\alpha[t]$ and $R_{\alpha^{-1}}[t]$ along $R_\alpha[t^{\pm 1}]$. This argument can not be extended to our setting of noninvertible twists. We follow an approach which is more similar to Waldhausen's splitting for generalised Laurent extensions [Wal78a].

The main tool we use in the proof is Land-Tamme's theory of K-theory of pushouts [LT23, Theorem 3.2]. Starting from the pushout square

$$\begin{tikzcd} \mathcal{C} \oplus \mathcal{C} \arrow[r, "\mathrm{id}\oplus\mathrm{id}"] \arrow[d, "\mathrm{id}\oplus\alpha"] & \mathcal{C} \arrow[d] \\ \mathcal{C} \arrow[r] & \mathcal{C}_{h\mathbb{N}} \end{tikzcd}$$

defining the categorical mapping torus, Land-Tamme show how to replace the upper left corner by a different category $\mathrm{Im}(\mathcal{C} \oplus \mathcal{C})$ such that the resulting commutative square becomes a pushout after applying the localising invariant E. We show that $\mathrm{Im}(\mathcal{C} \oplus \mathcal{C})$ admits semiorthogonal decomposition through the two Nil-categories $\mathrm{Nil}_\alpha(\mathcal{C})$ and $\overline{\mathrm{Nil}}_\alpha(\mathcal{C})$. By employing the splitting (1.1) and carefully analysing the resulting pushout square, we arrive at the claimed splitting.

Structure of the article

We begin by introducing the categories of twisted endomorphisms, automorphisms and nilpotent endomorphisms in §§2.1and 2.2 and study their basic properties, which will be needed later on. Before truning to the proofs of Theorems K and L in §§3.2and 3.3 we recall the necessary background on Land-Tamme's K-theory of pushouts in §3.1. In §3.4 we construct a t-structure on the category of twisted endomorphisms and prove Corollary M. In the first half of Chapter 4 we apply these results to obtain splittings for the K-theory tensor algebras and various kinds of (twisted) polynomial rings, as well as for A-theory. In the second half §4.3 we express some A-theoretic Nil-terms through free loop spaces.

Conventions

This article is written in the language of ∞–categories as set down in [Lur09; Lur17], and so by a *category* we will always mean an ∞–category unless stated otherwise. We also use the following notations throughout:

- Cat denotes the category of small categories and $\mathcal{S} \subseteq \mathrm{Cat}$ the category of spaces.
- $\mathrm{Cat}^{\mathrm{perf}}$ denotes the category of small idempotent complete stable categories (sometimes called perfect categories) and exact functors.
- We denote by $\mathrm{Map}_\mathcal{C}(x, y)$ the mapping space between objects in a category $\mathcal{C}$. If $\mathcal{C}$ is stable, $\mathrm{hom}_\mathcal{C}(x, y)$ denotes their mapping spectrum.
- $E\colon \mathrm{Cat}^{\mathrm{perf}} \to \mathcal{E}$ denotes a localising invariant with values in a stable category.

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Chapter 2

Three flavours of twisted endomorphisms

This section contains the construction of twisted endomorphism, automorphism and nilpotent endomorphism categories and some of their basic properties, which will be needed later on.

2.1 Twisted endomorphisms and automorphisms

Let us begin by recalling the construction of lax equalisers, whose basic properties can be found in [NS18, Proposition II.1.5].

Construction 2.1.1 (Lax equaliser)**.** Given two functors $f, g\colon \mathcal{C} \to \mathcal{D}$, the lax equaliser $\mathrm{laxeq}(f,g)$ is the category of pairs (x,r) consisting of an object $x \in \mathcal{C}$ and a map $r\colon f(x) \to g(x)$ in $\mathcal{D}$. Formally, it is defined as the pullback

$$\begin{tikzcd} \mathrm{laxeq}(f,g) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{D}^{\Delta^1} \arrow[d, "(s,t)"] \\ \mathcal{C} \arrow[r, "(f,g)"] & \mathcal{D} \times \mathcal{D}, \end{tikzcd} \tag{2.1}$$

where the right vertical map is restriction along the two inclusions $0,1\colon * \to \Delta^1$. From this, one obtains the formula for mapping spaces as the equaliser

$$\mathrm{Map}_{\mathrm{laxeq}(f,g)}((x,r),(y,s)) \simeq \mathrm{eq}\left(s_*f, r^*g\colon \mathrm{Map}_{\mathcal{C}}(x,y) \to \mathrm{Map}_{\mathcal{D}}(f(x),g(y))\right). \tag{2.2}$$

From now on, consider a category $\mathcal{C}$ together with an endofunctor $\alpha\colon \mathcal{C} \to \mathcal{C}$.

Definition 2.1.2 (Twisted endormophisms and automorphisms)**.** We define the categories of *twisted endomorphisms* as the lax equalisers

$$\mathrm{End}_\alpha(\mathcal{C}) = \mathrm{laxeq}(\mathrm{id}, \alpha\colon \mathcal{C} \to \mathcal{C}) \quad \text{and} \quad \overline{\mathrm{End}}_\alpha(\mathcal{C}) = \mathrm{laxeq}(\alpha, \mathrm{id}\colon \mathcal{C} \to \mathcal{C}).$$

Its objects are pairs (x, f) consisting of objects $x \in \mathcal{C}$ and maps $f\colon x \to \alpha(x)$ (resp. $f\colon \alpha(x) \to x$) in $\mathcal{C}$.

We also define the category $\mathrm{Aut}_\alpha(\mathcal{C})$ of *twisted automorphisms* as the equaliser $\mathrm{Aut}_\alpha(\mathcal{C}) = \mathrm{eq}(\mathrm{id}, \alpha\colon \mathcal{C} \to \mathcal{C})$. Its objects consist of pairs (x, f) of objects $x \in \mathcal{C}$ and equivalences $f\colon x \xrightarrow{\simeq} \alpha(x)$ in $\mathcal{C}$.

As a motivation for the definition and the following constructions, the reader should keep the example $\mathcal{C} = \mathrm{Mod}_R$ – the (derived) category of modules over a ring R – in mind. In the case $\alpha = \mathrm{id}$, one has $\overline{\mathrm{End}}_{\mathrm{id}}(\mathrm{Mod}_R) = \mathrm{End}_{\mathrm{id}}(\mathrm{Mod}_R) = \mathrm{Mod}_{R[t]}$ and $\mathrm{Aut}_{\mathrm{id}}(\mathrm{Mod}_R) = \mathrm{Mod}_{R[t^{\pm 1}]}$. The reader interested in more examples should skip ahead to Construction 4.1.2, where we will relate these notions to categories of modules over tensor algebras and twisted polynomial rings. The category $\mathrm{End}_\alpha(\mathcal{C})$ also appears in [NS18, Section II.5.] under the name α-coalgebras. Let us begin by studying some basic constructions on those categories.

Observation 2.1.3. The map $\mathcal{C} \xrightarrow{\mathrm{const}} \mathcal{C}^{\Delta^1}$ applied to the upper right corner of the pullback (2.1) induces a natural map $\mathrm{Aut}_\alpha(\mathcal{C}) \to \mathrm{End}_\alpha(\mathcal{C})$. By inspecting the limit formula (2.2) for mapping spaces in $\mathrm{End}_\alpha(\mathcal{C})$ and its analogue in $\mathrm{Aut}_\alpha(\mathcal{C})$, one sees that $\mathrm{Aut}_\alpha(\mathcal{C}) \subseteq \mathrm{End}_\alpha(\mathcal{C})$ is actually fully faithful. It identifies $\mathrm{Aut}_\alpha(\mathcal{C})$ as the full subcategory on all pairs $(x, f) \in \mathrm{End}_\alpha(\mathcal{C})$ for which $f\colon x \to \alpha(x)$ is an equivalence. Similarly, there is an inclusion $\mathrm{Aut}_\alpha(\mathcal{C}) \subseteq \overline{\mathrm{End}}_\alpha(\mathcal{C})$ identifying $\mathrm{Aut}_\alpha(\mathcal{C})$ as the full subcategory on all pairs $(y, g) \in \overline{\mathrm{End}}_\alpha(\mathcal{C})$ for which $g\colon \alpha(y) \to y$ is an equivalence.

Denote by

$$\mathrm{fgt}\colon \mathrm{End}_\alpha(\mathcal{C}) \to \mathcal{C}, (x, f) \mapsto x, \quad \mathrm{fgt}\colon \overline{\mathrm{End}}_\alpha(\mathcal{C}) \to \mathcal{C}, (x, f) \mapsto x \tag{2.3}$$

the *forgetful funtors*, obtained as the projection of the pullback defining lax equaliser (2.1) to the bottom left corner. They are conservative by [NS18, Proposition II.1.5 (ii)]. If $\mathcal{C}$ is a perfect (resp. presentable) category and α is exact (resp. a left adjoint, resp. a right adjoint), then $\mathrm{End}_\alpha(\mathcal{C})$, $\overline{\mathrm{End}}_\alpha(\mathcal{C})$ and $\mathrm{Aut}_\alpha(\mathcal{C})$ are perfect (resp. presentable) categories and the functors $\mathrm{fgt}\colon \mathrm{End}_\alpha(\mathcal{C}) \to \mathcal{C}$, $\mathrm{fgt}\colon \overline{\mathrm{End}}_\alpha(\mathcal{C}) \to \mathcal{C}$ and $\mathrm{Aut}_\alpha(\mathcal{C}) \to \mathrm{End}_\alpha(\mathcal{C})$ are exact (resp. a left adjoint, resp. a right adjoint). This directly follows from the fact that the forgetful functors $\mathrm{Cat}^{\mathrm{perf}}, \mathrm{Pr}^L, \mathrm{Pr}^R \to \mathrm{Cat}$ preserve limits.

Construction 2.1.4 (Twisted powers)**.** The functor α induces an endofunctor $\alpha\colon \mathrm{End}_\alpha(\mathcal{C}) \to \mathrm{End}_\alpha(\mathcal{C}), (x, f) \mapsto (\alpha x, \alpha f)$ constructed in [NS18, Construction

II.5.2], which we will recall. Formally, it is obtained from the commutative diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "(\mathrm{id},\alpha)"] \arrow[d, "\alpha"] & \mathcal{C} \times \mathcal{C} \arrow[d, "(\alpha,\alpha)"] & \mathcal{C}^{\Delta^1} \arrow[l, "(s,t)"'] \arrow[d, "\alpha"] \\ \mathcal{C} \arrow[r, "(\mathrm{id},\alpha)"] & \mathcal{C} \times \mathcal{C} & \mathcal{C}^{\Delta^1}. \arrow[l, "(s,t)"'] \end{tikzcd}$$

There is a natural transformation $[1]\colon \mathrm{id} \to \alpha$ of endofunctors of $\mathrm{End}_\alpha(\mathcal{C})$ given on objects by the commutative square

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "f"] & \alpha x \arrow[d, "\alpha f"] \\ \alpha x \arrow[r, "\alpha f"] & \alpha^2 x. \end{tikzcd} \tag{2.4}$$

Formally, it is a functor

$$\mathrm{End}_\alpha(\mathcal{C}) \to \mathrm{End}_\alpha(\mathcal{C})^{\Delta^1} \simeq \mathcal{C}^{\Delta^1} \times_{\mathcal{C}^{\Delta^1} \times \mathcal{C}^{\Delta^1}} \mathcal{C}^{\Delta^1 \times \Delta^1}$$

given by the projection $\mathrm{End}_\alpha(\mathcal{C}) \to \mathcal{C}^{\Delta^1}$ in the first component and the commutative square (2.4) in the second component. We can define the n-fold composite $[n] = (\alpha^{n-1}[1]) \circ (\alpha^{n-2}[1]) \circ \cdots \circ [1]\colon \alpha^n \to \mathrm{id}$ of $[1]$ and denote the image of the map $[n]\colon (x, f) \to \alpha^n(x, f)$ in $\mathcal{C}$ by

$$f^{(n)} := \left(x \xrightarrow{f} \alpha(x) \xrightarrow{\alpha(f)} \ldots \xrightarrow{\alpha^{n-1}(f)} \alpha^n(f) \right).$$

There are analogous constructions of an endofunctor $\alpha\colon \overline{\mathrm{End}}_\alpha(\mathcal{C}) \to \overline{\mathrm{End}}_\alpha(\mathcal{C})$ which comes with a transformation $[1]\colon \alpha \to \mathrm{id}$.

Lemma 2.1.5. *Suppose that $\alpha\colon \mathcal{C} \to \mathcal{C}$ admits a right adjoint α^R. Then there is an equivalence $\overline{\mathrm{End}}_\alpha(\mathcal{C}) \simeq \mathrm{End}_{\alpha^R}(\mathcal{C})$ sending a pair $(x, f\colon \alpha(x) \to x)$ to its adjoint pair $(x, f^R\colon x \to \alpha^R(x))$.*

Proof. Note that the adjunction unit $\eta\colon \mathrm{id} \to \alpha^R\alpha$ gives a commutative diagram

$$\begin{tikzcd} & \mathcal{C}^{\Delta^1} \arrow[d] \\ \mathcal{C} \arrow[ur, "\eta"] \arrow[r, "(\mathrm{id},\alpha^R\alpha)"'] & \mathcal{C} \times \mathcal{C}. \end{tikzcd}$$

We obtain a functor $\overline{\mathrm{End}}_\alpha(\mathcal{C}) \to \mathrm{End}_{\alpha^R}(\mathcal{C})$ as the composite

$$\begin{array}{rlr}
& \lim(\mathcal{C} \xrightarrow{(\alpha,\mathrm{id})} \mathcal{C}\times\mathcal{C} \xleftarrow{(s,t)} \mathcal{C}^{\Delta^1}) & (x, \alpha(x) \xrightarrow{f} x) \\
\xrightarrow{\alpha^R} & \lim(\mathcal{C} \xrightarrow{(\alpha^R\alpha,\alpha^R)} \mathcal{C}\times\mathcal{C} \xleftarrow{(s,t)} \mathcal{C}^{\Delta^1}) & (x, \alpha^R\alpha(x) \xrightarrow{\alpha^R f} \alpha^R(x)) \\
\xrightarrow{\eta} & \lim(\mathcal{C} \xrightarrow{(\mathrm{id},\alpha^R\alpha,\alpha^R)} \mathcal{C}\times\mathcal{C}\times\mathcal{C} \leftarrow \mathcal{C}^{\Delta^1}\times_{\mathcal{C}}\mathcal{C}^{\Delta^1}) & (x, x \xrightarrow{\eta} \alpha^R\alpha(x) \xrightarrow{\alpha^R f} \alpha^R(x)) \\
\xrightarrow{(\mathrm{pr}_0,\mathrm{pr}_2)} & \lim(\mathcal{C} \xrightarrow{(\mathrm{id},\alpha^R)} \mathcal{C}\times\mathcal{C} \xleftarrow{(s,t)} \mathcal{C}^{\Delta^1}) & (x, \alpha^R(x) \xrightarrow{\alpha^R f\circ\eta} \alpha^R(x)),
\end{array}$$

where we denote the effect of each functor on an element in the limit in the right column. The adjunction counit $\epsilon\colon \alpha\alpha^R \to \mathrm{id}$ induces a functor $\mathrm{End}_{\alpha^R}(\mathcal{C}) \to \overline{\mathrm{End}}_\alpha(\mathcal{C})$ in a similar way. Using the triangle identities $\epsilon\alpha \circ \alpha\eta \simeq \mathrm{id}$ and $\eta\alpha^R \circ \alpha^R\epsilon \simeq \mathrm{id}$, it is easy to check that they are inverse to each other. □

Crucial for us is the existence of free twisted endomorphisms on an object $x \in \mathcal{C}$. It should be viewed as a generalisation of induction along the ring homomorphism $R \to R[t]$.

Lemma 2.1.6 (Free twisted endomorphism)**.** *Suppose that $\mathcal{C}$ admits countable coproducts and α preserves those. Then $\mathrm{fgt}\colon \overline{\mathrm{End}}_\alpha(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint $\overline{\mathrm{free}}_\alpha$ given by the formula*

$$\overline{\mathrm{free}}_\alpha(x) = \left(\coprod_{n\geq 0} \alpha^n(x), \mathrm{shift}\right),$$

where $\mathrm{shift}\colon \alpha(\coprod_{n\geq 0}\alpha^n(x)) \simeq \coprod_{n\geq 1}\alpha^n(x) \to \coprod_{n\geq 0}\alpha^n(x)$ is the inclusion.

Proof. We claim that the projection $\mathrm{fgt}(\overline{\mathrm{free}}_\alpha(x)) = \coprod_{n\geq 0}\alpha^n(x) \to x$ onto the zeroth summand induces for any $(y, f) \in \overline{\mathrm{End}}_\alpha(\mathcal{C})$ and equivalence

$$\mathrm{Map}_{\overline{\mathrm{End}}_\alpha(\mathcal{C})}(\overline{\mathrm{free}}_\alpha(x), (y, f)) \simeq \mathrm{Map}_{\mathcal{C}}(x, y).$$

Using (2.2), this identifies with the projection of

$$\mathrm{eq}\left(f_*\alpha, i^*\colon \mathrm{Map}_{\mathcal{C}}(\coprod_{n\geq 0}\alpha^n(x), y) \to \mathrm{Map}_{\mathcal{C}}(\coprod_{n\geq 1}\alpha^n(x), y)\right),$$

i.e. the space of maps $g_n\colon \alpha^n(x) \to y$ and homotopies $h_n\colon f\circ\alpha(g_n) \simeq g_{n+1}$. Note that this is obtained as the limit over k of

$$\mathrm{eq}\left(f_*(j_k)^*\alpha, (i_k)^*\colon \mathrm{Map}_{\mathcal{C}}(\coprod_{0\leq n\leq k}\alpha^n(x), y) \to \mathrm{Map}_{\mathcal{C}}(\coprod_{1\leq n\leq k}\alpha^n(x), y)\right), \tag{2.5}$$

for $i_k\colon \coprod_{1\leq n\leq k}\alpha^n(x) \to \coprod_{0\leq n\leq k}\alpha^n(x)$ and $j_k\colon \coprod_{1\leq n\leq k}\alpha^n(x) \to \coprod_{1\leq n\leq k+1}\alpha^n(x)$ the inclusions. By induction over k it is easy to see that the projection of (2.5) to $\mathrm{Map}_{\mathcal{C}}(x, y)$ is an equivalence. □

Note that $\mathrm{fgt}\colon \mathrm{End}_\alpha(\mathcal{C}) \to \mathcal{C}$ only admits a left adjoint under the additional assumtption that α admits a left adjoint, using Lemma 2.1.5. This is the reason why many results in this article are formulated only for $\overline{\mathrm{End}}_\alpha(\mathcal{C})$.

Similarly, there is a way to localise any twisted endomorphism $x \to \alpha(x)$ to a twisted automorphis $x \xrightarrow{\simeq} \alpha(x)$. This generalises induction along the ring homomorphism $R[t] \to R[t^{\pm 1}]$.

Lemma 2.1.7 (Localisation)**.** *Suppose that $\mathcal{C}$ admits sequential colimits which are preserved by α. Then the inclusion $\mathrm{Aut}_\alpha(\mathcal{C}) \subseteq \mathrm{End}_\alpha(\mathcal{C})$ admits a left adjoint $\mathrm{loc}_\alpha\colon \mathrm{End}_\alpha(\mathcal{C}) \to \mathrm{Aut}_\alpha(\mathcal{C})$ given by the formula*

$$\mathrm{loc}_\alpha = \operatorname{colim}\left(\mathrm{id} \xrightarrow{[1]} \alpha \xrightarrow{\alpha[1]} \alpha^2 \to \dots\right). \tag{2.6}$$

Proof. This is shown in [NS18, Proposition II.5.3] under the hypothesis of presentability, but the same argument works under the assumptions of this lemma: Let us first show that the endofunctor loc_α of End_α defined by the colimit (2.6) is a left Bousfield localisation. For this, we need to argue that the structure map $\mathrm{id} \to \mathrm{loc}_\alpha$ of the colimit becomes an equivalence after pre- and postcomposing with loc_α, which holds as α preserves sequential colimits. It follows that loc_α is a localisation onto the loc_α-local objects. To identify the loc_α-local objects with precisely those pairs $(x, f\colon x \to \alpha(x)) \in \mathrm{End}_\alpha(\mathcal{C})$ for which f is an equivalence, one has to use that $[1]\colon \mathrm{id} \to \alpha$ induces an equivalence $\mathrm{loc}_\alpha \to \alpha\mathrm{loc}_\alpha$. □

We will later need the following generation result for categories of twisted endomorphisms and automorphisms. Recall that if $\mathcal{C}$ is a presentable stable ∞-category, then $\mathcal{C}^\omega$, its subcategory of compact objects, is perfect.

Lemma 2.1.8. *Suppose that $\mathcal{C}$ is a compactly generated stable presentable category. If $\alpha\colon \mathcal{C} \to \mathcal{C}$ preserves colimits, then $\overline{\mathrm{End}}_\alpha(\mathcal{C})$ is compactly generated and the image of $\mathcal{C}^\omega$ under $\overline{\mathrm{free}}_\alpha\colon \mathcal{C} \to \overline{\mathrm{End}}_\alpha(\mathcal{C})$ is a family of compact generators. If $\alpha\colon \mathcal{C} \to \mathcal{C}$ preserves limits and colimits, then $\mathrm{End}_\alpha(\mathcal{C})$ and $\mathrm{Aut}_\alpha(\mathcal{C})$ are compactly generated, with generators given by the image of $\mathcal{C}^\omega$ under $\mathrm{free}_\alpha\colon \mathcal{C} \to \mathrm{End}_\alpha(\mathcal{C})$ and $\mathrm{loc}_\alpha \circ \mathrm{free}_\alpha\colon \mathcal{C} \to \mathrm{Aut}_\alpha(\mathcal{C})$.*

Proof. The assumption guarantees that the left adjoint $\overline{\mathrm{free}}_\alpha\colon \mathcal{C} \to \overline{\mathrm{End}}_\alpha(\mathcal{C})$ to fgt exists. As fgt is colimits preserving its left adjoint preserves compact objects. To show that the image of $\mathcal{C}^\omega$ under $\overline{\mathrm{free}}_\alpha$ generates $\overline{\mathrm{End}}_\alpha(\mathcal{C})$, it suffices to show that if $(y, f) \in \overline{\mathrm{End}}_\alpha(\mathcal{C})$ such that $\mathrm{Map}_{\overline{\mathrm{End}}_\alpha(\mathcal{C})}(\overline{\mathrm{free}}_\alpha(x), (y, f)) \simeq 0$ for all $x \in \mathcal{C}^\omega$, then $(y, f) \simeq 0$. But as $\mathrm{Map}_{\overline{\mathrm{End}}_\alpha(\mathcal{C})}(\overline{\mathrm{free}}_\alpha(x), (y, f)) \simeq \mathrm{Map}_{\mathcal{C}}(x, y)$ this follows from $\mathcal{C}$ being compactly generated.

If α preserves limits and colimits, it admits a left adjoint α^L. It follows that $\mathrm{fgt}\colon \mathrm{End}_\alpha(\mathcal{C}) \to \mathcal{C}$ and $\mathrm{Aut}_\alpha(\mathcal{C}) \subseteq \mathrm{End}_\alpha(\mathcal{C})$ preserve colimits and admit a left adjoint by Observation 2.1.3, which implies that their left adjoints preserve compact objects. The argument showing that the images $\mathcal{C}^\omega$ under free_α and $\mathrm{loc}_\alpha\mathrm{free}_\alpha$ are generators is the same as above. □

Finally, let us relate the $\mathbb{N}$-orbits $(-)_{h\mathbb{N}}$ appearing in Theorem 3.2.5 to twisted automorphisms.

Recollection 2.1.9 ((Co)limits over $B\mathbb{N}$)**.** Let $\mathcal{D}$ be a category with finite colimits and consider an object $x \in \mathcal{D}$ together with an endomorphism $f\colon x \to x$. Denote by $B\mathbb{N}$ the category with a single object and $\mathbb{N}$ as its endomorpism monoid. The endomorphism f corresponds to a diagram $F\colon B\mathbb{N} \to \mathcal{D}$ characterised by $F(*) = x$ and $F(1) = f$. The colimit $x_{h\mathbb{N}} := \mathrm{colim}_{B\mathbb{N}} F$ in $\mathcal{D}$ then fits into the pushout

$$\begin{tikzcd} x \amalg x \arrow[r, "\mathrm{id}\amalg\mathrm{id}"] \arrow[d, "\mathrm{id}\amalg f"] \arrow[dr, phantom, "\ulcorner", very near end] & x \arrow[d] \\ x \arrow[r] & x_{h\mathbb{N}}. \end{tikzcd} \tag{2.7}$$

One way to see this is to represent $B\mathbb{N}$ as the pushout of the span $* \leftarrow * \amalg * \to \Delta^1$ and to apply [HY17, Corollary 1.3]. This has a dual version if $\mathcal{D}$ has finite limits. The limit of the diagram F is then given by the pullback

$$\begin{tikzcd} x^{h\mathbb{N}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & x \arrow[d, "(\mathrm{id},f)"] \\ x \arrow[r, "(\mathrm{id},\mathrm{id})"] & x \times x. \end{tikzcd} \tag{2.8}$$

Lemma 2.1.10. *Let $\alpha\colon \mathcal{C} \to \mathcal{C}$ be an exact endofunctor of a perfect category. Then there is an equivalence $\mathcal{C}_{h\mathbb{N}} \simeq (\mathrm{Aut}_{\alpha^R}(\mathrm{Ind}(\mathcal{C})))^{\omega}$, where $\alpha^R\colon \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C})$ denotes the right adjoint to* $\mathrm{Ind}(\alpha)$.

Proof. Recall that colimits in $\mathrm{Cat}^{\mathrm{perf}}$ are computed as the compact objects in the colimit of the Ind-completed diagram in Pr^L, or equivalently in the limit of the right adjoint diagram in Pr^R. We obtain

$$\mathcal{C}_{h\mathbb{N}} \simeq (\mathrm{Ind}(\mathcal{C})_{h\mathbb{N}})^{\omega} \simeq \left(\mathrm{Ind}(\mathcal{C})^{h\mathbb{N}}\right)^{\omega}$$

where the limit in the last step is formed over the diagram $B\mathbb{N} \to \mathrm{Cat}$ classified by $\alpha^R\colon \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C})$. Recollection 2.1.9 identifies $\mathrm{Ind}(\mathcal{C})^{h\mathbb{N}} \simeq \mathrm{Aut}_{\alpha^R}(\mathrm{Ind}(\mathcal{C}))$. $\square$

2.2 Twisted nilpotent endomorphisms

As the last ingredient for the main result of this article we will introduce twisted nilpotent endomorphisms, along with providing some equivalent charactersiations. These nilpotent endomorphisms are the source of the NE-terms appearing in Theorem K.

Let $\alpha\colon \mathcal{C} \to \mathcal{C}$ be an exact endomorphism of a perfect category. There are maps

$$\begin{aligned} \mathrm{triv}\colon \mathcal{C} &\to \mathrm{End}_\alpha(\mathcal{C}), \quad x \mapsto (x, 0\colon x \to \alpha(x)),\\ \overline{\mathrm{triv}}\colon \mathcal{C} &\to \overline{\mathrm{End}}_\alpha(\mathcal{C}), \quad x \mapsto (x, 0\colon \alpha(x) \to x) \end{aligned}$$

sending an object to the zero endomorphism on that object.

Definition 2.2.1. We define the category $\mathrm{Nil}_\alpha(\mathcal{C})$ of *twisted nilpotent endomorphisms* as the full perfect subcategory of $\mathrm{End}_\alpha(\mathcal{C})$ generated by the image of triv, i.e., the smallest full subcategory containing the image of triv which is closed under finite colimits and retracts. Similarly, we define the category $\overline{\mathrm{Nil}}_\alpha(\mathcal{C})$ as the full perfect subcategory of $\overline{\mathrm{End}}_\alpha(\mathcal{C})$ generated by the image of $\overline{\mathrm{triv}}$.

We call a twisted endomorphism $(x, f) \in \mathrm{End}_\alpha(\mathcal{C})$ (resp. $(x, f) \in \overline{\mathrm{End}}_\alpha(\mathcal{C})$) *homotopy nilpotent* if there is some $n \geq 1$ such that $f^{(n)} \simeq 0\colon x \to \alpha^n(x)$ (resp. $f^{(n)} \simeq 0\colon \alpha^n(x) \to x$) and write $\mathrm{Nil}_\alpha^{\mathrm{htpy}}(\mathcal{C}) \subseteq \mathrm{End}_\alpha(\mathcal{C})$ and $\overline{\mathrm{Nil}}_\alpha^{\mathrm{htpy}}(\mathcal{C}) \subseteq \overline{\mathrm{End}}_\alpha(\mathcal{C})$ for the full subcategories of homotopy nilpotent endomorphisms.

Let us show that in nilpotent endomorphism and homotopy nilpotent endomorphisms agree.

Theorem 2.2.2. *Consider a perfect category $\mathcal{C}$ with an exact endofunctor $\alpha\colon \mathcal{C} \to \mathcal{C}$. The following categories agree:*

(1) $\overline{\mathrm{Nil}}_\alpha(\mathcal{C})$*;*

(2) $\overline{\mathrm{Nil}}_\alpha^{\mathrm{htpy}}(\mathcal{C})$*;*

(3) the kernel of the composite

$$\overline{\mathrm{End}}_\alpha(\mathrm{Ind}(\mathcal{C}))^\omega \simeq \mathrm{End}_{\alpha^R}(\mathrm{Ind}(\mathcal{C}))^\omega \xrightarrow{\mathrm{loc}_{\alpha^R}} \mathrm{Aut}_{\alpha^R}(\mathrm{Ind}(\mathcal{C}))^\omega.$$

Before coming to the proof, let us remark that the analogous statement for $\mathrm{Nil}_\alpha(\mathcal{C})$ is not clear to us in general. If α admits a left adjoint $\alpha^L\colon \mathcal{C} \to \mathcal{C}$, we can use the equivalence $\overline{\mathrm{End}}_{\alpha^L}(\mathcal{C}) \simeq \mathrm{End}_\alpha(\mathcal{C})$ together with Theorem 2.2.2 to deduce $\mathrm{Nil}_\alpha(\mathcal{C}) \simeq \mathrm{Nil}_\alpha^{\mathrm{htpy}}(\mathcal{C})$. The problem is that the left adjoint to the forgetful functor $\mathrm{fgt}\colon \mathrm{End}_\alpha(\mathrm{Ind}(\mathcal{C})) \to \mathrm{Ind}(\mathcal{C})$ has no explicit description as in Lemma 2.1.6, unless $\mathrm{Ind}(\alpha)$ preserves limits.

Proof of Theorem 2.2.2. First note that as $\alpha\colon \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C})$ preserves compact objects, its right adjoint α^R preserves colimits. In particular, the inclusion $\mathrm{Aut}_{\alpha^R}(\mathrm{Ind}(\mathcal{C})) \subseteq \mathrm{End}_{\alpha^R}(\mathrm{Ind}(\mathcal{C}))$ admits a left adjoint by Lemma 2.1.7.

For the inclusion (2) $\subseteq$ (3), consider $(x, f) \in \overline{\mathrm{End}}_\alpha(\mathcal{C})$ and $n \in \mathbb{N}$ such that $f^{(n)} \simeq 0$. Denote by $f^R\colon x \to \alpha^R(x)$ the adjoint morphism to f. Now note that

$f^{(n)} \simeq 0\colon \alpha^n(x) \to x$ if and only if $(f^R)^{(n)} \simeq 0\colon x \to (\alpha^R)^n(x)$. Using cofinality of $n\mathbb{N} \subset \mathbb{N}$ as posets and the formula for loc_{α^R} from Lemma 2.1.7, we obtain

$$\begin{aligned}\mathrm{loc}_{\alpha^R}(x, f^R) &= \mathrm{colim}\left((x, f^R) \xrightarrow{f^R} \alpha^R(x, f^R) \xrightarrow{\alpha^R f^R} \dots\right) \\ &\simeq \mathrm{colim}\left((x, f^R) \xrightarrow{(f^R)^{(n)}} (\alpha^R)^n(x, f^R) \xrightarrow{(\alpha^R)^n(f^R)^{(n)}} \dots\right) \simeq 0.\end{aligned}$$

For the inclusion (3) $\subseteq$ (1), we first claim that for $(x, f) \in \overline{\mathrm{End}}_\alpha(\mathrm{Ind}(\mathcal{C}))^\omega$ and $k \geq 1$, the object $\mathrm{fib}([k]\colon \alpha^k(x, f) \to (x, f)) \in \overline{\mathrm{End}}_\alpha(\mathrm{Ind}(\mathcal{C}))$ lies in $\overline{\mathrm{Nil}}_\alpha(\mathcal{C})$. For $k = 1$, recall from Lemma 2.1.8 that $\overline{\mathrm{End}}_\alpha(\mathrm{Ind}(\mathcal{C}))^\omega$ is generated by the elements $\overline{\mathrm{free}}_\alpha(z)$ for $z \in \mathcal{C}$. As the claim is stable under retracts, shifts and fiber sequences, it suffices to consider (x, f) of this form. By the explicit formula $\overline{\mathrm{free}}_\alpha(z) = (\bigoplus_{n \geq 0} \alpha^n(z), \mathrm{shift})$ from Lemma 2.1.6, the endomorphism shift identifies with the inclusion $\bigoplus_{n \geq 1} \alpha^n(z) \to \bigoplus_{n \geq 0} \alpha^n(z)$. From this one obtains

$$\mathrm{fib}([1]\colon \alpha\overline{\mathrm{free}}_\alpha(z) \to \overline{\mathrm{free}}_\alpha(z)) \simeq \overline{\mathrm{triv}}(\Omega z) \in \overline{\mathrm{Nil}}_\alpha(\mathcal{C}).$$

The case of general k follows by induction using the fiber sequence

$$\mathrm{fib}([k-1]) \to \mathrm{fib}([k]) \to \mathrm{fib}(\alpha^{k-1}[1])$$

coming from $[k] \simeq \alpha^{k-1}[1] \circ [k-1]$.

We can now show (3) $\subseteq$ (1). Consider $(x, f) \in \overline{\mathrm{End}}_\alpha(\mathrm{Ind}(\mathcal{C}))^\omega$ such that (x, f^R) lies in the kernel of loc_{α^R}. By compactness of (x, f^R) and the formula for loc_{α^R} from Lemma 2.1.7 we obtain

$$\begin{aligned}0 &\simeq \mathrm{Map}_{\mathrm{Aut}_{\alpha^R}(\mathrm{Ind}(\mathcal{C}))}(\mathrm{loc}_{\alpha^R}(x, f^R), \mathrm{loc}_{\alpha^R}(x, f^R)) \\ &\simeq \mathrm{Map}_{\mathrm{End}_{\alpha^R}(\mathrm{Ind}(\mathcal{C}))}((x, f^R), \mathrm{loc}_{\alpha^R}(x, f^R)) \\ &\simeq \mathrm{colim}\Big(\mathrm{Map}_{\mathrm{End}_{\alpha^R}(\mathrm{Ind}(\mathcal{C}))}((x, f^R), (x, f^R)) \xrightarrow{[1]} \\ &\qquad\qquad \mathrm{Map}_{\mathrm{End}_{\alpha^R}(\mathrm{Ind}(\mathcal{C}))}((x, f^R), \alpha^R(x, f^R)) \xrightarrow{\alpha^R[1]} \dots\Big).\end{aligned}$$

In particular, $\mathrm{id}_{(x, f^R)}$ vanishes in a finite stage of the colimit. This shows that for large n we have $0 \simeq [n]\colon (x, f^R) \to (\alpha^R)^n(x, f^R)$, which is equivalent to $0 \simeq [n]\colon \alpha^n(x, f) \to (x, f)$. Its fiber $\Omega(x, f) \oplus \alpha^n(x, f)$ contains a shift of (x, f) as a retract and is contained in $\overline{\mathrm{Nil}}_\alpha(\mathcal{C})$ by the special case above. Thus, we obtain $(x, f) \in \overline{\mathrm{Nil}}_\alpha(\mathcal{C})$.

This also proves (3) $\subseteq$ (2): We just saw that for (x, f) in the kernel in (3) and large n we have $0 \simeq [n]\colon \alpha^n(x, f) \to (x, f)$, which reduces to $0 \simeq f^{(n)}\colon \alpha^n(x) \to x$

on underlying objects. It remains to argue that $x \in \mathcal{C}$. But this follows from the iinclusion $(3) \subseteq (1)$, which implies $(x, f) \in \overline{\mathrm{Nil}}_\alpha(\mathcal{C}) \subseteq \overline{\mathrm{End}}_\alpha(\mathcal{C})$.

Finally, for $(1) \subseteq (3)$, note that the category described in (3) contains the image of $\overline{\mathrm{triv}}$ by $(2) \subseteq (3)$. The kernel is a perfect subcategory of $\overline{\mathrm{End}}_\alpha(\mathrm{Ind}(\mathcal{C}))^\omega$ and is contained in $\overline{\mathrm{End}}_\alpha(\mathcal{C}) \subseteq \overline{\mathrm{End}}_\alpha(\mathrm{Ind}(\mathcal{C}))^\omega$ by $(3) \subseteq (2)$. Thus, it also contains $\overline{\mathrm{Nil}}_\alpha(\mathcal{C})$. $\square$

Chapter 3

The splitting results

In this section we will prove our main results Theorems K and L. Before doing so, let us recall some results on the K-theory of pushouts.

3.1 Recollection on K-theory of pushouts

In general, K-theory (or more general localising invariants) do not preserve pushouts of stable categories. This defect is studied in Land-Tamme's article [LT23, Section 3]. We summarise the results essential for our case, another exposition can also be found in [BL23, Section 4]. Let us begin by recalling the notion of a localising invariant.

Recollection 3.1.1 (Localising invariants)**.** A sequence $\mathcal{A} \xrightarrow{i} \mathcal{B} \xrightarrow{p} \mathcal{C}$ in $\mathrm{Cat}^{\mathrm{perf}}$ is called a *bifiber sequence* if it is both a fiber and cofiber sequence[1]. Equivalently, i is fully faithful and the induced map $\mathcal{B}/\mathcal{A} \to \mathcal{C}$ from the Verdier quotient becomes an equivalence after idempotent completion. A detailed discussion of bifiber sequences can be found in [CDH+23b, Appendix A]. For us, a *localising invariant* (sometimes also called Karoubi localising invariant) is a functor $E\colon \mathrm{Cat}^{\mathrm{perf}} \to \mathcal{E}$ with values in a stable category $\mathcal{E}$ sending bifiber sequences in $\mathrm{Cat}^{\mathrm{perf}}$ to fiber sequences in $\mathcal{E}$. Applying this to the bifiber sequence $0 \to 0 \to 0$, we see that localising invariants satisfy $E(0) \simeq 0$. Equivalent characteristations of this notion can be found in [HLS23]. Examples of localising invariants include nonconnective algebraic K-theory, topological Hochschild homology and topological cyclic homology, see [BGT13] for a proof.

Essential for Land-Tamme's construction is the concept of partially lax pullbacks as studied studied in [Tam18]. Let us recall their definition.

[1] This really is a property of such a sequence as $0 \in \mathrm{Cat}^{\mathrm{perf}}$ is a zero object.

Construction 3.1.2 (Partially lax pullback)**.** For a cospan $\mathcal{B} \xrightarrow{f} \mathcal{D} \xleftarrow{g} \mathcal{C}$ in Cat, the *partially lax pullbacks* $\mathcal{B} \overrightarrow{\times} \mathcal{C}$ is the category with objects given by triples (b, c, r) of objects $b \in \mathcal{B}$, $c \in \mathcal{C}$ and a morphism $r\colon f(b) \to g(c)$ in $\mathcal{D}$. Formally, it can be defined as the pullback

$$\begin{tikzcd} \mathcal{B} \overrightarrow{\times} \mathcal{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{D}^{\Delta^1} \arrow[d] \\ \mathcal{B} \times \mathcal{C} \arrow[r, "(f,g)"] & \mathcal{D} \times \mathcal{D}. \end{tikzcd}$$

Mapping spaces in $\mathcal{B} \overrightarrow{\times} \mathcal{C}$ are given by the pullback

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{B} \overrightarrow{\times} \mathcal{C}}((b,c,r),(b',c',r')) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}_{\mathcal{C}}(c,c') \arrow[d, "r^* \circ g"] \\ \mathrm{Map}_{\mathcal{B}}(b,b') \arrow[r, "r'_* \circ f"] & \mathrm{Map}_{\mathcal{D}}(f(b), g(c')). \end{tikzcd} \tag{3.1}$$

Note that the pullback $\mathcal{B} \times_{\mathcal{D}} \mathcal{C} \subseteq \mathcal{B} \overrightarrow{\times} \mathcal{C}$ identifies with the full subcategory on objects (b, c, r) for which $r\colon f(b) \to g(c)$ is an equivalence.

Observation 3.1.3. If $\mathcal{B} \xrightarrow{f} \mathcal{D} \xleftarrow{g} \mathcal{C}$ is a diagram in $\mathrm{Cat}^{\mathrm{perf}}$, then the partially lax pullback $\mathcal{B} \overrightarrow{\times} \mathcal{C}$ is again a perfect category. The pullback (3.1) then also is a pullback of mapping spectra. One essential property of the partially lax pullback is that $\mathcal{B}$ and $\mathcal{C}$, included via the maps $j_1\colon \mathcal{B} \to \mathcal{B} \overrightarrow{\times} \mathcal{C}, b \mapsto (b, 0, 0)$ and $j_2\colon \mathcal{C} \to \mathcal{B} \overrightarrow{\times} \mathcal{C}, c \mapsto (0, c, 0)$, form a semiorthogonal decomposition of $\mathcal{B} \overrightarrow{\times} \mathcal{C}$. In partiuclar, the projection $\mathcal{B} \overrightarrow{\times} \mathcal{C} \to \mathcal{B} \times \mathcal{C}$ becomes an equivalence after applying any localizing invariant.

Now consider a span $\mathcal{B} \xleftarrow{b} \mathcal{A} \xrightarrow{c} \mathcal{C}$ in $\mathrm{Cat}^{\mathrm{perf}}$ and let E be a localising invariant. The natural map $E(\mathcal{B}) \amalg_{E(\mathcal{A})} E(\mathcal{C}) \to E(\mathcal{B} \amalg_{\mathcal{A}} \mathcal{C})$ is usually not an equivalence. This defect can be measured by the following construction.

Construction 3.1.4 ($\odot$-product)**.** Consider a lax commutative diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, "c"] \arrow[d, "b"'] & \mathcal{C} \arrow[d, "q"] \\ \mathcal{B} \arrow[r, "p"'] \arrow[ur, Rightarrow, "h"] & \mathcal{D} \end{tikzcd} \tag{3.2}$$

in $\mathrm{Cat}^{\mathrm{perf}}$, meaning that $h\colon pb \to qc$ is a natural transformation. The transformation h induces a map $\mathcal{A} \to \mathcal{B} \overrightarrow{\times} \mathcal{C}$ and define the $\odot$-product $\mathcal{B} \odot_{\mathcal{A}}^{\mathcal{D}} \mathcal{C}$ as the cofiber of this map in $\mathrm{Cat}^{\mathrm{perf}}$. The map $\mathcal{B} \overrightarrow{\times} \mathcal{C} \to \mathcal{B} \odot_{\mathcal{A}}^{\mathcal{D}} \mathcal{C}$ is a Verdier localisation (see e.g. [NS18, Theorem I.3.3] for an explanation of this notion) and we denote by $\mathrm{Im}(\mathcal{A})$ its fiber.

Equivalently, it is the full perfect subcategory of $\mathcal{B}\overrightarrow{\times}\mathcal{C}$ generated by the image of $\mathcal{A}$. By the definition of localising invariants, we obtain the pushout diagram

$$\begin{tikzcd} E(\mathrm{Im}(\mathcal{A})) \arrow[r] \arrow[d] & E(\mathcal{C}) \arrow[d] \\ E(\mathcal{B}) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & E(\mathcal{B}\odot^{\mathcal{D}}_{\mathcal{A}}\mathcal{C}). \end{tikzcd} \tag{3.3}$$

To work with this construction, one has to understand the categories $\mathrm{Im}(\mathcal{A})$ and $\mathcal{B}\odot^{\mathcal{D}}_{\mathcal{A}}\mathcal{C}$.

Construction 3.1.5 ($\odot$-product associated to a pushout)**.** To actually come back to our original problem about the K-theory of pushouts, consider a span $\mathcal{B}\xleftarrow{b}\mathcal{A}\xrightarrow{c}\mathcal{C}$ in $\mathrm{Cat}^{\mathrm{perf}}$. Associated to it, we can construct a diagram of the shape (3.2) as follows

$$\begin{tikzcd} \mathcal{A} \arrow[r, "c"] \arrow[d, "b"'] & \mathcal{C} \arrow[d, "bc^R"] \\ \mathcal{B} \arrow[r] \arrow[ur, Rightarrow, "b\eta_c"] & \mathrm{Ind}(\mathcal{B}), \end{tikzcd} \tag{3.4}$$

where $\mathcal{B}\to\mathrm{Ind}(\mathcal{B})$ is the Yoneda embedding, $c^R\colon \mathrm{Ind}(\mathcal{C})\to\mathrm{Ind}(\mathcal{A})$ is a right adjoint to $c\colon \mathrm{Ind}(\mathcal{A})\to\mathrm{Ind}(\mathcal{C})$ and η_c denotes the unit of this adjunction.

The main technical theorem in [LT23] shows that the $\odot$-product is a pushout in this situation.

Theorem 3.1.6 ([LT23, Theorem 3.2])**.** *In the situation of Construction 3.1.5, the square*

$$\begin{tikzcd} \mathcal{A} \arrow[r, "c"] \arrow[d, "b"'] & \mathcal{C} \arrow[d, "j_2"] \\ \mathcal{B} \arrow[r, "\Omega j_1"] & \mathcal{B}\odot^{\mathrm{Ind}(\mathcal{B})}_{\mathcal{A}}\mathcal{C} \end{tikzcd}$$

is a pushout in $\mathrm{Cat}^{\mathrm{perf}}$*.*

Let us use this to give a description of the assembly map.

Observation 3.1.7. Consider the diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, "{(b,c)}"] \arrow[d] & \mathcal{B}\times\mathcal{C} \arrow[r, "{(\Omega p,q)}"] \arrow[d, "{(j_1,j_2)}"] & \mathcal{B}\coprod_{\mathcal{A}}\mathcal{C} \arrow[d, "\simeq"] \\ \mathrm{Im}(\mathcal{A}) \arrow[r] & \mathcal{B}\overrightarrow{\times}\mathcal{C} \arrow[r] & \mathcal{B}\odot^{\mathrm{Ind}(\mathcal{B})}_{\mathcal{A}}\mathcal{C}, \end{tikzcd}$$

where $p\colon \mathcal{B} \to \mathcal{B} \coprod_{\mathcal{A}} \mathcal{C}$ and $q\colon \mathcal{C} \to \mathcal{B} \coprod_{\mathcal{A}} \mathcal{C}$ are the structure maps and the bottom sequence is the Verdier sequence defining $\mathcal{B} \odot_{\mathcal{A}}^{\mathrm{Ind}(\mathcal{B})} \mathcal{C}$. The right square commutes by construction. The left square does not commute, but it commutes if we replace (j_1, j_2) by the projection $\mathcal{B} \overrightarrow{\times} \mathcal{C} \to \mathcal{B} \times \mathcal{C}$. The right vertical map is the equivalence from Theorem 3.1.6. Applying E turns the middle vertical map into an equivalence with inverse $\mathcal{B} \overrightarrow{\times} \mathcal{C} \to \mathcal{B} \times \mathcal{C}$ and the lower sequence into a fiber sequence. Together, this shows that the assembly map $E(\mathcal{B}) \coprod_{E(\mathcal{A})} E(\mathcal{C}) \to E(\mathcal{B} \coprod_{\mathcal{A}} \mathcal{C})$ can be identified with the map

$$E(\mathcal{B}) \coprod_{E(\mathcal{A})} E(\mathcal{C}) \to E(\mathcal{B}) \coprod_{E(\mathrm{Im}(\mathcal{A}))} E(\mathcal{C}),$$

and this identification is natural in the span $\mathcal{B} \leftarrow \mathcal{A} \to \mathcal{C}$.

3.2 K-theory of twisted automorphisms

In this subsection we will prove the main result of this article, Theorem K, about the splitting of $E(\mathcal{C}_{h\mathbb{N}})$ for an exact endofunctor $\alpha\colon \mathcal{C} \to \mathcal{C}$ of a perfect category and a localising invariant E. By Recollection 2.1.9, we have the pushout

$$\begin{tikzcd}
\mathcal{C} \oplus \mathcal{C} \arrow[r, "\mathrm{id}\oplus\mathrm{id}"] \arrow[d, "\mathrm{id}\oplus\alpha"] \arrow[dr, phantom, "\ulcorner" near end] & \mathcal{C} \arrow[d] \\
\mathcal{C} \arrow[r] & \mathcal{C}_{h\mathbb{N}}
\end{tikzcd}$$

in $\mathrm{Cat}^{\mathrm{perf}}$. Applying Land-Tamme's theory of K-theory of pushouts, more precisely Constructions 3.1.4 and 3.1.5 and Theorem 3.1.6, we obtain the commutative square

$$\begin{tikzcd}
\mathrm{Im}(\mathcal{C} \oplus \mathcal{C}) \arrow[r, "\mathrm{id}\oplus\mathrm{id}"] \arrow[d, "\mathrm{id}\oplus\alpha"] & \mathcal{C} \arrow[d] \\
\mathcal{C} \arrow[r] & \mathcal{C}_{h\mathbb{N}}
\end{tikzcd} \tag{3.5}$$

which becomes a pushout after applying any localising invariant. For the proof of Theorem 3.2.5 it remains to understand $E(\mathrm{Im}(\mathcal{C} \oplus \mathcal{C}))$. We will show that it admits a semiorthogonal decomposition by the categories $\mathrm{Nil}_\alpha(\mathcal{C})$ and $\overline{\mathrm{Nil}}_\alpha(\mathcal{C})$ from Definition 2.2.1.

Note that in the given situation, the right adjoint to $\mathrm{id} \oplus \mathrm{id}\colon \mathcal{C} \oplus \mathcal{C} \to \mathcal{C}$ already exists before Ind-completion and is given by the diagonal functor $\Delta\colon \mathcal{C} \to \mathcal{C} \oplus \mathcal{C}$. The corresponding square in (3.4) is thus given by

$$\begin{tikzcd}
\mathcal{C} \oplus \mathcal{C} \arrow[r, "\mathrm{id}\oplus\mathrm{id}"] \arrow[d, "\mathrm{id}\oplus\alpha"'] & \mathcal{C} \arrow[d, "\mathrm{id}+\alpha"] \\
\mathcal{C} \arrow[r, "\mathrm{id}"'] \arrow[ur, Rightarrow, "h"] & \mathcal{C},
\end{tikzcd}$$

where the homotopy h at an object $(x,y) \in \mathcal{C} \oplus \mathcal{C}$ is the transformation

$$h = (\mathrm{id}_x, 0, 0, \mathrm{id}_{\alpha(y)})\colon x \oplus \alpha(y) \to x \oplus y \oplus \alpha(x) \oplus \alpha(y).$$

The partially lax pullback $\mathcal{C} \overrightarrow{\times} \mathcal{C}$ then has objects given by triples (x,y,r) consisting of objects $x,y \in \mathcal{C}$ and a map $r\colon x \to y \oplus \alpha(y)$. As a reference for later let us record the following pullback for mapping spectra in $\mathcal{C} \overrightarrow{\times} \mathcal{C}$, specialising (3.1):

$$\begin{tikzcd}
\mathrm{map}_{\mathcal{C} \overrightarrow{\times} \mathcal{C}}((x,y,r),(x',y',r')) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{map}_{\mathcal{C}}(y,y') \arrow[d, "r^*(\mathrm{id}\oplus\alpha)"] \\
\mathrm{map}_{\mathcal{C}}(x,x') \arrow[r, "r'_*"] & \mathrm{map}_{\mathcal{C}}(x, y' \oplus \alpha(y'))
\end{tikzcd} \tag{3.6}$$

The induced map $(i_1, i_2)\colon \mathcal{C} \oplus \mathcal{C} \to \mathcal{C} \overrightarrow{\times} \mathcal{C}$ is given by

$$\begin{aligned} i_1(x) &= (x, x, (\mathrm{id}, 0)\colon x \to x \oplus \alpha(x)) \quad \text{and} \\ i_2(y) &= (\alpha(y), y, (0, \mathrm{id})\colon \alpha(y) \to y \oplus \alpha(y)). \end{aligned} \tag{3.7}$$

There are also maps

$$\begin{aligned} j_1&\colon \mathrm{End}_\alpha(\mathcal{C}) \to \mathcal{C} \overrightarrow{\times} \mathcal{C}, \quad (x, f\colon x \to \alpha(x)) \mapsto (x, x, (\mathrm{id}, f)\colon x \to x \oplus \alpha(x)), \\ j_2&\colon \overline{\mathrm{End}}_\alpha(\mathcal{C}) \to \mathcal{C} \overrightarrow{\times} \mathcal{C}, \quad (y, g\colon \alpha(y) \to y) \mapsto (\alpha(y), y, (g, \mathrm{id})\colon \alpha(y) \to y \oplus \alpha(y)). \end{aligned}$$

Lemma 3.2.1. *The functors j_1 and j_2 are full inclusions which identify $\mathrm{End}_\alpha(\mathcal{C})$ (and $\overline{\mathrm{End}}_\alpha(\mathcal{C})$) with the full subcategory of $\mathcal{C} \overrightarrow{\times} \mathcal{C}$ on objects (x,y,r) for which the first (resp. second) component of $r\colon x \to y \oplus \alpha(y)$ is an equivalence.*

Proof. We only prove the statement about j_1, with the other case being analogous. The pullback square in (3.6),

$$\begin{tikzcd}
\mathrm{map}_{\mathcal{C} \overrightarrow{\times} \mathcal{C}}(j_1(x,f), j_1(x',f')) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{map}_{\mathcal{C}}(x,x') \arrow[d, "(\mathrm{id}, f'_*)"] \\
\mathrm{map}_{\mathcal{C}}(x,x') \arrow[r, "(\mathrm{id}, f^*\alpha)"] & \mathrm{map}_{\mathcal{C}}(x,x') \times \mathrm{map}_{\mathcal{C}}(x, \alpha(x')),
\end{tikzcd}$$

identifies with

$$\mathrm{eq}(f^*\alpha, f'_*\colon \mathrm{map}(x,x') \to \mathrm{map}(x, \alpha(x'))) \simeq \mathrm{map}_{\mathrm{End}_\alpha(\mathcal{C})}((x,f),(x',f')).$$

For the description of the image, note that if $(x,y,r) \in \mathcal{C} \overrightarrow{\times} \mathcal{C}$ such that $r_1\colon x \to y$ is an equivalence, then $(x,y,r) \simeq j_1(x, \alpha(r_1)^{-1} r_2\colon x \to \alpha(x))$. $\square$

Let us briefly recall the notion of a semiorthogonal decomposition.

Recollection 3.2.2 (Semiorthogonal decomposition)**.** A *semiorthogonal decomposition* of a perfect category $\mathcal{D}$ consists of two full perfect subcategories $\mathcal{D}_0, \mathcal{D}_1 \subseteq \mathcal{D}$ such that $\mathcal{D}_0 \cup \mathcal{D}_1$ generates $\mathcal{D}$ as a perfect category and $\mathrm{Map}_{\mathcal{D}}(x_1, x_0) \simeq 0$ for all $x_i \in \mathcal{D}_i$. In this situation, the inclusion $\mathcal{D}_1 \subseteq \mathcal{D}$ admits a right adjoint p_1 and the inclusion $\mathcal{D}_0 \subseteq \mathcal{D}$ admits a left adjoint p_0. Importantly, the sequence $\mathcal{D}_0 \hookrightarrow \mathcal{D} \xrightarrow{p_1} \mathcal{D}_1$ is a Karoubi sequence. As $\mathcal{D}_0 \hookrightarrow \mathcal{D}$ admits the retraction p_0, we even see that for any localising invariant E the map $E(\mathcal{D}_0) \oplus E(\mathcal{D}_1) \to E(\mathcal{D})$ induced by the inclusions is an equivalence. More information on semiorthogonal decompositions can be found in [Lur18, Section II.7.2] or [Lur17, Section A.8].

Lemma 3.2.3. *The subcategories* $\mathrm{Nil}_\alpha(\mathcal{C})$ *and* $\overline{\mathrm{Nil}}_\alpha(\mathcal{C})$ *(included via j_1 and j_2) form a semiorthogonal decomposition of* $\mathrm{Im}(\mathcal{C} \oplus \mathcal{C})$*.*

Proof. By definition, $\mathrm{Im}(\mathcal{C} \oplus \mathcal{C})$ is the perfect subcategory of $\mathcal{C} \overrightarrow{\times} \mathcal{C}$ generated by the images of the functors $i_1, i_2 \colon \mathcal{C} \to \mathcal{C} \overrightarrow{\times} \mathcal{C}$ from (3.7). Notice that i_1 factors as the composite $\mathcal{C} \xrightarrow{\mathrm{triv}} \mathrm{Nil}_\alpha(\mathcal{C}) \overset{j_1}{\hookrightarrow} \mathcal{C} \overrightarrow{\times} \mathcal{C}$. Similarly, i_2 factors through $\overline{\mathrm{Nil}}_\alpha(\mathcal{C})$. Thus, $\mathrm{Nil}_\alpha(\mathcal{C})$ and $\overline{\mathrm{Nil}}_\alpha(\mathcal{C})$ generate $\mathrm{Im}(\mathcal{C} \oplus \mathcal{C})$.

To prove orthogonality, note that vanishing of the mapping spectrum $\mathrm{map}_{\mathcal{D}}(x_0, x_1)$ is stable under finite colimits, shifts and retracts in both variables, so it suffices to check it for the generating sets given by the image of i_1 and i_2. The pullback (3.1) for mapping spectra specialises for objects $x, y \in \mathcal{C}$ to the pullback

$$\begin{tikzcd}
\mathrm{map}_{\mathcal{C} \overrightarrow{\times} \mathcal{C}}(i_1(x), i_2(y)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{map}_{\mathcal{C}}(x, y) \arrow[d, "(\mathrm{id}_y, 0)"] \\
\mathrm{map}_{\mathcal{C}}(x, \alpha(y)) \arrow[r, "(0, \mathrm{id}_{\alpha(y)})"] & \mathrm{map}_{\mathcal{C}}(x, y \oplus \alpha(y)).
\end{tikzcd} \tag{3.8}$$

But as the pullback of the cospan $y \to y \oplus \alpha(y) \leftarrow \alpha(y)$ is trivial, this shows that $\mathrm{map}_{\mathcal{C} \overrightarrow{\times} \mathcal{C}}(i_1(x), i_2(y)) \simeq 0$. $\square$

One can check that $\mathrm{Nil}_\alpha(\mathcal{C})$ and $\overline{\mathrm{Nil}}_\alpha(\mathcal{C})$ even form an orthogonal decomposition of $\mathrm{Im}(\mathcal{C} \oplus \mathcal{C})$ if α is fully faithful, but we will not use this later. Let us come back to determining $E(\mathcal{C}_{h\mathbb{N}})$.

Definition 3.2.4. We define the Nil-terms by

$$\begin{aligned}
NE_\alpha(\mathcal{C}) &:= \Sigma\, \mathrm{cofib}(\mathrm{triv} \colon E(\mathcal{C}) \to E(\mathrm{Nil}_\alpha(\mathcal{C}))) \quad \text{and} \\
\overline{NE}_\alpha(\mathcal{C}) &:= \Sigma\, \mathrm{cofib}(\overline{\mathrm{triv}} \colon E(\mathcal{C}) \to E(\overline{\mathrm{Nil}}_\alpha(\mathcal{C}))).
\end{aligned}$$

Note that the functor $\mathrm{triv} \colon \mathcal{C} \to \mathrm{Nil}_\alpha(\mathcal{C})$ admits a retraction given by the forgetful functor $\mathrm{fgt} \colon \mathrm{Nil}_\alpha(\mathcal{C}) \to \mathcal{C}$. This gives us the split fiber sequence

$$\begin{tikzcd}
E(\mathcal{C}) \arrow[r, "\mathrm{triv}"'] & E(\mathrm{Nil}_\alpha(\mathcal{C})) \arrow[l, bend right, "\mathrm{fgt}"'] \arrow[r] & \Omega NE_\alpha(\mathcal{C}).
\end{tikzcd} \tag{3.9}$$

Similarly, the functor $\overline{\mathrm{triv}}\colon \mathcal{C} \to \overline{\mathrm{Nil}}_\alpha(\mathcal{C})$ admits a retraction $\mathrm{fgt}\colon \overline{\mathrm{Nil}}_\alpha(\mathcal{C}) \to \mathcal{C}$ which induces a splitting $E(\overline{\mathrm{Nil}}_\alpha(\mathcal{C})) \simeq E(\mathcal{C}) \oplus \Omega \overline{NE}_\alpha(\mathcal{C})$.

Theorem 3.2.5. *The assembly map $E(\mathcal{C})_{h\mathbb{N}} \to E(\mathcal{C}_{h\mathbb{N}})$ admits a splitting, natural in $\alpha \in \mathrm{Fun}(B\mathbb{N}, \mathrm{Cat}^{\mathrm{perf}})$, inducing an equivalence*

$$E(\mathcal{C}_{h\mathbb{N}}) \simeq E(\mathcal{C})_{h\mathbb{N}} \oplus NE_\alpha(\mathcal{C}) \oplus \overline{NE}_\alpha(\mathcal{C}).$$

Proof. Consider the commutative diagram

$$\begin{tikzcd}
E(\mathcal{C}) \arrow[d, equal] & E(\mathcal{C} \oplus \mathcal{C}) \arrow[l, "\mathrm{id}\oplus\alpha"'] \arrow[r, "\mathrm{id}\oplus\mathrm{id}"] \arrow[d] & E(\mathcal{C}) \arrow[d, equal] \\
E(\mathcal{C}) & E(\mathrm{Im}(\mathcal{C} \oplus \mathcal{C})) \arrow[l, "\mathrm{id}\oplus\alpha"'] \arrow[r, "\mathrm{id}\oplus\mathrm{id}"] & E(\mathcal{C}).
\end{tikzcd} \tag{3.10}$$

By Observation 3.1.7, the assembly map in the statement can naturally be identified with the map obtained by passing to horizontal pushouts. We have to show that it splits and identify its cofiber. The inclusions induce an equivalence $E(\mathrm{Nil}_\alpha(\mathcal{C})) \oplus E(\overline{\mathrm{Nil}}_\alpha(\mathcal{C})) \xrightarrow{\simeq} E(\mathrm{Im}(\mathcal{C} \oplus \mathcal{C}))$ by the semiorthogonal decomposition in Lemma 3.2.3, which also identifies the map $E(\mathcal{C} \oplus \mathcal{C}) \to E(\mathrm{Im}(\mathcal{C} \oplus \mathcal{C}))$ with the map $\mathrm{triv} \oplus \overline{\mathrm{triv}}\colon E(\mathcal{C}) \oplus E(\mathcal{C}) \to E(\mathrm{Nil}_\alpha(\mathcal{C})) \oplus E(\overline{\mathrm{Nil}}_\alpha(\mathcal{C}))$. Also note that the composite $\mathrm{Nil}_\alpha(\mathcal{C}) \xrightarrow{j_1} \mathcal{C} \overrightarrow{\times} \mathcal{C} \xrightarrow{\mathrm{pr}_i} \mathcal{C}$ is given by fgt for $i = 1, 2$ and the composite $\overline{\mathrm{Nil}}_\alpha(\mathcal{C}) \xrightarrow{j_2} \mathcal{C} \overrightarrow{\times} \mathcal{C} \xrightarrow{\mathrm{pr}_i} \mathcal{C}$ is given by α fgt for $i = 1$ and fgt for $i = 2$. Together with the splitting (3.9), this identifies (3.10) with

$$\begin{tikzcd}
E(\mathcal{C}) \arrow[d, equal] & E(\mathcal{C}) \oplus E(\mathcal{C}) \arrow[l, "\mathrm{id}\oplus\alpha"'] \arrow[r, "\mathrm{id}\oplus\mathrm{id}"] \arrow[d] & E(\mathcal{C}) \arrow[d, equal] \\
E(\mathcal{C}) & E(\mathcal{C}) \oplus E(\mathcal{C}) \oplus \Omega NE_\alpha(\mathcal{C}) \oplus \Omega \overline{NE}_\alpha(\mathcal{C}) \arrow[l, "{(\mathrm{id},\alpha,0,0)}"'] \arrow[r, "{(\mathrm{id},\mathrm{id},0,0)}"] & E(\mathcal{C}),
\end{tikzcd}$$

where the middle vertical map is the inclusion of the first two summands. The upper span is a retract of the bottom span by mapping the $\Omega NE_\alpha(\mathcal{C}) \oplus \Omega \overline{NE}_\alpha(\mathcal{C})$-terms to zero. Passing to horizontal pushouts, we obtain the split fiber sequence

$$E(\mathcal{C})_{h\mathbb{N}} \to E(\mathcal{C}_{h\mathbb{N}}) \to NE_\alpha(\mathcal{C}) \oplus \overline{NE}_\alpha(\mathcal{C})$$

as claimed. For naturality of the splitting, one observes that all the appearing inclusions and retractions are natural in $\alpha \in \mathrm{Fun}(B\mathbb{N}, \mathrm{Cat}^{\mathrm{perf}})$. $\square$

3.3 K-theory of twisted endomorphisms

In this section we will explain how the NE-terms appearing in the splitting in Theorem 3.2.5, defined as a certain summand in $E(\mathrm{Nil}_\alpha(\mathcal{C}))$, are related to

$E(\overline{\mathrm{End}}_\alpha(\mathrm{Ind}(\mathcal{C}))^\omega)$. Land-Tamme showed a variant of this for the K-theory of tensor algebras in [LT23, Corollary 4.5, Proposition 4.7]. Their proof relies on the quite technical computation of certain endomorphism rings, using their work on theory of K-theory of pullbacks [LT19]. We give a more direct argument by working one categorical level higher with the category of modules. This also extends Theorem 3.3.2 to categories not generated by a single element.

For all of this section suppose that $\alpha\colon \mathcal{C} \to \mathcal{C}$ is an exact endofunctor of a perfect category and denote by $\alpha^R\colon \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C})$ the right adjoint to $\mathrm{Ind}(\alpha)$. By abuse of notation, we will often not distinguish between $\mathrm{Ind}(\alpha)$ and α. We start by showing that $E(\overline{\mathrm{End}}_\alpha(\mathrm{Ind}(\mathcal{C}))^\omega)$ splits into two summands.

Lemma 3.3.1. *The functor* $\overline{\mathrm{free}}_\alpha\colon \mathcal{C} \to \overline{\mathrm{End}}_\alpha(\mathrm{Ind}(\mathcal{C}))^\omega$ *admits a retraction by*

$$\mathrm{cofib}\colon \overline{\mathrm{End}}_\alpha(\mathrm{Ind}(\mathcal{C}))^\omega \to \mathcal{C}, \quad (x, f\colon \alpha(x) \to x) \mapsto \mathrm{cofib}(f).$$

Similarly, $\overline{\mathrm{free}}_\alpha\colon \mathcal{C} \to \overline{\mathrm{End}}_{\alpha^R}(\mathrm{Ind}(\mathcal{C}))^\omega$ *admits a retraction given by*

$$\mathrm{cofib}\colon \overline{\mathrm{End}}_\alpha(\mathrm{Ind}(\mathcal{C}))^\omega \to \mathcal{C}, \quad (x, f\colon \alpha^R(x) \to x) \mapsto \mathrm{cofib}(f).$$

Proof. Let us first check that the functor cofib, formally given by the composite

$$\overline{\mathrm{End}}_\alpha(\mathrm{Ind}(\mathcal{C})) \to \mathrm{Ind}(\mathcal{C})^{\Delta^1} \xrightarrow{\mathrm{cofib}} \mathrm{Ind}(\mathcal{C})$$

is a retraction of $\overline{\mathrm{free}}_\alpha\colon \mathrm{Ind}(\mathcal{C}) \to \overline{\mathrm{End}}_\alpha(\mathrm{Ind}(\mathcal{C}))$. Using the explicit formula for $\overline{\mathrm{free}}_\alpha$ from Lemma 2.1.6, we see that $\mathrm{cofib}\,\overline{\mathrm{free}}_\alpha$ is given by

$$\mathrm{cofib}\left(\bigoplus_{n\geq 1} \alpha^n \to \bigoplus_{n\geq 0} \alpha^n\right) \simeq \mathrm{id}_{\mathrm{Ind}(\mathcal{C})}.$$

It remains to argue that cofib preserves compact objects. As $\overline{\mathrm{End}}_\alpha(\mathrm{Ind}(\mathcal{C}))$ is generated by the compact objects $\overline{\mathrm{free}}_\alpha(x)$ for $x \in \mathcal{C}$ as a stable presentable category by Lemma 2.1.8, it suffices to check that $\mathrm{cofib}(\overline{\mathrm{free}}_\alpha(x)) \in \mathcal{C}$ for $x \in \mathcal{C}$. But this follows from the first part.

The argument showing that $\overline{\mathrm{free}}_\alpha\colon \mathcal{C} \to \overline{\mathrm{End}}_{\alpha^R}(\mathrm{Ind}(\mathcal{C}))^\omega$ admits a retraction by the same formula is analogous. One just needs to observe that α^R preserves colimits as its left adjoint $\mathrm{Ind}(\alpha)$ preserves compact objects. □

We can now state the main theorem of this section.

Theorem 3.3.2. *There is a split fiber sequence*

$$\begin{tikzcd} E(\mathcal{C}) \arrow[r, "\overline{\mathrm{free}}_\alpha"'] & E(\overline{\mathrm{End}}_\alpha(\mathrm{Ind}(\mathcal{C}))^\omega) \arrow[l, bend right, "\mathrm{cofib}"'] \arrow[r] & NE_\alpha(\mathcal{C}). \end{tikzcd}$$

Similarly, there is a split fiber sequence

$$\begin{tikzcd} E(\mathcal{C}) \arrow[r, "\mathrm{free}_{\alpha^R}"'] & E(\overline{\mathrm{End}}_{\alpha^R}(\mathrm{Ind}(\mathcal{C}))^{\omega}) \arrow[l, bend right, "\mathrm{cofib}"'] \arrow[r] & \overline{NE}_{\alpha}(\mathcal{C}). \end{tikzcd}$$

If α is an equivalence, in which case $\alpha^R = \alpha^{-1}$, the second splitting can be rephrased more symmetrical as the splitting

$$E(\mathrm{End}_{\alpha}(\mathrm{Ind}(\mathcal{C}))^{\omega}) \simeq E(\mathcal{C}) \oplus \overline{NE}_{\alpha}(\mathcal{C}).$$

We will only prove the splitting for $E(\overline{\mathrm{End}}_{\alpha}(\mathrm{Ind}(\mathcal{C}))^{\omega})$, the argument for the splitting of $E(\overline{\mathrm{End}}_{\alpha^R}(\mathrm{Ind}(\mathcal{C}))^{\omega})$ being analogous. Our approach is similar to the proof of Theorem 3.2.5 by first presenting $\overline{\mathrm{End}}_{\alpha}(\mathrm{Ind}(\mathcal{C}))^{\omega}$ as a pushout in $\mathrm{Cat}^{\mathrm{perf}}$ and then applying Land-Tamme's machinery from §3.1 to it. Let us begin by recalling the tensoring construction for perfect categories.

Recollection 3.3.3 (Tensoring of $\mathrm{Cat}^{\mathrm{perf}}$ over Cat)**.** Given a category I, the functor $\mathrm{Fun}(I, -)\colon \mathrm{Cat}^{\mathrm{perf}} \to \mathrm{Cat}^{\mathrm{perf}}$ admits a left adjoint $- \otimes I$, the tensoring of $\mathrm{Cat}^{\mathrm{perf}}$ over Cat. The tensor $\mathcal{C} \otimes I$ is the initial perfect category which comes together with a functor $\mathcal{C} \times I \to \mathcal{C} \otimes I$ that is exact in the first variable and has an explicit description given by $\mathcal{C} \otimes I = \mathrm{Fun}(I^{\mathrm{op}}, \mathrm{Ind}(\mathcal{C}))^{\omega}$. Under this identification, for an object $i \in I$ the inclusion $a_i\colon \mathcal{C} \times \{i\} \to \mathcal{C} \otimes I$ is left adjoint to evaluation $\mathrm{ev}_i\colon \mathrm{Fun}(I^{\mathrm{op}}, \mathrm{Ind}(\mathcal{C})) \to \mathrm{Ind}(\mathcal{C})$. The functors a_i are fully faithful, which follows from $\mathrm{ev}_i\, a_i \simeq \mathrm{id}$, and their images generate $\mathcal{C} \otimes I$ as a perfect category. More details on the non idempotent complete version of this construction can be found in [CDH+23a, Remark 6.4.2] or [Sau23, Section 2]. Without making this precise, let us just mention that $\mathcal{C} \otimes I$ is also an instance of a lax colimit in the 2-category $\mathrm{Cat}^{\mathrm{perf}}$.

We will only be interested in the case $\mathcal{C} \otimes \Delta^1 \simeq \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ind}(\mathcal{C}))^{\omega}$. The left adjoints to the evaluation functors are explicitly given by $a_0(x) = (x \leftarrow 0)$ and $a_1(x) = (x \xleftarrow{\mathrm{id}} x)$, where we use leftwards pointing arrows to indicate that we are working in $(\Delta^1)^{\mathrm{op}}$. We can now give a presentation of $\overline{\mathrm{End}}_{\alpha}(\mathrm{Ind}(\mathcal{C}))^{\omega}$ as a pushout.

Lemma 3.3.4. *There is a pushout square in* $\mathrm{Cat}^{\mathrm{perf}}$ *of the form*

$$\begin{tikzcd} \mathcal{C} \oplus \mathcal{C} \arrow[r, "a_0 \oplus a_1"] \arrow[d, "\alpha \oplus \mathrm{id}"'] & \mathcal{C} \otimes \Delta^1 \arrow[d] \\ \mathcal{C} \arrow[r, "\overline{\mathrm{free}}_{\alpha}"] \arrow[ur, phantom, "\ulcorner"] & \overline{\mathrm{End}}_{\alpha}(\mathrm{Ind}(\mathcal{C}))^{\omega}. \end{tikzcd} \tag{3.11}$$

Proof. To calculate the pushout of the upper left span in (3.11) one instead passes to right adjoints after Ind completion and forms the pullback. The upper left span then becomes the cospan

$$\mathrm{Ind}(\mathcal{C}) \xrightarrow{(\alpha^R,\mathrm{id})} \mathrm{Ind}(\mathcal{C}) \times \mathrm{Ind}(\mathcal{C}) \xleftarrow{(\mathrm{ev}_0,\mathrm{ev}_1)} \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ind}(\mathcal{C})).$$

After the identification $\Delta^1 \simeq (\Delta^1)^{\mathrm{op}}$ this becomes the usual pullback square from (2.1) defining the lax equaliser

$$\begin{tikzcd} \mathrm{End}_{\alpha^R}(\mathrm{Ind}(\mathcal{C})) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Ind}(\mathcal{C})^{\Delta^1} \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \\ \mathrm{Ind}(\mathcal{C}) \arrow[r, "(\mathrm{id},\alpha^R)"] & \mathrm{Ind}(\mathcal{C}) \times \mathrm{Ind}(\mathcal{C}). \end{tikzcd} \tag{3.12}$$

The pushout of (3.11) is obtained by taking compact objects in this pullback, which is $\mathrm{End}_{\alpha^R}(\mathrm{Ind}(\mathcal{C}))^{\omega} \simeq \overline{\mathrm{End}}_{\alpha}(\mathrm{Ind}(\mathcal{C}))^{\omega}$. The left vertical arrow in (3.12) is given by fgt, so its left adjoint $\overline{\mathrm{free}}_{\alpha}$ appears in the diagram in the statement. □

Applying Land-Tamme's theory of K-theory of pushouts, more precisely Constructions 3.1.4 and 3.1.5 and Theorem 3.1.6, we obtain the commutative square

$$\begin{tikzcd} \mathrm{Im}(\mathcal{C} \oplus \mathcal{C}) \arrow[r, "i_0 \oplus i_1"] \arrow[d, "\alpha\oplus\mathrm{id}"] & \mathcal{C} \otimes \Delta^1 \arrow[d] \\ \mathcal{C} \arrow[r] & \overline{\mathrm{End}}_{\alpha}(\mathrm{Ind}(\mathcal{C}))^{\omega} \end{tikzcd} \tag{3.13}$$

which becomes an equivalence after applying any localising invariant. We again use the notation $\mathrm{Im}(\mathcal{C} \oplus \mathcal{C})$ even though it is different from the category denoted by the same symbol in §3.2. As before, the main difficulty is the description of this category. The lax square in (3.4) associated to the pushout (3.11) is given by

$$\begin{tikzcd} \mathcal{C} \oplus \mathcal{C} \arrow[r, "a_0 \oplus a_1"] \arrow[d, "\alpha\oplus\mathrm{id}"'] & \mathcal{C} \otimes \Delta^1 \arrow[d, "\alpha\,\mathrm{ev}_0 \oplus \mathrm{ev}_1"] \\ \mathcal{C} \arrow[r, "\mathrm{id}"'] \arrow[ur, Rightarrow, "h"] & \mathcal{C}. \end{tikzcd}$$

Note here that the Ind-right adjoint $(\mathrm{ev}_0, \mathrm{ev}_1)\colon \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ind}(\mathcal{C})) \to \mathrm{Ind}(\mathcal{C}) \times \mathrm{Ind}(\mathcal{C})$ to $a_0 \oplus a_1$ preserves compact objects as it sends the generators $a_0(x)$ and $a_1(x)$ of $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ind}(\mathcal{C}))$ for $x \in \mathcal{C}$ to compact objects in $\mathrm{Ind}(\mathcal{C})$, so it already exists before Ind-completion. The transformation h is given on (x,y) by the map $(\mathrm{id}_{\alpha(x)}, 0, \mathrm{id}_y)\colon \alpha(x) \oplus y \to \alpha(x) \oplus \alpha(y) \oplus y$. The corresponding partially lax pullback $\mathcal{C} \overrightarrow{\times} (\mathcal{C} \otimes \Delta^1)$ then has objects $(x, y \xleftarrow{f} z, r\colon x \to \alpha(y) \oplus z)$ with $x \in \mathcal{C}$ and

$f \in \mathcal{C} \otimes \Delta^1$. As a reference for later let us record the following pullback for mapping spectra in $\mathcal{C} \overrightarrow{\times} \mathcal{C}$, specialising (3.1):

$$\begin{tikzcd}
\mathrm{map}_{\mathcal{C} \overrightarrow{\times} \mathcal{C}}((x,f,r),(x',f',r')) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{map}_{\mathcal{C}\otimes\Delta^1}(f,f') \arrow[d, "r^*(\alpha\,\mathrm{ev}_0 \oplus \mathrm{ev}_1)"] \\
\mathrm{map}_{\mathcal{C}}(x,x') \arrow[r, "r'_*"] & \mathrm{map}_{\mathcal{C}}(x,\alpha(y') \oplus z').
\end{tikzcd} \tag{3.14}$$

The induced functor $(i_1, i_2)\colon \mathcal{C} \oplus \mathcal{C} \to \mathcal{C} \overrightarrow{\times} (\mathcal{C} \otimes \Delta^1)$ is given by

$$\begin{aligned}
i_1(x) &= (\alpha(x), a_0(x), (\mathrm{id}, 0)\colon \alpha(x) \to \alpha(x) \oplus 0), \\
i_2(x) &= (x, a_1(x), (0, \mathrm{id})\colon x \to \alpha(x) \oplus x).
\end{aligned}$$

There is also a functor

$$j\colon \mathrm{End}_\alpha(\mathcal{C}) \to \mathcal{C} \overrightarrow{\times} (\mathcal{C} \otimes \Delta^1), \quad (x, f\colon x \to \alpha(x)) \mapsto (x, a_1(x), (f, \mathrm{id})\colon x \to \alpha(x) \oplus x).$$

Lemma 3.3.5. *The functors i_1 and j are fully faithful.*

Proof. From (3.14) we get for $x, y \in \mathcal{C}$ the pullback square

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C} \overrightarrow{\times} (\mathcal{C}\otimes\Delta^1)}(i_1(x), i_1(y)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Map}_{\mathcal{C}\otimes\Delta^1}(a_0(x), a_0(y)) \arrow[d, "\alpha\,\mathrm{ev}_0"] \\
\mathrm{Map}_{\mathcal{C}}(\alpha(x), \alpha(y)) \arrow[r, "\mathrm{id}"] & \mathrm{Map}_{\mathcal{C}}(\alpha(x), \alpha(y)).
\end{tikzcd}$$

Note that $\mathrm{ev}_0\colon \mathrm{Map}_{\mathcal{C}\otimes\Delta^1}(a_0(x), a_0(y)) \to \mathrm{Map}_{\mathcal{C}}(x,y)$ is an equivalence from which the claim about i_1 follows. Similarly, we have for $(x,f), (y,g) \in \mathrm{End}_\alpha(\mathcal{C})$ the pullback square

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C} \overrightarrow{\times} (\mathcal{C}\otimes\Delta^1)}(j(x,f), j(y,g)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Map}_{\mathcal{C}\otimes\Delta^1}(a_1(x), a_1(y)) \arrow[d, "(f^*\alpha\,\mathrm{ev}_0) + \mathrm{ev}_1"] \\
\mathrm{Map}_{\mathcal{C}}(x,y) \arrow[r, "g_* + \mathrm{id}"] & \mathrm{Map}_{\mathcal{C}}(x, \alpha(y) \oplus y).
\end{tikzcd}$$

As $\mathrm{ev}_0, \mathrm{ev}_1\colon \mathrm{Map}_{\mathcal{C}\otimes\Delta^1}(a_1(x), a_1(y)) \to \mathrm{Map}_{\mathcal{C}}(x,y)$ are equivalences, this pullback identifies with

$$\mathrm{eq}(f^*\alpha, g_*\colon \mathrm{Map}_{\mathcal{C}}(x,y) \to \mathrm{Map}_{\mathcal{C}}(x, \alpha(y))) \simeq \mathrm{Map}_{\mathrm{End}_\alpha(\mathcal{C})}((x,f),(y,g)).$$

□

Lemma 3.3.6. *The subcategories* $\mathrm{Nil}_\alpha(\mathcal{C})$ *and* $\mathcal{C}$ *(included via* j *and* i_1*) form a semiorthogonal decomposition of* $\mathrm{Im}(\mathcal{C} \oplus \mathcal{C})$.

Proof. It is clear that the perfect subcategory of $\mathcal{C} \overrightarrow{\times} (\mathcal{C} \otimes \Delta^1)$ generated by $\mathrm{Nil}_\alpha(\mathcal{C})$ and $\mathcal{C}$ contains $\mathrm{Im}(\mathcal{C} \oplus \mathcal{C})$ as $i_2 \simeq j \circ \mathrm{triv}$. But $\mathrm{Nil}_\alpha(\mathcal{C})$ is also contained in $\mathrm{Im}(\mathcal{C} \oplus \mathcal{C})$ as it is generated by $\mathcal{C}$. Together, this shows that $\mathrm{Nil}_\alpha(\mathcal{C})$ and $\mathcal{C}$ generate $\mathrm{Im}(\mathcal{C} \oplus \mathcal{C})$. For semiorthogonality, (3.14) gives us for $(x, f) \in \mathrm{End}_\alpha(\mathcal{C})$ and $y \in \mathcal{C}$ the pullback

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C} \overrightarrow{\times} (\mathcal{C} \otimes \Delta^1)}(j(x,f), i_1(y)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Map}_{\mathcal{C} \otimes \Delta^1}(a_1(x), a_0(y)) \arrow[d, "f_* \alpha\, \mathrm{ev}_0"] \\
\mathrm{Map}_{\mathcal{C}}(x, \alpha(y)) \arrow[r, "\mathrm{id}"] & \mathrm{Map}_{\mathcal{C}}(x, \alpha(y)).
\end{tikzcd}$$

We have $\mathrm{Map}_{\mathcal{C} \otimes \Delta^1}(a_1(x), a_0(y)) \simeq \mathrm{Map}_{\mathcal{C}}(x, 0) \simeq 0$, using that a_1 is left adjoint to ev_1, which proves the desired vanishing. $\square$

As a final step towards the identification of Nil-terms, we need a splitting for $E(\mathcal{C} \otimes \Delta^1)$.

Lemma 3.3.7. *The functors* $a_0, a_1 \colon \mathcal{C} \hookrightarrow \mathcal{C} \otimes \Delta^1$ *form a semiorthogonal decomposition of* $\mathcal{C} \otimes \Delta^1$.

Proof. Recall from Recollection 3.3.3 that a_0 and a_1 are fully faithful and their images generate $\mathcal{C} \otimes \Delta^1$ as a perfect category. Semiorthogonality follows from $\mathrm{Map}_{\mathcal{C} \otimes \Delta^1}(a_1(x), a_0(y)) \simeq \mathrm{Map}_{\mathcal{C}}(x, \mathrm{ev}_1\, a_0(y)) \simeq 0$. $\square$

Proof of Theorem 3.3.2. By Theorem 3.1.6, applying E to the commutative square (3.13) gives the pushout

$$\begin{tikzcd}
E(\mathrm{Im}(\mathcal{C} \oplus \mathcal{C})) \arrow[r, "i_0 \oplus i_1"] \arrow[d, "\alpha \oplus \mathrm{id}"] \arrow[dr, phantom, "\ulcorner"] & E(\mathcal{C} \otimes \Delta^1) \arrow[d] \\
E(\mathcal{C}) \arrow[r, "\overline{\mathrm{free}}_\alpha"] & E(\overline{\mathrm{End}}_\alpha(\mathrm{Ind}(\mathcal{C}))^\omega).
\end{tikzcd}$$

Using the semiorthogonal decompositions from Lemmas 3.3.6 and 3.3.7 together with the splitting $E(\mathrm{Nil}_\alpha(\mathcal{C})) \simeq E(\mathcal{C}) \oplus \Omega NE_\alpha(\mathcal{C})$ from (3.9), we arrive at the pushout square

$$\begin{tikzcd}
E(\mathcal{C}) \oplus \Omega NE_\alpha(\mathcal{C}) \oplus E(\mathcal{C}) \arrow[r, "(\mathrm{id},0) \oplus 0 \oplus (0,\mathrm{id})"] \arrow[d, "\mathrm{id} \oplus 0 \oplus \alpha"'] \arrow[dr, phantom, "\ulcorner"] & E(\mathcal{C}) \oplus E(\mathcal{C}) \arrow[d] \\
E(\mathcal{C}) \arrow[r, "\overline{\mathrm{free}}_\alpha"] & E(\overline{\mathrm{End}}_\alpha(\mathrm{Ind}(\mathcal{C}))^\omega).
\end{tikzcd} \tag{3.15}$$

Let us explain why, under this decomposition, the top horizontal and left vertical map are the indicated ones. For the upper horizontal one, this follows as the composite $\mathrm{Nil}_\alpha(\mathcal{C}) \xrightarrow{j} \mathcal{C} \overrightarrow{\times} (\mathcal{C} \otimes \Delta^1) \to \mathcal{C} \otimes \Delta^1$ is equivalent to a_1 fgt and the composite $\mathcal{C} \xrightarrow{i_1} \mathcal{C} \overrightarrow{\times} (\mathcal{C} \otimes \Delta^1) \to \mathcal{C} \otimes \Delta^1$ is equialent to a_0. For the left vertical map, the composite $\mathrm{Nil}_\alpha(\mathcal{C}) \xrightarrow{j} \mathcal{C} \overrightarrow{\times} (\mathcal{C} \otimes \Delta^1) \to \mathcal{C}$ is equivalent to fgt and the composite $\mathcal{C} \xrightarrow{i_1} \mathcal{C} \overrightarrow{\times} (\mathcal{C} \otimes \Delta^1) \to \mathcal{C}$ is equivalent to α.

The cofibers of the upper and lower horizontal maps in (3.15) are equivalent. Combining this with the splitting of Lemma 3.3.1, we obtain the split fiber sequence

$$\begin{tikzcd} E(\mathcal{C}) \arrow[r, "\mathrm{free}_\alpha"'] & E(\overline{\mathrm{End}}_\alpha(\mathrm{Ind}(\mathcal{C}))^\omega) \arrow[l, bend right, "\mathrm{cofib}"'] \arrow[r] & NE_\alpha(\mathcal{C}). \end{tikzcd}$$

An analogous argument, using the pushout

$$\begin{tikzcd} \mathcal{C} \oplus \mathcal{C} \arrow[r, "a_0 \oplus a_1"] \arrow[d, "\mathrm{id} \oplus \alpha"'] & \mathcal{C} \otimes \Delta^1 \arrow[d] \\ \mathcal{C} \arrow[r, "\mathrm{free}_{\alpha^R}"] & \overline{\mathrm{End}}_{\alpha^R}(\mathrm{Ind}(\mathcal{C}))^\omega \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

instead of (3.11), provides a split fiber sequence

$$\begin{tikzcd} E(\mathcal{C}) \arrow[r, "\mathrm{free}_{\alpha^R}"'] & E(\overline{\mathrm{End}}_{\alpha^R}(\mathrm{Ind}(\mathcal{C}))^\omega) \arrow[l, bend right, dashed] \arrow[r] & \overline{NE}_\alpha(\mathcal{C}). \end{tikzcd}$$

□

3.4 Regularity and Nil-vanishing

In this section we want to generalise the classical vanishing result for Nil-terms saying that $NK_\alpha(R) \simeq 0$ if R is a regular ring and $\alpha\colon R \to R$ an automorphism, as shown for example in [Wal78a, Theorem 4]. When passing from rings to (derived) categories of modules, the analogue of regularity is the notion of a t-structure. Let us recall its definition.

Recollection 3.4.1 (t-structure)**.** A t-structure on a stable category $\mathcal{C}$ consists of two full subcategories $\mathcal{C}_{\leq 0}, \mathcal{C}_{\geq 0} \subseteq \mathcal{C}$ satisfying the following properties:

(1) for all $x \in \mathcal{C}_{\geq 0}$ and $y \in \mathcal{C}_{\leq 0}$ one has $\mathrm{Map}_{\mathcal{C}}(x, \Omega y) \simeq 0$;

(2) $\mathcal{C}_{\geq 0}$ is closed under Σ and $\mathcal{C}_{\leq 0}$ is closed under Ω;

(3) for any $x \in \mathcal{C}$ there is a fiber sequence $x' \to x \to x''$ with $x' \in \mathcal{C}_{\geq 0}$ and $x'' \in \Omega\mathcal{C}_{\leq 0}$.

In this situation, the inclusion $\mathcal{C}_{\leq 0} \subseteq \mathcal{C}$ has a left adjoint $\tau_{\leq 0}$ and the inclusion $\mathcal{C}_{\geq 0} \subseteq \mathcal{C}$ has a right adjoint $\tau_{\geq 0}$. A t-structure is bounded if for any $x \in \mathcal{C}$ there is $k \in \mathbb{N}$ with $\Sigma^k x \in \mathcal{C}_{\geq 0}$ and $\Omega^k x \in \mathcal{C}_{\leq 0}$. The heart of $\mathcal{C}$ is defined by $\mathcal{C}^\heartsuit = \mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq 0}$. More details on t-structures on stable categories can be found in [Lur17, Section 1.2.1].

As an important class of examples of bounded t-structures, consider a regular coherent discrete ring R. This means that any finitely generated left R-module is finitely presented. In this situation, the perfect derived category Mod_R^ω admits a bounded t-structure with (co)connective objects given by precisely those R-modules with homology in nonnegative (resp. nonpositive) degrees. For more details and examples of t-structures on module categories of ring spectra we refer the reader to [BL22].

Let us now state the general vanishing result for Nil-terms. It can be easily deduced Burklund-Levy's abstract dévissage result [BL23, Theorem 1.3]. Land-Tamme show the analogous result for tensor algebras in the case where $\mathcal{C}$ has a single generator in [LT23, Corollary 4.13].

Corollary 3.4.2. *Consider an exact endofunctor $\alpha\colon \mathcal{C} \to \mathcal{C}$ of a perfect category and assume that $\mathcal{C}$ admits a bounded t-structure. If α is left t-exact, meaning that $\alpha(\mathcal{C}_{\leq 0}) \subseteq \mathcal{C}_{\leq 0}$, then $\tau_{\geq 0} NK_\alpha(\mathcal{C}) \simeq 0$. If the heart $\mathcal{C}^\heartsuit$ is aditionally Noetherian, then $NK_\alpha(\mathcal{C}) \simeq 0$. The analougous vanishing results hold for $\overline{NK}_\alpha(\mathcal{C})$ if α is right t-exact.*

Proof. We want to apply [BL23, Theorem 1.3] to the functor $\mathrm{triv}\colon \mathcal{C} \to \mathrm{Nil}_\alpha(\mathcal{C})$. By definition, the image of triv generates $\mathrm{Nil}_\alpha(\mathcal{C})$ under finite colimits and retracts. It remains to check that the restriction of triv to $\mathcal{C}^\heartsuit$ is fully faithful. For $x, y \in \mathcal{C}^\heartsuit$ one has $\mathrm{Map}_{\mathrm{End}_\alpha(\mathcal{C})}((x,0),(y,0)) \simeq \mathrm{Map}_{\mathcal{C}}(x,y) \times \Omega \mathrm{Map}_{\mathcal{C}}(x,\alpha(y)) \simeq \mathrm{Map}_{\mathcal{C}}(x,y)$, where the last step uses that α is left t-exact. $\square$

The t-structure constructed on $\mathrm{Nil}_\alpha(\mathcal{C})$ in the proof of [BL23, Theorem 1.3] is very inexplicit. We can give a more direct construction of a t-structure on $\mathrm{End}_\alpha(\mathcal{C})$ and $\mathrm{Nil}_\alpha(\mathcal{C})$ which might be of independent interest.

Proposition 3.4.3. *Suppose that $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}$ admits a t-structure and that $\alpha\colon \mathcal{C} \to \mathcal{C}$ is left t-exact. Then the full subcategories $\mathrm{End}_\alpha(\mathcal{C})_{\geq 0}$ (resp. $\mathrm{End}_\alpha(\mathcal{C})_{\leq 0}$) on those objects (x, f) with $x \in \mathcal{C}_{\geq 0}$ (resp. $x \in \mathcal{C}_{\leq 0}$) define a t-structure on $\mathrm{End}_\alpha(\mathcal{C})$, called the pointwise t-structure. Similarly, if α is right exact, $\overline{\mathrm{End}}_\alpha(\mathcal{C})$ admits the pointwise t-structure.*

Proof. We have to verify the properties from Recollection 3.4.1. Recall that for objects $(x, f), (y, g) \in \mathrm{End}_\alpha(\mathcal{C})$ we have

$$\mathrm{Map}_{\mathrm{End}_\alpha(\mathcal{C})}((x,f),(y,g)) \simeq \mathrm{eq}(f^* \circ \alpha, g_*\colon \mathrm{Map}_{\mathcal{C}}(x,y) \to \mathrm{Map}_{\mathcal{C}}(x,\alpha(y))).$$

Now if $x \in \mathcal{C}_{\geq 0}$ and $y \in \mathcal{C}_{\leq -1}$, then $\alpha(y) \in \mathcal{C}_{\leq -1}$ showing that $\mathrm{Map}_{\mathcal{C}}(x,y) \simeq \mathrm{Map}_{\mathcal{C}}(x,\alpha(y)) \simeq 0$. This implies $\mathrm{Map}_{\mathrm{End}_\alpha(\mathcal{C})}((x,f),(y,g)) \simeq 0$ and verifies condition (1). As $\Sigma(x,f) \simeq (\Sigma x, \Sigma f)$ it follows that $\mathrm{End}_\alpha(\mathcal{C})_{\geq 0}$ is closed under Σ and $\mathrm{End}_\alpha(\mathcal{C})_{\leq 0}$ is closed under Ω which shows (2).

Finally, for (3) we have to show that every object $(x,f) \in \mathrm{End}_\alpha(\mathcal{C})$ sits in a fiber sequence with a 1-connective and 0-coconnective object. As α is only left t-exact, it generally does not commute with the truncation $\tau_{\leq 0}$. However, there is always a Beck-Chevalley transformation $\beta\colon \tau_{\leq 0}\alpha \to \alpha\tau_{\leq 0}$ associated to the commutative square

$$\begin{tikzcd}
\mathcal{C}_{\leq 0} \arrow[r, hook] \arrow[d, "\alpha"] & \mathcal{C} \arrow[d, "\alpha"] \\
\mathcal{C}_{\leq 0} \arrow[r, hook] & \mathcal{C},
\end{tikzcd}$$

where the horizontal left adjoints are precisely the truncations $\tau_{\leq 0}$. Denoting by $\eta\colon \mathrm{id} \to \tau_{\leq 0}$ the adjunction unit, we obtain the commutative diagram

$$\begin{tikzcd}
x \arrow[r, "\eta_x"] \arrow[d, "f"] & \tau_{\leq 0}x \arrow[d, "\tau_{\leq 0}f"] \arrow[dr, bend left] & \\
\alpha(x) \arrow[r, "\eta_{\alpha(x)}"] & \tau_{\leq 0}\alpha(x) \arrow[r, "\beta"] & \alpha(\tau_{\leq 0}x).
\end{tikzcd} \tag{3.16}$$

It follows from [CSY22, Lemma 2.2.4(3)] that $\beta \circ \eta_{\alpha(x)} \simeq \alpha\eta_x$. The outer quadrilateral of (3.16) defines a map $(x,f) \to (\tau_{\leq 0}x, \beta \circ \tau_{\leq 0}f)$ in $\mathrm{End}_\alpha(\mathcal{C})$ with underlying map $\eta_x\colon x \to \tau_{\leq 0}x$. The underlying object of the fiber of $(x,f) \to (\tau_{\leq 0}x, \beta\circ\tau_{\leq 0}f)$ is $\tau_{\geq 1}x \simeq \mathrm{fib}(x \to \tau_{\leq 0}x)$ and thus 1-connective.

The proof of the existence of the pointwise t-structure on $\overline{\mathrm{End}}_\alpha(\mathcal{C})$ is analogous. The truncation is now given by

$$\tau_{\geq 0}(x, f\colon \alpha(x) \to x) = (\tau_{\geq 0}x, \alpha(\tau_{\geq 0}x) \xrightarrow{\beta'} \tau_{\geq 0}\alpha(x) \xrightarrow{\tau_{\geq 0}f} \tau_{\geq 0}x),$$

using the Beck-Chevalley transformation $\beta'\colon \alpha\tau_{\geq 0} \to \tau_{\geq 0}\alpha$. □

This also gives us a t-structure on Nil-categories as follows.

Corollary 3.4.4. *Suppose that $\mathcal{C}$ admits a bounded t-structure and that $\alpha\colon \mathcal{C} \to \mathcal{C}$ is right t-exact. Then the pointwise t-structure on $\overline{\mathrm{End}}_\alpha(\mathcal{C})$ restricts to a t-structure on $\overline{\mathrm{Nil}}_\alpha(\mathcal{C})$.*

Similarly, if α is left t-exact and admits a left adjoint, then the pointwise t-structure on $\mathrm{End}_\alpha(\mathcal{C})$ restricts to a t-structure on $\mathrm{Nil}_\alpha(\mathcal{C})$

Proof. We only have to show that $\overline{\mathrm{Nil}}_\alpha(\mathcal{C}) \subseteq \overline{\mathrm{End}}_\alpha(\mathcal{C})$ is closed under truncation. Recall from Theorem 2.2.2 that $\overline{\mathrm{Nil}}_\alpha(\mathcal{C})$ can be identified with the homotopy nilpotent endomorphisms $(x,f) \in \overline{\mathrm{End}}_\alpha(\mathcal{C})$ for which $f^{(n)} \simeq 0$ for large

n. In the proof of Proposition 3.4.3 we saw the explicit formula for truncation $\tau_{\geq 0}(x, f) = (\tau_{\leq 0}x, \tau_{\geq 0}f \circ \beta')$, where $\beta' \colon \alpha\tau_{\geq 0} \to \tau_{\geq 0}\alpha$ is the Beck-Chevalley transformation. Now note that by naturality of β' the composite

$$\alpha^n\tau_{\geq 0}x \xrightarrow{\alpha^{n-1}\beta'} \alpha^{n-1}\tau_{\geq 0}\alpha(x) \xrightarrow{\alpha^{n-1}\tau_{\geq 0}f} \dots \xrightarrow{\beta'} \tau_{\geq 0}\alpha x \xrightarrow{\tau_{\geq 0}f} \tau_{\geq 0}x$$

defining $(\tau_{\geq 0}f \circ \beta')^{(n)}$ is equivalent to the composite

$$\alpha^n\tau_{\geq 0}x \xrightarrow{\alpha^{n-1}\beta'} \alpha^{n-1}\tau_{\geq 0}\alpha x \xrightarrow{\alpha^{n-2}\beta'} \dots \xrightarrow{\tau_{\geq 0}\alpha f} \tau_{\geq 0}\alpha x \xrightarrow{\tau_{\geq 0}f} \tau_{\geq 0}x$$

given by $\beta^{(n)} \circ \tau_{\leq 0}(f^{(n)})$. This vanishes for for large n, showing $\tau_{\leq 0}(x, f) \in \mathrm{Nil}_\alpha(\mathcal{C})$.

If α is left t-exact and has a left adjoint α^L, then α^L is right t-exact. The claim then follows from the first part together with the equivalences $\mathrm{Nil}_\alpha(\mathcal{C}) \simeq \overline{\mathrm{Nil}}_{\alpha^L}(\mathcal{C})$ and $\mathrm{End}_\alpha(\mathcal{C}) \simeq \overline{\mathrm{End}}_{\alpha^L}(\mathcal{C})$. □

Remark 3.4.5. If α is t-exact, one could prove Corollary 3.4.2 from Corollary 3.4.4 as follows: By Barwick's theorem of the heart [Bar15], it suffices to show that the map $\overline{\mathrm{triv}} \colon \mathcal{C}^\heartsuit \to \overline{\mathrm{Nil}}_\alpha(\mathcal{C})^\heartsuit$ becomes an equivalence on connective K-theory. This follows from Quillen's devissage theorem by filtering any homotopy nilpotent endomorphism $f \colon \alpha(x) \to x$ in $\overline{\mathrm{Nil}}_\alpha(\mathcal{C})^\heartsuit$ by its nilpotence degree. For this, note that the nilpotence degree of the endomorphism $\alpha(\ker(f)) \xrightarrow{f} \ker(f)$ is smaller than the one of f.

Chapter 4

Applications and Examples

4.1 K-theory of tensor algebras

In this section we will study various applications of Theorem 3.2.5 to obtain splittings for the K-theory of certain rings. We begin by relating the category of twisted endomorphisms studied in §2.1 to module categories over tensor algebras. For this, we need the following variant of [BCN24, Theorem B.2], which identifies tensor algebras with certain free monads.

Consider a presentable category $\mathcal{C}$. Composition makes $\mathrm{End}^L(\mathcal{C}) := \mathrm{Fun}^L(\mathcal{C}, \mathcal{C})$ into a presentably monoidal category. There is the corresponding adjunction

$$\begin{tikzcd} \mathrm{Alg}(\mathrm{End}^L(\mathcal{C})) \arrow[r, bend right, "U"'] & \mathrm{End}^L(\mathcal{C}), \arrow[l, bend right, "F"'] \end{tikzcd}$$

where U is the forgetful functor. Its left adjoint sends an endofunctor $\alpha \in \mathrm{End}^L(\mathcal{C})$ to the free monad on α.

Proposition 4.1.1. *For $\alpha \in \mathrm{End}^L(\mathcal{C})$, the inclusion $i\colon \alpha \to \coprod_{n\geq 0} \alpha^n \simeq \mathrm{fgt}\,\overline{\mathrm{free}}_\alpha$ in $\mathrm{End}^L(\mathcal{C})$ exhibits $\mathrm{fgt}\,\overline{\mathrm{free}}_\alpha$ as the free monad on α.*

Proof. First note that the adjunction $\overline{\mathrm{free}}_\alpha \dashv \mathrm{fgt}$ is monadic by the Barr-Beck-Lurie monadicity theorem [Lur17, Theorem 4.3.7.5], as fgt is conservative and colimit preserving.

Let us now turn to the proof of Proposition 4.1.1. We need to show that for any monad $T \in \mathrm{Alg}(\mathrm{End}^L(\mathcal{C}))$, restriction along i induces an equivalence

$$\mathrm{Map}_{\mathrm{Alg}(\mathrm{End}^L(\mathcal{C}))}(\mathrm{fgt}\,\mathrm{free}_\alpha, T) \xrightarrow{\simeq} \mathrm{Map}_{\mathrm{End}^L(\mathcal{C})}(\alpha, U(T)). \tag{4.1}$$

For this, recall that the functor $\mathrm{Alg}(\mathrm{End}(\mathcal{C}))^{\mathrm{op}} \to \mathrm{Cat}_{/\mathcal{C}}, T \mapsto (U_T\colon \mathrm{Mod}_T(\mathcal{C}) \to \mathcal{C})$ sending a monad to the forgetful functor on its category of modules is fully faithful by [Lur17, Remark 4.7.3.8]. Monadicity of $\overline{\mathrm{free}}_\alpha \dashv \mathrm{fgt}$ identifies the forgetful functor $\mathrm{Mod}_{\mathrm{fgt}\,\overline{\mathrm{free}}_\alpha}(\mathcal{C}) \to \mathcal{C}$ with $\mathrm{fgt}\colon \overline{\mathrm{End}}_\alpha(\mathcal{C}) \to \mathcal{C}$. Together, we obtain that the left side of (4.1) is equivalent to the space of factorisations

$$\begin{tikzcd} \mathrm{Mod}_T(\mathcal{C}) \arrow[r, dashed] \arrow[d, "U_T"] & \overline{\mathrm{End}}_\alpha(\mathcal{C}) \arrow[dl, "\mathrm{fgt}"] \\ \mathcal{C} & \end{tikzcd}.$$

Using the definition of $\overline{\mathrm{End}}_\alpha(\mathcal{C})$ as a lax equaliser, this is equivalent to the space of transformations $\alpha U_T \to U_T$. Any such dashed factorisation $\mathrm{Mod}_T(\mathcal{C}) \to \overline{\mathrm{End}}_\alpha(\mathcal{C})$ is automatically colimit preserving as U_T preserves colimits and fgt detects colimits. Unwinding the definitions of the map (4.1), this proves that it is an equivalence. □

Construction 4.1.2 (Tensor algebras)**.** For a ring spectrum $R \in \mathrm{Alg}(\mathrm{Sp})$ denote by $\mathrm{Mod}_R = \mathrm{Mod}_R(\mathrm{Sp})$ its category of left R-modules. Recall from [Lur17, Remark 4.8.4.9] that the map

$$\mathrm{BMod}(R,R) \to \mathrm{End}^L(\mathrm{Mod}_R), \quad M \mapsto M \otimes_R -$$

is an equivalence, where $\mathrm{BMod}(R,R) = \mathrm{BMod}(R,R)(\mathrm{Sp})$ is the category of (R,R)-bimodules. We can enhance it to a monoidal equivalence by endowing $\mathrm{BMod}(R,R)$ with the relative tensor product $\otimes_R$ and $\mathrm{End}^L(\mathrm{Mod}_R)$ with the monoidal structure given composition. The forgetful functor $\mathrm{Alg}(\mathrm{BMod}(R,R)) \to \mathrm{BMod}(R,R)$ has a left adjoint given by the *tensor algebra*

$$T_R\colon \mathrm{BMod}(R,R) \to \mathrm{Alg}(\mathrm{BMod}(R,R)), \quad M \mapsto \bigoplus_{n\geq 0} M^{\otimes_R^n}.$$

Combining Lemma 2.1.6 and Proposition 4.1.1, we obtain that

$$\mathrm{Mod}_{T_R(M)}(\mathrm{Sp}) \simeq \mathrm{Mod}_{T_R(M)}(\mathrm{Mod}_R) \simeq \overline{\mathrm{End}}_M(\mathrm{Mod}_R). \tag{4.2}$$

Now suppose that the right adjoint to $M \otimes_R -\colon \mathrm{Mod}_R \to \mathrm{Mod}_R$ preserves colimits, or equivalently that M is compact as a left R-module. Denote by $M^\vee$ its left dual, so that $M^\vee \otimes_R -$ is right adjoint to $M \otimes_R -$. $\mathrm{Aut}_{M^\vee}(\mathrm{Mod}_R)$ is also the module category of some type of tensor algebra, though its description is not as nice as for the generator of $\mathrm{End}_{M^\vee}(\mathrm{Mod}_R) \simeq \overline{\mathrm{End}}_M(\mathrm{Mod}_R)$. It follows from Lemma 2.1.8 that $\mathrm{Aut}_{M^\vee}(\mathrm{Mod}_R)$ is compactly generated with generator $\mathrm{loc}_{M^\vee}\overline{\mathrm{free}}_M(R)$. Lurie's version of the Schwede-Shipley theorem [Lur17, Theorem 7.1.2.1, Remark 7.1.2.3] shows that

$$\mathrm{Aut}_{M^\vee}(\mathrm{Mod}_R) \simeq \mathrm{Mod}_{T_R(M)[M^{-1}]}(\mathrm{Sp}),$$

where $T_R(M)[M^{-1}]$ is the endomorphism ring spectrum of $\mathrm{loc}_{M^\vee}\overline{\mathrm{free}}_M(R)$, which we call the *localised tensor algebra*. Forgetting its multiplicative structure, the underlying spectrum of $T_R(M)[M^{-1}]$ is given by

$$\begin{aligned} T_R(M)[M^{-1}] &\simeq \mathrm{map}_{\mathrm{Aut}_{M^\vee}(\mathrm{Mod}_R)}(\mathrm{loc}_{M^\vee}\overline{\mathrm{free}}_M(R), \mathrm{loc}_{M^\vee}\overline{\mathrm{free}}_M(R)) \\ &\simeq \mathrm{map}_{\mathrm{Mod}_R}(R, \mathrm{loc}_{M^\vee}\overline{\mathrm{free}}_M(R)) \\ &\simeq \mathrm{colim}\left(\bigoplus_{k\geq 0} M^{\otimes_R^k} \xrightarrow{\mathrm{coev}} \bigoplus_{k\geq 0} M^\vee \otimes_R M^{\otimes_R^k} \xrightarrow{\mathrm{coev}} \bigoplus_{k\geq 0} (M^\vee)^{\otimes_R^2} \otimes_R M^{\otimes_R^k} \xrightarrow{\mathrm{coev}} \dots\right), \end{aligned}$$

with maps in the colimit induced by the coevaluation $\mathrm{coev}\colon R \to M^\vee \otimes_R M$.

Example 4.1.3. Let $R \in \mathrm{Alg}(\mathrm{Sp})$ be a ring spectrum together with an endomorphism $\alpha\colon R \to R$. Induction induces an endomorphism $\alpha_!\colon \mathrm{Mod}_R^\omega \to \mathrm{Mod}_R^\omega$. In that case, the bimodule M from Construction 4.1.2 is given by $M = R$ with trivial left R-module structre and right R-module structure given by α. We write $R_\alpha[t] := T_R(M)$ and obtain as a left R-module

$$R_\alpha[t] \simeq \mathrm{fgt}\,\mathrm{free}_{\alpha^R}(R) \simeq \bigoplus_{n\geq 0} \alpha_!^n R \simeq \bigoplus_{n\geq 0} R.$$

The multiplication $\alpha_!^n R \times \alpha_!^m R \to \alpha_!^{n+m} R$ identifies with $R \times R \xrightarrow{\mathrm{id}\otimes\alpha^n} R \times R \xrightarrow{\otimes} R$. In particular, if the ring R is discrete, this recovers the classical twisted polynomial ring.

If α is an automorphism, the module $M^\vee$ is given by the (R,R)-bimodule R with trivial right R-module structure and left R-module structure given by α^{-1}. In that case, we write $R_\alpha[t^{\pm 1}] = T_R(M)[M^{-1}]$ and can identify

$$R_\alpha[t^{\pm 1}] \simeq \mathrm{colim}\left(\bigoplus_{k\geq 0} R \xrightarrow{\mathrm{shift}} \bigoplus_{k\geq 0} R \xrightarrow{\mathrm{shift}} \dots\right) \simeq \bigoplus_{n\in\mathbb{Z}} R.$$

Multiplication is again given on component $n \times m$ by $R \times R \xrightarrow{\mathrm{id}\otimes\alpha^n} R \times R \xrightarrow{\otimes} R$. If R is discrete, this recovers the classical ring of twisted Laurent polynomials.

Theorems 3.2.5 and 3.3.2 immediately specialise to the following result.

Corollary 4.1.4. *Consider a ring spectrum $R \in \mathrm{Alg}(\mathrm{Sp})$ together with a (R,R)-bimodule M which is compact as a left R-module. Then there is a natural splitting*

$$E(T_R(M)[M^{-1}]) \simeq E(R)_{h\mathbb{N}} \oplus NE_M(R) \oplus \overline{NE}_M(R).$$

Furthermore, there are natural splittings

$$E(T_R(M)) \simeq E(R) \oplus NE_M(R) \quad \textit{and} \quad E(T_R(M^\vee)) \simeq E(R) \oplus \overline{NE}_M(R),$$

where $M^\vee$ denotes a left dual to M. If M is induced by an automorphism α of R, this reduces to a splitting

$$E(R_\alpha[t^{\pm 1}]) \simeq E(R)_{h\mathbb{N}} \oplus NE_\alpha(R) \oplus \overline{NE}_\alpha(R).$$

Example 4.1.5 (Generators in nonzero degree)**.** Consider the action $\alpha = \Sigma^k \colon \mathcal{C} \xrightarrow{\simeq} \mathcal{C}$ through suspension on a perfect category $\mathcal{C}$ for some $k \in \mathbb{Z}$. Note that $E(\Sigma^k) \simeq \mathrm{id}_{E(\mathcal{C})}$ if k is even and $E(\Sigma^k) \simeq -\mathrm{id}_{E(\mathcal{C})}$ if k is odd. From this we obtain

$$E(\mathcal{C})_{h\mathbb{Z}} \simeq \begin{cases} E(\mathcal{C}) \oplus \Sigma E(\mathcal{C}), & \text{k even;} \\ E(\mathcal{C})/2, & \text{k odd} \end{cases}$$

for the middle term in the splitting in Theorem 3.2.5. If $\mathcal{C} = \mathrm{Mod}_R^\omega$ for some ring spectrum $R \in \mathrm{Alg}(\mathrm{Sp})$, then $(\mathrm{Mod}_R^\omega)_{h\mathbb{N}} \simeq \mathrm{Mod}^\omega_{R[x^{\pm 1}]}$, where $R[x^{\pm 1}]$ is the free associative algebra over R with an invertible generator in degree $|x| = k$.

If $\mathcal{C}$ admits a bounded t-structure and $k \leq 0$, then Σ^k is left t-exact. Corollary 3.4.2 shows that in this case $\tau_{\geq 0} NK_{\Sigma^k}(\mathcal{C}) \simeq 0$. It turns out that the other Nil-term $\overline{NK}_{\Sigma^k}(\mathcal{C})$ is generally not trivial, not even rationally. As an example, consider $\mathcal{C} = \mathrm{Mod}_R^\omega$ for a discrete ring R, which admits a bounded t-structure if R is regular. Land-Tamme compute in [LT23, Proposition 4.11]

$$\overline{NK}_{\Sigma^k}(R)_{\mathbb{Q}} \simeq \begin{cases} \bigoplus_{n \geq 1} \Sigma^{|k|n+1} \mathrm{HH}(R \otimes \mathbb{Q}), & k \text{ even;} \\ \bigoplus_{n \geq 1} \Sigma^{|k|(2n-1)+1} \mathrm{HH}(R \otimes \mathbb{Q}), & k \text{ odd.} \end{cases}$$

For example, for $R = \mathbb{Z}$ one obtains $\mathrm{HH}(\mathbb{Z} \otimes \mathbb{Q}) \simeq \mathbb{Q}[0]$. This also shows that the two Nil-terms $NE_\alpha(\mathcal{C})$ and $\overline{NE}_\alpha(\mathcal{C})$ are generally not isomorphic.

4.2 K-theory of mapping tori

As promised in the introduction, we can prove a splitting of Waldhausen's A-theory of mapping tori. Let us first recall the definition.

Recollection 4.2.1 (Waldhausen's A-theory)**.** For $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}$ and a space $X \in \mathcal{S}$, denote by $\mathcal{C}_X = \mathrm{colim}_X \, \mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}$ the colimit over the constant X-shaped diagram with value $\mathcal{C}$. There is an equivalence $\mathcal{C}_X \simeq \mathrm{Fun}(X, \mathrm{Ind}(\mathcal{C}))^\omega$ as colimits in $\mathrm{Cat}^{\mathrm{perf}}$ are computed by taking compact objects in the colimit of the Ind-completed diagram in Pr^L, or equivalently as limits in the right adjoint diagram in Pr^R, and using the equivalence $X \simeq X^{\mathrm{op}}$. Nonconnective A-theory of X is defined as

$$\mathbb{A}(X) = K((\mathrm{Sp}^\omega)_X).$$

If X is connected, then $\mathrm{Sp}_X^\omega \simeq \mathrm{Mod}^\omega_{\mathbb{S}[\Omega X]}$ so that $\mathbb{A}(X) \simeq K(\mathbb{S}[\Omega X])$. The classical (finitely dominated) version of A-theory constructed in [HKV+01] can be recovered from this as the connective cover $A(X) = \tau_{\geq 0}\mathbb{A}(X)$. It differs from Waldhausen's original definition of A theory in [Wal78b] only by π_0.

We obtain the following splitting result for mapping tori.

Corollary 4.2.2. *Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{perf}}$ and X be a space together with a selfmap $\alpha\colon X \to X$. For any localising invariant E, there is an equivalence*

$$E(\mathcal{C}_{X_{h\mathbb{N}}}) \simeq E(\mathcal{C}_X)_{h\mathbb{N}} \oplus NE_\alpha(\mathcal{C}_X) \oplus \overline{NE}_\alpha(\mathcal{C}_X).$$

In particular, there is a splitting

$$A(X_{h\mathbb{N}}) \simeq \tau_{\geq 0}(\mathbb{A}(X)_{h\mathbb{N}}) \oplus NA_\alpha(X) \oplus \overline{NA}_\alpha(X),$$

with Nil-*terms* $NA_\alpha(X) := \tau_{\geq 0} N\mathbb{A}_\alpha(X)$. *On 1-connective covers one has* $\tau_{\geq 1}(\mathbb{A}(X)_{h\mathbb{N}}) \simeq \tau_{\geq 1}(A(X)_{h\mathbb{N}})$

Proof. The first statement is a consequence of Theorem 3.2.5 together with the equivalence $\mathcal{C}_{X_{h\mathbb{N}}} \simeq (\mathcal{C}_X)_{h\mathbb{N}}$ coming from the usual composition formula for colimits. The statement about A-theory follows from this by taking $E = K$ and $\mathcal{C} = \mathrm{Sp}^\omega$ and passing to connective covers. The statement about 1-connective covers follows from the fact that for a span $T_1 \leftarrow T_0 \rightarrow T_2$ of spectra, the map $\tau_{\geq 0} T_1 \oplus_{\tau_{\geq 0} T_0} \tau_{\geq 0} T_2 \to T_1 \oplus_{T_0} T_2$ of pushouts is an equivalence after passing to 1-connective covers. □

Example 4.2.3 (HNN-extensions)**.** Let G be a discrete group and $f\colon G \to G$ a homomorphism. One has $(BG)_{h\mathbb{N}} \simeq B(G*_f)$, where $G*_f$ is a generalised HNN extension, explicitly given by the presentation $G*_f = \langle G, t | gt = tf(g) \text{ for } g \in G \rangle$.

As a special case of Corollary 4.2.2 we obtain the following.

Corollary 4.2.4. *Let $R \in \mathrm{Alg}(\mathrm{Sp})$ be a ring spectrum, G a group and $f\colon G \to G$ a homomorphism. Then there is an equivalence*

$$E(R[G*_f]) \simeq E(R[G])_{h\mathbb{N}} \oplus NE_f(R[G]) \oplus \overline{NE}_f(R[G]).$$

Proof. This is a combination of Corollary 4.2.2 and Example 4.2.3. □

4.3 A-theoretic Nil-terms

The goal of this subsection is to provide a guide to the computation of certain A-theoretic twisted Nil-terms, applying the work of Bökstedt-Hsiang-Madsen [BHM93]. We will achieve this through a comparison with topological cyclic homology. Let us first recall the situation for $X = *$. By work of Waldhausen [Wal78b], the map

$$NA(*) = \tau_{\geq 0} NK(\mathrm{Sp}^\omega) \to \tau_{\geq 0} NK(\mathrm{Mod}_{\mathbb{Z}}^\omega)$$

is a rational equivalence, but the target is rationally trivial as $\mathbb{Z}$ is a regular ring. On the other hand, lots is known about interesting torsion in the homotopy groups of $NK(\mathrm{Sp}^\omega)$ [GKM08].

For a space X we denote
$$\mathrm{TC}(X) := \mathrm{TC}(\mathrm{Sp}_X^\omega).$$
From now on, we will assume that X is connected and that X comes with a self-map $\alpha\colon X \to X$. Consider the following cube. To shorten notation, we will write $NA_\alpha^{\mathrm{tot}}(X)$ and $N\,\mathrm{TC}_\alpha^{\mathrm{tot}}(X)$ for the *total* Nil-*terms*, i.e., the cofiber of the respective horizontal assembly maps in the following cube.

$$\begin{tikzcd}[row sep=small, column sep=small]
& A(X)_{h\mathbb{N}} \arrow[rr] \arrow[dl] \arrow[dd] & & A(X_{h\mathbb{N}}) \arrow[dl] \arrow[dd] \\
A(*)_{h\mathbb{N}} \arrow[rr, crossing over] \arrow[dd] & & A(S^1) & \\
& \mathrm{TC}(X)_{h\mathbb{N}} \arrow[rr] \arrow[dl] & & \mathrm{TC}(X_{h\mathbb{N}}) \arrow[dl] \\
\mathrm{TC}(*)_{h\mathbb{N}} \arrow[rr] & & \mathrm{TC}(S^1) \arrow[from=uu, crossing over] &
\end{tikzcd}$$

From Corollary 4.2.2 we see that $NA_\alpha^{\mathrm{tot}}(X) \simeq NA_\alpha(\mathcal{C}) \oplus \overline{NA}_\alpha(\mathcal{C})$ after passing to 1-connective covers. Notice that $X_{h\mathbb{N}} \to *_{h\mathbb{N}} \simeq S^1$ and $X \to *$ induce π_0-isomorphisms after passing to spherical group rings, so the celebrated Dundas-Goodwillie-McCarthy-theorem [DGM13, Theorem 7.2.2.1] applies to show that the left face and the right face are cartesian. Thus, the whole cube is cartesian and the map
$$\mathrm{cofib}(NA_\alpha^{\mathrm{tot}}(X) \to NA^{\mathrm{tot}}(*)) \to \mathrm{cofib}(N\,\mathrm{TC}_\alpha^{\mathrm{tot}}(X) \to N\,\mathrm{TC}^{\mathrm{tot}}(*))$$
is an equivalence. Surpsisingly, the $N\,$TC-terms are quite accessible for computation.

For a space Y, denote by $LY = \mathrm{Map}(S^1, Y)$ its *free loop space*. It carries a natural S^1-action. Now suppose that Y comes with a map $f\colon Y \to S^1$. The typical example here will be the map $X_{n\mathbb{N}} \to *_{h\mathbb{N}} \simeq S^1$ from the space of homotopy orbits of a $\mathbb{N}$-action. It induces a map $Lf\colon LY \to LS^1$. Note that the degree identifies $\pi_0 LS^1$ with $\mathbb{Z}$. Let us denote $L(k)Y = LY \times_{\pi_0(LS^1)} \{k\}$ and $L(\neq k)Y = \coprod_{n \neq k} L(n)Y$. The main result of this subsection is the following.

Theorem 4.3.1. *For a connected based space X together with a selfmap $\alpha\colon X \to X$, there is a natural equivalence*
$$N\,\mathrm{TC}_\alpha^{\mathrm{tot}}(X) \simeq \Sigma(\Sigma_+^\infty L(\neq 0)(X_{h\mathbb{N}}))_{hS^1}$$
after p-completion at an arbitrary prime p.

A useful observation is that Theorem 4.3.1 can be reformulated as a splitting as an infinite direct sum after p-completion

$$N\,\mathrm{TC}^{\mathrm{tot}}_{\alpha}(X) \simeq \bigoplus_{k\neq 0} \Sigma(\Sigma^{\infty}_{+} L(k)(X_{h\mathbb{N}}))_{hS^1}.$$

The main ingredient is the following theorem of Bökstedt-Hsiang-Madsen [BHM93], which can also be found in [NS18, Thm. IV.3.6].

Theorem 4.3.2. *For a connected based space X, there is a natural pullback square*

$$\begin{tikzcd} \mathrm{TC}(\mathbb{S}[\Omega X]) \arrow[r] \arrow[d] & \Sigma(\Sigma^{\infty}_{+} LX)_{hS^1} \arrow[d] \\ \Sigma^{\infty}_{+} LX \arrow[r, "1-\widetilde{\varphi_p}"] & \Sigma^{\infty}_{+} LX \end{tikzcd}$$

after p-completion at an arbitrary prime p.

In the above theorem, the map $\widetilde{\varphi_p}$ is induced by the map $LX \to LX$ which precomposes with the p-fold selfcovering of S^1. Our goal is to understand how the pullback in Theorem 4.3.2 behaves with taking $\mathbb{N}$-orbits. We will start with an easy observation about loop spaces.

Lemma 4.3.3. *Let X be a space with $\mathbb{N}$-action. Then the assembly map*

$$(LX)_{h\mathbb{N}} \to L(X_{h\mathbb{N}})$$

can be identified with the inclusion of the path component

$$L(X_{h\mathbb{N}})(0) \to L(X_{h\mathbb{N}}).$$

Proof. The fiber of the map $X_{n\mathbb{N}} \to *_{h\mathbb{N}} \simeq S^1$ is given by the telescope $X_\infty = \mathrm{colim}(X \xrightarrow{\alpha} X \xrightarrow{\alpha} \dots)$. We have the following commutative diagram of fibre sequences

$$\begin{tikzcd} (LX)_\infty \arrow[r] \arrow[d] & (LX)_{h\mathbb{N}} \arrow[r] \arrow[d] & S^1 \arrow[d] \\ L(X_\infty) \arrow[r] & L(X_{h\mathbb{N}}) \arrow[r] & LS^1 \end{tikzcd}$$

where the right vertical map is the corresponding assembly map for $X = \{*\}$. Note that the left vertical map is an equivalence as S^1 is compact so L commutes with filtered colimits, showing that the right square is cartesian. There is an equivalence $LS^1 \simeq S^1 \times \mathbb{Z}$, induced by the evaluation at $1 \in S^1$ and the degree. Under this identification, the right vertical map is the inclusion of $S^1 \times \{0\}$. $\square$

We need one further observation about loop spaces for the proof of Theorem 4.3.1. Namely, we show that the square in Theorem 4.3.2 can be simplified dramatically for spaces of the form $X_{h\mathbb{N}}$.

Lemma 4.3.4. *The projection to the summand $\Sigma^\infty_+ L(X_{h\mathbb{N}}) \to \Sigma^\infty_+ L(X_{h\mathbb{N}})(0)$ induces the following pullback square.*

$$\begin{tikzcd}
\Sigma^\infty_+ L(X_{h\mathbb{N}}) \arrow[r, "1-\widetilde{\varphi_p}"] \arrow[d] & \Sigma^\infty_+ L(X_{h\mathbb{N}}) \arrow[d] \\
\Sigma^\infty_+ L(X_{h\mathbb{N}}) \arrow[r, "1-\widetilde{\varphi_p}"] & \Sigma^\infty_+ L(X_{h\mathbb{N}})(0)
\end{tikzcd}$$

Proof. Using the decomposition $\Sigma^\infty_+ L(X_{h\mathbb{N}}) = \bigoplus_{n\in\mathbb{Z}} \Sigma^\infty_+ L(X_{h\mathbb{N}})(n)$ and the order $0 < 1 < -1 < 2 < \dots$ on $\mathbb{Z}$, the map $1 - \widetilde{\varphi_p}$ is a triangular matrix whose diagonal entries are $1 - \widetilde{\varphi_p}$ at $(0,0)$ and 1 everywhere else. Thus, $1 - \widetilde{\varphi_p}$ is an equivalence when restricted to the fiber of the projection $\Sigma^\infty_+ L(X_{h\mathbb{N}}) \to \Sigma^\infty_+ L(X_{h\mathbb{N}})(0)$. $\square$

Remark 4.3.5. Some generalizations are possible. For example, let G be a group, whose conjugacy classes of elements admit a total order $\leq$ for which $[g] \leq [h]$ implies $[g^p] \leq [h^p]$. Suppose, X comes with a map to BG. Consider the collection $LX(1_G)$ of components of LX of loops mapping to the nullhomotopic loops in BG. Then we have a pullback square of the following form.

$$\begin{tikzcd}
\Sigma^\infty_+ LX \arrow[r, "1-\widetilde{\varphi_p}"] \arrow[d] & \Sigma^\infty_+ LX \arrow[d] \\
\Sigma^\infty_+ LX(1_G) \arrow[r, "1-\widetilde{\varphi_p}"] & \Sigma^\infty_+ LX(1_G)
\end{tikzcd}$$

Note that groups with a total order as required have to be p-torsion free. An obvious candidate for a map as above is the map $Y \to B\pi_1(Y, y)$.

Proof of Theorem 4.3.1. We start by considering the following commutative diagram

$$
\begin{tikzcd}
& \mathrm{TC}(X)_{h\mathbb{N}} \arrow[rr] \arrow[dl] \arrow[dd] & & (\Sigma(\Sigma^\infty_+ LX)_{hS^1})_{h\mathbb{N}} \arrow[dl] \arrow[dd] \\
\mathrm{TC}(X_{h\mathbb{N}}) \arrow[rr] \arrow[dd] & & \Sigma(\Sigma^\infty_+ L(X_{h\mathbb{N}}))_{hS^1} \arrow[dd] & \\
& \Sigma^\infty_+(LX)_{h\mathbb{N}} \arrow[rr, dashed] \arrow[dl] \arrow[dd] & & \Sigma^\infty_+(LX)_{h\mathbb{N}} \arrow[dl] \arrow[dd] \\
\Sigma^\infty_+ L(X_{h\mathbb{N}}) \arrow[rr] \arrow[dd] & & \Sigma^\infty_+ L(X_{h\mathbb{N}}) \arrow[dd] & \\
& \Sigma^\infty_+(LX)_{h\mathbb{N}} \arrow[rr, dashed] \arrow[dl] & & \Sigma^\infty_+(LX)_{h\mathbb{N}} \arrow[dl] \\
\Sigma^\infty_+ L(0)(X_{h\mathbb{N}}) \arrow[rr] & & \Sigma^\infty_+ L(0)(X_{h\mathbb{N}}), &
\end{tikzcd}
\tag{4.3}
$$

where the rightwards pointing maps in the bottom cube are all given by $1 - \widetilde{\varphi_p}$. We want to show that the topmost face is cartesian after p-completion. For this, note that the front and back face of the top cube are cartesian after p-completion by Theorem 4.3.2. The front face of the bottom cube is cartesian by Lemma 4.3.4. The vertical maps in the back face of the bottom cube are identities so it is also cartesian. Finally, the bottom face is cartesian by Lemma 4.3.3. Together this shows that the top face is cartesian after p-completion. We obtain an equivalence

$$\mathrm{cofib}(\mathrm{TC}(X)_{h\mathbb{N}} \to \mathrm{TC}(X_{h\mathbb{N}})) \simeq \mathrm{cofib}((\Sigma(\Sigma^\infty_+ LX)_{hS^1})_{h\mathbb{N}} \to \Sigma(\Sigma^\infty_+ L(X_{h\mathbb{N}}))_{hS^1}).$$

Using Lemma 4.3.3one more time, we see that this is equivalent to the cofiber of

$$(\Sigma(\Sigma^\infty_+(LX)_{h\mathbb{N}})_{hS^1}) \simeq (\Sigma(\Sigma^\infty_+ L(0)X)_{hS^1}) \to \Sigma(\Sigma^\infty_+ L(X_{h\mathbb{N}}))_{hS^1}.$$

This completes the proof. □

Part II

Equivariant Poincaré Duality

This part is reproduction of the authors work [HKK24b], joint with Kaif Hilman and Christian Kremer, with only minor corrections.

Chapter 5

Introduction

Poincaré duality has a distinguished history that goes back right to the birth of algebraic topology at the hands of Henri Poincaré. Broadly speaking, it says that there is often a hidden symmetry between homology and cohomology, and arguably beyond Poincaré's wildest dreams, it is a phenomenon that is not just endemic to algebraic topology but also pervasive in fields as far as algebraic geometry, arithmetic geometry, and even representation theory. Essentially, it tends to show up in any context in which homological algebra is present. Perhaps a reason as to why it is such a useful principle is that it may be exploited both computationally as well as theoretically: the former because it halves the amount of homological computations to be made and the latter because, for instance, it may be used to produce "wrong–way" maps which opens the way to powerful transfer arguments. No less importantly in the way of theoretical significance, it would also be remiss of us not to mention that Poincaré duality constitutes one of the starting points of the surgery theory of manifolds. In either case, it would be fair to summarise that Poincaré duality provides strong structural constraints on homological invariants which lends a rigidity not seen for a bare homotopy type.

On another front, group actions on manifolds have attracted the attention of topologists nearly since the beginning of the subject. A deep vein in this line of work is the hope of finding algebro–topological constraints on group actions in order to rule out the existence of certain group actions on manifolds. Smith theory, for example, predicts that if a p-group G acts on an $\mathbb{F}_p$-homology sphere, then the fixed points of the action must also be an $\mathbb{F}_p$-homology sphere. In a similar spirit, the Conner–Floyd conjecture, resolved affirmatively by Atiyah–Bott, predicts in a simple version that if p is an odd prime and the group C_{p^k} acts smoothly on a smooth, closed, orientable, positive–dimensional manifold, then the fixed point set of the action cannot consist of a single isolated point.

Given the motivations above, it should come as no surprise that a theory of equivariant Poincaré duality is desirable in order to incorporate the strong homological constraints aforementioned in the equivariant context. Indeed, so natural is this a question that there is a very rich corpus of contributions – too many to mention exhaustively – in this line of investigation coming from a wide variety of schools of thought. From our point of view, the strand of work that is most pertinent to us (and on which we build, either directly or indirectly), is the parametrised category theoretic one of Costenoble–Waner [CW92; CW16] and May–Sigurdsson [MS06], which were informed by the work of tom Dieck in [Die87]. More specifically, our work builds heavily upon Cnossen's work on twisted ambidexterity [Cno23], which is, in turn, built upon the insights of the preceding work. A more detailed account of the relationships between the present work and the ones just mentioned will be given at the end of the introduction.

The main goal of this article is to develop the theory of equivariant Poincaré duality for compact Lie groups from an ∞–categorical perspective. As a proof of concept, we will then apply it to a selection of concrete problems in equivariant geometric topology, some of which have been resolved through different methods before. Our categorical formalism of choice is the parametrised ∞–category theory of [Mar22a; Mar22b; MW22; MW24] and a central role will be played by equivariant stable homotopy theory as first systematically developed in [LMS86] and later in [GM95; MNN17]. The rest of the introduction will give an overview of our methods and highlighted results.

Notations and conventions: We work in the setting of ∞–categories as developed by Joyal and Lurie without referring to any particular model such as quasicategories. To avoid notational clutter, we will refer to ∞–categories as just categories, while classical categories (for which there is a set of morphisms between objects as opposed to a space of such) will be referred to as 1-categories. We fix three Grothendieck universes $U \in V \in W$ called small, large and very large. We denote the large category of small categories by Cat and the very large category of large categories by $\widehat{\mathrm{Cat}}$. The term "category" will be reserved for small categories. Furthermore, left adjoints will always be written on top of right adjoints in our diagrams.

Equivariant and parametrised homotopy theory

Let G be compact Lie group. It has long been understood that in order to have good access to inductive methods in equivariant homotopy theory, the fixed points spaces for all closed subgroups of G should be recorded as part of the structure of a G–space. The earliest categorical articulation of this principle is the theorem of Elmendorf's which says that there is an equivalence of (∞–)categories $\mathcal{S}_G \simeq \mathrm{Fun}(\mathcal{O}(G)^{\mathrm{op}}, \mathcal{S})$ between the category of G–spaces and presheaves on the orbit category of G.

In the same way that the category of spectra is the universal homology theory on spaces, it has been identified in [Seg70] and fully developed in [LMS86] that the appropriate replacement of spectra in the equivariant setting is the stable category Sp_G of *genuine G–spectra*. In the finite group case, we may even view Sp_G as G–Mackey functors valued in spectra. To each G–space $X \in \mathcal{S}_G$ we may associate an "equivariant stable homotopy type" $\Sigma^\infty_+ X \in \mathrm{Sp}_G$. From this, we may, among other things, recover the stable homotopy type of all fixed point spaces X^H for subgroups $H \leq G$ via the *geometric fixed points functor* $\Phi^H \colon \mathrm{Sp}_G \to \mathrm{Sp}$ to obtain $\Phi^H(\Sigma^\infty_+ X) \simeq \Sigma^\infty_+(X^H)$. In fact, more generally, there is a geometric fixed points functor associated to a *family* $\mathcal{F}$ of closed subgroups of G, and it should be thought as a functor which universally kills equivariant cells G/H where $H \in \mathcal{F}$.

It turns out to be fruitful to treat questions about G–spaces not only through a categorical lense, but rather work with equivariant versions of categories themselves. Two equivalent approaches have been developed, the first by Barwick–Dotto–Glasman–Nardin–Shah [BDG+16a; BDG+16b; Sha23; Nar17; NS22] and the second by Martini–Wolf [Mar22a; Mar22b; MW22; MW24]. Each formalism has their advantages, and for our purposes in this article, we have chosen to work mainly in the second one since it affords us the flexibility of working over an arbitrary topos: this will allow us to give uniform and streamlined proofs. In either case, the appropriate replacement for Cat in the equivariant setting is the category $\mathrm{Cat}_G := \mathrm{Fun}(\mathcal{O}(G)^{\mathrm{op}}, \mathrm{Cat})$ of *G–categories*. We will write $\underline{\mathcal{C}}$ for an object in Cat_G and $\mathcal{C}(G/H)$ for the evaluation of $\underline{\mathcal{C}}$ at G/H. Some G–categories of special interest to us, viewed as presheaves on the orbit category, are

$$\begin{aligned} \underline{\mathcal{S}}\colon &\quad G/H \mapsto \mathcal{S}_H && \text{the } G\text{–category of } G\text{–spaces;} \\ \underline{\mathrm{Sp}}\colon &\quad G/H \mapsto \mathrm{Sp}_H && \text{the } G\text{–category of } G\text{–spectra;} \\ \underline{\mathcal{P}\mathrm{ic}}(\underline{\mathrm{Sp}})\colon &\quad G/H \mapsto \mathcal{P}\mathrm{ic}(\mathrm{Sp}_H) && \text{the } G\text{–space of invertible } G\text{–spectra.} \end{aligned}$$

Crucially, for a large part of this article, we will rely upon a good theory of parametrised presentable categories, whereupon we may speak of, for instance, the category $\mathrm{Pr}^{L,G-\mathrm{st}}_G$ of G–stable presentable G–categories (in which $\underline{\mathrm{Sp}}_G$ is the symmetric monoidal unit). Now for a closed normal subgroup $N \leq G$ and writing $Q := G/N$ for the quotient group, there is a fully faithful inclusion of large Q–categories into large G–categories $\mathrm{incl}\colon \widehat{\mathrm{Cat}}_Q \hookrightarrow \widehat{\mathrm{Cat}}_G$ which admits a left adjoint $(-)^N$ given by forgetting all information from the subgroups of G that do not contain N. These two functors restrict to give functors $\mathrm{incl}\colon \mathrm{Pr}^{L,Q-\mathrm{st}}_Q \hookrightarrow \mathrm{Pr}^{L,G-\mathrm{st}}_G$ and $(-)^N\colon \mathrm{Pr}^{L,G-\mathrm{st}}_G \to \mathrm{Pr}^{L,Q-\mathrm{st}}_Q$ respectively. However, the adjunction on large categories does *not* descend to an adjunction on the presentable categories because the adjunction unit in $\widehat{\mathrm{Cat}}_G$ is not a morphism in $\mathrm{Pr}^{L,G-\mathrm{st}}_G$. Nevertheless, we show the following:

Theorem P (Theorem 6.2.26 and Proposition 6.2.29)**.** *Let G and Q be as above. Then the inclusion $\mathrm{Pr}^{L,Q-\mathrm{st}}_{Q} \hookrightarrow \mathrm{Pr}^{L,G-\mathrm{st}}_{G}$ admits a left adjoint Φ^N which is a smashing localisation.*

We call the functor Φ^N above the *Brauer quotient* functor, borrowing the term from classical Mackey functor theory. In fact, in the precise versions of the result, we prove it more generally for families and we also prove this for small G–stable categories when the group is finite. The result above should be viewed as a categorification of the geometric fixed points functors aforementioned. We may indeed recover the usual geometric fixed points functors by considering the adjunction unit evaluated on $\underline{\mathrm{Sp}}$. Using Theorem P, we may functorially construct geometric fixed points for *any* G–stable category, a construction that will be important to us in performing isotropy separation arguments for equivariant Poincaré duality, as we shall see below.

Equivariant and parametrised Poincaré duality

Poincaré duality is usually formulated as the statement that for a closed d-manifold M, there exists an infinite cyclic local coefficient system $\mathcal{O}$ on M and a class $[M] \in H_d(M;\mathcal{O})$ such that the the cap product with $[M]$ induces, for every local coefficient system η on M, isomorphisms

$$[M] \cap - \colon H^*(M;\eta) \longrightarrow H_{d-*}(M;\eta \otimes \mathcal{O}).$$

We will briefly recall a different formulation due to [Kle01] in terms of local systems of spectra (c.f. [Lan22, App. A] for a nice and detailed exposition of this point of view). It will let us arrive at an equivariant version (even a parametrised one, in general) with little creativity.

Following the notation of [Cno23], let $M \colon M \to *$ be the unique map. We get adjunctions

$$\begin{tikzcd} \mathrm{Sp}^M \arrow[rr, bend left, "M_!"] \arrow[rr, bend right, "M_*"'] & & \mathrm{Sp} \arrow[ll, "M^*"'] \end{tikzcd}$$

where $M_!$ (resp. M_*) associates to each local system $\xi \in \mathrm{Sp}^M$ its colimit (resp. limit), and the resulting spectrum should be viewed as the homology (resp. cohomology) of M with coefficients in ξ. For a smooth manifold M, the Spivak normal fibration can be used to construct a local system $D_M \in \mathcal{P}\mathrm{ic}(\mathrm{Sp})^M \subset \mathrm{Sp}^M$. The stable Pointryagin–Thom collapse map can be viewed as a map $c \colon \mathbb{S} \to M_! D_M$, which deserves the name "fundamental class" for M. It is possible[1] to describe the cap product with the fundamental class as a morphism in $\mathrm{Fun}(\mathrm{Sp}^M, \mathrm{Sp})$

$$c \cap - \colon M_*(-) \to M_!(- \otimes D_M) \tag{5.1}$$

[1]This is conveniently descibed in [Lur11] or [Lan22], which we generalise in §7.1.

and Poincaré duality may be interpreted as demanding that this transformation be an equivalence. It turns out that using general Morita theory, it is possible to construct a unique $D_X \in \mathrm{Sp}^X$ and $c\colon \mathbb{S} \to X_! D_X$ for *any compact space* X such that the associated map $c \cap -\colon X_*(-) \to X_!(- \otimes D_X)$ is an equivalence (since it is such a crucial property, we term the property of this map being an equivalence as *twisted ambidexterity*, inspired by [Cno23]). The local system D_X is referred to as the dualising sheaf of X. We call the compact space X a *Poincaré space* if the dualising sheaf D_X takes values in invertible spectra, i.e. in $\mathcal{P}\mathrm{ic}(\mathrm{Sp})$. Poincaré duality, as formulated above, shows that closed manifolds are Poincaré spaces. It bears mentioning that Wall, in his seminal paper [Wal67], introduced the notion of *Poincaré complexes* and we show in Example 7.2.16 that his notion coincides with Poincaré spaces as defined above (see also [Lan22, Prop. A.12] for a proof for finite spaces).

Now the theory of parametrised higher categories as introduced in [MW22; MW24] affords us the latitude of considering the situation just presented, but working internally to an arbitrary base topos $\mathcal{B}$ (e.g. the topos $\mathcal{S}_G$ of G–spaces). In this setting, one may speak of $\mathcal{B}$–functor categories, $\mathcal{B}$–adjunctions, $\mathcal{B}$–Kan extensions, $\mathcal{B}$–(co)limits, $\mathcal{B}$–Morita theory, etc. For example, in the equivariant situation, we may take parametrised colimits with respect to a diagram indexed by a G–category (and so in particular, diagrams indexed by G–spaces). In light of this, we may just transpose the discussions of the previous paragraph into the parametrised setting with relative ease and make sense of the notion of parametrised Poincaré duality.

However, in our theory, we have chosen to strictly generalise the well–known presentation above in two main ways: (1) we do not just consider the coefficient category of spectra (or rather $\mathcal{B}$–spectra), but we allow for arbitrary symmetric monoidal coefficient categories (which was also done in [Cno23] in the twisted ambidextrous setting); (2) we do not just consider *presentable* coefficient $\mathcal{B}$–categories, as in [Cno23], but also arbitrary $\mathcal{B}$–categories. As we shall see in our geometric applications later, both of these extra flexibilities will play important and conceptually natural roles. Crucially, point (2) precludes us from having access to Morita theory, by which token being a $\mathcal{B}$–Poincaré space is a property. Hence, we will have to declare more structures in order to be able to speak of Poincaré duality in arbitrary coefficient categories. We axiomatise this as follows:

Definition (Spivak data, Definition 7.1.1)**.** Let $\underline{X}$ be a $\mathcal{B}$–space and $\underline{\mathcal{C}}$ a symmetric monoidal $\mathcal{B}$–category admitting $\underline{X}$–shaped colimits. A *$\underline{\mathcal{C}}$–Spivak datum for $\underline{X}$* is defined to be a pair (ξ, c) where $\xi \in \underline{\mathrm{Fun}}(\underline{X}, \underline{\mathcal{C}})$ is called the *dualising sheaf* and $c\colon \mathbb{1}_{\underline{\mathcal{C}}} \to X_!\xi$ is a morphism in $\underline{\mathcal{C}}$ called the *fundamental class*.

From this datum, provided $\underline{\mathcal{C}}$ satisfies a standard condition called the $\underline{X}$–projection formula (c.f. Terminology 6.1.13), we may construct from (ξ, c) a transformation

$$c \cap_\xi -\colon X_*(-) \longrightarrow X_!(- \otimes \xi) \tag{5.2}$$

as in (5.1), called the capping transformation. This is a morphism in $\underline{\mathrm{Fun}}(\underline{\mathcal{C}}^{\underline{X}}, \underline{\mathcal{C}})$. We then say that the $\underline{\mathcal{C}}$–Spivak datum (ξ, c) is *twisted ambidextrous* if (5.2) is an equivalence, and we say that it is *Poincaré* if additionally $\xi\colon \underline{X} \to \underline{\mathcal{C}}$ factors through $\underline{\mathcal{P}\mathrm{ic}}(\underline{\mathcal{C}})$. As will become clear in the article, one advantage of studying such a structural axiomatisation of the situation is that it provides us with a finer control over the specific fundamental class and capping equivalences at play. Also note that this approach is very close to traditional formulations of Poincaré duality and even covers general duality groups in the sense of Bieri-Eckmann [BE73], which we hope helps clarify the relation of modern works such as Cnossen's [Cno23] with classical literature.

Having set up the primitive notions of the paper, we focus on the equivariant setting (i.e. working over the base topos $\mathcal{B} = \mathcal{S}_G$ of G–spaces) for the rest of the introduction and point out the more general parametrised versions along the way, as appropriate.

We now state one of the main theorems of our abstract equivariant Poincaré duality theory. For ease of statement in this introductory section, we only state it for presentable G–categories, where being Poincaré is a property of a G–space (i.e. the Spivak datum is unique, if it exists).

Theorem Q (Poincaré isotropy basechange, Theorem 8.2.7)**.** *Let G be a compact Lie group, N a closed normal subgroup, $\underline{X}$ a G–space, and $\underline{\mathcal{C}}$ a presentably symmetric monoidal fibrewise stable G–category. If $\underline{X}$ is Poincaré with coefficient $\underline{\mathcal{C}}$, then the G/N–fixed points space $\underline{X}^N$ is Poincaré with coefficient in the fibrewise stable Brauer quotient G/N–category $\Phi^N\underline{\mathcal{C}}$ of Theorem P.*

In the full statement, the result above works in the generality of a fixed family of closed subgroups and we also provide a version of the theorem for small categories with a weaker conclusion, but which is nevertheless strong enough for our applications in §9.1. Furthermore, Theorem Q will be the key tool for our categorified Smith–theoretic proof of Theorem W.

It should also be mentioned that the theorem above is an immediate consequence of a much more general set of basechange results for arbitrary base topoi (c.f. Theorems 7.3.5, 7.3.8 and 7.3.12). These general results constitute the main theorems in our theory of parametrised Poincaré duality. The operating philosophy of these results, and thus of the paper by extension, is that many important inductive manoeuvres on Poincaré duality may be casted as instances of basechanging the coefficient categories and basechanging the ambient topoi.

The most important coefficient category for us will be that of genuine G–spectra $\underline{\mathrm{Sp}}$, and we say that a G–space is G–Poincaré if it is Poincaré with respect to $\underline{\mathrm{Sp}}$. As a straightforward consequence of Theorem Q, we obtain the following result which says that being Poincaré is compatible with taking fixed points. It should be viewed as a spectral enhancement of the homological statement [CW17, Prop. 2.4] of Costenoble–Waner.

Theorem R (Theorem 8.2.9)**.** *Let G be a compact Lie group, $H \leq G$ a closed subgroup, and $N \leq G$ a closed normal subgroup. If $\underline{X}$ is a G–Poincaré space, then X^H is a Poincaré space and $\underline{X}^N$ is a G/N–Poincaré space.*

In fact, we also provide a conditional converse to the result above in Theorem 8.2.10 where we give a recognition principle for equivariant Poincaré duality in terms of nonequivariant Poincaré duality by way of the geometric fixed points functors. We warn the reader that the converse - that a compact G-space $\underline{X}$ is G-Poincaré if X^H is Poincaré for each closed subgroup $H \leq G$ - is *not* true. In fact, we also provide a conditional converse to the result above in Theorem 8.2.10 where we give a recognition principle for equivariant Poincaré duality in terms of nonequivariant Poincaré duality by way of the geometric fixed points functors. We warn the reader that the converse - that a compact G-space $\underline{X}$ is G-Poincaré if X^H is Poincaré for each closed subgroup $H \leq G$ - is *not* true. Using a contruction of Jones [Jon71] and general considerations on equivariant Poincaré spaces, we can construct a counterexample, see Corollary 9.1.13. Nevertheless, we provide a complete characterisation of C_p–Poincaré spaces in the companion article [HKK24a] by identifying an extra condition in addition to the fixed points spaces being nonequivariantly Poincaré.

Next, to justify the theory of equivariant Poincaré spaces, we first give a large collection of examples of such as encapsulated by the following (the smooth manifolds case is certainly not new and has been proven in various forms for example in [MS06; CW16; CW17; HKZ24]):

Theorem S (Proposition 8.4.2 and Theorem 8.4.8)**.** *Let G be a compact Lie group. Then smooth closed G–manifolds and tom Dieck's generalised homotopy representations are G–Poincaré spaces.*

Here, by tom Dieck's generalised homotopy representations, we mean a compact G–space $\underline{\mathcal{V}}$ such that all fixed points $\mathcal{V}^H$ have the homotopy type of a sphere of some dimension. They are a class of G–spaces strictly distinct from smooth G–manifolds. For example, Bredon [Bre72] gave an example of a generalised C_2–homotopy representation $\underline{\mathcal{V}}$ such that $\mathcal{V}^{C_2}$ and $\mathcal{V}^e$ are spheres of the same dimension, although the map $\mathcal{V}^{C_2} \to \mathcal{V}^e$ is not an equivalence. Of course, this cannot arise as the underlying C_2–space of a smooth closed C_2–manifold.

With plenty of naturally interesting examples in hand, we then provide a suite of construction principles to construct new examples of equivariant Poincaré spaces from old ones in §8.3. Among other things, we show that equivariant Poincaré duality is preserved under various standard equivariant operations such as inflations, restrictions, inductions, coinductions, and Borelifications. We also make contact with the nilpotence theory of Mathew–Naumann–Noel [MNN17; MNN19] and show that Poincaré duality interacts nicely with nilpotence with respect to families. Furthermore, we also show the following equivariant generalisation of Klein's well–known result [Kle01, Cor. F].

Theorem T (Theorem 8.3.12)**.** *Let $\underline{\mathcal{C}}$ be a presentably symmetric monoidal G–category and $f\colon \underline{X} \to \underline{Y}$ a map of G–spaces. If $\underline{Y}$ is $\underline{\mathcal{C}}$–Poincaré and for every closed subgroup $H \leq G$, the fibres of f over every H–point of $\underline{Y}$ is $\mathrm{Res}^G_H \underline{\mathcal{C}}$–Poincaré, then $\underline{X}$ is $\underline{\mathcal{C}}$–Poincaré too.*

Having set up a robust and nonempty abstract theory, we now ask ourselves: what does it all mean and what is it useful for?

Phenomena and applications

It turns out that equivariant Poincaré duality for a G–space $\underline{X}$ offers quite a lot "hidden" homotopical information about $\underline{X}$ that is not obvious from merely having all its fixed points satisfying Poincaré duality. To put it in a slogan, this is essentially because there is a global fundamental class which ties together the local fundamental classes of the various fixed points spaces in nontrivial ways. This is certainly not a new observation and is one that has been appreciated by many of the forerunners to this story. For the remainder of this introduction, we highlight three applications of a geometric flavour of our theory which exploit this principle in one form or another and which illustrate the rigidity of Poincaré spaces hinted at before. In a companion article [HKK24a], we give another application of our theory in the context of the Nielsen realisation problem. More precisely, we were able to employ the equivariant fundamental class theory developed here to remove a technical group homological condition stipulated by Davis–Lück [DL24] in their work on the aforementioned problem.

Let p be a prime and G a finite group. Now for a G–space $\underline{X}$, we may view the cohomology group $H^*(X^G;\mathbb{F}_p)$ as a count of the fixed points of $\underline{X}$ mod p. As aptly interpreted by Browder in [Bro87], if a map of G–spaces $f\colon \underline{X} \to \underline{Y}$ induces an injection on $H^*(-;\mathbb{F}_p)$, then we may "pull back" the mod p fixed points of $\underline{Y}$ to those of $\underline{X}$ and, among many things, he studied situations in which one can upgrade this cohomological statement to an actual surjection on the fixed points as topological spaces. Cohomological injection results of this type were first proved by Bredon as [Bre73, Thm. 5.1] for the group $G = C_p$ purely homotopy–theoretically and later on generalised by Browder as [Bro72, Thm. 1.1] to arbitrary finite abelian p–groups under stronger manifold assumptions. This question has also been studied for instance in [ES86; HP06]. In this line, we employ our categorical technology in the generality of Poincaré duality for small, non–presentable coefficient categories to prove the following version of the aforementioned results:

Theorem U (Theorem 9.1.1)**.** *Let A be an elementary abelian p–group. Let $f\colon \underline{X} \to \underline{Y}$ be a map of compact A–spaces. Suppose X^e and Y^e are $\mathrm{H}\mathbb{F}_p$–Poincaré spaces such that*

$f^e \colon X^e \to Y^e$ is of degree one. Then for any $\mathrm{H}\mathbb{F}_p$–local system $\zeta \in \mathrm{Fun}(Y^A, \mathrm{Perf}_{\mathrm{H}\mathbb{F}_p})$ for the fixed point space Y^A, the map f^A induces an injection $H^*(Y^A;\zeta) \to H^*(X^A; f^*\zeta)$.

Unlike the cited works above, our methods avoid manifold assumptions altogether and apart from one preliminary standard argument, we also avoid spectral sequences entirely and use instead formal categorical and stable homotopy theoretic manipulations. It should be noted also that our result works for arbitrary twisted coefficient systems, which as far as we know, is new and depends crucially on the categorical nature of our approach.

As an application of Theorem U (in fact, the version proved by Bredon suffices), we obtain the following rigidity result for equivariant Poincaré duality spaces.

Theorem V (Theorem 9.1.14)**.** *Let G be a solvable finite group (e.g. a group of odd order) and $\underline{X} \in \mathcal{S}_G^\omega$ a compact G–Poincaré space with $X^e \simeq * \in \mathcal{S}$. Then $\underline{X} \simeq \underline{*} \in \mathcal{S}_G$.*

This is a slightly surprising result in light of Bredon's examples mentioned after Theorem S which demonstrated that Poincaré spaces can be rather counterintuitive when the underlying space is noncontractible.

We mention one more application, whose investigation was one of the main goals of this project. In [CF64], Conner–Floyd made the following conjecture:

> *There cannot exist a periodic differentiable map of odd prime power period acting on a closed oriented manifold V^n, $n > 0$ preserving the orientation and possessing exactly one fixed point.*

The first proof of this statement (in fact, a slightly more general version) was given by Atiyah-Bott in [AB68] and soon after by Conner–Floyd [CF66] themselves. Many variations have been proven since then, and we mention [Lüc88; ABK92] as further examples. Atiyah-Bott's argument uses Atiyah–Singer's index theory, whereas Conner–Floyd's proof uses a particular bordism spectrum. In all these cases, local structures in the geometric settings were used in essential ways.

Inspired by the notion of a "gluing class" due to Lück which measures how the singular part of a G–space is glued into the whole space, we consider such a construction in our setting and use it to prove a very general version of the Conner–Floyd conjecture which in particular yields the theorem of Atiyah–Bott as an immediate corollary.

Theorem W (Theorem 9.2.2)**.** *Let p be an odd prime, $G = C_{p^k}$ for some k, and suppose $\underline{X} \in \mathcal{S}_G^\omega$ is a G–Poincare space such that the underlying space $X^e \in \mathcal{S}^\omega$ is connected, $\mathbb{Z}$–orientable, and has formal dimension (in the classical sense) $d > 0$. Then $X^G \not\simeq *$.*

Our proof uses categorified Smith–theoretic methods afforded to us by Theorem Q which reduces the problem to various forms of Tate cohomology considerations. In particular, it is fully homotopical and thus is of a "global" nature. We hazard a suggestion here that, apart from being a new generalisation of a very classical result, Theorem W locates the explanation of such phenomena in the global realm of homotopy theory as opposed to the local one of geometry. Finally, let us point out that when paired together, Theorems V and W give a curious partial dichotomy for C_{p^k}–Poincaré spaces delineated by whether or not the underlying space is contractible.

Before closing the introduction, we mention that since the theory of Poincaré duality here was developed in the generality of Martini–Wolf's parametrised category theory, it might be interesting to explore the theory presently developed in the context of topoi other than the equivariant ones. It could be said that the defining feature of our work is in exploiting various kinds of geometric morphisms of topoi central to equivariant homotopy theory, and one can imagine that this might also lead to fruitful lines of pursuit in other contexts.

Relations to other work

The following works are some of the milestones that made this article possible. Wall introduced the notion of a Poincaré space motivated by surgery theory, and developed their theory in [Wal67]. Klein built up an impressive amount of theory related to Poincaré spaces, one of his most influential concepts being that of the dualising spectrum [Kle01; Kle07]. His approach was revisited by Lurie [Lur11, Lecture 26], Nikolaus-Scholze [NS18, Sec. I.4.1.] and Land [Lan22, Appendix A], also providing an account of the "universality" of Klein's construction. An advantage of Klein's approach is that the stable Spivak fibration of a Poincaré space admits a categorically more natural description (as the dualising spectrum) than in Wall's original work, where it had to be constructed. The theory of dualising spectra in general was coined "twisted ambidexterity" by Cnossen in [Cno23].

Cnossen develops twisted ambidexterity in a general topos in terms of parametrised homotopy theory, and his approach is what we most closely follow. His motivation is a characterisation of the G-category of G-spectra as the initial presentable, fibrewise stable G-category in which all compact G-spaces are twisted ambidextrous. An important predecessor in the equivariant context is the book of May-Sigurdsson [MS06], which also gives an account of equivariant Poincaré duality. A more classical approach to equivariant Poincaré duality in terms of equivariant homology and cohomology (over the Burnside ring), much in the spirit of Wall's original definition can be found in the work of Costenoble-Waner [CW92; CW17]. For finite groups G, a nonabelian version of equivariant Poincaré duality for so–called "V–framed manifolds" has also been studied in [Zou23; HKZ24] which in particular implies the homological version of Poincaré duality for such objects, c.f. [HKZ24, Prop. 4.1.4]. An approach to Poincaré duality in the context of six-functor-formalisms is to be found for instance in [Sch23, Lecture V].

Organisation of the paper

In Chapter 6, we introduce and develop the categorical underpinnings that will support the later sections. In more detail, we recall in §6.1 the Martini–Wolf theory of parametrised higher categories and take the opportunity to record some elementary observations about geometric morphisms that we need. In §6.2, we specialise the preceding discussions to the equivariant context and recall the standard gamut of equivariant operations on categories; this will lead to the proof of Theorem P, categorifying the well–known geometric fixed points functor. This will be used later to articulate our results about fixed points of equivariant Poincaré spaces.

Having set up the requisite language, we turn to the matter of defining and studying Poincaré duality in Chapter 7 in the general context of parametrising over arbitrary topoi. We define and work out the basic properties of Spivak data in §7.1; we then use this structure to define twisted ambidexterity and Poincaré duality in §7.2 with respect to arbitrary coefficient $\mathcal{B}$–categories. In §7.3, we give several constructions one can perform on Spivak data and prove the main results of the section in the form of Theorems 7.3.5 and 7.3.8 on basechanging coefficient categories and Theorem 7.3.12 on basechanging the base topoi. Finally, we set up a theory of degrees for maps between Poincaré spaces in §7.4.

We specialise the general parametrised theory in Chapter 7 to the equivariant situation in Chapter 8 for compact Lie groups. After recording the specialisations of the notions in §8.1, we state and prove several isotropy separation statements including Theorems Q and R in §8.2, which will form our main suite of techniques for dealing with fixed points of equivariant Poincaré spaces. Following that, we provide a set of construction principles in §8.3 to generate new equivariant Poincaré spaces from old ones and we supply in §8.4 geometrically natural examples of Poincaré spaces. We then introduce the notion of gluing classes in §8.5 that will form the main obstruction class for our applications in §9.2, and we lay down a rudimentary theory of equivariant degrees in §8.6.

In the final Chapter 9, we use categorified Smith–theoretic methods supported by the abstract theory developed in the article to give two strands of applications: in §9.1, we use degree theory to show Theorem U, which is in turn used to show Theorem V; then, in §9.2, we use the gluing classes to prove Theorem W.

Beyond the main body of the article, we record in Chapter A several characterisations of G–stability for presentable categories when G is a compact Lie group, and we prove a standard observation about reflecting pushout squares in Chapter B.

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Chapter 6

Preliminaries

The present section reviews the techniques that are essential to our approach to parametrised Poincaré duality. In §6.1, we first recall some preliminaries on category theory parametrised by a topos $\mathcal{B}$. Special attention is given to presentable $\mathcal{B}$-categories, and basechange methods that allow us to switch the base topos along a geometric morphism. These basechange results will be essential for the isotropy separation arguments in the equivariant context.

After that, §6.2 specialises to topoi related to the category $\mathcal{S}_G$ of G-spaces, where G is a compact Lie group. This also features various change-of-group functors like induction, restriction and coinduction along a homomorphism of compact Lie groups $\alpha\colon H \to G$, as well as the theory of families. We give a quick recollection on the basics of G-spaces. After that, we record some facts on equivariant stability, preservation of equivariant stability under change-of-group functors, and multiplicative properties of these constructions. We then prove the main result of this section, namely Theorem P on Brauer quotients which categorifies the geometric fixed points functors. The section ends with some remarks on free actions that will be used later on.

6.1 Parametrised category theory

For the rest of this section, let $\mathcal{B}$ be a topos. For us, most topoi of interest will actually be presheaf topoi, and for the purpose of this article the most important such will be that of presheaves on the orbit category $\mathcal{O}(G)$ for a compact Lie group G. As category theory internal to $\mathcal{B}$ is essential to our considerations, we give a short recollection of the formalism developed by Martini and Martini-Wolf in the series of articles [Mar22b; Mar22a; MW22; MW24]. Let us mention here that the theory of categories parametrised by a topos was preceded by about a decade by the theory of categories parametrised by presheaf topoi, pioneered by Barwick–Dotto–Glasman–Nardin–Shah in [BDG+16b; BDG+16a; Nar16; Sha23].

Definition 6.1.1. A $\mathcal{B}$*–category* $\underline{\mathcal{C}}$ is a limit preserving functor $\underline{\mathcal{C}}\colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}$, i.e. a sheaf of categories on $\mathcal{B}$. Denote by $\mathrm{Cat}_{\mathcal{B}} \subseteq \mathrm{Psh}_{\mathrm{Cat}}(\mathcal{B})$ the full subcategory on $\mathcal{B}$–categories. Maps in $\mathrm{Cat}_{\mathcal{B}}$ are called $\mathcal{B}$*-functors*.

In [Mar22b], Martini produces an equivalence of categories

$$\mathrm{Psh}_{\mathrm{Cat}}(\mathcal{B}) \supseteq \{\text{Cat-valued sheaves on } \mathcal{B}\} \simeq \{\text{Complete Segal objects in } \mathcal{B}\} \subseteq s\mathcal{B}.$$

It is worthwhile to study $\mathcal{B}$–categories from both perspectives, the *parametrised* point of view on the left as well as the *internal* point of view on the right.

As our arguments will often require us to work with unparametrised categories and $\mathcal{B}$–categories at the same time, we follow the convention of underlining $\mathcal{B}$ categories, so a generic $\mathcal{B}$ category is denoted $\underline{\mathcal{C}}, \underline{\mathcal{D}}, \underline{\mathcal{E}}, \ldots$ and so on. For example, the category of G–spaces will be denoted by $\mathcal{S}_G$ while the G–category of G–spaces is written $\underline{\mathcal{S}}$ (or $\underline{\mathcal{S}}_G$ if we want to emphasise that this is happening for the group G).

Example 6.1.2 (Presheaf topoi)**.** For a small category T, consider the presheaf topos $\mathrm{Psh}(T) = \mathrm{Fun}(T^{\mathrm{op}}, \mathcal{S})$. We write $\mathrm{Cat}_T := \mathrm{Cat}_{\mathrm{Psh}(T)}$ and call it the category of T*–categories*. Restriction along the Yoneda embedding $T \hookrightarrow \mathrm{Psh}(T)$ induces an equivalence $\mathrm{Cat}_T \xrightarrow{\simeq} \mathrm{Fun}(T^{\mathrm{op}}, \mathrm{Cat})$ so a T–category is nothing but a functor $T^{\mathrm{op}} \to \mathrm{Cat}$. In particular for $T = *$, we see $\mathrm{Cat}_{\mathcal{S}} \simeq \mathrm{Cat}$ and an $\mathcal{S}$–category is just an ordinary (∞-)category. In the special case where G is a finite group (or a compact Lie group) and $T = \mathcal{O}(G)$ is its orbit category, the category of transitive G-sets (or homogeneous G–spaces), we obtain the category $\mathrm{Cat}_G := \mathrm{Cat}_{\mathcal{O}(G)}$ of G–categories.

Example 6.1.3 ($\mathcal{B}$-groupoids)**.** The Yoneda embedding $\mathcal{B} \to \mathrm{Fun}(\mathcal{B}^{\mathrm{op}}, \mathrm{Cat})$ restricts to a limit preserving fully faithful functor $\mathcal{B} \to \mathrm{Cat}_{\mathcal{B}}$. An object in the essential image will be referred to as a $\mathcal{B}$*-groupoid*.

Switching to the internal picture, Martini [Mar22b, Section 3.1] also characterised $\mathcal{B}$-groupoids as the complete Segal objects that are equivalent to constant simplicial objects in $\mathcal{B}$. We will not distinguish between $\mathcal{B}$-groupoids and objects in $\mathcal{B}$, so to avoid confusion we also denote both with an underline by $\underline{X}, \underline{Y}, \underline{Z}, \ldots$, except in the spectial case $\mathcal{B} = \mathcal{S}$.

With equivariant applications in mind, it will be important for us to change the base topos $\mathcal{B}$. This can be done along *geometric morphisms* of topoi, defined as:

Definition 6.1.4. A geometric morphism is an adjunction $f^*\colon \mathcal{B} \rightleftharpoons \mathcal{B}' : f_*$ between topoi whose left adjoint f^* is left exact, i.e. commutes with finite limits.

Construction 6.1.5 (Basechange along geometric morphisms)**.** A geometric morphism $f^*\colon \mathcal{B} \rightleftharpoons \mathcal{B}' : f_*$ of topoi induces an adjoint pair $f^*\colon \mathrm{Cat}_{\mathcal{B}} \rightleftharpoons \mathrm{Cat}_{\mathcal{B}'} : f_*$ where the right adjoint f_* is given by restriction along $(f^*)^{\mathrm{op}}\colon \mathcal{B}^{\mathrm{op}} \to (\mathcal{B}')^{\mathrm{op}}$.

In the internal picture, the functor $f^*\colon \mathcal{B} \to \mathcal{B}'$ applied entrywise induces a functor on simplicial objects $s\mathcal{B} \to s\mathcal{B}'$, which commutes with finite limits. By [Mar22b, Lem. 3.3.1], it restricts to a functor on complete Segal objects, and this is how to obtain $f^*\colon \mathrm{Cat}_{\mathcal{B}} \to \mathrm{Cat}_{\mathcal{B}'}$.

Lemma 6.1.6. *For a geometric morphism $f^*\colon \mathcal{B} \rightleftharpoons \mathcal{B}' : f_*$ of topoi, the functor $f^*\colon \mathrm{Cat}_{\mathcal{B}} \to \mathrm{Cat}_{\mathcal{B}}$ preserves finite limits. In fact, it preserves all limits if $f^*\colon \mathcal{B} \to \mathcal{B}'$ does.*

Proof. The induced functor $sf^*\colon s\mathcal{B} \to s\mathcal{B}'$ commutes with finite limits, and complete Segal objects are closed under limits. Thus, f^* preserves finite limits by being a restriction of a finite limit preserving functor to a category closed under (finite) limits. □

Example 6.1.7 (Geometric morphisms from presheaves)**.** A good supply of examples of geometric morphisms is given by considering a functor $f\colon S \to T$ of small categories. Then restriction and right Kan extension along $f^{\mathrm{op}}\colon S^{\mathrm{op}} \to T^{\mathrm{op}}$ induce a geometric morphism $f^*\colon \mathrm{Psh}(T) \rightleftharpoons \mathrm{Psh}(S) : f_*$.

Example 6.1.8 (Étale geometric morphisms)**.** If $\mathcal{B}$ is a topos and $\underline{X} \in \mathcal{B}$, then so is the slice category $\mathcal{B}_{/\underline{X}}$. We have an ajunction

$$(\pi_{\underline{X}})^*\colon \mathcal{B} \rightleftharpoons \mathcal{B}_{/\underline{X}} : (\pi_{\underline{X}})_* \tag{6.1}$$

whose left adjoint takes $\underline{A} \in \mathcal{B}$ to $\underline{A} \times \underline{X} \to \underline{X}$. Now $\mathcal{B}_{/\underline{X}}$ itself is a topos, and the adjunction above is in fact a geometric morphism of topoi. Geometric morphisms equivalent (in the category of topoi) equivalent to such of this kind are called *étale geometric morphisms*, see [Lur09, Sec. 6.3.5.] for a detailed account. A special feature of étale geometric morphisms is that in the adjunction (6.1) a further left adjoint exists, and so $(\pi_{\underline{X}})^*$ commutes with all limits. The further left adjoint forgets the map to $\underline{X}$ and is denoted by $(\pi_{\underline{X}})_!$. A useful characterisation of étale geometric morphisms is given in [Lur09, Prop. 6.3.5.11.].

Note that if a geometric morphism $f^*\colon \mathrm{Cat}_{\mathcal{B}} \rightleftharpoons \mathrm{Cat}_{\mathcal{B}'}\colon f_*$ is étale, then Lemma 6.1.6 shows that $f^*\colon \mathrm{Cat}_{\mathcal{B}} \to \mathrm{Cat}_{\mathcal{B}'}$ preserves all limits.

Example 6.1.9 (Constant categories and global sections)**.** Recall from [Lur09, Prop. 6.3.4.1] that there is a unique geometric morphism $\mathrm{Const}\colon \mathcal{S} \rightleftharpoons \mathcal{B}\colon \Gamma$. It induces an adjunction $\mathrm{Const}\colon \mathrm{Cat} \rightleftharpoons \mathrm{Cat}_{\mathcal{B}}\colon \Gamma$. We refer to Γ as the *global sections* functor. Explicitly, it is given by evaluation at the terminal object in $* \in \mathcal{B}$. Since geometric morphisms from $\mathcal{S}$ are unique, for any geometric morphism $f^*\colon \mathcal{B} \rightleftharpoons \mathcal{B}' : f_*$, we get a commuting triangle of adjunctions

$$\begin{tikzcd}
\mathrm{Cat} \arrow[d, shift right, "\mathrm{const}_{\mathcal{B}}"'] \arrow[drr, shift left, "\mathrm{const}_{\mathcal{B}'}"] & & \\
\mathrm{Cat}_{\mathcal{B}} \arrow[u, shift right, "\Gamma_{\mathcal{B}}"'] \arrow[rr, shift left, "f^*"] & & \mathrm{Cat}_{\mathcal{B}'} \arrow[ull, shift left, "\Gamma_{\mathcal{B}'}"] \arrow[ll, shift left, "f_*"]
\end{tikzcd}$$

In particular, note that $\mathrm{const}_{\mathcal{B}'} \simeq f^* \mathrm{const}_{\mathcal{B}}$ and $\Gamma_{\mathcal{B}} f_* \simeq \Gamma_{\mathcal{B}'}$.

Example 6.1.10 (Internal functor categories and 2-categorical structures)**.** The category $\mathrm{Cat}_{\mathcal{B}}$ is cartesian closed. This means that for any $\mathcal{B}$–category $\underline{\mathcal{C}}$ the product functor $- \times \underline{\mathcal{C}} \colon \mathrm{Cat}_{\mathcal{B}} \to \mathrm{Cat}_{\mathcal{B}}$ admits a right adjoint $\underline{\mathrm{Fun}}(\underline{\mathcal{C}}, -) \colon \mathrm{Cat}_{\mathcal{B}} \to \mathrm{Cat}_{\mathcal{B}}$. We call $\underline{\mathrm{Fun}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ the *$\mathcal{B}$–category of $\mathcal{B}$-functors* and denote its global sections (in the sense of Example 6.1.9) by $\mathrm{Fun}_{\mathcal{B}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$. Maps in $\mathrm{Fun}_{\mathcal{B}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ are called *$\mathcal{B}$-natural transformations*. $\mathrm{Fun}_{\mathcal{B}}$ can be enhanced to a Cat-enrichment of $\mathrm{Cat}_{\mathcal{B}}$ making $\mathrm{Cat}_{\mathcal{B}}$ into a 2-category, see [Mar22b, Remark 3.4.3].

Definition 6.1.11 (Adjoint functors)**.** Using the 2-categorical structure on $\mathrm{Cat}_{\mathcal{B}}$, one can define an adjoint pair of $\mathcal{B}$-functors as an internal adjunction in $\mathrm{Cat}_{\mathcal{B}}$. Explicitly, an adjunction consists of a pair of $\mathcal{B}$-functors $L \colon \underline{\mathcal{C}} \rightleftharpoons \underline{\mathcal{D}} \colon R$ as well as a pair of natural transformations $\eta \colon \mathrm{id}_{\underline{\mathcal{C}}} \to RL$, $\epsilon \colon LR \to \mathrm{id}_{\underline{\mathcal{D}}}$. satisfying the triangle identities in the sense that $\epsilon_L \circ L\eta$ and $R\epsilon \circ \eta_R$ are equivalent to the respective identities.

We recall here the key standard categorical concept that will underpin most this article.

Construction 6.1.12. Suppose we have a commuting square of $\mathcal{B}$–categories

$$\begin{tikzcd} \underline{\mathcal{C}} \arrow[r, "f"] \arrow[d, "g"'] & \underline{\mathcal{D}} \arrow[d, "g'"] \\ \underline{\mathcal{C}'} \arrow[r, "f'"] & \underline{\mathcal{D}'} \end{tikzcd}$$

such that f, f' admit a right adjoints h, h' respetively. Then we obtain a transformation

$$\mathrm{BC}_* \colon gh \xrightarrow{\eta_{gh}} h'f'gh \simeq h'g'fh \xrightarrow{h'g'\epsilon} h'g'$$

called the *right Beck–Chevalley transformation*. Similarly, if f, f' admit left adjoints ℓ, ℓ' respectively, then we obtain a transformation

$$\mathrm{BC}_! \colon \ell'g' \xrightarrow{\ell'g'\eta} \ell'g'f\ell \simeq \ell'f'g\ell \xrightarrow{\epsilon_{g\ell}} g\ell$$

called the *left Beck–Chevalley transformation*. We will often omit the words “left” and “right” when the context is clear. These transformations enjoy excellent functoriality properties, and we refer the reader to [CSY22, §2.2] for a good source on these matters.

The following is an important class of Beck–Chevalley transformations.

Terminology 6.1.13 (Projection formula)**.** Let $\underline{J}$ be a $\mathcal{B}$–category and $r \colon \underline{J} \to \underline{*}$ the unique map. We say that a symmetric monoidal $\mathcal{B}$–category $\underline{\mathcal{C}}$ *satisfies the $\underline{J}$-projection formula* if it admits $\underline{J}$-shaped colimits and the Beck–Chevalley transformation

$$\mathrm{PF}^{\underline{J}}_! \colon r_!(\xi \otimes r^*(-)) \to r_!\xi \otimes (-)$$

of functors $\underline{\mathcal{C}} \to \underline{\mathcal{C}}$ is an equivalence for all $\xi \in \underline{\mathcal{C}}^{\underline{I}}$.

In [MW24, Prop. 3.2.9.] it is shown, using work on relative adjunctions due to Lurie, that a functor $R\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ admits a left adjoint if and only if the following conditions are satisfied:

1. For every object $\underline{X} \in \mathcal{B}$ the map $R(\underline{X})\colon \mathcal{C}(\underline{X}) \to \mathcal{D}(\underline{X})$ admits a left adjoint $L(\underline{X})$.
2. For every map $f\colon X \to Y$ in $\mathcal{B}$ the Beck–Chevalley transformation $f^*L(\underline{X}) \to L(\underline{Y})f^*$ is an equivalence.

Example 6.1.14 (Limits and colimits)**.** A $\mathcal{B}$–category $\underline{\mathcal{C}}$ is said to admit *$\underline{I}$-shaped $\mathcal{B}$-(co)limits* if the restriction functor $I^*\colon \underline{\mathcal{C}} \to \underline{\mathrm{Fun}}(\underline{I}, \underline{\mathcal{C}})$ along $I\colon \underline{I} \to \underline{*}$ admits a right (resp. left) adjoint. The right adjoint will usually be denoted by I_*, the left adjoint by $I_!$. Note that for example, the adjunction unit for $I_! \dashv I^*$ produces for each $F \in \mathrm{Fun}_{\mathcal{B}}(\underline{I}, \underline{\mathcal{C}})$ a natural transformation

$$F \to I^* I_! F \in \mathrm{Fun}_{\mathcal{B}}(\underline{I}, \underline{\mathcal{C}})$$

which should be thought of as analogous to the diagram defining a colimit in unparametrised category theory.

Example 6.1.15 (Symmetric monoidal categories)**.** A *symmetric monoidal $\mathcal{B}$–category* is a commutative monoid in $\mathrm{Cat}_{\mathcal{B}}$. So $\mathrm{CMon}(\mathrm{Cat}_{\mathcal{B}})$ is the category of symmetric monoidal $\mathcal{B}$–categories and symmetric monoidal functors. Notice that $\mathrm{CMon}(\mathrm{Cat}_{\mathcal{B}})$ is equivalent to the category of $\mathrm{CMon}(\mathrm{Cat})$–valued sheaves on $\mathcal{B}$.

Geometric and étale morphisms of topoi

We record here further miscellaneous elementary observations about geometric and étale morphisms that will be relevant to us later. Since this will just be a litany of minor technical results, the reader is advised to skip this on first reading and return to it as needed.

Lemma 6.1.16. *Let $f^*\colon \mathcal{B} \rightleftharpoons \mathcal{B}' : f_*$ be a geometric morphism of topoi. There is an equivalence, natural in $\underline{X} \in \mathcal{B}$ and $\underline{\mathcal{C}} \in \mathrm{Cat}_{\mathcal{B}'}$ of functors $f_*\underline{\mathrm{Fun}}(f^*\underline{X}, \underline{\mathcal{C}}) \simeq \underline{\mathrm{Fun}}(\underline{X}, f_*\underline{\mathcal{C}})$. Moreover, if $\underline{\mathcal{C}}$ were a symmetric monoidal $\mathcal{B}$–category, then this equivalence naturally upgrades to a symmetric monoidal one.*

Proof. Note that the diagram

$$\begin{tikzcd} \mathrm{Cat}_{\mathcal{B}'} \arrow[r, "(\pi_{f^*\underline{X}})^*"] \arrow[d, "f_*"] & \mathrm{Cat}_{\mathcal{B}'_{/f^*\underline{X}}} \arrow[r, "(\pi_{f^*\underline{X}})_*"] \arrow[d, "f_*"] & \mathrm{Cat}_{\mathcal{B}'} \arrow[d, "f_*"] \\ \mathrm{Cat}_{\mathcal{B}} \arrow[r, "(\pi_{\underline{X}})^*"] & \mathrm{Cat}_{\mathcal{B}_{/\underline{X}}} \arrow[r, "(\pi_{\underline{X}})_*"] & \mathrm{Cat}_{\mathcal{B}} \end{tikzcd} \tag{6.2}$$

commutes as the corresponding diagram

$$\begin{tikzcd}
\mathcal{B}' \arrow[r, "(\pi_{f^*\underline{X}})^*"] \arrow[d, "f_*"] & \mathcal{B}'_{/f^*\underline{X}} \arrow[r, "(\pi_{f^*\underline{X}})_*"] \arrow[d, "f_*"] & \mathcal{B}' \arrow[d, "f_*"] \\
\mathcal{B} \arrow[r, "(\pi_{\underline{X}})^*"] & \mathcal{B}_{/\underline{X}} \arrow[r, "(\pi_{\underline{X}})_*"] & \mathcal{B}
\end{tikzcd}$$

of topoi commutes (this can be checked after passing to left adjoints everywhere where it is easy to see, see e.g. [Lur09, Remark 6.3.5.8.]). Now the composite $\mathrm{Cat}_{\mathcal{B}} \xrightarrow{(\pi_{\underline{X}})^*} \mathrm{Cat}_{\mathcal{B}_{/\underline{X}}} \xrightarrow{(\pi_{\underline{X}})_*} \mathrm{Cat}_{\mathcal{B}}$ sends $\underline{\mathcal{C}} \in \mathrm{Cat}_{\mathcal{B}}$ to $\underline{\mathrm{Fun}}(\underline{X}, \underline{\mathcal{C}}) = \lim_{\underline{X}} \underline{\mathcal{C}}$ which proves the first part of the statement.

For the part about symmetric monoidality, note that all functors in (6.2) are finite limit preserving. As the forgetful functors $\mathrm{CMon}(\mathrm{Cat}_{\mathcal{B}}) \to \mathrm{Cat}_{\mathcal{B}}$ are limit preserving, this shows that the equivalence $f_*\underline{\mathrm{Fun}}(f^*\underline{X}, \underline{\mathcal{C}}) \simeq \underline{\mathrm{Fun}}(\underline{X}, f_*\underline{\mathcal{C}})$ from the first part naturally refines to a symmetric monoidal one. $\square$

Lemma 6.1.17. *Let $f^* \colon \mathcal{B} \rightleftharpoons \mathcal{B}' : f_*$ be an étale morphism of topoi. There is an equivalence, natural in $\underline{X} \in \mathcal{B}$ and $\underline{\mathcal{C}} \in \mathrm{Cat}_{\mathcal{B}}$ of functors $f^*\underline{\mathrm{Fun}}(\underline{X}, \underline{\mathcal{C}}) \simeq \underline{\mathrm{Fun}}(f^*\underline{X}, f^*\underline{\mathcal{C}})$. Moreover, if $\underline{\mathcal{C}}$ were a symmetric monoidal $\mathcal{B}$–category, then this equivalence naturally upgrades to a symmetric monoidal one.*

Proof. The proof is similar to Lemma 6.1.16. As f is étale, it is equivalent to a functor of the form $(\pi_{\underline{Y}})^* \colon \mathcal{B} \rightleftharpoons \mathcal{B}_{/\underline{Y}} : (\pi_{\underline{Y}})_*$ for some $\underline{Y} \in \mathcal{B}$. Observe that there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Cat}_{\mathcal{B}} \arrow[r, "(\pi_{\underline{X}})^*"] \arrow[d, "(\pi_{\underline{Y}})^*"] & \mathrm{Cat}_{\mathcal{B}_{/\underline{X}}} \arrow[r, "(\pi_{\underline{X}})_*"] \arrow[d, "(\pi_{\underline{Y}})^*"] & \mathrm{Cat}_{\mathcal{B}} \arrow[d, "(\pi_{\underline{Y}})^*"] \\
\mathrm{Cat}_{\mathcal{B}_{/\underline{Y}}} \arrow[r, "(\pi_{\underline{X}})^*"'] & \mathrm{Cat}_{\mathcal{B}_{/\underline{X}\times\underline{Y}}} \arrow[r, "(\pi_{\underline{X}})_*"'] & \mathrm{Cat}_{\mathcal{B}_{/\underline{Y}}}
\end{tikzcd} \tag{6.3}$$

coming from the commutative diagram of topoi

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "(\pi_{\underline{X}})^*"] \arrow[d, "(\pi_{\underline{Y}})^*"] & \mathcal{B}_{/\underline{X}} \arrow[r, "(\pi_{\underline{X}})_*"] \arrow[d, "(\pi_{\underline{Y}})^*"] & \mathcal{B} \arrow[d, "(\pi_{\underline{Y}})^*"] \\
\mathcal{B}_{/\underline{Y}} \arrow[r, "(\pi_{\underline{X}})^*"'] & \mathcal{B}_{/\underline{X}\times\underline{Y}} \arrow[r, "(\pi_{\underline{X}})_*"'] & \mathcal{B}_{/\underline{Y}}.
\end{tikzcd}$$

The left square here obviously commutes and commutativity of the right square is easy to check after passing to left adjoints. Now the top right composite sends $\underline{\mathcal{C}}$ to $f^*\underline{\mathrm{Fun}}(\underline{X}, \underline{\mathcal{C}})$ while the bottom left composite sends it to $\underline{\mathrm{Fun}}(f^*\underline{X}, f^*\underline{\mathcal{C}})$. For the statement about symmetric monoidality, again note that all functors in (6.3) are product preserving and use the same argument as in the proof of Lemma 6.1.16. $\square$

Lemma 6.1.18 (Pushforward of parametrised (co)limits)**.** *Let $f^*\colon \mathcal{B} \rightleftharpoons \mathcal{B}' : f_*$ be a geometric morphism of topoi. Consider $X \in \mathcal{B}$ and a $\mathcal{B}'$–category $\underline{\mathcal{C}}$ which admits $f^*\underline{X}$-shaped limits and colimits. Then $f_*\underline{\mathcal{C}}$ admits $\underline{X}$-shaped limits and colimits. Furthermore, the equivalence from Lemma 6.1.16 induces an identification of adjoint triples*

$$\begin{tikzcd} f_*\underline{\mathrm{Fun}}(f^*\underline{X},\underline{\mathcal{C}}) \arrow[r, shift left=2, "f_*r_!"] \arrow[r, shift right=2, "f_*r_*"'] \arrow[d, equal] & f_*\underline{\mathcal{C}} \arrow[l, "f_*r^*"'] \arrow[d, equal] \\ \underline{\mathrm{Fun}}(\underline{X}, f_*\underline{\mathcal{C}}) \arrow[r, shift left=2, "r_!"] \arrow[r, shift right=2, "r_*"'] & f_*\underline{\mathcal{C}}. \arrow[l, "r^*"'] \end{tikzcd} \tag{6.4}$$

Proof. First note that (6.4) commutes with the leftwards pointing arrows. Since the functor $f_*\colon \mathrm{Cat}_{\mathcal{B}'} \to \mathrm{Cat}_{\mathcal{B}}$ preserves adjunctions (see e.g. [MW24, Cor. 3.1.9.]), it follows that $f_*r_!$ and f_*r_* define left and right adjoints to r^* and (6.4) also commutes with the rightwards pointing arrows. □

Lemma 6.1.19 (Pullback of parametrised (co)limits)**.** *Let $f^*\colon \mathcal{B} \rightleftharpoons \mathcal{B}' : f_*$ be an étale morphism of topoi. Consider $X \in \mathcal{B}$ and a $\mathcal{B}$–category $\underline{\mathcal{C}}$ which admits $\underline{X}$-shaped limits and colimits. Then $f^*\underline{\mathcal{C}}$ admits $f^*\underline{X}$-shaped limits and colimits. Furthermore, the equivalence from Lemma 6.1.17 induces an identification of adjoint triples*

$$\begin{tikzcd} f^*\underline{\mathrm{Fun}}(\underline{X},\underline{\mathcal{C}}) \arrow[r, shift left=2, "f^*r_!"] \arrow[r, shift right=2, "f^*r_*"'] \arrow[d, equal] & f^*\mathcal{C} \arrow[l, "f^*r^*"'] \arrow[d, equal] \\ \underline{\mathrm{Fun}}(f^*\underline{X}, f^*\underline{\mathcal{C}}) \arrow[r, shift left=2, "r_!"] \arrow[r, shift right=2, "r_*"'] & f^*\mathcal{C}. \arrow[l, "r^*"'] \end{tikzcd} \tag{6.5}$$

Proof. The proof is identical to Lemma 6.1.18, using Lemma 6.1.17 instead of Lemma 6.1.16. □

Lemma 6.1.20. *Let $f^*\colon \mathcal{B} \rightleftharpoons \mathcal{B}' : f_*$ be a geometric morphism of topoi. Let $\underline{J} \in \mathrm{Cat}_{\mathcal{B}}$ and $\underline{\mathcal{C}}, \underline{\mathcal{D}} \in \mathrm{Cat}_{\mathcal{B}'}$, and let $\alpha\colon \underline{\mathrm{Fun}}(f^*\underline{J}, \underline{\mathcal{C}}) \times \mathrm{const}_{\mathcal{B}'}\Delta^1 \to \underline{\mathcal{D}}$ be a natural transformation and $f_*\alpha \circ (\mathrm{id} \times \eta)\colon \underline{\mathrm{Fun}}(\underline{J}, f_*\underline{\mathcal{C}}) \times \mathrm{const}_{\mathcal{B}}\Delta^1 \to \underline{\mathrm{Fun}}(\underline{J}, f_*\underline{\mathcal{C}}) \times f_*f^*\mathrm{const}_{\mathcal{B}}\Delta^1 \xrightarrow{f_*\alpha} f_*\underline{\mathcal{D}}$ the associated transformation. Then α is a natural equivalence if and only if $f_*\alpha \circ (\mathrm{id} \times \eta)$ is a natural equivalence.*

Proof. By using that $\Gamma_{\mathcal{B}} f_* \simeq \Gamma_{\mathcal{B}'}$ from Example 6.1.9, the two putatively equivalent statements are equivalent to the condition that the natural transformation $\mathrm{Fun}_{\mathcal{B}'}(f^*\underline{J}, \underline{\mathcal{C}}) \times \Delta^1 \to \Gamma_{\mathcal{B}'}\underline{\mathcal{D}}$ of unparametrised categories is a natural equivalence. □

Notation 6.1.21. Recall the *Picard space functor* $\mathcal{P}\mathrm{ic}\colon \mathrm{CMon}(\mathrm{Cat}) \to \mathrm{CGrp}(\mathcal{S})$ which takes as input a symmetric monoidal category and outputs a space of invertible objects. This functor is right adjoint to the inclusion

$$\mathrm{CGrp}(\mathcal{S}) \hookrightarrow \mathrm{CMon}(\mathcal{S}) \hookrightarrow \mathrm{CMon}(\mathrm{Cat})$$

and is corepresented as $\mathcal{P}\mathrm{ic}(-) \simeq \mathrm{Map}_{\mathrm{CMon}(\mathrm{Cat})}(\Omega^\infty \mathbb{S}, -)$.

Lemma 6.1.22. *Let $f^*\colon \mathcal{B} \rightleftharpoons \mathcal{B}' :f_*$ be a geometric morphism. Then we have an equivalence of functors*

$$\underline{\mathcal{P}\mathrm{ic}}(f_*-) \simeq f_*\underline{\mathcal{P}\mathrm{ic}}(-)\colon \mathrm{CMon}(\mathrm{Cat}_{\mathcal{B}'}) \to \mathrm{CGrp}(\mathcal{B})$$

If $f^* \dashv f_*$ were moreover étale, then we also have an equivalence

$$f^*\underline{\mathcal{P}\mathrm{ic}}(-) \simeq \underline{\mathcal{P}\mathrm{ic}}(f^*-)\colon \mathrm{CMon}(\mathrm{Cat}_{\mathcal{B}}) \to \mathrm{CGrp}(\mathcal{B}').$$

Proof. The first part is an immediate consequence of the fact that the diagram of left adjoints

$$\begin{tikzcd}
\mathrm{CMon}(\mathrm{Cat}_{\mathcal{B}}) \arrow[d, "f^*"'] & \mathrm{CGrp}(\mathcal{B}) \arrow[l, hook'] \arrow[d, "f^*"] \\
\mathrm{CMon}(\mathrm{Cat}_{\mathcal{B}'}) & \mathrm{CGrp}(\mathcal{B}') \arrow[l, hook']
\end{tikzcd}$$

commutes, which is clear. For the second part, we note that $\underline{\mathcal{P}\mathrm{ic}}(-)\colon \mathrm{CMon}(\mathrm{Cat}_{\mathcal{B}}) \to \mathrm{CGrp}(\mathcal{B})$ is given by $\underline{\mathrm{Map}}_{\mathrm{Cat}_{\mathcal{B}}}(\mathrm{const}_{\mathcal{B}}\,\Omega^\infty\mathbb{S}, -)$. Thus, since $f^* \dashv f_*$ was étale, we get that $f^*\underline{\mathcal{P}\mathrm{ic}}(-) \simeq f^*\underline{\mathrm{Map}}_{\mathrm{Cat}_{\mathcal{B}}}(\mathrm{const}_{\mathcal{B}}\,\Omega^\infty\mathbb{S}, -) \simeq \underline{\mathrm{Map}}_{\mathrm{Cat}_{\mathcal{B}'}}(\mathrm{const}_{\mathcal{B}'}\,\Omega^\infty\mathbb{S}, f^*-) \simeq \underline{\mathcal{P}\mathrm{ic}}(f^*-)$. $\square$

Corollary 6.1.23. *Let $f^*\colon \mathcal{B} \rightleftharpoons \mathcal{B}' :f_*$ be a geometric morphism of topoi and consider $\underline{X} \in \mathcal{B}$, $\underline{\mathcal{D}} \in \mathrm{CMon}(\mathrm{Cat}_{\mathcal{B}'})$, and $\underline{\mathcal{E}} \in \mathrm{CMon}(\mathrm{Cat}_{\mathcal{B}})$.*

(1) A functor $\underline{X} \to f_*\underline{\mathcal{D}}$ has the property that it factors through $\underline{\mathcal{P}\mathrm{ic}}(f_*\underline{\mathcal{D}}) \hookrightarrow f_*\underline{\mathcal{D}}$ if and only if the associated functor $f^*\underline{X} \to \underline{\mathcal{D}}$ factors through $\underline{\mathcal{P}\mathrm{ic}}(\underline{\mathcal{D}}) \hookrightarrow \underline{\mathcal{D}}$,

(2) Suppose $f^* \dashv f_*$ is moreover étale. If a functor $\underline{X} \to \underline{\mathcal{E}}$ factors through $\underline{\mathcal{P}\mathrm{ic}}(\underline{\mathcal{E}})$, then $f^*\underline{X} \to f^*\underline{\mathcal{E}}$ factors through $\underline{\mathcal{P}\mathrm{ic}}(f^*\underline{\mathcal{E}})$.

Proof. Part (1) is an immediate consequence of the equivalence

$$\mathrm{Map}_{\mathrm{Cat}_{\mathcal{B}}}(\mathrm{const}_{\mathcal{B}}\,*, f_*\underline{\mathrm{Fun}}(f^*\underline{X}, \underline{\mathcal{P}\mathrm{ic}}(\underline{\mathcal{D}}))) \simeq \mathrm{Map}_{\mathrm{Cat}_{\mathcal{B}'}}(\mathrm{const}_{\mathcal{B}'}\,*, \underline{\mathrm{Fun}}(f^*\underline{X}, \underline{\mathcal{P}\mathrm{ic}}(\underline{\mathcal{D}})))$$

and the computation

$$\underline{\mathrm{Fun}}(\underline{X}, \underline{\mathcal{P}\mathrm{ic}}(f_*\underline{\mathcal{D}})) \simeq \underline{\mathrm{Fun}}(\underline{X}, f_*\underline{\mathcal{P}\mathrm{ic}}(\underline{\mathcal{D}})) \simeq f_*\underline{\mathrm{Fun}}(f^*\underline{X}, \underline{\mathcal{P}\mathrm{ic}}(\underline{\mathcal{D}}))$$

where the first equivalence is by Lemma 6.1.22 and the second by Lemma 6.1.16. For part (2), if we have a factorisation $\underline{X} \to \underline{\mathcal{P}\mathrm{ic}}(\underline{\mathcal{E}}) \hookrightarrow \underline{\mathcal{E}}$, then applying f^* to this and using the second part of Lemma 6.1.22 gives the required factorisation. $\square$

Proposition 6.1.24. *Let $f^*\colon \mathcal{B} \rightleftharpoons \mathcal{B}' : f_*$ be a geometric morphism of topoi, $\underline{X} \in \mathcal{B}$, $\underline{\mathcal{D}} \in \mathrm{CMon}(\mathrm{Cat}_{\mathcal{B}'})$, and $\underline{\mathcal{E}} \in \mathrm{CMon}(\mathrm{Cat}_{\mathcal{B}})$.*

(1) The symmetric monoidal $\mathcal{B}'$–category $\underline{\mathcal{D}}$ satisfies the $f^*\underline{X}$–projection formula if and only if the symmetric monoidal $\mathcal{B}$–category $f_*\underline{\mathcal{D}}$ satisfies the $\underline{X}$–projection formula,

(2) If f_* is fully faithful, then the colimit $X_!\colon \underline{\mathrm{Fun}}(\underline{X}, f_*\underline{\mathcal{D}}) \to f_*\underline{\mathcal{D}}$ (resp. limit X_*) exists if and only if the colimit $(f^*X)_!\colon \underline{\mathrm{Fun}}(f^*\underline{X}, \underline{\mathcal{D}}) \to \underline{\mathcal{D}}$ (resp. limit $(f^*X)_*$) does,

(3) If $f^* \dashv f_*$ is moreover étale, then if $\underline{\mathcal{E}}$ satisfies the $\underline{X}$–projection formula, then $f^*\underline{\mathcal{E}}$ satisfies the $f^*\underline{X}$–projection formula.

Proof. For (1), by the symmetric monoidal identification Lemma 6.1.16 and the identification of adjunctions Lemma 6.1.18, we see that for a fixed $A \in \underline{\mathcal{D}}$, applying f_* to the projection formula transformation on the left in

$$\begin{tikzcd} \underline{\mathrm{Fun}}(f^*\underline{X}, \underline{\mathcal{D}}) \arrow[rr, bend left, "(f^*X)_!(-\otimes(f^*X)^*A)"] \arrow[rr, bend right, "(f^*X)_!(-)\otimes A"'] \arrow[rr, phantom, "\Downarrow \mathrm{PF}"] & & \underline{\mathcal{D}} \end{tikzcd} \qquad \begin{tikzcd} \underline{\mathrm{Fun}}(\underline{X}, f_*\underline{\mathcal{D}}) \arrow[rr, bend left, "X_!(-\otimes X^*A)"] \arrow[rr, bend right, "X_!(-)\otimes f_*A"'] \arrow[rr, phantom, "\Downarrow \mathrm{PF}"] & & f_*\underline{\mathcal{D}} \end{tikzcd}$$

yields the projection formula transformation on the right. Thus, by Lemma 6.1.20, we see that the left projection formula transformation is an equivalence if and only if the right one is.

For (2), that the existence of $(f^*X)_!$ implies the existence of $X_!$ is by Lemma 6.1.18. For the converse, we use again the diagram Lemma 6.1.18 together with the fact f^* preserves adjunctions by [MW24, Cor. 3.1.9] and that $f^*f_* \simeq \mathrm{id}$ by fully faithfulness.

Part (3) is proved similarly as in (1), but using Lemma 6.1.17 and Lemma 6.1.19 instead. □

Presentability

Presentable categories are useful for many reasons, among them being that they have all (co)limits, fulfill the adjoint functor theorem, and have a symmetric monoidal structure coming from the Lurie tensor product. Presentability in the parametrised context was first studied in [Nar17] and later on in [Hil24b]. Subsequently, Martini-Wolf [MW22] introduced and developed a much more general theory for $\mathcal{B}$–categories, and this is the theory that we will use. Recall that presentable categories are usually large categories. To talk about presentable $\mathcal{B}$–categories, we define a *large $\mathcal{B}$–category* to be a sheaf of large categories on $\mathcal{B}$, i.e. a limit preserving functor $\mathcal{B}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}$. The very large category of large $\mathcal{B}$–categories will be denoted by $\widehat{\mathrm{Cat}}_{\mathcal{B}}$.

Definition 6.1.25. A $\mathcal{B}$–category $\underline{\mathcal{C}}$ is called *fibrewise presentable* if $\underline{\mathcal{C}}\colon \mathcal{B}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}$ factors through $\mathrm{Pr}^L \subset \widehat{\mathrm{Cat}}$. Furthermore, $\underline{\mathcal{C}}$ is called *presentable* if it is fibrewise presentable and the following conditions hold:

1. For any map $f\colon X \to Y$ in $\mathcal{B}$ the map $f^*\colon \mathcal{C}(\underline{Y}) \to \mathcal{C}(\underline{X})$ admits a left adjoint $f_!$.
2. For any pullback square
$$\begin{tikzcd} \underline{X'} \arrow[r, "g'"] \arrow[d, "f'"] & \underline{X} \arrow[d, "f"] \\ \underline{Y'} \arrow[r, "g"] & \underline{Y} \end{tikzcd} \tag{6.6}$$
in $\mathcal{B}$ the Beck–Chevalley transformation $f'_!(g')^* \to g^*f_!$ between functors $\mathcal{C}(\underline{X}) \to \mathcal{C}(\underline{Y'})$ is an equivalence.

The above definition was chosen because it is easy to state, but there are many useful ways to characterise presentable $\mathcal{B}$–categories, see [MW22, Thm. 6.2.4].

Definition 6.1.26. A map $F\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ between presentable $\mathcal{B}$–categories is said to *preserve $\mathcal{B}$-colimits* if it satisfies the following conditions:

1. For any object $\underline{X} \in \mathcal{B}$ the map $F(\underline{X})\colon \mathcal{C}(\underline{X}) \to \mathcal{D}(\underline{X})$ preserves colimits.
2. For any map $f\colon \underline{X} \to \underline{Y}$ in $\mathcal{B}$ the Beck–Chevalley transformation $f_!F(\underline{X}) \to F(\underline{Y})f_!$ is an equivalence.

If $\underline{\mathcal{C}}$ is presentable, then $f^*\colon \mathcal{C}(\underline{Y}) \to \mathcal{C}(\underline{X})$ also admits a right adjoint f_*. By passing to right adjoints one can see that for any pull back square (6.6) the Beck–Chevalley transformation $g^*f_* \to (f')_*(g')^*$ is an equivalence, see e.g. [Hai22, Observation 1.6.2].

Definition 6.1.27. Denote by $\mathrm{Pr}^L_{\mathcal{B}}$ the (nonfull) subcategory of $\widehat{\mathrm{Cat}}_{\mathcal{B}}$ of presentable $\mathcal{B}$–categories and $\mathcal{B}$-colimit preserving functors. We write $\underline{\mathrm{Fun}}^L_{\mathcal{B}}(\mathcal{C},\mathcal{D})$ for the full $\mathcal{B}$-subcategory of $\underline{\mathrm{Fun}}(\mathcal{C},\mathcal{D})$ of colimit preserving functors. The subcategories $\mathrm{Pr}^L_{\mathcal{B}_{/X}} \subset \widehat{\mathrm{Cat}}_{\mathcal{B}_{/X}}$ assemble into the $\mathcal{B}$–category $\underline{\mathrm{Pr}}^L_{\mathcal{B}} \subset \underline{\widehat{\mathrm{Cat}}}_{\mathcal{B}}$ of presentable $\mathcal{B}$–categories.

The $\mathcal{B}$–category $\underline{\mathrm{Pr}}^L_{\mathcal{B}}$ admits all $\mathcal{B}$-limits and $\mathcal{B}$-colimits [MW22, Cor. 6.4.11.]. Moreover, for two presentable $\mathcal{B}$–categories $\underline{\mathcal{C}}$ and $\underline{\mathcal{D}}$, their functor category $\underline{\mathrm{Fun}}^L(\underline{\mathcal{C}},\underline{\mathcal{D}})$ is presentable.

Construction 6.1.28 (Tensor product of presentable categories)**.** Given two presentable $\mathcal{B}$–categories $\underline{\mathcal{C}}$ and $\underline{\mathcal{D}}$, their tensor product $\underline{\mathcal{C}} \otimes \underline{\mathcal{D}}$ is a presentable $\mathcal{B}$–category together with a functor $\underline{\mathcal{C}} \times \underline{\mathcal{D}} \to \underline{\mathcal{C}} \otimes \underline{\mathcal{D}}$ which preserves colimits in each variable such that precomposition along it induces an equivalence $\mathrm{Fun}^L_{\mathcal{B}}(\underline{\mathcal{C}} \otimes \underline{\mathcal{D}}, \underline{\mathcal{E}}) \xrightarrow{\simeq} \mathrm{Fun}^L_{\mathcal{B}}(\underline{\mathcal{C}}, \underline{\mathrm{Fun}}^L(\underline{\mathcal{D}}, \underline{\mathcal{E}}))$ for any presentable $\mathcal{B}$–category $\underline{\mathcal{E}}$. This equips $\mathrm{Pr}^L_{\mathcal{B}}$ with the structure of a closed symmetric monoidal category. It can even be extended to a symmetric monoidal structure on the $\mathcal{B}$–category $\underline{\mathrm{Pr}}^L_{\mathcal{B}}$, see [MW22, Proposition 8.2.9]. Furthermore, the tensor product $- \otimes -\colon \underline{\mathrm{Pr}}^L_{\mathcal{B}} \times \underline{\mathrm{Pr}}^L_{\mathcal{B}} \to \underline{\mathrm{Pr}}^L_{\mathcal{B}}$ preserves $\mathcal{B}$-colimits in each variable.

Definition 6.1.29. A *presentably symmetric monoidal $\mathcal{B}$–category* is a commutative algebra $\underline{\mathcal{C}} \in \mathrm{CAlg}(\mathrm{Pr}^L_{\mathcal{B}})$. Explicitly, this means that $\underline{\mathcal{C}}$ is a symmetric monoidal $\underline{\mathcal{B}}$–category which is presentable such that the tensor product $- \otimes -\colon \underline{\mathcal{C}} \times \underline{\mathcal{C}} \to \underline{\mathcal{C}}$ preserves $\mathcal{B}$-colimits in both variables. The latter condition means that:

1. For all $\underline{X} \in \mathcal{B}$ the tensor product $\mathcal{C}(\underline{X}) \times \mathcal{C}(\underline{X}) \to \mathcal{C}(\underline{X})$ preserves colimits.
2. For all maps $f\colon \underline{X} \to \underline{Y}$ in $\mathcal{B}$ and all $A \in \mathcal{C}(\underline{X})$ and $B \in \mathcal{C}(\underline{Y})$ the Beck–Chevalley transformation $f_!(A \otimes f^*B) \to f_!A \otimes B$ is an equivalence.

To construct examples of presentable $\mathcal{B}$–categories, the following proposition is useful.

Proposition 6.1.30 ([MW22, Section 8.3])**.** *There is a symmetric monoidal colimit preserving fully faithful functor $- \otimes_{\mathcal{B}} \Omega\colon \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^L) \hookrightarrow \mathrm{Pr}^L_{\mathcal{B}}$ whose right adjoint Γ^{lin} refines the global sections functor $\Gamma\colon \mathrm{Pr}^L_{\mathcal{B}} \to \mathrm{Pr}^L$.*

Lemma 6.1.31 (Presentablility and basechange)**.** *Let $f^*\colon \mathcal{B} \rightleftharpoons \mathcal{B}' : f_*$ be a geometric morphism. Then the functor $f_*\colon \widehat{\mathrm{Cat}}_{\mathcal{B}'} \to \widehat{\mathrm{Cat}}_{\mathcal{B}}$ restricts to a functor $f_*\colon \mathrm{Pr}^L_{\mathcal{B}'} \to \mathrm{Pr}^L_{\mathcal{B}}$. It admits a unique lax symmetric monoidal refinement $f_*^{\otimes}\colon \mathrm{Pr}^{L,\otimes}_{\mathcal{B}'} \to \mathrm{Pr}^{L,\otimes}_{\mathcal{B}}$ lifting the lax symmetric monoidal functor $\mathrm{Pr}^{L,\otimes}_{\mathcal{B}'} \to \widehat{\mathrm{Cat}}^{\times}_{\mathcal{B}'} \xrightarrow{f_*} \widehat{\mathrm{Cat}}^{\times}_{\mathcal{B}}$ along the lax symmetric monoidal functor $\mathrm{Pr}^{L,\otimes}_{\mathcal{B}} \to \widehat{\mathrm{Cat}}^{\times}_{\mathcal{B}}$.*

Proof. Recall that $f_*\colon \widehat{\mathrm{Cat}}_{\mathcal{B}'} \to \widehat{\mathrm{Cat}}_{\mathcal{B}}$ is given by precomposition with $f^*\colon \mathcal{B}^{\mathrm{op}} \to (\mathcal{B}')^{\mathrm{op}}$. Hence, the first condition in Definition 6.1.25 is immediate. The second condition follows from the fact that $f^*\colon \mathcal{B}^{\mathrm{op}} \to (\mathcal{B}')^{\mathrm{op}}$ also preserves finite limits, and in particular pullbacks.

For the statement about lax symmetric monoidality, we will freely use the terminologies from [MW22]. First observe that as $f_*\colon \widehat{\mathrm{Cat}}_{\mathcal{B}'} \to \widehat{\mathrm{Cat}}_{\mathcal{B}}$ preserves products, it is symmetric monoidal with respect to the cartesian symmetric monoidal structure on both sides. Recall from [MW22, Section 8.2] that $\mathrm{Pr}^{L,\otimes}_{\mathcal{B}'} \looparrowright \widehat{\mathrm{Cat}}^{\times}_{\mathcal{B}'}$ is the subcategory generated by presentable $\mathcal{B}'$–categories and locally multilinear functors. We know from the first part that $f_*\colon \widehat{\mathrm{Cat}}_{\mathcal{B}'} \to \widehat{\mathrm{Cat}}_{\mathcal{B}}$ preserves presentable categories and multilinear functors between those. As $f^*\colon \widehat{\mathrm{Cat}}_{\mathcal{B}} \to \widehat{\mathrm{Cat}}_{\mathcal{B}'}$ preserves colimits and finite limits it preserves effective epimorphisms. From this it follows that f_* also preserves locally multilinear functors. $\square$

Lemma 6.1.32 (Presentability and fully faithful basechange)**.** *Let $f^*\colon \mathcal{B} \rightleftharpoons \mathcal{B}' : f_*$ be a geometric morphism of topoi and assume that f_* is fully faithful. Then the square*

$$\begin{tikzcd} \mathrm{Pr}^L_{\mathcal{B}'} \arrow[r, "f_*"] \arrow[d, hook] & \mathrm{Pr}^L_{\mathcal{B}} \arrow[d, hook] \\ \widehat{\mathrm{Cat}}_{\mathcal{B}'} \arrow[r, hook, "f_*"] & \widehat{\mathrm{Cat}}_{\mathcal{B}} \end{tikzcd}$$

is cartesian. In particular, $f_*\colon \mathrm{Pr}^L_{\mathcal{B}'} \to \mathrm{Pr}^L_{\mathcal{B}}$ is fully faithful.

If, in addition, the image of $f_*\colon \mathrm{Pr}^L_{\mathcal{B}'} \to \mathrm{Pr}^L_{\mathcal{B}}$ is closed under $\otimes$, then the structure maps $f_*\underline{\mathcal{C}} \otimes f_*\underline{\mathcal{D}} \to f_*(\underline{\mathcal{C}} \otimes \underline{\mathcal{D}})$ of the lax symmetric monoidal structure are equivalences.

Proof. We have to show that a $\mathcal{B}'$–category $\underline{\mathcal{C}}$ is presentable if $f_*\underline{\mathcal{C}}$ is presentable and similarly that a $\mathcal{B}'$-functor $F\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is $\mathcal{B}'$-colimit preserving if f_*F is $\mathcal{B}$-colimit preserving. Notice that the counit map $f^*f_* \to \mathrm{id}$ is an equivalence as f_* is fully faithful. This implies that for $\underline{X} \in \mathcal{B}$ we have $f_*\underline{\mathcal{C}}(f_*\underline{X}) = \mathcal{C}(f^*f_*\underline{X}) \simeq \mathcal{C}(\underline{X})$. The statements about presentablility and colimit preservation now directly follow from the definitions.

For the statement about the lax monoidal multiplication map, observe that for presentable $\mathcal{B}'$–categories $\underline{\mathcal{C}}, \underline{\mathcal{D}}, \underline{\mathcal{E}}$, the functor f_* induces an equivalence between $\mathcal{B}'$-multilinear functors $\underline{\mathcal{C}} \times \underline{\mathcal{D}} \to \underline{\mathcal{E}}$ and $\mathcal{B}$-multilinear functors $f_*\underline{\mathcal{C}} \times f_*\underline{\mathcal{D}} \to f_*\underline{\mathcal{E}}$: It is clear that if g is multilinear, then f_*g is multilinear while the converse follows from essential surjectivity of f^*. In particular, precomposition along $f_*\underline{\mathcal{C}} \otimes f_*\underline{\mathcal{D}} \to f_*(\underline{\mathcal{C}} \otimes \underline{\mathcal{D}})$ induces an equivalence

$$\mathrm{Fun}^L(f_*(\underline{\mathcal{C}} \otimes \underline{\mathcal{D}}), f_*\underline{\mathcal{E}}) \xrightarrow{\simeq} \mathrm{Fun}^L(f_*\underline{\mathcal{C}} \otimes f_*\underline{\mathcal{D}}, f_*\underline{\mathcal{E}})$$

from which the claim follows. □

Lemma 6.1.33 (Presentability and étale basechange)**.** *For $X \in \mathcal{B}$, the basechange adjunction $\pi_X^*\colon \widehat{\mathrm{Cat}}_{\mathcal{B}} \rightleftharpoons \widehat{\mathrm{Cat}}_{\mathcal{B}_{/X}} : (\pi_X)_*$ along the étale geometric morphism $\pi_X^*\colon \mathcal{B} \rightleftharpoons \mathcal{B}_{/X} : (\pi_X)_*$ restricts to an adjunction $\pi_X^*\colon \mathrm{Pr}^L_{\mathcal{B}} \rightleftharpoons \mathrm{Pr}^L_{\mathcal{B}_{/X}} : (\pi_X)_*$. The left adjoint $\pi_X^*\colon \mathrm{Pr}^L_{\mathcal{B}} \to \mathrm{Pr}^L_{\mathcal{B}_{/X}}$ admits a unique symmetric monoidal refinement which lifts the lax symmetric monoidal functor $\mathrm{Pr}^L_{\mathcal{B}} \to \widehat{\mathrm{Cat}}_{\mathcal{B}} \xrightarrow{\pi_X^*} \widehat{\mathrm{Cat}}_{\mathcal{B}_{/X}}$.*

Proof. That $\pi_X^*\colon \widehat{\mathrm{Cat}}_{\mathcal{B}} \rightleftharpoons \widehat{\mathrm{Cat}}_{\mathcal{B}_{/X}} : (\pi_X)_*$ restricts to an adjuction $\pi_X^*\colon \mathrm{Pr}^L_{\mathcal{B}} \rightleftharpoons \mathrm{Pr}^L_{\mathcal{B}_{/X}} \colon (\pi_X)_*$ is shown in [Cno23, Corollary 2.14]. For the statement about symmetric monoidality, note that $(\pi_X)_*\colon \widehat{\mathrm{Cat}}_{\mathcal{B}_{/X}} \to \widehat{\mathrm{Cat}}_{\mathcal{B}}$ is product preserving as it admits the left adjoint $(\pi_X)_!$. In particular, we obtain the symmetric monoidal unit map $\underline{\mathrm{Pr}}^L_{\mathcal{B}} \to (\pi_X)_*\pi_X^*\underline{\mathrm{Pr}}^L_{\mathcal{B}}$ which on global sections gives the desired symmetric monoidal refinement of $(\pi_X)^*\colon \mathrm{Pr}^L_{\mathcal{B}} \to \mathrm{Pr}^L_{\mathcal{B}_{/X}}$. □

$\underline{\mathcal{C}}$-linear categories

Here we recall some facts about $\underline{\mathcal{C}}$-linear categories and the classification of $\underline{\mathcal{C}}$-linear functors from [Cno23, Section 2.2].

Definition 6.1.34. ($\underline{\mathcal{C}}$-linear categories) Consider a presentably symmetric monoidal category $\underline{\mathcal{C}} \in \mathrm{CAlg}(\mathrm{Pr}^L_{\mathcal{B}})$. A *$\underline{\mathcal{C}}$-linear category* is a left $\underline{\mathcal{C}}$-module in $\mathrm{Pr}^L_{\mathcal{B}}$. The category of $\underline{\mathcal{C}}$-linear categories and $\underline{\mathcal{C}}$-linear functors is defined as the category $\mathrm{Mod}_{\underline{\mathcal{C}}}(\mathrm{Pr}^L_{\mathcal{B}})$.

The categories $\mathrm{Mod}_{\pi_X^*\underline{\mathcal{C}}}(\mathrm{Pr}^L_{\mathcal{B}_{/X}})$ assemble into the $\mathcal{B}$–category $\underline{\mathrm{Mod}}_{\underline{\mathcal{C}}}(\underline{\mathrm{Pr}}^L_{\mathcal{B}})$. The relative tensor product from [MW22, Proposition 7.2.7] equips this with the structure of a symmetric monoidal $\mathcal{B}$–category which is $\mathcal{B}$-complete and $\mathcal{B}$-cocomplete such that the tensor product is bilinear. This symmetric monoidal structure on $\mathrm{Mod}_{\underline{\mathcal{C}}}(\mathrm{Pr}^L_{\mathcal{B}})$ is closed and we denote the internal mapping object by $\underline{\mathrm{Fun}}_{\underline{\mathcal{C}}}(-,-)$. As in [Mar22b, Remark 3.4.3] it endows $\mathrm{Mod}_{\underline{\mathcal{C}}}(\mathrm{Pr}^L_{\mathcal{B}})$ with a 2-categorical structure. This allows us to talk about internal adjunctions in $\mathrm{Mod}_{\underline{\mathcal{C}}}(\mathrm{Pr}^L_{\mathcal{B}})$. The following lemma gives convenient criteria for a $\underline{\mathcal{C}}$-linear functor to be an internal left adjoint.

Lemma 6.1.35 ([Cno23], Lem. 2.21)**.** *A $\underline{\mathcal{C}}$-linear functor $F\colon \underline{\mathcal{D}} \to \underline{\mathcal{E}}$ is an internal left adjoint in $\mathrm{Mod}_{\underline{\mathcal{C}}}(\mathrm{Pr}^L_{\mathcal{B}})$ if and only if its right adjoint G preserves fibrewise colimits and satisfies the projection formula, which means that for each $\underline{X} \in \mathcal{B}$ and $e \in \underline{\mathcal{E}}(\underline{X})$ the map $\mathrm{PF}_*\colon c \otimes G(e) \to G(c \otimes e)$ is an equivalence.*

Definition 6.1.36 (Free and cofree categories, [Cno23, Definition 2.23])**.** For a category $\underline{\mathcal{D}} \in \mathrm{Mod}_{\underline{\mathcal{C}}}(\underline{\mathrm{Pr}}^L_{\mathcal{B}})$ and an object $\underline{X} \in \mathcal{B}$ we define the *cofree $\underline{\mathcal{C}}$-linear $\mathcal{B}$–category on $\underline{X}$* by $\underline{\mathcal{D}}^{\underline{X}} := \lim_{\underline{X}} \underline{\mathcal{D}}$ where the $\mathcal{B}$-limit is formed over the constant diagram in $\underline{\mathrm{Mod}}_{\underline{\mathcal{C}}}(\underline{\mathrm{Pr}}^L_{\mathcal{B}})$ with value $\underline{\mathcal{D}}$.

Note that after forgetting the $\underline{\mathcal{C}}$-linear structure, $\underline{\mathcal{D}}^{\underline{X}}$ is given by $\underline{\mathrm{Fun}}(\underline{X}, \underline{\mathcal{D}})$ as the forgetful functor $\underline{\mathrm{Mod}}_{\underline{\mathcal{C}}}(\underline{\mathrm{Pr}}^L_{\mathcal{B}}) \to \underline{\mathrm{Cat}}_{\mathcal{B}}$ preserves $\underline{\mathcal{B}}$-limits. If $\underline{\mathcal{D}} \in \mathrm{CAlg}(\mathrm{Mod}_{\underline{\mathcal{C}}}(\underline{\mathrm{Pr}}^L_{\mathcal{B}}))$ is a $\underline{\mathcal{C}}$-algebra, then $\underline{\mathcal{D}}^{\underline{X}}$ has a canonical pointwise symmetric monoidal structure as the forgetful functor $\underline{\mathrm{CAlg}}(\underline{\mathrm{Mod}}_{\underline{\mathcal{C}}}(\underline{\mathrm{Pr}}^L_{\mathcal{B}})) \to \underline{\mathrm{Mod}}_{\underline{\mathcal{C}}}(\underline{\mathrm{Pr}}^L_{\mathcal{B}})$ preserves $\mathcal{B}$-limits.

The following theorem is crucial for the development of Poincaré duality in a presentable context, even in classical terms. It should be viewed as a generalisation of the fact that for a small category for a space X there is an equivalence $\mathrm{Fun}^L(\mathcal{S}^X, \mathcal{S}) \simeq \mathcal{S}^X$, that is sometimes referred to as Morita theory.

Theorem 6.1.37 (Classification of $\underline{\mathcal{C}}$-linear functors, [Cno23, Theorem 2.32])**.** *Consider $\underline{\mathcal{C}} \in \mathrm{CAlg}(\mathrm{Pr}^L_{\mathcal{B}})$ and $\underline{X} \in \underline{\mathcal{C}}$. Then there is an equivalence of $\underline{\mathcal{C}}$-linear $\mathcal{B}$–categories*

$$\underline{\mathcal{C}}^{\underline{X}} \to \underline{\mathrm{Fun}}_{\underline{\mathcal{C}}}(\underline{\mathcal{C}}^{\underline{X}}, \underline{\mathcal{C}}), \quad \zeta \mapsto r_!(- \otimes \zeta).$$

Here, the map in the statement of the theorem is adjoint to the composite $\underline{\mathcal{C}}^{\underline{X}} \otimes \underline{\mathcal{C}}^{\underline{X}} \to \underline{\mathcal{C}}^{\underline{X}} \xrightarrow{r_!} \underline{\mathcal{C}}$.

Proposition 6.1.38 (Basechange of module categories, [MW22, Prop. 7.2.7]). *Suppose that* $f\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ *is a map in* $\mathrm{CAlg}(\mathrm{Pr}^L_{\mathcal{B}})$. *Then the restriction functor* $f^*\colon \underline{\mathrm{Mod}}_{\mathcal{D}}(\mathrm{Pr}^L_{\mathcal{B}}) \to \underline{\mathrm{Mod}}_{\mathcal{C}}(\mathrm{Pr}^L_{\mathcal{B}})$ *admits a symmetric monoidal left adjoint denoted by* $-\otimes_{\underline{\mathcal{C}}} \underline{\mathcal{D}}\colon \underline{\mathrm{Mod}}_{\mathcal{C}}(\mathrm{Pr}^L(\mathcal{B})) \to \underline{\mathrm{Mod}}_{\mathcal{D}}(\mathrm{Pr}^L(\mathcal{B}))$

Proof. Apply [MW22, Proposition 7.2.7] to the case $R = \underline{\mathcal{D}} \in \mathrm{CAlg}(\underline{\mathrm{Mod}}_{\mathcal{C}}(\mathrm{Pr}^L_{\mathcal{B}}))$. □

6.2 Equivariant categories and the theory of families

Change of group functors

We now specialise the previous general theory to the equivariant setting for a compact Lie group G. We set $\mathcal{S}_G := \mathrm{Psh}(\mathcal{O}(G))$, the *category of G–spaces*, where $\mathcal{O}(G)$ is the *orbit category of* G. This is a topos, and we write $\mathrm{Cat}_G := \mathrm{Cat}_{\mathcal{S}_G} \simeq \mathrm{Fun}(\mathcal{O}(G)^{\mathrm{op}}, \mathrm{Cat})$ for the category of G–categories. The value of a G-category $\underline{\mathcal{C}}$ at an orbit G/H will be denoted by $\mathcal{C}(\underline{G/H})$ or $\mathcal{C}^H$.

Recollection 6.2.1. Recall that the category of locally compact Hausdorff topological G–spaces is enriched over topological spaces by employing the compact–open topology on morphism sets. The full subcategory on the homogenous G–spaces, that is Hausdorff spaces with a transitive G-action, is equivalent to the full subcategory spanned by the orbits G/H, where $H \leq G$ is a closed subgroup. By $\mathcal{O}(G)$ we denote the associated (∞-) category which we call the *orbit category of* G.

Later we will need the following standard facts: For any morphism $\alpha\colon H \to G$ of compact Lie groups, there is an induction functor

$$\mathrm{Ind}^{\mathcal{O}}_{\alpha}\colon \mathcal{O}(H) \longrightarrow \mathcal{O}(G), \quad S \mapsto G \times_H S.$$

If $\alpha\colon H \longrightarrow G$ is an epimorphism, this admits the restriction functor

$$\mathrm{Res}^{\mathcal{O}}_{\alpha}\colon \mathcal{O}(G) \to \mathcal{O}(H)$$

as a fully faithful right adjoint. Both functors and the adjunction can be constructed on the level of topological categories. For a closed subgroup $H \leq G$, induction induces an equivalence of categories $\mathcal{O}(H) \xrightarrow{\simeq} \mathcal{O}(G)_{/(G/H)}$ whose inverse sends $T \to G/H$ to the homogeneous H-space given as the fibre over $eH \in G/H$.

More information on orbit categories of compact Lie groups can be found in [LNP22, Sec. 6] or [Bre72, Chapters I.3 and I.4].

Notation 6.2.2 (Restrictions, (co)inductions, and (co)inflations)**.** Consider a continuous homorphism $\alpha\colon K \to G$ of compact Lie groups. We obtain the two adjunctions

$$\begin{tikzcd}
\mathrm{Cat}_H \arrow[r, bend left, "\mathrm{Ind}_\alpha"] \arrow[r, bend right, "\mathrm{Coind}_\alpha"'] & \mathrm{Cat}_G, \arrow[l, "\mathrm{Res}_\alpha"']
\end{tikzcd}$$

called the *induction, restriction*, and *coinduction* functors, respectively. Here, Res_α is given by restriction along $\mathrm{Ind}_\alpha^{\mathcal{O}}\colon \mathcal{O}(K) \to \mathcal{O}(G)$ and Ind_α and Coind_α are given by left and right Kan extensions. The functors Res_α and Coind_α restrict to a geometric morphism of topoi $\mathrm{Res}_\alpha\colon \mathcal{S}_G \rightleftharpoons \mathcal{S}_K : \mathrm{Coind}_\alpha$. The two main classes of examples are:

(a) If α were an injection $\iota\colon H \rightarrowtail G$, then the geometric morphism $\mathrm{Res}_\iota\colon \mathcal{S}_G \rightleftharpoons \mathcal{S}_H : \mathrm{Coind}_\iota$ is étale. We will often also write $\mathrm{Ind}_\iota, \mathrm{Res}_\iota$, and Coind_ι as Ind_H^G, Res_H^G, and Coind_H^G respectively;

(b) If α were an epimorphism $\theta\colon G \twoheadrightarrow G/N =: Q$ (so that $N \leq G$ is a closed normal subgroup), then Coind_θ admits a further right adjoint which we write as Coinfl_θ given by right Kan extension along the fully faithful right adjoint $\mathrm{Res}_\theta^{\mathcal{O}}$ to $\mathrm{Ind}_\theta^{\mathcal{O}}$. In particular, $\mathrm{Res}_\theta = (\mathrm{Ind}_\theta^{\mathcal{O}})^* \simeq (\mathrm{Res}_\theta^{\mathcal{O}})_!$ and $\mathrm{Coinfl}_\theta = (\mathrm{Res}_\theta^{\mathcal{O}})_*$ are fully faithful in this epimorphic case. Note also that in this case, $\mathrm{Coind}_\theta \simeq (\mathrm{Res}_\theta^{\mathcal{O}})^*$, i.e. Coind_θ may be computed by restricting along $\mathrm{Res}_\theta^{\mathcal{O}}\colon \mathcal{O}(G) \to \mathcal{O}(K)$. We summarise in the following diagram the special notations we will also use in the epimorphic case as follows:

$$\begin{tikzcd}[column sep=huge]
\mathrm{Cat}_Q \arrow[r, hook, "\mathrm{Infl}_\theta := \mathrm{Res}_\theta" description] \arrow[r, hook, bend right=60, "\mathrm{Coinfl}_\theta"'] & \mathrm{Cat}_G, \arrow[l, bend right, "N\backslash(-) := \mathrm{Ind}_\theta"'] \arrow[l, bend left, "(-)^N := \mathrm{Coind}_\theta"]
\end{tikzcd}$$

The maps $N\backslash(-)$, Infl_θ, $(-)^N$, and Coinfl_θ are called the *genuine quotient, inflation, genuine fixed points,* and *coinflation*, respectively. We often also write Infl_θ and Coinfl_θ as Infl_G^Q and Coinfl_G^Q respectively.

Remark 6.2.3. From the left Kan extension formula defining the genuine quotient, we obtain

$$(N\backslash\underline{\mathcal{C}})(\underline{Q/H}) \simeq \operatorname*{colim}_{G/K,\ \underline{Q/H}\to N\backslash(\underline{G/K})} \mathcal{C}(\underline{G/K}). \tag{6.7}$$

Stability

In the parametrised theory, the theory of stability is more subtle than in the non-parametrised setting. The most naive version is the following, equivalent characterisations of which can be found in [MW22, Section 7.3].

Definition 6.2.4. A $\mathcal{B}$–category $\underline{\mathcal{C}}$ is called fibrewise pointed (resp. stable) if the functor $\underline{\mathcal{C}}\colon \mathcal{B}^{\mathrm{op}} \to \mathrm{Cat}$ factors through the subcategory $\mathrm{Cat}_* \subset \mathrm{Cat}$ of pointed categories and pointed functors (resp. $\mathrm{Cat}^{\mathrm{st}} \subset \mathrm{Cat}$ of stable categories and exact functors). We denote by $\mathrm{Cat}_{\mathcal{B}}^{\mathrm{ptd}}$ (resp. $\mathrm{Cat}_{\mathcal{B}}^{\mathrm{st}}$) the category of fibrewise pointed (resp. stable) categories and pointed (resp. exact) functors.

As a parametrised analogue of the category of spectra, there is the G-category of G-spectra $\underline{\mathrm{Sp}}_G$ whose value at an orbit G/H is given by the category $\underline{\mathrm{Sp}}_G(\underline{G/H}) = \mathrm{Sp}_H$ of genuine H-spectra together with restriction maps between them, see Definition A.1 for a definition. If G is clear form the context, we will also just write $\underline{\mathrm{Sp}}$ for $\underline{\mathrm{Sp}}_G$. In addition to being fibrewise stable, $\underline{\mathrm{Sp}}_G$ satsifies some form of the Wirthmüller isomorphism in the sense that indexed products and coproducts over orbits G/H are canonically equivalent. This was used by Nardin in [Nar17] to define the notion of G-stability for finite groups G. We will not recall the definition here and refer the interested reader to [Nar17] or [Hil24b, Section 4.1] for an exposition of this theory. In Chapter A we generalise this to define G-stability for presentable G-categories. This will be sufficient for our purposes.

Notation 6.2.5. For a compact Lie group G, we denote by $\mathrm{Pr}_G^{L,G-\mathrm{st}} \subseteq \mathrm{Pr}_G^{L,\mathrm{st}} \subseteq \mathrm{Pr}_G^L$ the full subcategories on G-stable and fibrewise stable presentable G-categories. For a finite group G, we also denote by $\mathrm{Cat}_G^{G-\mathrm{st}} \subset \mathrm{Cat}_G$ the subcategory of G-stable G-categories and G-exact functors.

Now we study the behaviour of G-stability with respect to standard equivariant operations.

Lemma 6.2.6 (Coinduction and stability)**.** *Let $\alpha\colon K \to G$ be a continous group homomorphism of compact Lie groups. The lax symmetric monoidal functor* $\mathrm{Coind}_\alpha\colon \mathrm{Pr}_K^L \to \mathrm{Pr}_G^L$ *from Lemma 6.1.31 restricts to a lax symmetric monoidal functor* $\mathrm{Coind}_\alpha\colon \mathrm{Pr}_K^{L,K-\mathrm{st}} \to \mathrm{Pr}_G^{L,G-\mathrm{st}}$

Proof. We apply Theorem A.5 (3) to show that Coind_α sends K-stable to G-stable categories. Suppose that $\underline{\mathcal{C}}$ is a K-stable category and V is a finite dimensional G-representation. By Lemma 6.1.18 we can identify the maps $S^V \otimes -$ and $\mathrm{Coind}_\alpha(S^{\mathrm{Res}_\alpha V} \otimes -)$ on $\mathrm{Coind}_\alpha \underline{\mathcal{C}}$. But as $\mathrm{Res}_\alpha V$ is a finite dimensional K-representation, K-stability of $\underline{\mathcal{C}}$ implies that the second map is invertible. □

Lemma 6.2.7 (Restriction and stability)**.** *For an injective continuous homomorphism $\alpha\colon H \rightarrowtail G$ of compact Lie groups, the adjunction* $\mathrm{Res}_\alpha\colon \mathrm{Pr}_G^L \rightleftharpoons \mathrm{Pr}_H^L : \mathrm{Coind}_\alpha$ *from*

Lemma 6.1.33 restricts to an adjunction $\mathrm{Res}_\alpha\colon \mathrm{Pr}_G^{L,G-\mathrm{st}} \rightleftharpoons \mathrm{Pr}_H^{L,H-\mathrm{st}} : \mathrm{Coind}_\alpha$ *with symmetric monoidal left adjoint*

Proof. By Lemma 6.2.6 Coind_H^G restricts to a functor between equivariantly stable categories. To show that restriction Res_H^G preserves equivariantly stable categories we employ Theorem A.5. Recall that, by the Peter-Weyl theorem, for any finite dimensional H-representation W there is a finite dimensional G-representaiton V such that W is a summand of $\mathrm{Res}_H^G V$. Now, if $\underline{\mathcal{C}}$ is a G-stable category, then $S^V \otimes -$ is invertible on $\underline{\mathcal{C}}$. By Lemma 6.1.19, we can identify the two maps $\mathrm{Res}_H^G(S^V \otimes -)$ and $S^{\mathrm{Res}_H^G V} \otimes -$ on $\mathrm{Res}_H^G \underline{\mathcal{C}}$. This shows that $S^{\mathrm{Res}_H^G V}$ and thus also S^W act invertibly on $\mathrm{Res}_H^G \underline{\mathcal{C}}$. □

Lemma 6.2.8 (Coinflation and stability)**.** *Let* $\theta\colon G \twoheadrightarrow Q = G/N$ *be a continuous epimorphism of compact Lie groups. Then the lax symmetric monoidal functor* $\mathrm{Coinfl}_\theta\colon \mathrm{Pr}_Q^L \to \mathrm{Pr}_G^L$ *from Lemma 6.1.31 restricts to a lax symmetric monoidal functor* $\mathrm{Coinfl}_\theta\colon \mathrm{Pr}_Q^{L,Q-\mathrm{st}} \to \mathrm{Pr}_G^{L,G-\mathrm{st}}$.

Proof. We apply Theorem A.5 to show that Coinfl_θ sends Q-stable to G-stable categories. Suppose that $\underline{\mathcal{C}}$ is a Q-stable category and V is a finite dimensional G-representation. By Lemma 6.1.18 we can identify the maps $S^V \otimes -$ and $\mathrm{Coinfl}_\theta(\mathrm{Coind}_\theta S^V \otimes -)$ on $\mathrm{Coinfl}_\theta \underline{\mathcal{C}}$. Note that $\mathrm{Coind}_\theta S^V \simeq S^{V^N}$ is the representation sphere of the finite dimensional Q-representation carrying the residual action. By Q-stability of $\underline{\mathcal{C}}$, this map is invertible. □

Construction 6.2.9 (Spectral restriction)**.** Let $\alpha\colon K \to G$ be a homomorphism of compact Lie groups. Lemma 6.2.6 endows $\mathrm{Coind}_\alpha\colon \mathrm{Pr}_K^{L,K-\mathrm{st}} \to \mathrm{Pr}_G^{L,G-\mathrm{st}}$ with a lax symmetric monoidal structure. This endows $\mathrm{Coind}_\alpha \underline{\mathrm{Sp}}_K$ with the structure of a commutative algebra in $\mathrm{Pr}_G^{L,G-\mathrm{st}}$. In particular, using that $\underline{\mathrm{Sp}}_G$ is the initial commutative algebra in $\mathrm{Pr}_G^{L,G-\mathrm{st}}$, we obtain the symmetric monoidal G-colimit preserving functor $\mathrm{Res}_\alpha\colon \underline{\mathrm{Sp}}_G \to \mathrm{Coind}_\alpha \underline{\mathrm{Sp}}_K$ called the *restriction map*. If $\alpha = \theta\colon G \twoheadrightarrow Q$ is an epimorphism, we also call $\mathrm{Res}_\theta = \mathrm{Infl}_\theta\colon \underline{\mathrm{Sp}}_Q \to \mathrm{Coind}_\theta \underline{\mathrm{Sp}}_G$ the *inflation map*.

Categorical isotropy separation

At various places in this article we will use isotropy separation arguments. For this, we recall here some constructions on G–categories given a family $\mathcal{F}$ of subgroups of G. Recall that a family of subgroups of a compact Lie group G is a collection of closed subgroups of G which is closed under subgroups and conjugation.

Note that conjugacy classes of subgroups of G correspond bijectively to isomorphism classes of objects in $\mathcal{O}(G)$. Given any collection S of closed subgroups of G that is closed under conjugacy, we set $\mathcal{O}_S(G) \subset \mathcal{O}(G)$ to be the full subcategory on those G/H with $H \in S$. One important example of this is the collection $S = \mathcal{F}^c$ given by the collection of all subgroups which lie in the complent of a family $\mathcal{F}$.

This never forms a family, except in the extreme cases of the empty family or the family of all subgroups.

Example 6.2.10 (A family for quotients)**.** Suppose that $N \leq G$ is a closed normal subgroup of G. An interesting family is provided by $\Gamma_N := \{H \leq G \mid N \nleq H\}$. Then Γ_N^c consists of those $H \leq G$ with $N \leq H$. Let $\alpha\colon G \to G/N$ denote the quotient homomorphism. Observe that the adjunction $\mathrm{Ind}_\alpha^{\mathcal{O}} \dashv \mathrm{Res}_\alpha^{\mathcal{O}}$ restricts to an equivalence of categories

$$\mathrm{Ind}_\alpha^{\mathcal{O}}\colon \mathcal{O}_{\Gamma_N^c}(G) \simeq \mathcal{O}(G/N)\colon \mathrm{Res}_\alpha^{\mathcal{O}}.$$

Example 6.2.11 (A family for free actions)**.** Suppose again that $N \leq G$ is a closed normal subgroup of G. Another family is given as $\mathcal{F}_N := \{H \subset G \mid H \cap N = \{1\}\}$. Note that when $N \neq \{1\}$, there is an inclusion of families $\mathcal{F}_N \subseteq \Gamma_N$. Thus, there is an inclusion $\Gamma_N^c \subseteq \mathcal{F}_N^c$.

Definition 6.2.12. Let G be a compact Lie group and S a collection of subgroups, closed under conjugacy. Then we write $\mathrm{Cat}_{G,S} := \mathrm{Fun}(\mathcal{O}_S(G)^{\mathrm{op}}, \mathrm{Cat})$ for the *category of S–categories*.

If $\mathcal{F}$ is a family of closed subgroups of G, we have the following variant of the standard isotropy separation sequence relating the categories Cat_G, $\mathrm{Cat}_{G,\mathcal{F}}$ and $\mathrm{Cat}_{G,\mathcal{F}^c}$. This will allow us to "separate" our problems into orthogonal pieces, one part concentrated in $\mathcal{F}^c$ and the part which is $\mathcal{F}$–local.

Construction 6.2.13 (Isotropy separation for G–categories)**.** Let $\mathcal{F}$ be a family of subgroups of a compact Lie group G and denote by $b\colon \mathcal{O}_{\mathcal{F}}(G) \hookrightarrow \mathcal{O}(G)$ and $s\colon \mathcal{O}_{\mathcal{F}^c}(G) \hookrightarrow \mathcal{O}(G)$ the inclusions. We obtain the adjoint triples

$$\begin{tikzcd} \mathrm{Cat}_{G,\mathcal{F}^c} \arrow[r, bend left, "s_!"] \arrow[r, bend right, "s_*"'] & \mathrm{Cat}_G \arrow[l, "s^*"'] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Cat}_G \arrow[r, "b^*"] & \mathrm{Cat}_{G,\mathcal{F}} \arrow[l, bend right, "b_!"'] \arrow[l, bend left, "b_*"] \end{tikzcd}$$

by restriction and Kan extension along s and b. Without making this precise, let us mention that these can be made into an unstable recollement using the cofibre sequence $\underline{E\mathcal{F}_+} \xrightarrow{b} \underline{S}^0 \to \underline{\widetilde{E\mathcal{F}}}$ of pointed G–spaces. For example, the map $b^*\colon \mathrm{Cat}_G \to \mathrm{Cat}_{G,\mathcal{F}}$ is equivalently given by taking the global sections on the map $b^*\colon \underline{\mathrm{Cat}} \to \underline{\mathrm{Fun}}(\underline{E\mathcal{F}}, \underline{\mathrm{Cat}})$.

Unwinding the right Kan extension formula, one obtains for example that s_* is given by

$$(s_*\mathcal{C})^H = \begin{cases} \mathcal{C}^H, & H \notin \mathcal{F} \\ *, & H \in \mathcal{F}. \end{cases}$$

Notice that the adjunction $b^* \dashv b_*$ is the basechange adjunction associated to the étale morphism $\pi_{E\mathcal{F}}^*\colon \mathcal{S}_G \rightleftharpoons (\mathcal{S}_G)_{/E\mathcal{F}} : (\pi_{E\mathcal{F}})_*$. In particular, it restricts to an adjunction $\mathrm{Pr}^L_G \rightleftharpoons \mathrm{Pr}^L_{G,\mathcal{F}}$ by Lemma 6.1.33. Similarly, the adjunction $s^* \dashv s_*$ is the basechange adjunction associated to the geometric morphism $s^* \colon \mathcal{S}_G \rightleftharpoons \mathcal{S}_{G,\mathcal{F}^c} : s_*$.

Example 6.2.14 (Isotropy separation and coinduction)**.** Consider a continuous epimorphism $\theta \colon G \twoheadrightarrow G/N$ of compact Lie groups. Recall from Example 6.2.10 that there is the functor $\mathrm{Res}^{\mathcal{O}}_{\theta} \colon \mathcal{O}(Q) \hookrightarrow \mathcal{O}(G)$ which restricts to an equivalence $\mathcal{O}(G/N) \simeq \mathcal{O}_{\Gamma^c_N}(G)$. This identifies the adjunctions $\mathrm{Coind}_\theta \colon \mathrm{Cat}_G \rightleftharpoons \mathrm{Cat}_Q : \mathrm{Coinfl}_\theta$ and $s^* \colon \mathrm{Cat}_G \rightleftharpoons \mathrm{Cat}_{G,\Gamma^c_N} : s_*$.

Construction 6.2.15 (Singular part)**.** Consider the inclusion $s \colon \mathcal{O}_{\mathcal{F}^c}(G) \hookrightarrow \mathcal{O}(G)$. Then we get the Bousfield colocalisation $s_! \colon \mathcal{S}_{G,\mathcal{F}^c} \rightleftharpoons \mathcal{S}_G : s^*$ such $s_! s^*(\underline{G/H}) = \underline{G/H}$ for every and $s_! s^*(\underline{G/H}) = \varnothing$. Since $s_! s^*$ picks out the isotropy of a G–space $\underline{X}$ not in $\mathcal{F}$, we shall also use the notations (which will be part of a larger notational package in Notation 6.2.28)

$$\underline{X}_{\mathcal{F}^c} := s^* \underline{X} \in \mathcal{S}_{\mathcal{F}^c} \qquad \underline{X}_{\widetilde{\mathcal{F}}} := s_! s^* \underline{X} = s_! \underline{X}_{\mathcal{F}^c} \in \mathcal{S}_G.$$

The adjunction counit $\epsilon \colon \underline{X}_{\widetilde{\mathcal{F}}} \to \underline{X}$ thus admits the classical interpretation as the inclusion of the $\mathcal{F}$–singular part of the G–space $\underline{X}$. It is the identity map on G/H for $H \notin \mathcal{F}$ and the map $\varnothing \to \underline{G/H}$ for $H \in \mathcal{F}$. We refer to $\epsilon \colon \underline{X}_{\widetilde{\mathcal{F}}} \to \underline{X}$ as the *inclusion of the $\mathcal{F}$–singular part of $\underline{X}$*.

Example 6.2.16. For $\mathcal{F} = \mathcal{P}$, the family of proper subgroups, $\underline{X}_{\widetilde{\mathcal{P}}}$ is given by the fixed points space X^G, considered as a G–space with trivial action. For $\mathcal{F} = \{e\}$, the trivial family, the intuition for $\underline{X}_{\widetilde{\mathcal{F}}}$ is that is gives the G–space of all points in $\underline{X}$ with nontrivial isotropy.

Having recounted the constructions relevant to the complementary part $\mathcal{F}^c$, we now recall some language associated to the $\mathcal{F}$–local part. Recall that for a family $\mathcal{F}$, we denoted by $b \colon \underline{E\mathcal{F}} \to \underline{*}$ the unique map.

Definition 6.2.17. We say that a G–category $\underline{\mathcal{C}}$ is *$\mathcal{F}$–Borel* if that the map $\underline{\mathcal{C}} \to b_* b^* \underline{\mathcal{C}}$ is an equivalence. A G–category $\underline{\mathcal{C}}$ will be called *$\mathcal{F}$–coBorel* if the map $b_! b^* \underline{\mathcal{C}} \to \underline{\mathcal{C}}$ is an equivalence.

Example 6.2.18 (Borel categories)**.** For the trivial family $\mathcal{F} = \{1\}$, we will also write $\underline{E\mathcal{F}}$ as $\underline{EG}$ and write $\underline{\mathrm{Bor}} := b_* \colon \mathrm{Cat}_{G,\{1\}} \simeq \mathrm{Cat}^{BG} \hookrightarrow \mathrm{Cat}_G$. We call $\underline{\mathrm{Bor}}(\mathcal{C})$ the *Borel-G-category* associated to $\mathcal{C}$. Explicitly, $\underline{\mathrm{Bor}}(\mathcal{C})(\underline{G/H}) = \mathcal{C}^{hH}$. In this case, the adjunctions Construction 6.2.13 produce the Borelification Bousfield (co)localisations studied in [Hil24a, §2.4]. While we will not need it in this article, we mention that there, it was shown in the case of finite groups G that $b^* \colon \mathrm{Cat}_G \to \mathrm{Cat}^{BG}$ naturally assemble to a G–symmetric monoidal Bousfield localisation and thus interacts well with the multiplicative norms.

Fact 6.2.19. An alternative description for $b_! b^* \underline{\mathcal{C}}$ and $b_* b^* \underline{\mathcal{C}}$ are $\underline{E\mathcal{F}} \times \underline{\mathcal{C}}$ and $\underline{\mathrm{Fun}}(\underline{E\mathcal{F}}, \underline{\mathcal{C}})$ respectively.

Notation 6.2.20. For a subgroup $K \leq G$, we write $\mathcal{F}_K$ for the family of subgroups of K which belong to $\mathcal{F}$. Note that, in particular, the equivalence $\mathcal{O}(G)_{/(G/K)} \simeq \mathcal{O}(K)$ induces an equivalence $\mathcal{O}_{\mathcal{F}}(G)_{/(G/K)} \simeq \mathcal{O}_{\mathcal{F}_K}(K)$.

Proposition 6.2.21 (Characterisations of (co)Borelness)**.** *Let $\underline{\mathcal{C}} \in \mathrm{Cat}_G$. Then:*

(a) $\underline{\mathcal{C}}$ is $\mathcal{F}$–coBorel if and only if $\mathcal{C}^H \simeq \varnothing$ for all $H \in \mathcal{F}^c$,

(b) $\underline{\mathcal{C}}$ is $\mathcal{F}$–Borel if and only if for all $K \leq G$, the canonical map induced by restrictions $\mathcal{C}^K \to \lim_{G/H \in \mathcal{O}_{\mathcal{F}_K}(K)^{\mathrm{op}}} \mathcal{C}^H$ is an equivalence.

Proof. Part (a) is immediate using the description $b_! b^* \underline{\mathcal{C}} \simeq \underline{E\mathcal{F}} \times \underline{\mathcal{C}}$. For part (b), the comma category used to compute the value of the right Kan extension $b_*\colon \mathrm{Fun}(\mathcal{O}_{\mathcal{F}}(G)^{\mathrm{op}}, \mathrm{Cat}) \to \mathrm{Fun}(\mathcal{O}(G)^{\mathrm{op}}, \mathrm{Cat})$ at G/K is

$$\left(\mathcal{O}_{\mathcal{F}}(G)^{\mathrm{op}}\right)_{(G/K)/} \simeq \left(\mathcal{O}_{\mathcal{F}}(G)_{/(G/K)}\right)^{\mathrm{op}} \simeq \mathcal{O}_{\mathcal{F}_K}(K)^{\mathrm{op}}$$

whence the claim. □

Example 6.2.22 (Modules over $\mathcal{F}$–nilpotent rings)**.** Suppose G is a finite group and $\mathcal{F}$ is a family of subgroups. By [MNN17, Prop. 6.38 (1), Thm. 6.42] and the concrete characterisation of $\mathcal{F}$–Borelness from Proposition 6.2.21 (b), we learn that if a ring spectrum $R \in \mathrm{CAlg}(\mathrm{Sp}_G)$ is $\mathcal{F}$–nilpotent, then $\underline{\mathrm{Mod}}_{\underline{\mathrm{Sp}}_G}(R)$ is an $\mathcal{F}$–Borel G–category.

Categorified Brauer quotients

Consider a family $\mathcal{F}$ of closed subgroups of G. The adjunction from Construction 6.2.13 $s^*\colon \mathrm{Cat}_G \rightleftharpoons \mathrm{Cat}_{G,\mathcal{F}^c} : s_*$ does *not* restrict to an adjunction between presentable or (fibrewise) stable categories as the adjunction unit does not preserve G-colimits. The main result of this section shows that its restriction to $s_*\colon \mathrm{Pr}^{L,\mathrm{st}}_{G,\mathcal{F}^c} \hookrightarrow \mathrm{Pr}^{L,\mathrm{st}}_G$ (which is fully faithful by Lemma 6.1.32) admits a symmetric monoidal left adjoint $\widetilde{s}^*$. We do this by showing that it is a smashing localisation.

Construction 6.2.23 ((Co)tensoring over pointed groupoids)**.** Let $\underline{\mathcal{E}}$ be a pointed $\mathcal{B}$–category admitting all parametrised (co)limits. Then $\underline{\mathcal{E}}$ is naturally tensored and cotensored over pointed $\mathcal{B}$–groupoids $\mathcal{B}_*$ as follows: for $\underline{*} \to \underline{X}$ in $\mathcal{B}_*$ and $E \in \underline{\mathcal{E}}$, we define

$$\underline{X} \wedge E := \mathrm{cofib}\left(E \simeq \underline{\operatorname{colim}}_{\underline{*}} E \to \underline{\operatorname{colim}}_{\underline{X}} E\right) \qquad \underline{\mathrm{hom}}_*(\underline{X}, E) := \mathrm{fib}\left(\underline{\lim}_{\underline{X}} E \to \underline{\lim}_{\underline{*}} E \simeq E\right)$$

These exhibit $\underline{\mathcal{E}}$ as being tensored and cotensored over $\mathcal{B}_*$, respectively, since for example, for a fixed $F \in \underline{\mathcal{E}}$, we have

$$\begin{aligned}\mathrm{Map}_{\underline{\mathcal{E}}}(\underline{X} \wedge E, F) &\simeq \mathrm{fib}\left(\mathrm{Map}_{\underline{\mathcal{E}}}(\underset{\underline{X}}{\mathrm{colim}}\, E, F) \to \mathrm{Map}_{\underline{\mathcal{E}}}(E, F)\right) \\ &\simeq \mathrm{Map}_{\mathcal{B}}(\underline{X}, \underline{\mathrm{Map}}_{\underline{\mathcal{E}}}(E, F)) \times_{\mathrm{Map}_{\mathcal{B}}(\underline{*}, \underline{\mathrm{Map}}_{\underline{\mathcal{E}}}(E,F))} \{*\} \\ &\simeq \mathrm{Map}_{\mathcal{B}_*}(\underline{X}, \underline{\mathrm{Map}}_{\underline{\mathcal{E}}}(E, F))\end{aligned}$$

Observe also that these constructions give us an adjunction $\underline{X} \wedge - \colon \underline{\mathcal{E}} \rightleftharpoons \underline{\mathcal{E}} : \underline{\mathrm{hom}}_*(\underline{X}, -)$. Moreover, it is easy to see that for $\underline{X}, \underline{Y} \in \mathcal{B}_*$, we have $\underline{X} \wedge (\underline{Y} \wedge E) \simeq (\underline{X} \wedge \underline{Y}) \wedge E$ where $\underline{X} \wedge \underline{Y} \simeq \mathrm{cofib}(\underline{X} \vee \underline{Y} \to \underline{X} \times \underline{Y})$.

Observation 6.2.24. Let $\mathcal{F}$ be a family of closed subgroups of G and $s^* \colon \mathrm{Cat}_{G,*} \rightleftharpoons \mathrm{Cat}_{\mathcal{F}^c,*} : s_*$ the associate Bousfield localisation of the geometric morphism $s^* \colon \mathcal{S}_G \rightleftharpoons \mathcal{S}_{\mathcal{F}^c}$. Let $\underline{X} \in \mathcal{S}_{G,*}$ and $\underline{\mathcal{C}} \in \mathrm{Cat}_{\mathcal{F}^c,*}$. Then there is an equivalence $\underline{\mathrm{hom}}_*(\underline{X}, s_*\underline{\mathcal{C}}) \simeq s_*\underline{\mathrm{hom}}_*(s^*\underline{X}, \underline{\mathcal{C}})$ by virtue of the following computation

$$\underline{\mathrm{hom}}_*(\underline{X}, s_*\underline{\mathcal{C}}) \simeq \mathrm{fib}(\lim_{\underline{X}} s_*\underline{\mathcal{C}} \to \lim_{\underline{*}} s_*\underline{\mathcal{C}}) \simeq s_*\,\mathrm{fib}(\lim_{s^*\underline{X}} \underline{\mathcal{C}} \to \lim_{\underline{*}} \underline{\mathcal{C}}) \simeq s_*\underline{\mathrm{hom}}_*(s^*\underline{X}, \underline{\mathcal{C}}).$$

Here we have used that s_* commutes with limits and the equivalence $\lim_{\underline{X}} s_*\underline{\mathcal{C}} \simeq s_* \lim_{s^*\underline{X}} \underline{\mathcal{C}}$ coming from the identifications of adjunctions in Lemma 6.1.18.

We introduce now the key notion of *Brauer quotients* of categories with respect to a fixed family. As will be clear from the next terminology, they will be a special case of the standard categorical construction of Verdier quotients. However, since they will play such a key role in this article and are so specific to the equivariant situation, we have chosen to dignify them with a special name, borrowing from the classical theory of Mackey functors.

Terminology 6.2.25 ($\mathcal{F}$–Brauer quotients)**.** For a finite group G, we define the $\mathcal{F}$–Brauer quotient $\underline{\mathcal{D}}/\langle\mathcal{F}\rangle$ of a small G–stable category $\underline{\mathcal{D}}$ as a G–stable category admitting a G–exact functor $\Phi^{\mathcal{F}} \colon \underline{\mathcal{D}} \to \underline{\mathcal{D}}/\langle\mathcal{F}\rangle$ which, for all G–stable categories $\underline{\mathcal{E}}$, induces an equivalence

$$(\Phi^{\mathcal{F}})^* \colon \underline{\mathrm{Fun}}^{\mathrm{ex}}(\underline{\mathcal{D}}/\langle\mathcal{F}\rangle, \underline{\mathcal{E}}) \xrightarrow{\simeq} \underline{\mathrm{Fun}}^{\mathrm{ex},\mathcal{F}=0}(\underline{\mathcal{D}}, \underline{\mathcal{E}})$$

where $\underline{\mathrm{Fun}}^{\mathrm{ex},\mathcal{F}=0}(\underline{\mathcal{D}}, \underline{\mathcal{E}}) \subseteq \underline{\mathrm{Fun}}^{\mathrm{ex}}(\underline{\mathcal{D}}, \underline{\mathcal{E}})$ is the full G–subcategory of G–exact functors $F \colon \underline{\mathcal{D}} \to \underline{\mathcal{E}}$ such that $\mathrm{Res}^G_H F \colon \mathrm{Res}^G_H \underline{\mathcal{D}} \to \mathrm{Res}^G_H \underline{\mathcal{E}}$ is the zero functor for all $H \in \mathcal{F}$. Observe that $\underline{\mathcal{D}}/\langle\mathcal{F}\rangle$ must be unique if it exists. We denote by $\mathrm{Cat}^{G-\mathrm{st}}_{G,\mathcal{F}^c} \subseteq \mathrm{Cat}^{G-\mathrm{st}}_G$ the full subcategory given by those G-stable categories lying in the image of $s_* \colon \mathrm{Cat}_{G,\mathcal{F}^c} \hookrightarrow \mathrm{Cat}_G$, i.e. those with value 0 on $\mathcal{O}_{\mathcal{F}}(G)$.

Analogously in the presentable setting, for a compact Lie group G and a family $\mathcal{F}$ of closed subgroups, we may define the $\mathcal{F}$–Brauer quotient of $\underline{\mathcal{C}} \in \mathrm{Pr}^{L,\mathrm{st}}_G$ as a presentable G–category $\underline{\mathcal{C}}/\langle\mathcal{F}\rangle$ equipped with a parametrised colimit–preserving functor $\Phi^{\mathcal{F}}: \underline{\mathcal{C}} \to \underline{\mathcal{C}}/\langle\mathcal{F}\rangle$ inducing for every fibrewise stable presentable G–category $\underline{\mathcal{E}}$ an equivalence

$$(\Phi^{\mathcal{F}})^*: \underline{\mathrm{Fun}}^L(\underline{\mathcal{C}}/\langle\mathcal{F}\rangle, \underline{\mathcal{E}}) \xrightarrow{\simeq} \underline{\mathrm{Fun}}^{L,\mathcal{F}=0}(\underline{\mathcal{C}}, \underline{\mathcal{E}})$$

Theorem 6.2.26 (Categorified Brauer quotients)**.** *Let G be a compact Lie group, H a finite group, $\mathcal{F}$ a family of closed subgroups of G and $\mathcal{E}$ a family of subgroups of H. Then the fully faithful inclusions*

$$s_*: \mathrm{Pr}^{L,\mathrm{st}}_{G,\mathcal{F}^c} \hookrightarrow \mathrm{Pr}^{L,\mathrm{st}}_{G} \qquad s_*: \mathrm{Cat}^{H-\mathrm{st}}_{H,\mathcal{E}^c} \hookrightarrow \mathrm{Cat}^{H-\mathrm{st}}_{H}$$

all admit symmetric monoidal left adjoints $\widetilde{s}^*$ which are smashing localisations at the idempotent algebra $\widetilde{E\mathcal{F}}$. In the first case, the induced lax symmetric monoidal structure on s_* agrees with the one from Lemma 6.1.31. Moreover, $\widetilde{s}^*$ satisfies the universal property of the $\mathcal{F}$–Brauer quotient.

Proof. We only prove the first case since the second one can be done entirely analogously. $\underline{\mathrm{Pr}}^{L,\mathrm{st}}_G$ is a pointed category admitting all parametrised (co)limits and is thus tensored over $\mathcal{S}_{G,*}$ by Construction 6.2.23 (in the presentable case, this also comes from pointedness being classified by the idempotent algebra $\mathcal{S}_{G,*} \in \mathrm{Pr}^L_G$).

We claim that the left adjoint to s_* is given by $\underline{\widetilde{E\mathcal{F}}} \wedge -$. To see that this functor does indeed take values in $\mathrm{Pr}^{L,\mathrm{st}}_{G,\mathcal{F}^c} \hookrightarrow \mathrm{Pr}^{L,\mathrm{st}}_{G}$ note that for all $K \in \mathcal{F}$ and $\underline{\mathcal{C}} \in \mathrm{Pr}^{L,\mathrm{st}}_G$ we have

$$\mathrm{Res}^G_K(\underline{\widetilde{E\mathcal{F}}} \wedge \underline{\mathcal{C}}) \simeq \mathrm{Res}^G_K \underline{\widetilde{E\mathcal{F}}} \wedge \mathrm{Res}^G_K \underline{\mathcal{C}} \simeq * \wedge \mathrm{Res}^G_K \underline{\mathcal{C}} \simeq 0.$$

To see that it is the left adjoint to s_*, let $\underline{\mathcal{D}} \in \mathrm{Pr}^{L,\mathrm{st}}_{G,\mathcal{F}^c}$. Observe that since $s^*\underline{\widetilde{E\mathcal{F}}} \simeq \underline{S}^0$, we have equivalences $\underline{\mathrm{hom}}_*(\underline{\widetilde{E\mathcal{F}}}, s_*\underline{\mathcal{D}}) \simeq s_*\underline{\mathrm{hom}}_*(\underline{S}^0, \underline{\mathcal{D}}) \simeq s_*\underline{\mathcal{D}}$, where the first equivalence is by Observation 6.2.24. Thus, the computation

$$\mathrm{Map}_{\mathrm{Pr}^{L,\mathrm{st}}_G}(\underline{\widetilde{E\mathcal{F}}} \wedge \underline{\mathcal{C}}, s_*\underline{\mathcal{D}}) \simeq \mathrm{Map}_{\mathrm{Pr}^{L,\mathrm{st}}_G}(\underline{\mathcal{C}}, \underline{\mathrm{hom}}_*(\underline{\widetilde{E\mathcal{F}}}, s_*\underline{\mathcal{D}})) \simeq \mathrm{Map}_{\mathrm{Pr}^{L,\mathrm{st}}_G}(\underline{\mathcal{C}}, s_*\underline{\mathcal{D}}).$$

shows that $\underline{\widetilde{E\mathcal{F}}} \wedge -$ is indeed the left adjoint to s_* as claimed.

Next, since $\underline{\widetilde{E\mathcal{F}}}$ is an idempotent algebra in $\mathcal{S}_{G,*}$, the left adjoint $\widetilde{s}^*(-) \simeq \underline{\widetilde{E\mathcal{F}}} \wedge -$ is a smashing localisation and in particular attains a canonical symmetric monoidal structure. To show that the induced lax symmetric monoidal structure on s_* is equivalent to the one from Lemma 6.1.31, by Lemma 6.1.32 we only have to show that the map $u: \widetilde{s}^*\mathbb{1}_{\mathrm{Pr}^{L,\mathrm{st}}_G} \to \mathbb{1}_{\mathrm{Pr}^{L,\mathrm{st}}_{G,\mathcal{F}^c}}$ adjoint to the lax unit $\mathbb{1}_{\mathrm{Pr}^{L,\mathrm{st}}_G} \to s_*\mathbb{1}_{\mathrm{Pr}^{L,\mathrm{st}}_{G,\mathcal{F}^c}}$ is an equivalence. But by construction of the lax symmetric monoidal structure on s_*, we have for $\underline{\mathcal{C}} \in \mathrm{Pr}^{L,\mathrm{st}}_{G,\mathcal{F}^c}$ the commutative diagram

$$\begin{tikzcd}
\underline{\mathrm{Fun}}^L_{\mathcal{F}}\left(\mathbb{1}_{\mathrm{Pr}^{L,\mathrm{st}}_{G,\mathcal{F}^c}}, \underline{\mathcal{C}}\right) \arrow[rr, "\simeq"] \arrow[d, "u^*"] & & \Gamma(\underline{\mathcal{C}}) \arrow[d, "\simeq"] \\
\underline{\mathrm{Fun}}^L_{\mathcal{F}}\left(\widetilde{s}^*\mathbb{1}_{\mathrm{Pr}^{L,\mathrm{st}}_G}, \underline{\mathcal{C}}\right) \arrow[r, "\simeq"] & \underline{\mathrm{Fun}}^L_G\left(\mathbb{1}_{\mathrm{Pr}^{L,\mathrm{st}}_G}, s_*\underline{\mathcal{C}}\right) \arrow[r, "\simeq"] & \Gamma(s_*\underline{\mathcal{C}})
\end{tikzcd}$$

showing that the left vertical map is an equivalence.

Finally, for the statement about $\mathcal{F}$–Brauer quotients, notice that the unit map $\underline{\mathcal{C}} \to \underline{\widetilde{E\mathcal{F}}} \wedge \underline{\mathcal{C}}$ has trivial restriction to each group $H \in \mathcal{F}$ as

$$\mathrm{Res}^G_H \underline{\widetilde{E\mathcal{F}}} \wedge \underline{\mathcal{C}} \simeq \mathrm{Res}^G_H \underline{\widetilde{E\mathcal{F}}} \wedge \mathrm{Res}^G_H \underline{\mathcal{C}} \simeq * \wedge \mathrm{Res}^G_H \underline{\mathcal{C}} \simeq 0.$$

We have to show that for all $\underline{\mathcal{D}} \in \mathrm{Pr}^{L,\mathrm{st}}_G$, the induced map $\underline{\mathrm{Fun}}^L(s_*\widetilde{s}^*\underline{\mathcal{C}}, \underline{\mathcal{D}}) \to \underline{\mathrm{Fun}}^{L,\mathcal{F}=0}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ is an equivalence. From the cofibre sequence $\underline{E\mathcal{F}}_+ \to \underline{S}^0 \to \underline{\widetilde{E\mathcal{F}}}$ in $\mathcal{S}_{G,*}$ we obtain the fibre sequence

$$\underline{\mathrm{Fun}}^L(\underline{\widetilde{E\mathcal{F}}} \wedge \underline{\mathcal{C}}, \underline{\mathcal{D}}) \to \underline{\mathrm{Fun}}^L(\underline{\mathcal{C}}, \underline{\mathcal{D}}) \to \underline{\mathrm{Fun}}^L(\underline{E\mathcal{F}}_+ \wedge \underline{\mathcal{C}}, \underline{\mathcal{D}})$$

in $\mathrm{Pr}^{L,\mathrm{st}}_G$. This shows that $\underline{\mathrm{Fun}}^L(\underline{\widetilde{E\mathcal{F}}} \wedge \underline{\mathcal{C}}, \underline{\mathcal{D}})$ is a full G-subcategory of $\underline{\mathrm{Fun}}^L(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ (this is true for any fibre sequence of stable categories). Now suppose that the functor $f\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ vanishes on $\mathcal{F}$. Denote by $\langle \mathrm{Im}(f) \rangle \subseteq \underline{\mathcal{D}}$ the full presentable fibrewise stable G-subcategory generated by the image of f. The assumption on f guarantees that $\mathrm{Res}^G_H \langle \mathrm{Im}(f) \rangle = 0$ whenever $H \in \mathcal{F}$, i.e. $\langle \mathrm{Im}(f) \rangle$ lies in the image of s_*. Consider the commutative diagram

$$\begin{tikzcd}
\underline{\mathcal{C}} \arrow[r, "f"] \arrow[d] & \langle \mathrm{Im}(f) \rangle \arrow[r, hook] \arrow[d, "\simeq"] & \underline{\mathcal{D}} \\
\underline{\widetilde{E\mathcal{F}}} \wedge \underline{\mathcal{C}} \arrow[r, "f"] & \underline{\widetilde{E\mathcal{F}}} \wedge \langle \mathrm{Im}(f) \rangle \arrow[ur, dashed]
\end{tikzcd}.$$

As $\langle \mathrm{Im}(f) \rangle$ lies in the image of s_*, the first part shows that the middle vertical arrow is an equivalence from which we obtain the dashed factorisation. This concludes the proof of the theorem. $\square$

Remark 6.2.27. In the setting of finite groups G, for a G–stable category $\underline{\mathcal{D}}$, since $\widetilde{s}^*\underline{\mathcal{D}}$ is the $\mathcal{F}$–Brauer quotient, it satisfies the universal property of the Verdier quotient articulated in [QS22, Thm. 5.23]. Thus, by [QS22, Def. 5.21], it may alternatively be described as a fibrewise Verdier quotient in the nonequivariant sense.

Notation 6.2.28. Now that we have all the fixed points functors that will concern us, let us collect and summarise them, introducing some new notations along the way. While the notations $\{s_!, s^*, s_*, \widetilde{s}^*\}$ are compact and lithe, useful to prove results, we believe that the notations presently introduced have more intuitive appeal. The starting point will be the inclusion $s\colon \mathcal{O}_{\mathcal{F}^c}(G)^{\mathrm{op}} \hookrightarrow \mathcal{O}(G)^{\mathrm{op}}$ from Construction 6.2.13.

(a) Recall the notations from Construction 6.2.15 which gives us the top adjunctions in

$$\begin{tikzcd}[column sep=huge]
\mathcal{S}_{\mathcal{F}^c} \arrow[r, hook, bend left, "(-)_{\widetilde{\mathcal{F}}}:=s_!"] \arrow[r, hook, bend right, "(-)^{\widetilde{\mathcal{F}}}:=s_*"'] \arrow[d, hook] & \mathcal{S}_G \arrow[l, "(-)^{\mathcal{F}^c}:=s^*" description] \arrow[d, hook] \\
\mathrm{Cat}_{G,\mathcal{F}^c} \arrow[r, hook, bend left, "(-)_{\widetilde{\mathcal{F}}}:=s_!"] \arrow[r, hook, bend right, "(-)^{\widetilde{\mathcal{F}}}:=s_*"'] & \mathrm{Cat}_G \arrow[l, "(-)^{\mathcal{F}^c}:=s^*" description]
\end{tikzcd}$$

and that we have the commuting squares of adjunctions is an easy check using that s^* commutes with the vertical maps and their adjoints. Since $(-)_{\widetilde{\mathcal{F}}}$ and $(-)^{\widetilde{\mathcal{F}}}$ are fully faithful, we also write $(-)_{\widetilde{\mathcal{F}}}$ and $(-)^{\widetilde{\mathcal{F}}}$ for $s_!s^*$ and s_*s^* respectively. In particular, for $\underline{X} \in \mathcal{S}_G$, the counit gives us a map $\epsilon\colon \underline{X}_{\widetilde{\mathcal{F}}} = s_!s^*\underline{X} \to \underline{X}$ as in Construction 6.2.15. Moreover, by Construction 6.2.13, we have for $\underline{\mathcal{C}} \in \mathrm{Cat}_{G,\mathcal{F}^c}$ the description

$$\underline{\mathcal{C}}^{\widetilde{\mathcal{F}}} = \begin{cases} \mathcal{C}(\underline{G/H}) & \text{if } H \in \mathcal{F}^c; \\ * & \text{if } H \in \mathcal{F}. \end{cases}$$

(b) We also have the following solid commuting squares

$$\begin{tikzcd}[column sep=huge]
\mathrm{Pr}^{\mathrm{st}}_G \arrow[r, dashed, shift left, "\Phi^{\mathcal{F}}(-):=\widetilde{s}^*"] \arrow[d, hook] & \mathrm{Pr}^{\mathrm{st}}_{G,\mathcal{F}^c} \arrow[l, hook, shift left, "(-)^{\Phi\widetilde{\mathcal{F}}}:=s_*"] \arrow[d, hook] \\
\widehat{\mathrm{Cat}}_G & \widehat{\mathrm{Cat}}_{G,\mathcal{F}^c} \arrow[l, hook', "(-)^{\widetilde{\mathcal{F}}}:=s_*"']
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=huge]
\mathrm{Cat}^{\mathrm{st}}_G \arrow[r, dashed, shift left, "\Phi^{\mathcal{F}}(-):=\widetilde{s}^*"] \arrow[d, hook] & \mathrm{Cat}^{\mathrm{st}}_{G,\mathcal{F}^c} \arrow[l, hook, shift left, "(-)^{\Phi\widetilde{\mathcal{F}}}:=s_*"] \arrow[d, hook] \\
\mathrm{Cat}_G & \mathrm{Cat}_{G,\mathcal{F}^c} \arrow[l, hook', "(-)^{\widetilde{\mathcal{F}}}:=s_*"']
\end{tikzcd}$$

where the top maps admit the dashed left adjoints. Here, the left diagram holds for general compact Lie groups G whereas the right diagram is only defined for finite groups G from Theorem 6.2.26. As above, since the functors $(-)^{\Phi\widetilde{\mathcal{F}}}$ are fully faithful, we will also write $(-)^{\Phi\widetilde{\mathcal{F}}}$ to denote $s_*\widetilde{s}^*$. The adjunction unit $\mathrm{id} \to s_*\widetilde{s}^*$ will be denoted by $\Phi^{\mathcal{F}}\colon (-) \to (-)^{\Phi\widetilde{\mathcal{F}}}$ or just $\Phi\colon (-) \to (-)^{\Phi\widetilde{\mathcal{F}}}$ when the family $\mathcal{F}$ is understood.

Stability for quotient groups

Let $N \leq G$ be a closed normal subgroup of the compact Lie group G and denote by $\theta\colon G \twoheadrightarrow G/N = Q$ the quotient map. We will use the categorified Brauer quotient from Theorem 6.2.26 for the family Γ_N from Example 6.2.10 to relate G– and Q–stable categories.

Proposition 6.2.29. *Suppose that $\theta\colon G \twoheadrightarrow G/N = Q$ is a continuous epimorphism of compact Lie groups. Then there is an adjunction*

$$\mathrm{Pr}_G^{L,G-\mathrm{st}} \underset{\mathrm{Coinfl}_\alpha}{\overset{\mathrm{Coind}_\alpha^\sim}{\rightleftarrows}} \mathrm{Pr}_Q^{L,Q-\mathrm{st}}$$

which is a smashing localisation. The lax symmetric monoidal structure on Coinfl_α *from Lemma 6.2.8 is equivalent to the lax symmetric monoidal structure from this smashing localisation. We thus may view G/N–stable presentable categories precisely as G–stable categories which vanish for all subgroups $H \leq G$ not containing N.*

Proof. By combining Example 6.2.14 and Theorem 6.2.26, $\mathrm{Coinfl}_\alpha\colon \mathrm{Pr}_Q^{L,\mathrm{st}} \hookrightarrow \mathrm{Pr}_G^{L,\mathrm{st}}$ admits a symmetric monoidal left adjoint $\mathrm{Coind}_\alpha^\sim = \mathrm{Coind}_\alpha(\underline{\widetilde{E\Gamma}_N} \wedge -)$ which is a smashing localisation. We only have to show that this restricts to an adjunction between G- and Q-stable categories. But this follows by combining Lemma 6.2.6, Lemma 6.2.8 and observing that $\underline{\widetilde{E\Gamma}_N} \wedge -$ preserves G-stable categories. $\square$

Corollary 6.2.30. *Writing $\theta\colon G \twoheadrightarrow G/N$ for the quotient map by a closed normal subgroup, the symmetric monoidal unit map* $\underline{\mathrm{Sp}}_{G/N} \to \mathrm{Coind}_\alpha^\sim \underline{\mathrm{Sp}}_G$ *is an equivalence.*

Proof. This is a direct consequence of symmetric monoidality of the adjunction in Proposition 6.2.29. $\square$

Construction 6.2.31 (Geometric fixed points)**.** Let $\theta_G\colon G \twoheadrightarrow 1$ be the quotient map. The symmetric monoidal G-colimit preserving unit map

$$\Phi^G\colon \underline{\mathrm{Sp}}_G \longrightarrow \mathrm{Coinfl}_{\theta_G} \mathrm{Coind}_{\theta_G}^\sim \underline{\mathrm{Sp}}_G \simeq \mathrm{Coinfl}_{\theta_G} \mathrm{Sp}$$

restricts to a symmetric monoidal colimit preserving functor $\Phi^G\colon \mathrm{Sp}_G \to \mathrm{Sp}$. There is an equivalence equivalence $\Phi^G \circ \Sigma_G^\infty(-) \simeq \Sigma^\infty(-)^G$ as, by construction, Φ_G is $\mathcal{S}_{G,*}$-linear and sends the unit to the unit. This shows that Φ^G recovers the classical geometric fixed points functor which is uniquely determined by these properties.

If $H \leq G$ is a closed subgroup, we have the symmetric monoidal G-colimit preserving functor $\Phi^H\colon \underline{\mathrm{Sp}}_G \to \mathrm{Coind}_H^G \underline{\mathrm{Sp}}_H \to \mathrm{Coind}_H^G \mathrm{Coinfl}_{\theta_H} \mathrm{Sp}$ which on global sections recovers the classical geometric fixed point functors $\Phi^H\colon \mathrm{Sp}_G \to \mathrm{Sp}$.

Definition 6.2.32. A collection of $\mathcal{B}$–functors $\{F_s\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}_s\}_{s\in S}$ is *jointly conservative* if for all $X \in \mathcal{B}$, the collection $\{F_s(X)\colon \mathcal{C}(\underline{X}) \to \mathcal{D}_s(\underline{X})\}_{s\in S}$ is jointly conservative.

Observation 6.2.33. Let $\{F_s \colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}_s\}_{s\in S}$ be a jointly conservative collection of $\mathcal{B}$–functors and $\underline{X} \in \mathcal{B}$. Then the collection $\{F_s \colon \underline{\mathrm{Fun}}(\underline{X}, \underline{\mathcal{C}}) \to \underline{\mathrm{Fun}}(\underline{X}, \underline{\mathcal{D}}_s)\}_{s\in S}$ is also a jointly conservative collection. This is an immediate consequence of the definition and that the evaluation at $Y \in \mathcal{B}$ for the $\mathcal{B}$–category $\underline{\mathrm{Fun}}(\underline{X}, \underline{\mathcal{C}})$ is $\mathcal{C}(\underline{Y} \times \underline{X})$.

Remark 6.2.34. If $\{\underline{X}_i\}_{i\in I}$ is a set of objects generating $\mathcal{B}$ under colimits, then a collection of $\mathcal{B}$–functors $\{F_s \colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}_s\}_{s\in S}$ is jointly conservative if the collection $\{F_s(X) \colon \mathcal{C}(\underline{X}_i) \to \mathcal{D}_s(\underline{X}_i)\}_{s\in S}$ is jointly conservative for each i. Indeed, let $\underline{X}$ be an object. Then $\mathcal{C}(\underline{X}) \simeq \lim_{(i,f\colon \underline{X}_i \to \underline{X})} \mathcal{C}(\underline{X}_i)$ and the collection

$$T = \{\mathcal{C}(\underline{X}) \xrightarrow{f^*} \mathcal{C}(\underline{X}_i) \mid i \in I, f \colon \underline{X}_i \to \underline{X}\}$$

is jointly conservative. Suppose that h is a morphism in $\mathcal{C}(\underline{X})$ which maps to an equivalence in $\mathcal{D}_s(\underline{X})$ for each $s \in S$. For $(i,f) \in T$, we observe that in the commutative diagram

$$\begin{tikzcd}
\mathcal{C}(\underline{X}) \arrow[r, "f^*"] \arrow[d] & \mathcal{D}_s(\underline{X}) \arrow[d] \\
\mathcal{C}(\underline{X}_i) \arrow[r] & \mathcal{D}_s(\underline{X}_i)
\end{tikzcd}$$

the morphism h maps to an equivalence in the lower right corner for each $s \in S$, so by assumption it mapped to an equivalence in the lower left corner. As that holds true for each $(i,f) \in T$ and the collection T was jointly conservative, we see that h was an equivalence to start with, as desired.

Proposition 6.2.35 (Joint conservativity of geometric fixed points)**.** *The collection of G–functors*

$$\left\{\Phi^H \colon \underline{\mathrm{Sp}} \xrightarrow{\eta} \mathrm{Coind}^G_H \mathrm{Res}^G_H \underline{\mathrm{Sp}} \xrightarrow{\mathrm{Coind}^G_H \mathrm{Res}^G_H \Phi^{\mathcal{P}_H}} \mathrm{Coind}^G_H \mathrm{Res}^G_H \underline{\mathrm{Sp}}^{\Phi\widetilde{\mathcal{P}_H}} \quad \middle| \quad H \leq G \text{ closed}\right\}$$

is jointly conservative.

Proof. By Remark 6.2.34, it suffices to show that the collection is a jointly conservative collection of functors when evaluated at each $\underline{G/K} \in \mathcal{S}_G$. Let $H \leq K$ be a subgroup. Since $\underline{G/H} \simeq \mathrm{Ind}^G_H \underline{*}$, by the triangle identity, the counit $\mathrm{Ind}^G_H \mathrm{Res}^G_H \underline{G/H} \xrightarrow{\epsilon_{G/H}} \underline{G/H}$ admits a section. Using this and the map $\underline{G/H} \to \underline{G/K}$ we obtain in total a map $h \colon \underline{G/H} \to \mathrm{Ind}^G_H \mathrm{Res}^G_H \underline{G/K}$. Hence, since we have $(\mathrm{Coind}^G_H \mathrm{Res}^G_H \underline{\mathcal{C}})(G/K) \simeq \mathrm{Fun}_G(\mathrm{Ind}^G_H \mathrm{Res}^G_H \underline{G/K}, \underline{\mathcal{C}})$ and $\mathcal{C}(G/H) \simeq \mathrm{Fun}_G(\underline{G/H}, \underline{\mathcal{C}})$ for any G–category $\underline{\mathcal{C}}$, we get a transformation $h^* \colon (\mathrm{Coind}^G_H \mathrm{Res}^G_H \underline{\mathcal{C}})(G/K) \to \mathcal{C}(G/H)$ natural in $\underline{\mathcal{C}}$. Therefore, for $H \leq K$, evaluating the functor Φ^H in the statement at G/K together with the transformation above gives the following commuting diagram

$$\begin{tikzcd}
\mathrm{Sp}_K \arrow[r, "\eta"] \arrow[d, equal] & (\mathrm{Coind}_H^G \mathrm{Res}_H^G \underline{\mathrm{Sp}})(G/K) \arrow[r, "\mathrm{Coind}_H^G \mathrm{Res}_H^G \Phi^{\mathcal{P}_H}"] \arrow[d, "h^*"] & (\mathrm{Coind}_H^G \mathrm{Res}_H^G \underline{\mathrm{Sp}}^{\widetilde{\Phi\mathcal{P}_H}})(G/K) \arrow[d, "h^*"] \\
\mathrm{Sp}_K \arrow[r, "\mathrm{Res}_H^K"] & \mathrm{Sp}_H \arrow[r, "\Phi^H"] & \mathrm{Sp}
\end{tikzcd}$$

But since the bottom compositions are jointly conservative when we let H vary over all closed subgroups of K (this is well–known, see for example [Sch18, Prop. 3.3.10]), we thus get similarly that the top compositions are too. This completes the proof. □

Free actions

Here we review a few geometric facts on G-spaces on which a normal subgroup N acts freely. It will be needed later on to argue for example, if $\underline{X}$ is a G-Poincaré space with free N-action, then also the quotient $N\backslash\underline{X}$ is G/N-Poincaré.

Definition 6.2.36 (Free actions)**.** Consider a group G together with a normal subgroup $N \leq G$. We say that the action of N on $\underline{X} \in \mathcal{S}_G$ is *free* if $\underline{X}$ is coBorel with respect to the family $\mathcal{F}_N$.

That is, N acts freely on $\underline{X}$ if whenever $N \cap H \neq \{1\}$ we have $X^H = \varnothing$ as is shown in Fact 6.2.19.

Remark 6.2.37 (Quotients of free G–spaces)**.** Consider $\underline{X} \in \mathcal{S}_G$ and isotropy separation with respect to the trivial family $\mathcal{F} = 1$ (which is the case $N = G$ in Definition 6.2.36). Recalling the operation of genuine quotients from Notation 6.2.2, note that for this family we obtain $G\backslash b_!(-) \simeq (-)_{hG}$ as the first functor is left adjoint to the composite $b^* \mathrm{Infl}_G^1 \colon \mathcal{S} \to \mathcal{S}^{BG}$ which is the restriction functor along the projection $BG \to *$. In particular, we obtain a map

$$X_{hG} \simeq G\backslash(b_! b^* \underline{X}) \to G\backslash\underline{X}.$$

This is an equivalence if G acts freely on $\underline{X}$ since $X \simeq b_! Y$ for some $Y \in \mathcal{S}^{BG}$.

We will now prove two lemmas about quotients by free actions. Together, they are useful in studying the fibres of the map $\underline{X} \to \mathrm{Infl}_G^Q N\backslash\underline{X}$, as we will see in Corollary 6.2.40.

Lemma 6.2.38. *Let G be a group and let $N \subset G$ be a closed normal subgroup and consider a map $f \colon \underline{X} \to \underline{Y}$ be a map of G–spaces, where $\underline{X}$ and $\underline{Y}$ have free N-actions. Then the square*

$$\begin{tikzcd}
\underline{X} \arrow[r] \arrow[d] & \underline{Y} \arrow[d] \\
\mathrm{Infl}_G^{G/N} N\backslash\underline{X} \arrow[r] & \mathrm{Infl}_G^{G/N} N\backslash\underline{Y}
\end{tikzcd} \tag{6.8}$$

is cartesian.

Proof. We want to check that the square (6.8) is cartesian and we will do so in three steps[1]. First, a computation shows that it is cartesian whenever $\underline{X}$ and $\underline{Y}$ are N-free G-orbits. Second, writing $\underline{Y} = \mathrm{colim}_{H \leq G,\, \underline{G/H} \to \underline{Y}} \underline{G/H}$ an application of [Lur09, 6.1.3.9.(4)] shows that the claim is true for $\underline{X}$ an N-free G-orbit and $\underline{Y}$ an N-free G–space. It is easy to see that the claim now also holds when $\underline{X}$ is a disjoint union of N-free G-orbits. For the general statement, note that by Lemma 6.2.39, we may find a collection of N-free G-orbits $\underline{G/H_i}$ together with maps $q_i \colon \underline{G/H_i} \to \underline{X}$ such that the map

$$\coprod \mathrm{Infl}_G^{G/N} N \backslash q_i \colon \coprod \mathrm{Infl}_G^{G/N} N \backslash \underline{G/H_i} \to \mathrm{Infl}_G^{G/N} N \backslash \underline{X}$$

induces a π_0-surjection on all fixed points[2]. In the diagram

$$\begin{tikzcd} \coprod \underline{G/H_i} \arrow[r] \arrow[d] & \underline{X} \arrow[r] \arrow[d] & \underline{Y} \arrow[d] \\ \coprod \mathrm{Infl}_G^{G/N} N \backslash \underline{G/H_i} \arrow[r] & \mathrm{Infl}_G^{G/N} N \backslash \underline{X} \arrow[r] & \mathrm{Infl}_G^{G/N} N \backslash \underline{Y} \end{tikzcd}$$

we know that the outer square and left square are cartesian. As the bottom left map induces a π_0-surjection on all fixed points, this implies that the right square is cartesian as well. □

Lemma 6.2.39. *Let $\underline{X}$ be a G–space with free N-action. Then, for each map $f \colon \underline{Q/H} \to N \backslash \underline{X}$ there exists a subgroup $K \in \mathcal{F}_N$ and a commutative diagram*

$$\begin{tikzcd} & \underline{G/K} \arrow[r] \arrow[d] & \underline{X} \arrow[d] \\ \mathrm{Infl}_G^Q \underline{Q/H} \arrow[r] & \mathrm{Infl}_G^Q N \backslash \underline{G/K} \arrow[r] & \mathrm{Infl}_G^Q N \backslash \underline{X} \end{tikzcd}$$

where the lower composition is $\mathrm{Infl}_G^Q(f)$.

Proof. Using the explicit formula from (6.7) we compute

$$\mathrm{Map}_{\mathcal{S}_Q}(\underline{Q/H}, N \backslash \underline{X}) \simeq \underset{G/K,\, Q/H \to N\backslash(G/K)}{\mathrm{colim}} \mathrm{Map}_{\mathcal{S}_G}(\underline{G/K}, \underline{X}).$$

The map from the left hand side takes $g \colon \underline{G/K} \to \underline{X}$, applies $N\backslash(-)$ to it and pre-composes with $Q/H \to N\backslash \underline{G/K}$. Thus, f eviently factors through some map $N\backslash \underline{G/K} \to N \backslash \underline{X}$ that is of the from $N\backslash g$ for some $g \colon \underline{G/H} \to \underline{X}$. Now as $\mathrm{Map}(\underline{G/H}, \underline{X})$ can only be nonempty if $H \in \mathcal{F}$, we have proved the assertion. □

[1] An (arguably shorter) proof is possible if one recalls the model from [Sch18, Prop. B.7.] and observes that given an N-free topological G-CW space $\mathcal{X}$, the map $\mathcal{X} \to \mathrm{Infl}_G^{G/N} N \backslash \mathcal{X}$ is a fibration with point-set fibre N.

[2] Such morphisms are effective epimorphisms.

Corollary 6.2.40. *Let $\underline{X}$ be a G–space on which the closed normal subgroup $N \leq G$ acts freely. Consider any map $f\colon \underline{G/H} \to \mathrm{Infl}_G^Q N\backslash\underline{X}$. Then there exists a cartesian diagram*

$$\begin{tikzcd}
\underline{G/H} \times_{\mathrm{Infl}_G^Q N\backslash\underline{X}} \underline{X} \arrow[r] \arrow[d, "proj"] & \underline{G/K_0} \arrow[d] \\
\underline{G/H} \arrow[r] & \underline{G/K_1}
\end{tikzcd}$$

where $K_0 \in \mathcal{F}$ and $\underline{G/K_1} \simeq \mathrm{Infl}\, N\backslash\underline{G/K_0}$.

Proof. Set $H' = H/(H \cap N) \subset Q$. The map $f\colon \underline{G/H} \to \mathrm{Infl}_G^Q N\backslash\underline{X}$ factors through the adjunction unit $\underline{G/H} \to \mathrm{Infl}_G^Q N\backslash\underline{G/H} \simeq \mathrm{Infl}_G^Q \underline{Q/H'}$. As the functor Infl_G^Q is fully faithful, we can apply Lemma 6.2.39 to the corresponding map $\underline{Q/H'} \to N\backslash\underline{X}$ and obtain a commutative diagram

$$\begin{tikzcd}
 & & \underline{G/K} \arrow[r] \arrow[d] & \underline{X} \arrow[d] \\
\underline{G/H} \arrow[r] & \mathrm{Infl}_G^Q \underline{Q/H} \arrow[r] & \mathrm{Infl}_G^Q N\backslash\underline{G/K} \arrow[r] & \mathrm{Infl}_G^Q N\backslash\underline{X}
\end{tikzcd}$$

in which the square is cartesian by Lemma 6.2.38. Completing the cospan involving $\underline{G/K}$ and $\underline{G/H}$ to a pullback gives the desired pullback. $\square$

Chapter 7

Parametrised Poincaré duality

In this section we start developing the basic formalism of Poincaré duality within the context of categories parametrised over a topos as summarised in §6.1. This general theory will later be specialised to the equivariant setting for compact Lie groups in Chapter 8.

As a motivation for the definitions appearing in this section recall that, for a closed smooth manifold M^d, an embedding $M \hookrightarrow \mathbb{R}^N$ gives rise to a collapse map

$$c\colon S^N \to \mathrm{Th}(\nu_{M\subset\mathbb{R}^N})$$

where $\nu_{M\subset\mathbb{R}^N}$ is the normal bundle of M in $\mathbb{R}^N$. It turns out that neither the stable homotopy type of the Thom space $\mathrm{Th}(\nu_{M\subset\mathbb{R}^N})$ nor the stable homotopy class of the collapse map c depend on the choice of embedding. The collapse map defines a class $[c] \in H_N(\mathrm{Th}(\nu_{M\subset\mathbb{R}^N})) \simeq H_d(M;\mathcal{O}_\nu)$, where the isomorphism is the Thom isomorphism and $\mathcal{O}_\nu$ denotes the orientation local system of ν. Classical Poincaré duality now says that

$$[c] \cap -\colon H^k(M) \to H_{k-d}(M;\mathcal{O}_\nu) \tag{7.1}$$

is an isomorphism.

We start by axiomatising in §7.1 such stable collapse maps as *Spivak data* with respect to a fixed coefficient category, upon which we may demand the further condition of being twisted ambidextrous and Poincaré in §7.2, generalising the situation sketched above. We then investigate in §7.3 various operations one can perform on Spivak data, proving along the way the main results of the section (c.f. Theorems 7.3.5 and 7.3.8) about basechanging coefficient categories, which will be the key inputs to our equivariant theory. We then end the section with a discussion of degree theory which will serve as the foundation for our theory of equivariant degrees in §8.6 and our geometric applications in Chapter 9.

7.1 Spivak data

For an object $\underline{X} \in \mathcal{B}$ we denote by $X\colon \underline{X} \to \underline{*}$ the map to the final object. Recall that a $\mathcal{B}$–category $\underline{\mathcal{C}}$ admits $\underline{X}$–shaped limits (resp. colimits) if $X^*\colon \underline{\mathcal{C}} \simeq \underline{\mathrm{Fun}}(\underline{*}, \underline{\mathcal{C}}) \to \underline{\mathrm{Fun}}(\underline{X}, \underline{\mathcal{C}})$ admits a right adjoint X_* (resp. left adjoint $X_!$).

Definition 7.1.1. Let $\underline{X} \in \mathcal{B}$ and $\underline{\mathcal{C}}$ a symmetric monoidal $\mathcal{B}$–category which admits $\underline{X}$-shaped colimits. A *$\underline{\mathcal{C}}$–Spivak datum for $\underline{X}$* consists of

(1) an object $\xi \in \underline{\mathrm{Fun}}(\underline{X}, \underline{\mathcal{C}})$ called the dualising sheaf;

(2) a map $c\colon \mathbb{1}_{\underline{\mathcal{C}}} \to X_!\xi$ in $\underline{\mathcal{C}}$, called the fundamental class (or collapse map).

The importance of Spivak data comes from the following construction, which allows us to compare the $\underline{X}$–shaped limit functor with a twisted $\underline{X}$–colimit functor. It is a generalisation of the map (7.1) given by capping with the fundamental class appearing in classical Poincaré duality.

Construction 7.1.2 (Capping map)**.** Let $\underline{\mathcal{C}}$ be a symmetric monoidal $\mathcal{B}$–category which admits $\underline{X}$–shaped limits and colimits and satisfies the $\underline{X}$–projection formula (c.f. Terminology 6.1.13). For each $\underline{\mathcal{C}}$–Spivak datum (ξ, c) on $\underline{X}$ we can construct a natural transformation

$$c \cap_\xi -\colon X_*(-) \xrightarrow{c\otimes -} X_!\xi \otimes X_*(-) \xleftarrow[\simeq]{\mathrm{PF}^X} X_!(\xi \otimes X^*X_*(-)) \xrightarrow{X_!(\mathrm{id}\otimes\epsilon)} X_!(\xi \otimes -)$$

which is a morphism in $\underline{\mathrm{Fun}}(\underline{\mathcal{C}}^{\underline{X}}, \underline{\mathcal{C}})$ where $\epsilon\colon X^*X_* \to \mathrm{id}$ denotes the adjunction counit. To avoid notational clutter, we will often omit the ξ from $c \cap_\xi -$ when the context is clear.

There is also a construction in the other direction, which produces a fundamental class for ξ from a natural transformation $X_*(-) \to X_!(\xi \otimes -)$.

Construction 7.1.3. Given a natural transformation $t\colon X_*(-) \to X_!(\xi \otimes -)$ and writing $\eta\colon \mathrm{id} \to X_*X^*$ for the adjunction unit, we obtain a collapse map as the composite

$$\mathrm{clps}_\xi(t)\colon \mathbb{1}_{\underline{\mathcal{C}}} \xrightarrow{\eta} X_*X^*\mathbb{1}_{\underline{\mathcal{C}}} \xrightarrow{t} X_!(\xi \otimes X^*\mathbb{1}_{\underline{\mathcal{C}}}) \simeq X_!\xi.$$

Lemma 7.1.4. *There is an equivalence* $\mathrm{clps}_\xi(c \cap_\xi -) \simeq c \in \mathrm{Map}_{\underline{\mathcal{C}}}(\mathbb{1}_{\underline{\mathcal{C}}}, X_!\xi)$*.*

Proof. Consider the commutative diagram

$$\begin{tikzcd}
& \mathbb{1}_{\underline{\mathcal{C}}} \arrow[r, "\eta"] \arrow[d, "c"] & X_*X^*\mathbb{1}_{\underline{\mathcal{C}}} \arrow[d, "c\otimes -"] & \\
& X_!\xi \arrow[r, "-\otimes\eta"] & X_!\xi \otimes X_*X^*\mathbb{1}_{\underline{\mathcal{C}}} & \\
X_!(\xi \otimes X^*\mathbb{1}_{\underline{\mathcal{C}}}) \arrow[ur, "\mathrm{PF}^X", "\simeq"'] \arrow[r, "X^*\eta"] & X_!(\xi \otimes X^*X_*X^*\mathbb{1}_{\underline{\mathcal{C}}}) \arrow[ur, "\mathrm{PF}^X", "\simeq"'] \arrow[r, "\epsilon_{X^*}"] & X_!(\xi \otimes X^*\mathbb{1}_{\underline{\mathcal{C}}}) \simeq X_!\xi. &
\end{tikzcd}$$

The composite $\mathbb{1}_{\underline{\mathcal{C}}} \to X_!\xi$ going through the upper right corner of the rectangle is by definition equal to $\mathrm{clps}_{\xi}(c \cap_{\xi} -)$. The composite $\mathbb{1}_{\underline{\mathcal{C}}} \to X_!\xi$ going through the bottom left corner of the rectangle is equivalent to c using the triangle identity $\epsilon_{X^*} \circ X^*\eta \simeq \mathrm{id}$. □

Intertwining capping with module maps

As we shall see throughout the article, the capping maps produced from Spivak data often intertwine the left and right Beck–Chevalley transformations. Our aim now is to give the first expression of this principle in the form of Proposition 7.1.9, the other one being Lemma 7.4.6.

Setting 7.1.5 (Module pushforwards from multiplicative basechanges)**.** Suppose we have:

- symmetric monoidal $\mathcal{B}$–categories $\underline{\mathcal{C}}, \underline{\mathcal{D}}$,
- a symmetric monoidal parametrised colimit–preserving functor $U\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ as well as a $\underline{\mathcal{C}}$–linear functor $F\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ using the $\underline{\mathcal{C}}$–linear structure on $\underline{\mathcal{D}}$ coming from U,
- a map $r\colon \underline{J} \to \underline{K}$ in $\mathrm{Cat}_{\mathcal{B}}$ (to disambiguate notations, we will write $\rho := r$ when we use it in the context of the category $\underline{\mathcal{D}}$),
- $\underline{\mathcal{C}}$ and $\underline{\mathcal{D}}$ admit left Kan extensions along $\underline{J} \to \underline{K}$.

For (ξ, c) a $\underline{\mathcal{C}}$–Spivak datum for r, we define

$$(\zeta,\ d) := \Big(U(\xi),\ U(c)\colon \mathbb{1}_{\underline{\mathcal{D}}} \to U(r_!\xi) \simeq \rho_!\zeta\Big)$$

as the associated $\underline{\mathcal{D}}$–Spivak datum for ρ. From the data above, we also obtain symmetric monoidal functors $U\colon \underline{\mathcal{C}}^{\underline{K}} \to \underline{\mathcal{C}}^{\underline{J}}$ and $U\colon \underline{\mathcal{D}}^{\underline{K}} \to \underline{\mathcal{D}}^{\underline{J}}$, using which we may upgrade the functors $F\colon \underline{\mathcal{C}}^{\underline{K}} \to \underline{\mathcal{D}}^{\underline{K}}, F\colon \underline{\mathcal{C}}^{\underline{J}} \to \underline{\mathcal{D}}^{\underline{J}}$ to a $\underline{\mathcal{C}}^{\underline{K}}$– and a $\underline{\mathcal{C}}^{\underline{J}}$–linear one, respectively. Note that by virtue of $\underline{\mathcal{C}}$–linearity in all its guises as explained in the previous sentence, we have for any $A \in \{\underline{\mathcal{C}}, \underline{\mathcal{C}}^{\underline{J}}, \underline{\mathcal{C}}^{\underline{K}}\}$ a natural map $UA \otimes F(-) \to F(A \otimes -)$ which is an equivalence. Furthermore, note also that we clearly have equivalences $\rho^*F \simeq Fr^*$. Since U was parametrised colimit preserving, we have an equivalence $Ur_! \simeq \rho_!U$.

Example 7.1.6. The following will be the examples of the abstract Setting 7.1.5 that will be important for us:

(a) In the case $F = U$, the $\underline{\mathcal{C}}$–linear structure on $F = U$ will be given by the symmetric monoidality structure $UA \otimes U(-) \xrightarrow{\simeq} U(A \otimes -)$;

(b) In the case when $\underline{\mathcal{D}} = \underline{\mathcal{C}}$, $U = \mathrm{id}_{\underline{\mathcal{C}}}$, and $F = a \otimes -$ for some fixed object $a \in \underline{\mathcal{C}}$, the $\underline{\mathcal{C}}$–linear structure on F is the tautological one given by $\mathrm{id}(A) \otimes a \otimes - \simeq a \otimes A \otimes -$ coming from the symmetric monoidal structure on $\underline{\mathcal{C}}$.

Lemma 7.1.7. *Suppose we are in the Setting 7.1.5. For an object $A \in \underline{\mathcal{C}}^{\underline{J}}$ let us write $B := U(A) \in \underline{\mathcal{D}}^{\underline{J}}$. We then have a commuting diagram*

$$\begin{tikzcd}
F(-) \otimes \rho_! B \arrow[r, "\text{linearity}", "\simeq"'] & F(- \otimes r_! A) \\
 & Fr_!(r^*(-) \otimes A) \arrow[u, "F(\mathrm{BC}_!)"'] \\
\rho_!(\rho^* F(-) \otimes B) \arrow[uu, "\mathrm{BC}_!"] \arrow[r, "\rho_!(\text{linearity})", "\simeq"'] & \rho_! F(r^*(-) \otimes A) \arrow[u, "\mathrm{BC}_!"']
\end{tikzcd}$$

Proof. Let $x \in \underline{\mathcal{C}}^{\underline{K}}$ be an arbitrary object. Consider the diagram

$$\begin{tikzcd}
 & & \underline{\mathcal{D}}^{\underline{K}} \arrow[dd, "\rho^*"] \\
\underline{\mathcal{C}}^{\underline{K}} \arrow[urr, "Fx \otimes U(-)"] \arrow[r, "x \otimes -"'] \arrow[dd, "r^*"'] & \underline{\mathcal{C}}^{\underline{K}} \arrow[ur, "F"'] \arrow[dd, "r^*"'] & \\
 & & \underline{\mathcal{D}}^{\underline{J}} \\
\underline{\mathcal{C}}^{\underline{J}} \arrow[urr, "\rho^* Fx \otimes U(-)"] \arrow[r, "r^* x \otimes -"'] & \underline{\mathcal{C}}^{\underline{J}} \arrow[ur, "F"'] &
\end{tikzcd}$$

where the commuting triangles come from the $\underline{\mathcal{C}}$–linearity of the functor F with the $\underline{\mathcal{C}}$–linear structure on $\underline{\mathcal{D}}$ coming from the symmetric monoidal colimit–preserving functor $U \colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$. By passing to the left adjoints $r_! \dashv r^*$ and $\rho_! \dashv \rho^*$ of the vertical functors and Beck–Chevalley pasting [CSY22, Lem. 2.2.4], we obtain the required commuting diagram. ☐

Observation 7.1.8. A funny consequence of the preceding lemma is that if we supposed that $\underline{\mathcal{C}}$ satisfied the r–projection formula and $\underline{\mathcal{D}}$ the ρ–projection formula so that the left vertical $\mathrm{BC}_!$ map and $F(\mathrm{BC}_!)$ are equivalences, then

$$\mathrm{BC}_! \colon \rho_! F(r^*(-) \otimes A) \to Fr_!(r^*(-) \otimes A)$$

is automatically an equivalence.

Proposition 7.1.9 (Linear intertwining principle). *Suppose we are as in Setting 7.1.5 and that $\underline{\mathcal{C}}$ and $\underline{\mathcal{D}}$ admit right Kan extensions along $\underline{J} \to \underline{K}$. Then we have a commuting square*

$$\begin{tikzcd}
Fr_* \arrow[rr, "F(c\cap -)"] \arrow[d, "\mathrm{BC}_*"'] & & Fr_!(\xi \otimes -) \\
\rho_* F \arrow[r, "d\cap F"'] & \rho_!(\zeta \otimes F-) \arrow[r, "\simeq", "\text{linearity}"'] & \rho_! F(\xi \otimes -) \arrow[u, "\mathrm{BC}_!"']
\end{tikzcd}$$

Proof. Consider the following large commuting diagram

$$\begin{tikzcd}
Fr_*(-) \arrow[r, "F(\mathrm{id}\otimes c)"] \arrow[ddd, "\mathrm{BC}_*"'] & F(r_*(-) \otimes r_!\xi) & Fr_!(r^*r_*(-) \otimes \xi) \arrow[l, "F(\mathrm{BC}_!)"', "\simeq"] \arrow[r, "Fr_!(\epsilon\otimes\mathrm{id})"] & Fr_!(- \otimes \xi) \\
 & \arrow[r, phantom, "\text{(A)}"] & \rho_! F(r^*r_*(-) \otimes \xi) \arrow[u, "\mathrm{BC}_!"] & \\
 & Fr_*(-) \otimes \rho_!\zeta \arrow[uu, "\text{linearity}", "\simeq"'] \arrow[d, "\mathrm{BC}_*\otimes\mathrm{id}"'] & \rho_!(\rho^* Fr_*(-) \otimes \zeta) = \rho_!(Fr^*r_*(-) \otimes \zeta) \arrow[l, "\mathrm{BC}_!"', "\simeq"] \arrow[u, "\text{linearity}", "\simeq"'] \arrow[d, "\rho_!(\rho^*\mathrm{BC}_*\otimes\mathrm{id})"'] \arrow[dr, "\rho_!(F\epsilon\otimes\mathrm{id})"] & \rho_! F(- \otimes \xi) \arrow[uu, "\mathrm{BC}_!"'] \\
\rho_* F(-) \arrow[r, "\mathrm{id}\otimes d"'] & \rho_* F(-) \otimes \rho_!\zeta & \rho_!(\rho^*\rho_* F(-) \otimes \zeta) \arrow[l, "\simeq"', "\mathrm{BC}_!"] \arrow[r, "\rho_!(\epsilon\otimes\mathrm{id})"'] \arrow[ur, phantom, "\text{(B)}"] & \rho_!(F(-) \otimes \zeta) \arrow[u, "\simeq", "\rho_!(\text{linearity})"']
\end{tikzcd}$$

where three of the squares clearly commute, square (A) commutes by Lemma 7.1.7, and triangle (B) commutes since the left triangle in the diagram

$$\begin{tikzcd}
\rho^* Fr_* \simeq Fr^* r_* \arrow[d, "\rho^*\mathrm{BC}_*"'] \arrow[dr, "F\epsilon"] & \\
\rho^*\rho_* F \arrow[r, "\epsilon_F"] & F
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
Fr_* \arrow[d, "\mathrm{BC}_*"'] \arrow[dr, "\mathrm{BC}_*"] & \\
\rho_* F \arrow[r, equal] & \rho_* F
\end{tikzcd}$$

is adjoint to the right one, which clearly commutes. Now we may take the outer square of the large diagram to conclude. ☐

7.2 Twisted ambidexterity and Poincaré duality

Our aim in this subsection is to introduce the notion of Poincaré duality for Spivak data. To this end, it would be beneficial first to isolate a property that we will demand Poincaré Spivak data to satisfy, namely that of *twisted ambidexterity*, i.e. that the associated capping map is an equivalence. This notion gives the equivalence of homology with cohomology necessary for Poincaré duality. While our definition makes sense in more generality – a level of flexibility we will need for some of our applications – we show in Remark 7.2.7 that our notion of twisted ambidexterity nevertheless coincides with the one given in [Cno23] for presentably symmetric monoidal coefficient categories.

For this subsection, we consider $\underline{X} \in \mathcal{B}$ and $\underline{\mathcal{C}}$ a symmetric monoidal $\mathcal{B}$–category which admits $\underline{X}$-shaped limits and colimits and satisfies the $\underline{X}$-projection formula. Notice that these conditions are satisfied whenever $\underline{\mathcal{C}}$ is a presentably symmetric monoidal $\mathcal{B}$–category.

Twisted ambidexterity

Definition 7.2.1. A $\underline{\mathcal{C}}$–Spivak datum (ξ, c) for $\underline{X}$ is *twisted ambidextrous* if the capping transformation $c \cap_\xi (-) \colon X_*(-) \to X_!(\xi \otimes -)$ from Construction 7.1.2 is an equivalence.

There is also the following relative version of this definition. Recall that associated to an object $\underline{Y} \in \mathcal{B}$ there is the basechange adjunction $\pi_Y^* \colon \mathcal{B} \rightleftharpoons \mathcal{B}_{/Y} : (\pi_Y)_*$.

Definition 7.2.2. (Twisted ambidextrous maps) Consider a map $f \colon \underline{X} \to \underline{Y}$ in $\mathcal{B}$ and a symmetric monoidal $\mathcal{B}$–category $\underline{\mathcal{C}}$ such that the $\mathcal{B}_{/Y}$–category $(\pi_Y)^*\underline{\mathcal{C}}$ admits f shaped limits and colimits and satisfies the f-projection formula. A *$\underline{\mathcal{C}}$-Spivak datum* for f is a $(\pi_Y)^*\underline{\mathcal{C}}$-Spivak datum for $f \in \mathcal{B}_{/Y}$. We say that such a Spivak datum exhibits f as a *$\underline{\mathcal{C}}$–twisted ambidextrous map* if it exhibits $f \in \mathcal{B}_{/Y}$ as $(\pi_Y)^*\underline{\mathcal{C}}$–twisted ambidextrous object.

We will see in Proposition 7.3.14 that f being $\underline{\mathcal{C}}$–twisted ambidextrous is closely related to the fibres of f being $\underline{\mathcal{C}}$–twisted ambidextrous, see also [Cno23, Prop. 3.13].

The situation simplifies significantly if $\underline{\mathcal{C}}$ is presentable. Then it is a property of X itself to be twisted ambidextrous and there exists a unique Spivak up to contractible choice. We also demonstrate that, in this case, our notion of twisted ambidexterity is equivalent to the one defined in [Cno23, Def. 3.4]. However, later in §9.1 it will be essential for us to work with nonpresentable coefficients, which is why we work in this generality for most of the article.

Lemma 7.2.3. *Let (ξ, c) be a twisted ambidextrous $\underline{\mathcal{C}}$–Spivak datum for $\underline{X} \in \mathcal{B}$. The adjunction $X^* \dashv X_*$ induces an adjunction $X^* \dashv X_!(\xi \otimes -)$ whose unit is given by*

$$\mathrm{id}(-) \xrightarrow{\mathrm{id}\otimes c} \mathrm{id}(-) \otimes X_!\xi \xleftarrow[\simeq]{\mathrm{BC}_!} X_!(X^*(-) \otimes \xi) = X_!(- \otimes \xi) \circ X^*(-),$$

Proof. It is clear that the the equivalence $X_*(-) \simeq X_!(\xi \otimes -)$ induces an adjunction $X^* \dashv X_!(\xi \otimes -)$. For the description of the adjunction unit, observe that we have the commuting diagram

$$\begin{tikzcd}
(-) \arrow[r, "\mathrm{id}\otimes c"] \arrow[d, "\eta"'] & (-) \otimes X_!\xi \arrow[d, "\eta\otimes\mathrm{id}"'] & X_!(X^*(-) \otimes \xi) \arrow[l, "\mathrm{BC}_!", "\simeq"'] \arrow[d, "X_!(X^*\eta\otimes\mathrm{id})"'] \arrow[dr, equal] & \\
X_*X^*(-) \arrow[r, "\mathrm{id}\otimes c"'] & X_*X^*(-) \otimes X_!\xi & X_!(X^*X_*X^*(-) \otimes \xi) \arrow[l, "\simeq", "\mathrm{BC}_!"'] \arrow[r, "r_!(\epsilon_{X^*}\otimes\mathrm{id})"'] & X_!(X^*(-) \otimes \xi)
\end{tikzcd}$$

where the bottom composite is the capping equivalence and the right triangle is by the triangle identity. This shows that the claimed map is compatible with the unit $\eta \colon \mathrm{id} \to X_*X^*$ under the capping equivalence $c \cap - \colon X_*(-) \xrightarrow{\simeq} X_!(\xi \otimes -)$ as required. $\square$

We also have the following converse construction. Let us begin with the following observation:

Observation 7.2.4. Suppose that $\mathcal{A}$ is a 2-category. Given an adjoint pair of morphisms $l\colon x \rightleftharpoons y : r$ and two maps $f\colon x \to z$, $g\colon y \to z$ in $\mathcal{A}$, the induced map

$$\mathrm{Map}(f, gl) \xrightarrow{-\circ r} \mathrm{Map}(fr, glr) \xrightarrow{\eta} \mathrm{Map}(fr, g) \tag{7.2}$$

is an equivalence.

Lemma 7.2.5. *Let $\underline{\mathcal{C}} \in \mathrm{CAlg}(\mathrm{Pr}^L_{\mathcal{B}})$ and suppose that X^* is an internal left adjoint in $\mathrm{Mod}_{\underline{\mathcal{C}}}(\mathrm{Pr}^L_{\mathcal{B}})$. Then the capping construction enhances for any $\xi \in \underline{\mathcal{C}}^X$ to an equivalence*

$$X_!\xi \simeq \mathrm{map}(\mathbb{1}_{\mathcal{C}}, X_!\xi) \xrightarrow{\simeq} \mathrm{map}(X_*(-), X_!(- \otimes \xi)),$$

where map *denotes the $\underline{\mathcal{C}}$-enriched mapping space. In partiuclar, the capping transformation $c \cap_\xi (-)\colon X_*(-) \to X_!(- \otimes \xi)$ refines to a $\underline{\mathcal{C}}$-linear transformation.*

Proof. This follows from Observation 7.2.4 applied to $\mathcal{A} = \mathrm{Mod}_{\underline{\mathcal{C}}}(\mathrm{Pr}^L_{\mathcal{B}})$, $l = X^*\colon \underline{\mathcal{C}} \to \underline{\mathcal{C}}^X$, $r = X_*$, $f = \mathrm{id}_{\underline{\mathcal{C}}}$ and $g = X_!(- \otimes \xi)$, using the projection formula $X_!(X^*(-) \otimes \xi) \simeq \mathrm{id}_{\underline{\mathcal{C}}} \otimes X_!\xi$. It is easy to check that the composite (7.2) is the capping map. □

If $X^*\colon \underline{\mathcal{C}} \to \underline{\mathcal{C}}^{\underline{X}}$ is an internal left adjoint in $\mathrm{Mod}_{\underline{\mathcal{C}}}(\mathrm{Pr}^L_{\mathcal{B}})$, then its right adjoint must be of the form $X_!(D_{\underline{X}} \otimes -)$ for a unique D_X by Theorem 6.1.37. We obtain the following characterisation of twisted ambidexterity in the presentable case:

Proposition 7.2.6 (The presentable case)**.** *Let $\underline{\mathcal{C}} \in \mathrm{CAlg}(\mathrm{Pr}^L_{\mathcal{B}})$ be a presentably symmetric monoidal $\mathcal{B}$–category and $\underline{X} \in \mathcal{B}$.*

(1) If X^* is an internal left adjoint in $\mathrm{Mod}_{\underline{\mathcal{C}}}(\mathrm{Pr}^L_{\mathcal{B}})$ with right adjoint $X_!(D_{\underline{X}} \otimes -)$, then the unit map $c : \mathbb{1}_{\underline{\mathcal{C}}} \to X_!(X^*\mathbb{1}_{\underline{\mathcal{C}}} \otimes D_{\underline{X}}) = X_!D_{\underline{X}}$ forms a $\underline{\mathcal{C}}$–twisted ambidextrous Spivak datum $(D_{\underline{X}}, c)$ for $\underline{X}$.

(2) If (ξ, c) is a $\underline{\mathcal{C}}$-twisted ambidextrous Spivak datum for $\underline{X}$, then the map

$$(-) \xrightarrow{\mathrm{id}\otimes c} (-) \otimes X_!\xi \simeq X_!(X^*(-) \otimes \xi)$$

is the unit map of a $\underline{\mathcal{C}}$-linear adjunction $X^* \dashv X_!(- \otimes \xi)$.

In particular, if (ξ, c) and (ξ', c') are twisted ambidextrous Spivak data, then there is an equivalence $\xi \simeq \xi'$ so that the composition $\mathbb{1}_{\underline{\mathcal{C}}} \xrightarrow{c} X_!\xi \simeq X_!\xi'$ is equivalent to c'.

Proof. Point (1) is an immediate consequence of Lemma 7.2.5.

Next, for point (2), suppose that (ξ, c) is a $\underline{\mathcal{C}}$-twisted ambidextrous Spivak datum for $\underline{X}$. Then X_* is $\underline{\mathcal{B}}$-colimit preserving. First, we check the condition in [Cno23, Prop. A.5] which guarantees that the adjunction $X^* \dashv X_*$ is $\underline{\mathcal{C}}$–linear. For this, we need to show that for $a \in \underline{\mathcal{C}}$ and $E \in \underline{\mathcal{C}}^{\underline{X}}$, the Beck–Chevalley map

$\mathrm{BC}_*\colon a \otimes X_*E \to X_*(X^*a \otimes E)$ is an equivalence. By the intertwining square in Proposition 7.1.9 applied to Example 7.1.6 (b), we see that BC_* is an equivalence because $\mathrm{BC}_!\colon X_!(X^*a \otimes E \otimes \xi) \to a \otimes X_!(E \otimes \xi)$ is an equivalence by presentably symmetric monoidality of $\underline{\mathcal{C}}$. By Lemma 7.2.5 the capping transformation $c \cap (-)\colon X_*(--) \to X_!(- \otimes \xi)$ refines to a $\underline{\mathcal{C}}$-linear equivalence from which we obtain a $\underline{\mathcal{C}}$-linear adjunction $X^* \dashv X_!(- \otimes \xi)$ The claimed description of the adjunction unit comes from Lemma 7.2.3.

For the final statement, since both $X_!(\xi \otimes -)$ and $X_!(\xi' \otimes -)$ are $\underline{\mathcal{C}}$-linear right adjoints to X^* by (2),we see by (1) that there is an equivalence $\xi \simeq D_X \simeq \xi'$. To see the coincidence of c and c', we use Lemma 7.1.4 to obtain the two commuting triangles in

$$\begin{tikzcd}
X_!\xi' & X_*X^*\mathbb{1} \arrow[l, "c' \cap_{\xi'} X^*\mathbb{1}"', "\simeq"] \arrow[r, "c \cap_{\xi} X^*\mathbb{1}", "\simeq"'] & X_!\xi \\
& \mathbb{1} \arrow[ul, "c'"] \arrow[u, "\eta"] \arrow[ur, "c"'] &
\end{tikzcd}$$

witnessing that $c \simeq c'$ as required. □

Remark 7.2.7. By combining Proposition 7.2.6 and [Cno23, Prop. 3.8], we see that $\underline{X}$ is $\underline{\mathcal{C}}$-twisted ambidextrous in the sense of Definition 7.2.1 if and only if it is so in the sense of [Cno23, Def. 3.4]. If that is the case, the twisted norm map $\widetilde{\mathrm{Nm}}_{\underline{X}}\colon X_!(-- \otimes D_{\underline{X}}) \to X_*(-)$ constructed in [Cno23, Def. 3.3] is an equivalence with inverse the map $\widetilde{\mathrm{Nm}}_{\underline{X}}^{-1}(\mathbb{1}) \cap_{D_{\underline{X}}} (-)$.

Definition 7.2.8. Let $\underline{\mathcal{C}}$ be a presentably symmetric monoidal $\mathcal{B}$–category. An object of $\mathcal{B}$ is called *$\underline{\mathcal{C}}$–twisted ambidextrous* if it admits a (necessarily unique) twisted ambidextrous Spivak datum with coefficients in $\underline{\mathcal{C}}$.

Notation 7.2.9. The twisted ambidextrous Spivak datum of a twisted $\underline{\mathcal{C}}$-ambidextrous object $\underline{X} \in \mathcal{B}$ will be denoted by $(D_{\underline{X}}^{\underline{\mathcal{C}}}, c)$. If $\underline{\mathcal{C}}$ is clear from the context, we will sometimes abbreviate this to $(D_{\underline{X}}, c)$.

Poincaré duality

We now come to the definition of Poincaré duality in the parametrised setting.

Definition 7.2.10. A Spivak datum (ξ, c) for $\underline{X}$ with coefficients in $\underline{\mathcal{C}}$ is *Poincaré* if it is twisted ambidextrous and ξ takes values in $\underline{\mathcal{P}\mathrm{ic}}(\underline{\mathcal{C}})$.

Definition 7.2.11. Let $\underline{\mathcal{C}}$ be a presentably symmetric monoidal $\mathcal{B}$–category. An object $\underline{X} \in \mathcal{B}$ is called *$\underline{\mathcal{C}}$–Poincaré* if it is twisted $\underline{\mathcal{C}}$–ambidextrous and the unique twisted ambidextrous Spivak datum $(D_{\underline{X}}, c)$ from Proposition 7.2.6 is Poincaré.

Remark 7.2.12. In [Qui72], Quinn defines the notion of a *normal space* to be a space together with (the unstable analog of) a Spivak datum (ξ, c), where ξ takes values in $\mathcal{P}\mathrm{ic}(\mathrm{Sp})$. He does not require the Spivak datum to be twisted ambidextrous though.

We again have the following relative version.

Definition 7.2.13. (Poincaré duality maps) Consider a map $f\colon X \to Y$ in $\mathcal{B}$ and a symmetric monoidal $\mathcal{B}$–category $\underline{\mathcal{C}}$ such that the $\mathcal{B}_{/Y}$–category $(\pi_Y)^*\underline{\mathcal{C}}$ admits f-shaped limits and colimits and satisfies the f-projection formula. We say that a $\underline{\mathcal{C}}$-Spivak datum for f exhibits f as a *$\underline{\mathcal{C}}$–Poincaré duality map* if it exhibits $f \in \mathcal{B}_{/Y}$ as a $(\pi_Y)^*\underline{\mathcal{C}}$–Poincaré duality object.

Using Costenoble-Waner duality, one can show the following standard result saying that dualisability of the dualising object implies its invertibility. We will not use it anywhere in the rest of this article but include it for completeness. In the setting of $\mathcal{B}$-categories, Costenoble-Waner duality was introduced in [Cno23, Section 3.3] and we follow the notation used there. In the nonparametrised context, the following result appears in [Lan22, Remark A.9].

Proposition 7.2.14 (Invertibility of dualising objects)**.** *Let $\underline{\mathcal{C}}$ be a presentable symmetric monoidal $\mathcal{B}$–category. Suppose that $\underline{X} \in \mathcal{B}$ is $\underline{\mathcal{C}}$-twisted ambidextrous and that $D_{\underline{X}} \in \underline{\mathcal{C}}^{\underline{X}}$ is dualisable. Then $D_{\underline{X}}$ is invertible, i.e. $\underline{X}$ is Poincaré.*

Proof. By [Cno23, Proposition 3.29], the unit $\mathbb{1}_X \in \mathcal{C}(\underline{X} \times *)$ is left Costenoble-Waner dualisable with left dual $D_{\underline{X}} \in \mathcal{C}(* \times \underline{X})$. By [Cno23, Proposition 3.30], this implies that for $F \in \mathcal{C}(\underline{X} \times Y)$ and $E \in \mathcal{C}(Y)$ we have equivalences

$$\begin{aligned}\mathrm{Map}(E, X_!F) &\simeq \mathrm{Map}(E, F \odot \mathbb{1}_X) \simeq \mathrm{Map}(E \odot D_{\underline{X}}, F)\\ &= \mathrm{Map}(X^*E \otimes D_{\underline{X}}, F) \simeq \mathrm{Map}(X^*E, D_{\underline{X}}^{\vee} \otimes F)\\ &\simeq \mathrm{Map}(E, X_*(D_{\underline{X}}^{\vee} \otimes F)) \simeq \mathrm{Map}(E, X_!(D_{\underline{X}} \otimes D_{\underline{X}}^{\vee} \otimes F)).\end{aligned}$$

giving a $\underline{\mathcal{C}}$–linear equivalence $X_!(-) \simeq X_!(- \otimes D_{\underline{X}} \otimes D_{\underline{X}}^{\vee})$. It now follows from Theorem 6.1.37 that $D_{\underline{X}} \otimes D_{\underline{X}}^{\vee} \simeq \mathbb{1}_X$ so that $D_{\underline{X}}$ is invertible. □

Example 7.2.15. The phenomenon of higher semiadditivity introduced by [HL13] provides many instances of Poincaré duality with trivial dualising sheaf.

(1) For any topos $\mathcal{B}$ and any symmetric monoidal $\mathcal{B}$–category $\underline{\mathcal{C}}$, the terminal object $\underline{*}$ has the tautological Poincaré $\underline{\mathcal{C}}$–Spivak datum $(\mathbb{1}, \mathrm{id}_{\mathbb{1}})$.

(2) If $\underline{\mathcal{C}}$ is pointed, then by [HL13, Rmk. 4.4.6], the map $\underline{\varnothing} \to \underline{X}$ in $\mathcal{B}$ is $\underline{\mathcal{C}}$–Poincaré.

(3) If $\underline{\mathcal{C}}$ is semiadditive, then by [HL13, Prop. 4.4.9], any finite fold map $\nabla\colon \coprod_{i=1}^{n} \underline{X} \to \underline{X}$ is $\underline{\mathcal{C}}$–Poincaré.

(4) More generally, a good supply of Poincaré spaces with trivial dualising sheaf comes from the theory of higher semiadditivity of [HL13; CSY22], as worked out in [Cno23].

Example 7.2.16 (Wall's Poincaré complexes)**.** Next, we recount some parts of the classical story that began from Wall's seminal paper [Wal67]. In this setting, our base topos $\mathcal{B}$ will be the category $\mathcal{S}$ of spaces. Wall defined a Poincaré complex (he used the word complex, because he worked with CW-complexes) to be a compact space X together a Spivak datum $(\xi \in \mathcal{P}\mathrm{ic}(\mathrm{Mod}_{\mathrm{H}\mathbb{Z}})^X, c\colon \mathrm{H}\mathbb{Z} \to X_!\xi)$ such that for each $\psi \in (\mathrm{Mod}_{\mathrm{H}\mathbb{Z}}^{\heartsuit})^X$ the map

$$c \cap_\xi \psi\colon X_*\psi \longrightarrow X_!(\xi \otimes \psi) \tag{7.3}$$

is an equivalence. As X was assumed to be compact, both sides of (7.3) commute with all (co)limits and so this also implies that the same transformation is an equivalence for arbitrary $\psi \in \mathrm{Mod}_{\mathbb{Z}}$. On the other hand, to compute the value of the Sp–dualising sheaf D_X of a space X at a point $x\colon * \to X$, one calculates

$$D_X(x) = x^*D_X \simeq X_!x_!x^*D_X \simeq X_!(D_X \otimes x_!\mathbb{S}) \simeq X_*x_!\mathbb{S}.$$

Note that $x_!$ preserves connective objects while X_* preserves bounded below objects if it is a retract of a space admitting a finite-dimensional cell structure . So we see that if X is compact (i.e. a retract of a space having a finite cell structure), then D_X is pointwise bounded below. This implies that if $D_X \otimes \mathbb{Z} \in \mathcal{P}\mathrm{ic}(\mathrm{Mod}_{\mathbb{Z}})^X$, then D_X is pointwise given by shifts of spheres and in particular, $D_X \in \mathcal{P}\mathrm{ic}(\mathrm{Sp})^X$. In conclusion, by combining the points above, a space X is a Poincaré complex in the sense of Wall if and only if it is compact and Sp–Poincaré in the sense of Definition 7.2.11. See also [Lan22, Prop. A.12] for a proof in the case of finite spaces.

Example 7.2.17 (Weak Poincaré spaces)**.** After Wall, some authors subsequently relaxed the compactness condition in the definition of Poincaré complexes. For example, in group theory it is not unusual to completely drop it. We say that a space X is weakly Poincaré if it admits a Spivak datum $(\xi \in \mathcal{P}\mathrm{ic}(\mathrm{Mod}_{\mathrm{H}\mathbb{Z}})^X, c\colon \mathrm{H}\mathbb{Z} \to X_!\xi)$ such that for each $\psi \in \mathrm{Mod}_{\mathrm{H}\mathbb{Z}}^{\heartsuit}$ the map in (7.3) is an equivalence. As X_* preserves coconnectivity, $X_!$ preserves connectivity, and both preserve fibre sequences, we see that they restrict to functors

$$X_*, X_!(- \otimes \xi)\colon (\mathrm{Mod}_{\mathrm{H}\mathbb{Z}}^b)^X \to \mathrm{Mod}_{\mathrm{H}\mathbb{Z}}^b$$

where $\mathrm{Mod}_{\mathbb{Z}}^b$ denotes the category of bounded $\mathbb{Z}$–chain complexes. Being weakly Poincaré is seen to be equivalent to admitting a Poincaré Spivak datum in the sense of Definition 7.2.10 with respect to the symmetric monoidal stable category $\mathrm{Mod}_{\mathrm{H}\mathbb{Z}}^b$.

Remark 7.2.18. Having now established the three notions and implications

$$\text{Sp–Poincaré and compact} \implies \text{Sp–Poincaré} \implies \text{weakly Poincaré},$$

we cannot give a conclusive answer about their precise relation. In [Bro72], Browder notes that if X is weakly Poincaré with finitely presented fundamental group, then it is even compact, and so by Example 7.2.16, also Sp–Poincaré. On the other hand, Davis shows in [Dav98] that there are weakly Poincaré spaces whose fundamental groups do not admit a finite presentation.

From here on, we will reserve the term *Poincaré space* for what we referred to as Sp-Poincaré spaces above. In particular, we slightly deviate from Wall's definition. It is useful to try and port concepts from manifold theory to the theory of Poincaré spaces. One concept that has a straighforward analog for Poincaré spaces is the dimension of a manifold.

Terminology 7.2.19 (Formal dimensions)**.** Let $X \in \mathcal{S}$ be a Poincaré space. We say that it has *formal dimension d* if for every point $x\colon * \to X$, we have $x^*D_X \simeq \Sigma^{-d}\mathbb{S}$. If for every point $x\colon * \to X$, we have $x^*D_X \simeq \Sigma^{-k}\mathbb{S}$ for some $0 \leq k \leq d$, then we will say that it has *formal dimension at most d*.

Fact 7.2.20. Here are some classical facts about nonequivariant Poincaré spaces that will be relevant to our investigations later.

(1) Let $X \in \mathcal{S}^\omega$ be a connected Poincaré space of formal dimension $d = 0$. Then by [Wal67, Thm. 4.2], we have $X \simeq *$. In fact, in the aforementioned theorem, Wall even provided classifications of Poincaré spaces up to formal dimension 3.

(2) Every connected Poincaré space has formal dimension a nonnegative number. This is since if X has formal dimension d, then taking $\mathbb{F}_2$–homology, we get $H_0(X;\mathbb{F}_2) \cong H^d(X;\mathbb{F}_2)$. Thus if $d < 0$, then $H_0(X;\mathbb{F}_2) = 0$, i.e. X was the empty space.

7.3 Constructions with Spivak data

This subsection constitutes the heart of our parametrised Poincaré duality theory. We begin by studying compositions of Spivak data. Next, we shall study two types of basechange results, namely basechanging coefficient categories (Theorems 7.3.5 and 7.3.8) and basechanging the underlying topos Theorem 7.3.12. These are the main abstract results of this article and they will play a fundamental role in much of our equivariant work in Chapters 8 and 9. We then end this subsection by proving a descent result for Poincaré duality.

Compositions

Construction 7.3.1 (Compositions of Spivak data)**.** Let $f\colon \underline{X} \to \underline{Y}$ and $g\colon \underline{Y} \to \underline{Z}$ be maps in $\mathcal{B}$ equipped with Spivak data

$$\left(\xi_f \in \underline{\mathcal{C}}^{\underline{X}}, \quad \mathbb{1}_{\underline{\mathcal{C}}^{\underline{Y}}} \xrightarrow{c_f} f_!\xi_f\right) \qquad \left(\xi_g \in \underline{\mathcal{C}}^{\underline{Y}}, \quad \mathbb{1}_{\underline{\mathcal{C}}^{\underline{Z}}} \xrightarrow{c_g} g_!\xi_g\right)$$

We may then define the *composition Spivak datum for the map* $gf\colon \underline{X} \to \underline{Z}$ as

$$\xi_{gf} \coloneqq \xi_f \otimes f^*\xi_g \in \underline{\mathcal{C}}^{\underline{X}}$$
$$c_{gf}\colon \mathbb{1}_{\underline{\mathcal{C}}^{\underline{X}}} \xrightarrow{c_g} g_!\xi_g \xrightarrow{g_!(c_f\otimes \mathrm{id})} g_!(f_!\xi_f \otimes \xi_g) \xleftarrow[\simeq]{\mathrm{BC}} (gf)_!(\xi_f \otimes f^*\xi_g)$$

Lemma 7.3.2 (Capping map for compositions)**.** *The capping map for the composition Spivak datum is equivalent to the composition of the constituent capping maps. That is, in the situation of Construction 7.3.1, we have a commuting diagram*

$$\begin{tikzcd}
(gf)_*(-) \arrow[r, "c_{gf}\cap(-)"] \arrow[d, "c_g\cap f_*(-)"'] & (gf)_!(\xi_f \otimes f^*\xi_g) \arrow[d, "\simeq"', "\mathrm{BC}"] \\
g_!(f_*(-) \otimes \xi_g) \arrow[r, "g_!((c_f\cap(-))\otimes \mathrm{id})"'] & g_!(f_!(- \otimes \xi_f) \otimes \xi_g)
\end{tikzcd}$$

Proof. First note that we have the commuting diagram

$$\begin{tikzcd}
g_*f_*(-) \arrow[d, "c_g\otimes \mathrm{id}"] & & \\
g_!\xi_g \otimes g_*f_*(-) \arrow[d, "g_!(c_f\otimes \mathrm{id})"'] & g_!(\xi_g \otimes g^*g_*f_*(-)) \arrow[l, "\mathrm{BC}", "\simeq"'] \arrow[r, "g_!(\mathrm{id}\otimes\epsilon)"] \arrow[d, "g_!(c_f\otimes \mathrm{id})"'] & g_!(\xi_g \otimes f_*(-)) \arrow[d, "g_!(c_f\otimes \mathrm{id})"] \\
g_!(\xi_g \otimes f_!\xi_f) \otimes g_*f_*(-) & g_!(\xi_g \otimes f_!\xi_f \otimes g^*g_*f_*(-)) \arrow[l, "\mathrm{BC}", "\simeq"'] \arrow[r, "g_!(\mathrm{id}\otimes\epsilon)"] & g_!(\xi_g \otimes f_!\xi_f \otimes f_*(-)) \\
 & g_!f_!(f^*\xi_g \otimes \xi_f \otimes f^*g^*g_*f_*(-)) \arrow[u, "g_!(\mathrm{BC})", "\simeq"'] \arrow[r, "g_!f_!(\mathrm{id}\otimes\epsilon)"] \arrow[dr, "g_!f_!(\mathrm{id}\otimes\epsilon)"'] & g_!f_!(f^*\xi_g \otimes \xi_f \otimes f^*f_*(-)) \arrow[u, "\simeq", "g_!(\mathrm{BC})"'] \arrow[d, "g_!f_!(\mathrm{id}\otimes\epsilon)"] \\
 & & g_!f_!(f^*\xi_g \otimes \xi_f \otimes -).
\end{tikzcd}$$

The required commuting square is then obtained by taking the outer diagram. □

Proposition 7.3.3 (Duality composition formula)**.** *Let $f\colon \underline{X} \to \underline{Y}$ and $g\colon \underline{Y} \to \underline{Z}$ be maps in $\mathcal{B}$ and $\underline{\mathcal{C}}$ be a symmetric monoidal $\mathcal{B}$–category satisfying the f– and g–projection formulas. Suppose f and g are equipped with Spivak data (ξ_f, c_f) and (ξ_g, c_g) respectively. Then under the composition Spivak datum on $gf\colon \underline{X} \to \underline{Z}$ from Construction 7.3.1 with dualising sheaf*

$$\xi_{gf} \coloneqq \xi_f \otimes f^*\xi_g,$$

we have that:

(1) *if f and g are twisted ambidextrous, then so is gf,*

(2) *if f, g, gf are all twisted ambidextrous and g is furthermore Poincaré duality, then f is Poincaré duality if and onlf if gf is.*

Proof. By Lemma 7.3.2, we have the commuting square

$$\begin{tikzcd}
(gf)_*(-) \arrow[rr, "c_{gf}\cap(-)"] \arrow[d, "c_g\cap f_*(-)"'] & & (gf)_!(\xi_f \otimes f^*\xi_g) \arrow[d, "\simeq"', "g_!\mathrm{BC}_!"] \\
g_!(f_*(-) \otimes \xi_g) \arrow[rr, "g_!((c_f\cap(-))\otimes\mathrm{id})"'] & & g_!(f_!(-\otimes \xi_f) \otimes \xi_g).
\end{tikzcd}$$

Hence, if the left vertical and bottom horizontal maps are equivalences, then so is the top horizontal map. It is also clear that from the formula $\xi_{gf} = \xi_f \otimes f^*\xi_g$ that if two ξ_g is invertible (and so also $f^*\xi_g$), then ξ_{gf} is invertible if and only if ξ_f is. $\square$

Change of coefficients

Construction 7.3.4 (Basechanging Spivak data)**.** Let $\underline{\mathcal{C}}, \underline{\mathcal{D}}$ be $\mathcal{B}$–categories admitting $\underline{X}$-shaped (co)limits and satisfying the $\underline{X}$-projection formula. Suppose $F\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is a symmetric monoidal functor of $\mathcal{B}$–categories which preserves $\underline{X}$–shaped colimits. We define a new $\underline{\mathcal{D}}$–Spivak datum for $\underline{X}$ as follows

$$F(\xi, c) := \big(F\xi\colon \underline{X} \xrightarrow{\xi} \underline{\mathcal{C}} \xrightarrow{F} \underline{\mathcal{D}},\ Fc\colon \mathbb{1}_{\underline{\mathcal{D}}} \simeq F(\mathbb{1}_{\underline{\mathcal{C}}}) \xrightarrow{Fc} F(X_!\xi) \simeq X_!(F\xi)\big).$$

Theorem 7.3.5 (Poincaré basechange - presentable version)**.** *Let $F\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ be a functor of presentably symmetric monoidal $\mathcal{B}$–categories. Suppose that (ξ, c) is a twisted ambidextrous Spivak datum with coefficients in $\underline{\mathcal{C}}$ for the object $\underline{X} \in \mathcal{B}$. Then $F(\xi, c)$ is a twisted ambidextrous Spivak datum with coefficients in $\underline{\mathcal{D}}$ for $\underline{X}$. In particular, if $\underline{X}$ is $\underline{\mathcal{C}}$–Poincaré, then $\underline{X}$ is also $\underline{\mathcal{D}}$–Poincaré.*

Proof. Recall from Proposition 6.1.38 that $-\otimes_{\underline{\mathcal{C}}} \underline{\mathcal{D}}\colon \underline{\mathrm{Mod}}_{\underline{\mathcal{C}}}(\underline{\mathrm{Pr}}^L_{\mathcal{B}}) \to \underline{\mathrm{Mod}}_{\underline{\mathcal{D}}}(\underline{\mathrm{Pr}}^L_{\mathcal{B}})$ is symmetric monoidal $\mathcal{B}$-colimit preserving. Using that $\underline{\mathcal{C}}^{\underline{X}} \in \mathrm{Mod}_{\underline{\mathcal{C}}}(\mathrm{Pr}^L(\mathcal{B}))$ is self dual (see [Cno23, Corollary 2.27]), one sees that the coassembly map $(\lim_{\underline{X}} \underline{\mathcal{C}}) \otimes_{\underline{\mathcal{C}}} \underline{\mathcal{D}} \to (\lim_{\underline{X}} \underline{\mathcal{C}}) \otimes_{\underline{\mathcal{C}}} \underline{\mathcal{D}}$ is an equivalence (even a symmetric monoidal one). The commutative diagram

$$\begin{tikzcd}
\underline{\mathcal{C}}^{\underline{X}} \arrow[r, "-\otimes_{\underline{\mathcal{C}}}\xi"] \arrow[d, "F"] & \underline{\mathcal{C}}^{\underline{X}} \otimes \underline{\mathcal{C}}^{\underline{X}} \arrow[r, "-\otimes_{\underline{\mathcal{C}}}-"] \arrow[d, "F"] & \underline{\mathcal{C}}^{\underline{X}} \arrow[d, "F"] \\
\underline{\mathcal{D}}^{\underline{X}} \arrow[r, "-\otimes_{\underline{\mathcal{D}}}F\xi"'] & \underline{\mathcal{D}}^{\underline{X}} \otimes \underline{\mathcal{D}}^{\underline{X}} \arrow[r, "-\otimes_{\underline{\mathcal{C}}}-"'] & \underline{\mathcal{D}}^{\underline{X}}
\end{tikzcd}$$

together with the equivalence $\mathrm{Fun}_{\underline{\mathcal{D}}}(\underline{\mathcal{D}}^{\underline{X}}, \underline{\mathcal{D}}^{\underline{X}}) \simeq \mathrm{Fun}_{\underline{\mathcal{C}}}(\underline{\mathcal{C}}^{\underline{X}}, \underline{\mathcal{D}}^{\underline{X}})$ then gives us an equivalence $(- \otimes \xi) \otimes_{\underline{\mathcal{C}}} \underline{\mathcal{D}} \simeq (- \otimes F\xi)$.

By standard arguments, the functor $- \otimes_{\underline{\mathcal{C}}} \underline{\mathcal{D}} \colon \mathrm{Mod}_{\underline{\mathcal{C}}}(\mathrm{Pr}^L_{\mathcal{B}}) \to \mathrm{Mod}_{\underline{\mathcal{D}}}(\mathrm{Pr}^L_{\mathcal{B}})$ preserves internal adjunctions. Hence, we see that $(X_{\underline{\mathcal{C}}})_! \otimes_{\underline{\mathcal{C}}} \underline{\mathcal{D}} \colon \underline{\mathcal{C}}^{\underline{X}} \otimes_{\underline{\mathcal{C}}} \underline{\mathcal{D}} \simeq \underline{\mathcal{D}}^{\underline{X}} \to \underline{\mathcal{C}} \otimes_{\underline{\mathcal{C}}} \underline{\mathcal{D}} \simeq \underline{\mathcal{D}}$ is an internal left adjoint of $(X_{\underline{\mathcal{D}}})^* = (X_{\underline{\mathcal{C}}})^* \otimes_{\underline{\mathcal{C}}} \underline{\mathcal{D}}$ from which we obtain an equivalence $(X_{\underline{\mathcal{C}}})_! \otimes_{\underline{\mathcal{C}}} \underline{\mathcal{D}} \simeq (X_{\underline{\mathcal{D}}})_!$. Together with the first part, the internal right adjoint $(X_{\underline{\mathcal{C}}})_!(\xi \otimes -)$ to $(X_{\underline{\mathcal{C}}})^*$ basechanges to an internal right adjoint

$$(X_{\underline{\mathcal{C}}})_!(\xi \otimes -) \otimes_{\underline{\mathcal{C}}} \underline{\mathcal{D}} \simeq (X_{\underline{\mathcal{D}}})_!(F\xi \otimes -) \colon \underline{\mathcal{D}}^{\underline{X}} \longrightarrow \underline{\mathcal{D}}$$

of $(X_{\underline{\mathcal{D}}})^*$. Because the internal adjunction $(X_{\underline{\mathcal{D}}})^* \dashv (X_{\underline{\mathcal{D}}})_!(F\xi \otimes -)$ on $\underline{\mathcal{D}}$ is basechanged from the internal adjunction $X^*_{\underline{\mathcal{C}}} \dashv (X_{\underline{\mathcal{C}}})_!(\xi \otimes -)$ on $\underline{\mathcal{C}}$, we see that the $\underline{\mathcal{D}}$–fundamental class, which is the unit of the former internal adjunction, is given by the composite

$$\mathbb{1}_{\underline{D}} \simeq F(\mathbb{1}_{\underline{\mathcal{C}}}) \xrightarrow{Fc} F(X_{\underline{\mathcal{C}}})_!\xi \simeq (X_{\underline{\mathcal{D}}})_! F\xi.$$

The final statement about Poincaré duality is clear since F is symmetric monoidal and so preserves invertibility. □

Corollary 7.3.6. *Let $\Phi \colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ be a symmetric monoidal functor of presentably symmetric monoidal $\mathcal{B}$–categories and let $\underline{X} \in \mathcal{B}$ be a $\underline{\mathcal{C}}$–twisted ambidextrous space. Then the Beck–Chevalley transformation $\mathrm{BC}_* \colon \Phi X_*(-) \to X_*\Phi(-)$ is an equivalence.*

Proof. To disambiguate notations, we will denote by $X_!$ and X_* for the $\underline{X}$–colimit and limit for the category $\underline{\mathcal{D}}$. Now, by Proposition 7.1.9 applied to the case of Example 7.1.6 (1), we obtain a commuting square

$$\begin{tikzcd} \Phi X_*(-) \arrow[rr, "\Phi(c\cap_{D_X} -)", "\simeq"'] \arrow[d, "\mathrm{BC}_*"] & & \Phi X_!(D_X \otimes -) \\ X_*\Phi(-) \arrow[r, "\simeq", "\Phi c \cap_{\Phi D_X} \Phi -"'] & X_!\Phi(D_X \otimes -) \simeq X_!(\Phi D_X \otimes \Phi(-)) \arrow[ur, "\mathrm{BC}_!"', "\simeq"] & \end{tikzcd}$$

where the top map is an equivalence by $\underline{\mathcal{C}}$–twisted ambidexterity, the bottom an equivalence by Theorem 7.3.5, and the right vertical is an equivalence since Φ preserves parametrised colimits by hypothesis. Thus the left vertical map is an equivalence too, as desired. □

In a limited sense, it is possible to exploit that an object $\underline{X} \in \mathcal{B}$ admits a Poincaré Spivak datum with coefficients in $\underline{\mathcal{C}}$ to get a Spivak datum with coefficients in $\underline{\mathcal{D}}$ with interesting properties. To this end, it would be convenient to establish the following terminology:

Terminology 7.3.7. Let $\underline{X} \in \mathcal{B}$, $\underline{\mathcal{D}} \in \mathrm{CMon}(\mathrm{Cat}_{\mathcal{B}})$ satisfying the $\underline{X}$–projection formula, and $\Phi\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ a functor of $\mathcal{B}$–categories. Suppose we have a $\underline{\mathcal{D}}$–Spivak datum (ζ, d) for $\underline{X}$. We say that the Spivak datum (ζ, d) is:

(a) Φ–*twisted ambidextrous* if the capping transformation

$$X_*\Phi(-) \xrightarrow{d\cap_\zeta \Phi-} X_!(\zeta \otimes \Phi(-))$$

of functors $\underline{\mathcal{C}}^{\underline{X}} \to \underline{\mathcal{D}}$ is an equivalence,

(b) Φ–*Poincaré duality* if it is Φ–twisted ambidextrous and ζ takes values in invertible objects.

Theorem 7.3.8 (Poincaré basechange - general version)**.** *Suppose $\Phi\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is a symmetric monoidal functor of $\mathcal{B}$–categories such that*

- *the $\mathcal{B}$–categories $\underline{\mathcal{C}}, \underline{\mathcal{D}}$ admit $\underline{X}$-shaped (co)limits and satisfy the $\underline{X}$-projection formula;*
- *the $\mathcal{B}$-functor Φ preserves $\underline{X}$-shaped limits and colimits.*

If (ξ, c) is a twisted ambidextrous $\underline{\mathcal{C}}$–Spivak datum for $\underline{X}$, then $\Phi(\xi, c)$ is a twisted ambidextrous Φ–Spivak datum for $\underline{X}$. In particular, if (ξ, c) is a Poincaré duality $\underline{\mathcal{C}}$–Spivak datum for $\underline{X}$, then $\Phi(\xi, c)$ is a Poincaré duality Φ–Spivak datum for $\underline{X}$.

Proof. By Proposition 7.1.9, we have a commuting square

$$\begin{tikzcd}
\Phi X_*(-) \arrow[rr, "\Phi(c\cap_\xi -)", "\simeq"'] \arrow[d, "\simeq"', "\mathrm{BC}_*"] & & \Phi X_!(\xi \otimes -) \\
X_*\Phi(-) \arrow[r, "\Phi c \cap_{\Phi\xi} \Phi-"'] & X_!\Phi(\xi \otimes -) \simeq X_!(\Phi\xi \otimes \Phi(-)) \arrow[ur, "\mathrm{BC}_!", "\simeq"'] &
\end{tikzcd}$$

from which the desired result is immediate. The statement about Poincaré duality comes immediately from the fact that Φ is symmetric monoidal and so preserves invertible objects. $\square$

We will exploit this later to reprove an injectivity result of Bredon and Browder as Theorem 9.1.1.

Change of base topoi

Notation 7.3.9. To state the next construction and result, it will be convenient to adopt the following notation: let $f^*\colon \mathcal{B} \rightleftharpoons \mathcal{B}' : f_*$ be a geometric morphism of topoi. If $\underline{\mathcal{C}} \xrightarrow{F} \underline{\mathcal{D}} \xrightarrow{G} \underline{\mathcal{E}}$ are functors of $\mathcal{B}'$–categories, we write $f_*[G \circ F]$ to mean the composite $f_*\underline{\mathcal{C}} \xrightarrow{f_*F} f_*\underline{\mathcal{D}} \xrightarrow{f_*G} f_*\underline{\mathcal{E}}$ and similarly for f^*. Furthermore, since both

f^* and f_* are product–preserving functors and so enhance to symmetric monoidal functors, we see that for an object $A \in \underline{\mathcal{D}}$, writing $f_*A \in f_*\underline{\mathcal{D}}$ under the equivalence $\mathrm{Map}_{\mathrm{Cat}_{\mathcal{B}}}(\mathrm{const}_{\mathcal{B}} *, f_*\underline{\mathcal{D}}) \simeq \mathrm{Map}_{\mathrm{Cat}_{\mathcal{B}'}}(\mathrm{const}_{\mathcal{B}'} *, \underline{\mathcal{D}})$, the map $- \otimes A \colon \underline{\mathcal{D}} \to \underline{\mathcal{D}}$ is sent to $- \otimes f_*A \colon f_*\underline{\mathcal{D}} \to f_*\underline{\mathcal{D}}$, and similarly in the case when we apply f^*.

Construction 7.3.10 (Pushing Spivak data along geometric morphisms)**.** Let $f^* \colon \mathcal{B} \rightleftharpoons \mathcal{B}' : f_*$ be a geometric morphism of topoi and consider $\underline{X} \in \mathcal{B}$ and $\underline{\mathcal{C}}$ a symmetric monoidal $\mathcal{B}'$–category which admits $f^*\underline{X}$–indexed colimits. By Lemma 6.1.18, we know that $f_*\underline{\mathcal{C}}$ admits $\underline{X}$–colimits.

Suppose we are given a $\underline{\mathcal{C}}$-Spivak datum (ξ, c) for $f^*\underline{X}$ and a $f_*\underline{\mathcal{C}}$–Spivak datum (ζ, d) for $\underline{X}$. Using the symmetric monoidal identification from Lemma 6.1.16, we obtain a $f_*\underline{\mathcal{C}}$–Spivak datum $f_*(\xi, c)$ for $\underline{X}$ and a $\underline{\mathcal{C}}$–Spivak datum $f^*(\zeta, d)$ for $f^*\underline{X}$. Observe in particular that, by construction, we have $f^*f_*(\xi, c) \simeq (\xi, c)$ and $f_*f^*(\zeta, d) \simeq (\zeta, d)$.

Here, for instance, $f_*\xi$ corresponds to ξ under the equivalence $f_*\underline{\mathrm{Fun}}(f^*\underline{X}, \underline{\mathcal{C}}) \simeq \underline{\mathrm{Fun}}(\underline{X}, f_*\underline{\mathcal{C}})$ and $f_*c \colon \mathbb{1}_{f_*\underline{\mathcal{C}}} \to X_! f_*\xi$ corresponds to c under the identification of adjunctions in (6.4). Explicitly, these new Spivak data are given by

$$f_*(\xi, c) \coloneqq \left(f_*\xi \colon \underline{X} \xrightarrow{\eta} f_*f^*\underline{X} \xrightarrow{f_*\xi} f_*\underline{\mathcal{C}},\ f_*c \colon \mathbb{1}_{f_*\underline{\mathcal{C}}} = f_*[\mathbb{1}_{\underline{\mathcal{C}}}] \to X_! f_*\xi = f_*[(f^*X)_!\xi] \right),$$

$$f^*(\eta, d) \coloneqq \left(f^*\underline{X} \xrightarrow{f^*\zeta} f^*f_*\underline{\mathcal{C}} \xrightarrow{\epsilon} \underline{\mathcal{C}},\ f^*d \colon \mathbb{1}_{\underline{\mathcal{C}}} = f^*[\mathbb{1}_{f_*\underline{\mathcal{C}}}] \to (f^*X)_! f^*\zeta = f^*[X_!\zeta] \right).$$

Construction 7.3.11 (Pushing Spivak data along étale morphisms)**.** Let $f^* \colon \mathcal{B} \rightleftharpoons \mathcal{B}' : f_*$ be an étale morphism of topoi, $\underline{\mathcal{C}} \in \mathrm{CMon}(\mathrm{Cat}_{\mathcal{B}})$, and $\underline{X} \in \mathcal{B}$. Suppose (ξ, c) is a $\underline{\mathcal{C}}$–Spivak datum for $\underline{X}$. Then we can construct a $f^*\underline{\mathcal{C}}$–Spivak datum $f^*(\xi, c)$ for $f^*\underline{X}$ given by

$$\left(f^*\xi \colon f^*\underline{X} \xrightarrow{f^*\xi} f^*\underline{\mathcal{C}},\ f^*c \colon \mathbb{1}_{f^*\underline{\mathcal{C}}} \simeq f^*[\mathbb{1}_{\underline{\mathcal{C}}}] \xrightarrow{f^*[c]} (f^*X)_! f^*\xi \simeq f^*[X_!\xi] \right)$$

where in the last equivalence, we have used $f^*[X_!] \simeq (f^*X)_!$ from Lemma 6.1.19.

For part (e) of the next result, see Terminology 7.3.7.

Theorem 7.3.12 (Omnibus geometric basechange of Spivak data)**.** *Let $f^* \colon \mathcal{B} \rightleftharpoons \mathcal{B}' : f_*$ be a geometric morphism of topoi, $\underline{X} \in \mathcal{B}$, $\underline{\mathcal{D}} \in \mathrm{CMon}(\mathrm{Cat}_{\mathcal{B}'})$ satisfying the $f^*\underline{X}$–projection formula, and $\underline{\mathcal{E}} \in \mathrm{CMon}(\mathrm{Cat}_{\mathcal{B}})$ satisfying the $\underline{X}$–projection formula. Let (ξ, c) be a $\underline{\mathcal{D}}$–Spivak datum for $f^*\underline{X}$ and (ζ, d) a $f_*\underline{\mathcal{D}}$–Spivak datum for $\underline{X}$. Then:*

(a) There is a commuting square of capping maps

$$\begin{tikzcd} X_*(-) \arrow[r, "f_*c \cap_{f_*\xi} -"] \arrow[d, "\simeq"'] & X_!(f_*\xi \otimes -) \arrow[d, "\simeq"] \\ f_*[(f^*X)_*(-)] \arrow[r, "f_*[c \cap_\xi -]"] & f_*[(f^*X)_!(\xi \otimes -)] \end{tikzcd}$$

of functors $\underline{\mathrm{Fun}}(\underline{X}, f_*\underline{\mathcal{D}}) \simeq f_*\underline{\mathrm{Fun}}(f^*\underline{X}, \underline{\mathcal{D}}) \to f_*\underline{\mathcal{D}}$

(b) *Suppose that* $f_*\colon \mathrm{Cat}_{\mathcal{B}'} \to \mathrm{Cat}_{\mathcal{B}}$ *is fully faithful (resp. that* $f^*\colon \mathcal{B} \rightleftharpoons \mathcal{B}' : f_*$ *is étale). Then there is a commuting square*

$$\begin{tikzcd} f^*\left[X_*(-)\right] \arrow[r, "f^*[d\cap_\zeta -]"] \arrow[d, "\simeq"'] & f^*\left[X_!(\zeta\otimes -)\right] \arrow[d, "\simeq"] \\ (f^*X)_*(-) \arrow[r, "f^*d\cap_{f^*\zeta}-"] & (f^*X)_!(f^*\zeta\otimes -) \end{tikzcd}$$

of functors $\underline{\mathrm{Fun}}(f^*\underline{X}, \underline{\mathcal{D}}) \simeq f^*\underline{\mathrm{Fun}}(\underline{X}, f_*\underline{\mathcal{D}}) \to \underline{\mathcal{D}}$ *(resp.* $\underline{\mathrm{Fun}}(f^*\underline{X}, f^*\underline{\mathcal{E}}) \to f^*\underline{\mathcal{E}}$*),*

(c) *If* (ξ, c) *is a twisted ambidextrous (resp. Poincaré)* $\underline{\mathcal{D}}$*–Spivak datum for* $f^*\underline{X}$*, then* $f_*(\xi, c)$ *is a twisted ambidextrous (resp. Poincaré)* $f_*\underline{\mathcal{D}}$*–Spivak datum for* $\underline{X}$*.*

(d) *Suppose either that* f_* *is fully faithful or that* $\underline{\mathcal{D}}$ *is presentably symmetric monoidal. If* (ζ, d) *is a twisted ambidextrous (resp. Poincaré)* $f_*\underline{\mathcal{D}}$*–Spivak datum for* $\underline{X}$*, then* $f^*(\zeta, d)$ *is a twisted ambidextrous (resp. Poincaré)* $\underline{\mathcal{D}}$*–Spivak datum for* $f^*\underline{X}$*.*

(e) *More generally: suppose* f_* *is fully faithful. Let* $\underline{\mathcal{C}} \in \mathrm{Cat}_{\mathcal{B}}$ *and* $\Phi\colon \underline{\mathcal{C}} \to f_*\underline{\mathcal{D}}$ *be a functor between* $\mathcal{B}$*–categories. Then* (ζ, d) *is* Φ*–twisted ambidextrous (resp. –Poincaré) for* $\underline{X}$ *if and only if* $f^*(\zeta, d)$ *is* $(f^*\Phi\colon f^*\mathcal{C} \to \underline{\mathcal{D}})$*–twisted ambidextrous (resp. –Poincaré) for* $f^*\underline{X}$*,*

(f) *Suppose the geometric morphism* $f^*\colon \mathcal{B} \rightleftharpoons \mathcal{B}' : f_*$ *is étale. If* (ζ, d) *is a twisted ambidextrous (resp. Poincaré)* $\underline{\mathcal{E}}$*–Spivak datum for* $\underline{X}$*, then* $f^*(\zeta, d)$ *is a twisted ambidextrous (resp. Poincaré)* $f^*\underline{\mathcal{E}}$*–Spivak datum for* $f^*\underline{X}$*.*

Proof. First note by Proposition 6.1.24 (1, 3) that $f_*\underline{\mathcal{D}}$ satisfies the $\underline{X}$–projection formula and $f^*\underline{\mathcal{E}}$ satisfies the $f^*\underline{X}$–projection formula, and so the squares in (a) and (b) make sense. Now to prove part (a), we have the commutative diagram

$$\begin{tikzcd} X_*(-) \arrow[r, "\simeq"] \arrow[d, "\mathrm{id}\otimes f_*c"'] & f_*\left[(f^*X)_*(-)\right] \arrow[d, "f_*[\mathrm{id}\otimes c]"] \\ X_*(-)\otimes X_! f_*\xi \arrow[r, "\simeq"] & f_*\left[(f^*X)_*(-)\otimes (f^*X)_!\xi\right] \\ X_!(X^*X_*(-)\otimes f_*\xi) \arrow[r, "\simeq"] \arrow[u, "\mathrm{BC}_!"', "\simeq"] \arrow[d, "\epsilon_*"'] & f_*\left[(f^*X)_!((f^*X)^*(f^*X)_*(-)\otimes \xi)\right] \arrow[u, "\simeq"', "f_*\mathrm{BC}_!"] \arrow[d, "f_*\epsilon_*"] \\ X_!(-\otimes f_*\xi) \arrow[r, "\simeq"] & f_*\left[(f^*X)_!(-\otimes \xi)\right] \end{tikzcd}$$

The horizontal arrows come from the identification in Lemma 6.1.18; the top square commutes by symmetric monoidality of the identification; the middle and bottom squares commute as the respective adjunction (co-)units are identified by (6.4). The required square is now obtained by extracting the outer square of the diagram above.

The proof for (b) in the case that f_* is fully faithful is done similarly as for (a), but using now the commuting squares of adjunctions obtained by applying f^* to Lemma 6.1.18 (f^* preserves adjunctions by [MW24, Cor. 3.1.9]) and that $f^*f_* \simeq \mathrm{id}$. The case of étale morphisms is also done similarly, using instead the squares of adjunctions from Lemma 6.1.19.

Next, we prove part (c). If (ξ, c) is twisted ambidextrous, then by Lemma 6.1.20, the bottom map in the square from (a) is an equivalence, and so the top map is an equivalence too, i.e. $f_*(\xi, c)$ is twisted ambidextrous. The statements about being Poincaré is a straightforward consequence of the twisted ambidexterity statements we just proved and the characterisation of factoring through invertible objects in Corollary 6.1.23 (1).

For the proof of (d), suppose now that f_* is fully faithful and that (ζ, d) is twisted ambidextrous. Then since $f^*f_* \simeq \mathrm{id}$, the top map in the square from (b) is an equivalence, and so the bottom map is an equivalence too, i.e. $f^*(\zeta, d)$ is twisted ambidextrous. Poincaré duality is then handled similarly as in (c).

Next, assume that $\underline{\mathcal{D}}$ is presentably symmetric monoidal. We show that the capping transformation $f^*d \cap_{f^*\zeta} (-)\colon \underline{\mathrm{Fun}}(f^*\underline{X}, \underline{\mathcal{D}}) \to \underline{\mathcal{D}}^{\Delta^1}$ is a natural equivalence in unparametrised categories when evaluated at every $W \in \mathcal{B}'$. Firstly, note that the $\underline{\mathcal{D}}$–Spivak datum for $f^*\underline{X}$ at level W is obtained via the symmetric monoidal biadjoint $\mathcal{B}'$–functor $W^*\colon \underline{\mathcal{D}} \to \underline{\mathcal{D}}^{\underline{W}}$, and so the transformation evaluated at $W \in \mathcal{B}'$ is given by applying global sections $\Gamma_{\mathcal{B}'}$ to

$$W^*(f^*d) \cap_{W^* \circ f^*\zeta} (-)\colon \underline{\mathrm{Fun}}(f^*\underline{X}, \underline{\mathcal{D}}^{\underline{W}}) \longrightarrow (\underline{\mathcal{D}}^{\underline{W}})^{\Delta^1}. \tag{7.4}$$

Next, applying Theorem 7.3.5 along $f_*[W^*]\colon f_*\underline{\mathcal{D}} \to f_*(\underline{\mathcal{D}}^{\underline{W}})$ shows that the $f_*(\underline{\mathcal{D}}^{\underline{W}})$–Spivak datum $f_*[W^*](\zeta, d)$ is twisted ambidextrous, i.e.

$$f_*[W^*](d) \cap_{f_*[W^*]\circ\zeta} (-)\colon \underline{\mathrm{Fun}}(\underline{X}, f_*(\underline{\mathcal{D}}^{\underline{W}})) \to (f_*(\underline{\mathcal{D}}^{\underline{W}}))^{\Delta^1}$$

is a natural equivalence. But then by the square in part (a), this capping transformation is equivalent to $f_*[W^*(f^*d) \cap_{W^*\circ f^*\zeta} (-)]\colon f_*\underline{\mathrm{Fun}}(f^*\underline{X}, \underline{\mathcal{D}}^{\underline{W}}) \to f_*((\underline{\mathcal{D}}^{\underline{W}})^{\Delta^1})$, where we have also used that $f_*f^*(\zeta, d) \simeq (\zeta, d)$ from Construction 7.3.10. Thus, by using that $\Gamma_{\mathcal{B}} f_* \simeq \Gamma_{\mathcal{B}'}$ from Example 6.1.9, we may apply $\Gamma_{\mathcal{B}}$ to $f_*[W^*(f^*d) \cap_{W^*\circ f^*\zeta} (-)]$ to get that applying $\Gamma_{\mathcal{B}'}$ to (7.4) yields a natural equivalence, as desired. And as usual, Poincaré duality is handled by Corollary 6.1.23 (1).

Now, the proof of parts (c, d) clearly goes through straightforwardly to yield a proof of (e). Finally, the proof of (f) in the twisted ambidexterity case is done similarly as in the proof of (c) using the square from (b), and the Poincaré duality case is handled by Corollary 6.1.23 (2). $\square$

Descent

For the next result, we briefly recall the notion of effective epimorphisms in a topos $\mathcal{B}$. Given a morphism $f\colon \underline{X} \to \underline{Y}$ in $\mathcal{B}$, its *Čech nerve* is the simplicial object

$$\check{C}(f)\colon \Delta^{\mathrm{op}} \to \mathcal{B}, \quad [n] \mapsto \underline{X} \times_{\underline{Y}} \underline{X} \times_{\underline{Y}} \cdots \times_{\underline{Y}} \underline{X} \quad (n+1 \text{ factors}).$$

Now f is called an *effective epimorphism* if the canonical map $\operatorname{colim}_{\Delta^{\mathrm{op}}} \check{C}(f) \to Y$ is an equivalence.

Example 7.3.13. A map in the topos $\mathcal{S}$ of spaces is an effective epimorphism if and only if it is a π_0-surjection (see e.g. [Lur09, Corollary 7.2.1.15]). Applying this criterion pointwise, one sees that a map $f\colon X \to Y$ in a presheaf topos $\mathrm{Psh}(T)$ over some category T is an effective epimorphism if and only if for all $t \in T$ the map $f(t)\colon X(t) \to Y(t)$ is a π_0-surjection. For example, a map of G–spaces $f\colon \underline{X} \to \underline{Y}$ is an effective epimorphism if and only if for each closed subgroup $H \leq G$ the map $f^H\colon X^H \to Y^H$ is a surjection on path components.

Proposition 7.3.14 (Poincaré duality and descent)**.** *Let $\underline{\mathcal{C}}$ be a presentably symmetric monoidal $\mathcal{B}$–category. Consider a pullback square*

$$\begin{tikzcd} P \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, "g"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

in $\mathcal{B}$. If f is $\underline{\mathcal{C}}$-twisted ambidextrous, then f' is $\underline{\mathcal{C}}$-twisted ambidextrous. Furthermore, there is an equivalence $(g')^*D_f \simeq D_{f'}$ where $D_f \in \mathcal{C}(X)$ denotes the dualising object. In particular, if f is a $\underline{\mathcal{C}}$–Poincaré duality map, then f' is a $\underline{\mathcal{C}}$–Poincaré duality map.

The converse to both statements is true if g is an effective epimorphism.

Proof. Basechange along g defines an étale morphism of topoi $g^*\colon \mathcal{B}_{/Y} \rightleftharpoons \mathcal{B}_{/Z} \colon g_*$ where g^* is given by pullback along g. Now suppose that f is $\underline{\mathcal{C}}$-twisted ambidextrous. This means that $f \in \mathcal{B}_{/Y}$ is $(\pi_Y)^*\underline{\mathcal{C}}$-twisted ambidextrous. Applying Theorem 7.3.12 shows that $g^*f = f' \in \mathcal{B}_{/Z}$ is $g^*\pi_Y^*\underline{\mathcal{C}} = \pi_Z^*\underline{\mathcal{C}}$-twisted ambidextrous with Spivak datum $(g^*\xi, g^*c)$.

If f is a $\underline{\mathcal{C}}$–Poincaré duality map, then D_f is invertible. By symmetric monoidality of the restriction map $(g')^*\colon \mathcal{C}(X) \to \mathcal{C}(P)$ we obtain that $D_{f'} = (g')^*D_f$ is invertible.

Now suppose that g is an effective epimorphism and that f' is $\underline{\mathcal{C}}$-twisted ambidextrous. It is shown in [Cno23, Proposition 3.13 (5)] that f is $\underline{\mathcal{C}}$-twisted ambidextrous. As $g'\colon P \to X$ is an effective epimorphism and the map $\Delta^{\mathrm{op}}_{\mathrm{inj}} \to \Delta^{\mathrm{op}}$ is colimit cofinal, the map $\operatorname{colim}_{\Delta^{\mathrm{op}}_{\mathrm{inj}}} \check{C}(g') \to X$ is an equivalence from which we obtain the symmetric monoidal equivalence $\mathcal{C}(X) \xrightarrow{\simeq} \lim_{\Delta_{\mathrm{inj}}} \mathcal{C}(\check{C}(g'))$. Next, suppose furthermore that f' is a Poincaré duality map. As invertibility in limits can be checked pointwise, we have to show that each restriction of D_f to $\mathcal{C}(\check{C}_n(g'))$ is invertible. Note that a restriction map $\mathcal{C}(X) \to \mathcal{C}(\check{C}_n(g'))$ factors into the symmetric monoidal restriction maps $\mathcal{C}(X) \to \mathcal{C}(P) \to \mathcal{C}(\check{C}_n(g'))$ and the first part shows the restriction $D_{f'} = (g')^* D_f$ to P is invertible. $\square$

Corollary 7.3.15 (Finite products)**.** *Let $\underline{X}, \underline{Y} \in \mathcal{B}$ and $\underline{\mathcal{C}}$ be a presentably symmetric monoidal $\mathcal{B}$–category. If $\underline{X}$ and $\underline{Y}$ are $\underline{\mathcal{C}}$–Poincaré, then $\underline{X} \times \underline{Y}$ is $\underline{\mathcal{C}}$–Poincaré and there is an equivalence*

$$D_{X\times Y} \simeq \mathrm{pr}_X^* D_X \otimes \mathrm{pr}_Y^* D_Y$$

where $\mathrm{pr}_X \colon \underline{X} \times \underline{Y} \to \underline{X}$ and $\mathrm{pr}_Y \colon \underline{X} \times \underline{Y} \to \underline{Y}$ denote the projections.

Proof. As $\underline{X}$ is $\underline{\mathcal{C}}$–Poincaré, it follows from Proposition 7.3.14 that the projection $\mathrm{pr}_Y \colon \underline{X} \times \underline{Y} \to \underline{Y}$ is a $\underline{\mathcal{C}}$–Poincaré map. Since $\underline{Y}$ is $\underline{\mathcal{C}}$–Poincaré, Proposition 7.3.3 implies that the composite $\underline{X} \times \underline{Y} \to \underline{Y} \to *$ is a $\underline{\mathcal{C}}$–Poincaré map showing that $\underline{X} \times \underline{Y}$ is $\underline{\mathcal{C}}$–Poincaré. Propositions 7.3.3 and 7.3.14 then give the identifications $D_{X\times Y} \simeq \mathrm{pr}_Y^* D_Y \otimes D_{\mathrm{pr}_Y} \simeq \mathrm{pr}_Y^* D_Y \otimes \mathrm{pr}_X^* D_X$. $\square$

Lemma 7.3.16. *Let $\underline{\mathcal{C}} \in \mathrm{CAlg}(\mathrm{Pr}^L_{\mathcal{B}})$ be a symmetric monoidal $\mathcal{B}$–category and $(f_i \colon \underline{X}_i \to \underline{Y}_i)_{i\in I}$ be a collection of maps in $\mathcal{B}$. Then the map $f = \coprod_i f_i \colon \coprod_i \underline{X}_i \to \coprod_i \underline{Y}_i$ is $\underline{\mathcal{C}}$-twisted ambidextrous (or $\underline{\mathcal{C}}$–Poincaré duality) if and only if for all $i \in I$ the map f_i is $\underline{\mathcal{C}}$-twisted ambidextrous (or $\underline{\mathcal{C}}$–Poincaré duality). If this is so, then under the identification $\mathcal{C}(\coprod_i X_i) \simeq \prod_i \mathcal{C}(X_i)$, we have an equivalence $D_f \simeq (D_{f_i})_i$.*

Proof. The "only if"-direction follows from Proposition 7.3.14. The "if"-direction in the twisted ambidexterity case is [Cno23, Proposition 3.13(3)]. If in addition all f_i are Poincaré duality maps, then $\coprod_i f_i$ is a Poincaré duality map as $D_f = (D_{f_i})_i$ under the equivalence $\mathcal{C}(\coprod_i \underline{X}_i) = \prod_i \mathcal{C}(\underline{X}_i)$. $\square$

Corollary 7.3.17. *Let $\underline{\mathcal{C}} \in \mathrm{CAlg}(\mathrm{Pr}^L_{\mathcal{B}})$ be a symmetric monoidal $\mathcal{B}$–category which is semiadditive and $\{\underline{X}_i\}_i$ a finite collection of objects in $\mathcal{B}$. Then $\coprod_i \underline{X}_i$ is $\underline{\mathcal{C}}$–Poincaré duality if and only if each $\underline{X}_i$ is $\underline{\mathcal{C}}$–Poincaré duality. In this case, under the identification $\mathcal{C}(\coprod_i X_i) \simeq \prod_i \mathcal{C}(X_i)$, we have an equivalence $D_{\coprod_i \underline{X}_i} \simeq (D_{\underline{X}_i})_i$.*

Proof. Suppose $\coprod_i \underline{X}_i$ is $\underline{\mathcal{C}}$–Poincaré duality. By Example 7.2.15 (1) and Lemma 7.3.16, we see that the inclusion $\underline{X}_j \hookrightarrow \coprod_i \underline{X}_i$ is Poincaré duality for each j. An immediate application of Proposition 7.3.3 using the triple of maps $\underline{X}_j \hookrightarrow \coprod_i \underline{X}_i, \coprod_i \underline{X}_i \to \underline{*}$, and $\underline{X}_j \to \underline{*}$ then shows that $\underline{X}_j$ is also Poincaré duality. Next, suppose each $\underline{X}_j$ is Poincaré duality. By semiadditivity and Example 7.2.15 (2), the map $\nabla \colon \coprod_i \underline{*} \to \underline{*}$ is $\underline{\mathcal{C}}$–Poincaré duality with dualising sheaf $(\mathbb{1}_{\underline{\mathcal{C}}})_i \in \prod_i \mathcal{C}(*)$. Thus, a simple combination of Proposition 7.3.3 and Lemma 7.3.16 using the triple of maps $\sqcup r_i \colon \coprod_i \underline{X}_i \to \coprod_i \underline{*}$, ∇, and $\nabla \circ (\sqcup_i r_i)$ yields the desired conclusion. $\square$

7.4 Degree theory

In this subsection we introduce the notion of the degree of a map between Poincaré spaces (or more generally objects with Spivak data). We use this to construct Umkehr squares which will important for our geometric applications. In §8.6 we will specialise this to the case of G–spaces for a finite group G which generalises classical constructions of the equivariant degree.

As a motivation for the definition, recall that given a map $f\colon X \to Y$ between closed connected manifolds of the same dimension, one can assign to it a degree if f is compatible with the orientation behaviour of X and Y: given an identification $\mathcal{O}_X \simeq f^*\mathcal{O}_Y$ of orientation local systems, the degree is given by the image of $[X]$ under $f_*\colon H_n(X;\mathcal{O}_X) \to H_n(Y;\mathcal{O}_Y) \simeq \mathbb{Z}$. In our setting, we will replace orientation local systems and the fundamental classes $[X]$ above with the dualising sheaves and fundamental classes from Definition 7.1.1.

The definition

Construction 7.4.1 ((Co)homological functoriality)**.** Consider a map $f\colon \underline{X} \to \underline{Y}$ in $\mathcal{B}$ and a $\mathcal{B}$–category $\underline{\mathcal{C}}$ which admits $\underline{X}$- and $\underline{Y}$-shaped limits and colimits. We obtain transformations

$$\mathrm{BC}^f_!\colon X_! f^* \longrightarrow Y_! \qquad\qquad \mathrm{BC}^f_*\colon Y_* \longrightarrow X_* f^*$$

of functors $\underline{\mathcal{C}}^{\underline{Y}} \to \underline{\mathcal{C}}$ coming from the left and right Beck–Chevalley transformations, respectively, associated to the commuting triangle

$$\begin{tikzcd} \underline{\mathcal{C}}^{\underline{X}} & \underline{\mathcal{C}}^{\underline{Y}} \arrow[l, "f^*"'] \\ \underline{\mathcal{C}} \arrow[u, "X^*"] \arrow[ur, "Y^*"'] & \end{tikzcd}$$

We call $\mathrm{BC}^f_!$ the *homological functoriality map* and BC^f_* the *cohomological functoriality map*.

Definition 7.4.2 (Degree of a map)**.** Consider a map $f\colon \underline{X} \to \underline{Y}$ in $\mathcal{B}$ and a symmetric $\mathcal{B}$–category $\underline{\mathcal{C}}$ which admits $\underline{X}$- and $\underline{Y}$-shaped limits and colimits and satisfies the $\underline{X}$- and $\underline{Y}$-projection formula. Suppose we are given Spivak data $(\xi_{\underline{X}}, c_{\underline{X}})$ for $\underline{X}$ and $(\xi_{\underline{Y}}, c_{\underline{Y}})$ for $\underline{Y}$. A *$\underline{\mathcal{C}}$-degree datum* for f consists of an equivalence $\alpha\colon \xi_{\underline{X}} \xrightarrow{\simeq} f^*\xi_{\underline{Y}}$ in $\mathrm{Fun}_{\mathcal{B}}(\underline{X}, \underline{\mathcal{C}})$. We define the *$\underline{\mathcal{C}}$-degree* of (f, α) as the point $\deg_{\underline{\mathcal{C}}}(f,\alpha) \in \mathrm{Map}(\mathbb{1}_{\underline{\mathcal{C}}}, Y_!\xi_{\underline{Y}})$ given by the composite

$$\mathbb{1}_{\underline{\mathcal{C}}} \xrightarrow{c_X} X_!\xi_{\underline{X}} \xrightarrow{\alpha} X_! f^*\xi_{\underline{Y}} \xrightarrow{\mathrm{BC}^f_!} Y_!\xi_{\underline{Y}}.$$

We say that an equivalence $c_{\underline{Y}} \simeq \deg_{\underline{\mathcal{C}}}(f,\alpha)$ exhibits f as a map of *$\underline{\mathcal{C}}$-degree one*.

Remark 7.4.3. Note that an equivalence $c_{\underline{Y}} \simeq \deg_{\underline{\mathcal{C}}}(f,\alpha)$ is the same datum as a homotopy rendering the following diagram commutative

$$\begin{tikzcd} \mathbb{1}_{\underline{\mathcal{C}}} \arrow[r, "c_{\underline{X}}"] \arrow[rrd, bend right, "c_{\underline{Y}}"'] & X_!\xi_{\underline{X}} \arrow[r, "X_!\alpha", "\simeq"'] & X_! f^*\xi_{\underline{Y}} \arrow[d, "\mathrm{BC}^f_!"] \\ & & Y_!\xi_{\underline{Y}}. \end{tikzcd}$$

Construction 7.4.4. If the Spivak datum $(\xi_{\underline{Y}}, c_{\underline{Y}})$ is $\underline{\mathcal{C}}$-twisted ambidextrous, then the equivalence $c_{\underline{Y}} \cap_{\xi_{\underline{Y}}} \mathbb{1}_{\underline{\mathcal{C}}} \colon Y_*Y^*\mathbb{1}_{\underline{\mathcal{C}}} \simeq Y_!\xi_{\underline{Y}}$ endows $Y_!\xi_{\underline{Y}}$ with the structure of a commutative algebra in $\underline{\mathcal{C}}$. This gives $\mathrm{Map}(\mathbb{1}_{\underline{\mathcal{C}}}, Y_!\xi_{\underline{Y}})$ the structure of a commutative monoid in $\mathcal{S}$ with unit $c_{\underline{Y}}$. This explains the name "degree one" in the previous definition.

Example 7.4.5. Here are some well-known sources of degree data in the case $\mathcal{B} = \mathcal{S}$ with respect to a presentably symmetric monoidal coefficients $\mathcal{C}$. Let $f\colon X \to Y$ be a map of connected Poincaré spaces of the same formal dimension d (c.f. Terminology 7.2.19). We consider situations when a degree datum exists for the map f with the Poincaré Spivak data (D_X, c_X) and (D_Y, c_Y) for X resp. Y.

(1) For $\mathcal{C} = \mathrm{Mod}_{\mathbb{F}_2}$, a degree datum exists uniquely since $\mathcal{P}\mathrm{ic}(\mathrm{Mod}_{\mathbb{F}_2}) \simeq \mathbb{Z} \times B\mathrm{Aut}(\mathbb{F}_2) \simeq \mathbb{Z} \times *$ has contractible components.

(2) For $\mathcal{C} = \mathrm{Mod}_{\mathbb{Z}}$, writing $w_1(-)$ for the first Stiefel–Whitney class of a space, a degree datum exists if and only if $f^*w_1(Y) = w_1(X) \in H^1(X;\mathbb{F}_2)$. On homotopy groups, the composite $X_!D_X \simeq X_!f^*D_Y \to Y_!D_Y$ then identifies with
$$f_*\colon H_{d+*}(X;\mathcal{O}_X) \to H_{d+*}(Y;\mathcal{O}_Y)$$
where $\mathcal{O}_X$ and $\mathcal{O}_Y$ denote the orientation local systems. The degree defined above is then given by $f_*[X] \in H_d(Y;\mathcal{O}_Y) \simeq \pi_0(\mathrm{Map}(\mathbb{1}_{\mathrm{Mod}_{\mathbb{Z}}}, Y_!D_Y))$ for $[X]$ the fundamental class of X, and agrees with the classical definition of the degree.

(3) In surgery theory, one is often provided with a *normal map*, i.e. a commuting diagram
$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] \arrow[ddr, bend right, "D_X"'] & & Y \arrow[dl] \arrow[ddl, bend left, "D_Y"] \\ & BO \times \mathbb{Z} \arrow[d, "J"] & \\ & \mathcal{P}\mathrm{ic}(\mathrm{Sp}) & \end{tikzcd}$$
and of course restricting to the outer commutative triangle gives rise to a degree datum.

Homological Umkehr squares

Classically, given a map $f\colon M \to N$ of degree one between closed oriented manifolds, one can construct a "homological Umkehr map" $f^!\colon H_*(N) \to H_*(M)$ going the "wrong way" using Poincaré duality. The following result is a generalisation of this.

Lemma 7.4.6 (Umkehr square)**.** *Consider a map $f\colon \underline{X} \to \underline{Y}$ in $\mathcal{B}$ and a symmetric monoidal $\mathcal{B}$–category $\underline{\mathcal{C}}$ which admits $\underline{X}$- and $\underline{Y}$-shaped limits and colimits and satisfies the $\underline{X}$- and $\underline{Y}$-projection formula. Suppose that there is a degree datum α for f which is of degree one. Then the diagram*

$$\begin{tikzcd}
Y_*(-) \arrow[rr, "\mathrm{BC}^f_*"] \arrow[d, "c_{\underline{Y}} \cap_{\xi_{\underline{Y}}} (-)"'] & & X_* f^*(-) \arrow[d, "c_{\underline{X}} \cap_{\xi_{\underline{X}}} f^*(-)"] \\
Y_!(\xi_{\underline{Y}} \otimes (-)) & X_!(f^*\xi_{\underline{Y}} \otimes f^*(-)) \arrow[l, "\mathrm{BC}^f_!"] & X_!(\xi_{\underline{X}} \otimes f^*(-)) \arrow[l, "\alpha"]
\end{tikzcd}$$

commutes.

Proof. Consider the diagram

$$\begin{tikzcd}
 & Y_*(-) \arrow[r, "\mathrm{BC}^f_*"] \arrow[dl, blue, "c_{\underline{Y}} \otimes -"'] \arrow[d, "\alpha c_{\underline{Y}} \otimes -"] & X_* f^*(-) \arrow[d, red, "\alpha c_{\underline{Y}} \otimes -"] \\
Y_! \xi_{\underline{Y}} \otimes Y_*(-) & X_! f^*\xi_{\underline{Y}} \otimes Y_*(-) \arrow[l, "\mathrm{BC}^f_! \otimes \mathrm{id}"] \arrow[r, "\mathrm{id} \otimes \mathrm{BC}^f_*"] & X_! f^*\xi_{\underline{Y}} \otimes X_* f^*(-) \\
Y_!(\xi_{\underline{Y}} \otimes Y^* Y_*(-)) \arrow[u, blue, "\mathrm{PF}^{\underline{Y}}_!", "\simeq"'] \arrow[d, blue, "\epsilon_{\underline{Y}}"] & X_!(f^*\xi_{\underline{Y}} \otimes X^* Y_*(-)) \arrow[u, "\mathrm{PF}^{\underline{X}}_!", "\simeq"'] \arrow[l, "\mathrm{BC}^f_!"] \arrow[r, "\mathrm{BC}^f_*"] \arrow[d, "\epsilon_{\underline{Y}}"] & X_!(f^*\xi_{\underline{Y}} \otimes X^* X_* f^*(-)) \arrow[u, red, "\mathrm{PF}^{\underline{X}}_!", "\simeq"'] \arrow[dl, red, "\epsilon_{\underline{X}}"] \\
Y_!(\xi_{\underline{Y}} \otimes (-)) & X_!(f^*\xi_{\underline{Y}} \otimes f^*(-)) \arrow[l, "\mathrm{BC}^f_!"] &
\end{tikzcd}$$

The degree one datum makes the top left triangle commute. The bottom right triangle commutes using the definition of the restriction map and the triangle identities. The top right and bottom left squares commute by naturality of BC. The middle two squares commute by naturality of By definition, the composite of the blue arrows is given by $c_{\underline{Y}} \cap_{\xi_{\underline{Y}}} (-)$ and the composite of the red arrows is given by $\alpha c_{\underline{X}} \cap_{f^*\xi_{\underline{Y}}} (-)$. To finish, observe that the diagram

$$\begin{tikzcd}
X_*(-) \arrow[d, "\alpha c_{\underline{X}} \cap_{f^*\xi_{\underline{Y}}} (-)"'] \arrow[dr, "c_{\underline{X}} \cap_{\xi_{\underline{X}}} (-)"] & \\
X_!(f^*\xi_{\underline{Y}} \otimes (-)) & X_!(\xi_{\underline{X}} \otimes (-)) \arrow[l, "\alpha"]
\end{tikzcd}$$

commutes. □

Basechange

Construction 7.4.7. Suppose that $F\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is a symmetric monoidal colimit preserving functor of presentably symmetric monoidal $\mathcal{B}$–categories. Consider a map $f\colon \underline{X} \to \underline{Y}$ in $\mathcal{B}$ and consider a $\underline{\mathcal{C}}$-degree datum α for f. We obtain a $\underline{\mathcal{D}}$-degree datum $F(\alpha)\colon F\xi_{\underline{X}} \xrightarrow{\simeq} Ff^*\xi_{\underline{Y}} \simeq f^*F\xi_Y$ for f and the Spivak data $F(\xi_{\underline{X}}, c_{\underline{X}})$ and $F(\xi_{\underline{Y}}, c_{\underline{Y}})$ from Construction 7.3.4.

Lemma 7.4.8. *In the situation of Construction 7.4.7, the image of* $\deg_{\underline{\mathcal{C}}}(f,\alpha)$ *under the map*

$$\mathrm{Map}(\mathbb{1}_{\underline{\mathcal{C}}}, Y_!\xi_{\underline{Y}}) \xrightarrow{F} \mathrm{Map}(\mathbb{1}_{\underline{\mathcal{D}}}, Y_!F\xi_{\underline{Y}}) \tag{7.5}$$

is equivalent to $\deg_{\underline{\mathcal{D}}}(f, F(\alpha))$*. Furthermore, if* $(\xi_{\underline{Y}}, c_{\underline{Y}})$ *is* $\underline{\mathcal{C}}$*-twisted ambidextrous, then the map (7.5) refines to a map of commutative monoids for the commutative monoid structures from Construction 7.4.4 and we have a commutative diagram*

$$\begin{tikzcd}
\mathrm{Map}(\mathbb{1}_{\underline{\mathcal{C}}}, Y_!\xi_{\underline{Y}}) \arrow[r, "F"] & \mathrm{Map}(\mathbb{1}_{\underline{\mathcal{D}}}, Y_!F\xi_{\underline{Y}}) \\
\mathrm{Map}(\mathbb{1}_{\underline{\mathcal{C}}}, Y_*Y^*\mathbb{1}_{\underline{\mathcal{C}}}) \arrow[r, "F"] \arrow[u, "\simeq"', "c_Y"] & \mathrm{Map}(\mathbb{1}_{\underline{\mathcal{D}}}, Y_*Y^*\mathbb{1}_{\underline{\mathcal{D}}}). \arrow[u, "\simeq"', "F(c_Y)"]
\end{tikzcd}$$

Proof. Note that as F is colimit preserving, it commutes with the homological functoriality constructed in Construction 7.4.1. Thus it sends

$$\deg_{\underline{\mathcal{C}}}(f,\alpha)\colon \mathbb{1}_{\underline{\mathcal{C}}} \xrightarrow{c_{\underline{X}}} X_!\xi_{\underline{X}} \underset{\simeq}{\xrightarrow{\alpha}} X_!f^*\xi_{\underline{Y}} \xrightarrow{\mathrm{BC}_!} Y_!\xi_{\underline{Y}}$$

$$\text{to} \qquad \deg_{\underline{\mathcal{D}}}(f,F(\alpha))\colon \mathbb{1}_{\underline{\mathcal{D}}} \xrightarrow{F(c_{\underline{X}})} X_!F\xi_{\underline{X}} \underset{\simeq}{\xrightarrow{\alpha}} X_!f^*F\xi_{\underline{Y}} \xrightarrow{\mathrm{BC}_!} Y_!F\xi_{\underline{Y}}$$

as desired. Next, suppose that $(\xi_{\underline{Y}}, c_{\underline{Y}})$ is $\underline{\mathcal{C}}$-twisted ambidextrous. By Proposition 7.1.9 applied to Example 7.1.6 (1), F sends the equivalence $Y_*Y^*\mathbb{1}_{\underline{\mathcal{C}}} \simeq Y_!\xi_{\underline{Y}}$ induced by $(\xi_{\underline{Y}}, c_{\underline{Y}})$ to the equivalence $Y_*Y^*\mathbb{1}_{\underline{\mathcal{D}}} \simeq Y_!F(\xi_{\underline{Y}})$ induced by $F(\xi_{\underline{Y}}, c_{\underline{Y}})$. Thus $Y_!F\xi_{\underline{Y}} \simeq FY_!\xi_{\underline{Y}}$ as commutative algebras in $\underline{\mathcal{D}}$. □

Construction 7.4.9. Let $f\colon \underline{X} \to \underline{Y}$ be a map in $\mathcal{B}$ and consider a symmetric monoidal $\mathcal{B}$–category $\underline{\mathcal{C}}$ which admits $\underline{X}$-shaped limits and colimits and satisfies the $\underline{X}$-projection formula. Furthermore, assume that the map $f^*\colon \underline{\mathrm{Fun}}(\underline{Y}, \underline{\mathcal{C}}) \to \underline{\mathrm{Fun}}(\underline{X}, \underline{\mathcal{C}})$ is an equivalence. It canonically refines to a symmetric monoidal equivalence. Hence, by Construction 7.4.1, we obtain canonical equivalences $\mathrm{BC}_!^f\colon X_!f^* \xrightarrow{\simeq} Y_!$ and $\mathrm{BC}_*^f\colon Y_* \xrightarrow{\simeq} X_*f^*$. Thus, for any Spivak datum $(\xi_{\underline{X}}, c_{\underline{X}})$ for $\underline{X}$ we obtain the Spivak datum

$$\left(\mathbb{1}_{\underline{\mathcal{C}}} \xrightarrow{c_{\underline{X}}} X_!\xi_{\underline{X}} \simeq Y_!(f^*)^{-1}\xi_{\underline{X}}\right)$$

for $\underline{Y}$. It is twisted ambidextrous (or Poincaré) if and only if $(\xi_{\underline{X}}, c_{\underline{X}})$ is. Furthermore, note that with respect to these Spivak data, the map f is clearly of degree one.

Lemma 7.4.10. *Consider a geometric morphism $f^*\colon \mathcal{B} \rightleftharpoons \mathcal{B}' : f_*$ of topoi and $\underline{\mathcal{C}}$ be a symmetric monoidal $\mathcal{B}'$-category. Suppose that we are given a map $g\colon \underline{X} \to \underline{Y}$ in $\underline{\mathcal{B}}$ together with $\underline{\mathcal{C}}$-Spivak data for $f^*\underline{X}$ and $f^*\underline{Y}$. Then a $\mathcal{C}$-degree (one) datum for $f^*g\colon f^*\underline{X} \to f^*\underline{Y}$ is equivalent to a $f_*\mathcal{C}$-degree (one) datum for g, where we endow $\underline{X}$ and $\underline{Y}$ with the $f_*\underline{\mathcal{C}}$-Spivak data from Construction 7.3.10.*

Proof. The equivalence of degree data follows from Lemma 6.1.16. The statement about degree one data being equivalent follows from Lemma 6.1.18. $\square$

Chapter 8

Equivariant Poincaré duality: elements

In this section we will apply the abstract theory of parametrised Poincaré duality developed in Chapter 7 to the topos $\mathcal{S}_G$ of G–spaces for a compact Lie group G and use this as our definition of equivariant Poincaré duality spaces. We begin in §8.1 by explaining the definition in this special case in more detail, and then come to one of the key components of the theory in §8.2, namely fixed points methods. After that, in §8.3 we study how Poincaré duality interacts with various kinds of equivariant and homotopical operations such as restrictions, inflations, (co)inductions, fibre sequences, and quotients, and we then give natural examples of G–Poincaré spaces in §8.4. Lastly, we will round off this section by explaining some geometrically meaningful ramifications of the theory of fundamental classes in §§8.5and 8.6.

The reader who is not too familiar with the abstract categorical language can in most situations safely replace G by a finite group and a presentably symmetric monoidal G–category $\underline{\mathcal{C}}$ by the G–category $\underline{\mathrm{Sp}}$ of genuine G–spectra or even $\underline{\mathrm{Mod}}_{\underline{A}(G)}(\underline{\mathrm{Sp}}) \simeq D(\underline{\mathrm{Mack}}_G(\mathrm{Ab}))$ (c.f. [GS14, §5.2] or [PSW22, §5] for this equivalence), the derived category of G-Mackey functors with values in abelian groups. As a sanity check for constructions not involving a change of groups, it might also be helpful to first read the statements for $G = 1$.

8.1 Setting the stage

We specialise the definitions in §§7.1and 7.2 from the abstract parametrised setting to the equivariant situation for a compact Lie group G. After giving the formal definitions, we will provide more intuition for them by unraveling what these notions mean in Remark 8.1.5.

Definition 8.1.1. Let $\underline{X} \in \mathcal{S}_G$ and $\underline{\mathcal{C}}$ a symmetric monoidal G–category admitting $\underline{X}$–shaped colimits. A $\underline{\mathcal{C}}$*–Spivak datum* for $\underline{X}$ is a pair (ξ, c) where $\xi \in \mathrm{Fun}_G(\underline{X}, \underline{\mathcal{C}})$

is called the dualising sheaf and $c\colon \mathbb{1}_{\underline{\mathcal{C}}} \to X_!\xi$ is a morphism in $\underline{\mathcal{C}}$ called the fundamental class.

Now let $\underline{\mathcal{C}}$ be a symmetric monoidal G–category and $\underline{X} \in \mathcal{S}_G$ a G–space. Suppose that $\underline{\mathcal{C}}$ admits $\underline{X}$–shaped limits and colimits and satisfies the $\underline{X}$–projection formula (c.f. Terminology 6.1.13). For example, if either: (a) the G–category $\underline{\mathcal{C}}$ were presentably symmetric monoidal, i.e. an object in $\mathrm{CAlg}(\mathrm{Pr}^L_G)$, or (b) if G were a finite group, $\underline{X} \in \mathcal{S}_G$ were compact, and $\underline{\mathcal{C}}$ were a small G–stably symmetric monoidal category, i.e. an object in $\mathrm{CAlg}(\mathrm{Cat}_G^{G-\mathrm{st}})$, then these conditions are satisfied. Under these conditions, given a $\underline{\mathcal{C}}$–Spivak datum, Construction 7.1.2 provides a a morphism in $\underline{\mathrm{Fun}}(\underline{\mathcal{C}}^{\underline{X}}, \underline{\mathcal{C}})$

$$c \cap_\xi - \colon X_*(-) \longrightarrow X_!(- \otimes \xi) \tag{8.1}$$

called the capping transformation. We refer the reader to the preamble of Chapter 7 for the motivation for these notations.

Definition 8.1.2. A $\underline{\mathcal{C}}$–Spivak datum (ξ, c) for $\underline{X}$ is *twisted ambidextrous* if the capping map (8.1) is an equivalence. It is *Poincaré* if additionally, ξ takes values in the subcategory $\underline{\mathcal{P}\mathrm{ic}}(\underline{\mathcal{C}})$.

If we take a presentably symmetric monoidal G–category $\underline{\mathcal{C}} \in \mathrm{CAlg}(\mathrm{Pr}^L_G)$ as coefficient system, then the situation again simplifies a little, for example because a twisted ambidextrous Spivak datum is unique, if it exists, by Proposition 7.2.6.

Definition 8.1.3. If $\underline{X}$ is a G–space and $\underline{\mathcal{C}}$ is a presentably symmetric monoidal G–category, we say that $\underline{X}$ is *$\underline{\mathcal{C}}$–twisted ambidextrous* if it admits a twisted ambidextrous $\underline{\mathcal{C}}$–Spivak datum $(D_{\underline{X}}, c)$. Furthermore, $\underline{X}$ is *$\underline{\mathcal{C}}$–Poincaré* if additionally $D_{\underline{X}}$ takes values in $\underline{\mathcal{P}\mathrm{ic}}(\underline{\mathcal{C}})$.

Terminology 8.1.4. In the special case where $\underline{\mathcal{C}} = \underline{\mathrm{Sp}}$, we just say that $\underline{X}$ is G–twisted ambidextrous or G–Poincaré.

Understanding the case of $\underline{\mathrm{Sp}}$–Poincaré duality is our main motivation for this article. Because of its importance, we give a few explanations about this particular space.

Remark 8.1.5 (Unraveling the definition of $\underline{\mathrm{Sp}}$-Poincaré duality)**.** To set up a good formalism, we needed to work in a generality that runs the risk of seeming overly abstract. We stress that the task of checking if a space $\underline{X}$ is G–Poincaré closely resembles classical Poincaré duality.

First, one has to find the correct analog of a local system with respect to which $\underline{X}$ is supposed to satisfy Poincaré duality. We required a $\xi \in \mathrm{Fun}_G(\underline{X}, \underline{\mathrm{Sp}})$ that lands in $\underline{\mathcal{P}\mathrm{ic}}(\underline{\mathrm{Sp}})$, which unravels to providing for each closed subgroup $H \leq G$ a local system of invertible H–spectra $\xi^H\colon X^H \to \mathcal{P}\mathrm{ic}(\mathrm{Sp}_H)$ together with compatibilities that amount to providing for each map $G/K \to G/H$ a homotopy in the diagram

$$\begin{tikzcd} X^K \arrow[r, "\xi^K"] \arrow[d, "\mathrm{Res}^K_H"'] & \mathcal{P}\mathrm{ic}(\mathrm{Sp}_K) \arrow[d, "\mathrm{Res}^K_H"] \\ X^H \arrow[r, "\xi^H"] & \mathcal{P}\mathrm{ic}(\mathrm{Sp}_H) \end{tikzcd}$$

plus higher coherences between these homotopies. Additionally, we required a fundamental class $c\colon \mathbb{S}_G \to X_!(\xi)$ which the reader should think of as an equivariant homology class of $\underline{X}$ with coefficients in the local system ξ. The capping map $c \cap_\xi -\colon X_*(-) \to X_!(- \otimes \xi)$ should then be thought of as the cap product with the homology class c. Recall from the preamble to Chapter 7 that classical Poincaré duality is the statement that capping with a certain "fundamental class" induces an isomorphism between cohomology and homology, and what we ask here is exactly the same condition.

In the presentable setting, we are in the pleasant situation where we can identify a large class of twisted ambidextrous objects.

Proposition 8.1.6. *Every compact G–space $\underline{X}$ is $\underline{\mathrm{Sp}}$–twisted ambidextrous. Consequently, every compact G–space $\underline{X}$ is $\underline{\mathcal{C}}$–twisted ambidextrous for any G-stable presentably symmetric monoidal G–category.*

Proof. The first part is an immediate consequence of [Cno23, Thm. 4.8 (5)] and Remark 7.2.7, and the second part is by Theorem 7.3.5. □

Next, as may be expected of a well–behaved equivariant notion, equivariant Poincaré duality is preserved under restrictions. To show this, first recall from Recollection 6.2.1 that for a closed subgroup $H \leq G$ there is an identification $\mathcal{O}(H) \simeq \mathcal{O}(G)_{/(G/H)}$ so that the induction $\mathcal{S}_H \to \mathcal{S}_G$ can be identified with the étale geometric morphism $\mathcal{S}_G \rightleftharpoons (\mathcal{S}_G)_{/\underline{(G/H)}}$.

Construction 8.1.7 (Pushing Spivak data along restrictions)**.** Consider $\underline{X} \in \mathcal{S}_G$, $\underline{\mathcal{C}} \in \mathrm{CMon}(\mathrm{Cat}_G)$, and (ξ, c) a $\underline{\mathcal{C}}$–Spivak datum for $\underline{X}$. Then by Construction 7.3.11, we obtain a $\mathrm{Res}^G_H\,\underline{\mathcal{C}}$–Spivak datum $\mathrm{Res}^G_H(\xi, c)$ for $\mathrm{Res}^G_H\,\underline{X}$ given by

$$\left(\mathrm{Res}^G_H\,\xi\colon\ \mathrm{Res}^G_H\,\underline{X} \xrightarrow{\mathrm{Res}^G_H\,\xi} \mathrm{Res}^G_H\,\underline{\mathcal{C}},\ \mathrm{Res}^G_H\,c\colon\ \mathrm{Res}^G_H\,\mathbb{1}_{\underline{\mathcal{C}}} \xrightarrow{\mathrm{Res}^G_H[c]} (\mathrm{Res}^G_H\,X)_!\,\mathrm{Res}^G_H\,\xi\right)$$

Proposition 8.1.8 (Restriction stability of Poincaré duality)**.** *Let $\underline{X} \in \mathcal{S}_G$, $\underline{\mathcal{C}}$ be a symmetric monoidal G–category, and (ξ, c) be a Poincaré $\underline{\mathcal{C}}$–Spivak datum. Then for all closed subgroups $H \leq G$, $\mathrm{Res}^G_H(\xi, c)$ is a Poincaré $\mathrm{Res}^G_H\,\underline{\mathcal{C}}$–Spivak datum for the H-space $\mathrm{Res}^G_H\,\underline{X}$.*

Proof. This is a direct consequence of part (e) of Theorem 7.3.12 applied to étale the geometric morphism $\mathcal{S}_G \rightleftharpoons (\mathcal{S}_G)_{/\underline{(G/H)}} \simeq \mathcal{S}_H$. □

8.2 Fixed points methods

Let G be a compact Lie group. In this subsection, we study how G–Poincaré duality for a G–space $\underline{X}$ relates to Poincaré duality for its fixed points. In fact, we shall build upon the theory set up in §6.2 and discuss these questions in the generality of isotropy separations with respect to a family of subgroups, of which the case of fixed points against a subgroup is a special case. Therefore, let us fix a family $\mathcal{F}$ of closed subgroups of G throughout this subsection. Recall the notational package from Notation 6.2.28.

An important family to keep in mind as an intuitional guide is the following:

Example 8.2.1 (Proper family)**.** Denote by $\mathcal{P}$ the family of proper closed subgroups of G, so that $\mathcal{P}^c = \{G\}$ and $s\colon * \simeq \mathcal{O}_{\mathcal{P}^c}(G)^{\mathrm{op}} \hookrightarrow \mathcal{O}(G)^{\mathrm{op}}$ is the inclusion of the orbit G/G. Note that for any $\underline{J} \in \mathrm{Cat}_G$, we thus have $\underline{J}^{\mathcal{P}^c} = s^*\underline{J} \simeq J^G \in \mathrm{Cat}$. In this special case, we know that the adjunction unit $\Phi\colon \underline{\mathrm{Sp}} \to \underline{\mathrm{Sp}}^{\Phi\widetilde{\mathcal{P}}} \simeq s_*\widetilde{s}^*\underline{\mathrm{Sp}}$ adjoints to the geometric fixed points functor $\Phi^G\colon s^*\underline{\mathrm{Sp}} = \underline{\mathrm{Sp}}^{\mathcal{P}^c} \simeq \mathrm{Sp}_G \to \Phi^{\mathcal{P}}\underline{\mathrm{Sp}} \simeq \mathrm{Sp}$.

Observation 8.2.2. Consider the case of the family of proper closed subgroups $\mathcal{P}$ of G. In particular, we have that $\underline{\mathrm{Fun}}(-,-)^{\mathcal{P}^c} \simeq \underline{\mathrm{Fun}}(-,-)^G \simeq \mathrm{Fun}_G(-,-)$. For a fixed $\mathcal{C} \in \mathrm{Cat}$ having the approparite (co)limits and a G-space $\underline{X}$, applying $(-)^{\mathcal{P}^c}$ to the commuting diagram in Lemma 6.1.18 and using that $(-)^{\widetilde{\mathcal{P}}}$ is fully faithful yields the left commuting diagram

$$\begin{tikzcd}
\mathrm{Fun}(X^G,\mathcal{C}) \arrow[r, shift left, "X^G_!"] \arrow[r, shift right, "X^G_*"'] \arrow[d, equal] & \mathcal{C} \arrow[d, equal] \\
\mathrm{Fun}_G(\underline{X},\mathcal{C}^{\widetilde{\mathcal{P}}}) \arrow[r, shift left, "X_!"] \arrow[r, shift right, "X_*"'] & \mathcal{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Fun}_G(\underline{X},\underline{\mathrm{Sp}}) \arrow[r, "X_!"] \arrow[d, "\Phi^G"'] & \mathrm{Sp}_G \arrow[d, "\Phi^G"] \\
\mathrm{Fun}(X^G,\mathrm{Sp}) \arrow[r, "X^G_!"] & \mathrm{Sp}.
\end{tikzcd}$$

That is, parametrised (co)limits in G–categories of the form $\mathcal{C}^{\widetilde{\mathcal{P}}}$ is given by the ordinary (co)limits of the fixed points of the indexing diagram. In particular, since $\Phi^G\colon \underline{\mathrm{Sp}} \to \underline{\mathrm{Sp}}^{\Phi\widetilde{\mathcal{P}}}$ preserves parametrised colimits, the identifications above yield the right commuting square in the diagram above.

Lemma 8.2.3. *Let $\underline{X} \in \mathcal{S}_G$ and consider a family $\mathcal{F}$ of subgroups of G. Let $\underline{\mathcal{C}} \in \mathrm{Cat}_{G,\mathcal{F}^c}$. Then the map $\epsilon^*\colon \underline{\mathrm{Fun}}(\underline{X},\underline{\mathcal{C}}^{\widetilde{\mathcal{F}}}) \to \underline{\mathrm{Fun}}(\underline{X}_{\widetilde{\mathcal{F}}},\underline{\mathcal{C}}^{\widetilde{\mathcal{F}}})$ is an equivalence.*

Proof. By Lemma 6.1.16, the equivalence $\underline{\mathrm{Fun}}_G(\underline{X}, s_*\underline{\mathcal{C}}) \simeq s_*\underline{\mathrm{Fun}}_{\mathcal{F}^c}(s^*\underline{X},\underline{\mathcal{C}})$ identifies restriction along $\epsilon\colon s_!s^*\underline{X} \to \underline{X}$ on the left side with restriction along $s^*\epsilon\colon s^*s_!s^*\underline{X} \to s^*\underline{X}$ on the right side. But $s^*\epsilon$ is an equivalence. □

Construction 8.2.4 (Isotropy separation for Spivak data)**.** Let $\underline{X} \in \mathcal{S}_G$ and $\underline{\mathcal{C}}$ a symmetric monoidal $\mathcal{F}^c$–category which admits $\underline{X}^{\mathcal{F}^c}$–indexed colimits. By

Lemma 6.1.18, we know that $\underline{\mathcal{C}}^{\widetilde{\mathcal{F}}}$ admits $\underline{X}$–colimits. Suppose we are given a $\underline{\mathcal{C}}$-Spivak datum (ξ, c) for $\underline{X}^{\mathcal{F}^c}$ and a $\underline{\mathcal{C}}^{\widetilde{\mathcal{F}}}$–Spivak datum (ζ, d) for $\underline{X}$. By Construction 7.3.10, we obtain a $\underline{\mathcal{C}}^{\widetilde{\mathcal{F}}}$–Spivak datum $(\xi, c)^{\widetilde{\mathcal{F}}}$ for $\underline{X}$ and a $\underline{\mathcal{C}}$–Spivak datum $(\zeta, d)^{\mathcal{F}^c}$ for $\underline{X}^{\mathcal{F}^c}$. Observe in particular that, by construction, we have $((\xi, c)^{\widetilde{\mathcal{F}}})^{\mathcal{F}^c} \simeq (\xi, c)$ and $((\zeta, d)^{\mathcal{F}^c})^{\widetilde{\mathcal{F}}} \simeq (\zeta, d)$.

Corollary 8.2.5 (Inclusion of singular part is degree one)**.** *Let $\underline{\mathcal{D}}$ be a symmetric monoidal $\mathcal{F}^c$–category and $\underline{X} \in \mathcal{S}_G$. Suppose $\underline{X}$ is equipped with a $\underline{\mathcal{D}}^{\widetilde{\mathcal{F}}}$–Spivak datum. Then $\underline{X}_{\widetilde{\mathcal{F}}} \in \mathcal{S}_G$ inherits a $\underline{\mathcal{D}}^{\widetilde{\mathcal{F}}}$–Spivak datum under which the inclusion $\epsilon\colon \underline{X}_{\widetilde{\mathcal{F}}} \to \underline{X}$ is of $\underline{\mathcal{D}}^{\widetilde{\mathcal{F}}}$–degree one.*

Proof. By Lemma 8.2.3, the map $\epsilon^*\colon \underline{\mathrm{Fun}}(\underline{X}, s_*\underline{\mathcal{D}}) \xrightarrow{\simeq} \underline{\mathrm{Fun}}(s_!s^*\underline{X}, s_*\underline{\mathcal{D}})$ is an equivalence. The result now follows immediately from Construction 7.4.9. □

Lemma 8.2.6. *Let $\underline{X} \in \mathcal{S}_G$, $\mathcal{F}$ be a family of closed subgroups of G, and $\underline{\mathcal{D}}$ a symmetric monoidal $\mathcal{F}^c$–category. Then a $\underline{\mathcal{D}}^{\widetilde{\mathcal{F}}}$–Spivak datum (ξ, c) for $\underline{X}$ is Poincaré if and only if the $\underline{\mathcal{D}}$–Spivak datum $(\xi, c)^{\mathcal{F}^c}$ for $\underline{X}_{\mathcal{F}^c}$ is Poincaré.*

Proof. This is a special case of Theorem 7.3.12 (c). □

We now come to the main result of this subsection which says that we may perform isotropy separation on equivariant Poincaré spaces by appropriately isotropy–separating the coefficient category. For the second part of the result, we will need to recall Terminology 7.3.7.

Theorem 8.2.7 (Poincaré isotropy basechange)**.** *Let $\underline{X} \in \mathcal{S}_G$, $\underline{Y} \in \mathcal{S}_G^{\omega}$, $\underline{\mathcal{C}}$ be a presentably symmetric monoidal fibrewise stable G–category, and $\underline{\mathcal{D}}$ be a G–stably symmetric monoidal category.*

(1) If $\underline{X}$ is $\underline{\mathcal{C}}$–Poincaré, then $\underline{X}^{\mathcal{F}^c}$ is $\Phi^{\mathcal{F}}\underline{\mathcal{C}}$–Poincaré;

(2) If (ξ, c) is a Poincaré $\underline{\mathcal{D}}$–Spivak datum for $\underline{Y}$, then the Spivak datum $(\Phi\xi, \Phi c)^{\mathcal{F}^c}$ is a Poincaré $(\Phi\colon \underline{\mathcal{D}}^{\mathcal{F}^c} \to \Phi^{\mathcal{F}}\underline{\mathcal{D}})$–Spivak datum for $\underline{Y}^{\mathcal{F}^c}$.

Proof. For (1), applying the basechange result Theorem 7.3.5 along the symmetric monoidal G-colimit preserving unit map $\underline{\mathcal{C}} \to \underline{\mathcal{C}}^{\Phi\widetilde{\mathcal{F}}}$ shows that $\underline{X}$ is $\underline{\mathcal{C}}^{\Phi\widetilde{\mathcal{F}}}$–Poincaré. Thus Lemma 8.2.6 shows that $\underline{X}^{\mathcal{F}^c}$ is $\Phi^{\mathcal{F}}\underline{\mathcal{C}}$–Poincaré. Point (2) is an immediate consequence of Theorem 7.3.8 and Lemma 8.2.6. □

Having set up a general theory of equivariant fixed points for Poincaré spaces, we now specialise to the most important coefficient category, namely the presentably symmetric monoidal G–stable category $\underline{\mathrm{Sp}}$ of genuine G–spectra.

Construction 8.2.8 (Pushing Spivak data along geometric fixed points)**.** Let $\underline{X} \in \mathcal{S}_G$, (ξ, c) a $\underline{\mathrm{Sp}}_G$–Spivak datum for $\underline{X}$, and $H \leq G$ a closed subgroup. By Construction 8.1.7, we obtain a $\underline{\mathrm{Sp}}_H$–Spivak datum $\mathrm{Res}^G_H(\xi, c)$ for $\mathrm{Res}^G_H \underline{X}$. On the other hand, we may apply Construction 8.2.4 to $\mathrm{Res}^G_H(\xi, c)$ along the symmetric monoidal map $\Phi^H \colon \underline{\mathrm{Sp}}_H \to s_*\mathrm{Sp}$ to get a nonequivariant Sp–Spivak datum $\Phi^H(\xi, c)$ for X^H. Explicitly, this is given by

$$\left(\Phi^H\xi \colon X^H \xrightarrow{\xi^H} \mathrm{Sp}_H \xrightarrow{\Phi^H} \mathrm{Sp},\ \Phi^H c \colon \mathbb{1}_{\mathrm{Sp}} = \Phi^H \mathbb{1}_{\mathrm{Sp}_H} \xrightarrow{\Phi^H c} \Phi^H(\mathrm{Res}^G_H X)_! \,\mathrm{Res}^G_H \xi \simeq X^H_! \Phi^H \xi\right)$$

where the last equivalence is by Observation 8.2.2.

Next, we unwind the general Theorem 8.2.7 (1) for the geometric fixed points functor on spectra to show that the fixed points of an equivariant Poincaré space are Poincaré with the residual Weyl group action (c.f. [CW17, Prop. 2.4] for the homological shadow of this).

Theorem 8.2.9 (Fixed points of Poincaré spaces)**.** *Suppose $\underline{X} \in \mathcal{S}_G$ is G–Poincaré. Then for any closed $H \leq G$, $\underline{X}^H \in \mathcal{S}_{W_G H}$ is a $W_G H$–Poincaré space. In particular, $X^H \in \mathcal{S}$ is a nonequivariant Poincaré space with dualising sheaf $X^H \xrightarrow{D_{\underline{X}}} \mathrm{Sp}_H \xrightarrow{\Phi^H} \mathrm{Sp}$.*

Proof. First consider the case where H is normal in G. We apply Theorem 8.2.7 (1) in the case $\underline{\mathcal{C}} = \underline{\mathrm{Sp}}_G$ and for the family $\Gamma_H := \{K \leq G \mid H \nleq K\}$ of subgroups of G not containing H. Thus, if $\underline{X}$ is a G–Poincaré space, then $s^*\underline{X}$ is a $\widetilde{s}^*\underline{\mathrm{Sp}}_G$ Poincaré space. In Proposition 6.2.29 we saw that $\mathrm{Coind} \colon \mathcal{S}_{G/H} \to \mathcal{S}_G$ induces an equivalence $\mathcal{S}_{G/H} \simeq \mathcal{S}_{\Gamma^c_H}$ endowing $s^*\underline{X}$ with a G/H action. It also follows from Corollary 6.2.30 that $\widetilde{s}^*\underline{\mathrm{Sp}}_G \simeq \underline{\mathrm{Sp}}_{G/N}$ which completes the proof of this case.

Now suppose that $H \leq G$ is a general subgroup. We can apply Proposition 8.1.8 to obtain that $\mathrm{Res}^G_{N_G H} \underline{X}$ is a $N_G H$–Poincaré duality space. Then the normal subgroup case from above shows that $\mathrm{Res}^G_H \underline{X} = \mathrm{Res}^{N_G H}_H \mathrm{Res}^G_{N_G H} \underline{X}$ is a $W_G H = N_G H/H$–Poincaré duality space.

The first part in particular shows that X^G is a Sp–Poincaré duality space. Applying this to $X^H \in \mathcal{S}_{W_G H}$, we see that X^H is a nonequivariant Sp–Poincaré duality space. □

To end our discussion on general fixed points methods, we provide a sort of converse to the previous statement. By Proposition 7.2.6, we know that in the presentable setting, a twisted ambidextrous Spivak datum is unique if it exists. Via the geometric fixed points functors, the following result gives a full characterisation for a candidate invertible Spivak datum to be the unique one for a twisted ambidextrous space in terms of nonequivariant Poincaré duality. It will be essential for constructing examples of equivariant Poincaré duality spaces in §8.4.

Theorem 8.2.10 (Fixed point recognition principle of Poincaré spaces)**.** *Suppose that $\underline{X} \in \mathcal{S}_G$ is a twisted ambidextrous G–space (e.g. a compact G–space) and let (ξ, c) be a*

$\underline{\mathrm{Sp}}_G$-Spivak datum for $\underline{X}$ such that $\xi\colon \underline{X} \to \underline{\mathrm{Sp}}_G$ takes values in $\underline{\mathcal{P}\mathrm{ic}}(\underline{\mathrm{Sp}}_G)$. Then (ξ, c) exhibits $\underline{X}$ as a G–Poincaré duality space if and only if for all closed subgroups $H \leq G$, the Spivak datum $\Phi^H(\xi, c)$ from Construction 8.2.8 exhibits X^H as a nonequivariant Sp*–Poincaré space.*

Proof. The "only if" direction is a consequence of Theorem 8.2.9. For the other direction, we have to show that the Spivak datum (ξ, c) is twisted ambidextrous as ξ is invertible by assumption. By Observation 6.2.33 and Proposition 6.2.35, the collection

$$\left\{ \underline{\mathrm{Fun}}(\underline{X}, \underline{\mathrm{Sp}}) \xrightarrow{\Phi^H} \underline{\mathrm{Fun}}(\underline{X}, \mathrm{Coind}_H^G s_* \widetilde{s}^* \underline{\mathrm{Sp}}) \simeq \prod_H^G s_* \,\mathrm{Fun}(X^H, \mathrm{Sp}) \mid H \leq G \text{ closed subgroups} \right\}$$

is jointly conservative. Thus, it suffices to show that the transformations

$$\Phi^H(c \cap_\xi -)\colon \Phi^H X_*(-) \to \Phi^H X_!(- \otimes \xi) \tag{8.2}$$

are equivalences. By passing to the adjoint $\underline{\mathrm{Fun}}(\mathrm{Res}_H^G \underline{X}, \mathrm{Res}_H^G \underline{\mathrm{Sp}}) \xrightarrow{\Phi^H} s_* \,\mathrm{Fun}(X^H, \mathrm{Sp})$ to consider everything as H–categories, we may without loss of generality just consider the case Φ^G. By Proposition 7.1.9 applied to the case of Example 7.1.6 (1), the symmetric monoidal functor of presentably symmetric monoidal G–categories $\Phi^G\colon \underline{\mathrm{Sp}} \to s_*\mathrm{Sp}$ yields a square

$$\begin{tikzcd}
\Phi^G X_*(-) \arrow[rr, "\Phi^G(c\cap_\xi -)"] \arrow[d, "\mathrm{BC}_*", "\simeq"'] & & \Phi^G X_!(\xi \otimes -) \\
X_*\Phi^G(-) \arrow[r, "\Phi^G c \cap_{\Phi^G\xi} \Phi^G -"'] & X_!\Phi^G(\xi \otimes -) \simeq X_!(\Phi^G\xi \otimes \Phi^G(-)) \arrow[ur, "\mathrm{BC}_!", "\simeq"']
\end{tikzcd}$$

where the vertical Beck–Chevalley maps are equivalences, the right one by Observation 8.2.2 and the left one by Corollary 7.3.6 since $\underline{X}$ was assumed to be twisted ambidextrous. By Observation 8.2.2, the bottom map identifies with $\Phi^G c \cap_{\Phi^G\xi} \Phi^G -\colon X_*^G\Phi^G(-) \to X_!^G(\Phi^G\xi \otimes \Phi^G -)$, which is an equivalence by hypothesis. Thus, in total, we see that the top horizontal map in the square above is an equivalence, as was to be shown. □

8.3 Construction principles

In this section we will study various results on how to build new Poincaré duality spaces out of old ones.

Change of groups

We begin by studying the effect of standard equivariant operations on $\underline{X}$. Recall the constructions and notations from Notation 6.2.2 and Construction 6.2.13.

Proposition 8.3.1 (Poincaré duality and restriction). *Suppose that $\alpha\colon H \to G$ is a continuous homomorphism of compact Lie groups and $\underline{X} \in \mathcal{S}_G$. If $\underline{X}$ is a G–Poincaré space, then $\operatorname{Res}_\alpha \underline{X}$ is a H–Poincaré space with Spivak datum $(\operatorname{Res}_\alpha c, \operatorname{Res}_\alpha D_{\underline{X}})$ where*

1. *the local system $\operatorname{Res}_\alpha D_{\underline{X}}$ is $\operatorname{Res}_\alpha \underline{X} \to \operatorname{Res}_\alpha \underline{\mathrm{Sp}}_G \to \underline{\mathrm{Sp}}_H$ and*
2. *the collapse map $\operatorname{Res}_\alpha c$ is $\mathbb{1}_{\mathrm{Sp}_H} = \operatorname{Res}_\alpha \mathbb{1}_{\mathrm{Sp}_G} \to \operatorname{Res}_\alpha X_! D_{\underline{X}} \simeq (\operatorname{Res}_\alpha X)_! \operatorname{Res}_\alpha D_{\underline{X}}$.*

Proof. If $\underline{X}$ is $\underline{\mathrm{Sp}}_G$–Poincaré, then applying Theorem 7.3.5 for the symmetric monoidal G-colimit preserving functor $\operatorname{Res}_\alpha\colon \underline{\mathrm{Sp}}_G \to \operatorname{Coind}_\alpha \underline{\mathrm{Sp}}_H$ from Construction 6.2.9 shows that $\underline{X}$ is $\operatorname{Coind}_\alpha \underline{\mathrm{Sp}}_H$–Poincaré. By Theorem 7.3.12 (d) we see that $\operatorname{Res}_\alpha \underline{X}$ is $\underline{\mathrm{Sp}}_H$–Poincaré with claimed Spivak datum. □

Proposition 8.3.2 (Poincaré duality and inflation). *Consider a closed normal subgroup $N \leq G$ and a G/N-space $\underline{X}$. Then $\underline{X}$ is a G/N–Poincaré duality space if and only if $\operatorname{Infl}_G^{G/N} \underline{X}$ is a G–Poincaré duality space.*

Proof. One direction is a consequence of Proposition 8.3.1 while the other one follows from Theorem 8.2.9. □

Proposition 8.3.3 (Poincaré duality and induction). *Let $\iota\colon H \to G$ be an injective homomorphism of compact Lie groups. If $\underline{X}$ is a H–Poincaré space, then $\operatorname{Ind}_H^G \underline{X}$ is G–Poincaré space.*

Proof. We first claim that the map $\operatorname{Ind}_H^G \underline{X} \to \underline{G/H}$ is a G–Poincaré map. Using the equivalence $(\mathcal{S}_G)_{/\underline{G/H}} \simeq \mathcal{S}_H$, this is equivalent to $\operatorname{Res}_H^G \operatorname{Ind}_H^G \underline{X}$ being a H–Poincaré space. Observe that $\operatorname{Res}_H^G \operatorname{Ind}_H^G \underline{X} \simeq \underline{X} \times \operatorname{Res}_H^G \underline{G/H}$. As $\underline{G/H}$ is a G–Poincaré space, the claim follows from Corollary 7.3.15 and Proposition 8.3.1. Now Proposition 7.3.3 implies that the composite $\operatorname{Ind}_H^G \underline{X} \to \underline{G/H} \to \underline{*}$ is a G–Poincaré map meaning that $\operatorname{Ind}_H^G \underline{X}$ is a G–Poincaré space. □

For the next result, we will need to restrict to the case of finite groups since we will need to invoke the theory of G–symmetric monoidal structures as introduced in [Nar17] and further developed in [NS22].

Recollection 8.3.4 (Multiplicative norms). Nardin constructed in [Nar17] a G–symmetric monoidal structure for the G–category of genuine G–spectra $\underline{\mathrm{Sp}}$, packaging the multiplicative norms of [GM97; HHR16] coherently. For a finite G–set, $U = \coprod_i G/H_i$, we write $\mathrm{Cat}_U := \prod_i \mathrm{Cat}_{H_i}$ and write $\underline{\mathrm{Sp}}_U := (\underline{\mathrm{Sp}}_{H_i})_i \in \mathrm{Cat}_U$. For a map of finite G–sets $f\colon U \to V$, we get an adjunction $f^*\colon \mathrm{Cat}_V \rightleftharpoons \mathrm{Cat}_U : f_*$ where f^* is given by restrictions and f_* is given by coinductions. As part of the G–symmetric monoidal structure on $\underline{\mathrm{Sp}}$, we have a map $f_\otimes\colon f_*\underline{\mathrm{Sp}}_U \to \underline{\mathrm{Sp}}_V$ encoding the multiplicative norm along f. For example, when f is the map $f\colon G/H \to G/G$, this encodes a map $f_\otimes\colon \operatorname{Coind}_H^G \underline{\mathrm{Sp}}_H \to \underline{\mathrm{Sp}}_G$, upon applying the functor $(-)^G$ to which yields the multiplicative norm $\mathrm{N}_H^G\colon \mathrm{Sp}_H \to \mathrm{Sp}_G$. By [NS22, §3.3], for a fixed

$\underline{X} \in \mathcal{S}_G$, we may obtain a pointwise G–symmetric monoidal structure on the functor category $\underline{\mathrm{Fun}}(\underline{X}, \underline{\mathrm{Sp}})$. From this, we ma for example extract the pointwise multiplicative norm functor

$$f_\otimes \colon f_*\underline{\mathrm{Fun}}(\underline{X}, f^*\underline{\mathrm{Sp}}) \longrightarrow f_\otimes \underline{\mathrm{Fun}}(\underline{X}, f^*\underline{\mathrm{Sp}}) \simeq \underline{\mathrm{Fun}}(f_*\underline{X}, \underline{\mathrm{Sp}})$$

where the equivalence is by [Hil24a, Cor. 2.2.20].

Proposition 8.3.5 (Poincaré duality and coinductions). *Let G be a finite group and $\{H_i\}_i$ a finite collection of subgroups of G. Suppose for each i, we have a H_i–Poincaré space $\underline{X}_i \in \mathcal{S}_{H_i}$ with dualising sheaf $D_{\underline{X}_i}$. Then $\prod_i \mathrm{Coind}_{H_i}^G \underline{X}_i \in \mathcal{S}_G$ is a G–Poincaré space with dualising sheaf $\bigotimes_i \mathrm{N}_{H_i}^G D_{\underline{X}_i} \in \underline{\mathrm{Fun}}(\prod_i \mathrm{Coind}_{H_i}^G \underline{X}, \underline{\mathrm{Sp}})$.*

Proof. We consider a map of finite G–sets $f \colon U = \coprod_i G/H_i \to V = G/G$ as in Recollection 8.3.4. Writing $\underline{X} := (\underline{X}_i)_i \in \mathrm{Cat}_U$, we have an equivalence of the two functors

$$\begin{tikzcd} \underline{\mathrm{Fun}}(\underline{X}, f^*\underline{\mathrm{Sp}}) \arrow[r, "D_{\underline{X}} \otimes -"'] \arrow[rr, bend left, "X_*"] & \underline{\mathrm{Fun}}(\underline{X}, f^*\underline{\mathrm{Sp}}) \arrow[r, "X_!"'] & f^*\underline{\mathrm{Sp}} \end{tikzcd} \tag{8.3}$$

by our hypothesis. Now writing $f_*X \colon f_*\underline{X} \to \underline{*}$ for the unique map, note that since X_* itself has a right adjoint, we may use [Hil24b, Lem. 4.4.3] to see that applying $f_\otimes$ preserves the adjunctions $X_! \dashv X^* \dashv X_*$ in the sense that we have the adjunctions

$$\begin{tikzcd} f_\otimes \underline{\mathrm{Fun}}(\underline{X}, f^*\underline{\mathrm{Sp}}) \arrow[r, bend left, "f_\otimes(X_!)"] \arrow[r, bend right, "f_\otimes(X_*)"'] & f_\otimes f^*\underline{\mathrm{Sp}} \arrow[l, "f_\otimes(X^*)"'] \end{tikzcd}$$

But then, since $\underline{\mathrm{Fun}}(-, \underline{\mathrm{Sp}}) \colon \underline{\mathrm{Cat}}_G^{\times} \to \underline{\mathrm{Pr}}_{L,\mathrm{st}}^{\underline{\otimes}}$ functorial in left Kan extensions is $\underline{\otimes}$–symmetric monoidal by [NS22, After Cor. 6.0.11] together with [Hil24a, Cor. 2.2.20], we get

$$f_\otimes(X_!) \simeq (f_*X)_! \colon f_\otimes \underline{\mathrm{Fun}}(\underline{X}, f^*\underline{\mathrm{Sp}}) \simeq \underline{\mathrm{Fun}}(f_*\underline{X}, \underline{\mathrm{Sp}}) \longrightarrow f_\otimes f^*\underline{\mathrm{Sp}} \simeq \underline{\mathrm{Sp}}$$

and thus consequently, also that $f_\otimes(X^*) \simeq (f_*X)^*$ and $f_\otimes(X_*) \simeq (f_*X)_*$. Next, note that the functor $D_{\underline{X}} \otimes -$ may be written as

$$f^*\underline{\mathrm{Sp}} \otimes \underline{\mathrm{Fun}}(\underline{X}, f^*\underline{\mathrm{Sp}}) \xrightarrow{D_{\underline{X}} \otimes \mathrm{id}} \underline{\mathrm{Fun}}(\underline{X}, f^*\underline{\mathrm{Sp}}) \otimes \underline{\mathrm{Fun}}(\underline{X}, f^*\underline{\mathrm{Sp}}) \xrightarrow{\otimes} \underline{\mathrm{Fun}}(\underline{X}, f^*\underline{\mathrm{Sp}})$$

Thus applying $f_\otimes$ to this composite and using that $f_\otimes$ is itself a symmetric monoidal functor, we get the identification of $f_\otimes(D_{\underline{X}} \otimes -)$ as

$$\underline{\mathrm{Sp}} \otimes f_\otimes \underline{\mathrm{Fun}}(\underline{X}, f^*\underline{\mathrm{Sp}}) \xrightarrow{f_\otimes D_{\underline{X}} \otimes \mathrm{id}} f_\otimes \underline{\mathrm{Fun}}(\underline{X}, f^*\underline{\mathrm{Sp}}) \otimes f_\otimes \underline{\mathrm{Fun}}(\underline{X}, f^*\underline{\mathrm{Sp}}) \xrightarrow{\otimes} f_\otimes \underline{\mathrm{Fun}}(\underline{X}, f^*\underline{\mathrm{Sp}})$$

That is, that $f_\otimes(D_{\underline{X}} \otimes -) \simeq f_\otimes D_{\underline{X}} \otimes -$. Therefore, all in all, applying $f_\otimes$ to the identification in (8.3), we obtain an equivalence

$$(f_*X)_* \simeq f_\otimes(X_*) \simeq f_\otimes\big(X_!(D_{\underline{X}} \otimes -)\big) \simeq (f_*X)_!(f_\otimes D_{\underline{X}} \otimes -)$$

as was to be shown. □

Proposition 8.3.6 (Poincaré duality and Borelification). *Let $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^L)$ and $\underline{X} \in \mathcal{S}_G$ such that X^e is nonequivariantly a $\mathcal{C}$–twisted ambidextrous (resp. Poincaré) space. Then $\underline{X}$ is a $\underline{\mathrm{Bor}}(\mathcal{C})$–twisted ambidextrous (resp. Poincaré) space.*

Proof. Since $* \to BG$ is an effective epimorphism, we may apply Proposition 7.3.14 to the fibre sequence $X^e \to X_{hG} \xrightarrow{\pi} BG$ to get that π is a $\mathcal{C}$–twisted ambidextrous (resp. Poincaré) map. Writing $s\colon BG \to \mathcal{O}(G)$ for the inclusion and using the identification $\mathcal{S}_{/BG} = \mathrm{Fun}(BG, \mathcal{S})$ under which π corresponds to $s^*\underline{X}$, this means by Definition 7.2.13 that $s^*\underline{X}$ is $\pi_{BG}^*\mathcal{C}$–twisted ambidextrous (resp. Poincaré), where $\pi_{BG}^*\colon \mathrm{Cat} \to \mathrm{Cat}_{BG}$ denotes the restriction functor. Now the basechange result Theorem 7.3.12 shows that $\underline{X}$ is $s_*\pi_{BG}^*\mathcal{C} = \underline{\mathrm{Bor}}(\mathcal{C})$–twisted ambidextrous (resp. Poincaré). □

Lemma 8.3.7 (Degree one data and Borelification). *Let $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^L)$ be a presentably symmetric monoidal category and $f\colon \underline{X} \to \underline{Y}$ a map of G–spaces such that X^e and Y^e are nonequivariantly $\underline{\mathcal{C}}$–twisted ambidextrous. Suppose that $\alpha\colon D_{X^e} \simeq f^*D_{Y^e}$ is a G–equivariant $\underline{\mathcal{C}}$–degree datum for $f^e\colon X^e \to Y^e$, i.e. α is an equivalence in $\mathrm{Fun}(X^e, \mathcal{C})^{hG}$. Then there is a $\underline{\mathrm{Bor}}(\mathcal{C})$–degree datum for $f\colon \underline{X} \to \underline{Y}$. If in addition the G–equivariant degree datum for f^e is G-equivariantly of degree one, i.e. there is an equivalence $c_{Y^e} \simeq f_!c_{X^e}$ in $\mathrm{Map}(\mathbb{1}_\mathcal{C}, (Y^e)_!D_{Y^e})^{hG}$, then the $\underline{\mathrm{Bor}}(\mathcal{C})$–degree datum for f is of degree one.*

Proof. With the notation from Proposition 8.3.6, the assumption on G-equivariance of the degree datum implies that means that α is a $\pi_{BG}^*\mathcal{C}$-degree datum for the map $f^e\colon X^e \to Y^e$ in $\mathcal{S}^{BG}$. If α is G-equivariantly of degree one, then the $\pi_{BG}^*\mathcal{C}$-degree datum for f^e is of degree one. Now Lemma 7.4.10 provides us with a $\underline{\mathrm{Bor}}(\mathcal{C})$ degree datum for $f\colon \underline{X} \to \underline{Y}$ (which is of degree one if α was G-equivariantly of degree one). □

Family nilpotence

We now study how Poincaré duality interacts with the $\mathcal{F}$–nilpotence theory of [MNN17].

Proposition 8.3.8. *Let $\mathcal{F}$ be a family of subgroups of G, $\underline{\mathcal{C}}$ a presentably symmetric monoidal G–category which is $\mathcal{F}$–Borel complete, and $\underline{X} \in \mathcal{S}_G$. Then $\underline{X}$ satisfies $\underline{\mathcal{C}}$–Poincare duality if and only if $\mathrm{Res}^G_H \underline{X}$ satisfies $\mathrm{Res}^G_H \underline{\mathcal{C}}$–Poincare duality for all $H \in \mathcal{F}$.*

Proof. Using that $\mathcal{C}$ is $\mathcal{F}$-Borel complete, Theorem 7.3.12 shows that $\underline{X}$ is a $\mathcal{C} \simeq b_*b^*\mathcal{C}$–Poincaré space if and only if $b^*\underline{X}$ is a $b^*\underline{\mathcal{C}}$–Poincaré space. But by definition, this is equivalent to the map $\underline{X} \to \underline{E\mathcal{F}}$ being a $\underline{\mathcal{C}}$–Poincaré duality map.

There is an effective epimorphism $\coprod_{H\in\mathcal{F}} \underline{G/H} \to \underline{E\mathcal{F}}$. The descent result Proposition 7.3.14 together with Lemma 7.3.16 now shows that $\underline{X} \to \underline{E\mathcal{F}}$ is a $\underline{\mathcal{C}}$–Poincaré duality map if and only if $\underline{G/H} \times_{\underline{E\mathcal{F}}} \underline{X} \to \underline{G/H}$ is a $\underline{\mathcal{C}}$–Poincaré duality map for all $H \in \mathcal{F}$. Note that under the equivalence $(\mathcal{S}_G)_{/\underline{G/H}} \simeq \mathcal{S}_H$ the map $\underline{G/H} \times_{\underline{E\mathcal{F}}} \underline{X} \to \underline{G/H}$ corresponds to $\mathrm{Res}^G_H \underline{X}$. Similarly, this equivalence identifies $\pi^*_{\underline{G/H}}\underline{\mathcal{C}}$ and $\mathrm{Res}^G_H \underline{\mathcal{C}}$. $\square$

Recall the notion of $\mathcal{F}$–nilpotent ring G–spectra from [MNN17, Def. 6.36].

Corollary 8.3.9. *Let G be a finite group, $\mathcal{F}$ a family of subgroups, $R \in \mathrm{CAlg}(\mathrm{Sp}_G)$ an $\mathcal{F}$–nilpotent ring G–spectrum, and $\underline{X} \in \mathcal{S}_G$. Then $\underline{X}$ is an R–Poincaré space if and only if for all $H \in \mathcal{F}$, $\mathrm{Res}^G_H \underline{X}$ is $\mathrm{Res}^G_H R$–Poincaré.*

Proof. By Example 6.2.22, we know that the presentably symmetric monoidal G–stable category $\underline{\mathrm{Mod}}_R(\underline{\mathrm{Sp}})$ is $\mathcal{F}$–Borel and so we may apply Proposition 8.3.8 to conclude. $\square$

Example 8.3.10. We collect here a small list of potentially interesting consequences of Corollary 8.3.9 using known nilpotence results from [MNN19, Table 2] for finite groups G. We invite the reader to consult the cited table for a quite exhaustive list of possibly interesting examples of coefficient ring G–spectra to consider.

(1) Writing KO_G and KU_G for Segal's equivariant topological K–theories, we see that a G–space $\underline{X}$ is KO_G– or KU_G–Poincaré if and only if $\mathrm{Res}^G_C \underline{X}$ is KO_C– or KU_C–Poincaré for all cyclic subgroups $C \leq G$.

(2) A G–space $\underline{X}$ is $\mathrm{Bor}_G(\mathrm{H}\mathbb{Z})$–Poincaré if and only if $\mathrm{Res}^G_E \underline{X}$ is $\mathrm{Bor}_E(\mathrm{H}\mathbb{Z})$–Poincaré for all elementary abelian p–subgroups $E \leq G$ for all primes p.

(3) A G–space $\underline{X}$ is $\mathrm{Bor}_G(\mathrm{MU})$–Poincaré if and only if $\mathrm{Res}^G_A \underline{X}$ is $\mathrm{Bor}_A(\mathrm{MU})$–Poincaré for all abelian p–subgroups $A \leq G$ for all primes p.

(4) A G–space $\underline{X}$ is $\mathrm{Bor}_G(\mathrm{MO})$–Poincaré if and only if $\mathrm{Res}^G_A \underline{X}$ is $\mathrm{Bor}_A(\mathrm{MO})$–Poincaré for all elementary abelian 2–subgroups $A \leq G$.

Poincaré integration

In this section we study the equivariant generalisation of a well known result of Klein [Kle01, Corollary F][1] which says that in a fibration $F \to E \to B$ of finitely

[1] In [Kle01], Klein mentioned that the result answered a question of Wall and also attributed the result to Quinn from an unpublished announcement and Gottlieb [Got79] who proved it in the manifolds setting.

dominated spaces, E is a Poincaré space if F and B are Poincaré spaces. Since we have $E \simeq \operatorname{colim}_B F$ by the straightening–unstraightening equivalence, the aforementioned result may be viewed as saying that integrating a Poincaré space along a diagram which is itself Poincaré yields a Poincaré space.

Terminology 8.3.11 (Fibrewise twisted ambidextrous and Poincaré maps)**.** Let $f\colon \underline{X} \to \underline{Y}$ be a map of G–spaces for G a compact Lie group and $\underline{\mathcal{C}}$ a presentably symmetric monoidal G-category. We say that it is a *fibrewise $\underline{\mathcal{C}}$–twisted ambidextrous (resp. Poincaré) map* if for all closed subgroups $H \leq G$ and all maps $y\colon \underline{G/H} \to \underline{Y}$, writing F_y for the pullback $\underline{G/H} \times_{\underline{Y}} \underline{X}$, the map $F_y \to \underline{G/H}$ is $\underline{\mathcal{C}}$–twisted ambidextrous (resp. Poincaré). Expanding Definitions 7.2.2 and 7.2.13, this means that viewed as an object in $\mathcal{S}_H \simeq (\mathcal{S}_G)_{/\underline{G/H}}$, F_y is $\operatorname{Res}^G_H \underline{\mathcal{C}}$–twisted ambidextrous (resp. Poincaré).

In the case where $\underline{\mathcal{C}} = \underline{\mathrm{Sp}}_G$ this means that F_y is $\operatorname{Res}^G_H \underline{\mathrm{Sp}}_G = \underline{\mathrm{Sp}}_H$–twisted ambidextrous (resp. Poincaré). Since $\underline{G/H}$ is G–Poincaré, we see by Proposition 7.3.3, Corollary 7.3.17, and Proposition 8.1.8 that in this case the preceding condition is also equivalent to F_y being G–twisted ambidextrous (resp. Poincaré).

Theorem 8.3.12 (Equivariant Poincaré integration)**.** *Let $f\colon \underline{X} \to \underline{Y}$ be a map of G–spaces and $\underline{\mathcal{C}}$ a presentably symmetric monoidal G-category. If $\underline{Y}$ is a $\underline{\mathcal{C}}$–Poincaré space and f is a fibrewise $\underline{\mathcal{C}}$–Poincaré map, then $\underline{X}$ is a $\underline{\mathcal{C}}$–Poincaré space. Furthermore, there is an equivalence $D_{\underline{X}} \simeq f^*D_{\underline{Y}} \otimes D_f$, where $D_f \in \mathrm{Fun}_G(\underline{X}, \underline{\mathcal{C}})$ such that $y^*D_f \simeq D_{F_y}$ is the dualising sheaf of the fibres.*

Conversely, suppose that $\underline{Y}$ is $\underline{\mathcal{C}}$–twisted ambidextrous and that f is fibrewise $\underline{\mathcal{C}}$-twisted ambidextrous. Furthermore assume that for all closed subgroups $H \leq G$ the map $f^H\colon X^H \to Y^H$ is a π_0-surjection. If $\underline{X}$ is a $\underline{\mathcal{C}}$–Poincaré space, then $\underline{Y}$ is also a $\underline{\mathcal{C}}$–Poincaré space and f is a fibrewise Poincaré map.

Proof. The map $\coprod_{H \leq G} \coprod_{\pi_0(Y^H)} \underline{G/H} \to \underline{Y}$ is a π_0 surjection on each fixed point space and thus an effective epimorphism in $\mathcal{S}_G$ (see Example 7.3.13). It then follows from Proposition 7.3.14 and Lemma 7.3.16 that f is a $\underline{\mathcal{C}}$–Poincaré map if and only if the map $F_y \to \underline{G/H}$ is $\underline{\mathcal{C}}$–Poincaré for all closed subgroups $H \leq G$ and all $y\colon \underline{G/H} \to \underline{Y}$. This is precisely what it means that f was fibrewise $\underline{\mathcal{C}}$–Poincaré. Moreover, Proposition 7.3.14 also provides an equivalence $y^*D_f \simeq D_{F_y}$. Since in addition $\underline{Y}$ is a $\underline{\mathcal{C}}$–Poincaré space, it follows from Proposition 7.3.3 that $\underline{X}$ is a $\underline{\mathcal{C}}$–Poincaré space and there is an equivalence $D_{\underline{X}} \simeq y^*D_{\underline{Y}} \otimes D_{F_y}$ as desired.

For the converse, as in the first we conclude from Proposition 7.3.14 and Proposition 7.3.3 that there is an equivalence $D_{\underline{X}} \simeq f^*D_{\underline{Y}} \otimes D_f$. If $\underline{X}$ is a $\underline{\mathcal{C}}$–Poincaré, then $D_{\underline{X}}$ is invertible which implies that $f^*D_{\underline{Y}}$ and D_f are invertible. The π_0 surjectivity hypothesis on f implies that $D_{\underline{Y}}$ is invertible so $\underline{Y}$ is a $\underline{\mathcal{C}}$–Poincaré space. From the equivalence $y^*D_f \simeq D_{F_y}$ we see that $(F_y \to \underline{G/H})$ is a $\operatorname{Res}^G_H \underline{\mathcal{C}}$–Poincaré space. $\square$

We now use the theorem above to obtain a characterisation of G–Poincaré duality for spaces with free actions in terms of Poincaré duality for a quotient group.

Corollary 8.3.13 (Poincaré duality and quotients by free actions)**.** *Let G be a compact Lie group, $N \leq G$ a closed normal subgroup and $Q := G/N$. If $\underline{X}$ is a G–space such that the action of N on $\underline{X}$ is free in the sense of Definition 6.2.36, then $\underline{X}$ is G–Poincaré duality space if and only if $N\backslash\underline{X}$ is a Q–Poincaré duality space.*

Proof. We will show that this follows from Theorem 8.3.12. To do so, it suffices to check that for each map $\underline{G/H} \to \mathrm{Infl}_G^Q N\backslash\underline{X}$ the space $\underline{G/H} \times_{\mathrm{Infl}_G^Q N\backslash\underline{X}} \underline{X}$ is a G–Poincaré space. But in Corollary 6.2.40 we have seen that there exists a cartesian square of the following form.

$$\begin{tikzcd} \underline{G/H} \times_{\mathrm{Infl}_G^Q N\backslash\underline{X}} \underline{X} \arrow[r] \arrow[d, "\mathrm{proj}"] & \underline{G/K_0} \arrow[d, "f"] \\ \underline{G/H} \arrow[r] & \underline{G/K_1} \end{tikzcd}$$

Let S denote the point-set fibre of the map of topological G–spaces $f\colon G/K_0 \to G/K_1$. Then S is a homogenous K_1-space, and $f = \mathrm{Ind}_{K_1}^G(S \to *)$. Now note that since the right map is a Poincaré duality map, so is the left one, and as $\underline{G/H}$ is G–Poincaré, Proposition 7.3.3 implies that $\underline{G/H} \times_{\mathrm{Infl}_G^Q N\backslash\underline{X}} \underline{X}$ is G–Poincaré, as desired. □

8.4 Examples

The next paragraphs will introduce two different sources of equivariant Poincaré spaces. First, we show that smooth G-manifolds are equivariantly Poincaré. Their study is one of the main motivations for a theory of equivariant Poincaré duality, and equivariant Poincaré spaces should be viewed as their homotopical analogue. Let us mention that while the proofs given here *depend* on the Wirthmüller isomorphism, the Wirthmüller isomorphism can also be proven using a different version of equivariant Poincaré duality, as is done for example in [MS06].

Our second source of examples are tom Dieck–Petrie's generalised homotopy representations. Here we will find what we consider to be the strangest equivariant Poincaré space we know: a C_p–Poincaré space $\underline{X}$ such that X^{C_p} and X^e are Poincaré of the same dimension, yet the map $X^{C_p} \to X^e$ is not an equivalence, see Example 8.4.10.

A general principle here is that Theorem 8.2.10 provides us with a clear strategy to deduce equivariant Poincaré duality from nonequivariant Poincaré duality of fixed points, provided an appropriate Spivak datum has been constructed.

Smooth G-manifolds

Let G be a compact Lie group. A *smooth G-manifold* is a smooth manifold on which G acts such that the action map $G \times M \to M$ is a smooth map. An *equivariant embedding* of smooth G-manifolds is a smooth embedding between smooth G-manifolds that is also equivariant. An *equivariant vector bundle* on M is a tuple $\xi = (E, p)$, where E is a smooth G–manifold and $p\colon E \to M$ is an equivariant map which is a vector bundle where G acts by bundle maps. For $x \in M^H$, the vector space $E_x := p^{-1}(x)$ carries an H-action by restriction. Smooth G-manifolds port nicely into our homotopical context by virtue of [Ill83, Cor. 7.2.] which guarantees that smooth G–manifolds admit the structure of G–CW complexes which is necessarily finite for compact manifolds. We recommend [Bre72, Chapter IV] for an introduction to the theory of smooth G-manifolds.

Fact 8.4.1. We collect here some basic facts from equivariant smooth manifold theory that we will need for our purposes.

(i) The tangent bundle of a smooth G-manifold can naturally be considered as an equivariant vector bundle [Bre72, p. 303]. If $f\colon M \to N$ is an equivariant embedding of smooth G-manifolds, then the equivariant tubular neighborhood theorem provides a smooth equivariant embedding of $\nu(f) = f^*TN/TM$ into N [Bre72, Thm. VI.2.2.].

(ii) Let us denote the underlying G-homotopy type of M by $\underline{M}$. Any G-vector bundle $p\colon E \to M$ over M defines a stable equivariant spherical fibration of the G-vector bundle $p\colon E \to M$. Furthermore, we can choose a G–invariant Riemannian metric for p from which we obtain an associated unit disc bundle $D(p) \subset E$ and unit sphere bundle $S(p) \subset E$. The fibrewise collapse maps $\underline{S^{E_x}} \to \mathrm{cofib}(\underline{S(p)}_x \to \underline{D(p)}_x)$ for each $x \in M$ then assemble into a G-equivalence
$$M_!(J(p)) \xrightarrow{\simeq} \Sigma^\infty \mathrm{cofib}\left[S(p) \to D(p)\right]$$
where $J(p)$ is the stable spherical fibration associated to the bundle p given by fibrewise open–point compactification and stabilisiation.

(iii) For each G-manifold M, there exists an equivariant embedding into some G-representation V. This is the content of the Mostow-Palais theorem, see [Pal57].

Proposition 8.4.2. *Let M be a closed smooth G–manifold. Then the underlying G–space $\underline{M}$ is a G–Poincaré space with dualizing object $J(TM)^{-1}$.*

Proof. Choose an embedding $f\colon M \to V$ into some G-representation. Denote the normal bundle of f by $\nu = (p\colon E \to M)$ and pick a tubular neighborhood of M in V.

Consider the Pontryagin-Thom collapse map

$$c\colon \mathbb{S}_G \xrightarrow{\simeq} \mathbb{S}^V \otimes \mathbb{S}^{-V} \to \Sigma^\infty \operatorname{cofib}\left[S^V \setminus (D(\nu) \setminus S(\nu)) \to S^V\right] \otimes \mathbb{S}^{-V}$$
$$\simeq \Sigma^\infty \operatorname{cofib}\left[S(\nu) \to D(\nu)\right] \otimes \mathbb{S}^{-V} \simeq M_!(J(\nu) \otimes \mathbb{S}^{-V}).$$

We claim that the Spivak datum $(J(\nu) \otimes \mathbb{S}^{-V}, c)$ is Poincaré. Since M is a G–compact space and $J(\nu)$ is invertible, by Theorem 8.2.10, it suffices to check that for every $H \subset G$, the Spivak datum $(\Phi^H J(\nu), \Phi^H c)$ is a Poincaré Spivak datum for M^H. Recall that $\Phi^H J(\nu)$ is

$$\Phi^H J(\nu)\colon M^H \to \mathcal{P}\mathrm{ic}(\mathrm{Sp}), \quad x \mapsto \Phi^H(J(\nu)(x)) = \Phi^H \Sigma^\infty S^{E_x} \simeq \Sigma^\infty S^{E_x^H}.$$

But this is just the underlying stable spherical fibration of the normal bundle of M^H in V^H. The collapse map $\Phi^H c$ identifies with the geometric Pontryagin-Thom collapse map of the smooth manifold M^H embedded in V^H. Thus, by [Lan22, Cor. A.11] the Spivak datum $(\Phi^H J(\nu), \Phi^H c)$ is Poincaré.

Now note that the equivalence $\mathrm{const}_V = TV|_M \simeq \nu \oplus TM$ shows that

$$J(\nu) \otimes \mathbb{S}^{-V} \simeq J(\nu) \otimes J(\mathrm{const}_V)^{-1} \simeq J(\mathrm{const}_V) \otimes J(TM)^{-1} \otimes J(\mathrm{const}_V)^{-1} \simeq J(TM)^{-1}$$

as claimed. □

Remark 8.4.3. We want to mention that versions of Proposition 8.4.2 are already contained in the literature so we do not claim any originality. In particular, May–Sigurdsson give an account of equivariant Poincaré duality and show that closed smooth G-manifolds satisfy Poincaré duality in their sense [MS06, Chapter 18.6.]. Depending on which proof of the Wirthmüller isomorphism the reader has in mind, the reader might complain that the proof of Proposition 8.4.2 is circular, as the Wirthmüller isomorphism for compact Lie groups itself was proved by showing that smooth G-manifolds are G-Poincaré. Another variant of Proposition 8.4.2 was given by Costenoble–Waner, see [CW17].

Generalised homotopy representations

We now turn our attention to another interesting source of equivariant Poincaré duality spaces, namely the class of generalised homotopy representations of tom Dieck–Petrie [DP82].

Definition 8.4.4. A *generalised homotopy representation* of a compact Lie group G is a compact G–space $\underline{\mathcal{V}}$ such that for each closed subgroup $H \leq G$ the space $\mathcal{V}^H$ is equivalent to $S^{n(H)}$ for some $n(H) \in \mathbb{N}$. The function $H \mapsto n(H)$ associated to a generalised homotopy representation is called its *dimension function*.[2]

[2] Beware that it is also common in the literature to shift the dimension function by one.

Examples of generalised homotopy representations are unit spheres of finite dimensional orthogonal G-representations or one–point compactifications of finite dimensional linear G-representations.

Remark 8.4.5. While it will not play a role in this article, let us mention that [DP82; Die86] have also studied what are called *homotopy representations*, namely generalised homotopy representations for which the fixed points have CW–dimensions those of the respective spheres. A special feature of homotopy representations is that they satisfy an equivariant Hopf degree theorem, i.e. G-homotopy classes of self maps are classified by their degree, an element in a Burnside ring.

To show that generalised G-homotopy representations are indeed G–Poincaré, we first recall a construction of a Poincaré Spivak datum for the nonequivariant spheres.

Observation 8.4.6 (Spivak data for spheres)**.** We construct a Spivak datum for S^d. Let $E := \mathrm{fib}(\Sigma^\infty_+ S^d \to \Sigma^\infty_+ * \simeq \mathbb{S})$. Then $E \simeq \mathbb{S}^d \in \mathcal{P}\mathrm{ic}(\mathrm{Sp})$. Consider the composition

$$c\colon \mathbb{S} \xrightarrow{\simeq} E \otimes E^\vee \to \Sigma^\infty_+ S^d \otimes E^\vee \simeq S^d_!(S^d)^* E^\vee.$$

We argue now that $((S^d)^* E^\vee, c)$ is a Poincaré Spivak datum for S^d. As S^d is stably parallelisable, we know that its dualising sheaf is constant with value $\mathbb{S}^{-d} \simeq E^\vee$. Assume $d \geq 1$, the case $d = 0$ being easier. Now $\pi_0 S^d_!(S^d)^* E^\vee \simeq \mathbb{Z}$, and the element $c \in \pi_0 S^d_!(S^d)^* E^\vee \simeq \mathbb{Z}$ gives the collapse map of a Poincaré Spivak datum if and only if it corresponds to a generator. This is indeed the case for $((S^d)^* E, c)$, so it is Poincaré as claimed.

Having this in mind, we can make an educated guess for the a Spivak datum of a generalised homotopy representation. To this end, the following terminology will be useful.

Definition 8.4.7. A *homotopical framing* for $\xi \in \underline{\mathrm{Fun}}(\underline{X}, \underline{\mathrm{Sp}})$ is a G–spectrum E together with an equivalence $\xi \xrightarrow{\simeq} X^* E$. A compact G–space $\underline{X}$ is *homotopically parallelisable* if its dualising sheaf $D_{\underline{X}} \in \underline{\mathrm{Fun}}(\underline{X}, \underline{\mathrm{Sp}})$ admits a homotopical framing.

Theorem 8.4.8. *The dualising sheaf of a generalised homotopy representation $\underline{\mathcal{V}}$ admits a canonical homotopical framing $D_{\underline{\mathcal{V}}} \xrightarrow{\simeq} \mathcal{V}^* \,\mathrm{fib}(\Sigma^\infty_+ \mathcal{V}^\vee \to \Sigma^\infty_+ * \simeq \mathbb{S}_G)^\vee$. In particular, generalised homotopy spheres are homotopically parallelisable G–Poincaré spaces.*

Proof. To prove the theorem, we will construct a Poincaré Spivak datum whose underlying parametrised spectrum is constant with value

$$E^\vee := \mathrm{fib}(\Sigma^\infty_+ \mathcal{V}^\vee \to \Sigma^\infty_+ * \simeq \mathbb{S}_G)^\vee.$$

As in Observation 8.4.6, we have a map

$$c\colon \mathbb{S}_G \to E \otimes E^\vee \to \Sigma^\infty_+ \mathcal{V} \otimes E^\vee \simeq \mathcal{V}_! \mathcal{V}^* E^\vee.$$

Upon taking geometric fixed points, Observation 8.4.6 identifies the composition

$$\Phi^H c\colon \Phi^H \mathbb{S}_G \to \Phi^H E \otimes \Phi^H E^\vee \to \Phi^H \Sigma^\infty_+ \underline{\mathcal{V}} \otimes \Phi^H E^\vee \simeq \mathcal{V}^H_!(\mathcal{V}^H)^* \Phi^H E^\vee.$$

as a Poincaré Spivak datum for $\mathcal{V}^H$. Thus, by Theorem 8.2.10, we get that $\mathcal{V}^* E^\vee$ is a Poincaré G–Spivak datum for $\underline{\mathcal{V}}$ and by Proposition 7.2.6 we get $D_{\underline{\mathcal{V}}} \simeq \mathcal{V}^* E^\vee$ as claimed. □

Lemma 8.4.9. *Suppose that $X \in \mathcal{S}^\omega_G$ is homotopically parallelisable and that X^H is a Poincaré space for all $H \leq G$. Then X is a G–Poincaré space.*

Proof. Since $\underline{X}$ was compact, note that $X_! D_{\underline{X}} \simeq X_* X^* \mathbb{S}_G$ is a compact G–spectrum. Now suppose that there is $E \in \mathrm{Sp}_G$ such that $D_{\underline{X}} \simeq X^* E$. As E is a retract of $X_! D_{\underline{X}} \simeq \Sigma^\infty_+ X \otimes E$ this implies that E is compact itself. If all fixed points of X are Poincaré spaces, then all geometric fixed points of E are invertible. Together this shows that E is invertible so that X is a G–Poincaré space. □

Example 8.4.10. In [Bre72, p. 391], Bredon constructs a curious example of a generalised homotopy representation. Namely, he constructs examples of compact C_p-spaces $\underline{X}$ which satisfy that $X^{C_p} \simeq X^e \simeq S^2$ such that the map $X^{C_p} \to X^e$ has degree $q = kp + 1$ for $k \in \mathbb{Z}$ arbitrary. Taking the unreduced suspension, examples of this type exist in arbitrary dimensions. From Smith theory we know that each generalised C_p-homotopy representation has the property that the dimension of the fixed point sphere does not exceed the dimension of the underlying space.

8.5 Gluing classes

Our next goal is to hint at nontrivial ways in which the fixed points interact. For this, we construct a certain homology class, the *gluing class*, that should be thought of as passing information between the fundamental class of a Poincaré space and fundamental classes of various fixed point spaces. The gluing class will be one of the main tools for our geometric applications. It is inspired by Lück's work on the Nielsen realisation problem, specifically by [Lüc22, Notation 1.8 (H) and Lemma 1.8 (5)]. Much of what we will present here will work for compact Lie groups too, but we nevertheless restrict our attention to finite groups G for this subsection which is sufficient for our geometric purposes later.

Construction 8.5.1 (Nonsingular part)**.** Fix $\underline{X} \in \mathcal{S}_G$, $\mathcal{F}$ a family of subgroups of G, and $\underline{\mathcal{C}}$ a G–stable category. Recall the adjunction counit $\epsilon\colon \underline{X}_{\widetilde{\mathcal{F}}} \to \underline{X}$ from Construction 6.2.15. This map then itself induces the adjunction $\epsilon_!\colon \underline{\mathrm{Fun}}(\underline{X}_{\widetilde{\mathcal{F}}}, \underline{\mathcal{C}}) \rightleftharpoons \underline{\mathrm{Fun}}(\underline{X}, \underline{\mathcal{C}}) : \epsilon^*$. The adjunction (co)unit of *this* adjunction then gives us functors

$$\underline{\mathrm{Fun}}(\underline{X}, \underline{\mathcal{C}}) \xrightarrow{c} \underline{\mathrm{Fun}}(\underline{X}, \underline{\mathcal{C}})^{\Delta^1} :: \xi \mapsto (\epsilon_! \epsilon^* \xi \to \xi),$$
$$\underline{\mathrm{Fun}}(\underline{X}, \underline{\mathcal{C}}) \xrightarrow{u} \underline{\mathrm{Fun}}(\underline{X}, \underline{\mathcal{C}})^{\Delta^1} :: \xi \mapsto (\xi \to \epsilon_* \epsilon^* \xi).$$

All in all, we can consider the compositions

$$\alpha\colon \underline{\mathrm{Fun}}(\underline{X},\underline{\mathcal{C}}) \xrightarrow{c} \underline{\mathrm{Fun}}(\underline{X},\underline{\mathcal{C}})^{\Delta^1} \xrightarrow{X_!} \underline{\mathcal{C}}^{\Delta^1} \xrightarrow{\mathrm{cofib}} \underline{\mathcal{C}},$$
$$\beta\colon \underline{\mathrm{Fun}}(\underline{X},\underline{\mathcal{C}}) \xrightarrow{u} \underline{\mathrm{Fun}}(\underline{X},\underline{\mathcal{C}})^{\Delta^1} \xrightarrow{X_*} \underline{\mathcal{C}}^{\Delta^1} \xrightarrow{\mathrm{fib}} \underline{\mathcal{C}}.$$

Concretely, these take ξ to the objects

$$\alpha(\xi) \simeq \mathrm{cofib}\left((X_{\widetilde{\mathcal{F}}})_!\epsilon^*\xi \longrightarrow X_!\xi\right), \qquad \beta(\xi) \simeq \mathrm{fib}\left(X_*\xi \longrightarrow (X_{\widetilde{\mathcal{F}}})_*\epsilon^*\xi\right).$$

Corollary 8.5.2 (Nonsingular vanishing)**.** *Let $\underline{X} \in \mathcal{S}_G^{\omega}$ and $\mathcal{F}$ a family of subgroups of G. Let $\nu\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ be a G–exact functor of G–stable categories such that for all $H \in \mathcal{F}$, functor $\mathrm{Res}^G_H\,\nu\colon\ \mathrm{Res}^G_H\,\underline{\mathcal{C}} \to \mathrm{Res}^G_H\,\underline{\mathcal{D}}$ is the zero map. Then the compositions*

$$\underline{\mathrm{Fun}}(\underline{X},\underline{\mathcal{C}}) \xrightarrow{\alpha} \underline{\mathcal{C}} \xrightarrow{\nu} \underline{\mathcal{D}} \qquad \underline{\mathrm{Fun}}(\underline{X},\underline{\mathcal{C}}) \xrightarrow{\beta} \underline{\mathcal{C}} \xrightarrow{\nu} \underline{\mathcal{D}}$$

have the property of being the zero functors.

Proof. First of all, since ν was G–exact, we have a commuting square

$$\begin{tikzcd} \underline{\mathrm{Fun}}(\underline{X},\underline{\mathcal{C}}) \arrow[r, "\nu"] \arrow[d, "\alpha"'] & \underline{\mathrm{Fun}}(\underline{X},\underline{\mathcal{D}}) \arrow[d, "\alpha"] \\ \underline{\mathcal{C}} \arrow[r, "\nu"] & \underline{\mathcal{D}}. \end{tikzcd}$$

Thus it suffices to show that $\alpha\colon \underline{\mathrm{Fun}}(\underline{X},\underline{\mathcal{D}}) \to \underline{\mathcal{D}}$ is the zero functor. By replacing $\underline{\mathcal{D}}$ by the G-stable subcategory generated by the image of ν we can assume that that $\mathcal{D}^H = 0$ for all $H \in \mathcal{F}$. Therefore, we have that $\underline{\mathcal{D}} \simeq \underline{\mathcal{D}}^{\Phi\widetilde{\mathcal{F}}}$ and Lemma 8.2.3 shows that the functor ϵ^* is an equivalence, and so the counit $\epsilon_!\epsilon^* \to \mathrm{id}$ and unit $\mathrm{id} \to \epsilon_*\epsilon^*$ are equivalences in $\underline{\mathrm{Fun}}(\underline{X},\underline{\mathcal{D}}) = \underline{\mathrm{Fun}}(\underline{X},\underline{\mathcal{D}}^{\Phi\widetilde{\mathcal{F}}})$. From this the claim directly follows. □

Notation 8.5.3. The family of relevance to us in this subsection will be the singleton family $\mathcal{T}$ consisting of the trivial subgroup. To reduce our notational cluttering, we will also write $\underline{X}^{>1}$ for $\underline{X}_{\widetilde{\mathcal{T}}}$, so that for example, for $\underline{X} \in \mathcal{S}_G$, we have the inclusion of the singular part $\epsilon\colon \underline{X}^{>1} \simeq \underline{X}_{\widetilde{\mathcal{T}}} \to \underline{X}$. The gluing class of $\underline{X}$ will live in $\pi_{-1}(X_!^{>1}\epsilon^* D_{\underline{X}})_{hG}$.

Construction 8.5.4. Let $\xi \in \mathrm{Fun}_G(\underline{X}, \underline{\mathrm{Sp}})$ and write $Q := \mathrm{cofib}\left(X_!^{>1}\epsilon^*\xi \to X_!\xi\right)$. Consider

$$\begin{tikzcd}
(X_!^{>1}\epsilon^*\xi)_{hG} \arrow[r] \arrow[d] & (X_!^{>1}\epsilon^*\xi)^{hG} \arrow[r] \arrow[d] & (X_!^{>1}\epsilon^*\xi)^{tG} \arrow[r, blue] \arrow[d, blue, "\simeq"] & \Sigma(X_!^{>1}\epsilon^*\xi)_{hG} \\
(X_!\xi)_{hG} \arrow[r] \arrow[d] \arrow[ur, phantom, "\ulcorner"] & (X_!\xi)^{hG} \arrow[r, blue] \arrow[d, red] & (X_!\xi)^{tG} & \\
Q_{hG} \arrow[r, red, "\simeq"] \arrow[d, red] & Q^{hG} & & \\
\Sigma(X_!^{>1}\epsilon^*\xi)_{hG} & & &
\end{tikzcd} \tag{8.4}$$

where the equivalence $Q_{hG} \to Q^{hG}$ is since $Q^{tG} \simeq 0$ by virtue of Corollary 8.5.2 applied to the functor $\nu \colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ given by $\widetilde{EG} \otimes F(EG_+, -) \colon \underline{\mathrm{Sp}} \to \underline{\mathrm{Mod}}_{\widetilde{EG} \otimes F(EG_+, \mathbb{S})}(\underline{\mathrm{Sp}})$ and the identification $(\widetilde{EG} \otimes F(EG_+, A))^G \simeq A^{tG}$. Observe that by Lemma B.1, up to a sign change, the red composite is equivalent to the blue composite in (8.4).

Construction 8.5.5 (Gluing classes)**.** Let $\underline{X} \in \mathcal{S}_G^{\omega}$ and $D_{\underline{X}} \in \mathrm{Fun}_G(\underline{X}, \underline{\mathrm{Sp}})$ its dualising sheaf (which in this generality, need not be invertible). From the fundamental class $\mathbb{S}_G \xrightarrow{c} X_! D_{\underline{X}}$ in Sp_G, we may extract a nonequivariant fundamental class $\mathbb{S} \xrightarrow{\mathrm{can}} \mathbb{S}_G^{hG} \xrightarrow{c^{hG}} (X_! D_{\underline{X}})^{hG}$ in Sp which we also denote by c. The *gluing class* is defined to be the composition

$$\mathbb{S} \xrightarrow{c} (r_! D_{\underline{X}})^{hG} \longrightarrow \Sigma(X_!^{>1}\epsilon^* D_{\underline{X}})_{hG}$$

obtained by postcomposing c with the blue route from (8.4).

Our goal now is to show Corollary 8.5.7 which says that under certain orientability assumptions, the gluing class "adds up to zero" in group homology. This supplies us with a useful obstruction class which will have meaningful geometric consequences as we shall in our applications in §9.2.

Lemma 8.5.6. *Let $\underline{X} \in \mathcal{S}_G$ and $\xi \simeq X^*W \in \mathrm{Fun}_G(\underline{X}, \underline{\mathrm{Sp}})$ for some $W \in \mathrm{Sp}_G$. Then the composition in $Q \longrightarrow \Sigma r_!^{>1}\epsilon^*\xi \simeq \Sigma X_!^{>1}(X^{>1})^*\xi \xrightarrow{\mathrm{BC}_!^{X^{>1}}} \Sigma W$ in Sp_G is nullhomotopic.*

Proof. By functoriality of colimits, we have the following map of cofibre sequences

$$\begin{tikzcd}
X_!^{>1}\epsilon^* X^* W \arrow[r, "\mathrm{BC}_!^{\epsilon}"] \arrow[d, "\mathrm{BC}_!^{X^{>1}}"'] & X_! X^* W \arrow[r] \arrow[d, "\mathrm{BC}_!^{X}"] & Q \arrow[d] \\
W \arrow[r, equal] & W \arrow[r] & 0
\end{tikzcd}$$

Thus taking the cofibre of the right horizontal maps gives a factorisation of the composition of interest through 0. □

Corollary 8.5.7. *Let $\underline{X} \in \mathcal{S}_G$ and $W \in \mathrm{Sp}_G$. Then the composition*

$$(X_! X^* W)^{hG} \xrightarrow{\text{red composite in (8.4)}} \Sigma(X_!^{>1}(X^{>1})^* W)_{hG} \xrightarrow{\mathrm{BC}_!^{X^{>1}}} \Sigma W_{hG}$$

is nullhomotopic.

Proof. This is an immediate combination of the fact from Construction 8.5.4 that the red and blue routes in (8.4) agrees up to a sign with Corollary 8.5.7. □

Remark 8.5.8. The gluing class really is an essential feature of equivariant Poincaré duality. It provides some information on how the "free part" of an equivariant Poincaré space is glued to the singular part. We will exploit it in the proof of Theorem 9.2.2 and plan on clarifying it and its role in relation to Lück's work [Lüc22] in the future.

8.6 Equivariant degree theory

A nice application of equivariant Poincaré duality is a theory of equivariant mapping degrees, as developed in [Lüc88]. For simplicity, we will assume that G is a finite group throughout this section.

Recollections on the Burnside ring

Our aim is to remind the reader of the classical connection between Burnside rings and the equivariant sphere spectrum.

Recollection 8.6.1 (Character maps on the Burnside ring)**.** The *Burnside ring of finite G-sets* $A(G)$ is the group–completion of the semiring of isomorphism classes of finite G-sets, with disjoint union as addition and the cartesian product as product. For each $H \leq G$, there is a unique ring homomorphism $\chi_H \colon A(G) \to \mathbb{Z}$ sending a finite G-set S to the order of the finite set S^H, and these assemble into a ring map, called the *character map*,

$$\chi \colon A(G) \to \prod_{(H) \leq G} \mathbb{Z} \tag{8.5}$$

where (H) runs through all conjugacy classes of finite subgroups.

The following classical theorem may be found for instance in [Die79, Prop. 1.3.5.].

Theorem 8.6.2. *The character map (8.5) is an injective ring homomorphism with finite cokernel. The image can be described through explicit congruences, the Burnside congruences.*

We abstain from recalling the Burnside congruences in full generality, the reader may find them in the reference mentioned above. To give some intuition, we describe them in the case of the group C_p.

Example 8.6.3. If $G = C_p$ then the image of the character homomorphism consists of those pairs $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ that satisfy the congruence

$$a \equiv b \mod p.$$

Indeed, for a finite G-set S, the orders of S and S^H agree modulo p. On the other hand, if $a + kp = b$ for integers a, b, c, then a copies of the point and k copies of C_p define an element in $A(G)$ mapping to (a, b).

Construction 8.6.4. We may obtain a similar character map for the ring $\pi_0^G \mathbb{S}_G$: for each subgroup $H \leq G$, using that $\Phi^H \mathbb{S}_G \simeq \mathbb{S} \in \mathrm{Sp}$, we may assemble the geometric fixed points functors Φ^H together with the identification $\deg\colon \pi_0 \operatorname{Map}_{\mathrm{Sp}}(\mathbb{S}, \mathbb{S}) \xrightarrow{\cong} \mathbb{Z}$ to obtain a ring map

$$\pi_0^G \mathbb{S}_G \cong \pi_0 \operatorname{Map}_{\mathrm{Sp}_G}(\mathbb{S}_G, \mathbb{S}_G) \longrightarrow \prod_{(H)} \mathbb{Z}, \quad f \mapsto \deg(\Phi^H f) \tag{8.6}$$

Theorem 8.6.5 (Segal)**.** *Let G be a finite group. The map (8.6) is an injective ring homomorphism whose image agrees with the image of the character map $\chi\colon A(G) \to \prod_{(H)} \mathbb{Z}$, yielding an identification $\pi_0^G \mathbb{S}_G \cong \pi_0 \operatorname{Map}_{\mathrm{Sp}_G}(\mathbb{S}_G, \mathbb{S}_G) \cong A(G)$ as commutative rings.*

Of course, this implies that the set of path components of the selfmaps of any $E \in \mathcal{P}\mathrm{ic}(\mathrm{Sp}_G)$ is equivalent to $A(G)$ by the equivalence $\operatorname{Map}_{\mathrm{Sp}_G}(E, E) \simeq \operatorname{Map}_{\mathrm{Sp}_G}(\mathbb{S}_G, \mathbb{S}_G)$.

The equivariant degree

We now want to specialise the abstract definition of the degree from §7.4 to the case of maps of G–Poincaré spaces which should roughly encode the mapping degrees on the various fixed points spaces. Recall that the definition of the degree of a map between G–spaces $f\colon \underline{X} \to \underline{Y}$ with Spivak data $(\xi_{\underline{X}}, c_{\underline{X}})$ and $(\xi_{\underline{Y}}, c_{\underline{Y}})$ depends on an equivalence $\xi_{\underline{X}} \xrightarrow{\simeq} f^* \xi_{\underline{Y}}$. The existence of such an equivalence is unreasonable to expect with coefficients in Sp_G but becomes more likely after linearising. Here we choose to work with coefficients in the Burnside Mackey functor $\underline{A}(G)$. Recall that for each subgroup $H \subset G$, restriction defines a ring homomorphism $A(G) \to A(H)$ and induction a transfer map $A(H) \to A(G)$. These assemble into a Mackey functor, and hence a G–spectrum $\underline{A}(G)$ which has values $\underline{A}(G)^H = A(H)$.

Definition 8.6.6. Let $\underline{X}$ and $\underline{Y}$ be Poincaré G–spaces. An $\underline{A}(G)$*–degree datum* is a pair (f, ψ) where $f\colon \underline{X} \to \underline{Y}$ is a map of G–spaces and $\psi\colon D_{\underline{X}} \otimes \underline{A}(G) \xrightarrow{\simeq} f^* D_{\underline{Y}} \otimes \underline{A}(G)$ is an equivalence of the $\underline{A}(G)$–linearised dualising sheaves.

In other words, a $\underline{A}(G)$–degree datum is a $\underline{\mathrm{Mod}}_{\underline{A}(G)}(\underline{\mathrm{Sp}})$–degree datum in the sense of Definition 7.4.2.

Definition 8.6.7. Let $\underline{X}, \underline{Y} \in \mathcal{S}_G$ be G–Poincaré and (f, ψ) a $\underline{A}(G)$–degree datum. We define the *equivariant degree* $\deg_G(f, \psi) \in \pi_0 \mathrm{Map}(\mathbb{S}_G, Y_* Y^* \underline{A}(G)) =: H^0(\underline{Y}; \underline{A})$ as the composite

$$\mathbb{S}_G \xrightarrow{c_X} X_!(D_{\underline{X}} \otimes \underline{A}(G)) \xrightarrow{\psi} X_!(f^* D_{\underline{Y}} \otimes \underline{A}(G)) \xrightarrow{\mathrm{BC}_!^f} Y_!(D_{\underline{Y}} \otimes \underline{A}(G)) \xleftarrow[\simeq]{c_Y \cap -} Y_* Y^* \underline{A}(G).$$

As explained in Construction 7.4.4, the commutative algebra structure on $Y_* Y^* \underline{A}(G)$ endows $H^0(\underline{Y}; \underline{A})$ with the structure of a commutative ring with unit c_Y.

Our goal for the rest of this discussion is to relate the equivariant degree of a map, which lives in $H^0(\underline{X}; \underline{A})$, to the various degrees induced on fixed points via a character map similar to (8.5) constructed from the geometric fixed points functors. To this end, first recall the Bousfield localisation $\pi_0 \colon \mathcal{S}_G \rightleftharpoons \mathrm{Set}_G : \mathrm{incl}$ from Construction 9.2.10. Notice that $\Omega^\infty \underline{A}(G)$ is levelwise 0–truncated with fixed points $(\Omega^\infty \underline{A}(G))^H = A(H)$.

Lemma 8.6.8. *For $\underline{X} \in \mathcal{S}_G$, we have an equivalence $H^0(\underline{X}; \underline{A}) \simeq \pi_0 \mathrm{Map}_{\mathcal{S}_G}(\tau_{\leq 0} \underline{X}, \Omega^\infty \underline{A}(G))$.*

Proof. Consider the computation of $H^0(\underline{X}; \underline{A})$ as

$$\pi_0 \mathrm{Map}_{\mathrm{Sp}_G}(\mathbb{S}_G, X_* X^* \underline{A}(G)) \simeq \mathrm{Map}_{\mathrm{Sp}_G}(\Sigma^\infty_+ \underline{X}, \underline{A}(G)) \simeq \mathrm{Map}_{\mathcal{S}_G}(\tau_{\leq 0} \underline{X}, \Omega^\infty \underline{A}(G))$$

where the last equivalence uses that $\Omega^\infty \underline{A}(G)$ is levelwise 0-truncated. □

Remark 8.6.9. This turns out to be quite simple to compute. Note that for two 0-truncated G–spaces $\underline{S}$ and $\underline{T}$, the map

$$\mathrm{Map}_{\mathcal{S}_G}(\underline{S}, \underline{T}) \longrightarrow \prod_{(H) \leq G} \mathrm{Map}_{\mathcal{S}}(S^H, T^H) \simeq \mathrm{Map}_{\mathrm{Set}}(S^H, T^H),$$

is injective with image given by all collections of maps $(f^H)_{(H)}$ compatible with the restrictions coming from inclusions $K \leq H$ or inner automorphsim $K \simeq H$. Specialising this to the case of interest, we obtain an injection

$$H^0(\underline{X}; \underline{A}) \hookrightarrow \prod_{(H)} \left(A(H)^{\pi_0(X^H)} \right)^{W_G H}.$$

For example, we have $\mathrm{Map}(\tau_{\leq 0} \underline{X}, \Omega^\infty \underline{A}(G)) \simeq A(G)$ if all fixed point sets of $\underline{X}$ are nonempty and connected. For a more complicated example, consider the C_2-action on S^1 given by complex conjugation. Then $(\tau_{\leq 0} \underline{X})^{C_2} \simeq * \coprod *$ while $(\tau_{\leq 0} \underline{X})^e \simeq *$. The set of equivalence classes of maps above identifies with the pullback $A(C_2) \times_{A(1)} A(C_2)$ where the two maps $A(C_2) \to A(1)$ are given by restriction along the group homomorphism $1 \to C_2$.

Recovering degrees on fixed points

Now we want to recover different degrees on fixed point spaces from the equivariant degree by base changing along the geometric fixed points functor. As it is not true that $\Phi^G \underline{A}(G) = \mathbb{Z}$, we need a small preparatory lemma. For this, denote by $\mathrm{Sp}_G^{\geq 0}$ (resp. $\mathrm{Sp}_G^{\leq 0}$) the full subcategory of all G–spectra X for which $X^H \in \mathrm{Sp}$ is connective (resp. coconnective) for each $H \leq G$. The pair $(\mathrm{Sp}_G^{\geq 0}, \mathrm{Sp}_G^{\leq 0})$ forms a t-structure on on Sp_G.

Lemma 8.6.10 (Geometric fixed points of Mackey functors)**.** *Let $X \in \mathrm{Sp}_G^{\geq 0}$. Then the canonical map $X \to \tau_{\leq 0} X$ induces an isomorphism $\pi_0 \Phi^G X \xrightarrow{\cong} \pi_0 \Phi^G \tau_{\leq 0} X$ of abelian groups.*

Proof. Recall that the geometric fixed points participates in an adjunction $\Phi^G \colon \mathrm{Sp}_G \rightleftharpoons \mathrm{Sp} : \Xi^G$ where the right adjoint Ξ^G is fully faithful and is given by the formula

$$(\Xi^G Y)^H = \begin{cases} Y & \text{if } H = G; \\ 0 & \text{if } H \lneq G. \end{cases}$$

Observe that Φ^G preserves connective objects. This is because Φ^G sends $\Sigma^\infty_+ G/G$ to $\mathbb{S}$ and $\Sigma^\infty_+ G/H$ to 0 for $H \lneq G$. Since connective G–spectra are built as colimits of the orbits $\{\Sigma^\infty_+ G/H\}_{H \leq G}$ and Φ^G preserves colimits, we see that connective G–spectra are sent to connective spectra. The formula for Ξ^G shows that it preserves connective and coconnective objects. In particular, both restrict to functors $\Xi^G \colon \mathrm{Sp}^\heartsuit \hookrightarrow \mathrm{Sp}_G^\heartsuit$ and $\tau_{\leq 0}\Phi^G \colon \mathrm{Sp}_G^\heartsuit \to \mathrm{Sp}^\heartsuit$ and we claim that those are adjoint. To see this, let $\underline{M} \in \mathrm{Sp}_G^\heartsuit$ and $N \in \mathrm{Sp}^\heartsuit$, and consider

$$\mathrm{Map}_{\mathrm{Sp}^\heartsuit}(\tau_{\leq 0}\Phi^G \underline{M}, N) \simeq \mathrm{Map}_{\mathrm{Sp}}(\Phi^G \underline{M}, N) \simeq \mathrm{Map}_{\mathrm{Sp}_G}(\underline{M}, \Xi^G N) \simeq \mathrm{Map}_{\mathrm{Sp}_G^\heartsuit}(\underline{M}, \Xi^G N).$$

To conclude, since the solid square in

$$\begin{tikzcd} \mathrm{Sp}_G^{\geq 0} \arrow[r, dashed, shift left, "\Phi^G"] \arrow[d, dashed, shift right, "\tau_{\leq 0}"'] & \mathrm{Sp}^{\geq 0} \arrow[l, hook, shift left, "\Xi^G"] \arrow[d, dashed, shift right, "\tau_{\leq 0}"'] \\ \mathrm{Sp}_G^\heartsuit \arrow[r, dashed, shift left, "\tau_{\leq 0}\Phi^G"] \arrow[u, hook, shift right] & \mathrm{Sp}^\heartsuit \arrow[l, hook, shift left, "\Xi^G"] \arrow[u, hook, shift right] \end{tikzcd}$$

commutes, so does the dashed square of left adjoints, as was to be shown. □

Remark 8.6.11. Theorem 8.6.5 gives an equivalence $\tau_{\leq 0}\mathbb{S}_G = \underline{A}(G)$. By Lemma 8.6.10, we have $\pi_0 \Phi^G \underline{A}(G) = \pi_0 \Phi^G \tau_{\leq 0} \mathbb{S}_G \cong \pi_0 \Phi^G \mathbb{S}_G \cong \pi_0 \mathbb{S} \cong \mathbb{Z}$. Now note that the diagram

$$
\begin{tikzcd}
\pi_0(\mathbb{S}_G)^G \arrow[r, "\chi^H"] \arrow[d, "\cong"] & \pi_0\Phi^G\mathbb{S}_G \arrow[r, "\simeq"] \arrow[d, "\simeq"] & \pi_0\tau_{\leq 0}\Phi^G\mathbb{S}_G \simeq \mathbb{Z} \arrow[d, "\simeq"] \\
\pi_0\underline{A}(G)^G \arrow[r] & \pi_0\Phi^G\underline{A}(G) \arrow[r] & \pi_0\tau_{\leq 0}\Phi^G\underline{A}(G) \simeq \mathbb{Z}
\end{tikzcd}
\tag{8.7}
$$

commutes. The lower horizontal composition thus agrees with the character map.

Now we come back to the problem of relating the equivariant degree to the degree on each fixed point space. We have the symmetric monoidal colimit preserving functor

$$
\begin{aligned}
\phi^H\colon \underline{\mathrm{Mod}}_{\underline{A}(G)}(\underline{\mathrm{Sp}}_G) &\xrightarrow{\Phi^H} \mathrm{Coind}_H^G \mathrm{Coinfl}_H^1 \mathrm{Mod}_{\Phi^H\underline{A}(H)}(\mathrm{Sp}) \\
&\to \mathrm{Coind}_H^G \mathrm{Coinfl}_H^1 \mathrm{Mod}_{\mathbb{Z}}(\mathrm{Sp})
\end{aligned}
$$

where Φ^H is the parametrised geometric fixed point functor constructed in Construction 6.2.31 and the second map is induced by the ring map $\Phi^H\underline{A}(H) \to \tau_{\leq 0}\Phi^H\underline{A}(H) \simeq \mathbb{Z}$.

Proposition 8.6.12. *For a G–Poincaré $\underline{Y} \in \mathcal{S}_G$, basechange along ϕ^H induces a ring map*

$$\chi^H\colon H^0(\underline{Y};\underline{A}) \longrightarrow H^0(Y^H,\mathbb{Z}).$$

Given a degree datum $(f\colon \underline{X} \to \underline{Y}, \psi)$, we have

$$\chi^H(\deg_{\underline{A}(G)}(f,\psi)) = \deg_{\mathbb{Z}}(f^H\colon X^H \to Y^H, \psi^H).$$

For $\underline{Y} = \underline{*}$, χ^H agrees with the character map from (8.5).

Proof. Note that we have identifications

$$
\begin{aligned}
\mathrm{Fun}_G(\underline{Y}, \mathrm{Coind}_H^G \mathrm{Coinfl}_H^1 \mathrm{Mod}_{\mathbb{Z}}(\mathrm{Sp})) &\simeq \mathrm{Fun}_H(\mathrm{Res}_H^G \underline{Y}, \mathrm{Coinfl}_H^1 \mathrm{Mod}_{\mathbb{Z}}(\mathrm{Sp})) \\
&\simeq \mathrm{Fun}(Y^H, \mathrm{Mod}_{\mathbb{Z}}(\mathrm{Sp})).
\end{aligned}
$$

Recall from Observation 8.2.2 that this identifies $Y_!$ with $Y_!^H$ (and also Y_* with Y_*^H as $\underline{Y}$ is Poincaré). Now applying Lemma 7.4.8 to basechange along ϕ^H, we obtain a ring map

$$H^0(\underline{Y};\underline{A}) = \pi_0 \mathrm{Map}_{\mathrm{Mod}_{\underline{A}(G)}(\mathrm{Sp}_G)}(\mathbb{1}, Y_*Y^*\mathbb{1}) \to \pi_0 \mathrm{Map}_{\mathrm{Mod}_{\mathbb{Z}}(\mathrm{Sp})}(\mathbb{1}, Y_*^H Y^{H*}\mathbb{1}) \simeq H^0(Y^H,\mathbb{Z}).$$

The statment about the degrees follows from Lemma 7.4.8. In the case $\underline{Y} = \underline{*}$, this map identifies with the character map by (8.7). $\square$

In the next corollary, we unravel Proposition 8.6.12 in a special case to illustrate how it can be used to deduce condruences between (nonequivariant) degrees between fixed point sets.

Corollary 8.6.13 (Congruences between degrees on fixed point sets)**.** *Suppose that $\underline{Y}$ is a G–Poincaré space and assume that Y^H is nonempty connected for all $H \leq G$. Given a degree datum $(f\colon \underline{X} \to \underline{Y}, \psi)$, the collection $(\deg_{\mathbb{Z}}(f^H, \psi^H))_{(H)}$ lies in the image of the character map*

$$\chi\colon A(G) \to \prod_{(H)} \mathbb{Z}.$$

Proof. Any map $f\colon \underline{X} \to \underline{Y}$ of G–Poincaré spaces induces a commutative diagram

$$\begin{tikzcd}
\pi_0 \operatorname{Map}_{\operatorname{Mod}_{\underline{A}(G)}(\mathrm{Sp}_G)}(\mathbb{1}, Y_* Y^* \mathbb{1}) \arrow[r, "\chi^H"] \arrow[d, "\mathrm{BC}_*"] & \pi_0 \operatorname{Map}_{\operatorname{Mod}_{\mathbb{Z}}(\mathrm{Sp})}(\mathbb{1}, Y^H_* Y^{H*} \mathbb{1}) \arrow[d, "\mathrm{BC}_*"] \\
\pi_0 \operatorname{Map}_{\operatorname{Mod}_{\underline{A}(G)}(\mathrm{Sp}_G)}(\mathbb{1}, X_* X^* \mathbb{1}) \arrow[r, "\chi^H"] & \pi_0 \operatorname{Map}_{\operatorname{Mod}_{\mathbb{Z}}(\mathrm{Sp})}(\mathbb{1}, X^H_* X^{H*} \mathbb{1}).
\end{tikzcd}$$

Applying this to the unique map $\underline{Y} \to \underline{*}$, the vertical maps become equivalences by the assumption on the fixed points of $\underline{Y}$. By Proposition 8.6.12, this identifies $\chi^H\colon H^0(\underline{Y}; \underline{A}) \to \mathbb{Z}$ with the character map $\chi^H\colon A(G) = H^0(\underline{G/G}; \underline{A}) \to \mathbb{Z}$. The statement about the degrees is now a consequence of Proposition 8.6.12. $\square$

Chapter 9

Equivariant Poincaré duality: applications

In this section, we employ the general theory developed in the article to investigate some problems of an equivariant geometric topological nature. In §9.1, we study cohomological injectivity statements for degree one maps and prove Theorem 9.1.1 along these lines; we then use it to obtain a rigidity result of equivariant Poincaré spaces in Theorem 9.1.14. Next, in §9.2, we prove the equivariant Poincaré generalisation of Atiyah–Bott and Conner–Floyd's theorem on single fixed points for group actions on smooth manifolds.

9.1 Pulling back twisted fixed points

Let $f\colon M \to N$ be a map closed, connected, oriented manifolds of the same dimension d. If the degree of f is nonzero, then f is surjective. The theory of equivariant degrees immediately gives an equivariant application: if $f\colon M \to N$ is a map of closed, conneced, smooth, oriented C_p-manifolds with connected fixed point sets of the same dimension, then if f is of degree coprime to p (when considered as a nonequivariant map), then also the degree of f^{C_p} is coprime to p. Thus, f^{C_p} is surjective as well.

To detect if the degree of f is coprime to p it of course suffices to check that $H^*(f^{C_p};\mathbb{F}_p)$ is nonzero in the top degree. This line of thought led Browder [Bro87] to interpret results about the injectivity of $H^*(f^{C_p};\mathbb{F}_p)$ as the "ability to pull back fixed points from N to fixed points of M". Browder's strategy is very successful to show actual surjectivity results on fixed points, even if one relaxes the conditions like smoothness, or takes G to be a more general group like an abelian p-group. Let us mention [HP06] for more information, and many interesting variations on this approach. In particular, see [HP06, Thm. 4] to see how to pass from cohomological injectivity results to surjectivity on fixed points. We will content ourselves with showing how our methods can be used to derive cohomological injectivity results of the following type, which was first studied by Bredon [Bre73] and generalised by Browder [Bro87] under stronger manifold assumptions:

Theorem 9.1.1 (Twisted Bredon–Browder injection). *Let A be an elementary abelian p–group $C_p^{\times r}$. Let $f\colon \underline{X} \to \underline{Y}$ be a map of compact A–spaces. Suppose X^e, Y^e are $\mathrm{Mod}_{\mathrm{H}\mathbb{F}_p}$–Poincaré spaces such that $f^e\colon X^e \to Y^e$ is of $\mathrm{Mod}_{\mathrm{H}\mathbb{F}_p}$–degree one (c.f. Definition 7.4.2). Then for any $\zeta \in \mathrm{Fun}(Y^A, \mathrm{Perf}_{\mathrm{H}\mathbb{F}_p})$, the map induces an injection $H^*(Y^A;\zeta) \to H^*(X^A; f^*\zeta)$.*

Our approach is by default homotopical, and point-set techniques are avoided. As our systematic treatment of Poincaré duality allows us to also derive consequences for homology with twisted coefficients, orientability assumptions may even be relaxed. We will also illustrate the usefulness of Browder's cohomological injectivity results to prove a structural result for G-Poincaré spaces where G is a solvable finite group as Theorem 9.1.14: if X^e is contractible, then so is X^H for any $H \leq G$. This in turn can be applied to give an example of a compact C_p-space all of whose fixed points are Poincaré spaces, while itself not being C_p-Poincaré.

The basic philosophy of our proof is similar to that of [HP06], namely, we proceed via equivariant localisations using the *proper Tate construction*. To this end, we first show that the fixed points of an equivariant space which is underlying Poincaré may most naturally be viewed as a Poincaré space with coefficients in the *stable module category*, which we now recall.

Construction 9.1.2 (Proper stable module categories). Let $R \in \mathrm{CAlg}(\mathrm{Sp})$ and G be a finite group. Consider the G–stable category $\underline{\mathrm{Bor}}(\mathrm{Perf}_R) \in \mathrm{Cat}_G^{G-\mathrm{st}}$ with value $\mathrm{Fun}(BH, \mathrm{Perf}_R)$ at G/H. Recall the notion and notations of Brauer quotients from §6.2. Using the family of proper subgroups $\mathcal{P}$ of G, we may construct a new G–stable category $\underline{\mathrm{stmod}}^{\mathcal{P}}(R) \in \mathrm{Cat}_G^{G-\mathrm{st}}$ defined as $s_*\widetilde{s}^*\underline{\mathrm{Bor}}(\mathrm{Perf})$. This G–category has value

$$\mathrm{stmod}_G^{\mathcal{P}}(R) := \mathrm{Fun}(BG, \mathrm{Perf}_R)/\langle R[G/H] \mid H \lneq G\rangle$$

at G/G and is trivial elsewhere. Furthermore, there exists a G–exact symmetric monoidal functor $\Phi\colon \underline{\mathrm{Bor}}(\mathrm{Perf}_R) \longrightarrow \underline{\mathrm{stmod}}^{\mathcal{P}}(R)$.

Construction 9.1.3 (Descending Poincaré duality from large to small coefficients). Let $\underline{X} \in \mathcal{S}_G^{\omega}$ such that X^e is an R–Poincaré space. Since $\underline{X}$ was a compact G–space, the adjunctions $X_! \dashv X^* \dashv X*\colon \underline{\mathrm{Fun}}(\underline{X}, \underline{\mathrm{Bor}}(\mathrm{Mod}_R)) \rightleftharpoons \underline{\mathrm{Bor}}(\mathrm{Mod}_R)$ restrict to adjunctions $X_! \dashv X^* \dashv X_*\colon \underline{\mathrm{Fun}}(\underline{X}, \underline{\mathrm{Bor}}(\mathrm{Perf}_R)) \rightleftharpoons \underline{\mathrm{Bor}}(\mathrm{Perf}_R)$ on the full subcategories. Now by Proposition 8.3.6, we know that $\underline{X}$ is $\underline{\mathrm{Bor}}(\mathrm{Mod}_R)$–Poincaré and we write $D_{\underline{X}} \in \underline{\mathrm{Fun}}(\underline{X}, \underline{\mathrm{Bor}}(\mathcal{P}\mathrm{ic}_R))$ for the dualising sheaf. Since $D_{\underline{X}} \in \underline{\mathrm{Bor}}(\mathrm{Perf}_R)$, we even obtain an equivalence

$$X_*(-) \simeq X_!(D_{\underline{X}} \otimes -)\colon \underline{\mathrm{Fun}}(\underline{X}, \underline{\mathrm{Bor}}(\mathrm{Perf}_R)) \longrightarrow \underline{\mathrm{Bor}}(\mathrm{Perf}_R)$$

and so $\underline{X}$ is also $\underline{\mathrm{Bor}}(\mathrm{Perf}_R)$–Poincaré. We write $D_X^G\colon X^G \to \mathrm{Perf}_R^{BG}$ for the dualising sheaf evaluated at the fixed points.

Via this construction, we may now prove the following as a simple consequence.

Proposition 9.1.4 (Proper stable module Poincaré duality)**.** *Let G be a finite group and $R \in \mathrm{CAlg}(\mathrm{Sp})$. If $\underline{X} \in \mathcal{S}_G^{\omega}$ such that the underlying space X^e is an R–Poincare duality space, then X^G is a partial $\mathrm{stmod}_G^{\mathcal{P}}(R)$–Poincare duality space, i.e. for any $\zeta \in \mathrm{Fun}(X^G, \mathrm{Perf}_R^{BG})$, we have an equivalence in $\mathrm{stmod}_G^{\mathcal{P}}(R)$*

$$\Phi c \cap_{\Phi D_X^G} \Phi\zeta\colon X_*^G(\Phi\zeta) \xrightarrow{\simeq} X_!^G(\Phi D_X^G \otimes \Phi\zeta)$$

Proof. By Construction 9.1.3, we know that $\underline{X}$ is $\underline{\mathrm{Bor}}(\mathrm{Perf}_R)$–Poincaré duality. The statement of the proposition is now an immediate consequence of Theorem 8.2.7 (2). □

Remark 9.1.5. While the proposition above looks restrictive and artificial, it already contains some interesting content since the map $\Phi\colon \mathrm{Fun}_G(\underline{X}, \underline{\mathrm{Bor}}(\mathrm{Perf}_R)) \to \mathrm{Fun}(X^G, \mathrm{stmod}_G^{\mathcal{P}}(R))$ is symmetric monoidal. In particular, it holds when ζ is the tensor unit $\mathbb{1}$. This will then recover the usual untwisted cohomology of X^G.

Example 9.1.6 (Underlying Poincaré duality does not imply Poincaré duality of the fixed points)**.** There exist piecewise linear C_2-actions on the sphere S^d whose fixed point sets are submanifolds M which are not homology spheres, see e.g. [FL04, p.5] for an exposition. Let $\underline{X}$ be the (unreduced) suspension of such an action. Then $X^e \simeq S^{d+1}$ which is a Poincaré duality space. However, X^{C_2} is the unreduced suspension of a manifold which is not a homology sphere, and hence clearly not Poincaré as Poincaré duality with integer coefficients must fail.

Next, we recall the proper Tate construction. The significance of this to our proof is that combining Proposition 9.1.4 with the degree theory from §7.4, we may obtain a version of the cohomological injection in proper Tate cohomology. We then extract the desired injection from this version by "finding" $\mathbb{F}_p$–cohomology inside proper Tate cohomology.

Recollection 9.1.7 (Proper Tate)**.** Let G be a finite group and $R \in \mathrm{CAlg}(\mathrm{Sp})$. One way to define the *R–based proper Tate functor* is as the lax symmetric monoidal composite

$$(-)^{t_{\mathcal{P}}G}\colon \mathrm{Fun}(BG, \mathrm{Mod}_R) \xhookrightarrow{b_*} \mathrm{Mack}_G(\mathrm{Mod}_R) \xrightarrow{\Phi^G} \mathrm{Mod}_R$$

This functor kills the proper induced terms, i.e. those $M \in \mathrm{Mod}_R^{BG}$ such that $M \simeq \mathrm{Ind}_H^G N$ for some $H \lneq G$ and $N \in \mathrm{Mod}_R^{BH}$ since Φ^G does. Furthermore, since $(-)^{t_{\mathcal{P}}G}$ is a lax symmetric monoidal functor, $R^{t_{\mathcal{P}}G}$ canonically attains an R–algebra structure.

Now let A be an elementary abelian p–group $A = C_p^{\times r}$. With the trivial action of A on $\mathrm{H}\mathbb{F}_p$, the A–proper Tate $\mathrm{H}\mathbb{F}_p^{t_{\mathcal{P}}A}$ is a nontrivial $\mathrm{H}\mathbb{F}_p$–algebra by [MNN19, Prop. 5.16]. Now let $T\colon \mathrm{stmod}_A^{\mathcal{P}}(\mathrm{H}\mathbb{F}_p) \longrightarrow \mathrm{Mod}_{\mathrm{H}\mathbb{F}_p}$ be the universal functor making the triangle

$$\begin{tikzcd} \mathrm{Fun}(BA, \mathrm{Perf}_{\mathrm{H}\mathbb{F}_p}) \arrow[r, "\mathrm{can}"] \arrow[dr, "(-)^{t_{\mathcal{P}}A}"'] & \mathrm{stmod}_A^{\mathcal{P}}(\mathrm{H}\mathbb{F}_p) \arrow[d, "T"] \\ & \mathrm{Mod}_{\mathrm{H}\mathbb{F}_p} \end{tikzcd}$$

commute, coming from the universal property of $\mathrm{stmod}_A^{\mathcal{P}}(\mathrm{H}\mathbb{F}_p)$.

Lemma 9.1.8 (Projection formula at dualisables)**.** *Let $\mathcal{A}, \mathcal{C}, \mathcal{D}$ be stably symmetric monoidal categories and $u\colon \mathcal{A} \to \mathcal{C}$ and $L\colon \mathcal{C} \to \mathcal{D}$ be symmetric monoidal exact functors. Suppose L admits a right adjoint R. Then for every $a \in \mathcal{A}$ dualisable and $d \in \mathcal{D}$, the canonical map $ua \otimes Rd \to R(Lua \otimes d)$ is an equivalence.*

Proof. Let $c \in \mathcal{C}$. By considering the equivalences

$$\begin{aligned} \mathrm{Map}_{\mathcal{C}}(c, ua \otimes Rd) &\simeq \mathrm{Map}_{\mathcal{C}}(c \otimes ua^{\vee}, Rd) \\ &\simeq \mathrm{Map}_{\mathcal{D}}(Lc \otimes Lua^{\vee}, d) \\ &\simeq \mathrm{Map}_{\mathcal{D}}(c, R(Lua \otimes d)), \end{aligned}$$

we obtain the desired conclusion by an application of Yoneda's lemma. □

Lemma 9.1.9. *Let $X \in \mathcal{S}^{\omega}$, $R \in \mathrm{CAlg}(\mathrm{Sp})$, and $\zeta \in \mathrm{Fun}(X, \mathrm{Perf}_R)$. Then viewing ζ as having the trivial G–action, we have an equivalence $(X_*\zeta)^{t_{\mathcal{P}}G} \simeq R^{t_{\mathcal{P}}G} \otimes_R X_*\zeta$.*

Proof. Since X was compact, we know that $X_*\zeta \in \mathrm{Perf}_R$, i.e. $X_*\zeta$ is a dualisable R–module. Setting $\mathrm{Infl}\colon \mathrm{Mod}_R \to \mathrm{Mack}_G(\mathrm{Mod}_R)$ and $b^*\colon \mathrm{Mack}_G(\mathrm{Mod}_R) \to \mathrm{Fun}(BG, \mathrm{Mod}_R)$ for the functors u and L in Lemma 9.1.8 (and writing $\mathrm{triv}_G\colon \mathrm{Mod}_R \to \mathrm{Fun}(BG, \mathrm{Mod}_R)$ for the composite), we see that by the lemma that

$$\begin{aligned} (\mathrm{triv}_G\, X_*R)^{t_{\mathcal{P}}G} &= \Phi^G b_*\big((\mathrm{triv}_G\, X_*R) \otimes_R R\big) \\ &\simeq \Phi^G\big((\mathrm{Infl}\, X)_*R \otimes_R b_*R\big) \\ &\simeq (X_*R) \otimes_R R^{t_{\mathcal{P}}G} \end{aligned}$$

as was to be shown. □

We now come to the main general proposition.

Proposition 9.1.10 (Injection after basechanging to proper Tate). *Consider a finite group G, $R \in \mathrm{CAlg}(\mathrm{Sp})$, and $f\colon \underline{X} \to \underline{Y}$ a map of compact G–spaces. Suppose X^e, Y^e are Mod_R–Poincaré spaces and $f\colon \underline{X} \to \underline{Y}$ is equipped with a $\underline{\mathrm{Bor}}(\mathrm{Mod}_R)$–degree one datum (c.f. Definition 7.4.2). Then for any $\zeta \in \mathrm{Fun}(Y^G, \mathrm{Perf}_R)$, the cohomological functoriality map in Mod_R*

$$R^{t_{\mathcal{P}}G} \otimes_R Y^G_* \zeta \longrightarrow R^{t_{\mathcal{P}}G} \otimes_R X^G_* f^*\zeta$$

is a π_*–split injection.

Proof. By Proposition 9.1.4 we know that X^G and Y^G are $\mathrm{stmod}^{\mathcal{P}}_G(R)$–partial Poincaré duality. In particular, viewing ζ as an object in $\mathrm{Fun}(Y^A, \mathrm{Perf}_R^{BG})$ under the symmetric monoidal functor $\mathrm{triv}_A\colon \mathrm{Perf}_R \to \mathrm{Perf}_R^{BG}$, we obtain using Lemma 7.4.6 the left commuting square

$$\begin{tikzcd} Y^A_*(\zeta) \arrow[r, "\mathrm{BC}^f_*"] \arrow[d, "\simeq"', "\mathrm{PD}"] & X^A_*(f^*\zeta) \arrow[d, "\simeq"', "\mathrm{PD}"] \\ Y^A_!(\Phi D_{Y^e} \otimes \Phi\zeta) & X^A_!(\Phi f^* D_{Y^e} \otimes \Phi f^*\zeta) \arrow[l, "\mathrm{BC}^f_!"'] \end{tikzcd}$$

in $\mathrm{stmod}^{\mathcal{P}}_G(R)$. Hence, the map $\mathrm{BC}^f_*\colon Y^G_*\zeta \to X^G_* f^*\zeta$ is a split inclusion. Finally, applying T to this map and using that $T \circ \Phi \simeq (-)^{t_{\mathcal{P}}G}$, we conclude from Lemma 9.1.9 that the map stated in the proposition is a split inclusion in Mod_R and in particular is a π_*–split injection. □

We would like to apply Proposition 9.1.10 to prove Theorem 9.1.1, and for this, a small preliminary calculation will be needed.

Lemma 9.1.11. *Let G be a p–group, $f\colon \underline{X} \to \underline{Y}$ a morphism in $\mathcal{S}_G$. Suppose that X^e and Y^e are $\mathrm{H}\mathbb{F}_p$–Poincaré and that $f\colon X^e \to Y^e$ is equipped with an $\mathrm{H}\mathbb{F}_p$–degree one datum. Then this degree one datum lifts to yield a $\underline{\mathrm{Bor}}(\mathrm{Mod}_{\mathrm{H}\mathbb{Z}})$–degree one datum for the map $f\colon \underline{X} \to \underline{Y}$.*

Proof. First recall from Proposition 8.3.6 that $\underline{X}$ and $\underline{Y}$ are indeed $\underline{\mathrm{Bor}}(\mathrm{Mod}_{\mathrm{H}\mathbb{F}_p})$–Poincaré. So by Lemma 8.3.7, we just need to find G–equivariant lifts of the equivalences $D_{X^e} \xrightarrow[\simeq]{\alpha} f^* D_{Y^e} \in \mathrm{Fun}(X^e, \mathcal{P}\mathrm{ic}(\mathrm{H}\mathbb{F}_p)) \simeq \mathrm{Map}(X^e, \mathcal{P}\mathrm{ic}(\mathrm{H}\mathbb{F}_p))$ and $c_Y \simeq \mathrm{BC}^f_! \circ \alpha \circ c_X \in Y^e_! D_{Y^e}$. That is, we would like to lift these equivalences to ones in $\mathrm{Map}(X^e, \mathcal{P}\mathrm{ic}(\mathrm{H}\mathbb{F}_p))^{hG}$ and $(Y^e_! D_{Y^e})^{hG}$ respectively. For the first problem, note that $\mathcal{P}\mathrm{ic}(\mathrm{H}\mathbb{F}_p) \simeq \mathbb{Z} \times B\mathrm{Aut}(\mathbb{F}_p) \simeq \mathbb{Z} \times B\mathbb{Z}/(p-1)$. Thus, by a standard analysis of the $(-)^{hG}$–spectral sequence

$$H^s(G; \pi_t \mathrm{Map}(X, \mathcal{P}\mathrm{ic}(\mathrm{H}\mathbb{F}_p))) \Rightarrow \pi_{t-s} \mathrm{Map}(X, \mathcal{P}\mathrm{ic}(\mathrm{H}\mathbb{F}_p))^{hG},$$

applying π_0 yields

$$\pi_0 \mathrm{Map}(X, \mathcal{P}\mathrm{ic}(\mathrm{H}\mathbb{F}_p))^{hG} \cong (\pi_0 \mathrm{Map}(X, \mathcal{P}\mathrm{ic}(\mathrm{H}\mathbb{F}_p)))^G \longrightarrow \pi_0 \mathrm{Map}(X, \mathcal{P}\mathrm{ic}(\mathrm{H}\mathbb{F}_p))$$

which in particular is an injection. Thus, since the G–equivariant lifts $D_{\underline{X}}$, $f^*D_{\underline{Y}}$ in the source get mapped to $D_{X^e} = f^*D_{Y^e} \in \pi_0 \operatorname{Map}(X, \mathcal{P}\mathrm{ic}(\mathrm{H}\mathbb{F}_p))$, we get that $D_{\underline{X}} = f^*D_{\underline{Y}}$ in the set $\pi_0 \operatorname{Map}(X, \mathcal{P}\mathrm{ic}(\mathrm{H}\mathbb{F}_p))^{hG}$. That is, the equivalence α lifts to a G–equivariant one, as required.

Next, note by Poincaré duality that $Y^e_! D_{Y^e} \simeq Y^e_* \mathbb{1}_{\mathrm{H}\mathbb{F}_p}$, and so since Y^e_* preserves coconnectivity, we learn that $Y^e_! D_{Y^e}$ is coconnective. Again, by looking at the spectral sequence $H^s(G; \pi_t Y^e_! D_{Y^e}) \Rightarrow \pi_{t-s}(Y^e_! D_{Y^e})^{hG}$, since no higher cohomologies may contribute to $\pi_0(Y^e_! D_{Y^e})^{hG}$ by coconnectivity, on π_0 the map $(Y^e_! D_{Y^e})^{hG} \to Y^e_! D_{Y^e}$ induces the map $(\pi_0 Y^e_! D_{Y^e})^G \to \pi_0 Y^e_! D_{Y^e}$, which is an injection. Thus by a similar argument as above, we obtain a G–equivariant lift of the equivalence $c_Y \simeq \mathrm{BC}^f_! \circ \alpha \circ c_X$, as wanted. □

We are now ready to assemble the pieces to prove the theorem.

Proof of Theorem 9.1.1. Since $\mathrm{H}\mathbb{F}_p$ was a field, we have the Künneth isomorphisms

$$\pi_{-*}(Y^A_* \zeta \otimes_{\mathrm{H}\mathbb{F}_p} \mathrm{H}\mathbb{F}_p^{t_{\mathcal{P}}A}) \cong H^*(Y^A; \zeta) \otimes_{\mathbb{F}_p} \pi_{-*}(\mathrm{H}\mathbb{F}_p^{t_{\mathcal{P}}A})$$

$$\pi_{-*}(X^A_* f^* \zeta \otimes_{\mathrm{H}\mathbb{F}_p} \mathrm{H}\mathbb{F}_p^{t_{\mathcal{P}}A}) \cong H^*(X^A; f^*\zeta) \otimes_{\mathbb{F}_p} \pi_{-*}(\mathrm{H}\mathbb{F}_p^{t_{\mathcal{P}}A}).$$

Now consider the commuting square

$$\begin{tikzcd}
H^*(Y^A;\zeta) \otimes_{\mathbb{F}_p} \pi_{-*}(\mathrm{H}\mathbb{F}_p^{t_{\mathcal{P}}A}) \arrow[r, tail, "f^*"] & H^*(X^A; f^*\zeta) \otimes_{\mathbb{F}_p} \pi_{-*}(\mathrm{H}\mathbb{F}_p^{t_{\mathcal{P}}A}) \\
H^*(Y^A;\zeta) \arrow[u, tail] \arrow[r, "f^*"] & H^*(X^A; f^*\zeta) \arrow[u, tail]
\end{tikzcd}$$

Here, the vertical arrows are induced by the injection $\mathbb{F}_p = \pi_{-*}(\mathrm{H}\mathbb{F}_p) \to \pi_{-*}(\mathrm{H}\mathbb{F}_p^{t_{\mathcal{P}}A})$ and so are themselves injections: this is since we are tensoring over a field and so all modules are flat. The top horizontal map is an injection by Proposition 9.1.10 and the fact that, by Lemma 9.1.11, we have a lift of the given nonequivariant degree one datum to a $\underline{\mathrm{Bor}}(\mathrm{Mod}_{\mathrm{H}\mathbb{F}_p})$–degree one datum for the map $f\colon \underline{X} \to \underline{Y}$. Therefore all in all, we see that the bottom map f^* is injective as desired. □

We end this subsection with an application of Theorem 9.1.1 where we show Theorem 9.1.14 that, when G is a solvable finite group, equivariant Poincaré spaces with contractible underlying spaces must already by G–contractible. Apart from perhaps being interesting in its own right, this result will also be a crucial ingredient in the inductive proof of the main theorem in the next subsection. We will need several preliminaries on orientations. It is easy to see that if an odd order group acts smoothly on an orientable manifold, then its fixed point set is also orientable. That is true in more generality.

Proposition 9.1.12 (Rigidity of orientability). *Let G be a finite group of odd order, and let $\underline{X}$ be a G-Poincaré space. Suppose the underlying space X^e is $\mathbb{Z}$-orientable. Then X^G is $\mathbb{Z}$-orientable as well.*

Proof. We check that, for each component of X^G, the first Stiefel Whitney class $w_1(X^G) \in H^1(X^G;\mathbb{Z}/2) \cong \hom(\pi_1(X^G),\mathbb{Z}/2)$ vanishes. Let $\gamma\colon S^1 \to X^{C_p}$ be a loop. The value of w_1 at the loop γ can be computed as the degree of $\mathrm{Mdrmy}_\gamma^{X^G}\colon D_{X^G}(\gamma(1)) \to D_{X^G}(\gamma(1)) \in \mathcal{P}\mathrm{ic}(\mathrm{Sp})$, the induced monodromy automorphism map.

We also have the automorphism $\mathrm{Mdrmy}_\gamma^{\underline{X}}\colon D_{\underline{X}}(\gamma(1)) \to D_{\underline{X}}(\gamma(1)) \in \mathcal{P}\mathrm{ic}(\mathrm{Sp}_G)$. Using Theorem 8.2.9, we see that $\Phi^G\mathrm{Mdrmy}_\gamma^{\underline{X}} \sim \mathrm{Mdrmy}_\gamma^{X^G}$. Automorphisms of $D_{\underline{X}}(\gamma(1))$ can be classified using the character homomorphism

$$\chi\colon \pi_0 \operatorname{Map}_{\mathrm{Sp}_G}(D_{\underline{X}}(\gamma(1)), D_{\underline{X}}(\gamma(1))) \to \prod_{H\leq G} \mathbb{Z}, \quad f \mapsto (\deg f^H)_H,$$

, see §8.6, where the product runs over all conjugacy classes of subgroups. Now χ is injective with image the subring satisfying the Burnside congruences. Being a ring map, it sends automorphisms to units. If G has odd order, then this ring has exactly two units, namely ± 1, see [Die79, Prop. 1.5.1.]. Thus, the degree of the monodromy automorphism on fixed points agrees with the degree of the monodromy automorphism on underlying spaces. □

This observation illustrates some nontrivial interaction between fixed point set and underlying space for equivariant Poincaré spaces. It allows us to give some very interesting non-examples for such.

Corollary 9.1.13. *Let p be an odd prime. There exists a compact C_p-space $\underline{X}$ with*

1. *the underlying space X^e is contractible and*
2. *the fixed point space X^{C_p} is Poincaré and*
3. *the C_p-space $\underline{X}$ is not C_p–Poincaré.*

Proof. Pick a noncontractible $\mathbb{F}_p$-acyclic Poincaré space K that is homotopy equivalent to a finite CW complex, for example $\mathbb{RP}^d$ for $d > 0$ an even number. By [Jon71, Thm. 1.1], we may pick a finite C_p–CW complex $\underline{X}$ with $X^e \simeq *$ and $X^{C_p} \simeq K$. By Proposition 9.1.12, we see that $\underline{X}$ can not be C_p–Poincaré, as then K would be orientable, contradicting $\mathbb{F}_p$-acyclicity. □

Theorem 9.1.14 (Poincaré rigidity of contractible underlying spaces). *Let G be a solvable group and $\underline{X} \in \mathcal{S}_G^\omega$ a compact G–Poincaré space with $X^e \simeq *$. Then $\underline{X} \simeq \underline{*}$.*

Proof. We prove this reducing to the case of $G = C_p$ using the solvability assumption. To wit, let us suppose we know the statement to be true for all solvable groups with size smaller than $|G|$. Choose a normal subgroup N of G such that $G/N = C_p$. By Proposition 8.1.8, we know that $\mathrm{Res}^G_N \underline{X}$ is N–Poincaré with $(\mathrm{Res}^G_N \underline{X})^e \simeq X^e \simeq *$, and so by induction, $\mathrm{Res}^G_N \underline{X} \simeq \underline{*}$. In particular, $X^N \simeq *$. Therefore, by Theorem 8.2.9, we have that $\underline{X}^N$ is a $G/N = C_p$–Poincaré space with $(\underline{X}^N)^e \simeq X^N \simeq *$. We are left to prove that for a C_p–Poincaré space $\underline{X}$, $X^e \simeq *$ implies $X^{C^p} \simeq *$.

Observe that $X^{C_p} \neq \varnothing$ as $\underline{EC}_p$ is not compact. Now pick a map $f : \underline{*} \to \underline{X}$. It is an equivalence on undelying spaces with C_p-action. By Theorem 9.1.1, f induces an injection on $\mathbb{F}_p$-cohomology

$$f^* : H^*(X^{C_p}; \mathbb{F}_p) \rightarrowtail H^*(*; \mathbb{F}_p). \tag{9.1}$$

In degree 0, this shows that X^{C_p} is connected. Furthermore, again by Theorem 8.2.9, X^{C_p} is a Poincaré space. To conclude, by the classification of zero-dimensional Poincaré spaces (Fact 7.2.20) it suffices to show that the formal dimension of X^{C_p} is zero. Note that X^{C_p} is $\mathbb{F}_p$-orientable. In the case $p = 2$ this is clear while in the case $p \neq 2$ this follows from Proposition 9.1.12. Now, injectivity of (9.1) implies that the formal dimension of X^{C_p} is zero, as zero is the highest degree in which $H^*(X^{C_p}; \mathbb{F}_p)$ does not vanish. $\square$

Remark 9.1.15. By Feit–Thompson's celebrated result, all finite groups of odd order are solvable. Hence, the Poincaré rigidity result above holds unconditionally for all odd finite groups.

9.2 The theorem of single fixed points

Throughout this subsection, we will fix an odd prime p.

In [CF64], Conner-Floyd conjectured that a smooth action by a cyclic group of odd prime power on a smooth, closed, orientable, positive–dimensional manifold cannot have exactly one fixed point. The first proof of this statement (in fact, a slightly more general version) was given by Atiyah-Bott in [AB68] and soon after by [CF66] themselves. Many variations have been proven since then, and we mention [Lüc88; ABK92] as further examples. Atiyah-Bott's argument uses Atiyah–Singer's index theory, whereas Conner–Floyd's proof used a particular bordism spectrum. In either case, and also in [Lüc88], local structures of smooth manifolds were used in essential ways. We exemplify such local arguments with the following corollary of Theorem 9.1.1 which answers the Conner–Floyd question for elementary abelian p–groups. As will be clear from the proof, the result holds more generally for locally smooth manifolds.

Corollary 9.2.1 (Conner–Floyd for elementary abelian groups)**.** *Let A be an elementary abelian p–group, and M a closed, orientable, positive–dimensional, smooth A–manifold. Then $M^A \neq *$.*

Proof. Suppose $M^A = *$. Writing $x \in M$ for this single fixed point, we may thus find an A–representation V equipped with a codimension zero equivariant embedding $V \subseteq M$ which sends $0 \in V$ to $x \in M$. Consider the collapse map $c\colon M \longrightarrow M/(M \setminus V) \simeq S^V$. It is a map of A–Poincaré spaces with $D_{M^e} \otimes \mathrm{H}\mathbb{Z} \simeq c^* D_{(S^V)^e} \otimes \mathrm{H}\mathbb{Z}$ as both are orientable. Thus by Theorem 9.1.1, we have an injection $H^*((S^V)^A; \mathbb{F}_p) \to H^*(M^A; \mathbb{F}_p)$. But note that $H^*((S^V)^A; \mathbb{F}_p) \simeq H^*(* \amalg *; \mathbb{F}_p)$ while $H^*(M^A; \mathbb{F}_p) \simeq H^*(*; \mathbb{F}_p)$. This is a contradiction. □

In this subsection, we will employ the theory of fundamental classes developed in this article to give a fully homotopical and global proof of the following generalisation of Atiyah–Bott and Conner–Floyd's theorem for C_{p^k}–Poincaré spaces. Philosophically, this says that the equivariant fundamental class packages enough structures so as to be able to provide a global obstruction to some naturally interesting geometric questions.

Theorem 9.2.2 (Generalised Atiyah–Bott–Conner–Floyd)**.** *Let p be an odd prime, $G = C_{p^k}$ for some k, and suppose $\underline{X} \in \mathcal{S}^{\omega}_G$ is G–Poincare such that the underlying space $X^e \in \mathcal{S}^{\omega}$ is connected, $\mathbb{Z}$–orientable, and has formal dimension $d > 0$. Then $X^G \not\simeq *$.*

We obtain the theorem of Atiyah–Bott and Conner–Floyd as an immediate consequence.

Corollary 9.2.3 ([AB68, Thm. 7.1], [CF66, p. 8.3])**.** *Let p be an odd prime and $G = C_{p^k}$. Let M be a closed connected orientable smooth manifold of positive dimension equipped with a smooth G–action. Then $M^G \neq *$.*

The orientability assumption is crucial, as illustrated by the following:

Example 9.2.4. For p odd, consider the suspension of the action of $C_p \subset S^1$ on S^2 which descends to an action of C_p on $\mathbb{R}\mathrm{P}^2$ with a single fixed point.

Consequently, we see that these no–go results for single fixed points cannot purely be a product of classical Smith theory since they must incorporate orientations in some fundamental way. From this perspective, our approach may be seen as a way to encode orientations by enconcing the discussion within the formalism of equivariant Poincaré duality, where Smith–theoretic fixed points methods are also available as afforded by §8.2.

Restricting to odd prime powers is essential as well, as the following example illustrates.

Example 9.2.5 ([CF64], Chapter 45)**.** The group C_4 acts on $\mathbb{C}\mathrm{P}^2$ with a single fixed point, by letting a generator act via $[z_0 : z_1 : z_2] \mapsto [\overline{z_0} : -\overline{z_2} : \overline{z_1}]$.

To start work on Theorem 9.2.2, we record some preliminaries on Tate cohomology which will be the computational input to our proof.

Recollection 9.2.6 (Group (co)homologies)**.** Let $n \geq 2$ be an integer and A an abelian group equipped with the trivial C_n–action. Then by definition, we have

$$\pi_d HA^{hC_n} \cong H^{-d}(C_n; A), \qquad \pi_d HA^{tC_n} \cong \widehat{H}^{-d}(C_n; A), \qquad \pi_d HA_{hC_n} \cong H_d(C_n; A).$$

Moreover, using the fibre sequence of spectra $HA_{hC_n} \longrightarrow HA^{hC_n} \longrightarrow HA^{tC_n}$, we get a long exact sequence

$$\cdots \to H^{-d}(C_n; A) \longrightarrow \widehat{H}^{-d}(C_n; A) \longrightarrow H_{d-1}(C_n; A) \longrightarrow H^{-(d-1)}(C_n; A) \to \cdots$$

giving us

$$\widehat{H}^{-d}(C_n; A) = \begin{cases} H^{-d}(C_n; A) \cong A/n & \text{if } d \leq -1 \text{ and } d \text{ even;} \\ H^{-d}(C_n; A) \cong 0 & \text{if } d \leq -1 \text{ and } d \text{ odd;} \\ A/n & \text{if } d = 0; \\ H_{d-1}(C_n; A) \cong A/n & \text{if } d \geq 1 \text{ and } d \text{ even;} \\ H_{d-1}(C_n; A) \cong 0 & \text{if } d \geq 1 \text{ and } d \text{ odd.} \end{cases}$$

It will be convenient to recall the notations of [GM95] to manipulate the various forms of the Tate constructions.

Lemma 9.2.7. *Let $H \leq G$ be a subgroup of a finite group G and $A \in \mathrm{Sp}_H$. Then we have $(\mathrm{Ind}_H^G A)^{tG} \simeq A^{tH}$.*

Proof. First observe that $\mathrm{Res}_H^G \widetilde{EG} \simeq \widetilde{EH}$ and $\mathrm{Res}_H^G EG_+ \simeq EH_+$. The required result is now obtained from the computation of $(\mathrm{Ind}_H^G A)^{tG}$ as

$$\big(\widetilde{EG} \otimes F(EG_+, \mathrm{Ind}_H^G A)\big)^G \simeq \big(\mathrm{Ind}_H^G \mathrm{Res}_H^G \widetilde{EG} \otimes F(EG_+, A)\big)^G \simeq \big(\widetilde{EH} \otimes F(EH_+, A)\big)^H = A^{tH}$$

where the equivalence $(\mathrm{Ind}_H^G -)^G \simeq (-)^H$ is since $\mathrm{Ind}_H^G \simeq \mathrm{Coind}_H^G$ and we have an equivalence of their left adjoints $\mathrm{Infl}_H^1 \simeq \mathrm{Res}_H^G \mathrm{Infl}_G^1$. ☐

Lemma 9.2.8. *Let $\underline{Y} \in \mathcal{S}^{\omega}_{C_{p^k}}$ such that $Y^{C_{p^k}} \simeq \varnothing$. Then the change of coefficients map $(Y_+ \otimes \mathrm{H}\mathbb{Z})^{tC_{p^k}} \to (Y_+ \otimes \mathrm{H}\mathbb{Z}/p^{k-1})^{tC_{p^k}}$ is an equivalence. In particular, the groups $\pi_n(Y_+ \otimes \mathrm{H}\mathbb{Z})^{tC_{p^k}}$ are p^{k-1}–torsion for all n.*

Proof. Note that the map being an equivalence is stable under retracts and finite colimits in the Y–variable. As any compact C_{p^k} space $\underline{Y}$ with $Y^{C_{p^k}} = \varnothing$ is a retract of a finite sequence of pushouts of orbits $\underline{C_{p^k}/C_{p^l}}$ with $l < k$, it thus suffices to show the desired equivalence for each of these orbits. By Lemma 9.2.7, we obtain for any $R \in \mathrm{CAlg}(\mathrm{Sp})$ the natural equivalence

$$\Big((C_{p^k}/C_{p^l})_+ \otimes R\Big)^{tC_{p^k}} = \Big(\mathrm{Ind}_{C_{p^l}}^{C_{p^k}} R\Big)^{tC_{p^k}} \simeq R^{tC_{p^l}}.$$

Thus, the claim follows from the fact that the quotient map $\mathrm{H}\mathbb{Z}^{tC_{p^l}} \to \left(\mathrm{H}\mathbb{Z}/p^{k-1}\right)^{tC_{p^l}}$ is an equivalence for $l < k$, see e.g. Recollection 9.2.6. □

For the next lemma, recall the cofibre sequence $EG_+ \to \mathbb{S}_G \to \widetilde{EG}$ in Sp_G.

Lemma 9.2.9. *Let G be an odd finite group and $P \in \mathcal{P}\mathrm{ic}(\mathrm{Sp}_G)$ with $P^e \simeq \Sigma^k \mathbb{S}$. Then there is a canonical equivalence $F(EG_+, \mathrm{H}\mathbb{Z}) \otimes P \simeq \Sigma^k F(EG_+, \mathrm{H}\mathbb{Z}) \in \mathrm{Sp}_G$. Consequently, the map*

$$(\widetilde{EG} \otimes F(EG_+, \mathrm{H}\mathbb{Z}) \otimes P)^G \longrightarrow (\Sigma EG_+ \otimes F(EG_+, \mathrm{H}\mathbb{Z}) \otimes P)^G$$

from the cofibre sequence $EG_+ \to \mathbb{S}_G \to \widetilde{EG}$ may be identified with the usual connecting map $\Sigma^k \mathrm{H}\mathbb{Z}^{tG} \longrightarrow \Sigma^{1+k}\mathrm{H}\mathbb{Z}_{hG}$.

Proof. We first show that the Borelification map

$$F(EG_+, \mathrm{H}\mathbb{Z}) \otimes P \to F(EG_+, \mathrm{H}\mathbb{Z} \otimes P^e)s$$

is an equivalence. To wit, let $Y \in \mathrm{Sp}_G$. Then

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Sp}_G}(Y, F(EG_+, \mathrm{H}\mathbb{Z}) \otimes P) &\simeq \mathrm{Map}_{\mathrm{Sp}_G}\left(Y \otimes P^{-1}, F(EG_+, \mathrm{H}\mathbb{Z})\right)\\
&\simeq \mathrm{Map}_{\mathrm{Sp}^{BG}}(Y^e \otimes (P^e)^{-1}, \mathrm{H}\mathbb{Z})\\
&\simeq \mathrm{Map}_{\mathrm{Sp}^{BG}}(Y^e, \mathrm{H}\mathbb{Z} \otimes P^e)\\
&\simeq \mathrm{Map}_{\mathrm{Sp}_G}\left(Y, F(EG_+, \mathrm{H}\mathbb{Z} \otimes P^e)\right)
\end{aligned}$$

as claimed. But then, since $\mathrm{H}\mathbb{Z} \otimes P^e \in \mathrm{Fun}(BG, \mathcal{P}\mathrm{ic}(\mathrm{Mod}_{\mathrm{Sp}}(\mathrm{H}\mathbb{Z})))$ and G was an odd group and $B\mathrm{Aut}(\mathrm{H}\mathbb{Z}) \simeq BC_2$, we know that $\mathrm{H}\mathbb{Z} \otimes P^e \simeq \mathrm{triv}_G \Sigma^k \mathrm{H}\mathbb{Z}$, whence the first statement. The second statement is then immediate from the equivalences $\left(\widetilde{EG} \otimes F(EG_+, E)\right)^G \simeq E^{tG}$ and $\left(EG_+ \otimes F(EG_+, E)\right)^G \simeq E_{hG}$ for all $E \in \mathrm{Sp}_G$. □

For the proof of the theorem, it will also be helpful to record the following:

Construction 9.2.10 (Orbit–component decompositions)**.** Let $\underline{X} \in \mathcal{S}_G$. By an easy adjunction computation, we have that $(\pi_0 X^e)/G \cong \pi_0(X_{hG})$. Let $S \sqcup T$ be a decomposition of $(\pi_0 X^e)/G \cong \pi_0(X_{hG})$. By considering the triple of adjunctions

$$\mathcal{S}_G \underset{\mathrm{incl}}{\overset{\pi_0}{\rightleftarrows}} \mathrm{Set}_G = \mathrm{Fun}(\mathcal{O}(G)^{\mathrm{op}}, \mathrm{Set}) \underset{b_*}{\overset{b^*}{\rightleftarrows}} \mathrm{Fun}(BG, \mathrm{Set}) \underset{r^*}{\overset{r_!}{\rightleftarrows}} \mathrm{Set}, \tag{9.2}$$

we may obtain a decomposition $\underline{X} \simeq \underline{Y} \sqcup \underline{Z} \in \mathcal{S}_G$ such that $\pi_0 Y_{hG} \cong S$ and $\pi_0 Z_{hG} \cong T$.

We now come to the proof of our generalisation of Atiyah–Bott and Conner–Floyd's theorem. For this, recall the notion of formal dimensions from Terminology 7.2.19.

Proof of Theorem 9.2.2. We prove this by induction on k, where the base case of $k = 0$ is trivial. Now suppose we know that it is true for $k - 1$. To prove the inductive step for the case of k, the strategy is to obtain a contradiction using the gluing class. For this, note first that $\underline{X}^{C_p}$ is a $\underline{\mathrm{Sp}}_{G/C_p}$–Poincaré space by Theorem 8.2.9. We claim that there is a decomposition $\underline{X}^{C_p} = \underline{*} \sqcup \underline{Y}$ of G/C_p-spaces, where $Y^{G/C_p} \simeq \varnothing$ necessarily since $* \simeq X^G = (X^{C_p})^{G/C_p} \simeq * \sqcup Y^{G/C_p}$. If the component of X^{C_p} containing $\underline{*}$ is of formal dimension larger than 0, then the induction hypothesis and Construction 9.2.10 gives such a decomposition as the G/C_p-space $\underline{X}^{C_p}$ also satisfies the conditions of the theorem. If the component of X^{C_p} containing $\underline{*}$ is of formal dimension 0, then it must be G/C_p–contractible by Fact 7.2.20 (1) and Theorem 9.1.14. Thus, again by Construction 9.2.10, we obtain the desired decomposition.

We will derive the contradiction by basechanging along $\underline{\mathrm{Sp}} \to \underline{\mathrm{Mod}}_{F(EG_+,\mathrm{H}\mathbb{Z})}$. Observe first that we may assume that $d > 0$ is even since we may replace $\underline{X}$ with $\underline{X} \times \underline{X}$ if necessary: this will still be a $\underline{\mathrm{Sp}}$–Poincaré space satisfying the hypotheses of the theorem with $(\underline{X} \times \underline{X})^G \simeq *$ and $X^e \times X^e$ having formal dimension $2d > 0$.

To set up notation, recall the map $\underline{X}^{>1} \xrightarrow{\epsilon} \underline{X}$ from Construction 6.2.15 and write $W := \Sigma^{-d}\mathrm{H}\mathbb{Z} \in \mathrm{Mod}_{\mathrm{H}\mathbb{Z}}$. We write $D^{\mathbb{Z}}_{\underline{X}} := F(EG_+, \mathrm{H}\mathbb{Z}) \otimes D_{\underline{X}} \in \underline{\mathrm{Mod}}^{\underline{X}}_{F(EG_+,\mathrm{H}\mathbb{Z})}$. By the hypothesis of $\mathbb{Z}$–orientability, we get $D^{\mathbb{Z}}_{X^e} \simeq \mathrm{H}\mathbb{Z} \otimes D_{X^e} \simeq X^*W \in \mathrm{Mod}^{X^e}_{\mathrm{H}\mathbb{Z}}$. By Proposition 9.1.12, the $\underline{\mathrm{Sp}}_{G/C_p}$–Poincaré space $\underline{X}^{C_p}$ has $\mathbb{Z}$–orientable underlying dualising sheaf. By Corollary 8.5.7, the composition giving the gluing class

$$\begin{tikzcd}
\mathrm{H}\mathbb{Z} \arrow[d, "c"] & (X_!^{>1}\epsilon^* D^{\mathbb{Z}}_{\underline{X}})^{tG} \simeq (X_!^{>1}(X^{>1})^*W)^{tG} \arrow[r, "\mathrm{can}"] \arrow[d, "\simeq"] & \Sigma(X_!^{>1}\epsilon^* D^{\mathbb{Z}}_{\underline{X}})_{hG} \simeq \Sigma(X_!^{>1}(X^{>1})^*W)_{hG} \arrow[d, "\mathrm{can}"] \\
(X_! D^{\mathbb{Z}}_{\underline{X}})^{hG} \arrow[r, "\mathrm{can}"] & (X_! D^{\mathbb{Z}}_{\underline{X}})^{tG} & \Sigma W_{hG}
\end{tikzcd} \tag{9.3}$$

is nullhomotopic. To achieve a contradiction, we show that this composition is also π_0–surjective onto a nontrivial group, assuming that $X^G \simeq *$. We do this in three steps.

(1) To this end, first note that since the composite $\underline{*} \hookrightarrow \underline{X}^{>1} \xrightarrow{\epsilon} \underline{X} \to \underline{*}$ is equivalent to the identity, by functoriality of colimits, the map in $\mathrm{Fun}(BG, \mathrm{Mod}_{\mathrm{H}\mathbb{Z}})$

$$W \longrightarrow X_!^{>1}\epsilon^* D^{\mathbb{Z}}_{X^e} \simeq X_!^{>1}(X^{>1})^*W \longrightarrow W$$

is also equivalent to the identity. Thus the rightmost vertical map can in (9.3) is (split) surjective on homotopy groups coming from the summand ΣW_{hG}

inside $\Sigma(X_!^{>1}\epsilon^*D_{\underline{X}}^{\mathbb{Z}})_{hG} \simeq \Sigma W_{hG} \oplus \Sigma(Y_!Y^*W)_{hG}$, where we have used that $\underline{X}^{>1} \simeq \mathrm{Infl}_G^{G/C_p}\,\underline{X}^{C_p} \simeq \underline{*} \sqcup \mathrm{Infl}_G^{G/C_p}\,\underline{Y}$ from the first paragraph of the proof.

(2) Next, the Tate–to–orbit canonical map breaks up to become

$$(X_!^{>1}\epsilon^*D_{\underline{X}}^{\mathbb{Z}})^{tG} \simeq W^{tG} \oplus (Y_!Y^*W)^{tG} \xrightarrow{\mathrm{can}\oplus\mathrm{can}} \Sigma W_{hG} \oplus \Sigma(Y_!Y^*W)_{hG}$$

By Recollection 9.2.6 and since $d-1$ is odd by our assumption, $\mathrm{can}\colon W^{tG} \to \Sigma W_{hG} \simeq \Sigma^{1-d}\mathrm{H}\mathbb{Z}_{hG}$ is a π_0–isomorphism onto $\mathbb{Z}/p^k$. Moreover, by Lemma 9.2.8, the image of $\mathrm{can}\colon \pi_0(Y_!Y^*W)^{tG} \to \pi_0\Sigma(Y_!Y^*W)_{hG}$ is p^{k-1}–torsion since $Y^{G/C_p} \simeq \varnothing$.

(3) Finally, consider the commuting diagram

$$\begin{tikzcd}
\mathrm{H}\mathbb{Z} \arrow[d, "c"'] \arrow[drr, "c"] & & \\
\left(\widetilde{EG}\otimes X_!D_{\underline{X}}^{\mathbb{Z}}\right)^G \arrow[d] & & \left(\widetilde{EG}\otimes X_!^{>1}\epsilon^*D_{\underline{X}}^{\mathbb{Z}}\right)^G \arrow[ll, "\epsilon_!"', "\simeq"] \arrow[d] \\
(X_!D_{\underline{X}}^{\mathbb{Z}})^{tG} & (X_!^{>1}\epsilon^*D_{\underline{X}}^{\mathbb{Z}})^{tG} \arrow[l, "\epsilon_!"', "\simeq"] & \simeq \left(\widetilde{EG}\otimes F(EG_+, X_!^{>1}\epsilon^*D_{\underline{X}}^{\mathbb{Z}})\right)^G
\end{tikzcd}$$

where the top right triangle involving the fundamental classes commutes since by Corollary 8.2.5 and Lemma 8.2.6, the map $\epsilon\colon \underline{X}^{>1} \to \underline{X}$ is $\underline{\mathrm{Mod}}_{\widetilde{EG}\otimes F(EG_+,\mathrm{H}\mathbb{Z})}$–degree one. Note importantly that the map

$$\mathrm{H}\mathbb{Z} \xrightarrow{c} (\widetilde{EG}\otimes X_!^{>1}\epsilon^*D_{\underline{X}}^{\mathbb{Z}})^G \simeq (\widetilde{EG}\otimes *_!\epsilon^*D_{\underline{X}}^{\mathbb{Z}})^G \oplus (\widetilde{EG}\otimes Y_!\epsilon^*D_{\underline{X}}^{\mathbb{Z}})^G$$

is the fundamental class of the $\widetilde{EG}\otimes F(EG_+,\mathrm{H}\mathbb{Z})$–Poincaré space $\underline{*}\sqcup \mathrm{Infl}_{G/C_p}^G\,\underline{Y}$, and so by Lemma 7.3.16, it hits the algebra unit 1 of $\pi_0(\widetilde{EG}\otimes *_!\epsilon^*D_{\underline{X}}^{\mathbb{Z}})^G$. Next, consider

$$\begin{tikzcd}
\left(\widetilde{EG}\otimes *_!\epsilon^*D_{\underline{X}}^{\mathbb{Z}}\right)^G \arrow[r] \arrow[d, "\simeq"'] & \left(\Sigma EG_+\otimes *_!\epsilon^*D_{\underline{X}}^{\mathbb{Z}}\right)^G \arrow[d, "\simeq"] \\
W^{tG} \simeq \left(\widetilde{EG}\otimes F(EG_+, *_!\epsilon^*D_{\underline{X}}^{\mathbb{Z}})\right)^G \arrow[r] & \left(\Sigma EG_+\otimes F(EG_+, *_!\epsilon^*D_{\underline{X}}^{\mathbb{Z}})\right)^G \simeq \Sigma W_{hG}.
\end{tikzcd}$$

where the vertical equivalences are by Lemma 9.2.9. Since the bottom horizontal map is a π_0–isomorphism onto $\mathbb{Z}/p^k$ as in step (2), the composition $\mathrm{H}\mathbb{Z} \xrightarrow{c} W^{tG} \to \Sigma W_{hG}$ is a π_0–surjection onto the abelian group $\mathbb{Z}/p^k$.

All in all, putting the three steps together, we see that the image of $1 \in \pi_0\mathrm{H}\mathbb{Z}$ under the composition map $\pi_0\mathrm{H}\mathbb{Z} \to \pi_0\Sigma W_{hG} \cong \mathbb{Z}/p^k$ in (9.3) is of the form

$1 + p \cdot a$ for some element $a \in \mathbb{Z}/p^k$, and hence is nonzero. This finishes the proof of the claim, and thus also of the theorem. $\square$

Remark 9.2.11. Step (3) in the proof above might seem labyrinthine at first glance, but the basic idea leading to it is quite simple. Namely, we know always from Corollary 8.5.2 that the map $\epsilon_!\colon (X_!^{>1}\epsilon^* D_{\underline{X}}^{\mathbb{Z}})^{tG} \to (X_! D_{\underline{X}}^{\mathbb{Z}})^{tG}$ is an equivalence. However, this equivalence has no control over the fundamental class $c\colon \mathrm{H}\mathbb{Z} \to (X_! D_{\underline{X}}^{\mathbb{Z}})^{tG}$, essentially because $(-)^{tG}$ is only a lax symmetric monoidal functor. In contrast, the functor $\widetilde{EG} \otimes -$ is a symmetric monoidal one, and so it is more suited to lift the fundamental class by virtue of the theory of degree one maps as encapsulated in Corollary 8.2.5.

Appendix A

G–stability for presentable G-categories

In this section we study presentable G-stable categories for compact Lie groups. In [Nar17], Nardin defines for a finite group G-stability as a property of fibrewise stable G–categories, that roughly translates to requiring certain Wirthmüller isomorphisms to hold. Instead of developing his theory for compact Lie groups in full generality, we take a different approach following the general phenomenon that certain properties of categories can be classified through idempotent algebras. For example, a presentable category is stable if and only if it is a module over the category of spectra, see [GGN15; CSY21] for more examples of this type. We define presentable G-stable categories as those presentable G–categories which are modules over the G–category $\underline{\mathrm{Sp}}_G$ of G–spectra.

Recall that we say that a map $u\colon \mathbb{1} \to A$ exhibits an object A in a symmetric monoidal category $\mathcal{C}$ as an idempotent object if the map $A \simeq \mathbb{1} \otimes A \xrightarrow{u \otimes A} A \otimes A$ is an equivalence. An idempotent object admits a unique structure of a commutative algebra in $\mathcal{C}$ with u as its unit map, see [Lur17, Proposition 4.8.2.9]. Now given an idempotent algebra A in a symmetric monoidal category $\mathcal{C}$, it is a property of an object $X \in \mathcal{C}$ to be a module over A, in the sense that the forgetful functor $\mathrm{Mod}_A(\mathcal{C}) \to \mathcal{C}$ is fully faithful. Its image is characterised by those $X \in \mathcal{C}$ for which the unit map $X \to A \otimes X$ is an equivalence, or equivalently admits a right inverse, see [Lur17, Proposition 4.8.2.10]. Taking this point of view, we observe that the G–category of G–spectra is an idempotent algebra in Pr^L_G and use this for the definition of presentable G-stable categories.

Definition A.1 ([GM23, Definition C.1, Corollary C.7], [Cno23, Definition 4.1])**.** The category of G–spectra is defined as the formal inversion

$$\mathrm{Sp}_G = \mathcal{S}_{G,*}[\{S^V\}^{-1}].$$

Here $\{S^V\}^{-1}$ denotes the collection in $\mathcal{S}_{G,*}$ consisting of representation spheres of all finite dimensional G-representations V.

This means that it comes together with a symmetric monoidal colimit preserving functor $\Sigma^\infty_G \colon \mathcal{S}_{G,*} \to \mathrm{Sp}_G$ sending all representation spheres S^V to invertible objects and is inital among those. More details on formal inversions of presentably symmetric monoidal categories can be found in [Rob15, Section 2] and [Hoy17, Section 6.1]. By [GM23, Corollary C.7], the canoncial map

$$\mathrm{Stab}_{\{S^V\}}(\mathcal{S}_{G,*}) \to \mathcal{S}_{G,*}[\{S^V\}^{-1}] \tag{A.1}$$

is an equivalence. Here, we denote for a presentably symmetric monoidal category $\mathcal{C}$ together with a small collection of objects $S \subseteq \mathcal{C}$ the stablisiation of $\mathcal{C}$ at S by $\mathrm{Stab}_S(\mathcal{C}) = \mathrm{colim}_{F \subseteq S \text{ finite}} \mathrm{Stab}_{\otimes F}(\mathcal{C})$, where for an element $x \in \mathcal{C}$ we denote

$$\mathrm{Stab}_x(\mathcal{C}) = \mathrm{colim}\left(\mathcal{C} \xrightarrow{-\otimes x} \mathcal{C} \xrightarrow{-\otimes x} \dots\right).$$

Definition A.2 ([Cno23, Definition 4.2])**.** We define the G–categories of pointed G–spaces and of (genuine) G–spectra as

$$\underline{\mathcal{S}}_{G,*} = \mathcal{S}_{G,*} \otimes_{\mathcal{S}_G} \Omega \qquad \text{and} \qquad \underline{\mathrm{Sp}}_G = \mathrm{Sp}_G \otimes_{\mathcal{S}_G} \Omega,$$

where $- \otimes_{\mathcal{S}_G} \Omega \colon \mathrm{Mod}_{\mathcal{S}_G}(\mathrm{Pr}^L) \to \mathrm{Pr}^L_G$ is the symmetric monoidal colimit preserving embedding from Proposition 6.1.30.

Lemma A.3. *The G–categories $\underline{\mathcal{S}}_{G,*}$ and $\underline{\mathrm{Sp}}_G$ are idempotent algebras in Pr^L_G.*

Proof. First note that $\mathcal{S}_{G,*} \in \mathrm{Mod}_{\mathcal{S}_G}(\mathrm{Pr}^L)$ is an idempotent algebra as the image of the idempotent algebra $\mathcal{S}_* \in \mathrm{Pr}^L$ under the symmetric monoidal functor $- \otimes \mathcal{S}_G \colon \mathrm{Pr}^L \to \mathrm{Mod}_{\mathcal{S}_G}$. It now follows from the definition of formal inversion that $\mathrm{Sp}_G \in \mathrm{Mod}_{\mathcal{S}_G}(\mathrm{Pr}^L)$ is an idempotent algebra. This proves the claim as the symmetric monoidal functor $- \otimes_{\mathcal{S}_G} \Omega \colon \mathrm{Mod}_{\mathcal{S}_G}(\mathrm{Pr}^L) \to \mathrm{Pr}^L_G$ preserves idempotent algebras. ☐

Having this at hand, we can now give our definition of G-stability.

Definition A.4. We say that a presentable G–category $\underline{\mathcal{C}}$ is *G-stable* if it is a module over the idempotent algebra $\underline{\mathrm{Sp}}_G \in \mathrm{CAlg}(\mathrm{Pr}^L_G)$. We denote by $\mathrm{Pr}^{L,G-\mathrm{st}}_G \subseteq \mathrm{Pr}^L_G$ the full subcategory on G-stable presentable G–categories. It is closed under all limits and colimits.

Our goal is to prove the following characterisation of presentable G-stable categories.

Theorem A.5 (Characterisation of G-stability)**.** *For a presentable G–category $\underline{\mathcal{C}}$ the following are equivalent:*

1. $\underline{\mathcal{C}}$ *is G-stable.*

2. $\underline{\mathcal{C}}$ *is fibrewise pointed and for all closed subgroups* $H \leq G$ *and all finite dimensional H-representations V tensoring with* $S^V \in \mathcal{S}_{H,*}$ *induces an equivalence -* $-\otimes S^V \colon \mathcal{C}^H \xrightarrow{\simeq} \mathcal{C}^H$.

3. $\underline{\mathcal{C}}$ *is fibrewise pointed and for all finite dimensional G-representation V tensoring with* $S^V \in \mathcal{S}_{G,*}$ *induces an equivalence* $-\otimes S^V \colon \underline{\mathcal{C}} \xrightarrow{\simeq} \underline{\mathcal{C}}$.

To clarify the statement, recall that the $\underline{\mathcal{S}}_G$-module structure on $\underline{\mathcal{C}}$ restricts to a $\mathcal{S}_H$-module structure on $\mathcal{C}^H$ which refines to a $\mathcal{S}_{H,*}$-module structure as $\mathcal{C}^H$ is pointed. The map $-\otimes S^V \colon \mathcal{C}^H \xrightarrow{\simeq} \mathcal{C}^H$ is now just the multiplication map induced by this module structure.

For the proof of Theorem A.5, we need the following preliminary result.

Lemma A.6. *Suppose that* $\underline{\mathcal{D}}$ *is a presentable G–category such that* $\mathcal{D}^G$ *is pointed and* $-\otimes S^V \colon \mathcal{D}^G \to \mathcal{D}^G$ *is an equivalence for any finite dimensional G-representation V. Then the restriction map* $\mathrm{Fun}^L_G(\underline{\mathrm{Sp}}_G, \mathcal{D}) \to \mathrm{Fun}^L_G(\underline{\mathcal{S}}_G, \mathcal{D})$ *is an equivalence.*

Proof. Using that $-\otimes \underline{\mathcal{S}}_G \colon \mathrm{Pr}^L \to \mathrm{Pr}^L_G$ is left adjoint to Γ, we obtain an equivalence

$$\mathrm{Fun}^L_G(\underline{\mathcal{S}}_{G,*}, \underline{\mathcal{D}}) \simeq \mathrm{Fun}^L(\mathcal{S}_*, \mathcal{D}^G) \xrightarrow{\simeq} \mathrm{Fun}^L(\mathcal{S}, \mathcal{D}^G) \simeq \mathrm{Fun}^L_G(\underline{\mathcal{S}}_G, \underline{\mathcal{D}})$$

where the middle equivalence uses that $\mathcal{D}^G$ is pointed. Similarly, the restriction map

$$\mathrm{Fun}^L_G(\underline{\mathrm{Sp}}_G, \underline{\mathcal{D}}) \simeq \mathrm{Fun}^L_{\mathcal{S}_{G,*}}(\mathrm{Sp}_G, \mathcal{D}^G) \xrightarrow{\simeq} \mathrm{Fun}^L_{\mathcal{S}_{G,*}}(\mathcal{S}_{G,*}, \mathcal{D}^G) \simeq \mathrm{Fun}^L_G(\underline{\mathcal{S}}_{G,*}, \underline{\mathcal{D}})$$

is an equivalence by employing the colimit description of $\mathrm{Sp}_G = \mathcal{S}_{G,*}[\{S^V\}^{-1}]$ from (A.1). □

Proof of Theorem A.5. 1 $\implies$ 2: Observe that $\underline{\mathrm{Sp}}_G$ is fibrewise pointed and satisfies the assumption on invertible actions of representations spheres as $\underline{\mathrm{Sp}}_G(G/H) = \mathrm{Sp}_H$ is the formal inversion of $\mathcal{S}_{H,*}$ at representation spheres of finite dimensional H-represenations. But this also holds for any G-stable category $\underline{\mathcal{C}}$ as $\mathcal{C}^H$ then is a module over Sp_H.

2 $\implies$ 3: Recall that, by the Peter-Weyl theorem, for any finite dimensional H-represenation W there is a finite dimensional G-representation V such that W is a summand of $\mathrm{Res}^G_H V$. In particular, if $-\otimes S^V \colon \mathcal{C}^H \xrightarrow{\simeq} \mathcal{C}^H$ is an equivalence, this implies that $-\otimes S^{\mathrm{Res}^G_H V} \colon \mathcal{C}^H \xrightarrow{\simeq} \mathcal{C}^H$ is an equivalence. But then also $-\otimes S^W$ is an equivalence

3 $\implies$ 1: We want to construct a right inverse $\underline{\mathcal{C}} \otimes \underline{\mathrm{Sp}}_G \to \underline{\mathcal{C}}$ to the unit map. By adjunction, this is equivalent to finding a factorisation of the unit map $\underline{\mathcal{S}}_G \to \underline{\mathrm{Fun}}^L_G(\underline{\mathcal{C}}, \underline{\mathcal{C}})$ through the unit map $\underline{\mathcal{S}}_G \to \underline{\mathrm{Sp}}_G$. For this, we apply Lemma A.6 for

$\underline{\mathcal{D}} = \underline{\mathrm{Fun}}^L_G(\underline{\mathcal{C}}, \underline{\mathcal{C}})$. It thus remains to show that $\mathrm{Fun}^L_G(\underline{\mathcal{C}}, \underline{\mathcal{C}}) = \mathcal{D}^G$ is pointed and tensoring with representation spheres is invertible. The assumption on $\underline{\mathcal{C}}$ being fibrewise pointed implies that $\mathrm{Fun}^L_G(\underline{\mathcal{C}}, \underline{\mathcal{C}})$ is pointed. Furthermore, S^V acts invertibly on $\mathrm{Fun}^L_G(\underline{\mathcal{C}}, \underline{\mathcal{C}})$ as it does so on $\underline{\mathcal{C}}$. □

Appendix B

Reflecting pushout squares

Let $A \to B \to B/A$ be a cofibre sequence in a stable category. Recall that there is a natural identification of the cofibre of $B \to B/A$ with ΣA, constructed as follows. Consider the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[rr, bend left, "\simeq 0"] & B \arrow[r] \arrow[rr, bend right, "\simeq 0"'] & B/A \arrow[r] & \mathrm{cofib}(B \to B/A) \end{tikzcd} \tag{B.1}$$

and note that the two nullhomotopies of bent arrows - coming from them being the structure of the cofibre sequences - define a map $\Sigma A \to \mathrm{cofib}(A \to B/A)$, which turns out to be an equivalence. This equivalence is natural in maps of cofibre sequences, and we will always use it to identify $\mathrm{cofib}(B \to B/A)$ with ΣA.

Lemma B.1. *Consider a pushout square*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

in a stable category. Then the two composites

$$\phi_C \colon D \to D/C \simeq B/A \to \Sigma A \quad \textit{and} \quad \phi_B \colon D \to D/B \simeq C/A \to \Sigma A,$$

coming from the following diagram, satisfy $\phi_B \simeq \pm\phi_C$.

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & B/A \arrow[r] \arrow[d, "\simeq"] & \Sigma A \\
C \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & D/C & \\
C/A \arrow[r, "\simeq"] \arrow[d] & D/B & & \\
\Sigma A & & &
\end{tikzcd}$$

Proof. To prove this, we consider the universal example of a span in a stable category. Denote by $\mathrm{Span}(\mathcal{C}) = \mathrm{Fun}(\bullet \leftarrow \bullet \rightarrow \bullet, \mathcal{C})$ the category of spans in the stable category $\mathcal{C}$. Note that there is an equivalence

$$\begin{aligned}
\mathrm{Span}(\mathcal{C}) &= \mathrm{Fun}(\bullet \leftarrow \bullet \rightarrow \bullet, \mathcal{C}) \\
&\simeq \mathrm{Fun}(\bullet \leftarrow \bullet \rightarrow \bullet, \mathrm{Fun}^{\mathrm{ex}}(\mathrm{Sp}^{\omega}, \mathcal{C})) \\
&\simeq \mathrm{Fun}^{\mathrm{ex}}(\mathrm{Sp}^{\omega} \otimes (\bullet \leftarrow \bullet \rightarrow \bullet), \mathcal{C}),
\end{aligned}$$

where $\mathrm{Sp}^{\omega} \otimes (\bullet \leftarrow \bullet \rightarrow \bullet)$ denotes the tensoring of $\mathrm{Cat}^{\mathrm{st}}$ over Cat. The construction of the tensoring in [CDH+23a, Section 6.4] shows that $\mathrm{Sp}^{\omega} \otimes (\bullet \leftarrow \bullet \rightarrow \bullet)$ is given by the stable subcategory $\mathrm{Cospan}(\mathrm{Sp})^{f}$ of $\mathrm{Cospan}(\mathrm{Sp}) = \mathrm{Fun}(\bullet \rightarrow \bullet \leftarrow \bullet, \mathrm{Sp})$ generated by the three objects in the span (B.2) (which are given as the values of the left Kan extensions of the inclusions of the individual objects in the category $(\bullet \rightarrow \bullet \leftarrow \bullet)$ at the sphere). Using this description, the equivalence $\mathrm{Fun}^{\mathrm{ex}}(\mathrm{Cospan}(\mathrm{Sp})^{f}, \mathcal{C}) \simeq \mathrm{Span}(\mathcal{C})$ is given by evaluation at the universal span

$$\begin{tikzcd}
\left(0 \rightarrow \mathbb{S} \leftarrow 0 \right) \arrow[r] \arrow[d] & \left(0 \rightarrow \mathbb{S} = \mathbb{S} \right) \\
\left(\mathbb{S} = \mathbb{S} \leftarrow 0 \right). &
\end{tikzcd} \tag{B.2}$$

It suffices to prove the claim in this specific case. Any span in $\mathcal{C}$ is the image of this universal span under an exact functor and thus also satisfies the statement of the lemma.

The possibilities for ϕ_B and ϕ_C are limited, since

$$\begin{aligned}
&\pi_0 \operatorname{Map}\left(\begin{tikzcd}[column sep=small] & \mathbb{S} \arrow[dl, equal] \arrow[dr, equal] & \\ \mathbb{S} & & \mathbb{S} \end{tikzcd}, \begin{tikzcd}[column sep=small] & \Sigma\mathbb{S} & \\ 0 \arrow[ur] & & 0 \arrow[ul] \end{tikzcd}\right) \\
\simeq &\pi_0 \operatorname{Map}\left(\mathbb{S}, \lim\left(\begin{tikzcd}[column sep=small] & \Sigma\mathbb{S} & \\ 0 \arrow[ur] & & 0 \arrow[ul] \end{tikzcd}\right)\right) \simeq \pi_0 \operatorname{Map}(\mathbb{S}, \mathbb{S}) \simeq \mathbb{Z},
\end{aligned}$$

so ϕ_B and ϕ_C identify with integers n_B and n_C. Note that if n_B is divisible by $k \in \mathbb{Z}$, then ϕ_B is divisible by k for any pushout in any stable category. But in the case of the following pushout in Sp

$$\begin{tikzcd} \mathbb{S} \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma\mathbb{S} \end{tikzcd} \tag{B.3}$$

the map ϕ_B is clearly an equivalence, which implies $n_B = \pm 1$. The same holds for ϕ_C, so by lack of alternatives we see $\phi_B = \pm\phi_C$. $\square$

Remark B.2. A more careful analysis of (B.3) in fact yields that $\phi_B = -\phi_C$.

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Index of symbols

$\mathcal{S}$	Category of spaces, homotopy types, aminae,...
Cat, $\widehat{\mathrm{Cat}}$	Category of small, resp. large, categories.
$\mathrm{Map}(x, y)$	Mapping space of two objects in a category.
$\mathrm{map}(x, y)$	Mapping spectrum of two objects in a stable category.
$\mathcal{C} \overrightarrow{\times} \mathcal{D}$	Partially lax pullback, Construction 3.1.2.
$\mathrm{Im}(\mathcal{A})$	Im-construction of Land-Tamme, Construction 3.1.4.
E	Localising invariant, Recollection 3.1.1.
$\mathrm{End}_\alpha(\mathcal{C})$	Category of twisted endomorphisms, Definition 2.1.2.
$\mathrm{Aut}_\alpha(\mathcal{C})$	Category of twisted automorphisms, Definition 2.1.2.
$NE_\alpha(\mathcal{C})$	Nil-term, Definition 3.2.4.
$NE_\alpha^{\mathrm{tot}}(\mathcal{C})$	Total Nil-term, Page 53.
$\mathrm{Nil}_\alpha(\mathcal{C})$	Category of twisted nilpotent endomorphisms, Definition 2.2.1.
free_α	Free twisted endomorphism, Lemma 2.1.6.
loc_α	Localising twisted endomorphisms to twisted automorphisms, Lemma 2.1.7.
triv_α	trivial twisted endomorphism.
$\mathcal{B}$	Base topos.
$\mathrm{Cat}_\mathcal{B}$	Category of $\mathcal{B}$-categories, Definition 6.1.1.
$\mathrm{Pr}^L_\mathcal{B}$	Category of presentable $\mathcal{B}$-categories, Definition 6.1.25.
$\mathrm{Cat}^{\mathrm{st}}_G$	Category of fiberwise stable G-categories, Notation 6.2.5.
$\mathrm{Cat}^{G-\mathrm{st}}_G$	Category of G-stable G-categories, Notation 6.2.5.
$\mathrm{Pr}^{L,\mathrm{st}}_G$	Category of fiberwise stable presentable G-categories, Notation 6.2.5.
$\mathrm{Pr}^{L,G-\mathrm{st}}_G$	Category of G-stable presentable G-categories, Notation 6.2.5.
$\underline{\mathcal{C}}$	Generic $\mathcal{B}$-category.
$\mathrm{Res}, \mathrm{Ind}, \mathrm{Coind}$	Restriction, induction, coinduction, Notation 6.2.2.