The mathematical properties of the radiative transfer equation

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Declaration

I declare that this thesis has been composed with no other help and resources than the ones indicated.

Summary

This thesis has as overall aim the development of a comprehensive mathematical theory for the radiative transfer equation. This is the kinetic equation describing the interaction of matter with electromagnetic waves. In particular, this thesis collects several results about different problems which study the behavior of the temperature distribution in a body where the heat is transferred by radiation and sometimes also by conduction. This work is a cumulative thesis which collects five articles produced by the author together with other collaborators and it is structured as follows.

Chapter 1 gives a detailed introduction of the radiative transfer equation. In particular, in the first part of this chapter the phenomenological derivation of this kinetic equation as well as the radiative heat transfer model are presented. Moreover, important features of the radiative transfer equation and the main mathematical strategies used in this thesis are introduced. Furthermore, an exhaustive summary of the studied problems and of the obtained results can be found here. Finally, the available mathematical literature concerning the problems considered in this thesis is summarized at the end of Chapter 1.

Chapter 2 is a summary of the article *"Compactness and existence theory for a general class of stationary radiative transfer equations"* [35], which can be found in Appendix A. It deals with the existence theory of the stationary radiative transfer equation when the absorption and the scattering coefficients depend non-trivially on the temperature. Furthermore, a new compactness result for operators containing exponentials of integrals along straight lines is developed.

In Chapter 3 the results of the article *"Equilibrium and Non-Equilibrium diffusion approximation for the radiative transfer equation"* [36], which can be found in Appendix B, are summarized. In this article the diffusion approximation of the radiative transfer equation is studied via matched asymptotic expansion. This problem arises when the mean free path of the photon is very small compared to the characteristic size of the domain. In particular, several reciprocal scalings between the absorption mean free path and the scattering mean free path are considered. Moreover, the concepts of equilibrium and non-equilibrium diffusion approximations are introduced and the condition of validity of these approximations are derived.

Chapter 4 studies rigorously the diffusion approximation of the stationary radiative transfer equation in the absence of scattering. This chapter summarizes the results of the published article *"On the diffusion approximation of the stationary radiative transfer equation with absorption and emission"* [37], which can be read in Appendix C. Specifically, using mainly maximum principle tools it is proved that, as the mean free path of the photons tends to zero, the radiation intensity converges to the Planck distribution of the temperature, which solves an elliptic Dirichlet problem.

In Chapter 5 a free boundary problem for the melting of ice is considered. Specifically, the well-posedness theory of a one-dimensional two-phases Stefan problem is developed. This problem models the phase transition between liquid and solid in the case in which the heat

is transferred by conduction in both phases and also by radiation in the solid phase of the material. This chapter is a summary of the article *"Well-posedness for a two-phase Stefan problem with radiation"* [39], whose latest version is in Appendix D.

Chapter 6 continues the study of the free boundary problem introduced in Chapter 5 and it summarizes the results achieved in the article *"Traveling waves for a Stefan problem with radiation"*, which can be found in Appendix E. Specifically, the existence of traveling wave solutions for this problem is obtained and the expected long-time asymptotic is derived.

Finally, Chapter 7 summarizes the main achievements obtained in this thesis and presents various open problems which give a possible future research direction in the study of the radiative transfer equation.

Appendices A to E include the articles upon which this thesis is based.

Contents

Chapter 1

Introduction

The interaction of electromagnetic waves, i.e. radiation, with matter is a fascinating problem which has been considered for long time both in mathematical and physical applications. It is indeed a phenomenon with vital consequences for everyday life. It is thanks to the interaction of sunlight with the atmosphere, for example, that the sky is blue and that the terrestrial temperature allows life on the planet. Also plant photosynthesis is made possible by the incoming radiation from the Sun. These are just few important examples of how electromagnetic radiation surrounds our world. Over the years many technological applications based on the radiative theory have been developed. For instance the results on the study of matter-photons interaction are used for the investigation of the ocean surfaces composition through remote sensing applied to oceanography, for non-invasive imaging in biomedical techniques, for the study of planets and galaxies in astrophysics, and for the correct design of steel furnaces in industrial and engineering applications. Mathematically, the interaction of matter with radiation can be described by a kinetic equation called radiative transfer equation.

This thesis studies several mathematical problems involving the radiative transfer equation and it provides solutions to those issues. In particular, in this work we present new results concerning the well-posedness theory for the stationary radiative transfer equation, its diffusion approximation, as well as a free boundary problem modeling the melting of ice in a situation where heat is transported also by radiation. Before describing with more details the problems under consideration, we give an introduction to the model, the derivation of the radiative transfer equation and its physical justification.

1.1 The radiative transfer equation

The radiative transfer equation is the kinetic equation which describes the behavior of photons interacting with matter. An extensive explanation of its derivation and of its form can be found in Chapter 6 of [108], Chapter 3 of [114] and Chapter 2 of [152], upon which this section is based. According to quantum mechanics, radiation is composed by photons. These have energy $h\nu$, where $\nu > 0$ is the frequency of the electromagnetic wave and $h = 6.62607015 \times 10^{-34}\ J \cdot s$ is the Planck constant. It is well-known that photons have a double nature. They can behave as (electromagnetic) waves or as (massless) particles. In this work we will always consider photons to be like particles. In particular, effects like diffraction and interference are ignored. Despite being massless, photons moving with speed of light $c = 2.997 \times 10^8\ m \cdot s^{-1}$ in direction $n \in \mathbb{S}^2$ have a momentum $\frac{h\nu}{c}n$. We will assume throughout this thesis that the photons have constant speed c. From a kinetic point of view one can describe the radiation by the distribution function $f(\nu, t, x, n)$ of photons with frequency $\nu > 0$ at position $x \in \mathbb{R}^3$ traveling at time $t > 0$ in direction $n \in \mathbb{S}^2$. Since photons move with speed c and have energy $h\nu$, the spectral radiation intensity, also known as specific radiation intensity, i.e. the radiative energy passing per unit of time through a unit surface perpendicular to the direction of motion n of photons with frequency $\nu > 0$, is given by

$$I_\nu(t,x,n) = h\nu c f(\nu,t,x,n).$$

Hence, both f and I_ν can be used in order to express the radiation field, which is determined by the radiative energy transported by photons as a function of time, frequency, position and direction. However, it is common in the mathematical literature to study the behavior of radiation through its spectral radiation intensity I_ν, which will be simply called radiation intensity throughout this thesis.

When radiation passes through matter, it interacts with it and it changes. Using the terminology of kinetic theory, photons colliding with matter can be lost due to their absorption or scattering by atoms, molecules or electrons. On the the other hand, there can be a gain of photons as a consequence of emission of radiation due to de-excitation of electrons as well as of scattering. The radiative transfer equation takes hence the form of the following transport equation

$$\frac{1}{c}\partial_t I_\nu(t,x,n) + n\cdot\nabla_x I_\nu(t,x,n) = (\delta I_\nu)_+ - (\delta I_\nu)_- \,, \tag{1.1}$$

where $(\delta I_\nu)_+$ and $(\delta I_\nu)_-$ are denoted as the gain and the loss terms, respectively. The structure of (1.1) is very common for kinetic equations. The gain term describes the increase of radiative energy resulting by emission and scattering of photons, the loss term expresses its reduction caused by absorption and scattering. As in [108,114,152] we assume that the attenuation of radiation is proportional to the radiation intensity with proportionality coefficient κ_ν. Thus, equation (1.1) can be written as

$$\frac{1}{c}\partial_t I_\nu(t,x,n) + n\cdot\nabla_x I_\nu(t,x,n) = \beta_\nu - \kappa_\nu I_\nu(t,x,n), \tag{1.2}$$

where $\kappa_\nu = \kappa_\nu^a + \kappa_\nu^s$ is the sum of the absorption and scattering coefficients and $\beta_\nu = e_\nu + s_\nu$ is the total emission parameter. The so-called emission parameter e_ν describes the creation of photons by de-excitation of electrons, while s_ν gives the amount of radiation of a given frequency ν and direction $n \in \mathbb{S}^2$ gained through scattering. In the following section we will give the physical justification for the form that the emission parameter takes as well as for the absorption and scattering coefficients.

Under the assumption of local thermal equilibrium, i.e. assuming that on every point of the material there is a well-defined temperature, or specifically, assuming that the fluid interacting with radiation is described by the Boltzmann, Maxwell or Saha distribution (cf. [114]), the radiative transfer equation takes the form

$$\begin{aligned}\frac{1}{c}\partial_t I_\nu(t,x,n) + n\cdot\nabla_x I_\nu(t,x,n) &= \alpha_\nu^a \left(B_\nu(T(t,x)) - I_\nu(t,x,n)\right)\\ &\quad + \alpha_\nu^s \left(\int_{\mathbb{S}^2} K_\nu(n,n')I_\nu(t,x,n')dn' - I_\nu(t,x,n)\right). \end{aligned}\tag{1.3}$$

The function $B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{\frac{h\nu}{kT}}-1}$ is the Planck distribution which satisfies the well-known Stefan-Boltzmann law

$$\int_0^\infty B_\nu(z)d\nu = \sigma z^4 \tag{1.4}$$

for $\sigma =: \frac{2\pi^4 k^4}{15h^3c^2}$, cf. [133]. Equation (1.3) is the radiative transfer equation that we will consider in this thesis.

1.2 Derivation of the gain and loss terms

In this section we give the physical intuition behind the form of the gain and loss terms appearing in (1.3). In particular, we are interested in the derivation of the emission term $\alpha_\nu^a B_\nu(T)$ and of the scattering operator $\alpha_\nu^s \int_{\mathbb{S}^2} K_\nu(n,n')I_\nu(t,x,n')dn'$. As we explained in the Section 1.1, radiation interacts with matter by emission-absorption processes and scattering processes. These two mechanisms have very different nature and they lead to distinct behaviors of the radiation-matter system. The following subsections contain the phenomenological derivation of the radiative transfer equation which is achieved by explaining these processes and they are based on Chapter 1 and 3 of [114], Section 6.3 of [108] and Chapter 2 of [152].

1.2.1 Emission and absorption

The emission-absorption process takes place whenever a photon is absorbed or emitted by an electron changing its quantum state. When a photon interacts with matter, it can indeed be absorbed by an electron, which is excited and passes to a higher energy level. Photons are consequently emitted whenever an electron de-excites. There are three kinds of electronic transitions that we have to take account of, which are known in the physical literature as bound-bound, bound-free and free-free transitions.

The bound-bound transition takes place in atoms, molecules or ions when an electron excites as a consequence of the absorption of a photon and jumps to a higher energy state. In this case the emission is a consequence of the de-excitation of the electron to its original energy state. The spectrum of transition energies for bound-bound transitions is discrete.

The bound-free transition arises when the energy of the photon absorbed by the electron is much higher than its binding energy. This results in the liberation of the electron, which corresponds to the well-known photoelectric effect (cf. [44]). In the bound-free transitions photons are emitted when the free electrons are caught by positive ions and are consequently bound to them. Unlike the case of bound-bound transitions, the spectrum of transition energies for the bound-free transitions is continuous.

Finally, free-free transitions occur mostly in plasma, i.e. the state of matter composed by ions and free electrons. In this case, when traveling near a positive ion, an electron can decelerate emitting a photon and reducing its kinetic energy (the so-called Bremsstrahlung). A free electron can also absorb radiation increasing its kinetic energy. Like in the case of bound-free transitions, the spectrum of transition energy is continuous.

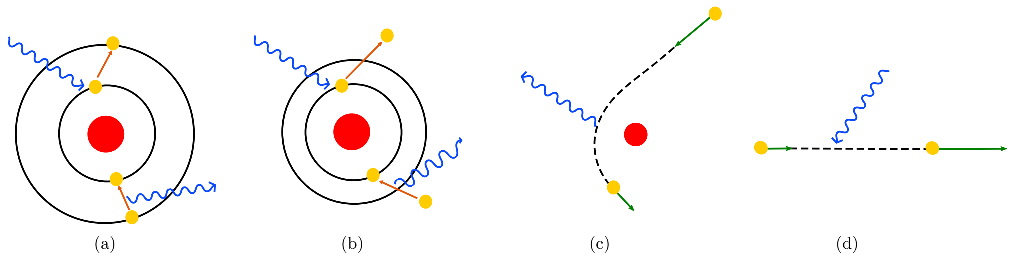

Figure 1.1: Schematic illustrations of the electronic transitions: (a) bound-bound transition, (b) bound-free transition, (c) Bremsstrahlung and (d) absorption in case of free-free transition.

The amount of absorbed radiation is proportional to the radiation intensity I_ν. Therefore, the loss term due to absorption takes the form

$$-\kappa_\nu^a I_\nu, \tag{1.5}$$

where κ_ν^a is the absorption coefficient.

The derivation of the emission term is more involved. First of all, the emission depends on the amount of electrons in the excited state. More precisely, the emission parameter is proportional to the number of such electrons. Moreover, the number of excited states increases with the temperature, cf. Section 7.10 in [152]. Hence, the emission parameter grows with the temperature. In order to derive carefully the emission parameter, besides the spontaneous emission process we have to examine the stimulated emission, whose theory is based on quantum field theory.

Specifically, the emission parameter e_ν has to take into account both the spontaneous and the stimulated emissions. The spontaneous emission of radiation depends uniquely on the physical and chemical properties of the irradiated material, such as its temperature and its atomic composition. In particular it is independent of the radiation present in the system.

The presence of a stimulated emission is due to quantum statistics, which combined with the well-known principle of detailed balance yields the exact form of the emission parameter. According to the principle of detailed balance, which is satisfied by the radiation processes, in a system at equilibrium each transition $i \to f$ from an initial state i to a final state f is compensated by its inverse transition, cf. Chapter 1 in [114]. It turns out however that the spontaneous emission is not strong enough to balance the absorption. The balance has to be understood in terms of equality of reaction rates, i.e. $[i \to f] = [f \to i]$. In particular, the reaction rate $[i \to f]$, which is proportional to the probability of transition $p(i \to f) = p(f \to i)$, depends strongly on the type of particles involved in the reaction. Specifically, $[i \to f]$ is proportional to the number of particles undertaking the transition and to the number of quantum states of the particles obtained after the reaction. However, when in the final state of the reaction photons are produced, which are bosons following Bose-Einstein statistics, the transition rate is proportional also to $(1+N)$, where N is the number of bosons present in the same quantum state of the considered gained photon. The quantum state of a photon is characterized by its momentum p and its polarization. Since $p = \frac{h\nu}{c} n$, the number of photons with a particular quantum state is

$$N = \frac{c^2}{2h\nu^3} I_\nu,$$

where the factor 2 is a consequence of the two possible linearly independent polarization states of a photon.

Thus, quantum statistic implies that the emission parameter is given by

$$e_\nu = \varepsilon_\nu \left(1 + \frac{c^2}{2h\nu^3} I_\nu\right), \tag{1.6}$$

where ε_ν is the spontaneous emission parameter, and $\varepsilon_\nu \frac{c^2}{2h\nu^3} I_\nu$ is the induced (or stimulated) emission term. The idea behind the stimulated emission is that the presence of radiation favors the emission of photons in the same quantum state.

A consequence of the principle of detailed balance is that in thermodynamic equilibrium, where $I_\nu = B_\nu(T)$, the emission e_ν and the absorption $\kappa_\nu^a I_\nu$ need to be equal. Thus,

$$\frac{\varepsilon_\nu}{\kappa_\nu^a} = \frac{B_\nu(T)}{1 + \frac{c^2}{2h\nu^3} B_\nu(T)} = \frac{2h\nu^3}{c^2} e^{-\frac{h\nu}{kT}}. \tag{1.7}$$

This is also known as Kirchhoff's Law, cf. [87]. In particular, in the case of local thermal equilibrium, which is the only case considered in this thesis, a further application of detailed balance shows that (1.7) holds also in this situation.

Finally, the spontaneous emission parameter can be written as

$$\varepsilon_\nu = \alpha_\nu^a B_\nu(T), \tag{1.8}$$

where

$$\alpha_\nu^a := \kappa_\nu^a - \frac{c^2}{2h\nu^3}\varepsilon_\nu \tag{1.9}$$

is the "phenomenological" absorption coefficient. The emission-absorption process is thus described by the following gain and loss terms

$$e_\nu - \kappa_\nu^a I_\nu = \alpha_\nu^a \left(B_\nu(T) - I_\nu\right).$$

We remark once more that we collected together the terms describing absorption and induced emission since they are proportional to the radiation intensity. This is the reason why α_ν^a is known in the literature also as phenomenological or effective absorption coefficient. A possible interpretation is to consider stimulated emission as negative absorption which attenuates the actual absorption of radiation.

1.2.2 Scattering

We now derive the scattering term in (1.3). In this thesis we consider only the most elementary model of scattering in which photons are scattered by particles at rest. We hence neglect any recoil of particles as well as any Doppler shift of the photons after the scattering. This is clearly the case when the velocity of the matter particles is much slower than the speed of light. This scattering model describes in a satisfactory way Rayleigh and Thomson scattering of radiation, in the cases in which I_ν is a slow varying function with respect to ν, as well as Rayleigh scattering for low enough temperatures, cf. [114]. Specifically, Rayleigh scattering describes the scattering of a photon by an atom or a molecule, while Thomson scattering is due to the collision of a photon with a free electron, cf. [108].

Under these assumptions the scattering process can be considered as the collision of a photon with an atom, a molecule, an ion or an electron which results in a pure deflection of the photon without changing its energy and hence its frequency $\nu > 0$. A central feature of the scattering process is that no change of energy of the photons is involved. We will see in the next section that this fact has a very important consequence in the heat transfer by radiation.

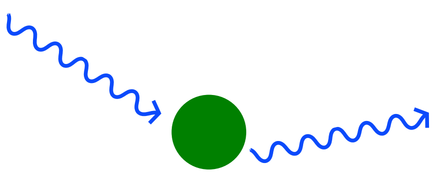

Figure 1.2: Schematic illustration of the scattering process.

Also for the scattering process we have to include both the spontaneous and the stimulated scattering. Nevertheless, the resulting gain of photons by the induced scattering is counterbalanced by the loss of radiation due to stimulated scattering, cf. Section 3.3.1 in [114]. This can be explained as follows.

We define by $K_\nu(n, n')$ the scattering kernel, i.e. the scattering rate of photons with frequency ν and initial direction of motion $n' \in \mathbb{S}^2$ and outgoing direction $n \in \mathbb{S}^2$. It is a non-negative symmetric function with total integral 1 with respect to the outgoing directions,

$$K_\nu(n,n') = K_\nu(n',n) \geq 0 \quad \forall \nu > 0,\ n,n' \in \mathbb{S}^2 \text{ and } \int_{\mathbb{S}^2} K_\nu(n,n')dn = 1 \quad \forall \nu > 0,\ n' \in \mathbb{S}^2. \tag{1.10}$$

Throughout this thesis we will assume the scattering kernel to be independent of the frequency. Moreover, we study only the situation in which the irradiated medium is isotropic, i.e. there is no preferred direction of scattering. Mathematically, we impose the scattering kernel to be invariant under rotations

$$K(n,n') = K(Rn,Rn') \quad \forall R \in SO(3),\ n,n' \in \mathbb{S}^2. \tag{1.11}$$

As it is shown later in Chapter 2 and in ([35], Appendix A), assumption (1.11) implies that the scattering kernel is symmetric with

$$K(n,n') = K(n',n) \quad \forall n,n' \in \mathbb{S}^2$$

as required in (1.10), since we consider radiation-matter systems occupying a portion of $\mathbb{R}^3$.

Turning back to the derivation of the scattering term in (1.3), we observe that using Bose-Einstein statistic the gain term due to scattering is

$$\alpha_\nu^s \int_{\mathbb{S}^2} K_\nu(n,n') I_\nu(t,x,n')dn' \left(1 + \frac{c^2}{2h\nu^3} I_\nu(t,x,n)\right), \tag{1.12}$$

where α_ν^s depends on the density of scattering particles and on the total scattering cross-section, but it is independent of the direction $n \in \mathbb{S}^2$.

Moreover, the loss term due to scattering is given by

$$\begin{aligned}\alpha_\nu^s I_\nu(t,x,n) &\int_{\mathbb{S}^2} K_\nu(n',n)\left(1 + \frac{c^2}{2h\nu^3} I_\nu(t,x,n')\right) dn' \\ &= \alpha_\nu^s I_\nu(t,x,n) + \alpha_\nu^s \frac{c^2}{2h\nu^3} I_\nu(t,x,n) \int_{\mathbb{S}^2} K_\nu(n,n') I_\nu(t,x,n')dn'. \end{aligned}\tag{1.13}$$

The formulations of the gain and loss terms (1.12) and (1.13) due to scattering are justified by the use of quantum statistic. As explained in Section 1.2.1, the reaction rate of a transition of photons is proportional to $(1+N)$, where $N = \frac{c^2}{2h\nu^3} I_\nu(t,x,n)$ is the number of photons before the reaction in the same quantum state of the considered final photon, i.e produced by the transition. Therefore, in the gain scattering term (1.12) we multiplied by $\left(1 + \frac{c^2}{2h\nu^3} I_\nu(t,x,n)\right)$, where $I_\nu(t,x,n)$ is the radiation intensity after the scattering. In the loss term (1.13) we multiplied by $\int_{\mathbb{S}^2} K_\nu(n',n)\left(1 + \frac{c^2}{2h\nu^3} I_\nu(t,x,n')\right) dn'$, which represents the radiation intensity produced by the scattering of photons moving in direction $n \in \mathbb{S}^2$. Hence, the scattering term in (1.3) is given by the combination of (1.12) and (1.13) as

$$\alpha_\nu^s \left(\int_{\mathbb{S}^2} K_\nu(n,n') I_\nu(t,x,n')dn' - I_\nu(t,x,n)\right), \tag{1.14}$$

Thus, in this model, induced scattering can be mathematically neglected. It is important to notice that this is not the case in the emission-absorption process. This is due to Bose-Einstein statistic. Indeed, in the absorption process no photons (and hence bosons) are produced. Therefore, there is no stimulated absorption and the loss term due to absorption is given by (1.5).

We remark that the properties (1.10) are natural since K is the rate of scattering. In particular, the probability of a photon moving in a given direction $n' \in \mathbb{S}^2$ to be scattered with any outgoing direction $n \in \mathbb{S}^2$ has to be 1. This implies in particular that the total integral on the sphere of directions $\mathbb{S}^2$ of the scattering terms (1.14) is zero. Indeed, changing the order of integration and using (1.10) we obtain easily that

$$\begin{aligned}\alpha_\nu^s \int_{\mathbb{S}^2} \left(\int_{\mathbb{S}^2} K_\nu(n,n') I_\nu(t,x,n') dn' - I_\nu(t,x,n) \right) dn & \\ = \alpha_\nu^s \left(\int_{\mathbb{S}^2} I_\nu(t,x,n') dn' - \int_{\mathbb{S}^2} I_\nu(t,x,n) dn \right) &= 0\end{aligned} \tag{1.15}$$

Moreover, the assumption of a non-isotropic material would lead to a different radiative transfer equation, where possibly also stimulated scattering appears. Indeed, if $K_\nu(n,n')$ is not symmetric, the combination of (1.12) and (1.13) does not have to imply (1.14) anymore.

As a matter of fact, scattering is mostly due to the interaction of photons with free electrons, cf. Section 2.3 in [152]. Moreover, in many applications dealing with the interaction of radiation with atmosphere under terrestrial conditions, the contribution of scattering is negligible compared to the emission absorption processes and thus it can be considered $\alpha_\nu^s \equiv 0$. However, this is not the case in astrophysics, where scattering can become more important than the emission and absorption processes. See [152] for more details.

This concludes the derivation of the radiative transfer equation as stated in (1.3). We remark at this point that even if the derivation of (1.3) is purely phenomenological, it describes in a precise way the interaction of matter with radiation. To the author's knowledge there have been some attempts in the derivation of the radiative transfer equation from the Maxwell's equation. While many of them remained on a formal level describing carefully only the scattering processes, as for example [110], a rigorous result starting from the wave equation satisfied by a body with very separated scatterers and for a radiation with wavelength much shorter than the scatterers' distances can be found in [10].

1.3 Radiative heat transfer

The problems analyzed in this thesis have as overall aim the description of the distribution of the temperature in a body where the heat, and hence the energy, is transferred among others by radiation. In this case the temperature of the underlying medium becomes a further unknown of the problem and it evolves according to the laws of thermodynamics, since the body is assumed to be in local thermodynamic equilibrium. Furthermore, we remark that all the studied systems are closed thermodynamic systems, i.e. they exchange energy (through radiation) but not matter with the surrounding, cf. [153]. In particular, according to the first law of thermodynamics the change of internal energy is due to the heat production.

On the one hand, the variation of energy is linked to the change of temperature by

$$\partial_t E = C_v \partial_t T, \tag{1.16}$$

where $C_v = \frac{\delta Q}{\delta T}$ is the volumetric heat capacity, also known as specific heat, which is defined as the amount of energy per unit of time and of volume required in order to raise the temperature by δT, cf. Chapter 1 in [108].

On the other hand, the total production energy rate due to radiation per unit of volume is

$$\int_0^\infty \int_{\mathbb{S}^2} \alpha_\nu^a \left(I_\nu(t,x,n) - B_\nu(T(t,x)) \right) dn\, d\nu, \tag{1.17}$$

cf. Section 2.9 in [152]. Notice that the body increases its energy absorbing radiation while it decreases its energy re-emitting photons. Moreover, we used also that during the process of scattering no exchange of energy between photons and scatterers takes place. Thus, equations (1.3), (1.15) and (1.17) yield that the production energy rate due to radiation can be written as

$$\int_0^\infty \int_{\mathbb{S}^2} \alpha_\nu^a \left(I_\nu(t,x,n) - B_\nu(T(t,x))\right) dn\, d\nu$$
$$= -\frac{1}{c}\int_0^\infty \int_{\mathbb{S}^2} \partial_t I_\nu(t,x,n) dn\, d\nu - \operatorname{div} \int_0^\infty \int_{\mathbb{S}^2} n\, I_\nu(t,x,n) dn\, d\nu. \quad (1.18)$$

Hence, the first law of thermodynamics implies that the temperature of a body where heat is transferred only by radiation evolves according to the following energy balance equation

$$C_v \partial_t T(t,x) + \frac{1}{c}\int_0^\infty \int_{\mathbb{S}^2} \partial_t I_\nu(t,x,n) dn\, d\nu + \operatorname{div} \int_0^\infty \int_{\mathbb{S}^2} n\, I_\nu(t,x,n) dn\, d\nu = 0, \quad (1.19)$$

where the radiation intensity I_ν solves the radiative transfer equation (1.3).

In most of the physical and engineering applications, the characteristic time needed by photons for traveling a length of the same order of the radiation mean free path is much shorter than the characteristic time necessary for temperature changes of order 1, cf. [108,152]. Thus, very often the radiation intensity can be considered quasi-static and the radiative transfer equation and the energy balance equation can be simplified to

$$n \cdot \nabla_x I_\nu(t,x,n) = \alpha_\nu^a \left(B_\nu(T(t,x)) - I_\nu(t,x,n)\right)$$
$$+ \alpha_\nu^s \left(\int_{\mathbb{S}^2} K_\nu(n,n') I_\nu(t,x,n') dn' - I_\nu(t,x,n)\right), \quad (1.20)$$

and

$$C_v \partial_t T(t,x) + \operatorname{div} \int_0^\infty \int_{\mathbb{S}^2} n\, I_\nu(t,x,n) dn\, d\nu = 0. \quad (1.21)$$

Notice that in this case the problem concerning the radiative heat transfer, i.e. the coupled equations (1.20) and (1.21), is not stationary. The radiation intensity depends on time through the temperature T, which evolves as time flows.

In some of the problems studied in this thesis we will consider the stationary problem of radiative heat transfer, i.e. we will assume that the radiation intensity and the temperature are time-independent. In this case the energy balance equation (1.19) reduces to the divergence-free condition for the flux of radiative energy as

$$\operatorname{div} \int_0^\infty \int_{\mathbb{S}^2} n\, I_\nu(x,n) dn\, d\nu = 0. \quad (1.22)$$

In the thermodynamical context, this condition corresponds to the assumption of pointwise radiative equilibrium, according to which at every point of the body the incoming and the outgoing energy flux is balanced. See Section 6.4 in [108].

Very interesting problems arise from the study of the evolution of the temperature in bodies where the heat is transferred by both conduction and radiation. The free boundary problem examined in Chapter 5 (resp. Appendix D) and in Chapter 6 (resp. Appendix E) is one of such examples. In this case, the total heat production rate takes into account both processes and it is given by

$$K \operatorname{div}(\nabla_x T(t,x)) + \int_0^\infty \int_{\mathbb{S}^2} \alpha_\nu^a \left(I_\nu(t,x,n) - B_\nu(T(t,x))\right) dn\, d\nu,$$

where we used the well-known Fourier law describing heat conduction and we denoted by denoting by K the volumetric conductivity of the material. Thus, under the quasi-static reduction for the radiation intensity, the temperature of the body evolves according to

$$C_v \partial_t T(t,x) - K\Delta_x T(x,t) + \operatorname{div} \int_0^\infty \int_{\mathbb{S}^2} n\, I_\nu(t,x,n) dn\, d\nu = 0. \tag{1.23}$$

Observe that the assumption of quasi-staticity of the radiation is a good approximation in the case of conductive and radiative heat transfer, as we explained above.

At the beginning of this Section 1.3 we remarked that a body where the heat is transported by radiation is an example of a so-called closed thermodynamical system, that is, a system which exchanges energy but not matter with its external environment. At this point we specify the mathematical assumptions on the body and its boundary conditions.

In the problems examined in chapters 2, 3 and 4 we consider an open bounded convex domain $\Omega \subset \mathbb{R}^3$ with sufficiently smooth boundary and strictly positive curvature.

In chapters 5 and 6 we study a free boundary problem for melting of ice. In that case we assume that the heat is transported by radiation and by conduction in the solid phase of the material which is the unbounded half-space $\Omega_t = (s(t), \infty) \times \mathbb{R}^2$. The moving interface is parameterized by $\{s(t)\} \times \mathbb{R}^2$ and it represents also the boundary of the domain Ω_t. Moreover, in the liquid part of the material $\mathbb{R}^3 - \bar{\Omega}_t$ the heat is transported by conduction only. For more details on the free boundary problem we refer to Section 1.6.3.

In all the problems considered in this thesis, the boundary conditions imposed to the radiation intensity are the so-called incoming boundary conditions. These constraints are defined as follows

$$I_\nu(t,x,n) = g_\nu(t,n) \quad \text{for } x \in \partial\Omega \text{ and } n \cdot n_x < 0, \tag{1.24}$$

where $n_x \in \mathbb{S}^2$ is the outer normal at $x \in \partial\Omega$. In all the problems analyzed in this work the source of radiation is independent of $x \in \partial\Omega$.

For the free boundary problem considered in chapters 5 and 6 the boundary conditions for the radiation intensity reduce to

$$I_\nu(t,x,n) = g_\nu(t,n) \quad \text{for } x \in \{s(t)\} \times \mathbb{R}^2 \text{ and } e_1 \cdot n_x > 0,$$

where $e_1 = (1,0,0)$.

1.4 The neutron transport equation

We now introduce some properties of the neutron transport equation, which in some cases has some analogies with the radiative transfer equation. This is the kinetic equation describing the interaction of neutron and matter which is usually composed by massive atoms. This equation is largely used in engineering and physical applications concerning nuclear reactions and the design of nuclear reactors. For an extensive derivation and explanation of the neutron transport theory we refer to [33, 88, 118], upon which this brief introduction is based.

The two main processes through which neutrons interact with heavy atoms are scattering and fission. Similarly as for the radiative transfer equation, scattering is described as the elastic collisions of neutrons with much heavier particles at rest. On the other hand, fission is a different mechanism than the one considered for the radiation. This nuclear reaction consists in the scission of a nucleus into two or more nuclei. It can be spontaneous or stimulated. For example, in a neutron-induced fission, as a result of the collision with a neutron, the massive atom splits into atoms producing in addition free neutrons, cf. [151]. The kinetic equation describing these processes is a transport equation for the neutron flux density and its velocity formulation has the form

$$\frac{1}{|v|}\partial t\varphi(x,t,v) - n\cdot\nabla_x\varphi(x,t,v) = S(x,t,v) - \Sigma_t(x,v)\varphi(x,t,v) + \Sigma_s(x,v)\left(\int_{\mathbb{R}^3} K(x,v,v')\varphi(x,t,v')dv'\right), \quad (1.25)$$

where φ is the neutron flux density, i.e. the density of neutrons with velocity $v\in\mathbb{R}^3$ passing through a unit surface at the point $x\in\mathbb{R}^3$ normal to the direction of motion $n=\frac{v}{|v|}$ at time $t>0$. As for the scattering of photons, the scattering kernel satisfies

$$\int_{\mathbb{R}^3} K(x,v,v')dv = 1 \quad \forall x\in\mathbb{R}^3,\ v'\in\mathbb{R}^3.$$

Moreover, Σ_s is the scattering coefficient and Σ_t is the collision coefficient which represents the rate in which neutrons are lost because of collisions with nuclei resulting in either fission or scattering. Finally, $S(x,t,v)$ is the source term which describes for instance the production of neutrons due to fission. Note that the gain term due to scattering in (1.25) is very similar to the one of the radiative transfer equation (1.3).

A largely studied approximation, which simplifies the transport equation (1.25) and it is used in technological applications, is the so-called one-speed approximation. This problem has been also extensively considered in the mathematical literature (cf. for instance [19, 147]). In this model it is assumed that all neutrons travel with the same speed $|v|$. Thus, in the absence of fission and of any other neutron sources, and assuming that the scattering coefficient does not depend on the direction of motion of the neutrons, the one-speed approximation of the neutron transport equation takes the form

$$\frac{1}{|v|}\partial t\varphi(x,t,v) - n\cdot\nabla_x\varphi(x,t,v) = \Sigma_s(x)\left(\int_{\mathbb{S}^2} K(x,n,n')\varphi(x,t,n')dn' - \varphi(x,t,n)\right), \quad (1.26)$$

where now $\Sigma_t=\Sigma_s$ since the only collision process occurring is the scattering. This equation is reminiscent to the radiative transfer equation. First of all, in the absence of the emission-absorption term, the one-speed neutron transport equation and the radiative transfer equation are equivalent pointwise for every frequency $\nu>0$.

Furthermore, the stationary one-speed neutron transport equation is equivalent to the stationary radiative transfer equation (1.20) coupled with the divergence-free condition of the radiation flux (1.22) in some approximation regimes, like the so called Grey approximation, i.e. assuming that the absorption and scattering coefficients are independent of the frequency.

Let indeed $\alpha_\nu^a=\alpha^a$ and $\alpha_\nu^s=\alpha^s$ be independent of $\nu>0$ satisfying $\alpha^a(x)+\alpha^s(x)>0$ for all $x\in\Omega\subset\mathbb{R}^3$, and let (I_ν,T) solve equations (1.20) and (1.22). Defining the total radiation intensity

$$u(x,n) = \int_0^\infty I_\nu(x,n)d\nu$$

and integrating (1.20) with respect to ν, we obtain the following system

$$\begin{cases} n\cdot\nabla_x u(x,n) = \alpha^a\left(\int_0^\infty B_\nu(T(x))d\nu - u(x,n)\right) + \alpha^s\left(\int_{\mathbb{S}^2} K(n,n')u(x,n')dn' - u(x,n)\right), \\ \operatorname{div}\int_{\mathbb{S}^2} nu(x,n)\,dn = 0. \end{cases}$$

The isotropy of $B_\nu(T)$ and the properties of the scattering kernel imply as for (1.15)

$$\int_0^\infty B_\nu(T(x))d\nu = ⨍_{\mathbb{S}^2} u(x,n).$$

Finally, defining

$$\alpha = \alpha^a + \alpha^s \quad \text{and} \quad \mathbb{K}(x,n,n') = \frac{\alpha^a(x)}{4\pi\alpha(x)} + \frac{\alpha^s(x)}{\alpha(x)} K(n,n'),$$

we conclude that the total radiation intensity u solves

$$n \cdot \nabla_x u(x,n) = \alpha(x) \left(\int_{\mathbb{S}^2} \mathbb{K}(x,n,n') u(x,n')\, dn' - u(x,n) \right),$$

which has the form of a stationary one-speed neutron transport equation (1.26).

Since they are equivalent in some regimes, several of the results obtained for the one-speed neutron transport equation apply also to the radiative transfer equation in the Grey approximation. This is the case for the diffusion approximation of the one-speed neutron transport equation in [19, 76, 146–149]. Nevertheless, it is important to notice that the radiative transfer equation and the one-speed neutron transport equation are in general very different. Therefore, the development of an independent mathematical theory for the transfer of radiation itself would be relevant.

To start with, we observe that when the absorption or the scattering coefficient depends explicitly on ν the one-speed neutron transfer and the radiative transfer equations are no longer equivalent.

Since the neutron transport equation and the radiative transfer equation describe physical phenomena that are fundamentally different, the problems involving these two equations are also mathematically distinct. Consider for instance the problem of radiative heat transfer, the main topic of this thesis. When emission-absorption processes take place, the temperature of the system changes. Thus, the temperature is a further unknown of the radiative transfer equation, which has to be coupled to the energy balance equation (1.19) in order to describe the evolution of the temperature due to the interaction with radiation. On the contrary, the neutron transport equation has as unique unknown the density flux of the neutrons. Therefore, any study concerning the radiative transfer is inevitably different. Besides, even in the cases where mathematically the one-speed neutron transport and the radiative transfer equation are equivalent, the radiation problem can be studied with different approaches in which the temperature and the coupling of the radiative transfer equation (1.1) with the energy balance equation (1.19) play a fundamental role. In other words, these different perspectives make the investigation's methods for the radiative transfer equation richer. An example is the study of the diffusion approximation as in Chapter 4.

1.5 Reduction to a non-local integral equation

Before introducing the main results of this thesis, we give an example of an approach for the radiative transfer equation which cannot be used in the study of neutron transport. The method that we are going to present is largely used in chapters 2, 4, 5 and 6 and it consists in the reduction of the radiative heat transfer system to a non-local equation for the temperature. We consider the problem given by the stationary radiative transfer equation (1.20) in the absence of scattering coupled with the energy balance equation (1.21) or with the divergence-free condition for the radiation flux (1.22). For simplicity we consider in this subsection the fully stationary case, hence the system

$$\begin{cases} n \cdot \nabla_x I_\nu(x,n) = \alpha_\nu \left(B_\nu(T(x)) - I_\nu(x,n) \right) & x \in \Omega, \\ \operatorname{div} \left(\int_0^\infty \int_{\mathbb{S}^2} n\, I_\nu(x,n) dn\, d\nu \right) = 0 & x \in \Omega, \\ I_\nu(t,x,n) = g_\nu(n) & x \in \partial\Omega \quad n \cdot n_x < 0, \end{cases} \tag{1.27}$$

where $\Omega \subset \mathbb{R}^3$ is open, bounded and convex. It is important to observe that this problem depends non-trivially on the frequency $\nu > 0$. Indeed, even though the radiative transfer equation can be solved knowing the temperature for any $\nu > 0$, the divergence-free condition for the radiation flux makes the dependence of the temperature, and consequently of the radiation intensity, on the frequency more involved.

We show now that problem (1.27) can be reduced to a non-local integral equation for a function which depends only on the temperature. The main idea is to solve the radiative transfer equation by characteristics and in a second step to apply the divergence-free condition to the characteristic formulation of the radiation intensity in order to obtain an equation for T. To this end we define for any $x \in \Omega$ and $n \in \mathbb{S}^2$ the boundary point $y(x,n) = x - s(x,n)n$, where $s(x,n) = |x - y(x,n)|$ is the distance of x to the boundary moving in direction $-n$, i.e. $y(x,n) = \{x - tn : t > 0\} \cap \partial\Omega$.

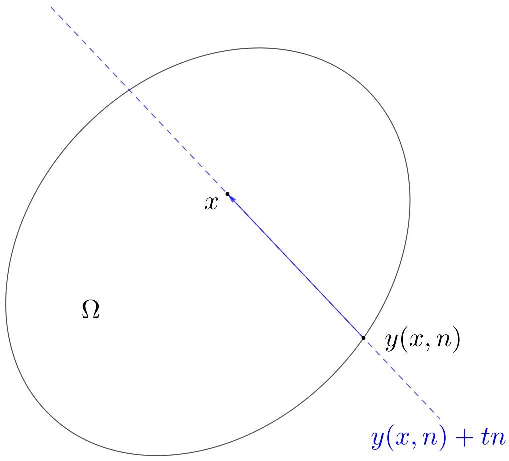

Figure 1.3: Representation of the backwards characteristics.

Solving (1.20) by characteristic we obtain

$$I_\nu(x,n) = g_\nu(n) \exp\left(\int_{[x,y(x,n)]} \alpha_\nu(\xi) d\xi\right) + \int_0^{s(x,n)} \alpha_\nu(x-tn) B_\nu(T(x-tn)) \exp\left(-\int_{[x,x-tn]} \alpha_\nu(\xi) d\xi\right) dt, \quad (1.28)$$

where we used the notation $\int_{[a,b]} f(\xi)d\xi = \int_0^{|a-b|} f\left(a - t\frac{a-b}{|a-b|}\right) dt$. Equation (1.22) implies now

$$\int_0^\infty \int_{\mathbb{S}^2} \alpha_\nu(x) \left(B_\nu(T(x)) - I_\nu(x,n)\right) dn\, d\nu = 0$$

and thus

$$4\pi \int_0^\infty \alpha_\nu(x) B_\nu(T(x))\, d\nu - \int_0^\infty \alpha_\nu(x) \int_{\mathbb{S}^2} g_\nu(n) \exp\left(-\int_{[x,y(x,n)]} \alpha_\nu(\xi) d\xi\right) dn\, d\nu$$
$$- \int_0^\infty \alpha_\nu(x) \int_{\mathbb{S}^2} \int_0^{s(x,n)} \alpha_\nu(x-tn) B_\nu(T(x-tn)) \exp\left(-\int_{[x,x-tn]} \alpha_\nu(\xi) d\xi\right) dt\, dn\, d\nu = 0. \quad (1.29)$$

This non-local integral equation can be further simplified. In particular, in the Grey approximation, i.e. assuming $\alpha_\nu(x) = \alpha(x)$, we obtain after a change of coordinates

$$T^4(x) - \int_\Omega \frac{\alpha(\eta)\exp\left(-\int_{[x,\eta]}\alpha_\nu(\xi)d\xi\right)}{4\pi|x-\eta|^2} T^4(\eta)d\eta = G(x), \tag{1.30}$$

where we used the Stefan-Boltzmann law for the Planck distribution (1.4) and we defined

$$G(x) = \frac{1}{\sigma}\int_0^\infty \fint_{\mathbb{S}^2} g_\nu(n)\exp\left(-\int_{[x,y(x,n)]}\alpha_\nu(\xi)d\xi\right) dn\, d\nu.$$

Observe that considering (1.30) is enough in order to study (1.27). This procedure is the key strategy for several of the problems presented in this thesis, for instance for the results of chapters 2, 4, 5 and 6. In (1.21) the system reduces to

$$C_v\partial_t T(x,t) + 4\pi\sigma\alpha(x)\left[T^4(x,t) - \int_\Omega \frac{\alpha(\eta)\exp\left(-\int_{[x,\eta]}\alpha_\nu(\xi)d\xi\right)}{4\pi|x-\eta|^2} T^4(\eta,t)d\eta - G(x,t)\right] = 0. \tag{1.31}$$

In a similar way, it is possible to obtain a non-local integral equation also if scattering is present. We refer to Chapter 4 for more details.

Finally, we remark that the left hand side of (1.30) is a non-local integral operator acting on T^4. As we will see, it behaves like a non-local elliptic operator and it has a (global) maximum principle. The latter property is due to the following estimate

$$\begin{aligned}\int_\Omega \frac{\alpha(\eta)\exp\left(\int_{[x,\eta]}\alpha(\xi)d\xi\right)}{4\pi|x-\eta|^2} d\eta =& -\fint_{\mathbb{S}^2}\int_0^{s(x,n)} \partial_t \exp\left(-\int_{[x,x-tn]}\alpha(\xi)d\xi\right) dt\, dn\\ =& \fint_{\mathbb{S}^2}\left(1-\exp\left(-\int_{[x,y(x,n)]}\alpha(\xi)d\xi\right)\right) dn\\ \leq& 1-\exp\left(-\operatorname{diam}(\Omega)\|\alpha\|_\infty\right) < 1,\end{aligned} \tag{1.32}$$

where we assumed $\Omega \subset \mathbb{R}^3$ bounded and $\alpha \in L^\infty\left(\mathbb{R}^3\right)$.

It is not difficult to see that equation (1.32) implies the global maximum principle of the non-local integral operator in (1.30). Let $u \in C^0(\bar{\Omega})$ and $U \subset \Omega$. Assume also that $u \leq 0$ on $\partial\Omega \cup U \subset \bar{\Omega}$ and that

$$u(x) - \int_\Omega \frac{\alpha(\eta)\exp\left(-\int_{[x,\eta]}\alpha_\nu(\xi)d\xi\right)}{4\pi|x-\eta|^2} u(\eta)d\eta \leq 0 \quad \text{ in } \Omega\setminus U.$$

It can be shown that $u \leq 0$ in Ω. Indeed, let us assume that $\max\limits_{\Omega} u > 0$. We argue by contradiction. Then, there exists $x_0 \in \Omega\setminus U$ such that $\max\limits_{\bar{\Omega}} u = u(x_0) > 0$. Estimate (1.32) implies the desired contradiction since

$$\begin{aligned}0 \geq u(x_0) &- \int_\Omega \frac{\alpha(\eta)\exp\left(-\int_{[x_0,\eta]}\alpha(\xi)d\xi\right)}{4\pi|x_0-\eta|^2} u(\eta)d\eta\\ &> \int_\Omega \frac{\alpha(\eta)\exp\left(-\int_{[x_0,\eta]}\alpha(\xi)d\xi\right)}{4\pi|x_0-\eta|^2}\left(u(x_0)-u(\eta)\right)d\eta > 0.\end{aligned}$$

We remark that a maximum principle exists also in the half-space $\Omega = (0,\infty)\times\mathbb{R}^2$.

The reduction of the radiative transfer equation coupled with the energy balance equation to a non-local integral equation for the temperature and the subsequent use of the maximum principle is a powerful tool and it is the main method used for the problems in chapters 2, 4, 5 and 6.

1.6 Description of the problems: main results of the thesis

In this section we give a description of the mathematical questions considered in this thesis, the main results and the principal strategy for their proofs. The problems presented in this work include the following. In Chapter 2 we summarize the article in Appendix A about the existence theory and a compactness result for the stationary radiative heat transfer problem (1.27) for a large class of absorption and scattering coefficients. Chapters 3 and 4 deal with the diffusion approximation for the radiative transfer equation and present the articles in Appendix B and Appendix C. The main results for a one-dimensional free boundary problem for melting of ice, where in the solid the heat is transferred also by radiation, are summarized in chapters 5 and 6. The original articles about this Stefan problem can be found in Appendix D and Appendix E. Finally, Chapter 7 concludes this thesis illustrating the main problems left open in the current work.

1.6.1 Existence theory for the stationary radiative transfer equation

In Chapter 2 and in the article [35], which can be found in Appendix A, we will study the existence theory for the stationary radiative transfer equation when the absorption and scattering coefficients depend on the temperature of the body. Specifically, we consider the stationary problem given by equation (1.20), (1.22) and (1.24), which can be formulated as

$$\begin{cases} n\cdot\nabla_x I_\nu(t,x,n) = \alpha_\nu^a\left(B_\nu(T(t,x)) - I_\nu(t,x,n)\right) & \\ \qquad\qquad +\alpha_\nu^s\left(\int_{\mathbb{S}^2} K_\nu(n,n')I_\nu(t,x,n')dn' - I_\nu(t,x,n)\right) & x\in\Omega, \\ \operatorname{div}\int_0^\infty\int_{\mathbb{S}^2} n\, I_\nu(x,n)dn\, d\nu = 0 & x\in\Omega, \\ I_\nu(x,n) = g_\nu(n) & x\in\partial\Omega,\ n\cdot n_x<0, \end{cases} \tag{1.33}$$

where the absorption and scattering coefficients have the form

$$\alpha_\nu^{a,s}(T(x)) = Q_{a,s}(\nu)\alpha^{a,s}(T(x)). \tag{1.34}$$

Specifically, we consider two types of absorption coefficients and scattering coefficients: the so-called Grey approximation where $Q_{a,s}(\nu)=1$ is constant and the "pseudo Grey" approximation where $Q_{a,s}(\nu)$ does not need to be constant, cf. [68]. Under the assumption of $\Omega\subset\mathbb{R}^3$ being bounded, convex with C^2-boundary and with strictly positive curvature, and under suitable assumptions on the incoming profile g_ν and on the coefficients $\alpha_\nu^{a,s}$, we prove the existence of a solution $(T,I_\nu)\in L^\infty(\Omega)\times L^\infty\left(\Omega, L^\infty\left(\mathbb{S}^2, L^1(\mathbb{R}_+)\right)\right)$. In contrast with the case in which $\alpha_\nu^s\equiv 0$ and $\alpha_\nu^a\equiv\alpha$ is constant, where the application of the Banach fixed-point theorem implies the existence of a unique solution, when the coefficients depend on the temperature the Banach fixed-point theorem can no longer be applied.

In the cases where $\alpha_\nu^s\equiv 0$ our strategy is to reduce (1.33) to the fixed-point equation for the temperature obtained by means of a change of variables from (1.29)

$$\int_0^\infty Q^a(\nu)B_\nu(T(x))d\nu$$
$$= \int_0^\infty (Q^a(\nu))^2 \int_\Omega \frac{\alpha^a(T(\eta))\exp\left(-Q^a(\nu)\int_{[x,\eta]}\alpha^a(T(\xi))d\xi\right)}{4\pi|x-\eta|^2} B_\nu(T(\eta))d\eta\, d\nu$$
$$+ \int_0^\infty Q^a(\nu)\int_{\mathbb{S}^2} g_\nu(n)\exp\left(-Q^a(\nu)\int_{[x,y(x,n)]}\alpha(T(\xi))d\xi\right) dn\, d\nu. \quad (1.35)$$

In this case, the right hand side of (1.35) is a self-map mapping $L^\infty(\Omega)$ to $L^\infty(\Omega)$. However, it is not compact, so that Schauder fixed-point theorem cannot be applied. This is due to the properties of the non-local operators containing exponentials of integrals along lines appearing in (1.35). Therefore, we will careful consider operators of the form

$$u \mapsto \int_{\mathbb{S}^2} \exp\left(-\int_{[x,x-\lambda n]}\kappa(u)(\xi)d\xi\right) dn \text{ for } \lambda > 0 \text{ and } x \in \Omega.$$

Hence, we will first obtain a sequence of regularized solutions whose existence is implied by an application of Schauder fixed-point theorem to a regularized version of (1.35). Afterwards, using a new L^2-compactness result for operators involving integrals along lines, we will show convergence to the solution of (1.35). This key result can be understood as some kind of "averaging lemma" which is different from the ones already available and largely used in kinetic theory, as for instance [40, 41, 70, 80, 144]. A compactness result for some similar operators arising in the study of the Boltzmann equation has been developed in [7]. However, the method used in this article cannot be used for our problem.

Finally, in the case of the full equation with both scattering and emission-absorption we adapt the previous result as follows. We first find an equivalent formulation (1.33) as a fixed-point equation for the temperature similar to (1.35) constructing suitable Green's functions for two different versions of the radiative transfer equation. Then, we use the recursive equations satisfied by the Green's functions in order to define a non-linear, non-local, integral equation which contains two Duhamel series, whose terms include exponentials of integrals along lines as in (1.35). We proceed then as we did in the absence of scattering as it is summarized in Chapter 2.

1.6.2 Diffusion approximation

In Chapter 3 and in Chapter 4 we will study in detail the problem of the diffusion approximation for the system obtained coupling (1.3) with the energy balance equation (1.19) on a convex bounded domain $\Omega \subset \mathbb{R}^3$, under the assumption that the mean free path of the photons tends to zero. This is the so-called diffusion approximation regime. In materials where the mean free path of the photons is very small compared to the characteristic length of the domain, the radiation processes become almost local. Indeed, in this case photons can move only very small distances, which by definition are of the order of the mean free path, before being absorbed or scattered. See [152]. Moreover, also the radiation reaching a point $x \in \Omega$ is emitted few mean free paths away. This yields that the radiation intensity can be approximated by a density function solving a (local) diffusion problem.

In Chapter 3, which deals with the results of the article [36] in Appendix B, using matched asymptotic expansions methods we derive several possible limit problems (depending on the assumptions) for both the stationary and the time-dependent cases. In Chapter 4, based on the work [37] presented in Appendix C, we prove that in the stationary case without scattering, the radiative energy density, which is proportional to T^4 (cf. Stefan-Boltzmann law), solves in the limit, as the mean free path of the photons tends to zero, an elliptic equation where the boundary value can be determined uniquely in terms of the outer source of radiation.

Specifically, we study the system given by (1.3), (1.19) and (1.24) in the framework of the diffusion approximation assuming that the mean free path of the photons is very small. To this end we define ℓ_A and ℓ_S to be the mean free paths of the emission-absorption process and of the scattering process, respectively. We study the limit of the following problem

$$\begin{cases} \frac{1}{c}\partial_t I_\nu(t,x,n) + \tau_h n\cdot\nabla_x I_\nu(t,x,n) = \frac{\alpha_\nu^a(x)\tau_h}{\ell_A}\left(B_\nu(T(t,x)) - I_\nu(t,x,n)\right) & \\ \qquad + \frac{\alpha_\nu^s(x)\tau_h}{\ell_S}\left(\int_{\mathbb{S}^2} K(n,n')I_\nu(t,x,n')\,dn' - I_\nu(t,x,n)\right) & x\in\Omega, t>0,\\ \partial_t T + \frac{1}{c}\partial_t\left(\int_0^\infty d\nu\int_{\mathbb{S}^2} dn\, I_\nu(t,n,x)\right) + \tau_h \operatorname{div}\left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, n I_\nu(t,n,x)\right) = 0 & x\in\Omega, t>0,\\ I_\nu(0,x,n) = I_0(x,n,\nu) & x\in\Omega,\\ T(0,x) = T_0(x) & x\in\Omega,\\ I_\nu(t,n,x) = g_\nu(t,n) & x\in\partial\Omega, n\cdot n_x<0, t>0, \end{cases} \tag{1.36}$$

as the total mean free path of the photons goes to zero, i.e. $\ell_M = \min\{\ell_A,\ell_S\} = \varepsilon \to 0$. Moreover, τ_h is the heat parameter, which represents the order of magnitude of times in which the change of the temperature takes place.

In Chapter 3 we examine both the time-dependent and the stationary version of (1.36). We consider all different possible relative scalings between ℓ_A, ℓ_S and $\operatorname{diam}(\Omega)\approx 1$ constructing via matched asymptotic expansions the limit problems for the leading order of the radiation intensity I_ν. Furthermore, the equations describing different boundary layers that yield the form of the solution in that regions are derived. Indeed, since the incoming radiation g_ν is in general not isotropic, and thus also not equal to $B_\nu(T)$, two nested boundary layers can appear. In these regions situated near the boundary the radiation intensity modifies its behavior, becoming for example isotropic or even approaching the Planck distribution of the temperature.

Although as $\ell_M = \varepsilon \to 0$ the radiation intensity is isotropic at the leading order, it is not always the Planck distribution. This is due to the fact that $\ell_M \to 0$ is caused by either $\ell_A \to 0$ or $\ell_S \to 0$. The different ratio between absorption and scattering mean free paths determines wether I_ν approaches the Planck distribution $B_\nu(T)$ or not. The case where the leading order of I_ν converges in the limit to the Planck equilibrium distribution of the temperature is denoted in the literature as equilibrium diffusion approximation, while the case in which the radiation intensity differs from the equilibrium distribution is called non-equilibrium diffusion approximation (see [108,152]). In Chapter 3 and in ([36], Appendix B) we give a careful mathematical classification of these notions. In particular we derive the conditions under which the equilibrium diffusion approximation holds and we find the regions which each one of these diffusion approximations is valid in the time-dependent and stationary problems given by (1.36).

While the derivation of the limit problems in Chapter 3 is formal, in Chapter 4 and in ([37], Appendix C) we prove rigorously that in absence of scattering the radiation intensity I_ν^ε solving the stationary version of the problem (1.36) converges as $\ell_A = \ell_M = \varepsilon \to 0$ to the Planck distribution (i.e. $I_\nu^\varepsilon \to B_\nu(T)$) of a suitable temperature $T(x)$, solution of an elliptic problem. Specifically, we consider $\ell_A = \varepsilon$, $\alpha^s = 0$ as well as $\alpha_\nu^a(x) = \alpha(x)$ independent of the frequency, i.e. the Grey approximation. Under these assumptions the stationary problem

(1.36) takes the form

$$\begin{cases} n \cdot \nabla_x I_\nu(x,n) = \frac{\alpha(x)}{\varepsilon}\left(B_\nu(T(x)) - I_\nu(x,n)\right) & x \in \Omega, \\ \operatorname{div}\left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\; n I_\nu(n,x)\right) = 0 & x \in \Omega, \\ I_\nu(n,x) = g_\nu(n) & x \in \partial\Omega, n \cdot n_x < 0. \end{cases} \tag{1.37}$$

We prove that under suitable hypotheses on Ω, on g_ν and on $\alpha(x)$, the solutions to (1.37) satisfy

$$(I_\nu^\varepsilon, T^\varepsilon) \to (B_\nu(T), T) \text{ as } \varepsilon \to 0$$

uniformly in every compact set, where T solves the Dirichlet problem

$$\begin{cases} -\operatorname{div}\left(\frac{4\sigma T^3(x)}{\alpha(x)} \nabla T(x)\right) = 0 & x \in \Omega, \\ T(p) = T_\Omega[g_\nu](p) & p \in \partial\Omega, \end{cases}$$

for $T_\Omega : L^\infty\left(\mathbb{S}^2, L^1(\mathbb{R}_+)\right) \to C(\partial\Omega)$ a functional which maps g_ν to a continuous function $T_\Omega[g_\nu](p)$ on the boundary $p \in \partial\Omega$.

As noticed in Section 1.4, the specific problem examined in Chapter 4 is equivalent to a one-speed neutron transport equation. Therefore, the existing results about the diffusion approximation of the one-speed neutron transport equation apply also to the considered problem (1.37). Nevertheless, in order to prove the (equilibrium) diffusion approximation of problem (1.37) we use a different strategy than the ones that have been used for the neutron transport equation. We refer for instance to the stochastic approach in [19] and to the functional analytical techniques in [147]. Our method consists in the reduction of the problem (1.37) to a non-local elliptic integral equation for T^4 similar to (1.30) and to the analysis of the new equivalent problem via maximum principles tools. More precisely, our proof is based on applying the maximum principle to suitable supersolutions, which will be constructed adapting particular solutions of the Laplace equation. With this method we proved the diffusion approximation of the problem (1.37) also for absorption coefficients which depend non-trivially on the spatial coordinate. This is a new result which is not covered in [76, 146–149].

1.6.3 Free boundary problem with radiation

In chapters 5 and 6 and in their associated articles [39, 134] in the appendices D and E we consider a free boundary problem describing the melting of ice in a situation in which heat is transferred by conduction in the whole liquid-solid system and additionally by radiation in the solid. Specifically, we study the case in which $\mathbb{R}^3$ is filled by the liquid and solid phase of a material. The temperature T at the contact surface between these two phases is the melting temperature T_M. This interface moves according to the specific phase change which is taking place. The temperature of the liquid is larger than T_M, while $T < T_M$ in the solid.

The assumption that radiative heat transfer occurs only in the solid can be interpreted from a physical point of view assuming that the liquid is transparent, i.e. it does not interact with radiation, while the solid is more opaque letting absorption and emission take place. In this time-dependent model we consider at the initial time $t = 0$ the situation in which the liquid fills $\mathbb{R}^3_- := \left\{x \in \mathbb{R}^3 : x_1 < 0\right\}$ and the solid fills $\mathbb{R}^3_+ := \left\{x \in \mathbb{R}^3 : x_1 > 0\right\}$. Moreover, the temperature is supposed to depend spatially only on the variable x_1. To some extents this means that T depends on the distance to the interface. We examine the two-phase Stefan-like problem under the further assumption of constant Grey approximation in the absence of scattering. It reads

$$\begin{cases} C_L\partial_t T(t,x_1) = K_L\partial^2_{x_1}T(t,x_1) & x_1 < s(t), \\ C_S\partial_t T(t,x_1) = K_S\partial^2_{x_1}T(t,x_1) - \operatorname{div}\left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, nI_\nu(t,x,n)\right) & x_1 > s(t), \\ n\cdot\nabla_x I_\nu(t,x,n) = \alpha\left(B_\nu(T(t,x_1)) - I_\nu(t,x,n)\right) & x_1 > s(t), \\ I_\nu(t,x,n) = 0 & x_1 = s(t),\ n_1 > 0, \\ T(t,s(t)) = T_M & x_1 = s(t), \\ T(0,x) = T_0(x) & x_1 \in \mathbb{R}, \\ \dot{s}(t) = \frac{1}{L}\left(K_S\partial_{x_1}T(t,s(t)^+) - K_L\partial_{x_1}T(t,s(t)^-)\right) & t > 0, \end{cases} \tag{1.38}$$

where C_L, C_S are the volumetric heat capacities of the liquid and solid, respectively, K_S, K_L are the conduction coefficients, L is the latent heat and $s(t) \in \mathbb{R}$ is the interface. Notice that we consider only the situation in which no extra source of radiation is present. We remark also that the radiation I_ν can be considered quasi-static since the characteristic time in which the radiation intensity stabilizes is much shorter than the characteristic time needed for changes of the temperature of order 1. This is due to the fact that photons travel at the speed of light. It is important to observe in addition that the Stefan condition for the velocity of the moving interface is the same as the one for the classical Stefan problem. This is due to fact that the heat flux is given in both phases by

$$-K_i\partial_{x_1}T(t,x_1) + \int_0^\infty \int_{\mathbb{S}^2} I_\nu(t,x,n)n\, dn\, d\nu \text{ for } i \in \{L,S\}.$$

We recall that we assume the liquid to be transparent. Hence, radiation is present also in the liquid without interacting with it. This implies that the radiation intensity is constant in the liquid yielding

$$\operatorname{div}\left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, nI_\nu(t,x,n)\right) = 0 \text{ for } x_1 < s(t).$$

In particular, in the liquid I_ν is the constant continuation of the radiation intensity at the free boundary, i.e.

$$I_\nu(t,x,n) = I_\nu\left(t, (s(t),x_2,x_3), n\right) \text{ for } x_1 < s(t).$$

Finally, the Stefan condition is due to the balance of the heat absorbed or released during solidification or melting, respectively, and the jump of the heat flux at the interface. We refer to Chapter 5 for more details about the Stefan condition.

We can reduce the equation describing the solid phase to a non-local integral equation for the temperature in the same spirit as (1.31) in Section 1.5. This yields, together with some rescaling, the following equivalent free boundary problem

$$\begin{cases} C\partial_t T(t,x) = K\partial^2_x T(t,x) & x < s(t), \\ \partial_t T(t,x) = \partial_x^2 T(t,x) - T^4(t,x) + \int_{s(t)}^\infty \frac{\alpha E_1(\alpha|x-\eta|)}{2}T^4(t,\eta)d\eta & x > s(t), \\ T(t,s(t)) = T_M & x = s(t), \\ T(0,x) = T_0(x) & x \in \mathbb{R}, \\ \dot{s}(t) = \frac{1}{L}\left(\partial_x T(t,s(t)^+) - K\partial_x T(t,s(t)^-)\right) & t > 0, \end{cases} \tag{1.39}$$

where $E_1(x) := \int_{|x|}^\infty \frac{e^{-t}}{t}dt$ is the normalized exponential integral.

Specifically, in Chapter 5 and in ([39], Appendix D) we will prove the local well-posedness of (1.39) applying the Banach fixed-point theorem together with classical parabolic theory.

Moreover, with the help of suitable subsolutions and supersolutions, an application of the maximum principle will imply the global in time well-posedness for a large class of initial temperatures. In particular, the solution is a classical solution of (1.39) and the temperature satisfies $T(t,x) \geq T_M$ for $x < s(t)$ and $0 \leq T(t,x) \leq T_M$ for $x > s(t)$ as long as it exists.

In Chapter 6 and in ([38], Appendix 6) we continue the study of the free boundary problem (1.39) considering the existence of traveling wave solutions and studying their properties. Specifically, we will prove that such traveling waves exist only in the case $\dot{s}(t) \leq 0$, i.e. when the ice is expanding. Notice that this is in contrast with the classical Stefan problem, where self-similar profiles exist but not traveling wave solutions. The existence of traveling waves solving for $s(t) = -ct$ and $c > 0$

$$\begin{cases} c\partial_y T_1(y) = \kappa \partial_y^2 T_1(y) & y < 0, \\ c\partial_y T_2(y) = \partial_y^2 T_2(y) - T_2^4(y) + \int_0^\infty \alpha \frac{E_1(\alpha(y-\eta))}{2} T_2^4(\eta) d\eta & y > 0, \\ T_2(0) = T_1(0) = T_M, \\ c = \frac{1}{L}\left(K\partial_y T_1(0^-) - \partial_y T_2(0^+)\right). \end{cases} \tag{1.40}$$

is based on a variational argument. We will also prove that the solutions to (1.40) satisfies

$$T_1(y) > T_M \text{ on } \mathbb{R}_- \quad \text{and} \quad 0 < \lambda \leq T_2(y) < T_M \text{ on } \mathbb{R}_+$$

for some $\lambda < 0$, as well as that the limits

$$\lim_{y \to -\infty} T_1(y) > T_M \quad \text{and} \quad \lim_{y \to \infty} T_2(y) > 0$$

exist. These properties will be proved using again maximum principle tools as well as arguments involving blowup limits. Finally, we will also present the expected long-time asymptotics for (1.39).

1.7 Overview of the mathematical literature

The study of the interaction of radiation with matter has been considered in mathematics, physics and engineering for a long time. Models of matter interacting with electromagnetic waves have been deeply studied starting from the early works of Compton [31], which considers the interaction of trapped resonance radiation and gas, and of Milne [109], which studies a one-dimensional model for the behavior of a confined gas hit by radiation. Both these early works model the interaction of radiation with a diffusion equation. Some years later Kenty [85] and Holstein [78] studied the same problem. In particular, the first work took into consideration also the change of frequency between absorption and emission of photons, while in the latter article the radiation problem is studied for the first time through a non-local equation. The radiative heat transfer was considered in those years by Spiegel [131], who derived the evolution equation for the temperature (1.19).

A detailed derivation of the radiative transfer equation, a review of its properties and of its physical and engineering applications as well as of the mathematical problems can be found in [29, 108, 114, 125, 152]. Specifically, in [108, 152] the difference between equilibrium and non-equilibrium diffusion approximation is explained. Moreover, in [114] a careful derivation of the radiative transfer equation is performed starting from the principle of detailed balance, and the entropy formula for radiation is presented.

The well-posedness problem for the radiative transfer equation (1.3) without scattering term coupled with the energy balance equation (1.19) has been studied in the time-dependent case in [13–15] under the assumption of non-increasing temperature-dependent absorption coefficient using the resulting m-accretiveness of the radiation operator as well as semigroup theory. Likewise, article [107] deals with the well-posedness of the time-dependent radiative transfer equation under accretiveness assumptions. In [16] the existence result for both the time-dependent and stationary radiative transfer equation without scattering has been studied for the Grey approximation in its equivalent formulation as a one-speed neutron transport equation with constant scattering kernel, cf. (1.26). Specifically, the considered stationary problem has the from

$$\lambda u + n \cdot \nabla_x u = \alpha\left(\bar{u}\right)\left(\bar{u} - u\right), \qquad \text{where } \bar{u} = \fint_{\mathbb{S}^2} u \, dn.$$

For very general coefficients α and for $\lambda > 0$ the existence of a bounded solution is obtained using Schauder fixed-point theorem, while for $\lambda = 0$ the existence is achieved by a limiting argument, under the further assumption that $\bar{u} \mapsto \alpha(\bar{u})$ is non-increasing.

Recent developments on the well-posedness of the problem (1.33) have been achieved in [83]. Specifically, existence and uniqueness of solutions to (1.33) in suitable L^p-spaces have been proved using the following strategy.

- Combining integration along characteristics and Banach fixed-point theorem, when $\alpha_\nu^a \equiv 0$.
- Reducing (1.33) to an integral equation for T^4 (cf. (1.30)) and using Banach fixed-point theorem, when $\alpha_\nu^s \equiv 0$ and $\alpha_\nu^a(T) \equiv \alpha$ is constant.
- Applying Scahuder fixed-point theorem to the integral equation equivalent to (1.33) and proving uniqueness in a second step, when $\alpha_\nu^s \equiv 0$ and $\alpha_\nu^a(T) = \alpha_\nu^a$ is independent of T.

In the case, where both α_ν^a and α_ν^s are non-trivial and independent of the temperature, the existence of a solution is shown applying Schauder fixed-point theorem to the integral equation for the temperature, which is obtained defining suitable Green's functions. This method is similar to the one we use in Chapter 2 and ([35], Appendix A).

It is important to remark at this point that the existence theory presented in [83] and consequently in this thesis in Chapter 2 does not assume any monotonicity constraints on the absorption coefficient. In the stationary problem (1.33) there is no extra term λI_ν and λT for $\lambda > 0$, which is crucial in the proof of the well-posedness for stationary problems including these terms as it is the case in [13–16, 107]. This is used also in order to show the existence of solutions to the time-dependent problem using semigroups. Nevertheless, as pointed out in [35, 83], the existence theory for the the time-dependent problem does not imply the existence of a stationary solution. Indeed, the problem (1.33) describes a closed but not isolated system which, in other words, allows exchange of energy but not of mass. Thus, there is no entropy dissipation, while the appearance of an entropy flux is possible. Moreover, the temperature could grow in time. Finally, even if the time-dependent solution is globally bounded, it could present an oscillatory behavior and hence it could not converge to a steady state.

Since in many physical applications the mean free path of the photons is very small and the radiative transfer equation has a strong non-local behavior, which makes its study more complex, its diffusion approximation has been largely used in Physics. Using matched asymptotic expansions it is indeed formally possible to show that when the mean free paths of the photons tends to zero the leading order of the radiation intensity (and sometimes of the temperature) solves an elliptic (or parabolic in the time-dependent case) equation, whose properties, such as existence, regularity and asymptotic behavior, are well studied. In the study of the diffusion approximation problem of the radiative transfer equation, as well as of other kinetic equations such as the neutron transport equation, also problems for the boundary layers need to be considered. These are the regions close to the boundary where the diffusion approximation fails. In the case of radiation the boundary layer is described by a stationary radiative transfer equation, which depends only on the distance to the boundary (i.e. it is one-dimensional) and on the properties of the original problem. In analogy to the one-dimensional problem studied by Milne (see [109]), the class of boundary layer equations are often referred to as Milne problems in the mathematical literature.

Many results available for the diffusion approximations are related to the study of the neutron transport equation (1.25), in particular for the one-speed approximation. An extensive overview of the neutron transport theory can be found in [33], where the equivalence between the one-speed approximation and the radiative transfer equation in the Grey approximation is analyzed. From the second half of the 70's the diffusion approximation of the one-speed neutron transport equation has been exhaustively studied (at least at a formal level) especially in the framework of the scattering eigenvalue problem. Indeed, the smallest size of the system for which there is an eigenfunction with eigenvalue 1 is denoted in the physical literature as critical size and it has an important application in the design of nuclear reactors. See [33]. The neutron transport equation has been considered by Larsen and coauthors in numerous works. Specifically, [99, 100, 103] deal with the diffusion approximation of the neutron transport equation both in its general form (1.25) and in its steady one-speed approximation (1.26). In all these works the presence of a source of neutron (e.g. due to a fission process) is considered and the asymptotic behavior of the solutions for small mean free paths is studied. In [66, 98, 101, 102, 104] Milne problems, i.e. one-dimensional versions of the steady neutron transport equation, are examined under several assumptions. Furthermore, in [97] the diffusion approximation for the radiative transfer equation is studied via matched asymptotic expansions considering both emission-absorption and scattering processes and avoiding the formation of boundary layers by setting $I_\nu = B_\nu(T)$ as initial and boundary condition.

As far as we know, [19] is the first mathematical work dealing rigorously with the diffusion approximation for the neutron transport equation. Specifically, the one-speed approximation (1.26) for a strict positive, bounded and rotationally symmetric scattering kernel is considered under different boundary conditions, like for instance the incoming boundary condition similar to (1.24) for the radiative transfer equation. This article uses stochastic methods in order to study the boundary layers and to prove the convergence of the solutions to a diffusion equation as the mean free path tends to zero.

In recent times the one-speed neutron transport equation for constant scattering kernel and constant scattering coefficient has been considered for several domains in the framework of the diffusion approximation by Guo and Wu in numerous works [76, 146–149]. Via suitable $L^2 - L^p - L^\infty$ estimates they proved rigorously, for both the stationary and the time-dependent equations, the diffusion approximation obtaining a geometric correction for the boundary layer. This method has been used also in kinetic theory for other equations, such as the Boltzmann equation and the Landau equation (cf. [75, 86]).

Moving back to the diffusion approximation for the radiative transfer equation, this theory has been studied in [13–15] for the time-dependent equation without scattering term under some monotonicity assumptions for the absorption coefficient using the resulting m-accretiveness of the operator. In [16] a similar result is shown for the radiative transfer equation in the case in which it is equivalent to a one-speed neutron transport equation (1.26) with constant scattering kernel. Another proof of the time-dependent diffusion approximation in the absence of scattering can be found in [72], where a more general equation for the internal energy is considered. In [73, 74] the authors study another class of diffusion approximations for the radiative transfer equation in a one-dimensional setting and in $\mathbb{R}^3$ when the scattering length tends to zero and the emission-absorption process is bounded. We remark also that in the study of the diffusion approximation the time-dependent and the stationary problem have to be analyzed separately, since they are fundamentally different.

Regarding the Milne problems, which describe the boundary layers in the diffusion approximation, [68] shows the well-posedness for this problem in the absence of scattering (i.e. $\alpha_\nu^s \equiv 0$) and for a large class of α_ν^a, namely for absorption coefficients which have the form $\alpha_\nu(T) = Q(\nu)\alpha(T)$. This proof relies on an iterative scheme combined with the accretiveness of the radiative operator. Moreover, the asymptotic behavior as $x \to \infty$ is considered and it is proved that the solution converges exponentially to a constant. Besides the already mentioned articles about the neutron transport equation (cf. [19, 76, 146–149]), also [17] studies the Milne problem for the one-speed approximation. The Milne problem for the radiative transfer equation including both scattering and emission-absorption terms has been examined only by Sentis [127]. While the considered absorption coefficient depends on frequency and temperature and it is not necessarily increasing with respect to the latter variable, the scattering coefficient and the scattering kernel are constant. The well-posedness result and the asymptotic behavior of the solution are treated in a similar way as in [68] using the accretiveness of the operators. Finally, the well-posedness theory for Milne problems arising from various transport equations has been studied in [12], while in [30] the one-dimensional problem for non-Grey radiative transfer is considered also numerically.

In chapters 5 and 6 we study a free boundary problem for the melting of ice assuming that the heat is transferred also by radiation in the solid. This problem can be considered as a modification of the well-known two-phase Stefan problem, which takes its name from J. Stefan, the person who first formulated such melting problems (cf. [134–137]). In the following we give a summary of the results available for the Stefan problem, focusing on the well-posedness theory for the one-dimensional case, which is the same situation we consider in chapters 5 and 6.

One version of the two-phase one-dimensional classical Stefan problem on a slab can be formulated as the following free boundary problem (cf. [123])

$$\begin{cases} C_L\partial_t T(t,x) = K_L\partial_x^2 T(t,x_1) & t>0,\ a<x<s(t), \\ C_S\partial_t T(t,x) = K_S\partial_x^2 T(t,x) & t>0,\ s(t)<x<b, \\ T(t,s(t)) = T_M & x=s(t), \\ T(0,x) = T_0(x) & x\in\mathbb{R}, \\ T(t,a) = g_a(t),\ T(t,b) = g_b(t) & t>0, \\ \dot{s}(t) = \frac{1}{L}\left(K_S\partial_x T(t,s(t)^+) - K_L\partial_x T(t,s(t)^-)\right) & t>0. \end{cases} \tag{1.41}$$

As we have introduced in Section 1.6.3, C_L and C_S are the volumetric heat capacities, K_L and K_S are the conduction coefficients and L is the latent heat. The case in which the temperature in one of the two-phases is kept constant to the melting temperature is denoted in the mathematical literature as one-phase Stefan problem.

Since the early results of J. Stefan, this free boundary problem has been exhaustively analyzed. First of all, different definitions of solutions has been introduced. Some of them are the so-called classical solutions and the weak enthalpy solutions, which are defined below.

Classical solutions of the Stefan problem are strong solutions of (1.41), where no further assumptions on the temperature of the liquid and of the solid are made. In other words, it is assumed that the change of phase takes place in a specific surface (i.e. the interface) where the heat is absorbed or released. This surface divides the domain in two regions, namely the liquid and the solid. Specifically, even if at initial time the temperature T is assumed to be $T > T_M$ in the liquid and $T < T_M$ in the solid, as time flows the temperature does not have to satisfy such inequalities in the liquid and in the solid regions. This allows the emergence of supercooled regions in the liquid, where $T \leq T_M$, or of superheated regions in the solid, where $T \geq T_M$. See [89, 123, 143] for further details.

Weak enthalpy solutions are defined as the weak solutions of the following equation (cf. [57])

$$\partial_t H(t,x) - \partial_x^2 \beta(T(t,x)) = 0, \tag{1.42}$$

where the enthalpy H and the function β are defined for the problem (1.41) by

$$H(t,x) = \begin{cases} C_L(T - T_M) + L & T > T_M, \\ C_S(T - T_M) & T < T_M, \end{cases} \quad \text{and} \quad \beta(T) = \begin{cases} K_L T & T > T_M, \\ K_S T & T < T_M, \end{cases}$$

where L is the volumetric latent heat, cf. [141]. The temperature can be also written as the inverted enthalpy by

$$T(t,x) = \begin{cases} \frac{H-L}{C_L} + T_M & H > L, \\ T_M & 0 \leq H \leq L, \\ \frac{H}{C_S} + T_M & H < 0. \end{cases} \tag{1.43}$$

In the weak formulation of the Stefan problem, the Stefan condition does not appear as an extra condition for the interface, which at time $t > 0$ is determined by $\partial\{x \in (a,b) : T(t,x) = T_M\}$. It is not difficult to see that in the absence of superheated or supercooled regions, the classical solutions are also weak enthalpy solutions. Indeed, the Stefan condition at the interface is equivalent to the Rankine-Hugoniot condition for the weak formulation of (1.42). On the other hand, the weak enthalpy solutions do not allow for superheated or supercooled regions. The liquid and the solid regions are indeed defined by their temperature being larger or smaller than T_M, respectively. If one considers (1.43) it is easy to see that the set $\{(t,x) \in (a,b) : T(t,x) = T_M\}$ could have positive measure. In that case the region $\{(t,x) \in (a,b) : T(t,x) = T_M\}$ is called mushy region. See [89, 117, 143].

Moving to the well-posedness theory for classical solutions to the free boundary problem (1.41), this can be found for instance in [123]. There, the existence of local solutions to the one and two-phase one-dimensional problem is proved via Picard approximations solving a fixed-point equation with Volterra-type integral terms. An application of the maximum principle implies the global in time well-posedness. A similar approach is used also in [55, 56], where the author solves the fixed-point integral equation using Banach fixed-point theorem. Moreover, in [56] well-posedness is proved using also the so-called Baiocchi transform (cf. [8]). A variational inequality is used in [59]. The same approach is considered for the one-phase problem in [60]. In [56, 106] the authors study the asymptotic behavior of solutions to the one-dimensional one-phase problem (and in [56] also to the two-phase one). It is proved that the temperature approaches a self-similar profile as $t \to \infty$, which depends on the prescribed behavior of the temperature at the fixed boundary and to the limits of the temperature as $|x| \to \infty$ in the two-phase problem for an unbounded domain. The well-posedness theory for classical solutions of the two-phase one-dimensional Stefan problem was also considered in [26, 27]. While in the second work the authors impose a smallness assumption on the initial data, in the first article well-posedness and regularity are proved combining the properties of the Green's function of the heat equation and Schauder fixed-point theorem. Concerning the classical solutions to higher dimensional problems, in [106] the well-posedness theory is proved. In addition to that, a notion of generalized solutions, the weak enthalpy solutions of (1.7), is introduced and the formation of mushy regions is studied. The theory of weak (enthalpy) solutions has been considered for the two-phase one-dimensional problem in [57], and for the higher dimensional problem in [58].

The problem of the formation of mushy regions, of superheated liquid or of supercooled solid is also of high interest and it has been extensively studied assuming the presence of volumetric heat sources. A precise characterization of classical and enthalpy solutions is considered in [50,51]. In these articles the so-called classical enthalpy solutions are introduced. These solutions are defined such that the temperature solves the heat equation with volumetric source in the liquid ($T > T_M$) and in the solid ($T < T_M$), in the mushy regions the enthalpy satisfies a time-dependent first order PDE and at the free boundaries the Stefan condition holds strongly. Moreover, for these solutions mushy regions may appear. The emergence of supercooled liquid or superheated solid for classical solutions to a one-dimensional two-phases Stefan problem is shown in [89] assuming an external volumetric heat source. This is an interesting result which indicates why we can expect superheated regions to appear, if we consider a positive external source of radiation heating the solid in problem (1.39), i.e. considering as boundary conditions

$$I_\nu\left(t, \left(s(t), x_2, x_3\right), n\right) = g_\nu(n) > 0 \text{ if } n_1 > 0.$$

A detailed review of strong and weak solutions as well as of models for supercooling, superheating and mushy regions can be found in [143]. Mushy regions are also consider in [20,90,117,141], while models for superheated and supercooled material are provided in [142].

Another problem concerning the Stefan problem, which has been extensively studied, is the regularity of the free boundary. In this thesis we consider only a one-dimensional free boundary problem, where the moving interface is a point. However, for the higher dimensional Stefan problems, the regularity of the moving interface has been studied in terms of parabolic obstacle problems for instance in [9, 24, 25, 43, 53].

The study of melting processes finds a straight application for example in engineering. Free boundary problems modeling the phase transition due to transport of heat by conduction and radiation are studied numerically in particular in [28, 124, 129, 130, 140], where different one-dimensional models describing one, two and three-phase free boundary problems are examined. The vaporization of dew can be also modeled as a free boundary problem, where both conduction and radiation are considered. In recent years several numerical articles have shown that in the phase transition of droplets the heat transfer by radiation plays a crucial role, cf. [2, 84, 92, 93, 126, 128, 145, 150].

We conclude the summary of the available literature giving a rough overview about other results concerning more involved interaction between radiation and matter. The interplay of electromagnetic waves in a moving fluid has been considered for instance in [69, 71, 108, 152]. We refer to [34,81,112–114,122] for the study of the behavior of systems composed by photons and a Boltzmann gas, whose colliding particles have two ore more energy levels and can absorb or emit photons. The analysis of heat transport by conduction and radiation has been considered also independently from the free boundary problems. Results in this direction can be found in [32, 42, 62–65, 95, 96, 115, 116, 138, 139]. Furthermore, homogenization in porous and perforated domains has been considered in the case where the heat is transported by conduction, radiation and even by convection. See [3–5, 22, 121] for further details. Finally, we refer to [110] for a formal derivation of a radiative transfer equation starting from the Maxwell equations and to [10] for a rigorous derivation of the radiative transfer equation involving only scattering processes obtained from Schrödinger's wave equation in a body with very separated scatterers.

1.8 Articles details

This work is a cumulative thesis containing five articles written during my PhD studies. The full papers can be found in the appendices in their most recent pre-print versions or in their accepted version if published in a peer-reviewed journal. These are the following:

1. **“Compactness and existence theory for a general class of stationary radiative transfer equations”**, joint work with J. W. Jang and J. J. L. Velázquez, submitted. ArXiv version `https://doi.org/10.48550/arXiv.2401.12828`, Chapter 2 and Appendix A;

2. **“Equilibrium and Non-Equilibrium diffusion approximation for the radiative transfer equation”**, joint work with J. J. L. Velázquez, submitted. ArXiv version `https://doi.org/10.48550/arXiv.2407.11797`, Chapter 3 and Appendix B;

3. **“On the Diffusion Approximation of the Stationary Radiative Transfer Equation with Absorption and Emission”**, joint work with J. J. L. Velázquez. Published in Annales Henri Poincaré and available in `https://doi.org/10.1007/s00023-025-01556-0`, Chapter 4 and Appendix C;

4. **“Well-posedness for a two-phase Stefan problem with radiation”**, joint work with J. J. L. Velázquez, submitted. ArXiv version `https://doi.org/10.48550/arXiv.2505.24602`, Chapter 5 and Appendix D.

5. **“Traveling waves for a two-phase Stefan problem with radiation”**, joint work with J. J. L. Velázquez, submitted. ArXiv version `https://doi.org/10.48550/arXiv.2506.01821`, Chapter 6 and Appendix E.

For the sake of readability and consistency, Chapters 2-6 summarize the main results and the proof's techniques of these articles, which can be found all in the appendices, both the accepted ones and the ones still in the review process.

Chapter 2

Existence theory

This chapter is based on the article *"Compactness and existence theory for a general class of stationary radiative transfer equations"* [35], which is joint work with J.W. Jang and J. J. L. Velázquez and whose most recent version can be found in Appendix A. All the authors contributed equally in this work.

As explained in the Introduction, in this chapter we develop an existence theory for the stationary system obtained by coupling the radiative transfer equation (1.20) with the divergence-free condition for the radiation energy flux (1.22) and imposing incoming boundary condition (1.24). The problem that we want to study is the following

$$\begin{cases} n\cdot\nabla_x I_\nu(x,n)=\alpha_\nu^a(T(x))\left(B_\nu(T(x)-I_\nu(x,n)\right) & \\ \qquad\qquad +\,\alpha_\nu^s(T(x))\left(\int_{\mathbb{S}^2}K(n,n')I_\nu(x,n')\,dn'-I_\nu(x,n)\right), & x\in\Omega,\ n\in\mathbb{S}^2,\\ \nabla_x\cdot\int_0^\infty\int_{\mathbb{S}^2}nI_\nu(x,n)\,dn\,d\nu=0, & x\in\Omega,\\ I_\nu(x,n)=g_\nu(n)\geq 0, & x\in\partial\Omega,\ n\cdot n_x<0, \end{cases} \tag{2.1}$$

where $\Omega\subset\mathbb{R}^3$ is an open convex bounded domain with C^2-boundary. We assume also that $\partial\Omega$ has strictly positive curvature and that K satisfies (1.10) and it is invariant under rotations, i.e.

$$K(Rn,Rn')=K(n,n')\quad \text{for all } n,n'\in\mathbb{S}^2,\ R\in SO(3).$$

This last property of the scattering kernel implies also that K is symmetric, i.e.

$$K(n,n')=K(n',n)\quad \text{for all } n,n'\in\mathbb{S}^2,$$

as required in (1.10). See Lemma A.2 for the proof.

The novelty and at the same time the complexity of the problem (2.1) lies in the general form of the considered absorption and scattering coefficients, which may depend on the temperature and on the frequency. Specifically, we show the existence of a solution $(T,I_\nu)\in L^\infty(\Omega)\times L^\infty\left(\Omega,L^\infty\left(\mathbb{S}^2,L^1(\mathbb{R}_+)\right)\right)$ in the pure emission-absorption case (i.e. $\alpha_\nu^s\equiv 0$) as well as in the presence of scattering (i.e. $\alpha_\nu^s\not\equiv 0$). Moreover, this existence theory has been developed for the Grey approximation (i.e. $\alpha_\nu^{a,s}(T)=\alpha^{a,s}(T)$) as well as for the "pseudo-Grey" approximation (i.e. $\alpha_\nu^{a,s}(T)=Q_{a,s}(\nu)\alpha^{a,s}(T)$). In all these situations the non-linear case is considered, namely the case where the coefficients depend non-trivially on the temperature.

We prove the following result, which summarizes the main theorems in ([35], Appendix A).

Theorem 2.1 (cf. [35], Theorem A.3, Theorem A.1, Theorem A.4 and Theorem A.2)**.** *Let $\Omega\subset\mathbb{R}^3$ be bounded and open with C^2-boundary and strictly positive curvature. Assume that one of the following conditions holds:*

a) $\alpha_\nu^s \equiv 0$ *and* $\alpha_\nu^a(T) = \alpha^a(T)$ *bounded and strictly positive;*

b) $\alpha_\nu^s \equiv 0$ *and* $\alpha_\nu^a(T) = Q_a(\nu)\alpha^a(T)$ *bounded and strictly positive;*

c) $\alpha_\nu^s(T) = \alpha^s(T)$ *and* $\alpha_\nu^a(T) = \alpha^a(T)$ *bounded and strictly positive;*

d) $\alpha_\nu^s(T) = Q_s(\nu)\alpha^s(T)$ *and* $\alpha_\nu^a(T) = \alpha^a(T)$ *bounded and strictly positive;*

Assume also that $Q_\ell \in C^1(\mathbb{R}_+)$ *and* $\alpha^\ell \in C^1(\mathbb{R}_+)$ *for* $\ell = a, s$ *(if applicable). Assume* $K \in C^1(\mathbb{S}^2 \times \mathbb{S}^2)$ *to be non-negative, rotationally symmetric, and independent of the frequency with the property* (1.10)*. Then there exists a solution* $(T, I_\nu) \in L^\infty(\Omega) \times L^\infty(\Omega, L^\infty(\mathbb{S}^2, L^1(\mathbb{R}_+)))$ *to the problem* (2.1)*, where the* I_ν *solves the radiative transfer equation in* (2.1) *in the sense of distribution.*

Notice that, under the assumptions a), b), c) and d), Theorem 2.1 corresponds to Theorem A.3, Theorem A.1, Theorem A.4 and Theorem A.2, respectively.

Before going into the details of the strategy of the proof, we study the easier case in which $\alpha_\nu^a \equiv 0$ and $\alpha_\nu^a \equiv \alpha = \text{const}$. This case is considered also in [83], where the existence of a unique solution $(T, I_\nu) \in L^4(\Omega)\times L^1(\Omega\times\mathbb{S}^2\times\mathbb{R}_+)$ is shown. In the following we prove that there exists a unique solution $(T, I_\nu) \in L^\infty(\Omega) \times L^\infty(\Omega, L^\infty(\mathbb{S}^2, L^1(\mathbb{R}_+)))$ to (2.1) for the pure emission-absorption case with constant absorption coefficient. To this end, we reformulate problem (2.1) as a fixed-point integral equation for the temperature, following the approach described in Section 1.5. Hence, solving the radiative transfer equation in (2.1) by characteristics, using the divergence free-condition of the radiation energy flux and simplifying the resulting terms by means of the Stefan-Boltzmann law (1.4), for this choice of coefficients we obtain

$$u(x) = \mathcal{B}(u)(x) = \int_\Omega \frac{\alpha e^{-\alpha|x-\eta|}}{4\pi|x-\eta|^2}u(\eta)d\eta + \int_{\mathbb{S}^2}\int_0^\infty g_\nu(n)e^{-\alpha s(x,n)}\,d\nu\,dn, \tag{2.2}$$

where $u = 4\pi\sigma T^4$. Moreover, $s(x, n)$ is the distance of $x \in \Omega$ to the boundary $\partial\Omega$ along direction $-n \in \mathbb{S}^2$, cf. Equation (1.30) and Figure 1.3. Let $M \geq 4\pi\|g_\nu\|_{L^\infty(\mathbb{S}^2,L^1(\mathbb{R}_+))}e^{\alpha\text{diam}(\Omega)}$. We now claim that there exists a unique solution u to (2.2) in the complete metric space

$$\mathcal{A} = \{u \in L^\infty(\Omega) :\ u \geq 0 \text{ and } \|u\|_{L^\infty} \leq M\}. \tag{2.3}$$

This can be proved applying the Banach fixed-point theorem. To prove that $\mathcal{B}$ is a self-map on $\mathcal{A}$ we first notice that both integral terms of $\mathcal{B}$ in (2.2) preserve the sign. Thus, if $u \in \mathcal{A}$ then $\mathcal{B}(u) \geq 0$. For the boundedness of the norm we compute changing to polar coordinates

$$\begin{aligned}\int_\Omega \frac{\alpha e^{-\alpha|x-\eta|}}{4\pi|x-\eta|^2}|u(\eta)|\,d\eta =& \fint_{\mathbb{S}^2}\int_0^{s(x,n)} \alpha e^{-\alpha r}|u(x-rn)|dr\,dn\\ \leq&\|u\|_{L^\infty}\fint_{\mathbb{S}^2}\int_0^{s(x,n)}\partial_r\left(-e^{-\alpha r}\right)dr\,dn\\ =&\|u\|_{L^\infty}\fint_{\mathbb{S}^2}\left(1-e^{-\alpha s(x,n)}\right)dn \leq \|u\|_{L^\infty}\left(1-e^{-\alpha\,\text{diam}(\Omega)}\right),\end{aligned} \tag{2.4}$$

where we used also $\sup\limits_{x,n\in\Omega\times\mathbb{S}^2} s(x, n) = \text{diam}(\Omega)$. On the other hand, we estimate

$$\int_{\mathbb{S}^2}\int_0^\infty g_\nu(n)e^{-\alpha s(x,n)}\,d\nu\,dn \leq 4\pi \sup_{n\in\mathbb{S}^2}\left(\int_0^\infty g_\nu(n)e^{-\alpha s(x,n)}\,d\nu\right) = 4\pi\|g_\nu\|_{L^\infty(\mathbb{S}^2,L^1(\mathbb{R}_+))}.$$

Therefore, $\|\mathcal{B}(u)\|_{L^\infty} \leq M\left(1-e^{-\alpha \operatorname{diam}(\Omega)}\right)+4\pi\|g_\nu\|_{L^\infty(\mathbb{S}^2,L^1(\mathbb{R}_+))} \leq M$ for all $u \in \mathcal{A}$, which implies that $\mathcal{B}$ is a self-map in $\mathcal{A}$. Finally, $\mathcal{B}$ is a contraction, since for $u_1,\ u_2 \in \mathcal{A}$ we estimate

$$\|\mathcal{B}(u_1)-\mathcal{B}(u_2)\|_{L^\infty} \leq \|u_1-u_2\|_{L^\infty} \sup_{x\in\Omega}\left(\int_\Omega \frac{\alpha e^{-\alpha|x-\eta|}}{4\pi|x-\eta|^2}d\eta\right) \leq \|u_1-u_2\|_{L^\infty} \underbrace{\left(1-e^{-\alpha \operatorname{diam}(\Omega)}\right)}_{<1},$$

where we changed to polar coordinates similarly as we did in (2.4).

It is important to observe that the independence of the absorption coefficient α on the temperature makes the integral operator $\mathcal{B}$ linear. This is crucial when proving contractivity. We will see in the non-linear case that, if α depends on T, the resulting non-linear version of $\mathcal{B}$ is still a self-map on $\mathcal{A}$, but in general it is not a contraction anymore. Finally, notice that the existence of a unique solution $T \in L^\infty(\Omega)$ implies the existence of a unique solution $(T, I_\nu) \in L^\infty(\Omega) \times L^\infty\left(\Omega, L^\infty\left(\mathbb{S}^2, L^1(\mathbb{R}_+)\right)\right)$, where $u = 4\pi\sigma T^4$ solves (2.2) and the radiation intensity is given by

$$I_\nu(x,n) = g_\nu(n)e^{-\alpha s(x,n)} + \int_0^{s(x,n)} \alpha e^{-\alpha t} B_\nu(T(x-tn))\, dt.$$

Remark. In the very same way we can show that there exists a unique solution $(T, I_\nu) \in C^0(\Omega) \times L^\infty\left(\Omega, L^\infty\left(\mathbb{S}^2, L^1(\mathbb{R}_+)\right)\right)$. Indeed, $\mathcal{B}$ is a contraction also in $\{u \in \mathcal{A} : u \in C^0(\Omega)\}$. Only the continuity of $\mathcal{B}(u)$ has to be proved. On the one hand, the regularity assumptions on $\partial\Omega$ imply the continuity of $x \mapsto s(x,n)$, cf. equation (A.33). On the other hand, the convolution of the integral function $\frac{\alpha e^{-\alpha|z|}}{|z|^2}$ with a bounded continuous function is a continuous function. This result will be used again in Chapter 4.

We now move to the main results presented in ([35], Appendix A) and the strategy of their proof. We start with the scattering-free case in Section 2.1 and we conclude with the full equation including both emission-absorption and scattering processes in Section 2.2.

2.1 Pure emission-absorption case

Let $\alpha_\nu^s \equiv 0$. As we have seen in Section 1.5, in this case the radiation intensity is given by

$$\begin{aligned} I_\nu(x,n) = g_\nu(n) \exp\left(\int_{[x,y(x,n)]} \alpha_\nu^a(T(\xi))d\xi\right) \\ + \int_0^{s(x,n)} \alpha_\nu^a(x-tn)B_\nu(T(x-tn)) \exp\left(-\int_{[x,x-tn]} \alpha_\nu^a(T(\xi))d\xi\right) dt, \end{aligned} \tag{2.5}$$

where T solves

$$\begin{aligned} 4\pi \int_0^\infty \alpha_\nu^a(x)B_\nu(T(x))\, d\nu = \int_0^\infty \alpha_\nu^a(x) \int_\Omega \frac{\alpha_\nu^a(\eta)\exp\left(-\int_{[x,\eta]} \alpha_\nu^a(T(\xi))d\xi\right)}{|x-\eta|^2} B_\nu(T(\eta))d\eta\, d\nu \\ + \int_0^\infty \alpha_\nu(x) \int_{\mathbb{S}^2} g_\nu(n) \exp\left(-\int_{[x,y(x,n)]} \alpha_\nu^a(T(\xi))d\xi\right) dn\, d\nu, \end{aligned} \tag{2.6}$$

where $\int_{[z,w]} f(\xi)d\xi$ is the integral along the line segment with endpoints $z \in \Omega$ and $w \in \Omega$. As for the Grey approximation with constant absorption coefficient considered at the beginning of this chapter, it is enough to show the existence of T solving (2.6), since equation (2.5) gives us the radiation intensity. In the following the strategy of the proof of Theorem 2.1 is presented first under the assumption a) and later under b).

2.1.1 Grey approximation

Besides $\alpha^s_\nu \equiv 0$, let us assume $\alpha^a_\nu(T) = \alpha^a(T)$ together with all the hypotheses of Theorem 2.1. We consider this case first because the computations are simpler, while the main strategy and the main steps for the proof of Theorem 2.1 are (up to adaptations) the same as for all other cases.

We aim to prove the existence of a fixed-point in $L^\infty(\Omega)$ to (2.6), which in this case reads

$$u(x) = \mathcal{B}(u)(x) = \int_\Omega \frac{\gamma(u(\eta)) \exp\left(-\int_{[x,\eta]} \gamma(u(\xi))d\xi\right)}{4\pi|x-\eta|^2} u(\eta)d\eta \\ + \int_{\mathbb{S}^2}\int_0^\infty g_\nu(n)\exp\left(-\int_{[x,y(x,n)]}\gamma(u(\xi))d\xi\right) d\nu\, dn, \quad (2.7)$$

where $u = 4\pi\sigma T^4$, $y(x,n) = \{x - tn : t > 0\} \cap \partial\Omega$ and $\gamma(z) = \alpha\left(\sqrt[4]{\frac{z}{4\pi\sigma}}\right)$ for $z \geq 0$, cf. (A.23). As for the case where $\alpha^a \equiv \alpha$, the operator $\mathcal{B}$ is a self-map on the same complete closed metric space $\mathcal{A}$, cf. (2.3). Nevertheless, $\mathcal{B}$ does not need to be a contraction. Also, $\mathcal{B}$ does not need to be compact in either $L^\infty(\Omega)$ nor $C^0(\Omega)$. These observations are due to the non-linear expressions $\exp\left(-\int_{[x,\eta]}\gamma(u(\xi))d\xi\right)$ appearing in both integral terms of $\mathcal{B}$, which have the same regularity as u. Thus, neither Banach fixed-point theorem nor Schauder fixed-point theorem can be directly used.

In order to overcome this problem we first regularize the operator $\mathcal{B}$ defining the following new operator

$$\mathcal{B}^\varepsilon(u)(x) =: \int_\Omega \left(\gamma(u) * \phi_\varepsilon\right)(\eta) u(\eta) \frac{\exp\left(-\int_{\mathbb{R}^3}\gamma(u(\xi))\delta_{[x,\eta]} * \phi_\varepsilon(\xi)d\xi\right)}{4\pi|x-\eta|^2}\, d\eta \\ + \int_{\mathbb{S}^2} \exp\left(-\int_{\mathbb{R}^3}\gamma(u(\xi))\delta_{[x,y(x,n)]} * \phi_\varepsilon(\xi)d\xi\right)\int_0^\infty g_\nu(n)\, d\nu\, dn \quad (2.8)$$

as in equation (A.25), where $\phi_\varepsilon \in C_c^\infty(\mathbb{R}^3)$ is a sequence of standard positive and radially symmetric mollifier and where we defined $\delta_{[x,\eta]} * \phi(\xi) = \int_{[x,\eta]}\phi(\xi - z)dz$ so that

$$\int_{\mathbb{R}^3}\gamma(u(\xi))\delta_{[x,\eta]} * \phi_\varepsilon(\xi)d\xi = \int_{[x,\eta]}\left(\gamma(u) * \phi_\varepsilon\right)(z)\, dz.$$

In Section A.3.1 we prove that $\mathcal{B}^\varepsilon$ is a compact continuous self-map on $\mathcal{A}$, which implies, applying Schauder fixed-point theorem, the existence of a fixed-point $u_\varepsilon = \mathcal{B}^\varepsilon(u_\varepsilon)$ for all $\varepsilon > 0$. This is due to the fact that $\gamma(u_\varepsilon) * \phi_\varepsilon$ has a higher regularity than u_ε. Moreover, the smoothness of the domain and the assumption on the curvature imply the Hölder continuity of $\mathcal{B}^\varepsilon(u_\varepsilon)$ and thus the compactness of the operator $\mathcal{B}^\varepsilon$. The existence of a fixed-point to the original equation (2.7) is given by the $L^2(\Omega)$-compactness of the sequence of regularized solutions u_ε. This yields the existence of a subsequence u_j converging to a function u both in $L^2(\Omega)$ and pointwise almost everywhere. By the boundedness of the sequence u_ε we obtain $u \in L^\infty(\Omega)$ and using Lebesgue dominated convergence theorem we conclude Theorem 2.1.

The compactness of the sequence $u_\varepsilon \in L^2(\Omega)$ is, besides the existence results, one of the most important contributions in [35].

2.1.2 Compactness result

Exponentials can be written as power series whose tails converge to zero. Moreover, u_ε satisfies using polar coordinates

$$u_\varepsilon(x) = \fint_{\mathbb{S}^2} dn \int_0^{s(x,n)} dt\ \left[(\gamma(u_\varepsilon) * \phi_\varepsilon)\, u_\varepsilon\right](x-tn) \exp\left(-\int_{[x,x-tn]} \gamma(u_\varepsilon) * \phi_\varepsilon(\xi) d\xi\right)$$
$$+ \int_{\mathbb{S}^2} dn \left(\int_0^\infty d\nu\ g_\nu(n)\right) \exp\left(-\int_{[x,y(x,n)]} \gamma(u_\varepsilon)) * \phi_\varepsilon(\xi) d\xi\right).$$

Then, the L^2-compactness of the sequence u_ε is implied by the L^2-compactness of operators of the form

$$\int_{\mathbb{S}^2} dn \left(\int_0^s d\tau\ f(x-\tau n)\right).$$

The main result is stated in Proposition A.1, whose summarized version is the following.

Proposition 2.1 (cf. [35], summary of Proposition A.1)**.** *Let $L > 0$ and $\Pi^3 = [-L, L]^3$. Consider a sequence of periodic functions $(\varphi_j)_{j\in\mathbb{N}} \in L^\infty\left(\Pi^3\right)$ with $\|\varphi_j\|_\infty \le M$. Let us define for $n \in \mathbb{S}^2$ and $m \in \mathbb{N}$ the operators L_n and T_m by*

$$L_n[\varphi](x) =: \int_{-L}^{L} d\lambda\ \varphi(x-\lambda n) \quad \textit{and} \quad T_m[\varphi](x) =: \int_{\mathbb{S}^2} dn \left(L_n[\varphi](x)\right)^m.$$

Then, the sequence $\left(T_m[\varphi_j]\right)_j$ is compact in $L^2\left(\Pi^3\right)$ for every $m \in \mathbb{N}$ and it satisfies a suitable equi-integrability condition.

This proposition will play an essential role in the proof of Theorem 2.1. Its proof is based on elementary measure theory. It relies on the properties of the two auxiliary measures $\mu_j^R, \nu_{n,j}^R \in \mathcal{M}_+(\mathbb{S}^2)$ associated to φ_j and to $L_n[\varphi_j]$, which are defined for $R > 0$ in the following way

$$\mu_j^R(\omega) = \sum_{\substack{k\in\frac{\pi}{L}\mathbb{Z}^3 \\ |k|>R}} \left|a_k^j\right|^2 \delta_{\frac{k}{|k|}}(\omega) \quad \text{and} \quad \nu_{n,j}^R(\omega) = \sum_{\substack{k\in\frac{\pi}{L}\mathbb{Z}^3 \\ |k|>R}} 4\left|a_k^j\right|^2 \left|\frac{\sin(L(k\cdot n))}{k\cdot n}\right|^2 \delta_{\frac{k}{|k|}}(\omega),$$

where we used the Fourier expansions

$$\varphi_j(x) = \sum_{k\in\frac{\pi}{L}\mathbb{Z}^3} a_k^j e^{ik\cdot x} \quad \text{and} \quad L_n[\varphi_j](x) = \sum_{k\in\frac{\pi}{L}\mathbb{Z}^3} 2a_k^j \frac{\sin(L(k\cdot n))}{k\cdot n} e^{ik\cdot x}.$$

Thus, by definition $\nu_{n,j}^R \le 4L^2\mu_j^R$ for all $n \in \mathbb{S}^2$ and $\mu_j^R(\mathbb{S}^2) \le 8L^3M^2$. Moreover,

$$\int_{\mathbb{S}^2} dn \int_{\mathbb{S}^2} d\nu_{n,j}^R(\omega)\mathbb{1}_{\{|\omega\cdot n|\ge\kappa\}} \le \frac{C(M,L)}{R^2\kappa^2} \underset{R\to\infty}{\longrightarrow} 0$$

as well as

$$\begin{aligned}\int_{\mathbb{S}^2} dn \int_{\mathbb{S}^2} d\nu_{n,j}^R(\omega)\mathbb{1}_{\{0\le|\omega\cdot n|<\kappa\}} &\le \int_{\mathbb{S}^2} dn \int_{\mathbb{S}^2} d\mu_j^R(\omega)\mathbb{1}_{\{0\le|\omega\cdot n|<\kappa\}} \\ &= \int_{\mathbb{S}^2} d\mu_j^R(\omega) \int_{\mathbb{S}^2} dn\mathbb{1}_{\{0\le|\omega\cdot n|<\kappa\}} \le C(M,L)\kappa \underset{\kappa\to 0}{\longrightarrow} 0\end{aligned} \tag{2.9}$$

Therefore, for any $\delta > 0$, choosing R large enough and κ small enough, we conclude

$$\|T_m[\varphi_j] - T_m[\varphi_j](\cdot + h)\|_{L^2(\Pi^3)} \le C(M, L, m) \left(\sum_{\substack{k \in \frac{\pi}{L}\mathbb{Z}^3 \\ |k| \le R}} |a_k^j|^2 |k|^2 |h|^2 + \int_{\mathbb{S}^2} dn \int_{\mathbb{S}^2} d\nu_{n,j}^R(\omega) \right) < \delta$$

for $|h|$ small enough. An active role in the proof of Proposition 2.1 is played by the integral of $(L_n[\varphi_j])^m$ on the sphere $\mathbb{S}^2$, specifically by the estimate (2.9). Indeed, while the term $(L_n[\varphi_j])^m$ is not compact for every $n \in \mathbb{S}^2$, the integral $\int_{\mathbb{S}^2} (L_n[\varphi_j])^m \, dn$ is compact in $L^2(\Omega)$. This is the reason why the compactness result of Proposition 2.1 can be considered as a new averaging lemma. Moreover, Proposition 2.1 can be extended for operators of the form

$$L_{n,t-s}[f](x) = \int_s^t d\lambda \, f(x - \lambda n)$$

for any $0 \le s < t \le \frac{L}{2}$, cf. Lemma A.1.

This result is used in order to prove the compactness of the sequence $u_\varepsilon = \mathcal{B}^\varepsilon(u_\varepsilon)$ in $L^2(\Omega)$ in the following way. By the boundedness of the domain we can define $\Omega \subset [-2\text{diam}(\Omega) - 2, 2\text{diam}(\Omega) + 2]^3 = \Pi^3$. We then extend in a suitable way both u_ε and $\gamma(u_\varepsilon)$ in Π^3 and periodically in $\mathbb{R}^3$. We expand the exponentials of integrals along lines and we use the generalized compactness result of Lemma A.1. Since the tails of the series converge, we need to apply this result only finitely many times in order to conclude the L^2-compactness of the sequence u_ε.

2.1.3 Pseudo-Grey approximation

Let assume $\alpha_\nu^s \equiv 0$, $\alpha_\nu^a(T) = Q_a(\nu)\alpha^a(T)$ bounded and strictly positive, and all other hypotheses of Theorem 2.1. Using the monotonicity of $0 \le z \mapsto B_\nu(z)$, we define the bounded functions u and F by

$$u(x) = 4\pi \int_0^\infty Q_a(\nu) B_\nu(T(x)) \, d\nu = F(T(x)), \tag{2.10}$$

which can be related by $T = F^{-1}(u)$ since F is monotone and invertible. In order to simplify the notation we denote also $\gamma = \alpha^a \circ F^{-1}$ and $f_\nu = B_\nu \circ F^{-1}$. Equation (2.6) can be re-written to the following fixed-point equation

$$\begin{aligned} u(x) = \mathcal{B}(u)(x) &= \int_0^\infty d\nu \int_\Omega d\eta \, \frac{Q_a(\nu)^2 \gamma(u(\eta)) f_\nu(u(\eta))}{|x - \eta|^2} \exp\left(-\int_{[x,\eta]} Q_a(\nu)\gamma(u(\xi)) d\xi \right) \\ &= \int_0^\infty d\nu \int_{\mathbb{S}^2} dn \, Q_a(\nu) g_\nu(n) \exp\left(-\int_{[x,y(x,n)]} Q_a(\nu)\gamma(u(\xi)) d\xi \right), \end{aligned}$$

cf. (A.54). In order to show the existence of a fixed-point we proceed as we did in Subsection 2.1.1. We regularize the operator $\mathcal{B}$ replacing $\gamma(u)$ by $\gamma(u) * \phi_\varepsilon$, where ϕ_ε is a sequence of standard mollifiers. Using that Q is bounded and that $\int_0^\infty Q(\nu) f_\nu(u(\eta)) d\eta \le \|u\|_\infty$, we conclude by applying finitely many times the compactness result of Lemma A.1 to the sequence of regularized solutions u_ε, whose existence is guaranteed again by Schauder fixed-point theorem.

2.2 Full equation

We consider now the case in which $\alpha_\nu^a(T)$ and $\alpha_\nu^s(T)$ are both positive. The steps for the proof of Theorem 2.1 under the assumptions c) and d) are the same as the ones considered in Section 2.1. We construct a fixed-point integral equation for the temperature and we regularize the integral operators obtaining a sequence of regularized fixed-point solutions. Finally, applying the compactness result in Proposition 2.1 we show the existence of a fixed-point of the original equation, which determines the temperature and hence the radiation intensity solving (2.1). The most significant difference with the proof of the pure emission-absorption case (where $\alpha_\nu^s \equiv 0$) is in the derivation of the fixed-point equation for the temperature. This is achieved through the definition of suitable Green's functions. We consider again first the case in which the coefficients are independent of the frequency, i.e. the Grey approximation.

2.2.1 Grey approximation

Under the assumption c) of Theorem 2.1 we construct the Green's functions $\tilde{I}(x,n;x_0)$ for $x, x_0 \in \Omega$ and $n \in \mathbb{S}^2$ and $\psi(x,n;x_0,n_0)$ for $x \in \Omega,\ x_0 \in \partial\Omega$ and $n, n_0 \in \mathbb{S}^2$. These solve

$$\begin{aligned}
n\cdot\nabla_x \tilde{I}(x,n;x_0) =& \alpha^s(T(x)) \int_{\mathbb{S}^2}, K(n,n')\tilde{I}(x,n';x_0)\,dn' \\
&- \left(\alpha^a\left(T(x)\right) + \alpha^s\left(T(x)\right)\right)\tilde{I}(x,n;x_0) + \delta(x-x_0),\ x,x_0\in\Omega,\ n\in\mathbb{S}^2,\\
\tilde{I}(x,n;x_0)\chi_{\{n\cdot n_x<0\}} =& 0,\ x\in\partial\Omega,\ x_0\in\Omega,\ n\in\mathbb{S}^2,\\
n\cdot\nabla_x\psi(x,n;x_0,n_0) =& \alpha^s(T(x))\int_{\mathbb{S}^2} K(n,n')\psi(x,n';x_0,n_0)\,dn' \\
&- \left(\alpha^a\left(T(x)\right) + \alpha^s\left(T(x)\right)\right)\psi(x,n;x_0,n_0),\ x\in\Omega,\ x_0\in\partial\Omega,\ n,n_0\in\mathbb{S}^2,\\
\psi(x,n;x_0,n_0)\chi_{\{n\cdot n_x<0\}} =& \delta_{\partial\Omega}(x-x_0)\frac{\delta^{(2)}(n,n_0)}{4\pi},\ x\in\Omega,\ x_0\in\partial\Omega,\ n_0\cdot N_{x_0}<0,
\end{aligned}$$

in the sense of distribution, cf. equations A.58 and (A.59). We refer to Sections A.4.1 and A.4.2 for more details of the definition and construction of such Green's functions. With the help of these auxiliary functions, the radiation intensity solving (2.1) is given by

$$I_\nu(x,n) = \int_\Omega dx_0\ \alpha^a(T(x_0))B_\nu(T(x_0))\tilde{I}(x,n;x_0) + \int_{\mathbb{S}^2} dn_0 \int_{\partial\Omega} dx_0\ g_\nu(n_0)\psi(x,n;x_0,n_0),$$

cf. (A.60). Using the divergence-free condition of the energy flux we obtain also in this case a fixed-point equation for $u = 4\pi\sigma T^4$ of the following form

$$u(x) = \mathcal{B}(u)(x) + \mathcal{C}(u)(x) = \sum_{i=1}^{\infty}\mathcal{V}_i(u)(x) + \sum_{i=1}^{\infty}\mathcal{U}_i(u)(x),$$

where $\mathcal{B}$ and $\mathcal{C}$ can be expanded in their Duhamel series similar to the ones in (A.86) and (A.82), respectively. The interpretation of such series is the following. Each term $\mathcal{V}_i(u)$ of $\mathcal{B}(u)$ (or $\mathcal{U}_i(u)$ of $\mathcal{C}(u)$) describes the interaction of photons emitted at the point $\eta \in \Omega$ (or $x_0 \in \partial\Omega$) scattered $(i-1)$-times before being absorbed at the point $x \in \Omega$. Therefore, each term $\mathcal{V}_i(u)$ and $\mathcal{U}_i(u)$ contains i exponentials of integrals along lines and $(i-1)$ scattering integrals. We refer to the equations (A.86) and (A.82) for the exact form of these integral terms.

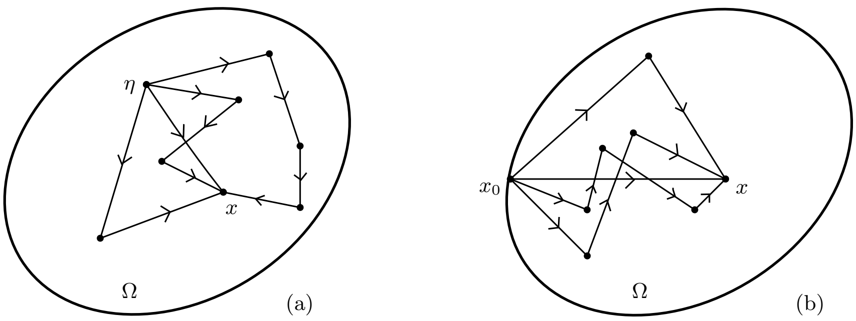

Figure 2.1: Interpretation of the terms in the Duhamel series: (a) represents the terms of the operator $\mathcal{B}$, (b) illustrates the terms of the operator $\mathcal{C}$.

We then proceed as follows. Replacing the non-linear coefficients $\alpha(T)$ by their convolution with the standard mollifiers ϕ_ε, we obtain a sequence of regularized solution u_ε via Schauder fixed-point theorem. While the compactness of the regularized operators are due to the convolution with the mollifiers, the self-map property is a consequence of

$$|\mathcal{B}_\varepsilon(u)(x)| \leq \|u\|_{\sup} \int_\Omega dx_0 \int_{\mathbb{S}^2} dn\, \frac{\alpha_\varepsilon^a(u(x_0))}{4\pi} I_\varepsilon(x,n;x_0) \leq \theta \|u\|_{\sup},$$

where I_ε is the Green's function for the regularized problem. In order to prove this inequality we show that $H_\varepsilon(x,n) = \int_\Omega dx_0 \alpha_\varepsilon^a(u(x_0)) I_\varepsilon(x,n;x_0)$ solves a differential equation with a maximum principle. See Section A.4.4 and Lemma A.3 for more details.

The L^2-compactness of the sequence u_ε is implied by an extension of the Proposition 2.1 to functions $\varphi \in C\left(\mathbb{S}^2, L^\infty(\Pi^3)\right)$ that are uniformly continuous with respect to $n \in \mathbb{S}^2$ (see Corollary A.2 for more details). This argument is used only a finite number of times since the series expansions of $\mathcal{B}(u)$ and of $\mathcal{C}(u)$ are absolutely convergent.

2.2.2 Pseudo-Grey approximation

The structure of the proof of Theorem 2.1 under the most general assumption d) is similar as the one in Section 2.2.1. We construct suitable Green's functions similar to the one for the Grey approximation, i.e. $Q_{a,s}(\nu) \equiv 1$. For u and F defined as in (2.10) we obtain a fixed-point equation which is then regularized via the convolution with mollifiers. Finally, we apply the appropriate version of the L^2-compactness result in Proposition 2.1.

Chapter 3

Equilibrium and non-equilibrium diffusion approximation

This chapter summarizes the results in the article *"Equilibrium and Non-Equilibrium diffusion approximation for the radiative transfer equation"* [36], which is joint work with J. J. L. Velázquez. In Appendix B the latest version of this paper can be found, to which both authors contributed equally.

This article studies using matched asymptotic expansions the diffusion approximation of the radiative heat transfer problem. Specifically, both the time-dependent and the stationary problems are considered. The first one is obtained coupling the radiative transfer equation (1.3) with the energy balance equation (1.19). The second one is constituted by the time-independent equation (1.3) and the divergence-free condition for the radiation energy flux (1.22). These problems are studied under the assumption that the total mean free paths of the photons is very small compared to the characteristic size of the domain. Using matched asymptotic expansions we formally derive, on the one hand, the limit problems that the radiation intensity I_ν and the temperature T should satisfy in the bulk. On the other hand, we also obtain the boundary and initial layer equations describing the behavior of the radiation close to the boundary and for small times. These are the regions where the diffusion approximation fails.

The main aim of this work is to give an accurate mathematical description of the so-called equilibrium and non-equilibrium diffusion approximation (cf. [108, 152]).

In the following sections we define the exact form of the problem we are considering and we introduce the concept of equilibrium and non-equilibrium diffusion diffusion approximation (Section 3.1). We also illustrate the method of matched asymptotic expansions largely used in the article [36] and we apply it to a particular example (Section 3.2). Finally, we summarize and discuss the limit problems obtained in the article.

3.1 Main results

Before writing the precise formulation of the equations studied in ([36], Appendix B), we need to define some particular length scales.

First of all, we consider a convex domain $\Omega \subset \mathbb{R}^3$, whose size is comparable in all space directions and such that $\operatorname{diam}(\Omega) = \sup_{x,y\in\Omega} |x-y| =: L$ is of order 1. We assume without loss of generality $L = 1$.

The mean free path of the photons is defined as the characteristic length that the photons can travel before being absorbed or scattered. The absorption mean free path (i.e. the mean free path associated to the emission-absorption process), which will be denoted by ℓ_A, and the scattering mean free path, denoted by ℓ_S, are mathematically defined by the order of magnitude of the reciprocal of the absorption and of the scattering coefficient, respectively.

The mean free path of the photons is defined by the Milne length

$$\ell_M = \min\{\ell_A, \ell_S\},$$

which in the diffusion approximation regime is considered to be very small compared to L, i.e. $\ell_M \ll L = 1$.

In order to fully describe the problem, another characteristic length is required, which is the so-called thermalization length, and it is defined by

$$\ell_T = \sqrt{\ell_A \ell_M} \gtrsim \ell_M.$$

Finally, it turns out that in the time-dependent case significant changes of the temperature occur at times of the same order of magnitude as the heat parameter, which is defined as

$$\tau_h = \frac{\ell_A}{\min\{\ell_T^2, 1\}} \gg 1.$$

It is important to notice that ℓ_A, ℓ_S, ℓ_M, ℓ_T are all non-dimensional. This is due to the choice of $L = 1$ and to the fact that all parameters are actually defined by $\frac{\ell}{L}$. Therefore, also τ_h is non-dimensional.

3.1.1 Formulation of the problem

We now replace in equations (1.3), (1.19), (1.21) and (1.22) the absorption and the scattering coefficient by $\frac{\alpha_\nu^a}{\ell_A}$ and by $\frac{\alpha_\nu^s}{\ell_S}$, respectively, and we rescale the time as $t \mapsto \tau_h t$. We recall that throughout this thesis the scattering kernel K is invariant under rotation and normalized.

The stationary problem that we want to study is

$$\begin{cases} n \cdot \nabla_x I_\nu(x,n) = \frac{\alpha_\nu^a(x)}{\ell_A} \left(B_\nu(T(x)) - I_\nu(x,n)\right) \\ \qquad\qquad + \frac{\alpha_\nu^s(x)}{\ell_S} \left(\int_{\mathbb{S}^2} K(n,n') I_\nu(x,n')\, dn' - I_\nu(x,n)\right) & x \in \Omega, n \in \mathbb{S}^2, \\ \operatorname{div}\left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, n I_\nu(n,x)\right) = 0 & x \in \Omega, n \in \mathbb{S}^2, \\ I_\nu(n,x) = g_\nu(n) & x \in \partial\Omega, n \cdot n_x < 0, \end{cases} \tag{3.1}$$

cf. (B.12). We also examine the time-dependent problem, which is given by

$$\begin{cases} \frac{1}{c}\partial_t I_\nu(t,x,n) + \tau_h n \cdot \nabla_x I_\nu(t,x,n) = \frac{\alpha_\nu^a(x)\tau_h}{\ell_A} \left(B_\nu(T(t,x)) - I_\nu(t,x,n)\right) \\ \qquad\qquad + \frac{\alpha_\nu^s(x)\tau_h}{\ell_S} \left(\int_{\mathbb{S}^2} K(n,n') I_\nu(t,x,n')\, dn' - I_\nu(t,x,n)\right) & x \in \Omega, n \in \mathbb{S}^2, t > 0, \\ \partial_t T + \frac{1}{c}\partial_t \left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, I_\nu(t,n,x)\right) \\ \qquad\qquad + \tau_h \operatorname{div}\left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, n I_\nu(t,n,x)\right) = 0 & x \in \Omega, n \in \mathbb{S}^2, t > 0, \\ I_\nu(0,x,n) = I_0(x,n,\nu) & x \in \Omega, n \in \mathbb{S}^2, \\ T(0,x) = T_0(x) & x \in \Omega, \\ I_\nu(t,n,x) = g_\nu(t,n) & x \in \partial\Omega, n \cdot n_x < 0, t > 0, \end{cases} \tag{3.2}$$

where the speed of light c will be consider to be of different orders of magnitude, cf. (B.10). In the case where we assume $c \to \infty$, (3.2) takes the following form

$$\begin{cases} n \cdot \nabla_x I_\nu(t,x,n) = \frac{\alpha_\nu^a(x)}{\ell_A}\left(B_\nu(T(t,x)) - I_\nu(t,x,n)\right) & \\ \qquad\qquad + \frac{\alpha_\nu^s(x)}{\ell_S}\left(\int_{\mathbb{S}^2} K(n,n')I_\nu(t,x,n')\,dn' - I_\nu(t,x,n)\right) & x \in \Omega, n \in \mathbb{S}^2, t > 0, \\ \partial_t T + \tau_h \operatorname{div}\left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, n I_\nu(t,n,x)\right) = 0 & x \in \Omega, n \in \mathbb{S}^2, t > 0, \\ T(0,x) = T_0(x) & x \in \Omega, \\ I_\nu(t,n,x) = g_\nu(t,n) & x \in \partial\Omega, n \cdot n_x < 0, t > 0, \end{cases} \tag{3.3}$$

cf. (B.11). Problems (3.1), (3.2) and (3.3) are studied considering all possible relative scalings between ℓ_A, ℓ_S and L under the assumption $\ell_M \to 0$.

3.1.2 Equilibrium and non-equilibrium diffusion approximation

Via matched asymptotic expansions we show that, as $\ell_M \to 0$, the radiation intensity solves a diffusion equation in the bulk. However, the exact form of this equation, and thus the behavior of I_ν, depends very much on the value of ℓ_A and ℓ_S. Indeed, $\ell_M \to 0$ either because $\ell_A \to 0$ or because $\ell_S \to 0$. The distinct nature of the emission-absorption process and of the scattering process is the reason of the different diffusion approximations that we obtain after considering all possible relative scalings between the characteristic lengths. Indeed, while the first process relates I_ν to $B_\nu(T)$ driving the radiation intensity towards the Planck equilibrium distribution, the latter one modifies I_ν making it isotropic but does not affect T.

These different features of the radiative processes are reflected in the fact that, according to the ratio between ℓ_A and ℓ_S, the radiation intensity approximates the Planck distribution $B_\nu(T)$ or not. The first case is defined in the literature as equilibrium diffusion approximation, and the second one as non-equilibrium diffusion approximation (cf. [108, 152]).

Instead of considering the lengths ℓ_A, ℓ_S and L, it is more convenient to examine the relative scalings between ℓ_M, ℓ_T and L. These lengths also describe the thickness of the two nested boundary layers emerging since I_ν does not need to be isotropic (nor $B_\nu(T)$) at the boundary $\partial\Omega$ or at time $t = 0$. These are the regions where the diffusion approximation is not satisfied and where I_ν modifies its behavior turning into an isotropic function when moving away form the boundary and in some cases even approaching $B_\nu(T)$.

The first boundary layer appearing is denoted as Milne layer and it is located in a region of thickness ℓ_M near $\partial\Omega$. In this layer I_ν becomes isotropic.

In the case of the equilibrium diffusion approximation another boundary layer emerges. This is the so-called thermalization layer of thickness ℓ_T. In this region the leading order of I_ν, which became isotropic in the Milne layer, changes until it approximates $B_\nu(T)$, where T is a further unknown of the problem. If $\ell_M \approx \ell_T$ this boundary layer coincides with the Milne layer.

	$\ell_M = \ell_T \ll L$	$\ell_M \ll \ell_T \ll L$	$\ell_M \ll \ell_T = L$	$\ell_M \ll L \ll \ell_T$
Milne layer	Milne =	Yes	Yes	Yes
Thermalization layer	Thermalization	Yes	$\approx$ Bulk	No
Bulk	Equilibrium diffusion approximation	Equilibrium diffusion approximation	Transition from equilibrium to non-equilibrium approximation	Non-equilibrium diffusion approximation

Table 3.1: Main results of ([36], Appendix B) (cf. Table B.1).

We presented Table B.1 of Appendix B (cf. also [36]), which summarizes the results obtained for the different relative scaling between the characteristic lengths. This table shows in which cases I_ν approaches $B_\nu(T)$ and when it does not happen. Moreover, it summarizes which boundary layer appears. Finally, we remark that in the time-dependent case two nested initial layers may also emerge.

3.2 Matched asymptotic expansions

We study equations (3.1), (3.2) and (3.3) via matched asymptotic expansions replacing $\ell_M = \varepsilon \ll 1$, $\ell_A = \varepsilon^{-\beta}$ and $\ell_S = \varepsilon^{-\gamma}$ for $\min\{\gamma, \beta\} = -1$. By definition we see that if $\ell_T = \varepsilon^{\frac{1-\beta}{2}} \lesssim 1$ then the heat parameter satisfies $\tau_h = \varepsilon^{-1}$, otherwise $\tau_h = \ell_A$. In the following section we explain how the method of matched asymptotic expansions has been used.

3.2.1 The description of the method of matched asymptotic expansions

The method of matched asymptotic expansions consists of the following steps. Using suitable asymptotic expansions and variables' rescalings, several approximations to the original problem are construted. The domain of validity of each of such approximate solutions is a subdomain of the original domain. Together, these solutions yield a final approximation valid in the original domain (cf. [94]). In this case, the subdomains that we shall consider are the *bulk* of Ω, where $\mathrm{dist}(\partial\Omega, x) \approx 1$ and $t \approx 1$, the *boundary layers*, where $\mathrm{dist}(\partial\Omega, x) \ll 1$ and $t \approx 1$, the *initial layers*, where $\mathrm{dist}(\partial\Omega, x) \approx 1$ and $t \ll 1$ and the *initial-boundary layers*, where $\mathrm{dist}(\partial\Omega, x) \ll 1$ and $t \ll 1$.

In order to derive the so-called outer problem, i.e. the approximate problem valid in the bulk of the domain, we use an expansion series. In our specific problem the radiation intensity is expanded as

$$I_\nu(t, x, n) = \phi_0(t, x, n, \nu) + \sum_{k\geq 0} \varepsilon^{\delta+1}\psi_{k+1}(t, x, n, \nu) + \sum_{l>0} \varepsilon^l \phi_l(t, x, n, \nu) \tag{3.4}$$

where $\delta >$ is chosen in a suitable way according to the values of ℓ_A and ℓ_S. Specifically, $\delta = 1 + \gamma > 0$ if $\beta = -1$ and $\delta = \beta - \lfloor\beta\rfloor > 0$ if $\gamma = -1$, cf. (B.20) and (B.21). We then plug (3.4) into the considered (initial-)boundary value problem and we compare the terms that are of the same order of magnitude. Finally, we derive a closed equation for the leading order of the radiation intensity I_ν and for the temperature T.

The boundary layer equations, i.e. the problems describing the boundary layers, are obtained in a similar way also using an expansion series. In this case, we also need first to rescale the spatial variable according to

$$y = -\frac{x - p}{\ell} \cdot n_p,$$

where $x \in \Omega$ is in a neighborhood of $p \in \partial\Omega$, $n_p \in \mathbb{S}^2$ is the outer normal at p and $\ell \in \{\ell_M, \ell_T\}$. Under the further assumption that in regions very close to the boundary I_ν and T depend only on the distance to the boundary, the resulting boundary layer equations are one-dimensional stationary problems describing the behavior of I_ν and T in a neighborhood of every point $p \in \partial\Omega$.

For every $p \in \partial\Omega$, the asymptotic $\lim_{y\to\infty} I_\nu$ gives the boundary condition that is valid for the next nested subdomain. This is the so-called matching.

The initial layers appearing in the time-dependent problem (3.2) are obtained similarly as we did for the boundary layers. In this case we rescale the time by $t = \frac{\varepsilon}{c\tau_h}$ for the initial

Milne layer, and by $t = \frac{\ell_T^2}{c}\tau$ for the initial thermalization layer if $\ell_M \ll \ell_T \ll 1$. First order ODEs are obtained expanding I_ν via asymptotic series. Once again, the limit $\lim_{\tau\to\infty} I_\nu$ gives the initial condition for the outer problem (or for the initial thermalization layer).

The equations describing the approximate solutions in the initial-boundary layers are obtained rescaling in a proper way both the space and the time variables and expanding the radiation intensity via expansion series. For more details we refer to the Sections B.4, B.5 and B.6 in Appendix B.

3.2.2 An application's example of matched asymptotic expansions

In this section we show an example of how the method of matched asymptotic expansions has been used in ([36], Appendix B). Let us consider the stationary problem (3.1) with $\ell_M = \ell_A = \varepsilon$ and $\ell_S = 1$. Then the system can be rewritten as

$$\begin{cases} n\cdot\nabla_x I_\nu(x,n) = \frac{\alpha_\nu^a(x)}{\varepsilon}\left(B_\nu(T(x)) - I_\nu(x,n)\right) & \\ \qquad + \alpha_\nu^s(x)\left(\int_{\mathbb{S}^2} K(n,n')I_\nu(x,n')\,dn' - I_\nu(x,n)\right) & x\in\Omega, n\in\mathbb{S}^2, \\ \operatorname{div}\left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\; nI_\nu(n,x)\right) = 0 & x\in\Omega, n\in\mathbb{S}^2, \\ I_\nu(n,x) = g_\nu(n) & x\in\partial\Omega, n\cdot n_x<0. \end{cases} \tag{3.5}$$

First of all, we expand I_ν as the following series

$$I_\nu(x,n) = \phi_0(x,n) + \varepsilon\phi_1(x,n) + \varepsilon^2\phi_2(x,n) + ...$$

and we plug it into the radiative transfer equation in (3.5). The comparison of the terms of order ε^{-1} gives

$$B_\nu(T(x)) = \phi_0(x,n).$$

Thus, I_ν approaches the Planck distribution in the Bulk and it is also isotropic. Moreover, the terms of order 1 imply, using that K is symmetric,

$$n\cdot\nabla_x B_\nu(T(x)) = -\alpha_\nu^a(x)\phi_1(x,n)),$$

so that

$$I_\nu(x,n) = B_\nu(T(x)) - \frac{\varepsilon}{\alpha_\nu^a(x)} n\cdot\nabla_x B_\nu(T(x)) + \mathcal{O}(\varepsilon^2).$$

Plugging this expression into the divergence-free condition in (3.5), we conclude that in the bulk I_ν approximate $B_\nu(T)$, where T solves

$$\operatorname{div}\left(\int_0^\infty \frac{\nabla_x B_\nu(T(x))}{\alpha_\nu^a(x)} d\nu\right) = 0. \tag{3.6}$$

Since $\ell_M = \ell_T$, there is only the Milne layer appearing. According to the rescaling $y = -\frac{x-p}{\varepsilon}\cdot n_p$, we obtain, as $\varepsilon\to 0$, the following boundary layer equation for $p\in\partial\Omega$ (cf. (B.26))

$$\begin{cases} -(n\cdot n_p)\partial_y I_\nu(y,n;p) = \alpha_\nu^a(p)(B_\nu(T(y,p)) - I_\nu(y,n;p)) & y>0\ , n\in\mathbb{S}^2, \\ \operatorname{div}\left(\int_0^\infty d\nu\int_{\mathbb{S}^2} dn\;(n\cdot n_p)I_\nu(y,n;p)\right) = 0 & y>0,\ n\in\mathbb{S}^2, \\ I_\nu(0,n;p) = g_\nu(n) & n\cdot n_p<0. \end{cases} \tag{3.7}$$

3.3 Summary of the results

We conclude this chapter with a summary and a discussion of the results obtained in ([37], Appendix B).

3.3.1 Stationary problem

We start collecting the outer problems obtained in the bulk for the stationary diffusion approximation of the problem (3.1). We refer to Section B.3.

As we have seen, in the case $\boxed{\ell_M = \ell_T \ll 1 \text{ and } \ell_A \ll \ell_S}$ the radiation intensity is at the leading order the Planck distribution $B_\nu(T)$ solving equation (3.6).

The outer problem for $\boxed{\ell_M = \ell_T = \ell_A = \ell_S \ll 1}$ is a different one. In this case the equilibrium diffusion approximation still holds, but T solves in the bulk

$$\operatorname{div}\left(\int_0^\infty d\nu \frac{1}{\alpha_\nu^a(x) + \alpha_\nu^s(x)} \left(\int_{\mathbb{S}^2} dn\, n \otimes \left(Id - A_{\nu,x}\right)^{-1}(n)\right) \nabla_x B_\nu(T(x))\right) = 0$$

for a suitable invertible operator $(Id - A_{\nu,x})$ related to the scattering operator, cf. (B.29).

The outer problem for $\boxed{\ell_M \ll \ell_T \ll 1}$ is very similar. It is given by

$$\operatorname{div}\left(\int_0^\infty d\nu \frac{1}{\alpha_\nu^s(x)} \left(\int_{\mathbb{S}^2} dn\, n \otimes \left(Id - H\right)^{-1}(n)\right) \nabla_x B_\nu(T(x))\right) = 0,$$

where $H[\varphi] = \int_{\mathbb{S}^2} K(\cdot, n')\varphi(n')dn'$.

The case $\boxed{\ell_M \ll \ell_T \approx 1}$ is the critical case in which the thermalization layer corresponds to the bulk and we obtain a non-equilibrium diffusion approximation. In this case the outer problem describes the isotropic leading order ϕ_0 of I_ν and the temperature T through the following screening equation

$$\begin{cases} \phi_0(x,\nu) - \frac{1}{4\pi\alpha_\nu^a(x)} \operatorname{div}\left(\frac{1}{\alpha_\nu^s(x)}\left(\int_{\mathbb{S}^2} n \otimes (Id - H)^{-1}(n)\, dn\right)\nabla_x \phi_0(x,\nu)\right) = B_\nu\left(T(x)\right) & x \in \Omega, \\ \int_0^\infty d\nu\, \alpha_\nu^a(x)\left(B_\nu(T(x)) - \phi_0(x,\nu)\right) = 0 & x \in \Omega. \end{cases} \tag{3.8}$$

The first equation is called screening equation because it describes the transition from the equilibrium to the non-equilibrium approximation. Indeed, if $\alpha_\nu^a(x)$ was much larger than 1, then $\phi_0(x,\nu) = B_\nu(T(x))$ leading to the equilibrium diffusion approximation. On the other hand, if $\alpha_\nu^a(x) \approx 0$, then the first equation of (3.8) reduces to the divergence term being equal to zero and thus ϕ_0 is independent of the temperature T.

Finally, when $\boxed{\ell_M \ll 1 \ll \ell_T}$ we obtain a non-equilibrium diffusion approximation where $I_\nu \to \phi_0$. Moreover, the leading order becomes isotropic and solves

$$\operatorname{div}\left(\frac{1}{\alpha_\nu^s(x)}\left(\int_{\mathbb{S}^2} n \otimes (Id - H)^{-1}(n)\, dn\right)\nabla_x \phi_0(x,\nu)\right) = 0,$$

while the temperature is recovered by the divergence-free condition as

$$\int_0^\infty d\nu\, \alpha_\nu^a(x)\left(B_\nu(T(x)) - \phi_0(x,\nu)\right) = 0.$$

We remark that in all the cases where $\ell_M = \ell_S$ the properties of the scattering operator have been used, which are studied in Proposition B.1 and in Proposition B.2.

3.3.2 Time-dependent problem

In ([36], Appendix B) we consider three different time-dependent problems according to the different relative values of the speed of light with respect to the characteristic quantities.

In Section (B.4) the limit problem are constructed for the time-dependent problem (3.3) when $c \to \infty$. It is interesting to notice that in this case no initial layer appears, while initial-boundary layers emerge.

In Section B.5 we consider the diffusion approximation for the problem (3.2) where $c = 1$. We obtain a complete picture analyzing also the initial layers. As expected, in the case where $\ell_M \ll \ell_T \ll 1$ also an initial thermalization layer of thickness ℓ_T^2 appears in which I_ν from being isotropic becomes $B_\nu(T)$. Initial Milne layers with thickness $\varepsilon\tau_h^{-1}$ also emerge. Here I_ν becomes isotropic.

A similar result is obtained in Section B.6 for (3.2) and $c = \varepsilon^{-\kappa}$ and $\kappa > 0$. It is interesting to notice that while in this case the outer problems correspond to the ones obtained in Section B.4 for $c = \infty$, initial layers also appear.

Chapter 4

Stationary diffusion approximation for absorption-emission process

This chapter is based on the article *"On the diffusion approximation of the stationary radiative transfer equation with absorption and emission"* [37], which is joint work with J. J. L. Velázquez. In Appendix C can be found the peer reviewed and published paper, in which both authors collaborated equally.

This work studies the diffusion approximation of the stationary radiative transfer equation (1.20) coupled to the divergence-free condition of the radiation energy flux. In particular, radiation takes place only through emission-absorption, i.e. in the absence of scattering processes. Furthermore, the particular case of the Grey approximation is considered, in which the absorption coefficient $\alpha_\nu(x) = \alpha(x)$ does not depend on the frequency $\nu > 0$. Moreover, the situations in which $\alpha(x)$ is independent of the temperature T and it is either constant or it depends on the spatial coordinate are taken into account. Assuming that the mean free path of the photons is much smaller than the characteristic size of the domain, we study the problem as $\varepsilon \to 0$

$$\begin{cases} n \cdot \nabla_x I_\nu(x,n) = \frac{\alpha(x)}{\varepsilon}\left(B_\nu(T(x)) - I_\nu(x,n)\right) & x \in \Omega, \\ \nabla_x \cdot \mathcal{F}(x) := \operatorname{div}_x\left(\int_0^\infty \int_{\mathbb{S}^2} n I_\nu(x,n)\, dn\, d\nu\right) = 0 & x \in \Omega, \\ I_\nu(x,n) = g_\nu(n) & x \in \partial\Omega \text{ and } n \cdot N_x < 0. \end{cases} \tag{4.1}$$

We remark that, as done in Chapter 3, we rescaled the original absorption coefficient so that $\alpha \approx 1$ in (4.1). Unlike in the rest of the thesis, in order to be consistent with the notation in [37], in this chapter we denote by N_x the outer normal on $x \in \partial\Omega$, instead of n_x.

Even though the scattering term is absent, i.e. $\alpha_\nu^s \equiv 0$, (4.1) is reminiscent to the stationary problem considered formally in Chapter 3 where the absorption length $\ell_A = \ell_M = \varepsilon \ll 1$ represents also the total mean free path of the photons while the scattering length satisfies $\ell_S \gg \ell_A$. According to the theory in Chapter 3, problem (4.1) is an example of equilibrium diffusion approximation. Indeed, the problem solved by (I_ν, T) as $\ell_A = \varepsilon \to 0$ is the same as the one derived in Section 3.2.2. In ([37], Appendix C) we prove rigorously the convergence to the limit problem obtained by matched asymptotic expansions as explained in Chapter 3. In particular the Milne problem (3.7) of the previous chapter is carefully studied.

As we observed in Section 1.4, the stationary radiative heat transfer in the Grey approximation, for instance problem (4.1), is equivalent to the one-speed neutron transport equation (1.26), which in this specific case has constant scattering kernel $\mathbb{K}(n,n') = \frac{1}{4\pi}$. The rigorous study of the diffusion approximation for this form of neutron transport equation has been considered by Bensoussan, Lions and Papanicolaou [19] with stochastic methods as well as by Guo and Wu [148] for constant coefficients $\alpha \equiv 1$ via $L^2 - L^p - L^\infty$ estimates.

The proof in ([37], Appendix C) is a new approach, which uses strongly the structure of the radiative transfer equation, for instance the fact that in (4.1) there are two unknowns, namely I_ν and T. Following the method explained in Section 1.5 we study the equivalent problem obtained reducing (4.1) to a non-local integral equation for the temperature using maximum-principle methods as main tool.

The main result that we prove is Theorem C.1 and it can be summarized as follows.

Theorem 4.1 (cf. [37], Summary of Theorem C.1). *Let $\alpha \in C^3(\Omega)$ with $0 < c_0 \leq \alpha \leq c_1$, $g_\nu \geq 0$ and $g_\nu \in L^\infty(\mathbb{S}^2, L^1(\mathbb{R}_+))$, $\Omega \subset \mathbb{R}^3$ bounded convex with C^3-boundary and strictly positive curvature. Let $(I_\nu^\varepsilon, T_\varepsilon)$ be the solution to problem (4.1). Then there exists a functional $T_\Omega : L^\infty\left(\mathbb{S}^2, L^1(\mathbb{R}_+)\right) \to C(\partial\Omega)$ which maps g_ν to a continuous function $T_\Omega[g_\nu](p)$ on the boundary $p \in \partial\Omega$ such that*

$$(I_\nu^\varepsilon(x,n), T_\varepsilon(x)) \to (B_\nu(T(x)), T(x))$$

uniformly in every compact subset of Ω as functions with values in $L^\infty\left(\mathbb{S}^2, L^1(\mathbb{R}_+)\right) \times \mathbb{R}_+$, where T is the solution to the Dirichlet problem

$$\begin{cases} -\operatorname{div}\left(\frac{4\pi\sigma}{\alpha}\nabla T^4\right) = 0 & x \in \Omega, \\ T(p) = T_\Omega[g_\nu](p) & p \in \partial\Omega. \end{cases}$$

In the following sections we outline the steps that we performed in order to prove Theorem 4.1. First of all, in Section 4.1 we derive using matched asymptotic expansions the outer problem and the boundary layer equation describing the approximate solution to (4.1). This is actually very similar to what we explained in Section 3.2.2. We will then study the boundary layer equation, also known as Milne problem, and we prove its well-posedness via the reduction to a non-local equation for the temperature. We also study the asymptotic behavior of the solutions to the Milne problem, cf. Section 4.2. Section 4.3 deals with the rigorous proof of the convergence to the diffusion approximation problem in the case of constant absorption coefficient. Finally, we summarize in Section 4.4 the main differences for the situation in which $\alpha(x)$ depends on $x \in \Omega$.

4.1 Derivation of the limit problem

Using matched asymptotic expansions as we did in Section 3.2.2, we observe that the leading order I_ν^ε solving (4.1) approaches the Planck distribution $B_\nu(T)$ of the temperature T, which is a further unknown of the problem and it solves the outer problem

$$-\operatorname{div}\left(\frac{1}{\alpha(x)}\nabla u\right) = 0, \tag{4.2}$$

where $u = 4\pi\sigma T^4$. In the case $\alpha \equiv 1$ the problem reduces to the Poisson equation $-\Delta u = 0$. According to the definition given in Chapter 3, this is a case of equilibrium diffusion approximation.

The boundary layer equation describing the Milne layer, in which the radiation intensity becomes at the same time isotropic and equal to the Planck distribution $B_\nu(T)$, is described by the following Milne problem obtained rescaling $y = \frac{x-p}{\varepsilon}$, for $p \in \partial\Omega$ and $x \in \Omega$.

$$\begin{cases} n \cdot \nabla_y I_\nu(y,n) = \alpha_\nu(p)\left(B_\nu(T(y)) - I_\nu(y,n)\right) & y \in \Pi_p, \\ \operatorname{div}_y\left(\int_0^\infty \int_{\mathbb{S}^2} n I_\nu(y,n)\, dn\, d\nu\right) = 0 & y \in \Pi_p, \\ I_\nu(y,n) = g_\nu(n) & y \in \partial\Pi_p \text{ and } n \cdot N_p < 0, \end{cases} \tag{4.3}$$

where $\Pi_p = \mathcal{R}_p^{-1}\left(\mathbb{R}_+ \times \mathbb{R}^2\right)$ and $\mathcal{R}_p = \mathrm{Rot}_p(\cdot - p)$ is the rigid motion defined by $\mathrm{Rot}_p(N_p) = -e_1$. Notice that after a suitable change of coordinates, (4.3) is equivalent to

$$\begin{cases} n \cdot \nabla_y I_\nu(y,n) = \alpha_\nu(0)\left(B_\nu(T(y)) - I_\nu(y,n)\right) & y \in \mathbb{R}_+ \times \mathbb{R}^2, \\ \mathrm{div}_y\left(\int_0^\infty \int_{\mathbb{S}^2} n I_\nu(y,n)\, dn\, d\nu\right) = 0 & y \in \mathbb{R}_+ \times \mathbb{R}^2, \\ I_\nu(y,n) = \overline{g}_\nu(n) := g_\nu\left(\mathrm{Rot}_p^{-1}(n)\right) & y \in \{0\} \times \mathbb{R}^2, \text{ and } n \cdot N < 0, \end{cases} \tag{4.4}$$

where $N = -e_1$ and with some abuse of notation we denote $I_\nu\left(\mathcal{R}_p^{-1}(y), \mathrm{Rot}_p^{-1}(n)\right)$ as $I_\nu(y,n)$. Moreover, we remark that, assuming that I_ν and T depend only on the distance to the boundary in direction N_p, the boundary layer equation (4.3) reduces to the Milne problem given in (3.7).

In the next section we will see that for the solution (I_ν, T) of (4.4) there exists a limit $\lim_{y\to\infty} T(y;p) = T_\Omega[g](p)$, where $p \in \partial\Omega$. The matching with the outer problem implies that $T_\Omega[g](\cdot)$ is the boundary value for the outer problem (4.2).

4.2 Boundary layer equation

Without loss of generality we can assume $\alpha(p) \equiv 1$. This can be obtained by the rescaling $y = \frac{x-p}{\varepsilon}\alpha(p)$. Moreover, solving by characteristics the first equation in (4.4), we get similarly to (1.28)

$$I_\nu(y,n;p) = \overline{g}_\nu(n) e^{s(y,n)} \mathbb{1}_{n\cdot N<0} + \int_0^{s(y,n)} e^{-t} B_\nu(T(y-tn;p))\, dt,$$

where $s(y,n) = |y - Y(y,n)|$ for $Y(y,n) = \{y - tn : t > 0\} \cap \{0\} \times \mathbb{R}^2$ if $n \cdot N < 0$ and $s(y,n) = \infty$ otherwise. We put the variable p in order to emphasize the dependence of the boundary layer solutions to the point $p \in \partial\Omega$. Notice that if $n \cdot N < 0$, then $s(y,n) = \frac{y_1}{|n\cdot N|}$. Thus, using the divergence free condition of the radiation flux, defining $\bar{u}(y,p) = 4\pi\sigma T^4(y,p)$ and assuming that (I_ν, T) depends only on y_1, we obtain

$$\bar{u}(y_1,p) - \int_{\mathbb{R}_+\times\mathbb{R}^2} \frac{e^{-|y-\eta|}}{4\pi|y-\eta|^2} \bar{u}(\eta_1,p) d\eta = \int_0^\infty d\nu \int_{n\cdot N<0} dn\, \overline{g}_\nu(n) e^{-\frac{y_1}{|n\cdot N|}},$$

where we also changed from spherical to Cartesian coordinates according to

$$\int_{\mathbb{S}^2} dn \int_0^{s(x,n)} dt f(y-tn) = \int_{\mathbb{R}_+\times\mathbb{R}^2} \frac{e^{-|y-\eta|}}{|y-\eta|^2} dy.$$

An application of Fubini theorem and another change of coordinates give for $y \in \mathbb{R}_+$

$$\bar{u}(y,p) - \int_0^\infty K(y-\eta)\bar{u}(\eta,p) d\eta = \int_0^\infty d\nu \int_{n\cdot N_p<0} dn\, g_\nu(n) e^{-\frac{y}{|n\cdot N_p|}} =: G_p(y), \tag{4.5}$$

where $K(x) = \frac{1}{2}\int_{|x|}^\infty \frac{e^{-t}}{t}\, dt$ is the normalized exponential integral, whose properties are collected in Proposition C.1 and Proposition C.2. In Section C.3 we study the well-posedness of (4.5) and the asymptotic behavior of the solution as $y \to \infty$.

4.2.1 Well-posedness theory

First of all we notice that the operator

$$\mathcal{L}[\bar{u}](y) := \bar{u}(y) - \int_0^\infty K(y-\eta)\bar{u}(\eta)\, d\eta$$

satisfies a maximum principle. This is shown in Lemma C.1 and it implies the non-negativity and the uniqueness of the solution to (4.5), since any bounded solution to $\mathcal{L}[u] = 0$ is trivial. This is shown in Theorem C.2 using the supersolution $(1+x)\mathbb{1}_{x>0}$. In Theorem C.3 the following is shown.

Theorem 4.2 (cf. [37], summary of Theorem C.3)**.** *Let $H \in C(\mathbb{R}_+)$ with $0 \leq H(x) \leq Ce^{-Ax}\chi_{\{x>0\}}$ for $C, A > 0$. Then there exists a unique bounded solution to*

$$\begin{cases} u(x) - \int_0^\infty dy\, K(x-y)\, u(y) = H(x) & x > 0, \\ u(x) = 0 & x < 0. \end{cases} \tag{4.6}$$

Moreover, u is continuous on $(0, \infty)$.

This theorem implies the well-posedness of (4.5), since $0 \leq G_p(y) \leq \|g\|_{L^\infty(\mathbb{S}^2, L^1(\mathbb{R}_+))} e^{-y}$. The result of Theorem 4.2 is shown using Fourier methods obtaining two suitable functions $u = \tilde{u} + v$. Indeed, the Fourier transform of the kernel K is explicit

$$\hat{K}(\xi) = \frac{1}{\sqrt{2\pi}} \frac{\arctan(\xi)}{\xi}.$$

This yields the construction of a function $\tilde{u}$, solution to

$$\tilde{u}(x) - K * \tilde{u}(x) = H(x)\mathbb{1}_{x>0} - H(-x)\mathbb{1}_{x<0}.$$

Finally, a function v solving

$$v(x) - K * v(x) = 0 \quad \text{if } x > 0 \quad \text{and} \quad v(x) = 0 \quad \text{if } x < 0$$

is determined via a method of sub- and supersolutions reminiscent to the Perron method for the Laplace equation.

4.2.2 Asymptotic behavior of the solution

Since the matching between the outer problem (4.2) and the boundary layer equation (4.5) is given by $\lim_{y\to\infty} \bar{u}(y,p)$, which is the boundary value for the limit problem, one needs to study the asymptotic behavior of $\bar{u}$. In Section C.3.3 we prove the following proposition.

Proposition 4.1 (cf. [37], Summary of Proposition C.3)**.** *Let $\bar{u}(y,p)$ be the unique non-negative solution to (4.5). Then*

(i) $\bar{u}$ is bounded and it is uniformly continuous with respect to $y \in \mathbb{R}_+$ and Lipschitz continuous with respect to $p \in \partial\Omega$;

(ii) The limit $\lim_{y\to\infty} \overline{u}(y,p) = \overline{u}_\infty(p)$ exists, it is Lipschitz continuous and it is uniquely determined by g_ν and $\overline{u}$;

(iii) $\overline{u}_\infty(p) > 0$ unless $\left|\left\{n \in \mathbb{S}^2 : n \cdot N_p < 0 \text{ and } \int_0^\infty d\nu g_\nu(n) \not\equiv 0\right\}\right| = 0$;

(iv) There exists $C > 0$ such that $\sup_{p\in\partial\Omega} |\overline{u}(y,p) - \overline{u}_\infty(p)| \leq Ce^{-y/2}$.

The most involved step in (i) is the proof of the Lipschitz continuity of $\overline{u}(y,p)$ in the second variable. The result is based on the estimate

$$|G_p(y) - G_q(y)| \leq C(\partial\Omega, g_\nu)|p-q| \quad \text{for } p, q \in \partial\Omega,$$

where the smoothness of $\partial\Omega$ and the property of the curvature, which is bounded from below, has been used. The existence of a limit $\overline{u}_\infty \geq 0$ (cf. (ii) and (iii)) can be proved via Fourier methods in the space of distributions using the Riemann-Lebesgue theorem. Finally, also the exponential decay, i.e. (iv), is obtained analyzing the behavior of the Fourier transform of $\overline{u}$ computing suitable contour integrals.

4.3 Diffusion approximation for $\alpha \equiv 1$

In order to prove Theorem 4.1 we first assume $\alpha \equiv 1$. Reformulating the problem (4.1) as indicated in Section 1.5 as a non-local integral equation for the temperature, we see that the solution $(I_\nu^\varepsilon, T_\varepsilon)$ solving (4.1) is the solution to the equivalent problem

$$I_\nu^\varepsilon(x,n) = g_\nu(n) e^{-\frac{s(x,n)}{\varepsilon}} + \frac{1}{\varepsilon}\int_0^{s(x,n)} e^{-\frac{t}{\varepsilon}} B_\nu\left(T\left(x-tn\right)\right)\, dt, \tag{4.7}$$

where $u^\varepsilon = 4\pi\sigma T_\varepsilon^4$ solves

$$u^\varepsilon(x) - \int_\Omega K_\varepsilon(y-\eta) u^\varepsilon(\eta)\, d\eta = \int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, g_\nu(n) e^{-\frac{s(x,n)}{\varepsilon}} \tag{4.8}$$

with $K_\varepsilon = \frac{e^{-\frac{|\cdot|}{\varepsilon}}}{4\pi\varepsilon|\cdot|^2}$, cf. (C.60). As proved in Chapter 2, equation (4.8) has a unique bounded non-negative solution $u^\varepsilon \in C(\Omega)$.

In order to prove Theorem 4.1 it is enough to show that $u^\varepsilon \to v$ uniformly in every compact set, where v solves the limit problem

$$\begin{cases} -\Delta v = 0 & \text{in } \Omega, \\ v = \overline{u}_\infty & \text{on } \partial\Omega, \end{cases} \tag{4.9}$$

which is obtained by matching the outer problem with the boundary layer solution.

The proof of Theorem 4.1 requires many steps and it is based only on maximum principle arguments. As shown in Section 1.5, the non-local operator $\mathcal{L}_\Omega^\varepsilon$ defined by the left-hand side of (4.8) satisfies a (global) maximum principle in the following sense, cf. Theorem C.5.

Theorem 4.3 (cf. [37], summary of Theorem C.5)**.** *Let $v \in C(\overline{\Omega})$ and $O \subset \Omega$ open. If $\mathcal{L}_\Omega^\varepsilon(v) \geq 0$ for all $x \in \Omega$ or if $v \geq 0$ on $\Omega \setminus \overline{O}$ and $\mathcal{L}_\Omega^\varepsilon(v) \geq 0$ for all $x \in O$, then $v \geq 0$.*

4.3.1 Uniformly boundedness

Constructing a suitable uniformly bounded family of supersolutions Φ_ε, we prove in Section C.4.2 that $\|u^\varepsilon\|_\infty \leq C(g,\Omega)$. As given in Theorem C.6, such supersolutions are for instance

$$\Phi^\varepsilon(x) = C_3\left(C_1 - |x|^2\right) + C_2\left[\left(1 - \frac{\gamma}{1+\left(\frac{d(x)}{\varepsilon}\right)^2}\right) \wedge \left(1 - \frac{\gamma}{1+\left(\frac{\mu R}{\varepsilon}\right)^2}\right)\right], \tag{4.10}$$

where R is the minimal radius of curvature and $d(x) = \operatorname{dist}(x,\partial\Omega)$. Moreover, $C_1,\ C_2,\ C_3,\ \mu \in (0,1),\ \gamma(\mu) \in (0,1/3)$ are suitable constants. Subdividing Ω and estimating carefully many integral terms it turns out that

$$\mathcal{L}_\Omega^\varepsilon(\Phi^\varepsilon)(x) \geq \|g\|_{L^1(\mathbb{S}^2\times\mathbb{R}_+)} e^{-d(x)/\varepsilon} \geq \int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, g_\nu(n) e^{-\frac{s(x,n)}{\varepsilon}}$$

for ε small enough. The idea behind the construction of Φ^ε relies on the properties of $\mathcal{L}_\Omega^\varepsilon$, for instance the approximation of $-\frac{\varepsilon^2}{3}\Delta$ by the operator $\mathcal{L}_\Omega^\varepsilon$. Indeed, $|x|^2$ and $\frac{1}{1+|x|^2}$ are supersolutions for the Laplace operator (at least for small $|x|>0$) with

$$-\Delta|x|^2 = 6 \quad \forall x \in \mathbb{R}^3 \quad \text{and} \quad -\Delta\frac{1}{1+|x|^2} \geq \frac{C}{(1+|x|^2)^3} > Ce^{-|x|} \quad \text{for } |x| < \frac{1}{2\sqrt{3}}.$$

On the other hand, another useful property is that

$$\mathcal{L}_\Omega^\varepsilon[1](x) = \int_{\Omega^c} K_\varepsilon(x-\eta)d\eta = \int_{-\infty}^{-d(x)/\varepsilon} K(z)dz > 0.$$

4.3.2 Estimates near the boundary

In order to prove that u^ε converges to the solution of the boundary value problem (4.9) we need to show that $u^\varepsilon - \overline{u}$ is small in regions near the boundary $\partial\Omega$. It is important to recall that the Milne layer has a thickness ε. Therefore, we cannot expect to be able to approximate u^ε by v in regions of distance of order ε to the boundary. In particular, we show that $u^\varepsilon - \overline{u}$ is small in regions of size close to $\varepsilon^{1/2}$ to the boundary. These regions are much greater than the Milne layer.

In Lemma C.9 we first estimate for $0 < \delta < \frac{1}{16}$

$$\left|\mathcal{L}_\Omega^\varepsilon \left(\overline{U}_\varepsilon(\cdot,p) - u^\varepsilon\right)(x)\right| \leq Ce^{-\frac{Ad(x)}{\varepsilon}} \begin{cases} \varepsilon^\delta & \text{if } |x-p| < \varepsilon^{\frac{1}{2}+2\delta}, \\ 1 & \text{if } |x-p| \geq \varepsilon^{\frac{1}{2}+2\delta}, \end{cases} \tag{4.11}$$

where $\overline{U}_\varepsilon(\cdot,p) = \overline{u}\left(\frac{\mathcal{R}_p(\cdot)\cdot e_1}{\varepsilon}, p\right)$ and $\mathcal{R}_p$ is the rigid motion defined before equation 4.4, so that $\Pi_p = \mathcal{R}_p^{-1}\left(\mathbb{R}_+ \times \mathbb{R}^2\right)$. This estimate has been achieved splitting the resulting integral terms in several integrals over appropriate regions and making large use of the boundary's approximations by suitable paraboloids. Indeed, using that $\bar{u}$ satisfies equation (4.5) we see that we have to estimate

$$\begin{aligned}\left|\mathcal{L}_\Omega^\varepsilon \left(\overline{U}_\varepsilon(\cdot,p) - u^\varepsilon\right)(x)\right| \leq & \int_{\Pi_p\setminus\Omega} d\eta K_\varepsilon(\eta - x)\overline{U}_\varepsilon(\eta,p) \\ & + \int_0^\infty d\nu \int_{\mathbb{S}^2} dn\; g_\nu(n) \left| e^{-\frac{\left|x - x_{\Pi_p}(x,n)\right|}{\varepsilon}} - e^{-\frac{\left|x - x_{\Omega}(x,n)\right|}{\varepsilon}} \right|,\end{aligned}$$

where $x_{\Pi_p}(x,n) \in \partial\Pi_p$ and $x_{\partial\Omega}(x,n) \in \partial\Omega$ connect x in direction n. Moreover, $|x - x_{\Pi_p}(x,n)| = \infty$ if $x \cdot N_p \geq 0$.

With the help of the key estimate (4.11), we apply the maximum principle in order to show that $|\overline{u} - u_\varepsilon|$ is very small in regions adjacent to the boundary of thickness $\varepsilon^{1/2+4\delta}$.

To this end, we construct in Proposition C.5 the family of supersolutions defined in (C.115) for $|\overline{u} - u^\varepsilon|$. Once more, the functions in (C.115) are obtained considering suitable combination of supersolutions for $-\frac{\varepsilon^2}{3}\Delta$ on the half space $\mathbb{R}_+ \times \mathbb{R}^2$. Indeed, in a small neighborhood of $p \in \partial\Omega$ the boundary can be approximated by the plane orthogonal to N_p at the point p. Specifically, we consider the harmonic function $\arctan\left(\frac{x_i}{x_1}\right)$ and the superharmonic functions $-\left(\frac{x_1}{\rho_i^2}\right)^2$ and $\left(\frac{x_1}{\rho_i^2}\right)^{\frac{1}{2}}$, where $\rho_i^2 = x_1^+ x_i^2$ for $i = 2, 3$. These functions are then combined in a proper way together with the supersolution defined in (4.10) in order to construct a new supersolution for (4.11), which is very small in a neighborhood of $p \in \partial\Omega$ of size $\varepsilon^{\frac{1}{2}+4\delta}$. We refer to Corollary C.4.

4.3.3 Uniform convergence

We conclude the proof of Theorem 4.1 for $\alpha \equiv 1$ showing via maximum principle that

$$|u^\varepsilon - v| \to 0 \quad \text{as } \varepsilon \to 0 \quad \text{in compact sets.}$$

This result has been proved decomposing Ω in several subdomains, as the following Figure 4.1 shows.

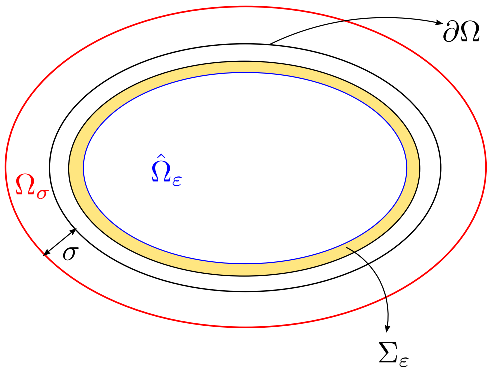

Figure 4.1: Decomposition of Ω in subdomains for the last step of the proof of the convergence of u^ε to v.

In the region Σ_ε, i.e at distances to the boundary of order $\varepsilon^{1/2+6\delta} < \operatorname{dist}(x,\partial\Omega) \leq \varepsilon^{1/2+4\delta}$ for $0<\delta<\frac{1}{16}$, the difference $|u^\varepsilon - \overline{u}_\infty|$ is very small. Moreover, in the region $\hat{\Omega}_\varepsilon$, i.e. for distances to the boundary of greater order, the maximum principle is applied to the function $|v_\sigma - u^\varepsilon|$, where v_σ solves the Poisson equation (4.9) on the bigger domain $\Omega_\sigma = \Omega \cup \{x \in \Omega^c : \operatorname{dist}(x,\partial\Omega) < \sigma\}$ for $\sigma \ll 1$ independent of ε.

Thus, letting first $\varepsilon \to 0$ and then $\sigma \to 0$ we conclude in Theorem C.7 the uniform convergence of u^ε to the unique solution of (4.9) in compact sets.

Finally, the convergence of $I_\nu^\varepsilon \to B_\nu(T)$, where $4\pi\sigma T^4$ solves the limit problem (4.9), is proved applying Lebesgue dominated convergence theorem to the equation (4.7). See Corollary C.5.

4.4 Diffusion approximation for spatially dependent absorption coefficient

In the case in which the absorption coefficient $\alpha(x)$ depends on $x \in \Omega$, we prove Theorem 4.1 under the assumption that it is bounded from above and from below, i.e. $0 < c_0 \leq \alpha(x) \leq \|\alpha\|_{C^3} = c_1 < \infty$ for $c_0, c_1 > 0$.

While in this situation the outer problem in the bulk is given for $u = 4\pi\sigma T^4$ by

$$-\operatorname{div}\left(\frac{1}{\alpha(x)}\nabla_x u\right) = 0$$

and $I_\nu^\varepsilon \to B_\nu(T)$, the boundary layer equation is the same as the one obtained for constant coefficients in (4.3). Thus, only the proof of the convergence has to be adjusted.

In Section C.5 we proceed refining the proof for constant coefficients for the case of spatially dependent absorption coefficient. It turns out that the supersolutions and the estimates that we obtained for $\alpha \equiv 1$ can be easily adapted and used in this situation. The most important change is in the definition of the supersolution for u^ε, i.e. Φ^ε of Theorem C.8, which are given by

$$\Phi^\varepsilon(x) = C_3\left[\left(e^{\lambda D} + C_1 - e^{\lambda x_1}\right) + C_2\left(\left(1 - \frac{\gamma}{1+\left(\frac{c_0 d(x)}{\varepsilon}\right)^2}\right) \wedge \left(1 - \frac{\gamma}{1+\left(\frac{c_0 \mu R}{\varepsilon}\right)^2}\right)\right)\right],$$

where $D = \operatorname{diam}(\Omega)$. Notice that we replace the term $(1-|x|^2)$ of the supersolution in (4.10) by the term $(1-e^{-\lambda x_1})$ for some $\lambda > 0$.

Since the remaining steps are very similar to what we did for $\alpha \equiv 1$ and only the estimates used for the numerous applications of the maximum principle need to be determined also for the spatially dependent coefficient, we refer to ([37], Appendix C) for the rest of the proof.

Chapter 5

Well-posedness theory for a Stefan problem with radiation

This chapter is based on the article *"Well-posedness for a two-phase Stefan problem with radiation"* [39], which is joint work with J. J. L. Velázquez. The most recent version of this work can be found in Appendix D. The results in [39] have been obtained by an equal collaboration between the two authors.

The paper [39] studies the well-posedness theory for a two-phase free boundary problem modeling the melting of ice assuming that the heat is transferred by conduction in both phases of the material and also by radiation in the solid phase. As we introduced in Section 1.6.3 it is assumed at initial time $t = 0$ that the liquid occupies $\mathbb{R}^3_- := \{x \in \mathbb{R}^3 : x_1 < 0\}$ with a temperature T larger than the melting temperature T_M and that the solid fills the region $\mathbb{R}^3_+ := \{x \in \mathbb{R}^3 : x_1 > 0\}$ with $T < T_M$. Initially the interface $\Gamma(t)$, i.e. the surface separating the two phases, is $\Gamma(0) = \{0\} \times \mathbb{R}^2$ and its temperature satisfies $T = T_M$.

The assumption of no radiative heat transfer in the liquid is equivalent to the assumption that the liquid phase is completely transparent. Thus, the radiation escaping from the solid (or going towards the solid if an external source of radiation is present) simply travels through the liquid without interacting with it. Under these hypotheses, the evolution of the temperature in the liquid is described by the heat equation

$$C_L \partial_t T = K_L \Delta T,$$

where C_L is the volumetric heat capacity of the liquid and K_L is the conductivity of the liquid.

On the other hand, according to the heat transfer theory presented in Section 1.3 the temperature's evolution of the solid is given by the following coupled equations

$$\begin{cases} C_S \partial_t T(t,x) = K_S \Delta T(t,x) - \operatorname{div}\left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, n I_\nu(t,x,n)\right), \\ n \cdot \nabla_x I_\nu(t,x,n) = \alpha \left(B_\nu(T(t,x)) - I_\nu(t,x,n)\right), \\ I_\nu(t,x,n)|_{x\in\Gamma(t)} = g_\nu(n) \qquad\qquad n \cdot n_x < 0, \end{cases}$$

where n_x is the outer normal at $x \in \Gamma(t)$, C_s and K_S are the volumetric heat capacity and the conductivity of the solid and the absorption coefficient $\alpha \equiv \text{const} > 0$ is assumed to be constant.

We remark that we are considering the quasi-static radiative transfer equation since the characteristic time in which the temperature has significant changes is much shorter than the characteristic time in which I_ν stabilizes. This is due to the fact that the photons travel at the speed of light. Moreover, in [39] and in the following article [38] we consider the situation in which scattering is negligible ($\alpha_\nu^s \equiv 0$) and in which there is no external source of radiation. We assume therefore from now on that

$$g_\nu(n) \equiv 0.$$

The evolution of the interface $\Gamma(t)$ is given by the so-called Stefan condition which we now introduce according to [6, 106, 123]. In general the Stefan condition can be written as

$$V \cdot n = \frac{1}{L}\left[\mathcal{F}_L|_{\Gamma(t)} - \mathcal{F}_S|_{\Gamma(t)}\right] \cdot n, \tag{5.1}$$

where $\mathcal{F}_L$ is the energy flux inside the liquid and $\mathcal{F}_S$ is the one inside the solid, V is the velocity of the moving interface, L is the latent heat and $n \in \mathbb{S}^2$ is the unit normal, which points from the liquid to the solid phase. The latent heat is defined as the amount of energy absorbed by the solid per unit of volume in order to melt at constant melting temperature, cf. [153]. During solidification, i.e. when the liquid becomes solid, the amount of energy released per unit of volume of during the phase transition equals to L. We remark that in this model any change of volume between the two phases during the phase transition is neglected.

Before giving the Stefan condition for the free boundary problem with radiation considered in [39], let us briefly explain the physical intuition behind (5.1).

Let us assume without loss of generality that in a short time interval $(t, t + \delta t)$ in a neighborhood of a point $x_0 \in \Gamma(t)$ the solid melts. If this is the case the interface moves towards the solid and thus $V \cdot n > 0$. During the process of melting (or of solidification) energy is absorbed (or released). The energy absorbed during the melting process equals the latent heat of an infinitesimal volume around the segment $x_0 + (V \cdot n)\delta t n$, i.e.

$$L(V \cdot n) dA \delta t,$$

where dA is the infinitesimal interface area perpendicular to n containing x_0. On the other hand, the absorbed energy is given by the difference between the energy flux entering in direction n this infinitesimal volume and the energy flux going out from it in the same time interval. This leads to the Stefan condition

$$L(V \cdot n) = \left[\mathcal{F}_L|_{\Gamma(t)} - \mathcal{F}_S|_{\Gamma(t)}\right] \cdot n.$$

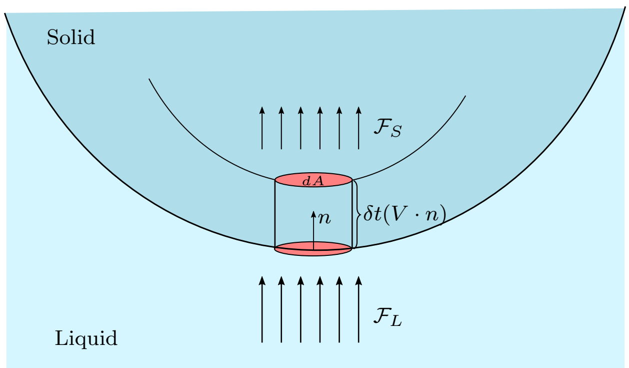

Figure 5.1: Interpretation of the Stefan condition.

According to the Fourier law, the heat flux due to conduction is

$$\mathcal{F}_{\text{conduction}}(t, x) = -K \nabla_x T(t, x),$$

where K is once more the conductivity of the material. On the other hand, the energy flux due to radiation is

$$\mathcal{F}_{\text{radiation}}(t,x) = \int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, nI_\nu(t,x,n).$$

It is important to notice that under the assumptions that we have made so far, the radiation energy flux is continuous at the interface. Indeed, even though the photons do not interact with the liquid, the radiation intensity I_ν is not zero in this phase. Precisely, it is given by the radiation intensity of the photons escaping from the solid, since there is no external source of radiation.

Therefore, the Stefan condition for the problem under consideration is the same as the Stefan condition for the classical Stefan problem, namely

$$(V\cdot n) = \frac{1}{L}\left(K_S\ \nabla_x T|_{\Gamma(t)} - K_L\ \nabla_x T|_{\Gamma(t)}\right)\cdot n.$$

Under the final assumption that the temperature depends only on x_1, which can be interpreted as the assumption that T depends only on the distance to the interface, the moving interface is the plane $\Gamma(t)=\{s(t)\}\times\mathbb{R}^2$. Hence, the Stefan condition can be rewritten for $n=e_1$ as

$$\dot{s}(t) = \frac{1}{L}\left(K_S\partial_{x_1}T(t,s(t)^+) - K_L\partial_{x_1}T(t,s(t)^-)\right).$$

Finally, the problem that we will study is

$$\begin{cases} C_L\partial_t T(t,x_1) = K_L\partial^2_{x_1}T(t,x_1) & x_1 < s(t),\\ C_S\partial_t T(t,x_1) = K_S\partial^2_{x_1}T(t,x_1) - \operatorname{div}\left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dn nI_\nu(t,x,n)\right) & x_1 > s(t),\\ n\cdot\nabla_x I_\nu(t,x,n) = \alpha\left(B_\nu(T(t,x_1)) - I_\nu(t,x,n)\right) & x_1 > s(t),\\ I_\nu(t,x,n) = 0 & x_1 = s(t),\ n_1>0,\\ T(t,s(t)) = T_M & x_1 = s(t),\\ T(0,x) = T_0(x) & x_1\in\mathbb{R},\\ \dot{s}(t) = \frac{1}{L}\left(K_S\partial_{x_1}T(t,s(t)^+) - K_L\partial_{x_1}T(t,s(t)^-)\right). \end{cases} \tag{5.2}$$

We study the well-posedness theory for (5.2) showing first the existence of a unique solution for small times, cf. Theorem D.1. Moreover, for a large class of initial values, defined by an upper bound on the temperature in the liquid, we prove also that there exists a unique global in time solution, cf. Theorem D.2.

In the following we will summarize the main results and proof's ideas developed in ([39], Appendix D). First of all we will briefly derive an equivalent form for (5.2), which we will study in the rest of this chapter, (cf. Section 5.1). In Section 5.2 we outline the main strategy for the local well-posedness theory, which is obtained using fixed-point arguments and classical parabolic theory. We conclude with Section 5.3 giving the key steps for the proof of the global well-posedness result, which is due to an application of the maximum principle.

5.1 Reduction to an equivalent problem for the temperature

Similarly as we did in Section 1.5 we reduce problem (5.2) to a one-dimensional free-boundary problem where the only unknowns are the temperature and the position of the interface $s(t)$. This can be done solving the stationary radiative transfer equation by characteristics. To this end, for $x_1 > s(t)$ and $n\in\mathbb{S}^2$ with $n_1>0$ we denote as usual by $y(t,x,n) = \{s(t)\}\times\mathbb{R}^2\cap\{x-\tau n:\tau>0\}$ the point at the interface connecting x with the interface in direction $-n$. Then we define $d(t,x,n) = |x - y(t,x,n)| = \frac{x_1 - s(t)}{n_1}$ if $n_1 > 0$ and $d(t,x,n) = \infty$ if $n_1 \leq 0$. Solving the radiative transfer equation by characteristic we obtain for $x_1 > 0$

$$I_\nu(t,x,n) = \int_0^{d(t,x,n)} d\tau \alpha \exp(-\alpha\tau) B_\nu(T(t, x_1 - \tau n_1)).$$

Thus, using the Stefan-Boltzmann law (1.4) and changing from spherical to Cartesian coordinates we have

$$\begin{aligned}
\operatorname{div}\left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dnnI_\nu(t,x,n)\right) &= \alpha\left[\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\ \left(B_\nu(T(t,x_1)) - I_\nu(t,x,n)\right)\right] \\
&= 4\pi\sigma\alpha\left[T^4(t,x_1) - \fint_{\mathbb{S}^2} dn \int_0^{d(t,x,n)} d\tau\alpha \exp(-\alpha\tau) T^4(t, x_1 - \tau n_1)\right] \\
&= 4\pi\sigma\alpha\left[T^4(t,x_1) - \int_{(s(t),\infty)\times\mathbb{R}^2} d\eta\ \frac{\alpha\exp(-\alpha|x-\eta|)}{4\pi|x-\eta|^2} T^4(t,\eta_1)\right] \\
&= 4\pi\sigma\alpha\left[T^4(t,x_1) - \int_{s(t)}^\infty d\eta_1\ \frac{\alpha E_1(\alpha(x_1-\eta_1))}{2} T^4(t,\eta_1)\right],
\end{aligned}$$

where $E_1(x) = \int_{|x|}^\infty \frac{e^{-t}}{t}$ is the exponential integral. Notice that this operator is similar to the one obtained in Chapter 4.

After a suitable time and space rescaling and changing to the non-moving coordinate system, the problem (5.2) is equivalent to

$$\begin{cases}
\partial_t T_1(t,y) - \dot{s}(t)\partial_y T_1(t,y) = \frac{K}{C}\partial_y^2 T_1(t,y) & y < 0, \\
\partial_t T_2(t,y) - \dot{s}(t)\partial_y T_2(t,y) = \partial_y^2 T_2(t,y) - T_2^4(t,y) + \int_0^\infty d\xi\ \frac{\alpha E_1(\alpha(y-\xi))}{2} T_2^4(t,\xi) & y > 0, \\
T_1(t,0) = T_2(t,0) = T_M & y = 0, \\
T(0,y) = T_0(y) & y \in \mathbb{R}, \\
\dot{s}(t) = \frac{1}{L}\left(\partial_y T_2(t,0) - K\partial_y T_1(t,0)\right),
\end{cases} \tag{5.3}$$

where K, C, α, L are new constants obtained from the original ones by the rescaling.

5.2 Local well-posedness theory

We outline here the local well-posedness result obtained in Theorem D.4 and in Proposition D.3, which can be summarized as follows

Theorem 5.1 (Local well-posedness, cf. [39], Theorem D.4 and Proposition D.3)**.** *Let $T_0 \in C^{0,1}(\mathbb{R})$ with $T_0(0) = T_M$, $T_0|_{\mathbb{R}_-} > T_M$ and $0 < T_0|_{\mathbb{R}_+} < T_M$. Assume also $T_0|_{\mathbb{R}_\pm} \in C^{2,\delta}(\mathbb{R}_\pm)$ for some $\delta \in (0,1/2)$. Then there exists $t^* > 0$ such that there exists a unique solution (T_1, T_2, s) to (5.3) with*

(i) $(T_1, T_2, \dot{s}) \in \mathcal{C}_{t,y}^{\delta/2,1+\delta}((0,t^*)\times\mathbb{R}_-) \times \mathcal{C}_{t,y}^{\delta/2,1+\delta}((0,t^*)\times\mathbb{R}_+) \times C^{\delta/2}([0,t^*])$;

(ii) $(T_1, T_2, s) \in \mathcal{C}_{t,y}^{1,2}((0,t^*)\times\mathbb{R}_-) \times \mathcal{C}_{t,y}^{1,2}((0,t^*)\times\mathbb{R}_+) \times C^1([0,t^*])$;

(iii) as long as (T_1, T_2, s) exists, the temperature satisfies $T_1 \geq T_M$ and $0 \leq T_2 \leq T_M$. In addition, the strict inequalities hold in open subsets of $\mathbb{R}_\pm$.

The Hölder spaces $\mathcal{C}_{t,y}^{\delta/2,1+\delta}((0,t)\times U)$ are defined in Section D.1.4 and are those spaces whose functions f are $\delta/2$-Hölder in time with $\partial_y f$ δ-Hölder in space.

For the proof of the Theorem 5.1 we actually consider $u_i = T_i - T_M$ solving (5.3) for $T_M = 0$ and $u(0,y) = u_0(y) = T_0(y) - T_M$.

5.2.1 Existence and uniqueness of solutions

In order to prove claims (i) and (ii) of Theorem 5.1 we first prove the existence of functions $(u_1, u_2, s) \in C^{0,1}_{t,y}((0,t^*) \times \mathbb{R}_-) \times C^{0,1}_{t,y}((0,t^*) \times \mathbb{R}_+) \times C^1((0,t^*))$ solving (5.3) in distributional sense. Later we prove the higher regularity stated in Theorem 5.1.

We consider the first two equations in (5.3) as heat equations on the half-spaces with external sources given by

$$F_1(t,y) = \dot{s}(t)\partial_y u_1(t,y) \text{ on } \mathbb{R}_-$$

and

$$F_2(t,y) = \dot{s}(t)\partial_y u_2(t,y) - (u_2(t,y)+T_M)^4 + \int_0^\infty d\xi \, \frac{\alpha E_1(\alpha(y-\xi))}{2}(u_2(t,\xi)+T_M)^4 \text{ on } \mathbb{R}_+.$$

Using the Green's function for the heat equation in the half space we obtain integral representation formulas for $u_1,\ u_2,\ \partial_y u_1,\ \partial_y u_2$ and $\dot{s}$, which are implicit since they depend on these functions. These equations define an operator on the space $X = C^{0,1}_{t,y}((0,t^*) \times \mathbb{R}_-) \times C^{0,1}_{t,y}((0,t^*) \times \mathbb{R}_+) \times C^0((0,t^*))$, which for times $t^* > 0$ small enough can be shown to be a contractive self-map on $\mathcal{A} = \left\{(u_1,u_2,\dot{s}) \in X :\ \|u_1\|_{0,1} \leq C_1,\ \|u_2\|_{0,1} \leq C_2,\ \|\dot{s}\|_{C^0} \leq C_3\right\}$, where the constant $C_1,\ C_2,\ C_3$ depend only on the norm of u_0 and of its piecewise defined derivative. For the exact definition of the norms considered for $\mathcal{A}$ we refer to the proof of Theorem D.3.

Thus, there exists a unique solution solving (5.3) in distributional sense. The strategy of using a fixed-point approach is similar to the idea used also by Rubenšteĭn [123] and by Friedman [55] for the classical Stefan problem.

We remark that $s \in C^1((0,t^*))$ satisfies the Stefan condition for $u_1,\ u_2$ in the classical sense. Hence, fixing $s \in C^1((0,t^*))$ as the position of the interface obtained for the given initial temperature u_0, we study the regularity of the parabolic problems

$$\begin{cases} \partial_t u_1(t,y = \frac{K}{C}\partial_y^2 u_1(t,y) + F_1(t,y) & y<0,\\ u_1(t,0)=0, & \\ u_1(0,y)=u_0(y) & y<0, \end{cases} \quad \text{and} \quad \begin{cases} \partial_t u_2(t,y) = \partial_y^2 u_2(t,y) + F_2(t,y) & y>0,\\ u_2(t,0)=0, & \\ u_2(0,y)=u_0(y) & y>0. \end{cases} \tag{5.4}$$

Classical parabolic regularity (cf. [91]) implies that $u_1,\ u_2$ have a Hölder regularity of the form $u_i \in \mathcal{C}^{\alpha/2,1+\beta}_{t,y}$ for any $\alpha,\beta \in (0,1)$. Thus, claim (i) of Theorem 5.1 is proved.

The properties of the Hölder space imply also that $\dot{s} \in C^{\delta/2}_t$. Moreover, the convolution of an Hölder function with the exponential integral is also Hölder continuous, so that we conclude $F_i \in \mathcal{C}^{\delta/2,\delta}_{t,y}$. A further application of classical parbolic regularity implies the desired regularity in claim (ii) of Theorem 5.1. Thus, (u_1,u_2,s) is the unique classical solution to (1.39). We refer to Theorem D.4 for more details.

5.2.2 Properties of the solutions

In order to prove the property (iii) of Theorem 5.1, we apply the maximum principle. Since the parabolic equations satisfied by u_1 and u_2 are defined on an unbounded domain, we need to consider a sequence of approximate solutions solving suitable parabolic equations on bounded domains and converging to the solutions of the original problem (5.4). Applying the maximum principle to these approximate solutions and using the convergence result we conclude the proof of claim (iii) of Theorem 5.1. We refer also to Lemma D.4, which shows that there are sequences $u_i^{R_n}$ whose maximal interval of existence approximates the one of the solution to the Stefan problem (5.3), and to Proposition D.3, which is about the application of the maximum principle.

5.3 Global well-posedness theory

In the classical Stefan problem, where there is no non-linear non-local integral operator as source of the heat equation, the existence of a unique classical solution for arbitrary times can be shown using the maximum principle. In this way, one obtains that the temperature and its derivative are bounded uniformly in time by constants, which only depend on the initial data. The method used by Rubenšteĭn [123] for the two-phase one-dimensional Stefan problem on a finite segment consists for instance in the application of the maximum principle to $\partial_y T_i$ in every interval of time (t_i, t_{i+1}) in which $\dot{s}(t)$ has a constant sign.

However, when we consider problem (5.3) this approach does not apply anymore. Indeed, even though the parabolic equation describing the temperature in the solid has a global maximum principle, its sub- and supersolution do not allow time-independent bounds on $\partial_y T_2$. Therefore, we use another strategy which also makes use of the maximum principle.

We assume that the maximal interval of existence of the unique solution (T_1, T_2, s) is finite, i.e. there exists $t^* < \infty$ such that the solution cannot be extended for $t > t^*$. In Theorem D.5 we prove that for a large class of initial data the norms of T_1, T_2, $\partial_y T_1$, $\partial_y T_2$ and $\dot{s}$ are bounded uniformly in $[0, t^*]$. Hence, according to Theorem 5.1, the solution can be extended for times $t > t^*$, so that $t^* = \infty$.

This can be done with the help of a time-independent function w, which is a supersolution for $y < 0$ and a subsolution for $y > 0$ of the problem (5.3). Moreover, $\partial_y w(0^-) > -\frac{L}{K} C_2$ and $\partial_y w(0^+) > -LC_1$ for suitable constants $C_1, C_2 > 0$. Finally, the auxiliary function w is constructed in a way, so that $w(y) > T_0(y)$ on $\mathbb{R}_-$ and $w(y) < T_0(y)$ on $\mathbb{R}_+$.

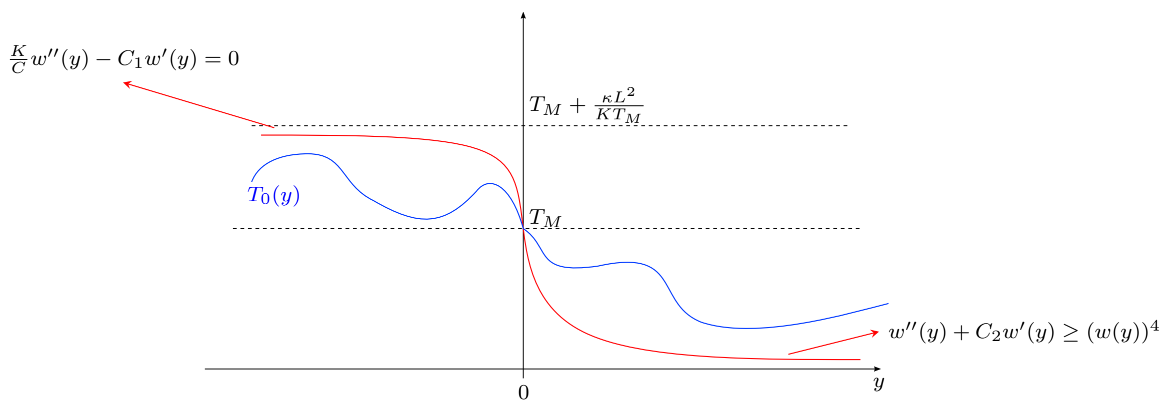

Figure 5.2: Representation of w.

Such a function w has been constructed in Lemma D.5 under the further assumption that the initial temperature satisfies in addition to the condition of Theorem 5.1 also the property

$$\sup_{\mathbb{R}_-} T_0 < T_M + \frac{\kappa L^2}{K T_M} \text{ and } \inf_{\mathbb{R}_+} T_0 > 0. \tag{5.5}$$

Finally, in Theorem D.5 we prove with the help of the maximum principle that

$$\|T_i\|_\infty \leq \max\{\|w\|_\infty, T_M\},$$

since $T_1(t, y) \leq w(y)$ for all $y < 0$ and $t \in [0, t^*]$ and $w(y) \leq T_2(t, y) \leq T_M$ for all $y > 0$ and $t \in [0, t^*]$. Since $0 \geq \partial_y T_i(0^\pm) > w'(0^\pm)$ a straight consequence is

$$\|\dot{s}\|_\infty \leq \max\{C_1, C_2\}.$$

Finally, in order to prove that also the norms $\|\partial_y T_i\|_\infty$ are bounded we apply the maximum principle to the parabolic equations satisfied by $\partial_y T_i$, which are obtained differentiating the original equations in (5.3) for T_i. Constructing suitable new sub- and supersolutions it turns out that

$$\|\partial_y T_1\|_\infty \leq |w'(0^-)|(1 + t^*) < \infty \text{ and } \|\partial_y T_2\|_\infty \leq C\left(|w'(0^+), T_M, C_1|\right) e^{4T_M^3 t^*} < \infty.$$

This concludes the proof of the global in time well-posedness of the Stefan problem (5.3) for the class of regular initial data satisfying the assumptions of Theorem 5.1 and the condition (5.5).

Chapter 6

Theory of traveling waves for a Stefan problem with radiation

This chapter summarizes the main results obtained in *"Traveling waves for a Stefan problem with radiation"* [38], which is joint work with J. J. L. Velázquez and in which both authors contributed equally. The most recent version of this article can be read in Appendix E.

In [38] the study of the one-dimensional two-phase free boundary problem introduced in [39] is extended. This problem models the phase transition of a material composed by a liquid and a solid phase at contact in the situation in which the heat is transported by conduction in both phases and it is transferred also by radiation only in the solid part. As we summarized in Chapter 5, in ([39], Appendix D) we developed a well-posedness theory for

$$\begin{cases} \partial_t T_1(t,x) = \kappa \partial_x^2 T_1(t,x) & x < s(t), \\ \partial_t T_2(t,x) = \partial_x^2 T_2(t,x) - T_2^4(t,x) + \int_{s(t)}^{\infty} d\xi \; \frac{\alpha E_1(\alpha(x-\xi))}{2} T_2^4(t,\xi) & y > 0, \\ T_1(t,(s(t))) = T_2(t,s(t)) = T_M > 0 & y = 0, \\ T(0,x) = T_0(x) & x \in \mathbb{R}, \\ \dot{s}(t) = \frac{1}{L}\left(\partial_x T_2(t,s(t)) - K \partial_x T_1(t,s(t))\right), \end{cases} \tag{6.1}$$

where $\kappa,\ K,\ L,\ \alpha > 0$ are given constants. For more details about the derivation of the model we refer to Chapter 5. Notice that, by a change of coordinate $y = x - s(t)$, (6.1) becomes (5.3), which is equivalent under suitable rescaling to the free boundary problem introduced in (1.38).

In ([38], Appendix E) the existence of traveling wave solutions to (6.1) is proved. This is a novelty and a difference with the classical Stefan problem, where traveling waves do not exist while self-similar solutions solving the free boundary problem can be constructed (cf. [56,106]).

Before giving more details about the results in ([38], Appendix E), let us consider the classical Stefan problem in the whole space, cf. [123].

$$\begin{cases} \partial_t T_1(t,x) = a \partial_x^2 T_1(t,x) & x < s(t), \\ \partial_t T_2(t,x) = \partial_x^2 T_2(t,x) & x > s(t), \\ T_1(s(t),t) = T_2(s(t),t) = T_M \\ \dot{s}(t) = \frac{1}{L}\left(\partial_x T_2(t,s(t)) - b \partial_x T_1(t,s(t))\right). \end{cases}$$

First of all we notice that there cannot exist bounded traveling wave solutions for which the interface moves linearly in time. Indeed, let $c \in \mathbb{R}$, then defining $\dot{s}(t) = -c$ and $T_i(t,x) =$

$f_i(x+ct) = f_i(y)$ we see that we need to solve

$$\begin{cases} f_1''(y) - \frac{c}{a} f_1'(y) = 0 & y < 0, \\ f_2''(y) - c f_2'(y) = 0 & y > 0, \\ f_1(0) = f_2(0) = T_M \\ c = \frac{1}{L}\left(b f_1'(0) - f_2'(0)\right). \end{cases}$$

Thus, solving these simple ODEs we obtain

$$f_1(y) = T_M + \frac{\alpha a}{c}\left(1 - e^{\frac{c}{a}y}\right) \text{ for } y < 0 \quad \text{ and } \quad f_2(y) = T_M - \frac{\beta}{c}\left(e^{cy} - 1\right) \text{ for } y > 0,$$

where $\alpha,\ \beta > 0$ in order to avoid superheated or supercooled solutions. By construction we observe that f_2 is unbounded if $c > 0$ and f_1 is unbounded if $c < 0$.

Nevertheless, self-similar solutions exist. Let indeed $s(t) = 2A\sqrt{t}$ and let $T_i(t,x) = f_i\left(\frac{x}{\sqrt{t}}\right) = f_i(z)$. We look for solutions to

$$\begin{cases} a f_1''(z) + \frac{z}{2} f_1'(z) = 0 & z < 2A, \\ f_2''(z) + \frac{z}{2} f_1'(z) = 0 & z > 2A, \\ f_1(2A) = f_2(2A) = T_M \\ A = \frac{1}{L}\left(f_2'(2A) - f_1'(2A)\right). \end{cases} \tag{6.2}$$

Such ODEs have an explicit solution given by $f_1(z) = T_M + \alpha \mathrm{Erf}\left(\frac{A}{\sqrt{a}}\right) - \alpha \mathrm{Erf}\left(\frac{z}{2\sqrt{a}}\right)$ for $y < 2A$ and by $f_2(z) = T_M + \beta \mathrm{Erf}(A) - \beta \mathrm{Erf}\left(\frac{z}{2}\right)$ for $y > 2A$, where Erf is the error function. Moreover, $\alpha,\ \beta > 0$ since the error function is odd and strictly increasing.

Thus, for any $f_{-\infty} > T_M$ and $0 < f_\infty < T_M$ there exists a unique A such that the functions $f_1^A,\ f_2^A$ as given above solve (6.2) with $\lim_{z\to-\infty} f_1(z) = f_{-\infty}$ and $\lim_{z\to\infty} f_2(z) = f_\infty$. It has been shown also rigorously that the long-time asymptotic of the classical Stefan problem is exactly given by error functions (cf. [56, 106]).

Let us consider now the Stefan problem with radiation as given in (6.1). By the structure of the equation we see that there are no self-similar profiles which can solve (6.1). However, the radiation operator behaves well under translations. Thus, we consider the traveling waves $T_i(t,x) = T_i(x+ct) = T_i(y)$ for $s(t) = -ct$ and $i \in \{1,2\}$ which solve

$$\begin{cases} c\partial_y T_1(y) = \kappa \partial_y^2 T_1(y) & y < 0, \\ c\partial_y T_2(y) = \partial_y^2 T_2(y) - T_2^4(y) + \int_0^\infty \alpha \frac{E_1(\alpha(y-\eta))}{2} T_2^4(\eta) d\eta & y > 0, \\ T_2(0) = T_1(0) = T_M \\ c = \frac{1}{L}\left(K\partial_y T_1(0^-) - \partial_y T_2(0^+)\right). \end{cases} \tag{6.3}$$

As we have seen in the case of the classical Stefan problem, if $c < 0$ there are no bounded traveling waves, since the solution in the liquid becomes unbounded. This means that the ice has to expand. Notice that this is coherent with the physical model. Since the liquid is transparent and there is no incoming radiation into the solid, the escaping radiation helps the ice to cool faster.

The existence of traveling wave solutions for the problem (6.1) solving (6.3) is shown in ([38], Appendix E). The result can be summarized as follows.

Theorem 6.1 (cf. [38], Theroem E.1, Theorem E.2, Theorem E.3, Theorem E.4 and Theorem E.7)**.** *It holds:*

(i) If $c<0$ problem (6.3) does not admit any bounded solution.

(ii) There exists $c_{\max}>0$ such that for any $c\in(0,c_{\max}]$ there exist $T_1\in C^{2,1/2}(\mathbb{R}_-)$ and $T_2\in C^{2,1/2}(\mathbb{R}_+)$ solving (6.3).

(iii) The solutions satisfy $T_1(y)>T_M$ on $\mathbb{R}_-$, $0<\lambda\leq T_2(y)<T_M$ on $\mathbb{R}_+$ for some $\lambda>0$ and both limits $\lim_{y\to-\infty} T_1(y)>T_M$ and $\lim_{y\to\infty} T_2(y)>0$ exist.

(iv) If $T_M=\varepsilon>0$ small enough, then the solution is unique.

Remark. $f\in C^{2,1/2}(U)$ for $U\subseteq\mathbb{R}$ has bounded norms $\max\{\|f\|_\infty,\ \|f'\|_\infty,\ \|f''\|_\infty\}<\infty$.

Notice that claim (i) has been already justified. Therefore, we will explain the strategy followed in ([38], Appendix E) in order to prove (ii)-(iv). First of all, we prove the existence of the traveling wave solutions, which for $y>0$ are monotone increasing with respect to the melting temperature, cf. Section 6.1. In Section 6.2 we prove claim (iv), which is a consequence of Banach fixed-point theorem. Moreover, we will see that for small melting temperatures there exists a limit as $y\to\infty$ and that the solution is strictly positive. This last result is used in Section 6.3 in order to prove the first part of (iii). In order to prove the existence of a limit as $y\to\infty$ for arbitrary melting temperatures we use maximum principle methods and blowup arguments. Finally, in Section 6.4 we summarize using formal arguments the expected long-time asymptotic of (6.3) which combines the traveling wave solutions with self-similar profiles.

6.1 Existence of the traveling waves

As we have seen for the classical Stefan problem, $T_1=T_M-\frac{\partial_y T_1(0^-)\kappa}{c}\left(1-e^{\frac{c}{\kappa}y}\right)$ on $\mathbb{R}_-$ for $\partial_y T_1(0^-)=\frac{Lc+\partial_y T_2(0^+)}{K}<0$. Therefore, it is enough to consider only the well-posedness of

$$\begin{cases}\partial_y^2 f(y)-c\partial_y f(y)-f^4(y)=-\int_0^\infty E(y-\eta)f^4(\eta)d\eta \quad y>0,\\ f(0)=T_M,\\ f\geq 0,\end{cases} \tag{6.4}$$

where $E(x)=\frac{E_1(x)}{2}$. It is also enough to prove (ii)-(iv) in Theorem 6.1 only for (6.4). Equation (6.4) is obtained rescaling T_2 of (6.3) by $T_2(y)=\alpha^{2/3}f(\alpha y)=\alpha^{2/3}f(\xi)$ and denoting with an abuse of notation $\frac{c}{\alpha}$ by c and $T_M\alpha^{-2/3}$ by T_M in (6.4).

We actually prove that for all $c>0$ the problem (6.4) has a solution. The condition $c\in(0,c_{\max}]$ in (ii) of Theorem 6.1 is due to the fact that if $c>c_{\max}$ then $\partial_y T_2(0^+)>-Lc$. This implies $\partial_y T_1(0^-)>0$ and thus the formation of supercooled liquid with $T_1(y)<T_M$.

In order to prove the existence of a solution to (6.4) we prove the existence of a sequence $f_n\in C^{2,1/2}(\mathbb{R}_+)$ defined by means of the recursive problem

$$\begin{cases}\partial_y^2 f_{n+1}(y)-c\partial_y f_{n+1}(y)-f_{n+1}^4(y)=-\int_0^\infty E(y-\eta)f_n^4(\eta)d\eta=:g_{n-1}(y) & y>0,\ n\geq 1,\\ f_0=0 & n=0,\\ f_{n+1}(0)=T_M,\\ f_{n+1}\geq 0,\end{cases} \tag{6.5}$$

and with $0\leq f_1\leq f_2\leq ...\leq f_n\leq f_{n+1}\leq T_M$, cf. Theorem E.3.

The existence of a function f solving the first equation in (6.5) for any $g \in C^{1/2}(\mathbb{R}_+)$ with $g < 0$ has been proved in Proposition E.1 with a variational argument since the Euler-Lagrange equation of the functional

$$I_g[f] = \int_{\mathbb{R}_+} e^{-cy}\left(\frac{|f'(y|^2)}{2} + \frac{(f(y))^5}{5} + f(y)g(y)dy\right)$$

is given by $-\partial_y\left(e^{-cy}\partial_y f(y)\right) + e^{-cy}\left(f(y)^4 + g(y)\right) = 0$. Using the weak maximum principle we show that the unique non-negative minimizer of I_g is actually positive for every $y > 0$ and it solves the Euler-Lagrange equation.

Turning back to (6.5), elliptic regularity and basic integral estimates imply together with another application of the maximum principle that

$$f_n \in C^{2,1/2}(\mathbb{R}_+) \text{ with } \|f_n\|_\infty \leq T_M,\ \|f_n'\|_\infty \leq \frac{T_M^4}{c} \text{ and } \|f_n''\|_\infty \leq T_M^4.$$

A key step is that, if $f_{n-1} \in C^{2,1/2}(\mathbb{R}_+)$, then $g_{n-1} \in C^{1/2}(\mathbb{R}_+)$ as a consequence of the convolution with the exponential integral.

Furthermore, a new application of the maximum principle shows $0 \leq f_n \leq f_{n+1} \leq T_M$. Thus, a solution f to (6.4) exists. We refer to Proposition E.1 and to Theorem E.3 for more details.

Finally, as shown in Lemma E.2, the maximum principle implies also that the functions f solving (6.4) and constructed with the recursive scheme of (6.5) are monotone with respect to the melting temperature, i.e. if $f_1(0) = \theta_1 < \theta_2 = f_2(0)$ then $f_1 \leq f_2$ on $\mathbb{R}_+$.

6.2 Traveling waves for small melting temperatures

In this section we summarize the strategy followed in order to show claim (iv) of Theorem 6.1. First of all, we use that if f solves (6.4) for $T_M = \varepsilon$, then $\tilde{f} = \frac{f}{\varepsilon}$ solves

$$\begin{cases} \partial_y^2 \tilde{f}(y) - c\partial_y \tilde{f}(y) - \varepsilon^3 \tilde{f}^4(y) = -\varepsilon^3 \int_0^\infty E(y-\eta)\tilde{f}^4(\eta)d\eta \quad y > 0, \\ \tilde{f}(0) = 1, \\ \tilde{f} \geq 0. \end{cases}$$

Moreover, notice that any solution $f \in C^{2,1/2}(\mathbb{R}_+)$ of (6.4) (or equivalently $\tilde{f}$), which has by the definition of the Hölder space all derivatives bounded, is a solution of the following fixed-point equation

$$\tilde{f}(y) = 1 + \varepsilon^3 \int_0^y e^{c\xi} \int_\xi^\infty e^{-c\eta}\left(\int_0^\infty E(\eta - z)\tilde{f}^4(z)dz - \tilde{f}^4(\eta)\right) d\eta\, d\xi. \tag{6.6}$$

Thus, it is enough to prove the uniqueness and the properties of $\tilde{f}$ solving (6.6).

First of all, Lemma E.3 shows that for $\varepsilon > 0$ small enough there exists a limit $\tilde{f}_\infty \in [0, \varepsilon]$ and a constant $A > 0$ such that the solution $\tilde{f}$ to (6.4) for $T_M = \varepsilon$ has an exponential decay as

$$\left|\tilde{f}(y) - \tilde{f}_\infty\right| \leq A\varepsilon^4 e^{-y/2}.$$

This can be shown studying the oscillations $\operatorname{osc}_{[R,R+1]} \tilde{f}$ for $R > 0$. Using the fixed-point equation (6.6) and the properties of the exponential integral we prove by induction that for $\varepsilon > 0$ small enough

$$\lambda(M) := \sup_{R \geq M} \operatorname*{osc}_{[R,R+1]} \tilde{f} \leq B\varepsilon^3 e^{-M/2},$$

where B depends only on c. This implies both the existence of a limit and the exponential decay.

Furthermore, a simple estimate shows in Lemma E.4 that for $\varepsilon > 0$ small enough, the solution f to (6.4) for $T_M = \varepsilon$ satisfies $f \geq c_0\varepsilon$ and $f_\infty \geq c_0\varepsilon$, where $c_0 > 0$.

Finally, an application of the Banach fixed-point theorem in the closed complete metric space

$$\mathcal{X}_{A,B} = \left\{ f \in C_b(\mathbb{R}_+) : |f(y)| \leq B,\ \exists f_\infty \text{ s.t. } |f(y) - f_\infty| \leq Ae^{-y/2} \right\}$$

equipped with the metric induced by the norm $\|f\|_{\mathcal{X}} = |f_\infty| + \sup_{y\in\mathbb{R}_+} e^{y/2}|f(y) - f_\infty|$ shows that $\tilde{f}$ solving (6.6) is unique. Hence, claim (*iv*) of Theorem 6.1 is true. We refer to Theorem E.4 for more details.

6.3 Existence of a limit

The theory of traveling wave solutions for small melting temperatures $T_M = \varepsilon > 0$ together with the monotonicity result (cf. Lemma E.2) implies that the solutions f of (6.4) constructed with the recursive method (6.5) in Theorem E.3 are larger than a positive constant $\lambda(T_M) > 0$ in $\mathbb{R}_+$. Moreover, $f < T_M$ in the interior of $\mathbb{R}_+$. This is due to a simple application of the maximum principle, since $f \leq T_M$ by construction, cf. Lemma E.1.

In order to prove the existence of a limit as $y \to \infty$ of a solution to (6.4) we first prove that for any increasing sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ with $x_n \to \infty$ as $n \to \infty$ the sequence of functions $f_n(y) := f(y + x_n)$ converges to a constant as $n \to \infty$ uniformly in every compact set. This outcome together with a stability result implies that $\lim_{y\to\infty} f(y)$ exists.

By the regularity of $f \in C^{2,1/2}(\mathbb{R}_+)$, every increasing sequence $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \to \infty$ as $n \to \infty$ has a subsequence such that

$$f_{n_k} = f(\cdot + x_{n_k}) \to \bar{f} \text{ as } k \to \infty \text{ uniformly in compact sets.}$$

Moreover, $\bar{f} \in C^{2,1/2}(\mathbb{R})$ solves

$$\begin{cases} \partial_y^2 \bar{f}(y) - c\partial_y \bar{f}(y) - \bar{f}^4(y) = -\int_{-\infty}^{\infty} E(y-\eta)\bar{f}^4(\eta)d\eta & y \in \mathbb{R}, \\ 0 < \lambda \leq \bar{f} \leq T_M, \end{cases} \tag{6.7}$$

cf. (E.46). Notice that the equation (6.7) solved by $\bar{f}$ is invariant under translations.

6.3.1 $\bar{f}$ is constant

We aim to show that any f solving (6.7) satisfies $\sup_{\mathbb{R}} f = \inf_{\mathbb{R}} f$, which implies that f is constant.

An application of the maximum principle shows that f does not attain is supremum and infimum at the interior of $\mathbb{R}$, unless f is constant, cf. Lemma (E.5).

Moreover, if f is not constant, the assumption that $\sup_{\mathbb{R}} f = \limsup_{y\to\infty} f = A$ or that $\inf_{\mathbb{R}} f = \liminf_{y\to\infty} f = B$ leads to a contradiction due to another application of the maximum principle. In particular, Lemma E.6 and Lemma E.7 show that f does not attain either supremum nor infinimum at $+\infty$, unless f is already constant.

Indeed, we can construct suitable subsolutions for $w_S = A - f \geq 0$ and $w_I = f - B \geq 0$, for which by definition $\liminf_{y\to\infty} W_I = 0$ and $\liminf_{y\to\infty} W_S = 0$. For the right choice of $\beta(c) > 0$,

$\theta > 0$, $\varepsilon(\beta,\theta) > 0$, $R(\varepsilon,\beta,\theta) > 0$ and of $\delta_0(\varepsilon,\theta,\beta) > 0$ we can prove that

$$\psi_\delta(y) = \begin{cases} 0 & y < -R, \\ \varepsilon - \delta e^{\beta y} & y \in [-R, 0), \\ \varepsilon\theta - \delta e^{\beta y} & y \in [0, R_\delta], \\ 0 & y > R_\delta, \end{cases}$$

is a subsolution on $[0, R_\delta]$, it satisfies $w_{S,I} \geq \psi_\delta$ on $\mathbb{R} \setminus (0, R_\delta)$ for all $\delta < \delta_0$ and $w_{S,I} > \psi_{\delta_0}$ for all $y \in \mathbb{R}$. The first property can be seen analyzing the linearized operators solved by w_S and w_I. In the case of w_S we see that

$$-w_S'' + cw_S' - (A - w_S)^4 + \int_{\mathbb{R}} E(\cdot - \eta)(A - w_S(\eta))^4 d\eta = 0.$$

Thus, we consider the linearized operator $\mathcal{L}(\psi) = -\psi'' + c\psi' + 4A^3\psi - 4A^3 \int_{\mathbb{R}} E(\cdot - \eta)\psi(\eta)d\eta$. A similar operator has to be studied in the case of w_I. Using that for $a < 1$ the exponential integral satisfies $\int_{\mathbb{R}} E(\eta - y)e^{a\eta}d\eta = e^{ay}\frac{\operatorname{artanh}(a)}{a}$, one can prove for $\psi(y) = e^{\beta y}$ that

$$-e^{-\beta y}\mathcal{L}(\psi)(y) = \beta^2 - c\beta + 4A^3\left(\frac{\operatorname{artanh}\beta}{\beta} - 1\right) \leq 0$$

is a convex function with two zeros, one for $\beta = 0$ and one for $\beta = \beta_0(c, A) > 0$. Thus, for $\beta \in (0, \beta_0)$ the function $\psi = -e^{\beta y}$ is a subsolution to the linearized operator with $\mathcal{L}(\psi) < 0$.

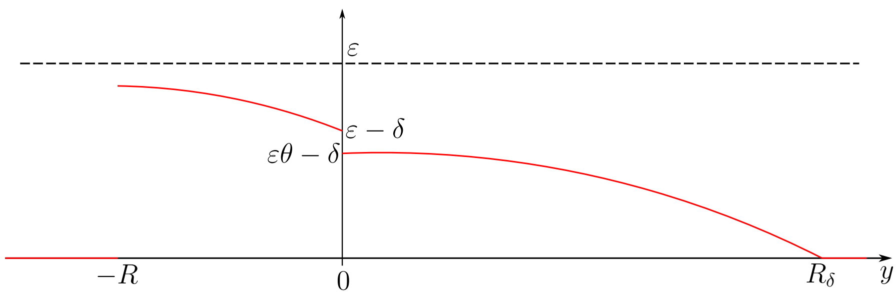

Figure 6.1: Sketch of the subsolution ψ_δ.

Finally, an application of the maximum principle implies that $W_{S,I} \geq \psi_\delta$ for all $\delta < \delta_0$. Therefore, since $R_\delta \to \infty$ as $\delta \to 0$ we conclude that $w_{S,I} \geq \varepsilon\theta > 0$ for all $y > 0$. This is a contradiction to the assumption of $A = \limsup_{y\to\infty} f(y)$ or $B = \liminf_{y\to\infty} f(y)$.

A direct consequence of Lemma E.6 and of Lemma E.7 is that

$$\sup_{\mathbb{R}} f = \limsup_{y\to-\infty} f \quad \text{and} \quad \inf_{\mathbb{R}} f = \liminf_{y\to-\infty} f.$$

We need now to show that $\limsup_{y\to-\infty} f = \liminf_{y\to-\infty} f$. This has been done developing a key stability result for the solution to (6.7), which can be understood as a Harnack-type inequality as follows.

Theorem 6.2 (cf. [38], summary of Theorem E.5)**.** *If f is a solution to* (6.7) *and it satisfies* $\operatorname{osc}_{[-L,L]} f < \varepsilon$ *for $\varepsilon < \varepsilon_0$ small enough and $L > L_0(\varepsilon)$ large enough, then* $\operatorname{osc}_{[L,\infty]} f < 3\varepsilon$.

This key result has been proved applying once more the maximum principle to a suitable family of subsolutions ψ_δ^L and to a suitable family of supersolutions ψ_γ^L, which have been constructed in a similar way as the one considered above.

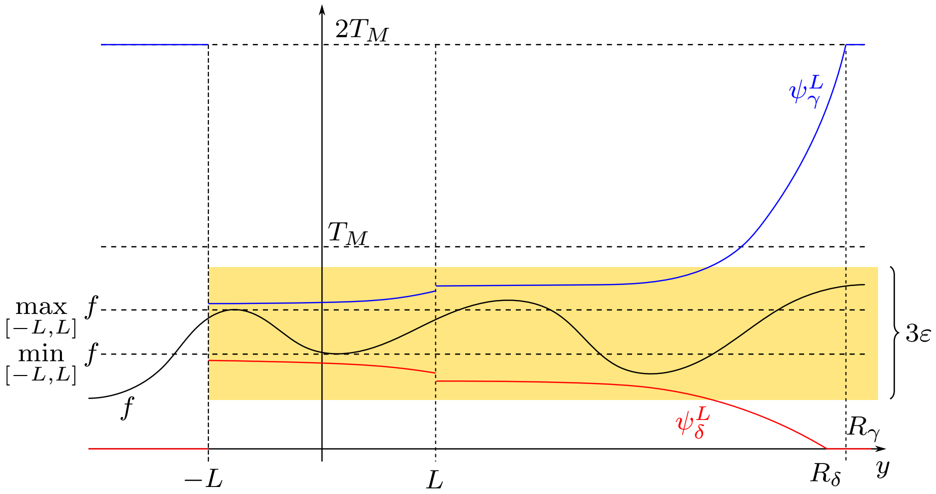

Figure 6.2: Sketch of the subsolutions ψ_δ^L and of the supersolutions ψ_γ^L. See (E.75) and (E.97).

For $\varepsilon > 0$ small enough and $L > 0$ large enough, these functions satisfy that $\psi_\delta^L < f < \psi_\gamma^L$ for $y < L$ and $y > R_\delta$ (respectively $y > R_\gamma$). Moreover, ψ_δ^L is a subsolution on $[L, R_\delta]$ and ψ_γ^L is a supersolution on $[L, R_\gamma]$. Since for some $\delta_0, \gamma_0 > 0$ also $\psi_{\delta_0}^L < f < \psi_{\gamma_0}^L$ in $\mathbb{R}$, the maximum principle yields $\psi_\delta^L < f < \psi_\gamma^L$ for all $y \in \mathbb{R}$. Thus, as $\delta, \gamma \to 0$ and $R_\delta, R_\gamma \to \infty$ we conclude $\min\limits_{[-L,L]} f - \varepsilon < f < \max\limits_{[-L,L]} f + \varepsilon$ for all $y > 0$ so that $\operatorname*{osc}\limits_{[L,\infty]} f < 3\varepsilon$.

Finally, as we prove in Lemma E.8, the monotone sequences $x_n \to -\infty$ and $\xi_n \to -\infty$ converging to $\limsup\limits_{y\to-\infty} f$ and to $\liminf\limits_{y\to-\infty} f$, respectively, have the property that, up to subsequences, $f(x_{n_k} + \cdot) \to \bar{g} = \sup\limits_{\mathbb{R}} f$ and $f(\xi_{n_k} + \cdot) \to \underline{g} = \inf\limits_{\mathbb{R}} f$ as $n \to \infty$ uniformly in compact sets.

This is a consequence of the regularity of f, of the translation invariance of (6.7) and of Lemma E.5, since in $y = 0$ the function $\bar{g}$ takes its supremum and $\underline{g}$ its infimum.

The stability result of Theorem 6.2 and the uniform convergence in compact sets, imply together $\sup\limits_{\mathbb{R}} f = \inf\limits_{\mathbb{R}} f$. Thus, any solution $\bar{f}$ to (6.7) is constant.

6.3.2 T_2 has a limit as $y \to \infty$

In Section 6.3.1 we summarized the strategy that we have used in order to prove that for any monotone sequence $x_n \to \infty$ as $n \to \infty$ the function f solving (6.4) satisfies, up to subsequences,

$$f(x_{n_k} + \cdot) \to \text{constant} \qquad \text{as } n \to \infty \text{ uniformly in compact sets.}$$

This holds especially for the sequences $\{x_n\}$, $\{\xi_n\} \subset \mathbb{R}_+$ with $\lim\limits_{n\to\infty} f(x_n) = \limsup\limits_{y\to\infty} f$ and $\lim\limits_{n\to\infty} f(\xi_n) = \liminf\limits_{y\to\infty} f$. Since the Harnack-type inequality of Theorem 6.2 holds also for the function $\tilde{f} = f(a + \cdot)$ solution to the equation

$$-\tilde{f}'' + c\tilde{f}' + \tilde{f}^4 - \int_a^\infty E(\cdot - \eta)\tilde{f}^4(\eta)d\eta = 0,$$

where $a > L_0(\varepsilon)$ large enough (cf. Corollary E.3), one can prove that $\limsup_{y\to\infty} f = \liminf_{y\to\infty} f$. Hence, claim (*iii*) of Theorem 6.1 holds. See Theorem E.7 for more details.

6.4 Expected long time asymptotic

We finish this summary of ([38], Appendix E) giving the expected behavior of the solution of the Stefan problem with radiation (6.1). As we have pointed out in Section E.4, the possible longtime asymptotic is given by the traveling wave constructed in Theorem 6.1 and by a self-similar solution describing the temperature of the solid in an additional layer far away form the interface. This is due to the fact that it is not possible to connect arbitrary values of the temperature as $x \to \infty$ and $x \to -\infty$ by means of solely traveling wave solutions. Indeed, we have proved in Theorem 6.1 that any traveling wave solution converges to a strictly positive temperature as $y \to \infty$. Moreover, the traveling wave T_1 defined for $y < 0$ is uniquely determined by T_2.

To be more precise, we expect that for any $T_{-\infty} \in [T_M, \infty]$ and $T_\infty \in [0, T_M]$ there exists $c \in [0, c_{\max}]$ such that as $t \to \infty$ the solution (T_1, T_2, s) of (6.1) is given by

(a) $s(t) = -ct$, hence ice is expanding;

(b) for $y < 0$ the solution satisfies $T_1(t,x) = T_1^c(x+ct)$, which solves the traveling wave equation (6.3) with $\lim_{y\to-\infty} T_1^c(y) = T_{-\infty}$;

(c) for $y > 0$ the function $T_2(t,x)$ is given by $T_2^c(x+ct)$ solving (6.3) with $\lim_{y\to\infty} T_2^c(y) = T_{\text{int}}^c > 0$ and for large distances the interface by the self-similar profile $F\left(\frac{x}{\sqrt{t}}\right) = F(z)$ solving

$$\begin{cases} -\frac{z}{2}F'(z) - F''(z) - \frac{1}{\alpha^2}\left(F^4(z)\right)'' = 0 \\ F(-\infty) = T_{\text{int}}^c \quad \text{and} \quad F(\infty) = T_\infty. \end{cases}$$

The fact that in the self-similar equation the integral operator describing the radiation simplifies into a porous-medium equation as $t \to \infty$, i.e.

$$-F^4\left(\frac{x}{\sqrt{t}}\right) + \int_{(s(t))}^{\infty} \frac{\alpha E_1(\alpha(x-\eta))}{2} F^4\left(\frac{\eta}{\sqrt{t}}\right) d\eta \sim -\frac{1}{\alpha^2}\partial_z^2 (F(z))^4 \text{ for } t \to \infty \text{ and } z = \frac{x}{\sqrt{t}},$$

is due to the diffusion approximation of the radiative transfer equation in regions very far from the interface.

Figure E.2, which is reported here, illustrates the expected form of the solution describing the asymptotic behavior for long times

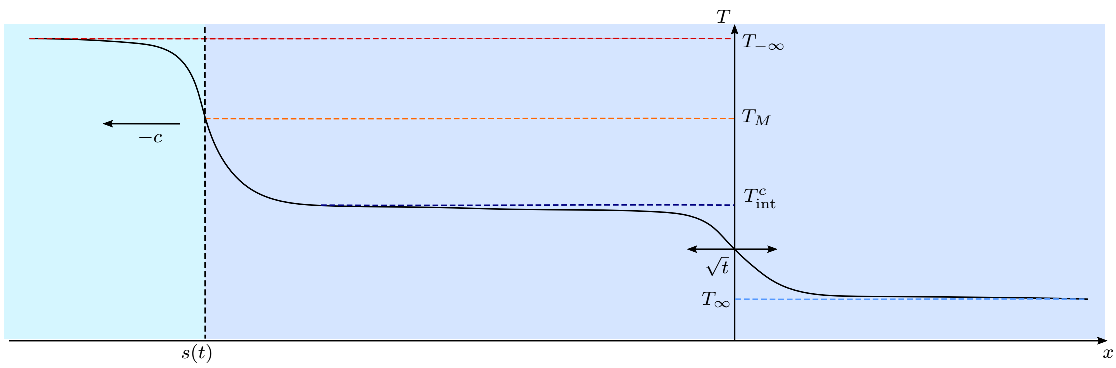

Figure 6.3: Illustration of the expected profile as $t \to \infty$. See Figure E.2.

Chapter 7

Concluding remarks and open problems

In this chapter we conclude summarizing the results obtained in this thesis and giving an outlook to open problems arising from the work that has been presented.

7.1 Well-posedness theory for the stationary radiative transfer equation

In Chapter 2 we presented the existence theory obtained in [35] for a large class of absorption and scattering coefficients. As we summarized in Theorem 2.1 we proved the existence of solutions to the stationary radiative transfer equation (1.20) coupled to divergence-free condition for the radiative energy (1.22) and satisfying the incoming boundary condition (1.24) in the cases in which the absorption and the scattering coefficients have the form $\alpha_\nu^{a,s}(T) = Q_{a,s}(\nu)\alpha^{a,s}(T)$. A key step in the proof of the existence theory was the development of a new compactness result for operators including exponential terms of integrals along straight lines (cf. Proposition A.1).

7.1.1 Uniqueness of solutions

The way in which the existence of solutions to the problem (2.1) was proven in ([35], Appendix A) does not imply the uniqueness of solutions, since it is based on a convergence result for a compact sequence of regularized solutions. Therefore, uniqueness is still an open problem. Another difficulty for the proof of the uniqueness of solutions is given by the non-linearity of the coefficients with respect to the temperature, due to which contractive estimates are difficult and perhaps even not possible to obtain.

Let us for example consider the pure emission-absorption case (i.e. $\alpha_\nu^s \equiv 0$) with $\alpha_\nu^a(T) = \alpha(T)$. Then, the fixed-point equation defining the temperature is given by

$$\begin{aligned} u(x) = &\int_\Omega \frac{\gamma(u(\eta)) \exp\left(-\int_{[x,\eta]} \gamma(u(\xi))d\xi\right)}{4\pi|x-\eta|^2} u(\eta)d\eta \\ &+ \int_{\mathbb{S}^2}\int_0^\infty g_\nu(n) \exp\left(-\int_{[x,y(x,n)]} \gamma(u(\xi))d\xi\right)\, d\nu\, dn, \end{aligned}$$

where $u = 4\pi\sigma T^4(x)$ and $\gamma(z) = \alpha\left(\sqrt[4]{\frac{z}{4\pi\sigma}}\right)$. Let us consider $u_1,\ u_2 \in L^\infty(\Omega)$ two different solutions to the fixed-point equation for the same source of radiation g_ν. Only for particular choices of γ (and hence α) and of g_ν we can prove that

$$\|u_1 - u_2\|_\infty \le \theta \|u_1 - u_2\|_\infty \text{ for } \theta < 1.$$

This would imply the uniqueness of solutions.

For example, if γ is differentiable with $\|\gamma'\|_\infty \le \frac{e^{-2D\|\gamma\|_\infty}}{3D(1+\|\gamma\|_\infty D)\|g_\nu\|_{L^1(\mathbb{S}^2\times\mathbb{R}_+)}}$, then one can prove that

$$\|u_1 - u_2\|_\infty \le \left(1 - \frac{1}{3}e^{-D\|\gamma\|_\infty}\right)\|u_1 - u_2\|_\infty,$$

where $D = \operatorname{diam}(\Omega)$. This estimate is similar to the one we obtained in (2.4). Indeed, on the one hand we have

$$\begin{aligned}
&\left|\int_\Omega \frac{\gamma(u_1(\eta))\exp\left(-\int_{[x,\eta]}\gamma(u_1(\xi))d\xi\right)}{4\pi|x-\eta|^2}u_1(\eta) - \frac{\gamma(u_1(\eta))\exp\left(-\int_{[x,\eta]}\gamma(u_1(\xi))d\xi\right)}{4\pi|x-\eta|^2}u_1(\eta)d\eta\right| \\
\le &\fint_{\mathbb{S}^2} dn \int_0^{s(x,n)} dr\gamma(u_1(x-rn))\exp\left(-\int_0^r \gamma(u_1(\xi))d\xi\right)|u_1-u_2|(x-rn) \\
&+\fint_{\mathbb{S}^2} dn \int_0^{s(x,n)} dr|u_2(x-rn)|\exp\left(-\int_0^r \gamma(u_1(\xi))d\xi\right)|\gamma(u_1)-\gamma(u_2)|(x-rn) \\
&+\fint_{\mathbb{S}^2} dn \int_0^{s(x,n)} dr|u_2(x-rn)|\gamma(u_2)(x-rn)\left|\int_0^r \gamma(u_1(\xi))-\gamma(u_1(\xi))d\xi\right| \\
\le &(1-e^{-D\|\gamma\|_\infty})\|u_1-u_2\|_\infty + e^{D\|\gamma\|_\infty}\|g_\nu\|_{L^1(\mathbb{S}^2\times\mathbb{R}_+)}D(1+D\|\gamma\|_\infty)\|\gamma'\|_\infty\|u_1-u_2\|_\infty,
\end{aligned}$$

where we used also $\|u_2\|_\infty \le e^{D\|\gamma\|_\infty}\|g_\nu\|_{L^1(\mathbb{S}^2\times\mathbb{R}_+)}$ as it is proved in (2.4). On the other hand, we can estimate

$$\begin{aligned}
&\left|\int_{\mathbb{S}^2}\int_0^\infty g_\nu(n)\exp\left(-\int_{[x,y(x,n)]}\gamma(u_1(\xi))d\xi\right) - \exp\left(-\int_{[x,y(x,n)]}\gamma(u_2(\xi))d\xi\right)\,d\nu\,dn\right| \\
&\quad \le \|g_\nu\|_{L^1(\mathbb{S}^2\times\mathbb{R}_+)}D\|\gamma'\|_\infty\|u_1-u_2\|_\infty.
\end{aligned}$$

However, for general coefficients uniqueness remains an open problem.

In [83] the entropy dissipation formula has been used in order to prove uniqueness of the solutions in the pure emission-absorption case when α_ν^a is independent of the temperature and the incoming radiation is at equilibrium with

$$g_\nu(n) = B_\nu(T_0)$$

for $T_0 =$ constant. This result could be adapted also in the case in which $\alpha_\nu^a(T)$ depends on T. The arguments in [83] seem indeed to work also in this situation. However, for general boundary conditions and in the presence of scattering, to prove uniqueness is more involved.

Since the maximum principle is a useful tool when the coefficients are independent of the temperature, one could also try, under suitable additional assumptions on the coefficients, to prove uniqueness of solutions via maximum principle.

7.1.2 Fully non-Grey coefficients

Another problem which is not covered in the results obtained in [35] is the existence of solutions in the more general case of fully non-Grey coefficients, i.e. when $\alpha_\nu^{a,s}(T)$ depends arbitrarily on ν and T.

In order to see where the arguments used in ([35], Appendix A) break down, we consider once again the case in which $\alpha_\nu^s \equiv 0$. As we have seen in (2.10), defining

$$u(x) = 4\pi \int_0^\infty \alpha_\nu^a(T(x))B_\nu(T(x))\, d\nu = F(T(x))$$

and assuming that F is invertible, which is the case for instance if $z \mapsto \alpha_\nu^a(z)B_\nu(z)$ is strictly monotone, we obtain that u satisfies the following fixed-point equation

$$\begin{aligned} u(x) = &\int_0^\infty \gamma_\nu(u(x))d\nu \int_\Omega d\eta\, \frac{\gamma_\nu(u(\eta))f_\nu(u_n(\eta))}{|x-\eta|^2} \exp\left(-\int_{[x,\eta]} \gamma_\nu(u(\xi))d\xi\right) \\ &+ \int_0^\infty d\nu \int_{\mathbb{S}^2} dn\; d\nu\; \gamma_\nu(u(x))g_\nu(n) \exp\left(-\int_{[x,y(x,n)]} \gamma_\nu(u(\xi))d\xi\right), \end{aligned} \tag{7.1}$$

where $\gamma_\nu(z) = \alpha_\nu^a(F^{-1}(z))$ and $f_\nu(z) = B_\nu(F^{-1}(z))$.

Hence, even if we could modify the L^2-compactness result for integrals along lines as given in Proposition A.1 for functions depending on the frequency, the term $\alpha_\nu^a(u(x))$ cannot be written as an integral along some straight line. Therefore, the compactness result obtained in ([35], Appendix A) cannot be used. We emphasize that the compactness has been shown for operators of the form

$$\int_{\mathbb{S}^2} dn \left(\int_0^s d\tau\; f(x-\tau n)\right),$$

for which the integral over all directions is a key feature. Without this kind of averaging the sequence $\int_0^\infty \gamma_\nu(u_\varepsilon) * \phi_\varepsilon(x)\, d\nu$ is in general not equi-integrable in $L^2(\Omega)$ unless the sequence u_ε is already equicontinuous. We recall that ϕ_ε is a sequence of standard non-negative radially symmetric mollifiers.

A possible approach in order to prove the existence of some kind of generalized solutions for general coefficients $\alpha_\nu^{a,s}(T)$ is to consider Young measure solutions. This would have the advantage that the compactness condition required in order to converge to such solutions is much weaker. Indeed, for any bounded sequence $\{f_k\}_{k\in\mathbb{N}} \in L^\infty(U,\mathbb{R}^m)$ there exist a subsequence $\{f_{k_j}\}_j$ and a Young measure $\{\nu_x\}_{x\in U}$ on $\mathbb{R}^m$ such that for all $F \in C^\infty(\mathbb{R}^m)$

$$F(f_{k_j}) \rightharpoonup^* \bar{F} \text{ in } L^\infty(U),$$

where $\bar{F}(x) := \int_{\mathbb{R}^m} F(y)d\nu_x(y)$ for almost every $x \in U$, cf. [48].

One possible strategy is to divide $\Omega \subset \bigcup_{0\le l\le L(k)} Q_l^k(x_l)$ in dyadic cubes of the form $Q_l^k(x_l) = \left[0, 2^{-k}\right]^3 + x_l$ and to define

$$u_k(x) = \sum_{0\le l\le L(k)} \mathbb{1}_{Q_l^k(x_l)}(x)u_k(x_l).$$

Then, $u_k(x_l)$ solves for any $x_l \in \Omega$

$$\begin{aligned} u_k(x_l) = &\int_0^\infty \gamma_\nu(u_k(x_l))d\nu \int_\Omega d\eta\, \frac{\gamma_\nu(u_k(\eta))f_\nu(u_k(\eta))}{|x-\eta|^2} \exp\left(-\int_{[x_l,\eta]} \gamma_\nu(u_k(\xi))d\xi\right) \\ &+ \int_0^\infty d\nu \int_{\mathbb{S}^2} dn\; \gamma_\nu(u_k(x_l))g_\nu(n) \exp\left(-\int_{[x_l,y(x_l,n)]} \gamma_\nu(u_k(\xi))d\xi\right). \end{aligned}$$

If $x_l \notin \Omega$ we set $u_k(x_l) = 0$. One can prove that the L^∞-estimate obtained for the coeffcient $\alpha_\nu^a = Q_a(\nu)\alpha^a(T)$ holds also in this case. Indeed, we can estimate

$$|u_k(x_l)| \leq \left(1 - e^{-\mathrm{diam}(\Omega)\|\gamma\|_\infty}\right) \int_0^\infty d\nu \int_{\mathbb{S}^2} dn\ \gamma_\nu(u_k(x_l)) B_\nu(F^{-1}(\|u_k\|_\infty)) \\ + \|\gamma_\nu\|_\infty \|g\|_{L^1(\mathbb{S}^2\times\mathbb{R}_+)},$$

which implies taking the maximum over all $\{x_l\}_{l\leq L(k)}$

$$\|u_k\|_\infty = \|\{u_k(x_l)\}_l\|_\infty \leq \|\gamma_\nu\|_\infty \|g\|_{L^1(\mathbb{S}^2\times\mathbb{R}_+)} e^{\mathrm{diam}(\Omega)\|\gamma\|_\infty} \text{ for all } k \in \mathbb{N}.$$

Therefore, Brouwer fixed-point theorem (cf. [49]) implies the existence of $\{u_k(x_l)\}_l$ and thus of the function $u_k(x)$ which solves

$$u_k(x) = \sum_{0\leq l\leq L(k)} \mathbb{1}_{Q_l^k(x_l)}(x) \int_0^\infty \gamma_\nu(u_k(x))d\nu \int_\Omega d\eta\ \frac{\gamma_\nu(u_k(\eta))f_\nu(u_k(\eta))}{|x-\eta|^2} \exp\left(-\int_{[x_l,\eta]} \gamma_\nu(u_k(\xi))d\xi\right) \\ + \sum_{0\leq l\leq L(k)} \mathbb{1}_{Q_l^k(x_l)}(x) \int_0^\infty d\nu \int_{\mathbb{S}^2} dn\ d\nu\ \gamma_\nu(u_k(x)) g_\nu(n) \exp\left(-\int_{[x_l,y(x_l,n)]} \gamma_\nu(u_k(\xi))d\xi\right). \tag{7.2}$$

By the uniform boundedness of $\{u_k\}_k \in L^\infty(\Omega)$ we obtain easily that up to subsequences u_k converges in the sense of Young measures to $\{\nu_x\}_{x\in\Omega}$. However, this is not enough in order to conclude the existence of a Young measure solution to (7.1) and a careful analysis of the convergence and of the properties of the Young measure $\{\nu_x\}_{x\in\Omega}$ has to be considered.

For instance, the line integrals appearing in (7.2) depends on x_l and not on x. Therefore, even if it would be possible to show

$$\exp\left(-\int_{[x,\eta]} \gamma_\nu(u_k(\xi))d\xi\right) \to \exp\left(-\int_{[x,\eta]}\int_{\mathbb{R}} \gamma_\nu(y)d\nu_\xi(y)d\xi\right),$$

we would need some uniform estimate for the integral terms

$$\int_0^\infty d\nu \int_\Omega d\eta\ ...\ \left|\exp\left(-\int_{[x_l,\eta]} \gamma_\nu(u_k(\xi))d\xi\right) - \exp\left(-\int_{[x,\eta]} \gamma_\nu(u_k(\xi))d\xi\right)\right|.$$

A possibility could be to adapt the compactness result of Proposition A.1 to this situation.

7.2 Diffusion approximation

In chapter 3 we summarized the results obtained in [36], where using matched asymptotic expansions we studied the diffusion approximation of the radiative transfer equation. In particular, we derive formally the approximate problems in the case in which the mean free path of the photons tends to zero. Also the different boundary and initial layer equations have been derived for all possible relative scalings between absorption length, scattering length and the characteristic size of the domain. Furthermore, a clear mathematical characterization of the equilibrium and non-equilibrium diffusion approximation has been presented. Many of the problems obtained formally in [36] were not considered before and they may therefore rigorously studied in the future.

In chapter 4 we considered the rigorous proof of the diffusion approximation for the stationary radiative transfer equation in the case where only emission-absorption processes take place. This theory has been developed in [37]. In particular, the absorption coefficient is independent of the frequency. The method presented in this article, which is based mainly in the application of the maximum principle, is different from the techniques used so far. Moreover, it represents an important step towards the proof of the stationary diffusion approximation for more general absorption coefficients.

7.2.1 Boundary and initial layer equations

The boundary layer equations describing the Milne and the thermalization layer as well as the initial layers and the initial-boundary layers derived in ([36], Appendix B) have been only partially studied so far.

While the Milne problem for the emission-absorption case (B.26) and the Milne problem for the scattering case (B.43) have been both extensively studied, for instance in [17,19,68,76,127], there are boundary layer equations that have not been considered. The Milne problem (B.34), which contains both emission-absorption and scattering terms, has been studied only for constant scattering coefficient and constant scattering kernel in [127]. Therefore, it would be interesting to consider more general scattering terms. Moreover, the thermalization layer equation (B.48) is a new problem developed in ([36], Appendix B), which has not been studied so far.

Also the well-posedness and the asymptotic behavior of the solutions of the initial layer equations given in Section B.5 are open problems. When the emission-absorption term occurs, a possible strategy could be to reduce the problems to an integro-differential equation for the temperature. For example, problem (B.72) is equivalent to the study of

$$\begin{aligned}\partial_\tau T(\tau,x)&+\int_0^\infty d\nu\int_{\mathbb{S}^2} dn\;\alpha_\nu^a(x)B_\nu(T(\tau,x))\\&-\int_0^\infty d\nu\int_{\mathbb{S}^2} dn\int_0^\tau ds\;(\alpha_\nu^a(x))^2e^{-\alpha_\nu^a(x)(\tau-s)}B_\nu(T(s,x))\\&=\int_0^\infty d\nu\int_{\mathbb{S}^2} dn\;\alpha_\nu^a(x)I_0(x,n,\nu)e^{-\alpha_\nu^a(x)\tau},\end{aligned}\tag{7.3}$$

where we used

$$\varphi_0(\tau,x,n,\nu)=I_0e^{-\alpha_\nu^a(x)\tau}+\int_0^\tau \alpha_\nu^a(x)e^{-\alpha_\nu^a(x)(\tau-s)}B_\nu(T(s,x))\;ds.$$

In the simpler case where $\alpha_\nu^a\equiv\alpha\equiv$ constant, equation (7.3) takes the form

$$\begin{aligned}\partial_\tau T(\tau,x)+4\pi\sigma\alpha\left(T^4(\tau,x)-\int_0^\tau \alpha e^{-\alpha(\tau-s)}T^4(s,x)\right)\\=\alpha e^{-\alpha\tau}\int_0^\infty d\nu\int_{\mathbb{S}^2} dn\;I_0(x,n,\nu).\end{aligned}$$

For (B.74) a similar equation to (7.3) can be obtained, which contains series of the form

$$\begin{aligned}\partial_\tau T(\tau,x)&+\int_0^\infty d\nu\int_{\mathbb{S}^2} dn\;\alpha_\nu^a(x)B_\nu(T(\tau,x))\\&-\int_0^\infty d\nu\int_{\mathbb{S}^2} dn\int_0^\tau ds\;(\alpha_\nu^a(x))^2e^{-\alpha_\nu^a(x)(\tau-s)}B_\nu(T(s,x))\\&=\int_0^\infty d\nu\int_{\mathbb{S}^2} dn\;\alpha_\nu^a(x)e^{-(\alpha_\nu^a(x)+\alpha_\nu^a(x))\tau}\sum_{k=0}^\infty\frac{(\alpha_\nu^s(x)\tau)^k}{k!}H^k[I_0](x,n,\nu).\end{aligned}$$

Also the initial thermalization layer equation (B.77) reduces to

$$\varphi_0(\tau, x, n, \nu) = \varphi(x,\nu)e^{-\alpha_\nu^a(x)\tau} + \int_0^\tau \alpha_\nu^a(x)e^{-\alpha_\nu^a(x)(\tau-s)}B_\nu(T(s,x))\,ds$$

and using the isotropy of $B_\nu(T)$ and of φ to

$$\begin{aligned}\partial_\tau T(\tau,x) &+ 4\pi\int_0^\infty d\nu\ \alpha_\nu^a(x)B_\nu(T(\tau,x))\\ &- 4\pi\int_0^\infty d\nu\int_0^\tau ds\ (\alpha_\nu^a(x))^2 e^{-\alpha_\nu^a(x)(\tau-s)}B_\nu(T(s,x))\\ =&4\pi\int_0^\infty \alpha_\nu^a(x)\varphi(x,\nu)e^{-\alpha_\nu^a(x)\tau}.\end{aligned}$$

Notice that all these equations are non-Markovian problems analogous to the one-dimensional problems describing the boundary layers. For this reason and since these equations are necessary in order to study some of the time-dependent diffusion approximations, it would be interesting to examine them.

It should be also possible to study problem (B.75) using standard spectral theory for the compact self-adjoint operator $H[\varphi](n) = \int_{\mathbb{S}^2} K(n,n')\varphi(n')\,dn'$.

The initial layer equation obtained in Section B.6.2 seems easier to study since the temperature is a constant and only the radiation intensity is an unknown.

Finally, also the initial-boundary layer equations of Section B.4, Section B.5 and Section B.6 need a careful study.

7.2.2 Rigorous proof of the diffusion approximation

The results obtained in [36] for the diffusion approximation of the radiative transfer equation are only formal. While the diffusion approximation has been rigorously studied in the pure emission-absorption case (cf. [13,16,37]) and in the pure stationary case in the framework of the one-speed neutron transport equation (cf. [19,76,146–148]), the problems containing both scattering and emission-absorption terms have not been rigorously studied so far.

Therefore, it would be interesting to prove rigorously the diffusion approximation results obtained in [36]. The various available results for the cases in which only emission-absorption or only scattering take place represent a promising starting point in order to tackle these new problems.

Particularly interesting is the non-equilibrium diffusion approximation, which is a novelty obtained in [36]. For example, the stationary screening equation (3.8) is a fascinating problem whose well-posedness should be studied.

7.2.3 Diffusion approximation for emission-absorption only

A problem which is not considered in [37] and which is currently still open is the rigorous proof of the diffusion approximation when scattering processes are neglected (i.e. $\alpha_\nu^s \equiv 0$) and the absorption coefficient depends on the frequency. Since the existence of a unique solution of the stationary radiative transfer equation coupled with the divergence-free condition of the radiative energy is proved only for coefficients independent on the temperature (cf. [83]), this is the first case that we should consider in order to extend the theory developed in [37] to the non-Grey case in which the absorption coefficient is $\alpha_\nu^a(x)$.

Since the maximum principle holds also in this case, the attempt to adapt the proof's ideas and the methods used in ([37], Appendix C) is worth it. Let us consider for instance the case in which $\alpha_\nu^a(x) = \alpha_\nu$ is independent of $x \in \Omega$. As we have pointed out in both articles [36,37], the diffusion approximation of the problem

$$\begin{cases} n \cdot \nabla_x I_\nu(x,n) = \frac{\alpha_\nu}{\varepsilon}\left(B_\nu(T(x)) - I_\nu(x,n)\right) & x \in \Omega, \\ \operatorname{div}\left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, n I_\nu(n,x)\right) = 0 & x \in \Omega, \\ I_\nu(n,x) = g_\nu(n) & x \in \partial\Omega, n \cdot n_x < 0, \end{cases}$$

is equivalent to the study of the convergence of the temperature $T_\varepsilon \to T$, where T_ε solves the integral equation

$$\begin{aligned} \mathcal{L}_\varepsilon(T_\varepsilon)(x) :=& 4\pi \int_0^\infty d\nu \alpha_\nu\, B_\nu(T_\varepsilon(x)) - \int_0^\infty d\nu\, \alpha_\nu \int_\Omega d\eta\, \frac{\alpha_\nu}{\varepsilon} \frac{e^{-\frac{\alpha_\nu |x-\eta|}{\varepsilon}}}{|x-\eta|^2} B_\nu(T_\varepsilon(\eta)) \\ =& \int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, g_\nu(n) \alpha_\nu e^{-\frac{\alpha_\nu s(x,n)}{\varepsilon}} \end{aligned} \tag{7.4}$$

and T solves $\Delta\left(\int_0^\infty \frac{B_\nu(T(x))}{\alpha_\nu} d\nu\right) = 0$ for the boundary conditions obtained from the corresponding Milne problem as the one in (B.26). We remark that the equation (7.4) has been derived solving the radiative transfer equation by characteristics in the same way as we did in Section 1.5 in order to obtain (1.29).

The non-local integral operator $\mathcal{L}_\varepsilon$ satisfies a maximum principle. Indeed, changing to spherical coordinates we compute

$$4\pi \int_0^\infty d\nu\, \alpha_\nu B_\nu(T(x)) = \int_0^\infty d\nu\, \alpha_\nu B_\nu(T(x)) \int_{\mathbb{R}^3} d\eta\, \frac{\alpha_\nu}{\varepsilon} \frac{e^{-\frac{\alpha_\nu |x-\eta|}{\varepsilon}}}{|x-\eta|^2}.$$

Thus, the monotonicity of the Planck distribution implies the following maximum principle for $\mathcal{L}_\varepsilon$:

If $\mathcal{L}_\varepsilon(v) \geq 0$ and $v\big|_{\partial\Omega} \geq 0$, then $v \geq 0$ in Ω for any $v \in C(\Omega)$.

This can be proved by contradiction assuming that there exists some $x_0 \in \Omega$, which is by assumption open, such that $\min_{\bar{\Omega}} v = v(x_0) < 0$, then by the monotonicity of $B_\nu(\cdot)$ also $B_\nu(v(x)) \geq B_\nu(v(x_0))$ for all $x \in \Omega$ and for all $\nu > 0$ and $B_\nu(v(x_0)) < 0$. Thus, we obtain the following contradiction

$$\begin{aligned} 0 \leq \mathcal{L}_\varepsilon(v)(x_0) =& \int_0^\infty d\nu\, \alpha_\nu B_\nu(v(x_0)) \int_{\Omega^c} d\eta\, \frac{\alpha_\nu}{\varepsilon} \frac{e^{-\frac{\alpha_\nu |x-\eta|}{\varepsilon}}}{|x-\eta|^2} \\ &+ \int_0^\infty d\nu\, \alpha_\nu \int_\Omega d\eta\, \frac{\alpha_\nu}{\varepsilon} \frac{e^{-\frac{\alpha_\nu |x-\eta|}{\varepsilon}}}{|x-\eta|^2} \left(B_\nu(v(x_0)) - B_\nu(v(\eta))\right) < 0. \end{aligned}$$

We finally remark that it seems to be possible to show under suitable condition on Ω that the L^∞-solution to (7.4) obtained in [83] is a continuous function.

7.2.4 More general domains

In [37] we considered the diffusion approximation for a convex domain. A natural question that arises concerns what would happen for a non-convex domain, where for instance cavities occur. This can be modeled assuming that the absorption coefficient is equal to 0 outside of the domain, i.e. also in the cavities.

Let us consider for example $\Omega \subset \mathbb{R}^3$ convex and let us define $\Omega_* = \Omega \setminus \overline{B_r(x_0)}$ for $B_r(x_0) \subset \Omega$. We can study the diffusion approximation of the problem

$$\begin{cases} n \cdot \nabla_x I_\nu(x,n) = \frac{\alpha}{\varepsilon}\left(B_\nu(T(x)) - I_\nu(x,n)\right) & x \in \Omega_*, \\ \operatorname{div}\left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, n I_\nu(n,x)\right) = 0 & x \in \Omega_*, \\ I_\nu(n,x) = g_\nu(n) & x \in \partial\Omega, n \cdot n_x < 0. \end{cases} \tag{7.5}$$

As it is remrked in [83], we expect that in this case interesting non-local interactions take place, which also determine the boundary condition at the boundary of the cavity $\partial B_r(x_0)$. Indeed, as we have proved in [37] the boundary condition for the diffusion problem at the interior of the domain Ω is determined by the incoming boundary profile $g_\nu(n)$ satisfied by I_ν at the "external" boundary $\partial\Omega$. In (7.5) there is no extra assumption on the boundary value of the radiation intensity at $\partial B_r(x_0)$, since I_ν is determined by the radiation crossing the cavity.

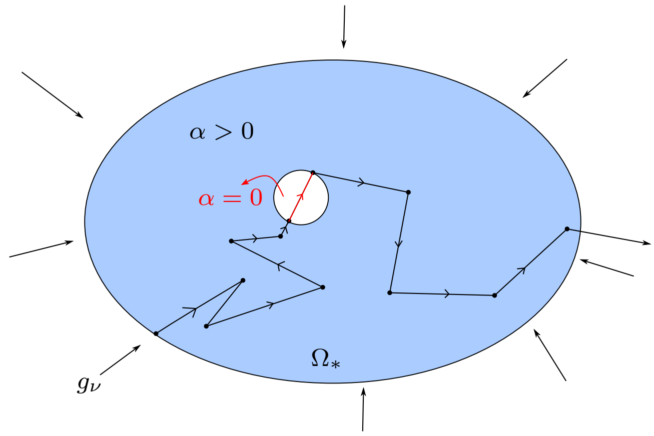

Figure 7.1: Representation of the interaction of radiation in a non-convex domain with a cavity.

7.3 Stefan problem with radiation

In Chapter 5 and in Chapter 6 we presented the results obtained for a one-dimensional two-phase Stefan problem modeling the phase transition in a body where the heat is transported by conduction in both phases of the material and also by radiation only in the solid phase, cf. equations (1.38) and (1.39).

In [39] we have developed a well-posedness theory for classical solutions in the case in which there is no external source of radiation. In [38] we have proved the existence of traveling wave solutions to the problem (1.39), for which the interface has to move towards the liquid yielding the expansion of ice.

The free boundary problem studied in the articles [38, 39] is a new problem which has not been considered before. The results that we have obtained are the first of many more that should be established in order to have a complete mathematical theory for this problem.

7.3.1 General global well-posedness result

In ([39], Appendix D) a global well-posedness result has been proved for a large class of initial temperatures satisfying precise bounds in the liquid phase, cf. (5.5). One problem that could be considered is the extension of the global well-posedness theory to all bounded initial temperatures, or the construction of a counterexample, i.e. of an initial temperature for which there is no bounded solution to (1.39) for all times $t > 0$ or for which the speed of the free boundary $\dot{s}(t)$ blows up in finite time.

As we have shown in the remark at the end of Section D.3, the class of initial temperatures constructed in Theorem D.5 applying the maximum principle to sub- and supersolutions is optimal for the equations satisfied by those auxiliary functions (cf. Figure 5.2). Thus, in order to obtain a more general global well-posedness theory we should argue differently than as is has been done in ([39], Appendix D).

7.3.2 Non-trivial external source of radiation

In ([39], Appendix D) and in ([38], Appendix E) the well-posedness theory and the traveling wave solutions have been studied only for the case in which there is no radiation entering the solid phase from the liquid one, i.e.

$$I_\nu(t,x,n) = g_\nu(n) = 0 \text{ for } x_1 = s(t),\ n_1 > 0.$$

Hence, a natural question arises concerning the case in which $g_\nu(n)$ is not trivial for $n_1 > 0$. In this situation the Stefan problem is described by

$$\begin{cases} C_L\partial_t T(t,x_1) = K_L\partial^2_{x_1}T(t,x_1) & x_1 < s(t),\\ C_S\partial_t T(t,x_1) = K_S\partial^2_{x_1}T(t,x_1) - \operatorname{div}\left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dn n I_\nu(t,x,n)\right) & x_1 > s(t),\\ n\cdot\nabla_x I_\nu(t,x,n) = \alpha\left(B_\nu(T(t,x_1)) - I_\nu(t,x,n)\right) & x_1 > s(t),\\ I_\nu(t,x,n) = g_\nu(n) & x_1 = s(t),\ n_1 > 0,\\ T(t,s(t)) = T_M & x_1 = s(t),\\ T(0,x) = T_0(x) & x_1 \in \mathbb{R},\\ \dot{s}(t) = \frac{1}{L}\left(K_S\partial_{x_1}T(t,s(t)^+) - K_L\partial_{x_1}T(t,s(t)^-)\right). \end{cases}$$

which reduces, similarly as we did for $g_\nu = 0$ in (5.3) and in (6.1), to the following equation

$$\begin{cases} C\partial_t T_1(t,x) = K\partial^2_x T_1(t,x) & x < s(t),\\ \partial_t T_2(t,x) = \partial_x^2 T_2(t,x) - T_2^4(t,x) + \int_{s(t)}^\infty d\xi\ \frac{\alpha E_1(\alpha(x-\xi))}{2} T_2^4(t,\xi) + G_\alpha(t,x) & y > 0,\\ T_1(t,(s(t))) = T_2(t,s(t)) = T_M > 0 & y = 0,\\ T(0,x) = T_0(x) & x \in \mathbb{R},\\ \dot{s}(t) = \frac{1}{L}\left(\partial_x T_2(t,s(t)) - K\partial_x T_1(t,s(t))\right), \end{cases} \tag{7.6}$$

where $E_1(x) = \int_{|x|}^\infty \frac{e^{-t}}{t}$ and where

$$G_\alpha(t,x) = \frac{1}{4\pi\sigma}\int_0^\infty d\nu \int_{n_1>0} dn\ g_\nu(n) e^{-\alpha\frac{x-s(t)}{n_1}}$$

is obtained solving the radiative transfer equation by characteristics.

Using the same strategy as in Section D.2, namely combining Banach fixed-point theorem for suitable integral equations obtained using the Green's functions for the half-plane for the Laplacian and classical parabolic regularity, it should be possible to show well-posedness for (7.6) for small times under suitable assumptions on the source g_ν.

The addition of a positive source in the evolution equation for the temperature in the solid makes the problem extremely intriguing. For instance, we expect the formation of superheated regions in the interior of the solid phase. Indeed, according to the result in [89], if positive volumetric heat sources are present, superheated regions appear. In order to prove a similar result for the problem (7.6), one could start considering a simpler source function of the form

$$H(t,x) = Ae^{-B|x-s(t)|}.$$

In a similar way one could proceed in order to obtain (if possible) a global well-posedness result for this equation.

We remark that for an external source of radiation which satisfies

$$0 < \int_0^\infty d\nu \int_{n_1>0} dn\, g_\nu(n) e^{-\alpha\frac{x-s(t)}{n_1}} \leq \|g_\nu\|_{L^1(\mathbb{S}^2\times\mathbb{R}_+)} e^{-\alpha|x-s(t)|},$$

different sub- and supersolutions have to be considered in order to show via maximum principle the bounds satisfied by the temperature, the formation of superheated regions, as well as a (possible) global well-posedness result.

If we obtain a global well-posedness result for solutions which may have superheated regions, we can proceed studying the long-time asymptotic of those functions. For instance, also in this case one could try to construct traveling wave solutions. Notice that the form of the source indicates that traveling waves are possible solutions. Indeed, $G(t,x) \approx Ae^{-B|x-s(t)|}$, which for $s(t) = -ct$ and $x + ct = y > 0$ reduces to $G(y) \approx Ae^{-By}$.

7.3.3 Long-time behavior

Turning back to the situation in which there is no external source of radiation, i.e. $g_\nu(n) = 0$ in (7.6), the long-time behavior of the solutions remains an open problem. In ([38], Appendix E) we proved the existence of traveling wave solutions and we presented the expected behavior of the solutions to the considered Stefan problem as $t \to \infty$. However, the rigorous proof of the long-time asymptotics has still to be developed. Regarding this issue, there are many problems that have to be considered.

First of all, we should study the well-posedness of the self-similar equation (cf. equation (E.111))

$$-\frac{z}{2}F'(z) - F''(z) - \frac{1}{\alpha^2}\left(F^4(z)\right)'' = 0 \tag{7.7}$$

for any $F(\pm\infty) \in [0, T_M]$.

Also the Problems E.4.1, E.4.2 and E.4.3 in Section E.4 should be considered. Indeed, they imply for any $T_{-\infty} \geq T_M$ and $T_\infty \in [0, T_M]$ the existence of a unique $c \geq 0$, T_1^c, T_2^c and F such that

$$\begin{cases} c\partial_y T_1^c(y) = \frac{K}{C}\partial_y^2 T_1^c(y) & y < 0, \\ c\partial_y T_2^c(y) = \partial_y^2 T_2^c(y) - (T_2^c(y))^4 + \int_0^\infty \alpha \frac{E_1(\alpha(y-\eta))}{2}\,(T_2^c(\eta))^c\, d\eta & y > 0, \\ T_2^c(0) = T_1^c(0) = T_M \\ c = \frac{1}{L}\left(K\partial_y T_1^c(0^-) - \partial_y T_2^c(0^+)\right), \\ T_1^c \geq T_M, \;\; 0 \leq T_2^c \leq T_M, \\ T_1^c(y) \underset{y\to-\infty}{\longrightarrow} T_{-\infty}, \end{cases}$$

and F satisfies (7.7) for $F(-\infty) = \lim\limits_{y\to\infty} T_2^c(y)$ as well as $F(\infty) = T_\infty$.

Finally, we should prove rigorously that as $t \to \infty$ the solution (T_1, T_2, s) of the problem (1.39) behaves as follows. The moving interface $s(t)$ becomes $-ct$ and the temperature in the liquid T_1 approaches T_1^c. Finally, the temperature in the solid T_2 is close to T_2^c near the interface and to the self-similar solution F far away from the free boundary.

Furthermore, also the global and local stability of the traveling waves should be studied. We could first try to use maximum principle methods in order to study the long-time behavior of the solutions. This is also the strategy used for the classical one-dimensional one-phase Stefan problem, cf. [106].

7.3.4 More general assumptions

We remark that there are plenty of more open problems concerning this type of Stefan problem. First of all, we could consider the case in which the heat is transferred by radiation and conduction in both phases of the material. Another possibility is to consider the case in which radiation interacts only with the liquid and not with the solid part of the body.

Under the assumptions that at time $t = 0$ the material fills the whole space $\mathbb{R}^3$, where the liquid region is $\mathbb{R}^3_-$ and the solid one is $\mathbb{R}^3_+$, and that the temperature depends only on the variable x_1, the free boundary problem under consideration reduces to a one-dimensional problem. We could also consider a more general case, where T depends on all variables and where the interface is not a plane. This would open to new problems regarding also the regularity of the free boundary as well as the stability of the planar interface.

Appendices

Appendix A

Compactness result and existence theory for a general class of stationary radiative transfer equations

Abstract: In this paper, we study the steady-states of a large class of stationary radiative transfer equations in a C^2 convex bounded domain. Namely, we consider the case in which both absorption-emission and scattering coefficients depend on the local temperature T and the radiation frequency ν. The radiative transfer equation determines the temperature of the material at each point. The main difficulty in proving existence of solutions is to obtain compactness of the sequence of integrals along lines that appear in several exponential terms. We prove a new compactness result suitable to deal with such a non-local operator containing integrals on a line segment. On the other hand, to obtain the existence theory of the full equation with both absorption and scattering terms we combine the compactness result with the construction of suitable Green functions for a class of non-local equations.

A.1 Introduction

In this paper, we study the *stationary radiative transfer equation* for a radiation intensity function $I_\nu : \Omega \times \mathbb{R}^3 \to [0, \infty)$ on a C^2 convex and bounded domain $\Omega \subset \mathbb{R}^3$, which takes the form

$$n \cdot \nabla_x I_\nu = \alpha_\nu^a(T)(B_\nu(T) - I_\nu) + \alpha_\nu^s(T) \left[\int_{\mathbb{S}^2} K(n', n) I_\nu(x, n') dn' - I_\nu \right], \tag{A.1}$$

where it factors in both the excitation and de-excitation processes of gas molecules alongside photon scattering. This formulation is underpinned by the presumption of local thermodynamic equilibrium (LTE) of gas molecules. Specifically, the components $\alpha_\nu^a I_\nu$, $\alpha_\nu^s I_\nu$, $\alpha_\nu^a B_\nu$, $\alpha_\nu^s \int_{\mathbb{S}^2} K(n', n) I_\nu(x, n') dn'$ represent absorption, scattering loss, emission from gas deexcitation, and scattering gain, respectively. Herein,

$$B_\nu = B_\nu(T) =: \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{kT}} - 1} \tag{A.2}$$

symbolizes the Planck emission from a black body, while $I_\nu = I_\nu(x, n)$ signifies the radiation intensity at frequency ν, located at position $x \in \Omega \subset \mathbb{R}^3$ and oriented in direction $n \in \mathbb{S}^2$.

Note that $B_\nu(T)$ is monotonically increasing in T for each ν and $B_\nu(T) = 0$ is equivalent to $T = 0$. By making a change of variables $\nu \mapsto \zeta =: \frac{h\nu}{kT}$, we obtain that

$$\int_0^\infty B_\nu(T)d\nu = \int_0^\infty \frac{2h\nu^3}{c^2}\frac{1}{e^{\frac{h\nu}{kT}}-1}d\nu = \int_0^\infty \frac{2hk^3T^3}{h^3c^2}\frac{\zeta^3}{e^\zeta - 1}\frac{kT}{h}d\zeta$$
$$= \frac{\pi^4}{15}\frac{2hk^3T^3}{h^3c^2}\frac{kT}{h} = \sigma T^4, \quad \text{(A.3)}$$

where we define $\sigma =: \frac{2\pi^4k^4}{15h^3c^2}$.

The radiation energy flux at frequency ν can be articulated as:

$$\mathcal{F}_\nu = \mathcal{F}_\nu(x) = \int_{\mathbb{S}^2} nI_\nu(x,n)dn \in \mathbb{R}^3.$$

The scattering kernel of the "non-local" gain term of scattering has the property

$$\int_{\mathbb{S}^2} K(n',n)dn = 1. \quad \text{(A.4)}$$

If the scattering is isotropic then it becomes simply $\alpha^s_\nu(T)I_\nu$ in (A.1). The class of models is for the LTE situation. The temperature T is well-defined at each point, and each coefficient $\alpha_\nu = \alpha_\nu(T)$ depends on the frequency of radiation ν and the local temperature T. The coefficient α_ν can be considered as the spectral lines for each ν or, more generally, the averages of these processes.

Throughout the paper, we will study the existence theory of the general model (A.1) with (A.4). The assumption (A.4) implies that the scattering does not modify the frequency. We will consider the general case where the absorption-emission and the scattering coefficients can depend not just on the radiation frequency $\nu > 0$ but also on the local temperature $T(x)$. Another main assumption in this model above is that the non-elastic mechanisms yielding LTE in the gas molecules' distributions are extremely fast, and therefore the scattering cannot modify much the Boltzmann ratio between the different energy levels at each point. For more details, see [34, 81]. We write the total flux of radiation energy with frequency ν at x as

$$\mathcal{F} = \mathcal{F}(x) =: \int_0^\infty d\nu \int_{\mathbb{S}^2} nI_\nu(x,n)dn.$$

At the Planck equilibrium, we have

$$I_\nu(x,n) = B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{\frac{h\nu}{kT}}-1}. \quad \text{(A.5)}$$

Then, the stationarity of the temperature at each point requires that the divergence of the total flux of radiation energy vanishes (cf. [108, 114]); i.e.,

$$\nabla_x \cdot \mathcal{F}(x) = 0, \text{ at any } x \in \Omega. \quad \text{(A.6)}$$

Hence, we will examine throughout this paper whether the temperature T at each point can be determined uniquely by (A.1) and a suitable boundary condition for the radiation at $\partial\Omega$, if we impose the divergence-free total-flux condition (A.6); see also [81] for the conservation law. The simplest boundary condition that one can impose is

$$I_\nu(x,n) = g_\nu(n) \geq 0 \text{ for } x \in \partial\Omega \text{ and } n \cdot n_x < 0, \quad \text{(A.7)}$$

where n_x is the outward normal vector at the boundary point $x \in \partial\Omega$ and Ω is with smooth boundary $\partial\Omega$.

Throughout the paper, we assume Ω to be a convex domain with C^2-boundary and strictly positive curvature.

The problem (A.1), (A.6), and (A.7) is considered in [83] in the case where α_ν^a and α_ν^s are independent of the temperature. The main novelty of this paper is that we were able to extend the proof of the existence of solutions to (A.1), (A.6), and (A.7) for a general class of coefficients α_ν^a and α_ν^s which also depend on the temperature T. To obtain this result we will derive a compactness result for a large class of non-local operators including terms with the form

$$T(\cdot) \mapsto \exp\left(-\int_{[x,\eta]} \beta(T(\xi))ds(\xi)\right), \ x, \eta \in \Omega, \tag{A.8}$$

where the integral is along the segment connecting x to η. The compactness result that we obtain in this paper to prove the existence has some analogies with the classical averaging lemmas that have been extensively used in kinetic theory [40,41,70,80,144]. Nevertheless, the currently available averaging lemmas including the treatment of line integrals in [7] do not seem to provide the compactness that we require. For this reason, we prove a new compactness result more suitable to deal with the non-local operator on a line segment with the form (A.8).

We now introduce some related works in the literature.

A.1.1 Summary of literature

The study of the distribution of the temperature within a body where the transfer of heat by means of radiation plays an important role has been extensively studied. Seminal works by Compton and Milne [31, 109] laid the foundation for understanding the interaction between radiation and gases. Subsequent papers by Holstein and Kenty provided further insights [78, 85]. Specifically, Holstein highlighted the necessity to approach heat transfer by radiation as a non-local issue. The study of the evolution of temperature over time in a bar where the heat transfer is strictly due to radiation was considered by Spiegel [131]. Detailed reviews on the physics of radiative transfer can be found in works by Mihalas, Oxenius and Rutten [108, 114, 125].

In recent times the mathematical properties of the radiative transfer equation have been examined in [12–14, 16, 107]. In several of these papers the authors studied the well-posedness of the time-dependent problem, usually using semi-group theory or the theory of m-accretive operators.

Another question that has been considered by several authors is the so-called Milne problem (cf. [30,68,127]). The Milne problem consists of describing the distribution of temperature in half space, a question which is motivated by the study of the distribution of the temperature near the boundary in the diffusion approximation limit. In this setting, the equation reduces to a one-dimensional problem.

Problems related to the diffusion approximation and to homogenization have been extensively studied as well as equations describing the distribution of temperature for bodies where the heat is transmitted by means of radiation and conduction have been considered by numerous authors with different boundary conditions, see for instance [22,32,42,62–65,72,95,96,116, 139]. We refer to [83] for more details. Moreover, other papers such as [34, 81, 112, 113, 122] consider the radiative transfer equation coupled to the Boltzmann equation.

Equations similar to (A.1) with the absorption-emission coefficient $\alpha_\nu^a = 0$, focusing solely on scattering, are widely examined in mathematical studies, especially about neutron diffusion as seen in references [19,76,148]. Similar equations appear also in the theory of Lorentz gases, cf. [18,21,61,105,111,132].

A recent paper by Arkeryd and Nouri [7] that considers the existence of mild solutions to normal discrete velocity Boltzmann equations with given incoming boundary values also requires a compactness theorem for line integrals having some analogies with the one derived in this paper.

We want to emphasize that although the time-dependent problem has been considered in various papers, the existence of a time-dependent solution, even a globally bounded one, does not imply the existence of a solution of the stationary problem.

A.1.2 Main theorems

In this paper, we consider the boundary value problem given by the system of equations (A.1), (A.6) and (A.7). We will consider two types of absorption coefficients and scattering coefficients. In the first case, the coefficients satisfy the so-called Grey approximation where $\alpha_\nu^a(T)=\alpha^a(T)$ and $\alpha_\nu^s(T)=\alpha^s(T)$ are independent of the frequency ν. We will also consider a particular choice of α's in the non-Grey case, namely where α_ν^a and α_ν^s can be written as the product of functions in ν and T, separately. We denote this case from now on as "pseudo Grey". A similar choice can be found in [68]. First, we study the case of pure emission and absorption where $\alpha_\nu^s=0$ and we will show the existence of a solution to this problem as stated in the following theorem. In the following theorem and throughout the rest of the paper, we denote $I_\nu\in L^\infty(\Omega,L^\infty(\mathbb{S}^2,L^1(\mathbb{R}_+)))$ when

$$\sup_{x\in\Omega}\sup_{n\in\mathbb{S}^2}\int_0^\infty I_\nu(x,n)d\nu<\infty.$$

Theorem A.1. *Let $\Omega\subset\mathbb{R}^3$ be bounded and open with C^2-boundary and strictly positive curvature. Suppose that the incoming boundary profile g_ν satisfies the bound*

$$\sup_{n\in\mathbb{S}^2}\int_0^\infty d\nu\ g_\nu(n)<\infty,$$

and that $\alpha_\nu^a(T(x))=Q(\nu)\alpha(T(x))$ is bounded, strictly positive and C^1 in T, where $Q:\mathbb{R}_+\to\mathbb{R}_+$ and $\alpha:\mathbb{R}_+\to\mathbb{R}_+$.
Then there exists a solution $(T,I_\nu)\in L^\infty(\Omega)\times L^\infty\left(\Omega,L^\infty\left(\mathbb{S}^2,L^1(\mathbb{R}_+)\right)\right)$, which solves the boundary value problem given by (A.1)*,* (A.6) *and* (A.7) *for $\alpha_\nu^s=0$, namely*

$$\begin{cases} n\cdot\nabla_x I_\nu(x,n)=Q(\nu)\alpha(T(x))\left(B_\nu(T(x)-I_\nu(x,n)\right),\ \text{for } x\in\Omega,\ n\in\mathbb{S}^2,\\ \nabla_x\cdot\mathcal{F}(x)=0,\ \text{at any } x\in\Omega,\\ I_\nu(x,n)=g_\nu(n)\geq 0,\quad \text{for } x\in\partial\Omega \text{ and } n\cdot n_x<0.\end{cases} \tag{A.9}$$

Here, I_ν is a solution to (A.1) *in the sense of distribution.*

We will prove Theorem A.1 using a fixed-point argument. As we will see, the main difficulty that arises in our proof is to show the compactness of the terms involving the exponential function of a line integral. As first step we will regularize the line integral in order to obtain a problem where it is possible to prove existence using the Schauder fixed-point theorem. We will then show the compactness of the solutions of such regularized problems uniformly in the regularizing parameter. To this end, we provide the following type of general L^2 compactness result for sequences of non-linear operators of line integrals based on the study of auxiliary measures on $\mathbb{S}^2$.

Proposition A.1 (Compactness result for line integrals). *Let $\Pi^3 = [-L, L]^3$ and $(\varphi_j)_{j\in\mathbb{N}} \in L^\infty\left(\Pi^3\right)$ be a sequence of periodic functions with $\|\varphi_j\|_\infty \le M$. For $n \in \mathbb{S}^2$ and $m \in \mathbb{N}$ we define the operators L_n and T_m by*

$$L_n[\varphi](x) =: \int_{-L}^{L} d\lambda\ \varphi(x - \lambda n) \quad and \quad T_m[\varphi](x) =: \int_{\mathbb{S}^2} dn\, (L_n[\varphi](x))^m .$$

Then for every $m \in \mathbb{N}$ the sequence $(T_m[\varphi_j])_j$ is compact in $L^2\left(\Pi^3\right)$. More precisely, the sequence $T_m[\varphi_j]$ satisfies the following equi-integrability condition: For any $\varepsilon > 0$ there exists some $h_0 > 0$ such that

$$\int_{\Pi^3} dx\ |T_m[\varphi_j](x) - T_m[\varphi_j](x+h)|^2 \le C_m \int_{\Pi^3} dx \int_{\mathbb{S}^2} dn\ |L_n[\varphi_j](x) - L_n[\varphi_j](x+h)|^2 < \varepsilon \quad \text{(A.10)}$$

for all $j \in \mathbb{N}$ and all $|h| < h_0$. The constant $C_m > 0$ depends only on $m \in \mathbb{N}$, M and L.

The proposition above will provide the compactness theory required to conclude the proof of the existence of solutions to the original problem (A.9). Finally, we study the existence of solutions for the full equation with both scattering and absorption-emission. In this case, we obtain the following existence theorem via the construction of suitable Green functions associated with the system.

Theorem A.2 (Full equations in the pseudo Grey case). *Let $\Omega \subset \mathbb{R}^3$ be bounded and open with C^2-boundary and strictly positive curvature. Let $\alpha_\nu^a(T(x)) = Q_a(\nu)\alpha^a(T(x))$ and $\alpha_\nu^s(T(x)) = Q_s(\nu)\alpha^s(T(x))$ be bounded and strictly positive. Assume that $Q_\ell \in C^1(\mathbb{R}_+)$ and $\alpha^\ell \in C^1(\mathbb{R}_+)$ for $\ell = a, s$. Assume $K \in C^1\left(\mathbb{S}^2 \times \mathbb{S}^2\right)$ be non-negative, rotationally symmetric, and independent of the frequency with the property* (A.4). *Then there exists a solution $(T, I_\nu) \in L^\infty(\Omega)\times L^\infty\left(\Omega, L^\infty\left(\mathbb{S}^2, L^1(\mathbb{R}_+)\right)\right)$ to the equation* (A.1) *coupled with* (A.6) *satisfying the boundary condition* (A.7)*, where the I_ν is a solution to* (A.1) *in the sense of distribution.*

A.1.3 Strategy of the proof and main estimates

In the case of pure absorption and emission, our main strategy for the proof of the existence to the boundary-value problem is to reduce the stationary radiation equation and the "divergence-free-radiation-flow" equation to a non-local non-linear elliptic equation for an explicit function u of the temperature in the presence of an external source. This will allow to reformulate the problem (A.1) (with $\alpha^s = 0$), (A.6) and (A.7) as

$$u(x) - \int_0^\infty \int_\Omega \frac{F(\nu, u(\eta))}{|x-\eta|^2} \exp\left(-\int_{[x,\eta]} \alpha_\nu^a(T(\xi))ds(\xi)\right) d\eta\, d\nu = S(x), \tag{A.11}$$

where $[x, \eta]$ indicates the segment from $x \in \Omega$ to $\eta \in \Omega$ and the exact form of u, F and S are given in Section A.2.1. To prove the existence of a solution to the system given by equations (A.1), (A.6) and (A.7) is equivalent to showing the existence of a solution to the non-local integral equation (A.11). When the emission-absorption coefficient α_ν^a depends on the local temperature and has the form $\alpha_\nu(x) = Q(\nu)\alpha(T(x))$, where $Q(\nu)$ is a function of the frequency which can be also constant, the strategy we use to prove the existence of a solution is the following. We consider first a regularized version of (A.11), for which the existence of a solution is guaranteed by the Schauder fixed-point theorem. With the compactness result of Proposition A.1, which is based on the study of some auxiliary measures defined on the sphere $\mathbb{S}^2$, it turns out that the sequence of regularized solutions is compact in L^2 and hence a subsequence converges pointwise almost everywhere to the solution of the original problem. We remark that obtaining L^∞-estimates for the function u solving (A.11) (or a regularized version of it) is not difficult using the structure of the integral operator. However, the main difficulty remains getting compactness. In the case of the problem including the scattering term, we use a similar strategy that however becomes more involved. To find a reformulation of (A.1), (A.6) and (A.7) analogous to (A.11) we construct suitable fundamental solutions for a problem that includes absorption and scattering. These fundamental solutions satisfy recursive equations that allow us to obtain information about their properties using Duhamel series which contain terms involving exponentials of some integrals along straight lines as in (A.11). Due to this we will regularize again the problem for which then solutions exist applying the Schauder fixed-point theorem. With the previous compactness result in Proposition A.1 applied to finitely many terms of the Duhamel expansion we will obtain the compactness of the sequence of regularized solution in $L^2(\Omega)$ and thus the convergence pointwise almost everywhere to the solution of the full problem.

A.1.4 Outline of the paper

The rest of the paper is organized as follows. In Section A.2, we provide the derivation of the non-local integral equation and the regularization of the equation which will be crucially used in the proof of the existence of solutions (Theorem A.1). Section A.3 is devoted to the study of the existence of a solution to (A.1) in the absence of scattering. In Subsection A.3.1 we prove the existence of solutions to the regularized problem. In Subsection A.3.2, we provide a L^2 compactness theory of non-linear operators of line integrals based on the study of some auxiliary measures defined on the sphere $\mathbb{S}^2$. This compactness theory will be used to obtain the compactness of the solution sequences of the regularized problem and this allows us to show the existence of the original problem stated in Theorem A.1 in Subsection A.3.3. In Section A.4 we show the existence of solutions to the full equation (A.1) taking into account also the scattering term, in particular we will prove Theorem A.2. This will be made in several steps starting from the study of the Grey case deriving a non-local equation for the temperature (Subsection A.4.1) and constructing suitable Green functions which encode the effect of the scattering (Subsection A.4.2). Subsections A.4.3 to A.4.6 are devoted to the proof of existence of solutions to the equation (A.1) in the Grey case. There, a regularized problem is solved by means of Schauder's fixed-point theorem and a weak maximum principle, while the compactness result of Subsection A.3.2 is used to conclude the existence of a solution for the Grey approximation. Finally, in Subsection A.4.7 we provide the proof of Theorem A.2.

A.2 Derivation of a non-local integral equation for the temperature and the regularization of the equation

A.2.1 Derivation of a non-local integral equation

In this section, we will first derive a non-local integral equation that is satisfied by the temperature. This equation is associated to the stationary equation

$$n \cdot \nabla_x I_\nu = \alpha_\nu^a(T)(B_\nu(T) - I_\nu). \tag{A.12}$$

Without loss of generality, we can assume $\sigma = 1$ by rescaling variables. We define for every $(x,n) \in \Omega \times \mathbb{S}^2$ a new coordinate system with variables $(y,s) = (y(x,n), s(x,n)) \in \partial\Omega \times \mathbb{R}_{\geq 0}$. These variables are defined in the following way. We consider for every $x \in \Omega$ and $n \in \mathbb{S}^2$

the backward trajectory starting from x in direction $-n$. Then $y(x,n) \in \partial\Omega$ is the boundary point that intersects with this straight line and $s(x,n)$ is its length, i.e. $s(x,n) = |x - y(x,n)|$ and $x = y + sn$. Therefore, using this notation, solving by characteristics I_ν and integrating (A.12), we obtain that the flow $\mathcal{F} = \int_0^\infty \int_{\mathbb{S}^2} nI_\nu(x,n)dnd\nu$ satisfies

$$\mathcal{F} = \int_0^\infty d\nu \int_{\mathbb{S}^2} dn\ ng_\nu(n) \exp\left(-\int_0^{s(x,n)} \alpha_\nu(T(y(x,n)+\zeta n))d\zeta\right)$$
$$+ \left[\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\ \int_0^{s(x,n)} d\xi \exp\left(-\int_\xi^{s(x,n)} \alpha_\nu(T(y(x,n)+\zeta n))d\zeta\right) n \right.$$
$$\left. \times\, \alpha_\nu\left(T(y(x,n)+\xi n)\right) B_\nu\left(T(y(x,n)+\xi n)\right)\right] =: \mathcal{F}_1 + \mathcal{F}_2. \quad \text{(A.13)}$$

Now we recall the *conservation of energy* (A.6) that yields $\nabla_x \cdot \mathcal{F} = 0$. In order to use this condition, we take the divergence of (A.13).

We first compute $\nabla_x \cdot \mathcal{F}_2$. We define new variables $\hat{\xi} =: s - \xi$ and $\hat{\zeta} =: s - \zeta$ and make a change of variables $\xi \mapsto \hat{\xi}$in the integral, then

$$\mathcal{F}_2 = \int_0^\infty d\nu \int_{\mathbb{S}^2} dn\ \int_0^{s(x,n)} d\xi \exp\left(-\int_\xi^{s(x,n)} \alpha_\nu(T(y(x,n)+\zeta n))d\zeta\right) n\alpha_\nu(T)B_\nu(T)$$
$$= \int_0^\infty d\nu \int_{\mathbb{S}^2} dn\ \int_0^{s(x,n)} d\hat{\xi} \exp\left(-\int_0^{\hat{\xi}} \alpha_\nu(T(x-\hat{\zeta}n))d\hat{\zeta}\right) n\alpha_\nu(T(x-\hat{\xi}n))B_\nu\left(T(x-\hat{\xi}n)\right)$$
$$= \int_\Omega d\eta \int_0^\infty d\nu \exp\left(-\int_0^{|x-\eta|} \alpha_\nu\left(T\left(x - \hat{\zeta}\frac{x-\eta}{|x-\eta|}\right)\right)d\hat{\zeta}\right) \frac{x-\eta}{|x-\eta|}\alpha_\nu(T(\eta))\frac{B_\nu(T(\eta))}{|x-\eta|^2},$$

since the Jacobian gives $\frac{\partial(\hat{\xi},n)}{\partial\eta} = \frac{1}{|x-\eta|^2}$ where $\eta =: x - \hat{\xi}n$ and $\hat{\xi} = (x-\eta)\cdot n = |x-\eta|$. Also note that $n = \frac{x-\eta}{|x-\eta|}$. Therefore, we have

$$\nabla_x \cdot \mathcal{F}_2 = \int_0^\infty d\nu \int_\Omega d\eta\ \alpha_\nu(T(\eta))B_\nu(T(\eta))\nabla_x \cdot (\varphi v), \quad \text{(A.14)}$$

where we define

$$\varphi(x,\eta) =: \exp\left(-\int_0^{|x-\eta|} \alpha_\nu\left(T\left(x-\hat{\zeta}\frac{x-\eta}{|x-\eta|}\right)\right)d\hat{\zeta}\right) \text{ and } v =: \frac{x-\eta}{|x-\eta|^3}.$$

We now use that

$$\nabla_x \cdot (\varphi v) = \nabla\varphi \cdot v + \varphi\nabla \cdot v,$$

where $\mathrm{div}(v) = 4\pi\delta(x-\eta)$, and

$$\nabla\varphi = -\varphi\alpha_\nu(T(\eta))\frac{x-\eta}{|x-\eta|}$$
$$-\varphi \int_0^{|x-\eta|} \frac{d\alpha_\nu}{dT}\left(T\left(x-\hat{\zeta}\frac{x-\eta}{|x-\eta|}\right)\right)\left(\nabla T\left(x-\hat{\zeta}\frac{x-\eta}{|x-\eta|}\right)\right)\cdot D_x\left(x-\hat{\zeta}\frac{x-\eta}{|x-\eta|}\right)d\hat{\zeta}.$$

Note that

$$\left(\frac{x-\eta}{|x-\eta|}\cdot D_x\left(\frac{x-\eta}{|x-\eta|}\right)\right)_l = \sum_{j=1}^3 \frac{(x-\eta)_j}{|x-\eta|}\left(\frac{\delta_{jl}}{|x-\eta|} - \frac{(x_l-\eta_l)}{|x-\eta|^3}(x_j-\eta_j)\right)$$
$$= \frac{(x-\eta)_l}{|x-\eta|}\frac{1}{|x-\eta|} - \frac{|x-\eta|(x_l-\eta_l)}{|x-\eta|^3} = 0. \quad \text{(A.15)}$$

Using $\varphi(\eta,\eta)=1$, we have

$$
\begin{aligned}
\nabla_x\cdot(\varphi v) &= 4\pi\delta(x-\eta)-\varphi\alpha_\nu(T(\eta))\frac{1}{|x-\eta|^2}\\
&\quad -\frac{\varphi}{|x-\eta|^2}\int_0^{|x-\eta|}\frac{d\alpha_\nu}{dT}(T\left(x-\hat\zeta\frac{x-\eta}{|x-\eta|}\right))\frac{x-\eta}{|x-\eta|}\cdot\nabla T\left(x-\hat\zeta\frac{x-\eta}{|x-\eta|}\right)d\hat\zeta\\
&= 4\pi\delta(x-\eta)-\varphi\alpha_\nu(T(\eta))\frac{1}{|x-\eta|^2}\\
&\quad +\frac{\varphi}{|x-\eta|^2}\int_0^{|x-\eta|}\frac{d\alpha}{dT}(T\left(x-\hat\zeta\frac{x-\eta}{|x-\eta|}\right))\frac{d}{d\hat\zeta}(T\left(x-\hat\zeta\frac{x-\eta}{|x-\eta|}\right))d\hat\zeta\\
&= 4\pi\delta(x-\eta)-\varphi\alpha_\nu(T(\eta))\frac{1}{|x-\eta|^2}+\frac{\varphi}{|x-\eta|^2}\left(\alpha_\nu(T(\eta)-\alpha_\nu(T(x))\right)\\
&= 4\pi\delta(x-\eta)-\frac{\varphi}{|x-\eta|^2}\alpha_\nu(T(x)).
\end{aligned}
$$

Therefore, by (A.14) we have

$$
\begin{aligned}
\nabla_x\cdot\mathcal{F}_2 &= 4\pi\int_0^\infty\alpha_\nu(T(x))B_\nu(T(x))d\nu\\
&\quad -\int_0^\infty d\nu\alpha_\nu(T(x))\int_\Omega d\eta\ \alpha_\nu(T(\eta))B_\nu(T(\eta))\frac{\exp\left(-\int_0^{|x-\eta|}\alpha_\nu(T\left(x-\hat\zeta\frac{x-\eta}{|x-\eta|}\right))d\hat\zeta\right)}{|x-\eta|^2}. \qquad (A.16)
\end{aligned}
$$

We will see that the integral operator in (A.16) is a contractive operator.

Now we compute $\nabla_x\cdot\mathcal{F}_1$. Using (A.13), we have

$$
\begin{aligned}
\nabla_x\cdot\mathcal{F}_1 &= \int_0^\infty\int_{\mathbb{S}^2}dn\ \exp\left(-\int_0^{s(x,n)}\alpha_\nu(T(y(x,n)+\zeta n))d\zeta\right)\\
&\quad\times n\cdot\left[-g_\nu(n)\nabla_x s\alpha_\nu(T(x))-g_\nu(n)\int_0^{s(x,n)}\nabla_x((\alpha_\nu\circ T)(y(x,n)+\zeta n))d\zeta\right].
\end{aligned}
$$

Here we observe that

$$
\begin{aligned}
n\cdot\nabla_x((\alpha_\nu\circ T)(y(x,n)+\zeta n)) &= \frac{d}{dt}((\alpha_\nu\circ T)(y(x+tn,n)+\zeta n))|_{t=0}\\
&= \frac{d}{dt}((\alpha_\nu\circ T)(y(x,n)+\zeta n))|_{t=0}=0,
\end{aligned}
$$

since $y(x+tn,n)=y(x,n)$. Also note that that $n\cdot\nabla_x s=1$. This holds by the following observation. For any $x_0\in\Omega$, we have

$$y(x_0+\zeta n,n)+s(x_0+\zeta n,n)n=x_0+\zeta n,$$

and hence

$$y(x_0,n)+s(x_0+\zeta n,n)n=x_0+\zeta n.$$

We can differentiate it with respect to ζ and we obtain

$$\frac{d}{d\zeta}(s(x_0+\zeta n,n)n)=(\nabla_x s(x_0+\zeta n,n)\cdot n)n=n.$$

Thus we have

$$\nabla_x \cdot \mathcal{F}_1 = -\int_0^\infty d\nu \int_{\mathbb{S}^2} dn \ \exp\left(-\int_0^{s(x,n)} \alpha_\nu(T(y(x,n)+\zeta n))d\zeta\right) \alpha_\nu(T(x)) g_\nu(n), \quad \text{(A.17)}$$

and note that $\nabla_x \cdot \mathcal{F}_1 \leq 0$ and $|\nabla_x \cdot \mathcal{F}_1|$ is bounded from above in L^∞, since $\alpha_\nu(\cdot)$ is bounded and $G = \int_0^\infty d\nu \ g_\nu(n) \in L^\infty(\mathbb{S}^2)$.

Combining (A.16) and (A.17) we finally obtain

$$\begin{aligned}\nabla_x \cdot \mathcal{F} = {} & 4\pi \int_0^\infty \alpha_\nu(T(x)) B_\nu(T(x)) d\nu \\ & - \int_0^\infty d\nu \alpha_\nu(T(x)) \int_\Omega d\eta \ \alpha_\nu(T(\eta)) B_\nu(T(\eta)) \frac{\exp\left(-\int_0^{|x-\eta|} \alpha_\nu(T\left(x - \hat{\zeta}\frac{x-\eta}{|x-\eta|}\right)) d\hat{\zeta}\right)}{|x-\eta|^2} \\ & - \int_0^\infty d\nu \int_{\mathbb{S}^2} dn \ \exp\left(-\int_0^{s(x,n)} \alpha_\nu(T(y(x,n)+\zeta n))d\zeta\right) \alpha_\nu(T(x)) g_\nu(n) = 0. \quad \text{(A.18)}\end{aligned}$$

In the pseudo Grey case as in Theorem A.1 the absorption coefficient has the form $\alpha_\nu(T(x)) = Q(\nu)\alpha(T(x))$ and it is strictly positive and bounded. Hence dividing by $\alpha(T(x))$ equation (A.18) reads

$$\begin{aligned} & 4\pi \int_0^\infty Q(\nu) B_\nu(T(x)) d\nu \\ & = \int_0^\infty d\nu \ Q^2(\nu) \int_\Omega d\eta \ \alpha(T(\eta)) B_\nu(T(\eta)) \frac{\exp\left(-Q(\nu)\int_0^{|x-\eta|} \alpha(T\left(x - \hat{\zeta}\frac{x-\eta}{|x-\eta|}\right)) d\hat{\zeta}\right)}{|x-\eta|^2} \\ & \quad + \int_0^\infty d\nu \int_{\mathbb{S}^2} dn \ \exp\left(-Q(\nu)\int_0^{s(x,n)} \alpha(T(y(x,n)+\zeta n))d\zeta\right) Q(\nu) g_\nu(n). \quad \text{(A.19)}\end{aligned}$$

Assuming now the Grey approximation, i.e. assuming that the absorption coefficient is strictly positive and independent of ν (i.e. $\alpha_\nu(T(x)) = \alpha(T(x))$), and using the Stefan Law (A.3) we obtain dividing (A.18) by $\alpha(T(x))$ and defining $G(n) = \int_0^\infty g_\nu(n) d\nu$,

$$\begin{aligned} 4\pi (T(x))^4 = {} & \int_\Omega d\eta \ \alpha(T(\eta))(T(\eta))^4 \frac{\exp\left(-\int_0^{|x-\eta|} \alpha(T\left(x - \hat{\zeta}\frac{x-\eta}{|x-\eta|}\right)) d\hat{\zeta}\right)}{|x-\eta|^2} \\ & + \int_{\mathbb{S}^2} dn \ \exp\left(-\int_0^{s(x,n)} \alpha(T(y(x,n)+\zeta n))d\zeta\right) G(n) \geq 0. \quad \text{(A.20)}\end{aligned}$$

A.2.2 Non-local integral equation in the Grey case

We will focus next on the Grey approximation and we will prove the following theorem.

Theorem A.3. *Let $\Omega \subset \mathbb{R}^3$ be bounded and open with C^2-boundary and strictly positive curvature. Suppose that the incoming boundary profile satisfies $\|G\|_{L^\infty(\mathbb{S}^2)} < \infty$, where $G(n) = \int_0^\infty g_\nu(n) d\nu$. In addition, suppose that the absorption coefficient $\alpha(\cdot)$ is bounded and strictly positive and assume $\alpha_s = 0$. Then there exists a solution $(T, I_\nu) \in L^\infty(\Omega) \times L^\infty\left(\Omega, L^\infty\left(\mathbb{S}^2, L^1\left(\mathbb{R}_+\right)\right)\right)$ which solves the boundary-value problem* (A.1)-(A.7) *coupled with the conservation of energy* (A.6). *I_ν is a solution to* (A.1) *in the sense of distribution.*

In order to prove Theorem A.3 we aim to use a fixed-point argument. To this end we first see that u satisfies an L^∞-estimate. Indeed, observe that Ω is bounded in $\mathbb{R}^3$ and

$$\begin{aligned}
&\int_\Omega d\eta\ \alpha(T(\eta))(T(\eta))^4 \frac{\exp\left(-\int_0^{|x-\eta|}\alpha(T\left(x-\hat{\zeta}\frac{x-\eta}{|x-\eta|}\right))d\hat{\zeta}\right)}{|x-\eta|^2} \\
&\quad = \int_{\mathbb{S}^2} dn \int_0^{s(x,n)} r^2 dr\ \alpha(T(\eta))(T(\eta))^4 \frac{\exp\left(-\int_0^r \alpha(T(x-\hat{\zeta}n))d\hat{\zeta}\right)}{r^2} \\
&\quad = \int_{\mathbb{S}^2} dn \int_0^{s(x,n)} dr\ (T(\eta))^4 \left(-\frac{d}{dr}\exp\left(-\int_0^r \alpha(T(x-\hat{\zeta}n))d\hat{\zeta}\right)\right) \\
&\quad \le \|T\|^4_{L^\infty(\Omega)} \int_{\mathbb{S}^2} dn \left(1-\exp\left(-\int_0^{s(x,n)} \alpha(T(x-\hat{\zeta}n))d\hat{\zeta}\right)\right) \\
&\quad \le 4\pi(1-\delta)\|T\|^4_{L^\infty(\Omega)}, \qquad \text{(A.21)}
\end{aligned}$$

where

$$\delta = \exp\left(-\|\alpha\|_{L^\infty} \max_{x\in\Omega,\ n\in\mathbb{S}^2} s(x,n)\right) > 0.$$

In addition, we observe

$$\begin{aligned}
&\left|\int_{\mathbb{S}^2} dn\ \exp\left(-\int_0^{s(x,n)} \alpha(T(y(x,n)+\zeta n))d\zeta\right) G(n)\right| \\
&\quad \le \int_{\mathbb{S}^2} G(n)dn = \|G\|_{L^1(\mathbb{S}^2)} \le 4\pi\|G\|_{L^\infty(\mathbb{S}^2)} < \infty. \qquad \text{(A.22)}
\end{aligned}$$

Let us now define $u(x) = 4\pi\sigma T^4(x)$. Hence, we write $\gamma(u(x)) = \alpha\left(\sqrt[4]{\frac{u(x)}{4\pi\sigma}}\right)$. In order to simplify the notation we also denote by

$$\int_{[x,\eta]} f(\xi)d\xi = \int_0^{|x-\eta|} f\left(x+t\frac{\eta-x}{|\eta-x|}\right)\, dt.$$

Then we obtain

$$\begin{aligned}
u(x) = &\int_\Omega d\eta\ \gamma(u(\eta))u(\eta)\frac{\exp\left(-\int_{[x,\eta]}\gamma(u(\zeta))d\zeta\right)}{4\pi|x-\eta|^2} \\
&+ \int_{\mathbb{S}^2} dn\ \exp\left(-\int_{[x,y(x,n)]}\gamma(u(\zeta))d\zeta\right) G(n). \qquad \text{(A.23)}
\end{aligned}$$

This completes the derivation of the non-local integral equation. In the next Subsection, we consider the regularization of the line integral in the non-local equation.

A.2.3 Regularization of the non-local equation

In order to prove the existence of a function u solving the non-local equation (A.23), we will consider a regularization of the line integral. For the the regularized problem we will apply Schauder's fixed-point theorem and show the existence of a solution. We obtain in this way a sequence of solutions to the regularized problem. We will hence show that the sequence of integral operators acting on that regularized solutions is compact in L^2. This implies the existence of a subsequence convergent pointwise almost everywhere to a function u. After an application of the dominated convergence theorem we will show that this limit function is a solution to the original problem (A.23).

Let $\phi_\varepsilon \in C_c^\infty(\mathbb{R}^3)$ be a standard positive and radially symmetric mollifier. Given a segment Γ we define $\int_{\mathbb{R}^3} \delta_\Gamma(y)\varphi(y)dy = \int_\Gamma \varphi(\xi)d\xi$. Hence, for $x, \eta \in \Omega$

$$\begin{aligned}\int_{\mathbb{R}^3} F(\xi)\delta_{[x,\eta]} * \phi_\varepsilon(\xi)d\xi &= \int_{\mathbb{R}^3} F(\xi)\int_0^{|\eta-x|} \phi_\varepsilon\left(\xi - x - \lambda\frac{\eta-x}{|\eta-x|}\right)d\lambda d\xi \\ &= \int_{\mathbb{R}^3}\int_0^{|\eta-x|} F\left(\xi + x + \lambda\frac{\eta-x}{|\eta-x|}\right)\phi_\varepsilon(\xi)d\lambda d\xi \\ &\overset{\varepsilon\to 0}{\longrightarrow} \int_0^{|\eta-x|} F\left(x + \lambda\frac{\eta-x}{|\eta-x|}\right)d\lambda. \quad (A.24)\end{aligned}$$

In order to have also the same type of L^∞ estimate we consider

$$\begin{aligned}u(x) = \mathcal{B}^\varepsilon(u)(x) =: &\int_\Omega d\eta\ \left(\gamma(u) * \phi_\varepsilon\right)(\eta)u(\eta)\frac{\exp\left(-\int_{\mathbb{R}^3}\gamma(u(\xi))\delta_{[x,\eta]} * \phi_\varepsilon(\xi)d\xi\right)}{4\pi|x-\eta|^2} \\ &+ \int_{\mathbb{S}^2} dn\ \exp\left(-\int_{\mathbb{R}^3}\gamma(u(\xi))\delta_{[x,y(x,n)]} * \phi_\varepsilon(\xi)d\xi\right)G(n). \quad (A.25)\end{aligned}$$

We remark that by the smoothness of γ and the continuity of the exponential function the integral operator $\mathcal{B}^\varepsilon$ is continuous. The interesting part of this regularization is that we can get the same type of L^∞-estimate as for the original problem. Indeed, using the symmetry of ϕ_ε and again the change of variables $\eta = x - rn$ we see that

$$\begin{aligned}&\frac{d}{dr}\exp\left(-\int_{\mathbb{R}^3}\gamma(u(\xi))\int_0^r \phi_\varepsilon(\xi - x + \lambda n)d\lambda d\xi\right) \\ &\quad= -\exp\left(-\int_{\mathbb{R}^3}\gamma(u(\xi))\int_0^r \phi_\varepsilon(\xi - x + \lambda n)d\lambda d\xi\right)\int_{\mathbb{R}^3}\phi_\varepsilon(x - rn - \xi)\gamma(u(\xi))d\xi \\ &\quad= -\exp\left(-\int_{\mathbb{R}^3}\gamma(u(\xi))\int_0^r \phi_\varepsilon(\xi - x + \lambda n)d\lambda d\xi\right)\left(\gamma(u) * \phi_\varepsilon\right)(x - rn). \quad (A.26)\end{aligned}$$

We can then argue as in (A.21) and (A.22) that $\|\mathcal{B}^\varepsilon(u)\|_\infty \leq (1-\delta)\|u\|_\infty + \|G\|_{L^1}$. We have hence obtained a suitable regularization of equation (A.23). In the next Subsection, we will prove the existence of a solution to the regularized problem.

A.3 Existence theory for the pure emission-absorption case

A.3.1 Existence of solutions to the regularized problem in the Grey case

We are now ready to prove the existence of a solution to the regularized problem (A.25) for the Grey approximation. We start with the L^∞-estimate and we proceed exactly as before. Hence, for $D = \operatorname{diam}(\Omega)$, passing to spherical coordinates and using (A.26) we obtain

$$\|\mathcal{B}^\varepsilon(u)\|_\infty \leq \|u\|_\infty\left(1 - e^{-D\|\gamma\|_\infty}\right) + \|G\|_{L^1}.$$

Thus, for $K > \|G\|_{L^1}e^{D\|\gamma\|_\infty}$ we see that the operator $\mathcal{B}^\varepsilon$ maps continuously the set

$$\{u \in L^\infty(\Omega) : u \geq 0,\ \|u\|_\infty \leq K\}$$

to itself. Actually, it is a compact operator mapping the non-negative continuous functions bounded by K to the Hölder continuous functions. This is relevant because it allows us to apply the Schauder fixed-point theorem (cf. [49]).

To this end we assume now $u \in C(\Omega)$ and $u \geq 0$. By definition, we can extend it continuously up to the boundary $\partial\Omega$. Moreover, we extend by zero both functions u and $\gamma(u)$ outside $\bar{\Omega}$ such that the convolution $\gamma(u) * \phi_\varepsilon$ is smooth and well-defined. Let $x \in \Omega$ and $h \in \mathbb{R}^3$ with $x + h \in \Omega$. We estimate

$$\begin{aligned}
&|\mathcal{B}^\varepsilon(u)(x) - \mathcal{B}^\varepsilon(u)(x+h)| \\
&\quad\leq \Bigg| \int_\Omega d\eta\ \left(\gamma(u) * \phi_\varepsilon\right)(\eta) u(\eta) \frac{\exp\left(-\int_{\mathbb{R}^3} \gamma(u(\xi))\delta_{[x,\eta]} * \phi_\varepsilon(\xi) d\xi\right)}{4\pi|x-\eta|^2} \\
&\qquad - \int_\Omega d\eta\ \left(\gamma(u) * \phi_\varepsilon\right)(\eta) u(\eta) \frac{\exp\left(-\int_{\mathbb{R}^3} \gamma(u(\xi))\delta_{[x+h,\eta]} * \phi_\varepsilon(\xi) d\xi\right)}{4\pi|x+h-\eta|^2} \Bigg| \\
&\qquad + \left| \int_{\mathbb{S}^2} dn \exp\left(-\int_{\mathbb{R}^3} \gamma(u(\xi))\delta_{[x,y(x,n)]} * \phi_\varepsilon(\xi) d\xi\right) G(n) \right. \\
&\qquad \left. - \int_{\mathbb{S}^2} dn \exp\left(-\int_{\mathbb{R}^3} \gamma(u(\xi))\delta_{[x+h,y(x+h,n)]} * \phi_\varepsilon(\xi) d\xi\right) G(n) \right| \\
&\quad\leq \frac{1}{4\pi} \int_\Omega d\eta\ \left(\gamma(u) * \phi_\varepsilon\right)(\eta) u(\eta) \exp\left(-\int_{\mathbb{R}^3} \gamma(u(\xi))\delta_{[x,\eta]} * \phi_\varepsilon(\xi) d\xi\right) \\
&\qquad\qquad \times \left| \frac{1}{|x-\eta|^2} - \frac{1}{|x+h-\eta|^2} \right| \\
&\qquad + \int_\Omega d\eta\ \frac{\left(\gamma(u) * \phi_\varepsilon\right)(\eta) u(\eta)}{4\pi|x+h-\eta|^2} \left| \exp\left(-\int_{\mathbb{R}^3} \gamma(u(\xi))\delta_{[x,\eta]} * \phi_\varepsilon(\xi) d\xi\right) \right. \\
&\qquad\qquad \left. - \exp\left(-\int_{\mathbb{R}^3} \gamma(u(\xi))\delta_{[x+h,\eta]} * \phi_\varepsilon(\xi) d\xi\right) \right| \\
&\qquad + \int_{\mathbb{S}^2} dn\ G(n) \left| \exp\left(-\int_{\mathbb{R}^3} \gamma(u(\xi))\delta_{[x,y(x,n)]} * \phi_\varepsilon(\xi) d\xi\right) \right. \\
&\qquad\qquad \left. - \exp\left(-\int_{\mathbb{R}^3} \gamma(u(\xi))\delta_{[x+h,y(x+h,n)]} * \phi_\varepsilon(\xi) d\xi\right) \right| \\
&\qquad =: I + II + III. \qquad \text{(A.27)}
\end{aligned}$$

In order to estimate the integral term I, we proceed splitting it in two integrals

$$\begin{aligned}
I &\leq C(\|\gamma\|_\infty, K) \int_\Omega d\eta \left| \frac{1}{|x-\eta|^2} - \frac{1}{|x+h-\eta|^2} \right| \\
&\leq C \int_{\Omega\cap\{|x-\eta|\leq 2|h|\}} d\eta \left| \frac{1}{|x-\eta|^2} - \frac{1}{|x+h-\eta|^2} \right| \\
&\qquad + C \int_{\Omega\cap\{|x-\eta|> 2|h|\}} d\eta \left| \frac{1}{|x-\eta|^2} - \frac{1}{|x+h-\eta|^2} \right|.
\end{aligned}$$

If $|x - \eta| \leq 2|h|$ then also $|x + h - \eta| \leq 3|h|$ and hence

$$\int_{\Omega\cap\{|x-\eta|\leq 2|h|\}} d\eta \left| \frac{1}{|x-\eta|^2} - \frac{1}{|x+h-\eta|^2} \right| \leq 2 \int_{B_{3|h|}(0)} dy \frac{1}{|y|^2} = 24\pi|h|.$$

On the other hand, if $|x - \eta| > 2|h|$, then for $0 < s < 1$,

$$\left| \frac{1}{|x-\eta|^2} - \frac{1}{|x+h-\eta|^2} \right| \leq \frac{|h|^2 + 2|h||x-\eta|}{|x-\eta|^2|x+h-\eta|^2} \leq 2^{s-2}|h|^s \frac{1}{|x-\eta|^s|x+h-\eta|^2}.$$

Choosing now $s = \frac{1}{2}$ we see that $\frac{1}{|x-\cdot|^s} \in L^4(\Omega)$ and $\frac{1}{|x+h-\cdot|^2} \in L^{\frac{4}{3}}(\Omega)$. Hence

$$\int_{\Omega\cap\{|x-\eta|>2|h|\}} d\eta \left| \frac{1}{|x-\eta|^2} - \frac{1}{|x+h-\eta|^2} \right| \le C(\Omega)|h|^{\frac{1}{2}}.$$

Summarizing we get for a sufficiently small $|h| < 1$,

$$I \le C(\Omega, \|\gamma\|_\infty, K)|h|^{\frac{1}{2}}. \tag{A.28}$$

For the second term II we use the following three estimates which are the consequence of the smoothness of ϕ_ε

$$\int_0^{|\eta-x|} d\lambda \left| \phi_\varepsilon \left(z - x - \lambda \frac{\eta - x}{|\eta - x|} \right) - \phi_\varepsilon \left(z - x - h - \lambda \frac{\eta - x}{|\eta - x|} \right) \right| \le C(\phi_\varepsilon)|h||\eta - x|; \tag{A.29}$$

$$\begin{aligned} \int_0^{|\eta-x|} d\lambda & \left| \phi_\varepsilon \left(z - x - h - \lambda \frac{\eta - x}{|\eta - x|} \right) - \phi_\varepsilon \left(z - x - h - \lambda \frac{\eta - x - h}{|\eta - x - h|} \right) \right| \\ & \le C(\phi_\varepsilon) \frac{|x-\eta|}{2} \frac{\big|(\eta - x \pm h)|\eta - x - h| - (\eta - x - h)|\eta - x|\big|}{|\eta - x - h|} \\ & \le C(\phi_\varepsilon) \frac{|x-\eta|}{2} \left(\big||\eta - x - h| - |x - h|\big| + |h| \right) \le C(\phi_\varepsilon)|h||\eta - x|; \end{aligned} \tag{A.30}$$

and

$$\left| \int_{|\eta-x|}^{|\eta-x-h|} d\lambda \; \phi_\varepsilon \left(z - x - h - \lambda \frac{\eta - x - h}{|\eta - x - h|} \right) \right| \le C(\phi_\varepsilon)|h|. \tag{A.31}$$

Now, using the well-known inequality $|e^{-a} - e^{-b}| \le |a - b|$ for $a, b \ge 0$ and the definition of the line integrals as in (A.24) we see

$$\begin{aligned} II \le C(\|\gamma\|_\infty, \phi_\varepsilon, K) & \int_\Omega d\eta \; \frac{1}{|\eta - x - h|^2} \int_{\mathbb{R}^3} dz \; \gamma(u)(z) \\ \times & \left| \int_0^{|\eta-x|} d\lambda \; \phi_\varepsilon \left(z - x - \lambda \frac{\eta - x}{|\eta - x|} \right) - \phi_\varepsilon \left(z - x - h - \lambda \frac{\eta - x - h}{|\eta - x - h|} \right) \right| \\ & \le C(\|\gamma\|_\infty, \phi_\varepsilon, K, \Omega)\|\gamma\|_\infty |h|, \end{aligned} \tag{A.32}$$

where in the last step we used all three estimates (A.29), (A.30) and (A.31).

The last integral term III is estimated in a similar way as we did for II. Since we assumed that $\partial\Omega$ is C^2 and has positive curvature, we notice that there exists a constant $C(\Omega)$ depending on the curvature of the domain, such that if $|h| < 1$ is sufficiently small then

$$|s(x, n) - s(x + h, n)| \le C(\Omega)|h|^{\frac{1}{2}}, \tag{A.33}$$

for all $n \in \mathbb{S}^2$. Estimate (A.33) is the result of a geometrical argument considering the worst case scenario when n is close to tangent to the boundary at the point $x - s(x, n)n$ or $x + h - s(x + h, n)n$ taking into account that the curvature of $\partial\Omega$ is strictly positive. Hence,

$$\begin{aligned} & \left| \int_0^{s(x,n)} d\lambda \; \phi_\varepsilon(z - x + \lambda n) - \int_0^{s(x+h,n)} d\lambda \; \phi_\varepsilon(z - x - h + \lambda n) \right| \\ & \quad \le \int_0^{\min(s(x,n),s(x+h,n))} d\lambda \, |\phi_\varepsilon(z - x + \lambda n) - \phi_\varepsilon(z - x - h + \lambda n)| \\ & \qquad + \int_{\min(s(x,n),s(x+h,n))}^{\max(s(x,n),s(x+h,n))} d\lambda \, |\phi_\varepsilon| \le C(\phi_\varepsilon, \Omega)|h|^{\frac{1}{2}}. \end{aligned}$$

Hence, we conclude

$$III \leq C(\|G\|_\infty, \Omega, \phi_\varepsilon)\|\gamma\|_\infty |h|^{\frac{1}{2}}. \tag{A.34}$$

Estimates (A.28), (A.32) and (A.34) together imply the estimate

$$|\mathcal{B}^\varepsilon(u)(x) - \mathcal{B}^\varepsilon(u)(x+h)| \leq C\left(G, \gamma, K, \phi_\varepsilon, \Omega\right)|h|^{\frac{1}{2}},$$

for all $x \in \Omega$ and $|h| < 1$ sufficiently small. We have just proved then that $\mathcal{B}^\varepsilon$ maps continuous functions to Hölder continuous functions. It is therefore a compact operator. As we have already noticed it is also a continuous operator. Then Schauder's fixed-point theorem implies the existence of a fixed-point $u_\varepsilon \in C(\Omega)$ with $0 \leq u_\varepsilon \leq K$ such that $u_\varepsilon = \mathcal{B}^\varepsilon(u_\varepsilon)$. This concludes the proof of the existence of a solution u_ε for the regularized problem (A.25). In the next section, we will provide a general L^2 compactness theory based on some auxiliary meausres defined on $\mathbb{S}^2$ to prove the existence of the original problem.

A.3.2 Compactness theory for operators defined by means of some line integrals

We prove now Proposition A.1.

Proof of Proposition A.1. Without loss of generality we can assume $L = 1$ and $M = 1$. We start writing φ_j in its Fourier series form as

$$\varphi_j = \sum_{k \in \pi\mathbb{Z}^3} a_k^j e^{ik\cdot x}.$$

We denote by μ_j the measure associated to φ_j and defined by

$$\mu_j = \sum_{k \in \pi\mathbb{Z}^3} \left|a_k^j\right|^2 \delta_{\frac{k}{|k|}} \in \mathcal{M}_+\left(\mathbb{S}^2\right).$$

We will work with the auxiliary measures defined on $\mathbb{S}^2$ given for $R > 0$ by

$$\mu_j^R = \sum_{\substack{k \in \pi\mathbb{Z}^3 \\ |k| > R}} \left|a_k^j\right|^2 \delta_{\frac{k}{|k|}} \in \mathcal{M}_+\left(\mathbb{S}^2\right).$$

Moreover we see that

$$\mu_j^R(\mathbb{S}^2) \leq \mu_j(\mathbb{S}^2) = \|\varphi_j\|_{L^2} \leq 8,$$

where we used that $\left|\Pi^3\right| = 8$.

We can now rewrite using the absolute convergence of the series and computing the integrals

$$\begin{aligned}
L_n[\varphi_j](x) &= \int_{-1}^{1} d\lambda \sum_{k \in \pi\mathbb{Z}^3} a_k^j e^{ik\cdot(x-\lambda n)} \\
&= \sum_{k \in \pi\mathbb{Z}^3} a_k^j e^{ik\cdot x} \int_{-1}^{1} d\lambda e^{-ik\cdot\lambda n} = \sum_{k \in \pi\mathbb{Z}^3} 2a_k^j \frac{\sin(k\cdot n)}{k\cdot n} e^{ik\cdot x} \\
&= \sum_{\substack{k \in \pi\mathbb{Z}^3 \\ |k| \leq R}} 2a_k^j \frac{\sin(k\cdot n)}{k\cdot n} e^{ik\cdot x} + \sum_{\substack{k \in \pi\mathbb{Z}^3 \\ |k| > R}} 2a_k^j \frac{\sin(k\cdot n)}{k\cdot n} e^{ik\cdot x}.
\end{aligned}$$

Since the first sum is finite, the first term on the right hand side is compact for every fixed $R > 0$. We now consider the contribution due to the second term. We define the auxiliary measure associated to $L_n[\varphi_j]$ that will be denoted by $\nu^R_{n,j}$. More precisely we define it by means of

$$\nu^R_{n,j}(\omega) =: \sum_{\substack{k\in\pi\mathbb{Z}^3\\ |k|>R}} 4\left|a^j_k\right|^2 \left|\frac{\sin(k\cdot n)}{k\cdot n}\right|^2 \delta_{\frac{k}{|k|}}(\omega). \tag{A.35}$$

Again we see $\nu^R_{n,j} \in \mathcal{M}_+\left(\mathbb{S}^2\right)$ with $\nu^R_{n,j}\left(\mathbb{S}^2\right) \leq 32$. Notice that $\nu^R_{n,j} \leq 4\mu^R_j$ for all $n \in \mathbb{S}^2$ as a measure.

We notice also that by definition $\nu^R_{n,j}\mid_{\{\omega\cdot n=0\}} = 4\mu^R_j\mid_{\{\omega\cdot n=0\}}$ since $\lim_{x\to 0}\frac{\sin(x)}{x} = 1$. Moreover, we can write

$$\nu^R_{n,j} = \nu^R_{n,j}\mid_{\{0\leq|\omega\cdot n|<\kappa\}} + \sum_{\substack{k\in\pi\mathbb{Z}^3\\ |k|>R}} 4\left|a^j_k\right|^2 \left|\frac{\sin(k\cdot n)}{k\cdot n}\right|^2 \delta_{\frac{k}{|k|}}(\omega)\mid_{\{|\omega\cdot n|\geq\kappa\}}. \tag{A.36}$$

On one hand we have

$$\nu^R_{n,j}\mid_{\{0\leq|\omega\cdot n|<\kappa\}} \leq 4\mu^R_j\mid_{\{0<|\omega\cdot n|<\kappa\}}$$

and also defining $f_\kappa(\omega, n) = \chi_{\{\omega:0\leq|\omega\cdot n|<\kappa\}}(\omega)$ we have

$$f_\kappa(\omega, n) = \chi_{\{(\omega,n):0\leq|\omega\cdot n|<\kappa\}}(\omega, n) = \chi_{\{n:0\leq|\omega\cdot n|<\kappa\}}(n)$$

hence we compute

$$\begin{aligned}\int_{\mathbb{S}^2} dn \int_{\mathbb{S}^2} d\nu^R_{n,j}(\omega)\mid_{\{0<|\omega\cdot n|<\kappa\}} &\leq 4\int_{\mathbb{S}^2} dn \int_{\mathbb{S}^2} d\mu^R_j(\omega)\chi_{\{(\omega,n):0<|\omega\cdot n|<\kappa\}}(\omega, n)\\ &= 4\int_{\mathbb{S}^2} d\mu^R_j(\omega)\int_{\mathbb{S}^2} dn\chi_{\{n:0\leq|\omega\cdot n|<\kappa\}}(n) \leq 128\pi\kappa \to 0 \end{aligned} \tag{A.37}$$

uniformly in $j \in \mathbb{N}$ and $R \in \pi\mathbb{N}$. For the first inequality we used that $\nu^R_{n,j} \leq 4\mu^R_j$, after that we changed the order of integration using the boundedness of the measures and we concluded using $\mu^R_j\left(\mathbb{S}^2\right) \leq 8$ as well as

$$\int_{\mathbb{S}^2} dn\chi_{\{0<|\omega\cdot n|<\kappa\}}(n) < 4\pi\kappa.$$

On the other hand we have for fixed $\kappa > 0$

$$\begin{aligned}&\int_{\mathbb{S}^2} dn \int_{\mathbb{S}^2} \sum_{\substack{k\in\pi\mathbb{Z}^3\\ |k|>R}} 4\left|a^j_k\right|^2 \left|\frac{\sin(k\cdot n)}{k\cdot n}\right|^2 \delta_{\frac{k}{|k|}}(\omega)\mid_{\{|\omega\cdot n|\geq\kappa\}}\\ &\qquad\leq \int_{\mathbb{S}^2} dn \int_{\mathbb{S}^2} \frac{4}{R^2\kappa^2} \sum_{\substack{k\in\pi\mathbb{Z}^3\\ |k|>R}} \left|a^{j_l}_k\right|^2 \delta_{\frac{k}{|k|}}(\omega)\mid_{\{|\omega\cdot n|\geq\kappa\}}\\ &\qquad\leq \int_{\mathbb{S}^2} dn \int_{\mathbb{S}^2} \frac{4}{R^2\kappa^2} d\mu_j(\omega)\mid_{\{|\omega\cdot n|\geq\kappa\}} \leq \frac{128\pi}{R^2\kappa^2} \underset{R\to\infty}{\longrightarrow} 0. \end{aligned} \tag{A.38}$$

uniformly in $j \in \mathbb{N}$. We used indeed that if $|k|\omega = k$, $|k| > R$ and $|\omega\cdot n| > \kappa$, then $|k\cdot n| > R\kappa$. Moreover, we can always bound the measure $\mu^R_j \leq \mu_j$ and $\mu_j(\mathbb{S}^2) \leq 8$.

Hence, we conclude $\int_{\mathbb{S}^2} dn \int_{\mathbb{S}^2} d\nu^R_{n,j_l}(\omega) \to 0$ as $R \to 0$. Indeed, let $\varepsilon > 0$. We chose $0 < \kappa_0 < \frac{1}{256\pi\varepsilon}$. Then testing according to (A.38) we define $R_0(\varepsilon) > \frac{16\sqrt{\pi}}{\kappa_0\sqrt{\varepsilon}}$ such that for all $R \geq R_0$ we have

$$\int_{\mathbb{S}^2} dn \int_{\mathbb{S}^2} d\nu^R_{n,j} \mid_{\{|\omega\cdot n|\geq\kappa_0\}} (\omega) < \frac{\varepsilon}{2}. \tag{A.39}$$

Combining (A.37) for κ_0 and (A.39) we obtain

$$0 \leq \int_{\mathbb{S}^2} dn \int_{\mathbb{S}^2} d\nu^R_{n,j}(\omega) < \varepsilon \tag{A.40}$$

for all $R \geq R_0$ and most importantly for all $j \in \mathbb{N}$.

We are now ready to show the compactness of the sequence $T_m[\varphi_j]$ in $L^2(\Pi^3)$. Since T_m is a bounded operator, Π^3 is a compact subset of $\mathbb{R}^3$, we only have to show the equi-integrability condition (cf. [23]). We recall that $\|L_n[\varphi]\|_\infty \leq 2$. Hence, let $x, h \in \Pi^3$ using Jensen's inequality we compute

$$\begin{aligned} |T_m[\varphi_j](x) - T_m[\varphi_j](x+h)|^2 &= \left|\int_{\mathbb{S}^2} dn\, [(L_n[\varphi_j](x))^m - (L_n[\varphi_j](x+h))^m]\right|^2 \\ &\leq \left(m2^{m-1}\right)^2 \left(\int_{\mathbb{S}^2} dn\, |L_n[\varphi_j](x) - L_n[\varphi_j](x+h)|\right)^2 \\ &\leq \left(4\pi m2^{m-1}\right)^2 \int_{\mathbb{S}^2} dn \left|\sum_{k\in\pi\mathbb{Z}^3} 2a^j_k \frac{\sin(k\cdot n)}{k\cdot n} e^{ik\cdot x}\left(1 - e^{ik\cdot h}\right)\right|^2. \end{aligned} \tag{A.41}$$

Since $\left\{\frac{1}{8}e^{ik\cdot x}\right\}_{k\in\pi\mathbb{Z}^3}$ is an orthonormal basis of Π^3 we obtain denoting by $C_m = 8^2 4\pi m 2^{m-1}$,

$$\begin{aligned} &\int_{\Pi^3} dx\, |T_m[\varphi_j](x) - T_m[\varphi_j](x+h)|^2 \\ &\qquad\leq \frac{C_m^2}{64}\int_{\Pi^3} dx \int_{\mathbb{S}^2} dn\ |L_n[\varphi_j](x) - L_n[\varphi_j](x+h)|^2 \\ &\qquad= C_m^2 \int_{\mathbb{S}^2} dn \sum_{k\in\pi\mathbb{Z}^3} 4\left|a^j_k\right|^2 \left|\frac{\sin(k\cdot n)}{k\cdot n}\right|^2 \left|\left(1 - e^{ik\cdot h}\right)\right|^2 \\ &\qquad\leq 32\pi C_m^2 \sum_{\substack{k\in\pi\mathbb{Z}^3 \\ |k|\leq R}} 4\left|a^j_k\right|^2 |k|^2|h|^2 + 32 C_m^2 \int_{\mathbb{S}^2} dn \int_{\mathbb{S}^2} d\nu^R_{n,j}(\omega). \end{aligned} \tag{A.42}$$

Let $\varepsilon > 0$. We have shown that there exists $R_0 > 0$ such that

$$\int_{\mathbb{S}^2} dn \int_{\mathbb{S}^2} d\nu^R_{n,j}(\omega) < \frac{\varepsilon}{64 C_m^2},$$

for all $R \geq R_0$ and for all $j \in \mathbb{N}$. Taking in (A.42) $R = R_0$ and $h_0 = \frac{\sqrt{\varepsilon}}{C_m 32\sqrt{2}R_0}$ we obtain the desired equi-integrability condition

$$\begin{aligned} &\int_{\Pi^3} dx\, |T_m[\varphi_j](x) - T_m[\varphi_j](x+h)|^2 \\ &\qquad\qquad\leq C_m \int_{\Pi^3} dx \int_{\mathbb{S}^2} dn\ |L_n[\varphi_j](x) - L_n[\varphi_j](x+h)|^2 < \varepsilon \end{aligned}$$

for all $|h| < h_0$. This concludes the proof of Proposition A.1. □

We can get a stronger result for the compactness of line integrals of functions depending also on the direction $n \in \mathbb{S}^2$. We will use it in the proof of existence of solution to the equation containing also the scattering term.

Corollary A.1. *Let* $\Pi^3 = [-L, L]^3$ *and* $(\varphi(x,n)_j)_{j\in\mathbb{N}} \in C\left(\mathbb{S}^2, L^2\left(\Pi^3\right) \cap L^\infty\left(\Pi^3\right)\right)$ *be a sequence of periodic functions with* $\sup_{n\in\mathbb{S}^2} \|\varphi_j(\cdot,n)\|_{L^\infty(\Pi^3)} \leq M$. *Assume also* $\|\varphi_j(\cdot,n_1) - \varphi_j(\cdot,n_2)\|_{L^\infty(\Pi^3)} \leq \sigma(d(n_1,n_2)) \to 0$ *uniformly in* $j \in \mathbb{N}$ *if* $d(n_1,n_2) \to 0$, *where* d *is the metric on the sphere and* $\sigma \in C(\mathbb{R}_+, \mathbb{R}_+)$ *with* $\sigma(0) = 0$ *is a uniform modulus of continuity. For* $n \in \mathbb{S}^2$ *and* $m \in \mathbb{N}$ *we define the operators* L_n *and* T_m *by*

$$L_n[\varphi](x,\omega) =: \int_{-1}^{1} d\lambda\ \varphi(x - \lambda n, \omega) \quad \textit{and} \quad T_m[\varphi](x) =: \int_{\mathbb{S}^2} dn\, (L_n[\varphi](x,n))^m$$

Then for every $m \in \mathbb{N}$ *the sequence* $(T_m[\varphi_j])_j$ *is compact in* $L^2\left(\Pi^3\right)$.

Proof. Without loss of generality we can assume again $L = 1$ and $M = 1$. This statement is a corollary to Proposition A.1 and the Besicovitch covering Lemma. Since $\mathbb{S}^2$ with the geodesic metric is a Riemannian Manifold of class greater than 2, it is also a directionally (1,C)-limited metric space for a fixed constant $C > 0$. See [52] for further reference. This implies that the Federer-Besicovitch covering Lemma (a generalization of the well-known Lemma in $\mathbb{R}^n$) applies. Hence, for any family $\mathcal{F}_\delta = \{B_\delta(n)\}_{n\in\mathbb{S}^2}$ of balls with radius $\delta < 1$ there exists subfamilies $\mathcal{G}_k \subset \mathcal{F}_\delta$ for $1 \leq k \leq 2C+1$ consisting of disjoint balls such that

$$\mathbb{S}^2 \subset \bigcup_{k=1}^{2C+1} \bigsqcup_{B\in\mathcal{G}_k} B,$$

where $\bigsqcup$ denotes the disjoint union. Since $\mathbb{S}^2$ is compact there exists also a finite cover, i.e., the subfamilies $\mathcal{G}_k$ are finite. Hence,

$$\mathbb{S}^2 \subset \bigcup_{k=1}^{2C+1} \bigsqcup_{1\leq i\leq N(k,\delta)} B_\delta(n_{k,i}).$$

Let now $\varepsilon > 0$ and $h \in \mathbb{R}^3$. Similarly as in equation (A.41) we estimate using first Jensen's inequality

$$\begin{aligned}
&\int_{\Pi^3} dx\, |T_m[\varphi_j](x) - T_m[\varphi_j](x+h)|^2 \\
&= \int_{\Pi^3} dx \left| \int_{\mathbb{S}^2} dn \left[\left(\int_{-1}^{1} d\lambda\ \varphi_j(x - \lambda n, n) \right)^m - \left(\int_{-1}^{1} d\lambda\ \varphi_j(x + h - \lambda n, n) \right)^m \right] \right|^2 \\
&\leq \frac{C_m}{4\pi} \int_{\Pi^3} dx \left[\int_{\mathbb{S}^2} dn \left| \int_{-1}^{1} d\lambda \left[\varphi_j(x - \lambda n, n) - \varphi_j(x + h - \lambda n, n)\right] \right| \right]^2 \\
&\leq C_m \int_{\Pi^3} dx \int_{\mathbb{S}^2} dn \left| \int_{-1}^{1} d\lambda \left[\varphi_j(x - \lambda n, n) - \varphi_j(x + h - \lambda n, n)\right] \right|^2 \\
&= C_m \int_{\Pi^3} dx \int_{\bigcup_{k=1}^{2C+1} \bigsqcup_{i=1}^{N(k,\delta)} B_\delta(n_{k,i})} dn \left| \int_{-1}^{1} d\lambda \left[\varphi_j(x - \lambda n, n) - \varphi_j(x + h - \lambda n, n)\right] \right|^2
\end{aligned} \tag{A.43}$$

$$
\begin{aligned}
&\le C_m \sum_{k=1}^{2C+1} \int_{\Pi^3} dx \int_{\bigsqcup_{i=1}^{N(k,\delta)} B_\delta(n_{k,i})} dn \left| \int_{-1}^{1} d\lambda \left[\varphi_j(x-\lambda n, n) - \varphi_j(x+h-\lambda n, n)\right] \right|^2 \\
&\le C_m \sum_{k=1}^{2C+1} \int_{\Pi^3} dx \sum_{i=1}^{N(k,\delta)} \int_{B_\delta(n_{k,i})} dn \left| \int_{-1}^{1} d\lambda \left[\varphi_j(x-\lambda n, n_{k,i}) - \varphi_j(x+h-\lambda n, n_{k,i})\right] \right|^2 \\
&\quad + C_m \sum_{k=1}^{2C+1} \int_{\Pi^3} dx \sum_{i=1}^{N(k,\delta)} \int_{B_\delta(n_{k,i})} dn 4\sigma(\delta)^2 \\
&\le C_m N(k,\delta) \sum_{k=1}^{2C+1} \sum_{i=1}^{N(k,\delta)} \int_{\Pi^3} dx \int_{\mathbb{S}^2} dn \left| \int_{-1}^{1} d\lambda \left[\varphi_j(x-\lambda n, n_{k,i}) - \varphi_j(x+h-\lambda n, n_{k,i})\right] \right|^2 \\
&\quad + 4C_m(2C+1) \int_{\Pi^3} dx \int_{\mathbb{S}^2} dn \sigma(\delta)^2,
\end{aligned}
$$

where in the last inequality we used that the balls $\{B_\delta(n_{k,i})\}_{1\le i\le N(k,\delta)}$ are disjoint. We choose thus $\delta_0 > 0$ such that $4C_m(2C+1)\sigma(\delta_0) < \frac{\varepsilon}{64}$. Lemma A.41 with equation (A.10) implies for any (k,i) with $1 \le k \le 2C+1$ and $1 \le i \le N(k,\delta_0)$ the existence of some $h_0(k,i)$ such that

$$
\int_{\Pi^3} dx \int_{\mathbb{S}^2} dn \left| \int_{-1}^{1} d\lambda \left[\varphi_j(x-\lambda n, n_{k,i}) - \varphi_j(x+h-\lambda n, n_{k,i})\right] \right|^2 < \frac{\varepsilon}{2C_m(2C+1)} \frac{1}{N(k,\delta_0)^2}
$$

for all $|h| < h_0(i,k)$ and for all $j \in \mathbb{N}$. Hence, choosing $h_0 = \min_{\substack{1\le k\le 2C+1 \\ 1\le i\le N(k,\delta_0)}} \{h_0(k,i)\}$ we conclude

$$
\int_{\Pi^3} dx \left| T_m[\varphi_j](x) - T_m[\varphi_j](x+h) \right|^2 < \varepsilon
$$

for all $|h| < h_0$ and all $j \in \mathbb{N}$. Hence, the sequence $(T_m[\varphi_j])_j$ is compact in $L^2(\Pi^3)$. □

We extend now Proposition A.1 to other more general type of operators involving line integrals. To this end we define for $\varphi \in L^\infty(\Pi^3)$ and $0 \le s < t \le \frac{L}{2}$, $x \in \Pi^3$ and $n \in \mathbb{S}^2$ the line integral

$$
L_{n,t-s}[\varphi](x) = \int_s^t d\lambda \varphi(x - \lambda n). \tag{A.44}
$$

Then the following lemma holds.

Lemma A.1. *Under the notation above let $(\varphi_j)_{j\in\mathbb{N}} \in L^\infty(\Pi^3)$ be a sequence of periodic functions with $\|\varphi_j\|_\infty \le M$. Let $\varepsilon > 0$, then there exists $h_0 > 0$ such that*

$$
\begin{aligned}
\int_{\Pi^3} dx &\left| ⨍_{\mathbb{S}^2} dn \left(L_{n,t-s}[\varphi_j](x) - L_{n,t-s}[\varphi_j](x+h) \right) \right|^2 \\
&\le \int_{\Pi^3} dx ⨍_{\mathbb{S}^2} dn \left| L_{n,t-s}[\varphi_j](x) - L_{n,t-s}[\varphi_j](x+h) \right|^2 < \varepsilon
\end{aligned} \tag{A.45}
$$

for all $|h| < h_0$ and $j \in \mathbb{N}$ and uniformly in t, s. This equi-integrability condition implies as in Proposition A.1 the compactness of any sequence $⨍_{\mathbb{S}^2} dn \left(L_{n,t-s}[\varphi_j](x)\right)^m$ in $L^2(\Pi^3)$ for any fixed $m \in \mathbb{N}$.

Proof. We expand the functions φ_j in their respectively Fourier series as

$$
\varphi_j(x) = \sum_{k\in\frac{\pi}{L}\mathbb{Z}^3} a_k^j e^{ik\cdot x},
$$

hence their associated auxiliary measures μ_j^R are given for $R \in \frac{\pi}{L}\mathbb{Z}$ by

$$\mu_j^R = \sum_{\substack{k \in \frac{\pi}{L}\mathbb{Z}^3 \\ |k|>R}} |a_k^j|^2 \delta_{\frac{k}{|k|}} \in \mathcal{M}_+\left(\mathbb{S}^2\right). \tag{A.46}$$

For any $s, t \in \left[0, \frac{L}{2}\right]$ with $s < t$ we compute for $k \in \frac{\pi}{L}\mathbb{Z}^3$ and $n \in \mathbb{S}^2$

$$\int_s^t d\lambda e^{-ik\cdot\lambda n} = \frac{e^{-it(k\cdot n)} - e^{-is(k\cdot n)}}{-ik\cdot n} = 2e^{-i(t+s)\frac{(k\cdot n)}{2}} \frac{\sin\left((t-s)\frac{(k\cdot n)}{2}\right)}{k\cdot n}.$$

Therefore, the auxiliary measures associated to the operators $L_{n,t-s}$ acting respectively on φ_j are given by

$$\nu_{n,j}^{t-s,R}(\omega) = \sum_{\substack{k \in \frac{\pi}{2D+2}\mathbb{Z}^3 \\ |k|>R}} |a_k^j|^2 4 \left| \frac{\sin\left((t-s)\frac{(k\cdot n)}{2}\right)}{k\cdot n} \right|^2 \delta_{\frac{k}{|k|}} \in \mathcal{M}_+\left(\mathbb{S}^2\right). \tag{A.47}$$

Since $\left|\frac{\sin(ax)}{x}\right| \leq a$ we notice that the auxiliary measures are uniformly bounded and satisfy

$$\nu_{n,j}^{t-s,R} \leq (t-s)^2 \mu_j^R \leq \frac{L^2}{4}\mu_j^R \leq \frac{L^2}{4}\mu_j.$$

Hence, exactly as we have argued in Proposition A.1 we see also in this case that for any $0 < \kappa < 1$

$$\int_{\mathbb{S}^2} dn \int_{\mathbb{S}^2} d\nu_{n,j}^{t-s,R}(\omega)\chi_{\{0\leq|\omega\cdot n|<\kappa\}} \leq \frac{L^2}{4}C(L,M)\kappa \to 0,$$

as $\kappa \to 0$ uniformly in $j \in \mathbb{N}$ and $t, s \in \left[0, \frac{L}{2}\right]$. Moreover, estimating the sine function by 1 we also have for fixed $\kappa > 0$

$$\int_{\mathbb{S}^2} dn \int_{\mathbb{S}^2} d\nu_{n,j}^{t-s,R}(\omega)\chi_{\{|\omega\cdot n|\geq\kappa\}} \leq \frac{C(L,M)}{R^2\kappa^2} \to 0,$$

s $R \to \infty$ uniformly in $j \in \mathbb{N}$ and $t, s \in \left[0, \frac{L}{2}\right]$. Thus, we conclude once again that for any $\varepsilon > 0$ there exists some $R_0(\varepsilon) > 0$ (independent of $j \in \mathbb{N}$ and $t, s \in \left[0, \frac{L}{2}\right]$) such that

$$\int_{\mathbb{S}^2} dn \int_{\mathbb{S}^2} d\nu_{n,j}^{t-s,R}(\omega) < \varepsilon,$$

for all $R \geq R_0$, for all $j \in \mathbb{N}$ and for all $t, s \in \left[0, \frac{L}{2}\right]$.

Let us define $C_L = \left|\Pi^3\right| = (2L)^3$. We can write for any $t, s \in \left[0, \frac{L}{2}\right]$ with $s < t$ using first

Jensen's inequality and secondly that $\left\{\frac{1}{C_L}e^{ik\cdot x}\right\}_{k\in\frac{\pi}{L}\mathbb{Z}^3}$ form an orthonormal basis of Π^3

$$\begin{aligned}\int_{\Pi_D^3}\left|\fint_{\mathbb{S}^2} dn\, L_{n,t-s}[\varphi_j](x)-L_{n,t-s}[\varphi_j](x+h)\right|^2 \\ \leq \int_{\Pi_D^3}\fint_{\mathbb{S}^2} dn\left|L_{n,t-s}[\varphi_j](x)-L_{n,t-s}[\varphi_j](x+h)\right|^2 \\ \leq C_L^2\sum_{\substack{k\in\frac{\pi}{L}\mathbb{Z}^3\\ |k|\leq R}}\fint_{\mathbb{S}^2} dn\left|a_k^j\right|^2 4\left|\frac{\sin\left((t-s)\frac{(k\cdot n)}{2}\right)}{k\cdot n}\right|^2\left|1-e^{ik\cdot h}\right|^2+4C_L^2\fint_{\mathbb{S}^2} dn\int_{\mathbb{S}^2} d\nu_{n,j}^{(t-s),R}(\omega) \\ \leq C_L^3M^2(t-s)^2R^2|h|^2+4C_L^2\fint_{\mathbb{S}^2} dn\int_{\mathbb{S}^2} d\nu_{n,j}^{(t-s),R}(\omega) \\ \leq C_L^3M^2\frac{L^2}{4}R^2|h|^2+4C_L^2\fint_{\mathbb{S}^2} dn\int_{\mathbb{S}^2} d\nu_{n,j}^{(t-s),R}(\omega).\end{aligned}$$

Hence, taking $R=R_0\left(\frac{\varepsilon}{8C_L^2}\right)$ and $h_0=\frac{2\sqrt{\varepsilon}}{MC_LR_0L\sqrt{C_L}}$ we conclude the desired equi-integrability result

$$\begin{aligned}\int_{\Pi_D^3}\left|\fint_{\mathbb{S}^2} dn\, L_{n,t-s}[\varphi_j](x)-L_{n,t-s}[\varphi_j](x+h)\right|^2 \\ \leq \int_{\Pi_D^3}\fint_{\mathbb{S}^2} dn\left|L_{n,t-s}[\varphi_j](x)-L_{n,t-s}[\varphi_j](x+h)\right|^2<\varepsilon\end{aligned}$$

for all $|h|<h_0$ uniformly in$j\in\mathbb{N}$ and $t,s\in\left[0,\frac{L}{2}\right]$. □

A.3.3 Proof of Theorems A.1 and A.3

We can now prove Theorems A.3 and A.1. A crucial step will be to adapt Proposition A.1 in order to show the compactness of the operators $\mathcal{B}^\varepsilon$ instead of the operator defined only by one line integral.

Proof of Theorem A.3. We first extend by 0 the function u_ε and $\gamma(u_\varepsilon)$. Assuming without loss of generality that $0\in\Omega$, since ϕ_ε has compact support in $B_\varepsilon(0)\subset B_1(0)$ for all $\varepsilon<1$ we see that $\gamma(u_\varepsilon)*\phi_\varepsilon$ and u_ε have both support contained in $[-D-1,D+1]^3$, where we denote by $D=\operatorname{diam}(\Omega)$. Let us extend periodically in $\mathbb{R}^3$ both functions u_ε and $\gamma(u_\varepsilon)*\phi_\varepsilon$ in $\Pi_D^3=:[-2D-2,2D+2]^3$. Then we see that

$$\int_\Omega dx\int_{\mathbb{S}^2} dn\int_0^{s(x,n)} u_\varepsilon(x-rn)dr=\int_\Omega dx\int_{\mathbb{S}^2} dn\int_0^D u_\varepsilon(x-rn)dr.$$

With the same notation of Lemma A.1 we consider the operators $L_{n,r-s}[\varphi](x)$ acting on $\varphi\in L^\infty(\Pi_D^3)$, $n\in\mathbb{S}^2$ and $x\in\Pi_D^3$ given by (A.44). In the case $s=0$ we simplify the notation by $L_{n,r-0}[\varphi](x)=L_{n,r}[\varphi](x)$. Using (A.24) and the radial symmetry of ϕ_ε we see

$$\int_{\mathbb{R}^3} d\xi\gamma(u_\varepsilon)(\xi)\delta_{[x,x-rn]}*\phi_\varepsilon(\xi)=\int_0^r d\lambda\left(\gamma(u_\varepsilon)*\phi_\varepsilon\right)(x-\lambda n)=L_{n,r}\left[\gamma(u_\varepsilon)*\phi_\varepsilon\right](x).$$

Thus, we can write the operator $\mathcal{B}^{\varepsilon}(u_{\varepsilon})$ in the following way changing the variables according to $\eta = x - rn$

$$\begin{aligned}\mathcal{B}^{\varepsilon}(u_{\varepsilon})(x) &= \int_0^D dr \fint_{\mathbb{S}^2} dn \left[(\gamma(u_\varepsilon) * \phi_\varepsilon) u_\varepsilon\right](x - rn) \exp\left(-L_{n,r}\left(\gamma(u_\varepsilon) * \phi_\varepsilon\right)(x)\right) \\ &\qquad + \int_{\mathbb{S}^2} dn\ G(n) \exp\left(-L_{n,s(x,n)}\left(\gamma(u_\varepsilon) * \phi_\varepsilon\right)(x)\right) \\ &= \int_0^D dr \fint_{\mathbb{S}^2} dn \left[(\gamma(u_\varepsilon) * \phi_\varepsilon) u_\varepsilon\right](x - rn) \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left(L_{n,r}\left(\gamma(u_\varepsilon) * \phi_\varepsilon\right)(x)\right)^m \\ &\qquad + \int_{\mathbb{S}^2} dn\ G(n) \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left(L_{n,s(x,n)}\left(\gamma(u_\varepsilon) * \phi_\varepsilon\right)(x)\right)^m . \end{aligned} \tag{A.48}$$

Since the sequence $\mathcal{B}^{\varepsilon}(u_{\varepsilon})$ is uniformly bounded by K in $L^{\infty}(\Omega)$, and thus by $|\Omega|^{\frac{1}{2}}K$ in $L^2(\Omega)$, for the compactness we need again to show only the equi-integrability. Let now $h \in \mathbb{R}^3$ with $|h| < \frac{1}{2}$ and $\varepsilon < \frac{1}{2}$. Since we extended by 0 the function $u_\varepsilon = \mathcal{B}^{\varepsilon}(u_{\varepsilon})$ outside Ω it is true that $\mathcal{B}^{\varepsilon}(u_{\varepsilon})(x+h) = \mathcal{B}^{\varepsilon}(u_{\varepsilon})(x+h)\chi_{\{x+h\in\Omega\}}(x)$. Hence, we multiply by this characteristic function also the integral definition of the operator as in (A.48), this guarantees the well-definiteness of the function at $x + h \notin \Omega$. We thus compute using Jensen's inequality

$$\begin{aligned}&\int_\Omega dx \left|\mathcal{B}^{\varepsilon}(u_{\varepsilon})(x) - \mathcal{B}^{\varepsilon}(u_{\varepsilon})(x+h)\right|^2 \\ &\leq C \int_\Omega dx \fint_{\mathbb{S}^2} dn \Bigg| \int_0^D dr \left[(\gamma(u_\varepsilon) * \phi_\varepsilon) u_\varepsilon\right](x - rn) \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left(L_{n,r}\left(\gamma(u_\varepsilon) * \phi_\varepsilon\right)(x)\right)^m \\ &- \int_0^D dr \chi_{\{x+h\in\Omega\}} \left[(\gamma(u_\varepsilon) * \phi_\varepsilon) u_\varepsilon\right](x + h - rn) \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left(L_{n,r}\left(\gamma(u_\varepsilon) * \phi_\varepsilon\right)(x+h)\right)^m \Bigg|^2 \\ &\qquad + C(G) \int_\Omega dx \int_{\mathbb{S}^2} dn\ G(n) \Bigg| \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left(L_{n,s(x,n)}\left(\gamma(u_\varepsilon) * \phi_\varepsilon\right)(x)\right)^m \\ &\qquad\qquad - \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \chi_{\{x+h\in\Omega\}} \left(L_{n,s(x+h,n)}\left(\gamma(u_\varepsilon) * \phi_\varepsilon\right)(x+h)\right)^m \Bigg|^2 .\end{aligned}$$

Applying now the triangle inequality we can further estimate

$$\begin{aligned}&\int_\Omega dx \left|\mathcal{B}^{\varepsilon}(u_{\varepsilon})(x) - \mathcal{B}^{\varepsilon}(u_{\varepsilon})(x+h)\right|^2 \\ &\quad\leq C\|\gamma\|_\infty^2 K^2 \int_\Omega dx \chi_{\{x+h\in\Omega\}} \int_0^D dr \fint_{\mathbb{S}^2} dn \Bigg| \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left(L_{n,r}\left(\gamma(u_\varepsilon) * \phi_\varepsilon\right)(x)\right)^m \\ &\qquad\qquad - \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left(L_{n,r}\left(\gamma(u_\varepsilon) * \phi_\varepsilon\right)(x+h)\right)^m \Bigg|^2 \\ &\qquad + C \int_\Omega dx \chi_{\{x+h\in\Omega\}} \fint_{\mathbb{S}^2} dn \Bigg| \int_0^D dr \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left(L_n, r\left(\gamma(u_\varepsilon) * \phi_\varepsilon\right)(x)\right)^m \\ &\qquad\qquad \times \left[\left((\gamma(u_\varepsilon) * \phi_\varepsilon) u_\varepsilon\right)(x - rn) - \left((\gamma(u_\varepsilon) * \phi_\varepsilon) u_\varepsilon\right)(x + h - rn)\right] \Bigg|^2 \end{aligned} \tag{A.49}$$

$$
\begin{aligned}
&+ C(G)\int_\Omega dx\chi_{\{x+h\in\Omega\}} \fint_{\mathbb{S}^2} dn \Bigg| \sum_{m=0}^\infty \frac{(-1)^m}{m!}\left(L_{n,D}\left(\gamma(u_\varepsilon)*\phi_\varepsilon\right)(x)\right)^m \\
&\qquad - \sum_{m=0}^\infty \frac{(-1)^m}{m!}\left(L_{n,D}\left(\gamma(u_\varepsilon)*\phi_\varepsilon\right)(x+h)\right)^m \Bigg|^2 \\
&+ C(G)\int_\Omega dx\chi_{\{x+h\in\Omega\}} \fint_{\mathbb{S}^2} dn \left\{ \left[e^{\left(-L_{n,s(x,n)}(\gamma(u_\varepsilon)*\phi_\varepsilon)(x)\right)} - e^{\left(-L_{n,D}(\gamma(u_\varepsilon)*\phi_\varepsilon)(x)\right)}\right]^2 \right. \\
&\qquad \left. + \left[e^{\left(-L_{n,s(x+h,n)}(\gamma(u_\varepsilon)*\phi_\varepsilon)(x+h)\right)} - e^{\left(-L_{n,D}(\gamma(u_\varepsilon)*\phi_\varepsilon)(x+h)\right)}\right]^2 \right\} \\
&\qquad\qquad + \left(C(G)+K^2\right)\left|\{x\in\Omega : x+h\notin\Omega\}\right|,
\end{aligned}
$$

where the term in the second line is obtained applying Jensen's inequality again and in the last term we estimate the exponential by 1. We notice that

$$
\|L_{n,r}[\gamma(u_\varepsilon)*\phi_\varepsilon]\|_\infty \le D\|\gamma\|_\infty,
$$

for any $0\le r\le D$ and any $n\in\mathbb{S}^2$. Hence, for any $\delta>0$ there exists some $M>0$ such that

$$
\left\| \sum_{m>M}^\infty \frac{(-1)^m}{m!}\left|L_{n,r}\left(\gamma(u_\varepsilon)*\phi_\varepsilon\right)(\cdot)\right|^m \right\|_\infty^2 < \frac{\delta}{2},
$$

for all $r\in[0,D]$ and $n\in\mathbb{S}^2$. Moreover, the smoothness of $\partial\Omega$ implies that there exists a constant $C(\Omega)>0$ such that

$$
|\{x\in\Omega : x+h\notin\Omega\}| \le C(\Omega)|h|.
$$

In addition to that the convexity of the domain Ω and a geometric argument implies that, since $\gamma(u_\varepsilon)*\phi_\varepsilon$ is supported in $\overline{\Omega+B_\varepsilon(0)}$, there exists a constant $C(\Omega)>0$ which depends on the curvature of Ω such that

$$
\sup_{\substack{x\in\Omega\\ n\in\mathbb{S}^2}} \int_{s(x,n)}^D \gamma(u_\varepsilon)*\phi_\varepsilon(x-\lambda n)d\lambda \le C(\Omega)\|\gamma\|_\infty\sqrt{\varepsilon},
$$

where $\sqrt{\varepsilon}$ is due to the set of directions $n\in\mathbb{S}^2$ that are tangent to the boundary $\partial\Omega$. Thus, using also the well-known inequality $|e^{-b}-e^{-a}|\le|a-b|$ for $a,b\ge0$ we compute

$$
\begin{aligned}
&\int_\Omega dx\left|\mathcal{B}^\varepsilon(u_\varepsilon)(x)-\mathcal{B}^\varepsilon(u_\varepsilon)(x+h)\right|^2 \\
&\qquad \le C(D,G)\|\gamma\|_\infty K^2 \sup_{n\in\mathbb{S}^2}\sup_{0\le r\le D} \left\| \sum_{m>M}^\infty \frac{(-1)^m}{m!}\left|L_{n,r}\left(\gamma(u_\varepsilon)*\phi_\varepsilon\right)(\cdot)\right|^m \right\|_\infty^2 \\
&\qquad\quad + C\|\gamma\|_\infty^2 K^2 \sum_{m=0}^M (M+1)\left(\frac{m}{m!}\right)^2 \left(\|\gamma\|_\infty D\right)^{2(m-1)} \int_{\Pi_D^3} dx \int_0^D dr \\
&\qquad\qquad \times \fint_{\mathbb{S}^2} dn \left|L_{n,r}\left(\gamma(u_\varepsilon)*\phi_\varepsilon\right)(x) - L_{n,r}\left(\gamma(u_\varepsilon)*\phi_\varepsilon\right)(x+h)\right|^2
\end{aligned}
\tag{A.50}
$$

$$+ C\int_{\Pi_D^3} dx \fint_{\mathbb{S}^2} dn \sum_{m=0}^{M} \frac{M+1}{(m!)^2}\|\gamma\|_\infty^{2m} D^m \int_0^D d\lambda_1 ... \int_0^D d\lambda_m \bigg| \int_{\max_{0\leq i\leq M}(\lambda_i)}^D dr$$
$$\times \left[\left(\left(\gamma(u_\varepsilon)*\phi_\varepsilon\right)u_\varepsilon\right)(x-rn) - \left(\left(\gamma(u_\varepsilon)*\phi_\varepsilon\right)u_\varepsilon\right)(x+h-rn)\right]\bigg|^2$$
$$+ C(G)\sum_{m=0}^{M}(M+1)\left(\frac{m}{m!}\right)^2 \left(\|\gamma\|_\infty D\right)^{2(m-1)} \int_{\Pi_D^3}\fint_{\mathbb{S}^2} dn \left| L_{n,D}\left(\gamma(u_\varepsilon)*\phi_\varepsilon\right)(x)\right.$$
$$\left. - L_{n,D}\left(\gamma(u_\varepsilon)*\phi_\varepsilon\right)(x+h)\right|^2 + C(G,\Omega)\left(\|\gamma\|_\infty^2\varepsilon + K^2|h|^2\right).$$

In order to obtain these last estimates we used also that $\Omega\subset\Pi_D^3$ since we are considering non-negative integrands. Moreover, in order to obtain the term containing

$$\int_0^D d\lambda_1 ... \int_0^D d\lambda_m \left| L_{n,D-\max(\lambda)}[\left(\gamma(u_\varepsilon)*\phi_\varepsilon\right)u_\varepsilon](x)\right.$$
$$\left. - L_{n,D-\max(\lambda)}[\left(\gamma(u_\varepsilon)*\phi_\varepsilon\right)u_\varepsilon](x+h)\right|^2$$

for $\max(\lambda)=\max_{0\leq i\leq M}(\lambda_i)$ we changed the order of integration applying Fubini's Theorem and we saw recursively that

$$\int_0^D dr\left(\int_0^r d\lambda\right)^m = \int_0^D dr \int_0^r d\lambda_1 ... \int_0^r d\lambda_m$$
$$= \int_0^D d\lambda_1 ... \int_0^D d\lambda_m \int_{\max_{0\leq i\leq M}(\lambda_i)}^D dr. \quad \text{(A.51)}$$

Hence, applying Fubini's Theorem and afterwards Jensen's inequality we conclude

$$\bigg|\int_0^D dr\left(\int_0^r d\lambda \left(\gamma(u_\varepsilon)*\phi_\varepsilon\right)(x-\lambda n)\right)^m$$
$$\times \left[\left(\left(\gamma(u_\varepsilon)*\phi_\varepsilon\right)u_\varepsilon\right)(x-rn) - \left(\left(\gamma(u_\varepsilon)*\phi_\varepsilon\right)u_\varepsilon\right)(x+h-rn)\right]\bigg|^2$$
$$= \bigg|\int_0^D d\lambda_1 \left(\gamma(u_\varepsilon)*\phi_\varepsilon\right)(x-\lambda_1 n) ... \int_0^D d\lambda_m \left(\gamma(u_\varepsilon)*\phi_\varepsilon\right)(x-\lambda_m n)$$
$$\times \int_{\max(\lambda)}^D dr \left[\left(\left(\gamma(u_\varepsilon)*\phi_\varepsilon\right)u_\varepsilon\right)(x-rn) - \left(\left(\gamma(u_\varepsilon)*\phi_\varepsilon\right)u_\varepsilon\right)(x+h-rn)\right]\bigg|^2$$
$$\leq D^m\|\gamma\|^{2m}\int_0^D d\lambda_1 ... \int_0^D d\lambda_m \bigg|\int_{\max(\lambda)}^D dr\left[\left(\left(\gamma(u_\varepsilon)*\phi_\varepsilon\right)u_\varepsilon\right)(x-rn)\right.$$
$$\left. - \left(\left(\gamma(u_\varepsilon)*\phi_\varepsilon\right)u_\varepsilon\right)(x+h-rn)\right]\bigg|^2.$$

We apply now the modification of Proposition A.1 as in Lemma A.1. Let us take the sequence $\varepsilon_j =: \frac{1}{j}$ for $j\in\mathbb{N}$. As we can notice in equation (A.50) we have to consider the operators $L_{n,t}$ and $L_{n,D-t}$ for some $0\leq t\leq D$ acting respectively on two different sequences of functions, i.e. $\left(\gamma(u_{\varepsilon_j})*\phi_{\varepsilon_j}\right)_j$ respectively $\left(\left[\gamma(u_{\varepsilon_j})*\phi_{\varepsilon_j}\right]u_{\varepsilon_j}\right)_j$. In order to simplify the notation we write f_j instead of f_{ε_j}. We recall that these sequences are uniformly bounded. Young's convolution inequality implies indeed

$$\sup_{j\geq 0}\|\gamma(u_j)*\phi_j\|_\infty \leq \|\gamma\|_\infty \quad\text{and}\quad \sup_{j\geq 0}\|\left(\gamma(u_j)*\phi_j\right)u_j\|_\infty \leq K\|\gamma\|_\infty.$$

Hence, we can apply Lemma A.1 to these sequences.

Let now $\beta > 0$ be arbitrarily small. We consider the terms appearing in (A.50). The convergence of the exponential implies the existence of $M_0(\beta) > 0$ such that

$$C(D,G)\|\gamma\|_\infty K^2 \sup_{n\in\mathbb{S}^2}\sup_{0\le r\le D}\left\|\sum_{m>M}^{\infty}\frac{(-1)^m}{m!}\left|L_{n,r}\left(\gamma(u_j)*\phi_j\right)(\cdot)\right|^m\right\|_\infty^2 < \frac{\beta}{4},$$

for any $M \ge M_0$. Moreover, Lemma A.1 applied to $L_{n,r}\left[\gamma(u_j)*\phi_j\right](x)$, $L_{n,D}\left[\gamma(u_j)*\phi_j\right](x)$, and $L_{n,D-\max(\lambda)}\left[\left(\gamma(u_j)*\phi_j\right)u_j\right](x)$ implies the existence of some $h_0(M_0,\beta) > 0$ such that

$$\begin{aligned}
C\|\gamma\|_\infty^2 K^2 \sum_{m=0}^{M_0}(M_0+1)\left(\frac{m}{m!}\right)^2\left(\|\gamma\|_\infty D\right)^{2(m-1)}\int_{\Pi_D^3}dx\int_0^D dr\\
\times \fint_{\mathbb{S}^2}dn\left|L_{n,r}\left(\gamma(u_j)*\phi_j\right)(x) - L_{n,r}\left(\gamma(u_j)*\phi_j\right)(x+h)\right|^2\\
+C\int_{\Pi_D^3}dx\fint_{\mathbb{S}^2}dn\sum_{m=0}^{M_0}\frac{M_0+1}{(m!)^2}\|\gamma\|_\infty^{2m}D^m\int_0^D d\lambda_1 ...\int_0^D d\lambda_m\left|\int_{\max_{0\le i\le M_0}(\lambda_i)}^D dr\right.\\
\left.\times\left[\left(\left(\gamma(u_j)*\phi_j\right)u_j\right)(x-rn) - \left(\left(\gamma(u_j)*\phi_j\right)u_j\right)(x+h-rn)\right]\right|^2\\
+C(G)\sum_{m=0}^{M_0}(M_0+1)\left(\frac{m}{m!}\right)^2\left(\|\gamma\|_\infty D\right)^{2(m-1)}\int_{\Pi_D^3}\fint_{\mathbb{S}^2}dn\left|L_{n,D}\left(\gamma(u_j)*\phi_j\right)(x)\right.\\
\left.-L_{n,D}\left(\gamma(u_j)*\phi_j\right)(x+h)\right|^2 < \frac{\beta}{4}. \quad \text{(A.52)}
\end{aligned}$$

This is true because we are applying Lemma A.1 finitely many times, since the sum is finite. Moreover, the equi-integrability result of Lemma A.1 is uniform with respect to the length of the line along which we are integrating. It applies hence to all terms appearing in (A.52). Taking now in (A.50)

$$J_0 = \frac{4C(D,G)\|\gamma\|_\infty^2}{\beta} \quad \text{and} \quad h_1 < \frac{\sqrt{\beta}}{2K\sqrt{C(G,\Omega)}},$$

we obtain

$$\int_\Omega dx\left|\mathcal{B}^j(u_j)(x) - \mathcal{B}^j(u_j)(x+h)\right|^2 < \beta$$

for all $j \ge J_0$ and for all $|h| < \min(h_0,h_1)$. The continuity of the functions u_j and the fact that for $j < J_0$ we have only finitely many elements of the sequence imply the existence of some $0 < H_0 \le \min(h_0,h_1)$ such that

$$\int_\Omega dx\left|\mathcal{B}^j(u_j)(x) - \mathcal{B}^j(u_j)(x+h)\right|^2 < \beta$$

for all $j \ge 0$ and all $|h| < H_0$. Hence, the sequence $\left(\mathcal{B}^j(u_j)\right)_{j\in\mathbb{N}}$ is compact in L^2 and there exists a subsequence $\left(\mathcal{B}^{j_l}(u_{j_l})\right)_{l\in\mathbb{N}}$ and a function $u \in L^2(\Omega)\cap L^\infty(\Omega)$ such that $u_{j_l} = \mathcal{B}^{j_l}(u_{j_l}) \to u$ both in L^2 and pointwise almost everywhere as $l \to \infty$.

The uniformly boundedness of u_{j_l} and also of $\gamma\left(u_{j_l}\right)$ implies the convergence in L^p for $p < \infty$ of $\gamma\left(u_{j_l}\right)*\phi_{j_l} \to \gamma(u)$ as $l \to \infty$ and hence for a subsequence (say still u_{j_l}) the convergence holds also pointwise almost everywhere. Indeed,

$$\begin{aligned}
&\|\gamma(u_{j_l})*\phi_{j_l}-\gamma(u)\|_p\\
&\qquad\le \|\gamma(u_{j_l})*\phi_{j_l}-\gamma(u)*\phi_{j_l}\|_p+\|\gamma(u)*\phi_{j_l}-\gamma(u)\|_p\\
&\qquad\le \|\gamma(u_{j_l})-\gamma(u)\|_p\|\phi_{j_l}\|_1+\|\gamma(u)*\phi_{j_l}-\gamma(u)\|_p\to 0 \quad \text{uniformly in } l,
\end{aligned}$$

where we used the Young's convolution inequality combined with the fact that ϕ_{j_l} are positive and with the dominated convergence. Finally another application of the dominated convergence theorem implies

$$u_{j_l}=\mathcal{B}^{j_l}(u_{j_l})\to u=\mathcal{B}(u) \tag{A.53}$$

pointwise almost everywhere as $l\to\infty$ and $u=\mathcal{B}(u)$ pointwise a.e. Hence, u is the desired solution to (A.23). □

A direct corollary of the proof of Theorem A.4 is the following.

Corollary A.2. *Let $\{\varphi_j\}_{j\in\mathbb{N}}$ and $\{\psi_j\}_{j\in\mathbb{N}}$ be two bounded sequences in $L^\infty(\Omega)$ for $\Omega\subset\mathbb{R}^3$ bounded with C^2-boundary and strictly positive curvature. Let also $f\in L^\infty(\mathbb{S}^2)$ be non-negative. Then the sequences*

$$\int_{\mathbb{S}^2} dn\int_0^D dr\varphi_j(x-rn)\exp\left(-\int_0^r\psi_j(x-\lambda n)d\lambda\right)$$

and

$$\int_{\mathbb{S}^2} dn\, f(n)\exp\left(-\int_0^D\psi_j(x-\lambda n)d\lambda\right)$$

are compact in $L^2(\Omega)$. In particular they are L^2-equiintegrable in the following way: For any $\varepsilon>0$ there exists some $h_0>0$ such that for all $j\in\mathbb{N}$ and all $|h|<h_0$ both estimates holds

$$\begin{aligned}
&\int_\Omega dx\left|\int_{\mathbb{S}^2}dn\int_0^D dr\left[\varphi_j(x-rn)\exp\left(-\int_0^r\psi_j(x-\lambda n)d\lambda\right)\right.\right.\\
&\qquad\qquad\left.\left.-\varphi_j(x+h-rn)\exp\left(-\int_0^r\psi_j(x+h-\lambda n)d\lambda\right)\right]\right|^2\\
&\qquad\le 4\pi\int_\Omega dx\int_{\mathbb{S}^2}dn\left|\int_0^D dr\left[\varphi_j(x-rn)\exp\left(-\int_0^r\psi_j(x-\lambda n)d\lambda\right)\right.\right.\\
&\qquad\qquad\qquad\left.\left.-\varphi_j(x+h-rn)\exp\left(-\int_0^r\psi_j(x+h-\lambda n)d\lambda\right)\right]\right|^2<\varepsilon
\end{aligned}$$

and

$$\begin{aligned}
&\int_\Omega dx\left|\int_{\mathbb{S}^2}dnf(n)\left[\exp\left(-\int_0^D\psi_j(x-\lambda n)d\lambda\right)\right.\right.\\
&\qquad\qquad\left.\left.-\exp\left(-\int_0^D\psi_j(x+h-\lambda n)d\lambda\right)\right]\right|^2\\
&\qquad\le 4\pi\|f\|_{L^\infty}\int_\Omega dx\int_{\mathbb{S}^2}dnf(n)\left|\exp\left(-\int_0^D\psi_j(x-\lambda n)d\lambda\right)\right.\\
&\qquad\qquad\qquad\left.-\exp\left(-\int_0^D\psi_j(x+h-\lambda n)d\lambda\right)\right|^2<\varepsilon.
\end{aligned}$$

Combining this result with Corollary A.1 we see that the following proposition holds.

Proposition A.2. *Let $\Omega \subset \mathbb{R}^3$ bounded with C^2-boundary and strictly positive curvature. Let also $\{\varphi_j(x,\omega)\}_{j\in\mathbb{N}} \subset C\left(\mathbb{S}^2, L^\infty(\Omega)\right)$ be uniformly bounded and satisfying the assumption of Corollary A.1; i.e.,*

$$\|\varphi_j(\cdot,n_1) - \varphi_j(\cdot,n_2)\|_{L^\infty(\Omega)} \leq \sigma(d(n_1,n_2)) \to 0,$$

uniformly in $j \in \mathbb{N}$ if $d(n_1,n_2) \to 0$, where d is the metric on the sphere and $\sigma \in C(\mathbb{R}_+,\mathbb{R}_+)$ with $\sigma(0) = 0$ is a uniform modulus of continuity.. Let $\{\psi_j(x)\}_{j\in\mathbb{N}} \subset L^\infty(\Omega)$ be a uniform bounded sequence. Then the new sequence

$$\int_{\mathbb{S}^2} dn \int_0^D dr \varphi_j(x-rn,n) \exp\left(-\int_0^r \psi_j(x-\lambda n)d\lambda\right)$$

is compact in $L^2(\Omega)$.

Proof. We apply Federer-Besicovitch covering Lemma for the sphere $\mathbb{S}^2$ as we did in Corollary A.1 to the result of Corollary A.2. □

This completes the proof of the existence of solutions to the case of Grey approximation. Then we can further use this result to prove our main theorem (Theorem A.1) for the pseudo Grey case as follows.

Proof of Theorem A.1. We expect for the pseudo Grey case the same result to hold. Let

$$\alpha_\nu(T(x)) = Q(\nu)h(T(x)),$$

for some non-negative bounded function $Q(\nu)$. Then define

$$u(x) = 4\pi \int_0^\infty d\nu \; Q(\nu)B_\nu(T(x)) = F(T(x)).$$

We notice that by the monotonicity in of $B_\nu(T)$ in T also F is monotone with respect to T and hence $T(x) = F^{-1}(u)(x)$. Denoting by $\gamma = h(F^{-1})$ and by $f_\nu = B_\nu(F^{-1})$ we see that u solves

$$\begin{aligned} u(x) = \int_0^\infty d\nu \int_\Omega d\eta \; \frac{Q(\nu)^2\gamma(u(\eta))f_\nu(u(\eta)}{|x-\eta|^2}) \exp\left(-\int_{[x,\eta]} Q(\nu)\gamma(u(\zeta))d\zeta\right) \\ + \int_0^\infty d\nu \int_{\mathbb{S}^2} dn \; \exp\left(-\int_{[x,y(x,n)]} Q(\nu)\gamma(u(\zeta))d\zeta\right) Q(\nu)g_\nu(n). \quad \text{(A.54)} \end{aligned}$$

We regularize this equation in the same way as in Section A.2.3 and (A.25) and obtain for ϕ_ε a standard positive and symmetric mollifier the following fixed-point equation.

$$\begin{aligned} u(x) = \mathcal{B}_\varepsilon(u)(x) = \int_0^\infty d\nu \int_\Omega d\eta \; \frac{Q(\nu)^2\left(\gamma(u) * \phi_\varepsilon\right)(\eta)f_\nu(u(\eta))}{|x-\eta|^2} \\ \times \exp\left(-\int_{[x,\eta]} Q(\nu)\left(\gamma(u) * \phi_\varepsilon\right)(\zeta)d\zeta\right) \\ + \int_0^\infty d\nu \int_{\mathbb{S}^2} dn \; \exp\left(-\int_{[x,y(x,n)]} Q(\nu)\left(\gamma(u) * \phi_\varepsilon\right)(\zeta)d\zeta\right) Q(\nu)g_\nu(n). \quad \text{(A.55)} \end{aligned}$$

Then the L^∞-estimate and the equicontinuity (or more precisely uniform Hölder continuity) of the right-hand side of (A.55) hold once more in the same way as in Subsection A.3.3. Also,

$\mathcal{B}_\varepsilon$ is a continuous operator. We hence have solutions u_ε. We need to show the compactness of the sequence of regularized solutions u_j, where $\varepsilon = \frac{1}{j}$. We consider similarly as in the proof of Theorem A.3 the line operators acting on some suitable sequence given for $r \in [0, D]$ by

$$Q(\nu)L_{n,r}\left(\gamma(u_j) * \phi_j\right)(x) \quad \text{and} \quad L_{n,D-r}\left(\gamma(u_j) * \phi_j \int_0^\infty Q(\nu)B_\nu\left(F^{-1}(u)\right)d\nu\right)(x).$$

By the definition of u obtain the following uniform estimate

$$\sup_{x\in\Omega}\left|\gamma(u_j) * \phi_j(x)\int_0^\infty Q(\nu)B_\nu\left(F^{-1}(u_j)\right)(x)d\nu\right| \le \|\gamma\|_\infty\|u\|_\infty,$$

where $\|u\|_\infty$ is the uniform upper bound of u_j. Hence, we can write as before the operator $\mathcal{B}_j$ in polar coordinates according to

$$\begin{aligned}\mathcal{B}_j(u_j)(x) = \int_0^\infty d\nu Q(\nu)\int_{\mathbb{S}^2} dn \int_0^{s(x,n)} dr\left[\left(\gamma(u_j) * \phi_j\right)Q(\nu)f_\nu(u_j)\right](x - rn)\\ \times \exp\left(-Q(\nu)L_{n,r}\left(\gamma(u_j) * \phi_j\right)(x)\right)\\ + \int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, \exp\left(-Q(\nu)L_{n,s(x,n)}\left(\gamma(u_j) * \phi_j\right)(x)\right)Q(\nu)g_\nu(n).\end{aligned}$$

Thus, using the boundedness of Q and the estimate $\int_0^\infty Q^{m+1}(\nu)f_\nu(x) \le \|Q\|_\infty^m\|u\|_\infty$ similarly as we did for equation (A.50) we can obtain

$$\begin{aligned}&\int_\Omega dx\left|\mathcal{B}_j(u_j)(x) - \mathcal{B}_j(u_j)(x+h)\right|^2\\ &\le C(D,\Omega,g)\|\gamma\|_\infty^2\|Q\|_\infty^2\|u_j\|_\infty^2 \sup_{\substack{\nu\ge0\\ n\in\mathbb{S}^2\\ 0\le r\le D}}\left\|\sum_{m>M}^\infty \frac{(-1)^m}{m!}Q^m(\nu)\left|L_{n,r}\left(\gamma(u_j) * \phi_j\right)(\cdot)\right|^m\right\|_\infty^2\\ &\quad + C\|\gamma\|_\infty^2\|u_j\|_\infty^2\sum_{m=0}^M (M+1)\left(\frac{m}{m!}\right)^2\left(\|\gamma\|_\infty D\right)^{2(m-1)}\|Q\|_\infty^{2m+2}\int_{\Pi_D^3} dx\int_0^D dr\\ &\quad \times\int_{\mathbb{S}^2} dn\left|L_{n,r}\left(\gamma(u_j) * \phi_j\right)(x) - L_{n,r}\left(\gamma(u_j) * \phi_j\right)(x+h)\right|^2\\ &\quad + C\int_{\Pi_D^3} dx\int_{\mathbb{S}^2} dn\sum_{m=0}^M\frac{M+1}{(m!)^2}\|\gamma\|_\infty^{2m}D^m\int_0^D d\lambda_1 ...\int_0^D d\lambda_m\left|\int_{\max_{0\le i\le M}(\lambda_i)}^D dr\int_0^\infty d\nu\right.\\ &\times\left.\left[\left(\left(\gamma(u_j) * \phi_j\right)Q^{m+2}(\nu)f_\nu(u_j)\right)(x - rn) - \left(\left(\gamma(u_j) * \phi_j\right)Q^{m+2}(\nu)f_\nu(u_j)\right)(x+h-rn)\right]\right|^2\\ &\quad + C(g_\nu)\sum_{m=0}^M (M+1)\left(\frac{m}{m!}\right)^2\left(\|\gamma\|_\infty D\right)^{2(m-1)}\|Q\|_\infty^{2m+2}\int_{\Pi_D^3}\int_{\mathbb{S}^2} dn\left|L_{n,D}\left(\gamma(u_j) * \phi_j\right)(x)\right.\\ &\quad \left. - L_{n,D}\left(\gamma(u_j) * \phi_j\right)(x+h)\right|^2 + C(g_\nu, Q, \Omega)\left(\|\gamma\|_\infty^2\frac{1}{j} + \|u\|_\infty^2|h|^2\right),\end{aligned}\tag{A.56}$$

where we used the triangle inequality as we did in (A.49). In addition, for the tails of the exponential terms in the estimate, we use the supremum norm and use that $\int_0^\infty Q(\nu)f_\nu(u_j) = u_j$. For the terms involving the finite difference of powers of line integrals we argue as we did in (A.50) taking the absolute value inside the integrals, estimating each term using the boundedness of $Q(\nu)$ and the integrability of f_ν and applying Jensen's inequality in the end. The term in the fifth line of (A.56) is obtained using the identity (A.51) given by Fubini's theorem and changing the order of integration so that the integral with respect to ν is the most interior one. Hence, we conclude with Jensen's inequality. The last term in (A.56) is obtained exactly as the last term in (A.50). We conclude the compactness of the sequence $u_j = \mathcal{B}_j(u_j)$ in L^2 as we did in the proof of Theorem A.3. We hence fix first of all the $M_0 > 0$ such that the first term in the right hand side of (A.56) is smaller than $\frac{\beta}{5}$ for an arbitrarily small $\beta > 0$. This is possible because

$$\sup_{\substack{\nu\geq 0,\ n\in\mathbb{S}^2\\ 0\leq r\leq D,\ x\in\Omega}} |Q(\nu)L_{n,r}\left(\gamma(u_j)*\phi_j\right)(x)| \leq D\|Q\|_\infty\|\gamma\|_\infty.$$

After that, since the sequence $\gamma(u_j)*\phi_j(x)\int_0^\infty Q^{m+2}(\nu)B_\nu\left(F^{-1}(u_j)\right)(x)d\nu$ is uniformly bounded for all $m \leq M_0+1$, all arguments in the proof of Theorem A.3 still apply and hence the line integral of this sequence is also equi-integrable. Hence, arguing in the same way as in the proof of Theorem A.3 we see that a subsequence u_{j_l} converges pointwise almost everywhere to the desired solution $u = \mathcal{B}(u)$. □

A.4 Full equation with both scattering and emission-absorption

In this section we consider the full equation with both scattering and emission-absorption terms. We study the case when the scattering coefficient and the absorption coefficient depend on the local temperature $T(x)$. The radiative transfer equation can be written as

$$\begin{aligned} n\cdot\nabla_x I_\nu(x,n) &= \alpha_\nu^a\left(T(x)\right)\left(B_\nu\left(T(x)\right)-I_\nu(x,n)\right)\\ &\quad + \alpha_\nu^s\left(T(x)\right)\left[\left(\int_{\mathbb{S}^2} dn'\ K(n,n')I_\nu(x,n')\right)-I_\nu(x,n)\right]. \end{aligned} \tag{A.57}$$

We consider as in the previous sections equation (A.57) coupled with the condition of divergence-free total flux in equation (A.6) and the incoming boundary condition (A.7). We will consider in this paper only the case of isotropic scattering, i.e. the case where the scattering kernel is invariant under rotation.

We notice first of all that the isotropic property of the scattering kernel implies its symmetry.

Lemma A.2. *Let $K(n,n')$ be rotation invariant, i.e. $K(n,n') = K(Rn,Rn')$ for all $n,n'\in\mathbb{S}^2$ and $R\in SO(3)$. Then $K(n,n') = K(n',n)$.*

Proof. Let $n,n'\in\mathbb{S}^2$. We denote by $\theta\in[0,\pi]$ the angle formed by n,n' on the plane spanned by these unit vectors. We denote moreover by $R_\theta\in SO(3)$ the rotation matrix defined by a rotation of π around the bisectrix of θ lying in the plane spanned by n and n'. Then we see that $R_\theta n = n'$ and $R_\theta n' = n$. Hence by assumption, $K(n,n') = K\left(R_\theta n', R_\theta n\right) = K(n',n)$. □

A.4.1 Main result in the case of the Grey approximation

We consider first the case of Grey approximation, i.e. we assume the coefficients and the scattering kernel to be independent of the frequency and we denote $\alpha_\nu^a = \alpha^a$ and $\alpha_\nu^s = \alpha^s$. We will now prove a theorem about the existence of solutions to (A.57) similar to Theorem A.3. The main difference with the setting of Theorem A.3 is the presence of the scattering operator.

In the pure emission-absorption case the motion of the photons between one emission and the next absorption is rectilinear. On the contrary, in the presence of scattering, the photons move along a polygonal path between emission and absorption events. In order to take it into account we will define suitable Green functions that incorporate the polygonal motion due to the scattering. Using these Green functions it will be possible to find a fixed-point equation for the temperature analogous to (A.20) that includes the non-rectilinear motion between emission and absorption events (cf. equation (A.61)). We show the following theorem.

Theorem A.4. *Let $\Omega \subset \mathbb{R}^3$ be bounded, convex and open with C^2-boundary and strictly positive curvature. Let α^a and α^s be positive and bounded C^1-functions of the temperature, independent of the frequency. Assume $K \in C^1\left(\mathbb{S}^2 \times \mathbb{S}^2\right)$ be non-negative, rotationally symmetric and independent of the frequency with the property* (A.4). *Then there exists a solution $(T, I_\nu) \in L^\infty(\Omega) \times L^\infty\left(\Omega, L^\infty\left(\mathbb{S}^2, L^1(\mathbb{R}_+)\right)\right)$ to the equation* (A.57) *coupled with* (A.6) *satisfying the boundary condition* (A.7)*, where the I_ν is a solution to* (A.57) *in the sense of distribution.*

For the proof we proceed in the following way. As indicated above we begin constructing a fixed-point equation for the temperature which contains information about the scattering processes. We will hence regularize the problem, similarly as we did in Section A.2.3 and will prove the existence of regularized solution using the Schauder fixed-point theorem. At the end we will use the compactness theory developed in Subsection A.3.2 in order to show the convergence of a subsequence of the regularized solutions to the desired solution.

We define for $x, x_0 \in \Omega$ and $n \in \mathbb{S}^2$ the fundamental solution $\tilde{I}(x, n; x_0)$ solving the following equation in distributional sense

$$\begin{aligned} n \cdot \nabla_x \tilde{I}(x, n; x_0) = \alpha^s(T(x)) \int_{\mathbb{S}^2} K(n, n') \tilde{I}(x, n'; x_0)\, dn' \\ - \left(\alpha^a\left(T(x)\right) + \alpha^s\left(T(x)\right)\right) \tilde{I}(x, n; x_0) + \delta(x - x_0) \end{aligned} \tag{A.58}$$

and the boundary condition for $x \in \partial\Omega$

$$\tilde{I}(x, n; x_0) \chi_{\{n \cdot n_x < 0\}} = 0,$$

where n_x denotes the normal outer vector to $\partial\Omega$ at x. Similar to the Poisson kernel for the Laplace equation, for $x \in \Omega$, $x_0 \in \partial\Omega$ and $n, n_0 \in \mathbb{S}^2$ we define the function $\psi(x, n; x_0, n_0)$ by the equation

$$\begin{aligned} n \cdot \nabla_x \psi(x, n; x_0, n_0) &= \alpha^s(T(x)) \int_{\mathbb{S}^2} K(n, n') \psi(x, n'; x_0, n_0)\, dn' \\ &\quad - \left(\alpha^a\left(T(x)\right) + \alpha^s\left(T(x)\right)\right) \psi(x, n; x_0, n_0), \ x \in \Omega, \qquad \text{(A.59)} \\ \psi(x, n; x_0, n_0) \chi_{\{n \cdot n_x < 0\}} &= \delta_{\partial\Omega}(x - x_0) \frac{\delta^{(2)}(n, n_0)}{4\pi}, \ x \in \Omega, \ n_0 \cdot N_{x_0} < 0, \end{aligned}$$

where we denoted by $\delta^{(2)}$ the two dimensional delta distribution on the sphere and by $\delta_{\partial\Omega}$ the two dimensional delta distribution on $\partial\Omega$. This allows to include the effect of the boundary. Before moving to the computations of such functions we see that the intensity of radiation can be expressed by these two functions as follows.

$$\begin{aligned} I_\nu(x, n) = \int_\Omega dx_0\, \alpha^a(T(x_0)) B_\nu(T(x_0)) \tilde{I}(x, n; x_0) \\ + \int_{\mathbb{S}^2} dn_0 \int_{\partial\Omega} dx_0\, g_\nu(n_0) \psi(x, n; x_0, n_0). \end{aligned} \tag{A.60}$$

Thus, plugging (A.60) into (A.6) and using (A.3) we obtain

$$\begin{aligned}
0 = \nabla_x \cdot \mathcal{F}(x) &= \operatorname{div}\Big(\int_0^\infty d\nu \int_{\mathbb{S}^2} dn \int_\Omega dx_0\ \alpha^a(T(x_0))B_\nu(T(x_0))n\tilde{I}(x,n;x_0) \\
&\quad + \int_0^\infty d\nu \int_{\mathbb{S}^2} dn \int_{\mathbb{S}^2} dn_0 \int_{\partial\Omega} dx_0\ g_\nu(n_0)n\psi(x,n;x_0,n_0)\Big) \\
&= \sigma \int_{\mathbb{S}^2} dn \int_\Omega dx_0\ \alpha^a(T(x_0))T^4(x_0)\left[\delta(x-x_0) - \alpha^a(T(x))\tilde{I}(x,n;x_0)\right] \\
&\quad + \sigma \int_{\mathbb{S}^2} dn \int_\Omega dx_0\ \alpha^a(T(x_0))T^4(x_0)\alpha^s(T(x))\left[\int_{\mathbb{S}^2} dn' K(n,n')\tilde{I}(x,n';x_0) - \tilde{I}(x,n;x_0)\right] \\
&\quad + \int_{\mathbb{S}^2} dn \int_{\mathbb{S}^2} dn_0 \int_{\partial\Omega} dx_0\ G(n_0)\Big[\alpha^s(T(x)) \int_{\mathbb{S}^2} K(n,n')\psi(x,n';x_0,n_0)\, dn' \\
&\qquad - (\alpha^s(T(x)) + \alpha^a(T(x)))\psi(x,n;x_0,n_0)\Big] \\
&= 4\pi\sigma\alpha^a(T(x))T^4(x) - \alpha^a(T(x))\sigma \int_{\mathbb{S}^2} dn \int_\Omega dx_0\ \alpha^a(T(x_0))T^4(x_0)\tilde{I}(x,n;x_0) \\
&\qquad - \alpha^a(T(x)) \int_{\mathbb{S}^2} dn \int_{\mathbb{S}^2} dn_0 \int_{\partial\Omega} dx_0\ G(n_0)\psi(x,n;x_0,n_0),
\end{aligned}$$

where we defined $G(n) =: \int_0^\infty d\nu\ g_\nu(n)$, and the last equality holds by the property (A.4) of the kernel K integrating first with respect to n. Hence, defining $u(x) = 4\pi\sigma T^4(x)$ and dividing by $\alpha^a(T(x))$ we get the following non-linear fixed-point equation

$$\begin{aligned}
u(x) = \int_\Omega dx_0 \int_{\mathbb{S}^2} dn\ \frac{\alpha^a(u(x_0))u(x_0)}{4\pi}\tilde{I}(x,n;x_0) \\
+ \int_{\mathbb{S}^2} dn_0 \int_{\mathbb{S}^2} dn \int_{\partial\Omega} dx_0\ G(n_0)\psi(x,n;x_0,n_0), \quad \text{(A.61)}
\end{aligned}$$

where by an abuse of notation we define $\alpha^a(\cdot) = \alpha^a\left(\sqrt[4]{\frac{\cdot}{4\pi\sigma}}\right)$.

A.4.2 Construction of the Green functions in the Grey case

Let us now construct the Green functions $\tilde{I}$ and ψ. We start with the first function. Denoting by $H(\cdot)$ the Heaviside function and by $P_n^\perp$ the projection $P_n^\perp = I - n\otimes n$, we see using the Fourier transform that the distribution $f_0(x,n;x_0) = H(n\cdot(x-x_0))\delta^{(2)}\left(P_n^\perp(x-x_0)\right)$ solves in distributional sense the equation

$$n\cdot\nabla_x f_0(x,n;x_0) = \delta(x-x_0)$$

with zero boundary condition.

Hence, the function $f_1(x,n;x_0) = f_0(x,n;x_0) + \int_{\mathbb{R}^3} dy\ F(y)f_0(x,n;y)$ solves in distributional sense the equation

$$n\cdot\nabla_x f_1(x,n;x_0) = F(x) + \delta(x-x_0).$$

Notice that by definition of f_0 we have $\int_{\mathbb{R}^3} dy\ F(y)f_0(x,n;y) = \int_0^{s(x,n)} dt F(x-tn)$. Moreover, the function $f_2(x,n;x_0) = f_0(x,n;x_0)\exp\left(-\int_{[x_0,x]}\alpha(\xi)ds(\xi)\right)$ solves in distributional sense the equation

$$n\cdot\nabla_x f_2(x,n;x_0) = \delta(x-x_0) - \alpha(x)f_2(x,n;x_0),$$

since $\frac{x-x_0}{|x-x_0|} \cdot \nabla_x \int_{[x_0,x]} \alpha(\xi) ds(\xi) = \alpha(x)$. Hence, we conclude that

$$f(x,n;x_0) = f_0(x,n;x_0) \exp\left(-\int_{[x_0,x]} \alpha(\xi) ds(\xi)\right) + \int_{\mathbb{R}^3} dy\ F(y) f_0(x,n;y) \exp\left(-\int_{[y,x]} \alpha(\xi) ds(\xi)\right),$$

solves in distributional sense the equation

$$n \cdot \nabla_x f(x,n;x_0) = \delta(x-x_0) - \alpha(x) f(x,n;x_0) + F(x).$$

Thus, with these considerations we write the Green function $\tilde{I}$ as

$$\begin{aligned} &\tilde{I}(x,n;x_0) \\ &= \chi_\Omega(x_0) \exp\left(-\int_{[x_0,x]} [\alpha^a(u(\xi)) + \alpha^s(u(\xi))]\, d\xi\right) H(n\cdot(x-x_0))\delta^{(2)}\left(P_n^\perp(x-x_0)\right) \\ &\quad + \int_0^{s(x,n)} dt\ \alpha^s(u(x-tn)) \exp\left(-\int_{[x-tn,x]} [\alpha^a(u(\xi)) + \alpha^s(u(\xi))]\, d\xi\right) \\ &\qquad \times \int_{\mathbb{S}^2} dn' K(n,n') \tilde{I}(x-tn,n';x_0). \end{aligned} \quad (A.62)$$

This is a recursive formula. After having regularized it we will write down the Duhamel series for this Green function.

Similarly, we can construct the function ψ. We notice first of all that for $x_0 \in \partial\Omega$ the distribution ψ solving the equation (A.59) is a solution to the equation

$$\begin{aligned} n \cdot \nabla_x W(x,n;x_0,n_0) &= \alpha^s(T(x)) \int_{\mathbb{S}^2} K(n,n') W(x,n';x_0,n_0)\ dn' \\ &\quad - \left(\alpha^a(T(x)) + \alpha^s(T(x))\right) W(x,n;x_0,n_0) + \delta(x-x_0) \frac{\delta^{(2)}(n,n_0)}{4\pi}. \end{aligned}$$

As we have computed above for f_0, as x approaches to x_0 the leading term of the distribution $W(x,n;x_0,n_0)$ is given by

$$W(x,n;x_0,n_0) \simeq H(n_0\cdot(x-x_0))\delta^{(2)}\left(P_{n_0}^\perp(x-x_0)\right) \frac{\delta^{(2)}(n,n_0)}{4\pi}.$$

As for the Poisson Kernel in the case of the Poisson equation, we expect W to differ from ψ only for a Jacobian as $x \to \overline{x} \in \partial\Omega$ with $n \cdot N_{\overline{x}} < 0$ and $n_0 \cdot N_{x_0} < 0$. We compute now the Jacobian. Hence, we consider $\varphi \in C_c^\infty(\partial\Omega)$ with $\operatorname{supp}(\varphi) \subset B_\varepsilon^{\partial\Omega}(\overline{x})$. We assume without loss of generality $\overline{x} = 0$, $N_{\overline{x}} = -e_1$ and $n \cdot e_3 = 0$. We compute

$$\begin{aligned} &\int_{\mathbb{S}^2} dn_0 \int_{\partial\Omega} dx_0\ \varphi(x_0) H(n_0\cdot(x-x_0))\delta^{(2)}\left(P_{n_0}^\perp(x-x_0)\right) \frac{\delta^{(2)}(n,n_0)}{4\pi} \\ &= \int_{\partial\Omega} dx_0\ \varphi(x_0) H(n\cdot(x-x_0))\delta^{(2)}\left(P_n^\perp(x-x_0)\right). \end{aligned} \quad (A.63)$$

For $\varepsilon > 0$ small enough we can approximate $\partial\Omega \cap B_\varepsilon^{\partial\Omega}(0)$ by $\mathbb{R}^3{}_+ \cap B_\varepsilon(0)$ and we define for $x_0^1 > 0$ the constant extension $\overline{\varphi}(x_0^1, x_0^2, x_0^3) = \varphi(x_0^2, x_0^3)$. Moreover we see that in the rotated coordinate system given by $y_1 \parallel n$ and $y_3 \parallel e_3$ we have

$$\begin{aligned}\delta^{(2)}(P_n^{\perp}(\overline{x}-x_0)) &= \delta(y_2)\delta(y_3)\\ &= \delta\left(-\sqrt{1-|n\cdot N_{\overline{x}}|^2}\left(\overline{x}^1-x_0^1\right)+|n\cdot N_{\overline{x}}|\left(\overline{x}^2-x_0^2\right)\right)\delta\left(x^3-x_0^3\right)\\ &= \frac{1}{|n\cdot N_{\overline{x}}|}\delta\left(-\tan(\theta)\left(\overline{x}^1-x_0^1\right)+\left(\overline{x}^2-x_0^2\right)\right)\delta\left(x^3-x_0^3\right), \end{aligned} \quad (A.64)$$

where θ is the angle between n and e_1, hence $|n\cdot N_{\overline{x}}|=\cos(\theta)$. Since $x\to\overline{x}$, we conclude our computation putting (A.64) into (A.63) and thus

$$\begin{aligned}&\int_{\mathbb{S}^2} dn_0 \int_{\partial\Omega} dx_0\ \varphi(x_0)H(n_0\cdot(x-x_0))\delta^{(2)}\left(P_{n_0}^{\perp}(x-x_0)\right)\frac{\delta^{(2)}(n,n_0)}{4\pi}\\ &= \int_{\mathbb{R}^3{}_+} dx_0\ \delta(\overline{x}^1-x_0^1)\frac{1}{|n\cdot N_{\overline{x}}|}\overline{\varphi}(x_0)\delta\left(-\tan(\theta)\left(\overline{x}^1-x_0^1\right)+\left(\overline{x}^2-x_0^2\right)\right)\delta\left(x^3-x_0^3\right)\\ &= \frac{\varphi(\overline{x})}{|n\cdot N_{\overline{x}}|}. \end{aligned} \quad (A.65)$$

We have just proved that in distributional sense we have

$$\begin{aligned}H(n_0\cdot(x-x_0))\delta^{(2)}\left(P_{n_0}^{\perp}(x-x_0)\right)\frac{\delta^{(2)}(n,n_0)}{4\pi} &\underset{x\to\overline{x}}{\longrightarrow} \frac{\delta_{\partial\Omega}(\overline{x}-x_0)}{|n\cdot N_{\overline{x}}|}\frac{\delta^{(2)}(n,n_0)}{4\pi}\\ &\overset{\mathcal{D}'}{=} \frac{\delta_{\partial\Omega}(\overline{x}-x_0)}{|n_0\cdot N_{x_0}|}\frac{\delta^{(2)}(n,n_0)}{4\pi}.\end{aligned}$$

Hence, as $x\to\overline{x}$ the distribution ψ is given at the leading order by

$$|n_0\cdot N_{x_0}|H(n_0\cdot(x-x_0))\delta^{(2)}\left(P_{n_0}^{\perp}(x-x_0)\right)\frac{\delta^{(2)}(n,n_0)}{4\pi}. \quad (A.66)$$

We note also, that $P_n^{\perp}(x-x_0)$ is not non-trivial only in a neighborhood of $y(x,n)\in\partial\Omega$ and $y(x,-n)\in\partial\Omega$. Moreover, $H(n\cdot(x-y(x,-n)))=0$ while $H(n\cdot(x-y(x,n)))=1$. Hence, with the same reasoning as in equations (A.63) and (A.65) we see that for any $x\in\Omega$

$$\begin{aligned}&\int_{\mathbb{S}^2} dn_0 \int_{\partial\Omega} dx_0\ |n_0\cdot N_{x_0}|\varphi(x_0)H(n_0\cdot(x-x_0))\delta^{(2)}\left(P_{n_0}^{\perp}(x-x_0)\right)\frac{\delta^{(2)}(n,n_0)}{4\pi}\\ &= \varphi(y(x,n)). \end{aligned} \quad (A.67)$$

We conclude the derivation of the Green function ψ integrating by characteristics the equation (A.59) with the boundary value given by (A.66) and we obtain

$$\begin{aligned}\psi(x,n;x_0,n_0) &= |n_0\cdot N_{x_0}|H(n_0\cdot(x-x_0))\\ &\times \delta^{(2)}\left(P_{n_0}^{\perp}(x-x_0)\right)\frac{\delta^{(2)}(n,n_0)}{4\pi}\exp\left(-\int_{[x_0,x]}\left[\alpha^a(u(\xi))+\alpha^s(u(\xi))\right]d\xi\right)\\ &+ \int_0^{s(x,n)} dt\ \alpha^s(u(x-tn))\exp\left(-\int_{[x-tn,x]}\left[\alpha^a(u(\xi))+\alpha^s(u(\xi))\right]d\xi\right)\\ &\times \int_{\mathbb{S}^2} dn' K(n,n')\psi(x-tn,n';x_0,n_0). \end{aligned} \quad (A.68)$$

A.4.3 Regularized fixed-point equation in the Grey case

We proceed now with the regularization of the fixed-point problem stated in (A.61). Similarly as we did in Section A.2 we regularized the fixed-point equation mollifying with a standard positive and rotationally symmetric mollifier the absorption and scattering coefficients. For $l \in \{a,s\}$ denote in order to simplify the notation $\alpha^l(u) * \phi_\varepsilon(x) = \alpha^l_\varepsilon(x)$. Notice that $\alpha^l_\varepsilon(x)$ still depends on the temperature. We recall also that

$$\int_{\mathbb{R}^3} f(\xi)\delta_{[x,y]} * \phi_\varepsilon(\xi)\, d\xi = \int_{[x,y]} f * \phi_\varepsilon(\xi)\, ds(\xi).$$

Hence, we define $I_\varepsilon(x,n;x_0)$ and $\psi_\varepsilon(x,n;x_0,n_0)$ solving the regularized equations

$$n\cdot\nabla_x I_\varepsilon(x,n;x_0) = \alpha^s_\varepsilon(x)\int_{\mathbb{S}^2} K(n,n')I_\varepsilon(x,n';x_0)\, dn' \\ -\left(\alpha^a_\varepsilon(x)+\alpha^s_\varepsilon(x)\right)I_\varepsilon(x,n;x_0)+\delta(x-x_0), \quad \text{(A.69)}$$

with zero incoming boundary conditions and

$$n\cdot\nabla_x\psi_\varepsilon(x,n;x_0,n_0) = \alpha^s_\varepsilon(x)\int_{\mathbb{S}^2} K(n,n')\psi_\varepsilon(x,n';x_0,n_0)\, dn' \\ -\left(\alpha^a_\varepsilon(x)+\alpha^a_\varepsilon(x)\right)\psi_\varepsilon(x,n;x_0,n_0),\ x\in\Omega \quad \text{(A.70)}$$
$$\psi_\varepsilon(x,n;x_0,n_0)\chi_{\{n\cdot n_x<0\}} = \delta_{\partial\Omega}(x-x_0)\frac{\delta^{(2)}(n,n_0)}{4\pi},\ x\in\Omega,\ n_0\cdot N_{x_0}<0.$$

Hence, the exact recursive formulas defining the regularized distributions are given by

$$I_\varepsilon(x,n;x_0) \\ = \chi_\Omega(x_0)\exp\left(-\int_{[x_0,x]}\left[\alpha^a_\varepsilon(\xi)+\alpha^s_\varepsilon(\xi)\right]d\xi\right)H(n\cdot(x-x_0))\delta^{(2)}\left(P^\perp_n(x-x_0)\right) \\ +\int_0^{s(x,n)} dt\ \alpha^s_\varepsilon(x-tn)\exp\left(-\int_{[x-tn,x]}\left[\alpha^a_\varepsilon(\xi)+\alpha^s_\varepsilon(\xi)\right]d\xi\right) \\ \times\int_{\mathbb{S}^2} dn' K(n,n')I_\varepsilon(x-tn,n';x_0), \quad \text{(A.71)}$$

and

$$\psi_\varepsilon(x,n;x_0,n_0) = |n_0\cdot N_{x_0}|H(n_0\cdot(x-x_0)) \\ \times\delta^{(2)}\left(P^\perp_{n_0}(x-x_0)\right)\frac{\delta^{(2)}(n,n_0)}{4\pi}\exp\left(-\int_{[x-tn,x]}\left[\alpha^a_\varepsilon(\xi)+\alpha^s_\varepsilon(\xi)\right]d\xi\right) \\ +\int_0^{s(x,n)} dt\ \alpha^s_\varepsilon(x-tn)\exp\left(-\int_{[x-tn,x]}\left[\alpha^a_\varepsilon(\xi)+\alpha^s_\varepsilon(\xi)\right]d\xi\right) \\ \times\int_{\mathbb{S}^2} dn' K(n,n')\psi_\varepsilon(x-tn,n';x_0,n_0). \quad \text{(A.72)}$$

Next we show the existence of regularized solutions u_ε to the equation

$$u_\varepsilon(x) = \int_\Omega dx_0\int_{\mathbb{S}^2} dn\ \frac{\alpha^a_\varepsilon(u_\varepsilon(x_0))u_\varepsilon(x_0)}{4\pi}I_\varepsilon(x,n;x_0) \\ +\int_{\mathbb{S}^2} dn_0\int_{\mathbb{S}^2} dn\int_{\partial\Omega} dx_0\ G(n_0)\psi_\varepsilon(x,n;x_0,n_0) \\ =: \mathcal{B}_\varepsilon(u_\varepsilon)(x)+\mathcal{C}_\varepsilon(u_\varepsilon)(x). \quad \text{(A.73)}$$

We aim to use Schauder fixed-point theorem. We start showing that the operator mapping $u \in \{C(\Omega) : 0 \le u \le M\}$ to the right-hand side of (A.73) is a self map taking M large enough. Similarly as in (A.21) we will first show that $\mathcal{B}^\varepsilon$ is a contractive operator. To this end we consider the function

$$H_\varepsilon(x,n) = \int_\Omega dx_0 \alpha_\varepsilon^a(u(x_0)) I_\varepsilon(x,n;x_0) \tag{A.74}$$

and we will show by means of the weak maximum principle formulated in the next Subsection that $0 \le H_\varepsilon(x,n) \le \theta < 1$, which will imply the contractivity as

$$|\mathcal{B}_\varepsilon(u)(x)| \le \|u\|_{\sup} \int_\Omega dx_0 \int_{\mathbb{S}^2} dn \, \frac{\alpha_\varepsilon^a(u(x_0))}{4\pi} I_\varepsilon(x,n;x_0) \le \theta \|u\|_{\sup}. \tag{A.75}$$

A.4.4 Weak maximum principle

In order to show that $\mathcal{B}^\varepsilon$ is a contractive operator we consider first the function H_ε defined in (A.74). Integrating with respect to x_0 the differential equation (A.69) satisfied by I_ε we obtain the differential equation satisfied by the function H_ε in the sense of distribution:

$$\begin{aligned} 0 = L_\varepsilon(H_\varepsilon)(x,n) - \alpha_\varepsilon^a(x) \\ &= n \cdot \nabla_x H_\varepsilon(x,n) - \alpha_\varepsilon^a(x) \left(1 - H_\varepsilon(x,n)\right) \\ &\quad - \alpha_\varepsilon^s(x) \int_{\mathbb{S}^2} dn' \, K(n,n') \left(H_\varepsilon(x,n) - H_\varepsilon(x,n')\right). \end{aligned} \tag{A.76}$$

With the following weak maximum principle we will show that $0 \le H_\varepsilon(x,n) \le 1$. To this end we consider the adjoint operator defined by

$$\begin{aligned} L_\varepsilon^*(\varphi)(x,n) = -n \cdot \nabla_x \varphi(x,n) + \left(\alpha_\varepsilon^a(x) + \alpha_\varepsilon^s(x)\right) \varphi(x,n) \\ - \alpha_\varepsilon^s(x) \int_{\mathbb{S}^2} dn' \, K(n,n') \varphi(x,n'). \end{aligned} \tag{A.77}$$

Lemma A.3 (Weak maximum principle)**.** *If a continuous bounded function $F(x,n)$ satisfies the boundary condition $F(x,n) \ge 0$ for $x \in \partial\Omega$ and $n \cdot n_x < 0$ and the inequality $\int_{\mathbb{S}^2} dn \int_\Omega dx \, L_\varepsilon^*(\varphi)(x,n) F(x,n) \ge 0$ for all non-negative $\varphi \in C^1\left(\bar{\Omega} \times \mathbb{S}^2\right)$ with $\varphi(x,n) = 0$ for $x \in \partial\Omega$ and $n \cdot n_x \ge 0$, then $F(x,n) \ge 0$ for all $x, n \in \Omega \times \mathbb{S}^2$.*

Remark. Before proving Lemma A.3 we notice that by definition H_ε is a continuous function which also satisfies $H_\varepsilon(x,n) = 0$ for $x \in \partial\Omega$ and $n \cdot n_x < 0$.

Proof. Assume that the claim of Lemma A.3 is not true. Then there exists an open set $U \subset \bar{\Omega} \times \mathbb{S}^2$ such that $F(x,n) < 0$ for every $(x,n) \in U$. Let $\xi \in C_c^1(U)$ with $\xi \ge 0$ and $\xi \ne 0$. We then consider the continuously differentiable function φ defined by

$$L_\varepsilon^*(\varphi)(x,n) = \xi(x,n). \tag{A.78}$$

Let us assume first that such φ exists. Then we can compute

$$\begin{aligned} 0 \le \int_{\mathbb{S}^2} dn \int_\Omega dx \, L_\varepsilon^*(\varphi)(x,n) F(x,n) = \int_{\mathbb{S}^2} dn \int_\Omega dx \, \xi(x,n) F(x,n) \\ = \int_U dn \, dx \, \xi(x,n) F(x,n) < 0. \end{aligned}$$

This contradiction implies the claim $F(x,n) \geq 0$ for all $x, n \in \Omega \times \mathbb{S}^2$.
We show now that such φ exists. Solving by characteristics the equation

$$L_\varepsilon^*(\varphi)(x,n) = \xi(x,n),$$

with boundary condition $\varphi(x,n) = 0$ for $x \in \partial\Omega$ and $n \cdot n_x > 0$ we obtain the following recursive formula

$$\begin{aligned}\varphi(x,n) = &\int_0^{s(x,-n)} \xi(x+tn,n) \exp\left(-\int_0^t \left[\alpha_\varepsilon^a(x+\tau n) + \alpha_\varepsilon^s(x+\tau n)\right] d\tau\right) dt \\ &+ \int_0^{s(x,-n)} dt\, \alpha_\varepsilon^s(x+tn) \exp\left(-\int_0^t \left[\alpha_\varepsilon^a(x+\tau n) + \alpha_\varepsilon^s(x+\tau n)\right] d\tau\right) \\ &\qquad \times \int_{\mathbb{S}^2} dn' K(n,n') \varphi(x+tn,n'),\end{aligned}$$

where $s(x,-n)$ is the length of the line connecting $x \in \Omega$ with the boundary $\partial\Omega$ in direction $n \in \mathbb{S}^2$. We still have to prove that φ is continuously differentiable and that it is non-negative. Since all functions α_ε^l, K and the exponential functions are non-negative and continuously differentiable we consider the Duhamel expansion of φ as

$$\begin{aligned}\varphi(x,n) = &\int_0^{s(x,-n)} \xi(x+tn,n) \exp\left(-\int_0^t \left[\alpha_\varepsilon^a(x+\tau n) + \alpha_\varepsilon^s(x+\tau n)\right] d\tau\right) dt \\ &+ \int_0^{s(x,-n)} dt\, \alpha_\varepsilon^s(x+tn) \exp\left(-\int_0^t \left[\alpha_\varepsilon^a(x+\tau n) + \alpha_\varepsilon^s(x+\tau n)\right] d\tau\right) \int_{\mathbb{S}^2} dn' K(n,n') \\ &\times \int_0^{s(x+tn,-n')} \xi(x+tn+t_1 n',n') \exp\left(-\int_0^t (\alpha_\varepsilon^a + \alpha_\varepsilon^s)(x+tn+\tau n') d\tau\right) dt_1 \\ &+ \cdots = \sum_{i=1}^\infty T_i(x,n). \qquad (A.79)\end{aligned}$$

Recursively, using

$$\int_0^D dr \; -\frac{d}{dr} \exp\left(-\int_0^r \left[\alpha_\varepsilon^a(z+rn) + \alpha_\varepsilon^s(z+rn)\right]\right) \leq 1 - e^{\|\alpha\|_\infty D} \theta < 1$$

for $D = \operatorname{diam}(\Omega)$ and $\|\alpha\|_\infty = \|\alpha^s + \alpha^a\|_\infty$, the symmetry of K so that

$$\int_{\mathbb{S}^2} dn' \; K(n,n') = 1,$$

we can estimate each term of the Duhamel expansion of φ by $|T_i(x,n)| \leq \|\xi\|_\infty D \theta^{i-1}$ and hence we obtain the absolute convergence of the Duhamel series since

$$\|\varphi\|_\infty \leq \|\xi\|_\infty D \sum_{i=0}^\infty \theta^i < \infty.$$

This implies the non-negativity and the continuity of φ. To prove that φ is differentiable one proceeds in the same way. We write the recursive formula for the derivative of φ and we estimate the Duhamel expansion similarly as we did for the boundedness of φ using this time also the uniformly boundedness of φ. We omit this computation since it is very similar to the one in (A.79). □

With this weak maximum principle we can carry on the proof of the contractivity of the operator $\mathcal{B}^\varepsilon$ in (A.73).

A.4.5 Existence of solution to the regularized problem

We can apply the weak maximum principle to $1 - H_\varepsilon(x,n)$. Indeed, $L_\varepsilon(1-H_\varepsilon) = 0$ in distributional sense. With an approximation argument we see that

$$\int_{\mathbb{S}^2} dn \int_\Omega dx\ L^*_\varepsilon(\varphi)(x,n)(1-H_\varepsilon(x,n)) \geq \int_{\mathbb{S}^2} dn \int_{\partial\Omega} dx\ \varphi(x,n)(1-H_\varepsilon(x,n)) n\cdot n_x \geq 0$$

for any non-negative $\varphi \in C^1\left(\bar{\Omega}\times\mathbb{S}^2\right)$ with $\varphi(x,n)=0$ if $x\in\partial\Omega$ and $n\cdot n_x \geq 0$. For similar arguments see also [82]. Therefore the weak maximum principle implies $H_\varepsilon(x,n)\leq 1$ for all $x,n\in\Omega\times\mathbb{S}^2$.

Hence, estimating then $H_\varepsilon(x,n)$ by 1 in the following equation obtained by integrating the equation (A.71) for I_ε we get

$$\begin{aligned}
H_\varepsilon(x,n) = &\int_\Omega dx_0\ \alpha^a_\varepsilon(u(x_0)) \exp\left(-\int_{[x_0,x]} [\alpha^a_\varepsilon(\xi)+\alpha^s_\varepsilon(\xi)]\,d\xi\right) \\
&\qquad \times H(n\cdot(x-x_0))\delta^{(2)}\left(P_n^\perp(x-x_0)\right) \\
&+ \int_0^{s(x,n)} dt\ \alpha^s_\varepsilon(x-tn)\exp\left(-\int_{[x-tn,x]}[\alpha^a_\varepsilon(\xi)+\alpha^s_\varepsilon(\xi)]\,d\xi\right) \\
\times \int_{\mathbb{S}^2} dn' K(n,n') H_\varepsilon(x-tn,n') = &\int_0^{s(x,n)} dt\ \alpha^a_\varepsilon(x-tn)\exp\left(-\int_{[x-tn,x]}[\alpha^a_\varepsilon(\xi)+\alpha^s_\varepsilon(\xi)]\,d\xi\right) \\
&+ \int_0^{s(x,n)} dt\ \alpha^s_\varepsilon(x-tn)\exp\left(-\int_{[x-tn,x]}[\alpha^a_\varepsilon(\xi)+\alpha^s_\varepsilon(\xi)]\,d\xi\right) \\
&\qquad \times \int_{\mathbb{S}^2} dn' K(n,n') H_\varepsilon(x-tn,n') \\
\leq &\int_0^{s(x,n)} dt\ \left(\alpha^a_\varepsilon(x-tn)+\alpha^s_\varepsilon(x-tn)\right)\exp\left(-\int_{[x-tn,x]}[\alpha^a_\varepsilon(\xi)+\alpha^s_\varepsilon(\xi)]\,d\xi\right) \\
&\qquad \leq \left(1-e^{-\|\alpha\|_\infty D}\right) = \theta < 1,
\end{aligned}$$

where D is the diameter of Ω, $\|\alpha\|_\infty = \|\alpha^a+\alpha^s\|_\infty$. The second equality is given solving the delta distribution together with the Heaviside function, while the first inequality is obtained by the isotropy of K so that $\int_{\mathbb{S}^2} K(n,n')\,dn' = 1$. Thus, equation (A.75) implies that $\mathcal{B}^\varepsilon$ is contractive with $|\mathcal{B}_\varepsilon(u)(x)| \leq \theta\|u\|_{\sup}$.

We move now to the estimate for the boundary term given by $\mathcal{C}_\varepsilon(u)$. It is enough to show that this term is uniformly bounded (say by a constant $C>0$), then for $M \geq \frac{C}{1-\theta}$ we have $|\mathcal{B}_\varepsilon(u)+\mathcal{C}_\varepsilon(u)| \leq M$ for all $x\in\Omega$ and $0\leq u\leq M$. In order to prove the boundedness we expand this boundary term in its Duhamel series taking as starting point the equation satisfied by $\mathcal{C}_\varepsilon(u)$. We simplify the notation denoting by $A^l_\varepsilon(y,z-y)$ the function

$$A^l_\varepsilon(y,z-y) = \frac{\alpha^l_\varepsilon(u(y))\exp\left(-\int_{[y,z]}[\alpha^a_\varepsilon(u)+\alpha^s_\varepsilon(u)]\,d\xi\right)}{|z-y|^2}, \tag{A.80}$$

for $l\in\{a,s\}$ and by $E_\varepsilon(z,\omega)$ the function of $\omega\in\mathbb{S}^2$ and $z\in\Omega$ given by

$$E_\varepsilon(z,\omega) = \exp\left(-\int_{[y(z,\omega),z]}[\alpha^a_\varepsilon(u)+\alpha^s_\varepsilon(u)]\,d\xi\right). \tag{A.81}$$

We put (A.72) into the definition of $\mathcal{C}_\varepsilon$ in (A.73), we use (A.67) and the notation above and we compute

$$\begin{aligned}\mathcal{C}_\varepsilon(u)(x) = &\int_{\mathbb{S}^2} dn\, G(n)E_\varepsilon(x,n)\\ &+\int_\Omega d\eta A^s_\varepsilon(\eta,x-\eta)\int_{\mathbb{S}^2} dn_0 G(n_0)E_\varepsilon(\eta,n_0)K\left(\frac{x-\eta}{|x-\eta|},n_0\right)\\ &+\int_\Omega d\eta A^s_\varepsilon(\eta,x-\eta)\int_\Omega d\eta_1\, A^s_\varepsilon(\eta_1,\eta-\eta_1)K\left(\frac{x-\eta}{|x-\eta|},\frac{\eta-\eta_1}{|\eta-\eta_1|}\right)\\ &\times\int_{\mathbb{S}^2} dn_0 G(n_0)E_\varepsilon(\eta_1,n_0)K\left(\frac{\eta-\eta_1}{|\eta-\eta_1|},n_0\right)+\cdots=\sum_{i=1}^\infty \mathcal{U}^\varepsilon_i(u)(x). \quad (A.82)\end{aligned}$$

We now estimate every term $\mathcal{U}^\varepsilon_i$. The first term is estimated by

$$|\mathcal{U}^\varepsilon_1(u)|\leq 4\pi\|G\|_\infty,$$

since the exponential term is bounded by 1. Estimating again the exponential term by 1 and $g(n)$ by $\|G\|_\infty$ and using the isotropy of the scattering kernel we compute

$$\begin{aligned}|\mathcal{U}^\varepsilon_2(u)|&\leq \|G\|_\infty\int_\Omega d\eta A^s_\varepsilon(\eta,x-\eta)\\ &\leq \|G\|_\infty\int_{\mathbb{S}^2} dn\int_0^D dr\left(-\frac{d}{dr}\exp\left(-\int_0^r dt\ [\alpha^a_\varepsilon(x-tn)+\alpha^s_\varepsilon(x-tn)]\right)\right)\\ &\leq 4\pi\theta\|G\|_\infty, \quad (A.83)\end{aligned}$$

where $\theta=1-e^{-\|\alpha\|_\infty D}$. For the next terms we proceed similarly.

$$\begin{aligned}|\mathcal{U}^\varepsilon_3(u)|&\leq \|G\|_\infty\int_\Omega d\eta A^s_\varepsilon(\eta,x-\eta)\int_\Omega d\eta_1\, A^s_\varepsilon(\eta_1,\eta-\eta_1)K\left(\frac{x-\eta}{|x-\eta|},\frac{\eta-\eta_1}{|\eta-\eta_1|}\right)\\ &\leq \theta\|G\|_\infty\int_\Omega d\eta\ A^s_\varepsilon(\eta,x-\eta)\leq 4\pi\theta^2\|G\|_\infty, \quad (A.84)\end{aligned}$$

where we estimated

$$\int_{\mathbb{S}^2} dn\int_0^D\left(-\frac{d}{dr}\exp\left(-\int_0^r dt\ [\alpha^a_\varepsilon(\eta-tn)+\alpha^s_\varepsilon(\eta-tn)]\right)\right)K\left(\frac{x-\eta}{|x-\eta|},n\right)\leq\theta.$$

Recursively, we conclude that the boundary term is uniformly bounded as

$$|\mathcal{C}_\varepsilon(u)(x)|\leq\sum_{i=1}^\infty|\mathcal{U}^\varepsilon_i(u)(x)|\leq 4\pi\|G\|_\infty\sum_{i=0}^\infty\theta^i=4\pi\|G\|_\infty\frac{1}{1-\theta}<\infty. \quad (A.85)$$

In a similar way, combining the fact that each term $\mathcal{U}^\varepsilon_i$ maps continuously bounded (continuous) maps to bounded (continuous) maps and the uniform absolute convergence of the Duhamel series we can conclude that $\mathcal{C}_\varepsilon$ is a continuous operator. Next we prove that $\mathcal{B}_\varepsilon(u)(x)+\mathcal{C}_\varepsilon(u)(x)$ is Hölder continuous. This will imply on the one hand that the operator is a self map and on the other hand that it is compact. Hence, Schauder fixed-point theorem concludes the existence of the regularize solutions satisfying (A.73). Before starting this proof we recall that we have shown in Section A.3.1 the Hölder continuity of all kind of operators given by

$$\int_\Omega d\eta\ \frac{\alpha^l_\varepsilon(u(\eta))\exp\left(-\int_{[\eta,x]}[\alpha^a_\varepsilon(u)+\alpha^s_\varepsilon(u)]\,d\xi\right)}{|x-\eta|^2},$$

for $l \in \{a, s\}$ and

$$\int_{\mathbb{S}^2} dn\ \exp\left(-\int_{[y(x,n),x]} [\alpha^a_\varepsilon(u) + \alpha^s_\varepsilon(u)]\, d\xi\right).$$

Moreover, in order to see the Hölder continuity of the interior term we have to expand the recursive formula of $\mathcal{B}_\varepsilon$ in its Duhamel series. We hence put (A.71) into the definition of $\mathcal{B}_\varepsilon$ in (A.73) and we compute

$$\begin{aligned}\mathcal{B}_\varepsilon(u)(x) =& \int_\Omega dx_0\ \frac{A^a_\varepsilon(x_0, x-x_0)u(x_0)}{4\pi}\\ &+ \int_\Omega d\eta \int_\Omega d\eta_1\ A^s_\varepsilon(\eta, x-\eta)\frac{A^a_\varepsilon(\eta_1, \eta-\eta_1)u(\eta_1)}{4\pi} K\left(\frac{x-\eta}{|x-\eta|}, \frac{\eta-\eta_1}{|\eta-\eta_1|}\right)\\ &+ \int_\Omega d\eta \int_\Omega d\eta_1 \int_\Omega d\eta_2\ A^s_\varepsilon(\eta, x-\eta)A^s_\varepsilon(\eta_1, \eta-\eta_1)\frac{A^a_\varepsilon(\eta_2, \eta_1-\eta_2)u(\eta_2)}{4\pi}\\ &\times K\left(\frac{x-\eta}{|x-\eta|}, \frac{\eta-\eta_1}{|\eta-\eta_1|}\right) K\left(\frac{\eta-\eta_1}{|\eta-\eta_1|}, \frac{\eta_1-\eta_2}{|\eta_1-\eta_2|}\right) + \cdots = \sum_{i=1}^\infty \mathcal{V}^\varepsilon_i(u)(x),\end{aligned} \tag{A.86}$$

where we used that

$$\int_{\mathbb{S}^2} dn\ H(n\cdot(x-x_0))\delta^{(2)}(P^\perp_n(x-x_0)) = \frac{1}{|x-x_0|^2},$$

in distributional sense.

Remark. Notice that (A.82) and (A.86) encode the fact that due to the scattering the photons move along a polygonal line.

Notice in addition that also $\mathcal{B}_\varepsilon$ maps continuously bounded (continuous) functions to bounded (continuous) functions. This is due to the uniform absolute convergence of the Duhamel series (similar calculation as for (A.85) and (A.89)) and the continuity of each term $\mathcal{V}^\varepsilon_i$ in (A.86).

We aim to show the Hölder continuity of the operators $\mathcal{B}^\varepsilon$ and $\mathcal{C}^\varepsilon$. We consider hence $u \in C(\Omega)$ with $0 \le u \le M$. As we did in Subsection A.3.1 we extend u continuously on the boundary $\partial\Omega$ and then u, $\alpha^a(u)$ and $\alpha^s(u)$ by zero outside $\bar{\Omega}$. We proceed now estimating term by term the following difference for $h \in \mathbb{R}^3$ and $x,\ x+h \in \mathbb{R}^3$

$$|\mathcal{B}_\varepsilon(u)(x) - \mathcal{B}_\varepsilon(u)(x+h)| \le \sum_{i=1}^\infty |\mathcal{V}^\varepsilon_i(x) - \mathcal{V}^\varepsilon_i(x+h)|\,.$$

For the first term we use the result in (A.28) and (A.32) and conclude

$$|\mathcal{V}^\varepsilon_1(x) - \mathcal{V}^\varepsilon_1(x+h)| \le C(\Omega, \|\alpha\|_\infty, \phi_\varepsilon)\|u\|_\infty\, |h|^{\frac{1}{2}}\,.$$

For the next order terms we need also to estimate expressions of the form

$$\left|K\left(\frac{x-\eta}{|x-\eta|}, \frac{\eta-\xi}{|\eta-\xi|}\right) - K\left(\frac{x+h-\eta}{|x+h-\eta|}, \frac{\eta-\xi}{|\eta-\xi|}\right)\right|.$$

Using the property of K being continuously differentiable in both variables and making use of the triangle inequality we know that there exists a constant $C_K > 0$ depending exclusively on K such that

$$\left|K\left(\frac{x-\eta}{|x-\eta|}, \frac{\eta-\xi}{|\eta-\xi|}\right) - K\left(\frac{x+h-\eta}{|x+h-\eta|}, \frac{\eta-\xi}{|\eta-\xi|}\right)\right| \le \frac{2C_K|h|^{\frac{1}{2}}}{|x+h-\eta|^{\frac{1}{2}}}. \tag{A.87}$$

We can hence proceed with the second term of the Duhamel series. We apply the triangle inequality first and then we combine the results for (A.28) and (A.32) with the estimate (A.87) and with the fact that $|x|^{-\frac{5}{2}} \in L^1(B_1(0))$. Then we have

$$\begin{aligned}
|\mathcal{V}_2^\varepsilon(x) - \mathcal{V}_2^\varepsilon(x+h)| \leq \|u\| \int_\Omega d\eta \int_\Omega d\eta_1 \; & \frac{A_\varepsilon^a(\eta_1, \eta-\eta_1)}{4\pi} K\left(\frac{x-\eta}{|x-\eta|}, \frac{\eta-\eta_1}{|\eta-\eta_1|}\right) \\
& \times |A_\varepsilon^s(\eta, x-\eta) - A_\varepsilon^s(\eta, x+h-\eta)| \\
+ \|u\| \int_\Omega d\eta \int_\Omega d\eta_1 \; & \frac{A_\varepsilon^a(\eta_1, \eta-\eta_1)}{4\pi} A_\varepsilon^s(\eta, x+h-\eta) \\
\times \left| K\left(\frac{x-\eta}{|x-\eta|}, \frac{\eta-\eta_1}{|\eta-\eta_1|}\right) - K\left(\frac{x+h-\eta}{|x+h-\eta|}, \frac{\eta-\eta_1}{|\eta-\eta_1|}\right) \right| \\
\leq \|u\|_\infty \theta \int_\Omega d\eta \, \frac{1}{4\pi} \, & |A_\varepsilon^s(\eta, x-\eta) - A_\varepsilon^s(\eta, x+h-\eta)| \\
+ \|u\|_\infty C_K |h|^{\frac{1}{2}} \theta \int_\Omega d\eta \, & \frac{A_\varepsilon^s(\eta, x+h-\eta)}{|x+h-\eta|^{\frac{1}{2}}} \\
& \leq \|u\|_\infty \theta \left(C(\Omega, \phi_\varepsilon, \|\alpha\|_\infty) + 4\pi C_K \|\alpha\|_\infty D^{\frac{1}{2}} \right) |h|^{\frac{1}{2}}, \quad \text{(A.88)}
\end{aligned}$$

where we estimated as we did above

$$\begin{aligned}
\int_\Omega d\eta_1 \; & A_\varepsilon^a(\eta_1, \eta-\eta_1) K\left(\frac{x-\eta}{|x-\eta|}, \frac{\eta-\eta_1}{|\eta-\eta_1|}\right) \\
& \leq \int_{\mathbb{S}^2} dn \int_0^D dr \; -\frac{d}{dr} \exp\left(-\int_0^r \ldots\right) K\left(\frac{x-\eta}{|x-\eta|}, n\right) \leq \theta,
\end{aligned}$$

with $\theta = 1 - e^{-\|\alpha\|_\infty D}$ and similarly also

$$\int_\Omega d\eta_1 \; A_\varepsilon^a(\eta_1, \eta-\eta_1) \frac{1}{4\pi} \leq \theta.$$

We can iterate this procedure for all terms in the Duhamel series and we obtain the following estimate

$$\begin{aligned}
& |\mathcal{B}_\varepsilon(u)(x) - \mathcal{B}_\varepsilon(u)(x+h)| \\
& \quad \leq \|u\|_\infty \left(C(\Omega, \phi_\varepsilon, \|\alpha\|_\infty) + 4\pi C_K \|\alpha\|_\infty D^{\frac{1}{2}} \right) |h|^{\frac{1}{2}} \sum_{i=0}^\infty \theta^i \\
& \quad = \|u\|_\infty \left(C(\Omega, \phi_\varepsilon, \|\alpha\|_\infty) + 4\pi C_K \|\alpha\|_\infty D^{\frac{1}{2}} \right) |h|^{\frac{1}{2}} \frac{1}{1-\theta}. \quad \text{(A.89)}
\end{aligned}$$

Using the result (A.34) combined with (A.87) we see in the same way that also the boundary term operator is Hölder continuous with

$$\begin{aligned}
& |\mathcal{C}_\varepsilon(u)(x) - \mathcal{C}_\varepsilon(u)(x+h)| \\
& \quad \leq 4\pi \|G\|_\infty \left(C(\Omega, \phi_\varepsilon, \|\alpha\|_\infty) + C_K D^{\frac{1}{2}} \|\alpha\|_\infty \right) \left(1 + \frac{1}{1-\theta} \right) |h|^{\frac{1}{2}}. \quad \text{(A.90)}
\end{aligned}$$

Indeed, we compute using (A.34) for the first term in the Duhamel series of the boundary term

$$|\mathcal{U}_1^\varepsilon(x) - \mathcal{U}_1^\varepsilon(x+h)| \;\leq\; \|G\|_\infty \int_{\mathbb{S}^2} dn \quad |E_\varepsilon(x,n) - E_\varepsilon(x+h,n)| \;\leq\; \|G\|_\infty C_\varepsilon C(\Omega) |h|^{\frac{1}{2}}.$$

Moreover, the estimates (A.28), (A.32) together with (A.87) gives for the second term

$$
\begin{aligned}
&|\mathcal{U}_2^\varepsilon(x) - \mathcal{U}_2^\varepsilon(x+h)| \\
&\quad \le \|G\|_\infty \int_{\mathbb{S}^2} dn_0 \int_\Omega d\eta\, K\left(\frac{x-\eta}{|x-\eta|}, n_0\right) |A_\varepsilon^s(\eta, x-\eta) - A_\varepsilon^s(\eta, x+h-\eta)| \\
&\quad + \|G\|_\infty \int_{\mathbb{S}^2} dn_0 \int_\Omega d\eta\, A_\varepsilon^s(\eta, x+h-\eta) \left| K\left(\frac{x-\eta}{|x-\eta|}, n_0\right) - K\left(\frac{x+h-\eta}{|x+h-\eta|}, n_0\right)\right| \\
&\quad \le \|G\|_\infty |h|^{\frac{1}{2}} \left(C_\varepsilon C(\Omega) + C_K 4\pi D^{\frac{1}{2}}\right).
\end{aligned}
$$

Similarly integrating first with respect to n_0, then with respect to η_1 and finally with respect to η we obtain

$$
\begin{aligned}
&|\mathcal{U}_3^\varepsilon(x) - \mathcal{U}_3^\varepsilon(x+h)| \\
&\quad \le \|G\|_\infty \int_\Omega d\eta \int_\Omega d\eta_1 \int_{\mathbb{S}^2} dn_0\, A_\varepsilon^s(\eta_1, \eta-\eta_1) K\left(\frac{x-\eta}{|x-\eta|}, \frac{\eta-\eta_1}{|\eta-\eta_1|}\right) \\
&\qquad \times K\left(\frac{\eta-\eta_1}{|\eta-\eta_1|}, n_0\right) |A_\varepsilon^s(\eta, x-\eta) - A_\varepsilon^s(\eta, x+h-\eta)| \\
&\quad + \|G\|_\infty \int_\Omega d\eta \int_\Omega d\eta_1 \int_{\mathbb{S}^2} dn_0\, A_\varepsilon^s(\eta_1, \eta-\eta_1) A_\varepsilon^s(\eta, x+h-\eta) K\left(\frac{\eta-\eta_1}{|\eta-\eta_1|}, n_0\right) \\
&\qquad \times \left| K\left(\frac{x-\eta}{|x-\eta|}, \frac{\eta-\eta_1}{|\eta-\eta_1|}\right) - K\left(\frac{x+h-\eta}{|x+h-\eta|}, \frac{\eta-\eta_1}{|\eta-\eta_1|}\right)\right| \\
&\quad \le \|G\|_\infty \theta \int_\Omega d\eta\, |A_\varepsilon^s(\eta, x-\eta) - A_\varepsilon^s(\eta, x+h-\eta)| \\
&\qquad + \|G\|_\infty \theta C_K |h|^{\frac{1}{2}} \int_\Omega d\eta \frac{A_\varepsilon^s(\eta, x+h-\eta)}{|x+h-\eta|^{\frac{1}{2}}} \\
&\quad \le \|G\|_\infty \theta |h|^{\frac{1}{2}} \left(C_\varepsilon C(\Omega) + C_K 4\pi D^{\frac{1}{2}}\right).
\end{aligned}
$$

Iterating this procedure we obtain the estimate (A.90).

Hence, (A.89) and (A.90) imply that the operator $\mathcal{B}_\varepsilon + \mathcal{C}_\varepsilon$ is a compact selfmap, mapping continuously uniformly bounded continuous functions to Hölder continuous functions. Thus, by the Schauder fixed-point theorem we obtain for every $\varepsilon > 0$ a solution u_ε to (A.73).

A.4.6 Compactness of the sequence of regularized solution and proof of Theorem A.4

Proof of Theorem A.4. In order to end the proof of Theorem A.4, we will show that the sequence of regularized solution u_ε to the equation (A.73) is compact in L^2. We already know that this is true in the case of pure absorption and emission, as we have seen in Subsection A.3.3. We will use the compactness result of Subsection A.3.3 in order to show that the same result holds also in the case of scattering. A crucial role is played in this proof by the result of Proposition A.2. Let us consider a sequence $\varepsilon_j = \frac{1}{j}$. In order to simplify the notation we define the sequence of regularized solutions $u_{\varepsilon_j} = u_j$, the coefficients $\alpha_{\varepsilon_j}^l(u_{\varepsilon_j}) = \alpha_j^l(u_j)$ as well as all kind of operators $\mathcal{B}_{\varepsilon_j} = \mathcal{B}_j$, $\mathcal{C}_{\varepsilon_j} = \mathcal{C}_j$, $A_{\varepsilon_j}^l(y, z+y) = A_j^l(y, z+y)$ and $E_{\varepsilon_j}(z, \omega) = E_j(z, \omega)$.

By the uniformly boundedness of the sequence and the boundedness of Ω we have only to show the equicontinuity, i.e. we want to prove that for any $\beta > 0$ there exists a $H_1(\beta) > 0$

such that

$$\int_\Omega dx\ |\mathcal{B}_j(u_j)(x) + \mathcal{C}_j(u_j)(x) - \mathcal{B}_j(u_j)(x+h) - \mathcal{C}_j(u_j)(x+h)|^2 < C(\alpha^s, \alpha^a, \Omega, M, G)\beta \quad \text{(A.91)}$$

for all $|h| < H_1$ and all $j \in \mathbb{N}$. Notice that the constant $C(\alpha^s, \alpha^a, \Omega, M, G)$ is independent of $j \in \mathbb{N}$, of $\beta > 0$ and of $h \in \mathbb{R}^3$. This would imply the L^2-compactness of the sequence $\mathcal{B}_j(u_j) + \mathcal{C}_j(u_j)$.

In order to prove this statement we start recalling the boundedness of the interior term $\mathcal{B}_j(u_j)$ and of the boundary term $\mathcal{C}_j(u_j)$ as

$$\sup_{x\in\Omega} |\mathcal{B}_j(u_j)(x)| \leq \sup_{x\in\Omega} \sum_{i=1}^{\infty} \left|\mathcal{V}_i^j(x)\right| \leq M\theta \sum_{i=0}^{\infty} (\theta)^i = \frac{M\theta}{1-\theta}, \quad \text{(A.92)}$$

where $\theta = 1 - e^{\|\alpha\|_\infty D} < 1$. The computation is similar to the one we did in (A.85) for the boundary term and to the Hölder estimate in (A.88). Moreover, (A.85) implies

$$\sup_{x\in\Omega} |\mathcal{C}_j(u_j)(x)| \leq \sup_{x\in\Omega} \sum_{i=1}^{\infty} \left|\mathcal{U}_i^j(x)\right| \leq \frac{4\pi}{1-\theta}\|G\|_\infty.$$

Hence, let $\beta > 0$. There exists an $N_0(\beta) > 0$ such that

$$\sup_{x\in\Omega} \left| \sum_{i=N_0}^{\infty} \left|\mathcal{V}_i^j(x)\right| + \left|\mathcal{U}_i^j(x)\right| \right|^2 < \beta. \quad \text{(A.93)}$$

Thus, using the triangle inequality we obtain

$$\begin{aligned}
&\int_\Omega dx\ |\mathcal{B}_j(u_j)(x) + \mathcal{C}_j(u_j)(x) - \mathcal{B}_j(u_j)(x+h) - \mathcal{C}_j(u_j)(x+h)|^2 \\
&\leq 2\beta|\Omega| + 2\int_\Omega dx \left| \sum_{i=1}^{N_0-1} \mathcal{V}_i^j(x) - \mathcal{V}_i^j(x+h) \right|^2 \\
&\quad + 2\int_\Omega dx \left| \sum_{i=1}^{N_0-1} \mathcal{U}_i^j(x) - \mathcal{U}_i^j(x+h) \right|^2 \\
&\leq 2\beta|\Omega| + 2N_0 \sum_{i=1}^{N_0-1} \int_\Omega dx\ \left|\mathcal{V}_i^j(x) - \mathcal{V}_i^j(x+h)\right|^2 \\
&\quad + 2N_0 \sum_{i=1}^{N_0-1} \int_\Omega dx\ \left|\mathcal{U}_i^j(x) - \mathcal{U}_i^j(x+h)\right|^2. \quad \text{(A.94)}
\end{aligned}$$

We aim to use for each term

$$\int_\Omega dx\ \left|\mathcal{V}_i^j(x) - \mathcal{V}_i^j(x+h)\right|^2 \text{ and } \int_\Omega dx\ \left|\mathcal{U}_i^j(x) - \mathcal{U}_i^j(x+h)\right|^2,$$

for $0 \leq 1 \leq N_0 - 1$ the results in Corollary A.2 and Proposition A.2 in order to show that they are equi-integrable.

Let us start considering the interior terms and we write each of them in spherical coordinates. We extend by 0 all the functions α^l and u_j which are defined only on the domain Ω. We denote by $D = \operatorname{diam}(\Omega)$ as usually. For $i = 1$ we have

$$\mathcal{V}_i^j(x) = \int_{\mathbb{S}^2} dn \int_0^D dr u_j(x - rn)\alpha_j^a\left(u_j(x - rn)\right) \times \exp\left(-\int_0^r \left[\alpha_j^a\left(u_j(x - \lambda n)\right) + \alpha_j^s\left(u_j(x - \lambda n)\right)\right] d\lambda\right).$$

We notice that taking $\varphi_j(x) = u_j(x)\alpha_j^a\left(u_j(x)\right)$ and $\psi_j(x) = \alpha_j^a\left(u_j(x)\right) + \alpha_j^s\left(u_j(x)\right)$ Corollary A.2 implies the compactness of the first interior term. Let us consider the second term. There we define

$$F_j^{(2)}(x, \omega) = \int_0^D d\lambda \int_{\mathbb{S}^2} dn\, K(\omega, n)u_j(x - rn)\alpha_j^a\left(u_j(x - \lambda n)\right) \times \exp\left(-\int_0^\lambda \left[\alpha_j^a\left(u_j(x - rn)\right) + \alpha_j^s\left(u_j(x - rn)\right)\right] dr\right)$$

We notice that $F_j^{(2)}$ is uniformly bounded in both variables and that it is uniformly continuous with respect to the second variable. Indeed, we can estimate on the one hand

$$\left|F_j^{(2)}(x, \omega)\right| \leq M\theta \int_{\mathbb{S}^2} dn\, K(\omega, n) = M\theta \tag{A.95}$$

and on the other hand also

$$\left|F_j^{(2)}(x, \omega_1) - F_j^{(2)}(x, \omega_2)\right| \leq M\theta \int_{\mathbb{S}^2} dn\ \left|K(\omega_1, n) - K(\omega_2, n)\right| \leq 4\pi M\theta C_K d(\omega_1, \omega_2). \tag{A.96}$$

Hence, defining also the error term

$$\mathcal{R}_j^{(2)}(x) = \fint_{\mathbb{S}^2} dn \int_{(s(x,n))}^D d\lambda F_j^{(2)}(x - \lambda n, n)\alpha_j^s\left(u_j(x - \lambda n)\right) \times \exp\left(-\int_0^\lambda \left[\alpha_j^a\left(u_j(x - rn)\right) + \alpha_j^s\left(u_j(x - rn)\right)\right] dr\right) \tag{A.97}$$

we can write the second term of the operator $\mathcal{B}_j$ as

$$\mathcal{V}_j^2(x) = -\mathcal{R}_j^{(2)}(x) + \fint_{\mathbb{S}^2} dn \int_0^D d\lambda F_j^{(2)}(x - \lambda n, n)\alpha_j^s\left(u_j(x - \lambda n)\right) \times \exp\left(-\int_0^\lambda \left[\alpha_j^a\left(u_j(x - rn)\right) + \alpha_j^s\left(u_j(x - rn)\right)\right] dr\right). \tag{A.98}$$

We notice that since $\alpha_j(u_j)$ is supported in $\Omega + \frac{1}{j}$ we can estimate the error term by

$$\left|\mathcal{R}_j^{(2)}(x)\right| \leq M\|\alpha\|_\infty \theta C(\Omega)\left(\frac{1}{j}\right)^{\frac{1}{2}}.$$

Hence, taking $\varphi_j(x, \omega) = F_j^{(2)}(x, \omega)\alpha_j^s\left(u_j(x)\right)$ and $\psi_j(x) = \alpha_j^a\left(u_j(x)\right) + \alpha_j^s\left(u_j(x)\right)$, Proposition A.2 implies the compactness in $L^2(\Omega)$ of $\mathcal{V}_j^2(x) + \mathcal{R}_j^{(2)}(x)$.

We are ready now for the generalization of this result. We define for $i \geq 2$ the functions $F_j^{(i)}(x,\omega)$ and $\mathcal{R}_j^{(i)}(x)$ by

$$F_j^{(i)}(x,\omega) = \int_0^D d\lambda_2 ... \int_0^D d\lambda_i \int_{\mathbb{S}^2} dn_2 ... \int_{\mathbb{S}^2} dn_i \, K(\omega, n_2)...K(n_{i-1}, n_i)$$
$$\times \alpha_j^s (u_j(x - \lambda_2 n_2)) \exp\left(- \int_0^{\lambda_2} \left[\alpha_j^a (u_j(x - rn_2)) + \alpha_j^s (u_j(x - rn_2)) \right] dr \right)$$
$$\times ... \times u_j(x - \lambda_2 n_2 + \cdots - \lambda_i n_i) \alpha_j^a (u_j(x - \lambda_2 n_2 + \cdots - \lambda_i n_i))$$
$$\times \exp\left(- \int_0^{\lambda_i} \left[\alpha_j^a (u_j(x - \lambda_2 n_2 + \cdots - rn_i)) + \alpha_j^s (u_j(x - \lambda_2 n_2 + \cdots - rn_i)) \right] dr \right) \quad \text{(A.99)}$$

and

$$\mathcal{V}_i^j(x) + \mathcal{R}_j^{(i)}(x) = \fint_{\mathbb{S}^2} dn \int_0^D d\lambda F_j^{(i)}(x - \lambda n, n) \alpha_j^s (u_j(x - \lambda n))$$
$$\times \exp\left(- \int_0^{\lambda} \left[\alpha_j^a (u_j(x - rn)) + \alpha_j^s (u_j(x - rn)) \right] dr \right). \quad \text{(A.100)}$$

Thus, we estimate

$$\left| F_j^{(i)}(x,\omega) \right| \leq M\theta^{i-1}, \; \left| \mathcal{R}_j^{(i)}(x) \right| \leq (i-1) M \|\alpha\|_\infty \theta^{i-1} C(\Omega) \left(\frac{1}{j} \right)^{\frac{1}{2}},$$

as well as

$$\left| F_j^{(i)}(x,\omega_1) - F_j^{(i)}(x,\omega_2) \right| \leq 4\pi M \theta^{i-1} C_K d(\omega_1, \omega_2).$$

Again, Proposition A.2 implies the L^2-compactness of $\mathcal{V}_i^j(x) + \mathcal{R}_j^{(i)}(x)$. The compactness of $\mathcal{V}_i^j(x) + \mathcal{R}_j^{(i)}(x)$ for $1 \leq i < N_0(\beta)$ implies the existence of an $h_0 > 0$ such that

$$\int_\Omega dx \left| \mathcal{V}_i^j(x) - \mathcal{V}_i^j(x+h) \right|^2 \leq \frac{\beta}{2N_0(\beta)} + |\Omega| N_0 M \|\alpha\|_\infty \theta^{i-1} C(\Omega) \left(\frac{1}{j} \right) \quad \text{(A.101)}$$

for all $|h| < h_0$, for all $j \geq 0$ and for all $1 \leq i < N_0$. Hence,

$$\int_\Omega dx \; |\mathcal{B}_j(u_j)(x) - \mathcal{B}_j(u_j)(x+h)|^2$$
$$\leq 2\beta|\Omega| + 2N_0 \sum_{i=1}^{N_0-1} \int_\Omega dx \; \left| \mathcal{V}_i^j(x) - \mathcal{V}_i^j(x+h) \right|^2$$
$$\leq \beta \left(2|\Omega| + 1 \right) + C(\Omega, \|\alpha\|_\infty, M) N_0^2 \frac{1}{j}, \quad \text{(A.102)}$$

for all $|h| < h_0$ and for all $j \geq 0$.

We examine now to the operator $\mathcal{C}_j(x)$ associated to the boundary term. We proceed similarly as for the interior term rewriting each expression $\mathcal{U}_i^j$ in spherical coordinates. We start as usual with $i = 1$, where we have

$$\mathcal{U}_1^j(x) = \int_{\mathbb{S}^2} dn \, G(n) \exp\left(- \int_0^{s(x,n)} \left[\alpha_j^a (u_j(x - rn)) + \alpha_j^s (u_j(x - rn)) \right] dr \right)$$
$$= \int_{\mathbb{S}^2} dn \, G(n) \exp\left(- \int_0^{D} \left[\alpha_j^a (u_j(x - rn)) + \alpha_j^s (u_j(x - rn)) \right] dr \right) + \mathcal{R}_j^{(1)}(x),$$

where

$$
\begin{aligned}
\left|\mathcal{R}_j^{(1)}(x)\right| &= \left|\int_{\mathbb{S}^2} dn\, G(n) \left[\exp\left(-\int_0^{s(x,n)} \left[\alpha_j^a\left(u_j(x-rn)\right) + \alpha_j^s\left(u_j(x-rn)\right)\right] dr\right)\right.\right. \\
&\qquad \left.\left. - \exp\left(-\int_0^{D} \left[\alpha_j^a\left(u_j(x-rn)\right) + \alpha_j^s\left(u_j(x-rn)\right)\right] dr\right)\right]\right| \\
&\le \int_{\mathbb{S}^2} dn\, G(n) \int_{s(x,n)}^{D} \left|\alpha_j^a\left(u_j(x-rn)\right) + \alpha_j^s\left(u_j(x-rn)\right)\right| dr \\
&\qquad\qquad \le \|G\|_{L^1}\|\alpha\|_\infty C(\Omega)\left(\frac{1}{j}\right)^{\frac{1}{2}}.
\end{aligned}
$$

Moreover, taking $\psi_j = \alpha_j^a(u_j) + \alpha_j^s(u_j)$ and $f = G$, Corollary A.2 implies the compactness of $\mathcal{U}_1^j - \mathcal{R}_j^{(1)}$ in $L^2(\Omega)$.

We proceed with $i = 2$. Here with the change of variables $\eta = x - \lambda_1 n_1$ we obtain

$$
\begin{aligned}
\mathcal{U}_2^j(x) &= \int_\Omega d\eta\ A_j^s(\eta, x-\eta) \int_{\mathbb{S}^2} dn_0 G(n_0) E_j(\eta, n_0) K\left(\frac{x-\eta}{|x-\eta|}, n_0\right) \\
&= \int_{\mathbb{S}^2} dn_1 \int_0^{s(x,n_1)} d\lambda_1 \int_{\mathbb{S}^2} dn_0 K(n_1, n_0) G(n_0) \alpha_j^s(u_j(x-\lambda_1 n_1)) \\
&\qquad \times \exp\left(-\int_0^{\lambda_1} \left[\alpha_j^a\left(u_j\right) + \alpha_j^s\left(u_j\right)\right](x - rn_1) dr\right) \\
&\quad \times \exp\left(-\int_0^{s(x-\lambda_1 n_1, n_0)} \left[\alpha_j^a\left(u_j\right) + \alpha_j^s\left(u_j\right)\right](x - \lambda_1 n_1 - rn_0) dr\right) \\
&= \mathcal{R}_j^{(2)}(x) + \int_{\mathbb{S}^2} dn_1 \int_0^D d\lambda_1 Q_j^{(2)}(x-\lambda_1 n_1, n_1) \alpha_j^s(u_j(x-\lambda_1 n_1)) \\
&\qquad\qquad \times \exp\left(-\int_0^{\lambda_1} \left[\alpha_j^a\left(u_j\right) + \alpha_j^s\left(u_j\right)\right](x - rn_1) dr\right),
\end{aligned}
$$

where

$$
Q_j^{(2)}(x,\omega) = \int_{\mathbb{S}^2} dn\ G(n) K(\omega, n) \exp\left(-\int_0^D \left[\alpha_j^a\left(u_j\right) + \alpha_j^s\left(u_j\right)\right](x - rn) dr\right)
$$

and

$$
\left|\mathcal{R}_j^{(2)}(x)\right| \le C(G,\Omega)\|\alpha\|_\infty (\theta + 1)\left(\frac{1}{j}\right)^{\frac{1}{2}}.
$$

Moreover, we see $\left|Q_j^{(2)}(x,\omega)\right| \le \|G\|_\infty$ and a similar computation to the one in (A.96) shows

$$
\left|Q_j^{(2)}(x,\omega_1) - Q_j^{(2)}(x,\omega_2)\right| \le \|G\|_{L^1} C_K d(\omega_1,\omega_2).
$$

Hence, Proposition (A.2) implies for $\varphi_j(x,\omega) = \alpha_j^s(u_j(x)) Q_j^{(2)}(x,\omega)$ and $\psi_j = \alpha_j^a(u_j) + \alpha_j^s(u_j)$ the L^2-compactness of $\mathcal{U}_2^j - \mathcal{R}_j^{(2)}$.

We proceed iteratively. We define for $i \geq 3$ the functions

$$\begin{aligned} Q_j^{(i)}(x,\omega) = \int_{\mathbb{S}^2} dn_2 \int_0^D & d\lambda_2 K(\omega,n_2)\alpha_j^s(u_j(x-\lambda_2 n_2)) \\ & \times \exp\left(-\int_0^{\lambda_2} \left[\alpha_j^a(u_j)+\alpha_j^s(u_j)\right](x-rn_2)dr\right) \\ & \times \ldots \times \int_{\mathbb{S}^2} dn_i G(n_i) K(n_{i-1},n_i) \\ & \times \exp\left(-\int_0^{D} \left[\alpha_j^a(u_j)+\alpha_j^s(u_j)\right](x-\lambda_2 n_2 - \ldots - rn_i)dr\right) \end{aligned} \quad \text{(A.103)}$$

and $\mathcal{R}_j^{(i)} = \mathcal{U}_i^j - Q_j^{(i)}$. Similarly as for the case $i=2$ we can estimate

$$\left|Q_j^{(i)}(x,\omega)\right| \leq \theta^{(i-2)}\|G\|_\infty$$

and

$$\left|Q_j^{(i)}(x,\omega_1) - Q_j^{(i)}(x,\omega_2)\right| \leq \theta^{(i-2)}\|G\|_{L^1} C_K d(\omega_1,\omega_2).$$

Moreover, the remainder statisfies

$$\left|\mathcal{R}_j^{(i)}(x)\right| \leq C(G,\Omega)\|\alpha\|_\infty \theta^{(i-2)} \left(\frac{1}{j}\right)^{\frac{1}{2}}.$$

Again, Proposition A.2 implies the L^2-compactness of $\mathcal{U}_i^j(x) - \mathcal{R}_j^{(i)}(x)$. The compactness of $\mathcal{U}_i^j(x) - \mathcal{R}_j^{(i)}(x)$ for $1 \leq i < N_0(\beta)$ implies the existence of an $h_1 > 0$ such that

$$\int_\Omega dx \left|\mathcal{U}_i^j(x) - \mathcal{U}_i^j(x+h)\right|^2 \leq \frac{\beta}{2N_0(\beta)} + |\Omega| N_0 C(G,\Omega)\|\alpha\|_\infty \theta^{(i-2)}\left(\frac{1}{j}\right) \quad \text{(A.104)}$$

for all $|h| < h_1$, for all $j \geq 0$ and for all $1 \leq i < N_0$. Hence,

$$\begin{aligned} \int_\Omega dx \ & |\mathcal{C}_j(u_j)(x) - \mathcal{C}_j(u_j)(x+h)|^2 \\ & \leq 2\beta|\Omega| + 2N_0 \sum_{i=1}^{N_0-1} \int_\Omega dx \ \left|\mathcal{U}_i^j(x) - \mathcal{U}_i^j(x+h)\right|^2 \\ & \leq \beta\left(2|\Omega|+1\right) + C(\Omega,\|\alpha\|_\infty,G) N_0^2 \frac{1}{j}, \end{aligned} \quad \text{(A.105)}$$

for all $|h| < h_1$ and for all $j \geq 0$. Putting equations (A.102) and (A.105) in (A.94) we obtain for $\beta > 0$, which was chosen arbitrary, the following estimate

$$\begin{aligned} \int_\Omega dx \ & |\mathcal{B}_j(u_j)(x) + \mathcal{C}_j(u_j)(x) - \mathcal{B}_j(u_j)(x+h) - \mathcal{C}_j(u_j)(x+h)|^2 \\ & \leq \beta\left(4|\Omega|+2\right) + C(\Omega,\|\alpha\|_\infty,G,M) N_0^2 \frac{1}{j}, \end{aligned} \quad \text{(A.106)}$$

for all $|h| < \min(h_0,h_1)$ and for all $j \geq 0$. Taking now $J_0 = \frac{2N_0(\beta)^2}{\beta}$ we obtain

$$\begin{aligned} \int_\Omega dx \ & |\mathcal{B}_j(u_j)(x) + \mathcal{C}_j(u_j)(x) - \mathcal{B}_j(u_j)(x+h) - \mathcal{C}_j(u_j)(x+h)|^2 \\ & \leq C(\Omega,\|\alpha\|_\infty,G,M)\beta, \end{aligned} \quad \text{(A.107)}$$

for all $|h| < \min(h_0, h_1)$ and for all $j \geq J_0$. Since $J_0 \in \mathbb{N}$ is finite and $\mathcal{B}_j(u_j)$ and $\mathcal{C}_j(u_j)$ are continuous there exists an $H_0 \leq \min(h_0, h_1)$ such that

$$\int_\Omega dx \ |\mathcal{B}_j(u_j)(x) + \mathcal{C}_j(u_j)(x) - \mathcal{B}_j(u_j)(x+h) - \mathcal{C}_j(u_j)(x+h)|^2 \leq \beta, \tag{A.108}$$

for all $|h| < H_0$ and $1 \leq j < J_0$. Hence, we have just proved that the uniformly bounded and tight sequence $\mathcal{B}_j(u_j) + \mathcal{C}_j(u_j)$ is equicontinuous in L^2 and thus compact. There exists hence a subsequence $u_{j_l} = \mathcal{B}_{j_l}(u_{j_l}) + \mathcal{C}_{j_l}(u_{j_l})$ and a function $u \in L^2(\Omega) \cap L^\infty(\Omega)$ such that $u_{j_l} = \mathcal{B}_{j_l}(u_{j_l}) + \mathcal{C}_{j_l}(u_{j_l}) \to u$ in $L^2(\Omega)$ and pointwise almost everywhere as $l \to \infty$.

The uniformly boundedness of u_{j_l} and also of $\alpha^a(u_{j_l})$ and $\alpha^s(u_{j_l})$ implies the convergence in L^p of $\alpha^a(u_{j_l}) * \phi_{j_l} \to \alpha^a(u)$ and $\alpha^s(u_{j_l}) * \phi_{j_l} \to \alpha^s(u)$ as $l \to \infty$ for $p < \infty$. Therefore for a subsequence (say still u_{j_l}) the convergence holds also pointwise almost everywhere. Finally a combination of the dominated convergence theorem for finitely many terms in terms in the Duhamel series and the convergence of such Duhamel series implies

$$u_{j_l} = \mathcal{B}_{j_l}(u_{j_l}) + \mathcal{C}_{j_l}(u_{j_l}) \to u = \mathcal{B}(u) + \mathcal{C}(u) \tag{A.109}$$

pointwise almost everywhere as $l \to \infty$ and $u = \mathcal{B}(u) + \mathcal{C}(u)$ pointwise almost everywhere. Hence, u is the desired solution to (A.61). □

A.4.7 Existence of solution for the pseudo Grey case

We want to show the existence of solutions also in the pseudo Grey case, i.e. when the absorption and scattering coefficient depends also on the frequency via the relation $\alpha_\nu^a(T(x)) = Q_a(\nu)\alpha^a(T(x))$ and $\alpha_\nu^s(T(x)) = Q_s(\nu)\alpha^s(T(x))$. We assume that $Q_i \in C^1(\mathbb{R}_+)$ and $\alpha^i \in C^1(\mathbb{R}_+)$ for $i = a, s$. It is not difficult to see that similarly as Theorem A.3 implies Proposition A.2 and the Corollary A.2, also Theorem A.1 and the Federer-Besicovitch covering's lemma implies the following Proposition.

Proposition A.3. *Let* $\{\varphi_j\}_{j\in\mathbb{N}} \subset L^\infty(\Omega, L^1(\mathbb{R}_+))$ *and* $\{\psi_j\}_{j\in\mathbb{N}} \subset L^\infty(\Omega, L^1(\mathbb{R}_+))$ *be two non-negative bounded sequences with* $\Omega \subset \mathbb{R}^3$ *bounded, convex with* C^2*-boundary and strictly positive curvature. Let also* $f \in L^\infty(\mathbb{S}^2, L^1(\mathbb{R}_+))$ *be non-negative. Then the sequences*

$$\int_{\mathbb{S}^2} dn \int_0^D dr \int_0^\infty d\nu \varphi_j(x - rn, \nu) \exp\left(-\int_0^r \psi_j(x - \lambda n, \nu) d\lambda\right)$$

and

$$\int_{\mathbb{S}^2} dn \int_0^\infty d\nu \ f(n,\nu) \exp\left(-\int_0^D \psi_j(x - \lambda n, \nu) d\lambda\right)$$

are compact in $L^2(\Omega)$*. If moreover* $\{\varphi_j\}_{j\in\mathbb{N}} \subset C(\mathbb{S}^2, L^\infty(\Omega, L^1(\mathbb{R}_+)))$ *with*

$$\|\varphi_j(\cdot,\cdot,\omega_1) - \varphi_j(\cdot,\cdot,\omega_2)\|_{L^\infty(\Omega, L^1(\mathbb{R}_+))} \leq \sigma(d(\omega_1,\omega_2)) \to 0$$

as $d(\omega_1,\omega_2) \to 0$*, where* d *is the metric on the sphere and* $\sigma \in C(\mathbb{R}_+, \mathbb{R}_+)$ *with* $\sigma(0) = 0$ *is a uniform modulus of continuity, then the sequence*

$$\int_{\mathbb{S}^2} dn \int_0^D dr \int_0^\infty d\nu \varphi_j(x - rn, \nu, n) \exp\left(-\int_0^r \psi_j(x - \lambda n, \nu) d\lambda\right)$$

is also compact in $L^2(\Omega)$*.*

Now we are ready to prove the existence of solution in the pseudo Grey case.

Proof of Theorem A.2. We proceed similarly as we did in the proof of Theorem A.4 indicating where differences arise. With the same notation as in Theorem A.4 we define therefore the Green functions $\tilde{I}_\nu(x, n\, x_0)$ $\psi_\nu(x, n; x_0, n_0)$ by

$$\begin{aligned} n \cdot \nabla_x \tilde{I}_\nu(x, n; x_0) = Q_s(\nu)\alpha^s(T(x)) \int_{\mathbb{S}^2} K(n, n')\tilde{I}_\nu(x, n'; x_0)\, dn' \\ - \left(Q_a(\nu)\alpha^a\left(T(x)\right) + Q_s(\nu)\alpha^s\left(T(x)\right)\right)\tilde{I}_\nu(x, n; x_0) + \delta(x - x_0) \end{aligned} \tag{A.110}$$

with boundary condition for $x \in \partial\Omega$

$$\tilde{I}_\nu(x, n; x_0)\chi_{\{n\cdot n_x<0\}} = 0$$

and

$$\begin{aligned} n \cdot \nabla_x \psi_\nu(x, n; x_0, n_0) = Q_s(\nu)\alpha^s(T(x)) \int_{\mathbb{S}^2} K(n, n')\psi_\nu(x, n'; x_0, n_0)\, dn' \\ - \left(Q_a(\nu)\alpha^a\left(T(x)\right) + Q_s(\nu)\alpha^s\left(T(x)\right)\right)\psi_\nu(x, n; x_0, n_0),\ x \in \Omega, \\ \psi_\nu(x, n; x_0, n_0)\chi_{\{n\cdot n_x<0\}} = \delta_{\partial\Omega}(x - x_0)\frac{\delta^{(2)}(n, n_0)}{4\pi},\ x \in \Omega,\ n_0 \cdot N_{x_0} < 0. \end{aligned} \tag{A.111}$$

Then the intensity of radiation can be expressed in terms of these two functions as follows.

$$\begin{aligned} I_\nu(x, n) = \int_\Omega dx_0\, Q_a(\nu)\alpha^a(T(x_0))B_\nu(T(x_0))\tilde{I}_\nu(x, n; x_0) \\ + \int_{\mathbb{S}^2} dn_0 \int_{\partial\Omega} dx_0\, g_\nu(n_0)\psi_\nu(x, n; x_0, n_0). \end{aligned} \tag{A.112}$$

Once again plugging in the definition of $I_\nu(x, n)$ into equation (A.6) we obtain the following fixed-point equation

$$\begin{aligned} u(x) = \int_{\mathbb{S}^2} dn \int_0^\infty d\nu \int_\Omega dx_0\, Q_a(\nu)^2\alpha^a(u(x_0))B_\nu\left(F^{-1}(u(x_0))\right)\tilde{I}_\nu(x, n\, x_0) \\ + \int_0^\infty d\nu \int_{\mathbb{S}^2} dn \int_{\mathbb{S}^2} dn_0 \int_\partial \Omega dx_0\, Q_a(\nu)g_\nu(n_0)\psi_\nu(x, n; x_0, n_0), \end{aligned} \tag{A.113}$$

where $u(x) = 4\pi \int_0^\infty Q_a(\nu)B_\nu(T(x)) = F(T(x))$. Since B_ν is a monotone function of the temperature, F is invertible.

Once more, we regularize the equation through a a sequence ϕ_ε of standard positive radial symmetric mollifiers. We hence define $I^\varepsilon_\nu(x, n\, x_0)$ and $\psi^\varepsilon_\nu(x, n; x_0, n_0)$ by

$$\begin{aligned} n \cdot \nabla_x I^\varepsilon_\nu(x, n; x_0) = Q_s(\nu)\alpha^s(T(\cdot)) * \phi_\varepsilon(x) \int_{\mathbb{S}^2} K(n, n')I^\varepsilon_\nu(x, n'; x_0)\, dn' \\ - \left(Q_a(\nu)\alpha^a\left(T(\cdot)\right) * \phi_\varepsilon(x) + Q_s(\nu)\alpha^s\left(T(\cdot)\right) * \phi_\varepsilon(x)\right)I^\varepsilon_\nu(x, n; x_0) + \delta(x - x_0) \end{aligned} \tag{A.114}$$

with boundary condition for $x \in \partial\Omega$

$$I^\varepsilon_\nu(x, n; x_0)\chi_{\{n\cdot n_x<0\}} = 0$$

and

$$\begin{aligned} n \cdot \nabla_x \psi^\varepsilon_\nu(x, n; x_0, n_0) = Q_s(\nu)\alpha^s(T(\cdot)) * \phi_\varepsilon(x) \int_{\mathbb{S}^2} K(n, n')\psi^\varepsilon_\nu(x, n'; x_0, n_0)\, dn' \\ - \left(Q_a(\nu)\alpha^a\left(T(\cdot)\right) * \phi_\varepsilon(x) + Q_s(\nu)\alpha^s\left(T(\cdot)\right) * \phi_\varepsilon(x)\right)\psi^\varepsilon_\nu(x, n; x_0, n_0),\ x \in \Omega, \\ \psi^\varepsilon_\nu(x, n; x_0, n_0)\chi_{\{n\cdot n_x<0\}} = \delta_{\partial\Omega}(x - x_0)\frac{\delta^{(2)}(n, n_0)}{4\pi},\ x \in \Omega,\ n_0 \cdot N_{x_0} < 0. \end{aligned} \tag{A.115}$$

The associated regularized fixed-point equation for

$$u(x) = \int_0^\infty d\nu\ Q_a(\nu) B_\nu(T(x)) = F(T(x)),$$

is defined for any $\varepsilon > 0$ by

$$\begin{aligned} u_\varepsilon(x) = &\int_0^\infty d\nu Q_a(\nu)^2 \int_\Omega dx_0 \int_{\mathbb{S}^2} dn\ \alpha_\varepsilon^a(x_0) B_\nu \left(F^{-1}(u_\varepsilon(x_0))\right) I_\nu^\varepsilon(x,n;x_0) \\ &+ \int_0^\infty d\nu Q_a(\nu) \int_{\mathbb{S}^2} dn_0 \int_{\mathbb{S}^2} dn \int_{\partial\Omega} dx_0\ G(n_0) \psi_\nu^\varepsilon(x,n;x_0,n_0) \\ &=: \mathcal{B}_\varepsilon(u_\varepsilon)(x) + \mathcal{C}_\varepsilon(u_\varepsilon)(x), \end{aligned} \tag{A.116}$$

where u_ε is the solution of the fixed-point equation for $\varepsilon > 0$ and we used the notation $\alpha_\varepsilon^i(x) = \alpha^i(u_\varepsilon(\cdot)) * \phi_\varepsilon(x)$ for $i = a, s$. The same reasoning and computations we did in Subsection (A.4.2) hold also in this case, so that we can write the explicit recursive formula for both I_ν^ε and ψ_ν^ε as

$$\begin{aligned} I_\nu^\varepsilon(x,n;x_0) = &\chi_\Omega(x_0) \exp\left(-\int_{[x_0,x]} [Q_a(\nu)\alpha_\varepsilon^a(\xi) + Q_s(\nu)\alpha_\varepsilon^s(\xi)]\, d\xi\right) \\ &\times H(n\cdot(x-x_0))\delta^{(2)}\left(P_n^\perp(x-x_0)\right) \\ &+ \int_0^{s(x,n)} dt\ Q_s(\nu)\alpha_\varepsilon^s(x-tn) \exp\left(-\int_{[x-tn,x]} [Q_a(\nu)\alpha_\varepsilon^a(\xi) + Q_s(\nu)\alpha_\varepsilon^s(\xi)]\, d\xi\right) \\ &\times \int_{\mathbb{S}^2} dn' K(n,n') I_\varepsilon(x-tn,n';x_0), \end{aligned} \tag{A.117}$$

and

$$\begin{aligned} \psi_\nu^\varepsilon(x,n;x_0,n_0) = &|n_0\cdot N_{x_0}| H(n_0\cdot(x-x_0)) \\ &\times \delta^{(2)}\left(P_{n_0}^\perp(x-x_0)\right) \frac{\delta^{(2)}(n,n_0)}{4\pi} \exp\left(-\int_{[x-tn,x]} [Q_a(\nu)\alpha_\varepsilon^a(\xi) + Q_s(\nu)\alpha_\varepsilon^s(\xi)]\, d\xi\right) \\ &+ \int_0^{s(x,n)} dt\ Q_s(\nu)\alpha_\varepsilon^s(x-tn) \exp\left(-\int_{[x-tn,x]} [Q_a(\nu)\alpha_\varepsilon^a(\xi) + Q_s(\nu)\alpha_\varepsilon^s(\xi)]\, d\xi\right) \\ &\times \int_{\mathbb{S}^2} dn' K(n,n') \psi_\varepsilon(x-tn,n';x_0,n_0). \end{aligned} \tag{A.118}$$

With these expressions we recover also the Duhamel representation of the bulk and boundary operators by

$$\begin{aligned} \mathcal{B}_\varepsilon(u)(x) = &\int_0^\infty d\nu Q_a(\nu) \int_\Omega dx_0\ \frac{Q_a(\nu)\alpha_\varepsilon^a(x_0) B_\nu(u(x_0))}{|x-x_0|^2} \\ &\times \exp\left(-\int_{[x_0,x]} [Q_a(\nu)\alpha_\varepsilon^a(\xi) + Q_s(\nu)\alpha_\varepsilon^s(\xi)]\, d\xi\right) \end{aligned} \tag{A.119}$$

$$
\begin{aligned}
&+\int_0^\infty d\nu Q_a(\nu)\int_\Omega d\eta\int_\Omega d\eta_1\, K\left(\frac{x-\eta}{|x-\eta|},\frac{\eta-\eta_1}{|\eta-\eta_1|}\right)\frac{Q_s(\nu)\alpha_\varepsilon^s(\eta)}{|x-\eta|^2}\\
&\quad\times\exp\left(-\int_{[\eta,x]}\left[Q_a(\nu)\alpha_\varepsilon^a(\xi)+Q_s(\nu)\alpha_\varepsilon^s(\xi)\right]d\xi\right)\frac{Q_a(\nu)\alpha_\varepsilon^a(\eta_1)B_\nu\left(u(\eta_1)\right)}{|\eta-\eta_1|^2}\\
&\qquad\times\exp\left(-\int_{[\eta_1,\eta]}\left[Q_a(\nu)\alpha_\varepsilon^a(\xi)+Q_s(\nu)\alpha_\varepsilon^s(\xi)\right]d\xi\right)\\
&+\int_0^\infty d\nu Q_a(\nu)\int_\Omega d\eta\int_\Omega d\eta_1\int_\Omega d\eta_2\, A_\varepsilon^s(\eta,x-\eta,\nu)A_\varepsilon^s(\eta_1,\eta-\eta_1,\nu)\\
&\times A_\varepsilon^a(\eta_2,\eta_1-\eta_2,\nu)B_\nu(u(\eta_2))K\left(\frac{x-\eta}{|x-\eta|},\frac{\eta-\eta_1}{|\eta-\eta_1|}\right)K\left(\frac{\eta-\eta_1}{|\eta-\eta_1|},\frac{\eta_1-\eta_2}{|\eta_1-\eta_2|}\right)\\
&\qquad\qquad+\cdots=\sum_{i=1}^\infty\mathcal{V}_i^\varepsilon(u)(x),
\end{aligned}
$$

where we used the definition

$$
A_\varepsilon^i(z,y-z,\nu)=\frac{Q_i(\nu)\alpha_\varepsilon^i(z)}{|y-z|^2}\exp\left(-\int_{[z,y]}\left[Q_a(\nu)\alpha_\varepsilon^a(\xi)+Q_s(\nu)\alpha_\varepsilon^s(\xi)\right]d\xi\right).
$$

For the boundary operator we obtain similarly

$$
\begin{aligned}
&\mathcal{C}_\varepsilon(u)(x)\\
&=\int_0^\infty d\nu\; Q_a(\nu)\int_{\mathbb{S}^2}dn\; g_\nu(n)\exp\left(-\int_{[y(x,n),x]}\left[Q_a(\nu)\alpha_\varepsilon^a(u)+Q_s(\nu)\alpha_\varepsilon^s(u)\right]d\xi\right)\\
&\quad+\int_0^\infty d\nu\; Q_a(\nu)\int_\Omega d\eta A_\varepsilon^s(\eta,x-\eta,\nu)\int_{\mathbb{S}^2}dn_0 g_\nu(n_0)\\
&\quad\times\exp\left(-\int_{[y(\eta,n_0),\eta]}\left[Q_a(\nu)\alpha_\varepsilon^a(u)+Q_s(\nu)\alpha_\varepsilon^s(u)\right]d\xi\right)K\left(\frac{x-\eta}{|x-\eta|},n_0\right)\\
&\quad+\int_0^\infty d\nu\; Q_a(\nu)\int_\Omega d\eta A_\varepsilon^s(\eta,x-\eta,\nu)\int_\Omega d\eta_1\; A_\varepsilon^s(\eta_1,\eta-\eta_1,\nu)\\
&\qquad\times K\left(\frac{x-\eta}{|x-\eta|},\frac{\eta-\eta_1}{|\eta-\eta_1|}\right)\int_{\mathbb{S}^2}dn_0 g_\nu(n_0)\\
&\quad\times\exp\left(-\int_{[y(\eta_1,n_0),\eta]}\left[Q_a(\nu)\alpha_\varepsilon^a(u)+Q_s(\nu)\alpha_\varepsilon^s(u)\right]d\xi\right)K\left(\frac{\eta-\eta_1}{|\eta-\eta_1|},n_0\right)\\
&\qquad\qquad+\cdots=\sum_{i=1}^\infty\mathcal{U}_i^\varepsilon(u)(x).
\end{aligned}
\tag{A.120}
$$

It can be shown, as we did in the pure Grey case, that the operator $\mathcal{B}_\varepsilon$ is a contraction, while the operator $\mathcal{C}_\varepsilon$ is bounded. For the first claim, we need to use a new version of the weak maximum-principle. We see that defining the function $H_\varepsilon(x,n,\nu)$ by

$$
H_\varepsilon(x,n,\nu)=\int_\Omega dx_0\; Q_a(\nu)\alpha_\varepsilon^a(x_0)I_\nu^\varepsilon\nu(u(x_0))
$$

it satisfies the equation

$$\begin{aligned} 0 = L_\varepsilon(H_\varepsilon)(x,n,\nu) - Q_a(\nu)\alpha_\varepsilon^a(x) \\ = n\cdot\nabla_x H_\varepsilon(x,n,\nu) - Q_a(\nu)\alpha_\varepsilon^a(x)\left(1 - H_\varepsilon(x,n,\nu)\right) \\ - Q_s(\nu)\alpha_\varepsilon^s(x)\int_{\mathbb{S}^2} dn'\; K(n,n')\left(H_\varepsilon(x,n,\nu) - H_\varepsilon(x,n',\nu)\right). \quad \text{(A.121)} \end{aligned}$$

Notice that also in this case by definition H_ε is non-negative, continuous and bounded, where the last assertions are due to its Duhamel expansion. Defining the adjoint operator by

$$\begin{aligned} L_\varepsilon^*(\varphi)(x,n,\nu) = -n\cdot\nabla_x\varphi(x,n,\nu) + \left(Q_a(\nu)\alpha_\varepsilon^a(x) + Q_s(\nu)\alpha_\varepsilon^s(x)\right)\varphi(x,n,\nu) \\ - Q_s(\nu)\alpha_\varepsilon^s(x)\int_{\mathbb{S}^2} dn'\; K(n,n')\varphi(x,n',\nu) \quad \text{(A.122)} \end{aligned}$$

we use the following weak-maximum principle

Lemma A.4. *If a continuous bounded $F(x,n,\nu)$ satisfies the boundary condition $F(x,n,\nu)\geq 0$ for $x\in\partial\Omega$ and $n\cdot n_x<0$ and the inequality*

$$\int_0^\infty d\nu\int_{\mathbb{S}^2} dn\;\int_\Omega dx\; L_\varepsilon^*(\varphi)(x,n,\nu)F(x,n,\nu)\geq 0$$

for all non-negative $\varphi\in C^1\left(\bar\Omega\times\mathbb{S}^2\times\mathbb{R}_+\right)$ with $\varphi(x,n,\nu)=0$ for $x\in\partial\Omega$ and $n\cdot n_x\geq 0$, then $F(x,n,\nu)\geq 0$ for all $x,n,\nu\in\Omega\times\mathbb{S}^2\times\mathbb{R}_+$.

Proof. We assume that Lemma A.4 is not true. Hence, there exists an open set $U\subset\bar\Omega\times\mathbb{S}^2\times\mathbb{R}_+$ such that $F(x,n,\nu)<0$ there. Taking then a function $\xi\in C_c^1(U)$ with $\xi\geq 0$ and $\xi\neq 0$ we define the non-negative continuously differentiable function $\varphi(x,n,\nu)$ by

$$L_\varepsilon^*(\varphi)(x,n,\nu) = \xi(x,n,\nu)$$

and with boundary condition $\varphi(x,n,\nu)=0$ for $x\in\partial\Omega$ and $n\cdot n_x>0$. As we did in the proof of Lemma A.3 one can show that $\varphi\geq 0$ and that it is continuously differentiable in all variables. Finally, one uses the constructed function in order to obtain a contradiction since

$$0\leq\int_0^\infty d\nu\int_{\mathbb{S}^2} dn\int_\Omega dx L_\varepsilon^*(\varphi)(x,n,\nu)F(x,n,\nu) = \int_U d\nu dn dx\xi(x,n,\nu)F(x,n,\nu)<0.$$

□

Using the fact that $L_\varepsilon(1-H_\varepsilon)(x,n,\nu)=0$ and the weak maximum principle in Lemma A.4 we conclude that $0\leq H_\varepsilon\leq 1$, where we used that by definition $H_\varepsilon(x,n,\nu)=0$ for $x\in\partial\Omega$ and $n\cdot n_x<0$. Once again, using the recursive formula for $H_\varepsilon(x,n,\nu)$ and the estimate

$$\int_0^D dr|f(x-rn)|\exp\left(-\int_0^r|f(x-tn)|dt\right)\leq 1-e^{-\|f\|_\infty D}<1,$$

we obtain, by defining $\theta = 1-e^{-\|\alpha_\nu\|D}<1$ for $\|\alpha_\nu\| = \|Q_a\alpha^a + Q_s\alpha^s\|_\infty$, the following estimate

$$
\begin{aligned}
0 \le H_\varepsilon(x,n,\nu) = & \int_0^{s(x,n)} dt\ \alpha_\varepsilon^a(x-tn)Q_a(\nu) \\
& \times \exp\left(-\int_0^t [Q_a(\nu)\alpha_\varepsilon^a(x-rn) + Q_s(\nu)\alpha_\varepsilon^s(x-rn)]\, dr\right) \\
+ \int_0^{s(x,n)} dt\ \alpha_\varepsilon^s(x-tn)Q_s(\nu) & \exp\left(-\int_0^t [Q_a(\nu)\alpha_\varepsilon^a(x-rn) + Q_s(\nu)\alpha_\varepsilon^s(x-rn)]\, dr\right) \\
& \times \int_{\mathbb{S}^2} dn' K(n,n')H_\varepsilon(x-tn,n',\nu) \\
\le & \int_0^{s(x,n)} dt\, (\alpha_\varepsilon^a(x-tn)Q_a(\nu) + \alpha_\varepsilon^s(x-tn)Q_s(\nu)) \\
& \times \exp\left(-\int_0^t [Q_a(\nu)\alpha_\varepsilon^a(x-rn) + Q_s(\nu)\alpha_\varepsilon^s(x-rn)]\, dr\right) \le \theta < 1.
\end{aligned}
$$

Hence, we conclude the contractivity of the bulk operator via

$$
\begin{aligned}
0 \le \mathcal{B}_\varepsilon(u)(x) = & \int_0^\infty d\nu Q_a(\nu)^2 \int_\Omega dx_0 \int_{\mathbb{S}^2} dn\ \alpha_\varepsilon^a(x_0)u_\varepsilon(x_0)I_\nu^\varepsilon(x,n;x_0) \\
& \le \int_0^\infty d\nu Q_a(\nu)B_\nu\left(F^{-1}(\|u\|_\infty)\right)H_\varepsilon(x,n,\nu) \\
& \qquad \le \theta F\left(F^{-1}(\|u\|_\infty)\right) = \theta\|u\|_\infty.
\end{aligned}
$$

On the other hand also the boundary term is bounded, indeed in the same way as we had in the pure grey case using the fact that

$$\int_0^\infty d\nu Q_a(\nu)\int_{\mathbb{S}^2} dn\ g_\nu(n) \le \|Q\|_\infty\|g\|,$$

we obtain

$$|\mathcal{U}_i^\varepsilon(x)| \le \|Q\|_\infty\|g\|\theta^{i-1},$$

for $\theta = 1 - e^{-\|\alpha_\nu\|D} < 1$. Hence, the Duhamel series is absolutely convergent and

$$|\mathcal{C}_\varepsilon(u)(x)| \le C(Q,\alpha,D,g) < \infty.$$

Moreover, the continuity of the operator $\mathcal{B}_\varepsilon + \mathcal{C}_\varepsilon$ can be shown using the convergence of the Duhamel expansions as we argued in Subsection A.4.5.

Thus, the operator $\mathcal{B}_\varepsilon + \mathcal{C}_\varepsilon$ is a continuous self-map on the set $\{u \in L^\infty(\Omega) : 0 \le u \le M\}$ for some $M > 0$ large enough. Moreover, in the same way as we have shown the Hölder continuity in the pure Grey case in Subsection A.4.5, using that

$$\int_0^\infty d\nu\ Q_a(\nu)B_\nu\left(F^{-1}(\|u\|_\infty)\right) \le \|u\|_\infty,$$

we can show that $\mathcal{B}_\varepsilon + \mathcal{C}_\varepsilon$ acting on $\{u \in C(\Omega) : 0 \le u \le M\}$ is a continuous self-map mapping continuous functions to Hölder continuous function, hence it is a compact continuous self-map. Schauder's fixed-point theorem implies the existence of regularized solutions u_ε to the equation (A.116).

We are ready for the last step of the proof. We want to show the compactness of the sequence of regularized solutions $u_{\frac{1}{j}} =: u_j$. To this end we will use Proposition A.3. We proceed in the same way as in the proof of the pure Grey case of Theorem A.4. We write the terms in the Duhamel expansion of the bulk and boundary operators in spherical coordinates. For the interior terms we replace the definition of F_j^i in (A.99) by

$$\begin{aligned} F_j^{(i)}(x,\omega,\nu) = &\int_0^D d\lambda_2 ... \int_0^D d\lambda_i \int_{\mathbb{S}^2} dn_2 ... \int_{\mathbb{S}^2} dn_i \; K(\omega,n_2)...K(n_{i-1},n_i) \\ &\times Q_s(\nu)\alpha_j^s(x-\lambda_2 n_2) \exp\left(-\int_0^{\lambda_2}\left[Q_a(\nu)\alpha_j^a(x-rn_2)+Q_s(\nu)\alpha_j^s(x-rn_2)\right]dr\right) \\ &\times ... \times B_\nu\left(F^{-1}(u_j(x-\lambda_2 n_2+..-\lambda_i n_i))\right) Q_a(\nu)\alpha_j^a(x-\lambda_2 n_2+..-\lambda_i n_i) \\ &\times \exp\left(-\int_0^{\lambda_i}\left[Q_a(\nu)\alpha_j^a(x-\lambda_2 n_2+..-rn_i)+Q_s(\nu)\alpha_j^s(x-\lambda_2 n_2+..-rn_i)\right]dr\right), \end{aligned}$$

for $i \geq 2$ and $F_j^{(1)}(x,\omega,\nu) = Q_a(\nu)B_\nu\left(F^{-1}(u)(x)\right)$. We notice that

$$0 \leq \int_0^\infty d\nu F_j^{(i)}(x,\omega,\nu) \leq \theta^{i-1}M.$$

Moreover, $F_j^{(i)}$ is also uniformly continuous with respect to the variable ω. Then, Proposition A.3 implies that all terms of the form

$$\begin{aligned} \tilde{\mathcal{V}}_j^i = \int_0^\infty d\nu \int_0^D dt \int_{\mathbb{S}^2} dn F_j^{(i)}(x-tn,n,\nu)Q_s(\nu)\alpha_j^s(x-tn) \\ \times \exp\left(-\int_0^t\left[Q_a(\nu)\alpha_j^a(x-rn_2)+Q_s(\nu)\alpha_j^s(x-rn_2)\right]dr\right), \end{aligned}$$

for $i \geq 2$ and

$$\begin{aligned} \tilde{\mathcal{V}}_j^1 = \int_0^\infty d\nu \int_0^D dt \int_{\mathbb{S}^2} dn F_j^{(1)}(x-tn,n,\nu)Q_a(\nu)^2\alpha_j^a(x-tn) \\ \times \exp\left(-\int_0^t\left[Q_a(\nu)\alpha_j^a(x-rn_2)+Q_s(\nu)\alpha_j^s(x-rn_2)\right]dr\right), \end{aligned}$$

are compact in $L^2(\Omega)$. Since the error terms can be still estimated by

$$\left|\mathcal{V}_j^i(x) - \tilde{\mathcal{V}}_j^i(x)\right| \leq C(M,\|\alpha\|,\|Q\|,\Omega)\theta^{i-1}\frac{1}{j}$$

and the Duhamel series is convergent, for any β there exists some $N_0 > 0$ and an $h_0 > 0$ such that

$$\int_\Omega dx \; |\mathcal{B}_j(u_j)(x) - \mathcal{B}_j(u_j)(x+h)|^2 \leq \beta\left(2|\Omega|+1\right) + C(\Omega,\|\alpha\|,\|Q\|,M)N_0^2\frac{1}{j}, \qquad \text{(A.123)}$$

for all $|h| < h_0$ and for all $j \geq 0$.

In a very similar way we consider the terms in the Duhamel expansion of the boundary operator written in spherical coordinates. Here we replace the definition of $Q_j^{(i)}$ in (A.103) by

$$
\begin{aligned}
Q_j^{(i)}(x,\omega,\nu) = &\int_{\mathbb{S}^2} dn_2 \int_0^D d\lambda_2 K(\omega,n_2) Q_s(\nu)\alpha_j^s(x-\lambda_2 n_2) \\
&\times \exp\left(-\int_0^{\lambda_2} \left[Q_a(\nu)\alpha_j^a(u_j) + Q_s(\nu)\alpha_j^s(u_j)\right](x - rn_2)dr\right) \\
&\times \ldots \times \int_{\mathbb{S}^2} dn_i Q_a(\nu) g_\nu(n_i) K(n_{i-1}, n_i) \\
&\times \exp\left(-\int_0^D \left[Q_a(\nu)\alpha_j^a(u_j) + Q_s(\nu)\alpha_j^s(u_j)\right](x - \lambda_2 n_2 - \ldots - rn_i)dr\right), \qquad \text{(A.124)}
\end{aligned}
$$

for $i \geq 2$. Again, $Q_j^{(i)}$ satisfies the assumption of Proposition A.3 with

$$
0 \leq \int_0^\infty d\nu\; Q_j^{(i)}(x,\omega,\nu) \leq \|Q\|_\infty \|g\|_{L^\infty(\mathbb{S}^2, L^1(\mathbb{R}_+))}\theta^{i-2}.
$$

Hence, all terms of the form

$$
\begin{aligned}
\tilde{\mathcal{U}}_j^i = \int_0^\infty d\nu \int_0^D dt \int_{\mathbb{S}^2} dn\; Q_j^{(i)}(x - tn, n, \nu) Q_s(\nu)\alpha_j^s(x - tn) \\
\times \exp\left(-\int_0^t \left[Q_a(\nu)\alpha_j^a(x - rn_2) + Q_s(\nu)\alpha_j^s(x - rn_2)\right] dr\right),
\end{aligned}
$$

for $i \geq 2$ and

$$
\begin{aligned}
\tilde{\mathcal{U}}_j^1 = \int_0^\infty d\nu \int_{\mathbb{S}} dn\; Q_a(\nu) g_\nu(n) \\
\times \exp\left(-\int_0^D \left[Q_a(\nu)\alpha_j^a(x - rn_2) + Q_s(\nu)\alpha_j^s(x - rn_2)\right] dr\right),
\end{aligned}
$$

are compact in $L^2(\Omega)$. Once more, the error terms can be still estimated by

$$
\left|\mathcal{U}_j^i(x) - \tilde{\mathcal{U}}_j^i(x)\right| \leq C(\|g\|, \|\alpha\|, \|Q\|, \Omega)\theta^{i-2}\frac{1}{j}
$$

and

$$
\left|\mathcal{U}_j^1(x) - \tilde{\mathcal{U}}_j^1(x)\right| \leq C(\|g\|, \|\alpha\|, \|Q\|, \Omega)\frac{1}{j}.
$$

This together with the absolute convergence of the Duhamel series implies for any β the existence of some $N_0 > 0$ and an $h_1 > 0$ such that

$$
\int_\Omega dx\; |\mathcal{C}_j(u_j)(x) - \mathcal{C}_j(u_j)(x+h)|^2 \leq \beta\,(2|\Omega| + 1) + C(\Omega, \|\alpha\|, \|Q\|, \|g\|) N_0^2 \frac{1}{j}, \qquad \text{(A.125)}
$$

for all $|h| < h_1$ and for all $j \geq 0$. Now we can conclude exactly as in the proof of Theorem A.4 that the sequence $u_j = \mathcal{B}_j(u_j) + \mathcal{C}_j(u_j)$ is compact in L^2. Extracting a subsequence converging pointwise almost every where to some u and arguing with the dominated convergence theorem and the absolute convergence of the Duhamel series we can show the existence of a solution to the fixed-point equation (A.113). □

Appendix B

Equilibrium and Non-equilibrium diffusion approximation for the radiative transfer equation

Abstract: In this paper we study the distribution of the temperature within a body where the heat is transported only by radiation. Specifically, we consider the situation where both emission-absorption and scattering processes take place. We study the initial boundary value problem given by the coupling of the radiative transfer equation with the energy balance equation on a convex domain $\Omega \subset \mathbb{R}^3$ in the diffusion approximation regime, i.e. when the mean free path of the photons tends to zero. Using the method of matched asymptotic expansions we will derive the limit initial boundary value problems for all different possible scaling limit regimes and we will classify them as equilibrium or non-equilibrium diffusion approximation. Moreover, we will observe the formation of boundary and initial layers for which suitable equations are obtained. We will consider both stationary and time dependent problems as well as different situations in which the light is assumed to propagate either instantaneously or with finite speed.

B.1 Introduction

The kinetic equation which describes the interaction of matter with photons is the radiative transfer equation. The radiative transfer equation can be written including absorption-emission processes and scattering processes in a rather general setting as

$$\begin{aligned}\frac{1}{c}\partial_t I_\nu(t,x,n) + n\cdot\nabla_x I_\nu(t,x,n) &= \alpha_\nu^e - \alpha_\nu^a I_\nu(t,x,n)\\ &\quad + \alpha_\nu^s\left(\int_{\mathbb{S}^2} K(n,n')I_\nu(t,x,n')\,dn' - I_\nu(t,x,n)\right).\end{aligned} \tag{B.1}$$

We denote by $I_\nu(t,x,n)$ the radiation intensity, i.e. the distribution of energy of photons moving at time $t>0$, at position $x\in\Omega\subset\mathbb{R}^3$ and in direction $n\in\mathbb{S}^2$ with frequency $\nu>0$. Moreover, c is the speed of light in the medium that will be assumed to be constant. The parameters α^e, α^a and α^s are respectively the emission, absorption and scattering coefficients. These are functions that can depend on the frequency ν, on the position x or in the case of local thermal equilibrium on the local temperature $T(x)$. The function K is the scattering kernel. It can be considered as the probability rate of a photon to be deflected from an incident direction $n'\in\mathbb{S}^2$ to a new direction $n\in\mathbb{S}^2$. The scattering kernel K can be assumed also to depend on the frequency $\nu \in \mathbb{R}_+$. However, in this paper we omit the dependence on ν in order to simplify the notation. Notice that all the results that we will present in this paper hold also in the case where K is also a function of ν.

In this paper we will study the heat transfer by means of radiation under some assumptions. First of all we consider only the case of local thermal equilibrium in which the temperature $T(t,x)$ is well-defined at any point $x \in \Omega$ and for any time $t > 0$. This is not necessarily the case in situations where the microscopic processes driving the system towards equilibrium are slow. Such problems arise in applications to astrophysics (cf. [114]). Under this assumption the emission coefficient takes a particular form. Indeed it is given by $\alpha_\nu^e = \alpha_\nu^a B_\nu(T(t,x))$, where $B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{\frac{h\nu}{kT}}-1}$ is the Planck distribution of a black body. We assume also that the considered material is isotropic without a preferred direction of scattering. Hence, the scattering kernel K is invariant under rotations.

We couple the radiative transfer equation with the energy balance equation

$$C\partial_t T(t,x) + \frac{1}{c}\partial_t\left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, I_\nu(t,x,n)\right) + \operatorname{div}\left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, nI_\nu(t,x,n)\right) = 0, \quad \text{(B.2)}$$

where $C > 0$ is the volumetric heat capacity of the material. The combined system (B.1) and (B.2) allows to determine the temperature of the system at any point when the heat is transferred only by means of radiation. Notice that in (B.2) we are not considering other heat transport processes such as conduction or convection. After a suitable time rescaling we can assume $C = 1$. As boundary condition we consider a source of radiation placed at infinity. Mathematically we impose

$$I_\nu(t,x,n) = g_\nu(t,n) \quad \text{if } x \in \partial\Omega \text{ and } n \cdot n_x < 0, \quad \text{(B.3)}$$

where $n_x \in \mathbb{S}^2$ is the outer normal to the boundary at point x. However, we could consider a more general setting with the incoming boundary profile $g_\nu(t,x,n)$ depending also on $x \in \partial\Omega$.

In this paper we will consider both time dependent and stationary cases. Assuming $\Omega \subset \mathbb{R}^3$ bounded and convex and as initial values the bounded functions $I_0(x,n,\nu)$ and $T_0(x)$, we consider the following initial-boundary value problem

$$\begin{cases} \frac{1}{c}\partial_t I_\nu(t,x,n) + n\cdot\nabla_x I_\nu(t,x,n) = \alpha_\nu^a(x)\left(B_\nu(T(t,x)) - I_\nu(t,x,n)\right) & \\ \qquad + \alpha_\nu^s(x)\left(\int_{\mathbb{S}^2} K(n,n')I_\nu(t,x,n')\,dn' - I_\nu(t,x,n)\right) & x\in\Omega, n\in\mathbb{S}^2, t>0 \\ \partial_t T + \frac{1}{c}\partial_t\left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, I_\nu(t,n,x)\right) & \\ \qquad + \operatorname{div}\left(\int_0^\infty d\nu\int_{\mathbb{S}^2} dn\, nI_\nu(t,n,x)\right) = 0 & x\in\Omega, n\in\mathbb{S}^2, t>0 \\ I_\nu(0,x,n) = I_0(x,n,\nu) & x\in\Omega, n\in\mathbb{S}^2 \\ T(0,x) = T_0(x) & x\in\Omega \\ I_\nu(t,n,x) = g_\nu(t,n) & x\in\partial\Omega, n\cdot n_x<0, t>0 \end{cases} \quad \text{(B.4)}$$

and the following stationary boundary value problem

$$\begin{cases} n\cdot\nabla_x I_\nu(x,n) = \alpha_\nu^a(x)\left(B_\nu(T(x)) - I_\nu(x,n)\right) & \\ \qquad + \alpha_\nu^s(x)\left(\int_{\mathbb{S}^2} K(n,n')I_\nu(x,n')\,dn' - I_\nu(x,n)\right) & x\in\Omega, n\in\mathbb{S}^2 \\ \operatorname{div}\left(\int_0^\infty d\nu\int_{\mathbb{S}^2} dn\, nI_\nu(n,x)\right) = 0 & x\in\Omega, n\in\mathbb{S}^2 \\ I_\nu(n,x) = g_\nu(n) & x\in\partial\Omega, n\cdot n_x<0. \end{cases} \quad \text{(B.5)}$$

Problems like (B.4) and (B.5) or similar equations related to radiative transfer are often studied in the framework of the so-called diffusion approximation (see [108, 152]). This approximation is valid when the mean free path of the photons is much smaller than the macroscopic size of the system. However, the mean free path of the photons can be small because either the scattering mean free path or the absorption mean free path is smaller than the size of the system. The main consequence for that is that, depending on the ratio between the different mean free paths, the radiation intensity can be approximated by the Planck distribution, i.e. $B_\nu(T)$, or it cannot be. The first case is denoted as equilibrium diffusion approximation while the second one is referred to as non-equilibrium diffusion approximation. These concepts have been extensively discussed in the physical literature on radiation (cf. [108, 152]). The goal of this paper is to obtain a precise mathematical characterization of these concepts, specifically to derive an accurate mathematical condition for the validity of the equilibrium diffusion approximation and to determine the regions where the equilibrium or non-equilibrium diffusion approximation holds for the specific problems (B.4) and (B.5). To this end, we will use perturbative methods and matched asymptotic expansions in order to study different scaling limits for the scattering and absorption mean free paths.

B.1.1 Scaling lengths and results

We study the solutions of the time dependent and stationary radiative transfer equations (B.4) and (B.5) under different scaling limits and we obtain suitable problems satisfied by the limit of the solutions of the original problems. For these problems we will obtain either the equilibrium or the non-equilibrium diffusion approximation. To this end we start defining some characteristic lengths.

We consider a convex domain $\Omega \subset \mathbb{R}^3$ with diameter of order 1 and such that the size of the domain is comparable in all directions of the space. Moreover, the characteristic macroscopic length L is assumed to be $L = 1$. We remark that many of the results obtained in this paper are valid also in non-convex domain. However, in non-convex domains we should take into account also the consequences of incoming radiation into cavities, an issue that we will not consider in this paper (see [83] for more details).

We will replace the absorption coefficient $\alpha_\nu^a(x)$ by

$$\frac{\alpha_\nu^a(x)}{\ell_A} \tag{B.6}$$

and the scattering coefficient $\alpha_\nu^s(x)$ by

$$\frac{\alpha_\nu^s(x)}{\ell_S}, \tag{B.7}$$

where now $\alpha_\nu^a(x) = \mathcal{O}(1)$ and $\alpha_\nu^s(x) = \mathcal{O}(1)$ are bounded by a constant of order one in both variables. We denote by ℓ_A the absorption length and by ℓ_S the scattering length. These are also the mean free paths of the absorption/emission processes and the scattering processes, respectively. In some physical applications it is convenient to assume $\alpha_\nu^a(x)$ or $\alpha_\nu^s(x)$ to tend to zero for large or small frequencies ν. The exact dependence of these functions on ν will be made after. Roughly speaking, we have to assume that they have to decay not too fast in order to obtain that some integrals arising in the analysis are convergent.

In many technological applications it can be assumed that $\alpha_\nu^s \ll \alpha_\nu^a$ (cf. [152]). However, there are also applications where the scattering plays a more important role than the absorption/emission process. This is the case for example in the analysis of planetary atmospheres, see [54, 114].

Another important scaling length that we should consider is the Milne length, which is given by the minimum between absorption and scattering length,

$$\ell_M = \min\{\ell_A, \ell_S\}. \tag{B.8}$$

The Milne length can be considered to be the effective mean free path of the whole radiative process. The key feature of the Milne length is that at distances of order ℓ_M to the boundary the radiation intensity becomes isotropic, i.e. independent of the direction $n \in \mathbb{S}^2$. Since we are interested in the diffusion approximation, we assume in the rest of this paper $\ell_M \ll L = 1$.

Another length which plays a crucial role in the analysis of this paper is the quantity that we will denote as thermalization length which is the geometrical mean of the absorption and the Milne length

$$\ell_T = \sqrt{\ell_A \ell_M}. \tag{B.9}$$

The thermalization length is the characteristic distance from the boundary in which the radiation intensity I_ν approaches the Planck equilibrium distribution of the temperature.

We now replace in (B.4) and (B.5) the absorption and scattering coefficients by the expression in (B.6) and (B.7). The changes of the temperature take place in times of order

$$\tau_h = \frac{\ell_A}{\min\{\ell_T^2, 1\}} \gg 1,$$

which will be denoted as heat parameter. Therefore, in order to obtain an equation that changes in times t of order 1 we will replace t by $\tau_h t$. Notice that, after this change of variable, the changes of times t of order 1 are associated to relevant changes of the temperature of order 1. We will use this notation throughout the paper, i.e. we will denote by t the time after the change of variable. Hence, (B.4) writes using $L = 1$

$$\begin{cases} \frac{1}{c}\partial_t I_\nu(t,x,n) + \tau_h n\cdot\nabla_x I_\nu(t,x,n) = \frac{\alpha_\nu^a(x)\tau_h}{\ell_A}\left(B_\nu(T(t,x)) - I_\nu(t,x,n)\right) & \\ \qquad + \frac{\alpha_\nu^s(x)\tau_h}{\ell_S}\left(\int_{\mathbb{S}^2} K(n,n')I_\nu(t,x,n')\,dn' - I_\nu(t,x,n)\right) & x\in\Omega, n\in\mathbb{S}^2, t>0 \\ \partial_t T + \frac{1}{c}\partial_t\left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, I_\nu(t,n,x)\right) & \\ \qquad + \tau_h \operatorname{div}\left(\int_0^\infty d\nu\int_{\mathbb{S}^2} dn\, nI_\nu(t,n,x)\right) = 0 & x\in\Omega, n\in\mathbb{S}^2, t>0 \\ I_\nu(0,x,n) = I_0(x,n,\nu) & x\in\Omega, n\in\mathbb{S}^2 \\ T(0,x) = T_0(x) & x\in\Omega \\ I_\nu(t,n,x) = g_\nu(t,n) & x\in\partial\Omega, n\cdot n_x<0, t>0. \end{cases} \tag{B.10}$$

We will also consider the case where the speed of light is infinite, i.e. $c = \infty$. This approximation is justified if the characteristic time for the temperature to change is much smaller than the time required for the light to cross the domain. In this case the equation will be

$$\begin{cases} n\cdot\nabla_x I_\nu(t,x,n) = \frac{\alpha_\nu^a(x)}{\ell_A}\left(B_\nu(T(t,x)) - I_\nu(t,x,n)\right) & \\ \qquad + \frac{\alpha_\nu^s(x)}{\ell_S}\left(\int_{\mathbb{S}^2} K(n,n')I_\nu(t,x,n')\,dn' - I_\nu(t,x,n)\right) & x\in\Omega, n\in\mathbb{S}^2, t>0 \\ \partial_t T + \tau_h \operatorname{div}\left(\int_0^\infty d\nu\int_{\mathbb{S}^2} dn\, nI_\nu(t,n,x)\right) = 0 & x\in\Omega, n\in\mathbb{S}^2, t>0 \\ T(0,x) = T_0(x) & x\in\Omega \\ I_\nu(t,n,x) = g_\nu(t,n) & x\in\partial\Omega, n\cdot n_x<0, t>0. \end{cases} \tag{B.11}$$

Similarly, the stationary problem (B.5) can be written as

$$\begin{cases} n \cdot \nabla_x I_\nu(x,n) = \frac{\alpha_\nu^a(x)}{\ell_A}\left(B_\nu(T(x)) - I_\nu(x,n)\right) \\ \qquad\qquad + \frac{\alpha_\nu^s(x)}{\ell_S}\left(\int_{\mathbb{S}^2} K(n,n')I_\nu(x,n')\,dn' - I_\nu(x,n)\right) & x \in \Omega, n \in \mathbb{S}^2 \\ \operatorname{div}\left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, n I_\nu(n,x)\right) = 0 & x \in \Omega, n \in \mathbb{S}^2 \\ I_\nu(n,x) = g_\nu(n) & x \in \partial\Omega, n \cdot n_x < 0. \end{cases} \tag{B.12}$$

It is important to remark that we assume $g_\nu(t,n)$ in (B.10) and (B.11) to change in times of order 1 after rescaling the time, i.e. we assume the incoming radiation g_ν to change in the same time scale as the one for meaningful changes of the temperature.

Notice that at the first glance the time τ_h does not seem to have units of time. However, we must take into account that since $L = 1$, omitted in all the equations, all quantities ℓ_A, ℓ_S, ℓ_M and ℓ_T are non-dimensional parameters that have to be understood as $\frac{\ell_A}{L}$, $\frac{\ell_S}{L}$, $\frac{\ell_M}{L}$ and $\frac{\ell_T}{L}$. In addition we recall that we have chosen a particular unit of time for which the heat capacity is $C = 1$. Hence, all the space and time variables appearing in (B.10)- (B.12) are non-dimensional. We will see in Sections B.4 to B.6 that the definition of the heat parameter, namely τ_h, is motivated by the behavior of the radiation intensity in the bulk and it is the order of time in which the temperature changes.

There are three characteristic lengths in (B.10)- (B.12), namely ℓ_A, ℓ_S and $L = 1$, and we can consider several relative scalings between them. Since $\ell_M \ll 1$ in the case of the diffusion approximation, the solutions can be described by means of different boundary layers. It turns out that the relative size and the structure of these boundary layers can be characterized using the relative scaling of ℓ_M (cf. (B.8)), ℓ_S (cf. (B.9)) and $L = 1$. In order to consider these different scalings, in the following sections we will set for the equations (B.10), (B.11) and (B.12) $\ell_M = \varepsilon \ll 1$ and we will choose ℓ_A, ℓ_S and c as power of ε.

Notice that the incoming radiation g_ν to the boundary of Ω is not necessarily isotropic and in general it is different from the Planck distribution, i.e. it is not in thermal equilibrium. This implies the onset (in principle) of two nested boundary layers near the boundary where the intensity I_ν changes its behavior. The thickness of these layers is ℓ_M and ℓ_T, respectively. In the first layer, which we call Milne layer, the radiative intensity I_ν becomes isotropic. In the latter, which we denote as thermalization layer, I_ν approaches the Planck distribution for a suitable temperature that has to be determined and it is one of the unknowns of the problem. Notice moreover that, since by definition $\ell_M \leq \ell_T$, the Milne layer appears always before the thermalization layer. On the other hand, if ℓ_M is comparable to ℓ_T both layers can coincide. It is worth to notice that beyond the thermalization layer the radiative intensity I_ν is given by a Planck distribution. In the time dependent problem besides the formation of boundary layers we observe the formation of initial layers in which the radiation intensity becomes isotropic or the equilibrium distribution, respectively.

Table B.1 summarizes the behavior of the solution (T, I_ν) to the equations (B.10)-(B.12) for different scaling limits yielding equilibrium or non-equilibrium diffusion approximation. Moreover, for any considered regime we observe the onset or not of Milne layers or thermalization layers. Finally, when ℓ_T is of the same order of the characteristic length L the thermalization, i.e. the transition of I_ν to the equilibrium distribution $B_\nu(T)$, takes place in the bulk of the domain Ω.

	$\ell_M = \ell_T \ll L$	$\ell_M \ll \ell_T \ll L$	$\ell_M \ll \ell_T = L$	$\ell_M \ll L \ll \ell_T$
Milne layer	Milne =	Yes	Yes	Yes
Thermalization layer	Thermalization	Yes	$\approx$ Bulk	No
Bulk	Equilibrium diffusion approximation	Equilibrium diffusion approximation	Transition from equilibrium to non-equilibrium approximation	Non-equilibrium diffusion approximation

Table B.1: Main results.

B.1.2 Revision of the literature

The problem concerning the distribution of temperature of a material interacting with electromagnetic waves is not only a relevant question in many physical applications but also it is the source of several interesting mathematical problems. The radiative transfer equation is the kinetic equation describing the interaction of photons with matter. Its derivation and its main properties are explained in [29, 108, 114, 125, 152]. In particular, the validity of the diffusion approximation and a discussion of the situations where the radiation intensity is expected to be or not to be given approximately by the Planck distribution are considered in [108, 152].

Starting from the seminal work of Compton [31], the interaction of matter and radiation has been widely studied both in the physical and mathematical literature. Some of the early results can be found in the paper of Milne [109], who considered a simplified model of monochromatic radiation depending only on one space variable.

When considering the diffusion approximation of the radiative transfer equation, a boundary layer near the boundary appears in which the distribution of radiation becomes isotropic. The specific equation describing this layer involves a radiative transfer equation depending on one space variable, whose details depend on the problem under consideration. This class of problems is known in the mathematical literature as Milne problems and they have been extensively studied at least for some particular choices of α_ν^a and α_ν^s.

While it is difficult to find explicit solutions of the radiative transfer equation, in the case of small photon's mean free path (i.e. in the diffusion approximation) this problem reduces to an elliptic (in the stationary case) or parabolic (in the time dependent case) problem. The mathematical properties of these problems are much better understood than the properties of the non-local radiative transfer equation (B.1). Due to this the diffusion approximation of the radiative transfer equation has been studied in great detail.

Before discussing the currently available mathematical results about the diffusion approximation and the Milne problems, it is worth to introduce an equation which is closely related to the radiative transfer equation (B.1). In the absence of emission-absorption processes, i.e. when $\alpha_\nu^a = 0$, and when α_ν^s is independent of the frequency ν the radiative transfer equation (B.1) reduces to

$$\partial_t u(t,x,n) + n \cdot \nabla_x u(t,x,n) = \alpha(x) \left(\int_{\mathbb{S}^2} K(n,n') u(t,x,n')\, dn' - u(t,x,n) \right), \tag{B.13}$$

where $u = \int_0^\infty I_\nu(t,x,n)\, d\nu$. This equation is mathematically identical to the one-speed neutron transport equation. Moreover, in the stationary case the radiative transfer equation reduces to (B.13) also in the presence of absorption-emission processes if both α^a and α^s are independent of the frequency. The case where both absorption and scattering coefficients are independent of the frequency is usually denoted in the literature as the Grey approximation. Therefore, the one-speed neutron transport equation and the radiative transfer equation for the Grey approximation are mathematically equivalent. See [33] for more details. As a matter of fact, the neutron transport equation, especially its diffusion approximation, was largely studied in the late 70's. The reason is that this problem is important in order to determine the critical size for neutron transport, i.e. the smallest size of the system for which the scattering eigenvalue problem has a stable solution. This is relevant in nuclear reactor engineering. For more details about this issue we refer to [33].

In several articles [66, 98, 100–104] Larsen and several coauthors studied many properties of the neutron transport equation and its diffusion approximation. Moreover, in [97] the authors studied via asymptotic analysis the diffusion approximation of the radiative transfer equation for both absorption and scattering taking as initial and boundary value the Planck distribution. This choice of boundary data simplifies the treatment of the problem because no boundary layers or initial transport problems arise at least to the leading order.

To the best of our knowledge the first mathematically rigorous article about the diffusion approximation for the neutron transport equation is [19]. In that article the authors studied equation (B.13) under different boundary conditions including also the absorbing boundary condition that we are considering in (B.3). In particular, using probabilistic methods they studied the Milne problem arising for the boundary layers and proved the convergence of the solution of the original neutron transport equation to the solution of a diffusive problem. Moreover, the scattering kernel considered is assumed to be strictly positive, bounded and rotationally symmetric.

More recently Guo and Wu studied in a series of papers [76, 146–149] both the stationary and time dependent diffusion approximation for the neutron transport equation with a constant scattering kernel and a constant scattering coefficient. They proved rigorously the convergence to such diffusion problem computing also a geometric correction for the boundary layer. Their method is based on the derivation of suitable $L^2 - L^p - L^\infty$ estimates, a method that has been extensively used in the study of kinetic equations (cf. [75, 86]).

The mathematical theory of the radiative transfer equation has been also extensively studied. The well-posedness and the diffusion approximation for the time dependent problem without scattering has been studied using the theory of m-accretive operators in [13–15].

In a recent paper [37] we developed an alternative method to derive the equilibrium diffusion approximation starting with the stationary radiative transfer equation. Specifically, in [37] the Grey approximation and the case of absence of scattering are considered. The procedure developed in [37] consists in reformulating the problem (B.12) as a non-local elliptic equation for the temperature for which maximum principles techniques are applicable.

As indicated before an important class of problems, which need to be studied in order to derive the boundary condition for the diffusion approximation, are the Milne problems.

In the case of pure absorption, namely when $\alpha_\nu^s = 0$, the well-posedness for the Milne problem can be found for instance in [68] and also in [37] using different methods. In particular in [68] well-posedness is shown for a very large class of absorption coefficients.

In the case of pure scattering radiative transfer equation for the Grey approximation (equivalently the neutron transport equation), the well-posedness of the Milne problem has been studied in [17, 19]. More recently, geometric corrections to the solution of the Milne problem have been obtained in [76, 146–149].

To our knowledge the only example of Milne problem involving both emission/absorption and scattering has been studied in [127]. The case considered in this paper is the one of constant scattering kernel and constant scattering coefficient and more general absorption coefficient. The proof relies on the accretiveness of the operators used similarly to the Perron method applied to solve boundary value problem for elliptic equations.

It is finally worth to mention that also for other kinetic equations, such as for example the Boltzmann equation, the diffusion limit and hence the boundary layer equations have been studied. The equations describing the boundary layers are also often denoted in the literature by Milne problems, see for instance [11, 45–47].

Besides the studies about the diffusion approximation, the radiative transfer equation has been analyzed in numerous works. In recent times there has been a graving interest of the study of problems involving the radiative transfer equation in different contexts. The well-posedness of the stationary equation (B.5) has been considered in [35, 83]. The authors proved the existence of solutions to the stationary radiative transfer equation with or without scattering in the cases of constant coefficients, coefficients depending on the frequency but not on the temperature of the system and finally coefficients depending on both the frequency and the temperature of the particular form $\alpha_\nu(T) = Q(\nu)\alpha(T)$.

Finally, the radiative transfer equation has been considered also for more complicated interactions between matter and photons. We refer to [69,71,108,152] for problems concerning the interaction of matter with radiation in a moving fluid. For the study of interaction of electromagnetic waves with a Boltzmann gas whose molecules have different energy levels we refer to [34, 81, 114, 122]. Several authors considered problems where the heat is transported in a body by means of both radiation and conduction, we refer to [62, 63, 95, 96, 116, 138, 139]. Finally, homogenization problems in porous and perforated domains where the heat is transported by conduction, radiation and possibly also convection are studied in [3–5, 121]. Specifically, in [121] the authors applied the method of multiple scales to a homogenization problem describing the heat transport in a porous medium. The heat transport is assumed to be due to the conduction in the solid part of the material and due to the radiation in the gas filled cavities.

Derivations of the scattering kernel for the radiative transfer equation taking as starting point the Maxwell equations has been also extensively studied in [110].

B.1.3 Structure of the paper

The paper is organized as follows. In Section B.2 we will study some of the mathematical properties of the scattering operator and of the absorption-emission process appearing in the radiative transfer equation. We will then proceed to the derivation of the limit problems in the diffusion approximation under different scaling limits. In Section B.3 we consider the stationary diffusion approximation for the radiative transfer equation and we derive using the method of matched asymptotic expansions the new limit boundary value problem as well as the boundary layer equations. Moreover, we will see for which choice of characteristic lengths the equilibrium diffusion approximation holds and for which ones it fails. We will then proceed with the study of the time dependent diffusion approximation, for which we will use again the method of matched asymptotic expansions. In Section B.4 the focus is on the case of infinite speed of light (i.e. instantaneously transport of the radiation in the domain), namely on the problem (B.11). Besides the construction of the limit problems and their classification as equilibrium and non-equilibrium diffusion approximations, we will also derive the initial layer and initial-boundary layer equations. In Section B.5 and in Section B.6 we proceed similarly to Section B.4 studying first the time dependent diffusion approximation in the case of finite speed of light, i.e. speed of light of order one, (cf. Section B.5) and later in the case where the speed of light is assumed to scale like a power law $c = \varepsilon^{-\kappa}$ for $\kappa > 0$ and $\varepsilon = \ell_M$ (cf. Section B.6).

B.2 Preliminary results

In this section we collect some properties of the scattering operator and absorption operator that will be used later in the analysis of the diffusion approximation.

B.2.1 Properties of the scattering operator

Before deriving suitable diffusion approximations according to the different values of ℓ_M and ℓ_T, we describe some properties of the scattering kernel and of the scattering operator.

We consider throughout the paper the kernel $K \in C\left(\mathbb{S}^2 \times \mathbb{S}^2\right)$ to be non-negative and satisfying
$\int_{\mathbb{S}^2} K(n,n')dn = 1$. We also assume in the whole article that the kernel K is invariant under rotations, i.e.

$$K(n,n') = K(\mathcal{R}n, \mathcal{R}n') \quad \text{for all } n,n' \in \mathbb{S}^2 \text{ and for any } \mathcal{R} \in SO(3).$$

Moreover, for any $n,\omega \in \mathbb{S}^2$ we define by $\mathcal{R}_{n,\omega} \in SO(3)$ the rotation of π around the bisectrix of the angle between n and ω lying in the plane containing both vectors. This rotation satisfies $\mathcal{R}_{n,\omega}(n) = \omega$ and $\mathcal{R}_{n,\omega}(\omega) = n$. As shown in [35], this implies that the scattering kernel K is symmetric. Notice that this is not true in two dimensions unless we assume K to be invariant also under reflections.

We define the scattering operator as the bounded linear operator given by

$$\begin{aligned} H : L^\infty\left(\mathbb{S}^2\right) &\to L^\infty\left(\mathbb{S}^2\right) \\ \varphi \mapsto H[\varphi] &= \int_{\mathbb{S}^2} K(\cdot,n')\varphi(n')\; dn'. \end{aligned} \tag{B.14}$$

With this notation we can formulate the following Proposition which contains the most important properties of the scattering operator.

Proposition B.1. *Let $K \in C\left(\mathbb{S}^2 \times \mathbb{S}^2\right)$, invariant under rotations, non-negative and satisfying*

$$\int_{\mathbb{S}^2} K(n,n')dn = 1.$$

Assume $\varphi \in L^\infty\left(\mathbb{S}^2\right)$ satisfies $H[\varphi] = \varphi$. Then

(i) φ is continuous,

(ii) φ is constant,

(iii) $Ran(Id - H) = \left\{\varphi \in L^\infty(\mathbb{S}^2) : \int_{\mathbb{S}^2} \varphi = 0\right\}$.

The proof of Proposition B.1 can be found in the Section B.8. A direct consequence of Proposition B.1 is the following Proposition for a continuous scattering kernel K with $K \in C\left(\mathbb{S}^2 \times \mathbb{S}^2 \times \Omega \times \mathbb{R}_+\right)$ invariant under rotations for each pair (x,ν).

Proposition B.2. *Let $K \in C\left(\mathbb{S}^2 \times \mathbb{S}^2 \times \Omega \times \mathbb{R}_+\right)$. For any $x,\nu \in \Omega \times \mathbb{R}_+$ we define $K_{x,\nu}(n,n') = K(n,n',x,\nu)$. Assume that for any $x,\nu \in \Omega \times \mathbb{R}_+$ the kernel $K_{x,\nu}$ is invariant under rotations, non-negative and satisfies $\int_{\mathbb{S}^2} K_{x,\nu}(n,n')dn = 1$. Then the following holds.*

(i) For any $x,\nu \in \Omega \times \mathbb{R}_+$ and $n,\omega \in \mathbb{S}^2$ there exist finitely many $n_1,\cdots,n_N \in \mathbb{S}^2$ such that (B.91) *holds for $K_{x,\nu}$;*

(ii) if $\varphi \in L^\infty\left(\mathbb{S}^2 \times \Omega \times \mathbb{R}_+\right)$ satisfies $H[\varphi]=\varphi$, then φ is continuous and it is constant for every $x,\nu \in \Omega\times\mathbb{R}_+$,

(iii) Ran$(Id-H)=\left\{\varphi(\cdot,x,\nu)\in L^\infty(\mathbb{S}^2) : \int_{\mathbb{S}^2}\varphi(n,x,\nu)\,dn=0\right\}$ for every $x,\nu\in\Omega\times\mathbb{R}_+$.

Proof. Apply Proposition B.1 to the continuous kernel $K_{x,\nu}$. □

Remark. In the following Sections we will consider the diffusion approximation for scattering kernels K independent of $x \in \Omega$ and $\nu \geq 0$. However, under the assumptions of Proposition B.2 the same results would apply for more general kernels depending continuously on x and ν.

Remark. The assumption of K being invariant under rotations is crucial for the validity of Proposition B.1 and Proposition B.2. Consider for example the following continuous function

$$k(n)=\frac{2}{3\pi}\left(\chi_{\{|n\cdot e_3|\leq\frac{1}{4}\}}(n)+(2-4|n\cdot e_3|)\chi_{\{\frac{1}{4}<|n\cdot e_3|<\frac{1}{2}\}}(n)\right).$$

Then the kernel $K(n,n')=k(n)\chi_{\mathbb{S}^2}(n')$ is continuous in both variables, is non-negative and satisfies

$$\int_{\mathbb{S}^2}K(n,n')\,dn=\int_{\mathbb{S}^2}k(n)\,dn=1.$$

However, K is not invariant under rotations. This kernel describes the scattering properties of a non-isotropic medium. It is easy to see that in this case $H[c](n)=ck(n)$, for $c\in\mathbb{R}$. Hence, the constant functions are not a solution to $H[\varphi]=\varphi$. Actually, all solutions of $H[\varphi]=\varphi$ satisfy $\varphi(n)=k(n)\int_{\mathbb{S}^2}\varphi(n')\,dn'$ and have hence the form $\varphi=\lambda k$ where $\lambda\in\mathbb{R}$ is an arbitrary constant. Therefore, the subspace of eigenvectors of H with eigenvalue 1 is one-dimensional.

Remark. As we noticed above, in two dimensions the invariance under rotations of K does not imply directly its symmetry under reflections. However, it is still possible to show that the only eigenfunctions of H with eigenvalue 1 are the constants. To check this we recall the well-known fact that the one-dimensional sphere $\mathbb{S}^1$ can be parameterized by $\theta\in[0,2\pi)$. Moreover, we can assume without loss of generality that any scattering kernel K invariant under rotations has the form $K(n,n')=K(\theta(n)-\theta(n'))$. Let now $f\in L^\infty(\mathbb{S}^1)$ an eigenfunction with eigenvalue 1 for H. We then see

$$\int_0^{2\pi}K(\theta-\varphi)f(\varphi)\,d\varphi=f(\theta).$$

This equation can be solved using Fourier series. We hence obtain the following identity for the Fourier coefficients

$$\hat{f}(n)\left(1-2\pi\hat{K}(n)\right)=0. \tag{B.15}$$

For $n=0$ we have $\hat{K}(0)=\frac{1}{2\pi}\int_0^{2\pi}K(\theta)\,d\theta=\frac{1}{2\pi}$. On the other hand, we obtain for $n\neq 0$

$$\left|\hat{K}(n)\right|<\frac{1}{2\pi}\int_0^{2\pi}K(\theta)\,d\theta=\frac{1}{2\pi}.$$

Therefore, the identity (B.15) is satisfied if and only if $\hat{f}(n)=0$ for all $n\neq 0$. This implies that f is constant.

B.2.2 Relation between the temperature and the radiation intensity

We derive here an identity that relates temperature and radiation intensity and that will be repeatedly used in the stationary problem, for instance in the stationary boundary layer equations.

Using the identity $\operatorname{div}\left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, nI_\nu(x,n)\right) = \int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, n\cdot\nabla_x I_\nu(x,n)$ and plugging the first equation of (B.5) into the second one we see that we have

$$\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, \alpha_\nu^a(x)\left(B_\nu(T(x)) - I_\nu(x,n)\right) = 0, \tag{B.16}$$

where we used also that the integral over the sphere $\mathbb{S}^2$ of the scattering term is 0 due to the symmetry of the kernel K. With this identity we can recover the value of the temperature given the radiation intensity. Let us define by $F : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ the following function

$$F(T,x) = \int_0^\infty \alpha_\nu^a(x) B_\nu(T)\, d\nu. \tag{B.17}$$

Since B_ν is monotone in T, the function $F(\cdot,x)$ is invertible. Hence, (B.16) implies that

$$T(x) = F^{-1}\left(\left(\int_0^\infty d\nu \fint_{\mathbb{S}^2} dn\, \alpha_\nu^a(x) I_\nu(x,n)\right), x\right), \tag{B.18}$$

where F^{-1} is the inverse with respect to the first variable, i.e. $F(T,x) = \xi$ implies $T = F^{-1}(\xi,x)$. Equations (B.16) and (B.18) will appear often in the following sections, in particular in the study of the boundary layers.

B.3 The stationary diffusion approximation: different scales

We first study the stationary diffusion regime for different scalings. We consider (B.12) for α_ν^a and α_ν^s strictly positive and bounded. Moreover, in the diffusion regime we have $\ell_M \ll 1$. Hence, in (B.12) we assume $\ell_M = \min\{\ell_A, \ell_S\} = \varepsilon$. Moreover, we impose $\ell_A = \varepsilon^{-\beta}$ and $\ell_S = \varepsilon^{-\gamma}$, for suitable choices of $\gamma, \beta \geq -1$ with $\min\{\gamma,\beta\} = -1$. Notice that at least one of β and γ is negative. This choice of ℓ_A and ℓ_S as an inverse power law of $\varepsilon > 0$ for $\beta, \gamma \geq -1$ will be convenient in order to make the computations simpler in the following subsections. Under these assumptions we rewrite equation (B.12) as

$$\begin{cases} n\cdot\nabla_x I_\nu(x,n) = \varepsilon^\beta \alpha_\nu^a(x)\left(B_\nu(T(x)) - I_\nu(x,n)\right) & \\ \qquad\qquad + \varepsilon^\gamma \alpha_\nu^s(x)\left(\int_{\mathbb{S}^2} K(n,n') I_\nu(x,n')\, dn' - I_\nu(x,n)\right) & x\in\Omega, n\in\mathbb{S}^2 \\ \operatorname{div}\left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, nI_\nu(n,x)\right) = 0 & x\in\Omega, n\in\mathbb{S}^2 \\ I_\nu(n,x) = g_\nu(n) & x\in\partial\Omega, n\cdot n_x < 0. \end{cases} \tag{B.19}$$

Moreover, we assume the scattering kernel $K \in C(\mathbb{S}^2\times\mathbb{S}^2)$ to be invariant under rotations, non-negative and with $\int_{\mathbb{S}^2} K(n,n')dn = 1$. We consider also $\Omega \subset \mathbb{R}^3$ to be a bounded convex domain with C^1-boundary. For $x \in \partial\Omega$ we denote by $n_x \in \mathbb{S}^2$ the outer normal to the boundary at x.

Before describing in details the limit diffusion problems for the different choices of scaling parameters, we shortly explain how we will use the method of matched asymptotic expansions to derive the limit problems for each case. In order to find the limit problem valid in the bulk, the so-called outer problem, we expand the radiation intensity as

$$I_\nu(x,n) = \phi_0(x,n,\nu) + \sum_{k\geq 0} \varepsilon^{\delta+k}\psi_{k+1}(x,n,\nu) + \sum_{l>0} \varepsilon^l \phi_l(x,n,\nu) \tag{B.20}$$

for a suitable $\delta > 0$ depending on the choice of the scaling parameters. To be more precise,

$$\delta = \begin{cases} \gamma + 1 & \text{if } \beta = -1 \text{ (i.e. } \ell_A = \ell_M), \\ \beta - \lfloor \beta \rfloor & \text{if } \gamma = -1 \text{ (i.e. } \ell_S = \ell_M). \end{cases} \tag{B.21}$$

We remark that if $-1 < \beta < 0$ by our definition $\delta = \beta + 1 > 0$. The choice of δ in (B.20) is due to the following observations. If $\ell_A = \ell_M$ the leading term of the radiative transfer equation is the emission-absorption term, so that

$$\alpha_\nu^a(x)(I_\nu(x,n) - B_\nu(T(x))) = \varepsilon n \cdot \nabla_x I_\nu(x,n) - \alpha_\nu^s(x)\varepsilon^{\gamma+1}(H - Id)[I_\nu(x,\cdot)](n),$$

where we used the notation of (B.14). Therefore, it is natural to look for a solution of this equation in form of a series of powers of ε with exponents 1 and $\gamma + 1$. On the other hand, if $\ell_S = \ell_M$ the leading term is the scattering term yielding

$$\alpha_\nu^s(x)(H - Id)[I_\nu(x,\cdot)](n) = \varepsilon n \cdot \nabla_x I_\nu(x,n) - \varepsilon^{\beta+1}\alpha_\nu^a(x)(I_\nu(x,n) - B_\nu(T(x))).$$

As we have seen in Proposition B.1, the solvability of this equation requires to impose a compatibility condition on the right hand side. More precisely, $(Id - H)$ is invertible in the space of functions with $\int_{\mathbb{S}^2} f(n)dn = 0$. This compatibility condition is provided by the transport term $\varepsilon n \cdot \nabla_x I_\nu(x,n)$. In particular, the relevant feature is that the problem

$$\alpha_\nu^s(x)(H - Id)[I_\nu(x,\cdot)](n) - \varepsilon n \cdot \nabla_x I_\nu(x,n) = f(x,n,\nu) \tag{B.22}$$

is not solvable if $\varepsilon = 0$, unless $\int_{\mathbb{S}} f(x,n,\nu)dn = 0$. On the contrary, in the case of $\varepsilon > 0$ and small, it turns out that problem (B.22) can be solved for general f. However, the solution becomes of the order $\varepsilon^{-2}\|f\|_\infty$. This explains why we have to add terms much larger than $\varepsilon^{\beta+1}$ in the expansion (B.20) for $\beta > 0$. We remark that the expansion (B.20) is used also in the time-dependent case. There, the value of δ when $\ell_S = \ell_M$ is justified by the behavior of the radiation intensity for smaller time scales and by the need to impose this orthogonality condition.

Having expansion (B.20), we proceed plugging it into the boundary value problem (B.12) and we compare all terms of the same order of magnitude. In this way we will obtain different diffusive equations solved by ϕ_0 in the interior of Ω that will yield the leading order of the radiation intensity I_ν.

However, to solve the resulting equation for ϕ_0 we need some boundary condition whose derivation requires to analyze boundary layer equations for (B.19). The resulting boundary layer problems are related to the description of the radiation intensity in the regions close to the boundary. The thickness of these layers is given by the Milne length and the thermalization length. Therefore, we will rescale the space variable according to ℓ_M and to ℓ_T and we will analyze the resulting one-dimensional problems.

The matching between the outer and the inner solutions will provide the boundary condition for the equation satisfied in the bulk.

B.3.1 Case 1.1: $\ell_M = \ell_T \ll \ell_S$ and $L = 1$. Equilibrium approximation

Since we set $\ell_M = \varepsilon \ll 1$, the case $\ell_M = \ell_T \ll \ell_S$ arises when $\ell_A = \varepsilon$ (i.e. $\beta = -1$) and $\ell_S = \varepsilon^{-\gamma}$ for $\gamma > -1$. Notice that in this case ℓ_S could be small, namely $\ell_S \ll L = 1$, but also large, e.g. if $\gamma > 0$.

In order to find the outer problem, we choose $\delta = \gamma + 1$ and we substitute (B.20) into the first equation in (B.19) and we identify all terms with the same power of ε, i.e. ε^{-1}, ε^γ (if $0 < |\gamma| < 1$) and ε^0. The terms of order ε^{-1} give

$$0 = \alpha_\nu^a(x)(B_\nu(T(x)) - \phi_0(x,n,\nu)).$$

Hence, the leading order satisfies $\phi_0(x,n,\nu) = B_\nu(T(x))$, where B_ν is the Planck distribution which is independent of $n \in \mathbb{S}^2$. This corresponds to the diffusion equilibrium approximation, since in the interior the radiation intensity is at the leading order the equilibrium Planck distribution.

The terms of order ε^γ imply $\psi_1 = 0$. Indeed, since $\phi_0(x,n,\nu) = B_\nu(T(x))$ is independent of $n \in \mathbb{S}^2$ we have $\int_{\mathbb{S}} K(n,n')\phi_0(x,\nu)dn' - \phi_0(x,\nu) = 0$, so that

$$\alpha_\nu^a \psi_1(x,n,\nu) = \int_{\mathbb{S}^2} K(n,n')\phi_0(x,\nu)dn' - \phi_0(x,\nu) = 0.$$

Finally, we compare all terms of order ε^0. In this case we have

$$n \cdot \nabla_x B_\nu(T(x)) = -\alpha_\nu^a(x)\phi_1(x,n,\nu),$$

where in the case $\gamma = 0$ we used again that $(H - Id)\,B_\nu(T) = 0$.

Therefore, we obtain the following expansion for I_ν

$$I_\nu(x,n) = B_\nu(T(x)) - \varepsilon\frac{1}{\alpha_\nu^a(x)} n \cdot \nabla_x B_\nu(T(x)) + \cdots\,, \tag{B.23}$$

where $T(x)$ is a function which is at this stage still unknown.

We now plug (B.23) into the second equation of (B.19). The term of order ε^0 cancels out because $B_\nu(T)$ is isotropic, hence

$$\operatorname{div}\left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\; nB_\nu(T(x))\right) = 0.$$

We find that the leading term is the one of order ε^1 and we obtain

$$\operatorname{div}\left(\int_0^\infty d\nu \frac{1}{\alpha_\nu(x)}\left(\int_{\mathbb{S}^2} dn\; n\otimes n\right)\nabla_x B_\nu(T(x))\right) = 0.$$

Finally, using that $\int_{\mathbb{S}^2} n\otimes n\; dn = \frac{4\pi}{3} Id$ we conclude that the limit problem solved at the interior by T is

$$\operatorname{div}\left(\int_0^\infty \frac{\nabla_x B_\nu(T(x))}{\alpha_\nu(x)} d\nu\right) = 0. \tag{B.24}$$

In order to obtain the behavior of I_ν close to the boundary $\partial\Omega$, we now derive a boundary value problem that can be written in a single variable. This boundary layer equation is known in the literature as Milne problem. The matching of the solution of the Milne problem with the outer solution will provide the boundary value for the equation (B.24) solved by the temperature T.

We take $p \in \partial\Omega$. Assuming that near the boundary the radiation intensity and the temperature only depend on the distance to the boundary, we can further assume that they depend only on the distance to the boundary in direction n_p. This is possible due to the smallness of the thickness of the boundary layer and the continuity of α. We hence define for $x \in \Omega$ in a neighborhood of p the new scalar rescaled variable

$$y = -\frac{x-p}{\varepsilon} \cdot n_p. \tag{B.25}$$

We recall that $-(x-p)\cdot n_p$ is non-negative, since $x-p$ points in the interior of the domain, and it is exactly the length of the cathetus with endpoint p of the triangle having as hypotenuse $x - p$ (cf. Figure B.1).

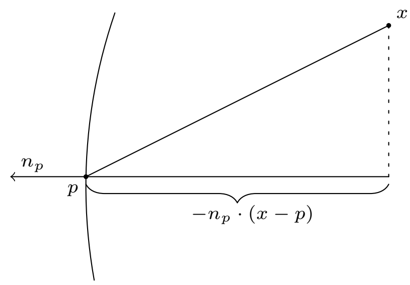

Figure B.1: Representation of the change of variables.

Defining $\mathcal{R}_p(x) = \mathrm{Rot}_p(x-p)$ as a rigid motion mapping p to zero with $\mathrm{Rot}_p(n_p) = -e_1$ we see that we can also write y as the first component of $y_1 = \mathcal{R}_p\left(\frac{x}{\varepsilon}\right)_1$. Hence, as $\varepsilon \to 0$ we obtain that both the absorption and scattering coefficients satisfy $\alpha_\nu^j(x) = \alpha_\nu^j(\varepsilon \mathrm{Rot}_p(x) + p) \to \alpha_\nu^j(p)$, $j \in \{a, s\}$.

We can now write the one-dimensional problem obtained by this new scaling and by the limit $\varepsilon \to 0$. Since $\varepsilon^{\gamma+1} \to 0$ as $\varepsilon \to 0$, the scattering term is negligible and we obtain for any $p \in \partial\Omega$

$$\begin{cases} -(n \cdot n_p)\partial_y I_\nu(y,n;p) = \alpha_\nu^a(p)(B_\nu(T(y,p)) - I_\nu(y,n;p)) & y > 0\ , n \in \mathbb{S}^2 \\ \operatorname{div}\left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\ (n \cdot n_p) I_\nu(y,n;p)\right) = 0 & y > 0,\ n \in \mathbb{S}^2 \\ I_\nu(0,n;p) = g_\nu(n) & n \cdot n_p < 0. \end{cases} \tag{B.26}$$

The Milne equation (B.26) is the equation describing the boundary layer for the diffusion approximation. In the pure absorption case the Milne problem was rigorously studied in [68]. The well-posedness of (B.26) is shown there for constant absorption coefficients and also for coefficients depending only on the frequency ν, as well as for coefficients depending on both frequency and temperature of the form $\alpha_\nu^a(p) = Q(\nu)\alpha(T(p))$. Moreover, also the asymptotic behavior of I_ν at infinity has been computed in this paper. It is indeed shown in [68] that as $y \to \infty$ the solution of the Milne problem converges to the Planck distribution, i.e.

$$\lim_{y\to\infty} I_\nu(y,n;p) = I_\nu^\infty(p) = B_\nu(T_\infty(p)),$$

for some $T_\infty(p)$ depending only on g_ν and p. Notice that $I_\nu^\infty(p)$ is independent of $n \in \mathbb{S}^2$.

Moreover, since in this case the thermalization length and the Milne length are the same this is the only boundary layer appearing. The radiation intensity I_ν becomes simultaneously isotropic and at equilibrium $B_\nu(T)$ in the same length scale. This gives a matching condition for the temperature that has to be used as boundary condition for the new limit problem. In particular, the temperature and the radiation intensity solving the Milne problem (B.26) are related by equation (B.16). In particular,

$$T_\infty(p) = \lim_{y\to\infty} F^{-1}\left(\left(\int_0^\infty d\nu \alpha_\nu^a(p) I_\nu^\infty(p)\right), p\right), \tag{B.27}$$

where F is defined in (B.17) and $g \mapsto I_\nu^\infty(p)$ is a functional that determines the limit intensity for each boundary point $p \in \partial\Omega$.

Summarizing, the limit problem for the stationary radiative transfer equation (B.12) in the case $\ell_M = \ell_T \ll \ell_S$ is given by the following boundary value problem

$$\begin{cases} \operatorname{div}\left(\int_0^\infty \frac{\nabla_x B_\nu(T(x))}{\alpha_\nu(x)} d\nu\right) = 0 & x \in \Omega \\ T(p) = T_\infty(p) & p \in \partial\Omega, \end{cases}$$

where $T_\infty(p)$ is given by (B.27).

B.3.2 Case 1.2: $\ell_M = \ell_T = \ell_S \ll L$. Equilibrium approximation

Due to the definitions $\ell_M = \varepsilon \ll 1$ and $\ell_T = \sqrt{\ell_A \ell_M}$ we have $\ell_M = \ell_S = \ell_T = \ell_A = \varepsilon$ in (B.12), i.e. $\beta = \gamma = -1$ in (B.19).

We consider the expansion (B.20) for $\delta = 0$, or equivalently without the expansion $\sum_{k\geq 0} \varepsilon^{\delta+k}\psi_{k+1}$. We plug (B.20) into (B.19) and we compare all terms of the same power of ε, namely ε^{-1} and ε^0. The term of order ε^{-1} yields

$$0 = \alpha_\nu^a(x)(B_\nu(T(x)) - \phi_0(x,n,\nu)) + \alpha_\nu^s(x)\left(\int_{\mathbb{S}^2} K(n,n')\phi_0(x,n',\nu)dn' - \phi_0(x,n,\nu)\right). \tag{B.28}$$

Notice that $\phi_0(x,n,\nu) = B_\nu(T(x))$ is a solution to (B.28). This follows from Proposition B.1 and the isotropy of $B_\nu(T)$. We show now that the solution to (B.28) is unique.

To this end for every $x \in \mathbb{R}^3$ and $\nu > 0$ we define $0 < \theta_{\nu,x} = \frac{\alpha_\nu^s(x)}{\alpha_\nu^a(x)+\alpha_\nu^s(x)} < 1$. Moreover, we define also the following operator which maps for every fixed x, ν non-negative continuous functions to non-negative continuous functions and given by

$$A_{\nu,x}[\varphi](n) = \theta_{\nu,x} \int_{\mathbb{S}^2} K(n,n')\varphi(n')\, dn'. \tag{B.29}$$

Then equation (B.28) can be rewritten as

$$\phi_0(x,n,\nu) = A_{\nu,x}[\phi_0](x,n,\nu) + \frac{\alpha_\nu^a(x)}{\alpha_\nu^a(x)+\alpha_\nu^s(x)} B_\nu(T(x)). \tag{B.30}$$

Since the maps $\phi_0 \mapsto A_{\nu,x}(\phi_0)$ is a linear contraction, the Banach fixed-point theorem implies that (B.30) has a unique solution for every $T(x) \in \mathbb{R}_+$. Hence, $\phi_0 = B_\nu(T)$. Therefore, also in this case we recover the equilibrium diffusion approximation.

We turn now to the terms of order ε^0. In this case we have

$$n \cdot \nabla_x B_\nu(T(x)) = -\alpha_\nu^a(x)\phi_1(x,n,\nu) - \alpha_\nu^s(x)\left(\int_{\mathbb{S}^2} K(n,n')\phi_1(x,n',\nu)\, dn' - \phi_1(x,n,\nu)\right).$$

Then, using the operator $A_{\nu,x}$ defined as in (B.29), we can rewrite this equation as

$$-\frac{1}{\alpha_\nu^a(x)+\alpha_\nu^s(x)} n \cdot \nabla_x B_\nu(T(x)) = (Id - A_{\nu,x})\, \phi_1(x,n,\nu). \tag{B.31}$$

The same argument as for the term of order ε^{-1} holds also in this case and Banach fixed-point theorem ensures the existence of a unique solution to (B.31) given by

$$\phi_1(x,n,\nu) = -\frac{1}{\alpha_\nu^a(x)+\alpha_\nu^s(x)} (Id - A_{\nu,x})^{-1}(n) \cdot \nabla_x B_\nu(T(x)),$$

where for any x, ν we used the notation

$$(Id - A_{\nu,x})^{-1}(n) = \begin{pmatrix} (Id - A_{\nu,x})^{-1}(n_1) \\ (Id - A_{\nu,x})^{-1}(n_2) \\ (Id - A_{\nu,x})^{-1}(n_3) \end{pmatrix},$$

which is well-defined due to the action of the linear operator $A_{\nu,x}$ only on the variable $n \in \mathbb{S}^2$.

Hence, we obtain the following expansion

$$I_\nu(x,n) = B_\nu(T(x)) - \varepsilon \frac{1}{\alpha_\nu^a(x)+\alpha_\nu^s(x)} (Id - A_{\nu,x})^{-1}(n) \cdot \nabla_x B_\nu(T(x)) + \varepsilon^2 \phi_2 + \cdots \tag{B.32}$$

Plugging (B.32) into the second equation of (B.12) and using that the Planck distribution is isotropic, we obtain the following limit problem solved in the domain Ω that yields the temperature $T(x)$ to the leading order

$$\begin{aligned} 0 &= \operatorname{div}\left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, n \frac{1}{\alpha_\nu^a(x)+\alpha_\nu^s(x)} \left(Id - A_{\nu,x}\right)^{-1}(n) \cdot \nabla_x B_\nu(T(x))\right) \\ &= \operatorname{div}\left(\int_0^\infty d\nu \frac{1}{\alpha_\nu^a(x)+\alpha_\nu^s(x)} \left(\int_{\mathbb{S}^2} dn\, n \otimes \left(Id - A_{\nu,x}\right)^{-1}(n)\right) \nabla_x B_\nu(T(x))\right). \end{aligned} \tag{B.33}$$

The behavior of I_ν close to the boundary $\partial\Omega$ is given again by a boundary layer equation which can be written in one variable. The derivation of the Milne problem for this case follows exactly the same steps as Subsection B.3.1 under the scaling (B.25). In this case both emission and scattering terms appear, since they are of the same order. Hence, for every $p \in \partial\Omega$ the Milne problem is given by

$$\begin{cases} -(n\cdot n_p)\partial_y I_\nu(y,n,p) = \alpha_\nu^a(p)(B_\nu(T(y,p)) - I_\nu(y,n,p)) & \\ \qquad\qquad +\alpha_\nu^s(p)\left(\int_{\mathbb{S}^2} K(n,n')I_\nu(y,n',p)\,dn' - I_\nu(y,n,p)\right) & y>0\,, n\in\mathbb{S}^2 \\ \operatorname{div}\left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\,(n\cdot n_p) I_\nu(y,n,p)\right) = 0 & y>0,\ n\in\mathbb{S}^2 \\ I_\nu(0,n,p) = g_\nu(n) & n\cdot n_p<0. \end{cases} \tag{B.34}$$

The mathematical properties of the Milne problem for both absorption and scattering processes have been considered in [127]. Although the results provided in [127] have been obtained only for the case of constant scattering kernel and constant scattering coefficient, the arguments there suggest that for more general choices of K and α_ν^s the solution I_ν of (B.34) converges to the Planck equilibrium distribution as $y \to \infty$.

Notice that in this case, the thermalization length and the Milne length are the same, hence the boundary layers coincide. Matching inner and outer solutions we obtain the following boundary condition for equation (B.33)

$$T_\infty(p) = \lim_{y\to\infty} F^{-1}\left(\left(\int_0^\infty d\nu \fint_{\mathbb{S}^2} dn\, \alpha_\nu^a(p) I_\nu(y,n,p)\right), p\right), \tag{B.35}$$

with F as in (B.17). Indeed as we have seen in Subsection B.2.2, the temperature T and the radiation energy I_ν satisfying the Milne problem (B.34) are related by the identity (B.16).

Summarizing, the limit problem for the stationary radiative transfer equation (B.12) in the case $\ell_M = \ell_T = \ell_S$ is given by the following boundary value problem

$$\begin{cases} \operatorname{div}\left(\int_0^\infty \frac{d\nu}{\alpha_\nu^a(x)+\alpha_\nu^s(x)} \left(\int_{\mathbb{S}^2} dn\, n\otimes \left(Id - A_{\nu,x}\right)^{-1}(n)\right) \nabla_x B_\nu(T(x))\right) = 0 & x\in\Omega \\ T(p) = T_\infty(p) & p\in\partial\Omega, \end{cases}$$

where T_∞ is defined as in (B.35) for the solution $I_\nu(y,n,p)$ to the Milne problem (B.34).

B.3.3 Case 2: $\ell_M \ll \ell_T \ll L$. Equilibrium approximation

The assumption $\ell_T = \sqrt{\ell_M \ell_A} \gg \ell_M$ implies $\ell_A > \ell_M$ and hence $\varepsilon = \ell_M = \ell_S$. We thus consider $\ell_A = \varepsilon^{-\beta}$ for $\beta > -1$. Moreover, since $\ell_T = \varepsilon^{\frac{1-\beta}{2}} \ll L = 1$ we restrict to the case $\ell_A = \varepsilon^{-\beta}$ for $\beta \in (-1,1)$.

Since $\ell_M = \ell_S \ll \ell_A$, the scattering process has a greater effect than the absorption-emission process. We expect hence the Milne problem to depend exclusively on the scattering process. In the bulk we expect also the scattering process to be present in the diffusive equation derived for the limit problem, but we will also show that at the interior the leading order of the radiation intensity is still the Planck distribution. Thus, we are again in the case of the equilibrium diffusion approximation. In this case the thermalization length is much larger than the Milne length but it is also still much smaller than the characteristic length of the domain. A second boundary layer, the so-called thermalization layer, will therefore appear. The equation describing this new layer will depend on both absorption-emission and scattering processes. Moreover, while the radiative energy becomes isotropic in the Milne layer, in the thermalization layer I_ν will approach the Planck distribution.

We use again the expansion (B.20) for the radiation intensity with $\delta = \beta - \lfloor\beta\rfloor$, i.e. $\delta = \beta + 1$ if $\beta < 0$ and $\delta = \beta$ if $\beta \geq 0$, and we plug it into the first equation in (B.12). We proceed as usual with the identification of the terms with the same power of ε.

Using the notation of (B.14) the terms of order ε^{-1} give

$$H[\phi_0(x,\cdot,\nu)](n) = \phi_0(x,n,\nu).$$

Proposition B.1 implies hence that ϕ_0 is independent of $n \in \mathbb{S}^2$ and hence $\phi_0 = \phi_0(x,\nu)$.

Next we consider $\beta < 0$. The terms of power ε^β give

$$\alpha_\nu^a(x)(B_\nu(T(x)) - \phi_0(x,n)) = \alpha_\nu^s(H - id)\psi_1(x,n,\nu).$$

An integration over $\mathbb{S}^2$ implies $B_\nu(T(x)) = \phi_0(x,\nu)$. Hence, as for ϕ_0 we conclude that $\psi_1 = \psi_1(x,\nu)$ is independent of $n \in \mathbb{S}^2$. The terms of power ε^0 give

$$n \cdot \nabla_x \phi_0(x,\nu) = \alpha_\nu^s(x)\left(H[\phi_2](x,n,\nu) - \phi_2(x,n,\nu)\right). \tag{B.36}$$

Now we consider $\beta > 0$. In this case $\delta = \beta$. The terms of power $\varepsilon^{\beta-1}$ give

$$H[\psi_1(x,\cdot,\nu)](n) = \psi_1(x,n,\nu),$$

which implies that $\psi_1(x,\nu)$ is independent of $n \in \mathbb{S}^2$. The terms of power ε^0 yield again equation (B.36), while the terms of power ε^β imply

$$n \cdot \nabla_x \psi_1(x,\nu) = \alpha_\nu^a(x)(B_\nu(T(x)) - \phi_0(x,n)) + \alpha_\nu^s(x)(H - id)[\psi_2(x,\cdot,\nu)](n), \tag{B.37}$$

for which an integration over $\mathbb{S}^2$ and the isotropy of both ϕ_0 and ψ_1 give $B_\nu(T(x)) = \phi_0(x,\nu)$.

Finally, it remains to study the case $\beta = 0$. In this case there is no expansion $\sum_{k\geq 0} \varepsilon^\delta \psi_{k+1}$. Therefore, the terms of order ε^0 give equation

$$n \cdot \nabla_x \phi_0(x,\nu) = \alpha_\nu^a(x)\left(B_\nu(T(x)) - \phi_0(x,\nu)\right) + \alpha_\nu^s(x)\left(H[\phi_2](x,n,\nu) - \phi_2(x,n,\nu)\right)$$

which integrated over $\mathbb{S}^2$ implies, due to the isotropy of ϕ_0, as for (B.37).

Hence, for all $\beta \in (-1,1)$ the identification of all terms of power ε^{-1}, ε^β, $\varepsilon^{\beta-1}$ (if $\beta > 0$) and ε^0 gives $\phi_0 = B_\nu(T)$, $\psi_1 = \psi_1(x,\nu)$ and

$$-\frac{1}{\alpha_\nu^s(x)} n \cdot \nabla_x B_\nu(T(x)) = (Id - H)[\phi_1(x,\cdot,\nu)](n). \tag{B.38}$$

We now study the equation (B.38). As we know from Proposition B.1 the kernel of the operator $(Id - H)$ is given by the constant functions and its range are all functions with zero mean integral, i.e. $\mathrm{Ran}(Id - H) = \{\varphi \in L^\infty(\mathbb{S}^2) : \int_{\mathbb{S}^2} \varphi = 0\}$. Hence, the following linear operator is bijective

$$(Id - H) : L^\infty(\mathbb{S}^2) \Big/ \mathcal{N}(Id - H) \to \mathrm{Ran}(Id - H),$$

where $L^\infty(\mathbb{S}^2) \big/ \mathcal{N}(Id-H)$ denotes the quotient space. Let $e_i \in \mathbb{R}^3$ be the unit vector, we consider the equation

$$n \cdot e_i = (Id-H)\varphi(n). \tag{B.39}$$

Since $n \cdot e_i \in \operatorname{Ran}(Id-H)$, for any $c \in \mathbb{R}$ the function $\varphi(n) = (Id-H)^{-1}(n\cdot e_i) + c$ is a solution to (B.39). Therefore, using the notation

$$(Id-H)^{-1}(n) = \begin{pmatrix} (Id-H)^{-1}(n\cdot e_1) \\ (Id-H)^{-1}(n\cdot e_2) \\ (Id-H)^{-1}(n\cdot e_3) \end{pmatrix}$$

and using the linearity of $(Id-H)$ we see that ϕ_2 is given by

$$\phi_2(x,n,\nu) = -\frac{1}{\alpha_\nu^s(x)}(Id-H)^{-1}(n)\cdot \nabla_x B_\nu(T(x)) + c(x,\nu) \tag{B.40}$$

where $c(x,\nu)$ is independent of $n \in \mathbb{S}^2$. The isotropic function $c(x,\nu)$ does not contribute in the divergence free condition of (B.12), therefore we will not compute the exact value of $c(x,\nu)$. Equation (B.40) implies that the first three terms in the expansion of I_ν are given for all $\beta \in (-1,1)$ by

$$I_\nu(x,n) = B_\nu(T(x)) + \varepsilon^{\beta-\lfloor\beta\rfloor}\psi_1(x,\nu) - \frac{\varepsilon}{\alpha_\nu^s(x)}(Id-H)^{-1}(n)\cdot\nabla_x B_\nu(T(x)) + \varepsilon c(x,\nu) + \cdots$$

The divergence free condition in (B.12) implies in the same manner as in the derivation of (B.33) the following equation, which yields the limit problem in the interior of the domain Ω

$$\operatorname{div}\left(\int_0^\infty d\nu \frac{1}{\alpha_\nu^s(x)}\left(\int_{\mathbb{S}^2} dn\, n\otimes (Id-H)^{-1}(n)\right)\nabla_x B_\nu(T(x))\right) = 0. \tag{B.41}$$

The behavior of I_ν close to the boundary $\partial\Omega$ is described by two nested boundary layer equations. As anticipated at the beginning of Subsection B.3.3, since $\ell_M \ll \ell_T \ll L$ we observe the formation of two distinct boundary layers. The first one, the Milne layer, has a thickness of size ℓ_M and it is described by the Milne problem, whose derivation is similar to the derivation of the Milne problems (B.26) and (B.34). The next boundary layer, which we will denote by thermalization layer, has a thickness of size ℓ_T and it is described by a new boundary layer equation, which we will denote as thermalization equation and which we will construct immediately after deriving the Milne problem.

Following the same procedure as in Subsection B.3.1 we can derive the Milne problem for this scaling limit under the rescaling (B.25). In this case we obtain a closed equation for I_ν which depends only on the scattering process, since this is the largest term. Indeed, rescaling the space variable we obtain

$$\begin{cases} -(n\cdot n_p)\partial_y I_\nu(y,n,p) = \alpha_\nu^s\left(p+\mathcal{O}(\varepsilon)\right)\left(\int_{\mathbb{S}^2} K(n,n')I_\nu(y,n',p)\,dn' - I_\nu(y,n,p)\right) & \\ \qquad\qquad +\varepsilon^{\beta+1}\alpha_\nu^a\left(p+\mathcal{O}(\varepsilon)\right)\left(B_\nu(T(y;p)) - I_\nu(y,n;p)\right) & y>0\,, n\in\mathbb{S}^2 \\ \int_0^\infty d\nu \int_{\mathbb{S}^2} dn\,(n\cdot n_p)\partial_y I_\nu(y,n,p) = 0 & y>0,\ n\in\mathbb{S}^2 \\ I_\nu(0,n,p) = g_\nu(n) & n\cdot n_p<0. \end{cases} \tag{B.42}$$

Hence, for every $p \in \partial\Omega$ the Milne problem is given by

$$\begin{cases} -(n\cdot n_p)\partial_y I_\nu(y,n,p) = \alpha_\nu^s(p)\left(\int_{\mathbb{S}^2} K(n,n')I_\nu(y,n',p)\,dn' - I_\nu(y,n,p)\right) & y>0\,, n\in\mathbb{S}^2 \\ I_\nu(0,n,p) = g_\nu(n) & n\cdot n_p<0. \end{cases} \tag{B.43}$$

On the other hand, we also obtain an equation for the temperature. Indeed, plugging the first equation of (B.42) into the second one, we obtain to the leading order

$$\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\ \alpha_\nu^a(p)\left(B_\nu(T(y;p)) - I_\nu(y,n;p)\right) = 0. \tag{B.44}$$

This equation has a steady distribution for the temperature T completely determined. At a first glance, this appears strange since in the limit equation (B.43) the absorption coefficient $\alpha_\nu^a(p)$ does not appear and the only processes able to modify the temperature are the absorption and emission of photons. However, the solution of this apparent paradox is that since (B.43) describes a stationary solution, it is implicitly understood that the system was running during an infinite amount of time before and the absorption/emission process had time to bring the system to a steady state, even when this process is very small.

The Milne problem for the pure scattering case has been studied in several papers such as [17, 19, 76, 127] in the context of neutron transport. Although all these results are actually obtained for functions α^s independent of the frequency, since the one-speed approximation for the neutron transport (cf. (B.13)) was considered, they are expected to hold pointwise for every frequency ν. For example, in [17] it is shown that there exists a unique solution to (B.43) for strictly positive bounded and rotational symmetric scattering kernels. Moreover, as $y \to \infty$ the solution approaches a function $I(\nu;p)$ independent of $n \in \mathbb{S}^2$. Hence, in the Milne layer the radiation intensity becomes isotropic.

We now turn to the thermalization layer. In this layer we expect the radiation intensity to approach the Planck equilibrium distribution. Moreover, the boundary value for the problem (B.41) can be also found analyzing the thermalization layer. In order to construct the new boundary layer equation, i.e. the thermalization equation, we rescale the space variable according to the one-dimensional variable $\eta = -\frac{x-p}{\ell_T}\cdot n_p$ for $p \in \partial\Omega$ and we obtain the following equation

$$\begin{cases} -\varepsilon^{\frac{1+\beta}{2}}(n\cdot n_p)\partial_\eta I_\nu(\eta,n,p) = \alpha_\nu^a\left(p+\varepsilon^{\frac{1-\beta}{2}}\mathrm{Rot}_p(\eta)\right)\varepsilon^{1+\beta}(B_\nu(T(\eta,p)) - I_\nu(\eta,n,p)) & \\ \qquad +\alpha_\nu^s\left(p+\varepsilon^{\frac{1-\beta}{2}}\mathrm{Rot}_p(\eta)\right)\left(\left(\int_{\mathbb{S}^2} dn' K(n,n')I_\nu(\eta,n',p)\right) - I_\nu(\eta,n,p)\right) & \eta>0\ , n\in\mathbb{S}^2 \\ \mathrm{div}\left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\ (n\cdot n_p) I_\nu(\eta,n,p)\right) = 0 & \eta>0,\ n\in\mathbb{S}^2 \\ I_\nu(0,n,p) = I(\nu;p) & p\in\partial\Omega, \end{cases} \tag{B.45}$$

where $I(\nu;p) = \lim\limits_{y\to\infty} I^M(y,n,\nu;p)$ for I^M the solution to the Milne problem (B.43). In order to find the thermalization equation we proceed in a way similar to the derivation of the outer problem. We hence expand the radiation intensity according to

$$I_\nu(\eta,n;p) = \varphi_0(\eta,n,\nu;p) + \varepsilon^{\frac{1+\beta}{2}}\varphi_1(\eta,n,\nu;p) + \varepsilon^{1+\beta}\varphi_2(\eta,n,\nu;p) + \cdots$$

and we identify in (B.45) all terms of the same power of ε, namely ε^0, $\varepsilon^{\frac{1+\beta}{2}}$ and $\varepsilon^{1+\beta}$. We remark first that the functions φ_i for $i \in \mathbb{N}$ could depend on ε. Moreover, the choice of the powers of ε in the expansion of I_ν is motivated by the order of magnitude of the sources in (B.45).

The terms of order ε^0 give

$$\int_{\mathbb{S}^2} K(n,n')\varphi_0 dn' = \varphi_0$$

and hence by Proposition B.1 $\varphi_0(\eta,n,\nu;p) = \varphi_0(\eta,\nu;p)$ is independent of the direction $n \in \mathbb{S}^2$. The isotropy of φ_0 was expected as it is matched with the solution of the Milne problem, which becomes isotropic. Moreover, we see also that φ_0 does not depend on ε.

The terms of order $\varepsilon^{\frac{1+\beta}{2}}$ give

$$(n \cdot n_p)\partial_\eta \varphi_0 = \alpha_\nu^s (p_\varepsilon) (Id - H)(\varphi_1),$$

where we defined $p_\varepsilon = p + \varepsilon^{\frac{1-\beta}{2}} \mathrm{Rot}_p(\eta)$. Thus, arguing as in the derivation of (B.40), Proposition B.1 implies

$$\varphi_1(\eta, n, \nu; p) = \frac{1}{\alpha_\nu^s (p_\varepsilon)} (Id - H)^{-1}(n) \cdot n_p \partial_\eta \varphi_0 + c(\eta, \nu),$$

for some function $c(\eta, \nu)$. Finally, identifying the terms of order $\varepsilon^{1+\beta}$ implies after an integration over $\mathbb{S}^2$

$$\begin{aligned} -\frac{1}{\alpha_\nu^s (p_\varepsilon)} \left(\fint_{\mathbb{S}^2} (n \cdot n_p)(Id - H)^{-1}(n) \cdot n_p \, dn \right) \partial_\eta^2 \varphi_0(\eta, \nu; p) \\ = \alpha_\nu^a (p_\varepsilon) \left(B_\nu(T(\eta; p)) - \varphi_0(\eta, \nu; p) \right). \end{aligned}$$

We now consider the limit as $\varepsilon \to 0$ and we obtain by the continuity of the absorption and scattering coefficient

$$-\frac{1}{\alpha_\nu^s (p)} \left(\fint_{\mathbb{S}^2} (n \cdot n_p)(Id - H)^{-1}(n) \cdot n_p \, dn \right) \partial_\eta^2 \varphi_0(\eta, \nu; p) = \alpha_\nu^a (p) \left(B_\nu(T(\eta; p)) - \varphi_0(\eta, \nu; p) \right). \tag{B.46}$$

Moreover, the second equation in (B.45) yields

$$\int_0^\infty d\nu \frac{1}{\alpha_\nu^s(p)} \left(\fint_{\mathbb{S}^2} (n \cdot n_p)(Id - H)^{-1}(n) \cdot n_p \, dn \right) \partial_\eta^2 \varphi_0(\eta, \nu; p) = 0, \tag{B.47}$$

where we again considered the limit $\varepsilon \to 0$. Thus, the thermalization layer equation is given for every $p \in \partial\Omega$ by

$$\begin{cases} \varphi_0(\eta, \nu; p) - \frac{1}{\alpha_\nu^a(p)\alpha_\nu^s(p)} \left(\fint_{\mathbb{S}^2} (n \cdot n_p)(Id - H)^{-1}(n) \cdot n_p \, dn \right) \partial_\eta^2 \varphi_0(\eta, \nu; p) & \\ \qquad = B_\nu(T(\eta; p)) & \eta > 0 \\ \int_0^\infty d\nu \; \alpha_\nu^a(p) B_\nu(T(\eta; p)) = \int_0^\infty d\nu \; \alpha_\nu^a(p) \varphi_0(\eta, \nu; p) & \eta > 0 \\ \varphi_0(0, \nu; p) = I(\nu; p) & p \in \partial\Omega, \end{cases} \tag{B.48}$$

where the second equation is implied by (B.47) taking the integral over the frequency of (B.46). As far as we know, the thermalization problem has not been studied so far in the literature and its well-posedness properties have not been described in detail. Nevertheless, we claim that the problem is well-posed under suitable assumptions and that the solution φ_0 to (B.48) converges to the Planck distribution, i.e.

$$\lim_{\eta \to \infty} \varphi_0(\eta, \nu; p) = \varphi(\nu, p) = B_\nu(T_\infty(p)).$$

From the second equation in (B.48) we recover the relation (B.16) between the temperature and the radiation intensity φ_0. Hence, $T(\eta; p) = F^{-1} \left(\left(\int_0^\infty d\nu \; \alpha_\nu^a(p) \varphi_0(\eta, \nu; p) \right), \eta; p \right)$ for F defined in (B.17). In particular,

$$T_\infty(p) = F^{-1} \left(\left(\int_0^\infty d\nu \; \alpha_\nu^a(p) \varphi(\nu, p) \right), p \right). \tag{B.49}$$

We remark that since $I(\nu;p)$, the limit as $y\to\infty$ of the solution $I^M(y,n,\nu;p)$ of the Milne problem (B.43), is a functional of the boundary condition g_ν, so are $\varphi(\nu,p)$ and $T_\infty(p)$ functionals of the boundary condition g_ν. Summarizing, in the case of $\ell_M \ll \ell_T \ll L$ the solution to (B.12) is expected to solve in the limit problem the following equilibrium diffusion approximation given by the stationary boundary value problem

$$\begin{cases} \operatorname{div}\left(\int_0^\infty d\nu \frac{1}{\alpha_\nu^s(x)}\left(\int_{\mathbb{S}^2} dn\, n\otimes (Id-H)^{-1}(n)\right)\nabla_x B_\nu(T(x))\right)=0 & x\in\Omega \\ T(p)=T_\infty(p) & p\in\partial\Omega, \end{cases}$$

where T_∞ is defined in (B.49)

B.3.4 Case 3: $\ell_M \ll \ell_T = L$. Transition from equilibrium to non-equilibrium.

Since $\ell_M=\varepsilon$ and $\ell_T=\sqrt{\varepsilon\ell_A}=L=1$, we have to consider $\ell_S=\varepsilon$ and $\ell_A=\varepsilon^{-1}$.

This case is intriguing, because as we will see it yields the transition between the equilibrium approximation and the non-equilibrium approximation, i.e. the case where in the limit the radiation intensity is not given by the Planck distribution at the leading order in the bulk of the domain Ω.

As usual we plug the expansion (B.20) for $\delta=0$, thus without terms ψ_k, into the first equation of (B.19) and we identify all terms of the same power of ε, namely ε^{-1}, ε^0 and ε^1.

The terms of order ε^{-1} give

$$\phi_0(x,n,\nu)=H[\phi_0(x,\cdot,\nu)](n),$$

and hence by Proposition B.1 the leading order is independent of $n\in\mathbb{S}^2$, i.e. $\phi_0=\phi_0(x,\nu)$.

The terms of order ε^0 give

$$n\cdot\nabla_x\phi_0(x,\nu)=\alpha_\nu^s(x)\left(H[\phi_1(x,\cdot,\nu)](n)-\phi_1(x,n,\nu)\right).$$

Due to the isotropy of ϕ_0, Proposition B.1 implies that ϕ_1 is given by

$$\phi_1(x,n,\nu)=-\frac{1}{\alpha_\nu^s(x)}\left(Id-H\right)^{-1}(n)\cdot\nabla_x\phi_0(x,\nu)+c(x,\nu),$$

where $c(x,\nu)$ is some function independent of $n\in\mathbb{S}^2$. As in subsection B.3.3 the isotropic function $c(x,\nu)$ will not contribute to the divergence free condition, hence it will not be explicitly computed.

Finally, the terms of order ε^1 yield, after an integration over $\mathbb{S}^2$

$$4\pi\alpha_\nu^a(x)\phi_0(x,\nu)-\operatorname{div}\left(\frac{1}{\alpha_\nu^s(x)}\left(\int_{\mathbb{S}^2} n\otimes(Id-H)^{-1}(n)\,dn\right)\nabla_x\phi_0(x,\nu)\right) \\ =4\pi\alpha_\nu^a(x)B_\nu(T(x)), \quad \text{(B.50)}$$

where we used the invariance under rotations of the scattering kernel K and the identity $n\cdot\nabla_x f=\operatorname{div}(nf)$.

Moreover, plugging the expansion

$$I_\nu(x,n)=\phi_0(x,\nu)-\frac{\varepsilon}{\alpha_\nu^s(x)}\left(Id-H\right)^{-1}(n)\cdot\nabla_x\phi_0(x,\nu)+\varepsilon c(x,\nu)+\varepsilon^2\cdots$$

into the divergence free equation in (B.19) we obtain at the leading order

$$\operatorname{div}\left(\int_0^\infty d\nu\,\frac{1}{\alpha_\nu^s(x)}\left(\int_{\mathbb{S}^2} n\otimes(Id-H)^{-1}(n)\,dn\right)\nabla_x\phi_0(x,\nu)\right)=0,$$

which implies integrating (B.50) the following equation for the temperature

$$\int_0^\infty d\nu\ \alpha_\nu^a(x)\phi_0(x,\nu)=\int_0^\infty d\nu\ \alpha_\nu^a(x)B_\nu(T(x)).$$

Hence, using the definition of F in (B.17) we obtain the limit problem for ϕ_0 in the interior, namely

$$\begin{aligned}\phi_0(x,\nu)-\frac{1}{4\pi\alpha_\nu^a(x)}\operatorname{div}\left(\frac{1}{\alpha_\nu^s(x)}\left(\int_{\mathbb{S}^2}n\otimes\left(Id-H\right)^{-1}(n)\,dn\right)\nabla_x\phi_0(x,\nu)\right)\\ =B_\nu\left(F^{-1}\left(\left(\int_0^\infty d\nu\ \alpha_\nu^a(x)\phi_0(x,\nu)\right),x\right)\right).\end{aligned}$$

Once more the boundary condition for the diffusion equation is given by the matching of the outer solution with the solution to a suitable boundary layer equation. Since $\ell_T=L=1$, the thermalization layer corresponds to the outer problem. Indeed, the radiation intensity is out of equilibrium in the limit as $\varepsilon\to 0$. Hence, there is only one boundary layer, namely the Milne layer. The Milne problem describing the boundary layer for (B.19) as $\ell_M\ll L=\ell_T$ is given once more by the (B.43). Indeed, the scattering term is the term of larger order with $\ell_M=\ell_S$. Therefore, the computations in Subsection B.3.3 hold in this case too. Summarizing, if $\ell_M\ll\ell_T=L$ the radiation intensity and the temperature satisfy the following equation

$$\begin{cases}\phi_0(x,\nu)-\frac{1}{4\pi\alpha_\nu^a(x)}\operatorname{div}\left(\frac{1}{\alpha_\nu^s(x)}\left(\int_{\mathbb{S}^2}n\otimes\left(Id-H\right)^{-1}(n)\,dn\right)\nabla_x\phi_0(x,\nu)\right)=B_\nu\left(T(x)\right) & x\in\Omega\\ \int_0^\infty d\nu\ \alpha_\nu^a(x)\left(B_\nu(T(x))-\phi_0(x,\nu)\right)=0 & x\in\Omega\\ \phi_0(p,\nu)=I_\nu^\infty(p) & p\in\partial\Omega,\end{cases}$$

where $I_\nu^\infty(p)=\lim\limits_{y\to\infty}I_\nu(y,n;p)$ for $I_\nu(y,n;p)$ the solution to (B.43) which converges to the isotropic function I_ν^∞. It is important to remark here that in this case we are not obtaining an equilibrium diffusion regime. Indeed, the leading order ϕ_0 is not the Planck distribution and therefore this case is an example of the non-equilibrium diffusion approximation.

B.3.5 Case 4: $\ell_M\ll L\ll\ell_T$. Non-Equilibrium approximation

Since $\ell_M=\varepsilon$, the case where $\ell_T=\sqrt{\varepsilon\ell_A}\gg L=1$ corresponds to $\ell_S=\varepsilon$ and $\ell_A=\varepsilon^{-\beta}$ for $\beta>1$. Under this assumption we obtain $\ell_T=\varepsilon^{\frac{1-\beta}{2}}\to\infty$ as $\varepsilon\to 0$. Therefore, in this last subsection we study the case when the thermalization length ℓ_T is growing to infinity as $\varepsilon\to 0$. In this case we do not expect the solution to (B.12) to approach at the interior the Planck distribution. We will indeed see that in this case we obtain the so called non-equilibrium diffusion approximation.

In order to derive the outer problem for (B.19), we plug expansion (B.20) with $\delta=\beta-1$ into the first equation of (B.19) and we identify all terms of the same power of ε, namely ε^{-1}, $\varepsilon^{\beta-2}$, ε^0, $\varepsilon^{\beta-1}$ and ε^1. The terms of order ε^{-1} and $\varepsilon^{\beta-\lfloor\beta\rfloor}$ yield $\int_{\mathbb{S}^2}K(n,n')f(n')dn'=f(n)$ for $f\in\{\phi_0,\psi_1\}$, respectively. Therefore, at the leading order the radiation intensity is isotropic, i.e. $\phi_0=\phi_0(x,\nu)$. Moreover, also $\psi_1=\psi_1(x,\nu)$.

The terms of power ε^0 give

$$-\frac{1}{\alpha_\nu^s(x)}n\cdot\nabla_x\phi_0=(Id-H)[\phi_1(x,\cdot,\nu)](n).$$

Hence, Proposition B.1 implies the existence of some function $c(x,\nu)$ independent of $n\in\mathbb{S}^2$ such that

$$\phi_1(x,n,\nu)=-\frac{1}{\alpha_\nu^s(x)}(Id-H)^{-1}(n)\cdot\nabla_x\phi_0+c(x,\nu).\tag{B.51}$$

Similar to the terms of order ε^0, the terms of power $\varepsilon^{\beta-\lfloor\beta\rfloor+1}$ give $\psi_2 = -\frac{1}{\alpha_\nu^s(x)}(Id - H)^{-1}(n) \cdot \nabla_x \psi_1 + c(x,\nu)$. As in subsection B.3.3 the isotropic function $c(x,\nu)$ does not contribute to the divergence free condition and it will not be explicitly computed.

Finally, the terms of order ε^1 imply

$$n \cdot \nabla_x \phi_1 = \alpha_\nu^s(x)(H - Id)[\phi_2(x,\cdot,\nu)](n).$$

Hence, using (B.51) and integrating over $\mathbb{S}^2$ we obtain the desired interior limit problem for ϕ_0

$$\operatorname{div}\left(\frac{1}{\alpha_\nu^s(x)}\left(\int_{\mathbb{S}^2} n \otimes (Id - H)^{-1}(n)\, dn\right) \nabla_x \phi_0(x,\nu)\right) = 0.$$

Plugging now the first equation of (B.12) into the second one we obtain also the following equation solved by the leading order of the temperature

$$\int_0^\infty d\nu\ \alpha_\nu^a(x)\left(B_\nu(T(x)) - \phi_0(x,\nu)\right) = 0.$$

We remark that ϕ_0 does not need to be the Planck distribution. This is also implied by the asymptotic expansion of the radiation intensity. Indeed, the comparison of the terms of order ε^β gives

$$n \cdot \nabla_x \psi_k(x,n,\nu) = \alpha_\nu^a(x)(B_\nu(T(x)) - \phi_0(x,\nu)) + \alpha_\nu^s(x)(H - Id)[\psi_{k+1}(x,\cdot,\nu)](n),$$

where $k = \lfloor\beta\rfloor + 1 \geq 2$. Since ψ_k does not need to be isotropic for $k \geq 2$, an integration over the sphere implies the orthogonality condition

$$\operatorname{div}\left(\fint_{\mathbb{S}^2} n\psi_{\lfloor\beta\rfloor+1}(x,n,\nu)\, dn\right) = \alpha_\nu^a(x)\left(B_\nu(T(x)) - \phi_0(x,\nu)\right).$$

As in Subsection B.3.3 the Milne problem for the Milne layer is given by (B.43). As in subsection B.3.4 there is no thermalization layer since the radiation intensity does not approach the equilibrium distribution. Hence, denoting by $I_\nu(y,n,p)$ the solution to (B.43) and by $I_\nu^\infty(p) = \lim_{y\to\infty} I_\nu(y,n;p)$ we obtain for this case the following limit stationary boundary value problem

$$\begin{cases} \operatorname{div}\left(\frac{1}{\alpha_\nu^s(x)}\left(\int_{\mathbb{S}^2} n \otimes (Id - H)^{-1}(n)\, dn\right) \nabla_x \phi_0(x,\nu)\right) = 0 & x \in \Omega \\ \int_0^\infty d\nu\ \alpha_\nu^a(x)\left(B_\nu(T(x)) - \phi_0(x,\nu)\right) = 0 & x \in \Omega \\ \phi_0(p,\nu) = I_\nu^\infty(p) & p \in \partial\Omega. \end{cases}$$

B.4 Time dependent diffusion approximation. The case of infinite speed of light ($c = \infty$)

We turn now to the time dependent case. In physical applications the order of magnitude of the speed of light c is so large compared with the speed of heat transfer that it is often considered infinite (cf. [152]). This approximation is valid if the distance traveled by the light in the time scale in which meaningful changes of the temperature take place is much larger than the characteristic length of the body L. We consider in this section the diffusion approximation for the time dependent radiative transfer equation (B.10) when $c = \infty$ and in the next sections we will consider other choices of c. Under this assumption the initial-boundary value problem (B.10) reduces to (B.11). This is the case when the radiation is instantaneously transported in the domain Ω. Notice that, since under this assumption in equation (B.11) there is no term containing $\partial_t I_\nu$, we do not need to impose any initial value for I_ν.

We recall that the diffusion regime holds if $\ell_M = \varepsilon \ll 1$. We will consider different choices of ℓ_A and ℓ_S given as powers of ε. We will construct the resulting initial-boundary value limit problems as follows. We will first derive the outer problems valid in the interior of Ω. Afterwards we will construct the initial layer problems describing the transient behavior of the radiation intensity for very small times. We will formulate also boundary layer equations describing I_ν near the boundary of Ω. It turns out that the latter are the Milne problems and the thermalization problems derived in Section B.3. Finally, the matching between the outer, the boundary layer and the initial layer solutions will provide the initial value and the boundary conditions for the limit problem in the diffusion approximation under consideration.

B.4.1 Outer problems

In this subsection we derive the outer problems arising from equation (B.11) under the assumption $\ell_M = \varepsilon \ll 1$ and for different choices of $\ell_A = \varepsilon^{-\beta}$ and $\ell_S = \varepsilon^{-\gamma}$. As in the stationary case analyzed in Section B.3 there are five different cases to be considered which yield five different diffusive problems.

In order to determine the outer problems yielding the form of the solutions in the bulk of Ω we use the expansion

$$I_\nu(t,x,n) = \phi_0(t,x,n,\nu) + \sum_{k\geq 0} \varepsilon^{\delta+k}\psi_{k+1}(t,x,n,\nu) + \sum_{l>0} \varepsilon^l \phi_l(t,x,n,\nu) \tag{B.52}$$

for δ defined as in (B.21) depending on ℓ_A and ℓ_S, plugging (B.52) into (B.11) and identifying all terms of the same power of ε. It turns out that the diffusive problems are in this case the time dependent version of the stationary outer problems of Section B.3. Indeed, since $c = \infty$ the first equation in (B.11) is a stationary equation for the intensity I_ν. Therefore, the same computations of Section B.3 show that for any choice of ℓ_A and ℓ_S the first order term ϕ_0 is isotropic and the next non-isotropic term arising in the expansion of I_ν is of order ε^1.

Hence, in the case $\ell_T \leq 1$, i.e. $\tau_h = \frac{1}{\varepsilon}$, since the time derivative of the temperature in the second equation of (B.11) is a term of order ε^0 which is balanced by the divergence of the flux of energy, we obtain the following outer problems

(i) for $\ell_M = \ell_T \ll \ell_S$

$$\partial_t T(t,x) - \frac{4\pi}{3} \operatorname{div}\left(\int_0^\infty \frac{\nabla_x B_\nu(T(t,x))}{\alpha_\nu(x)} d\nu\right) = 0, \tag{B.53}$$

(ii) for $\ell_M = \ell_T = \ell_S$

$$\partial_t T(t,x) = \operatorname{div}\left(\int_0^\infty d\nu \frac{1}{\alpha_\nu^a(x)+\alpha_\nu^s(x)}\left(\int_{\mathbb{S}^2} dn\, n\otimes \left(Id - A_{\nu,x}\right)^{-1}(n)\right)\nabla_x B_\nu(T(t,x))\right), \tag{B.54}$$

(iii) for $\ell_M \ll \ell_T \ll L$

$$\partial_t T(t,x) = \operatorname{div}\left(\int_0^\infty d\nu \frac{1}{\alpha_\nu^s(x)}\left(\int_{\mathbb{S}^2} dn\, n\otimes \left(Id - H\right)^{-1}(n)\right)\nabla_x B_\nu(T(t,x))\right), \tag{B.55}$$

(iv) for $\ell_M \ll L = \ell_T$

$$\begin{cases} -\frac{1}{\alpha_\nu^a(x)} \operatorname{div}\left(\frac{1}{\alpha_\nu^s(x)}\left(\fint_{\mathbb{S}^2} n \otimes (Id - H)^{-1}(n)\, dn\right)\nabla_x \phi_0(t,x,\nu)\right) = B_\nu(T(t,x)) - \phi_0(t,x,\nu) \\ \partial_t T(t,x) - \operatorname{div}\left(\int_0^\infty d\nu\, \frac{1}{\alpha_\nu^s(x)}\left(\fint_{\mathbb{S}^2} n \otimes (Id - H)^{-1}(n)\, dn\right)\nabla_x \phi_0(t,x,\nu)\right) = 0. \end{cases} \tag{B.56}$$

In the case $\ell_M \ll L \ll \ell_T$, namely when $\tau_h = \frac{1}{\varepsilon^\beta}$ for $\beta > 1$ the outer problem is

$$\begin{cases} \operatorname{div}\left(\frac{1}{\alpha_\nu^s(x)}\left(\fint_{\mathbb{S}^2} n \otimes (Id - H)^{-1}(n)\, dn\right)\nabla_x \phi_0(t,x,\nu)\right) = 0 \\ \partial_t T(t,x) - \int_0^\infty d\nu\, \alpha_\nu^a(x)\left(B_\nu(T(t,x)) - \phi_0(t,x,\nu)\right) = 0. \end{cases} \tag{B.57}$$

Indeed, plugging the expansion (B.52) with $\delta = \beta - 1$ into the first equation in (B.11) we obtain, arguing as in Section B.3.5, that the leading order ϕ_0 is isotropic and solves the stationary equation

$$\operatorname{div}\left(\frac{1}{\alpha_\nu^s(x)}\left(\fint_{\mathbb{S}^2} n \otimes (Id - H)^{-1}(n)\, dn\right)\nabla_x \phi_0(t,x,\nu)\right) = 0.$$

Moreover, plugging the second equation of (B.11) into the second one yields

$$\partial_t T(t,x) - \int_0^\infty d\nu\, \alpha_\nu^a(x)\left(B_\nu(T(t,x)) - \phi_0(t,x,\nu)\right) = 0.$$

These are the equations describing the radiation intensity and the temperature on the bulk away from the boundary and for positive times.

We remark that as for the stationary problem the regimes of equilibrium diffusion approximations are for $\ell_T \ll L$ and correspond to the problems (B.53), (B.54) and (B.55) while the regimes of non-equilibrium approximations are for $\ell_T \gtrsim L$ and are described by (B.56) and (B.57).

B.4.2 Initial layer equations and boundary layer equations

As in the case of the stationary diffusion approximation, the radiation intensity I_ν and the temperature T can change abruptly near the boundaries, i.e. boundary layers might arise. In addition, in the time dependent case also the behavior of (T, I_ν) could change quickly for small times. We will denote the latter as initial layers. In this subsection we construct the initial layers for distances to the boundary of order 1 and boundary layers for positive times of order 1. We denote by initial layer equations the problems derived for times $t \ll 1$ and solved at the interior of Ω. Similarly, the boundary layer equations are problems derived from rescaling the space variable only and solved for any $t > 0$.

In the considered case, i.e. $c = \infty$, there are no initial layers for the temperature appearing on the bulk, i.e. for distances to the boundary of order 1. To see this we have to consider two different cases. We recall that the second equation in (B.11) is

$$\partial_t T(t,x) + \tau_h \operatorname{div}\left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, n I_\nu(t,x,n)\right) = 0. \tag{B.58}$$

Hence, if $\ell_T \leq 1$ the heat parameter is $\tau_h = \frac{1}{\varepsilon}$. Therefore, in equation (B.58) the divergence of the flux of radiative energy is multiplied by ε^{-1}. As indicated before ϕ_0 is isotropic. In addition to that, since the first non-isotropic term is of order ε, it follows that in (B.58) the term containing the divergence is of order 1 in the bulk. Therefore, $\partial_t T$ is of order 1 and as a consequence $T \simeq T_0$ for small times $t \ll 1$ and no initial layer appears. On the other hand, in the case $\ell_T \gg 1$ the heat parameter is $\tau_h = \ell_A = \frac{1}{\varepsilon^\beta}$ for $\beta > 1$. In this case the leading term of the divergence of the total flux of energy is of order ε^β and it is given by

$$\varepsilon^\beta \operatorname{div}\left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, n\phi_3\right) = \varepsilon^\beta \int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, \alpha_\nu^a \left(B_\nu(T) - \phi_0\right),$$

where ϕ_3 is the term of order ε^β in the expansion (B.52) obtained for $\delta = \beta - 1$. This implies again that $\partial_t T$ is of order 1 and hence, there are also in this case no initial layers.

We now examine the boundary layers appearing for times of order 1. In this case, similarly as in the stationary case, Milne and thermalization layers arise. It turns out that the equations describing the radiation intensity near the boundary are given either by the stationary Milne problems (B.26), (B.34), (B.43), or by the thermalization problem (B.48) or by a combination of both of them depending on the choice of ℓ_A and ℓ_S.

We begin describing first the Milne layers. We rescale the space variable according to $y = -\frac{x-p}{\varepsilon} \cdot n_p$, where $\ell_M = \varepsilon$ and $p \in \partial\Omega$. We express also the absorption and scattering lengths according to $\ell_A = \varepsilon^{-\beta}$, $\ell_S = \varepsilon^{-\gamma}$ with $\min\{\beta, \gamma\} = -1$. With this notation, (B.11) becomes

$$\begin{cases} -(n \cdot n_p)\partial_y I_\nu(t, y, n; p) = \varepsilon^{\beta+1}\alpha_\nu^a \left(p + \mathcal{O}(\varepsilon)\right)\left(B_\nu(T) - I_\nu\right) & \\ \qquad\qquad +\varepsilon^{\gamma+1}\alpha_\nu^s \left(p + \mathcal{O}(\varepsilon)\right)\left(\int_{\mathbb{S}^2} K(n, n') I_\nu\, dn' - I_\nu\right) & y > 0 \\ \partial_t T(t, y; p) - \frac{\tau_h}{\varepsilon}\left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, (n \cdot n_p)\partial_y I_\nu\right) = 0 & y > 0 \\ T(0, y; p) = T_0(y; p) & y > 0 \\ I_\nu(t, 0, n; p) = g_\nu(t, n) & n \cdot n_p < 0. \end{cases} \tag{B.59}$$

Letting $\varepsilon \to 0$ we obtain different Milne problems for different choices of β and γ. With similar arguments as in Section B.3 we can see that the Milne problems are the same as the one derived for the stationary case, except for the fact that the unknowns depend also on the variable t. However, the variable t appears only as a parameter and the Milne problems are stationary. These are given by (B.26) in the case $\gamma > -1$, by (B.34) if $\gamma = \beta = -1$ and finally by (B.43) if $\beta > -1$. Notice that we are assuming that, if the incoming radiation g_ν depends on time, it does it only for times t of order one.

We remark that when $\beta > -1$ the Milne problem (B.43) is a closed problem involving only the radiation intensity I_ν. If $\ell_T \ll L$, in order to determine the temperature close to the boundary we have to solve the stationary equation

$$\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, \alpha_\nu^a(p)\left(B_\nu(T(t, y; p)) - I_\nu(t, y, n; p)\right) = 0.$$

This is the same equation that we obtained in the stationary case in (B.44). On the other hand, if $\ell_T \gtrsim L$ the temperature is related to the radiation intensity by a time dependent equation similar to the second one in (B.56) and (B.57), namely the equations describing the temperature in the bulk, i.e.

$$\partial_t T(t, y; p) + \int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, \alpha_\nu^a(p)\left(B_\nu(T(t, y; p)) - I_\nu(t, y, n; p)\right) = 0. \tag{B.60}$$

Besides the Milne layer, in the case $\ell_M \ll \ell_T \ll L$ we observe also the formation of a thermalization layer at distance ℓ_T to the boundary. The equation describing this layer is obtained with a change of variable $\eta = -\frac{x-p}{\ell_T} \cdot n_p$ for $p \in \partial\Omega$. Recall that in this case we consider $\ell_S = \varepsilon$ and $\ell_A = \varepsilon^{-\beta}$ for $\beta \in (-1,1)$ and hence $\ell_T = \varepsilon^{\frac{1-\beta}{2}}$ and $\tau_h = \frac{1}{\varepsilon}$. Thus, (B.11) becomes under this rescaling

$$\begin{cases} -\varepsilon^{\frac{1+\beta}{2}}(n\cdot n_p)\partial_\eta I_\nu(t,\eta,n;p) = \alpha_\nu^a\left(p+\mathcal{O}\left(\varepsilon^{\frac{1-\beta}{2}}\right)\right)\varepsilon^{\beta+1}(B_\nu(T)-I_\nu) \\ \qquad\qquad +\alpha_\nu^s\left(p+\mathcal{O}\left(\varepsilon^{\frac{1-\beta}{2}}\right)\right)\left(\int_{\mathbb{S}^2} K(n,n')I_\nu\, dn' - I_\nu\right) & \eta>0 \\ \partial_t T(t,\eta;p) - \varepsilon^{\frac{\beta-3}{2}}\left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, (n\cdot n_p)\partial_\eta I_\nu\right) = 0 & \eta > 0. \end{cases} \tag{B.61}$$

We see once more that the thermalization layer equation is equation (B.48), the equation constructed for the stationary problem in Section (B.3.3).

Finally, matching the solution of the boundary layer equations with the outer problem we can construct the boundary condition for the diffusive initial-boundary limit problem. We will summarize these problems in the following subsection.

B.4.3 Limit problems in the bulk

We summarize now the time dependent PDE problems that we obtain for the equation (B.11) as $\ell_M \to 0$ for all different choices of ℓ's. They are given by the outer problems (B.53)-(B.57) valid in the bulk for positive times. Since there are no initial layers appearing for times $t \ll 1$, the initial condition is $T(t,x) = T_0(x)$ for any choice of ℓ_A and ℓ_S. Moreover, the boundary condition is given by the matching of the solution of the boundary layer problems with the outer solution.

(i) If $\ell_M = \ell_T \ll \ell_S$ then the problem is given by

$$\begin{cases} \partial_t T(t,x) - \frac{4\pi}{3}\operatorname{div}\left(\int_0^\infty \frac{\nabla_x B_\nu(T(t,x))}{\alpha_\nu(x)} d\nu\right) = 0 & x\in\Omega,\ t>0 \\ T(0,x) = T_0(x) & x\in\Omega \\ T(t,p) = \lim_{y\to\infty} F^{-1}\left(\left(\int_o^\infty d\nu \alpha_\nu^a(p) I_\nu(t,y,n;p)\right), y, p\right) & p\in\partial\Omega,\ t>0, \end{cases} \tag{B.62}$$

where $I_\nu(y,n;p)$ is the solution to the Milne problem (B.26).

(ii) If $\ell_M = \ell_T \ll L$, we obtain the following limit problem

$$\begin{cases} \partial_t T(t,x) \\ \quad = \operatorname{div}\left(\int_0^\infty d\nu \frac{1}{\alpha_\nu^a(x)+\alpha_\nu^s(x)}\left(\int_{\mathbb{S}^2} dn\, n\otimes (Id - A_{\nu,x})^{-1}(n)\right)\nabla_x B_\nu(T(x))\right) & x\in\Omega,\ t>0 \\ T(0,x) = T_0(x) & x\in\Omega \\ T(t,p) = \lim_{y\to\infty} F^{-1}\left(\left(\int_0^\infty d\nu ⨍_{\mathbb{S}^2} dn\, \alpha_\nu^a(p) I_\nu(t,y,n,p)\right), y, p\right) & p\in\partial\Omega,\ t>0, \end{cases} \tag{B.63}$$

where $I_\nu(y,n,p)$ solves the Milne problem (B.34).

(iii) We turn now to the case $\ell_M \ll \ell_T \ll L$, which corresponds to the case $\ell_M = \varepsilon = \ell_S$ and $\ell_A = \varepsilon^{-\beta}$ for $\beta \in (-1,1)$. We obtain the following limit problem

$$\begin{cases} \partial_t T - \operatorname{div}\left(\int_0^\infty d\nu \frac{1}{\alpha_\nu^s(x)}\left(\int_{\mathbb{S}^2} dn\, n\otimes (Id - H)^{-1}(n)\right)\nabla_x B_\nu(T(x))\right) = 0 & x\in\Omega \\ T(0,x) = T_0(x) & x\in\Omega \\ T(t,p) = \lim_{\eta\to\infty} F^{-1}\left(\left(\int_0^\infty d\nu ⨍_{\mathbb{S}^2} dn\, \alpha_\nu^a(x)\varphi_0(t,\eta,\nu;p)\right), y, p\right) & p\in\partial\Omega,\ t>0, \end{cases} \tag{B.64}$$

where $\varphi_0(t,\eta,\nu;p)$ solves the thermalization equations (B.48) with boundary value $\varphi_0(t,0,\nu;p) = \lim_{y\to\infty} I_\nu(t,y,n,\nu;p)$ for I_ν the solution to the Milne problem (B.43) with boundary value $g_\nu(t,n)$.

(iv) We consider now the last two cases where $\ell_M \ll L \lesssim \ell_T$. The limit problem in the case $\ell_T = L$ is

$$\begin{cases} -\frac{1}{\alpha_\nu^a(x)} \operatorname{div}\left(\frac{1}{\alpha_\nu^s(x)}\left(⨍_{\mathbb{S}^2} n\otimes (Id-H)^{-1}(n)\,dn\right)\nabla_x\phi_0(t,x,\nu)\right) & \\ \qquad\qquad = B_\nu(T(t,x)) - \phi_0(t,x,\nu) & x\in\Omega,\ t>0 \\ \partial_t T(t,x) - \int_0^\infty d\nu\ ⨍_{\mathbb{S}^2} dn\ dn\ \alpha_\nu^a(x)\left(B_\nu(T(t,x)) - \phi_0(t,x,\nu)\right) = 0 & x\in\Omega,\ t>0 \\ T(0,x) = T_0(x) & x\in\Omega \\ \phi_0(t,p,\nu) = \lim_{y\to\infty} ⨍_{\mathbb{S}^2} I_\nu(t,y,n,p) & p\in\partial\Omega,\ t>0, \end{cases} \tag{B.65}$$

where $I_\nu(t,y,n,p)$ solves the Milne problem (B.43) for the boundary value $g_\nu(t,n)$. Notice that in the problem (B.43) the time t appears just as a parameter.

(v) Finally, if $L \ll \ell_T$ with the same notation as above the limit problem in this case is

$$\begin{cases} \operatorname{div}\left(\frac{1}{\alpha_\nu^s(x)}\left(⨍_{\mathbb{S}^2} n\otimes (Id-H)^{-1}(n)\,dn\right)\nabla_x\phi_0(t,x,\nu)\right) = 0 & x\in\Omega,\ t>0 \\ \partial_t T(t,x) + \int_0^\infty d\nu ⨍_{\mathbb{S}^2} \alpha_\nu^a(x)\left(B_\nu(T)(t,x) - I_\nu(t,x,n)\right) = 0 & x\in\Omega,\ t>0 \\ T(0,x) = T_0(x) & x\in\Omega \\ \phi_0(t,p,\nu) = \lim_{y\to\infty} ⨍_{\mathbb{S}^2} I_\nu(t,y,n,p) & p\in\partial\Omega,\ t>0. \end{cases} \tag{B.66}$$

Also for this case the boundary condition is obtained by the solution of the boundary layer described by the Milne problem (B.43).

B.4.4 Initial-boundary layers

It is important to notice that in regions very close to the boundary and for a times $t \ll 1$ new layers could appear. These are the regions where the radiation intensity I_ν and the temperature T change from the solution of the initial layer equation to the solution of the boundary layer equation. For this reason we denote these layers as initial-boundary layers. In this section we will derive the equations describing them for any choice of ℓ_A and ℓ_S. In the following we will always denote by p a point belonging to the boundary, i.e. $p\in\partial\Omega$.

(i) If $\ell_M = \ell_T \ll \ell_S$ we observe the formation of only one initial-boundary layer. It is described by an equation which can be constructed rescaling the space variable as $y = -\frac{x-p}{\varepsilon}\cdot n_p$ and the time by $t=\varepsilon^2\tau$. Indeed, since in this case $\beta=-1$ (because $\ell_A=\varepsilon$) and $\tau_h=\varepsilon^{-1}$ we see that the leading term of divergence of the flux of energy is of order ε^{-2} in the following equation

$$\partial_t T(t,y;p) + \tau_h\varepsilon^\beta \int_0^\infty d\nu \int_{\mathbb{S}^2} dn\ \alpha_\nu^a(p)\left(B_\nu(T(t,y;p)) - I_\nu(t,y,n;p)\right) = 0. \tag{B.67}$$

This equation is obtained plugging the first equation in (B.59) into the second one. We recall that equation (B.59) is obtained after a rescaling of only the space variable. Hence, the time rescaling $t=\varepsilon^2\tau$ gives a non-trivial equation for the temperature. Thus, the radiation intensity I_ν and the temperature T solve the following initial-boundary layer equation

$$\begin{cases} -(n\cdot n_p)\partial_y I_\nu(\tau,y,n;p) = \alpha_\nu^a(p)(B_\nu(T(\tau,y)) - I_\nu(\tau,y,n;p)) & y>0,\ \tau>0 \\ \partial_t T(\tau,y) - \left(\int_0^\infty d\nu ⨍_{\mathbb{S}^2} dn\ (n\cdot n_p)\partial_y I_\nu(\tau,y,n;p)\right) = 0 & y>0,\ \tau>0 \\ T(0,y;p) = T_0(p) & y>0 \\ I_\nu(\tau,0,n;p) = g_\nu(0,n) & n\cdot n_p<0,\ \tau>0. \end{cases}$$

(ii) In the case $\ell_M = \ell_T = \ell_S$ under the scaling $y = -\frac{x-p}{\ell_M} \cdot n_p$ and $t = \varepsilon^2 \tau$ we obtain as above the following initial-boundary layer equation

$$\begin{cases} -(n \cdot n_p)\partial_y I_\nu(\tau, y, n; p) = \alpha_\nu^a(p)(B_\nu(T(\tau, y)) - I_\nu(\tau, y, n; p)) & \\ \qquad\qquad + \alpha_\nu^s \left(\int_{\mathbb{S}^2} K(n, n') I_\nu(\tau, y, n'; p)\, dn' - I_\nu(\tau, y, n; p)\right) & y > 0,\ \tau > 0 \\ \partial_\tau T(\tau, y) + \operatorname{div}\left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, n I_\nu\right) = 0 & y > 0,\ \tau > 0 \\ T(0, y; p) = T_0(p) & y > 0 \\ I_\nu(\tau, 0, n; p) = g_\nu(0, n) & n \cdot n_p < 0,\ \tau > 0. \end{cases}$$

(iii) If $\ell_A \ll \ell_T \ll L$ we obtain two different initial-boundary layers. This is consistent with the fact that there are two boundary layers appearing, namely the Milne layer, in which I_ν becomes isotropic, and the thermalization layer, in which I_ν approaches to the Planck distribution. We now notice that rescaling the space variable by $y = -\frac{x-p}{\varepsilon} \cdot n_p$ and the time variable according to $t = \varepsilon^{1-\beta}\tau$ equation (B.67) gives the following initial-boundary Milne layer equation

$$\begin{cases} -(n \cdot n_p)\partial_y I_\nu(\tau, y, n; p) & \\ \qquad = \alpha_\nu^s(p) \left(\int_{\mathbb{S}^2} K(n, n') I_\nu(\tau, y, n'; p) dn' - I_\nu(\tau, y, n, \nu; p)\right) & y > 0,\ \tau > 0 \\ \partial_t T(\tau, y) + \int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, \alpha_\nu^a(p) \left(B_\nu(T)(\tau, y; p) - I_\nu(\tau, y, n; p)\right) = 0 & y > 0,\ \tau > 0 \\ T(0, y; p) = T_0(p) & y > 0 \\ I_\nu(\tau, 0, n; p) = g_\nu(0, n) & n \cdot n_p < 0,\ \tau > 0. \end{cases} \tag{B.68}$$

Moreover, rescaling the space variable according to $\eta = -\frac{x-p}{\ell_T} \cdot n_p$ and the time by $t = \varepsilon^{1-\beta}\tau$ from equation (B.61) we obtain the following initial-boundary thermalization layer equation

$$\begin{cases} \varphi_0(\tau, \eta, \nu; p) - \frac{1}{\alpha_\nu^a(p)\alpha_\nu^s(p)} \left(⨍_{\mathbb{S}^2} (n \cdot n_p)(Id - H)^{-1}(n) \cdot n_p\, dn\right) \partial_\eta^2 \varphi_0(\tau, \eta, \nu; p) & \\ \qquad = B_\nu(T(\tau, \eta; p)) & \eta > 0,\ \tau > 0 \\ \partial_\tau T - \int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, \alpha_\nu^a(p) \left(B_\nu(T)(\tau, \eta; p) - I_\nu(\tau, \eta, n; p)\right) = 0 & \eta > 0,\ \tau > 0 \\ T(0, \eta; p) = T_0(p) & y > 0 \\ \varphi_0(\tau, 0, \nu; p) = I(0, \nu; p) & p \in \partial\Omega,\ \tau > 0. \end{cases}$$

This is the initial-boundary layer equation describing the transition from the initial value to the boundary value in the limit problem (B.64).

(iv)+(v) Finally, in the last two considered case, namely when $\ell_T \gtrsim L$ we do not obtain a initial-boundary layer. However, under the space variable rescale $y = -\frac{x-p}{\varepsilon} \cdot n_p$ for the Milne problem (B.43) we obtained also an evolution equation for the temperature valid for all $t > 0$ given as we saw in (B.60) by

$$\begin{cases} -(n \cdot n_p)\partial_y I_\nu(t, y, n; p) & \\ \qquad = \alpha_\nu^s(p) \left(\int_{\mathbb{S}^2} K(n, n') I_\nu(t, y, n'; p) dn' - I_\nu(t, y, n, \nu; p)\right) & y > 0, t > 0 \\ \partial_t T(t, y) + \int_0^\infty d\nu \int_{\mathbb{S}^2} \alpha_\nu^a(p) \left(B_\nu(T)(t, y; p) - I_\nu(t, y, n; p)\right) = 0 & y > 0, t > 0 \\ T(0, y; p) = T_0(p) & y > 0 \\ I_\nu(t, 0, n; p) = g_\nu(t, n) & n \cdot n_p < 0, t > 0. \end{cases}$$

B.5 Time dependent diffusion approximation. The case of speed of light of order 1

In this section we construct the limit problem solved by the solution of the time dependent equation (B.10) when $\ell_M \to 0$ and the speed of light is finite. Without loss of generality we consider first the case $c = 1$. Physically this means that the characteristic time for the propagation of light is similar to the time of the heat transfer process. This situation can be expected to be relevant only in astrophysical applications. The strategy is the same as in Section B.4. We will first formulate the limit problem valid at the interior of the domain Ω for positive times. In Subsection B.5.2 we will consider the formation of initial and boundary layers. In this case we will obtain non-trivial initial layer equations. On the other hand, as in Section B.4 the boundary layer equations are stationary and are the same equations we constructed in Section B.3. Finally, in Subsections B.5.3 and B.5.4 we will summarize the initial boundary value problem that we have obtained and we will construct the initial-boundary layer equations that we have to consider in order to describe the behavior of the solution in a small neighborhood of the boundary for times $t \ll 1$.

B.5.1 Outer problems

We consider equation (B.10) in the case $c = 1$ and under the assumption $\ell_M = \varepsilon$ for the different choices of $\ell_A = \varepsilon^{-\beta}$ and $\ell_S = \varepsilon^{-\gamma}$. Expanding I_ν according to (B.52) and identifying in (B.10) all terms of the same order we conclude as we computed in Section B.3 and Section B.4 that the first order $\phi_0(t, x, n, \nu)$ of the intensity I_ν is isotropic and the first non-isotropic term is of order ε^1. Moreover, as long as $\ell_T \ll L$ we have $\phi_0(t, x, \nu) = B_\nu(T(t, x))$. The outer problems in the case $\ell_T \leq 1$, i.e. $\tau_h = \frac{1}{\varepsilon}$ are given

(i) for $\ell_M = \ell_T \ll \ell_S$ by

$$\partial_t T(t, x) + 4\pi\sigma\partial_t T^4(t, x) - \frac{4\pi}{3} \operatorname{div}\left(\int_0^\infty \frac{\nabla_x B_\nu(T(t, x))}{\alpha_\nu(x)} d\nu\right) = 0,$$

(ii) for $\ell_M = \ell_T = \ell_S$ by

$$\begin{aligned}&\partial_t T(t, x) + 4\pi\sigma\partial_t T^4(t, x)\\ &\quad = \operatorname{div}\left(\int_0^\infty d\nu \frac{1}{\alpha^a_\nu(x) + \alpha^s_\nu(x)} \left(\int_{\mathbb{S}^2} dn\, n \otimes (Id - A_{\nu,x})^{-1}(n)\right) \nabla_x B_\nu(T(t, x))\right),\end{aligned}$$

(iii) for $\ell_M \ll \ell_T \ll L$ by

$$\begin{aligned}&\partial_t T(t, x) + 4\pi\sigma\partial_t T^4(t, x)\\ &\qquad = \operatorname{div}\left(\int_0^\infty d\nu \frac{1}{\alpha^s_\nu(x)} \left(\int_{\mathbb{S}^2} dn\, n \otimes (Id - H)^{-1}(n)\right) \nabla_x B_\nu(T(t, x))\right),\end{aligned}$$

(iv) for $\ell_M \ll L = \ell_T$ by

$$\begin{cases} \partial_t \phi_0(t, x, \nu) - \operatorname{div}\left(\frac{1}{\alpha^s_\nu(x)} \left(⨍_{\mathbb{S}^2} n \otimes (Id - H)^{-1}(n)\, dn\right) \nabla_x \phi_0(t, x, \nu)\right) \\ \qquad\qquad = \alpha^a_\nu(x) \left(B_\nu(T(t, x)) - \phi_0(t, x, \nu)\right) \\ \partial_t T(t, x) + 4\pi \int_0^\infty d\nu\, \alpha^a_\nu(x) \left(B_\nu(T(t, x)) - \phi_0(t, x, \nu)\right) = 0. \end{cases} \tag{B.69}$$

In the case $\ell_T \gg 1$, i.e. $\tau_h = \ell_A = \varepsilon^{-\beta}$ for $\beta > 1$, a similar computation to the one for the derivation of the problem (B.57) yields

$$\begin{cases} \operatorname{div}\left(\frac{1}{\alpha_\nu^s(x)}\left(\fint_{\mathbb{S}^2} n \otimes (Id - H)^{-1}(n)\, dn\right) \nabla_x \phi_0(t,x,\nu)\right) = 0 \\ \partial_t T(t,x) + 4\pi \int_0^\infty d\nu\, \alpha_\nu^a(x) \left(B_\nu(T(t,x)) - \phi_0(t,x,\nu)\right) = 0. \end{cases} \tag{B.70}$$

Indeed, in the first equation of (B.10) the leading order of the term containing the time derivative of I_ν is of power ε^0 as the emission-absorption term. On the other hand, the leading order ϕ_0 of the radiation intensity is isotropic and the first non-isotropic term is of order ε^1. Therefore, the identification in the first equation of (B.10) of the terms of order $\varepsilon^{1-\beta} \gg \varepsilon^0$ gives the stationary equation in (B.70) solved by ϕ_0. Finally, plugging the first equation of (B.10) into the second one yields the equation for the temperature as in (B.70).

B.5.2 Initial layer equations and boundary layer equations

In this subsection we will describe the initial layers and the boundary layers appearing for time scales smaller than the heat parameter τ_h and for regions close to the boundary, respectively. We start with the initial layers and we will see that similarly as for the boundary layers considered in Sections B.3 and B.4 there are two nested initial layers appearing. Indeed, in a first layer, i.e. for a very small time scale, the radiation intensity becomes isotropic, while in a second initial layer it becomes eventually the Planck distribution for the temperature. We will denote the first layer as initial Milne layer and the second one as initial thermalization layer, due to their analogy with the boundary layers considered in Sections B.3 and B.4. We will also see that while the initial Milne layer appears for every choice of ℓ_A and ℓ_S, the initial thermalization layer coincides with the initial Milne layer (if $\ell_M = \ell_T$), appears after the initial Milne layer (if $\ell_M \ll \ell_T \ll L$) or it is not present at all (if $\ell_T \gtrsim L$).

We recall that under the assumption $\ell_A = \varepsilon^{-\beta}$ and $\ell_S = \varepsilon^{-\gamma}$ for $\min\{\beta,\gamma\} = -1$ equation (B.10) writes

$$\begin{cases} \partial_t I_\nu(t,x,n) + \tau_h n \cdot \nabla_x I_\nu(t,x,n) = \alpha_\nu^a(x)\varepsilon^\beta \tau_h \left(B_\nu(T(t,x)) - I_\nu(t,x,n)\right) \\ \qquad + \alpha_\nu^s(x)\varepsilon^\gamma \tau_h \left(\int_{\mathbb{S}^2} K(n,n') I_\nu(t,x,n')\, dn' - I_\nu(t,x,n)\right) & x \in \partial\Omega, n \in \mathbb{S}^2, t > 0 \\ \partial_t T + \partial_t \left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, I_\nu(t,n,x)\right) \\ \qquad + \tau_h \operatorname{div}\left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, n I_\nu(t,n,x)\right) = 0 & x \in \partial\Omega, n \in \mathbb{S}^2, t > 0 \\ I_\nu(0,x,n) = I_0(x,n,\nu) & x \in \partial\Omega, n \in \mathbb{S}^2 \\ T(0,x) = T_0(x) & x \in \partial\Omega \\ I_\nu(t,n,x) = g_\nu(t,n) & x \in \partial\Omega, n \cdot n_x < 0, t > 0. \end{cases} \tag{B.71}$$

Notice that the leading term in the first equation is of order $\frac{\varepsilon}{\tau_h}$. Therefore, under a time rescaling $t = \frac{\varepsilon}{\tau_h}\tau$ the first equation writes

$$\partial_\tau I_\nu = \varepsilon^{\beta+1}\alpha_\nu^a(x)\left(B_\nu(T) - I_\nu\right) + \varepsilon^{\gamma+1}\alpha_\nu^s(x)\left(\int_{\mathbb{S}^2} K(n,n') I_\nu\, dn' - I_\nu\right) + \varepsilon n \cdot \nabla_x I_\nu$$

while the second one is

$$\partial_\tau T + \partial_\tau \left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, I_\nu\right) + \varepsilon \operatorname{div}\left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, n I_\nu\right) = 0.$$

It is hence easy to see that for any choice of ℓ_M and ℓ_S there is an initial layer with thickness of order $\frac{\varepsilon}{\tau_h}$. Notice that as long as $\ell_T \lesssim 1$ (i.e. $\tau_h = \varepsilon^{-1}$) this initial layer has thickness of order ε^2, while in the case $\ell_T \gg 1$ (i.e. $\tau_h = \varepsilon^{-\beta}$ for $\beta > 1$) the order is $\varepsilon^{1+\beta}$. This layer plays the role of the Milne boundary layer in the time dependent case, as in this layer the radiation intensity becomes isotropic. For this reason we will denote it as the initial Milne layer.

(i) In the case $\ell_M = \ell_T \ll \ell_S$ the initial Milne layer is described by the following initial Milne equation for the leading order of the radiation intensity

$$\begin{cases} \partial_\tau \varphi_0(\tau,x,n,\nu) = \alpha_\nu^a(x)\left(B_\nu(T(\tau,x)) - \varphi_0(\tau,x,n,\nu)\right) & \tau > 0 \\ \partial_\tau T(\tau,x) + \int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, \alpha_\nu^a(x)\left(B_\nu(T(\tau,x)) - \varphi_0(\tau,x,n,\nu)\right) = 0 & \tau > 0 \\ \varphi_0(0,x,n,\nu) = I_0(x,n,\nu) \\ T(0,x) = T_0(x). \end{cases} \tag{B.72}$$

This equation plays the same role of the Milne problem and we expect $T \to T_\infty$ and $\varphi_0 \to B_\nu(T_\infty)$ as $\tau \to \infty$. Indeed, given a bounded solution to the equation (B.72), assuming $T_\infty(x) = \lim_{\tau\to\infty} T(\tau,x)$ and using simple ODE's arguments we have

$$\varphi_0(\tau,x,n,\nu) = I_0 e^{-\alpha_\nu^a(x)\tau} + \int_0^\tau \alpha_\nu^a(x) e^{-\alpha_\nu^a(x)(\tau-s)} B_\nu(T(s,x)\, ds \underset{\tau\to\infty}{\longrightarrow} B_\nu(T_\infty(x)). \tag{B.73}$$

(ii) We turn now to the case $\ell_M = \ell_T = \ell_S \ll L$. The initial Milne equation is

$$\begin{cases} \partial_\tau \varphi_0(\tau,x,n,\nu) = \alpha_\nu^a(x)\left(B_\nu(T(\tau,x)) - \varphi_0(\tau,x,n,\nu)\right) \\ \qquad\qquad + \alpha_\nu^s(x)\left(\int_{\mathbb{S}^2} K(n,n')\varphi_0(\tau,x,n',\nu)\, dn' - \varphi_0(\tau,x,n,\nu)\right) & \tau > 0 \\ \partial_\tau T(\tau,x) + \int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, \alpha_\nu^a(x)\left(B_\nu(T(\tau,x)) - \varphi_0(\tau,x,n,\nu)\right) = 0 & \tau > 0 \\ \varphi_0(0,x,n,\nu) = I_0(x,n,\nu) \\ T(0,x) = T_0(x). \end{cases} \tag{B.74}$$

Again, assuming $T_\infty(x) = \lim_{\tau\to\infty} T(\tau,x)$ for a bounded solution to (B.74) we can write an explicit formula for φ_0 and we also obtain

$$\begin{aligned} \varphi_0(\tau,x,n,\nu) =& I_0 e^{-(\alpha_\nu^a(x)+\alpha_\nu^s(x))\tau} + \int_0^\tau \alpha_\nu^a(x) e^{-(\alpha_\nu^a(x)+\alpha_\nu^s(x))(\tau-s)} B_\nu(T(s,x))\, ds \\ &+ \int_0^\tau \alpha_\nu^s(x) e^{-(\alpha_\nu^a(x)+\alpha_\nu^s(x))(\tau-s)} H[\varphi_0](\tau,x,n,\nu) \\ =& e^{-(\alpha_\nu^a(x)+\alpha_\nu^s(x))\tau} \sum_{n=0}^\infty \frac{(\alpha_\nu^s(x)\tau)^n}{n!} H^n[I_0](x,n,\nu) \\ &+ \int_0^\tau \alpha_\nu^a(x) e^{-\alpha_\nu^a(x)(\tau-s)} B_\nu(T(s,x))\, ds \\ &\underset{\tau\to\infty}{\longrightarrow} B_\nu(T_\infty(x)). \end{aligned}$$

(iii) For the case $\ell_M \ll \ell_T \ll L$, similarly as for the boundary layers, we expect the solution to the initial Milne layer equation to become isotropic but not necessarily to become the Planck distribution. In this case the initial Milne equation is

$$\begin{cases} \partial_\tau \varphi_0(\tau,x,n,\nu) = \alpha_\nu^s(x)\left(\int_{\mathbb{S}^2} K(n,n')\varphi_0(\tau,x,n',\nu)\, dn' - \varphi_0(\tau,x,n,\nu)\right) & \tau > 0 \\ \partial_\tau T(\tau,x) = 0 & \tau > 0 \\ \varphi_0(0,x,n,\nu) = I_0(x,n,\nu) \\ T(0,x) = T_0(x). \end{cases} \tag{B.75}$$

On one hand we have $T(\tau, x) = T_0(x)$ for all $\tau > 0$, on the other hand we have

$$\varphi_0(\tau, x, n, \nu) = \exp(-\alpha_\nu^s(x)\tau(Id - H))I_0.$$

Using standard spectral theory for $H \in \mathcal{L}\left(L^2(\mathbb{S}^2), L^2(\mathbb{S}^2)\right)$, a compact self-adjoint operator, we see that the greatest eigenvalue of H is 1 with eigenfunctions being the constants. Hence, an application of the spectral gap theory and of the continuous functional calculus (cf. [120]) yields the limit

$$\lim_{\tau\to\infty} \varphi_0(\tau, x, n, \nu) = \varphi(x, \nu),$$

where φ is independent of $n \in \mathbb{S}^2$. Moreover, $\varphi(x, \nu) = ⨍_{\mathbb{S}^2} I_0(x, n, \nu)$. Indeed, integrating over $\mathbb{S}^2$ the first equation of (B.75) we obtain using that $\int_{\mathbb{S}^2} K(n, n')dn = 1$ the equation

$$\begin{cases} \partial_\tau ⨍_{\mathbb{S}^2} \varphi_0(\tau, x, n, \nu)dn = 0 & \tau > 0 \\ ⨍_{\mathbb{S}^2} \varphi_0(0, x, n, \nu)dn = ⨍_{\mathbb{S}^2} I_0(x, n, \nu)dn. \end{cases}$$

Hence, we conclude by the isotropy of φ

$$⨍_{\mathbb{S}^2} I_0(x, n, \nu)dn = ⨍_{\mathbb{S}^2} \varphi_0(\tau, x, n, \nu)dn \underset{\tau\to\infty}{\longrightarrow} \varphi(x, \nu).$$

The study of the Milne initial layer described by (B.75) has been rigorously studied in the context of the one-speed neutron transport equation in [19] and in [147], i.e when α_ν^s is independent of ν. While in [19] the behavior of the neutron distribution for small times is analyzed for general kernels using stochastic methods, in [147] equation (B.75) is solved for a very specific scattering kernel, namely the constant kernel $K = \frac{1}{4\pi}$.

Moreover, there is also an initial thermalization layer. Indeed, under the rescaling $t = \varepsilon^{1-\beta}\tau$ for $\beta \in (-1, 1)$, $\gamma = -1$ and therefore $\tau_h = \frac{1}{\varepsilon}$ equation (B.71) becomes

$$\begin{cases} \partial_\tau I_\nu(\tau, x, n,) + \varepsilon^{-\beta} n \cdot \nabla_x I_\nu(\tau, x, n) = \alpha_\nu^a(x)\left(B_\nu(T(\tau, x)) - I_\nu(\tau, x, n)\right) \\ \qquad + \frac{\alpha_\nu^s(x)}{\varepsilon^{1+\beta}}\left(\int_{\mathbb{S}^2} K(n, n')I_\nu(\tau, x, n')\, dn' - I_\nu(\tau, x, n)\right) & \tau > 0 \\ \partial_\tau T(\tau, x) + \int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, \partial_\tau I_\nu(\tau, x, n) \\ \qquad + \varepsilon^{-\beta} \operatorname{div}\left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, I_\nu(\tau, x, n)n\right) = 0 & \tau > 0 \end{cases} \tag{B.76}$$

As we have seen several times, the leading order φ_0 of I_ν in (B.76) is isotropic. Moreover, for $\beta \geq 0$ also the term of order ε^β is isotropic. Hence, the initial thermalization layer equation for the leading order of the radiation intensity is given by

$$\begin{cases} \partial_\tau \varphi_0(\tau, x, \nu) = \alpha_\nu^a(x)\left(B_\nu(T(\tau, x)) - \varphi_0(\tau, x, \nu)\right) & \tau > 0 \\ \partial_\tau T(\tau, x) + \int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, \partial_\tau \varphi_0(\tau, x, \nu) = 0 & \tau > 0 \\ \varphi_0(0, x, n, \nu) = \varphi(x, \nu) = ⨍_{\mathbb{S}^2} I_0(x, n, \nu)dn \\ T(0, x) = T_0(x). \end{cases} \tag{B.77}$$

As for equation (B.72) arguing as in (B.73) we expect $\varphi_0(\tau, x, \nu) \to B_\nu(T_\infty(x))$ as $\tau \to \infty$ denoting by $T_\infty(x) = \lim_{\eta\to\infty} T(\tau, x)$.

(iv)+(v) Finally, in both cases $\ell_M \ll \ell_T = L$ and $\ell_M \ll L \ll \ell_T$, i.e. in the non-equilibrium diffusion case, we observe the formation of only the initial Milne layer in which the radiation intensity becomes isotropic. In both cases the initial Milne layer equation is once again (B.75).

We study now the boundary layers. We notice that in (B.71) $\partial_t I_\nu$ has relative order τ_h^{-1} compared to $n \cdot \nabla_x I_\nu$. Therefore, any rescaling of the space variable by $\xi = -\frac{x-p}{\varepsilon^\alpha} \cdot n_p$ for $\varepsilon^\alpha \in \{\ell_M = \varepsilon, \ell_T\} \ll L$ and $p \in \partial\Omega$ yields the boundary layer equations constructed in Section B.4.2. Indeed, under such procedure the system becomes

$$\begin{cases} \frac{\varepsilon^\alpha}{\tau_h}\partial_t I_\nu(t,\xi,n;p) - (n\cdot n_p)\partial_\xi I_\nu(t,\xi,n;p) = \alpha_\nu^a(p+\mathcal{O}(\varepsilon^\alpha))\varepsilon^{\beta+\alpha}\left(B_\nu(T(t,\xi;p)) - I_\nu(t,\xi,n;p)\right) \\ \qquad\qquad + \alpha_\nu^s(p+\mathcal{O}(\varepsilon^\alpha))\varepsilon^{\gamma+\alpha}\left(\int_{\mathbb{S}^2} K(n,n')I_\nu(t,\xi,n';p)\,dn' - I_\nu(t,\xi,n;p)\right) \\ \partial_t T(t,\xi;p) + \partial_t\left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, I_\nu(t,\xi,n;p)\right) \\ \qquad\qquad - \varepsilon^{-\alpha}\tau_h\partial_\xi\left(\int_0^\infty d\nu\int_{\mathbb{S}^2} dn\,(n\cdot n_p)I_\nu(t,\xi,n;p)\right) = 0 \\ I_\nu(0,\xi,n;p) = I_0(\xi,n,\nu;p) \\ T(0,\xi) = T_0(\xi) \\ I_\nu(t,0,n;p) = g_\nu(t,n) \quad \text{if} \quad n\cdot n_p < 0. \end{cases} \tag{B.78}$$

Under these rescalings we obtain namely the Milne problems (B.26) for $\ell_M = \ell_T \ll \ell_S$ and (B.34) for $\ell_M = \ell_T = \ell_S \ll L$. In the case $\ell_M \ll \ell_T \ll L$ there are two boundary layers appearing described by the Milne problem (B.43) and by the thermalization equation (B.48). Finally, if $\ell_M \ll L \lesssim \ell_T$ the Milne boundary layer is described by (B.43).

B.5.3 Limit problems in the bulk

We summarize now the PDEs which are expected to be solved by the solution of (B.10) in the limit $\ell_M = \varepsilon \to 0$ for any different choice of ℓ_T as the speed of light is finite, i.e. $c = 1$.

(i) In the case when $\ell_M = \ell_T \ll \ell_S$, the limit problem is given by

$$\begin{cases} \partial_t T(t,x) + 4\pi\sigma\partial_t T^4(t,x) - \frac{4\pi}{3}\operatorname{div}\left(\int_0^\infty \frac{\nabla_x B_\nu(T(t,x))}{\alpha_\nu(x)}d\nu\right) = 0 & t>0,\ x\in\Omega \\ T(0,x) = T_\infty(x) & x\in\Omega \\ T(t,x) = \lim\limits_{y\to\infty}\left(\int_0^\infty \alpha_\nu^a(p)I_\nu(t,y,n;p)\right) & p\in\partial\Omega, \end{cases}$$

where $I_\nu(t,y,n;p)$ is the solution to the Milne problem (B.26) for the boundary value $g_\nu(t,n)$ and $T_\infty(x) = \lim\limits_{\tau\to\infty} T(\tau,x)$ is defined as the limit of the solution to the initial layer (B.72).

(ii) If $\ell_M = \ell_T = \ell_S \ll L$, i.e. $\ell_S = \ell_A = \varepsilon$ and $\tau_h = \varepsilon^{-1}$, the limit problem that describes the temperature in the interior of Ω for positive times is

$$\begin{cases} \partial_t T(t,x) + 4\pi\sigma\partial_t T^4(t,x) \\ \quad = \operatorname{div}\left(\int_0^\infty d\nu \frac{1}{\alpha_\nu^a(x)+\alpha_\nu^s(x)}\left(\int_{\mathbb{S}^2} dn\, n\otimes\left(Id - A_{\nu,x}\right)^{-1}(n)\right)\nabla_x B_\nu(T(t,x))\right) & t>0, x\in\Omega \\ T(0,x) = T_\infty(x) = \lim\limits_{\tau\to\infty} T(\tau,x) & x\in\Omega \\ T(t,x) = \lim\limits_{y\to\infty}\left(\int_0^\infty \alpha_\nu^a(p)I_\nu(t,y,n;p)\right) & p\in\partial\Omega, \end{cases}$$

where $I_\nu(t,y,n;p)$ is the solution to the Milne problem (B.34) for the boundary value $g_\nu(t,n)$ and $T(\tau,x)$ the solution to the initial layer (B.74).

(iii) We move now to the case $\ell_M \ll \ell_T \ll L$, hence we consider $\ell_S = \varepsilon$ and $\ell_A = \varepsilon^{-\beta}$ for

$\beta \in (-1,1)$ and $\tau_h = \varepsilon^{-1}$. The limit problem is

$$\begin{cases} \partial_t T(t,x) + 4\pi\sigma\partial_t T^4(t,x) \\ \quad = \operatorname{div}\left(\int_0^\infty d\nu \frac{1}{\alpha_\nu^s(x)} \left(\int_{\mathbb{S}^2} dn\, n \otimes (Id - H)^{-1}(n)\right) \nabla_x B_\nu(T(t,x))\right) & t > 0,\ x \in \Omega \\ T(0,x) = T_\infty(x) = \lim\limits_{\tau\to\infty} T(\tau,x) & x \in \Omega \\ T(t,p) = \lim\limits_{y\to\infty} \left(\int_0^\infty d\nu\ \alpha_\nu^a(p)\varphi_0(t,\eta,\nu;p)\right) & p \in \partial\Omega, \end{cases}$$

where $T(\tau,x)$ solves the initial layer (B.77) and φ_0 is the solution to the thermalization problem (B.48).

(iv) If $\ell_M \ll L = \ell_T$ the limit problem is

$$\begin{cases} \partial_t \phi_0(t,x,\nu) - \operatorname{div}\left(\frac{1}{\alpha_\nu^s(x)} \left(⨍_{\mathbb{S}^2} n \otimes (Id - H)^{-1}(n)\, dn\right) \nabla_x \phi_0(t,x,\nu)\right) \\ \qquad\qquad = \left(B_\nu\left(T(t,x)\right) - \phi_0(t,x,\nu)\right) & x \in \Omega,\ t > 0 \\ \partial_t T(t,x) + 4\pi \int_0^\infty d\nu\ \alpha_\nu^a(x) \left(B_\nu(T(t,x)) - \phi_0(t,x,\nu)\right) = 0 & x \in \Omega,\ t > 0 \\ \phi(0,x,\nu) = \varphi(x,\nu) = ⨍_{\mathbb{S}^2} I_0(x,n,\nu) dn \\ T(0,x) = T_0(x) & x \in \Omega \\ \phi_0(t,p,\nu) = \lim\limits_{y\to\infty} ⨍_{\mathbb{S}^2} I_\nu(t,y,n,p) & p \in \partial\Omega,\ t > 0, \end{cases} \tag{B.79}$$

where $I_\nu(t,y,n,p)$ solves the Milne problem (B.43) for the boundary value $g_\nu(t,n)$ and $\varphi(p,\nu) = \lim\limits_{\tau\to\infty} \varphi_0(\tau,p,n,\nu)$ for the solution to (B.75).

(v) Finally, if $\ell_M \ll L \ll \ell_T$ the limit problem is with the same notation as in (B.79)

$$\begin{cases} \operatorname{div}\left(\frac{1}{\alpha_\nu^s(x)} \left(⨍_{\mathbb{S}^2} n \otimes (Id - H)^{-1}(n)\, dn\right) \nabla_x \phi_0(t,x,\nu)\right) = 0 & x \in \Omega,\ t > 0 \\ \partial_t T(t,x) + 4\pi \int_0^\infty d\nu\ \alpha_\nu^a(x) \left(B_\nu(T(t,x)) - \phi_0(t,x,\nu)\right) = 0 & x \in \Omega,\ t > 0 \\ T(0,x) = T_0(x) & x \in \Omega \\ \phi_0(t,p,\nu) = \lim\limits_{y\to\infty} ⨍_{\mathbb{S}^2} I_\nu(t,y,n,p) & p \in \partial\Omega,\ t > 0. \end{cases} \tag{B.80}$$

B.5.4 Initial-boundary layers

We conclude Section B.5 considering the initial-boundary layer equations, which can be found studying (B.78). This equation shows that on the one hand under the space rescale $\xi = -\frac{x-p}{\varepsilon^\alpha} \cdot n_p$ for $p \in \partial\Omega$ and $\varepsilon^\alpha \in \{\ell_M, \ell_T\}$ the time derivative term $\partial_t I_\nu$ becomes of the same order of $\partial_\xi I_\nu$ rescaling the time by $t = \frac{\varepsilon^\alpha}{\tau_h}\tau$, on the other hand it becomes of the same order of the absorption-emission term if we consider $t = \frac{\tau}{\varepsilon^\beta \tau_h}$. It is not difficult to see that rescaling the space variable according to the Milne length $\ell_M = \varepsilon$ we obtain a non-trivial equation of the leading order of I_ν in both time and space variables only rescaling the time by $t = \frac{\varepsilon}{\tau_h}\tau$. In the case $\ell_M \ll \ell_T \ll L$, i.e. when $\ell_S = \varepsilon$ and $\ell_A = \varepsilon^{-\beta}$ with $\beta \in (-1,1)$ and $\tau_h = \frac{1}{\varepsilon}$, a thermalization layer also appears. It is described for small times and for $x \in \Omega$ close to $\partial\Omega$ by the equation obtained rescaling the space variable by $\ell_T = \varepsilon^{\frac{1-\beta}{2}}$ and the time variable in a suitable way so that the resulting equation is non-trivial in both variables. This is the case when $t = \varepsilon^{1-\beta}\tau$.

(i) If $\ell_M = \ell_T \ll \ell_S$, i.e. if $\beta = -1$ and $\gamma > -1$ and $\tau_h = \varepsilon^{-1}$, rescaling the spatial variable by $y = -\frac{x-p}{\varepsilon} \cdot n_p$ for $p \in \partial\Omega$ and under the time rescaling $t = \varepsilon^2\tau$ we obtain the initial-boundary layer equation

$$\begin{cases} \partial_\tau I_\nu(\tau,y,n;p)-(n\cdot n_p)\partial_y I_\nu(\tau,y,n;p)= \\ \qquad\qquad \alpha_\nu^a(p)(B_\nu(T(\tau,y))-I_\nu(\tau,y,n;p)) & y>0,\ \tau>0 \\ \partial_t T(\tau,y)+\int_0^\infty d\nu\int_{\mathbb{S}^2}dn\ \partial_\tau I_\nu(\tau,y,n;p) \\ \qquad\qquad -\partial_y\left(\int_0^\infty d\nu\int_{\mathbb{S}^2}dn\ (n\cdot n_p)I_\nu(\tau,y,n;p)\right)=0 & y>0,\ \tau>0 \\ I_\nu(0,y,n;p)=I_0(p,n,\nu) & y>0 \\ T(0,y;p)=T_0(p) & y>0 \\ I_\nu(\tau,0,n;p)=g_\nu(0,n) & n\cdot n_p<0,\tau>0. \end{cases}$$

(ii) In the case $\ell_M=\ell_T=\ell_s$ we rescale again the variables according to $y=-\frac{x-p}{\ell_M}\cdot n_p$ for $p\in\partial\Omega$ and $t=\varepsilon^2\tau$ and we obtain the following initial-boundary layer equation

$$\begin{cases} \partial_\tau I_\nu(\tau,y,n;p)-(n\cdot n_p)\partial_y I_\nu(\tau,y,n;p)=\alpha_\nu^a(p)(B_\nu(T(\tau,y))-I_\nu(\tau,y,n;p)) \\ \qquad\qquad +\alpha_\nu^s\left(\int_{\mathbb{S}^2}K(n,n')I_\nu(\tau,y,n';p)\,dn'-I_\nu(\tau,y,n;p)\right) & y>0,\ \tau>0 \\ \partial_\tau T(\tau,y)\int_0^\infty d\nu\int_{\mathbb{S}^2}dn\ \partial_\tau I_\nu(\tau,y,n;p) \\ \qquad\qquad -\partial_y\left(\int_0^\infty d\nu\int_{\mathbb{S}^2}dn\ (n\cdot n_p)I_\nu(\tau,y,n;p)\right)=0 & y>0,\ \tau>0 \\ I_\nu(0,y,n;p)=I_0(p,n,\nu) & y>0 \\ T(0,y;p)=T_0(p) & y>0 \\ I_\nu(\tau,0,n;p)=g_\nu(0,n) & n\cdot n_p<0,\tau>0. \end{cases}$$

(iii) If $\ell_M\ll\ell_T\ll L$ there are two initial-boundary layers appearing. In order to find the initial-boundary layer equation describing the transition from T_∞ to the limit value $\lim\limits_{y\to\infty}\left(\int_0^\infty d\nu\ \alpha_\nu^a(p)\varphi_0(t,\eta,\nu;p)\right)$, we rescale first the space variable according to $\eta=\frac{x-p}{\ell_T}\cdot n_p$ for $p\in\partial\Omega$ with $\ell_T=\varepsilon^{\frac{1-\beta}{2}}$ and the time variable according to $t=\varepsilon^{1-\beta}\tau$ and following the same computations as we did in Section B.4 in equation (B.68) we obtain the initial-boundary layer equation

$$\begin{cases} \partial_\tau\varphi_0(\tau,\eta,\nu;p)-\frac{1}{\alpha_\nu^s(p)}\left(⨍_{\mathbb{S}^2}(n\cdot n_p)(Id-H)^{-1}(n)\cdot n_p\,dn\right)\partial_\eta^2\varphi_0(\tau,\eta,\nu;p) \\ \qquad\qquad =\alpha_\nu^a(p)\left(B_\nu(T(\tau,\eta;p))-\varphi_0(\tau,\eta,\nu;p)\right) & \eta>0,\ \tau>0 \\ \partial_\tau T(\tau,y;p)+\int_0^\infty d\nu\int_{\mathbb{S}^2}dn\ \alpha_\nu^a(p)\left(B_\nu(T(\tau,\eta;p))-\varphi_0(\tau,\eta,\nu;p)\right)=0 & \eta>0,\ \tau>0 \\ \varphi_0(0,\eta,\nu;p)=\varphi(p,\nu)=⨍_{\mathbb{S}^2}I_0(p,n,\nu)dn & \eta>0 \\ T(0,\eta;p)=T_0(p) & \eta>0 \\ \varphi_0(\tau,0,\nu;p)=I(0,\nu;p) & n\cdot n_p<0,\tau>0, \end{cases}$$

where we used $I(0,\nu;p)=\lim\limits_{y\to\infty}I_\nu(0,y,n;p)$ for the solution to the Milne problem (B.43) and also‘ $\varphi(p,\nu)=\lim\limits_{\tau\to\infty}\varphi_0(\tau,p,n,\nu)$ for the solution to (B.75).

Rescaling now both space and time variables according to $y=\frac{x-p}{\varepsilon}\cdot n_p$ for $p\in\partial\Omega$ and $t=\varepsilon^2\tau$ we obtain another initial-boundary layer equation which explains the transition from $I(0,\nu;p)$ to $\varphi(p,\nu)$. This is given by the following equation

$$\begin{cases} \partial_\tau I_\nu(\tau,y,n;p) - (n\cdot n_p)\partial_y I_\nu(\tau,y,n;p) = \\ \qquad\qquad + \alpha_\nu^s \left(\int_{\mathbb{S}^2} K(n,n')I_\nu(\tau,y,n';p)\,dn' - I_\nu(\tau,y,n;p)\right) & y>0,\ \tau>0 \\ \partial_\tau T(\tau,y) = 0 & y>0,\ \tau>0 \\ I_\nu(0,y,n;p) = I_0(p,n,\nu) & y>0 \\ T(0,y;p) = T_0(p) & y>0 \\ I_\nu(\tau,0,n;p) = g_\nu(0,n) & n\cdot n_p<0, \tau>0. \end{cases} \tag{B.81}$$

(iv)+(v) If $\ell_T \gtrsim L$ under the rescaling $y = \frac{x-p}{\varepsilon}\cdot n_p$ for $p\in\partial\Omega$ and $t=\varepsilon^2\tau$ we obtain the problem (B.81) as initial-boundary layer equation.

(v) Moreover, in the case $\ell_T \gg L$ we notice in equation (B.80) that the leading order ϕ_0 of the radiation intensity solves a stationary equation. The transition from the solution of a time dependent equation, as the one of the original problem, to the solution of a stationary equation happens in times of order $\varepsilon^{\beta-1}$. Indeed, under a time rescaling $t=\varepsilon^{\beta-1}\tau = \frac{\tau}{\tau_h\varepsilon}$ we obtain the following equation solved by the leading order ϕ_0 in the bulk

$$\begin{cases} \partial_\tau \phi_0(\tau,x,\nu) \\ \quad -\operatorname{div}\left(\frac{1}{\alpha_\nu^s(x)}\left(⨍_{\mathbb{S}^2} n\otimes (Id-H)^{-1}(n)\,dn\right)\nabla_x\phi_0(\tau,x,\nu)\right) = 0 & x\in\Omega,\ \tau>0 \\ \partial_\tau T(\tau,x) = 0 & x\in\Omega,\ \tau>0 \\ \phi(0,x,\nu) = \varphi(x,\nu) = ⨍_{\mathbb{S}^2} I_0(x,n,\nu)dn & x\in\Omega \\ T(0,x) = T_0(x) & x\in\Omega \\ \phi_0(\tau,p,\nu) = \lim\limits_{y\to\infty} ⨍_{\mathbb{S}^2} I_\nu(\tau,y,n,p) & p\in\partial\Omega,\ \tau>0, \end{cases} \tag{B.82}$$

where $\varphi(x,\nu)$ is defined by the initial layer equation (B.75). This equation can be derived in the same way as the outer problem (B.70) taking into account that under this time scale the term containing $\partial_\tau I_\nu$ is of order $\varepsilon^{1-\beta} \gg \varepsilon^0$. Moreover, also the second equation in (B.10) gives $\partial_\tau T = 0$ since the absorption emission terms are of order $\varepsilon^0 \ll \varepsilon^{1-\beta}$.

B.6 Time dependent diffusion approximation. The case of non-dimensional speed of light scaling as a power law of the Milne length

In this last section we repeat all the procedures used in Sections B.3, B.4 and B.5 and we construct the limit problem solved by the solution of the time dependent equation (B.10) when $\ell_M = \varepsilon \to 0$ and in the case in which the speed of light is a power-law of the form $c=\varepsilon^{-\kappa}$ for $\kappa>0$. The strategy is the same as in Section B.5. It will turn out that the limit problems valid at the interior of the domain Ω and for positive times are the same as the one we found in the case of infinite speed of light. On the other hand, differently from the case of infinite speed of light, in this case time layers appears also in regions far from the boundary. Similarly as in Section B.4 and B.5, the boundary layer equations are stationary and are the same equations constructed in Section B.3. Finally, we will summarize the initial boundary value problems that we have obtained and we will construct the initial-boundary layer equations that we have to consider in order to describe the behavior of the solution for small times in regions close to the boundary.

B.6.1 Outer problems

We consider equation (B.10) in the case $c = \varepsilon^{-\kappa}$, $\kappa > 0$. In order to find the outer problems solved in the limit we proceed as we did in the previous three sections. It turns out that the outer problems are the same evolution equations obtained for the infinite speed of light case. Indeed, under the assumption $c = \varepsilon^{-\kappa}$ and $\ell_A = \varepsilon^{-\beta}$, $\ell_S = \varepsilon^{-\gamma}$ with $\min\{\alpha, \gamma\} = -1$ equation (B.10) becomes

$$\begin{cases} \varepsilon^\kappa \partial_t I_\nu(t,x,n) + \tau_h n \cdot \nabla_x I_\nu(t,x,n) = \alpha_\nu^a(x)\varepsilon^\beta \tau_h \left(B_\nu(T(t,x)) - I_\nu(t,x,n)\right) \\ \qquad + \alpha_\nu^s(x)\varepsilon^\gamma \tau_h \left(\int_{\mathbb{S}^2} K(n,n')I_\nu(t,x,n')\, dn' - I_\nu(t,x,n)\right) \quad x \in \Omega, n \in \mathbb{S}^2, t > 0 \\ \partial_t T + \varepsilon^\kappa \partial_t \left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, I_\nu(t,n,x)\right) \\ \qquad + \tau_h \operatorname{div}\left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, n I_\nu(t,n,x)\right) = 0 \quad x \in \Omega, n \in \mathbb{S}^2, t > 0 \\ I_\nu(0,x,n) = I_0(x,n,\nu) \quad x \in \Omega, n \in \mathbb{S}^2 \\ T(0,x) = T_0(x) \quad x \in \Omega \\ I_\nu(t,n,x) = g_\nu(t,n) \quad x \in \partial\Omega, n \cdot n_x < 0, t > 0. \end{cases} \tag{B.83}$$

Then, plugging the usual expansion (B.52) for I_ν into equation (B.83) and identifying all terms of the same power of ε give the same results as in Section B.4. This is due to the fact that in the first equation of (B.83) the term involving the time derivative of the radiation intensity is of order ε^κ and hence it is much smaller than $\varepsilon^0 \ll \varepsilon^{-1} \ll \tau_h \varepsilon^{-1}$, i.e. the orders of magnitude which lead to the resulting first two terms in the expansion $I_\nu(t,x,n) = \phi_0(t,x,\nu) + \varepsilon\phi_1(t,x,n,\nu) + \cdots$. As we noticed in the previous sections, ϕ_0 is isotropic and as long as $\ell_T \ll L$ it is the Planck distribution $B_\nu(T)$. Since also in the second equation of (B.83) the leading term containing $\partial_t T$ is of order 1, the term $\varepsilon^\kappa \partial_t \int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, I_\nu$ is negligible. The outer problems are hence as in Section B.4 equation (B.53) for $\ell_M = \ell_T \ll \ell_S$, equation (B.54) for $\ell_M = \ell_T = \ell_S$, equation (B.55) for $\ell_M \ll \ell_T \ll L$, the system (B.56) for $\ell_M \ll L = \ell_T$ and the system (B.57) for $\ell_M \ll L \ll \ell_T$.

B.6.2 Initial layer equations and boundary layer equations

In contrast to Section B.4 (i.e. the case $c = \infty$), besides the formation of boundary layers also time layers appear. The equations describing them can be obtained similarly as in Section B.5. The first equation in (B.83) has leading order $\tau_h \varepsilon^{-1}$, hence a time rescaling $t = \frac{\varepsilon^{1+\kappa}}{\tau_h}\tau$ gives

$$\partial_\tau I_\nu = \varepsilon^{\beta+1}\alpha_\nu^a(x)\left(B_\nu(T) - I_\nu\right) + \varepsilon^{\gamma+1}\alpha_\nu^s(x)\left(\int_{\mathbb{S}^2} K(n,n')I_\nu\, dn' - I_\nu\right) - \varepsilon n \cdot \nabla_x I_\nu$$

and

$$\varepsilon^{-\kappa}\partial_\tau T + \partial_\tau\left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, I_\nu\right) + \varepsilon \operatorname{div}\left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, n I_\nu\right) = 0,$$

which implies $\partial_\tau T = 0$ at the leading order. Hence, an initial layer of thickness of order $\frac{\varepsilon^{1+\kappa}}{\tau_h}$ is appearing for any choice of ℓ_A and ℓ_S. This is the so called initial Milne layer.

(i) If $\ell_M = \ell_T \ll \ell_S$ the initial Milne layer is described by

$$\begin{cases} \partial_\tau \varphi_0(\tau,x,n,\nu) = \alpha_\nu^a(x)\left(B_\nu(T_0(x)) - \varphi_0(\tau,x,n\nu)\right) \quad \text{if } \tau > 0 \\ \varphi_0(0,x,n,\nu) = I_0(x,n,\nu). \end{cases} \tag{B.84}$$

Therefore, as $\tau \to \infty$ we obtain using a simple ODE argument $\lim_{\tau\to\infty} \varphi_0(\tau,x,n,\nu) = B_\nu(T_0(x))$.

(ii) In the case $\ell_M = \ell_T = \ell_S$ and hence $\tau_h = \frac{1}{\varepsilon}$ with the scaling $t = \tau\varepsilon^{2+\kappa}$ we obtain on one hand $\partial_\tau T = 0$ and on the other hand for the first order φ_0 the identity

$$\partial_\tau\varphi_0(\tau,x,n,\nu) = \alpha_\nu^a(x)\left(B_\nu(T_0(x)) - \varphi_0(\tau,x,n\nu)\right) + \alpha_\nu^s(x)\left(\int_{\mathbb{S}^2} K(n,n')\varphi_0(\tau,x,n',\nu)\,dn' - \varphi_0(\tau,x,n,\nu)\right).$$

Again, using semigroup theory we can write the solution as

$$\varphi_0 = e^{-\alpha_\nu^a(x)\tau}\left(e^{-\alpha_\nu^s\tau(Id-H)}I_0\right) + \left(1 - e^{\alpha_\nu^a(x)\tau}\right)B_\nu(T_0).$$

Hence, we have once more $\lim\limits_{\tau\to\infty}\varphi_0(\tau,x,n,\nu) = B_\nu(T_0(x))$.

(iii) For all cases $\ell_M \ll \ell_T \ll L$, i.e. $\ell_S = \varepsilon$ and $\ell_A = \varepsilon^{-\beta}$ for $\beta \in (-1,1)$ and $\tau_h = \frac{1}{\varepsilon}$, under the scaling $t = \tau\varepsilon^{2+\kappa}$ we have the initial Milne layer equation

$$\begin{cases} \partial_\tau\varphi_0(\tau,x,n,\nu) = \alpha_\nu^s(x)\left(\int_{\mathbb{S}^2} K(n,n')\varphi_0(\tau,x,n',\nu)\,dn' - \varphi_0(\tau,x,n,\nu)\right) & \tau > 0 \\ \partial_\tau T(\tau,x) = 0 & \tau > 0 \\ \varphi_0(0,x,n,\nu) = I_0(x,n,\nu) \\ T(0,x) = T_0(x). \end{cases} \tag{B.85}$$

This is exactly the same equation as (B.75). Thus, an application of spectral theory implies again
$\lim\limits_{\tau\to\infty}\varphi_0(\tau,x,n,\nu) = \varphi(x,\nu) = ⨍_{\mathbb{S}^2} I_0(x,n,\nu)dn.$

However, as for the finite speed of light case, there is also a thermalization layer appearing. Indeed, with a time rescaling $t = \varepsilon^{1-\beta+\kappa}\tau$ the term involving $\partial_t I_\nu$ becomes of the same order of the emission-absorption term according to

$$\begin{cases} \partial_\tau I_\nu(\tau,x,n) + \varepsilon^{-\beta} n\cdot\nabla_x I_\nu(\tau,x,n) = \alpha_\nu^a(x)\left(B_\nu(T(x)) - I_\nu(\tau,x,n)\right) \\ \qquad + \frac{\alpha_\nu^s(x)}{\varepsilon^{1+\beta}}\left(\int_{\mathbb{S}^2} K(n,n')I_\nu(\tau,x,n')\,dn' - I_\nu(\tau,x,n)\right) \\ \frac{1}{\varepsilon^\kappa}\partial_\tau T(\tau,x) + \left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\,\partial_\tau I_\nu(\tau,x,n)\right) + \varepsilon^{-\beta}\operatorname{div}\left(\int_0^\infty d\nu\int_{\mathbb{S}^2} dn\, nI_\nu(\tau,x,n)\right) = 0. \end{cases} \tag{B.86}$$

Hence, as we have seen in (B.76) the leading order φ_0 of I_ν in (B.86) is isotropic, as well as the term of order ε^β for $\beta \geq 0$. Moreover, once more the temperature T is just the initial temperature $T_0(x)$ to the leading order. This yields the initial thermalization layer equation

$$\begin{cases} \partial_\tau\varphi_0(\tau,x,\nu) = \alpha_\nu^a(x)\left(B_\nu(T_0(x)) - \varphi_0(\tau,x,\nu)\right) & \tau > 0 \\ \varphi_0(0,x,\nu) = \varphi(x,\nu). \end{cases}$$

Hence, similarly to (B.84) we have $\lim\limits_{\tau\to\infty}\varphi_0 = B_\nu(T_0(x))$ as $\tau\to\infty$.

(iv)+(v) For the cases $\ell_M \ll L \lesssim \ell_T$ the initial Milne layer equation is obtained again rescaling the time variable by $t = \frac{\varepsilon^{1+\kappa}}{\tau_h}\tau$ and it is given by equation (B.85).

For the boundary layer equations we argue similarly as in the case $c=\infty$ and c bounded. Rescaling the space variable by $\xi=-\frac{x-p}{\varepsilon^\alpha}\cdot n_p$ for $\varepsilon^\alpha\in\{\ell_M,\ell_T\}$ and $p\in\partial\Omega$ equation (B.83) becomes

$$\begin{cases}-(n\cdot n_p)\partial_\xi I_\nu(t,\xi,n;p)=\alpha_\nu^a\left(p+\mathcal{O}(\varepsilon^\alpha)\right)\frac{\varepsilon^\alpha}{\ell_A}(B_\nu(T)-I_\nu) & \\ \qquad +\frac{\varepsilon^\alpha}{\ell_S}\alpha_\nu^s\left(p+\mathcal{O}(\varepsilon^\alpha)\right)\left(\int_{\mathbb{S}^2}K(n,n')I_\nu\,dn'-I_\nu\right)-\frac{\varepsilon^{\alpha+\kappa}}{\tau_h}\partial_t I_\nu(t,\xi,n)+\varepsilon^{2\alpha}\cdots & \xi>0\\ \partial_t T(t,\xi;p)+\varepsilon^\kappa\left(\int_0^\infty d\nu\int_{\mathbb{S}^2}dn\partial_t I_\nu\right)+\varepsilon^{-\alpha}\tau_h\operatorname{div}\left(\int_0^\infty d\nu\int_{\mathbb{S}^2}dn\,nI_\nu\right)=0 & \xi>0\\ I_\nu(0,\xi,n;p)=I_0(p,n,\nu) & \\ T(0,\xi;p)=T_0(p) & \\ I_\nu(t,0,n;p)=g_\nu(t,n) & n\cdot n_p<0.\end{cases} \tag{B.87}$$

Therefore, the boundary layers are described by the same stationary equation we constructed in Section B.3. Indeed we obtain for $\ell_M=\ell_T\ll\ell_S$ the Milne problem (B.26) and for $\ell_M=\ell_T=\ell_S\ll L$ the Milne problem (B.34). The two boundary layers appearing in the case $\ell_M\ll\ell_T\ll L$ are described by the Milne problem (B.43) and by the thermalization equation (B.48). Finally, if $\ell_M\ll L\lesssim\ell_T$ the Milne problems are given by (B.43).

B.6.3 Limit problems in the bulk

We now summarize the PDE problems which are expected to be solved by the solution of (B.10) when $c=\varepsilon^{-\kappa}$, $\kappa>0$ in the limit $\ell_M=\varepsilon\to0$ for any choice of ℓ_A and ℓ_S. Matching the solution to the outer problems valid in the bulk for positive times $t>0$ with the solution to the initial layer equations and boundary layer equations, we obtain as limit equation exactly the same PDE problems in Section B.4. Indeed, on one hand the boundary layer problems are exactly the Milne and thermalization problems constructed for the stationary problem and valid also for the time dependent problem. On the other hand, in the initial layer equations derived in the previous subsection B.6.2 the temperature is constant, hence it is $T=T_0$, the same result we that obtained in the case $c=\infty$ in Subsection B.4.2. Therefore, since the outer problems coincides in both cases when $c=\infty$ and $c=\varepsilon^{-\kappa}$ with $\kappa>0$ and $\varepsilon\to0$, we conclude as in Section B.4 that the limit PDE problems are given by (B.62) if $\ell_M=\ell_T\ll\ell_S$, by (B.63) if $\ell_M=\ell_T=\ell_S$, by (B.64) if $\ell_M\ll\ell_T\ll L$, by (B.65) if $\ell_M\ll L=\ell_T$ and finally by (B.66) if $\ell_M\ll L\ll\ell_T$.

B.6.4 Initial-boundary layers

As in Sections B.4 and B.5 we will derive the initial-boundary layer equations, which describe the behavior of the solutions for very small times and in regions close to the boundary. The initial-boundary layer equations are obtained rescaling in a suitable way the space and time variables. Considering equation (B.87) resulting from the space rescale according to the Milne length or the thermalization length we notice that the term involving the time derivative of the radiation intensity has order $\frac{\varepsilon^{\alpha+\kappa}}{\tau_h}$. Hence, the initial-boundary Milne layer equation is obtained by the rescaling $y=-\frac{x-p}{\varepsilon}\cdot n_p$ and $t=\frac{\varepsilon^{1+\kappa}}{\tau_h}\tau$ for $p\in\partial\Omega$. In the case $\ell_M\ll\ell_T\ll L$ (i.e. when $\ell_A=\varepsilon^\beta$ for $\beta\in(-1,1)$, $\ell_S=\varepsilon$ and $\tau_h=\frac{1}{\varepsilon}$) the initial-boundary thermalization equation is obtained rescaling $\eta=-\frac{x-p}{\ell_T}\cdot n_p$ and $t=\varepsilon^{1-\beta+\kappa}$, where $\ell_T=\varepsilon^{-\frac{1-\beta}{2}}$ and $p\in\partial\Omega$.

(i) If $\ell_M=\ell_T\ll\ell_S$ rescaling the spatial variable by $y=-\frac{x-p}{\varepsilon}\cdot n_p$ for $p\in\partial\Omega$ and the time variable by $t = \varepsilon^{2+\kappa}\tau$ we see that the initial-boundary layer equation is given by

$$\begin{cases} \partial_\tau I_\nu(\tau, y, n; p) - (n \cdot n_p)\partial_y I_\nu(\tau, y, n; p) = \\ \qquad\qquad \alpha_\nu^a(p)(B_\nu(T_0(p)) - I_\nu(\tau, y, n; p)) & y > 0,\ \tau > 0 \\ I_\nu(0, y, n; p) = I_0(p, n, \nu) & y > 0,\ \tau > 0 \\ I_\nu(\tau, 0, n; p) = g_\nu(0, n) & n \cdot n_p < 0,\ \tau > 0, \end{cases}$$

where we used that equation

$$\begin{cases} \frac{1}{\varepsilon^\kappa}\partial_t T(\tau, y) + \partial_\tau \left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, I_\nu(\tau, y, n; p)\right) \\ \qquad\qquad = \partial_y \left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, (n \cdot n_p) I_\nu(\tau, y, n; p)\right) & y > 0,\ \tau > 0 \\ T(0, y; p) = T_0(p) & y > 0, \end{cases} \tag{B.88}$$

implies $T(\tau, y; p) = T_0(p)$.

(ii) If $\ell_M = \ell_T = \ell_S$ rescaling the variables according to $y = -\frac{x-p}{\varepsilon} \cdot n_p$ for $p \in \partial\Omega$ and $t = \varepsilon^{2+\kappa}\tau$ we obtain the following initial-boundary layer equation

$$\begin{cases} \partial_\tau I_\nu(\tau, y, n; p) - (n \cdot n_p)\partial_y I_\nu(\tau, y, n; p) = \alpha_\nu^a(p)(B_\nu(T_0(p)) - I_\nu(\tau, y, n; p)) \\ \qquad\qquad +\alpha_\nu^s \left(\int_{\mathbb{S}^2} K(n, n') I_\nu(\tau, y, n'; p)\, dn' - I_\nu(\tau, y, n; p)\right) & y > 0,\ \tau > 0 \\ I_\nu(0, y, n; p) = I_0(p, n, \nu) & y > 0 \\ I_\nu(\tau, 0, n; p) = g_\nu(0, n) & n \cdot n_p < 0,\ \tau > 0, \end{cases}$$

where we used (B.88) again.

(iii) If $\ell_M \ll \ell_T \ll L$ there are again two different initial-boundary layers. We consider first the thermalization problem. We hence rescale the space variable according to $\eta = \frac{x-p}{\varepsilon^{\frac{1-\beta}{2}}} \cdot n_p$ for $p \in \partial\Omega$ and the time variable according to $t = \varepsilon^{\kappa+1-\beta}\tau$ and following the same computations as we did in Section B.4 in equation (B.68) and using a similar argument as in (B.88) we obtain the initial-boundary layer equation as

$$\begin{cases} \partial_\tau \varphi_0(\tau, \eta, \nu; p) - \frac{1}{\alpha_\nu^s(p)} \left(\fint_{\mathbb{S}^2} (n \cdot n_p)(Id - H)^{-1}(n) \cdot n_p\, dn\right) \partial_\eta^2 \varphi_0(\tau, \eta, \nu; p) \\ \qquad\qquad = \alpha_\nu^a(p) \left(B_\nu(T_0(p)) - \varphi_0(\tau, \eta, \nu; p)\right) & \eta > 0,\ \tau > 0 \\ \varphi_0(0, \eta, \nu; p) = \varphi(p, \nu) & \eta > 0 \\ \varphi_0(\tau, 0, \nu; p) = I(0, \nu; p) & p \in \partial\Omega,\ \tau > 0, \end{cases}$$

where $I(0, \nu; p) = \lim_{y\to\infty} I_\nu(0, y, n; p)$ for the solution to the Milne problem (B.43) for the boundary value $g_\nu(t, n)$ and $\varphi(p, \nu) = \lim_{\tau\to\infty} \varphi_0(\tau, p, n, \nu)$ for the solution to (B.85).

As we have seen in Section B.5 there is another initial-boundary value equation which describes the transition from the initial value $\varphi(x, \nu)$ to the boundary value $I(0, \nu; p)$. This is obtained rescaling the space variable according to $y = \frac{x-p}{\varepsilon} \cdot n_p$ for $p \in \partial\Omega$ and the time variable according to $t = \varepsilon^{\kappa+2}\tau$. Using (B.88) we obtain hence

$$\begin{cases} \partial_\tau I_\nu(\tau, y, n; p) - (n \cdot n_p)\partial_y I_\nu(\tau, y, n; p) \\ \qquad\qquad = \alpha_\nu^s \left(\int_{\mathbb{S}^2} K(n, n') I_\nu(\tau, y, n'; p)\, dn' - I_\nu(\tau, y, n; p)\right) & y > 0,\ \tau > 0 \\ I_\nu(0, y, n; p) = I_0(p, n, \nu) & y > 0 \\ I_\nu(\tau, 0, n; p) = g_\nu(0, n) & n \cdot n_p < 0,\ \tau > 0. \end{cases}$$

(iv) In the case $\ell_M \ll \ell_T = L$ rescaling $\eta = \frac{x-p}{\varepsilon} \cdot n_p$ for $p \in \partial\Omega$ and $t = \varepsilon^{2+\kappa}\tau$ we obtain also the initial boundary layer equation for this case

$$\begin{cases} \partial_\tau I_\nu(\tau, y, n; p) - (n \cdot n_p)\partial_y I_\nu(\tau, y, n; p) = & \\ \qquad + \alpha_\nu^s \left(\int_{\mathbb{S}^2} K(n, n') I_\nu(\tau, y, n'; p)\, dn' - I_\nu(\tau, y, n; p)\right) & y > 0,\ \tau > 0 \\ I_\nu(0, y, n; p) = I_0(p, n, \nu) & y > 0 \\ I_\nu(\tau, 0, n; p) = g_\nu(0, n) & n \cdot n_p < 0,\ \tau > 0. \end{cases} \tag{B.89}$$

Similar to the case where $\ell_T \gg 1$ and $c = 1$ in Section B.5, we notice that the radiation intensity I_ν has a transition from a solution of a time dependent equation, as it was in the original problem (B.10), to a solution of a stationary equation, as it is in (B.65). This transition takes place at times of order ε^κ. Indeed, under the time rescaling $t = \varepsilon^\kappa \tau$ we obtain the following equation for the leading order ϕ_0 of I_ν for all $x \in \Omega$

$$\begin{cases} \partial_\tau \phi_0(\tau, x, \nu) - \operatorname{div}\left(\frac{1}{\alpha_\nu^s(x)} \left(⨍_{\mathbb{S}^2} n \otimes (Id - H)^{-1}(n)\, dn\right) \nabla_x \phi_0(\tau, x, \nu)\right) & \\ \qquad\qquad = \left(B_\nu\left(T(\tau, x)\right) - \phi_0(\tau, x, \nu)\right) & x \in \Omega,\ \tau > 0 \\ \partial_\tau T(\tau, x) = 0 & x \in \Omega,\ \tau > 0 \\ \phi(0, x, \nu) = \varphi(x, \nu) & x \in \Omega \\ T(0, x) = T_0(x) & x \in \Omega \\ \phi_0(\tau, p, \nu) = \lim_{y \to \infty} ⨍_{\mathbb{S}^2} I_\nu(0, y, n, p) & p \in \partial\Omega,\ \tau > 0, \end{cases} \tag{B.90}$$

where $I_\nu(0, y, n, p)$ solves the Milne problem (B.43) for the boundary value $g_\nu(0, n)$ and we used the notation $\varphi(x, \nu) = \lim_{\tau \to \infty} \varphi_0(\tau, x, n, \nu)$ for the solution to (B.85). In order to derive equation (B.90) we notice that under the time rescale $t = \varepsilon^\kappa \tau$ the term in the first equation of (B.10) containing $\partial_\tau I_\nu$ becomes of order ε^0 as the absorption-emission term. This implies the first equation in (B.90) as we did in Section B.5 for (B.69). On the other hand, in the second equation of (B.90) the leading term is $\partial_\tau T$ of order $\varepsilon^{-\kappa} \gg \varepsilon^0$.

(v) Finally, if $\ell_M \ll L \ll \ell_T$ the initial-boundary layer equation is again (B.89). Also for this last case we notice the leading order ϕ_0 of I_ν, which solves a time-dependent equation (B.10), solves in the limit a stationary equation (B.66). The transition from time-dependent solution to stationary solution takes place at time of order $\varepsilon^{\beta-1+\kappa}$. Under the time rescale $t = \varepsilon^{\beta-1+\kappa}\tau$ we derive in the same way as for equation (B.82) the equation solved by ϕ_0 in the bulk describing this transition. It turns out that it is exactly given by (B.82) for the initial condition $\phi(0, x, \nu) = \varphi(x, \nu)$ given by the solution to (B.85).

B.7 Concluding remarks

In this paper we considered the problem of describing the temperature distribution in a body where the heat is transported only by radiation. We considered the case where the mean free path of the radiative process tends to zero, i.e. $\ell_M \to 0$. Therefore, we coupled the radiative transfer equation (B.1) with the energy balance equation (B.2) and we studied the diffusion approximation for the time dependent equations (B.10) and (B.11) and the stationary equation (B.12).

For all different scaling limit regimes using the method of asymptotic expansions we derived the full limit models describing the temperature of the body and the radiation intensity.

The resulting models have been classified depending on the form of the radiation intensity at the leading order on the bulk of the domain. The cases where the isotropic leading order of the radiation intensity is given by the Planck distribution for the temperature yield the diffusion equilibrium approximation, while the models in which the radiation intensity is not approximated by the Planck distribution are denoted by diffusion non-equilibrium approximation. Notice that the diffusion approximation is valid only on the bulk of the domain Ω where the leading order of the radiation intensity is isotropic. On the other hand, at the boundary layers and at the initial layers the diffusion approximation fails. We also described for each considered case the boundary and initial layers appearing. Moreover, a summary of the available results about the diffusion approximation and the boundary layer problem for similar settings is included. Many of the derived problems in this article have to be still studied.

For the time dependent problem we studied three different cases. First we analyzed the problem for the speed of light assumed to be $c = \infty$, i.e. when the transport of radiation can be assumed to be instantaneous. We then considered the case where the speed of light is of order 1, i.e. when the time used by the light for spanning distances of order 1 is of the same order of the time needed by the temperature for having meaningful changes. Finally, we studied the case where the speed of light scales as a power law of the Milne length, i.e. $c = \varepsilon^{-\kappa}$ for $\kappa > 0$ and $\ell_M = \varepsilon$.

B.8 Appendix: Proof of Proposition B.2.1

We prove now Proposition B.1. To this end we need the following auxiliary Lemma.

Lemma B.1. *Let $K \in C\left(\mathbb{S}^2 \times \mathbb{S}^2\right)$, invariant under rotations, non-negative and satisfying*

$$\int_{\mathbb{S}^2} K(n, n')dn = 1.$$

Let $n, \omega \in \mathbb{S}^2$. Then there exists finitely many $n_1, \cdots, n_N \in \mathbb{S}^2$ such that

$$K(n_{i-1}, n_i) > 0 \text{ for all } i \in \{1, \cdots, N+1\}, \tag{B.91}$$

where we defined $n_0 = n$ and $n_{N+1} = \omega$.

Proof. Since $K \geq 0$ but it is not equal zero, there exists a pair $n', n'' \in \mathbb{S}^2$ such that $K(n', n'') > 0$. Hence, applying the rotation $\mathcal{R}_{n,n'}$ yields the existence of n_* such that $K(n, n_*) > 0$. By continuity the set $B_n = \{\tilde{n} \in \mathbb{S}^2 : K(n, \tilde{n}) > 0\}$ is open. Hence, there exists $\delta > 0$ and $n_1 \in \mathbb{S}^2$ such that $B_\delta(n_1) \subset B_n$. We remark that $\delta > 0$ is independent of the choice of $n \in \mathbb{S}^2$. Indeed, for any $n' \in \mathbb{S}^2$ there exists some $n'' \in \mathbb{S}^2$ such that $B_\delta(n'') \subset B_{n'}$. This is a consequence of the invariance under rotations of K. Indeed, it is not difficult to see that $\mathcal{R}_{n,n'}(B_n) = B_{n'}$ and so $\mathcal{R}_{n,n'}(B_\delta(n_1)) = B_\delta(\mathcal{R}_{n,n'}(n_1)) \subset B_{n'}$.

Let us consider the set

$$A_n = \{\tilde{n} \in \mathbb{S}^2 : \text{ there exist } n_1, ..., n_N \in \mathbb{S}^2 \text{ such that (B.91) holds for } n_0 = n \text{ and } n_{N+1} = \tilde{n}\}.$$

By the previous consideration we know that A_n is not empty. We claim now that $B_\delta(n') \subset A_n$ for any $n' \in A_n$. Indeed, let $\delta > 0$ as above. Since $n' \in A_n$, then $B_{n'}$ is not empty and there exists some $n_1 \in \mathbb{S}^2$ such that $B_\delta(n_1) \subset B_{n'}$. It is easy to see that $n_1 \in A_n$. Let now $\tilde{n} \in B_\delta(n')$, then $\mathcal{R}_{n',n_1}(\tilde{n}) \in B_\delta(n_1)$. Hence, $K(n', \mathcal{R}_{n',n_1}(\tilde{n})) = K(n_1, \tilde{n}) > 0$. Since also $K(n', n_1) > 0$, we conclude that $B_\delta(n') \subset A_n$ for all $n' \in A_n$. Hence, A_n is open and it is the whole sphere $\mathbb{S}^2$. Indeed, assume $A_n \neq \mathbb{S}^2$. Then, since A_n is open, the boundary $\partial A_n = \overline{A}_n \setminus A_n$ is not empty. Let $n^* \in \partial A_n$ and let $n_0 \in A_n$ with $d(n^*, n_0) < \frac{\delta}{3}$, where $d(n^*, n_0)$ is the distance on $\mathbb{S}^2$ between the two points $n^*, n_0 \in \mathbb{S}^2$. Since $n^* \in \partial A_n$, it is true that

$$B_{\frac{\delta}{3}}(n^*) \cap A_n \neq \emptyset \text{ and } B_{\frac{\delta}{3}}(n^*) \cap A_n^c \neq \emptyset.$$

On the other hand, we know that $B_\delta(n_0) \subset A_n$ and therefore

$$B_{\frac{\delta}{3}}(n^*) \subset B_{\frac{\delta}{2}}(n^*) \subset B_\delta(n_0) \subset A_n.$$

This contradiction concludes the proof of Lemma B.1. □

Proof of Proposition B.1. We first show that φ is continuous. Let $\varepsilon > 0$. By the continuity of the kernel K there exists some $\delta > 0$ such that

$$\left|K(n_1, n_1') - K(n_2, n_2')\right| < \frac{\varepsilon}{4\pi \|\varphi\|_\infty}$$

for all $n_1, n_2, n_1', n_2' \in \mathbb{S}^2$ with $d(n_1, n_2) + d(n_1', n_2') < \delta$. Let hence $n_1, n_2 \in \mathbb{S}^2$ with $d(n_1, n_2) < \delta$ then it is easy to see that φ is continuous since

$$\begin{aligned} |\varphi(n_1) - \varphi(n_2)| &= |H[\varphi](n_1) - H[\varphi](n_2)| \\ &\leq \int_{\mathbb{S}^2} \left|K(n_1, n') - K(n_2, n')\right| |\varphi(n')| \, dn' < \varepsilon. \end{aligned}$$

We move now to the proof of claim (ii). Let $M = \max_{n \in \mathbb{S}^2}(\varphi(n))$. By continuity there exists some $n_* \in \mathbb{S}^2$ such that $M = \varphi(n_*)$. We define the set $A_M = \left\{n \in \mathbb{S}^2 : \varphi(n) = M\right\}$. Thus, A_M is not empty and by continuity it is also closed. We claim that A_M is also open, which implies claim (ii). Let $n \in A_M$. Consider $B_n = \{\tilde{n} \in \mathbb{S}^2 : K(n, \tilde{n}) > 0\}$. Let $\varepsilon > 0$ and $B_n^\varepsilon = \{\tilde{n} \in B_n : \varphi(\tilde{n}) < M - \varepsilon\}$. We show $\varphi(\tilde{n}) = M$ for all $\tilde{n} \in B_n$. It is easy to see that this is true if $B_n^\varepsilon = \emptyset$ for all $\varepsilon > 0$. If not, let $\varepsilon > 0$ so that $B_n^\varepsilon \neq \emptyset$. Then

$$M = \varphi(n) = \int_{B_n^\varepsilon} K(n, n')\varphi(n')dn' + \int_{(B_n^\varepsilon)^c} K(n, n')\varphi(n')dn' < M - \varepsilon \int_{B_n^\varepsilon} K(n, n')dn' < M.$$

Arguing as in the proof of Lemma B.1 there exists a $\delta > 0$ such that $B_\delta(n_0) \subset B_n$ for some $n_0 \in B_n$. Hence, using the same argument, since $n_0 \in A_M$ it is also true that $\varphi(\tilde{n}) = M$ for all $\tilde{n} \in B_{n_0}$. Using the rotation invariance of the kernel analogously as we have done in Lemma B.1 we see that

$$\mathcal{R}_{n,n_0}\left(B_\delta(n_0)\right) = B_\delta(n) \subset \mathcal{R}_{n,n_0}(B_n) = B_{n_0} \subset A_M.$$

We have just proved that closed non-empty set A_M is open and hence it must be the whole sphere $\mathbb{S}^2$.

Finally, we prove claim (iii). To this end we notice that the linear operator H maps L^p-functions to continuous bounded functions. Analogously as in the proof of (i), this is a direct consequence of the Hölder inequality and the fact that the scattering kernel K is continuous. Hence, $(Id - H)_1 : L^1(\mathbb{S}^2) \to L^1(\mathbb{S}^2)$ given by $(Id - H)_1\varphi = \varphi - H[\varphi]$ is a well-defined operator which maps integrable functions to integrable functions. Since $H[\varphi] \in C(\mathbb{S}^2)$ for any $\varphi \in L^1(\mathbb{S}^2)$, if $(Id - H)_1\varphi = 0$ then also (ii) applies and hence $\varphi = \text{const}$. This means that the null space of $(Id - H)_1$ as an operator acting on $L^1(\mathbb{S}^2)$ is given by

$$\mathcal{N}\left((Id - H)_1\right) = \text{span}\langle 1 \rangle = \{f = c : \ c \in \mathbb{R}\}.$$

It is not difficult to see that the dual operator $(Id-H)_1^*: L^\infty(\mathbb{S}^2) \to L^\infty(\mathbb{S}^2)$ is exactly given by $(Id-H)$. Indeed, let $f \in L^1(\mathbb{S}^2)$ and $g \in L^\infty(\mathbb{S}^2)$. We compute using the invariance under rotations of the kernel K

$$\begin{aligned}\int_{\mathbb{S}^2} dn\, g(Id-H)_1[f] &= \int_{\mathbb{S}^2} dn\, g(n)f(n) - \int_{\mathbb{S}^2} dn \int_{\mathbb{S}^2} dn'\, K(n,n')g(n)f(n') \\ &= \int_{\mathbb{S}^2} dn\, g(n)f(n) - \int_{\mathbb{S}^2} dn' \int_{\mathbb{S}^2} dn\, K(n',n)g(n)f(n') = \int_{\mathbb{S}^2} dn\, (Id-H)[g]f.\end{aligned}$$

Therefore, by the orthogonality of the null-space to the range of the dual operator we conclude

$$\begin{aligned}\mathrm{Ran}(Id-H) &= \left\{\varphi \in L^\infty(\mathbb{S}^2) : \int_{\mathbb{S}^2} \varphi(n)f(n)\, dn = 0\ \forall f \in \mathcal{N}\left((Id-H)_1\right)\right\} \\ &= \left\{\varphi \in L^\infty(\mathbb{S}^2) : \int_{\mathbb{S}^2} \varphi(n)\, dn = 0\right\}.\end{aligned}$$

□

Appendix C

On the diffusion approximation of the stationary radiative transfer equation with absorption and emission

Abstract: In this paper we study the distribution of temperature of a body due to the transfer of radiation. Specifically the boundary value problem for the stationary radiative transfer equation is considered. In all the analysis we assume the so-called local thermal equilibrium (LTE), i.e. there is a well defined temperature of the body at each point. We consider the limit in which the mean free path of the photons is much smaller than the characteristic length of the domain. In this case we can approximate the solution by means of the so-called diffusion approximation. The analysis of this paper is restricted to the case in which the absorption coefficient is independent of the frequency ν (the so-called Grey approximation). We ignore also scattering effects. Under these assumptions we show that the density of radiative energy u, which is proportional to the fourth power of the temperature, solves in the limit an elliptic equation. The boundary values for that limit equation can be determined uniquely analyzing a suitable boundary layer problem. The method developed here allows to prove all the results using maximum principle arguments for a class of non-local elliptic equations.

C.1 Introduction

The radiative transfer equation is the kinetic equation which describes the distribution of energy and direction of motions of a set of photons, which can be absorbed and scattered by a medium. This equation can be used to describe the transfer of heat in a material due to radiative processes. The radiative transfer equation can be written in its more general form as

$$\frac{1}{c}\partial_t I_\nu(x,n,t)+n\cdot\nabla_x I_\nu(x,n,t) = \alpha_\nu^e-\alpha_\nu^a I_\nu(x,n,t)-\alpha_\nu^s I_\nu(x,n,t)+\alpha_\nu^s\int_{\mathbb{S}^2} K(n,n')I_\nu(x,n',t)dn'. \tag{C.1}$$

We denote by $I_\nu(x,n,t)$ the intensity of radiation (i.e. radiating energy) of frequency ν at position $x\in\Omega$ and in direction $n\in\mathbb{S}^2$ and at time $t\geq 0$. The coefficients α_ν^a, α_ν^e and α_ν^s are respectively the absorption, the emission and the scattering coefficient. In the scattering term the kernel is normalized such that $\int_{\mathbb{S}^2} K(n,n')dn' = 1$. The speed of light is indicated

by c. We remark that the radiation intensity and the emission, absorption and scattering coefficients are functions of the frequency $\nu \in \mathbb{R}_+$.
In this paper we focus on the stationary problem and on processes, where the scattering is negligible. Therefore the equation we will study reduces to

$$n \cdot \nabla_x I_\nu (x, n) = \alpha_\nu^e - \alpha_\nu^a I_\nu (x, n) . \tag{C.2}$$

In this article we consider the situation of local thermal equilibrium (LTE), which means that at every point $x \in \Omega$ there is a well-defined temperature $T(x) \geq 0$. This yields, according to the Kirchhoff's law (cf. [152]), the following relation for the absorption and emission coefficient

$$\alpha_\nu^e(x) = \alpha_\nu^a(x) B_\nu(T(x)),$$

where $B_\nu(T(x)) = \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{kT}} - 1}$ is the Planck emission of a black body at temperature $T(x)$ and k the Boltzmann constant. Moreover, it is well-known that

$$\int_0^\infty B_\nu (T(x)) \; d\nu = \sigma T^4(x), \tag{C.3}$$

where $\sigma = \frac{2\pi^4 k^4}{15 h^3 c^2}$ is the Stefan-Boltzmann constant. We will denote for simplicity from now on as the absorption coefficient α_ν^a as α_ν.

The solution $I_\nu(x, n)$ of (C.2) can be used to compute the flux of energy at each point $x \in \Omega$ of the material, which is given by

$$\mathcal{F}(x) := \int_0^\infty d\nu \int_{\mathbb{S}^2} dn \; n \, I_\nu (x, n) . \tag{C.4}$$

In the stationary case, if the temperature is independent of time at every point, the total incoming and outgoing flux of energy should balance. In mathematical terms this can be formulated by the condition for the flux of energy to be divergence free, i.e.

$$\nabla_x \cdot \mathcal{F}(x) = 0.$$

This situation is denoted in the physical literature by pointwise radiative equilibrium.

We study the situation when the radiation is coming from a very far source of infinite distance. This can be formalized in mathematical terms by means of the boundary condition

$$I_\nu (x, n) = g_\nu (n) \geq 0 \tag{C.5}$$

if $x \in \partial\Omega$ and $n \cdot N_x < 0$ for N_x the outer normal vector of the boundary at point $x \in \partial\Omega$. Throughout this paper we will consider $\Omega \subset \mathbb{R}^3$ to be a bounded convex domain with C^3-boundary.

We are concerned in this paper with the study of the diffusion approximation that arises in optically thick media, i.e. the case in which the mean free path of the photons is very small compared to the characteristic length of the system. Hence, we rescale and for $\varepsilon \ll 1$ we consider the following boundary value problem

$$\begin{cases} n \cdot \nabla_x I_\nu (x, n) = \frac{\alpha_\nu(x)}{\varepsilon} \left(B_\nu (T (x)) - I_\nu (x, n) \right) & x \in \Omega, n \in \mathbb{S}^2, \nu \in \mathbb{R}_+ \\ \nabla_x \cdot \mathcal{F} = 0 & x \in \Omega, \\ I_\nu (x, n) = g_\nu (n) & x \in \partial\Omega \text{ and } n \cdot N_x < 0, \nu \in \mathbb{R}_+. \end{cases} \tag{C.6}$$

Notice that this problem has two unknowns, namely the intensity of radiation $I_\nu(x,n)$ and the temperature $T(x)$. This feature is due to the presence of absorption-emission processes which also change the temperature of the body. This is not the case when scattering is considered in the radiative transfer equation but absorption-emission processes are ignored. Indeed, in that situation the intensity of radiation and the temperature change independently. Moreover, we remark that the divergence-free condition of the flux of energy $\mathcal{F}$ (cf. (C.4)) yields a non-trivial coupling between different frequencies $\nu \in \mathbb{R}_+$. Therefore the problem (C.6) cannot be solved independently on the frequency. For the solution to equation (C.6) we will prove that the intensity of radiation $I_\nu(x,n)$ is approximately the Planck distribution $B_\nu(T(x))$ with the local temperature at each point $x\in\Omega$, i.e. we will show

$$(I_\nu^\varepsilon(x,n), T_\varepsilon(x)) \to (B_\nu(T(x)), T(x)) \qquad \text{as } \varepsilon\to 0 \tag{C.7}$$

uniformly in every compact set $K\subset\Omega$ as functions with values in $L^\infty\left(\mathbb{S}^2, L^1(\mathbb{R}_+)\right)\times\mathbb{R}_+$. Notice however, that this approximation cannot be expected for points x that are close to the boundary $\partial\Omega$. The situations in which (C.7) holds are denoted in the physical literature as cases in which the diffusion equilibrium approximation holds (see e.g. [108] and [152]). More precisely, we will consider the limit problem when $\varepsilon\to 0$ and we will rigorously prove that it is given by a Dirichlet problem for the heat equation of the temperature with boundary value uniquely determined by the incoming source $g_\nu(n)$ and the outer normal N_x for $x\in\partial\Omega$. The main result we will prove in this paper is for the so called Grey approximation, i.e. the case when the absorption coefficient is independent of the frequency ν. The main reason for that is that some of the estimates are already in this case very technical. Hopefully, the type of methods we are developing in this paper can be extended to the non-Grey case.

Theorem C.1. *Let $\alpha_\nu(x)=\alpha(x)$ independent of ν, $\alpha\in C^3(\Omega)$, $g_\nu\geq 0$ with $\int_0^\infty g_\nu(n)\,d\nu \in L^\infty(\mathbb{S})$ in* (C.6)*, Ω bounded convex with C^3-boundary and strictly positive curvature. Let $(I_\nu^\varepsilon, T_\varepsilon)$ be the solution to the initial value problem* (C.6)*. Then there exists a functional $T_\Omega : L^\infty\left(\mathbb{S}^2, L^1(\mathbb{R}_+)\right)\to C(\partial\Omega)$ which maps g_ν to a continuous function $T_\Omega[g_\nu](p)$ on the boundary $p\in\partial\Omega$ such that*

$$T_\varepsilon(x)\to T(x)$$

uniformly in every compact subset of Ω, where T is the solution to the Dirichlet problem

$$\begin{cases} -\operatorname{div}\left(\frac{\sigma 4T^3}{\alpha}\nabla T\right)=0 & x\in\Omega,\\ T(p)=T_\Omega[g_\nu](p) & p\in\partial\Omega.\end{cases}$$

Moreover,

$$I_\nu^\varepsilon(x,n)\to B_\nu(T(x))$$

uniformly in every compact subset of Ω as a function with values in $L^\infty\left(\mathbb{S}^2, L^1(\mathbb{R}_+)\right)$.

C.1.1 Motivation and previous results

The computation of the distribution of temperature of matter interacting with radiation is an important issue in many physical application and in addition it rises interesting mathematical questions. The kinetic equation describing the interaction of matter with radiation is the radiative transfer equation. A detailed explanation of its derivation and its main properties can be found in [29,108,114,125,152]. In particular, in [108,152] there is an extensive discussion about the diffusion equilibrium approximation and the situations where this can be expected or not.

Since the earlier result by Compton [31] in 1922 the interaction of a gas with radiation has been extensively studied. Milne for example studied a simplified model, where the radiation is monochromatic and the gas density depends only on one space variable (cf. [109]).

A question which has been much studied in mathematical literature is the situation in which $\alpha_\nu^e = \alpha_\nu^a = 0$ in (C.1), i.e. the interaction between matter and radiation is due to scattering only. In this case the problem reduces to

$$\frac{1}{c}\partial_t I_\nu(x,n,t) + n\cdot\nabla_x I_\nu(x,n,t) = -\alpha_\nu^s(x) I_\nu(x,n,t) + \alpha_\nu^s(x)\int_{\mathbb{S}^2} K(n,n') I_\nu(x,n',t)dn'. \quad \text{(C.8)}$$

The same equation arises also in the study of neutron transport, a problem which has been extensively studied in mathematics.

It turns out that in the Grey approximation, i.e. when $\alpha_\nu(x) = \alpha(x)$, the problem (C.6) can be reduced exactly to the study of a particular case of neutron transport equation, namely the case when the kernel K is constant 1. Indeed, denoting by $J(x,n) = \int_0^\infty I_\nu(x,n)\, d\nu$ and combining the first two equations of (C.6) we obtain $\int_0^\infty B_\nu(T(x))\, d\nu = \fint_{\mathbb{S}^2} J(x,n)\, dn = \frac{1}{4\pi}\int_{\mathbb{S}^2} J(x,n)\, dn$. Hence, equation (C.6) is equivalent to the study of

$$\begin{cases} n\cdot\nabla_x J(x,n) = \frac{\alpha(x)}{\varepsilon}\left(\fint_{\mathbb{S}^2} J(x,n)\, dn - J(x,n)\right) & \text{if } x\in\Omega, \\ J(x,n) = \int_0^\infty g_\nu(n)\, d\nu & \text{if } x\in\partial\Omega,\ n\cdot N_x < 0. \end{cases} \quad \text{(C.9)}$$

However, the equivalence between (C.6) and (C.9) does not hold in the non-Grey case. The properties of equation (C.9) as well as the diffusion approximation limit have been studied for a long time, starting with the seminal paper [19] of 1979, where the stationary version of (C.8) was studied. In that work the authors proved the diffusion approximation for the neutron transport equation using a stochastic method. The result they obtained for J would imply in particular our main Theorem C.1. We emphasize that if in (C.6) the absorption coefficient $\alpha_\nu(x)$ has a non-trivial dependence on the frequency ν or if the neutron transport equation describes situations different from the so-called one-speed approximation, which is given by (C.9), the radiative transfer equation and the neutron transport equation would be far from equivalent from each other.

More recently, in a series of papers [76, 146–149] Yan Guo and Lei Wu have studied the diffusion approximation of both the stationary and the time dependent neutron transport equation (C.8) when $K \equiv 1$ and $\alpha_\nu^s(x) \equiv 1$, independent of x, for different classes of boundary conditions in 2 and 3 dimensions, in bounded convex domains or annuli (in 2D). In particular the result in paper [147] imply again the main Theorem C.1 when $\alpha \equiv 1$. Their proof relies on PDE methods and not on a stochastic approach. Moreover, they also computed the geometric approximation in the structure of the boundary layer.

The main goal of this paper is to develop a method which allows to obtain diffusive limit approximations like the one in Theorem C.1 for the radiative transfer equation (C.1) using PDE methods that rely only in maximum principle tools. This tools are different from those used by Guo and Wu. Specifically, the method in [76, 146–149] relies on the L^2-L^p-L^∞ estimates that were introduced for the analysis of kinetic equations by Yan Guo in [75]. In particular, the method is based on the estimates of the velocity distribution J. Our approach is based on the direct derivation of estimates for the temperature $T(x)$ associated to a given distribution of radiation $I_\nu(x,n)$. More precisely, equation (C.6) can be reformulated as a non-local integral equation for the temperature (cf. [83]). In the case of the Grey approximation we have the following equation for $u(x) = 4\pi\sigma T^4(x)$

$$u(x) - \int_\Omega K_\varepsilon(x,\eta)u(\eta)\, d\eta = S(x), \quad \text{(C.10)}$$

where the precise form of the kernel K_ε and of the source $S(x)$ are discussed in Sections 4 and 5.

Equation (C.10) can be thought as a non-local elliptic equation which in particular satisfies good properties, such as the maximum principle. Specifically, our proof relies only in finding suitable supersolutions and applying the maximum principle. The way in which we constructed these supersolutions is mimicking particular solutions of elliptic equations with constant coefficients. These supersolutions give also an insight of the behavior of the solution near the boundary $\partial\Omega$. Our hope is that the method developed in this paper could be extended to the non-Grey case, at least for some suitable choice of $\alpha_\nu(x)$. One reason why this should be possible is that [83] shows how to solve the non-local equation (C.10) for some class of non-Grey problems.

Another type of diffusion approximation for (C.1) is the one in [73,74] in which it has been considered the situation when $\alpha_\nu^s \to \infty$ while α_ν^e and α_ν^a remain bounded combined with the equation for balancing the energy either in the one dimensional case or in the whole space.

The well-posedness and the diffusion approximation of the time dependent problem (C.8) in the frame work of L^1-functions using the theory of m-accretive operators has ben studied in a series of papers [13, 16]. Seemingly, although the techniques in these papers allow to develop a theory for the time dependent problem, they do not provide information about the stationary solution.

Some versions of the stationary problem involving the radiative transfer equation can be found in [62, 63, 95, 96, 115, 138]. The problems studied in these papers include also heat conduction and different type of boundary condition of our model (for a more detailed discussion see [83]). Moreover, in [63] the authors consider the diffusive limit of a stationary radiative heat system, in which the radiative transfer equation with constant absorption coefficient and without scattering is coupled to the heat equation for the temperature. The convergence to the limit equation for this system is achieved with an $L^2 - L^\infty$ approach.

It is important to emphasize that equation (C.6) is very different in the non-Grey case from the scattering problem (C.8), in the sense that the system (C.6) provides an equation for the temperature. Specifically, the equation $\nabla_x \cdot \mathcal{F} = 0$ is automatic satisfied in the stationary version of (C.8). Physically, this is due to the fact that the radiation arriving at every point is just deflected. Equation (C.6) plays the same role as the Laplace equation in order to describe the stationary distribution of temperature in systems where the energy is transported by means of heat conduction. In the case of (C.6) the energy is transported by means of radiation which results in non-locality for determining the temperature distribution. The fact that the determination of the temperature in a body where the energy is transported by radiation is non-local was first formulated in [78]. Since the approximation (C.7) fails at the boundary, some boundary layers appears for which the intensity of radiation I_ν^ε differs from the Planck distribution $B_\nu(T)$. Hence, a careful analysis must be made for these boundary layers where the radiation is out of equilibrium. This will be essential in order to determine the functional T_Ω in Theorem C.1, which defines the temperature at every point of the boundary.

Finally, we mention that one can consider more complicated interactions between radiation and matter. For instance when the matter that interacts with radiation is a moving fluid. (cf. [69, 71, 108, 152]). The case when the interacting medium is a Boltzmann gas whose molecules can be in different energy layers has been considered in [34, 81, 114, 122].

C.1.2 Structure of the paper. Sketch of the proof. Notation

We aim to prove Theorem C.1. As first step in Section 2 we will derive the expected form of the solution using formal arguments from the theory of matched asymptotic expansions. In particular, this analysis shows that there exist thin boundary layers close to the boundary of the domain in which the radiation evolves from non-equilibrium (close to the boundary of the domain) to equilibrium (when moving towards the interior of the domain). Specifically, the intensity of radiation becomes close to the Planck distribution $B_\nu(T)$ at distances larger than ε from the boundary. In Section 3 we deal with the detailed mathematical study of this boundary layer equation which can be written as a linear integral equation in the half-line. In particular, we use as main tool Fourier analysis methods in order to prove well-posedness for this problem as well as to obtain the asymptotic behavior of the solution at points far from the boundary. Section 4 deal with the case of constant absorption coefficient $\alpha \equiv 1$. We prove that the energy densities $u_\varepsilon = T_\varepsilon^4$ converge to a harmonic function in the interior of the domain Ω as $\varepsilon \to 0$. Moreover, we can also obtain the boundary value for the limit function u using the detailed description of the boundary layer which has been developed in Section 3. Both in Section 3 and 4 we use in a fundamental way that we can reduce the problem (C.6) to a non-local elliptic equation for the temperature T. This allows us to use maximum principle arguments, something that we do extensively in Section 4. Specifically, we construct several sub- and super-solutions that we can use to estimate the functions u_ε and to characterize the boundary values of the limit function u. The way in which we obtain these boundary values is reminiscent to the barrier functions used in the Perron method for the Laplace equation. Finally, in Section 5 we extend the previous results to the case of non-constant coefficient $\alpha(x)$.

We introduce here some notation we will use throughout this paper. First of all, $\Omega \subset \mathbb{R}^3$ is an open bounded convex domain with C^3-boundary and strictly positive curvature. In order to avoid meaningless constants we assume without loss of generality that $0 \in \overline{\Omega}$. N_x indicates always the outer normal vector for a point $x \in \partial\Omega$.

We assume Ω to be convex in order to simplify some geometrical argument. First of all this assumption implies that for every point $p \in \partial\Omega$ the tangent plane to the boundary at p divided the space $\mathbb{R}^3$ in two disjoint half-spaces, one of them containing the whole domain Ω. This will be used several times in the definition for every point $p \in \partial\Omega$ of the isometric transformation mapping p to 0 and Ω in the positive half-space $\mathbb{R}_+ \times \mathbb{R}^2$. The assumption of convexity can be relaxed and the geometrical estimates should still hold, but we would need a more careful analysis of the geometry of the problem.
Moreover, for $g_\nu(n) \geq 0$ with $\int_0^\infty g_\nu(n)\, d\nu \in L^\infty\left(\mathbb{S}^2\right)$ we define the norms

$$\|g\|_1 := \int_0^\infty \int_{\mathbb{S}^2} g_\nu(n)\, d\nu\, dn \tag{C.11}$$

and

$$\|g\|_\infty := \sup_{n\in\mathbb{S}^2} \left(\int_0^\infty g_\nu(n)\, d\nu\right). \tag{C.12}$$

Remark. The reason why we are assuming the seemingly restrictive boundary condition (C.5) is because we are supposing that the source of radiation is placed at infinity. We can obtain analogous results to the one of the paper if we consider the more general boundary condition $g_\nu(n,x)$ depending also on $x \in \partial\Omega$. In addition to the assumption above we need to require $g_\nu(n,x)$ to be a C^1-function with respect to $x \in \partial\Omega$.

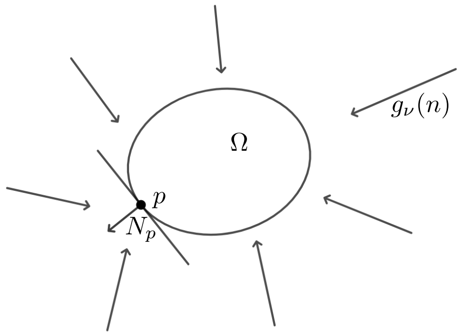

Figure C.1: Representation of the boundary value problem.

For any point $p \in \partial\Omega$ we choose a fixed isometry mapping p to 0 and the vector $p + N_p$ to $-e_1$. This rigid motion is denoted by $\mathcal{R}_p : \mathbb{R}^3 \to \mathbb{R}^3$ and can be defined as $\mathcal{R}_p(x) = \mathrm{Rot}_p(x-p)$, where $\mathrm{Rot}_p \in SO(3)$ is the chosen (and from now on fixed) rotation such that $\mathrm{Rot}_p(N_p) = -e_1$. Then the rigid motion $\mathcal{R}_p$ has the following properties:

$$\mathcal{R}_p(p) = 0 \qquad \text{and} \qquad \mathcal{R}_p(N_p + p) = -e_1. \tag{C.13}$$

Finally, we define by

$$\begin{aligned} \pi_{\partial\Omega} : \left\{x \in \mathbb{R}^3 : \mathrm{dist}(x, \partial\Omega) < \delta\right\} &\to \partial\Omega \\ x &\mapsto \pi_{\partial\Omega}(x) \end{aligned} \tag{C.14}$$

the projection to the unique closest point in the boundary $\partial\Omega$. This function is continuous and well-defined in small neighborhood of $\partial\Omega$, i.e. for $\delta > 0$ small enough.

C.2 Derivation of the limit problem

C.2.1 Formal derivation of the limit problem in the diffusive equilibrium approximation

We first remind how to obtain formally the equation for the interior in the limit problem. First of all we expand the intensity of radiation

$$I_\nu\left(x,n\right) = f_\nu^0\left(x,n\right) + \varepsilon f_\nu^1\left(x,n\right) + \varepsilon^2 ... \tag{C.15}$$

Substituting it in the first equation of (C.6) and identifying the terms containing ε^{-1} and ε^0 we see

$$f_\nu^0\left(x,n\right) = B_\nu\left(T\left(x\right)\right)$$

and

$$f_\nu^1\left(x,n\right) = -\frac{1}{\alpha_\nu(x)} n \cdot \nabla_x B_\nu\left(T\left(x\right)\right)$$

Using the second equation in (C.6) and the expansion in (C.15) we deduce

$$\begin{aligned} 0 &= \int_0^\infty d\nu \int_{\mathbb{S}^2} dn\; n \cdot \nabla_x I_\nu\left(x,n\right) \\ &= \mathrm{div}\left[\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\; n B_\nu\left(T\left(x\right)\right)\right] - \varepsilon\, \mathrm{div}\left[\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\; \left(n \otimes n\right) \frac{1}{\alpha_\nu(x)} \nabla_x B_\nu\left(T\left(x\right)\right)\right] \\ &= -\varepsilon \frac{4}{3}\pi\, \mathrm{div}\left(\left(\int_0^\infty d\nu \frac{1}{\alpha_\nu(x)} \nabla_x B_\nu\left(T\left(x\right)\right)\right)\right), \end{aligned}$$

where we used
$$\int_{\mathbb{S}^2} dn(n\otimes n)=\frac{4}{3}\pi\mathrm{Id}\qquad\text{and}\qquad\int_{\mathbb{S}^2} dn\, n=0.$$
Therefore,
$$\operatorname{div}\left(\kappa\left(T\right)\nabla_x T\right)=0,\tag{C.16}$$
where $\kappa\left(T\right):=\int_0^\infty d\nu\frac{\partial_T B_\nu(T(x))}{\alpha_\nu(x)}$. In the particular case of the Grey approximation when $\alpha_\nu(x)=1$ we have $\kappa(T)=4\sigma T^3(x)$. Then defining $u(x):=4\pi\sigma T^4(x)$ we obtain the following equation
$$\Delta u=0.$$
This is the limit problem we will study.

C.2.2 Formal derivation of boundary condition for the limit problem in the diffusive equilibrium approximation

In order to obtain the intensity of radiation closed to the boundary of Ω we derive a boundary layer equation, whose solution will be used to determine the value of the temperature at the boundary by means of a matched argument. Suppose that $x_0\in\partial\Omega$, without loss of generality we can assume $x_0=0$ and $N_{x_0}=N=-e_1$ using the rigid motion $\mathcal{R}_{x_0}$ defined in (C.13) and putting $\overline{g}_\nu(n):=g_\nu\left(\mathrm{Rot}_{x_0}^{-1}(n)\right)$. We rescale $x=\varepsilon y$, where $y\in\frac{1}{\varepsilon}\Omega$. Thus, at the leading order as $\varepsilon\to 0$ we obtain $\alpha_\nu(x)=\alpha_\nu(\varepsilon y)=\alpha_\nu(0)+\mathcal{O}(\varepsilon)$. Taking $\varepsilon\to 0$ we obtain that the intensity of radiation satisfies
$$\begin{cases} n\cdot\nabla_y I_\nu\left(y,n\right)=\alpha_\nu(0)\left(B_\nu\left(T\left(y\right)\right)-I_\nu\left(y,n\right)\right) & y\in\mathbb{R}_+\times\mathbb{R}^2\\ \nabla_y\cdot\mathcal{F}=0 & y\in\mathbb{R}_+\times\mathbb{R}^2\\ I_\nu\left(y,n\right)=\overline{g}_\nu\left(n\right) & y\in\{0\}\times\mathbb{R}^2 \text{ and } n\cdot N<0\end{cases}\tag{C.17}$$
This problem is also known in the literature as Milne problem. We will now derive an equivalent formulation as a non-local integral elliptic equation for the temperature T. To this end we solve the first equation in (C.17) for I_ν using the method of characteristics. Given $y\in\mathbb{R}_+\times\mathbb{R}^2$ and $n\in\mathbb{S}^2$ with $n\cdot N<0$ we call $Y(y,n)$ the unique point belonging to $\partial\left(\mathbb{R}_+\times\mathbb{R}^2\right)=\{0\}\times\mathbb{R}^2$ such that
$$y=Y(y,n)+s(y,n)n,$$
where $s(y,n)=|y-Y(y,n)|$. Notice that $s(y,n)$ is the distance to the first intersection point of the boundary $\{0\}\times\mathbb{R}^2$ with the half line $\{y-tn\,:\,t>0\}$. For $n\cdot N\geq 0$ we define $s(y,n)=\infty$. Solving the equation by characteristics we obtain
$$I_\nu\left(y,n\right)=\overline{g}_\nu(n)e^{-\alpha_\nu(0)s(y,n)}\chi_{n\cdot N<0}+\int_0^{s(y,n)} e^{-\alpha_\nu(0)t}\alpha_\nu(0)B_\nu\left(T\left(y-tn\right)\right)dt.$$
Using the second equation in the rescaled problem (C.17) we calculate
$$\begin{aligned}0=&\operatorname{div}\left[\int_0^\infty d\nu\int_{n\cdot N<0} dn\, n\overline{g}_\nu(n)e^{-\alpha_\nu(0)s(y,n)}\right.\\ &\left.+\int_0^\infty d\nu\int_{\mathbb{S}^2}dn\int_0^{s(y,n)}dt\, ne^{-\alpha_\nu(0)t}\alpha_\nu B_\nu\left(T\left(y-tn\right)\right)\right]\\ =&-\int_0^\infty d\nu\int_{n\cdot N<0}dn\,\overline{g}_\nu(n)\alpha_\nu n\cdot\nabla_y s(y,n)e^{-\alpha_\nu(0)s(y,n)}\\ &+\operatorname{div}\left(\int_0^\infty d\nu\int_{\mathbb{R}_+\times\mathbb{R}^2}d\eta\,\frac{y-\eta}{|y-\eta|^3}e^{-\alpha_\nu(0)|y-\eta|}\alpha_\nu(0)B_\nu\left(T\left(\eta\right)\right)\right)\end{aligned}\tag{C.18}$$

$$= -\int_0^\infty d\nu \int_{n\cdot N<0} dn\, \overline{g}_\nu(n)\alpha_\nu(0)e^{-\alpha_\nu(0)s(y,n)} + 4\pi \int_0^\infty d\nu(0)\ \alpha_\nu B_\nu\left(T\left(y\right)\right)$$
$$-\int_0^\infty d\nu \int_{\mathbb{R}_+\times\mathbb{R}^2} d\eta\ \frac{\alpha_\nu^2(0)}{|y-\eta|^2} e^{-\alpha_\nu(0)|y-\eta|} B_\nu\left(T\left(\eta\right)\right).$$

The second equality holds via the spherical change of variable

$$\mathbb{S}^2\times\mathbb{R}_+ \to \mathbb{R}_+\times\mathbb{R}^2$$
$$(n,t) \mapsto \eta = y - tn$$

so that $n = \frac{y-\eta}{|y-\eta|}$. For the third inequality we use on the one hand that $\operatorname{div}_y\left(\frac{y-\eta}{|y-\eta|^3}\right) = 4\pi\delta(y-\eta)$ and on the other hand that $n\cdot\nabla_y s(y,n) = 1$. The latter can be seen by the fact that

$$Y(y,n) + s(y+tn,n)n = y + tn = Y(y+tn,n) + s(y+tn,n)n.$$

This implies that $Y(y+tn,n)$ is t-constant and therefore $1 = \partial_t s(y+tn,n) = (\nabla_y s(y+tn,n))\cdot n$. We assume now that the temperature depends only on the first variable. This can be expected because we are considering limits for $\varepsilon \ll 1$ and hence the temperature can be considered to depend only on the distance to the point x_0, which is approximated by the first variable in this setting. After the change of variables $\xi = (y_2+\eta_2, y_3-\eta_3)$ and calling $y-\eta := y_1-\eta_1$ the last integral in (C.18) can be written as

$$\int_0^\infty d\nu \int_{\mathbb{R}_+} d\eta \int_{\mathbb{R}^2} d\xi\ \alpha_\nu^2(0)\frac{e^{-\alpha_\nu(0)\sqrt{(y-\eta)^2+|\xi|^2}}}{(y-\eta)^2+|\xi|^2} B_\nu\left(T\left(\eta\right)\right).$$

Using polar coordinates we obtain

$$\begin{aligned}\int_{\mathbb{R}^2} d\xi\ \frac{e^{-\alpha_\nu(0)\sqrt{(y-\eta)^2+|\xi|^2}}}{(y-\eta)^2+|\xi|^2} =& \pi \int_{|y-\eta|^2}^\infty dx\ \frac{e^{-\alpha_\nu(0)\sqrt{x}}}{x} \\ =& 2\pi \int_{\alpha_\nu(0)|y-\eta|}^\infty dt\ \frac{e^{-t}}{t} = 4\pi K(\alpha_\nu(0)|y-\eta|),\end{aligned} \tag{C.19}$$

where we will denote $K(x) = \frac{1}{2}\int_{|x|}^\infty dt\ \frac{e^{-t}}{t}$ as the normalized exponential integral.

Notice that $s(y,n) = \frac{y_1}{|n\cdot N|}$ if $n\cdot N<0$. We can summarize the equation the temperature satisfies in the non-Grey approximation as follows

$$\begin{aligned}&\int_0^\infty d\nu\ \alpha_\nu(0)B_\nu\left(T(y_1)\right) - \int_0^\infty d\nu \int_0^\infty d\eta\ \alpha_\nu^2(0)K\left(\alpha_\nu(0)\left|y_1-\eta_1\right|\right)B_\nu\left(T\left(\eta_1\right)\right) \\ =& \int_0^\infty d\nu \int_{n\cdot N<0} dn\ \overline{g}_\nu(n)\alpha_\nu(0)e^{-\alpha_\nu(0)\frac{y_1}{|n\cdot N|}}.\end{aligned} \tag{C.20}$$

In the particular case of the Grey approximation when $\alpha\equiv 1$ using that $u(y) = 4\pi\sigma T^4(y)$ we can simplify equation (C.20) by property (C.3)

$$u(y_1) - \int_0^\infty d\eta\ K(y_1-\eta)u(\eta) = \int_0^\infty d\nu \int_{n\cdot N<0} dn\ \overline{g}_\nu(n)e^{-\frac{y_1}{|n\cdot N|}}. \tag{C.21}$$

In some occasions, when the dependence of the boundary layer function u on the point $p\in\partial\Omega$ is needed, we will use the notation $\overline{u}(y_1,p)$, where this function solves according to the rigid motion $\mathcal{R}_p$ in (C.13)

$$\overline{u}(y_1,p) - \int_0^\infty d\eta\ K(y_1-\eta)u(\eta,p) = \int_0^\infty d\nu \int_{n\cdot N_p<0} dn\ g_\nu(n)e^{-\frac{y_1}{|n\cdot N_p|}}. \tag{C.22}$$

For the rest of Section 2 and Section 3 we will focus on the study of $\overline{u}(y_1,p)$ for an arbitrary given $p\in\partial\Omega$, hence we will call $u(y_1)=\overline{u}(y_1,p)$ and $N=N_p$. In order to simplify the reading from now on we set $G(x)=\int_0^\infty d\nu\int_{n\cdot N<0}dn\,\overline{g}_\nu(n)e^{-\frac{x}{|n\cdot N|}}\chi_{\{x>0\}}$ and if we want to stress out the dependence on $p\in\partial\Omega$ we write $G_p(x)=\int_0^\infty d\nu\int_{n\cdot N_p<0}\,g_\nu(n)e^{-\frac{x}{|n\cdot N_p|}}\chi_{\{x>0\}}$.

From now on until Section 5 we consider the case of constant absorption coefficient $\alpha\equiv 1$.

C.2.3 Some properties of the kernel

We consider the kernel K introduced in Section 2.2. We remark that $K(x)=\frac{1}{2}E_1(|x|)$, where E_1 is the standard exponential integral function. See [1]. We collect some properties of the normalized exponential integral.

Proposition C.1. *The function K satisfies $\int_{-\infty}^\infty dx\,K(x)=1$, $K\in L^1(\mathbb{R})\cap L^2(\mathbb{R})$ and the following estimate holds*

$$\frac{1}{4}e^{-|x|}\ln(1+\frac{2}{|x|})\leq K(x)\leq\frac{1}{2}e^{-|x|}\ln(1+\frac{1}{|x|}).$$

Moreover, the Fourier transform of K is $\hat{K}(\xi)=\frac{1}{\sqrt{2\pi}}\frac{\arctan(\xi)}{\xi}$.

Proof. Since K is even and non negative we can calculate, applying Tonelli's Theorem

$$\int_{-\infty}^\infty K(s)\,ds=2\int_0^\infty K(s)\,ds=\int_0^\infty\int_s^\infty\frac{e^{-t}}{t}\,dt\,ds=\int_0^\infty\frac{e^{-t}}{t}\int_0^t ds\,dt=\int_0^\infty e^{-t}\,dt=1.$$

This proves also that $K\in L^1(\mathbb{R})$.

For the square integrability we refer to equation 5.1.33 in [1] and see $\int_{\mathbb{R}}|K(x)|^2\,dx=\ln(2)$. Estimate 5.1.20 in [1] also implies $\frac{1}{4}e^{-|x|}\ln(1+\frac{2}{|x|})\leq K(x)\leq\frac{1}{2}e^{-|x|}\ln(1+\frac{1}{|x|})$.

We now move to the computation of the Fourier transform of the kernel K. The kernel is an even function, hence we compute

$$\begin{aligned}\hat{K}(\xi)=&\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-i\xi x}K(x)\,dx=\frac{1}{\sqrt{2\pi}}\int_0^\infty\cos(\xi x)\int_x^\infty\frac{e^{-t}}{t}\,dt\,dx\\=&\frac{1}{\sqrt{2\pi}}\frac{1}{\xi}\int_0^\infty\frac{e^{-t}}{t}\sin(\xi t)\,dt=\frac{1}{\sqrt{2\pi}}\frac{\arctan(\xi)}{\xi}.\end{aligned}$$

The last identity can be justified noticing that $F(\xi)=\int_0^\infty\frac{e^{-t}}{t}\sin(\xi t)\,dt$ has derivative $F'(\xi)=\frac{1}{\xi^2+1}$. ☐

The following calculation will also be very useful in the next section.

Proposition C.2. *Let $x>0$. Then we can compute*

$$\int_{-x}^\infty K(s)\,ds=1-\frac{e^{-x}}{2}+xK(x);\tag{C.23}$$

$$\int_x^\infty K(s)\,ds=\frac{e^{-x}}{2}-xK(x);\tag{C.24}$$

$$\int_{-x}^\infty sK(s)\,ds=\int_x^\infty sK(s)\,ds=\frac{xe^{-x}}{4}+\frac{e^{-x}}{4}-\frac{x^2}{2}K(x);\tag{C.25}$$

Proof. The proof relies on basic integral computations. We have to compute several integrals changing the order of integration applying Tonelli's Theorem and integrating by parts. We assume $x > 0$. We prove only (C.23), since all other formulas can be obtained in a similar way.

$$\begin{aligned}\int_{-x}^{\infty} K(s)\, ds =&\frac{1}{2}\int_{-x}^{0}\int_{|s|}^{\infty}\frac{e^{-t}}{t}\,dt\,ds+\frac{1}{2}\int_{0}^{\infty}\int_{s}^{\infty}\frac{e^{-t}}{t}\,dt\,ds\\ =&\frac{1}{2}\int_{0}^{x}\frac{e^{-t}}{t}\int_{0}^{t}\,ds\,dt+\frac{1}{2}\int_{x}^{\infty}\frac{e^{-t}}{t}\int_{0}^{x}\,ds\,dt+\frac{1}{2}\\ =&1-\frac{e^{-x}}{2}+xK(x).\end{aligned}$$

□

C.3 The boundary condition for the limit problem

We now start with the boundary layer analysis. This boundary layer problem, know in the literature as Milne problem, was studied with different approaches, e.g. [13,16,33,68,79]. We will present another proof of the boundary layer analysis for the equation (C.21) of the temperature, which is equivalent to the Milne problem (C.17). The proof uses a combination of comparison arguments and Fourier analysis. In addition, instead of considering the intensity of radiation the analysis is made directly for the temperature.

Our aim is now to solve equation (C.21). Indeed, according to the method of matched asymptotic expansions we expect the boundary condition for the limit problem to be the limit of u as $y \to \infty$ for every point $x \in \partial\Omega$. In order to simplify the notation we call $\mathcal{L}(u)(x) := u(x) - \int_0^\infty dy\, K(x-y)\, u(y)$ and $\overline{\mathcal{L}}(u)(x) := u(x) - \int_{-\infty}^\infty dy\, K(x-y)\, u(y)$.

C.3.1 The homogeneous equation

We start with the study of the homogeneous equation, i.e. (C.21) with $G(x) \equiv 0$. We will show using maximum principle that any bounded solution is the trivial solution $u \equiv 0$. We will use the following version of the maximum principle for the non-local operator $\mathcal{L}$.

Lemma C.1. *Let $\overline{u} \in C([0,\infty))$ with $\lim_{x\to\infty} \overline{u}(x) \in [0,\infty]$ be a supersolution of* (C.21), *i.e.*

$$\begin{cases} \overline{u}(x) - \int_0^\infty dy\, K(x-y)\,\overline{u}(y) \geq 0 & x > 0\\ \overline{u}(x) = 0 & x < 0\end{cases}$$

Then $u \geq 0$ for all $x \geq 0$.

Proof. Let us assume the contrary, i.e. that there exists some $x \in [0,\infty]$ such that $\overline{u}(x) < 0$. By assumption $x \in [0,\infty)$. Since $\overline{u}$ is continuous in $[0,\infty)$ and it has non-negative limit at infinity which is bounded or infinity, u attains its global minimum in $[0,\infty)$, i.e. there exists some $x_0 \in [0,\infty)$ such that $\overline{u}(x_0) = \inf_{x\in[0,\infty)} \overline{u}(x) < 0$. Since $\overline{u}$ is a super solution we can calculate

$$\begin{aligned}0 \leq&\mathcal{L}(\overline{u})(x_0) = \overline{u}(x_0) - \int_0^\infty dy\, K(x_0-y)\,\overline{u}(y)\\ =&\int_{-\infty}^\infty dy\, K(x_0-y)\,\overline{u}(x_0) - \int_0^\infty dy\, K(x_0-y)\,\overline{u}(y)\\ =&\int_{-\infty}^0 dy\, K(x_0-y)\,\overline{u}(x_0) + \int_0^\infty dy\, K(x_0-y)\,(\overline{u}(x_0)-\overline{u}(y)) < 0,\end{aligned}$$

where we used the positivity of $K(x_0 - y)$, the fact that the integral of the kernel K is 1 and the fact that $\overline{u}(x_0)$ is the minimum of $\overline{u}$ and it is strictly negative. This leads to a contradiction and thus we conclude the proof. □

With the maximum principle we can now show the following theorem on the triviality of the solution to the homogeneous equation.

Theorem C.2. *Assume u is a bounded solution to*

$$\overline{\mathcal{L}}(u)(x) = 0 \tag{C.26}$$

with $u(x) \equiv 0$ for $x < 0$. Then $u = 0$ for almost every $x \in \mathbb{R}$.

Proof. We will construct a supersolution $\overline{u}$ which converges to infinity and we will apply Lemma C.1 to the supersolutions $\overline{u} - u$ and $u + \overline{u}$. First of all we see that for $x > 0$ the bounded solution u is continuous, indeed $u(x) = K * u(x)$. Since $K \in L^1(\mathbb{R})$ and $u \in L^\infty(\mathbb{R})$ then the convolution is a continuous bounded function. Moreover we can extend continuously u in 0. Indeed, we define

$$u(0) = \lim_{x\to 0}\left[G(x) + \int_0^\infty dy\, K(x-y)\, u(y)\right].$$

This limit exists because G is continuous in $[0,\infty)$ and for the integral term we can apply the generalized dominated convergence theorem using that the sequence $K(x-y) \to K(y)$ as $x \to 0$ pointwise and in $L^1(\mathbb{R})$.

We consider now the function

$$\overline{u}(x) = \begin{cases} 1+x & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$\overline{u}$ is a supersolution. It is indeed possible to calculate $\mathcal{L}(\overline{u})(x)$. Let $x \geq 0$. Then $\mathcal{L}(\overline{u}) = \mathcal{L}(Id) + \mathcal{L}(1)$. By a simple calculation we get on the one hand

$$\mathcal{L}(Id)(x) = x - \int_0^\infty dy\, K(x-y)\, y = x - \int_{-x}^\infty dy\, K(y)\,(x+y) = \frac{x}{4}e^{-x} - \frac{e^{-x}}{4} - \frac{x^2}{2}K(x)$$

and on the other hand

$$\mathcal{L}(1)(x) = 1 - \int_0^\infty dy\, K(x-y) = 1 - \int_{-x}^\infty dy\, K(y) = \frac{e^{-x}}{2} - xK(x).$$

Therefore we want to show that the function $f(x) := \mathcal{L}(\overline{u})(x) = \frac{e^{-x}}{4}(1+x) - \frac{x}{2}K(x)(2+x)$ is non-negative for all $x \geq 0$. It is not difficult to see that $f(0) = \frac{1}{4} > 0$ and that $\lim_{x\to\infty} f(x) = 0$. Moreover, we can consider the derivative

$$f'(x) = \frac{1}{2}\left(e^{-x} - K(x)(2x+2)\right) \leq \frac{1}{2}\left(e^{-x} - \frac{e^{-x}}{2}\ln\left(1+\frac{2}{x}\right)(x+1)\right) \leq 0.$$

The first inequality is given by the estimate of Proposition C.1 and the second one is due to the well-know estimate $\ln(1+x) \geq \frac{2x}{2+x}$. The non-positivity of the derivative implies that f is monotonously decreasing, and therefore $\mathcal{L}(\overline{u})(x) = f(x) \geq 0$ for all $x \geq 0$.

Let now $\varepsilon > 0$ arbitrary. We know that u is bounded and $\overline{u}$ converges to infinity, moreover both u and $\overline{u}$ are continuous in $[0,\infty)$. Also u is a homogeneous solution of (C.21) and the operator $\mathcal{L}$ is linear. Therefore we can apply Lemma C.1 to the supersolutions $\varepsilon\overline{u} - u$ and $u + \varepsilon\overline{u}$ and get that the $\inf_{x\in[0,\infty)}[\varepsilon\overline{u}(x) - u(x)] \geq 0$ and $\inf_{x\in[0,\infty)}[\varepsilon\overline{u}(x) + u(x)] \geq 0$. This implies that for any $x \in \mathbb{R}$ the following holds

$$-\varepsilon\overline{u}(x) \leq u(x) \leq \varepsilon\overline{u}(x)$$

Since ε was arbitrary we conclude $u(x) = 0$ for all $x \in \mathbb{R}$. □

C.3.2 Well-posedness theory for the inhomogeneous equation

We can now move to the well-posedness theory for the inhomogeneous equation, for which the next theorem is the main result.

Theorem C.3. *Let $H : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function bounded by an exponential function, i.e. $|H(x)| \leq Ce^{-Ax}\chi_{\{x>0\}}$ for $C, A > 0$. Then there exists a unique bounded solution to the equation*

$$\begin{cases} u(x) - \int_0^\infty dy\, K\left(x-y\right) u(y) = H(x) & x > 0, \\ u(x) = 0 & x < 0. \end{cases} \tag{C.27}$$

Moreover, u is continuous on $(0, \infty)$.

Proof. The assumption on the exponential decay of H yields $H \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$. In order to find a bounded solution for (C.27) we will follow several steps. We will look for functions $\tilde{u}$ and v solutions of the following equations

$$\tilde{u}(x) - \int_{-\infty}^{\infty} K(x-y)\tilde{u}(y)\, dy = \overline{H}(x) := H(x) - H(-x) \qquad x \in \mathbb{R}$$

and

$$\begin{cases} v(x) - \int_{-\infty}^\infty K(x-y)v(v)\, dy = 0, & x > 0 \\ v(x) = -\tilde{u}(x) & x < 0. \end{cases} \tag{C.28}$$

Then $u = \tilde{u} + v$ will be the desired solution.

Step 1: Construction of $\tilde{u}$.
We can construct the solution $\tilde{u}$ via Fourier method. First of all we notice that any affine function is a solution to the homogeneous equation in the whole space $\mathbb{R}$. This is because $\int_{-\infty}^\infty K(x)\, dx = 1$ and $\int_{-\infty}^\infty xK(x)\, dx = 0$. Since by assumption $H \in L^2(\mathbb{R})$ also $\overline{H} \in L^2(\mathbb{R})$. We define for an integrable function f the kth-moment as $m_k(f) = \int_{-\infty}^\infty x^k f(x)\, dx$ assuming it exists. Then clearly by construction $m_0\left(\overline{H}\right) = 0$ while $m_1\left(\overline{H}\right) = \frac{2C}{A} > 0$. Moreover, since $\overline{H}$ has exponential decay, all moments $m_k\left(\overline{H}\right) < \infty$ are bounded.

We define also the function $F(x) = \overline{\mathcal{L}}\left(\frac{3}{2}\text{sgn}\right)(x)$. It can be compute that

$$F(x) = \frac{3}{2}\left(\text{sgn}(x) - \int_{-\infty}^{\infty} K(x-y)\text{sgn}(y)\, dy\right) = \frac{3}{2}\left(\text{sgn}(x) - \int_{-x}^{x} K(y)\, dy\right).$$

It is not difficult to see that $F(0) = 0$, $\lim\limits_{|x|\to\infty} F(x) = 0$ and that F is a stepwise continuous function with the discontinuity in 0. Therefore $F(x)$ is bounded. We proceed with the construction of $\tilde{u}$. We can write it as $\tilde{u} = u^{(1)} + u^{(2)} + a + bx$, where $u^{(1)}(x) = m_1\left(\overline{H}\right)\frac{3}{2}\text{sgn}(x)$ solves the equation

$$\mathcal{L}\left(u^{(1)}\right)(x) = m_1\left(\overline{H}\right)F(x) \qquad x \in \mathbb{R}$$

and $u^{(2)}$ solves

$$\mathcal{L}\left(u^{(2)}\right)(x) = \overline{H}(x) - m_1\left(\overline{H}\right)F(x) \qquad x \in \mathbb{R}. \tag{C.29}$$

applying now the Fourier transform to the equation (C.29), recalling the convolution rule and the Fourier transforms of the kernel K and of the sgn function we get first in distributional sense

$$\hat{u}^{(2)}(s)\left(\frac{s - \arctan(s)}{s}\right) = \mathcal{F}(\overline{H})(s) + \frac{3m_1\left(\overline{H}\right)}{\sqrt{2\pi}}\frac{i}{s}\frac{s-\arctan(s)}{s}. \tag{C.30}$$

The Fourier transform of $\overline{H}$ is in C^∞, since $\overline{H}$ has exponential decay and therefore it has all kth-moment finite. Therefore there exists a function $\tilde{H}$ such that $\tilde{H}(0) = \tilde{H}'(0) = 0$ such that $\mathcal{F}(\overline{H})(s) = -\frac{i}{\sqrt{2\pi}} m_1\left(\overline{H}\right) s + \tilde{H}(s)$, since $m_0\left(\overline{H}\right) = 0$ and by definition $\mathcal{F}(\overline{H})'(s)\big|_{s=0} = -\frac{i}{\sqrt{2\pi}} m_1\left(\overline{H}\right)$. We can therefore find first formally $u^{(2)}$ analyzing its Fourier transform

$$\begin{aligned}\hat{u}^{(2)}(s) =& \frac{s}{s-\arctan(s)}\mathcal{F}(\overline{H})(s) + \frac{3m_1\left(\overline{H}\right)}{\sqrt{2\pi}}\frac{i}{s}\\ =& -\frac{is^2}{s-\arctan(s)}\frac{m_1\left(\overline{H}\right)}{\sqrt{2\pi}} + \frac{3m_1\left(\overline{H}\right)}{\sqrt{2\pi}}\frac{i}{s} + \tilde{H}(s)\frac{s}{s-\arctan(s)}\\ =& \mathbb{H}(s).\end{aligned} \tag{C.31}$$

It is important to notice that $\lim\limits_{s\to 0} \frac{s^2}{s-\arctan(s)} - \frac{3}{s} = 0$, since $\frac{s}{s-\arctan(s)} = \frac{3}{s^2} + \frac{9}{5} + O(s^2)$ near zero. Using L'Hôpital rule we see also that $\lim\limits_{s\to 0} \tilde{H}(s)\frac{s}{s-\arctan(s)}$ is finite. On the other hand $\frac{s}{s-\arctan(s)}$ is bounded for $|s| > 1$. Since $\mathcal{F}(\overline{H})(s)$ and $\frac{1}{s}$ are both square integrable functions and since $\mathbb{H}$ is bounded near 0 we conclude that $\mathbb{H} \in L^2(\mathbb{R})$. Therefore also the in (C.31) defined $\hat{u}^{(2)}$ is square integrable. We can hence invert it

$$u^{(2)}(x) := \mathcal{F}^{-1}(\mathbb{H})(x) \in L^2(\mathbb{R}).$$

Since this function solves (C.30) not only in distributional sense but also pointwise almost everywhere, we can conclude rigorously that indeed the function in (C.31) is the desired $u^{(2)}$ solving (C.29). Moreover, $u^{(2)} = K * u^{(2)} + \overline{H} - F$ and since both K and $u^{(2)}$ itself are square integrable and both H and F are bounded, then also $u^{(2)}$ is bounded. We can conclude this step therefore defining

$$\tilde{u}(x) = \frac{3}{2} m_1\left(\overline{H}\right)\operatorname{sgn}(x) + a + bx + u^{(2)}(x). \tag{C.32}$$

Step 2: Construction of v.

We recall that the equation v shall solve (C.28). As we found out in the first step, $\tilde{u} = \frac{3}{2}m_1\left(\overline{H}\right)\operatorname{sgn}(x) + a + bx + u^{(2)}(x)$. As we already pointed out, affine solutions are always solution of the homogeneous equation in the whole space $\mathbb{R}$. Therefore, we shall look for a function of the form

$$v(x) = \frac{3}{2} m_1\left(\overline{H}\right) - a - bx + v^{(2)}(x) \tag{C.33}$$

where $v^{(2)}$ solves similarly as above

$$\begin{cases} v^{(2)}(x) - \int_{-\infty}^{\infty} K(x-y)v(v)\,dy = 0 & x > 0,\\ v^{(2)}(x) = -u^{(2)}(x) & x < 0.\end{cases} \tag{C.34}$$

We proceed now iteratively constructing the desired solution. We call $B > 0$ the constant such that $\left\|u^{(2)}\right\|_\infty \leq B$ and we define $\overline{v} = B$ and $\underline{v} = -B$. Inductively we define $v_0 := \underline{v}$ and for $k \geq 1$ we set

$$v_k(x) = \begin{cases} -u^{(2)}(x) & x < 0,\\ \int_{-\infty}^{\infty} K(x-y)\,v_{k-1}(y)\,dy & x > 0.\end{cases}$$

We claim that $\underline{v} = v_0 \leq v_1 \leq v_2 \leq ... \leq v_k \leq v_{k+1} \leq ...$ and that $v_k \leq \underline{v}$ for all $k \in \mathbb{N}$. Clearly for $k = 0$ both statements hold. On the one hand since $\int_{-\infty}^{\infty} K(x-y)\,v_0(y) = -B$ we see that

$$v_1(x) - v_0(x) = \begin{cases} -u^{(2)}(x) + B \geq 0 & x < 0,\\ 0 & x > 0,\end{cases}$$

on the other hand per definition we have $\overline{v} - \underline{v} = 2B \geq 0$. We see also that $\overline{v} - v_1 \geq 0$, indeed

$$\overline{v}(x) - v_1(x) = \begin{cases} B + u^{(2)}(x) \geq 0 & x < 0, \\ 2B & x > 0. \end{cases}$$

We now prove inductively that $v_k \geq v_{k-1}$ and $\overline{v} \geq v_k$. Hence, we assume that these inequalities are satisfied for k and we prove them for $k+1$. Indeed this just follows from the identities

$$v_{k+1}(x) - v_k(x) = \begin{cases} 0 & x < 0, \\ \int_{-\infty}^{\infty} K(x-y)\left(v_k(y) - v_{k-1}(y)\right) \geq 0 & x > 0, \end{cases}$$

$$\overline{v}(x) - v_{k+1}(x) = \begin{cases} B + u^{(2)}(x) \geq 0 & x < 0, \\ \int_{-\infty}^{\infty} K(x-y)\left(B - v_k(y)\right) \geq 0 & x > 0, \end{cases}$$

where we used again that the integral in the whole line of the kernel K is 1. Therefore the sequence $v_k(x)$ is increasing and bounded. This means that there exists a pointwise limit. By the dominated convergence theorem and by construction this will be also the desired solution of (C.34), i.e.

$$v^{(2)}(x) := \lim_{k\to\infty} v_k(x)$$

solves the equation (C.34) and it is by construction bounded.

Step 3: Properties of u.

Now we are ready to write down the whole solution. As we remarked at the beginning $u = \tilde{u} + v$, where $\tilde{u}$ solves as in Step 1 (C.3) and v solves as in Step 2 (C.28). Therefore by (C.33) and by (C.32)

$$u(x) = \begin{cases} 6m_1(H) + u^{(2)}(x) + v^{(2)}(x) & x > 0, \\ 0 & x < 0, \end{cases}$$

solves the initial problem (C.27) and it is by construction bounded. Moreover, since K is integrable and H is continuous in $[0,\infty)$ also $u = K * u + H$ is continuous in $[0,\infty)$.

Step 4: Uniqueness.

Let us assume that u_1 and u_2 are two bounded solution to the problem (C.27). Then $u_1 - u_2$ will be a bounded continuous solution to the homogeneous problem (C.26). Therefore by Theorem C.2 $u_1 - u_2 = 0$. Hence, there exists a unique bounded solution u to the inhomogeneous problem (C.27). This concludes the proof. □

Corollary C.1. *Let $p \in \partial\Omega$ and $G_p(x)$ as defined in* (C.22). *Let $g_\nu(n) \geq 0$ and assume that $\int_0^\infty d\nu\, g_\nu(n) \in L^\infty\left(\mathbb{S}^2\right)$. Then there exists a unique bounded solution to the equation*

$$\begin{cases} u(x) - \int_0^\infty dy\, K(x-y)\, u(y) = G_p(x) & x > 0, \\ u(x) = 0 & x < 0. \end{cases} \tag{C.35}$$

Moreover, u is continuous on $(0,\infty)$.

Proof. By assumption G_p is continuous for $x > 0$ and $|G_p(x)| \leq \|g\|_1 e^{-y}\chi_{\{x>0\}}$. Hence we can apply Theorem C.3. □

It is also possible to show, that the bounded solution u is non-negative

Lemma C.2. *Let u be the unique bounded solution to* (C.35)*. Then $u(x) \geq 0$ for all $x \in \mathbb{R}$.*

Proof. The proof is very similar to the proof of Theorem C.2. We consider the supersolution

$$\overline{u}(x) = \begin{cases} 1+x & x \geq 0 \\ 0 & x < 0 \end{cases}.$$

As we have seen before, $u = K * u + G$ is continuous in $[0, \infty)$. Moreover, since $G > 0$ as $x \geq 0$, u is a supersolution too. Let now $\varepsilon > 0$ be arbitrary. Let us consider the supersolution $\varepsilon\overline{u} + u$. This is continuous in $[0, \infty)$ and since u is bounded it converges to infinity as $x \to \infty$. Therefore Lemma C.1 implies that there exists no $x_0 \in [0, \infty)$ such that

$$\inf_{x \in [0,\infty)} (\varepsilon\overline{u}(x) + u(x)) = \varepsilon\overline{u}(x_0) + u(x_0) < 0.$$

Hence $u \geq -\varepsilon\overline{u}$ and since $\varepsilon > 0$ was arbitrary we conclude $u \geq 0$. □

Remark. Theorem C.3 can be proved also using the Wiener-Hopf solution formula for the problem (C.21) as given in [77]. It is true that in this way one obtains an explicit formula, which not only assures the well-posedness of the planar problem we are studying but also directly shows the existence of a limit for the solution u when $x \to \infty$. However, the Wiener-Hopf method produces a complicate formula which requires a careful analysis with complex variables in order to be understood. We have preferred to use this soft method approach which in particular allows us to prove some relevant properties of the solution, such as the positivity.

C.3.3 Asymptotic behavior of the bounded solution of the inhomogeneous equation

We were able to show that the equation for the boundary value in the Grey approximation has a unique bounded solution which is positive whenever $G > 0$. As we anticipated at the beginning of this section, we would like to study the limit as $x \to \infty$ of the solution $u(x)$. We will show, that such limit exists and is uniquely characterized by $g_\nu(n)$ and N. To this end we first prove that the function u is uniformly continuous.

Lemma C.3. *Let u be the unique bounded solution to the problem* (C.21)*. The u is uniform continuous on* $[0, \infty)$ *and it satisfies for* $x, y \in [0, \infty)$

$$\begin{aligned} |u(x) - u(y)| &\leq |G(x) - G(y)| \\ &+ \|u\|_\infty \left[\frac{|e^{-x} - e^{-y}|}{2} + 2\left(1 - e^{\frac{|x-y|}{2}}\right) + 4\left|\frac{y-x}{2}\right| K\left(\frac{y-x}{2}\right) + |xK(x) - yK(y)| \right]. \end{aligned} \tag{C.36}$$

Proof. This is a consequence of the uniform continuity of G and $xK(x)$. Clearly, since u solves the problem (C.35), we have the estimate

$$|u(x) - u(y)| \leq |G(x) - G(y)| + \int_0^\infty |K(\eta - x) - K(\eta - y)| \, u(\eta) \, d\eta. \tag{C.37}$$

Since G is continuous on $[0, 1]$, and therefore uniformly continuous on $[0, 1]$ and since G is Lipschitz continuous in $[1, \infty)$, G is uniform continuous in $[0, \infty)$. The latter affirmation is true, since

$$\sup_{x \geq 1} |G'(x)| \leq \int_0^\infty d\nu \int_{n \cdot N < 0} dn \, g_\nu(n) \frac{e^{-\frac{1}{|n \cdot N|}}}{|n \cdot N|} < \infty,$$

where the finiteness is due to the fact that $\lim_{|n\cdot N|\to 0} \frac{e^{-\frac{1}{|n\cdot N|}}}{|n\cdot N|} = 0$.

For the integral term in (C.37) we assume that $x < y$. Then we can calculate using the fact that for positive arguments the kernel K is decreasing

$$\begin{aligned}
&\int_0^\infty |K(\eta - x) - K(\eta - y)|\, u(\eta)\, d\eta \\
&= \int_0^{\frac{x+y}{2}} (K(\eta - x) - K(\eta - y))\, u(\eta)\, d\eta + \int_{\frac{x+y}{2}}^\infty (K(\eta - y) - K(\eta - x))\, u(\eta)\, d\eta \\
&\le \|u\|_\infty \left[\int_0^{\frac{x+y}{2}} (K(\eta - x) - K(\eta - y))\, d\eta + \int_{\frac{x+y}{2}}^\infty (K(\eta - y) - K(\eta - x))\, d\eta \right]
\end{aligned}$$

We can calculate explicitly the last two integrals using the result of Proposition C.2, indeed by a change of variable

$$\begin{aligned}
&\int_0^\infty |K(\eta - x) - K(\eta - y)|\, u(\eta)\, d\eta \\
&\le \|u\|_\infty \left[\int_{-x}^{\frac{y-x}{2}} K(\eta)\, d\eta - \int_{-y}^{\frac{x-y}{2}} K(\eta)\, d\eta + \int_{\frac{x-y}{2}}^\infty K(\eta)\, d\eta - \int_{\frac{y-x}{2}}^\infty K(\eta)\, d\eta \right] \\
&= \|u\|_\infty \left[\frac{e^{-y} - e^{-x}}{2} + 2\left(1 - e^{\frac{x-y}{2}}\right) + 4\frac{y-x}{2} K\left(\frac{y-x}{2}\right) + xK(x) - yK(y) \right].
\end{aligned}$$

Recalling that $x < y$ we get the estimate (C.36). From the well-known estimates $|e^{-x} - e^{-y}| \le |x - y|$ and $\left|1 - e^{\frac{x-y}{2}}\right| \le \frac{|x-y|}{2}$ we see that we shall only consider the function $f(x) = xK(x)$. Since $f(0) = 0$ and f is continuous, f is uniformly continuous on $[0, 1]$, on the other hand f is Lipschitz continuous on $[1, \infty]$. This is because

$$\sup_{x\ge 1} \left|f'(x)\right| = \sup_{x\ge 1} \left|K(x) - e^{-x}\right| \le \frac{1}{e} + K(1) < \infty.$$

Therefore f is uniform continuous on $[0, \infty)$. By the continuity of f in 0 we also now that given an $\varepsilon > 0$ there exists some δ such that $\frac{y-x}{2} K\left(\frac{y-x}{2}\right) < \varepsilon$ for all $|x - y| < \delta$. Hence, we conclude that u is uniform continuous. □

We want now to show that the limit $\lim_{y\to\infty} u(y)$ exists. To this end we proceed again using Fourier methods.

Theorem C.4. *Let u be the unique bounded solution to the problem* (C.35)*. Then $\lim_{x\to\infty} u(x)$ exists and it is uniquely determined by G and u itself. Moreover, the limit is positive if $\left\{n \in \mathbb{S} : n \cdot N < 0 \text{ and } \int_0^\infty d\nu \overline{g}_\nu(n) \not\equiv 0\right\}$ is not a zero measure set.*

Proof. Since u is the unique bounded solution, u solves for all $x \in \mathbb{R}$

$$u(x) - \int_{-\infty}^\infty K(y - x) u(y)\, dy = G(x)\, \chi_{\{x>0\}} - \int_0^\infty K(y - x) u(y)\, dy\, \chi_{\{x<0\}} \equiv W(x). \quad \text{(C.38)}$$

Indeed, this is equivalent to (C.35). This can be seen easily, since u solves for $x < 0$

$$u(x) - \int_{-\infty}^0 K(y - x) u(y)\, dy = 0$$

and since $u = 0$ for $x < 0$ is a possible solution, by uniqueness, this is the only possible solution. It is not only true that $W \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ but also that W has all moments bounded. This follows from the similar property of G (cf. Step 1 in Theorem C.3) as well as from the inequality $0 \le \int_0^\infty K(y-x)u(y)\,dy\chi_{\{x<0\}} \le \|u\|_\infty\,\chi_{\{x<0\}}\left(\frac{e^{-|x|}}{2} - |x|K(x)\right)$. Notice that $|x|K(x) \le \frac{e^{-|x|}}{2}$. Hence, finite moments and Riemann-Lebesgue Theorem imply that W has a Fourier transform $\hat{W} \in C_0(\mathbb{R}) \cap C^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$. Moreover, looking at the left hand side of (C.38) we recall as in [119] that in distributional sense for all $\phi \in \mathcal{S}(\mathbb{R})$

$$\langle \hat{u} - \mathcal{F}(u * K), \phi\rangle := \langle u - u * K, \hat{\phi}\rangle = \langle u, \mathcal{F}\left((1-\sqrt{2\pi}\hat{K})\phi\right)\rangle,$$

where the last equality is due to an elementary calculation involving the convolution and we define $\langle f, g\rangle = \int_{\mathbb{R}} f(x)g(x)\,dx$. We recall also that $1 - \sqrt{2\pi}\hat{K}(\xi) = \frac{\xi - \arctan(\xi)}{\xi} := F(\xi)$. Hence, for all $\phi \in \mathcal{S}(\mathbb{R})$ we have

$$\langle u, \mathcal{F}(\phi F)\rangle = \langle \hat{W}, \phi\rangle. \tag{C.39}$$

Now we consider for $\varepsilon > 0$ the sequence of standard mollifiers $\phi_\varepsilon(\xi) := \frac{1}{\varepsilon}\phi\left(\frac{\xi}{\varepsilon}\right) \in C_c^\infty(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$ such that in distributional sense $\phi_\varepsilon \rightharpoonup \delta$. The smoothness of $\hat{W}$ implies $\langle \hat{W}, \phi\rangle \to \hat{W}(0)$ as $\varepsilon \to 0$. It is our first aim to show that $\hat{W}(0)$ is zero. To this end we study the left hand side of (C.39). We calculate

$$\begin{aligned}\langle u, \mathcal{F}(\phi_\varepsilon F)\rangle &= \frac{1}{\sqrt{2\pi}}\int_0^\infty dx\; u(x)\int_{\mathbb{R}} d\xi\; \phi_\varepsilon(\xi)F(\xi)e^{-i\xi x}\\ &= \frac{1}{\sqrt{2\pi}}\int_0^1 dx\; u(x)\int_{\mathbb{R}} d\xi\; \phi_\varepsilon(\xi)F(\xi)e^{-i\xi x} - \frac{1}{\sqrt{2\pi}}\int_1^\infty dx\;\frac{u(x)}{x^2}\int_{\mathbb{R}} d\xi\; \left(\phi_\varepsilon(\xi)F(\xi)\right)'' e^{-i\xi x},\end{aligned}$$

where for the last equality we integrated twice by parts in ξ. By a change of coordinates and the dominated convergence theorem, since $F(0) = 0$ and $|F(\varepsilon\xi)\phi(\xi)| \le |\phi(\xi)|$ we see for the first term as $\varepsilon \to 0$

$$\left|\frac{1}{\sqrt{2\pi}}\int_0^1 dx\; u(x)\int_{\mathbb{R}} d\xi\; \phi_\varepsilon(\xi)F(\xi)e^{-i\xi x}\right| \le \frac{1}{\sqrt{2\pi}}\int_0^1 dx\; u(x)\int_{\mathbb{R}} d\xi\; |F(\varepsilon\xi)\phi(\xi)| \to 0.$$

Thus, we shall consider only the second term. We use the following well-known estimate $\left|e^{-i\xi x} - 1\right| \le 2|\xi|^\delta|x|^\delta$ for $0 < \delta < 1$ and $x \in \mathbb{R}$. Then using $\int_{\mathbb{R}}(\phi_\varepsilon F)'' = 0$

$$\begin{aligned}&\frac{1}{\sqrt{2\pi}}\int_1^\infty dx\;\frac{u(x)}{x^2}\int_{\mathbb{R}} d\xi\; \left(\phi_\varepsilon(\xi)F(\xi)\right)'' e^{-i\xi x}\\ &\qquad = \frac{1}{\sqrt{2\pi}}\int_1^\infty dx\;\frac{u(x)}{x^2}\int_{\mathbb{R}} d\xi\; \left(\phi_\varepsilon(\xi)F(\xi)\right)''\left(e^{-i\xi x} - 1\right),\end{aligned}$$

and hence

$$\left|\frac{1}{\sqrt{2\pi}}\int_1^\infty dx\;\frac{u(x)}{x^2}\int_{\mathbb{R}} d\xi\; \left(\phi_\varepsilon(\xi)F(\xi)\right)'' e^{-i\xi x}\right| \le \frac{1}{\sqrt{2\pi}}\int_1^\infty dx\;\frac{u(x)}{x^{2-\delta}}\int_{\mathbb{R}} d\xi\; \left|\left(\phi_\varepsilon(\xi)F(\xi)\right)''\right| 2|\xi|^\delta.$$

Now we notice that $\int_1^\infty dx\,\frac{u(x)}{x^{2-\delta}} < \infty$ and also we see that $F(\xi) \simeq \frac{\xi^2}{3}$ as $x \to 0$, similarly as $\xi \to 0$ also $F'(\xi) \simeq \frac{2}{3}\xi$ and $F''(\xi) \simeq \frac{2}{3}$. Hence, with a change of variables we see that

$$\begin{aligned}&\int_{\mathbb{R}} d\xi\; \left|\left(\phi_\varepsilon(\xi)F(\xi)\right)''\right| |\xi|^\delta\\ &\quad \le \int_{\mathbb{R}} d\xi\; \left[|\phi(\xi)||F''(\varepsilon\xi)|\varepsilon^\delta|\xi|^\delta + 2|\phi'(\xi)|\frac{|F'(\varepsilon\xi)||\xi|^\delta}{\varepsilon^{1-\delta}} + |\phi''(\xi)|\frac{|F(\varepsilon\xi)||\xi|^\delta}{\varepsilon^{2-\delta}}\right].\end{aligned}$$

With the consideration above about F and since $\phi \in C_c^\infty(\mathbb{R})$ we see that there exists a constant $C = 2\|\phi\|_{C_c^\infty(\mathbb{R})}\left(\max_{\mathrm{supp}\phi}|\xi|\right)^{2+\delta} < \infty$ such that

$$|\phi(\xi)||F''(\varepsilon\xi)|\varepsilon^\delta|\xi|^\delta + |\phi'(\xi)|\frac{|F'(\varepsilon\xi)||\xi|^\delta}{\varepsilon^{1-\delta}} + |\phi''(\xi)|\frac{|F(\varepsilon\xi)||\xi|^\delta}{\varepsilon^{2-\delta}} \leq C\varepsilon^\delta$$

for any $\xi \in \mathrm{supp}(\phi)$. Thus, again with the dominated convergence theorem we conclude

$$\left|\frac{1}{\sqrt{2\pi}}\int_1^\infty dx\, \frac{u(x)}{x^2}\int_{\mathbb{R}} d\xi\ \left(\phi_\varepsilon(\xi)F(\xi)\right)''\, e^{-i\xi x}\right| \to 0,$$

which implies the first claim, namely $\hat{W}(0) = 0$.

As next step we prove that the limit $\lim_{x\to\infty} u(x)$ exists. First of all we know that in distributional sense $\hat{u}$ solves the equation

$$F\hat{u} \overset{\mathcal{S}'}{=} \hat{W}. \tag{C.40}$$

Given any distributional solution $\hat{u}$ to (C.40) also $\hat{u} + \hat{u}_h$ is a solution, where $\hat{u}_h$ is the homogeneous solution to $F\hat{u}_h \overset{\mathcal{S}'}{=} 0$. Let us consider the tempered distribution given by $\hat{u}_h$ and let $\varphi \in \mathcal{S}(\mathbb{R})$ be any testfunction with support away from zero, i.e. $\mathrm{supp}(\varphi) \subset \mathbb{R}\setminus\{0\}$. Since $F(\xi) = 0$ if and only $\xi = 0$ and since it is bounded, the function $\frac{\varphi}{F} \in \mathcal{S}(\mathbb{R})$. Hence, $\int_{\mathbb{R}} \hat{u}_h\varphi = 0$. This implies (see [120]) that $\hat{u}_h \overset{\mathcal{S}'}{=} \sum_{0\leq\alpha<m} c_\alpha(D^\alpha\delta)$, for c_α constants and a suitable $m \in \mathbb{N}$. Since $c_\alpha F(D^\alpha\delta) \not\equiv 0$ for any $\alpha \geq 2$ we conclude

$$\hat{u}_h = c_0\delta + c_1\delta'$$

for suitable constants c_0, c_1. Using the smoothness of $\hat{W}$ we can write $\hat{W}(\xi) = \hat{W}'(0)\xi + H(\xi)$ where $\hat{W}'(0) = \frac{m_1(W)}{\sqrt{2\pi}i}$ and $H \in C^\infty(\mathbb{R})$ with $H(0) = H'(0) = 0$. Let us consider the behavior of F

$$F(\xi) \simeq \begin{cases} \frac{\xi^2}{3} - \frac{\xi^4}{5} + \mathcal{O}(\xi^6) & \xi \to .0, \\ 1 - \frac{\pi}{2\xi} + \mathcal{O}\left(\frac{1}{\xi^2}\right) & \xi \to \infty \end{cases} \tag{C.41}$$

Hence,

$$f(\xi) := \hat{W}(\xi) - \frac{3m_1(W)}{\sqrt{2\pi}i}\frac{F(\xi)}{\xi} \in L^2(\mathbb{R}) \tag{C.42}$$

and it also satisfies

$$f(\xi) \simeq H''(0)\xi^2 + \mathcal{O}(\xi^3) \qquad \text{as } \xi \to 0. \tag{C.43}$$

By the boundedness of F and given its behavior as in (C.41) we conclude that the function $\hat{h} := \frac{f}{F} \in L^2(\mathbb{R})$, in particular $\hat{h}$ is well-defined in zero. It is easy to see that $\hat{u}$ solves

$$F(\xi)\hat{u}(\xi) \overset{\mathcal{S}'}{=} \frac{3m_1(W)}{\sqrt{2\pi}i}\frac{F(\xi)}{\xi} + f(\xi). \tag{C.44}$$

Therefore, since $\hat{h} \in L^2(\mathbb{R})$ we have that $\hat{u}(\xi) = \frac{3m_1(W)}{\sqrt{2\pi}i}PV\left(\frac{1}{\xi}\right) + \hat{h}(\xi)$ is a solution to (C.44). We denote by $PV(\cdot)$ the principal value. Thus, adding the homogeneous solution we conclude

$$\hat{u}(\xi) \overset{\mathcal{S}'}{=} c_0\delta + c_1\delta' + \frac{3}{2i}m_1(W)\sqrt{\frac{2}{\pi}}PV\left(\frac{1}{\xi}\right) + \hat{h}(\xi),$$

which yields

$$u(x) \overset{\mathcal{S}'}{=} \frac{c_0}{\sqrt{2\pi}} - \frac{c_1 i}{\sqrt{\pi}} x + \frac{3}{2} m_1(W)\operatorname{sgn}(x) + h(x),$$

where $h \in L^2(\mathbb{R})$ is the inverse transform of $\hat{h}$. Since u is bounded and satisfies $u(x) = 0$ for all $x < 0$, we have in distributional sense

$$u(x) = \frac{3}{2} m_1(W) + \frac{3}{2} m_1(W)\operatorname{sgn}(x) + h(x).$$

Hence for $x > 0$ also $u(x) = 3m_1(W) + h(x)$ pointwise. Lemma C.3 implies also that h is uniformly continuous in the positive real line. Hence, $\lim_{x\to\infty} h(x) = 0$ and therefore the limit of u as $x \to \infty$ exists and is uniquely determined by $g_\nu(n)$ and N. This is true since

$$\lim_{y\to\infty} u(y) = 3m_1(W) = 3\left(\int_0^\infty dx\, xG(x) - \int_{-\infty}^0 dx\, x \int_0^\infty dy\, K(y-x)u(y)\right) \geq 0.$$

Also the positivity of the limit is guaranteed when $\left\{n \in \mathbb{S} : n \cdot N < 0 \text{ and } \int_0^\infty d\nu \overline{g}_\nu(n) \not\equiv 0\right\}$ is not a zero measure set. □

We will define $\overline{u}_\infty(p) := \lim_{y\to\infty} \overline{u}(y,p)$ for $p \in \partial\Omega$. We can also show that $\overline{u}$ converges to $\overline{u}_\infty$ with exponential rate.

Lemma C.4. *Let u be the unique bounded solution to the problem* (C.35) *and $u_\infty = \lim_{x\to\infty} u(x)$. Then there exists a constant $C > 0$ such that*

$$|u - u_\infty| \leq Ce^{-\frac{|x|}{2}}.$$

Proof. We use the same notation as in Theorem C.4. Hence, we know that

$$\hat{u}(\xi) \overset{\mathcal{S}'}{=} \frac{u_\infty}{2}\sqrt{2\pi}\delta + \frac{u_\infty}{2}\sqrt{\frac{2}{\pi}} PV\left(\frac{1}{\xi}\right) + \hat{h}(\xi), \tag{C.45}$$

with $F(\xi)\hat{h}(\xi) = \hat{W}(\xi) - \frac{3m_1(W)}{\sqrt{2\pi}i}\frac{F(\xi)}{\xi}$. By the definition of W we see

$$\lim_{x\nearrow 0} W(x) - \lim_{x\searrow 0} W(x) = W(0^+) - W(0^-) = u(0). \tag{C.46}$$

We recall that W has exactly one discontinuity in $x = 0$ and that $W\chi_{\{x<0\}} \in C^\infty(\mathbb{R}_-)$ and $W\chi_{\{x>0\}} \in C^\infty(\mathbb{R}_+)$. By the monotonicity of the two functions $W\chi_{\{x<0\}}$ and $W\chi_{\{x>0\}}$ and since $W \in L^\infty(\mathbb{R})$ we see that $W'\chi_{\{x<0\}} \in L^1(\mathbb{R}_-)$ and $W'\chi_{\{x>0\}} \in L^1(\mathbb{R}_+)$. Moreoevr, we have the asymptotics $\hat{W}(\xi) \simeq \frac{u(0)}{\sqrt{2\pi}i\xi} + \mathcal{O}\left(\frac{1}{\xi^{1+\delta}}\right)$ as $|\xi| \to \infty$ for $0 < \delta < 1$. Indeed, integrating by parts and using that $\lim_{|x|\to\infty} W(x) = 0$ we compute

$$\begin{aligned}
\sqrt{2\pi}\hat{W}(\xi) &= \int_{-\infty}^0 W(x)e^{-i\xi x}\, dx + \int_0^\infty W(x)e^{-i\xi x}\, dx \\
&= \frac{u(0)}{i\xi} + \frac{1}{i\xi}\left(\int_{-\infty}^0 W'(x)e^{-i\xi x}\, dx + \int_0^\infty W'(x)e^{-i\xi x}\, dx\right) \\
&= \frac{u(0)}{i\xi} - \frac{1}{i\xi}\left(\int_{-\infty}^{-1} dx \int_0^\infty dy\, \frac{e^{-(y-x)}u(y)}{2(y-x)} e^{-i\xi x} + \int_{-\infty}^0 dx \int_1^\infty dy\, \frac{e^{-(y-x)}u(y)}{2(y-x)} e^{-i\xi x}\right) \\
&\quad - \frac{1}{i\xi}\left(\int_{-1}^0 dx \int_0^1 dy\, \frac{e^{-(y-x)}u(y)}{2(y-x)} \frac{d}{dx}\frac{e^{-i\xi x}-1}{-i\xi}\right) \\
&\quad + \frac{1}{i\xi}\left(\int_1^\infty G'(x)e^{-i\xi x}\, dx + \int_0^1 G'(x)\frac{d}{dx}\frac{e^{-i\xi x}-1}{-i\xi}\right)
\end{aligned} \tag{C.47}$$

We conclude integrating by parts and applying the Riemann-Lebesgue Theorem in the following way. First of all, the function $\partial_x \frac{e^{-(y-x)}}{(y-x)}$ is integrable on $(-\infty,1)\times\mathbb{R}_+ \cup \mathbb{R}_- \times (1,\infty)$ and also $G''(x)$ is integrable in $(1,\infty)$. Moreover, using $\left|e^{-i\xi x}-1\right| \le 2|\xi|^\delta |x|^\delta$ for $0<\delta<1$ we have

$$\int_{-1}^{0} dx \int_{0}^{1} dy\, \partial_x \left(\frac{e^{-(y-x)}u(y)}{(y-x)}\right) |x|^\delta \le C \int_{-1}^{0} dx\, \left(|x|^{\delta-1}\right) < \infty$$

and

$$\int_0^1 dx\, G''(x)|x|^\delta \le C \int_0^1 \frac{e^{-x}}{x^{1-\delta}}\, dx < \infty.$$

For this last estimate we also used that $\frac{d}{d\theta} e^{\frac{x}{\cos(\theta)}} = x\frac{e^{\frac{x}{cos(\theta)}}}{\cos^2(\theta)}\sin(\theta)$, which implies $|G''(x)| \le 2\pi \|g\|_\infty \frac{e^{-x}}{x}$. Thus, by the definition of $\hat{h}$ and using (C.41) we have

$$\hat{h}(\xi) \simeq \begin{cases} \mathcal{O}(1) & |\xi| \to 0, \\ \frac{u(0)}{\sqrt{2\pi}}\frac{1}{i\xi} - \frac{u_\infty}{\sqrt{2\pi}}\frac{1}{i\xi} + \mathcal{O}\left(\frac{1}{\xi^{1+\delta}}\right) & |\xi| \to \infty. \end{cases} \tag{C.48}$$

By the definition of $\hat{u}$ in (C.45) we see

$$\hat{v}(\xi) := \hat{u}(\xi) - \frac{u_\infty}{2}\sqrt{2\pi}\delta - PV\left(\frac{1}{i\xi}\right)\left(\frac{u_\infty}{2}\sqrt{\frac{2}{\pi}}\frac{1}{1+\xi^2} + \frac{u(0)}{\sqrt{2\pi}}\frac{\xi^2}{1+\xi^2}\right) \in L^2(\mathbb{R}). \tag{C.49}$$

We claim that

(i) $\hat{v}$ is analytic in the strip $S = \{z \in \mathbb{C} : |\Im(z)| < \frac{3}{4}\}$;

(ii) $|\hat{v}(\xi)| \le \frac{C}{|1+\xi^{1+\delta}|}$;

(iii) $v(x) = u(x) - u_\infty + \frac{e^{-|x|}}{2}(u_\infty - u(0))$ for $x>0$ and $v(x) = \mathcal{F}^{-1}(\hat{v})(x)$.

A contour integral implies then the lemma. Indeed for $x>0$ we can compute

$$\begin{aligned} \sqrt{2\pi}|v(x)| &= \lim_{R\to\infty}\left|\int_{-R}^{R} \hat{v}(\xi)e^{i\xi x}\, d\xi\right| \\ &\le \lim_{R\to\infty}\left|i\int_0^{\frac{1}{2}} \hat{v}(R+it)e^{iRx}e^{-tx}\, dt\right| + \lim_{R\to\infty}\left|i\int_0^{\frac{1}{2}} \hat{v}(-R+it)e^{-iRx}e^{-tx}\, dt\right| \\ &\quad + \lim_{R\to\infty}\left|\int_{-R}^{R} \hat{v}\left(t+\frac{1}{2}i\right)e^{itx}e^{-\frac{x}{2}}\, dt\right| \\ &\le e^{-\frac{x}{2}} \lim_{R\to\infty}\int_{-R}^{R} \frac{C}{\frac{1}{2}+t^{1+\delta}}\, dt = \overline{C}e^{-\frac{x}{2}}, \end{aligned} \tag{C.50}$$

where for the first inequality we used the triangle inequality and the analycity of $\hat{v}$ by (i), the second inequality is due to dominate convergence and the claim (ii), finally the last integral is finite. Equation (C.50) and claim (iii) imply $|u(x)-u_\infty| \le Ce^{-\frac{x}{2}}$ for $x>0$.

We prove now the claims. To prove claim (i) it is enough to show that $\hat{h}$ is analytic in S. Then, (C.45) and (C.49) implies (i). First of all we recall that W has an exponential decay like $|W(x)| \le Ce^{-|x|}$, hence $|W(x)|e^{\frac{3}{4}|x|} \in L^1(\mathbb{R})$ and therefore Paley-Wiener Theorem implies that $\hat{W}$ is analytic in S. Since $\arctan(z) = \frac{1}{2i}\ln(\frac{1+iz}{1-iz})$ is analytic in $\{z\in\mathbb{C} : |\Im(z)|<1\}$ and since $F(z) = \frac{z-\arctan(z)}{z}$ has exactly one zero in $z = 0$, which is of degree 2, the definition of $\hat{h} = \frac{f}{F}$ together with (C.42) implies that $\hat{h}$ is analytic in S since (C.43) implies that 0 is a removable singularity.

For claim (ii) we just put together equations (C.45), (C.48) and (C.49). We notice also that the constant $C > 0$ of claim (ii) depends only on $\hat{W}$.

Claim (iii) is more involved. We have to consider again two different contour integrals in order to compute the inverse Fourier transform of $\hat{v}$. We start with considering the function $PV\,(f(\xi)) = PV\left(\frac{1}{i\xi(1+\xi^2)}\right)$. Let first of all $x > 0$ and let γ_1^+ the path around i given as in the following picture.

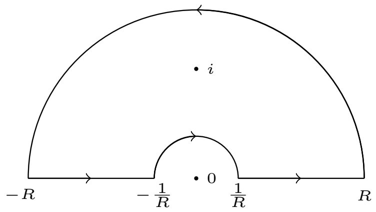

Figure C.2: sketch of γ_1^+.

Hence, we compute

$$\begin{aligned}
\mathcal{F}^{-1}\left(PV\left(\frac{1}{i\xi}\frac{1}{1+\xi^2}\right)\right)(x) &= \frac{1}{\sqrt{2\pi}}\lim_{R\to\infty}\left(\int_{-R}^{-\frac{1}{R}} f(\xi)e^{i\xi x}\,d\xi + \int_{\frac{1}{R}}^{R} f(\xi)e^{i\xi x}\,d\xi\right)\\
&=\frac{1}{\sqrt{2\pi}}\lim_{R\to\infty}\left(\int_{\gamma_1^+} f(\xi)e^{i\xi x}d\xi\right) + \frac{1}{\sqrt{2\pi}}\lim_{R\to\infty}\left(\int_0^{\pi} f\left(\frac{e^{i\theta}}{R}\right)\frac{ie^{i\theta}}{R}e^{-\frac{\sin(\theta)x}{R}}e^{\frac{i\cos(\theta)x}{R}}\,d\theta\right)\\
&\quad-\frac{1}{\sqrt{2\pi}}\lim_{R\to\infty}\left(\int_0^{\pi} f\left(Re^{i\theta}\right)Rie^{i\theta}e^{-R\sin(\theta)x}e^{iR\cos(\theta)x}\,d\theta\right)\\
&=\sqrt{\frac{\pi}{2}}\left(1-e^{-x}\right).
\end{aligned}$$

For the computation of these integrals we used the Cauchy's residue theorem and $\mathrm{Res}_i f(\xi)e^{i\xi x} = \frac{ie^{-x}}{2}$, the second integral converges to π as $R \to \infty$ and the third converges to zero, both limit are due to the Lebesgue dominated convergence theorem. Denoting by γ_1^- the mirrored path to γ_1^+ with respect to the real axis and arguing similarly we also get that for $x < 0$ the inverse Fourier transformation is $\mathcal{F}^{-1}\left(PV\left(\frac{1}{i\xi}\frac{1}{1+\xi^2}\right)\right)(x) = -\sqrt{\frac{\pi}{2}}\left(1-e^{-|x|}\right)$. Hence,

$$\mathcal{F}^{-1}\left(PV\left(\frac{1}{i\xi}\frac{1}{1+\xi^2}\right)\right)(x) = \mathrm{sgn}(x)\sqrt{\frac{\pi}{2}}\left(1-e^{-|x|}\right). \tag{C.51}$$

For the function $g(x)(\xi) = \frac{\xi}{i(1+\xi^2)}$ we consider again first of all $x > 0$ and the path γ_2^+ around i given by

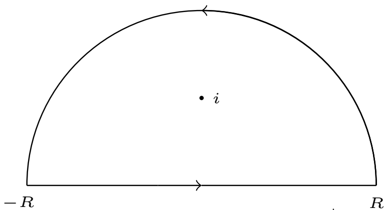

Figure C.3: sketch of γ_2^+.

Hence, the Cauchy's residue theorem and the dominated convergence imply

$$
\begin{aligned}
\mathcal{F}^{-1}\left(\frac{\xi}{i(1+\xi^2)}\right)(x) &= \frac{1}{\sqrt{2\pi}}\lim_{R\to\infty}\int_{-R}^{R} g(\xi)e^{i\xi x}\, d\xi \\
&= \frac{1}{\sqrt{2\pi}}\lim_{R\to\infty}\left(\int_{\gamma_2^+} g(\xi)e^{i\xi x} d\xi\right) - \frac{1}{\sqrt{2\pi}}\lim_{R\to\infty}\left(\int_0^\pi g\left(Re^{i\theta}\right) Rie^{i\theta}e^{-R\sin(\theta)x}e^{iR\cos(\theta)x}\, d\theta\right) \\
&= \sqrt{\frac{\pi}{2}}e^{-x},
\end{aligned}
$$

where we also used that $\text{Res}_i g(\xi)e^{-\xi x} = \frac{e^{-x}}{2i}$. Denoting similarly as before by γ_2^- the mirrored path to γ_2^+ with respect to the real axis we obtain $\mathcal{F}^{-1}(g)(x) = -\sqrt{\frac{\pi}{2}}e^{-|x|}$for $x < 0$ and thus

$$
\mathcal{F}^{-1}\left(\frac{\xi}{i(1+\xi^2)}\right)(x) = \text{sgn}(x)\sqrt{\frac{\pi}{2}}e^{-|x|}. \tag{C.52}
$$

Hence, the definition of u and equations (C.51), (C.52) imply claim (iii) for $x > 0$

$$
v(x) = u(x) - u_\infty + \frac{e^{-|x|}}{2}\left(u_\infty - u(0)\right).
$$

□

There are still two important properties of $\overline{u}(y,p)$ we will need for the rest of the paper and which are explained in the next two Lemmas. First of all $\overline{u}(y,p)$ is uniformly bounded in both variables.

Lemma C.5. *Let $\overline{u}(y,p)$ be the non-negative bounded solution to the problem* (C.22) *for $g_\nu(n)$ satisfying the assumption as in Theorem C.3. Then there exists a constant C such that*

$$
\sup_{y\in\mathbb{R},\, p\in\partial\Omega} \overline{u}(y,p) \le C < \infty.
$$

Proof. By definition $\overline{u}$ satisfies $\mathcal{L}(\overline{u})(y) = G_p(y)$ for $y > 0$ and $\overline{u}(y,p) = 0$ for $y < 0$. Moreover, recalling the norm as in (C.11) the source can be estimated by

$$
0 \le G_p(y) \le \|g\|_1 e^{-y},
$$

since $|n\cdot N_p| \le 1$.

Theorem C.3 assures us the existence of a unique bounded continuous (in the positive line) solution v of $\mathcal{L}(v)(y) = \|g\|_1 e^{-y}$ for $y > 0$ and $v(y) = 0$ for $y < 0$. Hence, we can apply the maximum principle of Theorem C.1 as we did in Lemma C.2 to the function $v - \overline{u}(\cdot,p) \in C\left([0,\infty]\right)$ and we conclude

$$
0 \le \overline{u}(y,p) \le v(y) \le \|v\|_\infty := C < \infty
$$

for all $y \in \mathbb{R}$ and $p \in \partial\Omega$. □

Also, the rate of convergence of $\overline{u}(y,p)$ to $\overline{u}_\infty(p)$ can be bounded independently of $p \in \partial\Omega$.

Corollary C.2. *There exists a constant $C > 0$ independent of $p \in \partial\Omega$ such that*

$$
|\overline{u}(y,p) - \overline{u}_\infty(p)| \le Ce^{-\frac{y}{2}}
$$

.

Proof. This is a consequence of Lemma C.4 and Lemma C.5. From Lemma C.5 we know that there exists a constant $C>0$ independent of $p\in\partial\Omega$ such that

$$|W(x)|\leq C\left(e^{-|x|}+|x|K(x)\chi_{\{x<0\}}\right)\in L^1(\mathbb{R})\cap L^2(\mathbb{R})\cap L^\infty(\mathbb{R}),$$

where W is the function defined in (C.38). Since $|x|K(x)\leq\frac{e^{-|x|}}{2}$ all moments of W are finite and for any $n\in\mathbb{N}$ there exists a constant $C_n>0$ independent of $p\in\partial\Omega$ such that

$$|m_n(W)|\leq C_n<\infty.$$

Hence, $\hat{W}\in C_0(\mathbb{R})\cap C^\infty(\mathbb{R})\cap L^2(\mathbb{R})$ and also all derivatives are uniformly bounded in $p\in\partial\Omega$ since $\left|\hat{W}^{(n)}(\xi)\right|\leq\frac{C_n}{\sqrt{2\pi}}$. Thus, the function $\hat{h}$ in (C.48) defined using (C.43) can be bounded independently of $p\in\partial\Omega$.

Moreover, we notice that in (C.47) as $|\xi|\to\infty$ we can bound $\left|\hat{W}(\xi)-\frac{u(0)}{\sqrt{(2\pi)}i\xi}\right|$ by $\frac{C}{|\xi^{1+\delta}|}$ with a constant $C>0$ independent of $p\in\partial\Omega$. Indeed, as we have seen in Lemma C.4 we have $|G''(x)|\leq 2\pi\|g\|_\infty\frac{e^{-x}}{x}$ and by Lemma C.5 we have also $|\overline{u}(y,p)|\leq C$.

Hence, we conclude as we did in Lemma C.4 that there exists a constant $C>0$ independent of $p\in\partial\Omega$ such that $|\hat{v}(\xi)|\leq\frac{C}{|1+\xi^{1+\delta}|}$, where $\hat{v}$ was defined in (C.49).

Arguing now exactly as in Lemma C.4 using also Lemma C.5 we conclude that there exists a constant $C>0$ independent of $p\in\partial\Omega$ such that $|\overline{u}(y,p)-\overline{u}_\infty(p)|\leq Ce^{-\frac{y}{2}}$. □

Next, using again the maximum principle we can also show that $\overline{u}(y,p)$ is Lipschitz continuous with respect to $p\in\partial\Omega$ uniformly in y.

Lemma C.6. *Let $g_\nu(n)$ be as in Theorem C.3 and let $\overline{u}$ be the unique bounded solution to* (C.22)*. Then $\overline{u}$ is uniformly continuous with respect the variable $p\in\partial\Omega$ uniformly in y. More precisely, it is Lipschitz continuous, i.e. there exists a constant $C>0$ such that for every $p,q\in\partial\Omega$*

$$\sup_{y\geq 0}|\overline{u}(y,p)-\overline{u}(y,q)|\leq C|p-q|:=\omega_1(|p-q|).$$

Proof. The proof is based on the maximum principle. We start taking $0<\tilde{\delta}<1$ sufficiently small and we consider $p,q\in\partial\Omega$ with $|p-q|<\tilde{\delta}$. We denote by $S_p(q)$ the plane defined by the vector $\overrightarrow{pq}$ and the unit vector N_p. Given that $\partial\Omega$ is a C^3-surface we can define ρ_p to be the radius of curvature of the curve $C_p(q):=S_p(q)\cap\partial\Omega$ at p. Since by assumption the curvature of $\partial\Omega$ is bounded from below by a positive constant, for $\tilde{\delta}$ small enough we can estimate

$$\frac{1}{2}\rho_p\theta_{pq}\leq|p-q|\leq 2\rho_p\theta_{pq}, \tag{C.53}$$

where θ_{pq} is the angle between N_p and N_q. This is true, because for $\tilde{\delta}$ sufficiently small the angle θ_{pq} is not zero and it is approximately the central angle between the rays connecting p and q with the center of the circle with radius ρ_p tangent to p. We denote by R the minimal radius of curvature of $\partial\Omega$, hence $\rho_p\geq R$. Now we consider the operator $\mathcal{L}$ acting on the difference $\overline{u}(y,p)-\overline{u}(y,q)$. We can estimate its absolute value by the sum of the following six terms

$$\begin{aligned}|\mathcal{L}(\ \overline{u}(y,p)-\overline{u}(y,q))|\leq&\int_{A_1}\int_0^\infty g_\nu(n)e^{-\frac{y}{|n\cdot N_p|}}\,d\nu\,dn+\int_{A_2}\int_0^\infty g_\nu(n)e^{-\frac{y}{|n\cdot N_q|}}\,d\nu\,dn\\&+\int_{A_3}\int_0^\infty g_\nu(n)\left|e^{-\frac{y}{|n\cdot N_p|}}-e^{-\frac{y}{|n\cdot N_q|}}\right|\,d\nu\,dn+\int_{A_4}\int_0^\infty g_\nu(n)\left|e^{-\frac{y}{|n\cdot N_p|}}-e^{-\frac{y}{|n\cdot N_q|}}\right|\,d\nu\,dn\\&+\int_{A_5}\int_0^\infty g_\nu(n)\left|e^{-\frac{y}{|n\cdot N_p|}}-e^{-\frac{y}{|n\cdot N_q|}}\right|\,d\nu\,dn+\int_{A_6}\int_0^\infty g_\nu(n)\left|e^{-\frac{y}{|n\cdot N_p|}}-e^{-\frac{y}{|n\cdot N_q|}}\right|\,d\nu\,dn,\end{aligned} \tag{C.54}$$

where we denote by A_i the following sets

$$A_1 := \left\{n \in \mathbb{S}^2 : n \cdot N_p < 0,\ n \cdot N_q \geq 0\right\}, \quad A_2 := \left\{n \in \mathbb{S}^2 : n \cdot N_p \geq 0,\ n \cdot N_q < 0\right\},$$
$$A_3 := \left\{n \in \mathbb{S}^2 : n \cdot N_p < 0,\ n \cdot N_q < 0,\ |n \cdot N_p| \geq |n \cdot N_q|, |; |n \cdot N_p| > \frac{4}{R}|p-q|\right\},$$
$$A_4 := \left\{n \in \mathbb{S}^2 : n \cdot N_p < 0,\ n \cdot N_q < 0,\ |n \cdot N_p| \geq |n \cdot N_q|, |; |n \cdot N_p| \leq \frac{4}{R}|p-q|\right\},$$
$$A_5 := \left\{n \in \mathbb{S}^2 : n \cdot N_p < 0,\ n \cdot N_q < 0,\ |n \cdot N_q| \geq |n \cdot N_p|, |; |n \cdot N_q| > \frac{4}{R}|p-q|\right\} \text{ and}$$
$$A_6 := \left\{n \in \mathbb{S}^2 : n \cdot N_p < 0,\ n \cdot N_q < 0,\ |n \cdot N_q| \geq |n \cdot N_p|, |; |n \cdot N_q| \leq \frac{4}{R}|p-q|\right\}.$$

By symmetry, we need to estimate only the first, the third and the fourth terms. We start with the first line of equation (C.54). The set A_1 is contained by the set given by all the n such that their angle with N_p is in the interval $(\frac{\pi}{2}, \frac{\pi}{2} + \theta_{pq})$. Using the fact that $\frac{y}{|n \cdot N_p|} > y$, we estimate the exponential by e^{-y} and hence we see

$$\int_{A_1}\int_0^\infty g_\nu(n) e^{-\frac{y}{|n\cdot N_p|}}\, d\nu\, dn \leq \|g\|_\infty 2\pi\theta_{pq} e^{-y} \leq \frac{4\pi}{R}\|g\|_\infty e^{-y}. \tag{C.55}$$

The second term in (C.54) is estimated similarly. For the third term of equation (C.54) we estimate the difference of the exponential as follows, assuming $|n \cdot N_p| \geq |n \cdot N_q|$

$$\left|e^{-\frac{y}{|n\cdot N_p|}} - e^{-\frac{y}{|n\cdot N_q|}}\right| \leq e^{-\frac{y}{|n\cdot N_p|}} y \left|\frac{1}{|n\cdot N_q|} - \frac{1}{|n \cdot N_p|}\right| \leq e^{-\frac{y}{|n\cdot N_p|}} y \left|\frac{|n\cdot N_p| - |n\cdot N_q|}{|n\cdot N_q||n\cdot N_p|}\right|,$$

where we used for $x > 0$ the inequality $1 - e^{-x} \leq x$. By definition $|n\cdot(N_p - N_q)| \leq \theta_{pq} \leq \frac{2}{R}|p-q|$ which implies

$$0 \leq |n\cdot N_p| - |n \cdot N_q| = |n\cdot(N_q - N_p)| \leq \frac{2}{R}|p-q|.$$

Since $|n \cdot N_p| > \frac{4}{R}|p-q|$ we see also that

$$|n\cdot N_q| \geq |n\cdot N_p| - \frac{2}{R}|p-q| \geq \frac{|n\cdot N_p|}{2}.$$

Hence,

$$\left|e^{-\frac{y}{|n\cdot N_p|}} - e^{-\frac{y}{|n\cdot N_q|}}\right| \leq e^{-\frac{y}{|n\cdot N_p|}} y \frac{4|p-q|}{R|n\cdot N_p|^2}.$$

Putting together these inequalities we compute

$$\begin{aligned}\int_{A_3}\int_0^\infty g_\nu(n)\left|e^{-\frac{y}{|n\cdot N_p|}} - e^{-\frac{y}{|n\cdot N_q|}}\right| d\nu\, dn &\leq \frac{4|p-q|}{R}\|g\|_\infty \int_{A_3} dn\, e^{-\frac{y}{|n\cdot N_p|}}\frac{y}{|n\cdot N_p|^2}\\ &\leq \frac{4|p-q|}{R}\|g\|_\infty 4\pi \int_0^{\frac{\pi}{2}} e^{-\frac{y}{\cos(\theta)}}\frac{y\sin(\theta)}{\cos^2(\theta)}\, d\theta = \frac{16\pi|p-q|}{R}\|g\|_\infty e^{-y},\end{aligned} \tag{C.56}$$

where we estimated the last integral in A_3 using polar coordinates in $\mathbb{S}^2$ using as reference N_p. It remains to estimate the integral on A_4. For this term we use the inclusion

$$\begin{aligned}A_4 &\subset \left\{n \in \mathbb{S}^2 : n\cdot N_p < 0,\ |n \cdot N_p| \leq \frac{4}{R}|p-q|\right\}\\ &\subset \left\{(\varphi,\theta) \in [0,2\pi]\times[0,\pi] : \theta \in \left(-\frac{\pi}{2}, -\frac{\pi}{2} + C(R)|p-q|\right) \cup \left(\frac{\pi}{2} - C(R)|p-q|, \frac{\pi}{2}\right)\right\},\end{aligned}$$

where the last inclusion is due to the smallness of $\frac{4}{R}|p-q| < 1$ and the expansion of the arc-cosine. Moreover, $C(R)$ is a constant depending only on R. Hence, as we estimated in (C.55) we have

$$\int_{A_4}\int_0^\infty g_\nu(n)\left|e^{-\frac{y}{|n\cdot N_p|}} - e^{-\frac{y}{|n\cdot N_q|}}\right| \,d\nu\,dn \le C(R)4\pi\|g\|_\infty|p-q|. \tag{C.57}$$

Now, with equations (C.55),(C.56) and (C.57) we estimate the operator by

$$|\mathcal{L}\left(\overline{u}(y,p)-\overline{u}(y,q)\right)| \le C(R)\|g\|_\infty|p-q|e^{-y},$$

where $C(R) > 0$ is a constant depending only on the minimal radius of curvature R. Theorem C.3 and the maximum principle imply the existence of a unique non-negative bounded continuous function V solution to the equation $\mathcal{L}(V)(y) = e^{-y}$ for $y \ge 0$. Hence, we apply the maximum principle of Theorem C.1 as in Lemma C.2 to the continuous functions $C(R)\|g\|_\infty|p-q|V - (\overline{u}(y,p)-\overline{u}(y,q))$ and $C(R)\|g\|_\infty|p-q|V - (\overline{u}(y,q)-\overline{u}(y,p))$. We conclude the uniformly continuity of $\overline{u}(y,p)$ in p uniformly in y

$$|\overline{u}(y,p)-\overline{u}(y,q)| \le C(R)\|g\|_\infty|p-q|.$$

The modulus of continuity ω_1 is hence defined by $\omega_1(r) = C(R)\|g\|_\infty r$. □

Corollary C.3. *The limit $\overline{u}_\infty$ is Lipschitz continuous in $p \in \partial\Omega$.*

Proof. This is a direct consequence of the previous Lemma C.6. The modulus of continuity of $\overline{u}_\infty$ is still the same ω_1 of $\overline{u}(y,p)$. □

Finally, we summarize all properties of $\overline{u}$ in the following proposition.

Proposition C.3. *Let $g_\nu(n)$ be as in Theorem C.3 and Ω as in the assumption. For every $p \in \partial\Omega$ there exists a unique non-negative bounded solution $\overline{u}(y,p)$ to* (C.22). *For every $p \in \partial\Omega$ the function $\overline{u}(\cdot,p)$ is uniformly continuous in $[0,\infty)$ and has a non-negative limit $\overline{u}_\infty(p) = \lim_{y\to\infty}\overline{u}(y,p)$, which is strictly positive if $\left\{n\in\mathbb{S} : n\cdot N_p < 0 \text{ and } \int_0^\infty d\nu g_\nu(n) \not\equiv 0\right\}$ is not a zero measure set. Moreover, $\overline{u}(y,p)$ is uniformly bounded in both variables and it is Lipschitz continuous with respect to $p \in \partial\Omega$ uniformly on $y \in \mathbb{R}_+$. Finally, $\overline{u}_\infty$ is Lipschitz continuous and there exists a constant $C > 0$ independent of $p \in \partial\Omega$ such that $|\overline{u}(y,p)-\overline{u}_\infty(p)| \le Ce^{-\frac{|y|}{2}}$.*

C.4 Rigorous proof of the diffusion equilibrium approximation for constant absorption coefficient

This section of the paper deals with the rigorous proof of the diffusion equilibrium approximation for the constant absorption coefficient case. We will show that the Stefan-Boltzmann law $u^\varepsilon(x) = 4\pi\sigma T_\varepsilon^4(x)$ for the temperature T_ε associated to the boundary value problem (C.6) converges pointwise as $\varepsilon \to 0$ to v, the solution to the Dirichlet problem

$$\begin{cases} -\Delta v = 0 & \text{in } \Omega, \\ v = \overline{u}_\infty & \text{on } \partial\Omega, \end{cases} \tag{C.58}$$

where $\overline{u}_\infty$ is defined as in Proposition C.3.

C.4.1 Derivation of the equation for u^ε

Let us call I_ν^ε the solution to the initial boundary value problem (C.6). We start with the derivation of the integral equation satisfied by $u^\varepsilon = 4\pi\sigma T_\varepsilon^4$. To this end we solve by characteristics the equation

$$n \cdot \nabla_x I_\nu(x,n) = \frac{1}{\varepsilon}\left(B_\nu(T(x)) - I_\nu(x,n)\right)$$

Let $x \in \Omega$ and $n \in \mathbb{S}^2$. The convexity of Ω implies the existence of a unique $x_\Omega(x,n) \in \partial\Omega$ connecting x in direction $-n$ with the boundary $\partial\Omega$. Hence, $\frac{x-x_\Omega(x,n)}{|x-x_\Omega(x,n)|} = n$ and we define $s(x,n) = |x - x_\Omega(x,n)|$. Then $x = x_\Omega(x,n) + s(x,n)n$. Integrating along the characteristics equation (C.6) we get

$$I_\nu^\varepsilon(x,n) = g_\nu(n)e^{-\frac{|x-x_\Omega(x,n)|}{\varepsilon}} + \frac{1}{\varepsilon}\int_0^{s(x,n)} e^{-\frac{t}{\varepsilon}} B_\nu\left(T(x-tn)\right)\, dt. \tag{C.59}$$

Using the heat equation, i.e. $\nabla_x \cdot \mathcal{F} = 0$ (see (C.6)), we calculate

$$0 = \int_0^\infty d\nu \int_{\mathbb{S}^2} dn\; n\cdot\nabla_x I_\nu^\varepsilon(x,n) = \frac{1}{\varepsilon}\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\; B_\nu(T_\varepsilon)(x) - I_\nu^\varepsilon(x,n).$$

We define $u^\varepsilon(x) = 4\pi\sigma T_\varepsilon^4(x) = \int_0^\infty d\nu \int_{\mathbb{S}^2} dn B_\nu\left(T_\varepsilon(x)\right)$ according to (C.3). Hence also $u^\varepsilon(x) = \int_0^\infty d\nu \int_{\mathbb{S}^2} dn\;\; I_\nu^\varepsilon(x,n)$. We integrate now the expression we got for the intensity and we conclude with the equation satisfied by u^ε as follows

$$\begin{aligned} u^\varepsilon(x) &= \int_0^\infty d\nu \int_{\mathbb{S}^2} dn\; g_\nu(n)e^{-\frac{|x-x_\Omega(x,n)|}{\varepsilon}} + \frac{1}{4\pi\varepsilon}\int_{\mathbb{S}^2} dn \int_0^{s(x,n)} e^{-\frac{t}{\varepsilon}} u^\varepsilon(x-tn)\, dt \\ &= \int_0^\infty d\nu \int_{\mathbb{S}^2} dn\; g_\nu(n)e^{-\frac{|x-x_\Omega(x,n)|}{\varepsilon}} + \frac{1}{4\pi\varepsilon}\int_\Omega \frac{e^{-\frac{|x-\eta|}{\varepsilon}}}{|x-\eta|^2} u^\varepsilon(\eta)\, d\eta, \end{aligned}$$

where the last equality is due to the change of variables $\mathbb{S}^2 \times (0,\infty) \to \Omega$ with $(n,t) \mapsto x - tn = \eta$. Hence the sequence u^ε of exact solutions solves

$$u^\varepsilon(x) - \int_\Omega \frac{e^{-\frac{|x-\eta|}{\varepsilon}}}{4\pi\varepsilon\,|x-\eta|^2} u^\varepsilon(\eta)\, d\eta = \int_0^\infty d\nu \int_{\mathbb{S}^2} dn\; g_\nu(n)e^{-\frac{|x-x_\Omega(x,n)|}{\varepsilon}}. \tag{C.60}$$

We define the kernel $K_\varepsilon(x) := \frac{e^{-\frac{|x|}{\varepsilon}}}{4\pi\varepsilon|x|^2}$ and we notice that its integral in $\mathbb{R}^3$ is 1.

Remark. There exists a unique solution u^ε continuous and bounded. We adapt the proof in [83]. The existence and uniqueness of a solution $u^\varepsilon \in L^\infty(\Omega)$ can be shown with the Banach fixed-point Theorem. We define for every given g and $\varepsilon > 0$ the self map $A_g^\varepsilon : L^\infty(\Omega) \to L^\infty(\Omega)$ by

$$A_g^\varepsilon(u)(x) = \int_\Omega K_\varepsilon(\eta - x)\, u(\eta)\, d\eta + \int_0^\infty d\nu \int_{\mathbb{S}^2} dn\; g_\nu(n)e^{-\frac{|x-x_\Omega(x,n)|}{\varepsilon}}.$$

Then since $\int_\Omega K_\varepsilon(\eta - x)\; d\eta < \int_{\mathbb{R}^3} K_\varepsilon(\eta-x)\; d\eta = 1$ we conclude that A_g^ε is a contraction, hence there is a unique fixed-point, which is the desired unique solution. Moreover, $G_{x_\Omega}^\varepsilon(x) := \int_0^\infty d\nu \int_{\mathbb{S}^2} dn\; g_\nu(n)e^{-\frac{|x-x_\Omega(x,n)|}{\varepsilon}}$ is continuous and since $u^\varepsilon \in L^\infty(\Omega)$ and $K_\varepsilon(x-\cdot) \in L^1\left(\mathbb{R}^3\right)$ we conclude that the convolution $\int_\Omega K_\varepsilon(\eta-x)u^\varepsilon(\eta)\; d\eta$ is continuous and bounded. Hence, u^ε is continuous and bounded. We can also extend continuously u^ε to the boundary $\partial\Omega$ defining $|x - x_\Omega(x,n)| = 0$ for $x \in \partial\Omega$ and $n \cdot N_x \leq 0$. Then using the generalized dominated convergence theorem we see that both integral terms in (C.60) are continuous up to the boundary. Hence, $u^\varepsilon \in C\left(\overline{\Omega}\right)$. Moreover, u^ε is non-negative. This is because of the maximum principle as stated in the following theorem.

Theorem C.5 (Maximum Principle). *Let v be bounded and continuous, $v \in C\left(\overline{\Omega}\right)$. Let $\mathcal{L}_\Omega^\varepsilon(v)(x) = v(x) - \int_\Omega K_\varepsilon(\eta - x)v(\eta)\, d\eta$. Assume v satisfies one of the following properties:*

(i) $\mathcal{L}_\Omega^\varepsilon(v)(x) \geq 0$ if $x \in \Omega$;

(ii) $\mathcal{L}_\Omega^\varepsilon(v)(x) \geq 0$ if $x \in O \subset \Omega$ open and $v(x) \geq 0$ if $x \in \Omega \setminus O$.

Then, $v \geq 0$.

Proof. Let $y \in \overline{\Omega}$ such that $v(y) = \min_{x\in\overline{\Omega}} v(x)$. Assume $v(y) < 0$.

Assume that property (i) holds. By continuity of the the operator we have that $\mathcal{L}_\Omega^\varepsilon(v)(x) \geq 0$ for all $x \in \overline{\Omega}$. Then

$$\begin{aligned} 0 \leq & \mathcal{L}_\Omega^\varepsilon(v)(y) = v(y) - \int_\Omega K_\varepsilon(\eta - y)v(\eta)\, d\eta \\ &= \int_\Omega K_\varepsilon(\eta - y)\left(v(y) - v(\eta)\right)\, d\eta + v(y)\int_{\Omega^c} K_\varepsilon(\eta - y)\, d\eta < 0, \end{aligned} \tag{C.61}$$

where we used the normalization of the kernel K_ε. Hence, this contradiction yields $v \geq 0$.

Assume now that (ii) holds. Then in this case $y \in \overline{O}$. Then again by the continuity of the operator we obtain exactly as in (C.61) a contradiction. Thus the Theorem is proved. □

C.4.2 Uniform boundedness of u^ε

In this section we will show that the sequence u^ε is uniformly bounded in ε. We will use the maximum principle again. Indeed, we will construct functions Φ^ε uniformly bounded such that $\mathcal{L}_\Omega^\varepsilon(\Phi^\varepsilon)(x) \geq \|g\|_1 e^{-\frac{\operatorname{dist}(x,\partial\Omega)}{\varepsilon}}$. We will use this to prove $\mathcal{L}_\Omega^\varepsilon\left(\Phi^\varepsilon - u^\varepsilon\right)(x) \geq 0$ which implies using the maximum principle $0 \leq u^\varepsilon \leq \Phi^\varepsilon$. The main result of this subsection is the following.

Theorem C.6. *There exists suitable constants $0 < \mu < 1$, $0 < \gamma(\mu) < \frac{1}{3}$, C_1, C_2, $C_3 > 0$ and there exists some $\varepsilon_0 > 0$ such that the function*

$$\Phi^\varepsilon(x) = C_3\|g\|_1\left(C_1 - |x|^2\right) + C_2\|g\|_1\left[\left(1 - \frac{\gamma}{1 + \left(\frac{d(x)}{\varepsilon}\right)^2}\right) \wedge \left(1 - \frac{\gamma}{1 + \left(\frac{\mu R}{\varepsilon}\right)^2}\right)\right],$$

for $a \wedge b = \min(a, b)$, $R > 0$ the minimal radius of curvature $R = \min_{x\in\partial\Omega} R(x)$ and $d(x) :=$ dist$(x, \partial\Omega)$, satisfies $\mathcal{L}_\Omega^\varepsilon\left(\Phi^\varepsilon\right)(x) \geq \|g\|_1 e^{-\frac{d(x)}{\varepsilon}}$ in Ω uniformly for all $\varepsilon < \varepsilon_0$. Moreover, the solutions u^ε of (C.60) *are uniformly bounded in ε.*

We split the proof of this theorem in two lemmas.

Lemma C.7. *Let $C_1 := 2\max_{x\in\overline{\Omega}}|x|^2 + 2\operatorname{diam}(\Omega)^2 + 4\operatorname{diam}(\Omega) + 4$, let $0 < \varepsilon < 1$. Then*

$$\mathcal{L}_\Omega^\varepsilon\left(C_1 - |x|^2\right) \geq 2\varepsilon^2.$$

Proof. We start computing the action of $\mathcal{L}_{\mathbb{R}^3}^\varepsilon$ on $|x|^2$.

$$\begin{aligned} \mathcal{L}_{\mathbb{R}^3}^\varepsilon\left[|\cdot|^2\right](x) &= |x|^2 - \int_{\mathbb{R}^3} K_\varepsilon(\eta - x)|\eta|^2\, d\eta \\ &= -\int_{\mathbb{R}^3} K_\varepsilon(\eta - x)|\eta - x|^2\, d\eta = -2\varepsilon^2, \end{aligned}$$

where we expanded $|\eta|^2 = |x + (\eta - x)|^2$ and we used that $\int_{\mathbb{R}^3} K_\varepsilon = 1$ and the symmetry of the kernel K_ε.

Le $D := \operatorname{diam}(\Omega)$ and let $B = 2\max_{x\in\overline{\Omega}} |x|^2$ and $\beta = 2D^2 + 4D + 4$. Thus, $C_1 = B + \beta$. Then

$$\begin{aligned}\mathcal{L}^\varepsilon_\Omega \left(K + \beta - |\cdot|^2\right)(x) &= (B+\beta)\int_{\Omega^c} K_\varepsilon(\eta - x)\ d\eta\ - \mathcal{L}^\varepsilon_{\mathbb{R}^3}\left[|\cdot|^2\right](x) - \int_{\Omega^c} K_\varepsilon(\eta - x)\,|\eta|^2\ d\eta \\ &\geq (B+\beta)\int_{\Omega^c} K_\varepsilon(\eta - x)\ d\eta + 2\varepsilon^2 - 2|x|^2\int_{\Omega^c} K_\varepsilon(\eta - x)\ d\eta - 2\int_{\Omega^c} K_\varepsilon(\eta - x)\,|\eta - x|^2\ d\eta,\end{aligned}$$

where we used $|\eta|^2 \leq 2|x|^2 + 2|\eta - x|^2$. Moreover using that $B - 2|x|^2 \geq 0$ and splitting for $x \in \Omega$ the complement of the domain as $\Omega^c = (\Omega^c \cap B_D(x)) \cup B^c_D(x)$ we obtain

$$\begin{aligned}\mathcal{L}^\varepsilon_\Omega \left(K + \beta - |\cdot|^2\right)(x) \geq& 2\varepsilon^2 + \int_{B^c_{D/\varepsilon}(0)} K_\varepsilon(\eta)\left(\beta - 2\varepsilon^2|\eta|^2\right)\ d\eta \\ =& 2\varepsilon^2 + \beta e^{-\frac{D}{\varepsilon}} - e^{-\frac{D}{\varepsilon}}\left(2D^2 + 4D\varepsilon + 4\varepsilon^2\right) \geq 2\varepsilon^2,\end{aligned}$$

where in the first inequality we used that $2|\eta - x|^2 \leq 2D^2 \leq \beta$ for $\eta, x \in B_D(x)$ and for the integral in $B^c_D(x)$ we changed variables $\frac{\eta - x}{\varepsilon} \mapsto \eta$ and we computed the resulting integral using also that $\varepsilon < 1$. □

In order to proceed further with the construction of the supersolution, we will use repeatedly the distance function and its relation to the curvature of the domain's boundary. All the properties of this function can be found in the Appendix "Boundary curvatures and distance functions" in [67]. It is well-know that if the boundary $\partial\Omega$ is C^3, then in a neighborhood of the boundary the distance function can be expanded by Taylor as

$$d(\eta) = d(x) + \nabla d(x)\cdot(\eta - x) + \frac{1}{2}(\eta - x)^\top \nabla^2 d(x)(\eta - x) + \mathcal{O}\left(|\eta - x|^3\right) \tag{C.62}$$

Moreover, the following proposition holds.

Proposition C.4. *For $x \in \Omega$ in a neighborhood of the boundary the gradient of the distance function is the inner normal, so that $|\nabla d(x)| = 1$. Moreover, denoting $R = \min_{x\in\partial\Omega} R(x) > 0$ the minimal radius of curvature and letting $\mu \in (0,1)$ we have*

$$\xi^\top \nabla^2 d(x)\xi \leq \frac{1}{(1-\mu)R} \tag{C.63}$$

for every $x \in \{y \in \Omega :\ d(y) < R\mu\}$ and $\|\xi\| = 1$.

Proof. See 14.6, Appendix "Boundary curvatures and distance functions" ([67]). □

Using these properties of the distance function we can prove the next lemma.

Lemma C.8. *Let $\psi(x) := \left(1 - \frac{\gamma}{1+\left(\frac{d(x)}{\varepsilon}\right)^2}\right) \wedge \left(1 - \frac{\gamma}{1+\left(\frac{\mu R}{\varepsilon}\right)^2}\right)$. Then there exists some $0 < \mu < 1$ small enough, $0 < \gamma(\mu) < \frac{1}{3}$, $0 < \varepsilon_1 < 1$ small enough and constants $C_0 := C_0(R, \Omega, \mu, \gamma) > 0$ and $c := c(R, \mu, \gamma) > 0$ such that for all $0 < \varepsilon \leq \varepsilon_1$*

$$\mathcal{L}^\varepsilon_\Omega(\psi)(x) \geq \begin{cases} C_0 e^{-\frac{d(x)}{\varepsilon}} & 0 < d(x) \leq \frac{R\mu}{2} \\ -c\varepsilon^2 & \frac{R\mu}{2} < d(x) < R\mu \\ 0 & d(x) \geq R\mu \end{cases} \tag{C.64}$$

Proof. We start with some preliminary consideration on the distance function. We define $\frac{d(\eta)}{\varepsilon} := d_\varepsilon(\eta)$. For every $x, \eta \in \{y \in \Omega : \ d(y) < R\mu\}$ we have using (C.62)

$$\begin{aligned} d_\varepsilon(\eta)^2 =& d_\varepsilon(x)^2 + \frac{2d(x)\nabla d(x)\cdot(\eta-x)}{\varepsilon^2} + \frac{d(x)(\eta-x)^\top \nabla^2 d(x)(\eta-x)}{\varepsilon^2} \\ &+ \frac{(\nabla d(x)\cdot(\eta-x))^2}{\varepsilon^2} + \mathcal{O}\left(\frac{d(x)}{\varepsilon^2}|\eta-x|^3\right). \end{aligned} \tag{C.65}$$

Then Taylor's expansion shows

$$\begin{aligned} \frac{1}{1+d_\varepsilon(\eta)^2} =& \frac{1}{\left(1+d_\varepsilon(x)^2\right)\left(1+\left[d_\varepsilon(\eta)^2 - d_\varepsilon(x)^2\right]\frac{1}{1+d_\varepsilon(x)^2}\right)} \\ =& Q_\varepsilon^{(1)}(x,\eta) + Q_\varepsilon^{(2)}(x,\eta) + Q_\varepsilon^{(3)}(x,\eta), \end{aligned} \tag{C.66}$$

where we the terms $Q_\varepsilon^{(i)}$ are defined as follows.

$$Q_\varepsilon^{(1)}(x,\eta) = \frac{1}{1+d_\varepsilon(x)^2} - \frac{2d(x)\nabla d(x)\cdot(\eta-x)}{\varepsilon^2\left(1+d_\varepsilon(x)^2\right)^2},$$

$$Q_\varepsilon^{(2)}(x,\eta) = -\frac{d(x)(\eta-x)^\top\nabla^2 d(x)(\eta-x)}{\varepsilon^2\left(1+d_\varepsilon(x)^2\right)^2} - \frac{(\nabla d(x)\cdot(\eta-x))^2}{\varepsilon^2\left(1+d_\varepsilon(x)^2\right)^2} + \frac{4d^2(x)(\nabla d(x)\cdot(\eta-x))^2}{\varepsilon^4\left(1+d_\varepsilon(x)^2\right)^3},$$

$$Q_\varepsilon^{(3)}(x,\eta) = \mathcal{O}\left(\frac{d(x)}{\varepsilon^2}\frac{|\eta-x|^3}{\left(1+d_\varepsilon(x)^2\right)^2}\right) + \mathcal{O}\left(\frac{d(x)}{\varepsilon^4}\frac{|\eta-x|^3}{\left(1+d_\varepsilon(x)^2\right)^3}\right).$$

We consider now the function $\psi(x)$ defined in the statement of Lemma C.8. We take $M = \frac{1}{\mu^2}$ for $0 < \mu < 1$ small enough and $0 < \varepsilon < 1$ also small enough such that $0 < M\varepsilon < \frac{R\mu}{2}$, i.e. $0 < \varepsilon < \frac{R\mu^3}{2}$, and we decompose Ω in four disjoint sets

$$\Omega = \{d(x) \geq R\mu\} \cup \{d(x) < M\varepsilon\} \cup \left\{M\varepsilon \leq d(x) \leq \frac{R\mu}{2}\right\} \cup \left\{\frac{R\mu}{2} < d(x) < R\mu\right\}.$$

We proceed estimating $\mathcal{L}_\Omega^\varepsilon(\psi)(x)$ for x in each of these regions of Ω.

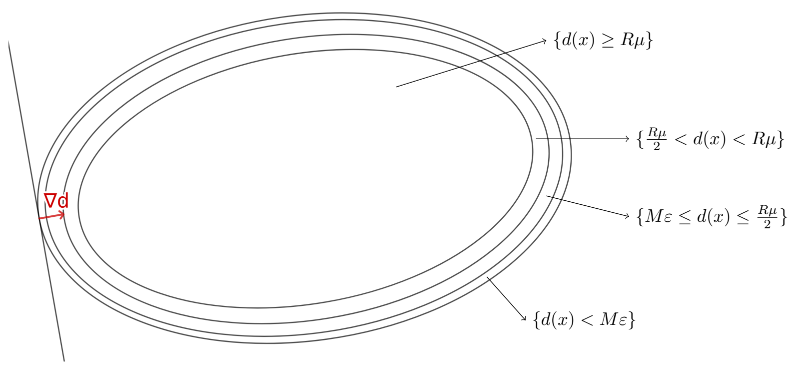

Figure C.4: Decomposition of Ω.

For further reference we write

$$\begin{aligned}\mathcal{L}_{\Omega}^{\varepsilon}\left(\psi\right)(x) =&\psi(x)-\int_{\Omega\cap\{d(\eta)<R\mu\}} d\eta K_{\varepsilon}(\eta-x)\left(1-\frac{\gamma}{1+d_{\varepsilon}\left(\eta\right)^{2}}\right)\\ &-\int_{\Omega\cap\{d(\eta)\geq R\mu\}} d\eta K_{\varepsilon}(\eta-x)\left(1-\frac{\gamma}{1+\left(\frac{\mu R}{\varepsilon}\right)^{2}}\right).\end{aligned} \tag{C.67}$$

In order to estimate $\mathcal{L}_{\Omega}^{\varepsilon}(\psi)(x)$ in the region $\{d(x)\geq R\mu\}$ we will use the fact that the minimum of supersolutions is again a supersolution. In the region where $d(x)<M\varepsilon$ we will use the explicit form of the kernel to see that the main contribution has the right sign. Finally, in the region $\{M\varepsilon\leq d(x)<R\mu\}$ the idea behind the arguments we will present is that $\mathcal{L}_{\Omega}^{\varepsilon}(\psi)(x)$ can be approximated by $-\varepsilon^{2}\Delta\psi$ using Taylor.

Step 1: $\{d(x)\geq R\mu\}$

First of all we notice that if $d(x)\geq R\mu$ then $\mathcal{L}_{\Omega}^{\varepsilon}(\psi)(x)\geq 0$. Indeed, $\psi(\eta)\leq\psi(x)=1-\frac{\gamma}{1+\left(\frac{\mu R}{\varepsilon}\right)^{2}}$ in the first integral of (C.67) since $d(\eta)<R\mu$ there. Hence

$$\mathcal{L}_{\Omega}^{\varepsilon}\left(\psi\right)(x)\geq\mathcal{L}_{\Omega}^{\varepsilon}\left(1-\frac{\gamma}{1+\left(\frac{\mu R}{\varepsilon}\right)^{2}}\right)\geq 0. \tag{C.68}$$

Step 2: $\{d(x)<M\varepsilon\}$

We consider now the region $\{d(x)<M\varepsilon\}$. After a suitable rigid motion we can assume $0\in\partial\Omega$ and $x=(d(x),0,0)$. Hence, $\Omega\subset\mathbb{R}_{+}\times\mathbb{R}^{2}$ and

$$\int_{\Omega^{c}}\frac{e^{-\frac{|\eta-x|}{\varepsilon}}}{4\pi\varepsilon\left|\eta-x\right|^{2}}\,d\eta\geq\int_{-\infty}^{-d(x)/\varepsilon}K\left(\eta\right)\,d\eta\geq\int_{-\infty}^{-M}K\left(\eta\right)\,d\eta:=\nu_{M}>0.$$

K is as usual the normalized exponential integral. On the other hand, using that $\frac{1}{1+d_{\varepsilon}(x)^{2}}\leq 1$ and choosing $\gamma<\frac{\nu_{M}}{2}$ we can conclude

$$\begin{aligned}\mathcal{L}_{\Omega}^{\varepsilon}\left(\psi\right)(x)=&-\frac{\gamma}{1+d_{\varepsilon}\left(x\right)^{2}}+\int_{\Omega^{c}}d\eta K_{\varepsilon}(\eta-x)\\ &+\gamma\int_{\Omega}d\eta K_{\varepsilon}(\eta-x)\left(\frac{1}{1+d_{\varepsilon}\left(\eta\right)^{2}}\vee\frac{1}{1+\left(\frac{\mu R}{\varepsilon}\right)^{2}}\right)\geq\frac{\nu_{M}}{2}\geq\frac{\nu_{M}}{2}e^{-d_{\varepsilon}(x)},\end{aligned} \tag{C.69}$$

where $a\vee b=\max(a,b)$.

Step 3: $\left\{M\varepsilon\leq d(x)\leq\frac{R\mu}{2}\right\}$

We consider now the set $\left\{M\varepsilon\leq d(x)\leq\frac{R\mu}{2}\right\}$. As first step we plug (C.66) into the right hand side of (C.67). To this end we define three integral terms $J_{1},\ J_{2},\ J_{3}$ as

$$\begin{aligned}J_{1}=&1-\frac{\gamma}{1+d_{\varepsilon}\left(x\right)^{2}}-\int_{\Omega\cap\{d(\eta)<R\mu\}}d\eta\;K_{\varepsilon}(\eta-x)\left(1-\gamma Q_{\varepsilon}^{(1)}(x,\eta)\right)\\ &-\int_{\Omega\cap\{d(\eta)\geq R\mu\}}d\eta\;K_{\varepsilon}(\eta-x)\left(1-\frac{\gamma}{1+\frac{R^{2}\mu^{2}}{\varepsilon^{2}}}\right),\end{aligned} \tag{C.70}$$

$$J_2 = \int_{\Omega\cap\{d(\eta)<R\mu\}} d\eta\; K_\varepsilon(\eta - x)\left(\gamma Q^{(2)}_\varepsilon(x,\eta)\right), \tag{C.71}$$

$$J_3 = \int_{\Omega\cap\{d(\eta)<R\mu\}} d\eta\; K_\varepsilon(\eta - x)\left(\gamma Q^{(3)}_\varepsilon(x,\eta)\right). \tag{C.72}$$

Hence, we have

$$\mathcal{L}^\varepsilon_\Omega(\psi)(x) = J_1 + J_2 + J_3. \tag{C.73}$$

The main contribution to these terms is due to J_2. Therefore we start with this term and we show that for $0<\mu<1$ small enough there exists a constant $\tilde{C}(\mu)>0$ independent of ε such that

$$J_2 \geq \frac{\tilde{C}(\mu)\gamma}{\left(1+d_\varepsilon(x)^2\right)^2}. \tag{C.74}$$

In order to prove this estimate we first notice that

$$\frac{4d_\varepsilon(x)^2}{\left(1+d_\varepsilon(x)^2\right)} - 1 = 3 - \frac{4}{\left(1+d_\varepsilon(x)^2\right)} \geq 3 - \frac{4}{(1+M^2)} \geq 0. \tag{C.75}$$

Hence, multiplying this inequality by $K_\varepsilon(\eta-x)\frac{\gamma(\nabla d(x)\cdot(\eta-x))^2}{\varepsilon^2\left(1+d_\varepsilon(x)^2\right)^2}$ and integrating on $\{d(\eta)<R\mu\}$ we obtain

$$\begin{aligned}
&\int_{\Omega\cap\{d(\eta)<R\mu\}} d\eta\; K_\varepsilon(\eta-x)\left(-\frac{\gamma\left(\nabla d(x)\cdot(\eta-x)\right)^2}{\varepsilon^2\left(1+d_\varepsilon(x)^2\right)^2} + \frac{4\gamma d^2(x)\left(\nabla d(x)\cdot(\eta-x)\right)^2}{\varepsilon^4\left(1+d_\varepsilon(x)^2\right)^3}\right)\\
&\qquad\geq \frac{\gamma\left(3-\frac{4}{1+M^2}\right)}{\left(1+d_\varepsilon(x)^2\right)^2}\int_{B_{M\varepsilon}(x)} d\eta\; K_\varepsilon(\eta-x)\frac{\left(\nabla d(x)\cdot(\eta-x)\right)^2}{\varepsilon^2}
\end{aligned} \tag{C.76}$$

$$= \frac{\gamma\left(3-\frac{4}{1+M^2}\right)}{\left(1+d_\varepsilon(x)^2\right)^2}\frac{1}{4\pi}\int_0^{2\pi} d\varphi\int_0^\pi d\theta\sin(\theta)\cos^2(\theta)\int_0^M dr\; e^{-r}r^2 = \frac{\gamma C(M)\left(3-\frac{4}{1+M^2}\right)}{\left(1+d_\varepsilon(x)^2\right)^2},$$

where used that $B_{M\varepsilon}(x)\subset\{d(\eta)<R\mu\}$ and we define the constant $C(M)=\frac{1}{3}\int_0^M dr\; e^{-r}r^2 = \frac{1}{3}(2-2e^{-M}-2Me^{-M}-M^2e^{-M})$ which depends on $M=\frac{1}{\mu^2}$. Notice that $C(M)\to\frac{2}{3}$ as $M\to\infty$ and hence for M sufficiently large we have also $C(M)\geq\frac{1}{2}$.

In order to conclude the estimate for J_2 we use the result (C.63) to estimate the Hessian of the distance function, thus

$$\frac{\gamma d(x)\,(\eta-x)^\top\,\nabla^2 d(x)\,(\eta-x)}{\varepsilon^2\left(1+d_\varepsilon(x)^2\right)^2} \leq \frac{\gamma\mu\,|\eta-x|^2}{\varepsilon^2(1-\mu)\left(1+d_\varepsilon(x)^2\right)^2} \tag{C.77}$$

and we conclude

$$-\int_{\Omega\cap\{d(\eta)<R\mu\}} d\eta\; K_\varepsilon(\eta-x)\frac{\gamma d(x)\,(\eta-x)^\top\,\nabla^2 d(x)\,(\eta-x)}{\varepsilon^2\left(1+d_\varepsilon(x)^2\right)^2} \geq -C\frac{\gamma\mu}{(1-\mu)\left(1+d_\varepsilon(x)^2\right)^2}, \tag{C.78}$$

for some constant $C > 0$.

Combining (C.76) and (C.78) we obtain (C.74).

We proceed now with the term J_1 in (C.70). Using the symmetry of the scalar product in $\mathbb{R}^3$ we write

$$J_1 = \int_{\Omega^c} d\eta\, K_\varepsilon(\eta - x)\left(1 - \gamma Q_\varepsilon^{(1)}(x,\eta)\right) + \int_{\Omega\cap\{d(\eta)\geq R\mu\}} d\eta\, K_\varepsilon(\eta - x)\left(\frac{\gamma}{1+\frac{R^2\mu^2}{\varepsilon^2}} - \gamma Q_\varepsilon^{(1)}(x,\eta)\right) = J_{1,1} + J_{1,2}. \tag{C.79}$$

We proceed with the estimate for $J_{1,1}$ in (C.79). By means of a suitable coordinate system we can assume again $0 \in \partial\Omega$ and $x = (d(x), 0, 0)$. We notice that if $\eta \in (-\infty, d(x)) \times \mathbb{R}^2$ then $\nabla d(x) \cdot (\eta - x) = \eta_1 - d(x) \leq 0$, while if $\eta \in (d(x), \infty) \times \mathbb{R}^2$ then $\nabla d(x) \cdot (\eta - x) \geq 0$. Hence, we obtain

$$J_{1,1} \geq \int_{\Omega^c\cap(-\infty,d(x))\times\mathbb{R}^2} d\eta\, K_\varepsilon(\eta - x)\left(1 - \gamma Q_\varepsilon^{(1)}(x,\eta)\right). \tag{C.80}$$

We now decompose the set $\Omega^c \cap \left((-\infty, d(x)) \times \mathbb{R}^2\right) = \left((-\infty, 0) \times \mathbb{R}^2\right) \cup \left(\Omega^c \cap \left((0, d(x)) \times \mathbb{R}^2\right)\right)$. Using that

$$\frac{d(x)}{\varepsilon^2\left(1+d_\varepsilon(x)^2\right)^2} = \frac{1}{d(x)\left(1+d_\varepsilon(x)^2\right)} - \frac{1}{d(x)\left(1+d_\varepsilon(x)^2\right)^2} \tag{C.81}$$

and since $\gamma < \frac{1}{3}$ we have $1 - \frac{\gamma}{1+d_\varepsilon(x)^2} > 0$ and therefore we obtain

$$\begin{aligned}
\int_{(-\infty,0)\times\mathbb{R}^2} d\eta\, K_\varepsilon(\eta - x)\left(1 - \gamma Q_\varepsilon^{(1)}(x,\eta)\right) &\geq \int_{(-\infty,0)\times\mathbb{R}^2} d\eta\, K_\varepsilon(\eta - x)\frac{2\gamma\nabla d(x)\cdot(\eta - x)}{d_\varepsilon(x)\,\varepsilon\left(1+d_\varepsilon(x)^2\right)} \\
&= -\frac{2\gamma}{d_\varepsilon(x)\left(1+d_\varepsilon(x)^2\right)}\int_{d_\varepsilon(x)}^{\infty} dz\, K(z)\, z \geq -\frac{\gamma}{2d_\varepsilon(x)}\frac{1+d_\varepsilon(x)}{1+d_\varepsilon(x)^2}e^{-d_\varepsilon(x)} \\
&\geq -\frac{\gamma C}{M}\frac{1}{\left(1+d_\varepsilon(x)^2\right)^2},
\end{aligned} \tag{C.82}$$

where we also changed variable $(d_\varepsilon(x) - z) \mapsto z$, we used the identity (C.25) for the normalized exponential integral in Proposition C.2, we estimated $d_\varepsilon(x) \geq M$ and finally we denote by C the constant such that $\frac{(1+x^2)^2}{2}e^{-|x|} \leq C$.

Concerning the integral in the set $\Omega^c \cap \left((0, d(x)) \times \mathbb{R}^2\right)$ we proceed similarly using again (C.81) and also the fact that if $z > 0$ then $z - d(x) > -d(x)$. Hence, we have

$$\begin{aligned}
&\int_{\Omega^c\cap((0,d(x))\times\mathbb{R}^2)} d\eta\, K_\varepsilon(\eta - x)\left(1 - \gamma Q_\varepsilon^{(1)}(x,\eta)\right) \\
&\geq \int_{\Omega^c\cap((0,d(x))\times\mathbb{R}^2)} d\eta\, K_\varepsilon(\eta - x)\left(1 - \frac{\gamma}{1+d_\varepsilon(x)^2} + \frac{2\gamma\nabla d(x)\cdot(\eta - x)}{d_\varepsilon(x)\,\varepsilon\left(1+d_\varepsilon(x)^2\right)}\right) \\
&= \int_{\Omega^c\cap((0,d(x)\times\mathbb{R}^2)} dz\, K_\varepsilon(\eta - d(x)e_1)\left(1 - \frac{\gamma}{1+d_\varepsilon(x)^2} + \frac{2\gamma(\eta_1 - d(x))}{d(x)\left(1+d_\varepsilon(x)^2\right)}\right) \geq 0
\end{aligned} \tag{C.83}$$

Hence, for $M\varepsilon \leq d(x) < R\mu$ and $\gamma < \frac{1}{3}$ we can summarize

$$J_{1,1} \geq -\frac{\gamma}{\left(1+d_\varepsilon(x)^2\right)^2}\frac{C}{M}. \tag{C.84}$$

Remark. Notice that the estimates (C.80)-(C.84) are valid in the whole region $\{M\varepsilon \leq d(x) < R\mu\}$.

We still have to consider the integral $J_{1,2}$ in (C.79). We notice that for all $\eta \in \Omega$ with $d(\eta) \geq R\mu$ we have on the one hand $|\eta - x| \geq \frac{R\mu}{2}$ and on the other hand $\nabla d(x) \cdot (\eta - x) \geq 0$ since $d(\eta) > d(x)$. We recall that $D := \operatorname{diam}(\Omega)$ and that $\Omega \cap \{d(\eta) \geq R\mu\} \subset B_D(x)$. Therefore, we estimate

$$\begin{aligned} J_{1,2} \geq& -\int_{\Omega\cap\{d(\eta)\geq R\mu\}} d\eta\; K_\varepsilon(\eta - x)\frac{\gamma}{1+d_\varepsilon(x)^2} \geq -\frac{\gamma e^{-\frac{R\mu}{2\varepsilon}}}{1+d_\varepsilon(x)^2}\int_{B_D(0)} dz\; \frac{1}{4\pi\varepsilon|z|^2} \\ \geq& -\gamma\frac{e^{-\frac{d_\varepsilon(x)}{2}}}{1+d_\varepsilon(x)^2}\frac{4D}{R\mu} \geq -\gamma C\frac{D}{R}\frac{\mu}{\left(1+d_\varepsilon(x)^2\right)^2} \end{aligned} \tag{C.85}$$

where we used the well-known estimate $xe^{-x} \leq e^{-1}$ combined with $e^{-\frac{R\mu}{4\varepsilon}} \leq e^{-\frac{d(x)}{2\varepsilon}}$ and we denoted by C the constant such that $4x(1+x^2)e^{-\frac{x}{2}} \leq C$ and finally the relation $M = \frac{1}{\mu^2}$.

Finally we estimate the term J_3 in (C.72). Here we have to estimate the integral term containing the error terms $Q_\varepsilon^{(3)}(x,\eta)$ of the Taylor expansion (C.65). If $M\varepsilon \leq d(x) \leq \frac{R\mu}{2}$ and if $\varepsilon < 1$ we use $\frac{x}{1+x^2} = \frac{1}{x} - \frac{1}{x(1+x^2)}$ and we calculate

$$\begin{aligned} &\gamma\int_{\Omega\cap\{d(\eta)<R\mu\}} d\eta\; K_\varepsilon(\eta - x)\left(\frac{d(x)}{\varepsilon^2}\frac{|\eta - x|^3}{\left(1+d_\varepsilon(x)^2\right)^2} + \frac{d(x)}{\varepsilon^4}\frac{|\eta - x|^3}{\left(1+d_\varepsilon(x)^2\right)^3}\right) \\ &\qquad\leq \int_{\mathbb{R}^3} d\eta\; \frac{\gamma e^{-|\eta|}}{4\pi}\frac{|\eta|}{\left(1+d_\varepsilon(x)^2\right)^2}\left(d(x)\varepsilon + \frac{1}{\frac{d(x)}{\varepsilon}} - \frac{1}{\frac{d(x)}{\varepsilon}\left(1+d_\varepsilon(x)^2\right)}\right) \\ &\qquad\leq \frac{C\gamma}{\left(1+d_\varepsilon(x)^2\right)^2}\left(\frac{R\mu}{2} + \mu^2\right). \end{aligned} \tag{C.86}$$

Hence, also $J_3 \geq -\frac{C\gamma}{\left(1+d_\varepsilon(x)^2\right)^2}\left(\frac{R\mu}{2} + \mu^2\right)$.

We conclude putting together estimates (C.74) (C.79), (C.84), (C.85) and (C.86) the existence of a constant $C(\Omega) > 0$ independent of μ, γ, ε such that

$$\mathcal{L}_\Omega^\varepsilon(\psi)(x) \geq \frac{\gamma}{\left(1+d_\varepsilon(x)^2\right)^2}\left[C(M)\left(3 - \frac{4}{1+M^2}\right) - C(\Omega)\frac{\mu}{1-\mu}\right]. \tag{C.87}$$

Choosing $0 < \mu < 1$ small enough, depending only on Ω, such that $C(M) > \frac{1}{3}$ and $C(\Omega)\frac{\mu}{1-\mu} < \frac{1}{6}$ we obtain

$$\mathcal{L}_\Omega^\varepsilon(\psi)(x) \geq \frac{\gamma}{6\left(1+d_\varepsilon(x)^2\right)^2} \geq Ce^{-\frac{d(x)}{\varepsilon}} \tag{C.88}$$

for $M\varepsilon \le d(x) \le \frac{R\mu}{2}$ and some constant C depending on Ω, R, γ, μ but independent of ε.

Step 4: $\left\{\frac{R\mu}{2} < d(x) < R\mu\right\}$

It remains to calculate the behavior of $\mathcal{L}^{\varepsilon}_{\Omega}(\psi)$ when $\frac{R\mu}{2} < d(x) < R\mu$. Here, we show that there exists a constant $c(R,\mu,\gamma)$ such that $\mathcal{L}^{\varepsilon}_{\Omega}(\psi)(x) \ge -c\varepsilon^2$. We can use several results we obtained in Step 3. We decompose again the operator $\mathcal{L}^{\varepsilon}_{\Omega}(\psi)(x) = J_1 + J_2 + J_3$ according to (C.73) using the integral terms defined in (C.70)-(C.72).

First of all (C.75) implies

$$\int_{\Omega\cap\{d(\eta)<R\mu\}} d\eta\, K_\varepsilon(\eta-x)\left(-\frac{\gamma\left(\nabla d(x)\cdot(\eta-x)\right)^2}{\varepsilon^2\left(1+d_\varepsilon(x)^2\right)^2}+\frac{4\gamma d^2(x)\left(\nabla d(x)\cdot(\eta-x)\right)^2}{\varepsilon^4\left(1+d_\varepsilon(x)^2\right)^3}\right)\ge 0$$

and hence we estimate J_2 using (C.77) and (C.78)

$$\begin{aligned} J_2 \ge& -\int_{\Omega\cap\{d(\eta)<R\mu\}} d\eta\, K_\varepsilon(\eta-x)\frac{\gamma d(x)\,(\eta-x)^\top\nabla^2 d(x)\,(\eta-x)}{\varepsilon^2\left(1+d_\varepsilon(x)^2\right)^2}\\ \ge& -C\frac{\gamma\mu}{(1-\mu)\left(1+d_\varepsilon(x)^2\right)^2}\ge -\frac{8\gamma C}{(1-\mu)R^3}\varepsilon^3, \end{aligned} \tag{C.89}$$

where we used $1+d_\varepsilon(x)^2 \ge d_\varepsilon(x)^2 \ge \left(\frac{R\mu}{2\varepsilon}\right)^2$ and $0<\varepsilon<\frac{R\mu^3}{2}$.

We now proceed to estimate J_1. To this end we use again the decomposition (C.79). The estimate (C.84) for $J_{1,1}$ is also valid in the region $\{\frac{R\mu}{2} < d(x) < R\mu\}$, as we indicated in the remark after (C.84). Hence we have for $\varepsilon < \frac{R\mu^3}{2}$

$$J_{1,1} \ge -\frac{\gamma\mu^2 C}{\left(1+d_\varepsilon(x)^2\right)^2} \ge -\frac{8\gamma C}{R^2}\varepsilon^3.$$

Concerning the term $J_{1,2}$ we have to argue slightly different than in Step 3. Using now the first inequality in (C.85) and $\int_{\mathbb{R}^3} d\eta K_\varepsilon(\eta-x)=1$ we compute

$$J_{1,2} \ge -\int_{\Omega\cap\{d(\eta)\ge R\mu\}} d\eta\, K_\varepsilon(\eta-x)\frac{\gamma}{1+d_\varepsilon(x)^2} \ge -\frac{\gamma}{1+d_\varepsilon(x)^2} \ge -\frac{4\gamma}{(R\mu)^2}\varepsilon^2. \tag{C.90}$$

Finally, we estimate J_3 as defined in (C.72). Arguing as in (C.86) and using $1+x^2 \ge x^2$ and $0<\varepsilon<\frac{R\mu^3}{2}$ we compute

$$\begin{aligned} &\int_{\Omega\cap\{d(\eta)<R\mu\}} d\eta\, K_\varepsilon(\eta-x)\left(\frac{d(x)}{\varepsilon^2}\frac{|\eta-x|^3}{\left(1+d_\varepsilon(x)^2\right)^2}+\frac{d(x)}{\varepsilon^4}\frac{|\eta-x|^3}{\left(1+d_\varepsilon(x)^2\right)^3}\right)\\ &\le \frac{\gamma\left(d(x)^2+1\right)}{4\pi\left(d_\varepsilon(x)\right)^5}\int_{\mathbb{R}^3} d\eta\, e^{-|\eta|}\,|\eta| \le \frac{2\gamma C\left(R^2+2\right)}{R^3}\varepsilon^2. \end{aligned} \tag{C.91}$$

Thus, also $J_3 \ge -\frac{2\gamma C\left(R^2+2\right)}{R^3}\varepsilon^2$.

Hence, (C.89),(C.8),(C.90) and (C.91) imply the existence of a constant $c(R,\mu,\gamma) > 0$ independent of ε such that

$$\mathcal{L}_{\Omega}^{\varepsilon}(\psi)(x) \geq -c\varepsilon^2 \tag{C.92}$$

for all $\frac{R\mu}{2} < d(x) < R\mu$.

We know summarize the results. Equations (C.68), (C.69), (C.88), (C.92) imply the claim in (C.64). We remark that μ, γ and ε_1 are chosen as follows. First of all μ is chosen according to Step 3 as in (C.87), then γ is taken according to Step 2 such that $0<\gamma<\frac{\nu_M}{2}$ and finally ε_1 satisfies $0<\varepsilon_1<\frac{R\mu^3}{2}$. This concludes the Lemma C.8. □

Using Lemma C.7 and C.8 we can now prove Theorem C.6.

(Proof of Theorem C.6). Let C_1 be the constant defined in Lemma C.7 and let γ, μ, C_0, c be as in Lemma C.8. We define $C_2 := \frac{1}{C_0}$ and $C_3 := \frac{C_0+c}{2C_0} > \frac{1}{2}$. Notice that all these constants are independent of ε. Hence, Lemma C.7 and C.8 imply

$$\mathcal{L}_{\Omega}^{\varepsilon}(\Phi^{\varepsilon})(x) \geq \|g\|_1 \begin{cases} e^{-\frac{d(x)}{\varepsilon}} + 2C_3\varepsilon^2 & 0<d(x)\leq \frac{R\mu}{2}, \\ \varepsilon^2 & \frac{R\mu}{2} < d(x) < R\mu, \\ 2C_3\varepsilon^2 & d(x)\geq R\mu, \end{cases} \geq \|g\|_1 \begin{cases} e^{-\frac{d(x)}{\varepsilon}} & 0<d(x)\leq \frac{R\mu}{2}, \\ \varepsilon^2 & \frac{R\mu}{2} < d(x) < R\mu, \\ \varepsilon^2 & d(x)\geq R\mu. \end{cases} \tag{C.93}$$

We define now $\varepsilon_0 := \min\{1,\ a, \varepsilon_1\}$ with a such that $2a\ln(\frac{1}{a}) < \frac{R\mu}{2}$ and $\varepsilon_1 > 0$ as in Lemma C.8. Then $\varepsilon^2 \geq e^{-\frac{R\mu}{2\varepsilon}} \geq e^{-\frac{d(x)}{\varepsilon}}$ for all $d(x) > \frac{R\mu}{2}$.

We now apply the maximum principle in Theorem C.5 to the function $\Phi^{\varepsilon} - u^{\varepsilon}$. This function satisfies the continuity and boundedness assumption. Indeed, for any $\varepsilon > 0$ the function u^{ε} is continuous and bounded as we have seen at the beginning of Section 4.1. Moreover, by construction Φ^{ε} is continuous and it is easy to see that it is even uniformly bounded since

$$0 \leq \Phi^{\varepsilon}(x) \leq \|g\|_1 (2C_3C_1 + C_2).$$

We also have

$$\mathcal{L}_{\Omega}^{\varepsilon}(\Phi^{\varepsilon} - u^{\varepsilon})(x) \geq \|g\|_1 e^{-\frac{d(x)}{\varepsilon}} - \int_0^{\infty} d\nu \int_{n\cdot N_{x_\Omega}<0} dn\, g_{\nu}(n) e^{-\frac{|x-x_\Omega(x,n)|}{\varepsilon}} \geq 0,$$

since $|x - x_\Omega(x,n)| \geq d(x)$. Hence, Theorem C.5 implies that $\Phi^{\varepsilon} - u^{\varepsilon} \geq 0$ and thus

$$0 \leq u^{\varepsilon} \leq \Phi^{\varepsilon} \leq \tilde{C} < \infty$$

uniformly in ε and $x \in \Omega$. □

C.4.3 Estimates of $u^{\varepsilon} - \overline{u}$ near the boundary $\partial\Omega$

In this subsection we will prove that for each point $p \in \partial\Omega$ the function $\overline{u}$ defined in (C.22) is a good approximation of u^{ε} in a neighborhood of size close to $\varepsilon^{\frac{1}{2}}$. Notice that this neighborhood is much greater than the region of size ε. We will do it by means of the maximum principle in Theorem C.5. Now we start estimating the action of the operator $\mathcal{L}_{\Omega}^{\varepsilon}$ on $\overline{u} - u^{\varepsilon}$.

Lemma C.9. *Let $p \in \partial\Omega$ and let $\mathcal{R}_p$ be the isometry defined in* (C.13)*. Then the following holds for $x \in \Omega$, $\delta > 0$ sufficiently small and independent of ε and a suitable* $0 < A < 1$ *and constant* $C > 0$

$$\left|\mathcal{L}_{\Omega}^{\varepsilon}\left(\overline{u}\left(\frac{\mathcal{R}_p(\cdot)\cdot e_1}{\varepsilon}, p\right) - u^{\varepsilon}\right)(x)\right| \leq Ce^{-\frac{Ad(x)}{\varepsilon}} \begin{cases} \varepsilon^{\delta} & \text{if } |x-p| < \varepsilon^{\frac{1}{2}+2\delta}, \\ 1 & \text{if } |x-p| \geq \varepsilon^{\frac{1}{2}+2\delta}. \end{cases} \tag{C.94}$$

Proof. Let us denote by Π_p the half space $\Pi_p := \mathcal{R}_p^{-1}\left(\mathbb{R}_+ \times \mathbb{R}^2\right)$. Then the function $\overline{U}_\varepsilon(x,p) := \overline{u}\left(\frac{\mathcal{R}_p(x)\cdot e_1}{\varepsilon}, p\right)$ is a continuous bounded function which maps $\Pi_p \times \partial\Omega$ to $\mathbb{R}_+$. Notice that $\overline{U}_\varepsilon(x,p)$ is the solution to the planar equation (C.22) before rescaling and rotating. Our plan is to approximate $\mathcal{L}^\varepsilon_\Omega\left(\overline{U}_\varepsilon\right)$ by $\mathcal{L}^\varepsilon_{\Pi_p}\left(\overline{U}_\varepsilon\right)$. Let $x \in \Pi_p$ and $p \in \Omega$. Using the definition of $\overline{u}$ in (C.22) we can compute

$$\int_0^\infty d\eta\, K\left(\eta - \frac{\mathcal{R}_p(x)\cdot e_1}{\varepsilon}\right)\overline{u}(\eta,p) = \int_{\mathbb{R}_+\times\mathbb{R}^2} d\eta\, \frac{e^{-\left|\eta - \frac{\mathcal{R}_p(x)}{\varepsilon}\right|}}{4\pi\left|\eta - \frac{\mathcal{R}_p(x)}{\varepsilon}\right|^2}\overline{u}\left(\eta_1,p\right)$$
$$= \int_{\mathbb{R}_+\times\mathbb{R}^2} d\eta\, \frac{e^{-\frac{|\eta-\mathcal{R}_p(x)|}{\varepsilon}}}{4\pi\varepsilon\left|\eta-\mathcal{R}_p(x)\right|^2}\overline{u}\left(\frac{\eta_1}{\varepsilon},p\right) = \int_{\Pi_p} d\eta\, K_\varepsilon(\eta-x)\overline{U}_\varepsilon(\eta,p),$$

where we used in the first equality the translation invariance of the integral with respect to the second and third variable, the definition of the planar kernel and the definition of y. For the second equality we used the change of variables $\tilde{\eta} = \varepsilon\eta$ and in the last identity the change of variables $\tilde{\eta} = \mathcal{R}_p^{-1}(\eta)$ gives the result. In order to write the value of $\mathcal{L}^\varepsilon_{\Pi_p}\left(\overline{U}_\varepsilon\right)$ we use once again equation (C.22) and we define $x_{\Pi_p}(x,n)$ as the point on the boundary of Π_p with $\frac{x-x_{\Pi_p}(x,n)}{\left|x-x_{\Pi_p}(x,n)\right|} = n$, i.e. $x = x_{\Pi_p}(x,n) + \left|x - x_{\Pi_p}(x,n)\right| n$ if $n\cdot N_p < 0$. By construction we see that $\frac{\mathcal{R}_p(x)\cdot e_1}{|n\cdot N_p|} = \left|x - x_{\Pi_p}(x,n)\right|$. Hence,

$$\mathcal{L}^\varepsilon_{\Pi_p}\left(\overline{U}_\varepsilon(\cdot,p)\right)(x) = \int_0^\infty d\nu \int_{n\cdot N_p<0} dn\, g_\nu(n) e^{-\frac{\left|x-x_{\Pi_p}(x,n)\right|}{\varepsilon}}.$$

We will hence estimate the two integrals terms on the right hand side of the following equation

$$\left|\mathcal{L}^\varepsilon_\Omega\left(\overline{U}_\varepsilon(\cdot,p) - u^\varepsilon\right)(x)\right| \leq \int_{\Pi_p\setminus\Omega} d\eta K_\varepsilon(\eta-x)\overline{U}_\varepsilon(\cdot,p)$$
$$+ \int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, g_\nu(n)\left|e^{-\frac{\left|x-x_{\Pi_p}(x,n)\right|}{\varepsilon}} - e^{-\frac{\left|x-x_\Omega(x,n)\right|}{\varepsilon}}\right| = S_1 + S_2, \tag{C.95}$$

where we put $\left|x - x_{\Pi_p}(x,n)\right| = \infty$ if $n\cdot N_p \geq 0$.

Step 1: estimate of S_1.

It is always possible to estimate S_1 by $e^{-\frac{d(x)}{\varepsilon}}$, indeed using $B_{d(x)}(x) \subset \Omega$ we compute

$$\int_{\Pi_p\setminus\Omega} d\eta K_\varepsilon(\eta-x)\overline{U}_\varepsilon(\eta,p) \leq \int_{B^c_{d(x)}(x)} d\eta K_\varepsilon(\eta-x)\overline{U}_\varepsilon(\eta,p) \leq Ce^{-\frac{d(x)}{\varepsilon}}, \tag{C.96}$$

where $C>0$ is the uniform bound on $\overline{u}$ that we have obtained in Lemma C.5.

Our goal is to obtain a better estimate for the region $|x-p| < \varepsilon^{\frac{1}{2}+2\delta}$ (cf. (C.94)). Therefore we will now assume $|x-p| < \varepsilon^{\frac{1}{2}+2\delta}$ and since $d(x) < |x-p|$ we can also assume $d(x) < \varepsilon^{\frac{1}{2}+2\delta}$.

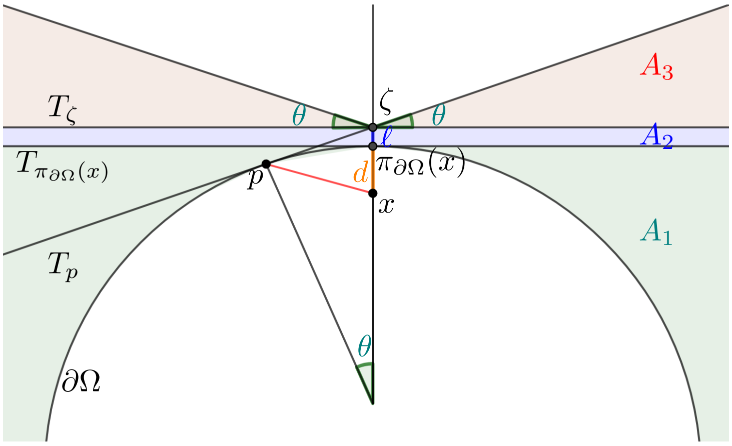

Figure C.5: Decomposition of $\Pi_p \setminus \Omega$.

Let $p \in \partial\Omega$ and Π_p the half space defined at the beginning of the proof. Let $x \in \Omega$ with $|x-p| < \varepsilon^{\frac{1}{2}+2\delta}$ for $\varepsilon > 0$ as small as needed. Let $\pi_{\partial\Omega}(x) \in \partial\Omega$ be the projection of x on the boundary as in (C.14). Then $x = \pi_{\partial\Omega}(x) - d(x)N_{\pi_{\partial\Omega}(x)}$. We denote further by $\theta(x)$ the angle between the normal vectors N_p and $N_{\pi_{\partial\Omega}(x)}$. Let T_p and $T_{\pi_{\partial\Omega}(x)}$ the tangent planes to $\partial\Omega$ containing p respectively $\pi_{\partial\Omega}(x)$. We define also $\zeta = T_p \cap \left\{x + tN_{\pi_{\partial\Omega}(x)} : t \geq 0\right\}$ and T_ζ the plane orthogonal to $N_{\pi_{\partial\Omega}(x)}$ containing ζ. We denote by $\ell(x) = \left|\zeta - N_{\pi_{\partial\Omega}(x)}\right|$ the distance between T_ζ and $T_{\pi_{\partial\Omega}(x)}$.

We decompose now $\Pi_p \setminus \Omega$ in three larger regions, i.e. $\Pi_p \setminus \Omega \subset A_1 \cup A_2 \cup A_3$. We define $A_1 := \Pi_{\pi_{\partial\Omega}(x)} \setminus \Omega$, A_2 is the region containing all points between the planes $T_{\pi_{\partial\Omega}(x)}$ and T_ζ and finally $A_3 := \left\{\zeta \in \mathbb{R}^3 :\ 0 \leq N_{\pi_{\partial\Omega}(x)} \cdot (\eta - \zeta) \leq \sin(\theta(x))|\eta - \zeta|\right\}$. The choice of these regions has been made in order to obtain integrals that are symmetric and easier to compute.

By standard differential geometry arguments we know that $\theta(x) \leq \frac{1}{R}\varepsilon^{\frac{1}{2}+2\delta}$ and that for some constant $C(\Omega) > 0$ also $\theta(x) \leq C(\Omega)|x-p|$. Moreover, denoting by ρ the radius of curvature of the curve given by the intersection of $\partial\Omega$ with the plane uniquely defined by N_p, $N_{\pi_{\partial\Omega}(x)}$ and containing p we obtain

$$\ell(x) \leq C\rho\theta(x)^2 \leq C(\Omega)|x-p|^2.$$

In case $\theta = 0$ and hence $N_{\pi_{\partial\Omega}(x)} = N_p$ we only consider A_1.

Region A_1

We begin with the integral on the set A_1. Elementary differential geometry implies that $A_1 \subseteq \Pi_{\pi_{\partial\Omega}(x)} \cap B^c_{\frac{R}{2}}\left(\pi_{\partial\Omega}(x) - \frac{R}{2}N_{\pi_{\partial\Omega}(x)}\right)$. Let us also denote by $B_r^{c,+}(0) := B_r^c(0) \cap \left(\mathbb{R}_+ \times \mathbb{R}^2\right)$. Moreover, using that

$$\left\{(\eta_1, \tilde{\eta}) \in [0, \frac{R}{2}] \times \mathbb{R}^2 :\ \left(\eta_1 - \frac{R}{2}\right)^2 + |\tilde{\eta}|^2 \geq \frac{R^2}{4}\right\} \subseteq \left\{(\eta_1, \tilde{\eta}) \in \mathbb{R}_+ \times \mathbb{R}^2 :\ |\tilde{\eta}|^2 \geq \frac{R}{2}\eta_1\right\}.$$

Hence, with a suitable change of variables we compute

$$\begin{aligned}\int_{A_1} d\eta K_\varepsilon(\eta-x) \leq & \int_{\mathbb{R}^2} d\tilde{\eta}\int_0^{2\frac{|\tilde{\eta}|^2}{R}} d\eta_1\, \frac{e^{-\frac{\sqrt{|d(x)-\eta_1|^2+|\tilde{\eta}|^2}}{\varepsilon}}}{4\pi\varepsilon\left(|d(x)-\eta_1|^2+|\tilde{\eta}|^2\right)}\\ &+\int_{B^{c,+}_{\frac{R}{2}}(0)} d\eta\, K_\varepsilon\left(\eta-\left(d(x)-\frac{R}{2}\right)e_1\right)\\ \leq & \int_{B_{d(x)}(0)} d\tilde{\eta}\int_0^{\frac{2|\tilde{\eta}|^2}{R}} d\eta_1\, \frac{e^{-\frac{\sqrt{|d(x)-\eta_1|^2+|\tilde{\eta}|^2}}{\varepsilon}}}{4\pi\varepsilon\left(|d(x)-\eta_1|^2+|\tilde{\eta}|^2\right)}\\ &+\int_{B^c_{d(x)}(0)} d\tilde{\eta}\int_0^{2\frac{|\tilde{\eta}|^2}{R}} d\eta_1\, \frac{e^{-\frac{\sqrt{|d(x)-\eta_1|^2+|\tilde{\eta}|^2}}{\varepsilon}}}{4\pi\varepsilon\left(|d(x)-\eta_1|^2+|\tilde{\eta}|^2\right)}+\int_{B^{c,+}_{\frac{R}{2}}(0)} d\eta\, \frac{e^{-\frac{|\eta|}{\varepsilon}}}{\pi\varepsilon R^2}\\ \leq & \int_{B_{d(x)}(0)} d\tilde{\eta}\frac{4|\tilde{\eta}|^2 e^{-\frac{d(x)}{2\varepsilon}}}{\pi\varepsilon R d^2(x)}+\int_{B^c_{d(x)}(0)} d\tilde{\eta}\frac{e^{-\frac{|\tilde{\eta}|}{\varepsilon}}}{2\pi R\varepsilon}+\frac{4\varepsilon^2 C}{R^2}\int_{\frac{R}{4\varepsilon}}^\infty dr\, e^{-r}\\ \leq & \frac{2d^2(x)}{R\varepsilon}e^{-\frac{d(x)}{2\varepsilon}}+C\varepsilon e^{-\frac{d(x)}{2\varepsilon}}+\frac{4\varepsilon^2 C}{R^2}e^{-\frac{R}{4\varepsilon}}\leq C(\Omega)\varepsilon e^{-\frac{d(x)}{4\varepsilon}}.\end{aligned} \tag{C.97}$$

We also used that if $(\eta_1,\tilde{\eta})\in\left[0,\frac{2|\tilde{\eta}|^2}{R}\right]\times B_{d(x)}(0)$ we can estimate

$$|d(x)-\eta_1| = d(x)-\eta \geq d(x)\left(1-\frac{2\varepsilon^{\frac{1}{2}+2\delta}}{R}\right)\geq\frac{d(x)}{2}, \tag{C.98}$$

since $d(x)<\varepsilon^{\frac{1}{2}+2\delta}$ and we combined (C.98) with $|d(x)-\eta_1|^2+|\tilde{\eta}|^2\geq|d(x)-\eta_1|^2$. If $(\eta_1,\tilde{\eta})\in\left[0,\frac{2|\tilde{\eta}|^2}{R}\right]\times B^c_{d(x)}(0)$ then we can estimate $|d(x)-\eta_1|^2+|\tilde{\eta}|^2\geq|\tilde{\eta}|^2$. Moreover, if $\eta\in B^{c,+}_{\frac{R}{2}}(0)$ then $\eta_1+\frac{R}{2}-d(x)\geq\eta_1$ and $\left|\eta-\left(d(x)-\frac{R}{2}\right)e_1\right|\geq\frac{R}{2}$. In the third inequality we computed the first two integrals on the 2 dimensional balls using also the fact that there exists a constant $C>0$ such that $e^{-x}x\leq Ce^{-\frac{x}{2}}$ if $x\geq 0$ and the last integral holds by the existence of a constant $C>0$ such that $x^2e^{-\frac{x}{2}}\leq C$ for $x\geq 0$. For the last estimate we notice first of all that $R\geq d(x)$ and we consider two different cases. If $d(x)\leq\varepsilon$ the result follows from the fact that $\frac{d^2(x)}{2R\varepsilon}\leq\frac{\varepsilon}{2R}$. If $d(x)\geq\varepsilon$ we use the well-known estimate $e^{-x}x^2\leq Ce^{-\frac{x}{2}}$ for $x\geq 0$.

Region A_2

We proceed with the integral on A_2. We compute using a change of variables

$$\begin{aligned}\int_{A_2} d\eta\, K_\varepsilon(\eta-x) &= \int_{\mathbb{R}^2} d\tilde{\eta}\int_{d(x)}^{d(x)+\ell(x)} d\eta\, \frac{e^{-\frac{\sqrt{\eta^2+|\tilde{\eta}|^2}}{\varepsilon}}}{4\pi\varepsilon\left(\eta^2+|\tilde{\eta}|^2\right)} = \int_{\mathbb{R}^2} d\tilde{\eta}\int_{\frac{d(x)}{\varepsilon}}^{\frac{d(x)+\ell(x)}{\varepsilon}} d\eta\, \frac{e^{-\sqrt{\eta^2+|\tilde{\eta}|^2}}}{4\pi\left(\eta^2+|\tilde{\eta}|^2\right)}\\ &= \int_{\frac{d(x)}{\varepsilon}}^{\frac{d(x)+\ell(x)}{\varepsilon}} d\eta\, K(\eta)\,,\end{aligned} \tag{C.99}$$

where we rescaled by ε and we used the definition of the normalized exponential integral K as in (C.19). The estimate of the last integral depends on the values for $d(x)$ and $\ell(x)$. We recall that $d(x) < \varepsilon^{\frac{1}{2}+2\delta}$ and that $\ell(x) \leq C(\Omega)\varepsilon^{1+4\delta}$. Proposition C.1 implies also the following estimate for the normalized exponential integral

$$K(\eta) \leq C \begin{cases} 1 + |\ln(\eta)| & \text{if } 0 \leq \eta \leq 2, \\ e^{-\eta} & \text{if } \eta \geq 1. \end{cases} \tag{C.100}$$

for some constant $C > 0$. Let us assume first $d(x) \geq \varepsilon$. Then (C.99) and (C.100) imply

$$\int_{A_2} d\eta\, K_\varepsilon(\eta - x) \leq C \int_{\frac{d(x)}{\varepsilon}}^{\frac{d(x)+\ell(x)}{\varepsilon}} e^{-\eta}\, d\eta \leq C(\Omega)\varepsilon^{4\delta} e^{-\frac{d(x)}{\varepsilon}}. \tag{C.101}$$

Let us assume now $d(x) < \varepsilon$. If $\ell(x) < d(x)$ we can use the monotonicity of the logarithm together with estimate (C.100). Thus,

$$\begin{aligned}\int_{A_2} d\eta\, K_\varepsilon(\eta - x) \leq& C \int_{\frac{d(x)}{\varepsilon}}^{\frac{d(x)+\ell(x)}{\varepsilon}} (1 + |\ln(\eta)|)\, d\eta \leq C\left(\frac{\ell(x)}{\varepsilon} + \frac{\ell(x)}{\varepsilon}\left|\ln\left(\frac{d(x)}{\varepsilon}\right)\right|\right) \\ \leq& C(\varepsilon^4 + \varepsilon^\delta) \leq C\varepsilon^\delta e^{-\frac{d(x)}{\varepsilon}},\end{aligned} \tag{C.102}$$

where we used the estimates $\sqrt{x}\,|\ln(x)| \leq \frac{2}{e} \leq 1$ for all $x \in [0,1]$ and $e^{-1} \leq e^{-\frac{d(x)}{\varepsilon}}$.

If $\ell(x) \geq d(x)$ we argue similarly as in (C.102) using also $(d(x), d(x)+\ell(x)) \subset (0, 2\ell(x))$ and we conclude

$$\begin{aligned}\int_{A_2} d\eta\, K_\varepsilon(\eta - x) \leq& C \int_0^{\frac{2\ell(x)}{\varepsilon}} (1 + |\ln(\eta)|)\, d\eta \leq C\left(\frac{\ell(x)}{\varepsilon} + \frac{2\ell(x)}{\varepsilon}\left|\ln\left(\frac{2\ell(x)}{\varepsilon}\right)\right|\right) \\ \leq& C(\varepsilon^4 + \varepsilon^{2\delta}) \leq C\varepsilon^{2\delta} e^{-\frac{d(x)}{\varepsilon}}.\end{aligned} \tag{C.103}$$

Region A_3

We are now ready for the estimate of the integral on the set A_3. We recall that for some constant $C(\Omega)$ we can estimate $\theta(x) \leq C(\Omega)|x - p| < C(\Omega)\varepsilon^{\frac{1}{2}+2\delta}$. Arguing similarly as in (C.97) we compute using $\tan(\theta(x)) \leq 2\theta(x)$ and a suitable change of variables

$$\begin{aligned}\int_{A_3} d\eta\, K_\varepsilon(\eta - x) =& \int_{\mathbb{R}^2} d\tilde{\eta} \int_{d(x)+\ell(x)}^{d(x)+\ell(x)+2\theta|\tilde{\eta}|} d\eta_1\, \frac{e^{-\frac{\sqrt{\eta_1^2+|\tilde{\eta}|^2}}{\varepsilon}}}{4\pi\varepsilon\left(\eta_1^2 + |\tilde{\eta}|^2\right)} \\ =& \int_{B_{d(x)+\ell(x)}(0)} d\tilde{\eta} \int_{d(x)+\ell(x)}^{d(x)+\ell(x)+2\theta|\tilde{\eta}|} d\eta_1\, \frac{e^{-\frac{\sqrt{\eta_1^2+|\tilde{\eta}|^2}}{\varepsilon}}}{4\pi\varepsilon\left(\eta^2 + |\tilde{\eta}|^2\right)} \\ &+ \int_{B^c_{d(x)+\ell(x)}(0)} d\tilde{\eta} \int_{d(x)+\ell(x)}^{d(x)+\ell(x)+2\theta|\tilde{\eta}|} d\eta_1\, \frac{e^{-\frac{\sqrt{\eta^2+|\tilde{\eta}|^2}}{\varepsilon}}}{4\pi\varepsilon\left(\eta^2 + |\tilde{\eta}|^2\right)} \\ \leq& \int_{B_{d(x)+\ell(x)}(0)} d\tilde{\eta} \frac{\theta|\tilde{\eta}|e^{-\frac{d(x)+\ell(x)}{\varepsilon}}}{2\pi\varepsilon\left(d(x)+\ell(x)\right)^2} + \int_{B^c_{d(x)+\ell(x)}(0)} d\tilde{\eta} \frac{\theta e^{-\frac{|\tilde{\eta}|}{\varepsilon}}}{2\pi|\tilde{\eta}|\varepsilon} \\ \leq& \theta\frac{d(x)+\ell(x)}{3\varepsilon} e^{-\frac{d(x)+\ell(x)}{\varepsilon}} + 2\theta e^{-\frac{d(x)+\ell(x)}{\varepsilon}} \leq C\theta e^{-\frac{d(x)}{2\varepsilon}} \leq C(\Omega)\varepsilon^{\frac{1}{2}+2\delta} e^{-\frac{d(x)}{2\varepsilon}},\end{aligned} \tag{C.104}$$

where we used in the first inequality that $\eta_1^2 + |\tilde{\eta}|^2 \geq \eta^2 \geq (d(x)+\ell(x))^2$ and also that $\eta_1^2 + |\tilde{\eta}|^2 \geq |\tilde{\eta}|^2$ and the well-know estimate $|x|e^{-\frac{|x|}{2}} \leq 1$.

Summarizing: estimate of S_1

Since Lemma C.5 implies $\overline{U}_\varepsilon \leq C(\Omega, g_\nu)$, then estimates (C.96), (C.97), (C.101), (C.102), (C.103) and (C.104) yield the existence of a constant $C>0$ independent of $\varepsilon, x, p, \delta$ such that

$$\int_{\Pi_p\setminus\Omega} d\eta K_\varepsilon(\eta - x)\overline{U}_\varepsilon(\eta,p) \leq C \begin{cases} \varepsilon^\delta e^{-\frac{d(x)}{4\varepsilon}} & |x-p| < \varepsilon^{\frac{1}{2}+2\delta} \\ e^{-\frac{d(x)}{4\varepsilon}} & |x-p| \geq \varepsilon^{\frac{1}{2}+2\delta} \end{cases}. \tag{C.105}$$

Step 2: estimate of S_2.

In order to end the proof for this lemma we now estimate the integral term S_2 of (C.95). If $|x-p| \geq \varepsilon^{\frac{1}{2}+2\delta}$ since $\left|x - x_{\Pi_p}(x,n)\right| \geq |x - x_\Omega(x,n)| \geq d(x)$ we have the estimate $S_2 \leq 8\pi \|g\|_\infty e^{-\frac{d(x)}{\varepsilon}}$. We now assume $|x-p| < \varepsilon^{\frac{1}{2}+2\delta}$. As before this implies $d(x) < \varepsilon^{\frac{1}{2}+2\delta}$. In order to estimate S_2 we will divide the integral on $\mathbb{S}^2$ in three integrals, which will be estimated using different approaches.

Figure 6 represents the decomposition we are going to consider. We denote θ_1 and θ_2 the angles given by $\tan(\theta_1) = \frac{\varepsilon^{\frac{1}{2}+2\delta}}{\varepsilon^{\frac{1}{2}+\delta}} = \varepsilon^\delta$ and $\tan(\theta_2) = 2\varepsilon^{\frac{1}{2}}$ and we denote by $\theta(n)$ the angle between $-n$ and N_p, i.e. $\theta(n) = \arg(\cos(-n\cdot N_p))$. We decompose the sphere in three different regions $\mathbb{S}^2 = \mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3$, were we define

$$\mathcal{U}_1 := \left\{ n \in \mathbb{S}^2 : \ -n\cdot N_p > \sin(\theta_1) \right\},$$
$$\mathcal{U}_2 := \left\{ n \in \mathbb{S}^2 : \ n\cdot N_p > \sin(\theta_2) \right\}$$

and

$$\mathcal{U}_3 := \left\{ n \in \mathbb{S}^2 : \ -\sin(\theta_1) \leq n\cdot N_p \leq \sin(\theta_2) \right\}.$$

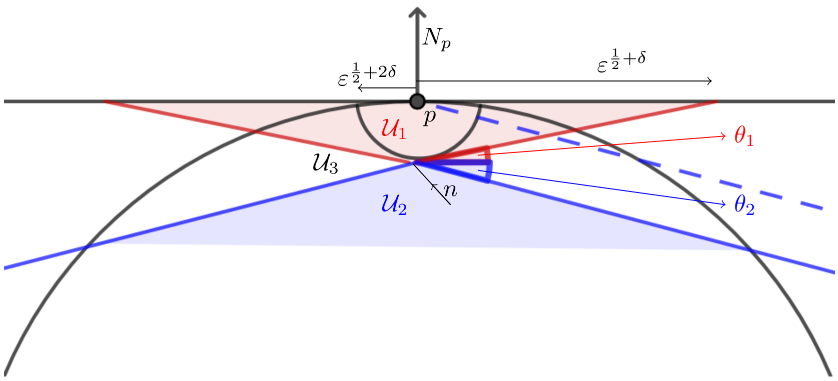

Figure C.6: Decomposition of S_2.

Region $\mathcal{U}_1$

Let us consider $n \in \mathcal{U}_1$ and $x = (y_1, y_2, y_3)^\top \in \Omega$ with $|x-p| < \varepsilon^{\frac{1}{2}+2\delta}$. Let us denote by $\overline{x}_\Omega$ the intersection point of the axis-symmetric paraboloid with curvature $\kappa = \frac{1}{R}$ approximating from the inside of the domain Ω the boundary $\partial\Omega$ in a neighborhood of the point p with the line connecting x and x_Ω. It is not difficult to see that $|x_{\Pi_p} - x_\Omega| \leq |x_{\Pi_p} - \overline{x}_\Omega|$. Without loss of generality we can assume $n = (n_1, n_2, 0)$ with $n_2 < 0$ and as usual $\Pi_p = \mathbb{R}_+ \times \mathbb{R}^2$. Hence, $x_{\Pi_p}(x,n) = (0, \sigma, y_3)$ and $\overline{x}_\Omega(x,n) = (x_1, \tilde{x}, y_3)$ and $|x| < \varepsilon^{\frac{1}{2}+2\delta}$. We see then that $\sigma \geq \tilde{x}$ and $|\sigma| \leq \frac{|x_1|}{\tan(\theta_1)} \leq \varepsilon^{\frac{1}{2}+\delta}$. Using the curvature of the boundary we know that the point $\overline{x}_\Omega$ satisfies the following system of equations

$$\begin{cases} x_1 = \frac{2}{R}\left(\tilde{x}^2 + y_3^2\right), \\ x_1 = \frac{\sigma - \tilde{x}}{\tan(\theta)}. \end{cases}$$

Hence we calculate

$$\left|x_{\Pi_p} - x_\Omega\right| \leq |x_{\Pi_p} - \overline{x}_\Omega| = \sqrt{|x_1|^2 + (\sigma - \tilde{x})^2} = \frac{\sigma - \tilde{x}}{\sin(\theta)} = \frac{2\left(\tilde{x}^2 + y_3^2\right)}{R\cos(\theta)} \leq \frac{\varepsilon^{1+2\delta}C}{R\varepsilon^\delta} = C(\Omega)\varepsilon^{1+\delta}.$$

Where we used that for $0 < \varepsilon < 1$ sufficiently small also $\tan\theta_1 \approx \sin\theta_1$. Thus, we estimate

$$\begin{aligned}
&\int_0^\infty d\nu \int_{\mathcal{U}_1} dn\, g_\nu(n) \left| e^{-\frac{\left|x - x_{\Pi_p}(x,n)\right|}{\varepsilon}} - e^{-\frac{\left|x - x_\Omega(x,n)\right|}{\varepsilon}} \right| \\
\leq& \|g\|_\infty \int_{\mathcal{U}_1} dn\, e^{-\frac{\left|x - x_\Omega(x,n)\right|}{\varepsilon}} \left| 1 - e^{-\frac{\left|x - x_{\Pi_p}(x,n)\right| - \left|x - x_\Omega(x,n)\right|}{\varepsilon}} \right| \\
\leq& 4\pi\|g\|_\infty e^{-\frac{d(x)}{\varepsilon}} \frac{\left|x - x_{\Pi_p}(x,n)\right| - \left|x - x_\Omega(x,n)\right|}{\varepsilon} = 4\pi\|g\|_\infty e^{-\frac{d(x)}{\varepsilon}} \frac{\left|x_\Omega - x_{\Pi_p}\right|}{\varepsilon} \\
\leq& 4\pi\|g\|_\infty C(\Omega)\varepsilon^\delta e^{-\frac{d(x)}{\varepsilon}},
\end{aligned} \tag{C.106}$$

where we used that x, x_{Π_p}, x_Ω lie all on the same line.

Region $\mathcal{U}_2$

Let us consider $n \in \mathcal{U}_2$ and $x \in \Omega$ with $|x - p| < \varepsilon^{\frac{1}{2}+2\delta}$. We see first of all that $n \cdot N_p \geq \sin(\theta_2) \geq 0$. Thus, by definition $e^{-\frac{\left|x - x_{\Pi_p}(x,n)\right|}{\varepsilon}} = 0$. In this case we have that $|x - x_\Omega(x,n)| \geq |x - x_\Omega(x,\tilde{n})|$, where $\tilde{n} \cdot N_p = \sin(\theta_2)$. We denote by $Q \in \partial\Omega$ the intersection of the line $\{x + tN_p : t > 0\}$ and the boundary $\partial\Omega$. As usual N_Q is the other normal at $Q \in \partial\Omega$. Since $|x - p| < \varepsilon^{\frac{1}{2}+2\delta}$ also $|p - Q| < \varepsilon^{\frac{1}{2}+2\delta}$ and hence there exists a constant $C > 0$ such that $\theta_{pQ} < C\varepsilon^{\frac{1}{2}+2\delta}$, where θ_{pQ} is the angle between N_p and N_Q. Let us also denote by $\tilde{\theta}$ the angle such that $\tilde{n} \cdot N_Q = \sin(\tilde{\theta})$. By a geometrical argument on the sphere it is not difficult to see that choosing ε sufficiently small, i.e. $0 < \varepsilon < \min\left(\frac{3}{24}, (4C)^{-\frac{1}{2\delta}}\right)$, we have $\tilde{\theta} \geq \theta_2 - \theta_{pQ} \geq \frac{3}{2}\varepsilon^{\frac{1}{2}}$. Choosing a suitable coordinate system we can assume $Q = (0,0,0)$, $N_Q = -e_1$ and $\tilde{n} = (-\sin(\tilde{\theta}), -\cos(\tilde{\theta}), 0)$. Let us denote by $\overline{x}_\Omega(x,\omega)$ the intersection point between the line $\{x - t\omega : t > 0\}$ and the axis-simmetric paraboloid with curvature $\kappa = \frac{1}{R}$ inside Ω tangent to $\partial\Omega$ at Q. Then, since now $x_\Omega(x,\tilde{n})$ lies outside this paraboloid we obtain $|x - x_\Omega(x,\tilde{n})| \geq |x - \overline{x}_\Omega(x,\tilde{n})|$. Moreover, notice that since the angle between the axis $-N_Q$ and the vector $x - Q$ is given by $\theta_{pQ} < C\varepsilon^{\frac{1}{2}+2\delta}$ we see that x lies inside the paraboloid choosing ε sufficiently small.

For $\varepsilon > 0$ small enough we also see that $\sin(\tilde{\theta}) \geq \varepsilon^{\frac{1}{2}}$. Hence, it is also true that for $\tilde{N} = \left(-\varepsilon^{\frac{1}{2}}, -\sqrt{1-\varepsilon}, 0\right)$ we have $|x - \overline{x}_\Omega(x,\tilde{n})| \geq |x - \overline{x}_\Omega(x,\tilde{N})|$. Thus, let us denote by $\overline{x}_\Omega(x,\tilde{N}) = (y_1, y_2, y_3)$ and $x = (x_1, x_2, x_3) \in \Omega$. To compute the position of $\overline{x}_\Omega(x,\tilde{N})$ we solve the following system.

$$\begin{cases}
y_1 = \frac{1}{R}\left(y_2^2 + y_3^2\right), \\
y_1 = x_1 + t\varepsilon^{\frac{1}{2}}, \\
y_2 = x_2 + t\sqrt{1-\varepsilon}, \\
y_3 = x_3.
\end{cases}$$

We want to estimate from below the value of $t > 0$. Hence we consider the quadratic equation

$$(1-\varepsilon)t^2 + t\left(2\sqrt{1-\varepsilon}x_2 - R\varepsilon^{\frac{1}{2}}\right) - \Delta = 0,$$

where $\Delta = Rx_1 - x_2^2 - x_3^2 \geq 0$ since $x \in \Omega$ inside the paraboloid. Moreover, since $x_2 < \varepsilon^{\frac{1}{2}+2\delta}$ for ε small enough (i.e. $\varepsilon < (R/4)^{1/2\delta}$) we have that $R\varepsilon^{\frac{1}{2}} - 2\sqrt{1-\varepsilon}x_2 \geq \frac{R}{2}\varepsilon^{\frac{1}{2}}$. Thus,

$$(1-\varepsilon)t^2 \geq \frac{R}{2}\varepsilon^{\frac{1}{2}}t + \Delta \geq \frac{R}{2}\varepsilon^{\frac{1}{2}}t$$

and therefore for $\varepsilon < 1$ we have

$$|x - x_\Omega(x,n)| \geq |x - x_\Omega(x,\tilde{n})| \geq |x - \overline{x}_\Omega(x,\tilde{N})| \geq t \geq \frac{R}{4}\varepsilon^{\frac{1}{2}} = C(\Omega)\varepsilon^{\frac{1}{2}}.$$

Hence, using the usual estimate $xe^{-\frac{x}{2}} \leq 1$ and that $|x - x_\Omega(x,n)| \geq d(x)$ we estimate

$$\begin{aligned}\int_0^\infty d\nu \int_{n\cdot N_p \geq \sin(\theta_2)} dn\, g_\nu(n) e^{-\frac{|x-x_\Omega(x,n)|}{\varepsilon}} \leq & \|g\|_\infty \int_{n\cdot N_p\geq\sin(\theta_2)} dn\, \frac{2\varepsilon}{|x-x_\Omega(x,n)|} e^{-\frac{|x-x_\Omega(x,n)|}{2\varepsilon}} \\ \leq & 2\pi C(\Omega)^{-1}\|g\|_\infty \varepsilon^\delta e^{-\frac{d(x)}{2\varepsilon}},\end{aligned} \tag{C.107}$$

since $\varepsilon^{\frac{1}{2}} \leq \varepsilon^\delta$ if $\delta \leq \frac{1}{2}$.

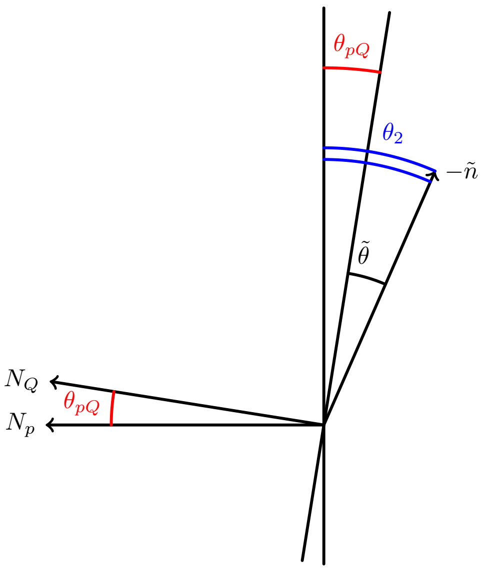

Figure C.7: Representation of the angles.

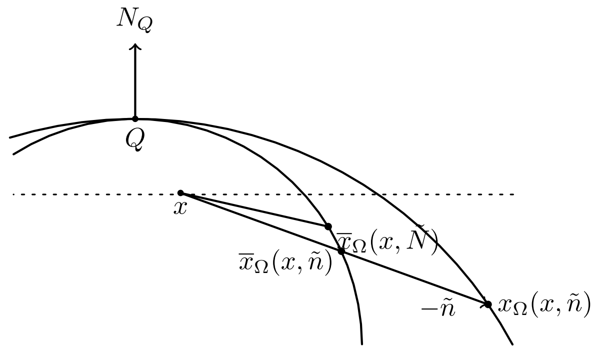

Figure C.8: Representation of the intersection points.

Region $\mathcal{U}_3$

Now, for the last estimate we notice that $|\mathcal{U}_3| \leq 2\pi(\theta_1+\theta_2) \leq 4\pi\varepsilon^\delta$ for $\delta < \frac{1}{2}$. Since it is always true that $\left|e^{-\frac{|x-x_{\Pi_p}(x,n)|}{\varepsilon}} - e^{-\frac{|x-x_\Omega(x,n)|}{\varepsilon}}\right| \leq 2e^{-\frac{d(x)}{\varepsilon}}$, we estimate

$$\int_0^\infty d\nu \int_{\mathcal{U}_3} dn\, g_\nu(n) \left| e^{-\frac{|x-x_{\Pi_p}(x,n)|}{\varepsilon}} - e^{-\frac{|x-x_\Omega(x,n)|}{\varepsilon}}\right| \leq C\|g\|_\infty \varepsilon^\delta e^{-\frac{d(x)}{\varepsilon}}. \tag{C.108}$$

Summarizing: estimate S_2

We put together equations (C.106),(C.107), (C.108) and we conclude

$$S_2 \leq C \begin{cases} \varepsilon^\delta e^{-\frac{d(x)}{2\varepsilon}} & |x-p| < \varepsilon^{\frac{1}{2}+2\delta}, \\ e^{-\frac{d(x)}{2\varepsilon}} & |x-p| \geq \varepsilon^{\frac{1}{2}+2\delta}, \end{cases} \tag{C.109}$$

for a constant $C > 0$ independent of $x, p, \varepsilon, \delta$.

Equations (C.109) and (C.105) imply the lemma. □

We are ready to construct a super-solution that will allow us to estimate $|u^\varepsilon - \overline{u}|$ near the boundary at a distance smaller than $\varepsilon^{\frac{1}{2}}$. We recall the rigid motion $\mathcal{R}_p$ defined in (C.13).

Proposition C.5. *Let $p \in \partial\Omega$, $0 < A < 1$ the constant of Lemma C.9. Let $L > 0$ large enough and $0 < \varepsilon < 1$ sufficiently small. Let $0 < \delta < \frac{1}{8}$. Then there exists a non negative continuous function $W_{\varepsilon,L} : \Omega \to \mathbb{R}_+$ such that*

$$\begin{cases} W_{\varepsilon,L} \geq C > 0 & \text{for } |\mathcal{R}_p(x) \cdot e_i| \geq \varepsilon^{\frac{1}{2}+3\delta}; \\ \mathcal{L}_\Omega^\varepsilon\left(W_{\varepsilon,L}\right)(x) \geq C\varepsilon^\delta e^{-\frac{Ad(x)}{\varepsilon}} & \text{for } |\mathcal{R}_p(x) \cdot e_i| < \varepsilon^{\frac{1}{2}+3\delta}; \\ 0 \leq W_{\varepsilon,L} \leq C\left(\varepsilon^\alpha + \frac{1}{\sqrt{L}}\right) & \text{for } |\mathcal{R}_p(x) \cdot e_i| < \varepsilon^{\frac{1}{2}+4\delta}, \end{cases} \tag{C.110}$$

for some constant $C > 0$ and $\alpha > 0$.

In order to construct this supersolution $W_{\varepsilon,L}$ we first need some definition for the geometrical setting. First of all we denote for simplicity $x_i = \mathcal{R}_p(x) \cdot e_i$. Let us define for $i = 2, 3$ the radii $\rho_i^\pm(x) = \sqrt{\left(x_1 + \frac{L}{2}\varepsilon\right)^2 + \left(x_i \pm \varepsilon^{\frac{1}{2}+3\delta}\right)^2}$ and the angles $\theta_i^\pm(x)$ given by $\cos\left(\theta_i^\pm\right) = \frac{1}{\rho_i^\pm(x)}\left(x_1 + \frac{L}{2}\varepsilon\right)$. The function $W_{\varepsilon,L}$ is constructed with the following auxiliary functions

$$F_i^\pm(x) = \frac{\pi}{2} \mp \arctan\left(\frac{x_i \pm \varepsilon^{\frac{1}{2}+3\delta}}{x_1 + \frac{L}{2}\varepsilon}\right), \tag{C.111}$$

$$G_i^\pm(x) = a\left(\frac{\cos\left(\theta_i^\pm(x)\right)}{\rho_i^\pm(x)/\varepsilon}\right)^{\frac{1}{2}}, \tag{C.112}$$

$$H_i^\pm(x) = -b\left(\frac{\cos\left(\theta_i^\pm(x)\right)}{\rho_i^\pm(x)/\varepsilon}\right)^2 \tag{C.113}$$

for $i = 2, 3$ and $a > b > 0$. Moreover, we define for $i = 2, 3$

$$W_i^\pm(x) = F_i^\pm(x) + G_i^\pm(x) + H_i^\pm(x). \tag{C.114}$$

We will prove that the desired supersolution of Proposition C.5 is given by

$$W_{\varepsilon,L}(x) = \sum_{i=2}^{3}\left(W_i^+(x) + W_i^-(x)\right) + \frac{\tilde{C}}{\sqrt{L}}\phi_{\frac{1}{8},\varepsilon} + C\varepsilon^\delta\phi_{A,\varepsilon}, \tag{C.115}$$

where $\phi_{A,\varepsilon} = \Phi^{\frac{\varepsilon}{A}}$ the supersolution defined in Theorem C.6 and $C, \tilde{C} > 0$ some suitable constants. We also define the following subsets of Ω for $i = 2, 3$.

$$\mathcal{C}_{i,2\delta}^+ := \left\{x \in \Omega : \ x_i \leq -\varepsilon^{\frac{1}{2}+3\delta} \text{ or } |x_i| < \varepsilon^{\frac{1}{2}+3\delta}, x_1 \geq \varepsilon^{\frac{1}{2}+3\delta}\right\}; \tag{C.116}$$

$$\mathcal{C}_{i,2\delta}^- := \left\{x \in \Omega : \ x_i \geq \varepsilon^{\frac{1}{2}+3\delta} \text{ or } |x_i| < \varepsilon^{\frac{1}{2}+3\delta}, x_1 \geq \varepsilon^{\frac{1}{2}+3\delta}\right\}; \tag{C.117}$$

$$\mathcal{C}_{3\delta} := \left\{x \in \Omega : \ x_1 < \varepsilon^{\frac{1}{2}+3\delta} \text{ and } |x_i| < \varepsilon^{\frac{1}{2}+3\delta} \text{ for } i = 2, 3\right\}; \tag{C.118}$$

$$\mathcal{C}_{i,4\delta} := \left\{x \in \Omega : \ |x_i| < \varepsilon^{\frac{1}{2}+4\delta} \text{ and } x_1 < \varepsilon^{\frac{1}{2}+4\delta}\right\}. \tag{C.119}$$

In order to prove Proposition C.5 we need the following computational lemma.

Lemma C.10. *Assume $p \in \Omega$, $0 < \varepsilon < 1$, L, δ as indicated in Proposition C.5. Let $x_i = \mathcal{R}_p(x) \cdot e_i$ for $i = 1, 2, 3$. Let $W_i^\pm$ as in* (C.114)*. Then there exist a constant $\alpha > 0$ depending only on δ and a constant $C > 0$ depending on Ω and g_ν but independent of ε and p and suitable $b > 0$ and $L > 0$ such that for $i = 2, 3$*

$$\begin{cases} W_i^\pm(x) \geq 0 & \text{in } \Omega & \text{(C.120)} \\ W_i^\pm(x) \geq \dfrac{\pi}{2} - \arctan(2) & \text{in } \mathcal{C}_{i,2\delta}^\pm & \text{(C.121)} \\ W_i^\pm(x) \leq C\varepsilon^\alpha & \text{in } \mathcal{C}_{i,4\delta} & \text{(C.122)} \\ \mathcal{L}_\Omega^\varepsilon\left(W_i^\pm\right)(x) \geq -\dfrac{C}{\sqrt{L}} e^{-\frac{d(x)}{8\varepsilon}} & \text{in } \mathcal{C}_{3\delta}, & \text{(C.123)} \end{cases}$$

where the sets $\mathcal{C}_{i,2\delta}^\pm$, $\mathcal{C}_{i,4\delta}$ and $\mathcal{C}_{3\delta}$ are defined in (C.116)*,* (C.117)*,* (C.119) *and* (C.118)*.*

Proof. Due to symmetry consideration it is enough to prove the lemma for $W = W_2^-$. For the sake of simplicity we write $\rho(x) = \rho_2^-(x)$ and $\theta(x) = \theta_2^-(x)$. Similarly we consider $F = F_2^-$, $G = G_2^-$ and $H = H_2^-$. We also denote by $\mathcal{C}_{j\delta}$ the sets $\mathcal{C}_{2,j\delta}^-$, $\mathcal{C}_{2,j\delta}$ and $\mathcal{C}_{j\delta}$ for $j = 2, 3, 4$.

First of all we notice that W is smooth on $x_1 > -\frac{L}{2}\varepsilon$. Moreover, since the arctangent is bounded from below by $\frac{\pi}{2}$ we have that $F \geq 0$. Since $x_1 \geq 0$ for $x \in \Omega$ we see that $\rho \geq \frac{L}{2}\varepsilon$ and hence for L big enough $\frac{\rho}{\varepsilon} > \frac{L}{2} > 1$. On the other hand $0 \leq \cos(\theta) \leq 1$ and hence for $a > b$ we have that

$$G + H = a\left(\frac{\cos\left(\theta_2^-(x)\right)}{\rho_2^-(x)/\varepsilon}\right)^{\frac{1}{2}} - b\left(\frac{\cos\left(\theta_2^-(x)\right)}{\rho_2^-(x)/\varepsilon}\right)^2 \geq (a-b)\left(\frac{\cos\left(\theta_2^-(x)\right)}{\rho_2^-(x)/\varepsilon}\right)^{\frac{1}{2}} \geq 0,$$

which yields (C.120).

Assume now $x_2 \geq \varepsilon^{\frac{1}{2}+3\delta}$. This implies $\frac{x_2 - \varepsilon^{\frac{1}{2}+3\delta}}{x_1 + \frac{L}{2}\varepsilon} \geq 0$ and thus $F(x) \geq \frac{\pi}{2}$. The non-negativity of $G + H$ yields $W(x) \geq \frac{\pi}{2}$.

Let us assume $x_1 \geq \varepsilon^{\frac{1}{2}+3\delta}$ and $|x_2| < \varepsilon^{\frac{1}{2}+3\delta}$. A similar computation as above shows $\frac{x_2 - \varepsilon^{\frac{1}{2}+3\delta}}{x_1 + \frac{L}{2}\varepsilon} \geq -2\frac{\varepsilon^{\frac{1}{2}+3\delta}}{\varepsilon^{\frac{1}{2}+3\delta}\left(1 + \frac{L}{2}\varepsilon^{\frac{1}{2}-3\delta}\right)} \geq -2$. Hence, $W(x) \geq F(x) \geq \frac{\pi}{2} - \arctan(2) > 0$ for $x \in \mathcal{C}_{2\delta}$ as in (C.121).

We move now to the proof of (C.122). Let therefore $x \in \mathcal{C}_{4\delta}$. First of all $W(x) \leq F + G$. Moreover, $x_2 - \varepsilon^{\frac{1}{2}+3\delta} < \varepsilon^{\frac{1}{2}+3\delta}(\varepsilon^\delta - 1) < 0$ and $x_1 + \frac{L}{2}\varepsilon < \varepsilon^{\frac{1}{2}+4\delta}\left(1 + \frac{L}{2}\varepsilon^{\frac{1}{2}-4\delta}\right) < \frac{3}{2}\varepsilon^{\frac{1}{2}+4\delta}$ if $\delta < \frac{1}{8}$, $L > 0$ large enough and $0 < \varepsilon < 1$ sufficiently small such that $L < \varepsilon^{-\beta}$ for $\beta = \frac{1-8\delta}{2}$. This computation implies

$$\frac{x_2 - \varepsilon^{\frac{1}{2}+3\delta}}{x_1 + \frac{L}{2}\varepsilon} < -\frac{2}{3}\frac{1}{\varepsilon^\delta}(1 - \varepsilon^\delta) < -\frac{1}{3}\frac{1}{\varepsilon^\delta}$$

for $\varepsilon > 0$ small enough. With an application of Taylor expansion for $y \to -\infty$ we conclude $F(x) \leq 3\varepsilon^\delta$, since $\frac{\pi}{2} + \arctan(y) \approx |y|^{-1} - \frac{1}{3|y|^3}$. Moreover, since $\frac{\rho}{\varepsilon} > 1$ and also $\rho \geq |x_2 - \varepsilon^{\frac{1}{2}+3\delta}| \geq \frac{\varepsilon^{\frac{1}{2}+3\delta}}{2}$ if ε small enough, we have that $\cos(\theta) = \frac{x_1 + \frac{L}{2}\varepsilon}{\rho} < \frac{3}{\varepsilon}^\delta$. Hence, $0 \leq \frac{\cos(\theta)}{\rho} < 3\varepsilon^\delta$ implies $G(x) < \sqrt{3}\varepsilon^{\delta/2}$. Taking then $\alpha = \frac{\delta}{2}$ we conclude $W \leq C\varepsilon^\alpha$.

It remains now to show, that W satisfies the estimate (C.123). The main idea for this proof is to approximate the operator $\mathcal{L}_\Omega^\varepsilon$ by a Laplacian expanding the function W by Taylor. Let us assume from now on that $x \in \mathcal{C}_{3\delta}$, i.e. $|x_3|, |x_2|, x_1 < \varepsilon^{\frac{1}{2}+3\delta}$. We first notice that for these x the function F is harmonic and the functions G and H are super-harmonic. In order to prove this we change the coordinates in cylinder coordinates: $(x_1, x_2, x_3) \mapsto (\rho(x_1, x_2), \theta(x_1, x_2), x_3)$. With this notation, since F, G, H are actually functions only of x_1 and x_2, we can write F as $F(\rho, \theta) = \frac{\pi}{2} - \theta$. Thus, since in cylinder coordinates the Laplacian can be written as $-\Delta = -\frac{1}{\rho}\partial_\rho(\rho\partial_\rho) - \frac{1}{\rho^2}\partial_\theta^2 - \partial_{x_3}^2$, we compute $-\Delta F = 0$. On the other hand we can also compute for H

$$\frac{\varepsilon^2}{\rho}\partial_\rho\left(\rho\partial_\rho\frac{\cos^2(\theta)}{\rho^2}\right) + \frac{\varepsilon^2}{\rho^2}\partial_\theta^2\frac{\cos^2(\theta)}{\rho^2} = \frac{4\varepsilon^2\cos^2(\theta)}{\rho^4} + \frac{2\varepsilon^2\sin^2(\theta)}{\rho^4} - \frac{2\varepsilon^2\cos^2(\theta)}{\rho^4} = \frac{2\varepsilon^2}{\rho^4}. \tag{C.124}$$

And similarly we have also for G

$$\begin{aligned} -\frac{\varepsilon^{\frac{1}{2}}}{\rho}\partial_\rho\left(\rho\partial_\rho\frac{\cos^{\frac{1}{2}}(\theta)}{\rho^{\frac{1}{2}}}\right) - \frac{\varepsilon^{\frac{1}{2}}}{\rho^2}\partial_\theta^2\frac{\cos^{\frac{1}{2}}(\theta)}{\rho^{\frac{1}{2}}} &= -\frac{\varepsilon^{\frac{1}{2}}\cos^{\frac{1}{2}}(\theta)}{4\rho^{\frac{5}{2}}} - \frac{\varepsilon^{\frac{1}{2}}}{2\rho^{\frac{5}{2}}}\left(-\cos^{\frac{1}{2}}(\theta) - \frac{\sin^2(\theta)}{2\cos^{\frac{3}{2}}(\theta)}\right) \\ &= \frac{\varepsilon^{\frac{1}{2}}}{4\cos^{\frac{3}{2}(\theta)}\rho^{\frac{5}{2}}}. \end{aligned} \tag{C.125}$$

We will use the (super-)harmonicity of these functions while applying the Taylor expansion on suitable domains.

Before moving to the exact estimate of the operator acting on W we estimate the derivatives of these functions. We start with analyzing F. As we have seen before $F = \frac{\pi}{2} - \theta(x_1, x_2)$ and hence we have $\partial_1 F(x_1, x_2) = \frac{\sin(\theta)}{\rho}$ and $\partial_2 F(x_1, x_2) = -\frac{\cos(\theta)}{\rho}$. Since the numerator contains only power laws of cosine and sine with exponent greater or equal 1 and the denominator also only power laws of ρ with exponent greater or equal 1, using the definition of derivatives in polar coordinates we see that there exists a constant $C_{F,n} > 0$ for $n \geq 1$ such that

$$|\nabla_x^n F(x)| \leq \frac{C_{F,n}}{\rho^n},$$

where we also estimated the cosine and the sine by 1.

Let us move to the function $H = -b\varepsilon^2\frac{\cos^2(\theta)}{\rho^2}$. We use a similar argument. We compute using polar coordinates $\partial_1 H(x) = \left(\cos(\theta)\partial_\rho - \frac{\sin(\theta)}{\rho}\partial_\theta)\right) H(x) = -2b\varepsilon^2\frac{\cos^3(\theta)}{\rho^3} + 2b\varepsilon^2\frac{\sin^2(\theta)\cos(\theta)}{\rho^3}$ and similarly $\partial_2 H(x) = \left(\sin(\theta)\partial_\rho + \frac{\cos(\theta)}{\rho}\partial_\theta)\right) H(x) = -2b\varepsilon^2\frac{\sin(\theta)\cos^2(\theta)}{\rho^3} - 2b\varepsilon^2\frac{\cos^2(\theta)\sin(\theta)}{\rho^3}$. Again, the numerator only contains power of cosine and sin of degrees greater or equal 1, while the denominator only power of ρ of degree 3. Hence, applying again the definition of derivative in polar coordinates and estimating cosine and sine by 1 we conclude again the existence of a constant $C_{H,n} > 0$ such that

$$|\nabla_x^n H(x)| \leq b\frac{C_{H,n}\varepsilon^2}{\rho^{n+2}}.$$

While the function F and H produce non singular derivatives for $x \in \Omega$, G produces sigular terms in the derivatives. This is because the denominator of this function contains a square root of the cosine, hence when differentiating by θ, it appears in the denominator, indeed

$$\partial_1 G(x) = \left(\cos(\theta)\partial_\rho - \frac{\sin(\theta)}{\rho}\partial_\theta)\right) G(x) = -a\varepsilon^{\frac{1}{2}}\frac{\cos^{\frac{3}{2}}(\theta)}{2\rho^{\frac{3}{2}}} + a\varepsilon^{\frac{1}{2}}\frac{\sin^2(\theta)}{2\rho^{\frac{3}{2}}\cos^{\frac{1}{2}}(\theta)};$$

$$\partial_2 G(x) = \left(\sin(\theta)\partial_\rho + \frac{\cos(\theta)}{\rho}\partial_\theta)\right) G(x) = -a\varepsilon^{\frac{1}{2}}\frac{\cos^{\frac{1}{2}}(\theta)\sin(\theta)}{\rho^{\frac{3}{2}}}.$$

Hence, the singular terms appear when differentiating with respect to x_1. Using that $\cos(\theta) < 1$, $\sin(\theta) < 1$ and that $\cos^{-\alpha}(\theta) > \cos^{-\beta}(\theta)$ for $\alpha > \beta \geq 0$ we conclude the existence of a constant $C_{G,n} > 0$ such that

$$|\nabla_x^n G(x)| \leq a \frac{C_{G,n}\varepsilon^{\frac{1}{2}}}{\cos^{n-\frac{1}{2}}(\theta)\rho^{\frac{1}{2}+n}}.$$

We remark that we used always that by construction $\cos(\theta) \geq 0$.

As we anticipated we will estimate $\mathcal{L}_\Omega^\varepsilon(W)(x)$ applying the Taylor expansion on F, G and H on suitable subsets of $\mathbb{R}^3$ where these functions are smooth. The functions F and H will be expanded until the third derivative and we will write

$$F(\eta) = F(x) + \nabla_x F(x)\cdot(\eta - x) + \frac{1}{2}(\eta - x)^\top \nabla_x^2 F(x)(\eta - x) + \sum_{|\alpha|=3} \frac{D^\alpha F(x)}{\alpha!}(\eta - x)^\alpha + E_F^4(\eta, x) \tag{C.126}$$

$$H(\eta) = H(x) + \nabla_x H(x)\cdot(\eta - x) + \frac{1}{2}(\eta - x)^\top \nabla_x^2 H(x)(\eta - x) + \sum_{|\alpha|=3} \frac{D^\alpha H(x)}{\alpha!}(\eta - x)^\alpha + E_H^4(\eta, x). \tag{C.127}$$

The function G will be expanded only until the second derivative, hence

$$G(\eta) = G(x) + \nabla_x G(x)\cdot(\eta - x) + \frac{1}{2}(\eta - x)^\top \nabla_x^2 G(x)(\eta - x) + E_G^3(\eta, x). \tag{C.128}$$

We recall also that for any smooth function $\varphi(x_1, x_2)$ the following is true

$$\begin{aligned}\int_{B_r^3(x)} d\eta\, \frac{e^{-\frac{|\eta - x|}{\varepsilon}}}{4\pi\varepsilon\,|\eta - x|^2}\frac{1}{2}(\eta - x)^\top \nabla_x^2\varphi(x)(\eta - x) &= \frac{1}{6}\Delta\varphi(x)\int_{B_r^3(x)} d\eta\, \frac{e^{-\frac{|\eta - x|}{\varepsilon}}}{4\pi\varepsilon\,|\eta - x|^2}|\eta - x|^2 \\ = \frac{1}{6}\Delta\varphi(x)\varepsilon^2\int_0^{\frac{r}{\varepsilon}} t^2 e^{-t}dt &= \frac{\varepsilon^2}{3}\Delta\varphi(x) - (r^2 + 2\varepsilon r + 2\varepsilon^2)e^{-\frac{r}{\varepsilon}}\frac{1}{6}\Delta\varphi(x).\end{aligned} \tag{C.129}$$

We can now move to the estimate of $\mathcal{L}_\Omega^\varepsilon(W)(x)$ for $|x_2|, |x_3| < \varepsilon^{\frac{1}{2}+3\delta}$ and $0 < x_1 < \varepsilon^{\frac{1}{2}+3\delta}$. We will consider three different cases: $\rho(x) < L\varepsilon$, $\rho(x) > L\varepsilon$ with $d(x) < \varepsilon$ and finally $\rho(x) > L\varepsilon$ with $d(x) > \varepsilon$.

Case 1: $\rho(x) < L\varepsilon$

Let us assume $\rho(x) < L\varepsilon$. Then, we remark first of all that if $\eta \in B_{\frac{\rho(x)}{4}}(x)$ then $\eta_1 \geq x_1 - \frac{\rho(x)}{4} > -\frac{L}{4}\varepsilon > -\frac{L}{2}\varepsilon$, which implies the smoothness of W on the whole ball $B_{\frac{\rho(x)}{4}}(x)$. Moreover, it is also true that $\cos(\theta) = \frac{x_1 + \frac{L}{2}\varepsilon}{\rho} \geq \frac{L\varepsilon}{2\rho} > \frac{1}{2}$. Hence, in this case the derivative of G is not singular and we can estimate $|\nabla_x^n G(x)| \leq \frac{C_{G,n}2^{n-\frac{1}{2}}}{\rho^{\frac{1}{2}+n}}$. Moreover, if $\eta \in B_{\frac{\rho(x)}{4}}(x)$, then from one hand we have $\frac{3}{4}\rho(x) < \rho(\eta) < \frac{5}{4}\rho(x)$ and on the other hand $\cos(\theta(\eta))\rho(\eta) = \eta_1 + \frac{L}{2}\varepsilon > \frac{L}{4}\varepsilon$, thus $\cos(\theta(\eta)) > \frac{1}{5}$

$$\sup_{\eta \in B_{\frac{\rho(x)}{4}}(x)}\left[\frac{1}{\cos^{\frac{5}{2}}(\theta(\eta))\,\rho^{\frac{7}{2}}(x)}\right] \leq 5^{\frac{5}{2}}\left(\frac{4}{3}\right)^{\frac{7}{2}}\left(\frac{1}{\rho(x)}\right)^{\frac{7}{2}} \leq \frac{2C}{\varepsilon L}\left(\frac{1}{\rho(x)}\right)^{\frac{5}{2}},$$

where at the end we used that $\rho(x) > \frac{L\varepsilon}{2}$ for all $x \in \Omega$. For the computation of the operator $\mathcal{L}_\Omega^\varepsilon$ for the function W we will use the Taylor expansion of this function on the ball $B_{\frac{\rho(x)}{4}}(x)$. The error terms as defined in (C.126), (C.127) and (C.128) satisfy then

$$\begin{gathered}\left|E_F^4(\eta,x)\right| \leq \left(\frac{4}{3}\right)^4 \frac{C_{F,4}}{\rho(x)^4}|x-\eta|^4, \quad \left|E_H^4(\eta,x)\right| \leq b\left(\frac{4}{3}\right)^6 \frac{C_{H,4}}{\rho(x)^6}|x-\eta|^4\varepsilon^2 \text{ and} \\ \left|E_G^3(\eta,x)\right| \leq a\frac{C_{G,3}}{L\rho(x)^{\frac{5}{2}}}\frac{|x-\eta|^3}{\varepsilon}\varepsilon^{\frac{1}{2}}.\end{gathered} \tag{C.130}$$

We can now proceed with the estimate for the operator. Applying the Taylor expansion we obtain

$$\begin{aligned}\mathcal{L}_\Omega^\varepsilon(W)(x) =& W(x) - \int_{B_{\frac{\rho(x)}{4}}(x)\cap\Omega} K_\varepsilon(\eta-x)\,W(\eta)\,d\eta - \int_{B^c_{\frac{\rho(x)}{4}}(x)\cap\Omega} K_\varepsilon(\eta-x)\,W(\eta)\,d\eta \\ \geq& W(x) - \int_{B_{\frac{\rho(x)}{4}}(x)} K_\varepsilon(\eta-x)\,W(\eta)\,d\eta - \int_{B^c_{\frac{\rho(x)}{4}}(x)\cap\Omega} K_\varepsilon(\eta-x)\,W(\eta)\,d\eta \\ =& W(x)\left(1 - \int_{B_{\frac{\rho(x)}{4}}(x)} K_\varepsilon(\eta-x)\;d\eta\right) - (\nabla_x W(x))\cdot\int_{B_{\frac{\rho(x)}{4}}(x)} K_\varepsilon(\eta-x)\,(\eta-x)\,d\eta \\ &- \frac{1}{2}\int_{B_{\frac{\rho(x)}{4}}(x)} K_\varepsilon(\eta-x)\,(\eta-x)^\top\nabla_x^2 W(x)(\eta-x)\,d\eta \\ &- \sum_{|\alpha|=3}\frac{D^\alpha(F+H)(x)}{\alpha!}\int_{B_{\frac{\rho(x)}{4}}(x)} K_\varepsilon(\eta-x)\,(\eta-x)^\alpha\,d\eta \\ &- \int_{B_{\frac{\rho(x)}{4}}(x)} K_\varepsilon(\eta-x)\,(E_F^4+E_H^4+E_G^3)(\eta,x)\,d\eta - \int_{B^c_{\frac{\rho(x)}{4}}(x)\cap\Omega} K_\varepsilon(\eta-x)\,W(\eta)\,d\eta\end{aligned} \tag{C.131}$$

For the terms of the Taylor expansion containing the first and third derivatives we use now the symmetry of the integral in $\mathbb{R}^3$, while for the second degree derivative terms we use the Laplacian identity as in equation (C.129), together with the fact that F is harmonic while H and G are super-harmonic as in equations (C.124) and (C.125). Denoting by C_F, C_G, C_H constants depending on F, G resp. H only and changing the coordinates $y \mapsto (\eta - x)$ we estimate using the estimates for the error terms as in (C.130)

$$\begin{aligned}\mathcal{L}_\Omega^\varepsilon(W)(x) \geq& -C_F\left(\frac{\varepsilon}{\rho}\int_{B^c_{\frac{\rho(x)}{4}}(0)}\frac{e^{-\frac{|y|}{\varepsilon}}}{4\pi\varepsilon^2|y|}\,dy + \frac{\varepsilon^3}{\rho^3}\int_{B^c_{\frac{\rho(x)}{4}}(0)}\frac{e^{-\frac{|y|}{\varepsilon}}|y|}{4\pi\varepsilon^4}\,dy\right) \\ &- C_H b\left(\frac{\varepsilon^3}{\rho^3}\int_{B^c_{\frac{\rho(x)}{4}}(0)}\frac{e^{-\frac{|y|}{\varepsilon}}}{4\pi\varepsilon^2|y|}\,dy + \frac{\varepsilon^4}{\rho^4}\int_{B^c_{\frac{\rho(x)}{4}}(0)}\frac{e^{-\frac{|y|}{\varepsilon}}}{4\pi\varepsilon^3}\,dy + \frac{\varepsilon^5}{\rho^5}\int_{B^c_{\frac{\rho(x)}{4}}(0)}\frac{e^{-\frac{|y|}{\varepsilon}}|y|}{4\pi\varepsilon^4}\,dy\right)\end{aligned} \tag{C.132}$$

$$
\begin{aligned}
&- C_G a \left(\frac{\varepsilon}{L^{\frac{1}{2}}\rho} \int_{B^c_{\frac{\rho(x)}{4}}(0)} \frac{e^{-\frac{|y|}{\varepsilon}}}{4\pi\varepsilon^2 |y|}\, dy + \frac{\varepsilon^2}{L^{\frac{1}{2}}\rho^2} \int_{B^c_{\frac{\rho(x)}{4}}(0)} \frac{e^{-\frac{|y|}{\varepsilon}}}{4\pi\varepsilon^3}\, dy \right) \\
&+ \frac{2b}{3}\frac{\varepsilon^4}{\rho^4} - C_F \frac{\varepsilon^4}{\rho^4} - C_H b \frac{\varepsilon^6}{\rho^6} + \frac{1}{12}\frac{a\varepsilon^{\frac{5}{2}}}{\cos^{\frac{3}{2}}(\theta)\rho^{\frac{5}{2}}} - a\frac{C_G \varepsilon^{\frac{5}{2}}}{L\rho(x)^{\frac{5}{2}}} - (\pi + a)e^{-\frac{\rho}{4\varepsilon}} \\
\geq & \frac{2}{3}\frac{\varepsilon^4}{\rho^4}\left(b - \frac{3}{2}(C_F + \pi\tilde{C}) - \frac{3C_H b}{2L^2} - \frac{3C_F\tilde{C}}{2L} - \frac{3bC_H\tilde{C}}{2L^3} \right) + \frac{a\varepsilon^{\frac{5}{2}}}{\rho^{\frac{5}{2}}}\left(\frac{1}{12} - \frac{C_G}{L} - \frac{\tilde{C}}{L^{\frac{3}{2}}} - \frac{C_G\tilde{C}}{L^3} \right).
\end{aligned}
$$

Moreover, we used that $\frac{\rho}{\varepsilon} > \frac{L}{2} > 1$ and the well-known estimates $\int_{\frac{r}{\varepsilon}}^{\infty} e^{-x}x^n \leq C_n e^{-\frac{r}{2\varepsilon}} \leq \tilde{C}\frac{\varepsilon^4}{r^4}$ for $n = 0, 1, 2, 3$, as well as $\frac{\varepsilon^n}{\rho^n} < \frac{\varepsilon}{\rho} < \frac{2}{L}$ and the fact that $0 < \cos(\theta) < 1$. Choosing

$$
b > 3C_F(1 + 2\tilde{C}\pi) \quad \text{and} \quad L > \max\{\tilde{C},\ \sqrt{6C_H},\ (24C_G + 12)^2\}. \tag{C.133}
$$

We can conclude

$$
\mathcal{L}^{\varepsilon}_{\Omega}(W)(x) \geq 0. \tag{C.134}
$$

Case 2: $\rho(x) > L\varepsilon$ and $d(x) < \varepsilon$

We consider now the case when $\rho(x) > L\varepsilon$ with $d(x) < \varepsilon$ for $|x_2|, |x_3| < \varepsilon^{\frac{1}{2}+3\delta}$ and $0 < x_1 < \varepsilon^{\frac{1}{2}+3\delta}$. First of all, we see that if $d(x) < \varepsilon$ then also $x_1 < 2\varepsilon$. This is true since for all these x the distance can be estimated $\varepsilon > d(x) = |x - z|$ for a unique $\pi_{\partial\Omega}(x) = z \in \partial\Omega$. Hence, $x_1 < \varepsilon + z_1$ since also by the convexity and for ε sufficiently small we have $z_1 \leq x_1$. Thus, if $z_1 \geq \varepsilon$ approximating the curvature by a sphere of radius R from the interior tangent to $\{0\} \times \mathbb{R}^2$ we see that $z_i \geq C(R)\varepsilon^{\frac{1}{2}}$ for an $i \in \{2, 3\}$ and hence $d(x) = |x - z| \geq \varepsilon^{\frac{1}{2}}(C(R) - \varepsilon^{3\delta}) > \varepsilon$ for $\varepsilon > 0$ sufficiently small. This implies a contradiction, and thus $x_1 < 2\varepsilon$. Let us consider now $\eta \in B_{\frac{\rho(x)}{4}}(x) \cap \Omega$, then $\frac{3}{4}\rho(x) < \rho(\eta) < \frac{5}{4}\rho(x)$. Hence, using the definition of cosine

$$
\begin{aligned}
\cos(\theta(x)) - \cos(\theta(\eta)) &= \frac{x_1 + \frac{L}{2}\varepsilon}{\rho(x)} - \frac{\eta_1 + \frac{L}{2}\varepsilon}{\rho(\eta)} \leq \frac{x_1 + \frac{L}{2}\varepsilon}{\rho(x)} - \frac{\frac{2L}{5}\varepsilon}{\rho(x)} \\
&= \frac{1}{\rho(x)}\left(x_1 + \frac{1}{10}L\varepsilon \right) = \frac{1}{\rho(x)}\left(\frac{x_1}{2} + \frac{L}{4}\varepsilon \right) + \frac{1}{\rho(x)}\left(\frac{x_1}{2} - \frac{3L}{20}\varepsilon \right) \leq \frac{\cos(\theta(x))}{2}
\end{aligned}
$$

if $L > \frac{20}{3}$. This implies that $\cos(\theta(\eta)) \geq \frac{1}{2}\cos(\theta(x))$ and therefore using $\cos(\theta)\rho = x_1 + \frac{L}{2}\varepsilon \geq \frac{L}{2}\varepsilon$ we obtain

$$
\begin{aligned}
\sup_{\eta \in B_{\frac{\rho(x)}{4}}(x) \cap \Omega} \left[\cos^{-\frac{5}{2}}(\theta(\eta))\, \rho^{-\frac{7}{2}}(x) \right] &\leq 5^{\frac{5}{2}} \left(\frac{4}{3} \right)^{\frac{7}{2}} \cos(\theta(x))^{-\frac{5}{2}} \rho(x)^{-\frac{7}{2}} \\
&\leq \frac{2C}{\varepsilon L} \cos(\theta(x))^{-\frac{3}{2}} \rho(x)^{-\frac{5}{2}}.
\end{aligned} \tag{C.135}
$$

We can now proceed similarly as we did in equations (C.131) and (C.132). We apply the

Taylor expansion on the set $B_{\frac{\rho(x)}{4}}(x) \cap \Omega$ where W is smooth.

$$\begin{aligned}
\mathcal{L}^{\varepsilon}_{\Omega}(W)(x) =& W(x) - \int_{B_{\frac{\rho(x)}{4}}(x)\cap\Omega} K_\varepsilon(\eta - x)\, W(\eta)\, d\eta - \int_{B^c_{\frac{\rho(x)}{4}}(x)\cap\Omega} K_\varepsilon(\eta - x)\, W(\eta)\, d\eta \\
\geq & W(x) - \int_{B_{\frac{\rho(x)}{4}}(x)\cap\Omega} K_\varepsilon(\eta - x)\, W(\eta)\, d\eta - \int_{B^c_{\frac{\rho(x)}{4}}(x)} K_\varepsilon(\eta - x)\, (F+G)(\eta)\, d\eta \\
=& W(x)\left(1 - \int_{B_{\frac{\rho(x)}{4}}(x)\cap\Omega} K_\varepsilon(\eta - x)\; d\eta\right) - (\nabla_x W(x))\cdot \int_{B_{\frac{\rho(x)}{4}}(x)\cap\Omega} K_\varepsilon(\eta - x)\,(\eta - x)\, d\eta \\
& - \frac{1}{2}\int_{B_{\frac{\rho(x)}{4}}(x)\cap\Omega} K_\varepsilon(\eta - x)\,(\eta - x)^\top \nabla^2_x W(x)(\eta - x)\, d\eta \\
& - \sum_{|\alpha|=3} \frac{D^\alpha(F+H)(x)}{\alpha!} \int_{B_{\frac{\rho(x)}{4}}(x)\cap\Omega} K_\varepsilon(\eta - x)\,(\eta - x)^\alpha\, d\eta \\
& - \int_{B_{\frac{\rho(x)}{4}}(x)\cap\Omega} \frac{e^{-\frac{|\eta - x|}{\varepsilon}}}{4\pi\varepsilon\,|\eta - x|^2}(E^4_F + E^4_H + E^3_G)(\eta, x)\, d\eta - \int_{B^c_{\frac{\rho(x)}{4}}(x)} K_\varepsilon(\eta - x)\,(F+G)(\eta)\, d\eta
\end{aligned} \tag{C.136}$$

Once again we use the symmetry for the first and third order term on the set $B_{\frac{d(x)}{4}}(x) \subset B_{\frac{\rho(x)}{4}}(x) \cap \Omega$ estimating the integral on $B_{\frac{\rho(x)}{4}}(x) \cap \left(\Omega \setminus B_{\frac{d(x)}{4}}(x)\right)$ by the integral on the larger set $B^c_{\frac{d(x)}{4}}(x)$. Once more we need the identity for the Laplacian on the set $B_{\frac{d(x)}{4}}(x)$ too. We estimate the error terms of F and H by (C.130) and the error term of G by the last equation (C.135). Hence, we estimate

$$\begin{aligned}
\mathcal{L}^{\varepsilon}_{\Omega}(W)(x) \geq & -C_F\left(\frac{\varepsilon}{\rho}\int_{B^c_{\frac{d(x)}{4}}(0)} \frac{e^{-\frac{|y|}{\varepsilon}}}{4\pi\varepsilon^2|y|}\, dy + \frac{\varepsilon^3}{\rho^3}\int_{B^c_{\frac{d(x)}{4}}(0)} \frac{e^{-\frac{|y|}{\varepsilon}}|y|}{4\pi\varepsilon^4}\, dy\right) \\
& - C_H b\left(\frac{\varepsilon^3}{\rho^3}\int_{B^c_{\frac{d(x)}{4}}(0)} \frac{e^{-\frac{|y|}{\varepsilon}}}{4\pi\varepsilon^2|y|}\, dy + \frac{\varepsilon^4}{\rho^4}\int_{B^c_{\frac{d(x)}{4}}(0)} \frac{e^{-\frac{|y|}{\varepsilon}}}{4\pi\varepsilon^3}\, dy + \frac{\varepsilon^5}{\rho^5}\int_{B^c_{\frac{d(x)}{4}}(0)} \frac{e^{-\frac{|y|}{\varepsilon}}|y|}{4\pi\varepsilon^4}\, dy\right) \\
& - C_G a\left(\frac{\varepsilon}{L^{\frac{1}{2}}\rho}\int_{B^c_{\frac{d(x)}{4}}(0)} \frac{e^{-\frac{|y|}{\varepsilon}}}{4\pi\varepsilon^2|y|}\, dy + \frac{\varepsilon}{L^{\frac{3}{2}}\rho}\int_{B^c_{\frac{d(x)}{4}}(0)} \frac{e^{-\frac{|y|}{\varepsilon}}}{4\pi\varepsilon^3}\, dy\right) \\
& + \frac{2b}{3}\frac{\varepsilon^4}{\rho^4} - C_F\frac{\varepsilon^4}{\rho^4} - C_H b\frac{\varepsilon^6}{\rho^6} + \frac{1}{12}\frac{a\varepsilon^{\frac{5}{2}}}{\cos^{\frac{3}{2}}(\theta)\rho^{\frac{5}{2}}} - a\frac{C_G\varepsilon^{\frac{5}{2}}}{L\cos^{\frac{3}{2}}(\theta)\rho(x)^{\frac{5}{2}}} - (\pi + a)e^{-\frac{\rho}{4\varepsilon}} \\
\geq & - Ce^{-\frac{d}{8\varepsilon}}\frac{\varepsilon}{\rho}\left(a + C_F + \frac{aC_G}{L^{\frac{1}{2}}} + \frac{bC_H}{L^2}\right) + \frac{2}{3}\frac{\varepsilon^4}{\rho^4}\left(b - \frac{3}{2}C_F - \frac{3C_H b}{2L^2}\right) \\
& + \frac{1}{12}\frac{a\varepsilon^{\frac{5}{2}}}{\cos^{\frac{3}{2}}(\theta)\rho^{\frac{5}{2}}}\left(1 - 12\frac{C_G}{L}\right).
\end{aligned} \tag{C.137}$$

In all the estimate above we used that $e^{-\frac{\rho}{4\varepsilon}} \leq \frac{C\varepsilon}{\rho} e^{-\frac{d}{8\varepsilon}}$, since $\rho > L\varepsilon > d(x)$. Hence, choosing

$$b > 3C_F \qquad \text{and} \qquad L > \max\left\{\frac{20}{3},\ 12C_G,\ \sqrt{3C_H}\right\} \tag{C.138}$$

we conclude using $L \geq \sqrt{L}$

$$\mathcal{L}^\varepsilon_\Omega(W)(x) \geq -\frac{C(a,\pi,b,C_F)}{\sqrt{L}} e^{-\frac{d(x)}{8\varepsilon}}. \tag{C.139}$$

Case 3: $\rho(x) > L\varepsilon$ **and** $d(x) > \varepsilon$

We can finish the proof of this Lemma by estimating the operator acting on W when $\rho(x) > L\varepsilon$ and $d(x) > \varepsilon$ for $x \in \mathcal{C}_{3\delta}$. We notice that by definition $d(x) \leq \rho(x)$. In this case, we estimate first the operator acting only on $F + H$ proceeding as for the derivation of (C.136) and (C.137) with the only difference that this time

$$\begin{aligned}(F+H)(x)\left(1 - \int_{B_{\frac{\rho(x)}{4}}(x)\cap\Omega} K_\varepsilon(\eta - x)\ d\eta\right) &\geq -\frac{b\varepsilon^2\cos(\theta)^2}{\rho^2}\int_{B^c_{\frac{d}{4}}(x)} K_\varepsilon(\eta - x)\ d\eta \\ &\geq -\frac{b\varepsilon^2}{\rho^2} e^{-\frac{d(x)}{4\varepsilon}} \geq -\frac{b}{L^2} e^{-\frac{d(x)}{4\varepsilon}}.\end{aligned}$$

Hence we get for $F + H$

$$\begin{aligned}\mathcal{L}^\varepsilon_\Omega(F+H)(x) \geq & -Ce^{-\frac{d(x)}{8\varepsilon}}\frac{1}{L}\left(C_F + \pi + \frac{bC_H}{L^2} + \frac{b}{L}\right) + \frac{2}{3}\frac{\varepsilon^4}{\rho^4}\left(b - \frac{3}{2}C_F - \frac{3C_H b}{2L^2}\right) \\ \geq & -Ce^{-\frac{d(x)}{8\varepsilon}}\frac{1}{L}(\pi + 2b),\end{aligned}$$

for b and L as in (C.138). We consider now the operator acting only on G and we compute

$$\begin{aligned}\mathcal{L}^\varepsilon_\Omega(G)(x) = & G(x) - \int_{B_{\frac{d(x)}{4}}(x)} K_\varepsilon(\eta - x)\, G(\eta)\ d\eta - \int_{B_{\frac{\rho(x)}{4}}(x)\cap\Omega\setminus B_{\frac{d(x)}{4}}(x)} K_\varepsilon(\eta - x)\, G(\eta)\ d\eta \\ & - \int_{B^c_{\frac{\rho(x)}{4}}(x)\cap\Omega} K_\varepsilon(\eta - x)\, G(\eta)\ d\eta \\ \geq & G(x) - \int_{B_{\frac{d(x)}{4}}(x)} K_\varepsilon(\eta - x)\, G(\eta)\ d\eta - \frac{2a}{\sqrt{3L}} e^{-\frac{d(x)}{4\varepsilon}} - \frac{Ca}{L} e^{-\frac{d(x)}{8\varepsilon}}.\end{aligned}$$

We used $\rho(\eta) \geq \frac{3}{4}\rho(x) \geq \frac{3}{4}L\varepsilon$ for $\eta \in B_{\frac{\rho(x)}{4}}(x) \cap \Omega \setminus B_{\frac{d(x)}{4}}(x)$, the integral on $B_{\frac{\rho(x)}{4}}(x) \cap \Omega \setminus B_{\frac{d(x)}{4}}(x)$ can be estimated from above by the one on $B^c_{\frac{d(x)}{4}}(x)$ and the last integral was estimated as in (C.136) and (C.137) using $e^{-\frac{\rho(x)}{4\varepsilon}} \leq \frac{C}{L} e^{-\frac{d(x)}{8\varepsilon}}$. In order to estimate the integral on $B_{\frac{d(x)}{4}}(x)$ we will expand G by Taylor and therefore we have to control the singularity $(\cos(\theta))^{-\frac{5}{2}}$ of the error term E^3_G defined in (C.128). Let hence $\eta \in B_{\frac{d(x)}{4}}(x)$ for $d(x) > \varepsilon$. Since $d(x) \leq x_1$, we know that $x_1 > \varepsilon$ and also that $\eta_1 > x_1 - \frac{d(x)}{4} > \frac{3}{4}x_1$. That $d(x) \leq x_1$ can be proved in the following way. Let $z = \{x - te_1 : t \geq 0\} \cap \partial\Omega$, hence $z_1 \geq 0$. Then $d(x) \leq |x - z| = x_1 - z_1 \leq x_1$. We can thus estimate

$$\begin{aligned}\cos(\theta(\eta)) = & \frac{\eta_1 + \frac{L}{2}\varepsilon}{\rho(\eta)} > \frac{x_1 - \frac{d(x)}{4} + \frac{L}{2}\varepsilon}{\rho(\eta)} > \frac{\frac{3}{4}x_1 + \frac{L}{2}\varepsilon}{\rho(\eta)} \\ = & \frac{3}{4}\frac{x_1 + \frac{L}{2}\varepsilon}{\rho(\eta)} + \frac{1}{8}\frac{L\varepsilon}{\rho(\eta)} > \frac{3}{4}\frac{x_1 + \frac{L}{2}\varepsilon}{\rho(\eta)} > \frac{3}{5}\frac{x_1 + \frac{L}{2}\varepsilon}{\rho(x)} = \frac{3}{5}\cos(\theta(x)),\end{aligned}$$

where we used that $\rho(\eta) < \rho(x) + \frac{d(x)}{4} < \frac{5}{4}\rho(x)$ since $\rho(x) \geq x_1 + \frac{L}{2\varepsilon} \geq d(x) + \frac{L}{2\varepsilon} > d(x)$. Similarly $\rho(\eta) > \frac{3}{4}\rho(x)$. Hence,

$$\begin{aligned}\sup_{\eta\in B_{\frac{d(x)}{4}}(x)} \left[\cos^{-\frac{5}{2}}(\theta(\eta))\,\rho^{-\frac{7}{2}}(x)\right] \leq & \left(\frac{5}{3}\right)^{\frac{5}{2}}\left(\frac{4}{3}\right)^{\frac{7}{2}} \cos(\theta(x))^{-\frac{5}{2}}\rho(x)^{-\frac{7}{2}} \\ \leq & \frac{2C}{\varepsilon L}\cos(\theta(x))^{-\frac{3}{2}}\rho(x)^{-\frac{5}{2}},\end{aligned}$$

where we used in addition $\cos(\theta)\rho = x_1 + \frac{L}{2}\varepsilon \geq \frac{L}{2}\varepsilon$. Now we are ready to conclude the estimate for G. We proceed as we did in (C.131) and in (C.136) using the Taylor expansion. We have then for L as in (C.138)

$$\begin{aligned}\mathcal{L}^\varepsilon_\Omega(G)(x) \geq & -\frac{2a}{\sqrt{3L}}e^{-\frac{d(x)}{4\varepsilon}} - \frac{Ca}{L}e^{-\frac{d(x)}{8\varepsilon}} - \frac{aC_G}{L^{\frac{3}{2}}}e^{-\frac{d(x)}{8\varepsilon}} + \frac{1}{12}\frac{a\varepsilon^{\frac{5}{2}}}{\cos^{\frac{3}{2}}(\theta)\rho^{\frac{5}{2}}}\left(1 - 12\frac{C_G}{L}\right) \\ \geq & -\frac{C(a)}{\sqrt{L}}e^{-\frac{d(x)}{8\varepsilon}},\end{aligned}$$

Thus, we obtain once more

$$\mathcal{L}^\varepsilon_\Omega(W)(x) \geq -\frac{C(a,\pi,C_F,b)}{\sqrt{L}}e^{-\frac{d(x)}{8\varepsilon}}. \tag{C.140}$$

Equations (C.134), (C.139) and (C.140) imply the last claim (C.123). Indeed, there exists a constant $C > 0$ independent of ε, L, δ such that for $L > L_0$ as in (C.133) and (C.138) and for $0 < \varepsilon < 1$ sufficiently small such that $L < \varepsilon^{-\frac{1-8\delta}{2}}$, $x \in \mathcal{C}_{3\delta}$ it holds

$$\mathcal{L}^\varepsilon_\Omega(W)(x) \geq -\frac{C}{\sqrt{L}}e^{-\frac{d(x)}{8\varepsilon}}.$$

This conclude the proof of the Lemma. □

We can now prove Proposition C.5.

Proof of Proposition C.5. Let $W_{\varepsilon,L}$ be as in (C.115). For $\overline{W} := \sum_{i=2}^3 \left(W_i^+(x) + W_i^-(x)\right)$ Lemma C.10 implies that for any $i = 1, 2, 3$

$$\begin{cases} \overline{W}(x) \geq 0 & \text{if } x \in \Omega; \\ \overline{W}(x) \geq \frac{\pi}{2} - \arctan(2) & \text{if } |x_i| \geq \varepsilon^{\frac{1}{2}+3\delta}; \\ \overline{W}(x) \leq C\varepsilon^\alpha & \text{if } |x_i| < \varepsilon^{\frac{1}{2}+4\delta}; \\ \mathcal{L}^\varepsilon_\Omega(\overline{W})(x) \geq -\frac{C}{\sqrt{L}}e^{-\frac{d(x)}{8\varepsilon}} & \text{if } |x_i| < \varepsilon^{\frac{1}{2}+3\delta}. \end{cases} \tag{C.141}$$

Moreover, Theorem C.6 imply that there exists a constant $\tilde{C}(\Omega, g_\nu)$ such that $\frac{\tilde{C}}{\sqrt{L}}\phi_{\frac{1}{8},\varepsilon}$ satisfies

$$\mathcal{L}^\varepsilon_\Omega\left(\frac{\tilde{C}}{\sqrt{L}}\phi_{\frac{1}{8},\varepsilon}\right)(x) \geq \frac{C}{\sqrt{L}}e^{-\frac{d(x)}{8\varepsilon}} \qquad \forall x \in \Omega,$$

where C is the constant in (C.141). Theorem C.6 implies also $0 \leq \phi_{A,\varepsilon}(x) \leq C(\Omega) < \infty$ for any $0 < A < 1$ and any $0 < A\varepsilon \leq \varepsilon_0$. Hence, $W_{\varepsilon,L}$ is non negative and we can conclude equation (C.110) for $\alpha = \frac{\delta}{2}$, for some constant $C(\Omega, g_\nu) > 0$ independent of $\delta, p \in \partial\Omega, \varepsilon, L$ and for $L > 0$ and $0 < \varepsilon < 1$ sufficiently small such that $L > \varepsilon^{-\frac{2}{1-8\delta}}$. □

The properties of $W_{\varepsilon,L}$ imply now the boundedness of $|u^\varepsilon - \overline{u}|$ near the boundary $\partial\Omega$.

Corollary C.4. *Let $0 < \delta < \frac{1}{16}$. There exists a constant $C > 0$, a large $L > 0$ and an $\alpha > 0$ independent of x, p, ε such that*

$$\left|\overline{u}\left(\frac{R_p(\cdot)\cdot e_1}{\varepsilon}, p\right) - u^\varepsilon\right|(x) \leq C\left(\varepsilon^\alpha + \frac{1}{\sqrt{L}}\right)$$

for all $|x-p| < \varepsilon^{\frac{1}{2}+4\delta}$.

Proof. Let $L > 0$ large as in Proposition C.5 and let $0 < \varepsilon < 1$ sufficiently small such that $L < \varepsilon^{-\frac{1}{\beta}}$ for $\beta = \frac{1-8\delta}{2}$. Let $\delta < \frac{1}{16}$ and $\alpha = \frac{\delta}{2}$. If $|x-p| < \varepsilon^{\frac{1}{2}+4\delta}$ with $x \in \Omega$ then $0 < x \cdot (-N_p) < \varepsilon^{\frac{1}{2}+4\delta}$ and $|\mathcal{R}_p(x)\cdot e_i| < \varepsilon^{\frac{1}{2}+4\delta}$ for $i = 2, 3$. Let us consider $W_{\varepsilon,L}$ as defined in (C.115). As we know, $\overline{u}\left(\frac{R_p(\cdot)\cdot e_1}{\varepsilon}, p\right)$ and u^ε are both uniformly bounded, independently of p. Let us call $K > 0$ this bound. Moreover, $W_{\varepsilon,L} \geq \frac{\pi}{2} - \arctan(2) := \tilde{C} > 0$. Hence, for all $|\mathcal{R}_p(x)\cdot e_i| \geq \varepsilon^{\frac{1}{2}+3\delta}$ we have that $\frac{K}{\tilde{C}} W_{\varepsilon,L}(x) - \left|\overline{u}\left(\frac{R_p(\cdot)\cdot e_1}{\varepsilon}, p\right) - u^\varepsilon\right|(x) \geq 0$. The function $\frac{K}{\tilde{C}} W_{\varepsilon,L}$ is also continuous and satisfies $\frac{K}{\tilde{C}} \mathcal{L}_\Omega^\varepsilon (W_{\varepsilon,L}) \geq C\varepsilon^\delta \frac{K}{\tilde{C}} e^{-\frac{Ad(x)}{\varepsilon}}$ for all $|\mathcal{R}_p(x)\cdot e_i| < \varepsilon^{\frac{1}{2}+3\delta}$. Using the maximum principle of Theorem C.5 and the estimate for the operator acting on $\overline{u}\left(\frac{R_p(\cdot)\cdot e_1}{\varepsilon}, p\right) - u^\varepsilon$ in Lemma C.9 we obtain

$$\left|\overline{u}\left(\frac{R_p(\cdot)\cdot e_1}{\varepsilon}, p\right) - u^\varepsilon\right|(x) \leq C\left(\varepsilon^\alpha + \frac{1}{\sqrt{L}}\right)$$

for all $|x-p| < \varepsilon^{\frac{1}{2}+4\delta}$. □

This is a key estimate for the proof of the convergence of the exact solutions u^ε up to the boundary $\partial\Omega$.

C.4.4 Convergence of u^ε to the solution of the new boundary value problem

This last section is devoted to the proof of the pointwise convergence of u^ε to the solution of the Laplace equation in Ω with boundary value $\overline{u}_\infty$ (cf. Proposition C.3). We proceed defining new regions. We define $\hat{\Omega}_\varepsilon := \left\{x \in \Omega : d(x) > \varepsilon^{\frac{1}{2}+4\delta}\right\}$, $\Sigma_\varepsilon := \left\{x \in \Omega : \varepsilon^{\frac{1}{2}+6\delta} < d(x) \leq \varepsilon^{\frac{1}{2}+4\delta}\right\}$ and their union $\Omega_\varepsilon = \hat{\Omega}_\varepsilon \cup \Sigma_\varepsilon$. We also define for $0 < \sigma \ll 1$ independent of ε the set $\Omega^\sigma := \Omega \cup \{x \in \Omega^c : d(x) < \sigma\}$. We recall the continuous projection $\pi_{\partial\Omega}$ as given in (C.14).

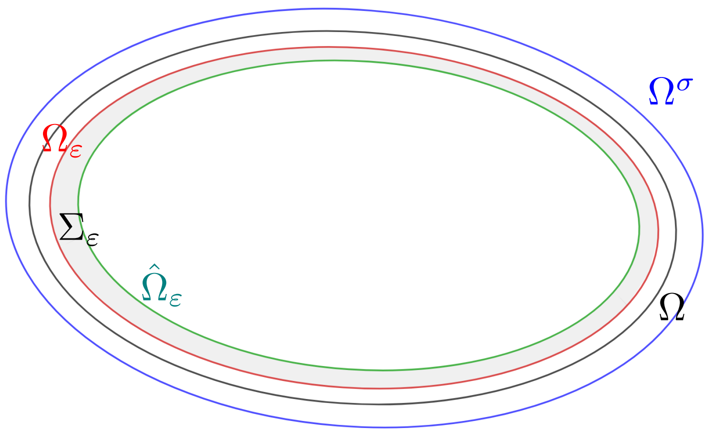

Figure C.9: Decomposition of Ω

Lemma C.11. *Let $0<\varepsilon<1$ and $0<\delta<\frac{1}{16}$. Let C,L,α be as in the Corollary C.4. Then it holds*

$$\sup_{x\in\Sigma_\varepsilon}\left|\overline{u}_\infty\left(\pi_{\partial\Omega}(x)\right)-u^\varepsilon(x)\right|\leq C\left(\varepsilon^\alpha+\frac{1}{\sqrt{L}}\right)+\tilde{\omega}_1\left(\varepsilon^{\frac{1}{2}-6\delta}\right), \tag{C.142}$$

where $\tilde{\omega}_1(r)=Ce^{-\frac{r}{2}}$ for a suitable constant $C>0$.

Proof. By Corollary C.2 there exists a constant $C>0$ independent of $p\in\partial\Omega$ such that $|\overline{u}(y,p)-\overline{u}_\infty(p)|\leq Ce^{-\frac{y}{2}}=\tilde{\omega}_1(y)$. Hence, let $x\in\Sigma_\varepsilon$. Then

$$\begin{aligned}&\left|\overline{u}_\infty\left(\pi_{\partial\Omega}(x)\right)-u^\varepsilon(x)\right|\\ \leq&\left|u^\varepsilon(x)-\overline{u}\left(\frac{R_{\pi_{\partial\Omega}(x)}(x)\cdot e_1}{\varepsilon},\pi_{\partial\Omega}(x)\right)\right|-\left|\overline{u}_\infty\left(\pi_{\partial\Omega}(x)\right)-\overline{u}\left(\frac{R_{\pi_{\partial\Omega}(x)}(x)\cdot e_1}{\varepsilon},\pi_{\partial\Omega}(x)\right)\right|\\ <&C\left(\varepsilon^\alpha+\frac{1}{\sqrt{L}}\right)+\tilde{\omega}_1\left(\varepsilon^{\frac{1}{2}-6\delta}\right),\end{aligned}$$

where we used the result of Corollary C.4 and the fact that since $\varepsilon^{\frac{1}{2}+6\delta}<d(x)<\varepsilon^{\frac{1}{2}+4\delta}$ then we have that $\varepsilon^{-\frac{1}{2}+6\delta}\leq\left|\frac{R_{\pi_{\partial\Omega}(x)}(x)\cdot e_1}{\varepsilon}\right|\leq\varepsilon^{-\frac{1}{2}+4\delta}$. Moreover, since $\delta<\frac{1}{16}$, we have that $\frac{1}{2}-6\delta>\frac{1}{8}>0$. □

We recall first the definition of v as solution to the Laplacian (cf. (C.58))

$$\begin{cases}-\Delta v(x)=0 & x\in\Omega,\\ v(p)=\overline{u}_\infty(p) & p\in\partial\Omega.\end{cases}$$

Clearly by the uniformly continuity of $\overline{u}_\infty$ we have that $v\in C^\infty(\Omega)\cap C\left(\overline{\Omega}\right)$. We call ω_2 its modulus of continuity. Further, we need to consider another function, harmonic on the larger domain Ω^σ

$$\begin{cases}-\Delta v_\sigma(x)=0 & x\in\Omega^\sigma,\\ v(x)=\overline{u}_\infty\left(\pi_{\partial\Omega}(x)\right) & x\in\partial\Omega^\sigma.\end{cases}$$

We recall that $\pi_{\partial\Omega}:\partial\Omega^\sigma\to\partial\Omega$ is a continuous bijection if $\sigma>0$ small enough and therefore $v_\sigma\in C^\infty(\Omega^\sigma)\cap C\left(\overline{\Omega^\sigma}\right)$. Denoting ω its modulus of continuity a simple application of the maximum principle for harmonic functions implies $\sup_{x\in\Omega}|v_\sigma(x)-v(x)|\leq\omega(\sigma)$. Indeed, $v_\sigma-v$ is harmonic on Ω and thus the maximum must be attained on $\partial\Omega$. Hence, using that $\pi_{\partial\Omega}$ is a bijection we obtain

$$v_\sigma(x)-v(x)\leq\max_{x\in\partial\Omega}\left(v_\sigma(x)-v(x)\right)=\max_{x\in\partial\Omega^\sigma}\left(v_\sigma(\pi_{\partial\Omega}(x))-v_\sigma(x)\right)\leq\omega(\sigma), \tag{C.143}$$

since $|x-\pi_{\partial\Omega}(x)|=\sigma$ for $x\in\partial\Omega^\sigma$. The same can be estimated for $v-v_\sigma$.

Lemma C.12. *Let $x\in\hat{\Omega}_\varepsilon$. Then*

$$\left|\mathcal{L}^\varepsilon_\Omega\left(v_\sigma-u^\varepsilon\right)(x)\right|\leq C(\Omega,g_\nu)e^{-\frac{Ad(x)}{\varepsilon}}\left(\varepsilon^\beta+\frac{\varepsilon}{\sigma}\right)+\frac{C}{\sigma^3}\varepsilon^3, \tag{C.144}$$

for some constant $C(\Omega,g_\nu)>0$ and $\varepsilon>0$ sufficiently small.

Proof. We already know that in general we always have the estimate $\left|\mathcal{L}^\varepsilon_\Omega\left(u^\varepsilon\right)(x)\right|\leq Ce^{-\frac{d}{\varepsilon}}$ (cf. Theorem C.6). Since for $x\in\hat{\Omega}_\varepsilon$ the distance to the boundary satisfies $d(x)>\varepsilon^{\frac{1}{2}+4\delta}$, we estimate for these points

$$\left|\mathcal{L}^\varepsilon_\Omega\left(u^\varepsilon\right)(x)\right|\leq Ce^{-\frac{d}{\varepsilon}}\leq Ce^{-\frac{d}{2\varepsilon}}2\frac{\varepsilon}{\varepsilon^{\frac{1}{2}+4\delta}}\leq Ce^{-\frac{d}{2\varepsilon}}\varepsilon^{\frac{1}{2}-4\delta}=Ce^{-\frac{d}{2\varepsilon}}\varepsilon^\beta, \tag{C.145}$$

where $\beta = \frac{1}{2} - 4\delta > 0$ for $\delta < \frac{1}{16}$.

Let us consider now the operator $\mathcal{L}^{\varepsilon}_{\Omega}$ acting on v_σ. We apply as usual the Taylor expansion to $v_\sigma(\eta) = v_\sigma(x)+\nabla_x v_\sigma(x)\cdot(\eta-x)+\frac{1}{2}(\eta-x)^\top\nabla^2_x v_\sigma(x)(\eta-x)+E^3(\eta,x)$. Since $x \in B_\sigma(x) \subset \Omega^\sigma$ for all $x \in \Omega$ by the harmonicity of v_σ we obtain

$$|\partial^\alpha v_\sigma(x)| \leq C(|\alpha|)\frac{\|v_\sigma\|_\infty \sigma^3}{\sigma^{3+|\alpha|}} = C(|\alpha|)\frac{\|\overline{u}_\infty\|_\infty}{\sigma^{|\alpha|}} \leq \frac{C}{\sigma^{|\alpha|}},$$

where we used that v_σ attains its maximum on the boundary and that $\overline{u}_\infty$ is uniformly bounded. Hence, we calculate for $x \in \hat{\Omega}_\varepsilon$ and $\varepsilon > 0$ sufficiently small

$$\begin{aligned}
&|\mathcal{L}^{\varepsilon}_{\Omega}\,(v_\sigma)\,(x)| \\
&\quad= \left|v_\sigma(x) - \int_\Omega K_\varepsilon\,(\eta-x)\left[v_\sigma(x)+\nabla_x v_\sigma(x)\cdot(\eta-x)+\frac{1}{2}(\eta-x)^\top\nabla^2_x v_\sigma(x)(\eta-x)\right]\,d\eta\right| \\
&\quad+\left|\int_\Omega K_\varepsilon\,(\eta-x)\,E^3(\eta,x)\,d\eta\right| \\
&\quad\leq v_\sigma(x)\int_{B^c_{d(x)(x)}} K_\varepsilon\,(\eta-x)\,\,d\eta + \frac{C\varepsilon}{\sigma}\int_{B^c_{d(x)(0)}}\frac{e^{-\frac{|y|}{\varepsilon}}}{4\pi\varepsilon^2|y|}\,dy + \frac{C\varepsilon^2}{\sigma^2}\int_{B^c_{d(x)(0)}}\frac{e^{-\frac{|y|}{\varepsilon}}}{4\pi\varepsilon^3}\,dy \\
&\quad+\frac{C\varepsilon^3}{\sigma^3}\int_{\mathbb{R}^3}\frac{e^{-\frac{|y|}{\varepsilon}}}{4\pi\varepsilon^3}\frac{|y|}{\varepsilon}\,dy \\
&\quad\leq Ce^{-\frac{d(x)}{\varepsilon}}\left(1+\frac{\varepsilon}{\sigma}\right)+\frac{C\varepsilon^3}{\sigma^3},
\end{aligned} \tag{C.146}$$

where we used the symmetry of the kernel for the first order term, the fact that v_σ is harmonic for the second order term, the boundedness of $\int_0^\infty e^{-r}r^3\,dr$ and the fact that $\Omega^c \subset B^c_{d(x)}(x)$. Equations (C.145) and (C.146) yield (C.144). □

Lemma C.13. *Let $x \in \Sigma_\varepsilon$ and $\varepsilon > 0$ small enough. Then the following uniform bound holds*

$$|v_\sigma(x) - u^\varepsilon(x)| \leq \omega(\sigma) + \omega_2\left(\varepsilon^{\frac{1}{2}+4\delta}\right) + C\left(\varepsilon^\alpha + \frac{1}{\sqrt{L}}\right) + \tilde{\omega}_1\left(\varepsilon^{\frac{1}{2}-6\delta}\right).$$

Proof. Let $x \in \Sigma_\varepsilon$. Then, $d(x) = |x - \pi_{\partial\Omega}(x)| < \varepsilon^{\frac{1}{2}+4\delta}$. Hence, $|v(x) - \overline{u}_\infty(\pi_{\partial\Omega}(x))| < \omega_2(\varepsilon^{\frac{1}{2}+4\delta})$. Thus using equation (C.143) and Lemma C.11 we conclude

$$\begin{aligned}
|v_\sigma(x) - u^\varepsilon(x)| &\leq |v_\sigma(x) - v(x)| + |v(x) - \overline{u}_\infty(\pi_{\partial\Omega}(x))| + |\overline{u}_\infty(\pi_{\partial\Omega}(x)) - u^\varepsilon(x)| \\
&\leq \omega(\sigma) + \omega_2\left(\varepsilon^{\frac{1}{2}+4\delta}\right) + C\left(\varepsilon^\alpha + \frac{1}{\sqrt{L}}\right) + \tilde{\omega}_1\left(\varepsilon^{\frac{1}{2}-6\delta}\right).
\end{aligned} \tag{C.147}$$

□

With an application of the maximum principle on $\Omega_\varepsilon := \hat{\Omega}_\varepsilon \cup \Sigma_\varepsilon$ we can prove the convergence of u^ε to the harmonic function v $\varepsilon \to 0$.

Theorem C.7. *u^ε converges to v uniformly in every compact set.*

Proof. As we mentioned above we will use the maximum principle for the operator $\mathcal{L}^{\varepsilon}_{\Omega_\varepsilon}$. We start estimating for $x \in \hat{\Omega}_\varepsilon$ this operator acting on $v_\sigma - u^\varepsilon$. Thus, for $\varepsilon > 0$ sufficiently small

$$\begin{aligned}\left|\mathcal{L}^{\varepsilon}_{\Omega_\varepsilon}\left(v_\sigma - u^\varepsilon\right)(x)\right| &\leq \left|\mathcal{L}^{\varepsilon}_{\Omega}\left(v_\sigma - u^\varepsilon\right)(x)\right| + \int_{\Omega\setminus\Omega_\varepsilon} K_\varepsilon\left(\eta - x\right)\left(v_\sigma(\eta) - u^\varepsilon(\eta)\right)\, d\eta \\ &\leq Ce^{-\frac{d(x)}{\varepsilon}}\left(1 + \frac{\varepsilon}{\sigma}\right) + \frac{C\varepsilon^3}{\sigma^3} + \frac{C}{\varepsilon}\exp\left(-\frac{\varepsilon^{\frac{8\delta-1}{2}}}{2}\right)\int_{\Omega\setminus\Omega_\varepsilon}\frac{1}{|\eta - x|^2}\, d\eta \\ &\leq Ce^{-\frac{d(x)}{2\varepsilon}}\left(\varepsilon^{\frac{1-8\delta}{2}} + \frac{\varepsilon}{\sigma}\right) + \frac{C\varepsilon^3}{\sigma^3} + C\left(\Omega\right)\varepsilon^3,\end{aligned} \tag{C.148}$$

where we used Lemma C.12, the fact that if $\eta \in \Omega \setminus \Omega_\varepsilon$ and $0 < \varepsilon < 2^{-\frac{1}{2\delta}}$ we have $|x-\eta| > \varepsilon^{\frac{1}{2}+4\delta}(1-\varepsilon^{2\delta}) > \frac{\varepsilon^{\frac{1}{2}+4\delta}}{2}$ and that $d(x) \geq \varepsilon^{\frac{1}{2}+4\delta}$. Moreover, we used that $\Omega\setminus\Omega_\varepsilon \subset \Omega$ and that for any $n \in \mathbb{N}$ there exists a constant C_n such that $|x|^n e^{-|x|} \leq C_n$. We chose here $\delta < \frac{1}{72}$ and $n = 9$.

We now set for $C > 0$ the maximum between the constants appearing in estimates (C.147) and (C.148)

$$K_\varepsilon = C\left(\varepsilon^{\frac{1-8\delta}{2}} + \frac{\varepsilon}{\sigma} + \frac{\varepsilon}{\sigma^3} + \omega(\sigma) + \omega_2\left(\varepsilon^{\frac{1}{2}+4\delta}\right) + C\left(\varepsilon^\alpha + \frac{1}{\sqrt{L}}\right) + \tilde{\omega}_1\left(\varepsilon^{\frac{1}{2}-6\delta}\right)\right).$$

Analogously to Theorem C.6 we define

$$\psi(x) = K_\varepsilon 2C_3\left(\left(C_1 - |x|^2\right) + C_2\left[\left(1 - \frac{\gamma}{1+\left(\frac{d_\varepsilon(x)}{2\varepsilon}\right)^2}\right)\wedge\left(1 - \frac{\gamma}{1+\left(\frac{\mu R}{4\varepsilon}\right)^2}\right)\right]\right),$$

where $C_1(\Omega) := 2\max_{x\in\overline{\Omega}}|x|^2 + 2\,\mathrm{diam}\,(\Omega)^2 + 4\,\mathrm{diam}\,(\Omega) + 4$ and $C_2(\Omega), C_3(\Omega)$ are independent of ε. We denote $d_\varepsilon := \mathrm{dist}(x, \partial\Omega_\varepsilon)$. We claim that ψ is a supersolution for $\mathcal{L}^{\varepsilon}_{\Omega_\varepsilon}$. This is true because the geometrical properties of Ω_ε, in particular its regularity and the estimate for the radii of curvature, are identical to those for Ω. More precisely, if $\varepsilon < \varepsilon_0$ sufficiently small the minimal radius of curvature R_ε for $\partial\Omega_\varepsilon$ satisfies $\frac{R}{2} < R_\varepsilon < R$. Moreover, $d_\varepsilon(x) \leq d(x)$ and therefore $\|\nabla^2_x d_\varepsilon\|_{op} \leq \frac{1}{(1-\mu)\frac{R}{2}}$ for $\mu \in (0,1)$ and $d_\varepsilon(x) < \frac{R}{2}\mu$. Hence, C_2, C_3, γ, μ can be chosen as in Theorem C.6. Thus, ψ has uniform upper and lower bound independently on ε and all calculations we performed in Lemma C.7 and Lemma C.8 apply also for $\mathcal{L}^{\varepsilon}_{\Omega_\varepsilon}(\psi)(x)$. Hence, since $d_\varepsilon(x) \leq d(x)$ we estimate using equation (C.93)

$$\mathcal{L}^{\varepsilon}_{\Omega_\varepsilon}\left(\psi\right)(x) \geq K_\varepsilon\left(e^{-\frac{d_\varepsilon(x)}{2\varepsilon}} + \varepsilon^2\right) \geq K_\varepsilon\left(e^{-\frac{d(x)}{2\varepsilon}} + \varepsilon^2\right). \tag{C.149}$$

We remark that $\psi \geq 8K_\varepsilon C_3$ by the definition of C_1. Hence, multiplying ψ by $K = \max\{1, \frac{1}{8C_3}\}$ we have $K\psi \geq K_\varepsilon$. We apply now use the maximum principle for $\mathcal{L}^{\varepsilon}_{\Omega_\varepsilon}$ acting on $K\psi - (v_\sigma - u^\varepsilon)$. Indeed, $K\psi(x) \geq |v_\sigma(x) - u^\varepsilon(x)|$ if $x \in \Sigma_\varepsilon$ by Lemma C.13 and the definition of K_ε. We notice also that estimates (C.148) and (C.149) imply $\mathcal{L}^{\varepsilon}_{\Omega_\varepsilon}(K\psi - (v_\sigma(x) - u^\varepsilon(x)))(x) \geq 0$ as well as $\mathcal{L}^{\varepsilon}_{\Omega_\varepsilon}\left(K\psi - (u^\varepsilon(x) - v_\sigma(x))\right)(x) \geq 0$ if $x \in \hat{\Omega}_\varepsilon$. Hence, using the maximum principle for the operator $\mathcal{L}^{\varepsilon}_{\Omega_\varepsilon}$ we conclude $|u^\varepsilon - v_\sigma| \leq K\psi \leq CK_\varepsilon$ for $x \in \Omega_\varepsilon$ and for some constant $C(\Omega, K, g_\nu)$ since ψ is bounded (cf. Theorem C.6). Thus,

$$CK_\varepsilon - v_\sigma(x) \leq u^\varepsilon(x) \leq CK_\varepsilon + v_\sigma(x) \tag{C.150}$$

for all $x \in \Omega_\varepsilon$. Since $\lim_{\varepsilon\to 0} K_\varepsilon = C\left(\omega(\sigma) + \frac{1}{\sqrt{L}}\right)$, we obtain

$$C\left(\omega(\sigma) + \frac{1}{\sqrt{L}}\right) - v_\sigma(x) = \liminf_{\varepsilon\to 0} u^\varepsilon(x) \le \limsup_{\varepsilon\to 0} u^\varepsilon(x) = C\left(\omega(\sigma) + \frac{1}{\sqrt{L}}\right) + v_\sigma(x).$$

Letting first $L \to \infty$ and then $\sigma \to 0$ we conclude

$$v(x) = \liminf_{\varepsilon\to 0} u^\varepsilon(x) \le \limsup_{\varepsilon\to 0} u^\varepsilon(x) = v(x)$$

and hence

$$\lim_{\varepsilon\to 0} u^\varepsilon(x) = v(x)$$

for all $x \in \Omega$. The convergence is not only pointwise but also uniform in every compact set. Let indeed $A \subset \Omega$ be compact. Then there exists an ε_0 such that $A \subset \Omega_\varepsilon$ for all $\varepsilon < \varepsilon_0$. Since CK_ε is independent of $x \in \Omega$ and $|v - v_\sigma| \le \omega(\sigma)$ uniformly in $x \in \Omega$ equation (C.150) implies

$$\sup_{x\in A} |u^\varepsilon(x) - v(x)| \le CK_\varepsilon + \omega(\sigma) \longrightarrow 0$$

as $\varepsilon \to 0$, $L \to \infty$ and $\sigma \to 0$ in this order. Hence, Theorem C.7 follows. □

We end this section with the following corollary which shows the convergence as $\varepsilon \to 0$ of the solution $(I_\nu^\varepsilon, T_\varepsilon)$ of the problem (C.6) to the vector $(B_\nu(T), T)$, where $4\pi\sigma T^4 = v$ is the solution to the boundary value problem (C.58). This result implies Theorem C.1.

Corollary C.5. *Let $(I_\nu^\varepsilon(x,n), T_\varepsilon(x))$ be the solution of the problem* (C.6) *and let $v(x)$ be the solution of* (C.58). *Then T_ε converges to $T = \left(\frac{v}{4\pi\sigma}\right)^{1/4}$ uniformly in every compact set. Moreover, $I_\nu^\varepsilon(x,n)$ converges to $B_\nu(T(x))$ uniformly in every compact set $K \subset \Omega$ as a function with values in $L^\infty(\mathbb{S}^2, L^1(\mathbb{R}_+))$.*

Proof. The convergence of T_ε to T is a direct implication of the previous Theorem (C.7) and of the definition of $u^\varepsilon = 4\pi\sigma T_\varepsilon^4$.

The convergence of the radiation intensity to the Planck distribution follows from an application of the dominated convergence theorem. Indeed, from equation (C.59), changing the variable $t \mapsto \frac{t}{\varepsilon}$ we see that

$$I_\nu^\varepsilon(x,n) = g_\nu(n) e^{-\frac{|x - x_\Omega(x,n)|}{\varepsilon}} + \int_0^{\frac{|x - x_\Omega(x,n)|}{\varepsilon}} e^{-t} B_\nu\left(T_\varepsilon(x - \varepsilon t n)\right)\, dt.$$

Hence, for any compact $K \in \Omega$ there exists some ε_0 such that $K \subset \Omega_\varepsilon$ for all $\varepsilon < \varepsilon_0$. Thus,

$$\frac{|x - x_\Omega(x,n)|}{\varepsilon} > \varepsilon^{-\frac{1}{2}+6\delta} > \varepsilon^{-\frac{1}{8}}$$

for all $x \in K$, $n \in \mathbb{S}^2$ and $\varepsilon < \varepsilon_0$. Moreover, for ε_0 small enough the set

$$A = \left\{ x - \tau n : x \in K, n \in \mathbb{S}^2, \tau \in \left[0, \varepsilon_0^{\frac{1}{2}+7\delta}\right] \right\} \subset \Omega$$

is compact. Therefore by the uniform convergence in compact sets of T_ε we conclude with the dominated convergence theorem that

$$\begin{aligned}
&\int_0^\infty |I_\nu^\varepsilon(x,n) - B_\nu(T(x))|\, d\nu \\
&\quad \le \int_0^\infty g_\nu(n) e^{-\frac{|x - x_\Omega(x,n)|}{\varepsilon}} d\nu + \int_0^{\frac{|x - x_\Omega(x,n)|}{\varepsilon}} e^{-t} |u^\varepsilon(x - \varepsilon t n) - v(x)|\, dt \\
&\quad \le e^{-\varepsilon^{-\frac{1}{8}}} \|g\|_\infty + \|u^\varepsilon - v\|_{C(A)} \int_0^{\varepsilon^{-\frac{1}{2}+7\delta}} e^{-t} dt + \left(\|u^\varepsilon\|_{C(\Omega)} + \|v\|_{C(\Omega)}\right) \int_{\varepsilon^{-\frac{1}{2}+7\delta}}^\infty e^{-t} dt \to 0
\end{aligned}$$

uniformly in $n \in \mathbb{S}^2$ and in the compact set $K \in \Omega$. Notice that we used also the uniform boundedness of the sequence u^ε. $\square$

C.5 Diffusion approximation for space dependent absorption coefficient

We could prove in the previous sections the convergence of the initial boundary value problem (C.6) to the solution of the Laplacian for constant absorption coefficient. In this section we will show an analogous result for the case, when the absorption coefficient depends on $x \in \Omega$, but it does not depend on the frequency.

C.5.1 The limit problem and the boundary layer equation

We assume, as we did throughout the paper, $\Omega \subset \mathbb{R}^3$ to be a convex bounded domain with C^3-boundary and strictly positive curvature. From now on we also assume $\alpha \in C^3(\overline{\Omega})$ with $0 < c_0 \leq \alpha(x) \leq \|\alpha\|_{C^3} := c_1 < \infty$. We define $\overline{\alpha} \in C^3_b(\mathbb{R})$ to be the extension of α in the whole space with $0 < c_0 \leq \overline{\alpha}(x) \leq c_1$ and $\overline{\alpha}\,|_\Omega = \alpha$. For conivience we will denote $\overline{\alpha}$ by α. We assume g_ν to satisfy the same assumption as in the rest of the paper, namely $g_\nu(n) \geq 0$ with $\int_0^\infty g_\nu(n)\, d\nu \in L^\infty\left(\mathbb{S}^2\right)$. We study the limit as $\varepsilon \to 0$ of the following boundary value problem

$$\begin{cases} n \cdot \nabla_x I_\nu\left(x,n\right) = \frac{\alpha(x)}{\varepsilon}\left(B_\nu\left(T\left(x\right)\right) - I_\nu\left(x,n\right)\right) & x \in \Omega, \\ \nabla_x \cdot \mathcal{F} = 0 & x \in \Omega, \\ I_\nu\left(x,n\right) = g_\nu\left(n\right) & x \in \partial\Omega \text{ and } n \cdot N_x < 0. \end{cases} \tag{C.151}$$

We proceed in the same way as in the case of constant absorption coefficient. Following the computation in Section 2.1 we obtain the limit problem in the interior Ω for $u(x) = 4\pi\sigma T^4(x)$ as the elliptic equation

$$-\operatorname{div}\left(\frac{1}{\alpha(x)}\nabla_x u(x)\right) = 0. \tag{C.152}$$

For the boundary layer equation we argue similarly as in Section 2.2. Let $x_0 \in \partial\Omega$ and let $\mathcal{R}_{x_0}$ be the rigid motion in (C.13). We rescale $x = \frac{\varepsilon}{\alpha(x_0)}\mathcal{R}^{-1}_{x_0}(y) + x_0$ for $y \in \frac{\alpha(x_0)}{\varepsilon}\mathcal{R}_{x_0}\left(\Omega\right)$ and we define $\overline{g}_\nu(n) = g_\nu\left(\mathrm{Rot}^{-1}_{x_0}(n)\right)$. Moreover, since α is a C^3-function we also have for ε sufficiently small that $\alpha(x) = \alpha(x_0) + \frac{\varepsilon}{\alpha(x_0)}\mathcal{O}(|y|)$, and hence taking $\varepsilon \to 0$ we obtain once again for $N = \mathrm{Rot}_{x_0}(N_{x_0})$

$$\begin{cases} n \cdot \nabla_y I_\nu\left(y,n\right) = \left(B_\nu\left(T\left(y\right)\right) - I_\nu\left(y,n\right)\right) & y \in \mathbb{R}_+ \times \mathbb{R}^2, \\ \nabla_y \cdot \mathcal{F} = 0 & y \in \mathbb{R}_+ \times \mathbb{R}^2, \\ I_\nu\left(y,n\right) = \overline{g}_\nu\left(n\right) & y \in \{0\} \times \mathbb{R}^2 \text{ and } n \cdot N < 0. \end{cases}$$

This implies, that the boundary layer equation for $\overline{u}(y_1,p) = 4\pi\sigma T^4(y)$ is also in this case given by the integral equation (C.22)

$$\overline{u}(y_1,p) - \int_0^\infty d\eta\; K(y_1-\eta)u(\eta,p) = \int_0^\infty d\nu \int_{n\cdot N_p<0} dn\; g_\nu(n) e^{-\frac{y_1}{|n\cdot N_p|}},$$

where K is the normalized exponential integral. Hence, the whole theory developed in Section 3 is still valid and can be summarized by the Proposition (C.3).

Before moving to the rigorous proof of the convergence to the solution of the elliptic equation given in (C.152) we remark that the function $\overline{U}_\varepsilon(x,p) := \overline{u}\left(\frac{\alpha(p)}{\varepsilon}\mathcal{R}_p(x)\cdot e_1, p\right)$ solves the integral equation

$$\overline{U}_\varepsilon(x,p) - \int_{\Pi_p} d\eta\ \frac{\alpha(p)e^{-\frac{\alpha(p)|x-\eta|}{\varepsilon}}}{4\pi\varepsilon|x-\eta|^2}\overline{U}_\varepsilon(\eta,p) = \int_0^\infty d\nu \int_{n\cdot N_p<0} dn\ g_\nu(n)e^{-\frac{\alpha(p)\left|x-x_{\Pi_p}(x,n)\right|}{\varepsilon}}. \tag{C.153}$$

C.5.2 Rigorous proof of the convergence: equation for u^ε and properties of the kernel

We can now move to the proof of the convergence of the solution to the problem (C.151) to the elliptic equation (C.152). We will follow all arguments given in Section 4 and change them were needed. Hence, first of all we find the integral equation that the sequence $u^\varepsilon(x) = 4\pi\sigma T_\varepsilon^4(x)$ associated to the solution $I_\nu^\varepsilon(x,n)$ of (C.151) satisfies. We follow the computations of Section 4.1. Let $x\in\Omega$, $n\in\mathbb{S}^2$, $x_\Omega(x,n)\in\partial\Omega$ the unique intersection point of the line $\{x-tn : t>0\}$ with the boundary. Let $s(x,n) = |x-x_\Omega(x,n)|$ and let us denote for $x,z\in\overline{\Omega}$ by $\int_{[x,z]}\alpha(\xi)\ ds(\xi)$ the integral along the line connecting x with z, i.e. $\int_{[x,x_\Omega(x,n)]}\alpha(\xi)\ ds(\xi) = \int_0^{s(x,n)}\alpha(x-tn)\ dt$. Solving the first equation in (C.151) by characteristics we obtain

$$I_\nu^\varepsilon(x,n) = g_\nu(n)e^{-\int_{[x,x_\Omega(x,n)]}\frac{\alpha(\xi)}{\varepsilon}\ ds(\xi)} + \int_0^{s(x,n)} e^{-\int_0^t\frac{\alpha(x-\tau n)}{\varepsilon}d\tau}\frac{\alpha(x-tn)}{\varepsilon}B_\nu\left(T\left(x-tn\right)\right)\ dt.$$

Therefore, analogously as in Section 4.1 using that the flux is divergence free together with the first equation in (C.151), equation (C.3), the characteristic solution of I^ε and changing from spherical coordinates to space coordinates we obtain the following integral equation for $u^\varepsilon(x) = 4\pi\sigma T_\varepsilon^4(x)$

$$u^\varepsilon(x) - \int_\Omega \frac{\alpha(\eta)e^{-\int_{[x,\eta]}\frac{\alpha(\xi)}{\varepsilon}ds(\xi)}}{4\pi\varepsilon|x-\eta|^2}u^\varepsilon(\eta)\ d\eta = \int_0^\infty d\nu\int_{\mathbb{S}^2} dn\ g_\nu(n)e^{-\int_{[x,x_\Omega(x,n)]}\frac{\alpha(\xi)}{\varepsilon}\ ds(\xi)}. \tag{C.154}$$

For $x,\eta\in\mathbb{R}^3$ we define the kernel K_ε by

$$K_\varepsilon(x;\eta) = \frac{\alpha(\eta)e^{-\int_{[x,\eta]}\frac{\alpha(\xi)}{\varepsilon}ds(\xi)}}{4\pi\varepsilon|x-\eta|^2}. \tag{C.155}$$

Notice that K_ε has been defined in the whole $\mathbb{R}^3$ extending α by $\overline{\alpha}$. We remark that $K_\varepsilon(x;\eta)$ is not symmetric, i.e. $K_\varepsilon(x;\eta)\neq K_\varepsilon(\eta;x)$. In the following we summarize some properties of the kernel K_ε.

Proposition C.6. *The kernel K_ε defined in* (C.155) *has integral equal* 1*. Moreover, it can be decomposed in the following three different ways:*

(i) Let $x,\eta\in\mathbb{R}^3$, then $K_\varepsilon(x;\eta) = K_\varepsilon^{\alpha(x)}(x-\eta) + R_\varepsilon^x(x;\eta)$, where

$$K_\varepsilon^{\alpha(x)}(x-\eta) = \frac{\alpha(x)e^{-\frac{\alpha(x)|x-\eta|}{\varepsilon}}}{4\pi\varepsilon|x-\eta|^2}.$$

Moreover, the remainder satisfies

$$|R_\varepsilon^x(x;\eta)| \leq C(c_0,c_1)\frac{c_0e^{-\frac{c_0|x-\eta|}{\varepsilon}}}{4\pi\varepsilon|x-\eta|^2}\begin{cases}\left(|x-\eta| + \frac{|x-\eta|^2}{\varepsilon}\right) & |x-\eta|<\sqrt{\varepsilon}\\ 1 & |x-\eta|\geq\sqrt{\varepsilon}.\end{cases}$$

(ii) Let $x,\eta\in\mathbb{R}^3$ and $p\in\partial\Omega$ with $|x-p|<\varepsilon^{\frac{1}{2}+2\delta}$ for $\delta>0$ very small. Then $K_\varepsilon(x;\eta)=K_\varepsilon^{\alpha(p)}(x-\eta)+R_\varepsilon^p(x;\eta)$, where the remainder satisfies

$$\begin{aligned}|R_\varepsilon^p(x;\eta)|\leq& C(c_0,c_1)\frac{c_0e^{-\frac{c_0|x-\eta|}{\varepsilon}}}{4\pi\varepsilon|x-\eta|^2}\\ &\times\begin{cases}\left(|x-p|+|x-\eta|+\frac{|x-\eta|^2}{\varepsilon}+\frac{|x-\eta|}{\varepsilon}|x-p|\right) & |x-\eta|<\varepsilon^{\frac{1}{2}+2\delta}\\ 1 & |x-\eta|\geq\varepsilon^{\frac{1}{2}+2\delta}.\end{cases}\end{aligned}$$

(iii) Let $x,\eta\in\mathbb{R}^3$, then $K_\varepsilon(x;\eta)=K_\varepsilon^{\alpha(x)}(x-\eta)+\mathcal{K}_\varepsilon^1(x-\eta)+\mathcal{K}_\varepsilon^2(x-\eta)+\tilde{R}_\varepsilon^x(x;\eta)$, where

$$\mathcal{K}_\varepsilon^1(x-\eta)=-\frac{1}{2}\frac{\alpha(x)e^{-\frac{\alpha(x)|x-\eta|}{\varepsilon}}}{4\pi\varepsilon|x-\eta|^2}\nabla_x\alpha(x)\cdot(\eta-x)\frac{|x-\eta|}{\varepsilon}$$

and

$$\mathcal{K}_\varepsilon^2(x-\eta)=\frac{e^{-\frac{\alpha(x)|x-\eta|}{\varepsilon}}}{4\pi\varepsilon|x-\eta|^2}\nabla_x\alpha(x)\cdot(\eta-x).$$

Moreover, the remainder satisfies

$$\left|\tilde{R}_\varepsilon^x(x;\eta)\right|\leq C(c_0,c_1)\begin{cases}\frac{c_0e^{-\frac{c_0|x-\eta|}{\varepsilon}}}{4\pi\varepsilon|x-\eta|^2}\left(\frac{|x-\eta|^3}{\varepsilon}+\frac{|x-\eta|^4}{\varepsilon^2}\right) & |x-\eta|<\sqrt{\varepsilon}\\ \frac{c_0e^{-\frac{c_0|x-\eta|}{2\varepsilon}}}{4\pi\varepsilon|x-\eta|^2}(1+\varepsilon+\varepsilon^2) & |x-\eta|\geq\sqrt{\varepsilon}.\end{cases}\tag{C.156}$$

Notice that (iii) is a refinement of (i).

Proof. We start proving that the integral of K_ε is 1. We compute changing to spherical coordinates as $\eta=x-rn$

$$\begin{aligned}\int_{\mathbb{R}^3}\frac{\alpha(\eta)e^{-\int_{[x,\eta]}\frac{\alpha(\xi)}{\varepsilon}ds(\xi)}}{4\pi\varepsilon|x-\eta|^2}\,d\eta&=\int_{\mathbb{S}^2}dn\int_0^\infty dr\,\frac{\alpha(x-rn)}{4\pi\varepsilon}e^{-\int_0^r\frac{\alpha(x-tn)}{\varepsilon}dt}\\ &=-\int_{\mathbb{S}^2}dn\int_0^\infty dr\,\frac{1}{4\pi}\frac{d}{dr}e^{-\int_0^r\frac{\alpha(x-tn)}{\varepsilon}dt}=\int_{\mathbb{S}^2}dn\,\frac{1}{4\pi}\left(1-e^{-\int_0^\infty\frac{\alpha(x-tn)}{\varepsilon}dt}\right)=1,\end{aligned}$$

since $\alpha\geq c_0>0$.

We now proceed with the decompositions of the kernel. We start with claim (i). To this end we consider first of all $|x-\eta|<\sqrt{\varepsilon}$. We can expand by Taylor and get

$$\alpha(\eta)=\alpha(x)+\mathcal{O}\left(|x-\eta|\right)\text{ and }\int_0^1\frac{\alpha(x-\tau(x-\eta))d\tau|x-\eta|}{\varepsilon}=\frac{\alpha(x)|x-\eta|}{\varepsilon}+\mathcal{O}\left(\frac{|x-\eta|^2}{\varepsilon}\right).\tag{C.157}$$

Since $|x-\eta|<\sqrt{\varepsilon}$ we can conclude

$$e^{-\int_{[x,\eta]}\frac{\alpha(\xi)}{\varepsilon}ds(\xi)}=e^{-\frac{\alpha(x)|x-\eta|}{\varepsilon}}\left(1+\mathcal{O}\left(\frac{|x-\eta|^2}{\varepsilon}\right)\right).\tag{C.158}$$

By assumptions $c_0\leq\alpha(x)\leq c_1$ and hence (C.157) and (C.158) imply claim (i) in the case $|x-\eta|<\sqrt{\varepsilon}$. In the case $|x-\eta|\geq\sqrt{\varepsilon}$ we use the rough estimate

$$\left|\alpha(\eta)e^{-\int_{[x,\eta]}\frac{\alpha(\xi)}{\varepsilon}ds(\xi)}-\alpha(x)e^{-\frac{\alpha(x)|x-\eta|}{\varepsilon}}\right|\leq 2c_1e^{-\frac{c_0|x-\eta|}{\varepsilon}}.\tag{C.159}$$

Concerning claim (ii) we argue similarly. Let $p \in \partial\Omega$ and $|x-p| < \varepsilon^{\frac{1}{2}+2\delta}$. Let us first of all consider $|x-\eta| < \varepsilon^{\frac{1}{2}+2\delta}$, then $|\eta - p| < \varepsilon^{\frac{1}{2}+\delta}$ for $\varepsilon > 0$ sufficiently small. We expand α again by Taylor around p. Hence, similarly as before

$$\begin{aligned}\alpha(\eta) = \alpha(p) + \mathcal{O}\left(|\eta - p|\right) \text{ and} \\ e^{-\int_{[x,\eta]} \frac{\alpha(\xi)}{\varepsilon} ds(\xi)} =& e^{-\frac{\alpha(p)|x-\eta|}{\varepsilon}} \left(1 + \mathcal{O}\left(\frac{|x-\eta|^2 + |x-p||x-\eta|}{\varepsilon}\right)\right).\end{aligned} \tag{C.160}$$

Using on one hand $|\eta - p| \leq |\eta - x| + |x-p|$ and on the other hand a rough estimate as in (C.159) we conclude also the proof of claim (ii).

It remains to show claim (iii). We assume again first of all $|x-\eta| < \sqrt{\varepsilon}$. We expand α using Taylor. Since $\alpha \in C_b^3\left(\mathbb{R}^3\right)$ all terms in the computation below are well-defined. Hence, we see

$$\alpha(\eta) = \alpha(x) + \nabla_x \alpha(x) \cdot (\eta - x) + \mathcal{O}\left(|x-\eta|^2\right) \tag{C.161}$$

and

$$\begin{aligned}\frac{1}{\varepsilon}\int_0^1 \alpha(x - \tau(x-\eta))d\tau|x-\eta| =& \frac{1}{\varepsilon}\int_0^1 \alpha(x) + \tau\nabla_x\alpha(x)\cdot(\eta - x) + \mathcal{O}\left(|x-\eta|^2\right) d\tau|x-\eta| \\ =& \frac{\alpha(x)|x-\eta|}{\varepsilon} + \frac{1}{2}\nabla_x\alpha(x)\cdot(\eta-x)\frac{|x-\eta|}{\varepsilon} + \mathcal{O}\left(\frac{|x-\eta|^3}{\varepsilon}\right).\end{aligned}$$

Thus, this implies for $|x-\eta| < \sqrt{\varepsilon}$ the estimate

$$e^{-\int_{[x,\eta]} \frac{\alpha(\xi)}{\varepsilon} ds(\xi)} = e^{-\frac{\alpha(x)|x-\eta|}{\varepsilon}} \left(1 - \frac{1}{2}\nabla_x\alpha(x)\cdot(\eta - x)\frac{|x-\eta|}{\varepsilon} + \mathcal{O}\left(\frac{|x-\eta|^3}{\varepsilon} + \frac{|x-\eta|^4}{\varepsilon^2}\right)\right). \tag{C.162}$$

Equations (C.161) and (C.162) imply the decomposition $K_\varepsilon(x;\eta) = K_\varepsilon^{\alpha(x)}(x-\eta) + \mathcal{K}_\varepsilon^1(x-\eta) + \mathcal{K}_\varepsilon^2(x-\eta) + \tilde{R}_\varepsilon^x(x;\eta)$ and the estimate on $\tilde{R}_\varepsilon^x$ when $|x-\eta| < \sqrt{\varepsilon}$, while a rough estimate similar to (C.159) and the well-known inequality $e^{-|x|}|x|^n \leq C_n e^{-\frac{|x|}{2}}$ imply the claim (iii) for $|x-\eta| \geq \sqrt{\varepsilon}$. □

This proposition is one of the key results which will allow us to generalize the results obtained in the previous section when the absorption coefficient depends smoothly enough on the space variable. There are two very important consequences. First of all, the Banach fixed point theorem guarantees a unique continuous bounded solution u^ε. Indeed, the continuity of the function $\frac{1}{|x|^2}$ and its integrability in Ω together with the uniform continuity in x and uniform boundedness of $e^{-\int_{[x,\eta]} \frac{\alpha(\xi)}{\varepsilon} ds(\xi)}$ imply that the operator $A_g^\varepsilon : C(\Omega) \to C(\Omega)$ is a selfmap, where we consider

$$A_g^\varepsilon(u)(x) = \int_\Omega d\eta K_\varepsilon(x;\eta)u(\eta) + \int_0^\infty d\nu \int_{\mathbb{S}^2} dn\ g_\nu(n) e^{-\int_{[x,x_\Omega(x,n)]} \frac{\alpha(\xi)}{\varepsilon} ds(\xi)}.$$

Moreover, since the integral on $\mathbb{R}^3$ of the kernel K_ε is 1, arguing as in Section 4.1 we conclude that A_g^ε is a contraction.

The second consequence is that we can prove for the integral operator $\mathcal{L}_\Omega^\varepsilon(u)(x) := u(x) - \int_\Omega d\eta\ K_\varepsilon(x;n)u(\eta)$ comparison properties identical to the one in Theorem C.5 (Maximum principle). Hence, u^ε in (C.154) is non-negative.

C.5.3 Rigorous proof of the convergence: uniform boundedness of u^ε

Next we generalize Section 4.2 for the case $\alpha(x) \in C^3(\overline{\Omega})$. We want to show that the sequence u^ε is uniform bounded in ε. As we have seen in Section 5.2 the maximum principle of Theorem C.5 holds. We want to apply it with a suitable supersolution. Indeed, using the notation $\varepsilon d_\varepsilon = d(x)$ as in Lemma C.8 we see $\int_{[x,x_\Omega(x,n)]} \frac{\alpha(\xi)}{\varepsilon} ds(\xi) \geq \frac{c_0|x-x_\Omega|}{\varepsilon} \geq c_0 d_\varepsilon(x)$ and thus $|\mathcal{L}_\Omega^\varepsilon(u^\varepsilon)(x)| \leq \|g\|_1 e^{-c_0 d_\varepsilon(x)}$.

The supersolution we will consider is similar to the one constructed in Theorem C.6. However, it is now not true anymore that there exists a constant $C > 0$ such that $\mathcal{L}_\Omega^\varepsilon(C - |x|^2) \geq 2\varepsilon^2$ for every $x \in \Omega$. This can be already been looking at the expected limit elliptic operator $L = -\operatorname{div}\left(\frac{1}{\alpha(x)}\nabla_x\right)$, for which the inequality $L\left(C - |x|^2\right) \geq 0$ holds for arbitrary functions α depending on x only for small values of $|x|$. We remark that the supersolution constructed in Theorem C.6 contains two parts, namely a multiple of $C - |x|^2$ and a function proportional to the function ψ constructed in Lemma C.8. The role of the function ψ is to control the value of the operator $\mathcal{L}_\Omega^\varepsilon$ near the boundary $\partial\Omega$. The contribution of ψ is relevant only in the region where $e^{-d_\varepsilon(x)} \geq \varepsilon^2$. We will see that the function ψ still gives a supersolution for the new operator $\mathcal{L}_\Omega^\varepsilon$ (up to an error of order ε^2). Therefore, in order to prove the analogous of Theorem C.6 we need to replace the function $C - |x|^2$ by exponential functions.

Theorem C.8. *There exist suitable constants $0 < \mu < 1$, $0 < \gamma(\mu) < \frac{1}{3}$, C_1, C_2, $C_3 > 0$, $\lambda > 0$ large enough and there exists some $\varepsilon_0 > 0$ such that the function*

$$\Phi^\varepsilon(x) = C_3\|g\|_1 \left[\left(e^{\lambda D} + C_1 - e^{\lambda x_1}\right) + C_2\left(\left(1 - \frac{\gamma}{1 + \left(\frac{c_0 d(x)}{\varepsilon}\right)^2}\right) \wedge \left(1 - \frac{\gamma}{1 + \left(\frac{c_0 \mu R}{\varepsilon}\right)^2}\right)\right)\right],$$

for $a \wedge b = \min(a,b)$, $R > 0$ the minimal radius of curvature, $D = \operatorname{diam}(\Omega)$ and $d(x) :=$ dist$(x, \partial\Omega)$, satisfies $\mathcal{L}_\Omega^\varepsilon(\Phi^\varepsilon)(x) \geq \|g\|_1 e^{-\frac{c_0 d(x)}{\varepsilon}}$ in Ω uniformly for all $\varepsilon < \varepsilon_0$. Moreover, the solutions u^ε of (C.154) *are uniformly bounded in ε.*

Proof. We will use the notation of Theorem C.6 and Lemma C.7 and C.8. Notice first of all that

$$\frac{c_0}{c_1} K_\varepsilon^{c_1}(x - \eta) \leq K_\varepsilon(x;\eta) \leq \frac{c_1}{c_0} K_\varepsilon^{c_0}(x - \eta), \tag{C.163}$$

where we used the notation of Proposition C.6. Moreover, using the decomposition of claim (i) in Proposition C.6 we can always estimate for any $n \in \mathbb{N}$ the integral $\int_{\mathbb{R}^3} |R_\varepsilon^x(x;\eta)||(\eta - x)|^n \, d\eta$ by

$$\begin{aligned}
\int_{\mathbb{R}^3} |R_\varepsilon^x(x;\eta)||(\eta - x)|^n \, d\eta \leq & C(c_0, c_1) \int_{|x-\eta|<\sqrt{\varepsilon}} K_\varepsilon^{c_0}(x - \eta) \left(\frac{|\eta - x|^{2+n}}{\varepsilon} + |x - \eta|^{1+n}\right) |d\eta \\
& + C(c_0, c_1) \int_{|x-\eta|\geq\sqrt{\varepsilon}} K_\varepsilon^{c_0}(x - \eta)|(\eta - x)|^n \, d\eta \\
\leq & C(c_0, c_1)\varepsilon^{n+1} \int_0^\infty e^{-r} r^{n+1}(1 + r) \, dr + C(c_0, c_1)\varepsilon^n \int_{\frac{c_0}{\sqrt{\varepsilon}}}^\infty e^{-r} r^n \, dr \\
\leq & C(c_0, c_1, n)\varepsilon^{n+1} + C(c_0, c_1, n)\varepsilon^n e^{-\frac{c_0}{2\sqrt{\varepsilon}}} \leq C(c_0, c_1, n)\varepsilon^{n+1},
\end{aligned} \tag{C.164}$$

where we used $e^{-|x|}|x|^m \leq C_m$. Let now $x \in \Omega$. Let $x_0 \in \partial\Omega$ be a point such that $|x - x_0| = d(x)$. Then we estimate with the help of Proposition C.6

$$\begin{aligned}\mathcal{L}^{\varepsilon}_{\Omega}(1) = \int_{\Omega^c} d\eta\, K_\varepsilon(x;\eta) \geq \int_{\Pi_{x_0}} d\eta\, K_\varepsilon(x;\eta) \geq \frac{c_0}{c_1}\int_{\mathbb{R}_-\times\mathbb{R}^2} K^{c_1}_\varepsilon(d(x)e_1 - \eta)\\ =\frac{c_0}{c_1}\int_{-\infty}^{-c_1 d_\varepsilon(x)} K(y)\, dy \geq \begin{cases} \frac{c_0}{c_1}\nu_{c_1 M_0} & d_\varepsilon(x) \leq M_0,\\ 0 & \forall x \in \Omega.\end{cases}\end{aligned}$$

This estimate will play a crucial role the proof. We proceed considering the function $\varphi(x) = e^{\lambda D} - e^{\lambda x_1}$. It is not difficult to see that for $x \in \Omega$ we get $\varphi(x) \geq 0$. Moreover, expanding by Taylor $e^{\lambda\eta_1} = e^{\lambda x_1} + \lambda e^{\lambda x_1}(\eta_1 - x_1) + \frac{\lambda^2}{2}e^{\lambda x_1}(\eta_1 - x_1)^2 + E_3(x;\eta)$ with $|E_3(x;\eta)| \leq \frac{\lambda^3}{6}e^{\lambda D}|x_1 - \eta_1|^3$ we compute

$$\begin{aligned}\mathcal{L}^{\varepsilon}_{\Omega}\left(e^{\lambda x_1}\right) &= e^{\lambda x_1} - \int_\Omega K_\varepsilon(x;\eta)e^{\lambda\eta_1}\, d\eta\\ &=e^{\lambda x_1}\int_{\Omega^c} K_\varepsilon(x;\eta)\, d\eta - \lambda e^{\lambda x_1}\int_\Omega K_\varepsilon(x;\eta)(\eta_1 - x_1)\, d\eta\\ &\quad - \frac{\lambda^2}{2}e^{\lambda x_1}\int_\Omega K_\varepsilon(x;\eta)(\eta_1 - x_1)^2\, d\eta - \int_\Omega K_\varepsilon(x;\eta)E_3(x;\eta)\, d\eta.\end{aligned} \tag{C.165}$$

The first term on the right hand side can be controlled by $e^{\lambda D}$. For the second term we use the decomposition of claim (i) in Proposition C.6, the symmetry of the operator $K^{\alpha(x)}_\varepsilon$ and the estimate (C.164). Moreover, we remind that $\Omega^c \subset B^c_{d(x)}(x)$. Hence,

$$\begin{aligned}-\lambda e^{\lambda x_1}&\int_\Omega K_\varepsilon(x;\eta)(\eta_1 - x_1)\, d\eta =\\ &\lambda e^{\lambda x_1}\int_{\Omega^c} K^{\alpha(x)}_\varepsilon(x-\eta)(\eta_1 - x_1)\, d\eta\lambda e^{\lambda x_1}\int_\Omega R_\varepsilon(x;\eta)(\eta_1 - x_1)\, d\eta\\ &\leq\lambda\frac{c_1}{c_0^2}\varepsilon e^{-\frac{c_0 d_\varepsilon(x)}{2}}e^{\lambda x_1} + C(c_0,c_1)\lambda e^{\lambda x_1}\varepsilon^2.\end{aligned} \tag{C.166}$$

For the third term in equation (C.165) we can proceed as follows using the decomposition of claim (i) of Proposition C.6 and (C.164)

$$\begin{aligned}\frac{\lambda^2}{2}e^{\lambda x_1}\int_\Omega K_\varepsilon(x;\eta)(\eta_1 - x_1)^2\, d\eta &= \frac{\lambda^2}{2}e^{\lambda x_1}\int_{\mathbb{R}^3} K^{\alpha(x)}_\varepsilon(x-\eta)(\eta_1 - x_1)^2\, d\eta\\ &\quad - \frac{\lambda^2}{2}e^{\lambda x_1}\int_{\Omega^c} K^{\alpha(x)}_\varepsilon(x-\eta)(\eta_1 - x_1)^2\, d\eta + \frac{\lambda^2}{2}e^{\lambda x_1}\int_\Omega R_\varepsilon(x;\eta)(\eta_1 - x_1)^2\, d\eta\\ &\geq\frac{2\varepsilon^2}{3c_1^2}\lambda^2 e^{\lambda x_1} - \frac{\varepsilon^2}{c_0^2}\lambda^2 e^{\lambda x_1}e^{-\frac{c_0 d_\varepsilon(x)}{2}} - c(c_0,c_1)\lambda^2 e^{\lambda x_1}\varepsilon^3.\end{aligned} \tag{C.167}$$

Finally, using the estimate (C.163) for K_ε we compute for the term containing the error $E_3(x;\eta)$

$$\int_\Omega K_\varepsilon(x;\eta)E_3(x;\eta)\, d\eta \leq \lambda^3 e^{\lambda D}C(c_0,c_1)\varepsilon^3\int_{\mathbb{R}^3} e^{-r}r^3 \leq C(c_0,c_1)\lambda^3 e^{\lambda D}C(c_0,c_1)\varepsilon^3. \tag{C.168}$$

Hence, (C.166), (C.167) and (C.168) imply for $\lambda > 0$ large enough and $0 < \varepsilon < 1$ sufficiently small

$$\begin{aligned}\mathcal{L}_\Omega^\varepsilon\left(e^{\lambda D}-e^{\lambda x_1}\right) \geq & \frac{2\varepsilon^2}{3c_1^2}\lambda^2 e^{\lambda x_1} - C(c_0,c_1)\lambda e^{\lambda x_1}\left(\varepsilon e^{-\frac{c_0 d_\varepsilon(x)}{2}}+\varepsilon^2+\varepsilon^2\lambda e^{-\frac{c_0 d_\varepsilon(x)}{2}}\right)\\ & - c(c_0,c_1,\lambda,D)\varepsilon^3\\ \geq & \begin{cases} -A_1\varepsilon e^{-\frac{c_0 d_\varepsilon(x)}{2}} & \text{if } d(x)<\frac{2\varepsilon}{c_0}\ln\left(\frac{1}{\varepsilon}\right),\\ A_2\varepsilon^2 & \text{if } d(x)\geq\frac{2\varepsilon}{c_0}\ln\left(\frac{1}{\varepsilon}\right),\end{cases}\end{aligned}$$

where $A_1(c_0,c_1,\lambda,D)>0$ and $A_2(c_0,c_1,\lambda,D)>0$ are constants independent of ε.

We proceed estimating the operator acting on $\psi(x)=\left(1-\frac{\gamma}{1+\left(\frac{c_0 d(x)}{\varepsilon}\right)^2}\right)\wedge\left(1-\frac{\gamma}{1+\left(\frac{c_0\mu R}{\varepsilon}\right)^2}\right)$. Arguing similarly as in Lemma C.8 we can show that for $\mu>0$ and $0<\gamma<\frac{1}{3}$ small enough and $\varepsilon<\varepsilon_1<\frac{R\mu^3}{2}$ sufficiently small there exists constants $A_3(R,\Omega,\mu,\gamma)>0$ and $A_4(R,\mu,\gamma)>0$ such that

$$\mathcal{L}_\Omega^\varepsilon(\psi)(x)\geq\begin{cases}A_3\frac{1}{(1+(c_0 d_\varepsilon(x))^2)^2} & \text{if } 0<d(x)\leq\frac{R\mu}{2},\\ -A_4\varepsilon^2 & \text{if } \frac{R\mu}{2}<d(x)<R\mu,\\ 0 & \text{if } d(x)\geq R\mu.\end{cases}$$

The proof works following the same steps of Lemma C.8. Indeed, for the regions $\{d(x)\geq R\mu\}$ and $\{0<d(x)<M\varepsilon\}$, for $M=\frac{1}{\mu^2}$, the proof does not change. We use the estimate (C.163) for K_ε and the fact that its integral in $\mathbb{R}^3$ is 1. The constant γ must be chosen sufficiently small so that $\gamma<\frac{\nu_{c_1 M}c_0}{2c_1}$. For the regions $\{M\varepsilon\leq d(x)\leq\frac{R\mu}{2}\}$ and $\{\frac{R\mu}{2}<d(x)<R\mu\}$ we use again the Taylor expansion, the estimates on K_ε, the fact that $\varepsilon<\frac{R\mu^2}{2}$ and $M=\frac{1}{\mu^2}$. Beside the fact that K_ε has integral 1 in $\mathbb{R}^3$ we use its decomposition $K_\varepsilon=K_\varepsilon^{\alpha(x)}+R_\varepsilon^x$ according to claim (i) in Proposition C.6 and the estimate (C.164) for the remainder. In every computation where we used the symmetry of the kernel, e.g. for the first term of the Taylor expansion, we decompose K_ε and apply the symmetry argument for $K_\varepsilon^{\alpha(x)}$ and estimate the remainder. We omit the details of the proof, since the computations are similar to those in Lemma C.8.

We now finish the proof of Theorem C.8. Let $\varepsilon_0\leq\varepsilon_1$ such that $\frac{R\mu}{2}>\frac{2\varepsilon}{c_0}\ln\left(\frac{1}{\varepsilon}\right)$ for all $0<\varepsilon<\varepsilon_0$. Let $C_2=\frac{A_2}{2A_4}$. Moreover, since $(1+x^2)^2 e^{-\frac{x}{2}}\to 0$ as $x\to\infty$ there exists some $M_0>0$ such that if $d_\varepsilon(x)\geq M_0$ then

$$-A_1\varepsilon e^{-\frac{c_0 d_\varepsilon(x)}{2}}+\frac{C_2 A_3}{\left(1+(c_0 d_\varepsilon(x))^2\right)^2}\geq\frac{C_2 A_3}{2\left(1+(c_0 d_\varepsilon(x))^2\right)^2}\geq\frac{C_2 A_3}{12}e^{-c_0 d_\varepsilon(x)}.$$

Let $C_1>0$ satisfy $C_1>\frac{A_1 c_1}{c_0\nu_{c_1 M_0}}$. Then for all $d_\varepsilon(x)<M_0$ we obtain

$$\mathcal{L}_\Omega^\varepsilon(C_1)\geq C_1\frac{c_0}{c_1}\nu_{c_1 M_0}>A_1\varepsilon e^{-\frac{c_0 d_\varepsilon(x)}{2}}.$$

Finally, taking $C_3^{-1}=\min\{\frac{A_2}{2},\frac{C_2 A_3}{12}\}$ we get the desired lower estimate

$$\mathcal{L}_\Omega^\varepsilon\left(\Phi^\varepsilon\right)(x)\geq\|g\|_1 e^{-\frac{c_0 d(x)}{\varepsilon}}.$$

We conclude the proof of this Theorem applying the maximum principle of Theorem C.5 to the continuous function $\Phi^\varepsilon-u^\varepsilon$.

□

Remark. We notice for further reference that we have obtained a stronger estimate than $\mathcal{L}_\Omega^\varepsilon\left(\Phi^\varepsilon\right)(x) \geq \|g\|_1 e^{-\frac{c_0 d(x)}{\varepsilon}}$, namely

$$\mathcal{L}_\Omega^\varepsilon\left(\Phi^\varepsilon\right)(x) \geq \|g\|_1 \begin{cases} \frac{6}{(1+(c_0 d_\varepsilon(\varepsilon))^2)^2} & \text{if } d(x) < \frac{2\varepsilon}{c_0}\ln\left(\frac{1}{\varepsilon}\right) \\ \varepsilon^2 + \frac{6}{(1+(c_0 d_\varepsilon(\varepsilon))^2)^2} & \text{if } \frac{2\varepsilon}{c_0}\ln\left(\frac{1}{\varepsilon}\right) \leq d(x) < \frac{R\mu}{2} \\ \varepsilon^2 & \text{if } d(x) \geq \frac{R\mu}{2} \end{cases} \geq \|g\|_1 \varepsilon^2, \tag{C.169}$$

since $\frac{6}{(1+(c_0 d_\varepsilon(\varepsilon))^2)^2} \geq e^{-\frac{c_0 d(x)}{\varepsilon}} \geq \varepsilon^2$ if $d(x) < \frac{2\varepsilon}{c_0}\ln\left(\frac{1}{\varepsilon}\right)$. This estimate will be important at the end of the paper.

C.5.4 Rigorous proof of the convergence: estimates of $u^\varepsilon - \overline{u}$ near the boundary $\partial\Omega$

In this section we will extend the results of Section 4.3 also for the case of space dependent absorption coefficient. We will show a slightly different result than the one in Section 4.3. We will prove that $\overline{U}_\varepsilon(x,p)$ as defined in (C.153) is a good approximation of $u^\varepsilon(x)$ in a neighborhood of $p \in \partial\Omega$ of size close to $\varepsilon^{\frac{2}{3}}$ istead of $\varepsilon^{\frac{1}{2}}$. We will use once more the maximum principle of Theorem C.5. We start with the estimate of $\mathcal{L}_\Omega^\varepsilon\left(\overline{U}_\varepsilon(\cdot,p) - u^\varepsilon\right)$.

Lemma C.14. *Let $p \in \partial\Omega$ and let $\mathcal{R}_p$ be the rigid motion defined in* (C.13). *Then the following holds for $x \in \Omega$, $\delta > 0$ sufficiently small and independent of ε and a suitable $0 < A < \frac{1}{4}$ and constant $C > 0$*

$$\left|\mathcal{L}_\Omega^\varepsilon\left(\overline{u}\left(\alpha(p)\frac{\mathcal{R}_p(\cdot)\cdot e_1}{\varepsilon}, p\right) - u^\varepsilon\right)(x)\right| \leq C e^{-A c_0 d_\varepsilon(x)} \begin{cases} 1, & \forall x \in \Omega \\ \varepsilon^\delta & \text{if } |x-p| < \varepsilon^{\frac{1}{2}+2\delta}. \end{cases} \tag{C.170}$$

Proof. We start with the rough estimate for $x \in \Omega$. The operator can be estimated by

$$\left|\mathcal{L}_\Omega^\varepsilon\left(\overline{U}_\varepsilon(x,p) - u^\varepsilon(x)\right)\right| \leq \left|\mathcal{L}_\Omega^\varepsilon\left(\overline{U}_\varepsilon(x,p)\right)\right| + \left|\mathcal{L}_\Omega^\varepsilon\left(u^\varepsilon(x)\right)\right|.$$

As we have seen in Section 5.3 it is always true that $\left|\mathcal{L}_\Omega^\varepsilon\left(u^\varepsilon(x)\right)\right| \leq \|g\|_1 e^{-c_0 d_\varepsilon(x)}$. Hence, we have to consider $\overline{U}_\varepsilon(x,p) = \overline{u}_\infty(p) + \overline{V}_\varepsilon(x,p)$ with $\overline{V}_\varepsilon(x,p) = \overline{U}_\varepsilon(x,p) - \overline{u}_\infty(p)$ and thus by Lemma C.4 $\left|\overline{V}_\varepsilon(x,p)\right| \leq C e^{-\frac{c_0|\mathcal{R}_p(x)\cdot e_1|}{2\varepsilon}}$.

By the geometry of the problem we can estimate $|\mathcal{R}_p(x)\cdot e_1| \geq d(x)$. Indeed, let $x_p \in \partial\Omega$ be the unique intersection point of the line $\{x + tN_p : t > 0\}$ with the boundary $\partial\Omega$, i.e. $x_p = x + t_p N_p$. Then, since Ω is convex,

$$|\mathcal{R}_p(x)\cdot e_1| = (x-p)\cdot(-N_p) \geq (x - x_p)\cdot(-N_p) = t_p = |x - x_p| \geq d(x). \tag{C.171}$$

Hence, using also that $|x-\eta| \geq |d(\eta) - d(x)|$ we compute

$$\begin{aligned}
\left|\mathcal{L}_\Omega^\varepsilon\;\left(\overline{U}_\varepsilon(x,p)\right)\right| \leq\, & \overline{u}_\infty(p)\int_{\Omega^c} K_\varepsilon(x;\eta)d\eta + C e^{-\frac{c_0}{2}d_\varepsilon(x)} + C\int_\Omega K^\varepsilon(x;\eta) e^{-\frac{c_0}{2}d_\varepsilon(\eta)}\,d\eta \\
\leq & C(c_0,c_1)\|\overline{u}_\infty\|_\infty e^{-\frac{c_0}{2}d_\varepsilon(x)}\int_0^\infty e^{-\frac{r}{2}}\,dr + C e^{-\frac{c_0}{2}d_\varepsilon(x)} \\
& + C(c_0,c_1)\int_{d(\eta)<d(x)} \frac{e^{-\frac{c_0|x-\eta|}{2\varepsilon}}}{4\pi\varepsilon|x-\eta|^2} e^{-\frac{c_0}{2}d_\varepsilon(x)}\,d\eta + C e^{-\frac{c_0}{2}d_\varepsilon(x)}\int_{d(\eta)\geq d(x)} K_\varepsilon(x;\eta)\,d\eta \\
\leq & C(c_0,c_1,\|\overline{u}_\infty\|_\infty) e^{-\frac{c_0}{2}d_\varepsilon(x)}.
\end{aligned}$$

We now prove the estimate when $|x-p| < \varepsilon^{\frac{1}{2}+2\delta}$. In this case we use the decomposition $K_\varepsilon = K_\varepsilon^{\alpha(p)} + R_\varepsilon^p$ in claim (ii) of Proposition C.6. Similarly as in (C.164) we can estimate

$$\int_{\mathbb{R}^3} |R_\varepsilon^p(x;\eta)||x-\eta|^n \, d\eta \le C(c_0,c_1)\varepsilon^n \varepsilon^{\frac{1}{2}+2\delta}. \tag{C.172}$$

Hence, using the decomposition above we compute for $|x-p| < \varepsilon^{\frac{1}{2}+2\delta}$

$$\begin{aligned}
&\left|\mathcal{L}_\Omega^\varepsilon \left(\overline{U}_\varepsilon(x,p) - u^\varepsilon(x)\right)\right| \\
\le & \left| \int_0^\infty d\nu \int_{\mathbb{S}^2} dn\, g_\nu(n) e^{-\int_{[x,x_\Omega(x,n)]} \frac{\alpha(\xi)}{\varepsilon}\, ds(\xi)} - \int_0^\infty d\nu \int_{n\cdot N_p<0} dn\, g_\nu(n) e^{-\frac{\alpha(p)\left|x - x_{\Pi_p}(x,n)\right|}{\varepsilon}} \right| \\
&+ \int_{\Pi_p\setminus\Omega} d\eta\, K_\varepsilon^{\alpha(p)}(x-p)\overline{U}_\varepsilon(\eta,p) + \int_\Omega d\eta\, |R_\varepsilon^p(x;\eta)|\overline{U}_\varepsilon(\eta) \\
=& I_1 + I_2 + I_3.
\end{aligned}$$

We estimate now these three terms. Analogous to Lemma C.9 we decompose $\mathbb{S}^2 = \mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3$. For the term I_1 we only have to notice that if $n \in \mathcal{U}_1$ then $|x_\Omega - x| \le \varepsilon^{\frac{1}{2}+\delta}(C(\Omega)\varepsilon^{\frac{1}{2}} + 1 + \varepsilon^\delta) \le 2\varepsilon^{\frac{1}{2}+\delta}$ for $\varepsilon > 0$ sufficiently small. Hence, $\frac{|\xi-p||x-x_\Omega|}{\varepsilon} < 4\varepsilon^{2\delta} < 1$ for any $\xi = x - t(x - x_\Omega)$, $0 \le t \le 1$ and $\varepsilon > 0$ sufficiently small. Thus, we obtain using the Taylor expansion on $\alpha(\xi)$ as we did in (C.160)

$$\begin{aligned}
I_1\,|_{\mathcal{U}_1} \le & \int_0^\infty d\nu \int_{\mathcal{U}_1} dn\, g_\nu(n) \left| e^{-\frac{\alpha(p)|x-x_\Omega(x,n)|}{\varepsilon}} - e^{-\frac{\alpha(p)\left|x-x_{\Pi_p}(x,n)\right|}{\varepsilon}} \right| \\
&+ \int_0^\infty d\nu \int_{\mathcal{U}_1} dn\, g_\nu(n) \left| e^{-\int_{[x,x_\Omega(x,n)]} \frac{\alpha(\xi)}{\varepsilon}\, ds(\xi)} - e^{-\frac{\alpha(p)|x-x_\Omega(x,n)|}{\varepsilon}} \right| \\
\le & C(\Omega)\|g\|_\infty \varepsilon^\delta e^{-c_0 d_\varepsilon(x)} + C\|g\|_\infty \int_{\mathcal{U}_1} dn\, e^{-\frac{\alpha(p)|x-x_\Omega(x,n)|}{\varepsilon}} \varepsilon^{2\delta} \le C(\Omega)\|g\|_\infty \varepsilon^\delta e^{-c_0 d_\varepsilon(x)},
\end{aligned}$$

where to estimate the first term in the first inequality we used Lemma C.9, specifically equation (C.106), and to estimate the second term we expanded $\alpha(\xi)$ at $\xi = p$. To estimate the contributions in the regions $\mathcal{U}_2$ and $\mathcal{U}_3$, the estimate $\int_{[x,x_\Omega(x,n)]} \frac{\alpha(\xi)}{\varepsilon}\, ds(\xi) \ge -c_0 \frac{|x-x_\Omega(x,n)|}{\varepsilon}$ together with the result of Lemma C.9 implies the bound on I_1. The term I_2 can be handled exactly as in Lemma C.9. Finally, for I_3 we use the uniform boundedness of $\overline{u}(y,p)$ and equation (C.172). This implies

$$\left|\mathcal{L}_\Omega^\varepsilon \left(\overline{U}_\varepsilon(x,p) - u^\varepsilon(x)\right)\right| \le C(\Omega,c_0,c_1,g)\varepsilon^\delta e^{-c_0 d_\varepsilon(x)} + C\varepsilon^{\frac{1}{2}+2\delta}.$$

We conclude by interpolation. Indeed, if $d(x) \le \frac{4\varepsilon}{c_0} \ln\left(\varepsilon^{-\frac{1}{2}-\delta}\right)$, then $\varepsilon^{\frac{1}{2}+2\delta} \le \varepsilon^\delta e^{-\frac{c_0 d_\varepsilon(x)}{4}}$, while if $d(x) > \frac{4\varepsilon}{c_0} \ln\left(\varepsilon^{-\frac{1}{2}-\delta}\right)$ we use the global estimate to get $e^{-\frac{c_0 d_\varepsilon(x)}{2}} \le \varepsilon^{\frac{1}{2}+\delta} e^{-\frac{c_0 d_\varepsilon(x)}{4}}$. □

It only remains to adapt the supersolution $W_{\varepsilon,L}$ of Proposition C.5. As we anticipated at the beginning of Section 5.4 we are going to prove that $\overline{u}$ is a good approximation of u^ε in a neighborhood of $p \in \partial\Omega$ of size close to $\varepsilon^{\frac{2}{3}}$. First of all we notice that if $\delta < \frac{1}{12}$ then $\frac{2}{3} > \frac{1}{2} + 2\delta$ and hence we have also that

$$\left|\mathcal{L}_\Omega^\varepsilon \left(\overline{u}\left(\frac{\mathcal{R}_p(\cdot)\cdot e_1}{\varepsilon}, p\right) - u^\varepsilon\right)(x)\right| \le Ce^{-Ac_0 d_\varepsilon(x)} \begin{cases} 1 & \text{if } |x-p| \ge \varepsilon^{\frac{2}{3}} \\ \varepsilon^\delta & \text{if } |x-p| < \varepsilon^{\frac{2}{3}}. \end{cases}$$

The result we prove is the following

Proposition C.7. *Let $p \in \partial\Omega$, $0 < A < \frac{1}{4}$ the constant of Lemma C.14. Let $L > 0$ large enough and $0 < \varepsilon < 1$ sufficiently small. Let $0 < \delta < \frac{1}{12}$. Then there exists a non negative continuous function $W_{\varepsilon,L} : \Omega \to \mathbb{R}_+$ such that*

$$\begin{cases} W_{\varepsilon,L} \geq C > 0 & \text{for } |\mathcal{R}_p(x) \cdot e_i| \geq \varepsilon^{\frac{2}{3}+\delta}; \\ \mathcal{L}^{\varepsilon}_{\Omega}\left(W_{\varepsilon,L}\right)(x) \geq C\varepsilon^{\delta} e^{-\frac{Ad(x)}{\varepsilon}} & \text{for } |\mathcal{R}_p(x) \cdot e_i| < \varepsilon^{\frac{2}{3}+\delta}; \\ 0 \leq W_{\varepsilon,L} \leq C\left(\varepsilon^{\alpha} + \frac{1}{\sqrt{L}}\right) & \text{for } |\mathcal{R}_p(x) \cdot e_i| < \varepsilon^{\frac{2}{3}+2\delta}, \end{cases}$$

for some constant $C > 0$ and $\alpha > 0$.

This proposition implies arguing as in Section 4.3, the following corollary (see Corollary C.4).

Corollary C.6. *There exists a constant $C > 0$, a large $L > 0$ and an $\alpha > 0$ independent of x, p, ε such that*

$$\left| \overline{u}\left(\frac{R_p(\cdot) \cdot e_1}{\varepsilon}, p\right) - u^{\varepsilon} \right|(x) \leq C\left(\varepsilon^{\alpha} + \frac{1}{\sqrt{L}}\right)$$

for all $|x - p| < \varepsilon^{\frac{2}{3}+2\delta}$.

In order to adapt the supersolution $W_{\varepsilon,L}$ of Proposition C.5 in this case we start considering a new slightly different geometrical setting. Once more we denote for simplicity $x_i = \mathcal{R}_p(x) \cdot e_i$. We define now for $i = 2, 3$ the radii $\rho_i^{\pm}(x) = \sqrt{\left(x_1 + \frac{L}{2}\varepsilon\right)^2 + \left(x_i \pm \varepsilon^{\frac{2}{3}+\delta}\right)^2}$ and the angles $\theta_i^{\pm}(x)$ given by $\cos\left(\theta_i^{\pm}\right) = \frac{1}{\rho_i^{\pm}(x)}\left(x_1 + \frac{L}{2}\varepsilon\right)$. We construct then the function $W_{\varepsilon,L}$ using now these definitions for $\rho_i^{\pm}(x)$ and $\theta_i^{\pm}(x)$ analogously to Section 4.3

$$W_{\varepsilon,L}(x) = \sum_{i=2}^{3}\left(W_i^+(x) + W_i^-(x)\right) + \frac{\tilde{C}}{\sqrt{L}}\phi_{\frac{1}{8},\varepsilon} + C\varepsilon^{\delta}\phi_{A,\varepsilon}, \tag{C.173}$$

where $\phi_{A,\varepsilon} = \Phi^{\frac{\varepsilon}{A}}$ the supersolution defined in Theorem C.8, $C, \tilde{C} > 0$ some suitable constants and $W_i^{\pm} = F_i^{\pm}(x) + G_i^{\pm}(x) + H_i^{\pm}(x)$ given by the auxiliary functions defined in (C.111),(C.112) and (C.113) adapted to the new geometrical setting. Analogously to Section 4.3 we define the following subsets of Ω for $i = 2, 3$.

$$\mathcal{C}_i^+ := \left\{x \in \Omega : \ x_i \leq -\varepsilon^{\frac{2}{3}+\delta} \text{ or } |x_i| < \varepsilon^{\frac{2}{3}+\delta}, x_1 \geq \varepsilon^{\frac{2}{3}+\delta}\right\};$$

$$\mathcal{C}_i^- := \left\{x \in \Omega : \ x_i \geq \varepsilon^{\frac{2}{3}+\delta} \text{ or } |x_i| < \varepsilon^{\frac{2}{3}+\delta}, x_1 \geq \varepsilon^{\frac{2}{3}+\delta}\right\};$$

$$\mathcal{C}_\delta := \left\{x \in \Omega : \ x_1 < \varepsilon^{\frac{2}{3}+\delta} \text{ and } |x_i| < \varepsilon^{\frac{2}{3}+\delta} \text{ for } i = 2, 3\right\};$$

$$\mathcal{C}_{i,2\delta} := \left\{x \in \Omega : \ |x_i| < \varepsilon^{\frac{2}{3}+2\delta} \text{ and } x_1 < \varepsilon^{\frac{2}{3}+2\delta}\right\}.$$

Also Lemma C.10 can be extended. We can prove using the geometrical setting above and the defined functions and sets the following Lemma.

Lemma C.15. *Assume $p \in \Omega$, $0 < \varepsilon < 1$, L, δ as indicated in Proposition C.7. Let $x_i = \mathcal{R}_p(x) \cdot e_i$ for $i = 1, 2, 3$. Let $W_i^{\pm}$ as above. Then there exist a constant $\alpha > 0$ depending only* *on δ and a constant $C>0$ depending on Ω, g_ν, c_0 and c_1 but independent of ε and p and suitable $b>0$ and $L>0$ such that for $i=2,3$*

$$\begin{cases} W_i^\pm(x)\geq 0 & \text{in } \Omega, & \text{(C.174)}\\ W_i^\pm(x)\geq \frac{\pi}{2}-\arctan(2) & \text{in } \mathcal{C}_i^\pm, & \text{(C.175)}\\ W_i^\pm(x)\leq C\varepsilon^\alpha & \text{in } \mathcal{C}_{i,2\delta}, & \text{(C.176)}\\ \mathcal{L}_\Omega^\varepsilon\left(W_i^\pm\right)(x)\geq -\frac{C}{\sqrt{L}}e^{-\frac{c_0 d_\varepsilon(x)}{8}} & \text{in } \mathcal{C}_\delta. & \text{(C.177)}\end{cases}$$

Proof. As in the proof of Lemma C.10 it is enough to prove it for $W=W_2^-$. Again we consider $\rho=\rho_2^-$, $\theta=\theta_2^-$, $F=F_2^-$, $G=G_2^-$, $H=H_2^-$, $\mathcal{C}_{2\delta}=\mathcal{C}_{2,2\delta}$ and $\mathcal{C}=\mathcal{C}_2$. We only have to show claim (C.177), since all other claims work exactly in the same way. The only thing that is needed is that $\varepsilon^{\frac{2}{3}+2\delta}=\varepsilon^\delta\varepsilon^{\frac{2}{3}+\delta}=\varepsilon^{2\delta}\varepsilon^{\frac{2}{3}}$. In this case we have $\beta=\frac{1-12\delta}{3}>0$ and $\varepsilon<\left(\frac{1}{L}\right)^{\frac{1}{\beta}}$.

In order to prove (C.177) we follow the strategy of Lemma C.10. We expand hence by Taylor the function W putting together equations (C.126), (C.127) and (C.128). Exactly as in Section 4.3 we consider for $x\in\mathcal{C}_\delta$ the three cases $\rho(x)<L\varepsilon$, $\rho(x)\geq L\varepsilon$ with $d(x)<\varepsilon$ and $\rho(x)\geq L\varepsilon$ with $d(x)\geq\varepsilon$. For each of these situations the same estimates of the error term $E^3(\eta,x)$ as in (C.130) holds. We substitute for $W(\eta)$ in the formulation of $\mathcal{L}_\Omega^\varepsilon(W)(x)$ the Taylor expansion. For all terms containing the first, second and third derivatives of W we argued in Lemma C.10 by the symmetry of the kernel. In this case is not possible anymore. Hence, for that terms, we decompose the kernel $K_\varepsilon(x;\eta)=K_\varepsilon^{\alpha(x)}(x-\eta)+R_\varepsilon^x(x;\eta)$ according to claim (i) of Proposition C.6. For $K_\varepsilon^{\alpha(x)}$ we use the same arguments as in Section 4. For the terms with the remainder R_ε^x we estimate in a different way. We notice first of all that if $x\in\mathcal{C}_\delta$, then

$$\rho(x)\leq 2\varepsilon^{\frac{2}{3}+\delta}<\varepsilon^{\frac{2}{3}} \tag{C.178}$$

taking $\varepsilon>0$ sufficiently small. Since $\rho>\frac{L}{2}\varepsilon$ and $\cos(\theta(x))\rho(x)\geq\frac{L}{2}\varepsilon$ we recall also that for $n\geq 1$

$$\varepsilon^n\left|\nabla_x^n W(x)\right|\leq C_F\frac{\varepsilon^n}{\rho^n(x)}+bC_H\frac{\varepsilon^{n+2}}{\rho^{n+2}(x)}+aC_G\frac{\varepsilon}{L^{n-\frac{1}{2}}\varepsilon^n\rho(x)}.$$

This implies, using the estimates (C.164) and (C.178) and $\rho>\frac{L}{2}\varepsilon$ that

$$\begin{aligned}\left|\nabla_x^n W(x)\right|\int_{\mathbb{R}^3}d\eta\,\left|R_\varepsilon^x(x;\eta)\right||x-\eta|^n\leq&\frac{C(c_0,c_1)\varepsilon^2}{\rho(x)}\left(\frac{C_F\varepsilon^{n-1}}{\rho^{n-1}}+bC_H\frac{\varepsilon^{n+1}}{\rho^{n+1}(x)}+\frac{aC_G}{L^{n-\frac{1}{2}}}\right)\\ =&\frac{C(c_0,c_1)\varepsilon^4}{\rho^4(x)}\left(\frac{C_F\varepsilon^{n-1}}{\rho^{n-1}}+bC_H\frac{\varepsilon^{n+1}}{\rho^{n+1}(x)}+\frac{aC_G}{L^{n-\frac{1}{2}}}\right)\varepsilon\left(\frac{\rho(x)}{\varepsilon}\right)^3\\ \leq&\frac{C(c_0,c_1)\varepsilon^4}{\rho^4(x)}\left(\frac{C_F\varepsilon^{n-1}}{\rho^{n-1}}+bC_H\frac{\varepsilon^{n+1}}{\rho^{n+1}(x)}+\frac{aC_G}{L^{n-\frac{1}{2}}}\right)\\ \leq&\frac{C(c_0,c_1)\varepsilon^4}{\rho^4(x)}\left(C_F+\frac{bC_H}{L^2}+\frac{aC_G}{\sqrt{L}}\right).\end{aligned}$$

Hence, since $\frac{\varepsilon^4}{\rho^4(x)}=\left(\frac{\varepsilon}{\rho(x)}\right)^{\frac{5}{2}}\left(\frac{\varepsilon}{\rho(x)}\right)^{\frac{3}{2}}$ all arguments and estimates we had in the proof of Lemma C.10 for the first case when $\rho(x)<L\varepsilon$ can be obtained also for this function W and for $\alpha\in C^3(\Omega)$. For the other two cases, when $\rho(x)\geq L\varepsilon$ with $d(x)<\varepsilon$ or $d(x)\geq\varepsilon$ we need also to add to the assumption $a > b$ the assumption $a = 2b$ (or even $a = b + 1$). Then the term coming from the integral

$$-\left|\nabla_x^n G(x)\right| \int_{\mathbb{R}^3} d\eta \left|R_\varepsilon^x(x;\eta)\right| |x-\eta|^n \geq -\frac{C(c_0,c_1) 2b C_G \varepsilon^4}{\sqrt{L}\rho^4(x)}$$

can be always absorbed taking L large enough by the term coming from the Laplacian of H when integrating the second derivative term of the Taylor expansion with the kernel $K_\varepsilon^{\alpha(x)}$, i.e. it is absorbed by $\frac{2b}{3\alpha^2(x)}\frac{\varepsilon^4}{\rho^4(x)} \geq \frac{2b}{3c_1^2}\frac{\varepsilon^4}{\rho^4(x)}$. The remaining arguments and computation are similar to the one in the proof of Lemma C.10. We thus refer to the that proof which implies Lemma C.15. □

Arguing as in Section 4.3, Lemma C.15 implies now Proposition C.14 for the supersolution $W_{\varepsilon,L}$ as given in equation (C.173). In this case we have to use Theorem C.8 instead of Theorem C.6.

C.5.5 Rigorous proof of the convergence of u^ε to the solution of the new boundary value problem

We are now ready conclude the proof of the convergence of u^ε to the function v, solution of the boundary value problem

$$\begin{cases} -\operatorname{div}\left(\frac{1}{\alpha(x)}\nabla_x v(x)\right) = 0 & x \in \Omega, \\ v(p) = \overline{u}_\infty(p) & p \in \partial\Omega. \end{cases} \tag{C.179}$$

To this end, we generalize Section 4.4 for the case $\alpha \in C^3\left(\overline{\Omega}\right)$. Again, we decompose Ω in new regions. In this case though, their distance from the boundary will be of the order $\varepsilon^{\frac{2}{3}}$. Since the results in this last section are analogous to those we obtained in Section 4.4 we use the same notation. Hence, we define in this case $\hat{\Omega}_\varepsilon := \left\{x \in \Omega : d(x) > \varepsilon^{\frac{2}{3}+2\delta}\right\}$, $\Sigma_\varepsilon := \left\{x \in \Omega : \varepsilon^{\frac{2}{3}+4\delta} < d(x) \leq \varepsilon^{\frac{2}{3}+2\delta}\right\}$ and their union $\Omega_\varepsilon = \hat{\Omega}_\varepsilon \cup \Sigma_\varepsilon$. We also define for $0 < \sigma < 1$ sufficiently small independent of ε the set $\Omega^\sigma := \Omega \cup \{x \in \Omega^c : d(x) < \sigma\}$. Recall the continuous projection $\pi_{\partial\Omega}$ as given in (C.14) and the estimate $\left|\mathcal{R}_{\pi_{\partial\Omega}(x)}(x) \cdot e_1\right| \geq d(x)$ as we have seen in (C.171). Then as in Lemma C.11 we can prove the following result.

Lemma C.16. *Let $0 < \varepsilon < 1$ sufficiently small, C, α, L, $0 < \delta < \frac{1}{12}$ according to Corollary C.6. Then*

$$\sup_{x\in\Sigma_\varepsilon} \left|\overline{u}_\infty\left(\pi_{\partial\Omega}(x)\right) - u^\varepsilon(x)\right| \leq C\left(\varepsilon^\alpha + \frac{1}{\sqrt{L}}\right) + C(c_0,c_1)\varepsilon^{\frac{1}{3}-4\delta}.$$

Proof. We combine the estimate in (C.171) with Lemma C.4, we use the Corollary C.6, the fact that $0 < \delta < \frac{1}{12}$ and the following estimate

$$\begin{aligned} \sup_{x\in\Sigma_\varepsilon} &\left|\overline{u}_\infty\left(\pi_{\partial\Omega}(x)\right) - u^\varepsilon(x)\right| \\ &\leq \sup_{x\in\Sigma_\varepsilon}\left|\overline{u}_\infty\left(\pi_{\partial\Omega}(x)\right) - \overline{U}_\varepsilon\left(x,\pi_{\partial\Omega}(x)\right)\right| + \sup_{x\in\Sigma_\varepsilon}\left|\overline{U}_\varepsilon\left(x,\pi_{\partial\Omega}(x)\right) - u^\varepsilon(x)\right|. \end{aligned}$$

□

Similarly to Section 4.4 we consider the function v_σ, solution to the boundary value problem

$$\begin{cases} -\operatorname{div}\left(\frac{1}{\overline{\alpha}(x)}\nabla_x v_\sigma(x)\right) = 0 & x \in \Omega^\sigma, \\ v(x) = \overline{u}_\infty(\pi_{\partial\Omega}(x)) & x \in \partial\Omega^\sigma, \end{cases} \tag{C.180}$$

where we consider the smooth extension of α as defined at the beginning of Section 5.1. Since $\overline{\alpha}\in C^3(\mathbb{R}^3)$, $\pi_{\partial\Omega}$ is a continuous bijection and $\overline{u}_\infty$ is Lipschitz, the theory on elliptic regularity assures that v_σ uniquely exists and it is also three times continuously differentiable, i.e. $v_\sigma\in C^3\left(\Omega^\sigma\right)\cap C\left(\overline{\Omega^\sigma}\right)$. For the same reason also the function v defined in (C.179) belongs to $C^3(\Omega)\cap C(\overline{\Omega})$. We denote again by ω the modulus of continuity of v_σ and by ω_2 the one of v. Moreover, the elliptic equation satisfies the maximum principle, hence for all $x\in\Omega$ we can estimate

$$|v(x)-v_\sigma(x)|\leq \max_{x\in\partial\Omega^\sigma}|v_\sigma\left(\pi_{\partial\Omega}(x)\right)-v_\sigma(x)|\leq\omega(\sigma).$$

We can now prove a suitable new version of Lemma C.12.

Lemma C.17. *Let $x\in\hat{\Omega}_\varepsilon$, $0<\delta<\frac{1}{12}$ and $\beta=\frac{1-6\delta}{3}>0$. Then*

$$\left|\mathcal{L}_\Omega^\varepsilon\left(v_\sigma-u^\varepsilon\right)(x)\right|\leq C(\Omega,g_\nu,c_0,c_1)e^{-\frac{c_0 d(x)}{2\varepsilon}}\left(\varepsilon^\beta+C_\sigma\varepsilon\right)+C(\Omega,\sigma,c_0,c_1)\varepsilon^3,$$

for some constants $C(\Omega,g_\nu,c_0,c_1)>0$ and $C(\Omega,\sigma,c_0,c_1)$ and $0<\varepsilon<1$ sufficiently small.

Proof. Since $x\in\hat{\Omega}_\varepsilon$ then $d(x)>\varepsilon^{\frac{2}{3}+2\delta}$. Hence, as we have seen at the beginning of Section 5.3 we obtain for β as in the Lemma

$$\left|\mathcal{L}_\Omega^\varepsilon\left(u^\varepsilon\right)(x)\right|\leq\|g\|_1 e^{-\frac{c_0 d}{\varepsilon}}\leq C(g_\nu,c_0)e^{-\frac{c_0 d}{2\varepsilon}}\varepsilon^\beta.$$

We consider now the operator acting on v_σ. Since $\Omega\subsetneq\Omega^\sigma$ there exists a constant $c_\sigma>0$ depending on v_σ such that $\sup\limits_{0\leq n\leq 3}\sup\limits_{x\in\Omega}\|\nabla_x^n v_\sigma\|_\infty\leq c_\sigma$. We expand v_σ in Ω with Taylor as $v_\sigma(\eta)=v_\sigma(x)+\nabla_x v_\sigma(x)\cdot(\eta-x)+\frac{1}{2}(\eta-x)^\top\nabla_x^2 v_\sigma(x)(\eta-x)+E^3(\eta,x)$, where $\left|E^3(\eta,x)\right|\leq c_\sigma|x-\eta|^3$. We want to use the expansion of v_σ together with the fact that this function solves the elliptic equation as given in (C.180). In the case of constant coefficient, the estimate on the operator was the result of the symmetry of the kernel. Indeed, the integral in $\mathbb{R}^3$ of the term with the first derivative in the Taylor expansion was zero and for the term with the second derivative we obtained the Laplacian, which was in that case also zero. As we have already noticed, when the absorption coefficient α is space dependent the kernel K_ε is no longer symmetric and moreover v_σ solves a more general elliptic equation.

Our strategy now is to find a decomposition of the kernel K_ε in such a way that we can recover the elliptic equation (C.180) and with a remainder which gives errors of the order ε^3. This decomposition is given by claim (iii) in Proposition C.6. We recall $K_\varepsilon(x;\eta)=K_\varepsilon^{\alpha(x)}(x-\eta)+\mathcal{K}_\varepsilon^1(x-\eta)+\mathcal{K}_\varepsilon^2(x-\eta)+\tilde{R}_\varepsilon^x(x;\eta)$. Analogously as we have computed in (C.164) we see in this case

$$\int_{\mathbb{R}^3}\tilde{R}_\varepsilon^x(x;\eta)|x-\eta|^n\,d\eta\leq C(c_0,c_1)\varepsilon^{n+2}. \tag{C.181}$$

With this decomposition we recover the elliptic equation. Indeed, using equation (C.129) we compute

$$\int_{\mathbb{R}^3}d\eta\left(K_\varepsilon^{\alpha(x)}(x-\eta)+\mathcal{K}_\varepsilon^1(x-\eta)+\mathcal{K}_\varepsilon^2(x-\eta)\right)\frac{1}{2}(\eta-x)^\top\nabla_x^2 v_\sigma(x)(\eta-x)=\frac{\varepsilon^2}{3\alpha^2(x)}\Delta v_\sigma(x), \tag{C.182}$$

where we used that $\mathcal{K}_\varepsilon^1$ and $\mathcal{K}_\varepsilon^2$ are antisymmetric while $(\eta-x)^\top\nabla_x^2 v_\sigma(x)(\eta-x)$ is symmetric.

Before moving to the term containing $\nabla_x v_\sigma(x)\cdot(\eta-x)$ we notice that for any symmetric function $F(x-\eta)$ we can compute

$$\begin{aligned}\int_{\mathbb{R}^3} d\eta\ F(x-\eta)\left(\nabla_x\alpha(x)\cdot(\eta-x)\right)\left(\nabla_x v_\sigma(x)\cdot(\eta-x)\right)\\ =&\sum_{i\neq j}\partial_i\alpha(x)\partial_j v_\sigma(x)\int_{\mathbb{R}^3} d\eta\ F(x-\eta)(\eta-x)_i(\eta-x)_j\\ &+\sum_{i=1}^3\partial_i\alpha(x)\partial_i v_\sigma(x)\int_{\mathbb{R}^3} d\eta\ F(x-\eta)(\eta-x)_i^2\\ =&\frac{1}{3}\nabla_x\alpha(x)\cdot\nabla_x v_\sigma(x)\int_{\mathbb{R}^3} d\eta\ F(x-\eta)|\eta-x|^2,\end{aligned}\tag{C.183}$$

where we used the symmetry of F. Hence, since $K_\varepsilon^{\alpha(x)}$ is symmetric using the definition of $\mathcal{K}_\varepsilon^1$ and $\mathcal{K}_\varepsilon^2$ and equation (C.183) we conclude

$$\begin{aligned}&\int_{\mathbb{R}^3} d\eta\ \left(K_\varepsilon^{\alpha(x)}(x-\eta)+\mathcal{K}_\varepsilon^1(x-\eta)+\mathcal{K}_\varepsilon^2(x-\eta)\right)\left(\nabla_x v_\sigma(x)\cdot(\eta-x)\right)\\ &=\int_{\mathbb{R}^3} d\eta\ \frac{\alpha(x)e^{-\frac{\alpha(x)|x-\eta|}{\varepsilon}}}{4\pi\varepsilon|x-\eta|^2}\left(\frac{1}{\alpha(x)}-\frac{|x-\eta|}{2\varepsilon}\right)\left(\nabla_x\alpha(x)\cdot(\eta-x)\right)\left(\nabla_x v_\sigma(x)\cdot(\eta-x)\right)\\ &=\frac{1}{3}\nabla_x\alpha(x)\cdot\nabla_x v_\sigma(x)\int_{\mathbb{R}^3} d\eta\ \frac{\alpha(x)e^{-\frac{\alpha(x)|x-\eta|}{\varepsilon}}}{4\pi\varepsilon|x-\eta|^2}\left(\frac{1}{\alpha(x)}-\frac{|x-\eta|}{2\varepsilon}\right)|\eta-x|^2\\ &=\frac{\varepsilon^2}{3\alpha^3(x)}\nabla_x\alpha(x)\cdot\nabla_x v_\sigma(x)\int_{\mathbb{R}^3} d\eta\ \frac{e^{-|\eta|}}{4\pi|\eta|^2}\left(1-\frac{|\eta|}{2}\right)|\eta|^2=-\frac{\varepsilon^2}{3\alpha^3(x)}\nabla_x\alpha(x)\cdot\nabla_x v_\sigma(x).\end{aligned}\tag{C.184}$$

Hence, equations (C.182) and (C.184) imply

$$\begin{aligned}&\int_{\mathbb{R}^3} d\eta\ \left(K_\varepsilon^{\alpha(x)}(x-\eta)+\mathcal{K}_\varepsilon^1(x-\eta)+\mathcal{K}_\varepsilon^2(x-\eta)\right)\\ &\qquad\times\left(\nabla_x v_\sigma(x)\cdot(\eta-x)+\frac{1}{2}(\eta-x)^\top\nabla_x^2 v_\sigma(x)(\eta-x)\right)\\ &=\frac{\varepsilon^2}{3\alpha(x)}\left(\frac{\Delta v_\sigma(x)}{\alpha(x)}-\frac{\nabla_x\alpha(x)}{\alpha(^2x)}\cdot\nabla_x v_\sigma(x)\right)=\frac{\varepsilon^2}{3\alpha(x)}\operatorname{div}\left(\frac{1}{\alpha(x)}\nabla_x v_\sigma(x)\right)=0.\end{aligned}\tag{C.185}$$

We are ready now to conclude the estimate of the operator acting on v_σ. Using indeed that $|x-\eta|>d(x)$ for $\eta\in\Omega^c$ we compute

$$\begin{aligned}&|\mathcal{L}_\Omega^\varepsilon\ (v_\sigma)\,(x)|\\ &\leq\left|v_\sigma(x)-\int_\Omega K_\varepsilon\,(\eta-x)\left[v_\sigma(x)+\nabla_x v_\sigma(x)\cdot(\eta-x)+\frac{1}{2}(\eta-x)^\top\nabla_x^2 v_\sigma(x)(\eta-x)\right]\ d\eta\right|\\ &+\left|\int_\Omega K_\varepsilon\,(\eta-x)\,E^3(\eta,x)\ d\eta\right|\\ &\leq v_\sigma(x)\int_{B^c_{d(x)}(x)} K_\varepsilon\,(\eta-x)\ d\eta+C(c_\sigma)\int_{\mathbb{R}^3} d\eta\ \left|\tilde{R}_\varepsilon^x(x;\eta)\right|\left(|x-\eta|+|x-\eta|^2\right)\\ &+C(c_0,c_1,\sigma)(\varepsilon+\varepsilon^2+\varepsilon^3)\int_{B^c_{d_\varepsilon(x)}(0)}\frac{e^{-|y|}}{4\pi|y|^2}\left(|y|+|y|^2+|y|^3+|y|^4\right)\ dy\\ &+C(c_0,c_1,\sigma)\varepsilon^3\int_{\mathbb{R}^3}\frac{e^{-\frac{|y|}{\varepsilon}}}{4\pi\varepsilon^3}\frac{|y|}{\varepsilon}\ dy\leq C(c_0,c_1,\sigma,\Omega)e^{-\frac{c_0 d(x)}{2\varepsilon}}\left(\varepsilon^\beta+\varepsilon\right)+C(c_0,c_1,\sigma)\varepsilon^3,\end{aligned}$$

where we first used $d(x) > \varepsilon^{\frac{2}{3}+2\delta}$, then we decomposed the kernel according to claim (iii) in Proposition C.6, we applied the result in (C.185), the estimate for the remainder $\tilde{R}_\varepsilon^x$ as given in (C.181) and finally the estimate $K_\varepsilon \leq C(c_0, c_1) K_\varepsilon^{c_0}$. This ended the proof of Lemma C.17. □

Similarly as in Lemma C.142, Lemma C.16, the maximum principle for elliptic operators and the uniform continuity of v imply the following result

Lemma C.18. *Let $x \in \Sigma_\varepsilon$ and $\varepsilon > 0$ small enough. Then the following uniform bound holds*

$$|v_\sigma(x) - u^\varepsilon(x)| \leq \omega(\sigma) + \omega_2\left(\varepsilon^{\frac{2}{3}+2\delta}\right) + C\left(\varepsilon^\alpha + \frac{1}{\sqrt{L}}\right) + C(c_0, c_1)\varepsilon^{\frac{1}{3}-4\delta}.$$

We have now all elements for completing the proof of the convergence of u^ε to v.

Theorem C.9. *u^ε converges to v uniformly in every compact set.*

Proof. We argue exactly as in Theorem C.7 applying the maximum principle to the operator $\mathcal{L}^\varepsilon_{\Omega_\varepsilon}$. To this end we see first of all that

$$\int_{\Omega\setminus\Omega_\varepsilon} d\eta\, K_\varepsilon(x\,\eta)\, |v_\sigma(\eta) - u^\varepsilon(\eta)| \leq \frac{C(c_0, c_1, g)}{\varepsilon} \exp\left(-\frac{\varepsilon^{\frac{6\delta-1}{3}}}{2}\right) \int_{\Omega\setminus\Omega_\varepsilon} d\eta\, \frac{1}{|x-\eta|^2} \leq C\varepsilon^3,$$

for $C = C(c_0, c_1, \Omega, g > 0$ and where we used $|x - \eta| > \frac{\varepsilon^{\frac{2}{3}+2\delta}}{2}$ for $\eta \in \Omega \setminus \Omega_\varepsilon$ and ε sufficiently small and the well-known estimate $|x|^n e^{-|x|} \leq C_n$ with $n = 13$ and $\delta < \frac{1}{78}$. Hence, Lemma C.17 implies for $x \in \hat{\Omega}_\varepsilon$ and $\beta = \frac{1-6\delta}{3}$

$$\left|\mathcal{L}^\varepsilon_{\Omega_\varepsilon}(v_\sigma - u^\varepsilon)(x)\right| \leq C(\Omega, g_\nu, c_0, c_1) e^{-\frac{c_0 d(x)}{2\varepsilon}} \left(\varepsilon^\beta + C_\sigma \varepsilon\right) + C(\Omega, \sigma, c_0, c_1)\varepsilon^3.$$

Moreover, Lemma C.18 assures that

$$|v_\sigma(x) - u^\varepsilon(x)| \leq \omega(\sigma) + \omega_2\left(\varepsilon^{\frac{2}{3}+2\delta}\right) + C\left(\varepsilon^\alpha + \frac{1}{\sqrt{L}}\right) + C(c_0, c_1)\varepsilon^{\frac{1}{3}-4\delta}$$

for $x \in \Sigma_\varepsilon$.

As we have seen in (C.169) the supersolution $\Phi^\varepsilon(x)$ satisfies also $\mathcal{L}^\varepsilon_\Omega(\Phi^\varepsilon)(x) \geq C\varepsilon^2$. Hence, we refer now to the proof Theorem C.7, which works here in the same way just replacing the supersolution with the suitable Φ^ε defined in Theorem C.8. Since the arguments are the same we omit the details. □

We conclude with the corollary about the convergence as $\varepsilon \to 0$ of the solution $(I^\varepsilon_\nu(x, n), T_\varepsilon(x))$ of the problem (C.6) to $(B_\nu(T), T)$, where $4\pi\sigma T = v$ is a solution to (C.179). This corollary implies once again Theorem C.1. We will omit the proof since it relies on the same arguments used in the proof of Corollary C.5.

Corollary C.7. *Let $(I^\varepsilon_\nu(x, n), T_\varepsilon(x))$ be the solution of the problem* (C.6) *and let $v(x)$ be the solution of* (C.179). *Then T_ε converges to $T = \left(\frac{v}{4\pi\sigma}\right)^{1/4}$ uniformly in every compact set. Moreover, $I^\varepsilon_\nu(x, n)$ converges to $B_\nu(T(x))$ uniformly in every compact set $K \subset \Omega$ as a function with values in $L^\infty(\mathbb{S}^2, L^1(\mathbb{R}_+))$.*

Appendix D

Well-posedness for a two-phase Stefan problem with radiation

Abstract: In this paper we consider a free boundary problem for the melting of ice where we assume that the heat is transported by conduction in both the liquid and the solid part of the material and also by radiation in the solid. Specifically, we study a one-dimensional two-phase Stefan-like problem which contains a non-local integral operator in the equation describing the temperature distribution of the solid. We will prove the local well-posedness of this free boundary problem combining the Banach fixed-point theorem and classical parabolic theory. Moreover, constructing suitable stationary sub- and supersolutions we will develop a global well-posedness theory for a large class of initial data.

D.1 Introduction

In this paper we study a one-dimensional two-phase free boundary problem which considers the melting of ice due to conduction and radiation. Specifically, we study the situation in which $\mathbb{R}^3$ is filled by a material in its liquid and solid phase. The moving interface is the surface of contact between the liquid and the solid and it changes its position according to the melting of the solid or the solidification of the liquid. The temperature of the interface equals the melting temperature T_M of the material, while the temperature of the liquid phase is larger than T_M and the temperature of the solid phase is smaller than T_M. We also assume that the heat is transferred by conduction in both phases of the material, similarly to the classical Stefan problem. In addition, we assume that in the solid phase the heat is transferred also by radiation. This is equivalent to the assumption of a transparent liquid phase, where the material does not interact with the radiation, and of an opaque solid phase, where both absorption and emission processes take place.

To be more precise, we consider a model in which at the initial time $t = 0$ the liquid phase of the material fills the negative half-space $\{x \in \mathbb{R}^3 : x_1 < 0\}$ and the solid phase fills the positive half-space $\{x \in \mathbb{R}^3 : x_1 > 0\}$. Hence, the interface is at time $t = 0$ the plane $\{0\} \times \mathbb{R}^2$. The model that we study is a one-dimensional free boundary problem obtained under the further assumption that the temperature depends only on the variable x_1 and that both phases have the same constant density.

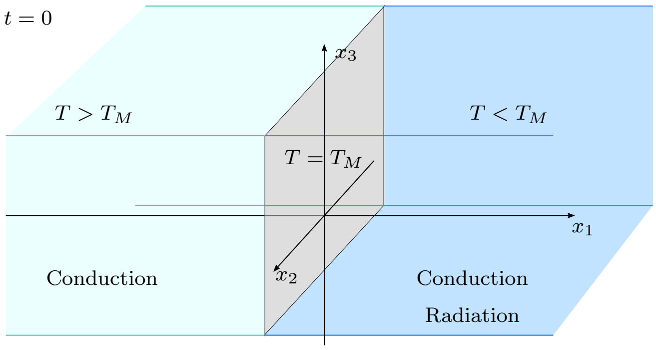

Figure D.1: Illustration of the considered model at the initial time $t = 0$.

In the case where the heat is transferred by conduction, the evolution equation for the temperature is given by the well-known heat equation

$$C\partial_t u = K\partial_x^2 u, \tag{D.1}$$

where $C > 0$ is the heat capacity of the material and $K > 0$ is the conductivity of the material.

When the heat is transferred also by radiation we have to include in the model, besides the terms describing heat conduction, the ones associated to the radiative transfer equation, i.e. the kinetic equation for the density of radiative energy. Let us consider first a body $\Omega \subset \mathbb{R}^3$ interacting with radiation. Defining by $I_\nu(t, x, n)$ the radiation intensity, i.e. the density of energy carried by photons with frequency $\nu > 0$, at position $x \in \Omega$, moving in direction $n \in \mathbb{S}^2$ at time $t > 0$, the radiative transfer equation is given by

$$\begin{aligned}\frac{1}{c}\partial_t I_\nu(t,x,n) + n\cdot\nabla_x I_\nu(t,x,n) &= \alpha_\nu^e - \alpha_\nu^a I_\nu(t,x,n)\\ &\quad + \alpha_\nu^s\left(\int_{\mathbb{S}^2}\mathbb{K}(n,n')I_\nu(t,x,n')\,dn' - I_\nu(t,x,n)\right),\end{aligned}$$

where $\mathbb{K}$ is the scattering kernel and α_ν^e, α_ν^a and α_ν^s are the emission parameter, the absorption and the scattering coefficient, respectively. They are used in order to describe the emission, absorption and scattering of photons, which are the processes involved in the interaction of radiation with the matter. In this paper we neglect the scattering process and we set $\alpha_\nu^s = 0$. Emission and absorption are the only processes through which the radiation changes the temperature of a material, although scattering could change the spatial distribution of radiation. Moreover, we consider local thermal equilibrium, i.e. we assume that at any time $t > 0$ and at any point x there is a well-defined temperature. Under this assumption, the emission parameter takes the specific form $\alpha_\nu^e = \alpha_\nu^a B_\nu(T(t,x))$, where $B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{\frac{h\nu}{kT}}-1}$ is the Planck distribution of a black body. Furthermore, we consider in this paper only the Grey approximation with constant absorption coefficient, i.e. we assume that the α_ν^a does not depend on the frequency ν nor on the space variable x.

Thus, defining $\alpha_\nu^a = \alpha$, the radiative transfer equation we will study takes the form

$$\frac{1}{c}\partial_t I_\nu(t,x,n) + n\cdot\nabla_x I_\nu(t,x,n) = \alpha\left(B_\nu(T(t,x)) - I_\nu(t,x,n)\right). \tag{D.2}$$

The evolution of the temperature due to the radiation process is given by the energy balance equation

$$C\partial_t T(t,x) + \frac{1}{c}\partial_t\left(\int_0^\infty d\nu\int_{\mathbb{S}^2} dn\; I_\nu(t,x,n)\right) + \operatorname{div}\left(\int_0^\infty d\nu\int_{\mathbb{S}^2} dn\; nI_\nu(t,x,n)\right) = 0 \tag{D.3}$$

coupled with the radiative transfer equation (D.2) and with suitable boundary conditions.

Turning back to the free boundary problem, we assume that the temperature depends only on x_1. Therefore, the interface is given by the plane $\{s(t)\} \times \mathbb{R}^2$ normal to the x_1-axis. In this paper we assume that there is no external source of radiation. Mathematically, we consider as boundary condition for the radiation at the interface

$$I_\nu\left(t,(s(t),x_2,x_3),n\right) = 0 \quad \text{if } n_1 > 0, \tag{D.4}$$

where $n_1 = n \cdot e_1$ for $n \in \mathbb{S}^2$. We emphasize that radiation can escape the solid, i.e. $I_\nu(t,s(t),n) \neq 0$ for $n_1 < 0$. Since the liquid is transparent, the escaped photons do not interact with the liquid and they do not change their direction. Specifically, the escaped radiation cannot return in the solid. This is the reason why in absence of external sources the incoming boundary condition at the interface is given by (D.4). Moreover, since we are studying a problem where the heat is transferred by both conduction and radiation we may assume that the radiation intensity solves the stationary radiative transfer equation and consequently that in (D.3) the time derivative of the total radiation energy is negligible. Indeed, since the photons travel with speed c, i.e. speed of light, the radiation intensity stabilizes in a much shorter time than the characteristic time required for significant changes of the temperature due to the transport of heat by both conduction and radiation. Finally, the evolution equation for the temperature in the case of heat transfer due to conduction and radiation takes into account the heat production rate due to both processes and it is a combination of the heat equation (D.36) and the energy balance equation (D.3) for the radiative transfer equation, i.e. in this case

$$C\partial_t T(t,x) - K\partial_x^2 T(t,x) + \operatorname{div}\left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dn\; n I_\nu(t,x,n)\right) = 0. \tag{D.5}$$

The interface moves according to the Stefan condition, i.e. the energy balance law at the interface, which is given by

$$\dot{s}(t) = \frac{1}{L}\left(K_S\partial_{x_1}T(t,s(t)^+) - K_L\partial_{x_1}T(t,s(t)^-)\right), \tag{D.6}$$

where K_S and K_L are the conductivity of the solid and the liquid, respectively, and L is the specific latent heat. Notice that the Stefan condition is the same as the one for the classical Stefan problem. This can be explained by the fact that the intensity of radiation I_ν in the liquid is given by the constant continuation of the radiation intensity at the interface. Indeed, the liquid is assumed to be transparent, i.e. the radiation is still present and it passes through the liquid region without interacting with it. In other words, in the liquid the temperature evolves also according to (D.5) for I_ν solving (D.2) with $\alpha = 0$. Note that in the case in which I_ν is constant, e.g. in the liquid, the divergence term disappears and (D.5) is equivalent to (D.1). Therefore, the Stefan condition, according to which the discontinuity of heat flux at the boundary is proportional to the speed of the motion of the interface, is given by (D.6) since the flux of radiating energy is continuous. Defining by C_S and C_L the heat capacity of the solid and the liquid and putting together equations (D.1), (D.2), (D.5), (D.4) and (D.6) we study the following free boundary problem

$$\begin{cases} C_L\partial_t T(t,x_1) = K_L\partial_{x_1}^2 T(t,x_1) & x_1 < s(t), \\ C_S\partial_t T(t,x_1) = K_S\partial_{x_1}^2 T(t,x_1) - \operatorname{div}\left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dnnI_\nu(t,x,n)\right) & x_1 > s(t), \\ n\cdot\nabla_x I_\nu(t,x,n) = \alpha\left(B_\nu(T(t,x_1)) - I_\nu(t,x,n)\right) & x_1 > s(t), \\ I_\nu(t,x,n) = 0 & x_1 = s(t),\ n_1 > 0, \\ T(t,s(t)) = T_M & x_1 = s(t), \\ T(0,x) = T_0(x) & x_1 \in \mathbb{R}, \\ \dot{s}(t) = \frac{1}{L}\left(K_S\partial_{x_1}T(t,s(t)^+) - K_L\partial_{x_1}T(t,s(t)^-)\right). \end{cases} \tag{D.7}$$

We emphasize that the main peculiarity of the model (D.7) is that only the solid is emitting radiation. This is due to the assumption of a perfectly transparent solid. Another interesting problem would be to consider in addition a non-trivial external source of radiation heating the solid from far away, i.e. to set as boundary conditions

$$(t,(s(t),x_2,x_3),n) = g_\nu(n) > 0 \quad \text{if } n_1 > 0.$$

In this case we expect to observe superheated solid, i.e. regions in the solid phase where the temperature is greater than the melting temperature.

Finally, we mention that in the upcoming paper [38] we continue the analysis of the problem presented in this article constructing traveling wave solutions for (D.7), which are the natural candidates to describe the long time asymptotics for the solution to (D.7).

D.1.1 Summary of previous results

In this paper we consider a free boundary problem similar to the classical two-phase Stefan problem modeling the melting of ice assuming in addition that the heat is transferred by radiation in the solid. The pioneer of the study of such free boundary problems was J. Stefan, after whom these problems are named, in the late 80's (cf. [135]). The same person worked also on heat radiation developing the well-known Stefan law, otherwise called Stefan-Boltzmann law, which states that the total radiation of a black body is proportional to the fourth power of the temperature (cf. [133]). In this subsection we revise important results on both the theory of free boundary problems and of radiative heat transfer.

Starting from the work of J. Stefan, the one and two-phase Stefan problem for the melting of ice has been extensively studied in both one-dimensional and higher dimensional versions, and different definitions of solutions were considered, such as classical solutions and weak enthalpy solutions. The first are defined as strong solutions of the Stefan problem, while the latter are defined as the weak solutions of the enthalpy formulation of the free boundary problem.

The well-posedness theory for classical solutions to the Stefan problem has been considered for example in [55,56,123] showing the local well-posedness via a fixed-point equation for integral equations of Volterra type. The global well-posedness is proved applying the maximum principle. In [56] another approach involving the Baiocchi transform is also considered. In [59,60] a well-posedness result based on the study of a suitable variational inequality is presented. Other results about the well-posedness theory for classical solutions can be found in [26,27,106]. In [56,106] the asymptotic behavior of the one-dimensional one-phase problem is considered and it is proved that the temperature is given by a self-similar profile as $t\to\infty$. Important results about the theory of weak (enthalpy) solutions can be found in [57], for the two-phase one-dimensional problem, and in [58], for the higher dimensional one and two-phase problem.

An important difference between classical and weak enthalpy solutions is in the context of supercooled, superheated and mushy regions. Regions of liquid (or solid) at a temperature $T < T_M$ (or $T > T_M$) are denoted in the classical theory supercooled (resp. superheated) regions. In the absence of such regions, weak and classical solutions are equivalent. In the enthalpy formulation the onset of mushy regions, i.e. $\{(t,x) : T(t,x) = T_M\}$ with positive measure, is allowed, whereas supercooled or superheated regions cannot appear.

Concerning the study of mushy regions for Stefan problems with volumetric heat sources, in [51] the authors give a clear distinction between classical and weak enthalpy solutions and introduce the notion of classical enthalpy solutions, which allow the formation of mushy regions. In [89] the authors consider the classical solutions to a one-dimensional two-phases Stefan problem with volumetric heat sources and show the formation of regions of supercooled liquid or superheated solid. Other examples of studies of the formation of mushy regions are [20, 50, 90, 117, 142, 143].

Moving to the transfer of heat by radiation, this problem has been widely studied starting form the seminal works of Compton [31] in 1922 and of Milne [109] in 1926. The mathematical theory behind the interaction of photons with matter deals with the study of the radiative transfer equation, as given in (D.2). The derivation and the main properties of this kinetic equation can be found for instance in [29, 108, 114, 125, 152].

Also in recent years many different problems were considered, such as well-posedness results, the diffusion approximation and the combination of radiative transfer with other existing models. In [35, 83] the authors proved the well-posedness theory for the stationary radiative transfer equation. Another extensively studied problem is the so-called diffusion approximation, i.e. the limit of the radiative transfer equation when the mean free path of the photons is very small. See for instance [13, 14, 36, 37] and the reference therein.

The radiative heat transfer has been also considered in problems concerning more involved interaction between radiation and matter. For instance, problems studying these interactions in a moving fluid can be found in [69, 71, 108, 152]. We refer to [34, 81] and the reference therein for problems considering models of coupled Boltzmann equations and radiative transfer. Moreover, also problems where the heat is transported in a body by conduction and radiation and homogenization problems in porous and perforated domains, where the heat is transported by conduction, by radiation and in some cases also by convection, have been studied in several works. We refer to the literature of our previous work [36].

Finally, models of melting processes assuming transport of heat by conduction and radiation has been considered numerically in some engineering applications, for instance in [28,124,129,130,140]. There, free boundary problems concerning phase transition due to both conduction and radiation are numerically analyzed and several one-dimensional models considering one, two, and three-phase Stefan-like problems are formulated based on experimental results. Another relevant numerical application, which has been extensively studied in recent years, is the analysis of free boundary problems modeling the vaporization of droplets where the heat is transported by radiation and conduction. For example in [2,84,92,126,128,145,150] numerical simulations show that the radiative heat transfer plays an important role in the vaporization of droplets.

D.1.2 Main results and plan of the paper

In this paper we study the well-posedness theory for problem (D.7) and it is structured as follows.

First of all, in the next subsection D.1.3 we will perform some rescaling obtaining an equivalent version of the problem (D.7) which we will consider for the rest of the paper, while in subsection D.1.4 we clarify some notations that we will use throughout this paper.

In the following Section D.2, using Banach fixed-point theorem and classical parabolic theory, we show a local well-posedness result, which can be summarized as the following

Theorem D.1. *Let $T_0 \in C^{0,1}(\mathbb{R})$ be the initial temperature satisfying the condition*

$$T_0(x) > T_M \quad \text{if } x < 0, \quad T_0(0) = T_M, \quad T_0(x) < T_M \quad \text{if } x > 0.$$

Under suitable assumptions on the regularity of T_0 in $\mathbb{R}_\pm$ there exists a time $t_* > 0$ such that there exists a unique solution to the problem (D.7) *for $t \in [0, t_*]$. Moreover, the temperature satisfies*

$$T_0(x) > T_M \quad \text{if } x < s(t), \quad T_0(s(t)) = T_M, \quad T_0(x) < T_M \quad \text{if } x > s(t).$$

In addition to the local well-posedness theory, in Section D.3 we will also prove a more general global well-posedness result, which applies for a large family of initial data. The following theorem will be proved constructing a suitable family of sub- and supersolutions and applying the maximum principle to the parabolic equations in (D.7).

Theorem D.2 (Global Well-posedness)**.** *Let T_0 as in Theorem D.1. There exists a large class of initial data for which there exists a unique global in time solution to the problem* (D.7)*.*

As we will see in Section D.3, the assumptions on the initial data concern the upper bound on the initial temperature of the liquid.

D.1.3 Some scaling

In this subsection we rescale the problem (D.7) obtaining an equivalent problem in order to reduce the number of parameters. In this paper we will not assuming a positive source of radiation. Nevertheless, we remark that the computations we perform in this subsection can be also adapted in the presence of a non-trivial source of radiation.

First of all we reduce the radiative term and the radiative transfer equation to a non-local integral term. To this end we solve

$$\begin{cases} n \cdot \nabla_x I_\nu(t,x,n) = \alpha \left(B_\nu(T(t,x_1)) - I_\nu(t,x,n)\right) & x_1 > s(t), \\ I_\nu(t,x,n) = 0 & x_1 = s(t),\ n_1 > 0, \end{cases}$$

by characteristics. This procedure is similar to the computation in Section 2.2 of [37].

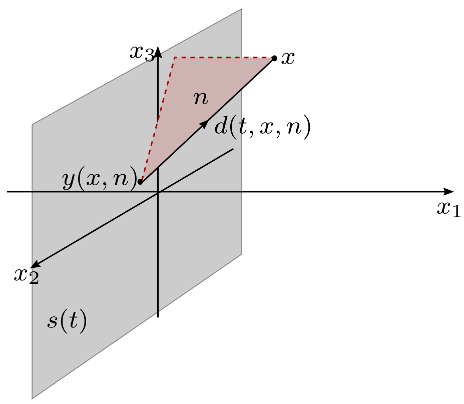

Figure D.2: Illustration of the characteristics.

We define for $(t,x,n) \in (0,\infty) \times \{\xi \in \mathbb{R}^3 : \xi_1 > s(t)\} \times \mathbb{S}^2$ the point $y(t,x,n) \in \{s(t)\} \times \mathbb{R}^2$ to be the intersection of the half-line starting from x and moving in direction $-n$ and we denote $d(t,x,n)$ to be the distance of x to the interface $\{s(t)\} \times \mathbb{R}^2$ in direction $-n$. We are hence considering

$$\begin{aligned} &y(t,x,n) = \{s(t)\} \times \mathbb{R}^2 \cap \{x - tn : t > 0\} \quad \text{and} \\ &d(t,,n) = |x - y(t,x,n)| \quad \text{such that} \quad x = y(t,x,n) + d(t,x,n)n. \end{aligned}$$

We also define $d(t,x,n) = \infty$ if $n_1 < 0$. An easy application of trigonometry shows also that

$$d(t,x,n) \cdot \cos(\theta(n,e_1)) = x_1 - s(t),$$

where $\theta(n,e_1)$ is the angle between the unit vectors n and $e_1 = (1,0,0)$. This implies that

$$d(t,x,n)n_1 = x_1 - s(t) \quad \text{if } n_1 > 0.$$

Solving the radiative transfer equation by characteristics we hence obtain for $x_1 > s(t)$

$$I_\nu(t,x,n) = \int_0^{d(t,x,n)} d\tau \alpha \exp(-\alpha\tau) B_\nu(T(t,x_1 - \tau n_1)).$$

As we pointed out above, I_ν is not zero on the liquid, i.e. for $x_1 < s(t)$, but is constant to the radiation intensity at the interface. Thus,for $x_1 < s(t)$ we have

$$I_\nu(t,x,n) = \mathbb{1}_{\{n_1 \leq 0\}} \int_0^\infty d\tau \alpha \exp(-\alpha\tau) B_\nu(T(t,s(t) - \tau n_1)).$$

This proves also the previous claim about the continuity of the radiative flux through the interface.

While $\operatorname{div}\left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dnnI_\nu(t,x,n)\right) = 0$ for $x_1 < s(t)$, a similar computation as in [37] shows for $x_1 > s(t)$

$$\operatorname{div}\left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dnnI_\nu(t,x,n)\right) = 4\pi\sigma\alpha T^4(t,x_1) - 4\pi\sigma\alpha \int_{s(t)}^\infty d\eta \frac{\alpha E_1(\alpha|x_1 - \eta|)}{2} T^4(t,\eta),$$

where $E_1(x) = \int_{|x|}^\infty \frac{e^{-t}}{t} dt$ is the exponential function. This can be proved using that

$$\begin{aligned} \int_{\mathbb{R}^2} d\xi \frac{e^{-\alpha\sqrt{y^2+\xi^2}}}{y^2+\xi^2} = 2\pi \int_0^\infty d\rho\, \rho \frac{e^{-\alpha\sqrt{y^2+\rho^2}}}{y^2+\rho^2} &= \pi \int_0^\infty dr \frac{e^{-\alpha\sqrt{y^2+r}}}{y^2+r} \\ &= 2\pi \int_{|y|}^\infty dz \frac{e^{-\alpha z}}{z} = 2\pi \int_{\alpha|y|}^\infty dz \frac{e^{-z}}{z}. \end{aligned}$$

Therefore, we can write the system (D.7) as follows

$$\begin{cases} C_L \partial_t T(t,x_1) = K_L \partial_{x_1}^2 T(t,x_1) & x_1 < s(t), \\ C_S \partial_t T(t,x_1) = K_S \partial_{x_1}^2 T(t,x_1) - 4\pi\sigma\alpha I_\alpha[T](t,x_1) & x_1 > s(t), \\ T(t,s(t)) = T_M & x_1 = s(t), \\ T(0,x) = T_0(x) & x_1 \in \mathbb{R}, \\ \dot{s}(t) = \frac{1}{L}\left(K_S \partial_{x_1} T(t,s(t)^+) - K_L \partial_{x_1} T(t,s(t)^-)\right), \end{cases}$$

where

$$I_\alpha[T](t,x_1) = T^4(t,x_1) - \int_{s(t)}^\infty d\eta \frac{\alpha E_1(\alpha|x_1 - \eta|)}{2} T^4(t,\eta).$$

Notice that we have obtained a system of equations depending only on the space variable x_1. In order to simplify the reading, we write x instead of x_1.

Next, we see that we can assume without loss of generality $C_S = K_S = 4\pi\sigma\alpha = 1$. To this end we define $\tau = \frac{4\pi\sigma\alpha}{C_S}t$ and $\xi = \sqrt{\frac{4\pi\sigma\alpha}{K_S}}x$. Let us also consider $T(t,x) = \tilde{T}(\tau,\xi)$ and $\tilde{s}(\tau) = \sqrt{\frac{4\pi\sigma\alpha}{K_S}}s(t)$. In this way we obtain

$$\partial_t T(t,x) = \frac{4\pi\sigma\alpha}{C_S}\partial\tau\tilde{T}(\tau,\xi) \quad \text{and} \quad \partial_x^2 T(t,x) = \frac{4\pi\sigma\alpha}{K_S}\partial_\xi^2\tilde{T}(\tau,\xi).$$

Defining $\tilde{\alpha} = \sqrt{\frac{K_S}{4\pi\sigma\alpha}}\alpha$ we see also that for the radiation term a change of variable gives

$$\begin{aligned} I_\alpha[T](t,x) =& T^4(t,x) - \int_{s(t)}^\infty d\eta \frac{\alpha E_1(\alpha|x-\eta|)}{2}T^4(t,\eta) \\ =& \tilde{T}^4(\tau,\xi) - \int_{s(t)}^\infty d\eta \frac{\alpha E_1\left(\alpha\sqrt{\frac{K_S}{4\pi\sigma\alpha}}\left|\sqrt{\frac{4\pi\sigma\alpha}{K_S}}(x-\eta)\right|\right)}{2}\tilde{T}^4\left(\tau, \sqrt{\frac{4\pi\sigma\alpha}{K_S}}\eta\right) \\ =& \tilde{T}^4(\tau,\xi)) - \int_{\tilde{s}(\tau)}^\infty d\zeta \frac{\tilde{\alpha}E_1\left(\tilde{\alpha}\left|\xi-\zeta\right|\right)}{2}\tilde{T}^4\left(\tau,\zeta\right) = I_{\tilde{\alpha}}[\tilde{T}](\tau,\xi). \end{aligned}$$

We see that $\dot{s}(t) = \partial_t\sqrt{\frac{K_S}{4\pi\sigma\alpha}}\tilde{s}\left(\frac{4\pi\sigma\alpha}{C_S}t\right) = \frac{\sqrt{K_S 4\pi\sigma\alpha}}{C_S}\dot{\tilde{s}}(\tau)$ and also

$$\frac{1}{L}\left(K_S\partial_{x_1}T(t,s(t)^+) - K_L\partial_{x_1}T(t,s(t)^-)\right) = \frac{K_S}{L}\sqrt{\frac{4\pi\sigma\alpha}{K_S}}\left(\partial_\xi\tilde{T}(\tau,\tilde{s}(\tau)^+) - \frac{K_L}{K_S}\partial_\xi\tilde{T}(\tau,\tilde{s}(\tau)^-)\right)$$

Moreover, we define also $K = \frac{K_L}{K_S}$, $C = \frac{C_L}{C_S}$, and finally $\tilde{L} = \frac{L}{C_S}$. With the change of variable above we obtain under this notation

$$\begin{cases} C\partial_\tau\tilde{T}(\tau,\xi) = K\partial_\xi^2\tilde{T}(\tau,\xi) & \xi < \tilde{s}(\tau), \\ \partial_\tau\tilde{T}(\tau,\xi) = \partial_\xi^2\tilde{T}(\tau,\xi) - I_{\tilde{\alpha}}[\tilde{T}](\tau,\xi) & \xi > \tilde{s}(t), \\ \tilde{T}(\tau,\tilde{s}(\tau)) = T_M & \xi = \tilde{s}(\tau), \\ \tilde{T}(0,\xi) = \tilde{T}_0(\xi) & \xi \in \mathbb{R}, \\ \dot{\tilde{s}}(\tau) = \frac{1}{\tilde{L}}\left(\partial_\xi\tilde{T}(\tau,\tilde{s}(\tau)^+) - K\partial_\xi\tilde{T}(\tau,\tilde{s}(\tau)^-)\right). \end{cases} \tag{D.8}$$

In order to simplify the notation we will write $\xi = x$, $t = \tau$, $\tilde{T} = T$, $\tilde{s} = s$, $\tilde{\alpha} = \alpha$, and finally $\tilde{L} = L$.

In the following we will study the problem (D.8) in a spatial coordinate system which is at rest. Therefore we now perform a change of variable. To this end we define $y = x - s(t)$ and we set $T(t,x) = \tilde{T}(t, x-s(t)) = \tilde{T}(t,y)$. The time derivative becomes

$$\partial_t T(t,x) = \partial_t\tilde{T}(t,y) - \dot{s}(t)\partial_y\tilde{T}(t,y).$$

Furthermore, the radiation term $I[T]$ is

$$\begin{aligned}
I_\alpha[T](t,x) =& T^4(t,x) - \int_{s(t)}^\infty d\eta \frac{\alpha E_1(\alpha|x-\eta|)}{2} T^4(t,\eta)\\
=& \tilde{T}^4(t,x-s(t)) - \int_{s(t)}^\infty d\eta \frac{\alpha E_1(\alpha|x-s(t)-(\eta-s(t))|)}{2} \tilde{T}^4(t,\eta-s(t))\\
=& \tilde{T}^4(t,x-s(t)) - \int_0^\infty d\xi \frac{\alpha E_1(\alpha|x-s(t)-\xi|)}{2} \tilde{T}^4(t,\xi)\\
=& \tilde{T}^4(t,y) - \int_0^\infty d\xi \frac{\alpha E_1(\alpha|y-\xi|)}{2} \tilde{T}^4(t,\xi) = I_\alpha[\tilde{T}](t,y).
\end{aligned}$$

In order to simplify the notation we will write $\tilde{T} = T$ and we obtain the following system

$$\begin{cases}
\partial_t T(t,y) - \dot{s}(t)\partial_y T(t,y) = \frac{K}{C}\partial_y^2 T(t,y) & y<0,\\
\partial_t T(t,y) - \dot{s}(t)\partial_y T(t,y) = \partial_y^2 T(t,y) - I_\alpha[T](t,y) & y>0,\\
T(t,0) = T_M & y=0,\\
T(0,y) = T_0(y) & y\in\mathbb{R},\\
\dot{s}(t) = \frac{1}{L}\left(\partial_y T(t,0^+) - K\partial_y T(t,0^-)\right).
\end{cases} \tag{D.9}$$

The rest of the paper is devoted to the study of the free boundary problem (D.7) in its equivalent formulation (D.9).

D.1.4 Some notation

Let $U \subseteq \mathbb{R}$. Throughout this article we will denote by $C^{k,\beta}(U)$ the space of k-times continuous differentiable functions f with

$$\|f\|_{k,\beta} = \max_{0\le j\le k}\left(\sup_U \left|\partial_x^j f\right|\right) + \sup_{x,y\in U} \frac{\left|\partial_x^k f(x) - \partial_x^k f(y)\right|}{|x-y|^\beta} < \infty.$$

We remark that $f \in C^{k,\beta}(U)$ has all k derivatives bounded.

In a similar way we consider the space $\mathcal{C}_{t,x}^{n,k}\left((0,\tau)\times U\right)$ to be the space of functions $f \in C^0\left((0,\tau)\times U\right)$ with continuous derivatives $\partial_t^j f \in C^0\left((0,\tau)\times U\right)$ and $\partial_x^l f \in C^0\left((0,\tau)\times U\right)$ for any $0\le j\le n$ and $0\le l\le k$. Notice that the functions and their derivatives are continuous up to the boundary but their norms have not to be bounded.

Moreover, for $0\le a<\tau$ the space $C_{t,x}^{n+\beta,k+\delta}\left([a,\tau]\times U\right)$ is the space of functions $f \in \mathcal{C}_{t,x}^{n,k}\left([a,\tau]\times U\right)$ with

$$\sup_{[a,\tau]\times U}\left|\partial_t^j f\right| < \infty \text{ for } 0\le j\le n \quad \text{and} \quad \sup_{[a,\tau] times U}\left|\partial_x^l f\right| < \infty \text{ for } 0\le l\le k$$

and with

$$\sup_{t,s\in(a,\tau),\, x\in U} \frac{\left|\partial_t^n f(t,x) - \partial_t^n f(s,x)\right|}{|t-s|^\beta} < \infty \quad \text{and} \quad \sup_{x,y\in U,\, t\in(a,\tau)} \frac{\left|\partial_x^k f(t,x) - \partial_x^k f(t,y)\right|}{|x-y|^\delta} < \infty.$$

In particular, if $2(n+\beta) = k+\delta$ the derivatives of the function $f \in C_{t,x}^{n+\beta,k+\delta}\left([a,\tau]\times U\right)$ satisfy also $\partial_t^j\partial_x^l f \in C^0\left([a,\tau]\times U\right)$ for all $2j+l<k$ and $\beta,\delta\in[0,1)$.

Finally, when we write that the domain of the functions v_i is $\mathbb{R}_\pm$ (or $[0,\pm R]$) we always mean that v_1 is a function on $\mathbb{R}_-$ (resp. on $[-R,0]$) and that v_2 is a function on $\mathbb{R}_+$ (resp. on $[0,R]$).

D.2 Local well-posedness

In this section we prove the local well-posedness theory for the free boundary problem (D.9). Later, in Section D.3 we will extend the result for a large class of initial data for which a global well-posedness result will be proved. In the following subsections we will show with a fixed-point argument the existence of a unique solution for small times. Further on we will show the regularity and some properties of the solutions.

D.2.1 Fixed-point method

We show the local well-posedness for the system (D.9). We will moreover denote by T_1 the temperature defined for $t>0$ and $y<0$ and by T_2 the temperature defined for $t>0$ and $y>0$. Hence the system can be rewritten as

$$\begin{cases} \partial_t T_1(t,y) - \dot{s}(t)\partial_y T_1(t,y) = \frac{K}{C}\partial_y^2 T_1(t,y) & y<0, \\ \partial_t T_2(t,y) - \dot{s}(t)\partial_y T_2(t,y) = \partial_y^2 T_2(t,y) - I_\alpha[T_2](t,y) & y>0, \\ T_1(t,0) = T_2(t,0) = T_M & y=0, \\ T_1(0,y) = T_0(y) \text{ and } T_2(0,y) = T_0(y) & y\in\mathbb{R}, \\ \dot{s}(t) = \frac{1}{L}\left(\partial_y T_2(t,0^+) - K\partial_y T_1(t,0^-)\right) & t>0 \\ s(0)=0. \end{cases} \tag{D.10}$$

Theorem D.3. *Let $T_0 \in C_b^{0,1}(\mathbb{R})$ be bounded and positive with $T_0(0)=T_M$, $T_0(y)>T_M$ if $y<0$ and $T_0(y)<T_M$ if $y>0$. Let also $T_0\mid_{\mathbb{R}_\pm} \in C^2(\mathbb{R}_\pm)$ with bounded first and second derivative. Then for a time $t^*>0$ small enough there exists a unique bounded solution $(T_1,T_2,s)\in \mathcal{C}^{0,1}_{t,y}((0,t^*)\times\mathbb{R}_-)\times \mathcal{C}^{0,1}_{t,y}((0,t^*)\times\mathbb{R}_+)\times C^1((0,t^*))$ solving the problem* (D.10) *in distributional sense.*

Proof. We follow the same strategy used by Rubenšteĭn in [123] and by Friedman in [55]. We will construct with the help of suitable Green's functions the (implicit) solution formula for T_1, T_2 and s and we will use a contraction argument in order to show the existence of a unique solution.

In order to simplify the computations we consider the equivalent problem for $u_1 := T_1 - T_M$ and $u_2 := T_2 - T_M$. Hence we consider the system

$$\begin{cases} \partial_t u_1(t,y) - \dot{s}(t)\partial_y u_1(t,y) = \frac{K}{C}\partial_y^2 u_1(t,y) & y<0, \\ \partial_t T_2(t,y) - \dot{s}(t)\partial_y u_2(t,y) = \partial_y^2 u_2(t,y) - I_\alpha[u_2+T_M](t,y) & y>0, \\ u_1(t,0) = u_2(t,0) = 0 & y=0, \\ u_1(0,y) = u_0(y) \text{ and } u_2(0,y) = u_0(y) & y\in\mathbb{R}, \\ \dot{s}(t) = \frac{1}{L}\left(\partial_y u_2(t,0^+) - K\partial_y u_1(t,0^-)\right) & t>0 \\ s(0)=0, \end{cases} \tag{D.11}$$

where $u_0 := T_0 - T_M$ satisfies $u_0(0)=0$ and $u_0>0$ if $y<0$ as well as $u_0<0$ if $y>0$.

Let

$$G(x,\xi,a(t-\tau)) := \Phi(x-\xi,a(t-\tau)) - \Phi(x+\xi,a(t-\tau))$$

be the Green's function for the half space $\mathbb{R}_+$, where $\Phi(z,s)=\frac{1}{\sqrt{4\pi z}}\exp\left(-\frac{z^2}{4s}\right)$ is the fundamental solution of the heat equation. Recall that with the help of the Green's function G the solution of the following Cauchy problem on $\mathbb{R}_+$ or on $\mathbb{R}_-$

$$\begin{cases} \partial_t v - a\partial_x^2 v = f & x > 0 \text{ resp. } x < 0, \\ v(0,t) = 0 & t > 0, \\ v(x,0) = g_0(x) & x > 0 \text{ resp. } x < 0 \end{cases}$$

has the integral representation

$$v(t,x) = \int_0^\infty g_0(\xi)G(x,\xi,at)d\xi + \int_0^\infty \int_0^t f(\xi,\tau)G(x,\xi,a(t-\tau))d\tau d\xi$$

on $\mathbb{R}_+$ and

$$v(t,x) = \int_{-\infty}^0 g_0(\xi)G(x,\xi,at)d\xi + \int_{-\infty}^0 \int_0^t f(\xi,\tau)G(x,\xi,a(t-\tau))d\tau d\xi$$

on $\mathbb{R}_-$. Hence, we obtain for u_1 and u_2 the following identities considering $\dot{s}(t)\partial_y u_1$ resp. $\dot{s}(t)\partial_y uu_2 - I_\alpha[u_2 + T_M]$ as sources

$$u_1(t,y) = \int_{-\infty}^0 u_0(\xi)G(y,\xi,\kappa t)d\xi + \int_{-\infty}^0 \int_0^t \dot{s}(\tau)\partial_\xi u_1(\tau,\xi)G(y,\xi,\kappa(t-\tau))d\tau d\xi, \qquad \text{(D.12)}$$

where we used the notation $\kappa = \frac{K}{C}$, and

$$\begin{aligned} u_2(t,y) = \int_0^\infty u_0(\xi)G(y,\xi,t)d\xi &+ \int_0^\infty \int_0^t \dot{s}(\tau)\partial_\xi u_2(\tau,\xi)G(y,\xi,(t-\tau))d\tau d\xi \\ &- \int_0^\infty \int_0^t I_\alpha[u_2 + T_M](\tau,\xi)G(y,\xi,(t-\tau))d\tau d\xi. \qquad \text{(D.13)} \end{aligned}$$

We now have to differentiate these expressions with respect to the spatial coordinate in order to find an expression for $\dot{s}$. We recall that

$$\partial_y G(y,\xi,a(t-\tau)) = -\partial_\xi \left(\Phi(y-\xi,a(t-\tau)) + \Phi(y+\xi,a(t-\tau))\right) = -\partial_\xi g(y,\xi,a(t-\tau)),$$

where $g(y,\xi,a(t-\tau)) = \Phi(y-\xi,a(t-\tau)) + \Phi(y+\xi,a(t-\tau))$. This implies on one hand

$$\partial_y u_1(t,y) = \int_{-\infty}^0 \partial_\xi u_0(\xi)g(y,\xi,\kappa t)d\xi - \int_{-\infty}^0 \int_0^t \dot{s}(\tau)\partial_\xi u_1(\tau,\xi)\partial_\xi g(y,\xi,\kappa(t-\tau))d\tau d\xi, \quad \text{(D.14)}$$

where we integrated by parts

$$-\int_{-\infty}^0 u_0(\xi)\partial_\xi g(y,\xi,\kappa t)d\xi = \int_{-\infty}^0 \partial_\xi u_0(\xi)g(y,\xi,\kappa t)d\xi$$

since $u_0(0) = 0$ and $g \to 0$ as $|\xi| \to \infty$ for every fixed y.

On the other hand we have also

$$\begin{aligned} \partial_y u_2(t,y) = \int_0^\infty \partial_\xi u_0(\xi)g(y,\xi,t)d\xi &- \int_0^\infty \int_0^t \dot{s}(\tau)\partial_\xi u_2(\tau,\xi)\partial_\xi g(y,\xi,(t-\tau))d\tau d\xi \\ &+ \int_0^\infty \int_0^t I_\alpha[u_2 + T_M](\tau,\xi)\partial_\xi g(y,\xi,(t-\tau))d\tau d\xi. \qquad \text{(D.15)} \end{aligned}$$

Hence, $\dot{s}(t)$ is given by

$$\dot{s}(t) = \frac{1}{L}\left(\int_0^\infty \partial_\xi u_0(\xi)g(0^+,\xi,t)d\xi - \int_0^\infty \int_0^t \dot{s}(\tau)\partial_\xi u_2(\tau,\xi)\partial_\xi g(0^+,\xi,(t-\tau))d\tau d\xi \right.$$
$$+ \int_0^\infty \int_0^t I_\alpha[u_2+T_M](\tau,\xi)\partial_\xi g(0^+,\xi,(t-\tau))d\tau d\xi - K\int_{-\infty}^0 \partial_\xi u_0(\xi)g(0^-,\xi,\kappa t)d\xi$$
$$\left. +K\int_{-\infty}^0 \int_0^t \dot{s}(\tau)\partial_\xi u_1(\tau,\xi)\partial_\xi g(0^-,\xi,\kappa(t-\tau))d\tau d\xi \right). \quad \text{(D.16)}$$

Equations (D.12), (D.13) and (D.16) define the operator $\mathcal{L}(u_1,u_2,\dot{s})$, for which we will show that there exists a unique fixed-point in a suitable set. This will conclude the proof of the existence of a unique solution for small times. Indeed, since $s(0)=0$ the solution to the problem (D.11) is given by $\left(u_1,u_2,\int_0^t \dot{s}(\tau)d\tau\right)$. Before defining the space in which we will work and proving the contraction property for the operator $\mathcal{L}$, we collect some key estimates.

First of all since $\partial_\xi \Phi(y-\xi,a(t-\tau)) = \frac{(\xi-y)}{2a(t-\tau)}\frac{1}{\sqrt{4\pi a(t-\tau)}}\exp\left(-\frac{|y-\xi|^2}{4a(t-\tau)}\right)$ we estimate

$$\int_0^\infty |\partial_\xi g(y,\xi,a(t-\tau))|\,d\xi \le \int_0^\infty \frac{|\xi-y|}{2a(t-\tau)}\frac{1}{\sqrt{4\pi a(t-\tau)}}\exp\left(-\frac{|y-\xi|^2}{4a(t-\tau)}\right)d\xi$$
$$+\int_0^\infty \frac{|\xi+y|}{2a(t-\tau)}\frac{1}{\sqrt{4\pi a(t-\tau)}}\exp\left(-\frac{|y+\xi|^2}{4a(t-\tau)}\right)d\xi$$
$$=\int_{-y}^\infty \frac{|\xi|}{2a(t-\tau)}\frac{1}{\sqrt{4\pi a(t-\tau)}}\exp\left(-\frac{|\xi|^2}{4a(t-\tau)}\right)d\xi$$
$$+\int_{y}^\infty \frac{|\xi|}{2a(t-\tau)}\frac{1}{\sqrt{4\pi a(t-\tau)}}\exp\left(-\frac{|\xi|^2}{4a(t-\tau)}\right)d\xi$$
$$=\int_0^\infty \frac{\xi}{a(t-\tau)}\frac{1}{\sqrt{4\pi a(t-\tau)}}\exp\left(-\frac{|\xi|^2}{4a(t-\tau)}\right)d\xi = \frac{1}{\sqrt{\pi a(t-\tau)}}, \quad \text{(D.17)}$$

where we used the change of coordinate $\xi'=\xi-y$ resp. $\xi'=\xi+y$ and the fact that the resulting function is even. In the very same way we can estimate also

$$\int_{-\infty}^0 |\partial_\xi g(y,\xi,a(t-\tau))|\,d\xi \le \frac{1}{\sqrt{\pi a(t-\tau)}}.$$

A direct consequence of these estimates are the following results

$$\int_0^\infty \int_0^t |\partial_\xi g(y,\xi,a(t-\tau))|\,d\xi d\tau \le \int_0^t \frac{1}{\sqrt{\pi a(t-\tau)}}d\tau = \frac{1}{\sqrt{a\pi}}\int_0^t \frac{1}{\sqrt{\tau}}d\tau = \frac{2\sqrt{t}}{\sqrt{a\pi}}, \quad \text{(D.18)}$$

where we used Fubini's theorem and (D.17). Analogously we also have

$$\int_{-\infty}^0 \int_0^t |\partial_\xi g(y,\xi,a(t-\tau))|\,d\xi d\tau \le \frac{2\sqrt{t}}{\sqrt{a\pi}}. \quad \text{(D.19)}$$

Another important estimates are the following ones

$$\int_0^\infty |G(y,\xi,a(t-\tau))|\,d\xi \le \int_0^\infty \Phi(y-\xi,a(t-\tau))+\Phi(y+\xi,a(t-\tau))d\xi$$
$$=2\int_{|y|}^\infty \Phi(\xi,a(t-\tau))d\xi + \int_{-|y|}^{|y|}\Phi(\xi,a(t-\tau))d\xi$$
$$=2\int_0^\infty \Phi(\xi,a(t-\tau))d\xi = \int_{-\infty}^\infty \Phi(\xi,a(t-\tau))d\xi = 1, \quad \text{(D.20)}$$

where we used the change of variables $\xi' = \xi - y$ resp. $\xi' = \xi + y$ and the fact that Φ is a non-negative even function with integral 1. The same holds on the negative real line

$$\int_{-\infty}^{0} |G(y,\xi,a(t-\tau))|\, d\xi \le 1. \tag{D.21}$$

Similarly, using the definition of $g(y,\xi,a(t-\tau))$ we compute

$$\int_{0}^{\infty} |g(y,\xi,a(t-\tau))|\, d\xi \le \int_{-\infty}^{\infty} \Phi(\xi,a(t-\tau))d\xi = 1 \tag{D.22}$$

and

$$\int_{-\infty}^{0} |g(y,\xi,a(t-\tau))|\, d\xi \le \int_{-\infty}^{\infty} \Phi(\xi,a(t-\tau))d\xi = 1. \tag{D.23}$$

We define now the metric space for which we will apply the Banach fixed-point theorem. Let us first introduce the notation that we will use throughout this section. On $\mathcal{C}^{0,1}_{t,y}\left((0,t^*)\times\mathbb{R}_\pm\right)$ we consider the norm $\|f\|_{0,1} := \max\{\|f\|_{C^0}, \|\partial_y f\|_{C^0}\}$. Let $\theta \in (0,1)$ and let

$$C_1,\ C_2 > \frac{\|u_0\|_1}{1-\theta} \text{ and } C_3 > \frac{1+K}{L}\frac{\|u_0\|_1}{1-\theta}.$$

We consider the following closed metric space

$$\begin{aligned}\mathcal{A}_{C_1,C_2,C_3} = \Big\{(u_1,u_2,\dot{s}) \in \mathcal{C}^{0,1}_{t,y}\left((0,t^*)\times\mathbb{R}_-\right)\times\mathcal{C}^{0,1}_{t,y}\left((0,t^*)\times\mathbb{R}_+\right)\times C^0((0,t^*)) :\\ \|u_1\|_{0,1}\le C_1,\ \|u_2\|_{0,1}\le C_2,\ \|\dot{s}\|_{C^0}\le C_3\Big\},\end{aligned}$$

with the metric induced by the norm $\|(u_1,u_2,\dot{s})\|_{\mathcal{A}} := \|u_1\|_{0,1} + \|u_2\|_{0,1} + \|\dot{s}\|_{C^0}$. We consider the following operator acting on $\mathcal{A}$

$$\begin{aligned}\mathcal{L}:\quad \mathcal{A}=\mathcal{A}_{C_1,C_2,C_3} &\to \mathcal{C}^{0,1}_{t,y}\left((0,t^*)\times\mathbb{R}_-\right)\times\mathcal{C}^{0,1}_{t,y}\left((0,t^*)\times\mathbb{R}_+\right)\times C^0((0,t^*))\\ (u_1,u_2,\dot{s}) &\mapsto \left(\mathcal{L}_1((u_1,u_2,\dot{s})),\mathcal{L}_2((u_1,u_2,\dot{s})),\mathcal{L}_2((u_1,u_2,\dot{s}))\right),\end{aligned}$$

where we defined according to (D.12), (D.13) and (D.16)

$$\mathcal{L}_1((u_1,u_2,\dot{s}))(t,y) = \int_{-\infty}^{0} u_0(\xi)G(y,\xi,\kappa t)d\xi + \int_{-\infty}^{0}\int_0^t \dot{s}(\tau)\partial_\xi u_1(\tau,\xi)G(y,\xi,\kappa(t-\tau))d\tau d\xi,$$

$$\begin{aligned}\mathcal{L}_2((u_1,u_2,\dot{s}))(t,y) = \int_0^\infty u_0(\xi)G(y,\xi,t)d\xi + \int_0^\infty\int_0^t \dot{s}(\tau)\partial_\xi u_2(\tau,\xi)G(y,\xi,(t-\tau))d\tau d\xi\\ -\int_0^\infty\int_0^t I_\alpha[u_2+T_M](\tau,\xi)G(y,\xi,(t-\tau))d\tau d\xi\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}_3((u_1,u_2,\dot{s}))(t) = \frac{1}{L}\Bigg(\int_0^\infty \partial_\xi u_0(\xi)g(0^+,\xi,t)d\xi\\ -\int_0^\infty\int_0^t \dot{s}(\tau)\partial_\xi u_2(\tau,\xi)\partial_\xi g(0^+,\xi,(t-\tau))d\tau d\xi\\ +\int_0^\infty\int_0^t I_\alpha[u_2+T_M](\tau,\xi)\partial_\xi g(0^+,\xi,(t-\tau))d\tau d\xi - K\int_{-\infty}^0 \partial_\xi u_0(\xi)g(0^-,\xi,\kappa t)d\xi\\ +K\int_{-\infty}^0\int_0^t \dot{s}(\tau)\partial_\xi u_1(\tau,\xi)\partial_\xi g(0^-,\xi,\kappa(t-\tau))d\tau d\xi\Bigg).\end{aligned}$$

With the help of (D.14) and (D.15) and using the properties of the Green's functions, it is not difficult to see that the operator is well-defined. We prove next that choosing a time $t^* > 0$ small enough and for $C_1,\ C_2,\ C_3$ large enough the operator is a self-map. To this end, we need to estimate the norms of the three components of $\mathcal{L}$.

We assumed

$$\|u_0\|_1 = \max\left\{\|u_0\|_{C^0}, \|\partial_y u_0\|_{C^0(\mathbb{R}_-)}, \|\partial_y u_0\|_{C^0(\mathbb{R}_+)}\right\} < \infty.$$

Recall that $\alpha > 0$, $K > 0$, $L > 0$ and $\kappa > 0$ are all constants. Combining the triangle inequality as well as Hölder's inequality with the estimate (D.21) we can conclude

$$\begin{aligned}\|\mathcal{L}_1((u_1,u_2,\dot{s}))\|_{C^0} \leq &\|u_0\|_{C^0} \sup_{0<t\leq t^*} \int_{-\infty}^0 |G(y,\xi,\kappa t)| d\xi \\ &+ \|u_1\|_{0,1}\|\dot{s}\|_{C^0} \sup_{0<t\leq T} \int_{-\infty}^0 \int_0^t |G(y,\xi,\kappa(t-\tau))| d\tau d\xi \\ \leq &\|u_0\|_1 + t^*\|u_1\|_{0,1}\|\dot{s}\|_{C^0}.\end{aligned} \tag{D.24}$$

Before moving to the estimate for the second component of the operator $\mathcal{L}$ we have to consider the radiation term I_α. Using $\int_{-\infty}^{\infty} \frac{E_1(\xi)}{2} d\xi = 1$ and $|a+b|^4 \leq 8|a|^4 + 8|b|^4$ we obtain

$$\begin{aligned}\|I_\alpha[u_2+T_M]\|_{C^0} = &\sup_{0<t\leq t^*, y>0} \left| \int_0^\infty \frac{\alpha E_1(\alpha(y-\eta))}{2} (u_2(t,\eta)+T_M)^4 d\eta - (u_2(t,y)+T_M)^4 \right| \\ \leq &\|(u_2+T_M)^4\|_{C^0} \sup_{y>0} \left(\int_0^\infty \frac{\alpha E_1(\alpha(y-\eta))}{2} d\eta + 1 \right) \\ \leq &16\left(\|u_2\|^4 + T_M^4\right).\end{aligned} \tag{D.25}$$

Hence, using now (D.20) we obtain similarly as in (D.24)

$$\|\mathcal{L}_2((u_1,u_2,\dot{s}))\|_{C^0} \leq \|u_0\|_1 + t^*\left(\|u_2\|_{0,1}\|\dot{s}\|_{C^0} + 16\|u_2\|_{0,1}^4 + 16T_M^4\right). \tag{D.26}$$

We now estimate the norm of the derivative of the first two component of $\mathcal{L}$. Notice that $\partial_y\mathcal{L}_1(u_1,u_2,\dot{})$ is given by the right hand side of (D.14), while $\partial_y\mathcal{L}_2(u_1,u_2,\dot{})$ is given by the right hand side of (D.15). Hence, using this time (D.23) and (D.19), we obtain in a similar manner as (D.24)

$$\|\partial_y\mathcal{L}_1((u_1,u_2,\dot{s}))\|_{C^0} \leq \|u_0\|_1 + \frac{2}{\sqrt{\kappa\pi}}\sqrt{t^*}\|u_1\|_{0,1}\|\dot{s}\|_{C^0}. \tag{D.27}$$

Analogously, (D.22), (D.18) and (D.25) imply

$$\|\partial_y\mathcal{L}_2((u_1,u_2,\dot{s}))\|_{C^0} \leq \|u_0\|_1 + \frac{2}{\sqrt{\pi}}\sqrt{t^*}\left(\|u_2\|_{0,1}\|\dot{s}\|_{C^0} + 16\|u_2\|_{0,1}^4 + 16T_M^4\right). \tag{D.28}$$

Finally, combining (D.27) and (D.28) we have

$$\begin{aligned}\|\mathcal{L}_3((u_1,u_2,\dot{s}))\|_{C^0} \leq \frac{1}{L}\Bigg(&(1+K)\|u_0\|_1 \\ &+ \frac{2}{\sqrt{\pi}}\sqrt{t^*}\left(\|u_2\|_{0,1}\|\dot{s}\|_{C^0} + \frac{K}{\sqrt{\kappa}}\|u_1\|_{0,1}\|\dot{s}\|_{C^0} + 16\|u_2\|_{0,1}^4 + 16T_M^4\right)\Bigg).\end{aligned} \tag{D.29}$$

Therefore, for $(u_1, u_2, \dot{s}) \in \mathcal{A}$ combining (D.24), (D.26), (D.27), (D.28) and (D.29) we obtain

$$\|\mathcal{L}_1((u_1, u_2, \dot{s}))\|_{0,1} \leq \|u_0\|_1 + \left(t^* + \frac{2}{\sqrt{\kappa\pi}}\sqrt{t^*}\right) C_1 C_3,$$

$$\|\mathcal{L}_2((u_1, u_2, \dot{s}))\|_{0,1} \leq \|u_0\|_1 + \left(t^* + \frac{2}{\sqrt{\pi}}\sqrt{t^*}\right) \left(C_2 C_3 + 16C_2^4 + 16T_M^4\right),$$

and finally

$$\|\mathcal{L}_3((u_1, u_2, \dot{s}))\|_{C^0} \leq \frac{1}{L}\left((1+K)\|u_0\|_1 + \frac{2}{\sqrt{\pi}}\sqrt{t^*}\left(C_2 C_3 + \frac{K}{\sqrt{\kappa}} C_1 C_3 + 16C_2^4 + 16T_M^4\right)\right).$$

Then defining

$$t_1 = \frac{1}{2}\min\left\{\frac{\theta}{C_3}, \frac{\theta^2 \kappa\pi}{8C_3^2}\right\},$$

$$t_2 = \frac{1}{6}\min\left\{\frac{\theta}{C_3}, \frac{\theta}{16C_2^3}, \frac{\theta C_2}{16T_M^4}, \frac{1}{6}\left(\frac{\sqrt{\pi}\theta}{2C_3}\right)^2, \frac{1}{6}\left(\frac{\sqrt{\pi}\theta}{32C_2^3}\right)^2, \frac{1}{6}\left(\frac{\sqrt{\pi}\theta C_2}{32T_M^4}\right)^2\right\}$$

and

$$t_3 = \frac{L^2\pi\theta^2}{64}\min\left\{\left(\frac{1}{C_2}\right)^2, \left(\frac{\sqrt{\kappa}}{KC_1}\right)^2, \left(\frac{C_3}{16C_2^4}\right)^2, \left(\frac{C_3}{16T_M^4}\right)^2\right\}$$

we conclude for $t^* \leq \min\{t_1, t_2, t_3\}$ that

$$\|\mathcal{L}_1((u_1, u_2, \dot{s}))\|_{0,1} \leq C_1, \quad \|\mathcal{L}_2((u_1, u_2, \dot{s}))\|_{0,1} \leq C_2 \quad \text{and} \quad \|\mathcal{L}_3((u_1, u_2, \dot{s}))\|_{C^0} \leq C_3,$$

and hence $\mathcal{L}$ maps $\mathcal{A}$ into itself. We show now that for $t^* > 0$ small enough $\mathcal{L}$ is also a contraction. To this end we assume $(u_1, u_2, \dot{s}), (\bar{u}_1, \bar{u}_2, \dot{\bar{s}}) \in \mathcal{A}$. First of all we consider the radiation term. Using that $a^4 - b^4 = (a-b)(a^3 + a^2 b + ab^2 + b^3) = (a-b)p_3(a,b)$ and that $p_3(a,b) \leq 2(|a|^3 + |b|^3)$ we estimate

$$\begin{aligned}
&\|I_\alpha[u_2 + T_M] - I_\alpha[\bar{u}_2 + T_M]\|_{C^0} \\
=& \sup_{0<t\leq t^*, y>0}\left|\int_0^\infty \frac{\alpha E_1(\alpha(y-\eta))}{2}(u_2(t,\eta) - \bar{u}_2(t,\eta))p_3(u_2 + T_M, \bar{u}_2 + T_M)(t,\eta)d\eta\right. \\
&\qquad \left. - (u_2(t,y) - \bar{u}_2(t,y))p_3(u_2 + T_M, \bar{u}_2 + T_M)(t,y)\right| \\
\leq& 2\|u_2 - \bar{u}_2\|_{0,1}\left(\||u_2 + T_M|^3\|_{0,1} + \||\bar{u}_2 + T_M|^3\|_{0,1}\right)\sup_{y>0}\left(\int_0^\infty \frac{\alpha E_1(\alpha(y-\eta))}{2}d\eta + 1\right) \\
\leq& 32\left(C_2^3 + T_M^3\right)\|u_2 - \bar{u}_2\|_{0,1}.
\end{aligned} \tag{D.30}$$

Hence, using triangle inequality as well as (D.21) and (D.19) we see

$$\begin{aligned}
&\|\mathcal{L}_1((u_1, u_2, \dot{s})) - \mathcal{L}_1((\bar{u}_1, \bar{u}_2, \dot{\bar{s}}))\|_{0,1} \\
&\leq \sup_{0<t\leq t^*, y>0}\left|\int_{-\infty}^0 \int_0^t \left(\dot{s}(\tau)\partial_\xi u_1(\tau,\xi) - \dot{\bar{s}}(\tau)\partial_\xi u_1(\bar{\tau},\xi)\right) G(y,\xi,\kappa(t-\tau))d\tau d\xi\right| \\
&\quad + \sup_{0<t\leq t^*, y>0}\left|\int_{-\infty}^0 \int_0^t \left(\dot{s}(\tau)\partial_\xi u_1(\tau,\xi) - \dot{\bar{s}}(\tau)\partial_\xi u_1(\bar{\tau},\xi)\right) \partial_\xi g(y,\xi,\kappa(t-\tau))d\tau d\xi\right| \\
&\leq \left(t^* + \frac{2}{\sqrt{\kappa\pi}}\sqrt{t^*}\right)\left(C_1\|\dot{s} - \dot{\bar{s}}\|_{C^0} + C_3\|u_1 - \bar{u}_1\|_{0,1}\right).
\end{aligned}$$

Similarly, using (D.30), (D.20) and (D.18) we have

$$\|\mathcal{L}_2((u_1,u_2,\dot{s})) - \mathcal{L}_2((\bar{u}_1,\bar{u}_2,\dot{\bar{s}}))\|_{0,1} \le \left(t^* + \frac{2}{\sqrt{\pi}}\sqrt{t^*}\right)(C_2\|\dot{s}-\dot{\bar{s}}\|_{C^0} + C_3\|u_2-\bar{u}_2\|_{0,1}) + 32(C_2^3+T_M^3)\left(t^* + \frac{2}{\sqrt{\pi}}\sqrt{v}\right)\|u_2-\bar{u}_2\|_{0,1}.$$

Finally, combining the results for the derivatives we obtain

$$\|\mathcal{L}_3((u_1,u_2,\dot{s})) - \mathcal{L}_3((\bar{u}_1,\bar{u}_2,\dot{\bar{s}}))\|_{C^0} \le \frac{2}{\sqrt{\kappa\pi}L}\sqrt{t^*}\,(C_1\|\dot{s}-\dot{\bar{s}}\|_{C^0} + C_3\|u_1-\bar{u}_1\|_{0,1}) + \frac{2}{\sqrt{\pi}L}\sqrt{t^*}\left(C_2\|\dot{s}-\dot{\bar{s}}\|_{C^0} + C_3\|u_2-\bar{u}_2\|_{0,1} + 32(C_2^3+T_M^3)\|u_2-\bar{u}_2\|_{0,1}\right).$$

Hence, we have

$$\begin{aligned}\|\mathcal{L}((u_1,u_2,\dot{s})) - \mathcal{L}((\bar{u}_1,\bar{u}_2,\dot{\bar{s}}))\|_{\mathcal{A}} \le{}& \left(t^* + \frac{2}{\sqrt{\kappa\pi}}\sqrt{t^*} + \frac{2}{\sqrt{\kappa\pi}L}\sqrt{t^*}\right)C_3\|u_1-\bar{u}_1\|_{0,1}\\ &+\left(t^* + \frac{2}{\sqrt{\pi}}\sqrt{t^*} + \frac{2}{\sqrt{\pi}L}\sqrt{t^*}\right)\left(C_3 + 32(C_2^3+T_M^3)\right)\|u_2-\bar{u}_2\|_{0,1}\\ &+\left[\left(t^* + \frac{2}{\sqrt{\kappa\pi}}\sqrt{t^*} + \frac{2}{\sqrt{\kappa\pi}L}\sqrt{t^*}\right)C_1 + \left(t^* + \frac{2}{\sqrt{\pi}}\sqrt{t^*} + \frac{2}{\sqrt{\pi}L}\sqrt{t^*}\right)C_2\right]\|\dot{s}-\dot{\bar{s}}\|_{C^0}.\end{aligned}$$

Let $\lambda \in (0,1)$. We take

$$\begin{aligned}t^* = \min\Bigg\{& t_1, t_2, t_3, \frac{\lambda}{3C_3}, \left(\frac{\lambda\sqrt{\kappa\pi}}{6C_3}\right)^2, \left(\frac{\lambda\sqrt{\kappa\pi}L}{6C_3}\right)^2, \frac{\lambda}{3\left(C_3+32(C_2^3+T_M^3)\right)},\\ &\left(\frac{\lambda\sqrt{\pi}}{6\left(C_3+32(C_2^3+T_M^3)\right)}\right)^2, \left(\frac{\lambda\sqrt{\pi}L}{6\left(C_3+32(C_2^3+T_M^3)\right)}\right)^2,\\ &\frac{\lambda}{3(C_1+C_2)}, \left(\frac{\lambda\sqrt{\pi}}{6(C_1\kappa^{-1}+C_2)}\right)^2, \left(\frac{\lambda\sqrt{\pi}L}{6(C_1\kappa^{-1}+C_2)}\right)^2\Bigg\}.\end{aligned}$$

It is now easy to see that $\mathcal{L}$ is a contraction with

$$\|\mathcal{L}((u_1,u_2,\dot{s})) - \mathcal{L}((\bar{u}_1,\bar{u}_2,\dot{\bar{s}}))\|_{\mathcal{A}} \le \lambda\|(u_1,u_2,\dot{s}) - (\bar{u}_1,\bar{u}_2,\dot{\bar{s}})\|_{\mathcal{A}}.$$

Thus, an application of Banach fixed-point theorem yields the existence of a unique solution $(u_1,u_2,\dot{s}) \in \mathcal{A}_{C_1,C_2,C_3}$ to the fixed-point system defined by the equations (D.12), (D.13) and (D.16). Moreover, this is the unique solution in the sense that if $(\bar{u}_1,\bar{u}_2,\dot{\bar{s}}) \in \mathcal{A}_{\bar{C}_1,\bar{C}_2,\bar{C}_3}$ is another fixed-point solution on $[0,\bar{t}^*]$ for $(C_1,C_2,C_3) \neq (\bar{C}_1,\bar{C}_2,\bar{C}_3)$, then $(u_1,u_2,\dot{s}) = (\bar{u}_1,\bar{u}_2,\dot{\bar{s}})$ for all $t \le \{t^*,\bar{t}^*\}$. We notice that by construction $\left(u_1,u_2,\int_0^t \dot{s}(\tau)d\tau\right)$ solves the problem (D.11) in distributional sense. Notice that $\dot{s}$ satisfies the Stefan condition strongly. Thus, the theorem is proved. □

In the next subsection we prove that the unique distributional solution found in Theorem D.3 is also a classical solution with Hölder regularity.

D.2.2 Regularity

In this section we show that the local distributional solutions $u_1 \in \mathcal{C}^{0,1}_{t,y}([0,T]\times\mathbb{R}_-)$ and $u_2 \in \mathcal{C}^{0,1}_{t,y}([0,T]\times\mathbb{R}_+)$ of the parabolic equations

$$\begin{cases} \partial_t u_1(t,y) - \dot{s}(t)\partial_y u_1(t,y) = \frac{K}{C}\partial_y^2 u_1(t,y) & y<0,\\ \partial_t u_2(t,y) - \dot{s}(t)\partial_y u_2(t,y) = \partial_y^2 u_2(t,y) - I_\alpha[u_2+T_M](t,y) & y>0,\\ u_1(t,0)=u_2(t,0)=0 & y=0,\\ u_1(0,y)=u_0(y) \text{ and } u_2(0,y)=u_0(y) & y\in\mathbb{R}. \end{cases} \tag{D.31}$$

are strong solutions, i.e. $u_1 \in \mathcal{C}^{1,2}_{t,y}((0,t^*]\times\mathbb{R}_-)$ and $u_1 \in \mathcal{C}^{1,2}_{t,y}((0,t^*]\times\mathbb{R}_+)$. Moreover, we will show that these solutions have also locally Hölder regularity in the sense that $u_i \in C^{1+\delta/2,2+\delta}([\varepsilon,t^*]\times\mathbb{R}_\pm)$ for any $\varepsilon>0$ and for some $\delta\in(0,1)$. To this end, we will use the following classical result for the heat equation (cf. [91] p.273).

Proposition D.1. *Let $\Phi(x,t)=\frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}\mathbb{1}_{t>0}$ be the fundamental solution of the heat equation. Then for $F\in\mathcal{C}^{\delta/2,\delta}_{t,y}([0,t^*]\times\mathbb{R})$, $\varphi\in C^{2,\delta}(\mathbb{R})$ and $g\in C^{1,\delta/2}([-\infty,t^*])$ the following estimates are true*

$$\|\Phi * F\|_{1+\delta/2,2+\delta} \le c\|F\|_{\delta/2,\delta},$$

$$\|\Phi *_y \varphi\|_{1+\delta/2,2+\delta} \le c\|\varphi\|_\delta \tag{D.32}$$

and

$$\|\partial_y\Phi *_t g\|_{1+\delta/2,2+\delta} \le c\|g\|_{1+\delta}, \tag{D.33}$$

where in (D.32) *we mean*

$$\Phi *_y \varphi(t,y) = \int_{\mathbb{R}} \varphi(\xi)\Phi(y-\xi,t)d\xi$$

and in (D.33)

$$\partial_y\Phi *_t g(t,y) = \int_{-\infty}^t g(\tau)\partial_y\Phi(y,t-\tau)d\tau.$$

Moreover, if $g\in C^{1,\delta/2}([0,t^*])$, then $\partial_y\Phi *_t g \in \mathcal{C}^{1+\delta/2,2+\delta}_{t,y}([\varepsilon,t^*]\times\mathbb{R})$ for all $\varepsilon>0$.

We remark that the last statement can be proved following the estimates in [91]. The proof of the regularity of the solutions u_1 and u_2 follows from classical parabolic theory. Nevertheless, we recall the key estimates that we will use.

Lemma D.1. *Let $\varphi\in C^{1,1}(\mathbb{R})$. Then*

$$\varphi *_y \Phi(\cdot,t) \in \mathcal{C}^{1/2,1+1}_{t,y}([0,t^*]\times\mathbb{R}) \tag{D.34}$$

with

$$[\varphi *_y \Phi]_{t,1/2} \le \|\varphi'\|_\infty \text{ and } [\varphi' *_y \Phi]_{y,Lip} \le \|\varphi'\|_\infty.$$

Moreover,

$$\varphi *_y \Phi(\cdot,t) \in \mathcal{C}^{1+1/2,2+1}_{t,y}([\varepsilon,t^*]\times\mathbb{R})$$

for any $\varepsilon>0$.

Proof. We see that by a change of variables $\eta = \frac{\xi - y}{\sqrt{t}}$ we obtain

$$\varphi *_y \Phi(y,t) = \int_{\mathbb{R}} \varphi(y + \sqrt{t}\eta) \frac{e^{-\eta^2/4}}{\sqrt{4\pi}} d\eta.$$

A simple computations shows (D.34) as well as the estimates for the Hölder seminorms. We remark that $|\sqrt{a} - \sqrt{b}| \leq |a-b|^{1/2}$ for all $a, b > 0$. In a similar way we can also see that for any $\varepsilon > 0$ the function $\varphi *_y \Phi$ has the claimed higher Hölder regularity on $[\varepsilon, t^*] \times \mathbb{R}$. For example, we see integrating by parts that

$$\begin{aligned} \left| \partial_y^2 \int_{\mathbb{R}} \varphi(\xi) \Phi(y - \xi, t) d\xi \right| &= \left| \partial_y \int_{\mathbb{R}} \varphi'(\xi) \Phi(y - \xi, t) d\xi \right| \\ &= \left| \int_{\mathbb{R}} \varphi'(y + \sqrt{t}\eta) \frac{\eta e^{-\eta^2/4}}{\sqrt{4\pi t}} d\xi \right| \leq \frac{\sqrt{2}\|\varphi'\|_\infty}{\sqrt{\varepsilon}}. \end{aligned}$$

In a similar way we can prove the estimates for $\partial_t \varphi *_y \Phi$ as well as the one for the Hölder seminorms. $\square$

Further we will use also the following result

Lemma D.2. *Let $G(y, \xi, at) = \Phi(y - \xi, t) - \Phi(y + \xi, at)$ be the fundamental solution of the heat equation in the half-space. Let also $f_\pm \in C^{0,0}_{t,y}([0,t^*] \times \mathbb{R}_\pm)$. Then*

$$f_\pm * G \in \mathcal{C}^{\alpha/2, 1+\beta}_{t,y}([0,t^*] \times \mathbb{R}_\pm)$$

for any $\alpha,\ \beta \in (0,1)$, where we define

$$f_\pm * G(t,y) = \int_0^t \int_{\mathbb{R}_\pm} f_\pm(\tau, \xi) G(y, \xi, a(t - \tau)) d\xi d\tau.$$

Moreover, the norm of $f_\pm * G$ in the space $\mathcal{C}^{\alpha/2,1+\beta}_{t,y}([0,t^*] \times \mathbb{R}_\pm)$ depends only on $\|f\|_\infty$, t^*, α, β.

Proof. It is enough to prove this Lemma only for $f = f_+ \in \mathcal{C}^{0,0}_{t,y}([0,t^*] \times \mathbb{R}_+)$ and $a = 1$. Using that $-\partial_\xi g(y,\xi,t) = \partial_y G(y,\xi,t)$ and equation (D.18) we have already seen that $f * G \in \mathcal{C}^{0,1}_{t,y}([0,t^*] \times \mathbb{R}_+)$ with norm bounded by $\|f * G\|_{0,1} \leq C \max\{t^*, \sqrt{t*}\}\|f\|_\infty$. Now we only need to show the Hölder regularity of $f * G$. Let hence $0 < s < t < t^*$. Since if $s < t - s$, then $|t| + |s| \leq 3|t - s|$ so that $|f * G(t,y) - f * G(s,y)| \leq 3|t-s|\|f\|_\infty$ by (D.20), it is enough to consider $s > t - s$.

$$\begin{aligned} &\left| \int_0^t \int_{\mathbb{R}_+} f(\tau,\xi) \Phi(y \pm \xi, t - \tau) d\xi d\tau - \int_0^s \int_{\mathbb{R}_+} f(\tau,\xi) \Phi(y \pm \xi, s - \tau) d\xi d\tau \right| \\ \leq & \|f\|_\infty \int_{s-(t-s)}^t \int_{\mathbb{R}_+} |\Phi(y \pm \xi, t - \tau)| \, d\xi d\tau + \|f\|_\infty \int_{s-(t-s)}^s \int_{\mathbb{R}_+} |\Phi(y \pm \xi, s - \tau)| \, d\xi d\tau \\ &+ \|f\|_\infty \int_0^s \int_{\mathbb{R}_+} |\Phi(y \pm \xi, t - \tau) - \Phi(y \pm \xi, s - \tau)| \, d\xi d\tau \\ \leq & 3\|f\|_\infty |t - s| \\ &+ \|f\|_\infty \int_0^{s-(t-s)} \int_{\mathbb{R}_+} \Phi(y \pm \xi, t - \tau) \frac{|y \pm \xi|^2}{4(t - \tau)} \frac{t - s}{s - \tau} + \Phi(y \pm \xi, s - \tau) \frac{\sqrt{t-s}}{\sqrt{t - \tau}} d\xi d\tau \\ \leq & \|f\|_\infty \left(4 + \sqrt{2}\right) \sqrt{t^*} |t - s|^{1/2}, \end{aligned}$$

where we used that $t-s \leq s-\tau$ if $\tau < s-(t-s)$.We turn to the Hölder continuity of the spatial derivative. Let us consider $0 \leq x < y$. Since $\partial_y f * G$ is uniformly bounded in $[0,t^*]\times\mathbb{R}_+$, we have only to show the Hölder condition for $|x-y|<1$. Let hence $|x-y|<1$. If $t \leq |y-x|^2$ then by (D.18) we see that

$$\left|\int_0^t \int_{\mathbb{R}_+} f(\tau,\xi)\left(\partial_\xi g(y,\xi,t-\tau)-\partial_\xi g(x,\xi,t-\tau)\right) d\xi d\tau t\right| \leq C\|f\|_\infty \sqrt{t} \leq C\|f\|_\infty |x-y|.$$

Let now $t > |x-y|^2$. Using that if $|y\pm\xi| > |x\pm\xi|$, the following estimate holds

$$\begin{aligned}|\partial_\xi \Phi(y\pm\xi,\tau)-\partial_\xi\Phi(x\pm\xi,\tau)| \leq & \frac{|x-y|}{2\tau}\Phi(y\pm\xi,\tau)+|x-y|\frac{|x\pm\xi}{2\tau}\Phi(x\pm\xi,\tau)\frac{|y\pm\xi|+|x\pm\xi|}{4\tau} \\ \leq & \frac{|x-y|}{2\tau}\Phi(y\pm\xi,\tau)+\Phi\left(\frac{x\pm\xi}{\sqrt{2}},\tau\right)\left(\frac{|x-y|^2}{4\tau^{3/2}}+\frac{|x-y|}{\tau}\right).\end{aligned}$$

We estimate for any $\beta \in (0,1)$

$$\begin{aligned}&\int_0^t \int_{\mathbb{R}_+} |f(t-\tau,\xi)|\,|\partial_\xi\Phi(y\pm\xi,\tau)-\partial_\xi\Phi(x\pm\xi,\tau)|\,d\xi d\tau \\ &\quad\leq \int_0^{|x-y|^2}\int_{\mathbb{R}_+} |f(t-\tau,\xi)|\,|\partial_\xi\Phi(y\pm\xi,\tau)-\partial_\xi\Phi(x\pm\xi,\tau)|\,d\xi d\tau \\ &\quad\quad+\int_{|x-y|^2}^t\int_{\mathbb{R}_+} |f(t-\tau,\xi)|\,|\partial_\xi\Phi(y\pm\xi,\tau)-\partial_\xi\Phi(x\pm\xi,\tau)|\,d\xi d\tau \\ &\quad\quad\leq 2\|f\|_\infty \int_0^{|x-y|^2}\int_{\mathbb{R}} |\partial_\xi\Phi(\xi,\tau)|\,d\xi d\tau + 2C\|f\|_\infty|x-y|\int_{|x-y|^2}^t \frac{1}{\tau}d\tau \\ &\quad\leq 2\|f\|_\infty|x-y| + 2C\|f\|_\infty|x-y|^\beta \int_{|x-y|^2}^t \tau^{-\frac{1+\beta}{2}}d\tau \\ &\quad\leq \tilde{C}\frac{\|f\|_\infty}{1-\beta}\max\left\{1,(t^*)^{(1-\beta)/2}\right\}|x-y|^\beta,\end{aligned}$$

where we also used that $|x-y|<1$. □

Finally, we will also use the following result, which can be found in [91].

Proposition D.2. *Let $E \in \{\mathbb{R},\mathbb{R}_\pm\}$. Let $u \in \mathcal{C}^{\alpha,1+\beta}_{t,y}([0,t^*]\times E)$. Then $\partial_y u$ is $\frac{\alpha\beta}{1-\beta}$-Hölder in time.*

Proof. We refer to Lemma 3.1, Chapter II of [91]. This proposition can be proved in an easier way using that for any $x,y\in E$ and any $t,s\in[0,t^*]$ we have the following estimates

$$u(t,y) = u(t,x)\partial_x u(t,x)(y-x)+\mathcal{O}(|x-y|^{1+\beta}) \text{ for any } t\in[0,t^*]$$

and

$$|u(t,x)-u(s,x)| \leq C|s-t|^\alpha \text{ for any } x\in E.$$

Thus,we conclude

$$|\partial_x u(t,x)-\partial_x u(s,x)| \leq C_1\frac{|t-s|^\alpha}{|x-y|}+C_2|x-y|^\beta \leq C|t-s|^{\frac{\alpha\beta}{1+\beta}}$$

choosing $|x-y| = |t-s|^{\frac{\alpha}{1+\beta}}$. □

We are now ready to prove the following Theorem.

Theorem D.4. *Let* $u_0 \in C^{0,1}(\mathbb{R})$ *be bounded with* $u_0(0) = 0$, $u_0(y) > 0$ *if* $y < 0$ *and* $u_0(y) < 0$ *if* $y > 0$. *Moreover, let* $\delta \in \left(0, \frac{1}{2}\right)$ *and* $u_0 \left|_{\mathbb{R}_\pm}\right. \in C^{2,\delta}(\mathbb{R}_\pm)$. *Then, for a time* $t^* > 0$ *small enough there exists a unique solution*

$$(u_1, u_2, s) \in \mathcal{C}^{1,2}_{t,y}\left((0,t^*) \times \mathbb{R}_-\right) \times \mathcal{C}^{1,2}_{t,y}\left((0,t^*) \times \mathbb{R}_+\right) \times C^1([0,t^*])$$

to the problem (D.11). *Moreover,*

$$(u_1, u_2, \dot{s}) \in \mathcal{C}^{\delta/2,1+\delta}_{t,y}\left((0,t^*) \times \mathbb{R}_-\right) \times \mathcal{C}^{\delta/2,1+\delta}_{t,y}\left((0,t^*) \times \mathbb{R}_+\right) \times C^{\delta/2}([0,t^*])$$

for $\delta < \frac{1}{2}$. *Furthermore, for any* $\varepsilon > 0$ *it is also true that* $u_1 \in \mathcal{C}^{1+\delta/2,2+\delta}_{t,y}([\varepsilon,t^*] \times \mathbb{R}_-)$ *as well as* $u_2 \in \mathcal{C}^{1+\delta/2,2+\delta}_{t,y}([\varepsilon,t^*] \times \mathbb{R}_+)$.

Proof. We have to show that the fixed-point solution (u_1, u_2, s) found in Theorem D.3 has the desired regularity. We already know that the interface $s(t) = \int_0^t \dot{s}(\tau)d\tau$ solves in a classical sense the equation

$$\dot{s}(t) = \frac{1}{L}\left(\partial_y u_2(t,0^+) - K\partial_y u_1(t,0^-)\right)$$

with initial value $s(0) = 0$. Moreover, $s \in C^1([0,t^*])$.

We will now show that $(u_1, u_2) \in \mathcal{C}^{1,2}_{t,y}\left((0,t^*) \times \mathbb{R}_-\right) \times \mathcal{C}^{1,2}_{t,y}\left((0,t^*) \times \mathbb{R}_+\right)$. This will imply that they solve the parabolic equations (D.31) strongly in $(0,t^*] \times \mathbb{R}_\pm$. For the fixed-point solution $(u_1, u_2, \dot{s})$ of Theorem D.3 we define the sources

$$F_1(t,y) = \dot{s}(t)\partial_y u_1(t,y) \text{ for } y < 0$$

and

$$F_2(t,y) = \dot{s}(t)\partial_y u_2(t,y) + I_\alpha[u_2 + T_M] \text{for } y > 0.$$

We will show that $F_i \in C^{\delta/2,\delta}([0,t^*],\mathbb{R}_\pm)$, this will imply the regularity of the functions u_1, u_2.

We first show that u_1, u_2 are $\frac{\alpha}{2}$-Hölder in time. To this end we define

$$\bar{u}_{0,1}(y) = \begin{cases} u_0(y) & y \leq 0 \\ -u_0(-y) & y > 0 \end{cases} \quad \text{and} \quad \bar{u}_{0,2}(y) = \begin{cases} u_0(y) & y \geq 0 \\ -u_0(-y) & y < 0 \end{cases}.$$

Since the continuous function u_0 satisfies $u_0 \in C^{2,\delta}(\mathbb{R}_\pm)$, then $\bar{u}_{0,i} \in C^{1,1}(\mathbb{R}_\pm)$. Hence, Lemma D.1 implies that $\bar{u}_{0,i} *_y \Phi \in \mathcal{C}^{1/2,1+1}_{t,y}([0,t^*] \times \mathbb{R}_\pm)$ as well as $\bar{u}_{0,i} *_y \Phi \in \mathcal{C}^{1+1/2,2+1}_{t,y}([\varepsilon,t^*] \times \mathbb{R}_\pm)$ for any $\varepsilon > 0$. We also know that $F_i \in \mathcal{C}^{0,0}_{t,y} \in ([0,t^*],\mathbb{R}_\pm)$. Therefore Lemma D.2 implies that $F_i * G \in \mathcal{C}^{\alpha/2,1+\beta}_{t,y}([0,t^*],\mathbb{R}_+)$ for any $\alpha, \beta \in (0,1)$. Thus,

$$u_i = u_{0,i} *_y \Phi + F_i * G \in \mathcal{C}^{\alpha/2,1+\beta}_{t,y}([0,t^*],\mathbb{R}_\pm) \text{ for any } \alpha, \beta \in (0,1).$$

This result has two important consequences. First of all, Proposition D.2 implies that the functions $\partial_y u_i$ are $\frac{\delta}{2}$-Hölder in time for some $\delta \in \left(0, \frac{1}{2}\right)$. Indeed, it is not difficult to see that for any $0 < \delta < \frac{1}{2}$ there exists $\alpha, \beta \in (0,1)$ such that $\frac{\alpha\beta}{1+\beta} = \delta$. This implies also that the derivatives $\partial_y u_i(t,0)$ are $\frac{\delta}{2}$-Hölder in time and thus by definition $\dot{s}(t) \in C^{\delta/2}([0,t^*])$. This yields further a better regularity for F_1, indeed $F_1 \in \mathcal{C}^{\delta/2,1+\delta}([0,t^*],\mathbb{R}_-)$.

A similar result can be shown also for F_2. We first of all remark that since u_2 is bounded, then $(u_2 + T_M) \in \mathcal{C}^{\alpha/2,1+\beta}_{t,y}([0,t^*],\mathbb{R}_+)$ for any $\alpha, \beta \in (0,1)$. Thus, we have to show that

$I_\alpha[u_2+T_M] \in \mathcal{C}^{\delta/2,\delta}([0,t^*]\times\mathbb{R}_+)$. Clearly, it is $\delta/2$-Hölder in time, since u_2 is so. For the space variable we use that $E_1 \in L^1(\mathbb{R})\cap L^2(\mathbb{R})$, hence by interpolation also $E_1 \in L^q(\mathbb{R})$ for any $q\in[1,2]$. Let $b>a>0$, then for any $\delta\in[0,1/2]$ we have

$$\int_a^b \frac{\alpha}{2}E_1(\alpha\eta)d\eta \le \frac{1}{2}|a-b|^\delta \|E_1\|_{L^{\frac{1}{1-\delta}}}.$$

Therefore, for $v\in C^{0,\delta}(\mathbb{R}_+)$ and $y>x>0$ we estimate

$$\begin{aligned}
&\left|\int_0^\infty \frac{\alpha}{2}v^4(\eta)\left(E_1(\alpha(y-\eta))-E_1(\alpha(x-\eta))\right)d\eta\right| \\
&\qquad= \left|\int_{-y}^\infty \frac{\alpha}{2}v^4(\eta+y)E_1(\alpha\eta)d\eta - \int_{-x}^\infty \frac{\alpha}{2}v^4(\eta+x)E_1(\alpha\eta)d\eta\right| \\
&\le \left|\int_{-x}^\infty \frac{\alpha}{2}E_1(\alpha\eta)\left(v^4(\eta+y)-v^4(\eta+x)\right)d\eta\right| + \left|\int_{-y}^{-x}\frac{\alpha}{2}E_1(\alpha\eta)v^4(\eta+y)d\eta\right| \\
&\le \|v^4\|_\delta|x-y|^\delta + \frac{1}{2}\|v^4\|_{C^0}\|E_1\|_{L^{\frac{1}{1-\delta}}}|x-y|^\delta. \qquad \text{(D.35)}
\end{aligned}$$

Hence, we can conclude that $I_\alpha[u_2+T_M]\in\mathcal{C}^{\delta/2,\delta}([0,t^*]\times\mathbb{R}_+)$ and consequently that $F_2\in\mathcal{C}^{\delta/2,\delta}([0,t^*]\times\mathbb{R}_+)$.

In order to prove finally that (u_1,u_2) is a classical solution to (D.31) we use that any bounded solution $w_i\in\mathcal{C}^{1,2}_{t,y}([0,t^*]\times\mathbb{R}_\pm)$ of the heat equation

$$\begin{cases}\partial_t w_i(t,y)-\partial_y^2 w_i(t,y)=F_i(t,y) & (0,t^*)\times\mathbb{R}_\pm\\ w_i(0,y)=u_0(y) & y\in\mathbb{R}_\pm\\ w_i(t,0)=0 & t\in[0,t^*]\end{cases} \qquad \text{(D.36)}$$

can be written both by

$$w_i=\bar{u}_{0,i}*_y\Phi+F_i*G$$

and by the sum $w_i=v_i+h_i$ of two functions v_i,h_i solutions to an inhomogeneous heat equation in the whole space and a homogeneous equation in the half-space, respectively.

Let us consider the even extensions of F_i

$$\bar{F}_1(t,y)=\begin{cases}F_1(t,y) & y\le 0\\ F_1(t,-y) & y>0\end{cases} \text{ and } \bar{F}_2(t,y)=\begin{cases}F_2(t,y) & y\ge 0\\ F_2(t,-y) & y<0.\end{cases}$$

Then, $\bar{F}_i\in\mathcal{C}^{\delta/2,\delta}([0,t^*],\mathbb{R})$. By Proposition D.1 and Lemma D.1 we see that

$$v_i(t,y):=\left(\bar{u}_{0,i}*_y\Phi\right)(t,y)+\left(\bar{F}_i*\Phi\right)(t,y)\in\mathcal{C}^{1+\delta/2,2+\delta}_{t,y}([\varepsilon,t^*]\times\mathbb{R})$$

for any $\varepsilon>0$. Thus, $v_i\in\mathcal{C}^{1,2}_{t,y}((0,t^*]\times\mathbb{R})$ is a strong solution to

$$\begin{cases}\partial_t v_i(t,y)-\partial_y^2 v_i(t,y)=\bar{F}_i(t,y) & (0,t^*)\times\mathbb{R}\\ v_i(0,y)=\bar{u}_{0,i}(y) & y\in\mathbb{R}\end{cases}$$

Moreover, since $v_i(t,0)=\left(\bar{F}_i*\Phi\right)(t,0)$ by Proposition D.1 we see that $v_i(t,0)\in\mathcal{C}^{1,\delta/2}_{t,y}([0,t^*])$. Thus, by Proposition D.1 we can conclude that

$$h_i(t,y):=2\left(\partial_y\Phi*_t v(\cdot,0)\right)(t,y)\in\mathcal{C}^{1+\delta/2,2+\delta}_{t,y}([\varepsilon,t^*]\times\mathbb{R}_\pm)$$

for all $\varepsilon > 0$. Using the property of the double-layer potential (cf. [91]) we obtain that $h_i \in \mathcal{C}^{1,2}_{t,y}((0,t^*),\mathbb{R}_\pm)$ is a strong solution to

$$\begin{cases} \partial_t h_i(t,y) - \partial_y^2 h_i(t,y) = 0 & (0,t^*)\times\mathbb{R}_\pm \\ h_i(0,y) = 0 & y \in \mathbb{R}_\pm \\ h_i(t,0) = -v_i(t,0) & t \in [0,t^*]. \end{cases}$$

Thus, it is not difficult to see that $w_i = v_i + h_i$ is a strong solution of (D.36) in the interior $(0,t^*)\times\mathbb{R}_\pm$. Moreover, w_i is the unique bounded solution of (D.36) and it has the same integral representation of u_i. Hence $u_i = w_i$ and using the regularity of v_i, h_i we conclude

$$u_i \in \mathcal{C}^{1,2}_{t,y}((0,t^*),\mathbb{R}_\pm) \text{ and } u_i \in \mathcal{C}^{1+\delta/2,2+\delta}_{t,y}([\varepsilon,t^*]\times\mathbb{R}_\pm) \text{ for all } \varepsilon > 0.$$

□

D.2.3 Maximum principle

The local solutions u_1, u_2 were obtained as $u_i = T_i - T_M$, where T_M is the melting temperature and T_1, T_2 are the solutions to (D.10). Since the temperature is a non-negative quantity, we want to show that the solutions T_i are non-negative as long as they are defined. Moreover, we expect that in the liquid, i.e. $y < 0$, the temperature satisfies $T_1 > T_M$ while in the solid ($y > 0$) the temperature satisfies $T_2 < T_M$. In bounded domains the maximum principle yields these results. Since we are working on an unbounded domain we will first consider some suitable problems in bounded domains, where the maximum principle assures the desired properties of the temperature, and then we will show that their solutions converge to u_1 and u_2, the solutions to the problem in the whole space.

For $R \geq 2$ we fix $\eta_R \in C^\infty(\mathbb{R})$ with the property that $\eta_R \equiv 1$ if $|y| \leq R-1$ and $\eta_R \equiv 0$ if $|y| \geq R$ and $|\eta| \leq 1$. Moreover, we choose η_R with $\|\eta_R'\|_\infty \leq 2$ as well as $\max\{\|\eta_R''\|_\infty, \|\eta_R'''\|_\infty\} \leq C$ for a fixed constant $C > 0$ independent of R. We will consider u_1^R and u_2^R solutions to

$$\begin{cases} \partial_t u_1^R - \dot{s}\partial_y u_1^R = \kappa\partial_y^2 u_1^R & \\ u_1^R(t,0) = 0 & (0,t_*)\times(-R,0) \\ u_1^R(t,-R) = 0 & \\ u_1^R(0,y) = u_0(y)\eta_R(y) & \end{cases} \text{ and } \begin{cases} \partial_t u_2^R - \dot{s}\partial_y u_2^R = \partial_y^2 u_2^R - I_\alpha^R[u_2^R + T_M] & \\ u_2^R(t,0) = 0 & (0,t_*)\times(0,R) \\ u_2^R(t,R) = 0 & \\ u_2^R(0,y) = u_0(y)\eta_R(y) & \end{cases} \tag{D.37}$$

$I_\alpha^R[v](t,y) = v^4(t,y) - \int_0^R \frac{\alpha}{2}E_1(\alpha(y-\eta))v^4(t,\eta)d\eta$ and $\dot{s}$ is the time derivative of the moving interface s, which is together with u_1, u_2 of Theorem D.4 the unique solution of problem (D.11) for $t \in (0,t_*)$. We will show the following Lemma.

Lemma D.3. *Let $R \geq 2$. Let u_0 be as in Theorem D.4 and let η_R as above. Then there exist a time $0 < t_* \leq t^*$ small enough and independent of R such that there exist unique solutions $u_1^R \in \mathcal{C}^{1,2}_{t,y}((0,t_*)\times[-R,0])$ and $u_2^R \in \mathcal{C}^{1,2}_{t,y}((0,t_*)\times[0,R])$ to* (D.37) *which satisfy the Hölder regularity*

$$u_1^R \in \mathcal{C}^{\delta/2,1+\delta}_{t,y}([0,t_*]\times[-R,0]) \text{ and } u_1^R \in \mathcal{C}^{\delta/2,1+\delta}_{t,y}([0,t_*]\times[0,R])$$

with uniformly bounded Hölder norms.

Let also u_i be the solutions to (D.11) *of Theorem D.4, then there exist two subsequences $u_i^{R_n}$ which converge to u_i as $n\to\infty$ uniformly in every compact set and pointwise everywhere.*

Proof. We consider the Green's function for the heat equation on the interval $[-R,0]$ and $[0,R]$ given by

$$\tilde{G}_R(y,\xi,at)=\sum_{n\in\mathbb{Z}}\Phi(y-\xi-2nR,at)-\Phi(y+\xi-2nR,at).$$

Let $F_1^R(t,y)=\dot{s}(t)\partial_y u_1^R(y)$ and $F_2^R(t,y)=\dot{s}(t)\partial_y u_1^R(t,y)-I_\alpha^R[U_2^R+T_M](t,y)$. We obtain the following fixed-point representations for the solutions to (D.37)

$$\begin{aligned}u_1^R(t,y)=&\mathcal{L}_1^R(u_1^R)(t,y)\\ =&\int_{-R}^0 u_0(\xi)\eta_R(\xi)\tilde{G}_R(y,\xi,\kappa t)d\xi+\int_{-R}^0\int_0^t F_1^R(\tau,\xi)\tilde{G}_R(y,\xi,\kappa(t-\tau))d\tau d\xi,\end{aligned}\tag{D.38}$$

and

$$u_2^R(t,y)=\mathcal{L}_2^R(u_2^R)(t,y)=\int_0^R u_0(\xi)\eta_R(\xi)\tilde{G}_R(y,\xi,t)d\xi+\int_0^R\int_0^t F_2^R(\tau,\xi)\tilde{G}_R(y,\xi,(t-\tau))d\tau d\xi.\tag{D.39}$$

We will prove the existence of a unique fixed-point for the operators defined by $\mathcal{L}_1^R:\mathcal{A}_1^R\to\mathcal{C}_{t,y}^{0,1}([0,t_*]\times[0,R])$ and $\mathcal{L}_2^R:\mathcal{A}_2^R\to\mathcal{C}_{t,y}^{0,1}([0,t_*]\times[-R,0])$ for all $R\geq 2$, where

$$\mathcal{A}_1^R=\left\{u\in\mathcal{C}_{t,y}^{0,1}([0,t_*]\times[-R,0]):\ \|u\|_{0,1}\leq C_1\right\}$$

and

$$\mathcal{A}_2^R=\left\{u\in\mathcal{C}_{t,y}^{0,1}([0,t_*]\times[0,R]):\ \|u\|_{0,1}\leq C_2\right\}$$

for constants $(1-\theta)C_i>4\|u_0\|_1$ for a fixed $\theta\in(0,1)$.

First of all we see that it is possible to extend in an odd manner $u_0\eta_R$ to the whole real line. Indeed, we can consider as usual the odd extension of the initial value as

$$\tilde{u}_{0,1}^R(y)=\begin{cases}u_0(y)\eta_R(y) & -R\leq y\leq 0\\ -u_0(-y)\eta_R(-y) & 0<y\leq R\end{cases}\ \text{and}\ \tilde{u}_{0,2}^R(y)=\begin{cases}u_0(y)\eta_R(y) & 0\leq y\leq R\\ -u_0(-y)\eta_R(-y) & -R\leq y<0.\end{cases}$$

We define

$$\bar{u}_{0,1}^R(y)=\bar{u}_{0,1}^R(y+2nR)\ \text{and}\ \tilde{u}_{0,2}^R(y)=\bar{u}_{0,2}^R(y+2nR)\ \text{for}\ y\in[-(2n+1)R,(-2n+1)R]$$

One can easily see then that

$$\left(\bar{u}_{0,1}^R*\Phi(\cdot,\kappa\cdot)\right)(t,y)=\int_{-R}^0 u_0(\xi)\eta_R(\xi)\tilde{G}_R(y,\xi,\kappa t)d\xi$$

and

$$\left(\bar{u}_{0,2}^R*\Phi(\cdot,\cdot)\right)(t,y)=\int_0^R u_0(\xi)\eta_R(\xi)\tilde{G}_R(y,\xi,t)d\xi.$$

Similarly, it is not difficult to see that with a change of coordinate

$$\int_0^{\pm R}\left|\tilde{G}_R(y,\xi,at)\right|d\xi\leq\sum_{n\in\mathbb{Z}}\int_{2nR}^{(2n+1)R}|\Phi(y-\xi,at)|+|\Phi(y+\xi,at)|\,d\xi\leq 2\int_{\mathbb{R}}|\Phi(\xi,at)|d\xi\leq 2.\tag{D.40}$$

The same estimate holds for the integral in $[-R,0]$. In the same way we have also

$$\int_0^{\pm R} \left|\partial_y \tilde{G}_R(y,\xi,at)\right| d\xi \leq \sum_{n\in\mathbb{Z}} \int_{2nR}^{(2n+1)R} |\partial_\xi \Phi(y-\xi,at)| + |\partial_\xi \Phi(y+\xi,at)|\, d\xi$$
$$\leq 2\int_{\mathbb{R}} |\partial_\xi \Phi(\xi,at)| d\xi \leq \frac{2\sqrt{2}}{\sqrt{at}}. \quad \text{(D.41)}$$

Thus, for $u_i, v_i \in \mathcal{A}_i^R$ we have

$$\begin{aligned} \|\mathcal{L}_1^R(u_1)\|_{0,1} \leq & 4\|u_0\|_1 + 2\|\dot{s}\|_\infty C_1 \left(t_* + \frac{\sqrt{2t_*}}{\sqrt{\kappa}}\right) \quad \text{and} \\ \|\mathcal{L}_1^R(u_2)\|_{0,1} \leq & 4\|u_0\|_1 + 2\left(\|\dot{s}\|_\infty C_2 + 2\left(C_2+T_M\right)^4\right)\left(t_* + \sqrt{2t_*}\right) \end{aligned} \quad \text{(D.42)}$$

and

$$\|\mathcal{L}_1^R(u_1) - \mathcal{L}_1^R(v_1)\|_{0,1} \leq 2\|\dot{s}\|_\infty \|u_1 - v_1\|_{0,1} \left(t_* + \frac{\sqrt{2\bar{t}}}{\sqrt{\kappa}}\right) \quad \text{(D.43)}$$

as well as

$$\|\mathcal{L}_1^R(u_2) - \mathcal{L}_1^R(v_2)\|_{0,1} \leq 2\|u_2 - v_2\|_{0,1}\left(\|\dot{s}\|_\infty + 16\left(C_2^3 + T_M^3\right)\right)\left(t_* + \sqrt{2\bar{t}}\right). \quad \text{(D.44)}$$

Since the estimates (D.42), (D.43) and (D.44) do not depend on R, there exists $0 < t_* \leq t^*$ independent of R and small enough such that by the Banach fixed-point Theorem there exist unique fixed-points $u_1^R \in \mathcal{C}_{t,y}^{0,1}([0,t_*]\times[-R,0])$ and $u_2^R \in \mathcal{C}_{t,y}^{0,1}([0,t_*]\times[0,R])$ of the operators $\mathcal{L}_1^R$ and $\mathcal{L}_2^R$ of (D.38) and (D.39), respectively.

Adapting Lemma D.2 to the Green's function $\tilde{G}_R$ one can prove that for any function $f_\pm \in \mathcal{C}_{t,y}^{0,0}([0,t_*],[0,\pm R])$ the convolution $f_\pm * \tilde{G}_R \in \mathcal{C}_{t,y}^{\alpha/2,1+\beta}([0,t_*],[0,\pm R])$ for any $\alpha,\beta \in (0,1)$. We omit the proof since it an easy calculation based on the proof of Lemma D.2 and on a suitable change of variables as in estimates (D.40) and (D.41). This result together with Lemma D.1 applied to the odd extensions $\bar{u}_{0,i}^R$ for $i=1,2$ implies the desired Hölder regularity

$$u_1^R \in \mathcal{C}_{t,y}^{\delta/2,1+\delta}([0,t_*]\times[-R,0]) \text{ and } u_2^R \in \mathcal{C}_{t,y}^{\delta/2,1+\delta}([0,t_*]\times[0,R])$$

for any $\delta \in (0,1)$. Moreover, the Hölder norms are uniformly bounded in R. Indeed, $\|F_1^R\|_\infty \leq \|C_1\dot{s}\|_\infty$ and $\|F_2^R\|_\infty \leq \|C_2\dot{s}\|_\infty + 2(C_2+T_M)^4$ and hence Lemma D.1 and Lemma D.2 yield

$$\|u_i^R\|_{\delta/2,1+\delta} \leq \max\left\{C_i, \|u_0\|_1, C\left(t_*, t_*^{(1-\delta)/2}\right)\frac{\|F_i^R\|_\infty}{1-\delta}\right\} \quad \text{for } i=1,2,$$

where $C\left(t_*, t_*^{(1-\delta)/2}\right) > 0$ does not depend on R.

Before proving that the functions u_i^R are also classical solutions to (D.37), we prove the convergence result. Let us extend u_i^R for $|y|>R$ by a function $\bar{u}_i^R \in \mathcal{C}_{t,y}^{\delta/2,1+\delta}([0,t_*],\mathbb{R}_\pm)$ with norm $\|\bar{u}_i^R\|_{\delta/2,1+\delta} \leq 2\|u_i^R\|_{\delta/2,1+\delta}$. Then the uniform boundedness of the Hölder norm implies that $\bar{u}_i^R$ is a compact sequence in $\mathcal{C}_{t,y}^{\alpha/2,1+\alpha}([0,t_*],[a,b])$ for every compact set $[a,b] \in \mathbb{R}_\pm$ and for $\alpha < \delta$. Therefore, for any sequence $R_n \to \infty$ a diagonal argument yields the existence of a subsequence R_{n_k}, which we will denote for simplicity by R_n, and of a function $\bar{u}_i \in \mathcal{C}_{t,y}^{0,1}([0,t_*]\times\mathbb{R}_\pm) \cap C_{\text{loc}}^{\alpha/2,1+\alpha}([0,t_*]\times\mathbb{R}_\pm)$ such that

$$\bar{u}_i^{R_n} \to \bar{u}_i \text{ and } \partial_y \bar{u}_i^{R_n} \to \partial_y \bar{u}_i \text{ as } n \to \infty$$

uniformly in every compact set and pointwise everywhere.

We show now $\bar{u}_i = u_i$. It is not difficult to see that $u_0\eta_{R_n} \to u_0$ pointwise for $y \in \mathbb{R}$. Notice that also $u_i^{R_n}(y) \to \bar{u}_i(y)$ and $\partial_y u_i^{R_n}(y) \to \partial_y \bar{u}_i(y)$ for $R_n > |y|$ and $n \to \infty$, and that

$$\left| \int_0^{R_n} \frac{\alpha}{2} E_1(\alpha(\eta-\xi)) \left(u_2^{R_n} + T_M\right)^4 d\eta - \int_0^\infty \frac{\alpha}{2} E_1(\alpha(\eta-\xi)) \left(\bar{u}_2 + T_M\right)^4 d\eta \right| \to 0 \text{ as } n \to \infty.$$

Hence, we see that

$$F_1^{R_n}(t,y) \to \bar{F}_1(t,y) = \dot{s}(t)\partial_y \bar{u}_1(t,y) \text{ for } y<0 \text{ and } n \to \infty$$

and

$$F_2^{R_n}(t,y) \to \bar{F}_2(t,y) = \dot{s}(t)\partial_y \bar{u}_2(t,y) + I_\alpha[\bar{u}_2 + T_M] \text{for } y>0 \text{ and } n \to \infty.$$

Using that

$$\tilde{G}_{R_n}(y,\xi,at) = G(y,\xi,at) + \sum_{|n|\geq 1} G(y-2nR,\xi,at) \to G(y,\xi,at) \text{ as } n \to \infty$$

we can conclude using Lebesgue dominated convergence theorem that the functions $\bar{u}_i$ solve the integral equations

$$\bar{u}_i(t,y) = \int_{\mathbb{R}_\pm} u_0(\xi) G(y,\xi,a_i t) d\xi + \int_0^t \int_{\mathbb{R}_\pm} \bar{F}_i(\tau,\xi) G(y,\xi,a_i(t-\tau)) d\tau d\xi, \tag{D.45}$$

where $a_1 = \kappa$ and $a_2 = 0$. An easy application of Banach fixed-point theorem shows that (D.45) has a unique fixed-point for bounded functions in $\mathcal{C}^{0,1}_{t,y}\left([0,t_*],\mathbb{R}_\pm\right)$ with bounded derivative. Thus, since u_i solves also (D.45), we can conclude that $\bar{u}_i = u_i$.

Finally, we prove that the functions u_i^R are classical solutions to the heat equations (D.37). Clearly, by Lemma D.1 we have that $\bar{u}^R_{0,i} * \Phi \in \mathcal{C}^{1+\delta/2,2+\delta}_{t,y}([\varepsilon,t_*]\times[0,\pm R])$. We need to prove the differentiability of $F_i^R * \tilde{G}_R$. First of all we notice that because of $\partial_t \tilde{G}_R(y,\xi,at) = a\partial_y^2 \tilde{G}_R(y,\xi,at)$ we only need to show that there exists the second spatial derivative since also

$$\lim_{\varepsilon\to 0} \int_0^{\pm R} F_i^R(t-\varepsilon,\xi)\tilde{G}_R(y,\xi,a\varepsilon)d\xi = \lim_{\varepsilon\to 0}\int_0^{\pm R} F_i^R(t-\varepsilon,\xi)\Phi(y-\xi,a\varepsilon)d\xi = F_i^R(t,y)$$

for any $y \in (-R,0)$ or $y \in (0,R)$. Thus, we compute using the change of coordinates as in (D.40) and (D.41)

$$\begin{aligned}
&\left|\int_0^t \int_0^{\pm R} F_i^R(t-\tau,\xi)\partial_y^2\tilde{G}_R(y,\xi,a\tau)d\xi d\tau\right| \\
&\quad\leq \left|\int_0^t \int_0^{\pm R} \left(F_i^R(t-\tau,\xi) - F_i^R(t-\tau,y)\right)\partial_y^2 \tilde{G}_R(y,\xi,a\tau)d\xi d\tau\right| \\
&\quad\quad+ \left|\int_0^t F_i^R(t-\tau,y)\int_0^{\pm R}\partial_y^2\tilde{G}_R(y,\xi,a\tau)d\xi d\tau\right| \\
&\quad\leq 2\|F_i^R\|_\delta \int_0^t \int_{\mathbb{R}} |y-\xi|^\delta \left|\partial_\xi^2 \Phi(\xi,a\tau)\right| + \left|\int_0^t F_i^R(t-\tau,y)\int_0^{\pm R}\partial_\xi^2 \tilde{G}_R(y,\xi,a\tau)d\xi d\tau\right| < \infty.
\end{aligned}$$

We remark that we obtain the first term since $|y-(\eta+2nR)| \leq \min\{|y-\eta|,|y+\eta|\}$ for any $\eta \in [-2nR,(-2n+1)R]$. Moreover, the boundedness of the first term is a well-known property of the fundamental solution of the heat equation. The boundedness of the second term is a classical result in parabolic theory which combines the fact that

$$\left|\int_0^t F_i^R(t-\tau,y)\int_0^{\pm R}\partial_\xi^2\Phi(y\pm\xi-2nR,a\tau)d\xi d\tau\right|$$
$$=\left|\int_0^t F_i^R(t-\tau,y)\left(\partial_\xi\Phi(y\pm R-2nR,a\tau)-\partial_\xi\Phi(y-2nR,a\tau)\right)d\xi d\tau\right|\leq C\|F_i^R\|_\infty$$

and the fact that the tail of the series $\int_0^t\int_0^{\pm R}\left|\partial_\xi^2\tilde{G}_R(y,\xi,a\tau)\right|d\xi d\tau$ converges. Finally, the Hölder continuity of F_i^R is due to the Hölder regularity of u_i^R and of $\dot{s}$ as well as to the convolution with the exponential integral as in (D.35). Thus, we conclude that $u_1^R\in\mathcal{C}_{t,y}^{1,2}((0,t_*)\times[-R,0])$ and $u_1^R\in\mathcal{C}_{t,y}^{1,2}((0,t_*)\times[0,R])$ are classical solutions of (D.37). □

This approximation result will be used in order to show that $u_1>0$ as well as $0<u_2<-T_M$. Before applying the maximum principle though, we need to show that the maximal interval of existence of the solutions for the original equation (D.11) can be approximated by the one of the solutions to (D.37). This is due to the uniform convergence in compact domain of any subsequence of solutions to (D.37). Thus, the norms of the convergent sequence are uniformly bounded in time.

Lemma D.4. *Let $[0,t^*]$ be the maximal interval of existence for the solution $(u_1,u_2,\dot{s})$ to the problem (D.11). For any $\varepsilon>0$ there exists a sequence $\left(u_1^{R_n^\varepsilon},u_2^{R_n^\varepsilon}\right)$ solving (D.37) on $[0,t^*-\varepsilon]$ with*

$$u_i^{R_n^\varepsilon}\to u_i \text{ as } n\to\infty$$

uniformly in every compact set and for $i=1,2$.

Proof. We argue by contradiction. For any sequence $\{R_n\}_{n\in\mathbb{N}}$ with $R_n\to\infty$ as $n\to\infty$ we define the maximal time of existence of the sequence $u_i^{R_n}$ by

$$t_*(R_n):=\sup\left\{t_*>0: u_i^{R_n}\text{ exists in }[0,t_*],\text{ for all }n\text{ and }i=1,2\right\}.$$

By the convergence result of Lemma D.3 we know that $t_*(R_n)\leq t^*$. Hence we consider

$$\bar{t}:=\sup\{t_*(R_n):R_n\text{ is an increasing diverging sequence}\}\leq t^*.$$

If $\bar{t}=t^*$ then Lemma D.4 is proved. Indeed, by the definition of $\bar{t}$ for any $\varepsilon>0$ there exists an increasing diverging sequence R_n^ε such that $t_*(R_n^\varepsilon)>\bar{t}-\varepsilon=t^*-\varepsilon$. Thus, taking a suitable subsequence we conclude the lemma.

Let us assume that $\bar{t}<t^*$. Let also $\delta>0$ and let R_n^δ be an increasing sequence such that $t_*\left(R_n^\delta\right)>\bar{t}-\delta$. By Lemma D.3 there exists a subsequence $R_{n_k}^\delta$ such that $u_i^{R_{n_k}^\delta}\to u_i$ uniformly in every compact set. We hence see by the convergence result that the norms of $u_i^{R_{n_k}^\delta}$ are uniformly bounded on $[0,t_*\left(R_n^\delta\right)]$ by the (bounded) norm of u_i on the larger interval time $[0,\bar{t}]$. Thus, the solutions $u_i^{R_{n_k}^\delta}$ can be extended for larger times so that

$$t_*\left(R_{n_k}^\delta\right)\geq t_*\left(R_n^\delta\right)+\theta(\bar{t})>\bar{t}-\delta+\theta(\bar{t}),$$

where $\theta(\bar{t})>0$ depends on the norm of u_i on $[0,\bar{t}]$ and not on δ. Since δ is arbitrary we obtain the contradiction $\bar{t}\geq\bar{t}+\theta(\bar{t})$. This concludes the proof of this lemma. □

We can now prove with the maximum principle the following proposition.

Proposition D.3 (Properties of the solution). *Let $u_0 \in C_b^0(\mathbb{R})$ be as in the assumptions of Theorem D.4. Let $(u_1, u_2, \dot{s})$ be the local solution to* (D.11) *of Theorem D.4 for $t \in [0, t^*]$, which is the maximal interval of existence. Then $u_1 \geq 0$ and $-T_M \leq u_2 \leq 0$. Moreover, $u_1(t, y) > 0$ for all $y \in (-a, b) \subset (-\infty, 0]$ and $-T_M < u_2(t, y) < 0$ for all $y \in (-a, b) \subset [0, \infty)$.*

Proof. As we have seen in Lemma D.4 for any $\varepsilon > 0$ there exists an increasing diverging sequence $\{R_n^\varepsilon\}_n$ such that the solutions $u_i^{R_n^\varepsilon}$ of (D.37) exist on the interval $[0, t^* - \varepsilon]$ and converge to the solutions u_i uniformly in every compact set.

First of all, for any $R > 0$ we apply the classical maximum principle to the functions u_1^R and u_2^R solving the parabolic problem (D.37) on the bounded domains $[0, t_*(R)] \times [-R, 0]$ and $[0, t_*(R)] \times [0, R]$. Where for the sake of readability we denote $t_* = t_*(R)$ the maximal time of existence for the solutions u_1^R, u_2^R.

Let us first consider u_1^R. Then, since $u_0(y)\eta_R(y) > 0$ for all $y \in (-R, 0)$ and $u_1^R(t, 0) = u_1^R(t-, R) = 0$ for all $t \in (0, t_*)$, the strong maximum principle for the parabolic equation solved by u_1^R implies that the minimum is attained only at the parabolic boundary, i.e. $u_1^R(t, y) > 0$ for all $(t, y) \in (0, t_*] \times (-R, 0)$. Let now $\varepsilon > 0$. Using Lemma D.4 and the pointwise convergence result of Lemma D.3 we obtain $u_1(t, y) \geq 0$ for all $(t, y) \in [0, t^* - \varepsilon] \times \mathbb{R}_-$. Thus, since $\varepsilon > 0$ is arbitrary, we conclude $u_1(t, y) \geq 0$ in $[0, t^*] \times \mathbb{R}_-$. Let us now consider $(a, b) \in \mathbb{R}_-$. On one hand by assumption we know that $u_0(y) > 0$ for all $y \in (a, b)$, on the other hand we have just seen that $u_1(t, a), u_1(t, b) \geq 0$. Applying once more the strong maximum principle to the parabolic equation solved by solution u_1 on $(0, t^*) \times (a, b)$ we can conclude that also $u_1(t, y) > 0$ for all $y \in (a, b)$.

We now pass to the analysis of u_2^R. Let us assume that $u_2^R(t, y) \leq -T_M$ for some $(t, y) \in [0, t_*] \times [0, R]$. Then, since $u_0(y)\eta_R(y) > -T_M$, there exists a $t_0 \in (0, t_*]$, the first time such that a $y_0 \in (0, R)$ exist with $u_2^R(t_0, y_0) = -T_M$. Hence, $u_2^R(t, y) > -T_M$ for all $0 \leq t < t_0$ and $y \in [0, R]$. This implies $\partial_t u_2^R(t_0, y_0) \leq 0$, $\partial_y u_2^R(t_0, y_0) = 0$ and also $\partial_y^2 u_2^R(t_0, y_0) \geq 0$. Moreover, on the one hand $\left(u_2^R(t_0, y_0) + T_M\right)^4 = 0$ and on the other hand there exists an interval $(0, y_1) \in (0, R)$ such that $u_2^R(t_0, y) > -T_M$. Hence,

$$
\begin{aligned}
0 = \partial_t u_2^R(t_0, y_0) - \dot{s}(t_0)\partial_y u_2^R(t_0, y_0) - \partial_y^2 u_2^R(t_0, y_0) + I_\alpha^R[u_2^R + T_M](t_0, y_0) \\
\leq -\int_0^R \frac{\alpha}{2} E_1(\alpha(y - \eta)) \left(u_2^R(t_0, y) + T_M\right)^4 d\eta < 0.
\end{aligned}
$$

This contradiction implies $u_2^R(t, y) > -T_M$ for all $[0, t_*] \times [0, R]$.

Let now $(t_0, y_0) \in [0, t_*] \times [0, R]$ be such that $\max\limits_{[0,t_*]\times[0,R]} u_2^R(t, y) = u_2^R(t_0, y_0)$. Assume first that $(t_0, y_0) \in (0, t_*] \times (0, R)$. Then, $\partial_t u_2^R(t_0, y_0) \geq 0$, $\partial_y u_2^R(t_0, y_0) = 0$ and $\partial_2 u_2^R(t_0, y_0) \leq 0$. Moreover, since $u_2^R > -T_M$ we have also $\left(u_2^R(t_0, y_0) + T_M\right)^4 \geq \left(u_2^R(t, y) + T_M\right)^4$.

This implies

$$\begin{aligned}
0 &= \partial_t u_2^R(t_0,y_0) - \dot{s}(t_0)\partial_y u_2^R(t_0,y_0) - \partial_y^2 u_2^R(t_0,y_0) + I_\alpha^R[u_2^R + T_M](t_0,y_0) \\
&\geq \left(\int_{-\infty}^0 \frac{\alpha}{2}E_1(\alpha(y-\eta))d\eta + \int_R^\infty \frac{\alpha}{2}E_1(\alpha(y-\eta))d\eta\right)\left(u_2^R(t_0,y_0)+T_M\right)^4 \\
&\quad + \int_R^0 \frac{\alpha}{2}E_1(\alpha(y-\eta))\left[\left(u_2^R(t_0,y_0)+T_M\right)^4 - \left(u_2^R(t_0,y)+T_M\right)^4\right]d\eta \\
&\geq \left(\int_{-\infty}^0 \frac{\alpha}{2}E_1(\alpha(y-\eta))d\eta + \int_R^\infty \frac{\alpha}{2}E_1(\alpha(y-\eta))d\eta\right)\left(u_2^R(t_0,y_0)+T_M\right)^4 > 0.
\end{aligned}$$

This contradiction yields that the maximum is attained at the parabolic boundary, i.e.

$$u_2^R(t,y) < \max\{u_0(y)\eta_R(y),0\} = 0 \quad \text{for all} \quad (t,y)\in(0,t_*]\times(0,R).$$

by the initial condition $u_2^R(0,y) = \eta_R(y)u_0(y) < 0$ for all $y \in (0,R)$ we conclude that $u_2^R(t,y) < 0$ for all $(t,y) \in (0,t_*]\times(0,R)$. Let $\varepsilon > 0$. Using once more Lemma D.4 and the pointwise convergence result of Lemma D.3 we obtain $-T_M \leq u_2(t,y) \leq 0$ for all $(t,y) \in [0,t^*-\varepsilon]\times\mathbb{R}_+$. The erbitrary choice of $\varepsilon > 0$ implies again that $-T_M \leq u_2(t,y) \leq 0$ for all $(t,y) \in [0,t^*]\times\mathbb{R}_+$. To prove that also $-T_M < u_2(t,y) < 0$ for all $y \in (a,b) \subset [0,\infty)$ we apply the maximum principle to the function u_2 again. Let $R > b$ and assume that $\min\limits_{[0,t^*]\times[0,R]} u_2 = -T_M$. Since $u_0(y) > -T_M$ for all $y \in [0,R]$ and $u_2(t,0) = 0$ and $u_2(t,R) \geq -T_M$, there exists $t_0 > 0$ the first time for which there exists some $y_0 \in (0,R]$ such that $u_2(t_0,y_0) = -T_M$. Hence,

$$\begin{aligned}
0 &= \partial_t u_2(t_0,y_0) - \dot{s}(t_0)\partial_y u_2(t_0,y_0) - \partial_y^2 u_2(t_0,y_0) + I_\alpha[u_2+T_M](t_0,y_0) \\
&\leq -\int_0^\infty \frac{\alpha}{2}E_1(\alpha(y-\eta))\left(u_2(t_0,y)+T_M\right)^4 d\eta < 0,
\end{aligned}$$

where we used that $u_2 \geq -T_M$ and that the strict inequality holds in a set of positive measure. This contradiction implies that $u_2(t,y) > -T_M$ for all $y \in (a,b)$. Let us assume now that $\max\limits_{[0,t^*]\times[a,b]} u_2 = u_2(t_0,y_0) = 0$ for a $(t_0,y_0) \in (0,t^*]\times(a,b)$. Since $u_2(t,a), u_2(t,b) \leq 0$ for $t > 0$ we see that

$$\begin{aligned}
0 &= \partial_t u_2(t_0,y_0) - \dot{s}(t_0)\partial_y u_2(t_0,y_0) - \partial_y^2 u_2(t_0,y_0) + I_\alpha[u_2+T_M](t_0,y_0) \\
&\geq T_M^4 - \int_0^\infty \frac{\alpha}{2}E_1(\alpha(y-\eta))\left(u_2(t_0,y)+T_M\right)^4 d\eta \geq T_M^4\left(1 - \int_{-b}^\infty \frac{\alpha}{2}E_1(\alpha\eta)d\eta\right) > 0,
\end{aligned}$$

where we also used $y_0 < b$. Thus, since also $u_0(y) < 0$ for $y \in (a,b)$ we conclude that $u_2(t,y) < 0$ for all $y \in (a,b)$.

□

D.3 Global well-posedness

In this section we will show that for a class of initial data, the system (D.11) has a unique global solution in time. Our aim is to construct a function $w \in C^{0,1}(\mathbb{R})$ twice continuously differentiable in $\mathbb{R}_\pm$ such that $w(0) = 0$ and such that $u_1 \leq w$ on $\mathbb{R}_-$ and $u_2 \geq w$ on $\mathbb{R}_+$. This would imply global well-posedness. Instead of considering the shifted temperature u_i we will now study the original temperature $T_i = u_i + T_M$.

Theorem D.5 (Global Well-posedness)**.** *Let $T_0 \in C^{0,1}(\mathbb{R})$ with $T_0 |_{\mathbb{R}_\pm} \in C^{2,\delta}(\mathbb{R}_\pm)$. Let also $T_0(0) = T_M$, $T_M < T_0(y)$ for $y < 0$ and $0 < T_0(y) < T_M$ for $y > 0$. Then, if in addition $\sup\limits_{\mathbb{R}_-} T_0 < T_M + \frac{\kappa L^2}{K T_M}$ and $\inf\limits_{\mathbb{R}_+} T_0 > 0$, there is a unique global solution (T_1, T_2, s) of* (D.10)*.*

Theorem D.5 will be proved applying in a suitable way the maximum principle. We will also need the following Lemma, which will be proved at the end of this section.

Lemma D.5. *Let T_0 be as in the assumption of Theorem D.5. Let $w \in C^{0,1}(\mathbb{R})$ be defined by*

$$w(y) = \begin{cases} T_M - \frac{\alpha\kappa}{C_1}\left(1 - \exp\left(\frac{C_1}{\kappa} y\right)\right) & \text{for} \quad y < 0 \\ T_M & \text{for } y = 0 \\ w(y) = T_M e^{-C_2 y}\left(1 - \frac{T_M^3}{12 C_2^2} + \frac{T_M^3 e^{-3C_2 y}}{12 C_2^2}\right) & \text{for } y > 0. \end{cases} \tag{D.46}$$

Then there exist $C_1, C_2 > 0$ satisfying $C_2 > \frac{T_M^{3/2}}{2\sqrt{3}}$, $C_1 > \left(\frac{T_M^5 + 1}{L^2}\right)^{1/2}$ with $\Gamma^-(C_1) < C_2 < \Gamma^+(C_1)$, where $\Gamma^\pm(C_1) = \frac{L C_1 \pm \sqrt{L^2 C_1^2 - T_M^5}}{2 T_M}$, and $-\frac{L}{K} C_2 < \alpha < 0$ such that $T_0 < w$ on $\mathbb{R}_-$, $T_0 > w$ on $\mathbb{R}_+$ and $\sup\limits_{\mathbb{R}_\pm} |\partial_y T_0(y)| < |\partial_y w(0^\pm)|$. Moreover, for any $R > 0$ there exists $a > 0$ such that $|T_0(y) - w(y)| > a$ for all $|y| > R$.

We prove now Theorem D.5 with the help of Lemma D.5

Proof of Theorem D.5. Let $C_1, C_2 > 0$ satisfying $C_2 > \frac{T_M^{3/2}}{2\sqrt{3}}$, $C_1 > \left(\frac{T_M^5 + 1}{L^2}\right)^{1/2}$ such that $\Gamma^-(C_1) < C_2 < \Gamma^+(C_1)$, where $\Gamma^\pm(C_1) = \frac{L C_1 \pm \sqrt{L^2 C_1^2 - T_M^5}}{2 T_M}$. Let us also consider a solution w to

$$\begin{cases} \kappa \partial_y^2 w - C_1 \partial_y w = 0 & y < 0 \\ \partial_y^2 w + C_2 \partial_y w \geq w^4 & y > 0 \\ w(0) = T_M \\ \partial_y w(0^-) > -\frac{L}{K} C_2 \\ \partial_y w(0^+) > -L C_1 \\ w \geq 0 \end{cases}$$

A simple ODE argument, solving the first equation for $v = w'$ and integrating, shows that on the negative real line w is given by

$$w(y) = T_M - \frac{\alpha\kappa}{C_1}\left(1 - \exp\left(\frac{C_1}{\kappa} y\right)\right) \quad \text{for} \quad y < 0,$$

for some $\alpha \in \mathbb{R}$ with $-\frac{L}{K} C_2 < \alpha < 0$. Hence, $\partial_y w(y) = \alpha e^{C_1/\kappa} < 0$

For $y > 0$ we consider the function

$$w(y) = T_M e^{-C_2 y}\left(1 - \frac{T_M^3}{12 C_2^2} + \frac{T_M^3 e^{-3C_2 y}}{12 C_2^2}\right).$$

We see that $w \leq T_M e^{-C_2 y}$ as well as $w \geq 0$, since $C_2 > \frac{T_M^{3/2}}{2\sqrt{3}}$. Moreover, the function w satisfies

$$\begin{aligned} &w''(y) + C_2 w'(y) \\ &\quad = C_2^2 T_M e^{-C_2 y}\left(1 - \frac{T_M^3}{12 C_2^2}\right) + \frac{4}{3} T_M^4 e^{-4C_2 y} - C_2^2 T_M e^{-C_2 y}\left(1 - \frac{T_M^3}{12 C_2^2}\right) - \frac{1}{3} T_M^4 e^{-4C_2 y} \\ &\qquad = T_M^4 e^{-4C_2 y} = \left(T_M e^{-C_2 y}\right)^4 \geq w^4. \end{aligned}$$

Notice in addition that w is monotonically decreasing, since

$$w'(y) = -C_2 T_M e^{-C_2 y}\left(1 - \frac{T_M^3}{12C_2^2}\right) - \frac{1}{3C_2} T_M^4 e^{-4C_2 y} < 0.$$

Finally, we see that $\partial_y w(0^+) > -LC_1$. Indeed, we need to show that

$$w'(0) = -C_2 T_M + \frac{T_M^4}{12C_2} - \frac{1}{3C_2} T_M^4 = -\frac{4C_2^2 T_M + T_M^4}{4C_2} > -LC_1,$$

which is equivalent to show that

$$4C_2^2 T_M - 4LC_1 C_2 + T_M^4 < 0.$$

Since the two roots are given by $\Gamma^{\pm}(C_1) = \frac{LC_1 \pm \sqrt{L^2 C_1^2 - T_M^5}}{2T_M}$ and since by assumption $C_1 > \left(\frac{T_M^5+1}{L^2}\right)^{1/2}$ we see that $\Gamma^{\pm}(C_1)$ are well defined with $\Gamma^{+}(C_1) > \frac{T_M^{3/2}}{2} > \frac{T_M^{3/2}}{2\sqrt{3}}$. Hence, we conclude $w'(0) > -LC_1$ using that by assumption

$$\Gamma^{-}(C_1) < C_2 < \Gamma^{+}(C_1).$$

Let us now consider $T_i = T_M + u_i$ the solutions of (D.10) considered in Theorem D.3 and D.4 and in Proposition D.3 on the maximal time interval $[0, t^*]$. Let us assume that $t^* < \infty$ and that $(T_1, T_2, \dot{s})$ cannot be extended for $t > t^*$, otherwise it is already the global in time solution. We will show that $\|T_i\|_\infty \leq C(w) < \infty$, $\|\partial_y T_i\|_\infty < C(w, t^*) < \infty$ and $\|\dot{s}\|_\infty < C(w) < \infty$, where the sup-norms are taken on $[0, t^*]$. This will imply that the solutions can be extended for $t > t^*$ as we did in Theorem D.3 and D.4, and hence $t^* = \infty$.

Lemma D.5 implies that for the initial value T_0 as in the assumption of the Theorem there are $C_1, C_2 > 0$ satisfying the prescribed conditions such that $T_0(y) < w(y)$ for $y < 0$ and that $T_0(y) > w(y)$ for $y > 0$ and such that $|w'(0^{\pm})| > \sup_{\mathbb{R}_\pm} |\partial_y T_0(y|)$. Thus, by the uniform continuity of $T_i - w \in \mathcal{C}^{1/2,1/2}_{t,y}([0, t_*] \times \mathbb{R}_\pm)$, by the positivity $\partial_y(T_i(0, y - w))|_{y=0} > 0$ as well as by the fact that for any $R > 0$ there exists $a > 0$ such that $|T_0(y) - w(y)| > a$ for all $|y| > R$, there exists a positive time $t_0 \leq t^*$ such that $T_1(t, y) < w(y)$ for $y < 0$ and $T_2(t, y) > w(y)$ for $y > 0$ and $0 \leq t < t_0$. Let us define

$$t_0 = \inf\{t \in [0, t^*] : \exists y \neq 0 \text{ such that } T_1(t, y) = w(y) \text{ if } y < 0 \text{ or } T_2(t, y) = w(y) \text{ if } y > 0\}.$$

Let us assume that $t_0 < t^*$. Then, since $T_1(t, 0) = T_M = w(0) = T_2(t, 0)$ we have $0 \geq \partial_y T_1(t, 0^-) \geq \partial_y w(0^-)$ as well as $0 \geq \partial_y T_2(t, 0^+) \geq \partial_y w(0^+)$ for $t \in [0, t_0]$. Hence, for $t \in [0, t_0]$ we obtain that $-C_1 < \dot{s}(t) < C_2$. Let us also denote by $\mathcal{L}_1(v) = \partial_t v - \dot{s}(t)\partial_y v - \kappa \partial_y^2 v$ for $y < 0$ and $\mathcal{L}_2(v) = \partial_t v - \dot{s}(t)\partial_y v - \partial_y^2 v + v^4$ for $y > 0$. We note that,

$$\mathcal{L}_1(w) = -(\dot{s}(t) + C_1)\partial_y w(y) > 0.$$

Hence, $\mathcal{L}_1(T_1 - w)(t, y) < 0$ for all $(t, y) \in (0, t_0] \times \mathbb{R}_-$ and $T_1(t, y) - w(y) \leq 0$ for $(t, y) \in \{0\} \times \mathbb{R}_- \cup (0, t_0) \times \{a, b\}$, where $(a, b) \subset \mathbb{R}_-$. An application of the maximum principle to the bounded function $T_1 - w$ on domains $[0, t_0] \times [a, b]$ for any $[a, b] \in \mathbb{R}_-$ shows that $T_1(t, y) < w(t, y)$ for all $(t, y) \in (0, t_*] \times (a, b)$ for any $(a, b) \in \mathbb{R}_-$.

Moreover, for $y > 0$ we see

$$\mathcal{L}_2(w) = -(w''(y) + C_2 w'(y) - w^4(y)) + (C_2 - \dot{s}(t))\partial_y w(y) \leq (C_2 - \dot{s}(t))\partial_y w(y) < 0.$$

This implies

$$\mathcal{L}_2(T_2)(t,y) - \mathcal{L}_2(w)(t,y) > \int_0^\infty \frac{\alpha}{2} E_1(\alpha(y-\eta))T_2^4(t,\eta)d\eta > 0.$$

Hence, if for $R > 0$ there exists some $y_0 \in (0,R)$ such that $T_2(t_0,y_0) - w(y_0) = 0$ we obtain the contradiction

$$0 < \mathcal{L}_2(T_2)(t_0,y_0) - \mathcal{L}_2(w)(t_0,y_0) \leq 0.$$

This is because $\partial_t T_2(t_0,y_0) \leq 0$ as well as $T_2(t_0,y_0) - w(y_0) = 0$ would be a minimum in space. Thus, $T_2(t,y) > w(y)$ for all $(t,y) \in (0,t_0] \times (a,b)$ for every $(a,b) \in \mathbb{R}_+$. Hence, by the definition of t_0 follows that $t_0 = t^*$ and

$$\|T_i\|_\infty \leq \max\{\|w\|_\infty, T_M\} < \infty.$$

Moreover, since $T_i(t,0) = T_M = w(0)$ we have that $0 \geq \partial_y T_i(t,0^\pm) \geq \partial_y w(0^\pm)$ and hence by construction

$$\|\dot{s}\|_\infty \leq \max\{C_1, C_2\} < \infty.$$

We will now show that also the norms of $\partial_y T_i$ are bounded. We will use the maximum principle applied to the equation solved by $\partial_y T_i$. Before considering those equations we shall argue that $\partial_y T_i$ are indeed twice differentiable. This follows using classic parabolic theory. Let $\varepsilon > 0$ and $\gamma > 0$ be arbitrary. We have already shown in Theorem D.4 that $T_i \in \mathcal{C}^{1+\delta/2,2+\delta}_{t,y}([\frac{\varepsilon}{2},t^*]\times\mathbb{R}_\pm) \cap \mathcal{C}^{\alpha/2,1+\beta}_{t,y}([0,t^*],\mathbb{R}_\pm)$, for $\delta \in \left(0,\frac{1}{2}\right]$ and $\alpha,\beta \in (0,1)$. One can prove that $T_i \in \mathcal{C}^{1+\delta/2,3+\delta}_{t,y}([\varepsilon,t^*]\times[\pm\gamma,\pm\infty))$ since $\dot{s}\partial_y^2 T_1 \in \mathcal{C}^{\delta/2,\delta}_{t,y}\left([\frac{\varepsilon}{2},t^*]\times\mathbb{R}_-\right)$ and

$$\begin{aligned}\dot{s}\partial_y^2 T_2 - &4T_2^3\partial_y T_2 + \frac{\alpha}{2}T_M^4 E_1(\alpha\cdot)\\ &+4\int_0^\infty \frac{\alpha}{2}E_1(\alpha(y-\eta))T_2^3(\eta)\partial_\eta T_2(\cdot,\eta)d\eta \in C^{\delta/2,\delta}\left(\left[\frac{\varepsilon}{2},t^*\right]\times\left[\frac{\gamma}{2},\infty\right)\right).\end{aligned}$$

Since the computations are similar to the one in Proposition D.1, Lemma D.1 and Lemma D.2 we omit the details. Hence, differentiating the equations satisfied by T_i, we obtain that $\partial_y\partial_t T_i$ exists and it is continuous. Furthermore, differentiating the representation formulas and using classic parabolic theory again, we conclude that the derivatives $\partial_t\partial_y T_i$ exist and that they are continuous for every $t,y \in [\varepsilon,t^*]\times[\pm\gamma,\pm\infty)$. Thus, $\partial_y\partial_t T_i = \partial_y\partial_t T_i$ at the interior of $(0,t^*]\times\mathbb{R}_\pm$. Differentiating the operators $\mathcal{L}_1$ and $\mathcal{L}_2$ we see that $\partial_y T_i$ solve

$$\begin{gathered}\mathcal{L}_1(\partial_y T_1)(t,y) = 0 \text{ for } t > 0,\ y < 0,\\ \mathcal{L}_2^1(\partial_y T_2)(t,y) = \frac{\alpha}{2}T_M^4 E_1(\alpha y) > 0 \text{ and } \mathcal{L}_2^2(\partial_y T_2)(t,y) = 0 \text{ for } t > 0, y > 0,\end{gathered}$$

where we defined

$$\mathcal{L}_2^1(v) = \partial_t v - \dot{s}(t)\partial_y v - \partial_y^2 v + 4T_2^3 v - 4\int_0^\infty \frac{\alpha}{2}E_1(\alpha(\cdot-\eta))T_2^3(\eta)v(\cdot,\eta)d\eta$$

as well as

$$\mathcal{L}_2^2(v) = \partial_t v - \dot{s}(t)\partial_y v - \partial_y^2 v + 4T_2^3 v - 4\int_0^\infty \frac{\alpha}{2}E_1(\alpha(\cdot-\eta))T_2^3(\eta)v(\cdot,\eta)d\eta - \frac{\alpha}{2}T_M^4 E_1(\alpha\cdot).$$

Let us consider for $t \geq 0$ the functions $\psi_\pm(t) = \mp\partial_y w(0^-)(1+t)$. It is easy to see that $\mathcal{L}_1(\psi_\pm) = \mp\partial_y w(0^-)$ and therefore ψ_+ is a supersolutions while ψ_- is a subsolution for $\mathcal{L}_1$.

Moreover, $\psi_- < \partial_y T_1(0,y) < \psi_+$ as well as $\psi_- < \partial_y T_1(t,0^-) < \psi_+$ for $t \in [0,t^*]$ and $y < 0$. We define

$$t_0 = \inf\left\{t_0 \in [0,t^*] : \exists y_0 < 0 \text{ s.t. } \partial_y T_1(t_0,y_0) = \psi_+(t_0) \text{ or } \partial_y T_1(t_0,y_0) = \psi_-(t_0)\right\}.$$

Let us assume that $t_0 < t^*$. Then, by the uniform continuity of $\partial_y T_1 - \psi_\pm \in \mathcal{C}^{1/2,1/2}_{t,y}([0,t^*]\times\mathbb{R}_-)$ and since $\sup_{y\in\mathbb{R}_-} \partial_y T_1(0,y) > \partial_y w(0^-)$, we know that $t_0 > 0$ as well as $y_0 < 0$. Let us assume that $y_0 < 0$ is the smallest such that $\partial_y T_1(t_0,y_0) = \psi_+(t_0)$. Then, $\partial_t(\partial_y T_1 - \psi_+)(t_0,y_0) \geq 0$ as well as $\partial_y T_1(t_0,\cdot)$ has a maximum in y_0. Thus,

$$0 > \mathcal{L}_1(\partial_y T_1 - \psi_+)(t_0,y_0) \geq 0.$$

A similar contradiction is obtained applying the maximum principle to $\partial_y T_1 - \psi_-$ assuming that $\partial_y T_1(t_0,y_0) - \psi_-(t_0) = 0$. Hence, we conclude that $t_0 = t^*$ so that

$$\|\partial_y T_1\|_\infty \leq |\partial_y w(0^-)|(1+t^*) < \infty.$$

We now consider $\partial_y T_2$. Let us define $\varphi_- = \partial_y w(0^+)e^{4T_M^3 t} < 0$. Then, since $0 < \int_0^\infty \frac{\alpha}{2}E_1(\alpha(y-\eta))T_2^3(\eta)d\eta \leq T_M^3$ we see that

$$\mathcal{L}_2^1(\varphi_-)(t,y) \leq 4T_2^3(t,y)\varphi_-(t) < 0.$$

Moreover, $\varphi_-(0) < \sup_{y\in\mathbb{R}_+} \partial_y T_2(0,y)$ for $y > 0$ as well as $\varphi_-(t) < \partial_y T_2(t,0^-) \leq 0$ for $t \in [0,t^*]$. We remark that $\partial_y T_2 - \varphi_- \in \mathcal{C}^{1/2,1/2}_{t,y}([0,t^*]\times\mathbb{R}_+)$. Defining again via uniform continuity

$$t_0 = \inf\left\{t_0 \in [0,t^*] : \exists y_0 > 0 \text{ s.t. } \partial_y T_2(t_0,y_0) = \varphi_-(t_0)\right\} > 0,$$

assuming $t_0 < t^*$ and applying the maximum principle to $\mathcal{L}_2^1(\partial_y T_2 - \varphi_-)(t,y) > 0$ at (t_0,y_0) we obtain a contradiction. Indeed, $(\partial_y T_2 - \varphi_-)(t_0,y) \geq 0$ so that $(\partial_y T_2 - \varphi_-)(t_0,y_0)$ is a minimum. Hence,

$$0 < \mathcal{L}_2^1(\partial_y T_2 - \varphi_-)(t_0,y_0) \leq -4\int_0^\infty \frac{\alpha}{2}E_1(\alpha(y-\eta))T_2^3(\eta)(\partial_y T_2 - \varphi_-)(t_0,\eta)d\eta < 0.$$

This contradiction implies $\partial_y T_2(t,y) \geq \partial_y w(0^+)e^{4T_M^3 t^*}$ for all $t \in [0,t_*]$. Let us now define

$$g(y) = -T_M^4 \int_0^y d\xi \int_0^\xi dz e^{-(\xi-z)}\frac{\alpha}{2}E_1(\alpha(z))\eta(z),$$

where $\eta \in C^\infty([0,\infty))$ with $0 \leq \eta \leq 1$, $\eta(z) \equiv 1$ for $y \in \left[0,\frac{1}{2}\right]$ and $\eta(z) \equiv 0$ for $y \geq 1$. A simple computation shows $-T_M^4 \leq g(y) \leq 0$ as well as $-\frac{T_M^4}{2} \leq g'(y) \leq 0$. We remark that $g \in C^{0,1/2}(\mathbb{R}_+)$. Moreover, for $y > 0$ the function g solves

$$-g''(y) - g'(y) = T_M^4\frac{\alpha}{2}E_1(\alpha(y))\eta(y).$$

We also consider the function $h \in C^{0,1/2}([0,t^*])$ given by

$$\begin{aligned} h(t) &= \left[|\partial_y w(0^+)| + T_M^4 + \frac{T_M}{4}\left(1+\frac{C_1+1}{2}+4T_M^3\right)\right]e^{4T_M^3 t} - \frac{T_M}{4}\left(1+\frac{C_1+1}{2}+4T_M^3\right) \\ &\geq \left[|\partial_y w(0^+)| + T_M^4\right]e^{4T_M^3 t} \geq \left[|\partial_y w(0^+)|\right]e^{4T_M^3 t} + |g(y)| > 0. \end{aligned}$$

Using the estimates $\frac{\alpha}{2}E(\alpha(y)) \leq 1$ for all $y \geq \frac{1}{2}$ and $-\dot{s}(t) \leq C_1$ we compute for $\varphi_+(t,y) = h(t)+g(y) \geq |\partial_y w(0^+)|e^{4T_M^3 t} > \sup_{y\in\mathbb{R}_+} |\partial_y T_2(0,y)| > 0$ the following

$$\begin{aligned}\mathcal{L}_2^2(\varphi_+)(t,y) =& \partial_t h(t) + 4T_2^3(t,y)(\varphi_+)(t,y) - \int_0^\infty \frac{\alpha}{2}E_1(\alpha(y-\eta))T_2^3(\eta)\varphi_+(t,\eta)d\eta \\ &+ (-\dot{s}(t)+1)\partial_y g(y) + \frac{\alpha}{2}T_M^4 E_1(\alpha y)(\eta(y)-1) \\ >& \partial_t h(t) - 4T_M^3 T_M^4 - (C_1+1)\frac{T_M^4}{2} - T_M^4 > 0.\end{aligned}$$

For this estimate we used that $g \leq 0$ and $h > 0$. We also notice that by construction $\partial_y T_2(0,y) < \varphi_+(t,y)$ as well as $\partial_y T_2(t,0) \leq 0 < \varphi_+(t,y)$ for $t \in [0,t^*]$ and $y > 0$. Using the uniform continuity of $\partial_y T_2 - \varphi_+ \in \mathcal{C}_{t,y}^{1/2,1/2}([0,t^*]\times\mathbb{R}_+)$ we define

$$t_0 = \inf\{t_0 \in [0,t^*] : \exists y_0 < 0 \text{ s.t. } \partial_y T_2(t_0,y_0) = \varphi_+(t_0,y_0)\} > 0.$$

Assuming $t_0 < t^*$, arguing by continuity and applying once more the maximum principle to $\mathcal{L}_2^2(\partial_y T_2 - \varphi_+)(t,y) < 0$ at (t_0,y_0) we obtain a contradiction. Indeed, we use that $(\partial_y T_2 - \varphi_+)(t_0,y) \leq 0$ and therefore $(\partial_y T_2 - \varphi_+)(t_0,y_0)$ is a maximum. Thus,

$$0 > \mathcal{L}_2^2(\partial_y T_2 - \varphi_+)(t_0,y_0) \geq 0.$$

We finally conclude that

$$\|\partial_y T_2\|_\infty \leq \left[|\partial_y w(0^+)| + T_M^4 + \frac{T_M}{4}\left(1 + \frac{C_1+1}{2} + 4T_M^3\right)\right]e^{4T_M^3 t^*}.$$

Therefore, $(T_1,T_2,\dot{s})$ can be extended for $t > t^*$. □

Finally, we prove Lemma D.5.

Proof of Lemma D.5. By our assumptions on T_0 we can fix some $\theta \in (0,1)$ such that $\sup_{\mathbb{R}_-} T_0 \leq T_M + \frac{L^2\kappa}{T_M K}\frac{(1-\theta)}{4}$. Let us also define

$$C_2^0 > \max\left\{\frac{T_M^{3/2}}{2\sqrt{3}}, \Gamma^-\left(\left(\frac{T_M^5+1}{L^2}\right)^{1/2}\right), \frac{T_M^{3/2}}{2}, \frac{(T_M^5+1)^{1/2}}{2T_M}, \sup_{\mathbb{R}_+}\frac{|\partial_y T_0|}{T_M}, \sup_{\mathbb{R}_-}\frac{|\partial_y T_0|K}{L(1-\theta)}\right\}$$

and let us denote $f_-^{C_2^0}(y) = T_M + \frac{(1-\theta)L^2\kappa}{2T_M K}\left(1 - e^{\frac{2T_M C_2^0}{L\kappa}y}\right)$ for $y < 0$ and $f_+^{C_2^0}(y) = T_M e^{-C_2^0 y}$ for $y > 0$. Then, since $0 \geq \partial_y T_0(0^-) > \partial_y f_-^{C_2^0}(0^-)$ and $0 \geq \partial_y T_0(0^+) > \partial_y f_+^{C_2^0}(0^+)$, by monotonicity there exist constants $\delta_1(C_2^0), \delta_2(C_2^0) > 0$ such that

$$T_0(y) < f_-^{C_2^0}(y) \text{ for } y \in (-\delta_1(C_2^0),0) \text{ and } f_+^{C_2^0}(y) < T_0(y) \text{ for } y \in (0,\delta_2(C_2^0)).$$

We remark that since $C_2 \mapsto f_-^{C_2}(y)$ is increasing in $C_2 > 0$ for $y < 0$ and since $C_2 \mapsto f_+^{C_2}(y)$ is decreasing in $C_2 > 0$ for $y > 0$ the estimates $T_0(y) < f_-^{C_2}(y)$ and $f_+^{C_2}(y) < T_0(y)$ are valid in the intervals $(-\delta_1(C_2^0),0)$ and $(0,\delta_2(C_2^0))$, respectively. Thus, defining

$$C_2 = \max\left\{C_2^0, \frac{L\kappa}{2T_M\delta_1(C_2^0)}\ln\left(\frac{1}{2}\right), \frac{1}{\delta_2(C_2^0)}\ln\left(\frac{2T_M}{\inf_{\mathbb{R}_+} T_0}\right)\right\}$$

we see that $f_-^{C_2}(y) > T_0(y)$ for $y < 0$ as well as $T_0(y) > f_+^{C_2}(y)$ for $y > 0$. Moreover, we notice that $C_2 < \infty$ by assumption on T_0. We now choose $\alpha = -\frac{(1-\theta)LC_2}{K}$ and $C_1 = \frac{2T_M}{L}C_2$. Notice that by definition $C_1 > \left(\frac{T_M^5+1}{L^2}\right)^{1/2}$. Then, for the chosen constants C_1, C_2, α the function w defined by (D.46) satisfies

$$w(y) = f_-^{C_2}(y) \text{ for } y < 0 \text{ and } w(y) < f_+^{C_2}(y) \text{ for } y > 0.$$

Thus, we have found constants $C_1, C_2 > 0$ such that

$$w(y) < T_0(y) \text{ for } y < 0 \text{ and } T_0(y) > 0 \text{ for } y > 0.$$

Notice that by construction for any $R > 0$ there exists $a > 0$ such that $|T_0(y) - w(y)| > a$ for all $|y| > R$. Moreover, by definition

$$|\partial_y w(0^-)| = |\alpha| = \frac{(1-\theta)LC_2}{K} > \sup_{\mathbb{R}_-} |\partial_y T_0| \text{ and } |\partial_y w(0^+)| > C_2 T_M > \sup_{\mathbb{R}_+} |\partial_y T_0(y)|.$$

Finally, since $C_2 > \frac{T_M^{3/2}}{2}$ we conclude

$$\partial_y w(0^+) = -C_2 T_M \left(1 + \frac{T_M^3}{4C_2^2}\right) > -2C_2 T_M = -LC_1,$$

which implies $\Gamma^-(C_1) < C_2 < \Gamma^+(C_1)$ as well as the fact

$$w(0^-) = \alpha = -\frac{(1-\theta)LC_2}{K} > -\frac{L}{K}C_2.$$

We also remark that the considered class of initial data is optimal for the argument in Theorem D.5 involving w as a barrier function defined in (D.46). Since we can take C_2 arbitrary large as $C_2 = \frac{1}{\varepsilon} \to \infty$ we obtain

$$w(y) \le T_M e^{-\frac{y}{\varepsilon}} \text{ for } y > 0 \text{ and } \partial_y w(0^+) = -\frac{T_M}{\varepsilon}\left(1 + \frac{T_M^3 \varepsilon^2}{4}\right) \to -\infty \text{ as } \varepsilon \to 0.$$

Moreover, $C_1 > \frac{T_M}{L\varepsilon}\left(1 + \frac{T_M^3\varepsilon^2}{4}\right)$ so that $\partial_y w(0^-) = -|\alpha| > -\frac{K}{L\varepsilon} \to -\infty$ as $\varepsilon \to 0$ and

$$\begin{aligned} w(y) <& T_M + \frac{\kappa L}{KC_1}\frac{1}{\varepsilon}\left(1 - \exp\left(\frac{C_1}{\kappa}y\right)\right) \\ <& T_M + \frac{\kappa L^2}{KT_M}\frac{1}{\left(1 + \frac{T_M^3\varepsilon^2}{4}\right)}\left(1 - \exp\left(\frac{C_1}{\kappa}y\right)\right) \to T_M + \frac{\kappa L^2}{KT_M} \end{aligned}$$

as $\varepsilon \to 0$ and $y < 0$. Moreover, $\dot{s}(t) \in (-\infty, \infty)$.

□

Remark. Observe also that the class of initial values T_0 considered in Theorem D.5 is optimal for the argument presented in the proof of the Theorem. Indeed, instead of considering for $y > 0$ the subsolution $w'' + C_2 w' \ge w^4$, we could have considered the solution to

$$\begin{cases} w''(y) + C_2 w'(y) = w^4(y) & y > 0 \\ w(0) = T_M & \\ w(y) \to 0 & \text{as } y \to \infty \\ w \ge 0 & \end{cases} \tag{D.47}$$

That such a solution exists can be proven considering the variational problem given by the functional

$$I(w) = \int_0^\infty e^{C_2 y}\left(\frac{|\partial_y w(y|^2)}{2} + \frac{w^5(y)}{5}\right) dy.$$

By the direct method of calculus of variations one can prove that there exists a unique minimizer of I on the closed convex set

$$\mathcal{A} = \left\{g \geq 0 : g \in W^{1,2}\left(\mathbb{R}_+; e^{C_2 y} dy\right) \cap L^5\left(\mathbb{R}_+; e^{C_2 y} dy\right),\ g(0) = T_M\right\}.$$

Since then $e^{\frac{C_2}{2}y} g \in L^\infty \cap C^{0,1/2}(\mathbb{R}_+)$ it also follows that if $g \in \mathcal{A}$ then $\lim_{y\to\infty} g = 0$. The unique minimizer $w \in \mathcal{A}$ is also bounded by T_M, since $I[\min\{w, T_M\}] \leq I[w] \leq I[\min\{w, T_M\}]$. Moreover, w solves weakly the following variational inequality

$$-\partial_y\left(e^{C_2 y}\partial_y w(y)\right) + w^4(y) e^{C_2 y} \geq 0,$$

and is a weak solution of $-\partial_y\left(e^{C_2 y}\partial_y w(y)\right) + w^4(y) e^{C_2 y} = 0$ in the region $w > 0$. With the weak maximum principle it can be also shown that $\{y > 0 : w(y) > 0\} = \mathbb{R}_+$. This implies that the unique minimizer w is a weak solution of

$$-\partial_y\left(e^{C_2 y}\partial_y w(y)\right) + w^4(y) e^{C_2 y} = 0 \text{ in } \mathbb{R}_+.$$

Using elliptic regularity we obtain easily that since $w \in L^\infty \cap C^{0,1/2}(\mathbb{R}_+)$ also $w^4 \in L^\infty \cap C^{0,1/2}(\mathbb{R}_+)$ and hence locally $w \in C^{2+1/2}_{\text{loc}}(\mathbb{R}_+)$ so that iterating this argument we have $w \in C^\infty(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$. Thus, w is a strong solution solving the boundary problem (D.47).

Finally, the solution w of (D.47) is unique. This is a consequence of the strong maximum principle.

Let us assume now again that $C_2 = \frac{1}{\varepsilon}$ is arbitrarily large. Then using the rescaling $y = \varepsilon\eta$ and $w(y) = w(\varepsilon\eta) = \tilde{w}(\eta)$ we see that the leading order of $\tilde{w}$ solves as $\varepsilon \to 0$

$$\begin{cases} \tilde{w}''(\eta) + \tilde{w}'(\eta) = 0 & \eta > 0 \\ \tilde{w}(0) = T_M & \\ \tilde{w}(\eta) \to 0 & \text{as } \eta \to \infty \\ \tilde{w} \geq 0 & \end{cases}$$

Hence, $\tilde{w}(\eta) = T_M\left(e^{-\eta}\right)$. Thus, $w(y) = T_M\left(e^{-y/\varepsilon}\right)$ at the leading order, so that we need to take

$$C_1 > \frac{T_M}{\varepsilon L},$$

which implies for $y < 0$ as above that

$$w(y) < T_M + \frac{\kappa L}{K C_1}\frac{1}{\varepsilon} < T_M + \frac{\kappa L^2}{K T_M} \text{ as } \varepsilon \to 0.$$

Appendix E

Traveling waves for a two-phase Stefan problem with radiation

Abstract: In this paper we study the existence of traveling wave solutions for a free-boundary problem modeling the phase transition of a material where the heat is transported by both conduction and radiation. Specifically, we consider a one-dimensional two-phase Stefan problem with an additional non-local non-linear integral term describing the situation in which the heat is transferred in the solid phase also by radiation, while the liquid phase is completely transparent, not interacting with radiation. We will prove that there are traveling wave solutions for the considered model, differently from the case of the classical Stefan problem in which only self-similar solutions with the parabolic scale $x \sim \sqrt{t}$ exist. In particular we will show that there exist traveling waves for which the solid expands. The properties of these solutions will be studied using maximum-principle methods, blow-up limits and Liouville-type Theorems for non-linear integro-differential equations.

E.1 Introduction

In this paper we continue the study of the free-boundary problem presented in [39] considering a one-dimensional Stefan-like problem which describes the melting of ice (resp. the solidification of water) under the assumption that the heat is transported by conduction in both phases of the material and additionally by radiation in the solid. To be more precise, we are studying the problem in which $\mathbb{R}^3$ is divided in two regions, one liquid region with a temperature T greater then the melting temperature T_M and one solid region with $0 < T < T_M$. At the contact surface between the two phases the temperature satisfies $T = T_M$. This surface moves as the solid melts or the liquid solidifies and it is thus called moving interface. Analogously to the classical Stefan problem, the heat is transferred by conduction in both the liquid and the solid phase. In addition we assume that the heat is transported also by radiation only in the solid. Equivalently, we assume the liquid to be perfectly transparent not allowing any interaction with radiation.

At the initial time $t = 0$, the liquid is considered to fill the half-space $\mathbb{R}^3_- = \{x \in \mathbb{R}^3 : x_1 < 0\}$ and the solid to fill $\mathbb{R}^3_+ = \{x \in \mathbb{R}^3 : x_1 > 0\}$. Thus, the interface is initially the plane $\{0\} \times \mathbb{R}^2$. Furthermore, we assume the temperature to depend only on the first variable, i.e. $T(t, x) = T(t, x_1)$. This implies that the interface is described by the plane $\{s(t)\} \times \mathbb{R}^2$ for all $t \geq 0$ and the problem reduces to the study of a one-dimensional model.

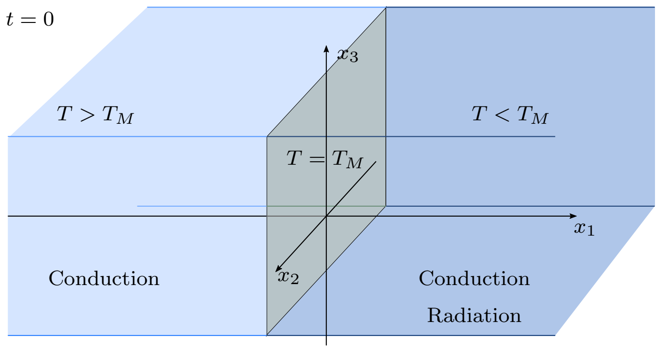

Figure E.1: Illustration of the considered model at initial time $t = 0$.

We also assume the material to satisfy local thermal equilibrium (i.e. there exists a well-defined temperature for all $t > 0$, $x \in \mathbb{R}^3$) and we consider the case in which the scattering process is negligible. Hence, the interaction of photons with matter is described in the solid phase by the (stationary) radiative transfer equation, which under the further assumption of constant Grey approximation (i.e. $\alpha \equiv$const.) writes

$$n \cdot \nabla_x I_\nu(t,x,n) = \alpha \left(B_\nu(T(t,x)) - I_\nu(t,x,n)\right) \quad \nu > 0,\ x_1 > s(t),\ n \in \mathbb{S}^2,\ t > 0, \tag{E.1}$$

where I_ν is the radiation intensity , i.e. the energy of photons with frequency $\nu > 0$, at position $x \in \Omega$, moving in direction $n \in \mathbb{S}^2$ at time $t > 0$, and $B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{\frac{h\nu}{kT}}-1}$ is the Planck distribution of a black body.

In the transport term of equation (E.1) the term containing the time derivative of I_ν, i.e. $\frac{1}{c}\partial_t I_\nu(t,x,n)$ has been neglected since the characteristic time scale required in order to obtain significant changes of the temperature is much larger than the time scale in which the radiation intensity becomes stable. This is due to the fact that photons travel with the speed of light.

In this paper it is assumed also the absence of external sources of radiation. Thus, since the photons do not interact with the liquid phase, at the interface the radiation intensity has to satisfy

$$I_\nu\left(t,x,n\right) = 0 \quad \text{if } x_1 = s(t),\ n_1 > 0. \tag{E.2}$$

We remark that the transparency of the liquid implies that the radiation escaping the solid (i.e. traveling with direction $n_1 < 0$) passes through the liquid without interacting with it and hence without any possibility to return in the solid phase. Thus, radiation helps the solid to cool faster.

Under all these assumptions, the two-phase free boundary problem that we study in this paper is given by

$$\begin{cases} C_L\partial_t T(t,x_1) = K_L \partial^2_{x_1} T(t,x_1) & x_1 < s(t), \\ C_S\partial_t T(t,x_1) = K_S \partial^2_{x_1} T(t,x_1) - \operatorname{div}\left(\int_0^\infty d\nu \int_{\mathbb{S}^2} dn n I_\nu(t,x,n)\right) & x_1 > s(t), \\ n\cdot\nabla_x I_\nu(t,x,n) = \alpha\left(B_\nu(T(t,x_1)) - I_\nu(t,x,n)\right) & x_1 > s(t), \\ I_\nu(t,x,n) = 0 & x_1 = s(t),\ n_1 > 0, \\ T(t,s(t)) = T_M & x_1 = s(t), \\ T(0,x) = T_0(x) & x_1 \in \mathbb{R}, \\ \dot{s}(t) = \frac{1}{L}\left(K_S\partial_{x_1}T(t,s(t)^+) - K_L\partial_{x_1}T(t,s(t)^-)\right), \end{cases} \tag{E.3}$$

where C_S, C_L are the volumetric heat capacities of the solid and liquid, K_S, K_L the conductivities of the two phases and L is the latent heat. Notice that for simplicity we are assuming that the two phases have the same constant density. For a more detailed explanation of the derivation of (E.3) and in particular of the Stefan condition for the moving interface we refer to [39]. As remarked there, the main feature of system (E.3) is that there is no external source of radiation and only the solid is emitting radiation. The addition of a non-trivial external source of radiation heating the solid from far away is another very interesting problem that could be studied, not only developing a well-posedness theory but also examining the possible existence of traveling waves. In this case we would consider as boundary condition

$$(t,(s(t),x_2,x_3),n) = g_\nu(n) > 0 \quad \text{if } n_1 > 0.$$

Moreover, in our previous article [39] we showed that reducing the radiative transfer equation to a non-local non-linear integral operator for T^4 and performing suitable rescalings, the system (E.3) is equivalent to

$$\begin{cases} \partial_t T(t,x) = \kappa \partial_x^2 T(t,x) & x < s(t), \\ \partial_t T(t,x) = \partial_x^2 T(t,x) - I_\alpha[T](t,x) & x > s(t), \\ T(t,s(t)) = T_M & \\ T(0,x) = T_0(x) & x \in \mathbb{R}, \\ \dot{s}(t) = \frac{1}{L}\left(\partial_x T(t,s(t)^+) - K\partial_x T(t,s(t)^-)\right), & \end{cases} \tag{E.4}$$

where

$$I_\alpha[T](t,x_1) = T^4(t,x_1) - \int_{s(t)}^\infty d\eta \frac{\alpha E_1(\alpha|x_1-\eta|)}{2} T^4(t,\eta)$$

for $E_1(x) = \int_{|x|}^\infty \frac{e^{-t}}{t}$ being the exponential integral. Given a solution T of (E.4), the intensity of radiation is obtained solving by characteristics the radiative transfer equation (E.1) with boundary conditions (E.2) as

$$I_\nu(t,x,n) = \int_0^{d(x,n)} d\tau \alpha \exp(-\alpha\tau) B_\nu(T(t,x_1-\tau n_1)) \quad \text{ for } x_1 > 0,$$

where $d(x,n)$ is the distance of $x \in \mathbb{R}^3$ to the interface $\{s(t)\} \times \mathbb{R}^2$ in direction $-n \in \mathbb{S}^2$ and it is possibly infinity. In [39] a local and global well-posedness theory for (E.4) has been developed. Thus, a natural question that arises concerns the asymptotic behavior of the solutions to (E.4) as $t \to \infty$. In this paper we construct traveling waves of (E.4) and we study their properties. Therefore, considering solutions of the form $T(t,x) = T(x - s(t))$ and $s(t) = -ct$ for $c \in \mathbb{R}$, in this article we study the system

$$\begin{cases} c\partial_y T_1(y) = \kappa \partial_y^2 T_1(y) & y < 0 \\ c\partial_y T_2(y) = \partial_y^2 T_2(y) - T_2^4(y) + \int_0^\infty \alpha \frac{E_1(\alpha(y-\eta))}{2} T_2^4(\eta) d\eta & y > 0 \\ T_2(0) = T_1(0) = T_M & \\ c = \frac{1}{L}\left(K\partial_y T_1(0^-) - \partial_y T_2(0^+)\right), & \end{cases} \tag{E.5}$$

where we changed the variables according to $y = x - ct$.

E.1.1 Summary of previous results

This paper studies a problem arising from the combination of a classical Stefan problem with the radiative transfer equation. It is therefore worth revising the most important results for these two particular problems, which as far as we know were consider together rigorously firstly in our previous paper [39].

Starting from the seminal work [135] of J. Stefan, who also discovered the well-known Stefan-Boltzmann law for the total emission of a black body (cf. [133]), the Stefan problem for melting of ice has been comprehensively studied in both the one-phase and the two-phase formulations, in the case of classical (i.e. strong) and weak enthalpy solutions, i.e. the weak solutions of the enthalpy formulation of the problem.

The well-posedness theory for classical solutions to the Stefan problem has been considered in many works, like for instance [26, 27, 55, 56, 59, 60, 106, 123], using among other methods fixed-point equations for Volterra-type integrals and the maximum principle, the Baiocchi transform, a variational inequality. Concerning the long time behavior of the one-dimensional, one-phase Stefan problem, [56,106] prove that the temperature approaches a self-similar profile as $t \to \infty$, which is given by an error function. The works [57,58] deal with the well-posedness theory of weak (enthalpy) solutions for the one and two-phase free boundary problem.

Another interesting question emerging for the higher dimensional local and non-local (cf. fractional Laplacian) Stefan problem concerns the regularity of the free boundary, which can be studied through its formulation as a parabolic obstacle problem. This has been considered in [9, 24, 25, 43, 53].

Finally, an important class of results addresses of the formation of supercooled liquid (i.e. liquid regions where $T < T_M$) or superheated solid (i.e. $T > T_M$) for the classical solutions of the Stefan problem (cf. [89]) as well as the creation of mushy regions (i.e. where $T = T_M$) of positive measure for the weak enthalpy formulation of the freee boundary problem, cf. [20, 50, 51, 90, 117, 142, 143].

Besides the theory of free boundary problems, this paper deals also with the theory of radiative transfer, an issue extensively studied starting from the pioneer works of Compton [31] in 1922 and of Milne [109] in 1926. The kinetic equation describing the interaction of photons with matter is the radiative transfer equation, whose derivation and main properties can be found in [29, 108, 114, 125, 152].

In recent years, several different problems concerning the study of the distribution of temperature due to radiation have been considered, such as well-posedness results for the stationary radiative transfer equation as in [35,83], diffusion approximation (see [13,14,36,37] and the references therein), the interaction of radiation and fluids (for instance in [69, 71, 108, 152]) and in Boltzmann gases (cf. [34, 81] and the reference therein). Also the study of heat transfer due to conduction and radiation as well as homogenization problems have been studied, we refer to the literature of our previous article [36].

Finally, we want to mention that free boundary problems where heat is transported by conduction and radiation have been considered numerically in engineering applications in terms of melting problems (see for instance [28,124,129,130,140]) and in numerical applications in the context of vaporization problems for droplets (cf. [2, 84, 92, 126, 128, 145, 150])

E.1.2 Main results, plan of the paper and notation

In this paper we will study problem (E.3) and we will see that the addition of the radiative operator to the one-dimensional two-phase Stefan problem yields interesting phenomena which differs from the well-known results for the classical Stefan problem. Specifically, we will also show that there exist traveling wave solutions for the problem (E.3). This is very different from the classical two-phase Stefan problem, for which self-similar profiles exist while traveling wave solutions are impossible to obtain. We will show also that the interface moves towards the liquid region, i.e. in our case $\dot{s} = -c < 0$, implying that the traveling wave solutions exist only when the solid expands.

Theorem E.1. *There exists $c_{\max} > 0$ such that for any $c \in (0, c_{\max}]$ there exists a solution to the system* (E.3) *(without initial condition), such that the interface satisfies $s(t) = -ct$ and the temperature is a traveling wave defined by $T(y) = T(x - s(t))$ with $T > T_M$ for $y < 0$ and $T < T_M$ for $y > 0$. Moreover, for $c < 0$ there exists no solution with $s(t) = -ct$ such that T is a traveling wave. Finally, for any $c \in (0, c_{\max}]$ and for T_M small enough the traveling waves are unique.*

We will see that also for $c = 0$ traveling waves exist. However, in this case the asymptotic behavior as $y \to \infty$ is more involved and it has not been considered yet.

The results of Section E.2 and of Section E.3 will imply Theorem E.1. Specifically, in Section E.2 we will show the existence of traveling wave solutions in the case of negative speed of the interface, i.e. when the ice is expanding. While the traveling waves in the liquid are given by the well-known solution to the ODE $\ddot{y} = \lambda \dot{y}$ for $x < 0$, the existence of traveling wave solutions in the solid is more involved. By a variational argument we will prove the existence of such traveling waves (cf. subsection E.2.1), which will be shown to be monotonically increasing with respect to the melting temperature ((cf. subsection E.2.2)). In Subsection E.2.3 we will also show that for very small melting temperatures there exists a unique strictly positive traveling wave solution, which also converges with exponential rate to a positive constant as $x \to \infty$. In Section E.3 the analysis of the traveling wave is carried on. In particular several applications of the maximum principle will be used together with blow-up limits, Liouville-type theorems, and Harnack-type arguments in order to show that the traveling waves have a limit as $x \to \infty$. Finally, in Section E.4 we will conclude this paper using asymptotic arguments with a formal picture of the long time asymptotic of the solutions to (E.3) for arbitrary values of $\lim_{y\to-\infty} T(y) = T(-\infty)$ and $\lim_{y\to\infty} T(y) = T(\infty)$.

Throughout this article we will denote by $C^{k,\beta}(U)$, where $U \subseteq \mathbb{R}$ is possibly unbounded, the space of k-times continuous differentiable functions f with

$$\|f\|_{k,\beta} = \max_{0\leq j\leq k} \left(\sup_U \left|\partial_x^j f\right| \right) + \sup_{x,y\in U} \frac{\left|\partial_x^k f(x) - \partial_x^k f(y)\right|}{|x-y|^\beta} < \infty.$$

Notice that $f \in C^{k,\beta}(U)$ has all k derivatives bounded.

E.2 Existence of traveling wave solutions

In the following section we construct traveling wave solutions solving (E.5) and we will prove some important properties satisfied by such functions. First of all, we will see that the traveling waves propagate necessarily with negative velocity. Hence, the interface is moving towards the liquid part and the ice is expanding. This behavior is intriguing at a first glance. However, it can be expected. Indeed, while in the liquid the heat is transferred only by conduction, in the solid the heat is also transferred by radiation. Since a radiative source is absent in this problem, the radiation escaping from the solid is helping the ice to cool faster.

Recall that the system (E.5) has been obtained considering solutions to the original problem (E.4) of the form $T(t, x) = T(t, x - s(t)) := T(y)$ and $s(t) = -ct$ with $c \in \mathbb{R}$. First of all we see that in order to obtain the existence of bounded solutions to the problem (E.5) c must be positive, thus since $\dot{s}(t) = -c < 0$ the ice is expanding. Indeed, if $c < 0$ then the temperature of the liquid should satisfy

$$\begin{cases} c\partial_y T_1(y) = \kappa \partial_y^2 T_1(y) & y < 0 \\ T_1(0) = T_M \end{cases} \tag{E.6}$$

and hence

$$T_1(y) = T_M + \left|\frac{A}{c}\right| \kappa \left(e^{-\frac{|c|}{\kappa} y} - 1\right) \to \infty \quad \text{as } y \to -\infty. \tag{E.7}$$

Let thus $c > 0$. In the next subsections we will prove the following theorem.

Theorem E.2. *For $c < 0$ the problem (E.5) does not admit any bounded solution. However, there exists $c_{\max} > 0$ such that for any $c \in (0, c_{\max}]$ there exists traveling waves T_1, T_2 solving (E.5). Moreover, for $c \in (0, c_{\max}]$ the solutions satisfy $T_1(y) > T_M$ for $y < 0$ and $0 < T_2(y) < T_M$ for $y > 0$, and the limits $\lim_{y\to-\infty} T_1(y)$ and $\lim_{y\to\infty} T_2(y)$ exist.*

Proof. As we have seen in (E.6) and in (E.7), if $c < 0$ the problem (E.5) does not have any bounded solution. Thus, we set $c > 0$ and we see that for any c and any $\alpha \in \mathbb{R}$ the solution to

$$\begin{cases} c\partial_y T_1(y) & y < 0 \\ T_1(0) = T_M \\ \partial_y T_1(0) = -A \end{cases}$$

is given by

$$T_1(y) = T_M + \frac{A}{c}\kappa \left(1 - e^{\frac{c}{\kappa} y}\right).$$

Moreover, $\lim_{y\to-\infty} T_1(y) = T_M + \frac{A}{c}\kappa$. Since T_1 describes the temperature in the liquid, we are interested only in $A \geq 0$.

In the following subsections we will study

$$\begin{cases} \partial_y^2 f(y) - c\partial_y f(y) - f^4(y) = -\int_0^\infty \alpha \frac{E_1(\alpha(y-\eta))}{2} f^4(\eta) d\eta & y > 0 \\ f(0) = T_M \\ f \geq 0 \end{cases} \tag{E.8}$$

We will prove the existence of functions $f \in C^{2,1/2}(\mathbb{R}_+)$ solving the problem (E.8). We will show also that there exists $c_{\max} > 0$ such that $\partial_y f(0^+) \leq -Lc$ for all $0 < c < c_{\max}$. Then for $c \in (0, c_{\max})$ and $A = -\frac{Lc + \partial_y T_2(0^+)}{K}$ the functions $T_1(y) = T_M + \frac{A}{c}\kappa\left(1 - e^{\frac{c}{\kappa} y}\right)$ and $T_2 := f$ are traveling waves solving (E.5). □

Before moving to the existence theory for the solutions to (E.8) we do the following remark. It is enough to prove that the traveling wave solutions in the solid exists, that they are bounded from below and have a limit only for $\alpha = 1$ and $c > 0$. Indeed, let $\alpha > 0$, $c > 0$ and $T_M > 0$ and let f solve (E.8). Then the function $\tilde{f}$ defined by

$$f(y) := \alpha^{2/3} \tilde{f}(\alpha y) = \alpha^{2/3} \tilde{f}(\eta)$$

satisfies the following equation

$$\begin{cases} \partial_\eta^2 \tilde{f}(\eta) - c\alpha^{-1}\partial_\eta \tilde{f}(\eta) - \tilde{f}^4(\eta) = -\int_0^\infty \frac{E_1(\eta-\xi)}{2} \tilde{f}^4(\xi) d\xi & \eta > 0 \\ \tilde{f}(0) = T_M \alpha^{-2/3} \\ \tilde{f} \geq 0 \end{cases} \tag{E.9}$$

This is true since $\partial_y f(y) = \alpha^{5/3} \partial_\eta \tilde{f}(\eta)$ as well as $\partial_y^2 f(y) = \alpha^{8/3} \partial_\eta^2 \tilde{f}(\eta)$. Notice also that f and $\tilde{f}$ have the same regularity. Moreover, using that $\eta = \alpha y$ and changing the variable $\alpha\xi = z$ we have

$$\int_0^\infty \alpha \frac{E_1(\alpha(y-\xi))}{2} f^4(\xi) d\xi = \int_0^\infty \alpha \frac{E_1(\eta - \alpha\xi)}{2} \alpha^{8/3} \tilde{f}^4(\alpha\xi) d\xi = \alpha^{8/3} \int_0^\infty \frac{E_1(\eta - z)}{2} \tilde{f}^4(z) dz.$$

Hence, we see that defining $E(x) = \frac{E_1(x)}{2}$ it is enough to consider the solutions to

$$\begin{cases} \partial_y^2 f(y) - c\partial_y f(y) - f^4(y) = -\int_0^\infty E(y-\eta) f^4(\eta) d\eta \quad y > 0 \\ f(0) = T_M \\ f \geq 0 \end{cases} \tag{E.10}$$

E.2.1 Existence of traveling wave solutions for $y > 0$

Before proving the existence of traveling wave solutions for $y > 0$ we prove the following technical proposition.

Proposition E.1. *Let $c > 0$ and $g \in C^{0,1/2}(\mathbb{R}_+)$ with $-A^4 < g \leq 0$ for some $A > 0$. Let also*

$$\mathcal{A}_{A,c} = \left\{ f \geq 0 \text{ measurable s.t. } f \in W^{1,2}\left(e^{-cy}dy, \mathbb{R}_+\right) \cap L^5\left(e^{-cy}dy, \mathbb{R}_+\right),\ f(0) = A > 0 \right\}.$$

Then the functional

$$I_g[f] = \int_0^\infty e^{-cy}\left(\frac{(\partial_y f(y))^2}{2} + \frac{f(y)^5}{5} + g(y)f(y)\right) dy$$

has a unique minimizer $f \in \mathcal{A}_{A,c}$. Moreover, $0 < f \leq A$ for $y \in [0,\infty)$. Finally, $f \in C^{2,1/2}(\mathbb{R}_+)$ solves the ODE

$$\partial_y\left(e^{-cy}\partial_y f(y)\right) = \left(f^4(y) + g(y)\right)e^{-cy}$$

and satisfies the bounds

$$|f'(y| \leq \frac{A^4}{c},\ |f''(y)| \leq A^4 \text{ and } [f'']_{1/2} \leq \max\left\{2A^4, 2A^4 c + \frac{4A^7}{c} + [g]_{1/2}\right\}.$$

Proof. Let us define the measure μ given by the density $d\mu(y) = e^{-cy}dy$. First of all we notice that if $f \in W^{1,2}(\mu,\mathbb{R}_+) \cap L^5(\mu,\mathbb{R}_+)$, then $fe^{-c/2y} \in W^{1,2}(\mathbb{R}_+)$. Thus, by Morrey's embedding theorem $fe^{-c/2y} \in C^{0,1/2}(\mathbb{R}_+)$. Hence, if $f \in \mathcal{A}$, then f is continuous. This implies that the condition for $f \in \mathcal{A}_{A,c}$ to be $f \geq 0$ holds everywhere in $\mathbb{R}_+$ as well as the boundary condition $f(0) = A$, which for general functions in $W^{1,2}(\mu,\mathbb{R}_+)$ is to be intended as trace condition, holds pointwise. These observations yield that $\mathcal{A}_{A,c}$ is a closed and convex subset of $W^{1,2}(\mu,\mathbb{R}_+) \cap L^5(\mu,\mathbb{R}_+)$. We also remark that the trace operator for $\partial\mathbb{R}_+ = \{0\}$ is a continuous operator with respect to the norm $\|\cdot\|_{W^{1,2}(\mu\mathbb{R}_+)}$.

Further, we notice that I_g is well-defined for $f \in \mathcal{A}_{A,c}$ with

$$|I_g[f]| \leq \frac{1}{2}\|f\|^2_{W^{1,2}(\mu)} + \frac{1}{5}\|f\|^5_{L^5(\mu)} + \frac{1}{2c}\|g\|^2_\infty.$$

Moreover, $I_g[f]$ is bounded from below and coercive. Indeed, using both Young's inequality

$$|g(y)|f(y) \leq \frac{8\cdot 2^{1/4}}{5c}|g(y)|^{5/4} + \frac{f^5(y)}{10}$$

and Hölder's inequality

$$\left(\int_0^\infty e^{-cy}|f(y)|^2 dy\right)^{5/2} \leq \frac{1}{c^{3/2}}\|f\|^5_{L^5(\mu)}$$

we estimate

$$I_0[f] \geq \min\left\{\frac{c^{3/2}}{10}, \frac{1}{2}\right\}\left(\|\partial_y f\|^2_{L^2(\mu)} + \|f\|^5_{L^2(\mu)}\right) + \frac{1}{10}\|f\|^5_{L^5(\mu)} \to \infty \quad \text{as } \|f\|_{\mathcal{A}} \to \infty$$

if $g \equiv 0$ and

$$I_g[f] \geq \min\left\{\frac{c^{3/2}}{20}, \frac{1}{2}\right\}\left(\|\partial_y f\|^2_{L^2(\mu)} + \|f\|^5_{L^2(\mu)}\right) + \frac{1}{20}\|f\|^5_{L^5(\mu)} - \frac{8\cdot 2^{1/4}}{5c}\|g\|^{4/5}_\infty \to \infty$$

as $\|f\|_{\mathcal{A}} \to \infty$ if $g \not\equiv 0$. Moreover, $I_0[f] \geq 0$ as well as $I_g[f] \geq -\frac{8\cdot 2^{1/4}}{5c}\|g\|^{4/5}_\infty$.

Therefore, there exists a bounded minimizing sequence $f_k \in \mathcal{A}_{A,c}$ such that $I_g[f_k] \to \inf_{f\in\mathcal{A}_{A,c}} I[f]$ as $k \to \infty$. The boundedness of this sequence, the uniqueness of the weak and strong limit as well as the fact that $L^2(\mu) \subset L^{5/4}(\mu) = \left(L^5(\mu)\right)^*$ imply the existence of a common subsequence f_{k_j} such that

$$f_{k_j} \xrightarrow[\text{ptw. a.e.}]{L^2(\mu)} f \in L^2(\mu) \quad \text{and} \quad f_{k_j} \xrightarrow[\text{weak } L^5(\mu)]{\text{weak } W^{1,2}(\mu)} f \in W^{1,2}(\mu) \cap L^5(\mu) \quad \text{as } j \to \infty.$$

The closedness and the convexity of $\mathcal{A}_{A,c}$ imply also $f \in \mathcal{A}_{A,c}$. Moreover, the pointwise convergence almost everywhere and the weak lower semicontinuity of the L^2 norm imply the weak lower semicontinuity of the functional I_g. Hence, f is a minimizer of I_g, i.e. $I_g[f] = \inf_{f\in\mathcal{A}_{A,c}} [f]$. In addition to that, since the functional I_g is strictly convex for non-negative functions, the minimizer is unique.

We remark that $f \in C^{0,1/2}_{\text{loc}}(\mathbb{R}_+)$ with $f(y) \geq 0$ for $y \geq 0$ and $f(0) = A$. Next we prove that $f \leq T_M$ if $g \equiv 0$ and that $f \leq 5A$ if $g \not\equiv 0$. Both claims are a consequence of the uniqueness of the minimizer of I_g in $\mathcal{A}_{A,c}$. If $g \equiv 0$ let us consider $h_0 = \min\{f,\ A\} \in \mathcal{A}_{A,c}$, since the minimum of two Sobolev functions is a Sobolev function. Then the functional I_0 acting on h_0 gives

$$\begin{aligned} I_0[h_0] &= \int_0^\infty e^{-cy}\left(\frac{|\partial_y f|^2}{2}\mathbb{1}_{\{f\leq A\}} + \frac{h_0^5}{5}\right)dy \\ &\leq \int_0^\infty e^{-cy}\left(\frac{|\partial_y f|^2}{2} + \frac{f^5}{5}\right)dy = I_0[f] = \inf_{f\in\mathcal{A}_{A,c}} I_0[f], \end{aligned}$$

where we used that $0 \leq h_0 \leq f$. By uniqueness we conclude $0 \leq f \leq A$. In a similar way, if $g \not\equiv 0$ we consider $h_1 = \min\{f, 5A\}$. It is not difficult to see that

$$\begin{aligned} \mathbb{1}_{\{f>5A\}}\left(\frac{h_1^5}{5} - |g|h_1\right) &= \mathbb{1}_{\{f>5A\}}\left(5^3A^4 - |g|\right)5A \\ &< \mathbb{1}_{\{f>5A\}}\left(\frac{f^4}{5} - |g|\right)5A < \mathbb{1}_{\{f>5A\}}\left(\frac{f^5}{5} - |g|f\right). \end{aligned}$$

For this chain of inequalities we used the definition of h_1 and the fact that $|g| < A^4$. Therefore $0 < \left(5^3A^4 - |g|\right) < \left(\frac{f^4}{5} - |g|\right)$ in the set $\{f > 5A\}$. We conclude

$$\begin{aligned} I_g[h_1] &= \int_0^\infty e^{-cy}\left(\frac{|\partial_y f|^2}{2}\mathbb{1}_{\{f\leq 5A\}} + \frac{h_1^5}{5} + gh_1\right)dy \\ &\leq \int_0^\infty e^{-cy}\left(\frac{|\partial_y f|^2}{2} + \frac{f^5}{5} + gf\right)dy = I_g[f] = \inf_{f\in\mathcal{A}_{A,c}} I_g[f]. \end{aligned}$$

Hence, $f = h \leq 5A$. These results show that $f \in C_b(\mathbb{R}_+)$.

We now study the Euler-Lagrange equations associated to the functional I_g. It turns out that the minimizer f is the weak solution of the following inequality

$$-\partial_y \left(e^{-cy}\partial_y f\right) + e^{-cy} f^4(y) + e^{-cy} g \geq 0. \tag{E.11}$$

Hence, f satisfies

$$0 \leq \int_0^\infty e^{-cy} \left(\partial_y f \partial_y \psi + f^4(y)\psi(y) + g(y)\psi(y)\right) dy, \tag{E.12}$$

for all $\psi \geq 0$, $\psi \in C_c^\infty(\mathbb{R}_+)$ or also $\psi \in W_0^{1,2}(\mu) \cap L^5(\mu)$. Moreover, on the open set $\{f > 0\}$ the minimizer f is a weak solution of the equation

$$-\partial_y \left(e^{-cy}\partial_y f\right) + e^{-cy} f^4(y) + e^{-cy} g(y) = 0. \tag{E.13}$$

Indeed, on the open set $\{f > 0\}$ for any $\psi \in C_c^\infty(\{f > 0\})$ the function $f + \varepsilon\psi \in \mathcal{A}$ for $\varepsilon > 0$ small enough. Hence

$$0 = \partial_\varepsilon I[f + \varepsilon\psi]\big|_{\varepsilon=0} = \int_0^\infty e^{-cy} \left(\partial_y f \partial_y \psi + f^4(y)\psi(y) + g(y)\psi(y)\right) dy, \tag{E.14}$$

for all $\psi \in C_c^\infty(\{f > 0\})$ or also $\psi \in W_0^{1,2}(\mu, \{f > 0\}) \cap L^5(\mu, \{f > 0\})$. We remark that equations (E.11)-(E.14) hold for both $g \equiv 0$ and $g \not\equiv 0$.

We aim to show that actually the minimizer f is a strong solution to (E.13) in the whole real line. To this end we will first show that $\{f > 0\} = \mathbb{R}_+$, which implies that f is a weak solution of (E.13) in $\mathbb{R}_+$, and finally we will use elliptic regularity theory for (E.13).

Let us assume that $\{f > 0\} \subsetneq \mathbb{R}_+$. Then there exists $a \in \mathbb{R}_+$ such that $f(y) > 0$ for all $y < a$ and $f(a) = 0$. We have to consider two cases: first the case where $f(y) \equiv 0$ in an interval $(a, a+r)$ for some $r > 0$ and second the case where $f(y) \not\equiv 0$ on the interval $(a, a+r)$ for any $r > 0$.

Let us assume first that there exists $r > 0$ such that $f(y) = 0$ for all $y \in (a, a + r)$. Since f is continuous there exists $0 < \varepsilon < \min\left\{r, \frac{c}{2}\right\}$ small enough such that $f(a - \varepsilon) = \delta \ll 1$ as well as $f(a + \varepsilon) = 0$. Let us define for $y \in [a - \varepsilon, a + \varepsilon]$ the following function

$$\bar{f}(y) = \delta \left(1 - \frac{y - (a - \varepsilon)}{2\varepsilon}\right).$$

It is easy to see that $0 < \bar{f} < \delta < 1$ for $y \in (a - \varepsilon, a + \varepsilon)$, $\bar{f}(a - \varepsilon) = f(a - \varepsilon)$ as well as $\bar{f}(a + \varepsilon) = f(a + \varepsilon) = 0$. Moreover, $\bar{f}(a) = \frac{\delta}{2} > 0$. Finally, since $\bar{f}^4 \leq \delta$ and $2\varepsilon < c$, an easy computation shows

$$-\bar{f}''(y) + c\bar{f}'(y) + \bar{f}'(y) = -\frac{c\delta}{2\varepsilon} + \bar{f}^4(y) \leq \delta \left(1 - \frac{c}{2\varepsilon}\right) < 0. \tag{E.15}$$

Thus, $-\left(e^{-cy}\bar{f}'(y)\right)' + e^{-cy}\bar{f}^4(y) < 0$. Since $f(a) = 0 < \bar{f}(a)$, there exists an interval $(y_0, y_1) \subseteq (a-\varepsilon, a+\varepsilon)$ such that $f(y_0) = \bar{f}(y_0)$, $f(y_1) = \bar{f}(y_1)$ and $f(y) < \bar{f}(y)$ for $y \in (y_0, y_1)$. Using the weak maximum principle we show now that this is not possible. Therefore, we test (E.15) with a suitable test function $\psi \geq 0$. Let us consider the smooth solution to

$$\begin{cases} \partial_y^2 \psi(y) - c\partial_y \psi(y) = -1 \quad (y_0, y_1); \\ \psi(y_0) = \psi(y_1) = 0 \end{cases} \tag{E.16}$$

The solution is given by the explicit formula $\psi(y) = \frac{y-y_0}{c} - \frac{y_1-y_0}{c}\frac{e^{c(y-y_0)}-1}{e^{c(y_1-y_0)}-1}$. By a simple application of the maximum principle we see that $\psi > 0$ in (y_0, y_1). Indeed, if ψ would have a minimum at $y^* \in (y_0, y_1)$ on that point ψ would not solve the equation, since $\psi''(y^*) - c\psi'(y^*) \geq 0$. Hence, let us consider $\bar{\psi}$ as the extension by 0 of ψ in the whole positive real line, i.e.

$$\bar{\psi}(y) = \begin{cases} \psi(y) & y \in (y_0, y_1) \\ 0 & \text{else.} \end{cases}$$

Clearly $\bar{\psi} \in W_0^{1,2}(\mathbb{R}_+, \mu) \cap L^5(\mathbb{R}_+, \mu)$. Then,

$$-\left(e^{-cy}\bar{f}'(y)\right)'\bar{\psi}(y) + e^{-cy}\bar{f}^4(y)\bar{\psi}(y) \leq 0,$$

where we used that $\bar{\psi} \equiv 0$ on $\mathbb{R}_+ \setminus (a-\varepsilon, a+\varepsilon)$. Therefore, using also that the weak derivative of $\bar{\psi}$ is supported also on $[a-\varepsilon, a+\varepsilon]$ we obtain

$$\int_0^\infty e^{-cy}\left(\bar{f}'(y)\bar{\psi}'(y) + \bar{f}^4(y)\bar{\psi}(y)\right)dy \leq 0. \tag{E.17}$$

Hence, using (E.12), (E.17) and the definition of $\bar{\psi}$ we have

$$\begin{aligned} 0 &\leq \int_0^\infty e^{-cy}\left(\partial_y(f-\bar{f})\partial_y\bar{\psi} + \left(f^4 - \bar{f}^4\right)\bar{\psi} + g\bar{\psi}\right)dy \\ &= \int_{y_0}^{y_1} (f-\bar{f})\partial_y\left(-e^{-cy}\partial_y\psi\right) + e^{-cy}\left(f^4-\bar{f}^4\right)\psi + e^{-cy}g\psi dy \\ &= \int_{y_0}^{y_1} e^{-cy}\left((f-\bar{f})\left(-\partial_y^2\psi + c\partial_y\psi\right) + \left(f^4-\bar{f}^4\right)\psi + g\psi\right)dy < 0 \end{aligned} \tag{E.18}$$

where we used also that $\left(f-\bar{f}\right)|_{\{y_0,y_1\}} = 0$, $0 \leq f < \bar{f}$ on (y_0, y_1) as well as $g \leq 0$. This contradiction implies that $f(y) \geq \bar{f}(y) > 0$ on $(a-\varepsilon, a+\varepsilon)$. But since we assumed $f(a) = 0 < \bar{f}(a)$ we conclude that there cannot exist any $r > 0$ such that $f(y) = 0$ for $y \in (a, a+r)$.

Hence, we assume that $f(y) \not\equiv 0$ for $y \in (a, a+r)$ and $r > 0$. Since $f(a) = 0$ by continuity there exist $0 < \varepsilon_1, \varepsilon_2 < \min\{r, \frac{c}{4}\}$ small enough such that $f(a-\varepsilon_1) = \delta \ll 1$ and $f(a+\varepsilon_2) = \frac{\delta}{2}$. We then define for $y \in [a-\varepsilon_1, a+\varepsilon_2]$ the function

$$\bar{f}(y) = \delta\left(1 - \frac{y-(a-\varepsilon_1)}{2(\varepsilon_1+\varepsilon_2)}\right).$$

Also in this case $\bar{f}$ satisfies $0 < \frac{\delta}{2} < \bar{f} < \delta < 1$ for $y \in (a-\varepsilon_1, a+\varepsilon_2)$, $\bar{f}(a-\varepsilon_1) = f(a-\varepsilon_2)$, $\bar{f}(a+\varepsilon_2) = f(a+\varepsilon_2)$, $\bar{f}(a) \geq \frac{\delta}{2} > 0$, as well as

$$-\bar{f}''(y) + c\bar{f}'(y) + \bar{f}'(y) = -\frac{c\delta}{2(\varepsilon_1+\varepsilon_2)} + \bar{f}^4(y) \leq \delta\left(1 - \frac{c}{2(\varepsilon_1+\varepsilon_2)}\right) < 0.$$

We now argue as in the case $f(a+\varepsilon_2) = 0$. As we have seen before, since $f(a) = 0 < \bar{f}(a)$, there exists an interval $(y_0, y_1) \subseteq (a-\varepsilon, a+\varepsilon)$ such that $f(y_0) = \bar{f}(y_0)$, $f(y_1) = \bar{f}(y_1)$ and $f(y) < \bar{f}(y)$ for $y \in (y_0, y_1)$. Then, testing $f - \bar{f}$ against the function $\bar{\psi}$ defined as the zero extension of ψ in (E.16) we obtain the following contradiction as for (E.18)

$$0 \leq \int_0^\infty e^{-cy}\left(\partial_y(f-\bar{f})\partial_y\bar{\psi} + \left(f^4-\bar{f}^4\right)\bar{\psi} + g\bar{\psi}\right)dy < 0.$$

This contradiction yields that $\{f > 0\} = \mathbb{R}_+$. Thus, f is a weak solution to (E.13).

In the case where $g \not\equiv 0$, we proved that $f \leq 5A$. We now prove that also $f \leq A$ holds. To this end we consider for $R > 0$ the function $\phi_R(y)$ defined by $\phi_R(y) = A + 4Ae^{c(y-R)} \geq A$. We see that $\phi_R(0) > A = f(0)$ as well as $\phi_R(R) = 5A \geq f(R)$. By continuity we know that there exists some $x_0 \in [0, R]$ such that $\min_{[0,R]} \psi_R - f = \psi_R(x_0) - f(x_0)$. Hence, let us assume that $\min_{[0,R]} \psi_R - f = \psi_R(x_0) - f(x_0) < 0$. Since $\phi_R - f\mid_{\{0,R\}} \geq 0$, there exists an interval $x_0 \in (a, b) \subset [0, R]$ in which $\phi_R - f < 0$ and $\phi_R - f(a) = \phi_R(b) - f(b) = 0$. We also see that ϕ_R is a supersolution for the operator $\mathcal{L}[\phi] = -\phi'' + c\phi' + \phi^4 + g$ on $[0, R]$. Indeed

$$\mathcal{L}[\phi_R] = \phi_R^4 + g > A^4 - A^4 = 0.$$

Let us consider once again the zero extension $\bar{\psi}$ of the function $\psi > 0$ given by (E.16) on the interval (a, b). Then we see that

$$0 \leq \int_0^\infty e^{-cy} \left(\partial_y \phi_R \partial_y \bar{\psi} + \phi_R^4 \psi + g(y)\bar{\psi}(y) \right) dy.$$

Therefore we obtain the following contradiction using once more that $(f - \phi_R)\mid_{\{a,b\}} = 0$, that $0 < \phi_R < f$ on (a, b), and that f is a weak solution solving (E.13)

$$\begin{aligned} 0 &\leq \int_0^\infty e^{-cy} \left(\partial_y(\phi_R - f)\partial_y \bar{\psi} + \left(\phi_R^4 - f^4\right) \bar{\psi} \right) dy \\ &= \int_a^b e^{-cy} \left((\phi_R - f) \left(-\partial_y^2 \psi + c\partial_y \psi\right) + \left(\phi_R^4 - f^4\right) \psi \right) dy \\ &= \int_a^b e^{-cy} \left((\phi_R - f) + \left(\phi_R^4 - f^4\right) \psi \right) dy < 0. \end{aligned}$$

Hence, for any $y \in [0, R]$ we have $f(y) \leq A + 4Ae^{c(y-R)}$. Letting now $R \to \infty$, we conclude that $0 \leq f \leq A$.

We finish the proof of Proposition E.1 showing that f is also a strong solution to (E.13). This can be proved using the elliptic Schauder regularity. Indeed, since $f \in \mathcal{A}_{A,c}$ is bounded and continuous, we have that $f \in W^{1,2}(\mu) \cap L^\infty(\mathbb{R}_+)$. Hence, $f \in W^{1,2}_{\text{loc}}(\mathbb{R}_+, dy) \cap L^\infty(\mathbb{R}_+)$, so that also $f^4 e^{-cy} \in W^{1,2}_{\text{loc}}(\mathbb{R}_+, dy) \cap L^\infty(\mathbb{R}_+)$. Morrey's embedding theorem implies that $f \in C^{0,1/2}_{\text{loc}}(\mathbb{R}_+)$, which yields also $f^4 e^{-cy} \in C^{0,1/2}_{\text{loc}}(\mathbb{R}_+)$. Applying now the elliptic regularity theory to the equation (E.13) we obtain that $f \in C^{2,1/2}_{\text{loc}}(\mathbb{R}_+)$ since also $ge^{-cy} \in C^{0,1/2}(\mathbb{R}_+)$. Thus, $f \in C^2(\mathbb{R}_+)$ is a strong solution to (E.13).

We now show that f has also bounded first and second derivative. This is due to the fact that also $f' \in W^{1,2}(\mu)$. Indeed,

$$\begin{aligned} \int_0^\infty e^{-cy} \left(|f''|^2 + |f'|^2\right) dy &\leq \int_0^\infty e^{-cy} \left(|f^4 + cf' + g|^2 + |f'|^2\right) dy \\ &\leq C(A, c) \left(\|f\|_{W^{1,2}(\mu)} + \frac{A^8}{c} \right). \end{aligned}$$

Hence, $e^{-\frac{c}{2}y} f' \in W^{1,2}(\mathbb{R}_+, dy)$, which implies that $e^{-cy}(f')^2$ is bounded since its derivative $2e^{cy} f' f'' - ce^{-cy}(f')^2$ is integrable. Thus, the consequent boundedness of $e^{-\frac{c}{2}y} |f'|$ implies that

$$\lim_{y \to \infty} e^{-cy} |f'|(y) = 0. \tag{E.19}$$

Since f solves (E.13), using (E.19) and integrating in (y, ∞) we obtain the desired estimate

$$|f'|(y) \le e^{cy} \int_y^\infty e^{-c\xi} \left|f^4(\xi) + g(\xi)\right| d\xi \le \frac{A^4}{c}.$$

Moreover, multiplying (E.13) by e^{cy} we conclude that f is a C^2-solution to

$$f'' - cf' = f^4 + g \text{ on } \mathbb{R}_+.$$

This yields the boundedness of the second derivative of f as

$$|f''|(y) \le c|f'|(y) + \left|f^4(y) + g(y)\right| \le A^4,$$

where we used also $0 \le f^4 \le A^4$ and $-A^4 \le g \le 0$. These estimates imply that $f \in C^{1,1}(\mathbb{R}_+)$ with bounded first and second derivatives. Since $cf' + f^4 + g \in C^{0,1/2}(\mathbb{R}_+)$ we conclude that $f \in C^{2,1/2}(\mathbb{R}_+)$ with Hölder seminorm bounded by

$$\begin{aligned}[f'']_{1/2} &\le \max\left\{2\|f''\|_\infty, c\|f''\|_\infty + 4\|f\|_\infty^3\|f'\|_\infty + [g]_{1/2}\right\} \\ &\le \max\left\{2A^4, 2A^4c + \frac{4A^7}{c} + [g]_{1/2}\right\}.\end{aligned}$$

□

Let us now consider the sequence $f_n \in C^2(\mathbb{R}_+)$ with $f_n \ge 0$ such that

$$\begin{cases} \partial_y^2 f_{n+1}(y) - c\partial_y f_{n+1}(y) - f_{n+1}^4(y) = -\int_0^\infty \alpha E(y-\eta) f_n^4(\eta) d\eta & y > 0; \quad n \ge 1 \\ f_0 = 0 & n = 0 \\ f_{n+1}(0) = T_M \\ f_{n+1} \ge 0 \end{cases} \tag{E.20}$$

We prove the following theorem

Theorem E.3. *Let $T_M, c > 0$. Then there exists a solution $f \in C^{2,1/2}(\mathbb{R}_+)$ with $f > 0$ at the interior of $\mathbb{R}_+$ solving* (E.10)*. Moreover, f is obtained as the limit of the monotone increasing bounded sequence*

$$0 \le f_1 \le f_2 \le ... \le f_n \le f_{n+1} \le ... \le T_M$$

with $(f_n)_{n\in\mathbb{N}} \in C^{2,1/2}(\mathbb{R}_+)$ with $\|f_n\|_{2,1/2}$ uniformly bounded and with $f_n > 0$ in the interior of $\mathbb{R}_+$ solving the recursive system (E.20)*.*

Proof. We start considering the function f_1 solving the problem

$$\begin{cases} \partial_y^2 f_1(y) - c\partial_y f_1(y) - f_1^4(y) = 0 & y > 0; \\ f_1(0) = T_M \\ f_1 \ge 0 \end{cases} \tag{E.21}$$

The differential equation is equivalent to the elliptic ODE

$$\left(e^{-cy} f'\right)' = e^{-cy} f^4.$$

Hence, we consider the minimization problem of the functional

$$I_0[f] = \int_0^\infty e^{-cy} \left(\frac{(\partial_y f(y))^2}{2} + \frac{f(y)^5}{5}\right) dy$$

on the set

$$\mathcal{A}_{T_M,c} = \left\{f \geq 0 \text{ measurable s.t. } f \in W^{1,2}\left(e^{-cy}dy, \mathbb{R}_+\right) \cap L^5\left(e^{-cy}dy, \mathbb{R}_+\right),\ f(0) = T_M\right\}. \tag{E.22}$$

Proposition E.1 shows that there exists a unique $f_1 \in \mathcal{A}_{T_M,c}$ minimizing the functional I_0. Moreover, f_1 solves (E.21) and satisfies $0 < f_1(y) \leq T_M$ for $y \geq 0$. In addition to that, $f_1 \in C^{2,1/2}(\mathbb{R}_+)$ has bounded first and second derivative according to

$$|f_1'(y)| \leq \frac{T_M^4}{c} \text{ and } |f_2''(y)| \leq T_M^4$$

and Hölder seminorm bounded by

$$[f_1'']_{1/2} \leq \max\{2T_M^4, 2T_M^4 c + \frac{4T_M^7}{c}\}.$$

We now show the existence of the solutions $f_n \in C^{2,1/2}(\mathbb{R}_+)$ of the equation (E.20) for $n \geq 2$. We do the proof only for $n = 2$, since the very same arguments will work recursively for all $n \geq 2$. Let us define $g = -\int_0^\infty E(y-\eta)f_1^4(\eta)d\eta$. We readily see that $-T_M^4 < g < 0$. Moreover, since $f_1^4 \in C^1(\mathbb{R}_+)$ with bounded derivative we conclude that $g \in C^{0,1/2}(\mathbb{R}_+)$ with the seminorm $[\cdot]_{1/2}$ bounded by

$$[g]_{1/2} \leq \max\{2\|g\|_\infty, 4\|f_1\|_\infty^3\|f_1'\|_\infty + \|f_1\|_\infty^4\|E\|_{L^2}\} \leq \max\left\{2T_M^4, \frac{4T_M^7}{c} + T_M^4\|E\|_{L^2}\right\}.$$

Indeed, the normalized exponential integral has the property that $E \in L^q(\mathbb{R})$for any $q \in [1,2]$, since $E \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. This yields together with the Hölder 's inequality that for $b > a > 0$ and $\delta \in [0, 1/2]$

$$\int_a^b E(\eta)d\eta \leq |a-b|^\delta \|E\|_{L^{\frac{1}{1-\delta}}}.$$

Therefore, for $v \in C^{0,\delta}(\mathbb{R}_+)$ and $y > x > 0$ we estimate

$$\begin{aligned}\left|\int_0^\infty v^4(\eta)\left(E((y-\eta)) - E((x-\eta))\right)d\eta\right| &= \left|\int_{-y}^\infty v^4(\eta+y)E(\eta)d\eta - \int_{-x}^\infty v^4(\eta+x)E(\eta)d\eta\right| \\ &\leq \left|\int_{-x}^\infty E(\eta)\left(v^4(\eta+y) - v^4(\eta+x)\right)d\eta\right| + \left|\int_{-y}^{-x} E(\eta)v^4(\eta+y)d\eta\right| \\ &\leq [v^4]_\delta|x-y|^\delta + \|v^4\|_\infty\|E\|_{L^{\frac{1}{1-\delta}}}|x-y|^\delta. \end{aligned} \tag{E.23}$$

We remark that if $v \in C^1(\mathbb{R}_+)$ with bounded derivative and if $|x-y| < 1$, one can estimate

$$\left|\int_0^\infty v^4(\eta)\left(E((y-\eta)) - E((x-\eta))\right)d\eta\right| \leq \|(v^4)'\|_\infty|x-y|^\delta + \|v^4\|_\infty\|E\|_{L^{\frac{1}{1-\delta}}}|x-y|^\delta$$

since also $|(y+\eta) - (x+\eta)| < 1$.

Similarly as for the function f_1, we will consider a suitable minimization problem for which the unique minimizer will be f_2. Let us consider the minimization problem associated to the functional

$$I_g[f] = \int_0^\infty e^{-cy}\left(\frac{(\partial_y f(y))^2}{2} + \frac{f(y)^5}{5} + gf\right)dy$$

on the set $\mathcal{A}_{T_M,c}$ defined in (E.22). Another application of Proposition E.1 shows that there exists $f_2 \in C^{2,1/2}(\mathbb{R}_+)$ solution to (E.20) for $n = 2$ with

$$|f_2'(y)| \leq \frac{T_M^4}{c}, \quad |f_2''(y)| \leq T_M^4 \quad \text{and}$$

$$[f_2'']_{1/2} \leq \max\left\{2T_M^4, 2T_M^4 c + \frac{4T_M^7}{c} + \max\left\{2T_M^4, \frac{4T_M^7}{c} + T_M^4\|E\|_{L^2}\right\}\right\}.$$

Moreover, $0 < f_2(y) \leq T_M$ for $y \geq 0$. A recursive application of Proposition E.1 shows the existence of a sequence $(f_n)_{n\in\mathbb{N}} \in C^{2,1/2}(\mathbb{R}_+)$ with $f_n > 0$ in the interior of $\mathbb{R}_+$ solving the recursive system (E.20). Moreover, for all $n \geq 1$ we have the uniform bounds

$$f_n(y) \leq T_M,\ |f_n'(y)| \leq \frac{T_M^4}{c}, |f_n''(y)| \leq T_M^4$$

and

$$[f_n'']_{1/2} \leq \max\left\{2T_M^4, 2T_M^4 c + \frac{4T_M^7}{c} + \max\left\{2T_M^4, \frac{4T_M^7}{c} + T_M^4\|E\|_{L^2}\right\}\right\},$$

where the uniform bound of the Hölder seminorm is a consequence of the uniform bounds of f_{n-1}, f_{n-1}' and f_{n-1}''. We will now prove that the solutions form a monotonous sequence such that $0 \leq f_1 \leq f_2 \leq ... \leq f_n \leq f_{n+1} \leq T_M$. We only need to show that $f_n \leq f_{n+1}$ for all $n \in \mathbb{N}$. We prove it by induction. Let us consider $n = 1$. Then we define $\varphi = f_2 - f_1$ and

$$a_1(y) = f_1^3(y) + f_2^3(y) + f_1^2(y)f_2(y) + f_1(y)f_2^2(y) > 0.$$

The strict positivity is due to the fact that by construction $f_n > 0$ in any open set of $\mathbb{R}_+$ and in $y = 0$. Let $R > 0$. Then $\varphi(0) = 0$ as well as $|\varphi(R)| \leq T_M$. Moreover,

$$\varphi'' - c\varphi' - a_1(y)\varphi(y) \leq 0.$$

Let us consider now $\psi_R(y) = -T_M e^{c(y-R)}$. Then we have on one hand that $\varphi(0) - \psi_R(0) > 0$ as well as $\varphi(R) - \psi_R(R) \geq 0$ and on the other hand that

$$\psi_R'' - c\psi_R' - a_1(y)\psi_R = -a_1(y)\psi_R \geq 0.$$

Hence, an application of the maximum principle to the function $\varphi - \psi_R$ shows that there is no negative minimum on $[0, R]$ since

$$(\varphi - \psi_R)'' - c(\varphi - \psi_R)' - a_1(\varphi - \psi_R) \leq 0.$$

Therefore, $f_2(y) - f_1(y) \geq -T_M e^{c(y-R)}$ for all $y \leq R$. Hence, for $R \to \infty$ we conclude $f_2 \geq f_1$.

Let us assume now that for $n \in \mathbb{N}$ it is true that $f_{n-1} \leq f_n$. We shall now show that $f_n \leq f_{n+1}$. We define $\varphi_n = f_{n+1} - f_n$ and $a_n(y) = f_n^3(y) + f_{n+1}^3(y) + f_n^2(y)f_{n+1}(y) + f_n(y)f_{n+1}^2(y) > 0$. Moreover, since by induction $0 < f_{n-1} \leq f_n$ we also have that

$$\int_0^\infty E(y-\eta)\left(f_n^4(\eta) - f_{n-1}^4(\eta)\right) d\eta \geq 0.$$

Hence, we have once more that $\varphi_n(0) - \psi_R(0) > 0$ and $\varphi_n(R) - \psi_R(R) \geq 0$ as well as

$$(\varphi_n - \psi_R)'' - c(\varphi_n - \psi_R)' - a_n(\varphi_n - \psi_R) \leq 0$$

on $[0, R]$. We can conclude with the maximum principle that $f_n - f_{n+1} \geq -T_M e^{c(y-R)}$ for all $y \leq R$. This yields the claim $f_n \geq f_{n+1}$.

This concludes the proof of the existence $f_n \in C^{2,1/2}(\mathbb{R}_+)$ with uniformly bounded $C^{2,1/2}$-norm solving the recursive system (E.20) and satisfying

$$0 \leq f_1 \leq f_2 \leq ... \leq f_n \leq f_{n+1} \leq T_M.$$

We now prove the existence of a solution to (E.10). Let $f(y) = \lim_{n\to\infty} f_n(y)$. This function exists, since the sequence is monotone and bounded. Moreover, on any compact set $[0, R]$ the sequence converges also uniformly in $C^{2,1/4}([0, R])$ to the function f. Hence, Lebesgue dominated convergence theorem assures that

$$\int_0^\infty E(y-\eta) f_n^4(\eta) d\eta \to \int_0^\infty E(y-\eta) f^4(\eta) d\eta \quad \text{as } n \to \infty$$

and the C^2-uniform convergence in compact sets implies that $f \in C^2(\mathbb{R}_+) \cap C^{1,1}(\mathbb{R}_+) \cap C^{2,1/2}_{\text{loc}}(\mathbb{R}_+)$ solves (E.10), where the $C^{2,1/2}-$regularity is once again a consequence of elliptic regularity theory. Finally, we prove that $f \in C^{2,1/2}(\mathbb{R}_+)$ globally. Indeed, $f \in C^{1,1}(\mathbb{R}_+)$ solves strongly (E.10). Thus,

$$f'' = cf' + f^4 - \int_0^\infty E(\cdot - \eta) f^4(\eta) d\eta \in C^{0,1/2}(\mathbb{R}_+),$$

where we used that the convolution of a Hölder continuous function with the exponential integral E is Hölder continuous as we have proven in (E.23). This concludes the proof of the existence of traveling wave for (E.5) if $y > 0$. Moreover, the monotonicity of the sequence f_n implies also $f(y) > 0$ for any $y > 0$. □

In order to finish the proof of Theorem (E.2) we have to show the existence of such $c_{\max} > 0$. This will be done in the following Lemma and Corollary.

Lemma E.1. *Let $T_M > 0$ and $c > 0$. Let $f \in C^2(\mathbb{R}_+) \cap C_b(\mathbb{R}_+)$ be a solution to* (E.10) *with $|f| \leq T_M$. Then $f(y) > T_M$ for all $y > 0$ and $\partial_y f(0^+) < 0$.*

Proof. The proof is an adaptation of the proof of Hopf-Lemma. First of all, we notice that by the maximum principle $f(y) < T_M$ for any $y \in (0, R)$ with $R > 0$. Indeed, by assumption we have $\max_{[0,R]} f = T_M$. If we assume that there exists $y_0 \in (0, R)$ such that $f(y_0) = T_M$, we obtain the following contradiction

$$0 = f''(y_0) - cf'(y_0) - T_M^4 + \int_0^\infty E(\eta - y_0) f^4(\eta) d\eta \leq T_M^4 \left(-1 + \int_{-R}^\infty E(\eta) d\eta\right) < 0.$$

Thus, since $f^4(0) - \int_0^\infty E(\eta) f^4(\eta) \geq \frac{T_M^4}{2} > 0$ by continuity there exists $\delta > 0$ such that $f(\delta) < T_M$ and $f^4(y) - \int_0^\infty E(\eta - y) f^4(\eta) d\eta > 0$ for all $y \in (0, \delta)$.

Let us now consider the operator $\mathcal{L} = \partial_y^2 - c\partial_y$. By construction we see $\mathcal{L}(f)(y) > 0$ for all $y \in (0, \delta)$. For $\alpha > c$ and $0 < \varepsilon < \frac{T_M - f(\delta)}{e^{\alpha\delta} - 1}$ we define the auxiliary function $z(y) = e^{\alpha y} - 1$. Then a simple computation shows

$$\mathcal{L}(f + \varepsilon z)(y) > 0 \text{ for all } y \in (0, \delta) \text{ as well as } f(0) + \varepsilon z(0) = T_M > f(\delta) + \varepsilon(\delta).$$

Hence, the maximum principle for $\mathcal{L}$ implies that $f(y) + \varepsilon z(y) \leq T_M$ for all $y \in (0, \delta)$. This yields that

$$f'(0^+) + \varepsilon z'(0^+) = f'(0^+) + \varepsilon\alpha \leq 0$$

and therefore since $\alpha > 0$ we conclude $\partial_y f(0^+) < 0$. □

A direct consequence of Lemma E.1 is the following Corollary.

Corollary E.1. *There exists $c_{\max} > 0$ such that for any $c \in (0, c_{\max})$ the solution f^c of* (E.8) *constructed as in Theorem E.3 satisfies $\partial_y f^c(0^+) < -Lc$.*

Proof. Let $c > 0$ and let $f^c \in C^{2,1/2}(\mathbb{R}_+)$ be the solution to (E.8) given by $f = \alpha^{2/3}\tilde{f}^{\tilde{c}}(\alpha y)$, where $\tilde{f}^{\tilde{c}}$ is the solution of (E.10) of Theorem E.3 for $\tilde{c} = \frac{c}{\alpha}$ and melting temperature $\tilde{T}_M = \frac{T_M}{\alpha^{2/3}}$. Using the bound of the first derivative obtained in Theorem E.3 and the definition of the rescaling, we conclude

$$\|\partial_y f^c\|_\infty = \alpha^{5/3}\|\partial_\eta \tilde{f}^{\tilde{c}}\|_\infty \leq \alpha^{5/3}\frac{\tilde{T}_M}{\tilde{c}} = \frac{T_M^4}{c}.$$

Lemma E.1 implies $\partial_y f^c(0^+) < 0$. Thus, the set $\{c > 0 : \partial_y f^c(0^+) < -Lc\}$ is not empty. We hence define

$$c_{\max} := \sup\{c > 0 : \partial_y f^c(0^+) < -Lc\}.$$

□

In the next section we will prove that in the solid the traveling waves are bounded from below by a positive constant and they converge to a positive constant as $y \to \infty$.

E.2.2 Monotonicity with respect to the melting temperature of the traveling wave solutions for $y > 0$

In this section we will show that for $y > 0$ to the traveling waves constructed in the previous section are monotone increasing with respect to the melting temperature, i.e. if $f_1(0) = \theta_1$ and $f_2(0) = \theta_2$ with $\theta_1 < \theta_2$ and f_1, f_2 solve (E.8), then $f_1 \leq f_2$. We prove the following Lemma

Lemma E.2. *Let $0 < \theta_1 < \theta_2$ and let $f_1, f_2 \in C^{2,1/2}(\mathbb{R}_+)$ be the two solutions of* (E.8) *constructed with the iterative scheme in Theorem E.3 for $T_M = \theta_1$ and $T_M = \theta_2$, respectively. Then $f_1 \leq f_2$.*

Proof. Let f_1, f_2 be given by the limit of the monotone bounded sequences $f_i^n \in C^{2,1/2}(\mathbb{R}_+)$ solving the recursive problem

$$\begin{cases} \partial_y^2 f_i^{n+1}(y) - c\partial_y f_i^{n+1}(y) - \left(f_i^{n+1}(y)\right)^4 = g_i^n(y) & y > 0; \quad n \geq 1 \\ f_i^0 = 0 & n = 0 \\ f_i^{n+1}(0) = \theta_i \\ f_i^{n+1} \geq 0 \end{cases}$$

where

$$g_i^n(y) = -\int_0^\infty E(y-\eta)\left(f_i^n(\eta)\right)^4 d\eta.$$

We show by induction that $f_1^n \leq f_2^n$ for all $n \in \mathbb{N}$. This will imply the lemma, since $f_i(y) := \lim_{n\to\infty} f_i^n(y)$.

Let us define $\varphi_n = f_2^n - f_1^n$. Then $\varphi_0 = 0$ and for $n \geq 1$ it solves

$$\begin{cases} \partial_y^2\varphi_n(y) - c\partial_y\varphi_n(y) - a_n(y)\varphi_n(y) = h_{n-1}(y) & y > 0; \quad n \geq 1 \\ \varphi_n(0) = \theta_2 - \theta_1 > 0 \\ \varphi_n \in [-\theta_1, \theta_2] \end{cases}$$

where $h_{n-1}(y) = g_2^{n-1}(y) - g_1^{n-1}(y) = \int_0^\infty E(y-\eta)\left(f_1^{n-1}(\eta)^4 - f_2^{n-1}(\eta)^4\right) d\eta$ and

$$a_n(y) = \frac{f_2^n(y)^4 - f_1^n(y)^4}{f_2^n(y) - f_1^n(y)} = f_2^n(y)^3 + f_1^n(y)^3 + f_2^n(y)^2 f_1^n(y) + f_2^n(y) f_1^n(y)^2 > 0.$$

The positivity of $a_n(y)$ is given by the strict positivity of f_i^n in the interior of $\mathbb{R}_+$ and in $y=0$ as shown before. Moreover, $\varphi_n \in [-\theta_1, \theta_2]$ since $0 \leq f_i^n \leq \theta_i$ for $i \in 1,2$ by the construction in Theorem E.3. We show inductively that $\varphi_n \geq 0$ for all $n \geq 1$. To this end we consider for $R>0$ the function $\psi_R = -\theta_1 e^{c(y-R)}$. It satisfies $\psi_R(0) \geq -\theta_1$ as well as $\psi_R(R) = -\theta_1$. Hence, on $[0,R]$ we have

$$\begin{cases} \partial_y^2 \left(\varphi_n(y) - \psi_R(y)\right) - c\partial_y \left(\varphi_n(y) - \psi_R(y)\right) - a_n(y) \left(\varphi_n(y) - \psi_R(y)\right) \\ \qquad = h_{n-1}(y) + a_n(y)\psi_R(y) \leq h_{n-1}(y) & y \in [0,R]; \quad n \geq 1 \\ \varphi_n(0) - \psi_R(0) > 0 \\ \varphi_n(R) - \psi_R(R) \geq 0 \end{cases}$$

Let us now consider $n=1$. Since $h_0 = 0$ the supersolution $\varphi_1 - \psi_R$ solves

$$\partial_y^2 \left(\varphi_1(y) - \psi_R(y)\right) - c\partial_y \left(\varphi_1(y) - \psi_R(y)\right) - a_1(y) \left(\varphi_1(y) - \psi_R(y)\right) \leq 0.$$

An application of the maximum principle assuming the existence of a negative minimum, gives $\varphi_1 = f_2^1 - f_1^1 \geq -\theta_1 e^{c(y-R)}$ for $y \in [0,R]$. Thus, letting $R \to \infty$ we conclude $f_2^1 \geq f_1^1$.

Let us now assume that for $n \in \mathbb{N}$ we know that $f_2^{n-1} \geq f_1^{n-1}$. We show that $f_2^n \geq f_1^n$. First of all we see that by the induction step we have $h_{n-1} \leq 0$, since $\left(f_1^{n-1}\right)^4 \leq \left(f_2^{n-1}\right)^4$. Then the maximum principle applied to the supersolution $\varphi_n - \psi_R$ solving

$$\partial_y^2 \left(\varphi_n(y) - \psi_R(y)\right) - c\partial_y \left(\varphi_n(y) - \psi_R(y)\right) - a_n(y) \left(\varphi_n(y) - \psi_R(y)\right) \leq 0$$

implies as before $f_2^n \leq f_1^n$. This concludes the proof of the lemma. □

In the following we aim to show that for $y>0$ the constructed traveling wave solutions are bounded from below by a positive constant. This can be proved using the monotonicity property of the traveling wave solutions with respect to the melting temperature. We will indeed show that for very small melting temperature the traveling wave solutions are unique, strictly positive and with a positive limit.

E.2.3 Traveling wave solutions for small melting temperatures for $y>0$

In this section we will show that for any $T_M = \varepsilon < \varepsilon_0$ with $\varepsilon_0 > 0$ small enough there exists a unique solution f to (E.8) which converges to a positive constant with exponential rate $y \to \infty$. Moreover, f is bounded from below by a positive constant. We will show it in several steps. We will first prove that any solution f obtained in Theorem E.3 for $T_M = \varepsilon$ small enough has a limit f_∞ as $y \to \infty$ and converges to f_∞ with exponential rate. Afterwards, we will prove that both f and f_∞ are positive and bounded from below by a positive constant. Finally, we will prove that for $T_M = \varepsilon$ small enough there exists a unique solution to (E.8) converging with exponential rate to a constant.

Lemma E.3. *Let f be a solution to* (E.10) *as in Theorem E.3. Then for $T_M = \varepsilon > 0$ small enough there exists $A>0$, $\alpha \in (0,1)$ and $f_\infty \in [0, T_M]$ such that*

$$|f(y) - f_\infty| \leq \varepsilon^4 A e^{-\alpha y}.$$

Proof. Let f be the function obtained in Theorem E.3. First of all we notice that it is equivalent to consider f solving the equation

$$\begin{cases} \partial_y^2 f(y) - c\partial_y f(y) - \varepsilon^3 f^4(y) = -\varepsilon^3 \int_0^\infty E(y-\eta) f^4(\eta) d\eta & y>0 \\ f(0) = 1 \\ f \geq 0 \end{cases} \tag{E.24}$$

Indeed, (E.24) is obtained considering $\tilde{f}$ defined by $\varepsilon\tilde{f}(y) = f(y)$. Clearly, if $\tilde{f}$ converges with exponential rate to a constant $\tilde{f}_\infty$ as $y \to \infty$, then also f converges with same rate to $f_\infty = \varepsilon\tilde{f}_\infty$. Therefore, we will show the lemma for $\tilde{f}$. In order to simplify the notation we will consider in this proof $f = \tilde{f}$ solving (E.24).

Since f is bounded and it solves strongly (E.24), then it solves also

$$\left(e^{-cy}f'\right)' = \varepsilon^3 e^{-cy}\left(f^4 - \int_0^\infty E(y-\eta)f^4(\eta)d\eta\right).$$

Hence, using that by the boundedness of the first derivative we have $\lim_{y\to\infty} e^{-cy}f'(y) = 0$, we obtain integrating in (y,∞)

$$f'(y) = -\varepsilon^3 e^{cy}\int_y^\infty e^{-c\eta}\left(f^4(\eta) - \int_0^\infty E(\eta - z)f^4(z)dz\right)d\eta.$$

Integrating once more in $(0, y)$, we conclude that f solves also the following fixed-point equation

$$f(y) = 1 + \varepsilon^3\int_0^y e^{c\xi}\int_\xi^\infty e^{-c\eta}\left(\int_0^\infty E(\eta - z)f^4(z)dz - f^4(\eta)\right)d\eta d\xi. \tag{E.25}$$

We define now

$$\operatorname*{osc}_{(R,R+1)} f = \sup_{y_1,y_2\in(R,R+1)} |f(y_1) - f(y_2)|.$$

Since f is non-negative and it is bounded by 1, we know that $\operatorname*{osc}_{(R,R+1)} f \leq 1$ for all $R > 0$. For $M > 0$ we also define

$$\lambda(M) = \sup_{R\geq M}\operatorname*{osc}_{(R,R+1)} f.$$

Notice that $\lambda(M)$ is decreasing with $\lambda(M) \leq \lambda(0) \leq 1$. We will show that $\lambda(M)$ decays like $e^{-\frac{M}{2}}$. To this end we consider for $M > 0$ and $R \geq M$ the points $y_1, y_2 \in [R, R+1]$ (w.l.o.g. $y_1 \leq y_2$) and we compute

$$\begin{aligned}
|f(y_1) - f(y_2)| \leq& \varepsilon^3\int_{y_1}^{y_2} e^{c\xi}\int_\xi^\infty e^{-c\eta}\left|\int_0^\infty E(\eta - z)f^4(z)dz - f^4(\eta)\right| d\eta d\xi\\
\leq& \varepsilon^3\int_{y_1}^{y_2} e^{c\xi}\int_{y_1}^\infty e^{-c\eta}\left|\int_0^\infty E(\eta - z)f^4(z)dz - f^4(\eta)\right| d\eta d\xi\\
=& \varepsilon^3\frac{e^{cy_2} - e^{cy_1}}{c}\int_{y_1}^\infty e^{-c\eta}\left|\int_0^\infty E(\eta - z)f^4(z)dz - f^4(\eta)\right| d\eta,
\end{aligned}$$

where in the first inequality we used the triangle inequality, in the second we used that $\xi \geq y_1$ and the last equality is given by integrating with respect to ξ. We use now that $0 \leq y_2 - y_1 \leq 1$, so that

$$\frac{e^{cy_2} - e^{cy_1}}{c} = e^{cy_2}\frac{1 - e^{-c(y_2-y_1)}}{c} \leq e^{cy_2}|y_2 - y_1| \leq e^{cy_2} \leq \exp(c)e^{cy_1}.$$

Thus,we can further estimate

$$
\begin{aligned}
|f(y_1) - f(y_2)| &\leq \varepsilon^3 \exp(c) \int_{y_1}^{\infty} e^{-c(\eta-y_1)} \left| \int_0^{\infty} E(\eta - z) f^4(z) dz - f^4(\eta) \right| d\eta \\
=&\varepsilon^3 \exp(c) \int_{y_1}^{\infty} e^{-c(\eta-y_1)} \left| \int_{-\eta}^{\infty} E(z) f^4(z+\eta) dz - f^4(\eta) \right| d\eta \\
=&\varepsilon^3 \exp(c) \int_{y_1}^{\infty} e^{-c(\eta-y_1)} \left| \int_{-\eta}^{\infty} E(z) \left(f^4(z+\eta) - f^4(\eta)\right) dz - \left(\int_{\eta}^{\infty} E(z) dz \right) f^4(\eta) \right| d\eta \\
\leq&\varepsilon^3 \frac{\exp(c)}{2} \int_{y_1}^{\infty} e^{-c(\eta-y_1)} e^{-\eta} d\eta + \varepsilon^3 \exp(c) \int_{y_1}^{\infty} e^{-c(\eta-y_1)} \left| \int_{-\eta}^{\infty} E(z) \left(f^4(z+\eta) - f^4(\eta)\right) dz \right| d\eta \\
\leq&\varepsilon^3 \frac{\exp(c)}{2} e^{-M} + \varepsilon^3 \exp(c) \int_{y_1}^{\infty} e^{-c(\eta-y_1)} \left| \int_{-\eta}^{\infty} E(z) \left(f^4(z+\eta) - f^4(\eta)\right) dz \right| d\eta
\end{aligned}
\tag{E.26}
$$

where the first equality follows by a change of coordinates $z \to z - \eta$ using the symmetry of the kernel E and the second one is a consequence of the normalization of the kernel E. Moreover, the last inequality uses the boundedness of $f \leq 1$ and the estimate

$$
\int_a^{\infty} E(z) dz \leq \frac{e^{-a}}{2} \tag{E.27}
$$

for any $a > 0$. Finally, we considered $\eta - y_1 \geq 0$ as well as $y_1 \geq M$.

We now estimate the second term in the last line of (E.26). First of all, using that $|f^4(a) - f^4(b)| \leq 4|f(a) - f(b)| \leq 4$ we can rewrite it as the sum of three integrals

$$
\begin{aligned}
&\varepsilon^3 \exp(c) \int_{y_1}^{\infty} e^{-c(\eta-y_1)} \left| \int_{-\eta}^{\infty} E(z) \left(f^4(z+\eta) - f^4(\eta)\right) dz \right| d\eta \\
&\leq 4\varepsilon^3 \exp(c) \int_{y_1}^{\infty} e^{-c(\eta-y_1)} \int_{-\eta}^{-M} E(z) dz d\eta \\
&\quad + 4\varepsilon^3 \exp(c) \int_{y_1}^{\infty} e^{-c(\eta-y_1)} \int_0^{\infty} E(z) \left|(f(z+\eta) - f(\eta))\right| dz d\eta \\
&\quad + 4\varepsilon^3 \exp(c) \int_{y_1}^{\infty} e^{-c(\eta-y_1)} \int_{-M}^{0} E(z) \left|(f(z+\eta) - f(\eta))\right| dz d\eta \\
&\leq A_1 + A_2 + A_3
\end{aligned}
\tag{E.28}
$$

The first integral term can be estimated easily by

$$
A_1 \leq 2\varepsilon^3 \frac{\exp(c)}{c} e^{-M}, \tag{E.29}
$$

where we used (E.27) and we solved $\int_{y_1}^{\infty} e^{-c(\eta-y_1)} d\eta = \frac{1}{c}$. For the terms A_2 and A_3 we will argue in a different way. We recall that $\lambda(M)$ is decreasing. Hence, if $z \in (0,1)$ for $\eta \geq y_1 \geq M$ we have $|f(\eta) - f(\eta + z)| \leq \lambda(\eta) \leq \lambda(M)$ as well as $|f(\eta) - f(\eta - z)| \leq \lambda(\eta - 1) \leq \lambda(M - 1)$.

Thus, using a telescopic sum for $\eta \geq y_1 \geq M$ we compute

$$\begin{aligned}\int_0^\infty E(z)\left|f(z+\eta)-f(\eta)\right|dz =& \sum_{n=0}^\infty \int_n^{n+1} E(z)\left|f(z+\eta)-f(\eta)\right|dz \\ \leq& \sum_{n=0}^\infty \int_n^{n+1} E(z)\left(\left|f(\eta+z)-f(n+\eta)\right| + \sum_{k=1}^n \left|f(\eta+k)-f(\eta+k-1)\right|\right)dz \\ \leq& \lambda(M)\sum_{n=0}^\infty \int_n^{n+1} E(z)(n+1)dz \leq \lambda(M)\sum_{n=0}^\infty \int_n^{n+1} E(z)(z+1)dz \\ =& \lambda(M)\int_0^\infty E(z)(z+1)dz \leq \lambda(M),\end{aligned} \tag{E.30}$$

where at the end we used also $E(a)a \leq \frac{e^{-a}}{2}$ for all $a > 0$. Thus, (E.30) implies

$$A_2 \leq 4\varepsilon^3 \frac{\exp(c)}{c}\lambda(M). \tag{E.31}$$

Similarly as we did in (E.30), using again a telescopic sum and estimating $\lambda(0) \leq 1$, we estimate for $\eta \geq y_1 \geq M$

$$\begin{aligned}&\int_{-M}^0 E(z)\left|f(z+\eta)-f(\eta)\right|dz \\ &\quad= \int_0^M E(z)\left|f(\eta)-f(\eta-z)\right|dz = \sum_{n=1}^M \int_{n-1}^n E(z)\left|f(\eta)-f(\eta-z)\right|dz \\ &\leq \sum_{n=1}^M \int_{n-1}^n E(z)\left(\left|f(\eta-z)-f(\eta-(n-1))\right| + \sum_{k=1}^{n-1}\left|f(\eta-(k-1))-f(\eta-k)\right|\right)dz \\ &\leq \sum_{n=1}^M \int_{n-1}^n E(z)\left(\lambda(M-n)+\sum_{k=1}^{n-1}\lambda(M-k)\right)dz \leq \sum_{n=1}^M \lambda(M-n)\int_{n-1}^n E(z)ndz \\ &\leq \sum_{n=1}^M \lambda(M-n)\int_{n-1}^n E(z)(z+1)dz \leq \sum_{n=1}^M \lambda(M-n)\int_{n-1}^\infty E(z)(z+1)dz \\ &\leq \sum_{n=1}^M \lambda(M-n)e^{-(n-1)} \leq e^{-(M-1)} + e\sum_{n=1}^{M-1} e^{-n}\lambda(M-n).\end{aligned} \tag{E.32}$$

Hence, we have also the following estimate

$$A_3 \leq 4\varepsilon^3 \frac{\exp(c+1)}{c}\left[e^{-M} + \sum_{n=1}^{M-1} e^{-n}\lambda(M-n)\right]. \tag{E.33}$$

Finally, putting together (E.26), (E.28), (E.29), (E.31) and (E.33) we obtain for $M \leq R \leq y_1 \leq y_2 \leq R+1$

$$\begin{aligned}&\operatorname*{osc}_{(R,R+1)} f \leq \left|f(y_1)-f(y_2)\right| \\ &\leq \varepsilon^3\exp(c)\left(\frac{1}{2}+\frac{2}{c}+\frac{4e}{c}\right)e^{-M} + 4\varepsilon^3\frac{\exp(c)}{c}\lambda(M) + 4\varepsilon^3\frac{\exp(c+1)}{c}\sum_{n=1}^{M-1} e^{-n}\lambda(M-n).\end{aligned} \tag{E.34}$$

Let us take

$$\varepsilon < \varepsilon_1(c) = \sqrt[3]{\frac{c}{8\exp(c)}} \tag{E.35}$$

and let us define $B(c) = \exp(c)\left(1 + \frac{4+8e}{c}\right)$. Then taking the supremum over all $R \geq M$ we have

$$\lambda(M) \leq B\varepsilon^3 e^{-M} + B\varepsilon^3 \sum_{n=1}^{M-1} e^{-n}\lambda(M-n). \tag{E.36}$$

We now show by induction that $\lambda(M) \leq 2B\varepsilon^3 e^{-M/2}$ for all $\varepsilon < \min\{\varepsilon_1(c), \varepsilon_2(c)\}$, where

$$\varepsilon_2(c) = \sqrt[3]{\frac{1}{2B\gamma}} \tag{E.37}$$

for $\gamma = 5\frac{1/2}{1-e^{-1/2}} = \frac{5}{2}\sum_{n=0}^{\infty} e^{-n/2}$. Moreover, since $\frac{1/2}{1-e^{-1/2}} > \frac{1}{2}$ we have $\gamma > 2$. This implies also that $B\varepsilon^3 < \frac{1}{2\gamma} < \frac{1}{4}$. First of all we see that if $M = 0$ the estimates (E.32) and (E.33) reduce to $A_3 = 0$. Hence, using (E.26), (E.28), (E.29), (E.31) we obtain

$$\lambda(0) \leq \varepsilon^3\left(\frac{\exp(c)}{2} + \frac{2\exp(c)}{c}\right) + 4\varepsilon^3\frac{\exp(c)}{c}\lambda(0).$$

Thus, for $\varepsilon < \varepsilon_1$ we have

$$\lambda(0) \leq B\varepsilon^3 \leq 2B\varepsilon^3.$$

Let us consider $M = 1$. In this case (E.32) and (E.33) reduce to $A_3 = \frac{4\varepsilon^3\exp(c)}{c}\lambda(0)e^{-0} \leq \frac{4\varepsilon^3\exp(c+1)}{c}\lambda(0)e^{-1}$, where we used $\lambda(0) \leq 1$. Thus, we obtain once more for $\varepsilon < \min\{\varepsilon_1, \varepsilon_2\}$

$$\lambda(1) \leq B\varepsilon^3 e^{-1} \leq 2B\varepsilon^3 e^{-1/2}.$$

Let us now consider $M = 2$. In this case the sum on the right hand side of (E.34) is non-zero. We compute using (E.36)

$$\lambda(2) \leq B\varepsilon^3 e^{-2} + B\varepsilon^3\lambda(1)e^{-1}.$$

Using now the estimate for $\lambda(1)$ and that $\varepsilon < \varepsilon_2$ and so that $B\varepsilon^3 < 1/4$ we have

$$\lambda(2) \leq B\varepsilon^3\left(e^{-2} + \frac{e^{-3/2}}{2}\right) \leq 2B\varepsilon^3 e^{-1}.$$

Let us now assume that $\lambda(k)$ satisfies

$$\lambda(k) \leq 2B\varepsilon^3 e^{-k/2}$$

for $k = 2, ..., M \in \mathbb{N}$. We show that also

$$\lambda(M+1) \leq 2B\varepsilon^3 e^{-(M+1)/2}.$$

This is a consequence of the choice of ε_2 depending on γ. Indeed, by (E.36) we have

$$\begin{aligned}\lambda(M+1) &\leq B\varepsilon^3 e^{-(M+1)} + B\varepsilon^3\sum_{n=1}^{M} e^{-n}\lambda(M+1-n)\\ &\leq B\varepsilon^3 e^{-(M+1)} + B\varepsilon^3 e^{-(M+1)/2}\sum_{n=1}^{M} 2B\varepsilon^3 e^{-n/2}\\ &\leq B\varepsilon^3 e^{-(M+1)} + B\varepsilon^3 e^{-(M+1)/2}\frac{2B\varepsilon^3}{1-e^{-1/2}} < 2B\varepsilon^3 e^{-(M+1)/2},\end{aligned}$$

where at the end we used the definition of γ as well as $B\varepsilon^3\gamma < 1/2$ for $\varepsilon < \varepsilon_2$. This concludes the proof of the exponential decay of $\lambda(M)$. We will use this result in order to prove the convergence at exponential rate of f. Let us consider $x, y \in \mathbb{R}_+$ with $x < y$. Then there exists $A > 0$ such that $|f(x) - f(y)| \leq \varepsilon^3 A e^{-x/2}$. Indeed we have

$$\begin{aligned}|f(x) - f(y)| &\leq |f(x) - f(\lfloor x\rfloor)| + |f(y) - f(\lfloor y\rfloor)| + |f(\lfloor x\rfloor) - f(\lfloor y\rfloor)| \\ &\leq 2B\varepsilon^3 e^{1/2}\left(e^{-x/2} + e^{-y/2}\right) + \sum_{n=\lfloor x\rfloor}^{\lfloor y\rfloor -1} |f(n) - f(n+1)| \\ &\leq 4B\varepsilon^3 e^{1/2} e^{-x/2} + 2B\varepsilon^3 \frac{e^{-\lfloor x\rfloor/2} - e^{-\lfloor y\rfloor/2}}{1 - e^{-1/2}} \leq \varepsilon^3 A e^{-x/2},\end{aligned}$$

where $A = 4Be^{1/2} + 2B\frac{e^{1/2}}{1-e^{-1/2}}$. Therefore, $|f(x) - f(y)| \leq \varepsilon^3 A e^{-x/2} \to 0$ as $x, y \to \infty$. This implies that for any increasing sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ with $\lim\limits_{n\to\infty} x_n = \infty$, the sequence $f(x_n)$ is a Cauchy sequence and hence has a limit as $n \to \infty$. Indeed,

$$|f(x_n) - f(x_m)| \leq \varepsilon^3 A e^{-\frac{\min\{x_n, x_m\}}{2}} \to 0 \quad \text{as } n, m \to \infty.$$

Let hence, $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ and $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ be two increasing sequences with $x_n, y_n \to \infty$ as $n \to \infty$ and such that

$$f_{\infty-} = \liminf_{y\to\infty} f(y) = \lim_{n\to\infty} f(y_n) \leq \lim_{n\to\infty} f(x_n) = \limsup_{y\to\infty} f(y) = f_{\infty+}.$$

Let $\delta > 0$. Then there exists some $N_0 \in \mathbb{N}$ such that

$$\varepsilon^3 A e^{-\frac{\min\{x_n, y_n\}}{2}} < \frac{\delta}{3} \quad \text{for all } n \geq N_0$$

and

$$|f_{\infty+} - f(x_n)| < \frac{\delta}{3} \quad \text{as well as} \quad |f_{\infty-} - f(y_n)| < \frac{\delta}{3} \quad \text{for all } n \geq N_0.$$

Hence, for all $n \geq N_0$ we conclude

$$|f_{\infty-} - f_{\infty+}| \leq |f_{\infty+} - f(x_n)| + |f_{\infty-} - f(y_n)| + |f(x_n) - f(y_n)| < \delta.$$

This implies that f has a limit for $y \to \infty$ which is denoted by

$$\liminf_{y\to\infty} f(y) = \limsup_{y\to\infty} f(y) = \lim_{y\to\infty} f(y) = f_\infty.$$

A consequence of the existence of a limit is that any sequence $\{f(x_n)\}_{n\in\mathbb{N}}$ defined by an increasing diverging sequence $\{x_n\}_{n\in\mathbb{N}}$ has to converge to f_∞. Hence, also for $y \in \mathbb{R}_+$ we have $\lim\limits_{n\to\infty} f(y+n) = f_\infty$.

Finally, let $y \in \mathbb{R}_+$. We show that f converges to f_∞ with an exponential rate.

$$|f(y) - f_\infty| = \sum_{n=0}^{\infty} |f(y+n) - f(y+n+1)| \leq A\varepsilon^3 e^{-\frac{y}{2}} \sum_{n=0}^{\infty} e^{-\frac{n}{2}} = \frac{A\sqrt{e}\varepsilon^3}{\sqrt{e}-1} e^{-\frac{y}{2}}.$$

□

We continue the theory for small melting temperatures showing that the solution f of theorem E.3 is bounded from below by a positive constant. This will imply that also the limit f_∞ is strictly positive. We prove the following lemma.

Lemma E.4. *Let f be a solution to* (E.10) *as in Theorem E.3. Then for $T_M = \varepsilon > 0$ small enough there exists $c_0 > 0$ such that*

$$f(y) \geq c_0\varepsilon \quad \textit{for all } y \in \mathbb{R}_+.$$

This implies also $f_\infty \geq c_0\varepsilon$.

Proof. As for Lemma (E.3) we consider $f = \varepsilon\tilde{f}$, where $\tilde{f}$ solves (E.24). We will show that $\tilde{f}(y) \geq c_0$ for all $y \in \mathbb{R}_+$. This implies clearly the claim of Lemma E.4. In order to simplify the notation, we will denote in this proof $\tilde{f}$ by f. By Lemma E.3 there exist f_∞ and $A > 0$ such that $|f(y) - f_\infty| \leq A\varepsilon^3 e^{-y/2}$ for $\varepsilon > 0$ small enough. As we have seen in Lemma E.3 the solution f to (E.24) solves the fixed-point equation (E.25). This can be rewritten as

$$f(y) = 1 + \varepsilon^3 \int_0^y e^{c\xi} \int_\xi^\infty e^{-c\eta} \left(\int_0^\infty E(\eta - z) \left[(f(z) - f_\infty) + f_\infty\right]^4 dz - \left[(f(\eta) - f_\infty) + f_\infty\right]^4 \right) d\eta d\xi.$$

We recall that

$$[(f - f_\infty) + f_\infty]^4 = (f - f_\infty)^4 + 4(f - f_\infty)^3 f_\infty + 6(f - f_\infty)^2 f_\infty^2 + 4(f - f_\infty) f_\infty^3 + f_\infty^4.$$

Hence, using on the one hand that $0 \leq f_\infty \leq 1$, $|f - f_\infty| \leq 1$ and that $|f(y) - f_\infty| \leq A\varepsilon^3 e^{-y/2}$ we see easily that

$$[(f(y) - f_\infty) + f_\infty]^4 \leq f_\infty^4 + 15\varepsilon^3 A e^{-y/2}. \tag{E.38}$$

On the other hand, using in addition that $(f - f_\infty)^4 \geq 0$ as well as $(f - f_\infty)^2 f_\infty^2 \geq 0$ we have

$$[(f(y) - f_\infty) + f_\infty]^4 \geq f_\infty^4 - 8\varepsilon^3 A e^{-y/2}. \tag{E.39}$$

We can hence estimate from below f as

$$\begin{aligned}
f(y) \geq\, & 1 - \varepsilon^3 f_\infty^4 \int_0^y e^{c\xi} \int_\xi^\infty e^{-c\eta} \int_\eta^\infty E(z) dz d\eta d\xi \\
& - 8\varepsilon^6 A \int_0^y e^{c\xi} \int_\xi^\infty e^{-(c+\frac{1}{2})\eta} \int_{-\eta}^\infty E(z) e^{-\frac{z}{2}} dz d\eta d\xi - 15\varepsilon^6 A \int_0^y e^{c\xi} \int_\xi^\infty e^{-(c+\frac{1}{2})\eta} d\eta d\xi \\
\geq & 1 - \frac{\varepsilon^3}{2} \int_0^y e^{c\xi} \int_\xi^\infty e^{-(c+1)\eta} d\eta d\xi - \varepsilon^6 A \left(16 \operatorname{artanh}\left(\frac{1}{2}\right) + 15\right) \int_0^y e^{c\xi} \int_\xi^\infty e^{-(c+\frac{1}{2})\eta} d\eta d\xi \\
= & 1 - \frac{\varepsilon^3}{2(c+1)} \int_0^y e^{-\xi} d\xi - \frac{\varepsilon^6 A}{c + \frac{1}{2}} \left(16 \operatorname{artanh}\left(\frac{1}{2}\right) + 15\right) \int_0^y e^{-\frac{\xi}{2}} d\xi \\
\geq & 1 - \varepsilon^3 \left(\frac{1}{2(c+1)} + \frac{4\varepsilon^3 A}{2c+1} \left(16 \operatorname{artanh}\left(\frac{1}{2}\right) + 15\right)\right).
\end{aligned} \tag{E.40}$$

We used for the second inequality the fact that $0 \leq f_\infty \leq 1$, as well as (E.27). Moreover, for the equality we used that for any $a \in [0, 1)$

$$\int_{-\infty}^\infty E(z) e^{-az} dz = \int_0^\infty E_1(z) \cosh(az) dz = \frac{\operatorname{artanh}(a)}{a}.$$

Equation (E.40) implies that for any $c_0 \in (0, 1)$ defining

$$\varepsilon_3 = \min \left\{ 1, \sqrt[3]{(1 - c_0) \left(\frac{1}{2(c+1)} + \frac{4A}{2c+1} \left(16 \operatorname{artanh}\left(\frac{1}{2}\right) + 15\right)\right)^{-1}} \right\} \tag{E.41}$$

and choosing $\varepsilon < \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ according to (E.35), (E.37) and (E.41), we conclude that the function f satisfies

$$f(y) \geq c_0.$$

This concludes the proof of the lemma. □

Lemma E.2 and Lemma E.4 imply the following Corollary.

Corollary E.2. *Let $T_M > 0$ and let f be a solution to* (E.10) *as in Theorem E.3. Then there exists $\lambda > 0$ such that $f(y) \geq \lambda > 0$ for all $y \geq 0$.*

Finally, we show that if $T_M = \varepsilon$ small enough the solution of (E.10) of Theorem E.3 is also unique. Indeed, we show that there is a unique solution of the fixed-point equation (E.25) converging to a constant with exponential rate. This is stated in the following theorem.

Theorem E.4. *Let $T_M = \varepsilon$. Then, for $\varepsilon < \varepsilon_0$ small enough there exists a unique solution $f \in C^{2,1/2}(\mathbb{R}_+)$ of* (E.10) *with $\lim_{y\to\infty} f(y) = f_\infty$ and $|f(y) - f_\infty| \leq Ae^{-y/2}$. Moreover, $f(y) \geq c_0\varepsilon$ as well as $f_\infty \geq c_0\varepsilon$ for $c_0 \in (0,1)$.*

Proof. First of all we remark that it is enough to prove the existence and uniqueness of the solution $\tilde{f}$ to the equation (E.24). Indeed, then $f = \varepsilon\tilde{f}$ is the desired unique solution of Theorem E.4. We will indeed prove the theorem for $\tilde{f}$, which is denoted in the rest of the proof by the sake of simplicity $\tilde{f} = f$.

Moreover, it is enough also to show the existence and uniqueness of the solution to the fixed-point equation (E.25). Indeed, any strong solution f to (E.24) satisfies $f \in C^{2,1/2}(\mathbb{R}_+)$ and it solves (E.25).

Let us consider for $B > 1$ and for $A > 0$ the following space

$$\mathcal{X} = \left\{ f \in C_b(\mathbb{R}_+) : |f(y)| \leq B,\ \exists f_\infty \text{ s.t. } |f(y) - f_\infty| \leq Ae^{-y/2} \right\}$$

equipped with the metric $d_{\mathcal{X}}$ induced by the following norm

$$\|f\|_{\mathcal{X}} = |f_\infty| + \sup_{y\in\mathbb{R}_+} e^{y/2}|f(y) - f_\infty|.$$

We also define the following seminorm

$$[f]_{\mathcal{X}} = \sup_{y\in\mathbb{R}_+} e^{y/2}|f(y) - f_\infty|$$

so that $\|f\|_{\mathcal{X}} = |f_\infty| + [f]_{\mathcal{X}}$. One can prove that $(\mathcal{X}, d_{\mathcal{X}})$ is a complete metric space. We omit the elementary proof.

We will now prove that the map

$$\mathcal{L}[f](y) = 1 + \varepsilon^3 \int_0^y e^{c\xi} \int_\xi^\infty e^{-c\eta} \left(\int_0^\infty E(\eta - z) f^4(z)dz - f^4(\eta) \right) d\eta d\xi$$

is a selfmap $\mathcal{L} : \mathcal{X} \to \mathcal{X}$ and that it is a contraction for $\varepsilon < \varepsilon_4$ small enough. The Banach fixed-point theorem will imply the existence of a unique fixed-point f solving (E.24).

Let now $f \in \mathcal{X}$. We observe that if $f \in C_b(\mathbb{R}_+)$ then $\mathcal{L}[f]$ is continuous. We move on proving that for $f \in \mathcal{X}$ also $\mathcal{L}[f]$ is bounded. Indeed, using that $|f| \leq B$ as well as $|f - f_\infty| \leq 2B$, we obtain similarly as for (E.38) and for (E.39) that

$$[(f(y) - f_\infty) + f_\infty]^4 \leq f_\infty^4 + 40B^3Ae^{-y/2}$$

and

$$[(f(y) - f_\infty) + f_\infty]^4 \geq f_\infty^4 - 20B^3Ae^{-y/2}.$$

Thus, we estimate similarly as in (E.40)

$$|\mathcal{L}[f](y)| \leq 1 + \varepsilon^3 B^3 \left(\frac{B}{2(c+1)} + \frac{4AB}{2c+1} \left(40 \operatorname{artanh}\left(\frac{1}{2}\right) + 20 \right) \right). \tag{E.42}$$

Hence, defining by $c_1(A,B) = \left(\frac{B}{2(c+1)} + \frac{4AB}{2c+1} \left(40 \operatorname{artanh}\left(\frac{1}{2}\right) + 20 \right) \right)$ and taking

$$\varepsilon_5 = \frac{1}{B} \sqrt[3]{\frac{B-1}{c_1(A,B)}} \tag{E.43}$$

we see that for $\varepsilon < \varepsilon_5$ we have

$$|\mathcal{L}[f](y)| \leq B.$$

We have now to show that $\mathcal{L}[f]$ has also a limit as $y \to \infty$, which we will call $\mathcal{L}_\infty[f]$. Moreover, we shall show that $|\mathcal{L}[f](y) - \mathcal{L}_\infty[f]| \leq Ae^{-y/2}$. This is the consequence of the convergence of f to f_∞ with exponential rate. Let us define

$$\mathcal{L}_\infty[f] = 1 + \varepsilon^3 \int_0^\infty e^{c\xi} \int_\xi^\infty e^{-c\eta} \left(\int_0^\infty E(\eta - z) f^4(z) dz - f^4(\eta) \right) d\eta d\xi.$$

By (E.42) we know that $\mathcal{L}_\infty[f]$ is bounded. Moreover, using that

$$\left| f^4(y) - f_\infty^4 \right| \leq 4B^3 \left| f(y) - f_\infty \right| \leq 4AB^3 e^{-y/2}$$

we can estimate

$$\begin{aligned}
|\mathcal{L}\ [f](y) - \mathcal{L}_\infty[f]| \leq& \varepsilon^3 \int_y^\infty e^{c\xi} \int_\xi^\infty e^{-c\eta} \left| \left(\int_0^\infty E(\eta - z) f^4(z) dz - f^4(\eta) \right) \right| d\eta d\xi \\
=& \varepsilon^3 \int_y^\infty e^{c\xi} \int_\xi^\infty e^{-c\eta} \left| \left(\int_0^\infty E(\eta - z) \left(f^4(z) - f_\infty^4 \right) dz - \left(f^4(\eta) - f_\infty^4 \right) \right) - \int_\eta^\infty E(z) dz f_\infty^4 \right| d\eta d\xi \\
\leq& \varepsilon^3 4AB^3 \int_y^\infty e^{c\xi} \int_\xi^\infty e^{-(c+1/2)\eta} \left(\int_{-\infty}^\infty E(z) e^{-z/2} dz + 1 \right) d\eta d\xi + \varepsilon^3 \frac{B^4}{2} \int_y^\infty e^{c\xi} \int_\xi^\infty e^{-(c+1)\eta} d\eta d\xi \\
\leq& \varepsilon^3 B^3 \left[4A \frac{2 \operatorname{artanh}\left(\frac{1}{2}\right) + 1}{2c+1} \right] e^{-y/2} + \varepsilon^3 \frac{B^4}{2(c+1)} e^{-y}.
\end{aligned}$$

Hence, defining

$$\varepsilon_6 = \left[\frac{B^3}{A} \left(4A \frac{2 \operatorname{artanh}\left(\frac{1}{2}\right) + 1}{2c+1} + \frac{B}{2(c+1)} \right) \right]^{-\frac{1}{3}} \tag{E.44}$$

we can conclude that there exists a limit

$$\lim_{y \to \infty} \mathcal{L}[f](y) = \mathcal{L}_\infty[f]$$

such that

$$|\mathcal{L}[f](y) - \mathcal{L}_\infty[f]| \leq Ae^{-y/2}$$

for all $\varepsilon < \min\{\varepsilon_5, \varepsilon_6\}$, defined in (E.43) and (E.44). This concludes the proof of $\mathcal{L}$ being a self-map. We now finish the proof of the theorem showing that $\mathcal{L}$ is also a contraction map.

We first prove that there exists a constant $c_2(A,B)$ such that if $f, g \in \mathcal{X}$, then

$$\|f^4 - g^4\|_{\mathcal{X}} \leq c_2(A,B)\|f-g\|_{\mathcal{X}}.$$

We recall that if $g \in \mathcal{X}$, then $[g]_{\mathcal{X}} \leq A$ and $|g| \leq B$. Hence, we have the estimate

$$\begin{aligned}
e^{y/2}\left| \left(f(y)^4 - g(y)^4\right) - \left(f_\infty^4 - g_\infty^4\right)\right| &= e^{y/2}\left|\left(f(y) - f_\infty + f_\infty\right)^4 - \left(g(y) - g_\infty + g_\infty\right)^4 - \left(f_\infty^4 - g_\infty^4\right)\right| \\
&\leq e^{y/2}\left|\left(f(y)-f_\infty\right)^4 - \left(g(y)-g_\infty\right)^4\right| + e^{y/2}\left|4f_\infty\left(f(y)-f_\infty\right)^3 - 4g_\infty\left(g(y)-g_\infty\right)^3\right| \\
&\quad + e^{y/2}\left|6f_\infty^2\left(f(y)-f_\infty\right)^2 - 6g_\infty^2\left(g(y)-g_\infty\right)^2\right| + e^{y/2}\left|4f_\infty^3\left(f(y)-f_\infty\right) - 4g_\infty^3\left(g(y)-g_\infty\right)\right| \\
&\leq (16+24+12)\,AB^2|f_\infty - g_\infty| + (32+48+24+4)\,B^3[f-g]_{\mathcal{X}} \\
&= 52AB^2|f_\infty - g_\infty| + 108B^3[f-g]_{\mathcal{X}}.
\end{aligned}$$

Thus, using that $|f_\infty^4 - g_\infty^4| \leq 4B^3|f_\infty - g_\infty|$ and defining $c_2(A,B) = \max\{52AB^2 + 4B^3, 108B^3\}$ we conclude that

$$\|f^4 - g^4\|_{\mathcal{X}} \leq c_2(A,B)\|f-g\|_{\mathcal{X}}.$$

Moreover, we see that $[f^4-g^4]_{\mathcal{X}} \leq c_2(A,B)\|f-g\|_{\mathcal{X}}$, which implies

$$\left|\left(f(y)^4 - g(y)^4\right) - \left(f_\infty^4 - g_\infty^4\right)\right| \leq c_2(A,B))\|f-g\|_{\mathcal{X}}e^{-y/2}.$$

Hence, we estimate

$$\begin{aligned}
|\mathcal{L}_\infty[f] - \mathcal{L}_\infty[g]| &\leq \varepsilon^3\int_0^\infty e^{c\xi}\int_\xi^\infty e^{-c\eta}\left|\left(\int_0^\infty E(\eta-z)\left(f^4(z)-g^4(z)\right)dz - \left(f^4(\eta)-g^4(\eta)\right)\right)\right|d\eta d\xi \\
&= \varepsilon^3\int_0^\infty e^{c\xi}\int_\xi^\infty e^{-c\eta}\left|\left(\int_0^\infty E(\eta-z)\left(f^4(z)-f_\infty^4-\left(g^4(z)-g_\infty^4\right)\right)dz\right.\right. \\
&\qquad \left.\left. - \left(f^4(\eta)-f_\infty^4-\left(g^4(\eta)-g_\infty^4\right)\right)\right) - \int_\eta^\infty E(z)dz\left(f_\infty^4-g_\infty^4\right)\right|d\eta d\xi \\
&\leq \varepsilon^3[f^4-g^4]_{\mathcal{X}}\left(2\operatorname{artanh}\left(\frac{1}{2}\right)+1\right)\int_0^\infty e^{c\xi}\int_\xi^\infty e^{-(c+1/2)\eta}d\eta d\xi \\
&\quad + \varepsilon^3\frac{|f_\infty^4-g_\infty^4|}{2}\int_0^\infty e^{c\xi}\int_\xi^\infty e^{-(c+1)\eta}d\eta d\xi \\
&\leq \varepsilon^3 c_2(A,B)\left(4\frac{2\operatorname{artanh}\left(\frac{1}{2}\right)+1}{2c+1} + \frac{1}{c+1}\right)\|f-g\|_{\mathcal{X}}.
\end{aligned}$$

In a similar way we can estimate

$$\begin{aligned}
&|\mathcal{L}[f](y) - \mathcal{L}[g](y) - (\mathcal{L}_\infty[f] - \mathcal{L}_\infty[g])| \\
&\quad\leq \varepsilon^3\int_y^\infty e^{c\xi}\int_\xi^\infty e^{-c\eta}\left|\left(\int_0^\infty E(\eta-z)\left(f^4(z)-g^4(z)\right)dz - \left(f^4(\eta)-g^4(\eta)\right)\right)\right|d\eta d\xi \\
&\quad\leq \varepsilon^3[f^4-g^4]_{\mathcal{X}}\left(2\operatorname{artanh}\left(\frac{1}{2}\right)+1\right)\int_y^\infty e^{c\xi}\int_\xi^\infty e^{-(c+1/2)\eta}d\eta d\xi \\
&\qquad + \varepsilon^3\frac{|f_\infty^4-g_\infty^4|}{2}\int_y^\infty e^{c\xi}\int_\xi^\infty e^{-(c+1)\eta}d\eta d\xi \\
&\quad\leq \varepsilon^3 c_2(A,B)\left(4\frac{2\operatorname{artanh}\left(\frac{1}{2}\right)+1}{2c+1}\right)\|f-g\|_{\mathcal{X}}e^{-y/2} + \varepsilon^3\frac{c_2(A,B)}{c+1}|f_\infty - g_\infty|e^{-y}.
\end{aligned}$$

This estimate implies easily

$$[\mathcal{L}[f](y) - \mathcal{L}[g](y)]_{\mathcal{X}} \leq \varepsilon^3 c_2(A,B)\left(4\frac{2\operatorname{artanh}\left(\frac{1}{2}\right)+1}{2c+1} + \frac{1}{c+1}\right)\|f-g\|_{\mathcal{X}}.$$

Therefore, taking $\theta \in (0,1)$ and

$$\varepsilon_7 = \left[\frac{2c_2(A,B)}{\theta}\left(4\frac{2\,\mathrm{artanh}\left(\frac{1}{2}\right)+1}{2c+1} + \frac{1}{c+1}\right)\right]^{-\frac{1}{3}} \tag{E.45}$$

we conclude that the map $\mathcal{L} : \mathcal{X} \to \mathcal{X}$ is a contraction self-map for $\varepsilon < \min\{\varepsilon_5, \varepsilon_6, \varepsilon_7\} = \varepsilon_4$, given in (E.43), (E.44) and (E.45). Hence, there exists a unique fixed-point $\tilde{f}$ of the equation (E.25), which solves also (E.24). Finally, $f = \varepsilon\tilde{f} \in C^{2,1/2}(\mathbb{R}_+)$ solves (E.10). Taking now $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$, for ε_i defined in (E.35), (E.37), (E.41) and above, Lemma E.3 and Lemma E.4 imply Theorem E.4. □

E.3 Existence of the limit of the traveling wave solutions as $y \to \infty$

We have proved in Theorem E.3 the existence for any $c > 0$ of a traveling wave f in $\mathbb{R}_+$ solving (E.10) and with the property that that $f \in C^{2,1/2}(\mathbb{R}_+)$. Moreover, as we have seen in Corollary E.2, f is bounded from below by a positive constant as long as $T_M > 0$. In this section we will show that f has a limit as $y \to \infty$.

We will proceed as follows. We will show that for any sequence $\{y_n\}_{n\in\mathbb{N}}$ increasing and diverging, the sequence $f_n(y) = f(y + y_n)$ has a subsequence converging to a function, which will be denoted by an abuse of notation as ω-limit. This definition relies on the similarity with the notion of ω-limit point for dynamical systems. Analogously, the ω-limit set is given in this setting by all the existing limit functions $\lim\limits_{k\to\infty} f(y + y_k)$, i.e

$$\omega(f) := \left\{ \bar{f} : \mathbb{R} \to \mathbb{R} \text{ such that } \quad \begin{matrix} \exists \{y_k\}_k,\ y_k < y_{k+1},\ y_k \to \infty \text{ as } k \to \infty, \\ \text{satisfying } \lim\limits_{k\to\infty} f(y + y_k) = \bar{f}(y) \end{matrix} \right\}.$$

We will prove that any ω-limit is a constant function. This will be used in the end in order to show that f has a limit.

E.3.1 Elementary properties of the ω-limits of the traveling waves

Let us consider $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ any increasing sequence with $\lim\limits_{n\to\infty} y_n = \infty$ and let us consider $f_n(y) := f(y + y_n)$. Then $f_n : [-y_n, \infty) \to \mathbb{R}_+$ solves for $\lambda > 0$ small enough

$$\begin{cases} \partial_y^2 f_n(y) - c\partial_y f_n(y) - f_n^4(y) = -\int_{-y_n}^{\infty} E(y-\eta) f_n^4(\eta)d\eta & \text{for } y > -y_n \\ f(-y_n) = T_M \\ f \geq \lambda > 0. \end{cases}$$

Since $f_n \in C^{2,1/2}[-y_n, \infty)$, by compactness a diagonal argument shows that there exists a subsequence f_{n_k} such that $f_{n_k} \to \bar{f}$ in $C^{2,\alpha}([-R,R])$ for $\alpha \in \left(0, \frac{1}{2}\right)$ and for any $R > 0$. Therefore $\bar{f} \in C^2(\mathbb{R})$ and by the uniform boundedness of f_n' and f_n'' we also have $\|\bar{f}'\|_\infty \leq \frac{2T_M^4}{c}$ and $\|\bar{f}''\|_\infty \leq 4T_M^4$. Moreover, an application of the dominated convergence theorem yields that $\bar{f}$ solves

$$\begin{cases} \partial_y^2 \bar{f}(y) - c\partial_y \bar{f}(y) - \bar{f}^4(y) = -\int_{-\infty}^{\infty} E(y-\eta)\bar{f}^4(\eta)d\eta & y \in \mathbb{R} \\ 0 < \lambda \leq \bar{f} \leq T_M. \end{cases} \tag{E.46}$$

Hence, regularity theory implies $\bar{f} \in C^{2,1/2}(\mathbb{R})$, since the convolution $E * \bar{f}^4 \in C^{0,1/2}(\mathbb{R})$.

Lemma E.5. *Let f solve* (E.46)*. Then f does not attain its supremum and infimum at the interior, unless f is constant.*

Proof. The proof is a direct consequence of the maximum principle. Let us assume that f is not constant and that there exists $y_m \in \mathbb{R}$ or $y_M \in \mathbb{R}$ such that $\sup_{\mathbb{R}} f = f(y_M)$ or $\inf_{\mathbb{R}} f = f(y_m)$. Then by the positivity of f we see that $f^4(y) - f^4(y_M) \leq 0$ as well as $f^4(y) - f^4(y_m) \geq 0$. Moreover, f differs from its maximum and minimum in sets of positive measures, since f is continuous and non-constant. Hence, we obtain the following contradictions

$$0 = f''(y_M) - cf'(y_M) + \int_{\mathbb{R}} E(\eta - y)\left[f^4(\eta) - f^4(y_M)\right] d\eta < 0$$

if the supremum is attained at the interior or

$$0 = f''(y_m) - cf'(y_m) + \int_{\mathbb{R}} E(\eta - y)\left[f^4(\eta) - f^4(y_m)\right] d\eta > 0$$

if the infimum is attained at the interior. This concludes the proof of the lemma. □

This result implies that, if $\bar{f}$ is not constant, it have to attain its supremum and infimum at $+\infty$ or $-\infty$. We will prove that this is not possible. We start showing that $\bar{f}$ does not attain its supremum and infimum at $+\infty$.

Lemma E.6. *Let f solve* (E.46)*. Then f does not attain its supremum at $+\infty$, that is $\limsup_{y\to\infty} f(y) < \sup_{\mathbb{R}} f$, unless f is constant.*

Proof. The proof is based once again on the maximum principle. Let us assume that f is not constant and that $\limsup_{y\to\infty} f(y) = \sup_{\mathbb{R}} f =: A$. We consider the function $\omega = A - f \geq 0$. Moreover, since f is not constant also $\omega > 0$ at the interior by Lemma E.5. Hence, ω solves

$$-\omega''(y) + c\omega'(y) - (A - \omega(y))^4 + \int_{\mathbb{R}} E(y - \eta)(A - \omega(\eta))^4 d\eta = 0. \tag{E.47}$$

We will show that $\omega(y) > 0$ as $y \to \infty$, which is a contradiction with the assumption of f attaining its supremum at $+\infty$. To this end we construct a suitable family of subsolutions $\psi_\delta(y)$ with the property $f \geq \psi_\delta$ and such that $\psi_\delta > 0$ for $y \in [0, R_\delta)$ for a suitable $R_\delta \to \infty$ as $\delta \to 0$.

We define the following constants. First of all we take $\theta = \frac{1}{5}$ and $R > 0$ fixed so that

$$\int_y^{R+y} E(\eta)d\eta > \int_{R+y}^{\infty} E(\eta)d\eta \qquad \text{for all } y > 0. \tag{E.48}$$

Moreover, we define $c_0 = \min\{1, c\}$ and we take $\beta \in \left(0, \frac{c_0}{4}\right)$ fixed so that

$$\frac{\operatorname{artanh}(4\beta)}{4\beta} < \frac{3}{2} \qquad \text{and} \qquad \beta^2 - \frac{c_0}{4}\beta + 4A^3\left(\frac{\operatorname{artanh}(\beta)}{\beta} - 1\right) \leq 0. \tag{E.49}$$

For a suitable constant $C(\beta, A, \theta) > 0$, which will be computed later, we also fix

$$\varepsilon < \min\left\{\min_{\left[-R, \frac{\ln(2)}{\beta}\right]}\{A - f(y)\},\ C(\beta, A, \theta)\right\}. \tag{E.50}$$

Finally, for $\delta_0 = \frac{\varepsilon\theta}{2}$ we consider the following family of subsolutions

$$\psi_\delta(y) = \begin{cases} 0 & y < -R \\ \varepsilon - \delta e^{\beta y} & y \in [-R, 0) \\ \varepsilon\theta - \delta e^{\beta y} & y \in [0, R_\delta] \\ 0 & y > R_\delta, \end{cases} \tag{E.51}$$

where $R_\delta = \frac{1}{\beta} \ln\left(\frac{\varepsilon\theta}{\delta}\right) \to \infty$ as $\delta \to 0$ as well as $\varepsilon\theta - \delta e^{\beta R_\delta} = 0$. By construction, $\psi_\delta \leq \omega$ for $y \in \mathbb{R} \setminus (0, R_\delta)$. We will show that on $(0, R_\delta)$ the family ψ_δ consists of subsolutions to (E.47). However, before moving to the proof of this claim we show that equations (E.48) and (E.49) are well-defined. We first show the function

$$h(y) = \int_y^{R+y} E(\eta) d\eta - \int_{R+y}^{\infty} E(\eta) d\eta$$

is a decreasing function. Using the definition of the kernel E, we notice

$$h(0) = \frac{1}{2} - e^{-R} + 2RE(R) > 0 \qquad \text{for } R > 0 \text{ large enough.}$$

Moreover, $\lim\limits_{y \to \infty} h(y) = 0$. We compute also for $R > \max\{1, \ln(2)\} = 1$

$$h'(y) = 2E(R+y) - E(y) \quad \text{and} \quad h''(y) = \frac{e^{-y}}{2y} - \frac{e^{-(y+R)}}{y+R} > \frac{e^{-y}}{y}\left(\frac{1}{2} - e^{-R}\right) > 0.$$

Since $\lim\limits_{y \to 0} h'(y) = -\infty$ and $\lim\limits_{y \to \infty} h'(y) = 0$, we conclude $h'(y) < 0$. This implies that h is monotonically decreasing for $R > 1$. Therefore, there exists an $R > 0$ such that (E.48) holds.

We move to the existence of $\beta \in \left(0, \frac{c_0}{4}\right)$ solving (E.49). First of all, let us define $g(y) = \frac{\operatorname{artanh}(y)}{y}$. Then $g : (0,1) \to \mathbb{R}_+$. Hence, $\beta \in \left(0, \frac{c_0}{4}\right)$ is well-defined. Moreover, elementary calculus implies

$$\lim_{y \to 0} g(y) = 1, \quad \lim_{y \to 1} g(y) = \infty, \quad g'(y) \geq 0 \text{ with } \quad g'(0) = 0, \quad \text{and} \quad g''(y) \geq 0.$$

Therefore, g is a convex monotone non-decreasing function with $g(0) = 1$. Hence, there exists $\beta_0 \in \left(0, \frac{c_0}{4}\right)$ such that $g(4\beta) < \frac{3}{2}$ for all $\beta < \beta_0$. Moreover, the function $k(\beta) = \beta^2 - \frac{c_0}{4}\beta + 4A^3(g(\beta) - 1)$ is convex as sum of convex functions. Since $k(0) = 0$, $k\left(\frac{c_0}{4}\right) > 0$ as well as

$$k'(\beta) = -\frac{c_0}{4} + [2\beta + 4A^3 g'(\beta)] \underset{\beta \to 0}{\longrightarrow} -\frac{c_0}{4} < 0$$

we conclude the existence of a β satisfying (E.49).

We prove now that ψ_δ are subsolutions to (E.47) for $y \in (0, R_\delta)$, where the functions are smooth. We compute for $y \in (0, R_\delta)$

$$\begin{aligned}
&-\psi_\delta''(y) + c\psi_\delta'(y) - (A - \psi_\delta(y))^4 + \int_{\mathbb{R}} E(y-\eta)\,(A - \psi_\delta(\eta))^4\, d\eta \\
&= -\left(c - \frac{c_0}{4}\right)\beta\delta e^{\beta y} + \delta e^{\beta y}\left(\beta^2 - \frac{c_0}{4}\beta\right) - \left(A - \varepsilon\theta + \delta e^{\beta y}\right)^4 + \int_{-\infty}^{-R} E(y-\eta) A^4 d\eta \\
&\quad + \int_{-R}^{0} E(y-\eta)\left(A - \varepsilon + \delta e^{\beta\eta}\right)^4 d\eta + \int_0^{R_\delta} E(y-\eta)\left(A - \varepsilon\theta + \delta e^{\beta\eta}\right)^4 d\eta \\
&\quad + \int_{R_\delta}^{\infty} E(y-\eta)\left(A - \varepsilon\theta + \delta e^{\beta R_\delta}\right)^4 d\eta \\
&\leq -\frac{3c_0}{4}\beta\delta e^{\beta y} + \delta e^{\beta y}\left(\beta^2 - \frac{c_0}{4}\beta\right) - \left(A - \varepsilon\theta + \delta e^{\beta y}\right)^4 + \int_{-\infty}^{-R} E(y-\eta)\left(A + \delta e^{\beta\eta}\right)^4 d\eta \\
&\quad + \int_{-R}^{0} E(y-\eta)\left(A - \varepsilon + \delta e^{\beta\eta}\right)^4 d\eta + \int_0^{R_\delta} E(y-\eta)\left(A - \varepsilon\theta + \delta e^{\beta\eta}\right)^4 d\eta \\
&\quad + \int_{R_\delta}^{\infty} E(y-\eta)\left(A - \varepsilon\theta + \delta e^{\beta\eta}\right)^4 d\eta,
\end{aligned} \tag{E.52}$$

where we used the definition of c_0, the fact that $A^4 \leq \left(A + \delta e^{\beta\eta}\right)^4$ as well as that $e^{\beta R_\delta} \leq e^{\beta\eta}$ for $\eta > R_\delta$. Expanding the power-law, ordering terms together and using that

$$\int_{\mathbb{R}} E(\eta - y)e^{\alpha\eta} d\eta = \frac{\operatorname{artanh}(\alpha)}{\alpha} e^{\alpha y} \tag{E.53}$$

for all $|\alpha| < 1$, we compute

$$-\psi_\delta''(y) + c\psi_\delta'(y) - (A - \psi_\delta(y))^4 + \int_{\mathbb{R}} E(y-\eta)\,(A - \psi_\delta(\eta))^4\, d\eta \tag{E.54}$$

$$\leq -\frac{3c_0}{4}\beta\delta e^{\beta y} + \delta e^{\beta y}\left(\beta^2 - \frac{c_0}{4}\beta + 4A^3\left(\frac{\operatorname{artanh}(\beta)}{\beta} - 1\right)\right) \tag{I_1^1}$$

$$+ 4A^3\varepsilon\left[\theta - \int_{-R}^{0} E(y-\eta)d\eta - \theta\int_0^{\infty} E(y-\eta)d\eta\right] \tag{I_2^1}$$

$$+ 4A\varepsilon^3\left[\theta^3 - \int_{-R}^{0} E(y-\eta)d\eta - \theta^3\int_0^{\infty} E(y-\eta)d\eta\right] \tag{I_3^1}$$

$$- 6A^2\varepsilon^2\left[\theta^2 - \int_{-R}^{0} E(y-\eta)d\eta - \theta^2\int_0^{\infty} E(y-\eta)d\eta\right] \tag{I_4^1}$$

$$- \varepsilon^4\left[\theta^4 - \int_{-R}^{0} E(y-\eta)d\eta - \theta^4\int_0^{\infty} E(y-\eta)d\eta\right] \tag{I_5^1}$$

$$+ 4\delta^3 e^{3\beta y}\varepsilon\left[\theta - \int_{-R}^{0} E(y-\eta)e^{3\beta(\eta-y)}d\eta - \theta\int_0^{\infty} E(y-\eta)e^{3\beta(\eta-y)}d\eta\right] \tag{I_6^1}$$

$$+ 4\delta e^{\beta y}\varepsilon^3\left[\theta^3 - \int_{-R}^{0} E(y-\eta)e^{\beta(\eta-y)}d\eta - \theta^3\int_0^{\infty} E(y-\eta)e^{\beta(\eta-y)}d\eta\right] \tag{I_7^1}$$

$$- 6\delta^2 e^{2\beta y}\varepsilon^2\left[\theta^2 - \int_{-R}^{0} E(y-\eta)e^{2\beta(\eta-y)}d\eta - \theta^2\int_0^{\infty} E(y-\eta)e^{2\beta(\eta-y)}d\eta\right] \tag{I_8^1}$$

$$- 6A^2\delta^2 e^{2\beta y}\left[1 - \int_{\mathbb{R}} E(\eta-y)e^{2\beta(\eta-y)}d\eta\right] \tag{I_9^1}$$

$$- 4A\delta^3 e^{3\beta y}\left[1 - \int_{\mathbb{R}} E(\eta - y) e^{3\beta(\eta - y)} d\eta\right] \tag{I^1_{10}}$$

$$- \delta^4 e^{4\beta y}\left[1 - \int_{\mathbb{R}} E(\eta - y) e^{4\beta(\eta - y)} d\eta\right] \tag{I^1_{11}}$$

$$+ 12A^2\varepsilon\delta e^{\beta y}\left[\theta - \int_{-R}^0 E(y-\eta) e^{\beta(\eta - y)} d\eta - \theta\int_0^\infty E(y-\eta) e^{\beta(\eta - y)} d\eta\right] \tag{I^1_{12}}$$

$$+ 12A\varepsilon\delta^2 e^{2\beta y}\left[\theta - \int_{-R}^0 E(y-\eta) e^{2\beta(\eta - y)} d\eta - \theta\int_0^\infty E(y-\eta) e^{\beta(2\eta - y)} d\eta\right] \tag{I^1_{13}}$$

$$- 12A\varepsilon^2\delta e^{\beta y}\left[\theta^2 - \int_{-R}^0 E(y-\eta) e^{\beta(\eta - y)} d\eta - \theta^2\int_0^\infty E(y-\eta) e^{\beta(\eta - y)} d\eta\right]. \tag{I^1_{14}}$$

We proceed now estimating all different terms in (E.54). By the choice of β in (E.49) we have

$$(I^1_1) \leq -\frac{3c_0}{4}\beta\delta e^{\beta y}. \tag{E.55}$$

We now proceed estimating the terms (I^1_2)-(I^1_5). Using the symmetry of E, the definition of R and the choice of $\theta = \frac{1}{5}$, we compute

$$\begin{aligned}(I^1_2) =&4A^3\varepsilon\left[\theta\int_y^\infty E(\eta)d\eta - \int_y^{R+y} E(\eta)d\eta\right] = 4A^3\varepsilon\left[\theta\int_{R+y}^\infty E(\eta)d\eta - (1-\theta)\int_y^{R+y} E(\eta)d\eta\right]\\ =&4A^3\varepsilon\left[-\theta\left(\int_y^{R+y} E(\eta)d\eta - \int_{R+y}^\infty E(\eta)d\eta\right) - 3\theta\int_y^{R+y} E(\eta)d\eta\right]\\ \leq& - 12A^3\varepsilon\theta\int_y^{R+y} E(\eta)d\eta.\end{aligned} \tag{E.56}$$

Similarly, since $1-\theta^3 = 124\theta^3$ we have

$$(I^1_3) \leq - 492A\varepsilon^3\theta^3\int_y^{R+y} E(\eta)d\eta. \tag{E.57}$$

Choosing $\varepsilon < 2A\theta$, which by the choice of θ implies that $\varepsilon < 492A\theta^3$, we obtain

$$\begin{aligned}(I^1_4) =& - 6A^2\varepsilon^2\left[\theta^2\int_y^\infty E(\eta)d\eta - \int_y^{R+y} E(\eta)d\eta\right]\\ \leq&6A^2\varepsilon^2\int_y^{R+y} E(\eta)d\eta \leq 12A^3\varepsilon\theta\int_y^{R+y} E(\eta)d\eta\end{aligned} \tag{E.58}$$

and

$$(I^1_5) = -\varepsilon^4\left[\theta^4\int_y^\infty E(\eta)d\eta - \int_y^{R+y} E(\eta)d\eta\right] \leq \varepsilon^4\int_y^{R+y} E(\eta)d\eta \leq 492A\varepsilon^3\theta^3\int_y^{R+y} E(\eta)d\eta. \tag{E.59}$$

Hence, (E.56)-(E.59) imply

$$(I^1_2) + (I^1_3) + (I^1_4) + (I^1_5) \leq 0. \tag{E.60}$$

Besides the choice of β as in (E.49) we use in the remaining estimates the fact that for $y \in (0, R_\delta)$ the following holds true

$$\delta e^{\beta y} \leq \delta e^{yR_\delta} = \varepsilon\theta < \varepsilon.$$

Hence, we see that

$$(I_6^1) \leq 4\delta^3 e^{3\beta y}\varepsilon\theta \leq 4\varepsilon^3\delta e^{\beta y}, \qquad (I_7^1) \leq 4\delta e^{\beta y}\varepsilon^3\theta^3 \leq 4\varepsilon^3\delta e^{\beta y}, \tag{E.61}$$

and

$$\begin{aligned}(I_8^1) \leq 6\delta^2 e^{2\beta y}\varepsilon^2 &\left(\int_{-(R+y)}^{-y} E(\eta)e^{2\beta\eta}d\eta + \theta^2\int_{-y}^{\infty} E(\eta)e^{2\beta\eta}d\eta\right)\\ &\leq 6\delta^2 e^{2\beta y}\varepsilon^2\int_{-\infty}^{\infty} E(\eta)e^{2\beta\eta}d\eta 6\delta^2 e^{2\beta y}\varepsilon^2\frac{\operatorname{artanh}(2\beta)}{2\beta} \leq 9\varepsilon^3\delta e^{\beta y},\end{aligned} \tag{E.62}$$

where we used also that $\beta \mapsto \frac{\operatorname{artanh}(\beta)}{\beta}$ is monotonically increasing. Finally, for the last six terms we estimate

$$(I_9^1) = 6A^2\delta^2 e^{2\beta y}\left(\frac{\operatorname{artanh}(2\beta)}{2\beta} - 1\right) \leq 3A^2\delta^2 e^{2\beta y} \leq 3\varepsilon A^2\delta e^{\beta y}, \tag{E.63}$$

$$(I_{10}^1) = 4A\delta^3 e^{3\beta y}\left(\frac{\operatorname{artanh}(3\beta)}{3\beta} - 1\right) \leq 2\varepsilon^2 A\delta e^{\beta y}, \tag{E.64}$$

$$(I_{11}^1) = \delta^4 e^{4\beta y}\left(\frac{\operatorname{artanh}(4\beta)}{4\beta} - 1\right) \leq \frac{1}{2}\varepsilon^3\delta e^{\beta y}, \tag{E.65}$$

$$(I_{12}^1) \leq 12A^2\varepsilon\delta e^{\beta y}, \qquad (I_{13}^1) \leq 12A\varepsilon^2\delta e^{\beta y}, \qquad \text{and} \qquad (I_{14}^1) \leq 18A\varepsilon^2\delta e^{\beta y}. \tag{E.66}$$

Therefore, defining the constant in equation (E.50)

$$C(\beta, A, \theta) = \min\left\{1, 2A\theta, \frac{c_0\beta}{2(18 + 15A^2 + 32A)}\right\}$$

and combining the estimates (E.55),(E.60)-(E.66) we conclude that

$$-\psi_\delta''(y) + c\psi_\delta'(y) - (A - \psi_\delta(y))^4 + \int_{\mathbb{R}} E(y-\eta)\,(A - \psi_\delta(\eta))^4\,d\eta \leq -\frac{c_0}{4}\beta\delta e^{\beta y} < 0$$

for all $y \in (0, R_\delta)$.

We now notice that by the choice of δ_0 we have $\psi_{\delta_0} \leq \omega$ on $\mathbb{R}$. In particular, since $R_{\delta_0} = \frac{1}{\beta}\ln(2)$ the definition of ε in (E.50) implies that $\psi_{\delta_0}(y) \leq \psi_{\delta_0}(0) = \frac{\varepsilon\theta}{2} < \omega$ on $[0, R_{\delta_0}]$ as well as $\inf_{y>0}(\omega - \psi_{\delta_0}) \geq \frac{\varepsilon\theta}{2} > 0$. We remark that on $\{y > 0\}$ the functions ψ_δ are continuous.

We aim to show that $\psi_\delta \leq \omega$ on $[0, R_\delta]$ for all $\delta \leq \delta_0$. To this end we assume the contrary, i.e. we assume that there exists some $0 < \delta < \delta_0$ such that

$$\inf_{y>0}(\omega - \psi_\delta) < 0. \tag{E.67}$$

By construction this yields that $\inf_{y>0}(\omega - \psi_\delta) = \min_{[0,R_\delta]}(\omega - \psi_\delta) < 0$. The uniform continuity of $[\delta, \delta_0] \ni \bar{\delta} \mapsto \psi_{\bar{\delta}}$ as functions on $[0, R_\delta]$ and their monotonicity ($\delta \mapsto \psi_\delta$ is increasing) imply that there exists

$$\delta^* := \sup\left\{\delta < \delta^* < \delta_0 : \min_{[0,R_\delta]}(\omega - \psi_{\delta^*}) < 0\right\} \tag{E.68}$$

such that

$$\min_{[0,R_\delta]}(\omega - \psi_{\delta^*}) = \omega(y_0) - \psi_{\delta^*} = 0$$

for some $y_0 \in (0, R_{\delta^*})$. Indeed, by construction $\psi_{\delta^*} < \omega$ on $y \geq R_{\delta^*}$ as well as $\psi_{\delta^*}(0) = \varepsilon\theta - \delta^* < \omega(0)$. Hence, we can apply the maximum principle for (E.47) at the point y_0 since on $(0, R_{\delta^*})$ the function ψ_{δ^*} is smooth. We obtain the following contradiction

$$\begin{aligned} 0 <& -(\omega - \psi_{\delta^*})''(y_0) + c(\omega - \psi_{\delta^*})'(y_0) - (A - \omega(y_0))^4 + (A - \psi_{\delta^*})^4 \\ &+ \int_{\mathbb{R}} E(y-\eta)(A-\omega)^4 d\eta - \int_{\mathbb{R}} E(y-\eta)(A - \psi_{\delta^*})^4 d\eta \\ \leq& \int_{\mathbb{R}} E(y-\eta)(A-\omega)^4 d\eta - \int_{\mathbb{R}} E(y-\eta)(A-\psi_{\delta^*})^4 d\eta \leq 0, \end{aligned} \tag{E.69}$$

since by construction $0 \leq \psi_{\delta^*} \leq \omega$ for all $y \in \mathbb{R} \setminus (0, R_{\delta^*})$. Moreover, $0 \leq \psi_{\delta^*} \leq \omega$ for $y \in (0, R_{\delta^*})$. Thus, $(A - \psi_{\delta^*}) \geq (A - \omega) \geq 0$ on $\mathbb{R}$. This contradiction implies that such δ^* as in (E.68) and consequently such δ satisfying (E.67) do not exist. Therefore we conclude that

$$\inf_{y>0} (\omega - \psi_\delta) \geq 0$$

for all $\delta < \delta_0$.

This implies that for all $y \in [0, R_\delta]$ we can estimate $w(y) \geq \varepsilon\theta - \delta e^{\beta y}$ for all $\delta < \delta_0$. Thus, taking the pointwise limit as $\delta \to 0$ we conclude

$$A - f(y) = w(y) \geq \varepsilon\theta > 0.$$

This is clearly a contradiction to the assumption that $\limsup_{y\to\infty} f(y) = A$. Hence, f does not attain its supremum at $+\infty$. □

A similar argument shows that $\bar{f}$, solution to (E.46), does not attain its infimum at $+\infty$, unless it is constant.

Lemma E.7. *Let f solve* (E.46). *Then f does not attain its infimum at $+\infty$, i.e. $\liminf_{y\to\infty} f(y) < \inf_{\mathbb{R}} f$, unless f is constant.*

Proof. We assume again that f is not constant and that $\liminf_{y\to\infty} f(y) = \inf_{\mathbb{R}} f =: B > 0$. We consider the function $\omega = f - B \geq 0$. Moreover, since f is not constant also $\omega > 0$ at the interior by Lemma E.5. Hence, ω solves

$$-\omega''(y) + c\omega'(y) + (B + \omega(y))^4 - \int_{\mathbb{R}} E(y-\eta)(B + \omega(\eta))^4 d\eta = 0. \tag{E.70}$$

As we did in Lemma E.6 we will show that $\omega(y) > 0$ as $y \to \infty$, which is a contradiction to the assumption of $\liminf_{y\to\infty} f(y) = \inf_{\mathbb{R}} f$. We will consider the family of functions ψ_δ defined as in (E.51) for $\theta = \frac{1}{5}$, β as in (E.49) and R defined in (E.48). Moreover, we take $\varepsilon > 0$ satisfying

$$\varepsilon < \min\left\{ \min_{\left[-R, \frac{\ln(2)}{\beta}\right]} \{f(y) - B\},\ C(\beta, B, \theta) \right\}, \tag{E.71}$$

where $C(\beta, B, \theta) > 0$ is a constant that will be computed later. Finally, we consider $\delta < \delta_0 = \frac{\varepsilon\theta}{2}$.

By construction we see that $\psi_\delta \leq \omega$ on $\mathbb{R} \setminus (0, R_\delta)$. Moreover, it is important to remark that for $y \in (0, R_\delta)$ the functions ψ_δ are smooth, as well as ψ_δ are continuous on $y \geq 0$. Hence, we can compute for $y \in (0, R_\delta)$ the following

$$\begin{aligned}
&-\psi_\delta''(y) + c\psi_\delta'(y) + (B + \psi_\delta(y))^4 - \int_{\mathbb{R}} E(y-\eta)\,(B+\psi_\delta(\eta))^4\,d\eta \\
&= -\left(c - \frac{c_0}{4}\right)\beta\delta e^{\beta y} + \delta e^{\beta y}\left(\beta^2 - \frac{c_0}{4}\beta\right) + \left(B + \varepsilon\theta - \delta e^{\beta y}\right)^4 - \int_{-\infty}^{-R} E(y-\eta)B^4 d\eta \\
&\quad - \int_{-R}^{0} E(y-\eta)\left(B + \varepsilon - \delta e^{\beta\eta}\right)^4 d\eta - \int_0^{R_\delta} E(y-\eta)\left(B + \varepsilon\theta - \delta e^{\beta\eta}\right)^4 d\eta \\
&\quad - \int_{R_\delta}^{\infty} E(y-\eta)\left(B + \varepsilon\theta - \delta e^{\beta R_\delta}\right)^4 d\eta \\
&\leq -\frac{3c_0}{4}\beta\delta e^{\beta y} + \delta e^{\beta y}\left(\beta^2 - \frac{c_0}{4}\beta\right) + \left(B + \varepsilon\theta - \delta e^{\beta y}\right)^4 - \int_{-\infty}^{-R} E(y-\eta)\left(B - \delta e^{\beta\eta}\right)^4 d\eta \\
&\quad - \int_{-R}^{0} E(y-\eta)\left(B + \varepsilon - \delta e^{\beta\eta}\right)^4 d\eta - \int_0^{R_\delta} E(y-\eta)\left(B + \varepsilon\theta - \delta e^{\beta\eta}\right)^4 d\eta \\
&\quad - \int_{R_\delta}^{\infty} E(y-\eta)\left(B + \varepsilon\theta - \delta e^{\beta R_\delta}\right)^4 d\eta.
\end{aligned} \tag{E.72}$$

As for (E.52) we used here the definition of c_0 as well as the fact that $B^4 \geq \left(B - \delta e^{\beta\eta}\right)^4$ for $\eta < -R$. Expanding the power-law, ordering terms together and using (E.53), we compute

$$-\psi_\delta''(y) + c\psi_\delta'(y) + (B + \psi_\delta(y))^4 - \int_{\mathbb{R}} E(y-\eta)\,(B+\psi_\delta(\eta))^4\,d\eta \tag{E.73}$$

$$\begin{aligned}
&\leq -\frac{3c_0}{4}\beta\delta e^{\beta y} + \delta e^{\beta y}\left(\beta^2 - \frac{c_0}{4}\beta + 4B^3\left(\frac{\operatorname{artanh}(\beta)}{\beta} - 1\right)\right) && (I_1^2) \\
&\quad + 4B^3\varepsilon\left[\theta - \int_{-R}^0 E(y-\eta)d\eta - \theta\int_0^\infty E(y-\eta)d\eta\right] && (I_2^2) \\
&\quad + 4B\varepsilon^3\left[\theta^3 - \int_{-R}^0 E(y-\eta)d\eta - \theta^3\int_0^\infty E(y-\eta)d\eta\right] && (I_3^2) \\
&\quad + 6B^2\varepsilon^2\left[\theta^2 - \int_{-R}^0 E(y-\eta)d\eta - \theta^2\int_0^\infty E(y-\eta)d\eta\right] && (I_4^2) \\
&\quad + \varepsilon^4\left[\theta^4 - \int_{-R}^0 E(y-\eta)d\eta - \theta^4\int_0^\infty E(y-\eta)d\eta\right] && (I_5^2) \\
&\quad - 4\delta^3 e^{3\beta y}\varepsilon\left[\theta - \int_{-R}^0 E(y-\eta)e^{3\beta(\eta-y)}d\eta - \theta\int_0^\infty E(y-\eta)e^{3\beta(\eta-y)}d\eta\right] && (I_6^2) \\
&\quad - 4\delta e^{\beta y}\varepsilon^3\left[\theta^3 - \int_{-R}^0 E(y-\eta)e^{\beta(\eta-y)}d\eta - \theta^3\int_0^\infty E(y-\eta)e^{\beta(\eta-y)}d\eta\right] && (I_7^2) \\
&\quad + 6\delta^2 e^{2\beta y}\varepsilon^2\left[\theta^2 - \int_{-R}^0 E(y-\eta)e^{2\beta(\eta-y)}d\eta - \theta^2\int_0^{R_\delta} E(y-\eta)e^{2\beta(\eta-y)}d\eta\right] && (I_8^2) \\
&\quad - 6\delta^2 e^{2\beta y}\varepsilon^2\theta^2\int_{R_\delta}^\infty E(y-\eta)e^{2\beta(R_\delta-y)}d\eta
\end{aligned}$$

$$
\begin{aligned}
&+ 6B^2\delta^2 e^{2\beta y}\left[1 - \int_{-\infty}^{R_\delta} E(\eta - y)e^{2\beta(\eta-y)}d\eta - \int_{R_\delta}^{\infty} E(\eta - y)e^{2\beta(R_\delta - y)}d\eta\right] && (I_9^2)\\
&- 4B\delta^3 e^{3\beta y}\left[1 - \int_{\mathbb{R}} E(\eta - y)e^{3\beta(\eta-y)}d\eta\right] && (I_{10}^2)\\
&+ \delta^4 e^{4\beta y}\left[1 - \int_{-\infty}^{R_\delta} E(\eta - y)e^{4\beta(\eta-y)}d\eta - \int_{R_\delta}^{\infty} E(\eta - y)e^{4\beta(R_\delta - y)}d\eta\right] && (I_{11}^2)\\
&- 12B^2\varepsilon\delta e^{\beta y}\left[\theta - \int_{-R}^{0} E(\eta - y)e^{\beta(\eta-y)}d\eta - \theta\int_{0}^{\infty} E(\eta - y)e^{\beta(\eta-y)}d\eta\right] && (I_{12}^2)\\
&- 12B\varepsilon^2\delta e^{\beta y}\left[\theta^2 - \int_{-R}^{0} E(\eta - y)e^{\beta(\eta-y)}d\eta - \theta^2\int_{0}^{\infty} E(\eta - y)e^{\beta(\eta-y)}d\eta\right] && (I_{13}^2)\\
&+ 12B\varepsilon\delta^2 e^{2\beta y}\left[\theta - \int_{-R}^{0} E(\eta - y)e^{2\beta(\eta-y)}d\eta - \theta\int_{0}^{R_\delta} E(\eta - y)e^{2\beta(\eta-y)}d\eta\right] && (I_{14}^2)\\
&- 12B\theta\varepsilon\delta^2 e^{2\beta y}\int_{R_\delta}^{\infty} E(\eta - y)e^{2\beta(R_\delta - y)}d\eta,
\end{aligned}
$$

where we used that $e^{n\beta R_\delta} \leq e^{n\beta\eta}$ for any $n = 1, 2, 3, 4$ and $\eta \geq R_\delta$.

Arguing as in the proof of (E.55), (E.56) and (E.57) we see that also

$$
(I_1^2) \leq -\frac{3c_0}{4}\beta\delta e^{\beta y}, \qquad (I_2^2) \leq 0 \qquad \text{and} \qquad (I_3^2) \leq 0.
$$

As we argued for (E.56) and (E.57) using that $1 - \theta^2 = 24\theta^2$ and $1 - \theta^4 = 624\theta^4$ we estimate

$$
(I_4^2) \leq -138B^2\varepsilon^2\theta^2 \int_y^{R+y} E(\eta)d\eta < 0 \qquad \text{and} \qquad (I_5^2) \leq -623\varepsilon^4\theta^4 \int_y^{R+y} E(\eta)d\eta < 0. \tag{E.74}
$$

Finally, estimating only the positive terms, using that $\delta e^{\beta y} \leq \varepsilon\theta$ for $y < R_\delta$ and using the definition of β in (E.49), we compute

$$
\begin{aligned}
(I_6^2) &\leq 4\varepsilon\frac{\operatorname{artanh}(3\beta)}{3\beta}\delta^3 e^{3\beta y} \leq 6\varepsilon^3\delta e^{\beta y}, \; (I_7^2) \leq 4\varepsilon^3\frac{\operatorname{artanh}(\beta)}{\beta}\delta e^{\beta y} \leq 6\varepsilon^3\delta e^{\beta y}, \quad (I_8^2) \leq 6\varepsilon^3\delta e^{\beta y}\\
(I_9^2) &\leq 6\varepsilon B^2\delta e^{\beta y}, \qquad (I_{10}^2) \leq 6\varepsilon^2 B\delta e^{\beta y}, \qquad (I_{11}^2) \leq \varepsilon^3\delta e^{\beta y},\\
(I_{12}^2) &\leq 18B^2\varepsilon\delta e^{\beta y}, \qquad (I_{13}^2) \leq 18B\varepsilon^2\delta e^{\beta y}, \text{ and } \qquad (I_{14}^2) \leq 12B\varepsilon^2\delta e^{\beta y}.
\end{aligned}
$$

Hence, choosing in the definition (E.71) of ε the constant $C(\beta, B, \theta) > 0$ as

$$
C(\beta, B, \theta) = \min\left\{1, \frac{c_0\beta}{2(18 + 36B + 24B^2)}\right\},
$$

we conclude that

$$
-\psi_\delta''(y) + c\psi_\delta'(y) + (B + \psi_\delta(y))^4 - \int_{\mathbb{R}} E(y - \eta)\left(B + \psi_\delta(\eta)\right)^4 d\eta = -\frac{c_0}{4}\beta\delta e^{\beta y} < 0.
$$

We see once more that by the choice of all the parameters we have $\psi_{\delta_0} \leq \omega$ on $\mathbb{R}$ as well as $\psi_{\delta_0} < \omega$ on $[0, R_{\delta_0}]$. Moreover, for all $\delta < \delta_0$ it is true that $\psi_\delta \leq \omega$ on $\mathbb{R} \setminus (0, R_\delta)$ as well as $\psi_R(0) < \omega$ and $\psi_\delta(R_\delta) < \omega$. Hence, arguing as in the proof of Lemma E.6 we see that assuming the existence of some $\delta < \delta_0$ with

$$
\inf_{y>0}(\omega - \psi_\delta) = \min_{[0,R_\delta]}(\omega - \psi_\delta) < 0
$$

there exists also some $\delta < \delta^* < \delta_0$ defined by $\delta^* := \sup \left\{ \delta < \delta^* < \delta_0 : \min_{[0,R_\delta]} (\omega - \psi_{\delta^*}) < 0 \right\}$ such that

$$\min_{[0,R_\delta]} (\omega - \psi_{\delta^*}) = \omega(y_0) - \psi_{\delta^*} = 0$$

for some $y_0 \in (0, R_{\delta^*})$. However, the application of the maximum principle for the equation (E.70) to the functions ω and ψ_{δ^*} yields as in (E.69) the contradiction $0 < -\int_{\mathbb{R}} E(y - \eta) \left[(B + \omega(\eta))^4 - (B + \psi_{\delta^*}(\eta))^4 \right] d\eta < 0$. Therefore, we conclude that

$$\inf_{y>0} (\omega - \psi_\delta) \geq 0$$

for all $\delta < \delta_0$, so that $w(y) \geq \varepsilon\theta - \delta e^{\beta y}$ for all $\delta < \delta_0$ and all $y \in [0, R_\delta]$. Thus, taking the pointwise limit as $\delta \to 0$ we establish

$$f(y) - B = w(y) \geq \varepsilon\theta > 0,$$

which contradicts the assumption that $\liminf_{y\to\infty} f(y) = B$. Hence, f does not attain its infimum at $+\infty$. $\square$

E.3.2 The ω-limits of the traveling waves are constant

Lemma E.5, Lemma E.6 and Lemma E.7 imply that the limit function $\bar{f}$ solving (E.46) is either constant or it takes the supremum and infimum at $-\infty$, i.e.

$$\inf_{\mathbb{R}} \bar{f} = \liminf_{y\to-\infty} \bar{f}(y) < \limsup_{y\to-\infty} \bar{f}(y) = \sup_{\mathbb{R}} \bar{f}.$$

We will show that $\bar{f}$ is constant, showing that $\liminf_{y\to-\infty} \bar{f}(y) = \limsup_{y\to-\infty} \bar{f}(y)$. We start proving the following Theorem, which is a fundamental stability result.

Theorem E.5. *Let f solve* (E.46) *for $0 < \lambda < T_M$. Then there exists an $\varepsilon_0 = \varepsilon_0(T_M, \lambda, c) > 0$ such that for all $\varepsilon < \varepsilon_0$ there exists $L_0(\varepsilon, T_m, \lambda, c) > 0$ with the property that if*

$$\operatorname*{osc}_{[-L,L]} f < \varepsilon$$

then also

$$\operatorname*{osc}_{[L,\infty)} f < 3\varepsilon$$

for all $L > L_0$.

Proof. Let us assume that f satisfy $\operatorname*{osc}_{[-L,L]} f < \varepsilon$ for some $L > 0$ and some $\varepsilon > 0$. We show that for $\varepsilon > 0$ small enough and for $L > 0$ large enough this assumption implies $\operatorname*{osc}_{[L,\infty)} f < 3\varepsilon$. In the course of the proof we will also define ε_0 and $L_0(\varepsilon)$.

If $\operatorname*{osc}_{[-L,L]} f < \varepsilon$, then the maximum $\max_{[-L,L]} f =: M_L < T_M$ and the minimum $\min_{[-L,L]} f =: m_L > \lambda$ satisfy

$$M_L - m_L < \varepsilon.$$

We now construct two suitable families of subsolutions and supersolutions, for which the maximum principle will show the claim in a similar way as in the proofs of Lemma E.6 and of Lemma E.7. Let us consider the following functions

$$\psi_\delta^L(y) = m_L - \varepsilon + \begin{cases} -(m_L - \varepsilon) & y < -L \\ \varepsilon - \delta e^{\beta(y-L)} & -L \le y < L \\ \varepsilon\theta - \delta e^{\beta(y-L)} & y \in [L, R_\delta] \\ -(m_L - \varepsilon) & y > R_\delta, \end{cases} \tag{E.75}$$

where $R_\delta = \frac{1}{\beta} \ln\left(\frac{m_L-(1-\theta)\varepsilon}{\delta}\right) + L$ is so defined that ψ_δ^L is continuous on $[L, \infty)$. Moreover, we notice that ψ_δ^L is smooth in (L, R_δ). We consider $\varepsilon < \lambda$ as well as $\delta < m_L - \lambda$, so that $m_L - \varepsilon > 0$ and $R_\delta > L$, and we study the family of functions ψ_δ^L for $\delta < \delta_0$, where $\delta_0(\varepsilon, L) > 0$ will be specified later. We also fix $\theta = \frac{1}{5}$ and $c_0 = \min\{c, 1\}$. In addition we choose

$$\beta < \min\left\{\frac{1}{24}, \frac{c_0}{2}, \frac{c_0}{2T_M^3}, \left(\frac{10}{8}\frac{c_0}{77T_M^3}\right)^{24}\right\} \tag{E.76}$$

satisfying also

$$\beta^2 - \frac{c_0}{2}\beta + 4T_M^3\left(\frac{\operatorname{artanh}(\beta)}{\beta} - 1\right) + 4T_M^3\left(\frac{\operatorname{artanh}(3\beta)}{3\beta} - 1\right) \le 0. \tag{E.77}$$

For $c_1 = 6 \operatorname{artanh}\left(\frac{1}{6}\right)$ we take $\varepsilon < \varepsilon_1$ defined by

$$\varepsilon_1 = \min\left\{1, \lambda, \frac{c_0\beta}{8(4c_1 + 16c_1T_M^2 + 12T_M^2 + 12T_Mc_1 + 6T_M)}\right\}. \tag{E.78}$$

We also consider a fixed $L > L_1(\varepsilon)$ satisfying

$$\left(1 + \frac{T_M}{\varepsilon\theta} + \frac{T_M^3}{(\varepsilon\theta)^3}\right)\int_{L_1+y}^{\infty} E(z)dz < \int_{y-L_1}^{L_1+y} E(z)dz \text{ for all } y > L_1,\ e^{-L_1} < \beta^2 \text{ and } L_1 > \frac{1}{\sqrt{\beta}}. \tag{E.79}$$

We remark that β given by (E.77) and L_1 defined by (E.79) are well-defined. For β one argues similarly as for (E.49), while for L_1 we need to adapt the proof for (E.48). This adaptation however is easy. As we proved in (E.48), one can readily see that for any $A > 0$

$$A\int_{N+y}^{\infty} E(z)dz < \int_0^{N+y} E(z)dz \text{ for } N = N(A) > 0 \text{ large enough and for all } y > 0.$$

Taking $A = 1 + \frac{T_M}{\varepsilon\theta} + \frac{T_M^3}{(\varepsilon\theta)^3}$ and $L_1 = \frac{N(A)}{2}$, we conclude (E.79). Moreover, we remark that the function

$$N \mapsto A\int_{N+y}^{\infty} E(z)dz - \int_0^{N+y} E(z)dz$$

is monotonically decreasing for $N > N(A)$. Hence, (E.79) holds also for all $L > L_1(\varepsilon)$.

Finally, we set

$$\delta_0 = (m_L - (1-\theta)\varepsilon)e^{-\beta R_\varepsilon} > 0, \tag{E.80}$$

where R_ε is the distance such that

$$f(y) - m_L \ge -\frac{1-\theta}{2}\varepsilon \quad \text{for all} \quad y \in [L - R_\varepsilon, L + R_\varepsilon]. \tag{E.81}$$

It is important to notice that R_ε is independent of L. This can be proved using the uniform continuity of f, according to which there exists R_ε such that $f(y) - f(x) \geq -\frac{1-\theta}{2}\varepsilon$ for all $|x-y| < R_\varepsilon$. Finally, $x = L$ and $f(L) \geq m_L$ implies (E.81). Thus, with δ_0 defined in (E.80) we see that

$$R_{\delta_0} = \frac{1}{\beta}\ln\left(e^{\beta R_\varepsilon}\right) = R_\varepsilon + L.$$

Hence, for all $y \in [L, R_{\delta_0}]$ we have by construction $\psi_{\delta_0}^L \leq m_L - (1-\theta)\varepsilon$ as well as $f(y) \geq m_L - \frac{1-\theta}{2}\varepsilon$. This implies

$$f(y) - \psi_{\delta_0}^L \geq \frac{1-\theta}{2}\varepsilon > 0.$$

for $y \in [L, R_{\delta_0}]$.

Moreover, by definition we know that $\psi_\delta^L < f(y)$ for $y \in \mathbb{R} \setminus (L, R_\delta)$ and for all $\delta \leq \delta_0$. We remark also that $\psi_\delta^L(L) = m_L - (1-\theta)\varepsilon < m_L \leq f(L)$ as well as $\psi_\delta^L(R_\delta) = 0 < f(R_\delta)$. Hence, $\psi_{\delta_0}^L < f$ in $\mathbb{R}$.

We will now show that ψ_δ^L is a subsolution to the equation (E.46) for $y \in (L, R_\delta)$, where the function is also smooth.

Let us first of all assume that $y \in \left[R_\delta - \frac{1}{\sqrt{\beta}}, R_\delta\right) \cap (L, R_\delta)$. We compute

$$\delta e^{\beta(y-L)} \geq \delta e^{\beta(R_\delta - L)} e^{-\sqrt{\beta}} = (m_L - (1-\theta)\varepsilon)e^{-\sqrt{\beta}}.$$

Hence,

$$0 \leq \psi_\delta^L(y) \leq (m_L - (1-\theta)\varepsilon)\left(1 - e^{-\sqrt{\beta}}\right).$$

This implies that

$$\begin{aligned}
-\left(\psi_\delta^L\right)''(y) &+ c\left(\psi_\delta^L\right)'(y) + \left(\psi_\delta^L(y)\right)^4 - \int_{\mathbb{R}} E(\eta - y)\left(\psi_\delta^L(\eta)\right)^4 d\eta \\
&< (m_L - (1-\theta)\varepsilon)(\beta^2 - c\beta) + (m_L - (1-\theta)\varepsilon)^4\left(1 - e^{-\sqrt{\beta}}\right)^4 \\
&\leq (m_L - (1-\theta)\varepsilon)^4\left[\frac{\beta^2 - c_0\beta}{(m_L - (1-\theta)\varepsilon)^3} + \beta^2\right] \\
&\leq (m_L - (1-\theta)\varepsilon)^4\left[-\frac{c_0\beta}{2(m_L - (1-\theta)\varepsilon)^3} + \beta^2\right] \leq (m_L - (1-\theta)\varepsilon)^4\left[-\frac{c_0\beta}{2T_M^3} + \beta^2\right] < 0.
\end{aligned} \tag{E.82}$$

We used besides the definition of β in (E.76) also that $(m_L - (1-\theta)\varepsilon) \leq T_M$, $c \leq c_0$ as well as $1 - e^{-|x|} \leq |x|$.

It remains to show that ψ_δ^L is a subsolution also for $y \in \left(L, R_\delta - \frac{1}{\sqrt{\beta}}\right)$. Without loss of generality we assume $\left(L, R_\delta - \frac{1}{\sqrt{\beta}}\right) \neq \emptyset$, since this is true for δ small enough. Moreover, for all $\delta \leq \delta_0$ with $\left[R_\delta - \frac{1}{\sqrt{\beta}}, R_\delta\right) \cap (L, R_\delta) = (L, R_\delta)$ estimate (E.82) gives the result about ψ_δ^L being a subsolution. We collect many estimates similar to the ones made for (E.54) and (E.73). For the following computation we use $c \geq c_0$ and that $e^{\beta R_\delta} < e^{\beta\eta}$ for $\eta > R_\delta$, we expand the power law, and we order similar terms together.

$$
\begin{aligned}
&-\left(\psi_\delta^L\right)''(y) + c\left(\psi_\delta^L\right)'(y) + \left(\psi_\delta^L(y)\right)^4 - \int_{\mathbb{R}} E(\eta-y)\left(\psi_\delta^L(\eta)\right)^4 d\eta && \text{(E.83)}\\
&\leq -\frac{c_0}{2}\beta\delta e^{\beta(y-L)} && (I_1^3)\\
&\quad + \delta e^{\beta(y-L)}\left(\beta^2 - \frac{c_0}{4}\beta - 4(m_L-\varepsilon)^3 + 4(m_L-\varepsilon)^3\int_{-L}^{\infty} E(\eta-y)e^{\beta(\eta-y)}d\eta\right)\\
&\quad - 4(m_L-\varepsilon)\delta^3 e^{3\beta(y-L)}\left(1 - \int_{-L}^{\infty} E(\eta-y)e^{3\beta(\eta-y)}d\eta\right) && (I_2^3)\\
&\quad + 6(m_L-\varepsilon)^2\delta^2 e^{2\beta(y-L)}\left(1 - \int_{-L}^{R_\delta} E(\eta-y)e^{2\beta(\eta-y)}d\eta\right) && (I_3^3)\\
&\quad - 6(m_L-\varepsilon)^2\delta^2\int_{R_\delta}^{\infty} E(\eta-y)e^{2\beta(R_\delta-L)}\\
&\quad + \delta^4 e^{4\beta(y-L)}\left(1 - \int_{-L}^{R_\delta} E(\eta-y)e^{4\beta(\eta-y)}d\eta\right) - \delta^4\int_{R_\delta}^{\infty} E(\eta-y)e^{4\beta(R_\delta-L)} && (I_4^3)\\
&\quad + 4(m_L-\varepsilon)^3\left[\varepsilon\theta + \int_{-\infty}^{-L} E(\eta-y)(m_L-\varepsilon)d\eta - \varepsilon\int_{-L}^{L} E(\eta-y)d\eta\right] && (I_5^3)\\
&\quad - 4(m_L-\varepsilon)^3\varepsilon\theta\int_{L}^{\infty} E(\eta-y)d\eta\\
&\quad + 4(m_L-\varepsilon)\left[(\varepsilon\theta)^3 + \int_{-\infty}^{-L} E(\eta-y)(m_L-\varepsilon)^3 d\eta - \varepsilon^3\int_{-L}^{L} E(\eta-y)d\eta\right] && (I_6^3)\\
&\quad - 4(m_L-\varepsilon)(\varepsilon\theta)^3\int_{L}^{\infty} E(\eta-y)d\eta\\
&\quad + 6(m_L-\varepsilon)^2\left[(\varepsilon\theta)^2 - \int_{-\infty}^{-L} E(\eta-y)(m_L-\varepsilon)^2 d\eta - \varepsilon^2\int_{-L}^{L} E(\eta-y)d\eta\right] && (I_7^3)\\
&\quad - 6(m_L-\varepsilon)^2(\varepsilon\theta)^2\int_{L}^{\infty} E(\eta-y)d\eta\\
&\quad + (\varepsilon\theta)^4 - \int_{-\infty}^{-L} E(\eta-y)(m_L-\varepsilon)^4 d\eta - \varepsilon^4\int_{-L}^{L} E(\eta-y)d\eta - (\varepsilon\theta)^4\int_{L}^{\infty} E(\eta-y)d\eta && (I_8^3)\\
&\quad - 4\varepsilon^3\delta e^{\beta(y-L)}\left(\theta^3 - \int_{-L}^{L} E(\eta-y)e^{\beta(\eta-y)}d\eta - \theta^3\int_{L}^{\infty} E(\eta-y)e^{\beta(\eta-y)}d\eta\right) && (I_9^3)\\
&\quad - 4\varepsilon\delta^3 e^{3\beta(y-L)}\left(\theta^3 - \int_{-L}^{L} E(\eta-y)e^{3\beta(\eta-y)}d\eta - \theta\int_{L}^{\infty} E(\eta-y)e^{3\beta(\eta-y)}d\eta\right) && (I_{10}^3)\\
&\quad + 6\varepsilon^2\delta^2 e^{2\beta(y-L)}\left(\theta^2 - \int_{-L}^{L} E(\eta-y)e^{2\beta(\eta-y)}d\eta - \theta^2\int_{L}^{R_\delta} E(\eta-y)e^{2\beta(\eta-y)}d\eta\right) && (I_{11}^3)\\
&\quad - 6(\varepsilon\theta\delta)^2 e^{2\beta(R_\delta-L)}\int_{R_\delta}^{\infty} E(\eta-y)d\eta\\
&\quad - 12(m_L-\varepsilon)^2\varepsilon\delta e^{\beta(y-L)}\left(\theta - \int_{-L}^{L} E(\eta-y)e^{\beta(\eta-y)}d\eta - \theta\int_{L}^{\infty} E(\eta-y)e^{\beta(\eta-y)}d\eta\right) && (I_{12}^3)\\
&\quad - 12(m_L-\varepsilon)\varepsilon^2\delta e^{\beta(y-L)}\left(\theta^2 - \int_{-L}^{L} E(\eta-y)e^{\beta(\eta-y)}d\eta - \theta^2\int_{L}^{\infty} E(\eta-y)e^{\beta(\eta-y)}d\eta\right) && (I_{13}^3)\\
&\quad + 12(m_L-\varepsilon)\varepsilon\delta^2 e^{2\beta(y-L)}\left(\theta - \int_{-L}^{L} E(\eta-y)e^{2\beta(\eta-y)}d\eta - \theta\int_{L}^{R_\delta} E(\eta-y)e^{2\beta(\eta-y)}d\eta\right) && (I_{14}^3)\\
&\quad - 12(m_L-\varepsilon)\varepsilon\delta^2 e^{2\beta(R_\delta-L)}\int_{R_\delta}^{\infty} E(\eta-y)d\eta.
\end{aligned}
$$

Next, using (E.53) and estimating $(m-\varepsilon) \leq T_M$ as well as $\delta e^{\beta(y-L)} \leq m-(1-\theta)\varepsilon \leq T_M$ we can compute

$$\begin{aligned}(I_1^3)+(I_2^3) \leq & -\frac{c_0}{2}\beta\delta e^{\beta(y-L)} \\ & + \delta e^{\beta(y-L)}\left[\beta^2 - \frac{c_0}{4}\beta + 4T_M^3\left(\frac{\operatorname{artanh}(\beta)}{\beta}-1\right)+4T_M^3\left(\frac{\operatorname{artanh}(3\beta)}{3\beta}-1\right)\right] \\ \leq & -\frac{c_0}{2}\beta\delta e^{\beta(y-L)}\end{aligned} \tag{E.84}$$

by the choice of β as in (E.77). Next we consider (I_7^3) and (I_8^3). Here we use (E.56), (E.57) and (E.74), obtaining

$$(I_7^3) \leq 6(m_L-\varepsilon)^2\left[(\varepsilon\theta)^2 - \varepsilon^2\int_{-L}^{L}E(\eta-y)d\eta - (\varepsilon\theta)^2\int_{L}^{\infty}E(\eta-y)d\eta\right] \leq 0 \tag{E.85}$$

as well as

$$(I_8^3) \leq (\varepsilon\theta)^4 - \varepsilon^4\int_{-L}^{L}E(\eta-y)d\eta - (\varepsilon\theta)^4\int_{L}^{\infty}E(\eta-y)d\eta \leq 0 \tag{E.86}$$

Using $\theta=\frac{1}{5}$ and $L \geq L_1$ as defined in (E.79), we also estimate

$$\begin{aligned}(I_5^3) \leq & 4(m_L-\varepsilon)^3\left[\varepsilon\theta\int_{-\infty}^{L}E(\eta-y)d\eta + T_M\int_{-\infty}^{-L}E(\eta-y)d\eta - \varepsilon\int_{-L}^{L}E(\eta-y)d\eta\right] \\ \leq & 4(m_L-\varepsilon)^3\left[\varepsilon\theta\int_{-\infty}^{-L}E(\eta-y)d\eta + T_M\int_{-\infty}^{-L}E(\eta-y)d\eta - 4\varepsilon\theta\int_{-L}^{L}E(\eta-y)d\eta\right] \\ = & 4(m_L-\varepsilon)^3\varepsilon\theta\left[\left(1+\frac{T_M}{\varepsilon\theta}\right)\int_{-\infty}^{-L}E(\eta-y)d\eta - 4\int_{-L}^{L}E(\eta-y)d\eta\right] \leq 0\end{aligned} \tag{E.87}$$

and

$$\begin{aligned}(I_6^3) \leq & 4(m_L-\varepsilon)\left[(\varepsilon\theta)^3\int_{-\infty}^{L}E(\eta-y)d\eta + T_M^3\int_{-\infty}^{-L}E(\eta-y)d\eta - \varepsilon^3\int_{-L}^{L}E(\eta-y)d\eta\right] \\ \leq & 4(m_L-\varepsilon)(\varepsilon\theta)^3\left[\left(1+\frac{T_M^3}{(\varepsilon\theta)^3}\right)\int_{-\infty}^{-L}E(\eta-y)d\eta - 124\int_{-L}^{L}E(\eta-y)d\eta\right] \leq 0.\end{aligned} \tag{E.88}$$

Using $\frac{\operatorname{artanh}(n\beta)}{n\beta} \leq c_1$ for $n \leq 4$, $\theta=\frac{1}{5}$ as well as the estimate $\delta e^{\beta(y-L)} \leq T_M$ for $y \leq R_\delta$ we compute furthermore

$$(I_9^3) \leq 4\varepsilon^3\delta e^{\beta(y-L)}\frac{\operatorname{artanh}(\beta)}{\beta} \leq 4\varepsilon^3 c_1\delta e^{\beta(y-L)}, \tag{E.89}$$

$$(I_{10}^3) \leq 4\varepsilon\delta^3 e^{3\beta(y-L)}\frac{\operatorname{artanh}(3\beta)}{3\beta} \leq 4\varepsilon T_M^2 c_1\delta e^{\beta(y-L)}, \tag{E.90}$$

$$(I_{11}^3) \leq 6\varepsilon^2 T_M\delta e^{\beta(y-L)}, \text{ and } (I_{12}^3)+(I_{13}^3)+(I_{14}^3) \leq 12\varepsilon T_M\delta e^{\beta(y-L)}\left(T_M(c_1+1)+c_1\right). \tag{E.91}$$

Finally, we have to estimate the remaining terms (I_3^3) and (I_4^3). To this end we recall that we are considering the case for which $R_\delta - y \geq \frac{1}{\sqrt{\beta}}$ and that we have chosen $L > L_1$ such that $e^{-L_1} \leq \beta^2$. Additionally, we also use that

$$e^{-\frac{1}{\sqrt{\beta}}} < \beta^{\frac{5}{4}} \text{ for all } \beta > 0. \tag{E.92}$$

We hence estimate

$$\begin{aligned}(I_3) \leq & 6(m_L-\varepsilon)^2\delta^2 e^{2\beta(y-L)}\left(1-\int_{-L}^{R_\delta} E(\eta-y)e^{2\beta(\eta-y)}d\eta\right)\\ =& 6(m_L-\varepsilon)^2\delta^2 e^{2\beta(y-L)}\left(1-\frac{\operatorname{artanh}(2\beta)}{2\beta}+\int_{-\infty}^{-L} E(\eta-y)e^{2\beta(\eta-y)}d\eta\right)\\ &+6(m_L-\varepsilon)^2\delta^2 e^{2\beta(y-L)}\int_{R_\delta}^{\infty} E(\eta-y)e^{2\beta(\eta-y)}d\eta\\ \leq & 6T_M^3\delta e^{\beta(y-L)}\left(\int_{-\infty}^{-(L+y)} E(\eta)e^{2\beta\eta}d\eta+\int_{R_\delta-y}^{\infty} E(\eta)e^{2\beta\eta}d\eta\right)\\ \leq & 6T_M^3\delta e^{\beta(y-L)}\left(\frac{e^{-(L+y)}}{2}+\int_{\frac{1}{\sqrt{\beta}}}^{\infty}\frac{e^{-(1-2\beta)\eta}}{2}d\eta\right)\leq 6T_M^3\delta e^{\beta(y-L)}\left(\frac{\beta^2}{2}+\frac{3}{5}e^{-\frac{5}{6\sqrt{\beta}}}\right)\\ \leq & 6T_M^3\delta e^{\beta(y-L)}\left(\frac{\beta^2}{2}+\frac{3}{5}\beta\beta^{\frac{1}{24}}\right).\end{aligned} \tag{E.93}$$

We also used in the second inequality that $\frac{\operatorname{artanh}(a)}{a}\geq 1$, as well as in the third inequality the estimate $e^{2\beta(\eta)}\leq 1$ for $\eta\leq -(L+y)\leq 0$, the estimate (E.27) and the inequality $E(z)\leq \frac{e^{-|z|}}{2}$ for $|z|>1$ since $\beta^{-\frac{1}{2}}>1$. For the fourth inequality we used $(1-2\beta)\geq \frac{5}{6}$ since $\beta<\frac{1}{24}<\frac{1}{12}$ and we concluded the fifth estimate with (E.92). In a very similar way, using again that $(1-4\beta)\geq\frac{5}{6}$ since $\beta<\frac{1}{24}$, we also have the estimate

$$\begin{aligned}(I_4) \leq & \ \delta^4 e^{4\beta(y-L)}\left(1-\int_{-L}^{R_\delta} E(\eta-y)e^{4\beta(\eta-y)}d\eta\right)\\ =&\delta^4 e^{4\beta(y-L)}\left(1-\frac{\operatorname{artanh}(4\beta)}{4\beta}+\int_{-\infty}^{-L} E(\eta-y)e^{4\beta(\eta-y)}d\eta+\int_{R_\delta}^{\infty} E(\eta-y)e^{4\beta(\eta-y)}d\eta\right)\\ \leq & T_M^3\delta e^{\beta(y-L)}\left(\int_{-\infty}^{-(L+y)} E(\eta)e^{4\beta\eta}d\eta+\int_{R_\delta-y}^{\infty} E(\eta)e^{4\beta\eta}d\eta\right)\leq T_M^3\delta e^{\beta(y-L)}\left(\frac{\beta^2}{2}+\frac{3}{5}\beta\beta^{\frac{1}{24}}\right).\end{aligned} \tag{E.94}$$

Putting now together all estimates (E.84)-(E.91) and (E.93)-(E.94), and using that $\beta<\left(\frac{10}{8}\frac{c_0}{77T_M^3}\right)^{24}$ and $\varepsilon<1$ we conclude for $y\in\left(L,R_\delta-\frac{1}{\sqrt{\beta}}\right)$

$$\begin{aligned}&-\left(\psi_\delta^L\right)''(y)+c\left(\psi_\delta^L\right)'(y)+\left(\psi_\delta^L(y)\right)^4-\int_{\mathbb{R}} E(\eta-y)\left(\psi_\delta^L(\eta)\right)^4d\eta\\ &\quad\leq \delta e^{\beta(y-L)}\left(-\frac{c_0}{2}\beta+\varepsilon\left(4c_1+16c_1T_M^2+12T_M^2+12c_1T_M+6T_M\right)+\frac{77T_M^3}{10}\beta\beta^{\frac{1}{24}}\right)\\ &\quad\leq \delta e^{\beta(y-L)}\left(-\frac{c_0}{2}\beta+\frac{c_0}{8}\beta+\frac{c_0}{8}\beta\right)=-\frac{c_0}{4}\beta\delta e^{\beta(y-L)}<0,\end{aligned} \tag{E.95}$$

where at the end we used the choice of $\varepsilon<\varepsilon_1$ and of β according to (E.78) and (E.76), respectively.

Estimates (E.82) and (E.95) show that for all $\delta<\delta_0$ and for all $y\in(L,R_\delta)$ the functions ψ_δ^L are subsolutions, i.e.

$$-\left(\psi_\delta^L\right)''(y)+c\left(\psi_\delta^L\right)'(y)+\left(\psi_\delta^L(y)\right)^4\leq 0.$$

Since by construction $\psi_{\delta_0}^L < f$ in $\mathbb{R}$ with $\psi_{\delta_0}^L - f \le -\frac{1-\theta}{2}\varepsilon < 0$ for all $y \ge L$, as well as $\psi_\delta^L \le f$ for $y \in \mathbb{R} \setminus (L, R_\delta)$ with $\psi_\delta^L\mid_{\{L,R_\delta\}} < f\mid_{\{L,R_\delta\}}$, applying the maximum principle in the same way as we did in the proof of Lemma E.6 and Lemma E.7 and using the uniform continuity and the increasing monotonicity of $\delta \mapsto \psi_\delta^L$ on compact sets as well as the fact that ψ_δ^L are subsolutions on (L, R_δ) we conclude that

$$\psi_\delta^L(y) \le f(y) \quad \text{for all } y \in \mathbb{R} \text{ and } \delta < \delta_0.$$

Hence, for any $y > L$ we have for $\delta < \delta_0$ small enough

$$f(y) \ge m_L - (1-\theta)\varepsilon - \delta e^{\beta(y-L)}.$$

Taking $\delta \to 0$ and thus $R_\delta \to \infty$ yields

$$f(y) \ge m_L - (1-\theta)\varepsilon \quad \text{for all } y > L. \tag{E.96}$$

In a similar way we show now that $f(y) \le M_L + (1-\theta)\varepsilon$ for $y > L$. We consider a similar family of functions called $\{\psi_\gamma^L\}$ which we will prove to be supersolutions. In this case we define

$$\psi_\gamma^L(y) = M_L + \varepsilon + \begin{cases} 2T_M - (M_L+\varepsilon) & y < -L \\ \gamma e^{\zeta(y-L)} - \varepsilon & -L \le y < L \\ \gamma e^{\zeta(y-L)} - \varepsilon\theta & L \le y \le R_\gamma \\ 2T_M - (M_L+\varepsilon) & y > R_\gamma, \end{cases} \tag{E.97}$$

where $R_\gamma = \frac{1}{\zeta}\ln\left(\frac{2T_M - (M_L + (1-\theta)\varepsilon)}{\gamma}\right) + L$. We consider also $\varepsilon < T_M$ and $\gamma < T_M - M_L$, so that $2T_M - M_L - \varepsilon > 0$ as well as $R_\gamma > L$. We remark that since f does not take supremum and infimum at the interior, $T_M - M_L > 0$. Moreover, we notice that this family of functions is continuous on (L,∞) as well as smooth on (L, R_γ). For a $\gamma_0(\varepsilon, L) > 0$ defined later we study the family of functions $\left\{\psi_\gamma^L\right\}$ for $\gamma < \gamma_0$. We also fix as usual $\theta = \frac{1}{5}$ and $c_0 = \min\{1, c\}$. Additionally, we choose

$$\zeta < \min\left\{\frac{1}{4}, \frac{c_0}{2}\right\}$$

such that

$$\frac{\operatorname{artanh}(4\zeta)}{4\zeta} < \frac{3}{2} \quad \text{and} \quad \frac{c_0}{2}\zeta - \zeta^2 - 15(2T_M)^3\left(\frac{\operatorname{artanh}(4\zeta)}{4\zeta} - 1\right) > 0. \tag{E.98}$$

We also consider $\varepsilon < \varepsilon_2$ defined by

$$\varepsilon_2 = \min\left\{1, T_M, \frac{\zeta c_0}{4}\frac{1}{4 + 27(2T_M) + 28(2T_M)^2}, \frac{2}{5}\lambda\right\}. \tag{E.99}$$

We notice that ζ depends only on c, so that $\varepsilon_2 = \varepsilon_2(c, \lambda, T_M)$. Moreover, we study the family of functions for $L > L_2(\varepsilon, T_M)$ satisfying

$$\left(1 + \frac{2T_M}{\varepsilon\theta} + \frac{(2T_M)^2}{(\varepsilon\theta)^2} + \frac{(2T_M)^3}{(\varepsilon\theta)^3} + \frac{(2T_M)^4}{(\varepsilon\theta)^4}\right)\int_{L_2+y}^{\infty} E(z)dz < \int_{y-L_2}^{y+L_2} E(z)dz \text{ for all } y > L_2. \tag{E.100}$$

We remark that such ζ as in (E.98) and such L_2 as in (E.100) exist, as we have seen already several times. Moreover, (E.100) holds true for all $L > L_2$.

We will also consider $\gamma_0 = (2T_M - (M_L + (1-\theta)\varepsilon))\, e^{-\zeta R_\varepsilon}$, where R_ε is once again the distance such that $M_L - f(y) \geq -\frac{1-\theta}{2}\varepsilon$ for all $y \in [L - R_\varepsilon, L + R_\varepsilon]$. By the uniform continuity of f and since $f(L) \leq M_L$ we know that such R_ε exists and it is independent of L. Moreover, by definition we obtain

$$R_{\gamma_0} = R_\varepsilon,$$

which implies that for all $y \in [L, R_{\gamma_0}]$ we have

$$\psi^L_{\gamma_0}(y) - f(y) > \frac{1-\theta}{2}\varepsilon > 0,$$

since $\psi^L_{\gamma_0} \geq M_L + (1-\theta)\varepsilon$ and $f(y) \leq M_L + \frac{1-\theta}{2}\varepsilon$.

We also remark that by construction we have that for all $\gamma \leq \gamma_0$

$$\psi^L_\gamma(y) > f(y) \qquad \text{for all } y \in \mathbb{R} \setminus (L, R_\gamma)$$

and also $\psi^L_\gamma(L) > M_L \geq f(y)$ as well as $\psi^L_\gamma(R_\delta) = 0 < f(R_\gamma)$. Thus, $\psi^L_{\gamma_0} > f$ in $\mathbb{R}$.

We now show that the functions ψ^L_γ are supersolutions to the equation (E.46) for $y \in (L, R_\gamma)$, the interval where the functions are smooth. This will be done in the spirit of (E.54), (E.72) and (E.83). We use that $-e^{\zeta(\eta-L)} \leq -e^{\zeta(R_\gamma - L)}$ for all $\eta > R_\gamma$, we expand the power law, and we rearrange the terms. Moreover, using also $c \geq c_0$ we obtain

$$
\begin{aligned}
&- \left(\psi^L_\gamma\right)''(y) + c\left(\psi^L_\gamma\right)'(y) + \left(\psi^L_\gamma(y)\right)^4 - \int_{\mathbb{R}} E(\eta - y)\left(\psi^L_\gamma(\eta)\right)^4 d\eta && \text{(E.101)}\\
\geq& \frac{c_0}{2}\zeta\gamma e^{\zeta(y-L)} + \gamma e^{\zeta(y-L)}\left(\frac{c_0}{2}\zeta - \zeta^2 + 4(M_L+\varepsilon)^3 - 4(M_L+\varepsilon)^3\int_{-L}^{\infty} E(\eta-y)e^{\zeta(\eta-y)}d\eta\right) && (I_1^4)\\
&+ 4(M_L+\varepsilon)\gamma^3 e^{3\zeta(y-L)}\left(1 - \int_{-L}^{\infty} E(\eta-y)e^{3\zeta(\eta-y)}d\eta\right) && (I_2^4)\\
&+ 6(M_L+\varepsilon)^2\gamma^2 e^{2\zeta(y-L)}\left(1 - \int_{-L}^{\infty} E(\eta-y)e^{2\zeta(\eta-y)}d\eta\right) && (I_3^4)\\
&+ \gamma^4 e^{4\zeta(y-L)}\left(1 - \int_{-L}^{\infty} E(\eta-y)e^{4\zeta(\eta-y)}d\eta\right) && (I_4^4)\\
&- 4(M_L+\varepsilon)^3\left[\varepsilon\theta + \int_{-\infty}^{-L} E(\eta-y)\left(2T_M - (M_L+\varepsilon)\right)d\eta - \varepsilon\int_{-L}^{L} E(\eta-y)d\eta\right] && (I_5^4)\\
&+ 4(M_L+\varepsilon)^3\varepsilon\theta\int_{L}^{\infty} E(\eta-y)d\eta\\
&- 4(M_L+\varepsilon)\left[(\varepsilon\theta)^3 + \int_{-\infty}^{-L} E(\eta-y)\left(2T_M - (M_L+\varepsilon)\right)^3 d\eta - \varepsilon^3\int_{-L}^{L} E(\eta-y)d\eta\right] && (I_6^4)\\
&+ 4(M_L+\varepsilon)(\varepsilon\theta)^3\int_{L}^{\infty} E(\eta-y)d\eta\\
&+ 6(M_L+\varepsilon)^2\left[(\varepsilon\theta)^2 - \int_{-\infty}^{-L} E(\eta-y)\left(2T_M - (M_L+\varepsilon)\right)^2 d\eta - \varepsilon^2\int_{-L}^{L} E(\eta-y)d\eta\right] && (I_7^4)\\
&- 6(M_L+\varepsilon)^2(\varepsilon\theta)^2\int_{L}^{\infty} E(\eta-y)d\eta\\
&+ (\varepsilon\theta)^4 - \int_{-\infty}^{-L} E(\eta-y)\left(2T_M - (M_L+\varepsilon)\right)^4 d\eta - \varepsilon^4\int_{-L}^{L} E(\eta-y)d\eta && (I_8^4)\\
&- (\varepsilon\theta)^4\int_{L}^{\infty} E(\eta-y)d\eta
\end{aligned}
$$

$$- 4\varepsilon^3\gamma e^{\zeta(y-L)}\left(\theta^3 - \int_{-L}^{L} E(\eta-y)e^{\zeta(\eta-y)}d\eta - \theta^3\int_{L}^{\infty} E(\eta-y)e^{\zeta(\eta-y)}d\eta\right) \qquad (I_9^4)$$
$$+ 4\varepsilon\theta\gamma^3 e^{3\zeta(R_\gamma-L)}\int_{R_\gamma}^{\infty} E(\eta-y)d\eta$$
$$- 4\varepsilon\gamma^3 e^{3\zeta(y-L)}\left(\theta - \int_{-L}^{L} E(\eta-y)e^{3\zeta(\eta-y)}d\eta - \theta\int_{L}^{\infty} E(\eta-y)e^{3\zeta(\eta-y)}d\eta\right) \qquad (I_{10}^4)$$
$$+ 4^3\gamma^3 e^{\zeta(R_\gamma-L)}\int_{R_\gamma}^{\infty} E(\eta-y)d\eta$$
$$+ 6\varepsilon^2\gamma^2 e^{2\zeta(y-L)}\left(\theta^2 - \int_{-L}^{L} E(\eta-y)e^{2\zeta(\eta-y)}d\eta - \theta^2\int_{L}^{R_\gamma} E(\eta-y)e^{2\zeta(\eta-y)}d\eta\right) \qquad (I_{11}^4)$$
$$- 6(\varepsilon\theta\gamma)^2 e^{2\zeta(R_\gamma-L)}\int_{R_\gamma}^{\infty} E(\eta-y)d\eta$$
$$- 12(M_L+\varepsilon)^2\varepsilon\gamma e^{\zeta(y-L)}\left(\theta - \int_{-L}^{L} E(\eta-y)e^{\zeta(\eta-y)}d\eta - \theta\int_{L}^{R_\gamma} E(\eta-y)e^{2\zeta(\eta-y)}d\eta\right) \qquad (I_{12}^4)$$
$$+ 12(M_L+\varepsilon)^2\varepsilon\theta\gamma e^{\zeta(R_\gamma-L)}\int_{R_\gamma}^{\infty} E(\eta-y)d\eta$$
$$- 12(M_L+\varepsilon)\varepsilon\gamma^2 e^{2\zeta(y-L)}\left(\theta - \int_{-L}^{L} E(\eta-y)e^{2\zeta(\eta-y)}d\eta - \theta\int_{L}^{R_\gamma} E(\eta-y)e^{2\zeta(\eta-y)}d\eta\right) \qquad (I_{13}^4)$$
$$+ 12(M_L+\varepsilon)\varepsilon\theta\gamma^2 e^{2\zeta(R_\gamma-L)}\int_{R_\gamma}^{\infty} E(\eta-y)d\eta$$
$$+ 12(M_L+\varepsilon)\varepsilon^2\gamma e^{\zeta(y-L)}\left(\theta^2 - \int_{-L}^{L} E(\eta-y)e^{\zeta(\eta-y)}d\eta - \theta^2\int_{L}^{R_\gamma} E(\eta-y)e^{\zeta(\eta-y)}d\eta\right) \qquad (I_{14}^4)$$
$$- 12(M_L+\varepsilon)\varepsilon^2\theta^2\gamma 2e^{\zeta(R_\gamma-L)}\int_{R_\gamma}^{\infty} E(\eta-y)d\eta.$$

We now proceed to estimate all the terms. First of all, using the identity (E.53), the estimate $(M_L+\varepsilon) \leq 2T_M$ as well as the definition of ζ in (E.98) we compute

$$\begin{aligned}
(I_1^4) + (I_2^4) + (I_3^4) + (I_4^4) &\geq \frac{c_0}{2}\zeta\gamma e^{\zeta(y-L)} + \gamma e^{\zeta(y-L)}\left(\frac{c_0}{2}\zeta - \zeta^2 - 4(2T_M)^3\left(\frac{\operatorname{artanh}(\zeta)}{\zeta} - 1\right)\right) \\
&\quad - 4(2T_M)\gamma^3 e^{3\zeta(y-L)}\left(\frac{\operatorname{artanh}(3\zeta)}{3\zeta} - 1\right) \\
&\quad - 6(2T_M)^2\gamma^2 e^{2\zeta(y-L)}\left(\frac{\operatorname{artanh}(2\zeta)}{2\zeta} - 1\right) + \gamma^4 e^{4\zeta(y-L)}\left(\frac{\operatorname{artanh}(4\zeta)}{4\zeta} - 1\right) \\
&\geq \frac{c_0}{2}\zeta\gamma e^{\zeta(y-L)} + \gamma e^{\zeta(y-L)}\left(\frac{c_0}{2}\zeta - \zeta^2 - 4(2T_M)^3\left(\frac{\operatorname{artanh}(4\zeta)}{4\zeta} - 1\right)\right) \\
&\quad -4(2T_M)^3\gamma e^{\zeta(y-L)}\left(\frac{\operatorname{artanh}(4\zeta)}{4\zeta} - 1\right) \\
&\quad -6(2T_M)^3\gamma e^{\zeta(y-L)}\left(\frac{\operatorname{artanh}(4\zeta)}{4\zeta} - 1\right) + (2T_M)^3\gamma e^{\zeta(y-L)}\left(\frac{\operatorname{artanh}(4\zeta)}{4\zeta} - 1\right) \\
&= \frac{c_0}{2}\zeta\gamma e^{\zeta(y-L)} + \gamma e^{\zeta(y-L)}\left(\frac{c_0}{2}\zeta - \zeta^2 - 15(2T_M)^3\left(\frac{\operatorname{artanh}(4\zeta)}{4\zeta} - 1\right)\right) \geq \frac{c_0}{2}\zeta\gamma e^{\zeta(y-L)},
\end{aligned} \tag{E.102}$$

where we used also that $\zeta \mapsto \frac{\operatorname{artanh}(\zeta)}{\zeta} - 1$ is a monotonically increasing non-negative function and that $\gamma e^{\zeta(y-L)} \leq 2T_M$ for $y \leq R_\gamma$.

We estimate the terms (I_5^4), (I_6^4), (I_7^4) and (I_8^4). We compute using $\theta = \frac{1}{5}$ and the choice of $L > L_2$ as in (E.100)

$$\begin{aligned}
(I_5^4) + (I_7^4) \geq & -4(M_L+\varepsilon)^3 \left[\varepsilon\theta \int_{-\infty}^{-L} E(\eta-y)d\eta + \int_{-\infty}^{-L} E(\eta-y)\left(2T_M - (M_L+\varepsilon)\right)d\eta\right] \\
& + 16(M_L+\varepsilon)^3\theta\varepsilon \int_{-L}^{L} E(\eta-y)d\eta + 6(M_L+\varepsilon)^2(\varepsilon\theta)^2 \int_{-\infty}^{L} E(\eta-y)d\eta \\
& - 6(M_L+\varepsilon)^2 \left[\int_{-\infty}^{-L} E(\eta-y)\left(2T_M - (M_L+\varepsilon)\right)^2 d\eta + \varepsilon^2 \int_{-L}^{L} E(\eta-y)d\eta\right] \\
\geq & - 4(M_L+\varepsilon)^3\varepsilon\theta \left[\left(1 + \frac{2T_M}{\varepsilon\theta}\right)\int_{-\infty}^{-L} E(\eta-y)d\eta - 4\int_{-L}^{L} E(\eta-y)d\eta\right] \\
& + 6(M_L+\varepsilon)^2 \left[(\varepsilon\theta)^2 \int_{-\infty}^{L} E(\eta-y)d\eta - (2T_M)^2 \int_{-\infty}^{-L} E(\eta-y)d\eta - \varepsilon^2 \int_{-L}^{L} E(\eta-y)d\eta\right] \\
\geq & 4(M_L+\varepsilon)^3 \left[3\theta\varepsilon \int_{-L}^{L} E(\eta-y)d\eta\right] - 6(M_L+\varepsilon)^2 \left[\varepsilon^2 \int_{-L}^{L} E(\eta-y)d\eta\right] \\
= & 6(M_L+\varepsilon)^2\varepsilon \int_{-L}^{L} E(\eta-y)d\eta \left(\frac{2}{5}(M_L+\varepsilon) - \varepsilon\right) > 0
\end{aligned} \tag{E.103}$$

for $\varepsilon < \varepsilon_2$ as in (E.99) since $M_L + \varepsilon > \lambda$. Since $\frac{492}{125} > \frac{2}{5}$, in a similar way we can estimate

$$\begin{aligned}
(I_6^4) + (I_8^4) \geq & - 4(M_L+\varepsilon) \left[(\varepsilon\theta)^3 \int_{-\infty}^{-L} E(\eta-y)d\eta + \int_{-\infty}^{-L} E(\eta-y)\left(2T_M\right)^3 d\eta\right] \\
& - 124(4(M_L+\varepsilon))(\theta\varepsilon)^3 \int_{-L}^{L} E(\eta-y)d\eta + (\varepsilon\theta)^4 \int_{-\infty}^{L} E(\eta-y)d\eta \\
& - \int_{-\infty}^{-L} E(\eta-y)\left(2T_M\right)^4 d\eta - \varepsilon^4 \int_{-L}^{L} E(\eta-y)d\eta \\
\geq & 4(M_L+\varepsilon)(\varepsilon\theta)^3 \left[123(\theta\varepsilon)^3 \int_{-L}^{L} E(\eta-y)d\eta\right] - \varepsilon^4 \int_{-L}^{L} E(\eta-y)d\eta,
\end{aligned}$$

so that

$$(I_6^4) + (I_8^4) \geq \varepsilon^3 \int_{-L}^{L} E(\eta-y)d\eta \left(\frac{492}{125}(M_L+\varepsilon) - \varepsilon\right) > 0. \tag{E.104}$$

We estimate the last terms using $\frac{\operatorname{artanh}(4\zeta)}{4\zeta} < \frac{3}{2}$ as well as $\gamma e^{\zeta(y-L)} \leq 2T_M$.

$$(I_9^4) \geq -4\varepsilon^3\theta^3\gamma e^{\zeta(y-L)} \geq -4\varepsilon\gamma e^{\zeta(y-L)}, \qquad (I_{10}^4) \geq -4\varepsilon\theta(2T_M)^2\gamma e^{\zeta(y-L)}, \tag{E.105}$$

$$(I_{11}^4) \geq -6\varepsilon^2\gamma^2 e^{2\zeta(y-L)}\frac{\operatorname{artanh}(2\zeta)}{2\zeta} \geq -9\varepsilon(2T_M)\gamma e^{\zeta(y-L)} \tag{E.106}$$

and similarly

$$(I_{12}^4) + (I_{13}^4) + (I_{14}^4) \geq -12(2T_M)\varepsilon\gamma e^{\zeta(y-L)}\left(2(2T_M) + 18\right) \tag{E.107}$$

Finally, using $\varepsilon < \varepsilon_2$ as given in (E.99) and combining the equations (E.101)-(E.107) we conclude that ψ_γ^L are supersolutions in (L, R_γ), i.e.

$$-\left(\psi_\gamma^L\right)''(y) + c\left(\psi_\gamma^L\right)'(y) + \left(\psi_\gamma^L(y)\right)^4 - \int_{\mathbb{R}} E(\eta-y)\left(\psi_\gamma^L(\eta)\right)^4 d\eta \geq \frac{c_0}{4}\zeta\gamma e^{\zeta(y-L)} > 0$$

for all $y \in (L, R_\gamma)$.

We recall that by construction $\psi^L_{\gamma_0} > f$ in $\mathbb{R}$ with $\psi^L_{\gamma_0} - f \geq \frac{1-\theta}{2}\varepsilon > 0$ for all $y \geq L$ and $\psi^L_\gamma \geq f$ for $y \in \mathbb{R} \setminus (L, R_\gamma)$ with $\psi^L_\gamma \mid_{\{L,R_\gamma\}} > f \mid_{\{L,R_\gamma\}}$. Hence, once again arguing with the maximum principle as we did in the proof of Lemma E.6 and of Lemma E.7 we conclude, by the uniform continuity of $\gamma \mapsto \psi^L_\gamma$ on compact sets and their decreasing monotonicity, that

$$\psi^L_\gamma(y) \geq f(y) \quad \text{for all } y \in \mathbb{R} \text{ and } \gamma < \gamma_0,$$

since ψ^L_γ are supersolutions on (L, R_γ). Hence, for any $y > L$ we have for $\gamma < \gamma_0$ small enough

$$f(y) \leq M_L + (1-\theta)\varepsilon + \gamma e^{\zeta(y-L)}.$$

Finally, taking $\gamma \to 0$ and thus $R_\gamma \to \infty$ we conclude

$$f(y) \leq M_L + (1-\theta)\varepsilon \quad \text{for all } y > L. \tag{E.108}$$

Let us now define $\varepsilon_0(T_M, \lambda, c) = \min\{\varepsilon_1, \varepsilon_2\}$ for ε_1 and ε_2 defined in (E.78) and (E.99), respectively. For any given $\varepsilon < \varepsilon_0$ we define also $L_0(\varepsilon, T_M, \lambda, c) = \max\{L_1, L_2\}$, where $L_1,\ L_2$ are given in (E.79) and (E.100), respectively, and $\theta = \frac{1}{5}$.

The estimates (E.96) and (E.108) yield the proof of Theorem E.5. Indeed, we have just proved that, if f solves (E.46), there exists some $\varepsilon_0(T_M, \lambda, c) > 0$ such that, if

$$\operatorname*{osc}_{[-L,L]} f < \varepsilon$$

for $\varepsilon < \varepsilon_0$ and for $L > L_0(\varepsilon, T_M, \lambda, c)$, then

$$\operatorname*{osc}_{[L,\infty)} f \leq M_L + (1-\theta)\varepsilon - m_L + (1-\theta)\varepsilon = (3-2\theta)\varepsilon < 3\varepsilon,$$

where we also use that $m_L \leq f(L) \leq M_L$. □

In order to use Theorem E.5 we need to have functions satisfying the oscillation assumption. The next lemma shows that there exist sequences of functions satisfying both (E.46) and the oscillation condition.

Lemma E.8. *Let f solve* (E.46) *for $0 < \lambda < T_M$. Let us assume that f is not constant. Then there exist $\{x_n\}_{n\in\mathbb{N}}$ and $\{\xi_k\}_{k\in\mathbb{N}}$ monotonically decreasing sequences with $\lim_{n\to\infty} x_n = -\infty$ as well as $\lim_{k\to\infty} \xi_k = -\infty$ satisfying*

$$\lim_{n\to\infty} f(x_n) = \sup_{\mathbb{R}} f \quad \text{and} \quad \lim_{k\to\infty} f(\xi_k) = \inf_{\mathbb{R}} f.$$

Moreover, they satisfy

$$\lim_{n\to\infty} \operatorname*{osc}_{[-L,L]} f(x_n + \cdot) = 0 \quad \text{and} \quad \lim_{k\to\infty} \operatorname*{osc}_{[-L,L]} f(\xi_k + \cdot) = 0$$

for all $L > 0$.

Proof. Since f is not constant, according to Lemma E.5, Lemma E.6 and Lemma E.7 it has to attain its supremum and infimum at $-\infty$. Hence, there exist monotonically decreasing sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{\xi_k\}_{k\in\mathbb{N}}$ satisfying

$$\lim_{n\to\infty} x_n = -\infty, \quad \lim_{k\to\infty} \xi_k = -\infty, \quad \lim_{n\to\infty} f(x_n) = \sup_{\mathbb{R}} f \quad \text{and} \quad \lim_{k\to\infty} f(\xi_k) = \inf_{\mathbb{R}} f.$$

We now prove the statement for the supremum. We define $f_n = f(x_n + \cdot)$. Then f_n solves the same equation (E.46) by the translation invariance of this equation. Moreover, $f_n \in C^{2,1/2}(\mathbb{R})$. Thus, by compactness for $\alpha \in \left(0, \frac{1}{2}\right)$ there exists a subsequence $f_{n_j} = f\left(x_{n_j} + \cdot\right) \to g$ in $C^{2,\alpha}(\mathbb{R})$ in every compact set and hence uniformly everywhere. Moreover, g solves also (E.46). By regularity theory we see that $g \in C^{2,1/2}(\mathbb{R})$.

It is important to notice also that

$$g(0) = \lim_{j\to\infty} f\left(x_{n_j}\right) = \sup_{\mathbb{R}} f \geq \sup_{\mathbb{R}} g \geq g(0).$$

Since g attains its supremum at the interior, it is constant according to Lemma E.5. Thus, $f_{n_j} \to g = \sup_{\mathbb{R}} f$ uniformly in every compact set.

Let $\varepsilon > 0$ and $L > 0$. By the uniform convergence in $[-L, L]$ there exists $N_0(\varepsilon, L) > 0$ such that for all $j \geq N_0$ we have

$$\|(f_{n_j} - g)\mid_{[-L,L]}\|_\infty < \frac{\varepsilon}{2}.$$

We thus conclude that

$$\operatorname*{osc}_{[-L,L]} f_{n_j} = \max_{[-L,L]} f_{n_j} - \min_{[-L,L]} f_{n_j} < \frac{\varepsilon}{2} + g - g + \frac{\varepsilon}{2} = \varepsilon.$$

This proves $\lim_{j\to\infty} \operatorname*{osc}_{[-L,L]} f(x_{n_j} + \cdot) = 0$ for all $L > 0$. Thus, the sequence $\{\tilde{x}_j\}_{j\in\mathbb{N}} = \{x_{n_j}\}_{j\in\mathbb{N}}$ satisfies the statement of Lemma E.8 concerning the supremum.

Using that any solution to (E.46) which attains its infimum at the interior is constant according to Lemma E.5, we conclude the proof of this lemma repeating the same arguments for the sequence $f_k = f(\xi_k + \cdot)$, for which a subsequence converges uniformly in every compact set to $g = \inf_{\mathbb{R}} f$. □

Finally, Lemma E.8 and Theorem E.5 together with the previous results in Lemma E.5, Lemma E.6 and Lemma E.7 imply that the solution f to (E.46) is constant.

Theorem E.6. *Let f solve* (E.46) *for $0 < \lambda < T_M$. Then f is constant.*

Proof. Let f solve (E.46). Let us assume that f is not constant. By Lemma E.5, Lemma E.6 and Lemma E.7 there exist $\{x_n\}_{n\in\mathbb{N}}$ and $\{\xi_k\}_{k\in\mathbb{N}}$ monotonically decreasing sequences with $\lim_{n\to\infty} x_n = -\infty$ as well as $\lim_{k\to\infty} \xi_k = -\infty$ satisfying

$$\lim_{n\to\infty} f(x_n) = \sup_{\mathbb{R}} f \quad \text{ and } \quad \lim_{k\to\infty} f(\xi_k) = \inf_{\mathbb{R}} f.$$

Let also $\varepsilon < \varepsilon_0$ be arbitrary and $L > L_0(\varepsilon)$ for ε_0 and $L_0(\varepsilon)$ as in Theorem E.5. According to Lemma E.8 there exists $N_0(\varepsilon, L)$ such that

$$\operatorname*{osc}_{[-L,L]} f(x_n + \cdot) < \varepsilon \quad \text{ for all } n \geq N_0.$$

Since by the translation invariance $f(x_n + \cdot)$ solve (E.46) with $\lambda \leq f(x_n + \cdot) \leq T_M$, Theorem E.5 implies that

$$\operatorname*{osc}_{[L,\infty)} f(x_n + \cdot) < 3\varepsilon \quad \text{ for all } n \geq N_0.$$

Thus,

$$\operatorname*{osc}_{[-L,\infty)} f(x_n + \cdot) < 4\varepsilon \quad \text{ for all } n \geq N_0.$$

Similarly, there exists $K_0(\varepsilon, L) > 0$ such that

$$\operatorname*{osc}_{[-L,\infty)} f(\xi_k + \cdot) < 4\varepsilon \quad \text{for all } k \geq K_0.$$

Hence, for any $n \geq N_0$ and $k \geq K_0$ it is either $x_n - \xi_k > 0$ or $\xi_k - x_n > 0$. In the first case we estimate $|f(\xi_k) - f(x_n)| \leq \operatorname*{osc}_{[-L,\infty)} f(\xi_k + \cdot) < 4\varepsilon$, while in the latter situation $|f(x_n) - f(\xi_k)| \leq \operatorname*{osc}_{[-L,\infty)} f(x_n + \cdot) < 4\varepsilon$. Therefore,

$$|f(x_j) - f(\xi_j)| < 4\varepsilon \quad \text{for all } j \geq \max\{N_0, K_0\}.$$

Taking now the limit as $j \to \infty$ we conclude

$$\sup_{\mathbb{R}} f - \inf_{\mathbb{R}} f \leq 4\varepsilon.$$

Since $\varepsilon < \varepsilon_0$ was arbitrary, this implies that $\sup_{\mathbb{R}} f = \inf_{\mathbb{R}} f$ and hence f is constant. $\square$

E.3.3 Existence of a positive limit of the traveling waves as $y \to \infty$

We now finish this section proving that any traveling wave solving (E.10) for $y > 0$ has a limit as $y \to \infty$. We first of all need to show a corollary to the stability result in Theorem E.5.

Corollary E.3. *Let f solve* (E.10) *according to Theorem E.3 for $0 < \lambda \leq f \leq T_M$ and $c > 0$. Let $\varepsilon < \varepsilon_0(c, \lambda, T_M)$ and $L_0(\varepsilon, \lambda, T_M, c)$ be as in Theorem E.5. Let also $a > L_0(\varepsilon)$ and $\tilde{f}(y) := f(a + y)$. Then $\tilde{f} : [-a, \infty) \to \mathbb{R}_+$ solves*

$$-\tilde{f}''(y) + c\tilde{f}'(y) + \tilde{f}^4(y) - \int_{-a}^{\infty} E(\eta - y)\tilde{f}^4(\eta)d\eta = 0 \tag{E.109}$$

with $\tilde{f}(-a) = T_M$ and $0 < \lambda \leq \tilde{f} \leq T_M$. Moreover, if

$$\operatorname*{osc}_{[-L,L]} \tilde{f} < \varepsilon \quad \textit{for } L_0(\varepsilon) < L < a$$

then

$$\operatorname*{osc}_{[L,\infty)} \tilde{f} < 3\varepsilon.$$

Proof. It is easy to see that $\tilde{f}$ solves (E.109). In order to simplify the reading we use the same notation as in Theorem E.5. Let hence m_L and M_L being the minimum and the maximum of $\tilde{f}$ on $[-L, L]$, respectively. Moreover, β, $\zeta > 0$, $\delta < \delta_0$ and $\gamma < \gamma_0$ are defined for $\tilde{f}$ as in Theorem E.5. Let finally ψ_δ^L as in (E.75) and ψ_γ^L as in (E.97). We argue that $\psi_\delta^L \mathbb{1}_{[-a,\infty)}$ and $\psi_\gamma^L \mathbb{1}_{[-a,\infty)}$ are subsolutions and supersolutions for the equation (E.109) on (L, R_δ) and (L, R_γ), respectively.

Indeed, by definition $\psi_\delta^L \mathbb{1}_{[-a,\infty)} = \psi_\delta^L$ since $L_0(\varepsilon) < L < a$ and $\psi_\delta^L = 0$ for $y < -L$. This implies that

$$\int_{-a}^{\infty} E(\eta - y) \left(\psi_\delta^L(\eta)\right)^4 d\eta = \int_{-L}^{\infty} E(\eta - y) \left(\psi_\delta^L(\eta)\right)^4 d\eta = \int_{-\infty}^{\infty} E(\eta - y) \left(\psi_\delta^L(\eta)\right)^4 d\eta.$$

Hence, for $y \in (L, R_\delta)$ the functions $\psi_\delta^L \mathbb{1}_{[-a,\infty)}$ are subsolutions for the equation (E.109) with $\psi_\delta^L \mathbb{1}_{[-a,\infty)} \leq \tilde{f}$ on $[-a, \infty) \setminus (L, R_\delta)$, $\psi_\delta^L\big|_{\{L,R_\delta\}} < \tilde{f}\big|_{\{L,R_\delta\}}$, as well as $\psi_{\delta_0}^L \mathbb{1}_{[-a,\infty)} < \tilde{f}$ on

$[-a, \infty)$.

Similarly, $\psi_\gamma^L \mathbb{1}_{[-a,\infty)} \leq \psi_\gamma^L$ since $L_0(\varepsilon) < L < a$ and $\psi_\gamma^L = 2T_M$ for $y < -L$. This implies that

$$\int_{-a}^{\infty} E(\eta - y) \left(\psi_\gamma^L(\eta)\right)^4 d\eta \leq \int_{-\infty}^{\infty} E(\eta - y) \left(\psi_\gamma^L(\eta)\right)^4 d\eta.$$

Thus, the functions $\psi_\gamma^L \mathbb{1}_{[-a,\infty)}$ are supersolutions for the equation (E.109) for $y \in (L, R_\gamma)$. Moreover, they satisfy $\psi_\gamma^L \mathbb{1}_{[-a,\infty)} \geq \tilde{f}$ on $[-a\infty) \setminus (L, R_\gamma)$, $\psi_\gamma^L \big|_{\{L,R_\gamma\}} > \tilde{f} \big|_{\{L,R_\gamma\}}$, as well as $\psi_\gamma^L \mathbb{1}_{[-a,\infty)} > \tilde{f}$ on $[-a, \infty)$.

Hence, as we saw in Theorem E.5, an application of the maximum principle and of the uniform continuity on compact sets of the families of sub- and supersolutions with respect of δ and γ, respectively, implies

$$\operatorname*{osc}_{[L,\infty)} \tilde{f} < 3\varepsilon.$$

□

Finally, we can prove the convergence of the traveling wave to a positive constant as $y \to \infty$.

Theorem E.7. *Let f solve* (E.10) *according to Theorem E.3 for $T_M > 0$ and $c > 0$. Then there exists a limit*

$$\lim_{y \to \infty} f(y) =: f_\infty > 0.$$

Proof. By Theorem E.3, Lemma E.2 and Theorem E.4 we know that $f \geq \lambda > 0$ for some $\lambda > 0$. Let us take $\{x_n\}_{n \in \mathbb{N}}$ and $\{\xi_n\}_{n \in \mathbb{N}}$ two diverging monotone increasing sequences such that

$$\lim_{n \to \infty} f(x_n) = \limsup_{y \to \infty} f(y) =: \overline{f_\infty} \quad \text{and} \quad \lim_{n \to \infty} f(\xi_n) = \liminf_{y \to \infty} f(y) =: \underline{f_\infty}.$$

We notice that $\overline{f_\infty}, \underline{f_\infty} \in [\lambda, T_M]$. Up to subsequences we know that $f(x_n + \cdot)$ and $f(\xi_n + \cdot)$ converge to constant functions, as we have proved in Theorem E.6. We denote these subsequences x_n and ξ_n. Hence, we have

$$\lim_{n \to \infty} f(x_n + \cdot) = \overline{f_\infty} \text{ and } \lim_{n \to \infty} f(\xi_n + \cdot) = \underline{f_\infty}$$

uniformly on compact sets. Therefore, for all $L > 0$ there exists $N_0(L)$ such that $x_n, \xi_n > L$ for all $n \geq N_0(L)$ and such that

$$\operatorname*{osc}_{[-L,L]} f(x_n + \cdot) \to 0 \quad \text{and} \quad \operatorname*{osc}_{[-L,L]} f(\xi_n + \cdot) \to 0 \quad \text{as } n \to \infty \text{ and } n \geq N_0(L).$$

Let now $\varepsilon < \varepsilon_0(c, \lambda, T_M)$ and $L_0(\varepsilon, c, \lambda, T_M)$ as defined in Theorem E.5 and in Corollary E.3. Then there exists $N_1(\varepsilon, L_0(\varepsilon)) > 0$ such that $x_n, \xi_n > L_0(\varepsilon)$ for all $n \geq N_1$. Let also $L \in (L_0(\varepsilon), \min\{x_{N_1}, \xi_{N_1}\})$. Then there exists $N_2(\varepsilon, L) \geq N_1(\varepsilon)$ such that

$$\operatorname*{osc}_{[-L,L]} f(x_n + \cdot) < \varepsilon \quad \text{and} \quad \operatorname*{osc}_{[-L,L]} f(\xi_n + \cdot) < \varepsilon \quad \text{for all } n \geq N_2(\varepsilon, L).$$

We remark that $L_0(\varepsilon) < L < \min\{x_n, \xi_n\}$ for all $n \geq N_2(\varepsilon, L)$. Then by the Corollary E.3 we can conclude that

$$\operatorname*{osc}_{[L,\infty)} f(x_n + \cdot) < 3\varepsilon \quad \text{and} \quad \operatorname*{osc}_{[L,\infty)} f(\xi_n + \cdot) < 3\varepsilon \quad \text{for all } n \geq N_2(\varepsilon, L).$$

This implies that

$$|f(x_n) - f(\xi_n)| < 4\varepsilon \text{ for all } n \geq N_2(\varepsilon, L). \tag{E.110}$$

Indeed, let $n \geq N_2(\varepsilon, L)$. If $x_n - \xi_n > 0$ we compute

$$|f(\xi_n) - f(x_n)| \leq \underset{[-L,\infty)}{\operatorname{osc}} f(\xi_n + \cdot) < 4\varepsilon,$$

while if $\xi_n - x_n > 0$

$$|f(x_n) - f(\xi_n)| \leq \underset{[-L,\infty)}{\operatorname{osc}} f(x_n + \cdot) < 4\varepsilon.$$

Taking the limit $n \to \infty$ in (E.110) we obtain

$$0 \leq \overline{f_\infty} - \underline{f_\infty} \leq 4\varepsilon,$$

which implies that f has a limit, since $\varepsilon < \varepsilon_0$ is arbitrarily small, i.e.

$$\overline{f_\infty} = \underline{f_\infty} = f_\infty.$$

□

E.4 Formal description of the long time asymptotic for arbitrary values of $T(\pm\infty)$

In this last section we conclude giving the expected behavior of the solution to the Stefan problem (E.4) as $t \to \infty$. We remark that what we present here is formal.

Theorem E.1 shows the existence of $c_{\max} > 0$ such that for any $c \in (0, c_{\max})$ there exists traveling waves $T_1(x + ct) =: T_1^c(y)$ and $T_2(x + ct) =: T_2^c(y)$ solving the Stefan problem for $s(t) = -ct$. The first problem we should solve concerns the uniqueness of the traveling waves.

Problem E.4.1. Prove or disprove that for any $c \in (0, c_{\max})$ and $T > 0$ the traveling waves T_1^c, T_2^c solving (E.5) are unique.

Notice that it is enough to have the uniqueness of the traveling wave in the solid solving (E.8).

Recall that $\lim_{y\to\infty} T_2^c > 0$ and $\lim_{y\to-\infty} T_1^c = T_M - \frac{cL+\partial_y T_2^c(0+)}{LK}$. Moreover, we notice that also for $c = c_{\max}$ there exist traveling waves. Indeed, $T_2^{c_{\max}}$ exists by Theorem E.3. By definition $\partial_y T_2^{c_{\max}}(0^+) = -Lc_{\max}$. Thus, since $\partial_y T_1^{c_{\max}}(0^-) = 0$, in this case the traveling wave is constant in the liquid part, i.e. $T_1^{c_{\max}} = T_M$.

Also the existence of a traveling wave T_2^0 solving (E.8) for $c = 0$ is an important problem that should be considered.

Problem E.4.2. Prove or disprove that there exists a unique traveling wave T_2^0 solving (E.8) for $c = 0$. Moreover, T_2^0 converges to a positive constant as $y \to \infty$.

Remark. The existence of T_2^0 can be proved as follows using an iterative argument. First of all the function

$$f_1(y) = \frac{A}{(B+y)^{\frac{2}{3}}}, \text{ where } A = \frac{2}{\sqrt[3]{9}} \text{ and } B = \frac{1}{3}\left(\frac{2}{T_M}\right)^{\frac{2}{3}},$$

is a solution to $f_1'' - f_1^4 = 0$ on $\mathbb{R}_+$ with $f_1(0) = T_M$. Moreover, f_1 is monotonically decreasing with $\lim_{y\to\infty} f_1(y) = 0$. It is also possible to show the existence of a monotone sequence $0 \leq$

$f_1 \leq f_2 \leq ... \leq f_n \leq f_{n-1} \leq ... \leq T_M$ solving for $n \geq 2$ equation (E.10) for $c = 0$. In this case though, the variational principle method we used in Proposition E.1 does not work. Nevertheless, knowing for $n \geq 2$ the existence of $f_{n-1} \in C^{0,1/2}(R_+)$ with $f_1 \leq f_{n-1} \leq T_M$, the method of sub- and supersolutions (c.f [49]) can be implemented in order to find for any $R > 0$ a solution $f_n^R \in C^{2,1/2}([0,R])$ of the boundary value problem

$$\begin{cases} -\left(f_n^R\right)''(y) + \left(f_n^R(y)\right)^4 = \int_0^\infty E(y-\eta) f_{n-1}^4(\eta) d\eta & y \in (0,R) \\ f_n^R(0) = T_M \\ f_n^R(R) = f_1(R). \end{cases}$$

Indeed, f_1 and T_M are sub- and supersolutions of the operator $L(u) = -\partial_y^2 u + 4T_M^3 u$ and the function $\lambda \mapsto -\lambda^4 + 4T_M^3\lambda$ is increasing for $\lambda \in [0, T_M]$. Moreover, since $\|f_n^R\|_\infty \leq T_M$ as well as $\|\partial_y^2 f_n^R\|_\infty \leq T_M^4$ we conclude that $f_n^R \in C^{2,1/2}([0,R])$ with uniformly bounded norm with respect to R. Hence, taking the limit we prove the existence of a function $f_n \in C^{2,1/2}(\mathbb{R}_+)$ solution to (E.10) for $c = 0$. Since the monotonicity argument in Theorem E.3 applies also in this case, such a monotone sequence exists. This implies the existence of a traveling wave solving (E.8) for $c = 0$ and $y > 0$. However, the uniqueness and the existence of a positive limit are more involved problems.

This remark shows that T_2^0 exists, moreover, $\partial_y T_2^0(0^+) < 0$ by the Hopf-principle. Hence, in the liquid the traveling wave T_1^0 solves $\partial_y^2 T_1^0 = 0$ with $T_1^0(0) = T_M$ and $\partial_y T_1^0(0^-) = \frac{\partial_y T_2^0(0^+)}{K}$. Thus, we obtain

$$\lim_{c\to 0} T_1^c(y) = T_M - \frac{\partial_y T_2^0(0^+)}{K} y \text{ with } \lim_{y\to-\infty} T_1^0(y) = \infty.$$

These observations lead to the following open problem.

Problem E.4.3. Prove or disprove that for any $T_{-\infty} \in [T_M, \infty]$ there exists a unique $c \in [0, c_{\max}]$ such that in the liquid the traveling wave T_1^c of Theorem E.2 satisfies

$$\lim_{y\to-\infty} T_1^c(y) = T_{-\infty}.$$

On the contrary, in the solid we already know that there exists $\theta > 0$ such that for any $c \geq 0$ the traveling waves satisfy $\lim_{y\to\infty} T_2^c = T_{\text{int}}^c \geq \theta$. Therefore, we cannot expect that T_{int}^c can attain all the values in $[0, T_M]$ for $c \in [0, c_{\max}]$. Nevertheless, we can reach any value in $[0, T_M]$ if we include an additional layer in which the radiative transfer equation is approximated using the diffusion approximation. More precisely, we expect to approximate the evolution equation of the temperature by an equation of the form

$$T_t = T_{xx} + \left(T^4\right)_{xx}$$

in a domain $x > s(t)$ where T changes in a length scale much larger than 1.

We conclude the final picture of the asymptotic of the solution (T_1, T_2, s) to the Stefan problem (E.4) as $t \to \infty$ with the following claim.

Given $T_{-\infty} \in [T_M, \infty]$ and $T_\infty \in [0, T_M]$ there exist $c \in [0, c_{\max}]$ and functions T_1^c, T_2 with the following properties:

(i) $s(t) = -ct$;

(ii) T_1^c is the traveling wave of Theorem (E.3) for $y < 0$ with $\lim_{y\to-\infty} T_1^c(y) = T_{-\infty}$;

(iii) T_2 is given by the traveling wave T_2^c of Theorem (E.3) for $y > 0$ and by a self-similar profile F connecting T_{int}^c to T_∞, which solves

$$\begin{cases} -\frac{z}{2}F'(z) - F''(z) - \frac{1}{\alpha^2}\left(F^4(z)\right)'' = 0 \\ F(-\infty) = T_{\text{int}}^c \\ F(\infty) = T_\infty. \end{cases} \tag{E.111}$$

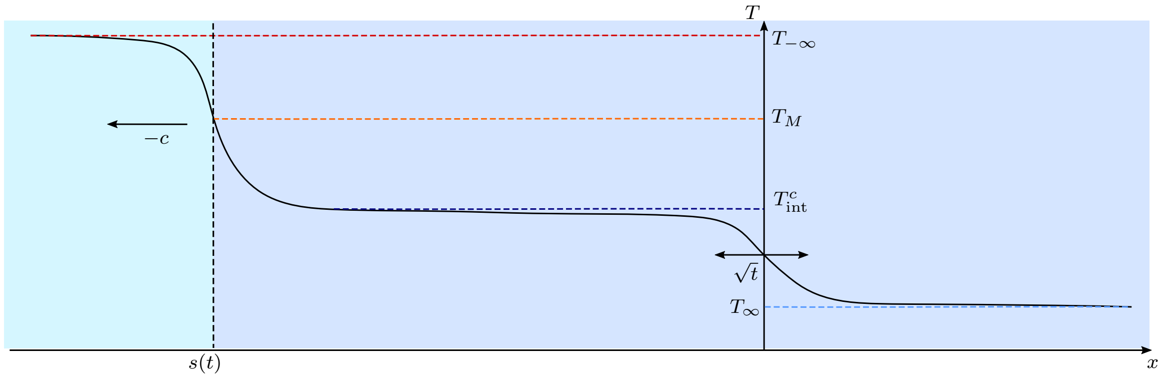

Figure E.2: Illustration of the expected profile as $t \to \infty$.

Remark. The self-similar profile F and equation (E.111) can be expected due to the diffusion approximation of the radiative transfer equation. Indeed, let us define $T_2(x,t) = F\left(\frac{x}{\sqrt{t}}\right) := F(z)$ for $x > -ct$ as $t \to \infty$. Then, using the Hölder regularity of T_2 we compute for the radiation term

$$\begin{aligned} &F\left(\frac{x}{\sqrt{t}}\right)^4 - \int_{-ct}^{\infty} \frac{\alpha E_1(\alpha(\eta - x))}{2} F\left(\frac{\eta}{\sqrt{t}}\right)^4 d\eta \\ &=F(z)^4 - \int_{-ct}^{\infty} \frac{\alpha E_1(\alpha(\eta - \sqrt{t}z))}{2} F\left(z + \frac{\eta - \sqrt{t}z}{\sqrt{t}}\right)^4 d\eta \\ &=F(z)^4 - \int_{-ct-z\sqrt{t}}^{\infty} \frac{\alpha E_1(\alpha\eta)}{2}\left[F^4(z) + \partial_z F^4(z)\frac{\eta}{\sqrt{t}} + \frac{\partial_z^2 F^4(z)}{2}\frac{\eta^2}{t} + \mathcal{O}\left(\frac{|\eta|^{2+\delta}}{t^{1+\delta/2}}\right)\right] \\ &=F^4(z)\int_{ct+z\sqrt{t}}^{\infty} \frac{\alpha E_1(\alpha\eta)}{2} d\eta - \frac{\partial_z F^4(z)}{\sqrt{t}} \int_{ct+z\sqrt{t}}^{\infty} \frac{\alpha E_1(\alpha\eta)}{2}\eta d\eta \\ &\quad + \frac{\partial_z^2 F^4(z)}{2t}\int_{-ct-z\sqrt{t}}^{\infty} \frac{\alpha E_1(\alpha\eta)}{2}\eta^2 d\eta + \mathcal{O}\left(\frac{\int_{-ct-z\sqrt{t}}^{\infty} \frac{\alpha E_1(\alpha\eta)}{2}|\eta|^{2+\delta} d\eta}{t^{1+\delta/2}}\right). \end{aligned}$$

Using that

$$t\int_{\alpha(ct+z\sqrt{t})}^{\infty} E(\eta)d\eta \sim te^{-\alpha(ct+z\sqrt{t})} \underset{t\to\infty}{\longrightarrow} 0, \quad \int_{-\infty}^{\infty} E(\eta)|\eta|^{2+\delta}d\eta < \infty \quad \text{and}$$

$$\sqrt{t}\int_{\alpha(ct+z\sqrt{t})}^{\infty} E(\eta)\eta d\eta \sim \sqrt{t}(\alpha(ct+z\sqrt{t})+1)e^{-\alpha(ct+z\sqrt{t})} \underset{t\to\infty}{\longrightarrow} 0,$$

we conclude multiplying by t and letting $t \to \infty$ that

$$t\left(F\left(\frac{x}{\sqrt{t}}\right)^4 - \int_0^{\infty} E(\eta - x)F\left(\frac{\eta}{\sqrt{t}}\right)^4 d\eta\right) \underset{t\to\infty}{\longrightarrow} \frac{\partial_z^2 F^4(z)}{2\alpha^2}\int_{-\infty}^{\infty} E(\eta)\eta^2 d\eta = \frac{1}{\alpha^2}\partial_z^2 F^4(z).$$

Finally, we recover (E.111) observing that $\partial_t F\left(\frac{x}{\sqrt{t}}\right) = -\frac{z}{t}F'(z)$ and $\partial_x^2 F\left(\frac{x}{\sqrt{t}}\right) = \frac{1}{t}F''(z)$.

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